**I-INTERNAL – FUZZY SETS AND ITS APPLICATIONS**

**KEY ANSWER**

**PART-A**

**1.Crisp set**

**2.disjoint set**

**3.normal**

**4.1-A(x)**

**5. c(a)≥ $c(b)$**

**6.a**

**7. 1**

**8. cardinality**

**9. countable set**

**10. family of sets**

**11. logic operation**

**12. threshold**

**13. relation**

**14. X**

**15. infinite**

**16. Archimedean t-norm**

**17. equilibrium**

**18. a**

**19. nested crisp sets**

**20. support**

**PART-B**

**21.a) Fuzzy logic:**

*Logic* is the study of the methods and principles of *reasoning* in all its possible forms. Classical logic deals with *propositions* that are required to be either *true* or *false*. Each proposition has its opposite, which is usually called a *negation* of the proposition. A proposition and its negation are required to assume opposite truth values.

**Logic formulas are then defined recursively as follows:**

1. if $v$ denotes a logic variable, then $v$ and $\overline{v}$ are logic formulas;
2. if $a$ and $b$ denote logic formulas, then $a \wedge b$ and $a \vee b$ are also logic formulas;
3. the only logic formulas are those defined by the previous two rules.

Various forms of tautologies can be used for making deductive inferences. They are referred to as *inference rules*. Examples of some tautologies frequently used as inference rules are

$$(a \wedge (a \Rightarrow b)) \Rightarrow b \ (\textit{modus ponens}),$$

$$(\overline{b} \wedge (a \Rightarrow b)) \Rightarrow \overline{a} \ (\textit{modus tollens}),$$

$$((a \Rightarrow b) \wedge (b \Rightarrow c)) \Rightarrow (a \Rightarrow c) \ (\textit{hypothetical syllogism}).$$

**b)**

**Theorem 3.2.** Every fuzzy complement has at most one equilibrium.

***Proof:*** Let $c$ be an arbitrary fuzzy complement. An equilibrium of $c$ is a solution of the equation

$$c(a) - a = 0,$$

where $a \in [0, 1]$. We can demonstrate that any equation $c(a) - a = b$, where $b$ is a real constant, must have at most one solution, thus proving the theorem. In order to do so, we assume that $a_1$ and $a_2$ are two different solutions of the equation $c(a) - a = b$ such that $a_1 < a_2$. Then, since $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$, we get

$$c(a_1) - a_1 = c(a_2) - a_2. \tag{3.7}$$

However, because $c$ is monotonic nonincreasing (by Axiom c2), $c(a_1) \geq c(a_2)$ and, since $a_1 < a_2$,

$$c(a_1) - a_1 > c(a_2) - a_2.$$

This inequality contradicts (3.7), thus demonstrating that the equation must have at most one solution. ■

22.a.Proof:

The equilibrium $e_c$ of a fuzzy complement c is the solution of the equation c(a)-a=0 where a∈ $[0,1]$.

The demonstrates the necessary existences of an equillibrium value for a continuous function.

Uniqeness:

Let c(a)-a=b with a1<a2.

Since c is monotonoic non increasing c(a1)≥ c(a2), atmost one solution of equation c(a)-a=b with a1<a2 the equilibrium of a fuzzy complement is unique.

b. Proof:

Assume A is convex and let $\alpha = A(x1) \leq A(x2)$.

By the convexity of A ,then

A$[\lambda x1 + (1-\boldsymbol{\lambda})\boldsymbol{x2}] \geq \boldsymbol{\alpha} = \boldsymbol{A}(\boldsymbol{x1})$

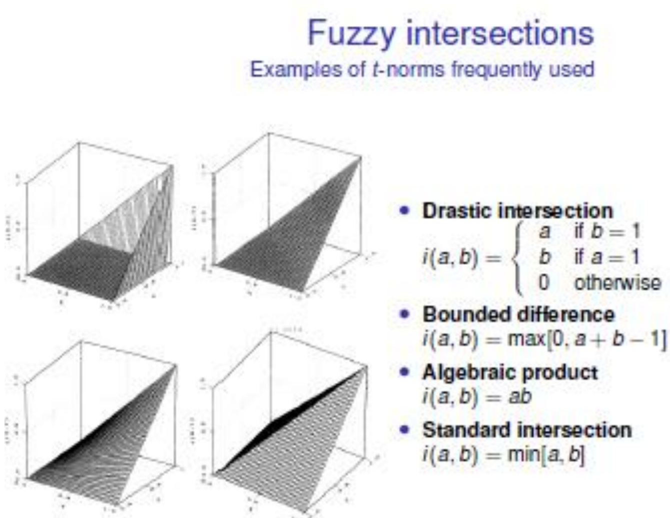
A$[\lambda x1 + (1-\boldsymbol{\lambda})\boldsymbol{x2}] \geq \boldsymbol{min}[\boldsymbol{A}(\boldsymbol{x1}), \boldsymbol{A}(\boldsymbol{x2})]$

**Conversly,**

A$[\lambda x1 + (1-\boldsymbol{\lambda})\boldsymbol{x2}] \geq \boldsymbol{min}[\boldsymbol{A}(\boldsymbol{x1}), \boldsymbol{A}(\boldsymbol{x2})]$**=**$\boldsymbol{\alpha}$

A is convex.

23. a) Fuzzy intersection:



- $i_{\min}(a, b) \leq \max(0, a + b - 1) \leq ab \leq \min(a, b)$.
- For all $a, b \in [0, 1]$, $\quad i_{\min}(a, b) \leq i(a, b) \leq \min[a, b]$.

The membership function $\mu_{\tilde{C}}(x)$ of the *intersection* $\tilde{C} = \tilde{A} \cap \tilde{B}$ is pointwise defined by

$$\mu_{\tilde{C}}(x) = \min\,\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\}, \quad x \in X$$

b. Proof:

If a= $e_c$ , then by definition of equilibrium c(a)=a.

If $d_{a=e_c}$ , then c($d_{a)-d_a}$=0

a= $d_{a=e_c}$. Henc e the equilibrium of any fuzzy complement is its own dual point.

**II- INTERNAL – FUZZY SETS AND ITS APPLICATIONS**

**KEY ANSWER**

**PART-A**

1.degree of association

2. fuzzy preorder relation

3. infinite universe

4.characteristic function

5. associative

6. real number in [0,1]

7. product space

8. crisp set

9. i(a,b)=min(a,b)

10. X xY

11. degree

12. parameters

13. cylindrical extension

14.max & min

15. i(a,b) $\leq$ min(a,b)

16. presence

17. reasoning

18. crisp

19. u

20. u(a,b)=max(a,b)

**PART-B**

**21.a proof:**

Let us consider associative union

u(u(a,b),c)= u(a,u(b,c)) we have b=0 & c=0

u(u(a,0),0)=u(a,u(0,0)

Then by boundary condition, u(u(a,0),0)=u(a,0)=a

Monotonicity we have ,

u(a,0)$\leq u(a,b)$

a$\leq u(a,b)$-------- 1

by commutative property, u(a,b)=u(b,a) we have b$\leq u(a,b)$------ 2

from 1 & 2 we get, u(a,b) $\geq \max(a,b)$ for all a,b$\in [0,1]$

b.i) if a or b=0

ie) {1,b} or {1,a}

ii) if a=b

$\lim_{w\to\infty} \min[1, (a^w + b^w)^{1/w}$ ]= max(a,b)

If a$\neq$ b then we take Q = $(a^w + b^w)^{1/w}$ and takeing log on both sides we have,

log Q= log $(a^w + b^w)^{1/w}$

apply L' Hospital rule,

$\lim_{w\to\infty} \log Q = \lim_{w\to\infty} (a^w + b^w)^{1/w}$ =log b

Thus $\lim_{w\to\infty} \min[1, (a^w + b^w)^{1/w}]$= max(a,b)

22. a. Compositions of Fuzzy Relations

Fuzzy relations in different product spaces can be combined with each other by the operation "composition." Different versions of "composition" have been suggested, which differ in their results and also with respect to their mathematical properties. The max-min composition has become the best known and the most frequently used one. However, often the so-called max-product or max-average compositions lead to results that are more appealing.

*Max-min composition*: Let $\tilde{R}_1(x, y), (x, y) \in X \times Y$ and $\tilde{R}_2(y, z), (y, z) \in Y \times Z$ be two fuzzy relations. The max-min composition $\tilde{R}_1$ max-min $\tilde{R}_2$ is then the fuzzy set

$$\tilde{R}_1 \circ \tilde{R}_2 = \{[(x, z), \max_y \{\min \{\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z)\}\}] \mid x \in X, y \in Y, z \in Z\}$$

$\mu_{\tilde{R}_1 \circ \tilde{R}_2}$ is again the membership function of a fuzzy relation on fuzzy sets

b) Max-product composition:

Let $\tilde{R}_1$ and $\tilde{R}_2$ be defined as in definition 6–7. The *max-* composition* of $\tilde{R}_1$ and $\tilde{R}_2$ is then defined as

$$\tilde{R}_1 \overset{\circ}{*} \tilde{R}_2 = \{[(x, z), \max_y (\mu_{\tilde{R}_1}(x, y) * \mu_{\tilde{R}_2}(y, z))] \mid x \in X, y \in Y, z \in Z\}$$

If * is an associative operation that is monotonically nondecreasing in each argument, then the max-* composition corresponds essentially to the max-min composition.

23.a) Charecteristic components of symmetric, reflexive and transitive relation:

## Reflexivity

Let $\tilde{R}$ be a fuzzy relation in $X \times X$.

1. $\tilde{R}$ is called *reflexive* [Zadeh 1971] if

$$\mu_{\tilde{R}}(x, x) = 1 \quad \forall x \in X$$

2. $\tilde{R}$ is called *ε-reflective* [Yeh 1975] if

$$\mu_{\tilde{R}}(x, x) \geqslant \varepsilon \quad \forall x \in X$$

3. $\tilde{R}$ is called *weakly reflexive* [Yeh 1975] if

$$\left.\begin{array}{l}\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x) \\ \mu_{\tilde{R}}(y, x) \leq \mu_{\tilde{R}}(x, x)\end{array}\right\} \forall x, y \in X.$$

### *Example*

Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.
The following relation "$y$ is close to $x$" is reflexive:

| $\tilde{R}$: | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | 1 | 0 | .2 | .3 |
| $x_2$ | 0 | 1 | .1 | 1 |
| $x_3$ | .2 | .7 | 1 | .4 |
| $x_3$ | 0 | 1 | .4 | 1 |

## Symmetry

A fuzzy relation $\tilde{R}$ is called *symmetric* if $\tilde{R}(x, y) = \tilde{R}(y, x)\ \forall x, y \in X$.

A relation is called *antisymmetric* if for

$$x \neq y \quad \left.\begin{array}{ll}\text{either} & \mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x) \\ \text{or} & \mu_{\tilde{R}}(x, y) = \mu_{\tilde{R}}(y, x) = 0\end{array}\right\} \forall x, y \in X$$

## Transitivity

A fuzzy relation $\tilde{R}$ is called (max-min) *transitive* if

$$\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$$

# FUZZY MEASURES

b)

Given a universal set $X$ and a nonempty family $\mathcal{C}$ of subsets of $X$, a ***fuzzy measure*** on $\langle X, \mathcal{C}\rangle$ is a function

$$g : \mathcal{C} \rightarrow [0, 1]$$

that satisfies the following requirements:

(g1) $g(\varnothing) = 0$ and $g(X) = 1$ (*boundary requirements*);
(g2) for all $A, B \in \mathcal{C}$, if $A \subseteq B$, then $g(A) \leq g(B)$ (*monotonicity*);
(g3) for any increasing sequence $A_1 \subset A_2 \subset \ldots$ in $\mathcal{C}$, if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$, then

$$\lim_{i \to \infty} g(A_i) = g\left(\bigcup_{i=1}^{\infty} A_i\right)$$

(*continuity from below*);

(g4) for any decreasing sequence $A_1 \supset A_2 \supset \ldots$ in $\mathcal{C}$, if $\bigcap_{i=1}^{\infty} A_i \in \mathcal{C}$, then

$$\lim_{i \to \infty} g(A_i) = g\left(\bigcap_{i=1}^{\infty} A_i\right)$$

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(15MMU604B)

**KARPAGAM ACADEMY OF HIGHER EDUCATION**

**Coimbatore- 641 021**

**DEPARTMENT OF MATHEMATICS**

**Sixth semester**

**Model Examination - Mar'18**

**Fuzzy sets and its applications**

**Class: III B.Sc Mathematics-A Time : 3 Hours**

**Date: 22.03.2018(FN) Max. Marks : 60**

**PART – A(20 × 1 = 20 marks)**

**ANSWER ALL THE QUESTIONS**

1. The set of all levels $\alpha \in [0,1]$ , the distinct $\alpha -$ cuts of a fuzzy set is called a_____
   (a) Level set (b) crisp set
   (c) fuzzy set (d) proper subset
2. Any two sets that have no common members are called___
   (a) disjoint set (b) joint set
   (c) subset (d) proper subset
3. A fuzzy set A is called ___ when h(a) = 1
   (a) normal (b) subnormal
   (c) convex (d) membership function
4. A fuzzy operation $\overline{A(x)} =$ ____
   (a) 1- $A(x)$ (b) $A(x)$
   (c) $A(x) - 1$ (d) 1
5. If a ≤ b, for all $a, b \in [0,1]$ then ___
   (a) $c(a) \geq c(b)$ (b) $c(a) > c(b)$
   (c) $c(a) \leq c(b)$ (d) $c(a) < c(b)$
6. If c is involutive then $c\big(c(a)\big) =$ ___
   (a) a (b) $c(a)$
   (c) b (d) $c\overline{(a)}$
7. If c is a fuzzy complement then $c(0) =$____
   (a) 0 (b) 1
   (c) c (d) $c(\overline{0})$
8. The number of members of a finite set A is ______
   (a) cardinality (b) index
   (c) union (d) intersection
9. A set whose members can be labeled by the positive integers is called _____set
   (a) countable set (b) uncountable set (c) relations (d) universal set
10. A set whose elements are themselves sets is called ___
   (a) family of sets (b) indexed set (c) involutive (d) Boolean
11. Logic functions of one or two variables are called ____
   (a) logic operation (b) logic variable
   (c) Boolean (d) relation
12. If $a \in [0,1]$ $and$ $t \in [0,1)$ ,$c(a) = \begin{cases} 1 \; for \; a \leq t \\ 0 \; for \; a > t \end{cases}$ then t is _______
   (a) threshold (b) characteristic function (c) membership function (d) logic function
13. Subsets of cartesian products are called_____
   (a) relation (b) function
   (c) nested family (d) denumerable
14. The property $A \cup \bar{A} =$ _______
   (a) A (b) X (c) $\emptyset$ (d) $\overline{A}$
15. Every uncountable set is ________
   (a) finite (b) infinite (c) disjoint (d) joint
16. A continuous t-norm that satisfies subidempotency is called _______
   (a) Archimedean t-norm (b) idempotent t-norm
   (c)monotonic (d) boundary
17. Elements of X , $A(x) = \overline{A(x)}$ are called ___

(a) equilibrium (b) operator
(c) membership function (d) function

18. The boundary condition $i(a,1)$ =______
(a) a (b) $i(1,a)$ (c) 1
(d) $i(1)$

19. All $\alpha -$ cuts and all strong $\alpha -$ cuts of any fuzzy set form two distinct families of ____ sets.
(a) nested crisp sets (b) crisp sets
(c) fuzzy sets (d) crisp sets

20. The ___ of a fuzzy set A within a universal set x is the crisp set that contains all the elements of X that have nonzero membership grades in A.
(a) Support (b) nested
(c) subnormal (d) core

**PART – B (5× 8 = 40 marks)**

**ANSWER ALL THE QUESTIONS**

21. (a) Explain operations of fuzzy sets

**(OR)**

(b) Prove that every fuzzy complement has at most one equilibrium

22. a) Show that $u(a,b) \geq \max(a,b)$ if for all $a,b \in [0,1]$

**(OR)**

b) Prove that $u(a,b) = \max(a,b)$ is the only continuous and idempotent fuzzy set union.

23. a) Compute $\tilde{R}\circ\tilde{R}$ for the fuzzy relation $\tilde{R}$ is defined as

| | $X_1$ | $X_2$ | $X_3$ | X4 |
|---|---|---|---|---|
| $X_1$ | 0.2 | 1 | 0.4 | 0.4 |
| $X_2$ | 0 | 0.6 | 0.3 | 0 |
| $X_3$ | 0 | 1 | 0.3 | 0 |
| $X_4$ | 0.1 | 1 | 1 | 0.1 |

**(OR)**

b) Compute i) min-max composition,

ii) max-prod composition

iii) max-av composition of $\widetilde{R_1}$ and $\widetilde{R_2}$ if

$\widetilde{R_1}$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ |
|---|---|---|---|---|---|
| $X_1$ | 0.1 | 0.2 | 0 | 1 | 0.7 |
| $X_2$ | 0.3 | 0.5 | 0 | 0.2 | 1 |
| X3 | 0.8 | 0 | 1 | 0.4 | 0.3 |

$\widetilde{R_2}$ :

| | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ |
|---|---|---|---|---|
| $Y_1$ | 0.9 | 0 | 0.3 | 0.4 |
| $Y_2$ | 0.2 | 1 | 0.8 | 0 |
| $Y_3$ | 0.8 | 0 | 0.7 | 1 |
| $Y_4$ | 0.4 | 0.2 | 0.3 | 0 |
| $Y_5$ | 0 | 1 | 0 | 0.8 |

24. a) Prove that a belief measure Bel on a finite power set P(x) is a probability measure if and only if its basic assignment m is given by m({x})=Bel ({x}) and m(A) =0 for all subsets of x that are not singletons.

**(OR)**

b) Derive Dempster's rule of combination

25. a) Describe about fuzzy decision making.

**(OR)**

b) Describe about individual decision making with example

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**KARPAGAM ACADEMY OF HIGHER EDUCATION**
**Coimbatore- 641 021**
**DEPARTMENT OF MATHEMATICS**
**sixth semester**
**I Internal Test - Jan'18**
**Fuzzy sets and its applications**

**Class: III B.Sc Mathematics** **Time : 2 Hours**
**Date: 20.01.2018 FN** **Max. Marks : 50 Marks**

**PART – A(20 × 1 = 20 marks)**
**ANSWER ALL THE QUESTIONS**

1. The set of all levels $\alpha \in [0,1]$ , the distinct $\alpha -$ cuts of a fuzzy set is called a____
   (a) Level set (b) crisp set
   (c) fuzzy set (d) proper subset
2. Any two sets that have no common members are called___
   (a) disjoint set (b) joint set
   (c) subset (d) proper subset
3. A fuzzy set A is called ___ when h(a) = 1
   (a) normal (b) subnormal
   (c) convex (d) membership function
4. A fuzzy operation $\overline{A(x)} =$ ____
   (a) 1- $A(x)$ (b) $A(x)$ (c) $A(x) - 1$ (d) 1
5. If a ≤ b, for all $a, b \in [0,1]$ then ___
   (a) $c(a) \geq c(b)$ (b) $c(a) > c(b)$
   (c) $c(a) \leq c(b)$ (d) $c(a) < c(b)$
6. If c is involutive then $c\big(c(a)\big) =$ ___
   (a) a (b) $c(a)$ (c) b (d) $c\overline{(a)}$
7. If c is a fuzzy complement then $c(0) =$____
   (a) 0 (b) 1 (c) c (d) $c(\overline{0})$
8. The number of members of a finite set A is _____
   (a) cardinality (b) index (c) union (d) intersection
9. A set whose members can be labeled by the positive integers is called _____set
   (a) countable set (b) uncountable set
   (c) relations (d) universal set
10. A set whose elements are themselves sets is called ___
    (a) family of sets (b) indexed set
    (c) involutive (d) Boolean
11. Logic functions of one or two variables are called ____
    (a) logic operation (b) logic variable
    (c) Boolean (d) relation
12. if $a \in [0,1]$ *and* $t \in [0,1)$ ,$c(a) = \begin{cases} 1 \; for \; a \leq t \\ 0 \; for \; a > t \end{cases}$ then t is _______
    (a) threshold (b) characteristic function
    (c) membership function (d) logic function
13. Subsets of cartesian products are called_____
    (a) relation (b) function
    (c) nested family (d) denumerable
14. The property $A \cup \bar{A} =$ _______
    (a) A (b) X (c) ∅ (d) $\overline{A}$
15. Every uncountable set is ________
    (a) finite (b) infinite (c) disjoint (d) joint
16. A continuous t-norm that satisfies subidempotency is called _______
    (a) Archimedean t-norm (b) idempotent t-norm
    (c)monotonic (d) boundary
17. Elements of X , $A(x) = \overline{A(x)}$ are called ___
    (a) equilibrium (b) operator
    (c) membership function (d) function
18. The boundary condition $i(a, 1) =$______
    (a) a (b) $i(1, a)$ (c) 1 (d) $i(1)$

19. All $\alpha -$ cuts and all strong $\alpha -$ cuts of any fuzzy set form two distinct families of ____ sets.
   (a) nested crisp sets (b) crisp sets
   (c) fuzzy sets (d) crisp sets
20. The ___ of a fuzzy set A within a universal set x is the crisp set that contains all the elements of X that have nonzero membership grades in A.
   (a) Support (b) nested (c) subnormal (d) core

PART – B **(3 × 10 = 30 marks)**

ANSWER ALL THE QUESTIONS

21. (a) Write briefly about on the fuzzy logic

(or)

(b) Prove that every fuzzy complement has atmost one equilibrium.

22. (a) If ‘c’ is a continuous fuzzy complement then prove that ‘c’ has a unique equilibrium.

(or)

(b) Prove that a fuzzy set A on $\mathcal{R}$ is convex if and only if $A\big(\lambda x_1 + (1-\lambda x_2)\big) \geq \min[A(x_1), A(x_2)]$

23. (a) Prove that the standard fuzzy intersection is the only idempotent t - norm.

(or)

(b) If a complement ‘c’ has an equilibrium $e_c$, then prove that $d_{e_c'} = e_c$

**UNIT-IV**

**SYLLABUS**

Fuzzy measures, general discussion, belief and plausibility measures, probability measures, possibility and necessity measures.

*Fuzzy measure theory*, which is now well developed [Wang and Klir, 1992], is not of primary interest in this text. However, we need to introduce the concept of a fuzzy measure for at least two reasons. First, the concept will provide us with a broad framework within which it is convenient to introduce and examine possibility theory, a theory that is closely connected with fuzzy set theory and plays an important role in some of its applications. Second, it will allow us to explicate differences between fuzzy set theory and probability theory.

# FUZZY MEASURES

## Definition

Given a universal set $X$ and a nonempty family $\mathcal{C}$ of subsets of $X$, a *fuzzy measure* on $\langle X, \mathcal{C}\rangle$ is a function

$$g : \mathcal{C} \to [0, 1]$$

that satisfies the following requirements:

(g1) $g(\varnothing) = 0$ and $g(X) = 1$ (*boundary requirements*);
(g2) for all $A, B \in \mathcal{C}$, if $A \subseteq B$, then $g(A) \le g(B)$ (*monotonicity*);
(g3) for any increasing sequence $A_1 \subset A_2 \subset \ldots$ in $\mathcal{C}$, if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$, then

$$\lim_{i\to\infty} g(A_i) = g\left(\bigcup_{i=1}^{\infty} A_i\right)$$

(*continuity from below*);
(g4) for any decreasing sequence $A_1 \supset A_2 \supset \ldots$ in $\mathcal{C}$, if $\bigcap_{i=1}^{\infty} A_i \in \mathcal{C}$, then

$$\lim_{i\to\infty} g(A_i) = g\left(\bigcap_{i=1}^{\infty} A_i\right)$$

The boundary requirements (g1) state that, regardless of our evidence, we always know that the element in question definitely does not belong to the empty set and definitely does belong to the universal set. The empty set, by definition, does not contain any element; hence, it cannot contain the element of our interest, either; the universal set, on the other hand, contains all elements under consideration in each particular context; therefore, it must contain our element as well.

Requirements (g3) and (g4) are clearly applicable only to an infinite universal set. They can therefore be disregarded when the universal set is finite. The requirements state that for every infinite sequence $A_1, A_2, \ldots$ of nested (monotonic) subsets of $X$ which converge to the set $A$, where $A = \bigcup_{i=1}^{\infty} A_i$ for increasing sequences and $A = \bigcap_{i=1}^{\infty} A_i$ for decreasing sequences, the sequence of numbers $g(A_1), g(A_2), \ldots$ must converge to the number $g(A)$. That is, $g$ is required to be a continuous function. Requirements (g3) and (g4) may also be viewed as requirements of *consistency*: calculation of $g(A)$ in two different ways, either as the limit of $g(A_i)$ for $i \to \infty$ or by application of the function $g$ to the limit of $A_i$ for $i \to \infty$, is required to yield the same value.

$$g(A \cap B) \leq \min[g(A), g(B)]$$

for any three sets $A, B, A \cap B \in \mathcal{C}$. Similarly, since both $A \subseteq A \cup B$ and $B \subseteq A \cup B$ for any two sets, the monotonicity of fuzzy measures implies that every fuzzy measure $g$ satisfies the inequality

$$g(A \cup B) \geq \max[g(A), g(B)]$$

for any three sets $A, B, A \cup B \in \mathcal{C}$.

**Belief and plausibility:**

*Evidence theory* is based on two dual nonadditive measures: belief measures and plausibility measures. Given a universal set $X$, assumed here to be finite, a *belief measure* is a function

$$\text{Bel} : \mathcal{P}(X) \to [0, 1]$$

such that $\text{Bel}(\varnothing) = 0$, $\text{Bel}(X) = 1$, and

$$\begin{aligned}\text{Bel}(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_j \text{Bel}(A_j) - \sum_{j<k} \text{Bel}(A_j \cap A_k) \\ + \ldots + (-1)^{n+1} \text{Bel}(A_1 \cap A_2 \cap \ldots \cap A_n)\end{aligned} \tag{7.3}$$

for all possible families of subsets of $X$. Due to the inequality (7.3), belief measures are called *superadditive*. When $X$ is infinite, function Bel is also required to be *continuous from above*.

For each $A \in \mathcal{P}(X)$, Bel $(A)$ is interpreted as the *degree of belief* (based on available evidence) that a given element of $X$ belongs to the set $A$. We may also view the subsets of $X$ as answers to a particular question. We assume that some of the answers are correct, but we do not know with full certainty which ones they are.

When the sets $A_1, A_2, \ldots, A_n$ in (7.3) are pair-wise disjoint, the inequality requires that the degree of belief associated with the union of the sets is not smaller than the sum of the degrees of belief pertaining to the individual sets. This basic property of belief measures is thus a weaker version of the additivity property of probability measures. This implies that probability measures are special cases of belief measures for which the equality in (7.3) is always satisfied.

We can easily show that (7.3) implies the monotonicity requirement (g2) of fuzzy measures. Let $A \subseteq B (A, B \in \mathcal{P}(X))$ and let $C = B - A$. Then, $A \cup C = B$ and $A \cap C = \varnothing$. Applying now $A$ and $C$ to (7.3) for $n = 2$, we obtain

$$\mathrm{Bel}(A \cup C) = \mathrm{Bel}(B) \geq \mathrm{Bel}(A) + \mathrm{Bel}(C) - \mathrm{Bel}(A \cap C).$$

Since $A \cap C = \varnothing$ and $\mathrm{Bel}(\varnothing) = 0$, we have

$$\mathrm{Bel}(B) \geq \mathrm{Bel}(A) + \mathrm{Bel}(C)$$

and, consequently, $\mathrm{Bel}(B) \geq \mathrm{Bel}(A)$.

Let $A_1 = A$ and $A_2 = \overline{A}$ in (7.3) for $n = 2$. Then, we can immediately derive the following fundamental property of belief measures:

$$\mathrm{Bel}(A) + \mathrm{Bel}(\overline{A}) \leq 1. \tag{7.4}$$

Associated with each belief measure is a *plausibility measure*, Pl, defined by the equation

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\overline{A}) \tag{7.5}$$

for all $A \in \mathcal{P}(X)$. Similarly,

$$\mathrm{Bel}(A) = 1 - \mathrm{Pl}(\overline{A}). \tag{7.6}$$

Belief measures and plausibility measures are therefore mutually dual. Plausibility measures, however, can also be defined independent of belief measures.

A plausibility measure is a function

$$\mathrm{Pl} : \mathcal{P}(X) \rightarrow [0, 1]$$

such that $\mathrm{Pl}(\varnothing) = 0$, $\mathrm{Pl}(X) = 1$, and

$$\begin{aligned} \mathrm{Pl}(A_1 \cap A_2 \cap \ldots \cap A_n) \leq & \sum_j \mathrm{Pl}(A_j) - \sum_{j<k} \mathrm{Pl}(A_j \cup A_k) \\ & + \ldots + (-1)^{n+1} \mathrm{Pl}(A_1 \cup A_2 \cup \ldots \cup A_n) \end{aligned} \tag{7.7}$$

for all possible families of subsets of $X$. Due to (7.7), plausibility measures are called *subadditive*. When $X$ is infinite, function Pl is also required to be *continuous from below*.

Let $n = 2, A_1 = A$, and $A_2 = \overline{A}$ in (7.7). Then, we immediately obtain the following basic inequality of plausibility measures:

$$\mathrm{Pl}(A) + \mathrm{Pl}(\overline{A}) \geq 1. \tag{7.8}$$

Belief and plausibility measures can conveniently be characterized by a function

$$m : \mathcal{P}(X) \rightarrow [0, 1],$$

such that $m(\varnothing) = 0$ and

$$\sum_{A \in \mathcal{P}(X)} m(A) = 1. \tag{7.9}$$

This function is called a *basic probability assignment*.

Upon careful examination of the definition of the basic assignment, we observe the following:

1. it is not required that $m(X) = 1$;
2. it is not required that $m(A) \leq m(B)$ when $A \subseteq B$; and
3. no relationship between $m(A)$ and $m(\overline{A})$ is required.

It follows from these observations that the basic assignments are not fuzzy measures. However, given a basic assignment $m$, a belief measure and a plausibility measure are uniquely determined for all set $A \in \mathcal{P}(X)$ by the formulas

$$\mathrm{Bel}(A) = \sum_{B \mid B \subseteq A} m(B), \tag{7.10}$$

$$\mathrm{Pl}(A) = \sum_{B \mid A \cap B \neq \varnothing} m(B). \tag{7.11}$$

The inverse procedure is also possible. Given, for example, a belief measure Bel, the corresponding basic probability assignment $m$ is determined for all $A \in \mathcal{P}(X)$ by the formula

$$m(A) = \sum_{B \mid B \subseteq A} (-1)^{|A-B|} \mathrm{Bel}(B). \tag{7.12}$$

The relationship between $m(A)$ and $\mathrm{Bel}(A)$, expressed by (7.10), has the following meaning: while $m(A)$ characterizes the degree of evidence or belief that the element in question belongs to the set $A$ alone (i.e., exactly to set $A$), $\mathrm{Bel}(A)$ represents the total evidence or belief that the element belongs to $A$ as well as to the various special subsets of $A$. The plausibility measure $\mathrm{Pl}(A)$, as defined by (7.11), has a different meaning: it represents not only the total evidence or belief that the element in question belongs to set $A$ or to any of its subsets, but also the additional evidence or belief associated with sets that overlap with $A$. Hence,

$$\mathrm{Pl}(A) \geq \mathrm{Bel}(A) \tag{7.13}$$

for all $A \in \mathcal{P}(X)$.

**Probability measures:**

A special branch of evidence theory that deals only with bodies of evidence whose focal elements are nested is referred to as *possibility theory*. Special counterparts of belief measures and plausibility measures in possibility theory are called *necessity measures* and *possibility measures*, respectively.

**Theorem 7.1.** Let a given finite body of evidence $(\mathcal{F}, m)$ be nested. Then, the associated belief and plausibility measures have the following properties for all $A, B \in \mathcal{P}(X)$:

(i) $\text{Bel}(A \cap B) = \min[\text{Bel}(A), \text{Bel}(B)]$;
(ii) $\text{Pl}(A \cup B) = \max[\text{Bel}(A), \text{Bel}(B)]$.

***Proof:*** (i) Since the focal elements in $\mathcal{F}$ are nested, they may be linearly ordered by the subset relationship. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$, and assume that $A_i \subset A_j$ whenever $i < j$. Consider now arbitrary subsets $A$ and $B$ of $X$. Let $i_1$ be the largest integer $i$ such that $A_i \subseteq A$, and let $i_2$ be the largest integer $i$ such that $A_i \subseteq B$. Then, $A_i \subseteq A$ and $A_i \subseteq B$ iff $i \leq i_1$ and $i \leq i_2$, respectively. Moreover, $A_i \subseteq A \cap B$ iff $i \leq \min(i_1, i_2)$. Hence,

$$\begin{aligned}\text{Bel}(A \cap B) &= \sum_{i=1}^{\min(i_1,i_2)} m(A_i) = \min\left[\sum_{i=1}^{i_1} m(A_i), \sum_{i=1}^{i_2} m(A_i)\right] \\ &= \min[\text{Bel}(A), \text{Bel}(B)].\end{aligned}$$

(ii) Assume that (i) holds. Then, by (7.5),

$$\begin{aligned}\text{Pl}(A \cup B) &= 1 - \text{Bel}(\overline{A \cup B}) = 1 - \text{Bel}(\overline{A} \cap \overline{B}) \\ &= 1 - \min[\text{Bel}(\overline{A}), \text{Bel}(\overline{B})] \\ &= \max[1 - \text{Bel}(\overline{A}), 1 - \text{Bel}(\overline{B})] \\ &= \max[\text{Pl}(A), \text{Pl}(B)]\end{aligned}$$

for all $A, B \in \mathcal{P}(X)$.

**Possibility and necessity measures:**

**Definition 7.2.** Let Nec denote a fuzzy measure on $(X, \mathcal{C})$. Then, Nec is called a *necessity measure* iff

$$\text{Nec}\left(\bigcap_{k \in K} A_k\right) = \inf_{k \in K} \text{Nec}(A_k) \tag{7.21}$$

**Theorem 7.3.** Every possibility measure Pos on a finite power set $\mathcal{P}(X)$ is uniquely determined by a *possibility distribution function*

$$r : X \to [0, 1]$$

via the formula

$$\text{Pos}(A) = \max_{x \in A} r(x) \tag{7.28}$$

for each $A \in \mathcal{P}(X)$.

***Proof:*** We prove the theorem by induction on the cardinality of set $A$. Let $|A| = 1$. Then, $A = \{x\}$, where $x \in X$, and (7.28) is trivially satisfied. Assume now that (7.28) is satisfied for $|A| = n - 1$, and let $A = \{x_1, x_2, \ldots, x_n\}$. Then, by (7.20),

$$\begin{aligned}
\text{Pos}(A) &= \max[\text{Pos}(\{x_1, x_2, \ldots, x_{n-1}\}), \text{Pos}(\{x_n\})] \\
&= \max[\max[\text{Pos}(\{x_1\}), \text{Pos}(\{x_2\}), \ldots, \text{Pos}(\{x_{n-1}\})], \text{Pos}(\{x_n\})] \\
&= \max[\text{Pos}(\{x_1\}), \text{Pos}(\{x_2\}), \ldots, \text{Pos}(\{x_n\})] \\
&= \max_{x \in A} r(x).
\end{aligned}$$

**POSSIBLE QUESTIONS**

**PART-B (5 x 8 =40 Marks)**

**Answer all the questions**

1. Prove that a belief measure Bel on a finite power set P(x) is a probability measure if and only if its basic assignment m is given by m({x})=Bel ({x}) and m(A) =0 for all subsets of x that are not singletons.
2. Derive Dempster’s rule of combination
3. Given a consonant body of evidence (f,m).Prove that the associated consonant belief and probability measure s process the following properties.
   i) Bel(A∩B)=min {Bel (A), Bel (B)}
   ii) Pl(A U B)=max {Pl(A),Pl(B)} for all A,B ∈ $\varphi(x)$.
4. Prove that every possibility measure POS on a finite power set P(X) is unequally determined by a possibility distribution function.
5. Discuss similarities and difference between probability theory and possibility theory.
6. Show that the function Bel(A)=$\sum_{B/B\subseteq A} m(B)$ for any given basic assignment m is a plausibility measure.
7. Explain possibility distribution for a fuzzy proposition by an examples.
8. Explain about belief and plausibility measures.

**KARPAGAM ACADEMY OF HIGHER EDUCATION**
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**Coimbatore – 641 021.**

# LECTURE PLAN
# DEPARTMENT OF MATHEMATICS

STAFF NAME: S.INDHU
SUBJECT NAME: FUZZY SETS AND ITS APPLICATIONS
SUB.CODE:15MMU604B

SEMESTER: VI CLASS: III B.SC MATHEMATICS-A

| S.No | Lecture Duration Period | Topics to be Covered | Support Material/Page Nos |
|---|---|---|---|
| | | **UNIT-I** | |
| 1 | 1 | Introduction on crisp sets | T1:ch1.1:pg 1-3 |
| 2 | 1 | Definitions on crisp sets | T1:ch1.1:pg 4-5 |
| 3 | 1 | Basic concepts on crisp set | T1:ch1.1:pg 5-6 |
| 4 | 1 | Continuation of basic concepts on crisp set | T1:ch1.1:pg 6-8 |
| 5 | 1 | Fundamental properties of crisp set | T1:ch1.2:pg 8 |
| 6 | 1 | Introduction to fuzzy sets | R1:ch2.4:pg 34 |
| 7 | 1 | Basic definitions on fuzzy sets | T1:ch1.3:pg 11 |
| 8 | 1 | Basic concepts on fuzzy sets | T1:ch1.4:pg 19-21 |
| 9 | 1 | Continuation of basic concepts of fuzzy sets | T1:ch1.4:pg 21-33 |
| 10 | 1 | Types of operations | T1:ch3.1:pg 50 |
| 11 | 1 | Fuzzy complements | T1:ch3.2:pg 51 |
| 12 | 1 | First characterization theorem of fuzzy complement | T1:ch3.2:pg 59-60 |
| 13 | 1 | Second charecterization theorem of fuzzy complements | T1:ch3.2:pg 60-61 |

| | | | |
|---|---|---|---|
| 14 | 1 | Introduction on fuzzy logic | T1:ch8.1:pg 212-215 |
| 15 | 1 | Continuation of basic concepts on fuzzy logic | T1:ch8.1:pg 215-217 |
| 16 | 1 | Continuation of basic concepts on fuzzy logic | T1:ch8.1:pg 217-220 |
| 17 | 1 | Fuzzy prepositions | T1:ch8.3:pg 220 |
| 18 | 1 | Fuzzy quantifiers | T1:ch8.4:pg 225 |
| 19 | 1 | Linguistic hedges | T1:ch8.5:pg 229-231 |
| 20 | 1 | Recapitulation and discussion on possible questions | |
| | **Total No of Hours Planned For Unit 1=20** | | |
| | | **UNIT-II** | |
| 1 | 1 | Operations on fuzzy sets | T1:ch3.4:pg 76 |
| 2 | 1 | Fuzzy unions | R2:ch2.3:pg 45 |
| 3 | 1 | Fuzzy union theorems | T1:ch3.4:pg 77 |
| 4 | 1 | Continuation of theorems on fuzzy union | T1:ch3.4:pg 78-79 |
| 5 | 1 | Continuation of theorems on fuzzy union | T1:ch3.4:pg 79-80 |
| 6 | 1 | Fuzzy intersections | R2:ch2.4:pg 50 |
| 7 | 1 | t-norms on fuzzy intersection | T1:ch3.3:pg 61-62 |
| 8 | 1 | Theorems on fuzzy intersection | R2:ch2.4:pg 50-52 |
| 9 | 1 | Fuzzy intersection characterization theorem of t-norms | T1:ch3.3:pg 68-69 |
| 10 | 1 | Continuation of fuzzy intersection characterization theorem of t-norms | T1:ch3.3:pg 69-70 |
| 11 | 1 | Further operations on fuzzy sets | R3:ch3.2:pg 28 |
| 12 | 1 | Combination of operations | T2:ch2.5:pg 58 |
| 13 | 1 | Continuation of theorems on combinations of operations | T1:ch3.5:pg 83-85 |
| 14 | 1 | Continuation of theorems on | T1:ch3.5:pg 85-86 |

| | | | |
|---|---|---|---|
| | | combinations of operations | |
| 15 | 1 | Continuation of theorems on combinations of operations | T1:ch3.5:pg 86-88 |
| 16 | 1 | Recapitulation and discussion of possible questions | |
| | **Total No of Hours Planned For Unit II=16** | | |
| | | **UNIT-III** | |
| 1 | 1 | Introduction on fuzzy relations | R3:ch6:pg 69 |
| 2 | 1 | Introduction on fuzzy graphs | R3:ch6.1:pg 69-70 |
| 3 | 1 | Fuzzy relations on sets and fuzzy sets | R3:ch6.1:pg 70-71 |
| 4 | 1 | Continuations of fuzzy relations on sets and fuzzy sets | R3:ch6.1:pg 71-73 |
| 5 | 1 | Composition of fuzzy relations | R3:ch6.1:pg 74 |
| 6 | 1 | Binary fuzzy relations | T1:ch5.3:pg 124-126 |
| 7 | 1 | Continuous of fuzzy relations on sets and fuzzy sets | T1:ch5.3:pg 126-128 |
| 8 | 1 | Binary relations on a single set | T1:ch5.4:pg 128-129 |
| 9 | 1 | Examples on binary relations on a single set | T1:ch5.4:pg 130-131 |
| 10 | 1 | Fuzzy equivalence relations | T1:ch5.5:pg 132-135 |
| 11 | 1 | Fuzzy compatibility relations | T1:ch5.6:pg 135-136 |
| 12 | 1 | Fuzzy ordering relations | T1:ch5.7:pg 136-137 |
| 13 | 1 | Properties of the min-max compositions | T1:ch6.1.2:pg 77-78 |
| 14 | 1 | Fuzzy graphs | R3:ch6.2:pg 82 |
| 15 | 1 | Special fuzzy graphs | R3:ch6.2:pg 85 |
| 16 | 1 | Recapitulation and discussion of possible questions | |
| | **Total No of Hours Planned For Unit III=16** | | |
| | | **UNIT-IV** | |

| | | | |
|---|---|---|---|
| 1 | 1 | Introduction on fuzzy measures | R2:ch4:pg 107 |
| 2 | 1 | Continuation of fuzzy measures | R2:ch4:pg 107-108 |
| 3 | 1 | Theorems on fuzzy measures | R2:ch4.1:pg 108-109 |
| 4 | 1 | Belief and plausibility of measures | R2:ch4.2:pg 109-110 |
| 5 | 1 | Continuation of belief and plausibility of measures | R2:ch4.3:pg 111-113 |
| 6 | 1 | Continuation of belief and plausibility of measures | R2:ch4.2:pg 113-114 |
| 7 | 1 | Continuation of belief and plausibility of measures | R2:ch4.2:pg 114-116 |
| 8 | 1 | Probability measures | R2:ch4.2:pg 118-120 |
| 9 | 1 | Continuation of probability measures | R2:ch4.3:pg 120-121 |
| 10 | 1 | Possibility and necessity measures | R2:ch4.4:pg 121-125 |
| 11 | 1 | Recapitulation and discussion of possible questions | |
| | **Total No of Hours Planned For Unit IV=11** | | |
| | | **UNIT-V** | |
| 1 | 1 | Fuzzy decision making | R1:ch9:pg 276 |
| 2 | 1 | Fuzzy general discussions | T1:ch15.1:pg 390 |
| 3 | 1 | Individual decision making | T1:ch15.1:pg 391-392 |
| 4 | 1 | Examples on individual decision making | T1:ch15.2:pg 393-395 |
| 5 | 1 | Fuzzy ranking methods | T1:ch15.6:pg 405 |
| 6 | 1 | Examples on fuzzy ranking methods | T1:ch15.7:pg 407-408 |
| 7 | 1 | Fuzzy linear programming | T1:ch15.7:pg 409-410 |
| 8 | 1 | Examples on fuzzy linear programming | T1:ch15.7:pg 410-411 |
| 9 | 1 | Discussion of previous ESE question paper | |

| | | | |
|---|---|---|---|
| 10 | 1 | Discussion of previous ESE question paper | |
| 11 | 1 | Discussion of previous ESE question paper | |
| 12 | 1 | Recapitulation and discussion of possible questions | |
| | **Total No of Hours Planned for unit V=12** | | |
| Total Planned Hours | **75** | | |

## TEXT BOOK

1. 1. George J. Klir and Bo Yuan, 1995.Fuzzy sets and fuzzy logic theory and applications, Prentice-Hall of India private limited, New Delhi.

## REFERENCES

1. Timothy J. Ross, 2000. Fuzzy logic with Engineering Applications, McGrawHill, Inc. New Delhi.

2. George J.Klir, Tina.A Folger, 2008. Fuzzy sets, uncertainty and information, Prentice Hall of

India Pvt Ltd, New Delhi.
3. H.J. Zimmermann, 2006. Fuzzy set theory and its applications, Second Edition, Springer New

**MODEL – FUZZY SETS AND ITS APPLICATIONS**

**KEY ANSWER**

**PART-A**

1.crisp set

2. disjoint set

3. normal

4. 1-A(x)

5.C(a) $\geq$c(b)

6. a

7. 1

8. cardinality

9. countable set

10. family of sets

11. logic operation

12. threshold

13. relation

14. X

15. infinite

16. Archimedean t-norm

17. equilibrium

18. a

19. nested crisp sets

20. support

**PART-B**

**21. a)**

## Operations on Classical Sets

Let A and B be two sets on the universe X. The union between the two sets, denoted A ∪ B, represents all those elements in the universe that reside in (or belong to) the set A, the set B, or both sets A and B. (This operation is also called the *logical or*; another form of the union is the *exclusive or* operation. The *exclusive or* will be described in Chapter 5.) The intersection of the two sets, denoted A ∩ B, represents all those elements in the universe X that simultaneously reside in (or belong to) both sets A and B. The complement of a set A, denoted $\overline{A}$, is defined as the collection of all elements in the universe that do not reside in the set A. The difference of a set A with respect to B, denoted A | B, is defined as the collection of all elements in the universe that reside in A and that do not reside in B simultaneously. These operations are shown below in set-theoretic terms.

*Union* $$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x)$$

*Intersection* $$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x)$$

*Complement* $$\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x)$$

Any fuzzy set $\underset{\sim}{A}$ defined on a universe X is a subset of that universe. Also by definition just as with classical sets, the membership value of any element $x$ in the null set ∅ is 0

b)

**Theorem 3.2.** Every fuzzy complement has at most one equilibrium.

***Proof:*** Let $c$ be an arbitrary fuzzy complement. An equilibrium of $c$ is a solution of the equation

$$c(a) - a = 0,$$

where $a \in [0, 1]$. We can demonstrate that any equation $c(a) - a = b$, where $b$ is a real constant, must have at most one solution, thus proving the theorem. In order to do so, we assume that $a_1$ and $a_2$ are two different solutions of the equation $c(a) - a = b$ such that $a_1 < a_2$. Then, since $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$, we get

$$c(a_1) - a_1 = c(a_2) - a_2. \quad (3.7)$$

However, because $c$ is monotonic nonincreasing (by Axiom c2), $c(a_1) \geq c(a_2)$ and, since $a_1 < a_2$,

$$c(a_1) - a_1 > c(a_2) - a_2.$$

This inequality contradicts (3.7), thus demonstrating that the equation must have at most one solution. ■

**22. a)**proof:

For fuzzy union associative condition is

u(u(a,0),0)=u(a,u(0,0))

since from the boundary condition is

u(u(a,0),0)=u(a,0)------ 1

assume u(a,0)=∝ which is contradiction.

Therefore u(a,0)=a.

Then monotonicity condition, u(a,0) $\leq$ u(a,b)

a≤u(a,b) and b≤ $u(b,a)$ we have

u(a,b) ≥max(a,b) , for a,b∈ $[0,1]$.

b) proof:

we know that u(a,b) ≥max(a,b) , for a,b∈ $[0,1]$

and similarly we have

u(a,b) ≤max(a,b) , for a,b∈ $[0,1]$

Thus u(a,b) = max(a,b).

23. a)

$$\tilde{R}_1 \circ \tilde{R}_2 = \{[(x, z), \max_y \{\min \{\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z)\}\}] \mid x \in X, y \in Y, z \in Z\}$$

Max{.2,0,.1}= .2

b)min-max composition:

.2  0  .3  .2

.4  0   .3  .2

.2  .4  .3  0

Max-prod composition:

4   7   8  .5

.2 5 .4 8

8 3 7 1

Max avg composition:

1 1 1 1

1 1 1 1

1 1 1 1

24. a)

Dempster's rule of combination:

The standard fuzzy evidence of the way is

$M_{1,2}$ (A)= $\sum_{B\cap C=A} M1(B).\frac{M2(C)}{1} - 1 - K\ for\ A \neq \emptyset. where\ K = \sum_{B\cap C\neq\emptyset} M1(B).M2(C), M1{,}2(\emptyset) = 0$ . This is referrd to the dempster rule of combination.

25.a) Fuzzy decision making:

The subject of decision making is, as the name suggests, the study of how decisions are actually made and how they can be made better or more successfully. That is, the field is concerned, in general, with both descriptive theories and normative theories. Much of the focus in developing the field has been in the area of management, in which the decision-making process is of key importance for functions such as inventory control, investment, personnel actions, new-product development, and allocation of resources, as well as many others. Decision making itself, however, is broadly defined to include any choice or selection of alternatives, and is therefore of importance in many fields in both the "soft" social sciences and the "hard" disciplines of natural sciences and engineering.

Making decisions is undoubtedly one of the most fundamental activities of human beings. We all are faced in our daily life with varieties of alternative actions available to us and, at least in some instances, we have to decide which of the available actions to take. The beginnings of decision making, as a subject of study, can be traced, presumably, to the late 18th century, when various studies were made in France regarding methods of election and social choice. Since these initial studies, decision making has evolved into a respectable and rich field of study. The current literature on decision making, based largely on theories and methods developed in this century, is enormous.

# INDIVIDUAL DECISION MAKING

b)

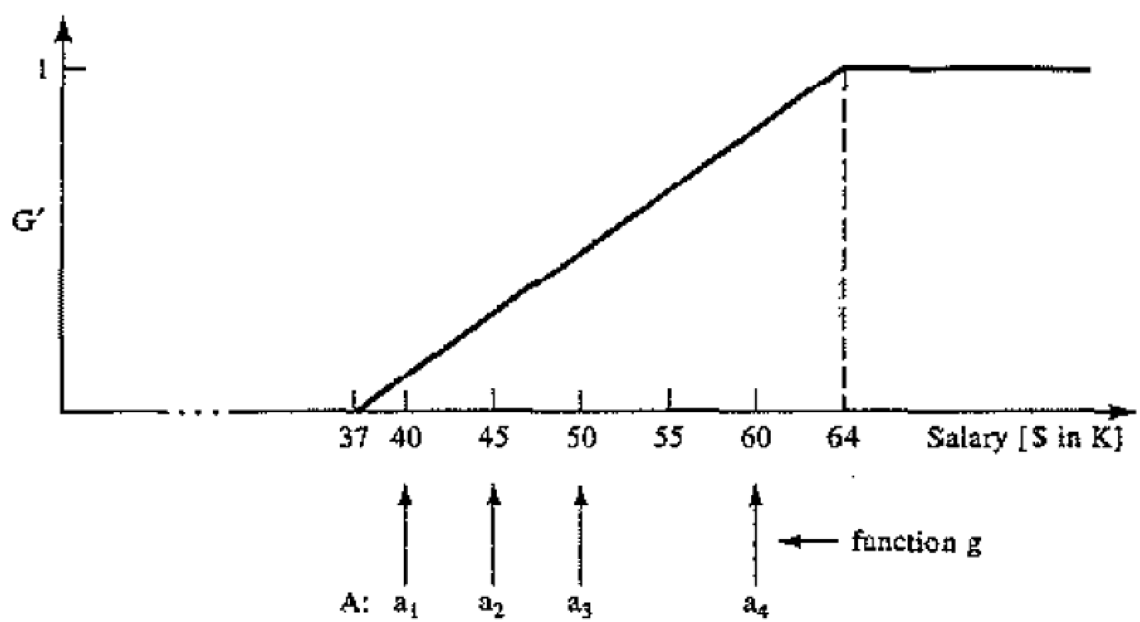
Fuzziness can be introduced into the existing models of decision models in various ways. In the first paper on fuzzy decision making, Bellman and Zadeh [1970] suggest a fuzzy model of decision making in which relevant goals and constraints are expressed in terms of fuzzy sets, and a decision is determined by an appropriate aggregation of these fuzzy sets. A decision situation in this model is characterized by the following components:

- a set $A$ of *possible actions*;
- a set of *goals* $G_i (i \in \mathbb{N}_n)$, each of which is expressed in terms of a fuzzy set defined on $A$;
- a set of *constraints* $C_j (j \in \mathbb{N}_m)$, each of which is also expressed by a fuzzy set defined on $A$.

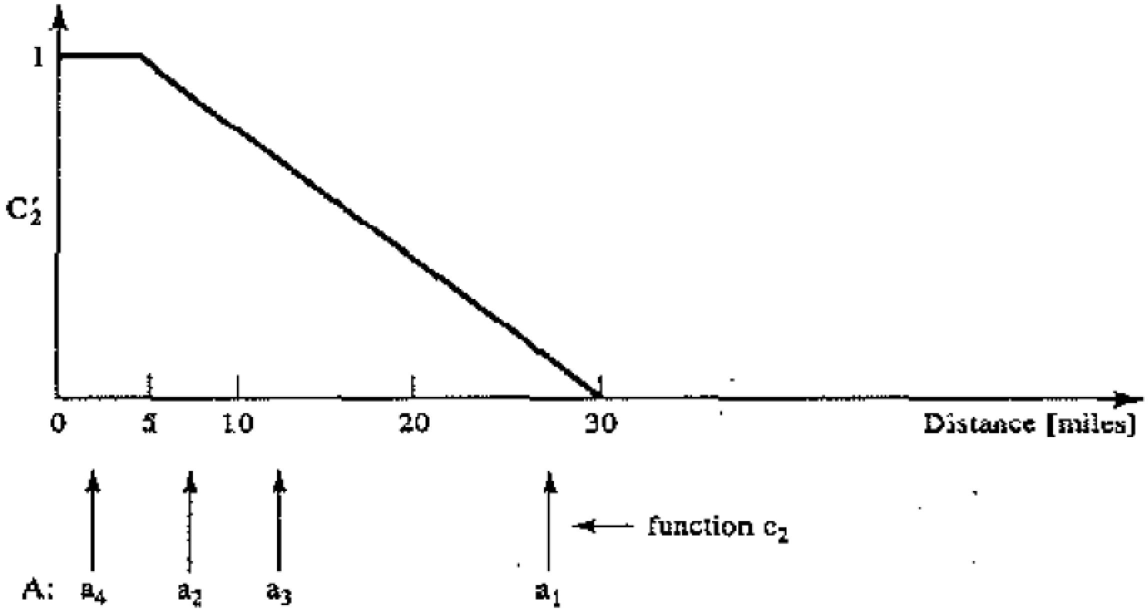
**Example:**

Suppose that an individual needs to decide which of four possible jobs, $a_1, a_2, a_3, a_4$, to choose. His or her goal is to choose a job that offers a high salary under the constraints that the job is interesting and within close driving distance. In this case, $A = \{a_1, a_2, a_3, a_4\}$, and the fuzzy sets involved represent the concepts of *high salary*, *interesting job*, and *close driving distance*. These concepts are highly subjective and context-dependent, and must be defined by the individual in a given context. The goal is expressed in monetary terms, independent of the jobs available. Hence, according to our notation, we denote the fuzzy set expressing the goal by $G'$. A possible definition of $G'$ is given in Fig. 15.1a, where we assume, for convenience, that the underlying universal set is $\mathbb{R}^+$. To express the goal in terms of set $A$, we need a function g: $A \to \mathbb{R}^+$, which assigns to each job the respective salary. Assume the following assignments:






(a) Goal $G'$: High salary.




**(b) Constraint $C_2'$: Close driving distance.**

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**Coimbatore –641 021**

**Subject: FUZZY SETS AND ITS APPLICATIONS** **Subject Code: 15MMU604B**

**Class : III - B.Sc. Mathematics-A** **Semester : VI**

# Unit I
# Crisp sets and fuzzy sets

**Part A (20x1=20 Marks)**
**(Question Nos. 1 to 20 Online Examinations)**

**Possible Questions**

| Question | Choice 1 | Choice 2 | Choice 3 | Choice 4 | Answer |
|---|---|---|---|---|---|
| The set of all levels α∈[0,1] , the distinct α- cuts of a fuzzy set is called a____ | Level set | crisp set | fuzzy set | proper subset | crisp set |
| Any two sets that have no common members are called___ | disjoint set | joint set | subset | proper subset | disjoint set |
| A fuzzy set A is called ___ when h(a) = 1 | normal | subnormal | convex | membership function | normal |
| A fuzzy operation $\overline{(A(x))}$ =_____ | 1- A(x) | A(x) | A(x)-1 | 1 | 1- A(x) |
| The________ of a path n>0 is the number of nodes contained in the path. | length | height | width | long | a) length |
| The ________ of the path is min{μ_G(x_(i,) x_(i+1))} for all nodes contained in the path. | length | length | strength | height | strength |
| A path is called a _______ if x0 = xn and n≥3 | path | cycle | strength | height | cycle |
| Two nodes that are joined by a path are called _______________ | connected nodes | unconnected | path | line | connected nodes |
| The fuzzy truth value of A’ with respect to A , denoted by | R | RT | RA/T | RT( A^'/A ) | RT( A^'/A ) |
| If f:R2 →R such that f(w1,w2)=w for the variables of product function | f(w1, w2) = w1 w2 | b) f(w1, w2)= w1, w2 | c) f(w1, w2)= f(w1 w2) | d) f(w1, w2)= f(w1 )f(w2) | f(w1, w2) = w1 w2 |

| | | | | | |
|---|---|---|---|---|---|
| The fuzzy measure assigns a value to each crisp set of the ………… | empty set | universal set | null set | super set | universal set |
| The sequence is said to be monotonic then ……… | $\lim_{i\to\infty} g(A_i) = g(\lim_{i\to\infty} A_i)$ | {0,0,0,0…} | ∞ | empty set | $\lim_{i\to\infty} g(A_i) = g(\lim_{i\to\infty} A_i)$ |
| The________ of a path n>0 is the number of nodes contained in the path. | length | height | width | long | length |
| The ________ of the path is min{$\mu\_G(x\_(i,) \ x\_(i+1))$} for all nodes contained in the path. | length | width | strength | height | strength |
| A path is called a _______ if $x0 = xn$ and $n\geq3$ | path | cycle | strength | height | cycle |
| Two nodes that are joined by a path are called _______________ | connected nodes | unconnected | path | line | connected nodes |
| What is the Methodology for fuzzy decision Making ---------- | Evaluate | Alternative | Evaluating Alternative | Determining | Evaluating Alternative |
| Multi attribute evaluation is carried out on the basis of ---------- | Equations | non-linear | linear | Polynomial | linear |
| Fuzzy concept is an example of ------ | Light | traffic | amber traffic light | Signal | amber traffic light |
| Individual Decision Making only ----------- no of peraons for taking decision | Many | Minimum | Single | few | Single |
| The crisp set is defined in such way to dichotomize the individuals in some given universe of discourse into ----------- | one group | two groups | three groups | five groups | two groups |
| To distinguish between fuzzy sets and classical sets,we refer to the latter as ----------- | empty set | crisp set | fuzzy logic | fuzzy complements | crisp set |
| Sets are denoted in this text by -------- letters | upper case | lower case | either upper or lower | both cases | upper case |
| Their members by ---------- letters | upper case | lower case | middle | both cases | lower case |
| The set that contains no members is called the----- | fuzzy set | empty set | crisp set | classical set | empty set |
| The characteristic function of a crisp set assigns a value of | 1 | 0 | 2 | either 1 or 0 | either 1 or 0 |
| Membership function of the-----fuzzy sets | 1 | 2 | 3 | 4 | 3 |
| A fuzzy set A is called normal when-------- | h(A)<1 | h(A)=1 | h(A)>1 | h(A)=0 | h(A)=1 |
| A subnormal of the fuzzy set | h(A)<1 | h(A)=1 | h(A)>1 | h(A)=0 | h(A)<1 |

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**Subject: FUZZY SETS AND ITS APPLICATIONS** **Subject Code: 15MMU604B**

**Class : III - B.Sc. Mathematics-A** **Semester : VI**

# Unit III
# Fuzzy Relations

**Part A (20x1=20 Marks)**
**(Question Nos. 1 to 20 Online Examinations)**

**Possible Questions**

| Question | Choice 1 | Choice 2 | Choice 3 | Choice 4 | Answer |
|---|---|---|---|---|---|
| A set whose members can be labeled by the positive integers is called _____set | countable set | uncountable set | relations | universal set | countable set |
| A set whose elements are themselves sets is called ___ | family of sets | indexed set | involutive | Boolean | family of sets |
| Logic functions of one or two variables are called ____ | logic operation | logic variable | Boolean | relation | logic operation |
| If a∈[0,1] and t∈[ 0,1) , c(a)={1 for a≤t, 0 for a>t} then t is _______ | threshold | characteristic function | membership function | logic function | threshold |
| Fuzzy relations are fuzzy subsets of _________ | map | perfect order | X x Y | X+Y | X x Y |
| Fuzzy relations is a ______ from fuzzy sets contained in the universal sets into the unit interval | map | order | linear | composition | map |
| Fuzzy relations are fuzzy sets in ________ | product space | linear space | binary space | composition | product space |
| The largest relation is called the ___________ of the projection relation. | product space | linear space | cylindrical extension | set | cylindrical extension |
| If A⊂A∪B and B ⊂ A∪B , we have Max[g(A),g(B)] is…………… | ≤g(A∪B) | ≥ g(A∪B) | none | < g(A∪B) | ≤g(A∪B) |
| If A⊂A∪B and B ⊂ A∪B , we have g(A∪B) is…………… | ≥Min[g(A),g(B)] | )=[g(A),g(B)] | ≤ Min[g(A),g(B)] | either b or c | ≤ Min[g(A),g(B)] |

| | | | | | |
|---|---|---|---|---|---|
| The fuzzy measure assigns a value to each________ of the universal set | empty set | universal set | crisp set | super set | crisp set |
| If A∈B and B∈B then A∪B∈B ,then the set B is called…. | superset field | field | Borel field | sunfield | Borel field |
| Fuzzy relations are fuzzy subsets of _________ | map | perfect order | X x Y | X+Y | X x Y |
| Fuzzy relations is a ______ from fuzzy sets contained in the universal sets into the unit interval | map | order | linear | composition | map |
| Fuzzy relations are fuzzy sets in ________ | product space | linear space | binary space | composition | product space |
| The largest relation is called the ___________ of the projection relation | product space | linear space | cylindrical extension | set | cylindrical extension |
| Truth of a Fuzzy Proposition is a matter of ---------- | degree | Power | order | no of elements | degree |
| Fuzzy LPP values of the -------------- of LPP | parameters | Types | constraints | conjunction | parameters |
| $\bar{A}(x)$ = ------------ | A( x) | 1 | U (x) | 1 - A( x) | 1 - A( x) |
| Every fuzzy complement has atmost ---------- | many equilibrium | one equilibrium | two equilibrium | equilibrium | |
| A crisp relation represents the | presence | association | grades | fuzzy relation | presence |
| Which can be represented by membershipgrades in fuzzy relation | degrees of association | commutative | degrees of associative | idempotent | degrees of association |
| The cartesian product of two crisp sets X and Y is denoted by | X x Y | X = Y | X or Y | X/Y | X x Y |
| A relation between two sets is called---------- | binary | ternary | quaternary | n-ary | binary |
| A fuzzy relation is a fuzzy set defined on--------of crisp sets. | n-tuples | cartesian product | closed intervals | n-dimension | cartesian product |
| A fuzzy relation represented by--------membership array. | n-dimensional | one dimensional | two dimensional | three dimensional | n-dimensional |
| An inverse to the projection is called------- | cylindric extension | cylindric closure | either b or c | cylindric | cylindric extension |
| The cylindric extension clearly produces the -------that is compatible with the projection. | smallest fuzzy relation | largest fuzzy relation | equal fuzzy relation | not equal the fuzzy relation | largest fuzzy relation |
| The resulting relation is usually called a-------- | cylindric extension | cylindric closure | either b or c | cylindric | cylindric extension |
| The rang of crisp binary relation R(X,Y) is denoted by------ | ran R(X,Y) | dom R(X,Y) | (X,Y) | ran(X,Y) | ran R(X,Y) |

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**Class : III - B.Sc. Mathematics-A** **Semester : VI**

# Unit II
# Fuzzy union,intersection and operations

**Part A (20x1=20 Marks)**
**(Question Nos. 1 to 20 Online Examinations)**

**Possible Questions**

| Question | Choice 1 | Choice 2 | Choice 3 | Choice 4 | Answer |
|---|---|---|---|---|---|
| If a ≤ b, for all a,b∈[0,1] then ___ | c(a)≥c(b) | c(a)>c(b) | c(a)≤c(b) | c(a)<c(b) | c(a)≥c(b) |
| If c is involutive then c(c(a) )=____ | a | c(a) | b | c((a) )¯ | a |
| If c is a fuzzy complement then c(0)=____ | 0 | 1 | c | c(0¯) | 1 |
| The number of members of a finite set A is ______ | cardinality | index | union | intersection | cardinality |
| A _____________ is a fuzzy relation μ_s(.) that is reflexive, symmetrical and max-min transitive. | similarity relation | order relation | path | node | similarity relation |
| A fuzzy relation that is (max-min) transitive and reflexive is called a _____________ | order relation | fuzzy preorder relation | relation | set | fuzzy preorder relation |
| A fuzzy relation that is (min-max) transitive,reflexive and antisymmetric is called a __________ | fuzzy order relation | order relation | relation | set | fuzzy order relation |
| If the relation is perfectly antisymmetrical, it is called a __________ | map | perfect fuzzy order relation | product space | linear space | perfect fuzzy order relation |
| The axiom one states that …………. | evidence | one | degree of evidence | monotonic | degree of evidence |
| The axiom three is clearly applicable only to an …………. | finite set | universal set | infinite set | infinite universal | infinite universal |

| | | | | | |
|---|---|---|---|---|---|
| If fuzzy measure is often defined more generally as a function g:B→[0,1] where BⅭp(x) is a | family of sunsets | Null set | Universal set | super set | family of sunsets |
| The sequence of numbers g(A1) ,g(A2) ….must converge to the number…… | 1 | 0 | ∞ | g(A) | g(A) |
| A _____________ is a fuzzy relation μ_s(.) that is reflexive, symmetrical and max-min transitive | similarity relation | order relation | path | node | similarity relation |
| A fuzzy relation that is (max-min) transitive and reflexive is called a _____________ | order relation | fuzzy preorder relation | relation | set | fuzzy preorder relation |
| A fuzzy relation that is (min-max) transitive,reflexive and antisymmetric is called a _______ | fuzzy order relation | order relation | relation | set | fuzzy order relation |
| . If the relation is perfectly antisymmetrical, it is called a __________ | map | perfect fuzzy order relation | product space | linear space | perfect fuzzy order relation |
| In fuzzy total no of decision Makers ------------ | n+1 | n | ∞ | n-1 | n |
| The Characteristic function of a crisp set assign values ------------ | 0 | 1 | (0,1) | either 0 or 1 | either 0 or 1 |
| The membership function of a fuzzy set A is denoted by -------------- | μ | μ A | λ | λ A | μ A |
| Fuzzy affect the types of ---------- | Verbal | Complications | Reasoning | Individuality | Reasoning |
| Fuzzy intersection and fuzzy union do not cover all operations by which--- | fuzzy set | empty set | classical set | crisp set | fuzzy set |
| Fuzzy union and fuzzy intersection cover all aggregating operations that are ---- | associative | idempotent | continuous | additive | associative |
| The classical intersection with crisp sets is | i(1,1)=1 | i(0,0)=0 | i(1,2)=1 | i(0,3)=1 | both b and c |
| The fuzzy union that satisfy the | axiomatic skeleton | axiom 1 | axiom 2 | axiom 3 | axiomatic skeleton |
| Max,min,and any of the sugeno complements satisfy | associative | Demorgans law | commutative | idempotent | Demorgans law |
| The only continuous and idempotent fuzzy union | u(a,b)=min(a,b) | u(a,b)=max(a,b) | u(a,b)>min(a,b) | u(a,b)>max(a,b) | u(a,b)=max(a,b) |
| The only continuous and idempotent fuzzy intersection | i(a,b)=min(a,b) | i(a,b)>min(a,b) | i(a,b)>max(a,b) | i(a,b)=max(a,b) | i(a,b)=min(a,b) |
| For all a,b in [0,1] , then | i(a,b)=min(a,b) | i(a,b)>min(a,b) | i(a,b)>max(a,b) | i(a,b)<=min(a,b) | i(a,b)<=min(a,b) |
| If u is idempotent,then | u(a,a)=a | u(a,a)=0 | u(a,a)=1 | u(a,b)=1 | u(a,a)=a |
| The fuzzy union of axiom1 satisfy the | commutative | boundary condition | associative | continuous | boundary condition |

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**Subject: FUZZY SETS AND ITS APPLICATIONS** **Subject Code: 15MMU604B**

**Class : III - B.Sc. Mathematics-A** **Semester : VI**

# Unit IV
# Fuzzy measures

**Part A (20x1=20 Marks)**
**(Question Nos. 1 to 20 Online Examinations)**

**Possible Questions**

| Question | Choice 1 | Choice 2 | Choice 3 | Choice 4 | Answer |
|---|---|---|---|---|---|
| Subsets of cartesian products are called_____ | relation | function | nested family | denumerable | relation |
| The property AUA¯= _______ | A | X | Ø | ( A)¯ | X |
| Every uncountable set is ________ | finite | infinite | disjoint | joint | infinite |
| A continuous t-norm that satisfies subidempotency is called _______ | Archimedean t-norm | idempotent t-norm | monotonic | boundary | Archimedean t-norm |
| Fuzzy relation in different product spaces can be combined with each other by the operation__________ | composition | closure | associative | union | composition |
| ( R)˜ is called _______ if μ_R˜ (x,x)= 1 for all xϵX | symmetric | transitive | reflexive | union | reflexive |
| R˜ is called _______ if μ_R˜ (x,x)≥ϵ for all xϵX | ϵ-reflexive | symmetric | transitive | union | symmetric |
| The shortest term basic assignment instead of the usual full term---------- | basic probability assignment | probability | normal | plausibility measure | basic probability assignment |
| If pl(AUB)………………… | >Max[pl(A),pl(B)] | ≤ Max[pl(A),pl(B)] | ≥ Max[pl(A),pl(B)] | [pl(A),pl(b)] | ≤Max[pl(A),pl(B)] |

| | | | | | |
|---|---|---|---|---|---|
| If π denote the necessity measure and a possibility measure on p(x) then π (A∪B)=…. | max{π(A) ,π(B)} | A | B | 0 | max{π(A) ,π(B)} |
| If A ≠∅ The total ignorance in terms of the associated plausibility measures pl(∅)=⋯ | 1 | ∅ | π | 0 | 0 |
| If A≠∅ The total ignorance in terms of the associated plausibility measures pl(A)=⋯ | 1 | ∅ | π | A | 1 |
| Fuzzy relation in different product spaces can be combined with each other by the operation___________ | composition | closure | associative | union | composition |
| A belief measure of a function is----------- | Bel : p(X)→[0,1] | pl : p(X)→[0,1] | p(X)→[0,1] | p(x)→(0,1) | Bel : p(X)→[0,1] |
| The basis assignment not required that------- | m(A)=m(B) | m(A)≤m(B) | m(A)≥m(B) | m(A)>m(B) | m(A)≤m(B) |
| The basis element is not required that____ | m(X)=1 | m(X)<1 | m(X)>1 | m(X)=0 | m(X)=1 |
| A fuzzy set A is called normal if ------------ | n( A) = 0 | n( A) = 1 | n( A) = n | n( A) = m | n( A) = 1 |
| A fuzzy set is convex if ------------ | no α level convex | all α level convex | n- α level convex | min no of α level convex | all α level convex |
| A fuzzy set A is called subnormal if --------- | n( A) = 0 | n( A) < 1 | n( A) < n | n( A) = m | n( A) < 1 |
| CFL is a branch of fuzzy with modified rules for ----------- | conjunction | disjunction | conjunction and Disjunction | conjunction or disjunction | conjunction and Disjunction |
| The value would indicate the degree of evidence or certainity of the elements membership in such representation of uncertainity is known as --------- | fuzzy measure | fuzzy crisp sets | Universal set | fuzzy relation | fuzzy measure |
| A plausibility measure is a function --------- | Pl : p(x)→[0,1] | Pl : p(x)→(0,1) | Pl→[0,1] | p:x→(0,1) | Pl : p(x)→[0,1] |
| The function m : p(x)→[0,1], m is usually called a---- | basic probability assignment | probability distribution function | basic assignment | normal | basic probability assignment |
| The basis assignments that possess this property are called----- | normal | extension | dimensional | probability | normal |
| Every set Aϵp(x) for which m(A)>0 is usually called -------- of m | n-element | focal element | zero element | n-dimensional | focal element |
| Total --------- is expressed in terms of the basis assignment by m(X)=1 and m(A)=0 | body of evidence | basis assignment | ignorance | focal element | ignorance |
| A basis assignment m is said to be a simple suport function focused at A if | m(A)=s | m(A)>s | m(A)<s | m(A)≤s | m(A)=s |

| | | | | | |
|---|---|---|---|---|---|
| An important property of probability measures is that each of them can uniquely represented by a function defined on element of ----- | universel set | subsequence | fuzzy measure | subset | universel set |
| The most fundamental property of fuzzy measure is their ----- with respect to the subset relation ship | possibility measure | monotonicity | distributive function | believe measure | monotonicity |
| Each pair consisting of a belief measure and its dual plausibility measure is represented by--- | single function | double function | triple function | either b or c | single function |

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**Class : III - B.Sc. Mathematics-A** **Semester : VI**

# Unit V
# Fuzzy decision making

**Part A (20x1=20 Marks)**
**(Question Nos. 1 to 20 Online Examinations)**

**Possible Questions**

| Question | Choice 1 | Choice 2 | Choice 3 | Choice 4 | Answer |
|---|---|---|---|---|---|
| Elements of X , A(x)=(A(x) )¯are called ___ | equilibrium | operator | membership function | function | equilibrium |
| The boundary condition i(a,1)=______ | a | i(1,a) | 1 | i(1) | a |
| All α- cuts and all strong α- cuts of any fuzzy set form two distinct families of ____ sets. | nested crisp sets | crisp sets | fuzzy sets | crisp sets | nested crisp sets |
| The ___ of a fuzzy set A within a universal set x is the crisp set that contains all the elements of X that have nonzero membership grades in A. | Support | nested | subnormal | core | Support |
| A fuzzy relation is called symmetric if ( R _1 )˜(x,y)= (R_2 )˜(x,y) for all x,y ϵ X | symmetric | reflexive | asymmetric | transitive | symmetric |
| A fuzzy relation R˜ is called (max-min) _______ if R˜°R˜ • R˜ | symmetric | reflexive | asymmetric | transitive | transitive |
| If R˜ is reflexive and transitive, then ___________ | R˜°R˜ =R˜ | R˜°R˜ ≤R˜ | R˜°R˜ ≥R˜ | R˜°R˜ ≠R˜ | R˜°R˜ =R˜ |
| Any ∝ - cut of a fuzzy linear order is a _____________ | partial order | crisp linear order | order | relation | crisp linear order |
| Basic assignments is said to be normal if m(∅)=………….. | 1 | 0 | π | empty set | 0 |
| The single property of additivity expressed by the equation Bel(A∪B)=……….. | Bel(A+B) | Bel(A)+Bel(B) | Bel(A∩B) | Bel(A*B) | Bel(A)+Bel(B) |

| | | | | | |
|---|---|---|---|---|---|
| Subsets that are assigned non-zero degrees of evidence are called……………. | focal elements | Bel(A) | basic probability | dual measure | focal elements |
| When all focal elements are …… the crpnding belief measure is equal to its dual plausibity measure. | singletons | dual | zero | empty | singletons |
| A fuzzy relation is called symmetric if ( R _1 )̃(x,y)=(R_2 )̃(x,y) for all x,y ϵ X. | symmetric | reflexive | asymmetric | transitive | symmetric |
| A fuzzy relation R̃ is called (max-min) _______ if R̃ °R̃ • R̃ | symmetric | reflexive | asymmetric | transitive | transitive |
| If R̃ is reflexive and transitive, then ___________ | R̃°R̃ =R̃ | R̃°R̃ ≤R̃ | R̃°R̃ ≥R̃ | R̃°R̃ ≠R̃ | R̃°R̃ =R̃ |
| Any ∝ - cut of a fuzzy linear order is a _____________ | partial order | crisp linear order | order | relation | crisp linear order |
| The conjunction is the --------- | A.M | G.M | H.M | Arithmetic Process | G.M |
| Which one of the following Technique used in fuzzy ranking ? | Degree | Degree of Optimality | Decision Analysis | Multi Decision Analys | Degree of Optimality |
| The term attribute referred to as ----------- | Criterion | Condition | goal | Ideas | goal |
| A Chain is only as strong as its link Weakest is called ------------- | Logic | Reasoning | attribute | Links | Logic |
| classical decision making generally deals with a -------- comprising the decision space. | set of alternatives | certainity | fuzzy model | uncertainity | set of alternatives |
| The decision may involve the simple optimization of a------- | linear function | utility function | membership function | nonlinear function | utility function |
| Decisions may, for instance, be considered to occur in a-----stages | single | multiple | either b or c | triple | either b or c |

Reg. No................................

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(For the candidates admitted from 2015 onwards)

## B.Sc., DEGREE EXAMINATION, APRIL 2018

Sixth Semester

MATHEMATICS

FUZZY SETS AND ITS APPLICATIONS

Time: 3 hours                                        Maximum : 60 marks

**PART – A (20 x 1 = 20 Marks) (30 Minutes)**
**(Question Nos. 1 to 20 Online Examinations)**

**PART B (5 x 8 = 40 Marks) (2 ½ Hours)**
**Answer ALL the Questions**

21.a. Prove that for each $a \in [0,1], d_a = c(a)$ if and only if $c(c(a)) = a$ that is which the complement is involutive.

Or

b. Explain briefly about fuzzy complement.

22.a. Prove that $\lim_{w \to \infty} i_w = \lim_{w \to \infty}\left[1 - \min\left(1, \left[(1-a)^w + (1-b)^w\right]^{\frac{1}{w}}\right)\right]$

Or

b. If for all $a, b \in [0,1]$ then prove that $i(a,b) \geq i_{\min}(a,b)$

23.a. Compute i) $\tilde{R} \cup \tilde{Z}$   ii) $\tilde{R} \cap \tilde{Z}$

If $\tilde{R}$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ |
|---|---|---|---|---|
| $X_1$ | 0.8 | 1 | 0.1 | 0.7 |
| $X_2$ | 0 | 0.8 | 0 | 0 |
| $X_3$ | 0.9 | 1 | 0.7 | 0.8 |



and

$\tilde{Z}$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | Y4 |
|---|---|---|---|---|
| $X_1$ | 0.4 | 0 | 0.9 | 0.6 |
| X2 | 0.9 | 0.4 | 0.5 | 0.7 |
| X3 | 0.3 | 0 | 0.8 | 0.5 |

Or

b. Find i)first projection
ii) second projection
iii)Total projection
if $\tilde{R}$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | Y5 | Y6 |
|---|---|---|---|---|---|---|
| $X_1$ | 0.1 | 0.2 | 0.4 | 1 | 1 | 0.8 |
| $X_2$ | 0.2 | 0.4 | 0.8 | 0.8 | 0.8 | 0.6 |
| $X_3$ | 0.4 | 0.8 | 1 | 0.4 | 0.4 | 0.2 |

24.a. Given a consonant body of evidence (f,m).Prove that the associated consonant belief and probability measure s process the following properties.
i) Bel(A∩B)=min {Bel (A), Bel (B)}
ii) Pl(A U B)=max {Pl(A),Pl(B)} for all A,B $\in \varphi(x)$.

Or

b. Prove that every possibility measure POS on a finite power set P(X) is unequally determined by a possibility distribution function.

25.a. Describe about fuzzy ranking methods with example.

Or

b. Discuss the types of fuzzy Decision making.

-----------

(c) i(a,b) < max(a,b)  (d) i(a,b) ≥ min(a,b)

16. A crisp relation represents the___
    (a) presence (b) association (c) grades (d) fuzzy relation

17. Fuzzy affect the types of ___
    (a) Verbal (b) Complications (c) Reasoning
    (d) Individuality

18. The fuzzy measure assigns a value to each ___set
    (a) crisp (b) fuzzy union (c) fuzzy intersection
    (d) empty

19. In fuzzy total no of decision makers ___
    (a) n+1 (b) n (c) ∞ (d) n-1

20. The only continuous and idempotent fuzzy union is___
    (a) u(a,b)=min(a,b) (b) u(a,b)=max(a,b)
    (c) u(a,b)>min(a,b) (d) u(a,b)>max(a,b)

**PART – B (3 × 10 = 30 marks)**
**ANSWER ALL THE QUESTIONS**

21.(a) Show that $u(a,b) \geq \max(a,b)$ if for all $a, b \in [0,1]$

**(OR)**

(b) ) Prove that $\lim_{w\to\infty} \min\left[1, (a^w + b^w)^{\frac{1}{w}}\right] = \max(a,b)$

22. a) Define fuzzy relation and explain about composition of fuzzy relation.

**(OR)**

b) Explain about max-product composition.

23. a) Explain about Charecteristic components of symmetric, reflexive and transitive relation.

**(OR)**

b) Define fuzzy measure and explain the axioms for fuzzy measures. .

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# KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE-21

Department of Mathematics

Sixth Semester

II Internal Test- FEBRUARY 2018

Elective-II :Fuzzy Sets and its Applications

Date: 01.03.2018(FN) Time: 2 Hours

Class: III B.Sc Mathematics-A Maximum Marks: 50

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## PART – A (20 x 1 = 20 Marks)

**Answer all the Questions:**

1. Fuzzy relation describes the ___ of elements.
   (a) degree of association (b) membership grades
   (c) Cartesian product (d) binary relation
2. A fuzzy relation that is (max-min) transitive and reflexive is called a ______
   (a) order relation (b) fuzzy preorder relation
   (c) relation (d) set
3. The axiom three is clearly applicable only to an ______
   (a) finite set (b) universal set
   (c) infinite set (d)infinite universe
4. Each crisp relation R can be defined by a______
   (a) characteristic function (b) membership function
   (c)fuzzy preorder relation (d) order relation
5. Fuzzy union and fuzzy intersection cover all aggregating operations that are ___
   (a) associative (b) idempotent (c) continuous
   (d) additive
6. The membership grade is usually represented by a _____
   (a) real number in [0,1] (b) complex number in[0,1]
   (c) ) membership array (d) n-dim membership array
7. Fuzzy relations are fuzzy sets in ________
   (a) product space (b) linear space (c) binary space
   (d) composition
8. The fuzzy measure assigns a value to each________ of the universal set
   (a) empty set (b) universal set (c) crisp set
   (d) super set
9. _______ is the only continuous and idempotent fuzzy set intersection.
   (a) i(a,b)=min(a,b) (b) i(a,b)>min(a,b) (c) i(a,b)<min(a,b)
   (d) i(a,b)=max(a,b)
10. Fuzzy relations are fuzzy subsets of _________
    (a) map (b) perfect order (c) X x Y (d) X+Y
11. Truth of a fuzzy proposition is a matter of _________
    (a) degree (b) Power (c) order (d) no of elements
12. Fuzzy LPP values of the ---------- of LPP
    (a) parameters (b) Types (c) constraints
    (d) conjunction
13. The largest relation is called the __________ of the projection relation.
    (a) product space (b) linear space (c) cylindrical extension (d) set
14. The standard ____operations denoted as extinct pair of all the possible pairs of fuzzy union and intersection.
    (a) combinations (b) max & min (c) logic (d) fuzzy
15. For all a,b∈ [0,1], then ____
    (a) $i(a,b) \leq \min(a,b)$ (b) $i(a,b) < \min(a,b)$

**SEMESTER – FUZZY SETS AND ITS APPLICATIONS**

**KEY ANSWER**

**PART-B**

**21. a.** let $d_a$ = c (a).

$d_a$ is dual point of the fuzzy complement c with membership grade 'a'

We have c ($d_a$) $- d_a$ = a- c(a) ------- 1

Replace $d_a$ by c (a) in 1

We get c (c(a)) =a

Fuzzy complent is involutive.

Let c(c(a))=a

Since every fuzzy complement and membership grade has atleast one dual point we have,

C($d_a$) $- d_a = a - c(a)$------ 2

Replace a by c(c(a)) in 2 we get, c($d_a$) $- d_a = c\big(c(a)\big) - c(a)$

$$d_a = c(a).$$

b. Fuzzy complement:

A complement of a fuzzy set A is specified by c[0,1]→[0,1] which assigns a value c($u_A$(x)) to each membership grade $u_A$(x).

Axiom 1:

C(0) =1, c(1)=0

(ie) c behaves as the ordinary complement for crisp sets.

Axiom 2:

C is monotonic non-increasing.

Axiom 3:

C is continuous function.

Axiom 4:

C is involutive.

C(c(a))=a.

**22. a. Proof:**

We know yagar class of fuzzy unions, thus

$i_w$ (a,b) = 1-max[1-a ,1-b]

Ie) $i_\infty$ (a,b) = min (a,b)

**b. proof:**

for all a,bϵ[0,1] , i(a,b) $\leq$ min(a,b), then i(a,b)=a the theorem is satisfied.

Similarly commutativity, then i(a,b)=b. and i(a,0)=i(0,b)=0.

Then by monotonicity,i(a,b)$\geq$ i(0,b)=i(a,0)=0.

23. a.i)

Let $\tilde{R}$ and $\tilde{Z}$ be two fuzzy relations in the same product space. The *union/intersection* of $\tilde{R}$ with $\tilde{Z}$ is then defined by

$$\mu_{\tilde{R}\cup\tilde{Z}}(x, y) = \max\ \{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, \quad (x, y) \in X \times Y$$

$$\mu_{\tilde{R}\cap\tilde{Z}}(x, y) = \min\ \{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, \quad (x, y) \in X \times Y$$

$\tilde{R} \cup \tilde{Z}$:

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | .8 | 1 | .9 | .7 |
| $x_2$ | .9 | .8 | .5 | .7 |
| $x_3$ | .9 | 1 | .8 | .8 |

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $\tilde{R} \cap \tilde{Z}$: $x_1$ | .4 | 0 | .1 | .6 |
| $x_2$ | 0 | .4 | 0 | 0 |
| $x_3$ | .3 | 0 | .7 | .5 |

ii)

$$\tilde{R}^{(1)} = \{(x, \max_{y} \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$$

b)

The *second projection* is defined as

$$\tilde{R}^{(2)} = \{(y, \max_{x} \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$$

and the *total projection* as

$$\tilde{R}^{(T)} = \max_{x} \max_{y} \{\mu_{\tilde{R}}(x, y) \mid (x, y) \in X \times Y\}$$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | First projection $[\mu_{\tilde{R}^{(1)}}(x)]$ |
|---|---|---|---|---|---|---|---|
| $\tilde{R}$: $x_1$ | .1 | .2 | .4 | .8 | 1 | .8 | 1 |
| $x_2$ | .2 | .4 | .8 | 1 | .8 | .6 | 1 |
| $x_3$ | .4 | .8 | 1 | .8 | .4 | .2 | 1 |
| Second projection: $[\mu_{\tilde{R}^{(2)}}(x)]$ | .4 | .8 | 1 | 1 | 1 | .8 | 1 Total projection |

24. a)

***Proof:*** (i) Since the focal elements in $\mathcal{F}$ are nested, they may be linearly ordered by the subset relationship. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$, and assume that $A_i \subset A_j$ whenever $i < j$. Consider now arbitrary subsets $A$ and $B$ of $X$. Let $i_1$ be the largest integer $i$ such that $A_i \subseteq A$, and let $i_2$ be the largest integer $i$ such that $A_i \subseteq B$. Then, $A_i \subseteq A$ and $A_i \subseteq B$ iff $i \leq i_1$ and $i \leq i_2$, respectively. Moreover, $A_i \subseteq A \cap B$ iff $i \leq \min(i_1, i_2)$. Hence,

$$\begin{aligned}\mathrm{Bel}\,(A \cap B) &= \sum_{i=1}^{\min(i_1,i_2)} m(A_i) = \min\left[\sum_{i=1}^{i_1} m(A_i), \sum_{i=1}^{i_2} m(A_i)\right] \\ &= \min[\mathrm{Bel}\,(A), \mathrm{Bel}\,(B)].\end{aligned}$$

(ii) Assume that (i) holds. Then, by (7.5),

$$\begin{aligned}\mathrm{Pl}\,(A \cup B) &= 1 - \mathrm{Bel}\,(\overline{A \cup B}) = 1 - \mathrm{Bel}\,(\overline{A} \cap \overline{B}) \\ &= 1 - \min[\mathrm{Bel}\,(\overline{A}), \mathrm{Bel}\,(\overline{B})] \\ &= \max[1 - \mathrm{Bel}\,(\overline{A}), 1 - \mathrm{Bel}\,(\overline{B})] \\ &= \max[\mathrm{Pl}\,(A), \mathrm{Pl}\,(B)]\end{aligned}$$

for all $A, B \in \mathcal{P}(X)$.

b)

***Proof:*** We prove the theorem by induction on the cardinality of set $A$. Let $|A| = 1$. Then, $A = \{x\}$, where $x \in X$, and (7.28) is trivially satisfied. Assume now that (7.28) is satisfied for $|A| = n - 1$, and let $A = \{x_1, x_2, \ldots, x_n\}$. Then, by (7.20),

$$\begin{aligned}\mathrm{Pos}\,(A) &= \max[\mathrm{Pos}\,(\{x_1, x_2, \ldots, x_{n-1}\}), \mathrm{Pos}\,(\{x_n\})] \\ &= \max[\max[\mathrm{Pos}\,(\{x_1\}), \mathrm{Pos}\,(\{x_2\}), \ldots, \mathrm{Pos}\,(\{x_{n-1}\})], \mathrm{Pos}\,(\{x_n\})]\end{aligned}$$

$$= \max[\mathrm{Pos}\,(\{x_1\}), \mathrm{Pos}\,(\{x_2\}), \ldots, \mathrm{Pos}\,(\{x_n\})]$$

$$= \max_{x \in A} r(x).$$

25. a)

# FUZZY RANKING METHODS

In many fuzzy decision problems, the final scores of alternatives are represented in terms of fuzzy numbers. In order to express a crisp preference of alternatives, we need a method for constructing a crisp total ordering from fuzzy numbers. Unfortunately, the lattice of fuzzy numbers, $\langle \mathcal{R}, \mathrm{MIN}, \mathrm{MAX}\rangle$, is not linearly ordered, as discussed in Sec. 4.5. Thus, some fuzzy numbers are not directly comparable.

Numerous methods for total ordering of fuzzy numbers have been suggested in the literature. Each method appears to have some advantages as well as disadvantages. In the context of each application, some methods seem more appropriate than others. However, the issue of choosing a proper ordering method in a given context is still a subject of active research. To illustrate the problem of total ordering of fuzzy numbers, we describe three simple methods and illustrate them by examples.

The first method is based upon defining the *Hamming distance* on the set $\mathcal{R}$ of all fuzzy numbers. For any given fuzzy numbers $A$ and $B$, the Hamming distance, $d(A, B)$, is defined by the formula

$$d(A, B) = \int_{\mathbb{R}} |A(x) - B(x)| dx. \tag{15.16}$$

have been suggested in the literature. A simple variation of this methods proceeds as follows. Given fuzzy numbers $A$ and $B$ to be compared, we select a particular value of $\alpha \in [0, 1]$ and determine the $\alpha$-cuts ${}^{\alpha}A = [a_1, a_2]$ and ${}^{\alpha}B = [b_1, b_2]$. Then, we define

$$A \leq B \text{ if } a_2 \leq b_2.$$

This definition is, of course, dependent on the chosen value of $\alpha$. It is usually required that $\alpha > 0.5$. More sophisticated methods based on $\alpha$-cuts, such as the one developed by Mabuchi [1988], aggregate appropriately defined degrees expressing the dominance of one fuzzy number over the other one for all $\alpha$-cuts.

The third method is based on the extension principle. This method can be employed for ordering several fuzzy numbers, say $A_1, A_2, \ldots, A_n$. The basic idea is to construct a fuzzy set $P$ on $\{A_1, A_2, \ldots, A_n\}$, called a *priority set*, such as $P(A_i)$ is the degree to which $A_i$ is ranked as the greatest fuzzy number. Using the extension principle, $P$ is defined for each $i \in \mathbb{N}_n$ by the formula

$$P(A_i) = \sup \min_{k \in \mathbb{N}_n} A_k(r_k), \tag{15.17}$$

where the supremum is taken over all vectors $\langle r_1, r_2, \ldots, r_n\rangle \in \mathbb{R}^n$ such that $r_i \geq r_j$ for all $j \in \mathbb{N}_n$.

Example:

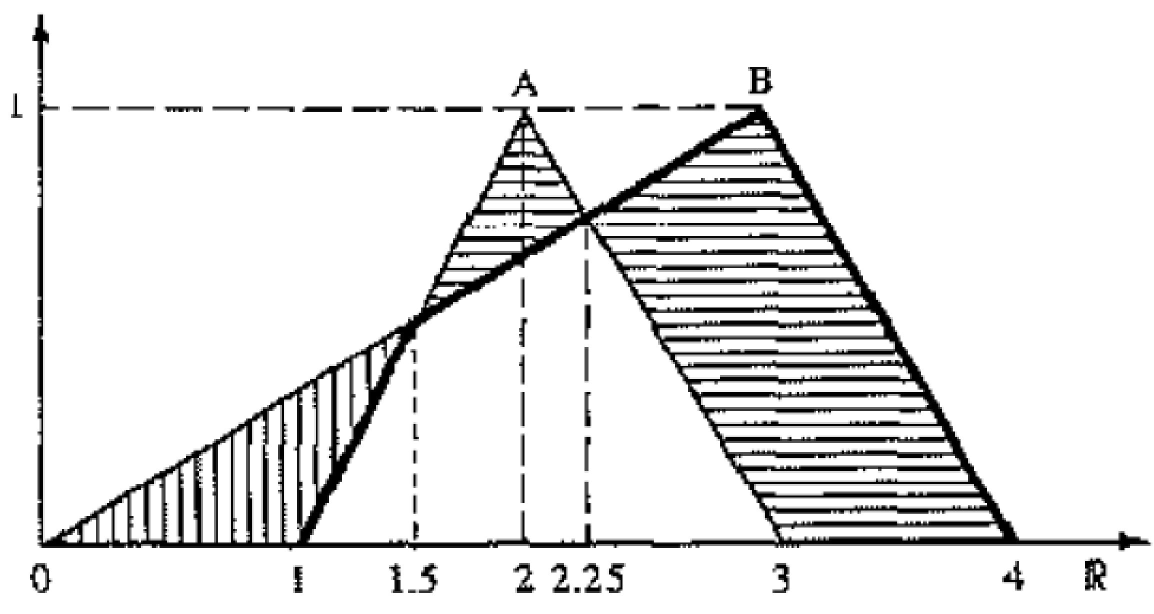
$$d(\mathrm{MAX}(A,B),A) = \int_{1.5}^{2}[x-1-\frac{x}{3}]dx + \int_{2}^{2.25}[-x+3-\frac{x}{3}]dx$$

$$+\int_{2.25}^{3}[\frac{x}{3}+x-3]dx + \int_{3}^{4}[4-x]dx$$

$$= \frac{1}{12}+\frac{1}{24}+\frac{3}{8}+\frac{1}{2} = 1$$

$$d(\mathrm{MAX}(A,B),B) = \int_{0}^{1.5}\frac{x}{3}dx - \int_{1}^{1.5}[x-1]dx$$

$$= \frac{3}{8}-\frac{1}{8} = 0.25.$$




(a)

b) The types of fuzzy decision making:

Making decisions is undoubtedly one of the most fundamental activities of human beings. We all are faced in our daily life with varieties of alternative actions available to us and, at least in some instances, we have to decide which of the available actions to take. The beginnings of decision making, as a subject of study, can be traced, presumably, to the late 18th century, when various studies were made in France regarding methods of election and social choice. Since these initial studies, decision making has evolved into a respectable and rich field of study. The current literature on decision making, based largely on theories and methods developed in this century, is enormous.

## INDIVIDUAL DECISION MAKING

Fuzziness can be introduced into the existing models of decision models in various ways. In the first paper on fuzzy decision making, Bellman and Zadeh [1970] suggest a fuzzy model of decision making in which relevant goals and constraints are expressed in terms of fuzzy sets, and a decision is determined by an appropriate aggregation of these fuzzy sets. A decision situation in this model is characterized by the following components:

- a set $A$ of *possible actions*;
- a set of *goals* $G_i (i \in \mathbb{N}_n)$, each of which is expressed in terms of a fuzzy set defined on $A$;
- a set of *constraints* $C_j (j \in \mathbb{N}_m)$, each of which is also expressed by a fuzzy set defined on $A$.

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## FUZZY LINEAR PROGRAMMING

The *classical linear programming problem* is to find the minimum or maximum values of a linear function under constraints represented by linear inequalities or equations. The most typical linear programming problem is:

$$
\begin{aligned}
&\text{Minimize (or maximize)} && c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
&\text{Subject to} && a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
& && a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
& && \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& && a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
& && x_1, x_2, \ldots, x_n \ge 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{Min} \quad & z = \mathbf{c}\mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \qquad (15.18) \\
& \mathbf{x} \ge 0,
\end{aligned}
$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is a *vector of variables,* and s.t. stands for "subject to." The set of vectors $\mathbf{x}$ that satisfy all given constraints is called a *feasible set.* As is well known, many practical problems can be formulated as linear programming problems.

# KARPAGAM ACADEMY OF HIGHER EDUCATION

*(Deemed to be University Established Under Section 3 of UGC Act 1956)*

**Coimbatore – 641 021.**

**SYLLABUS**

**ELECTIVE-II**

| | | Semester - VI |
|---|---|---|
| | | L T P C |
| **15MMU604B** | **FUZZY SETS AND ITS APPLICATIONS** | 5 0 0 5 |

**Scope:** To make the students understand the notion of fuzzy sets and to make them to understand the importance , applications of these techniques in real time problems.

**Objectives:** To enable the students to understand the basic concepts of Fuzzy sets and its applications.

**UNIT I**

Crisp sets and fuzzy sets - basic concept of fuzzy set - fuzzy logic - operations on fuzzy sets - general discussion fuzzy complements.

**UNIT II**

Fuzzy union - fuzzy intersection - combinations operations.

**UNIT III**

Fuzzy relations and fuzzy graphs - fuzzy relation on sets and fuzzy sets - composition of fuzzy relations - properties of the min-max composition - fuzzy graphs - special fuzzy relations.

**UNIT IV**

Fuzzy measures - general discussion - belief and plausibility measures - probability measures - possibility and necessity measures.

**UNIT V**

Fuzzy decision making - individual decision making - fuzzy ranking methods - fuzzy linear programming.

**SUGGESTED READINGS**

**TEXTBOOK**

1. George J. Klir and Bo Yuan, 1995.Fuzzy sets and fuzzy logic theory and applications, Prentice-Hall of India private limited, New Delhi.

**REFERENCES**

1. Timothy J. Ross, 2000. Fuzzy logic with Engineering Applications, McGrawHill, Inc. New Delhi.

2. George J.Klir, Tina.A Folger, 2008. Fuzzy sets, uncertainty and information, Prentice Hall of

India Pvt Ltd, New Delhi.

3. H.J. Zimmermann, 2006. Fuzzy set theory and its applications, Second Edition, Springer New

## UNIT-V

## SYLLABUS

Fuzzy decision making, individual decision making, fuzzy ranking methods, fuzzy linear programming.

The subject of decision making is, as the name suggests, the study of how decisions are actually made and how they can be made better or more successfully. That is, the field is concerned, in general, with both descriptive theories and normative theories. Much of the focus in developing the field has been in the area of management, in which the decision-making process is of key importance for functions such as inventory control, investment, personnel actions, new-product development, and allocation of resources, as well as many others. Decision making itself, however, is broadly defined to include any choice or selection of alternatives, and is therefore of importance in many fields in both the "soft" social sciences and the "hard" disciplines of natural sciences and engineering.

**Fuzzy decision making:**

## GENERAL DISCUSSION

Making decisions is undoubtedly one of the most fundamental activities of human beings. We all are faced in our daily life with varieties of alternative actions available to us and, at least in some instances, we have to decide which of the available actions to take. The beginnings of decision making, as a subject of study, can be traced, presumably, to the late 18th century, when various studies were made in France regarding methods of election and social choice. Since these initial studies, decision making has evolved into a respectable and rich field of study. The current literature on decision making, based largely on theories and methods developed in this century, is enormous.

## INDIVIDUAL DECISION MAKING

Fuzziness can be introduced into the existing models of decision models in various ways. In the first paper on fuzzy decision making, Bellman and Zadeh [1970] suggest a fuzzy model of decision making in which relevant goals and constraints are expressed in terms of fuzzy sets, and a decision is determined by an appropriate aggregation of these fuzzy sets. A decision situation in this model is characterized by the following components:

- a set $A$ of *possible actions*;
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- a set of *constraints* $C_j (j \in \mathbb{N}_m)$, each of which is also expressed by a fuzzy set defined on $A$.

It is common that the fuzzy sets expressing goals and constraints in this formulation are not defined directly on the set of actions, but indirectly, through other sets that characterize relevant states of nature. Let $G'_i$ and $C'_j$ be fuzzy sets defined on sets $X_i$ and $Y_j$, respectively, where $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$. Assume that these fuzzy sets represent goals and constraints expressed by the decision maker. Then, for each $i \in \mathbb{N}_n$ and each $j \in \mathbb{N}_m$, we describe the meanings of actions in set $A$ in terms of sets $X_i$ and $Y_j$ by functions

$$g_i : A \to X_i,$$

$$c_j : A \to Y_j,$$

and express goals $G_i$ and constraints $C_j$ by the compositions of $g_i$ with $G'_i$ and the compositions of $c_j$ and $C'_j$; that is,

$$G_i(a) = G'_i(g_i(a)), \tag{15.1}$$

$$C_j(a) = C'_j(c_j(a)) \tag{15.2}$$

for each $a \in A$.

Given a decision situation characterized by fuzzy sets $A$, $G_i (i \in \mathbb{N}_n)$, and $C_j (j \in \mathbb{N}_m)$, a *fuzzy decision*, $D$, is conceived as a fuzzy set on $A$ that simultaneously satisfies the given goals $G_i$ and constraints $C_j$. That is,

$$D(a) = \min[\inf_{i \in \mathbb{N}_n} G_i(a), \inf_{j \in \mathbb{N}_m} C_j(a)] \tag{15.3}$$

for all $a \in A$, provided that the standard operator of fuzzy intersection is employed.

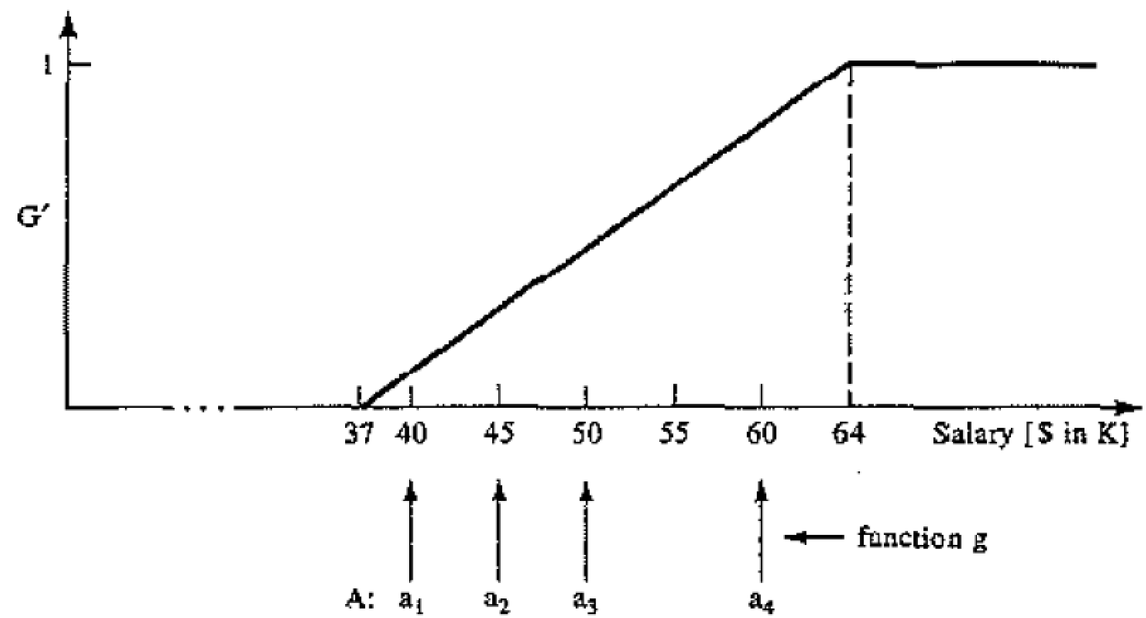
Once a fuzzy decision has been arrived at, it may be necessary to choose the "best" single crisp alternative from this fuzzy set. This may be accomplished in a straightforward manner by choosing an alternative $\hat{a} \in A$ that attains the maximum membership grade in $D$. Since this method ignores information concerning any of the other alternatives, it may not be desirable in all situations. When $A$ is defined on $\mathbb{R}$, it is preferable to determine $\hat{a}$ by an appropriate defuzzification method (Sec. 12.2).

Before discussing the various features of this fuzzy decision model and its possible modifications or extensions, let us illustrate how it works by two simple examples.

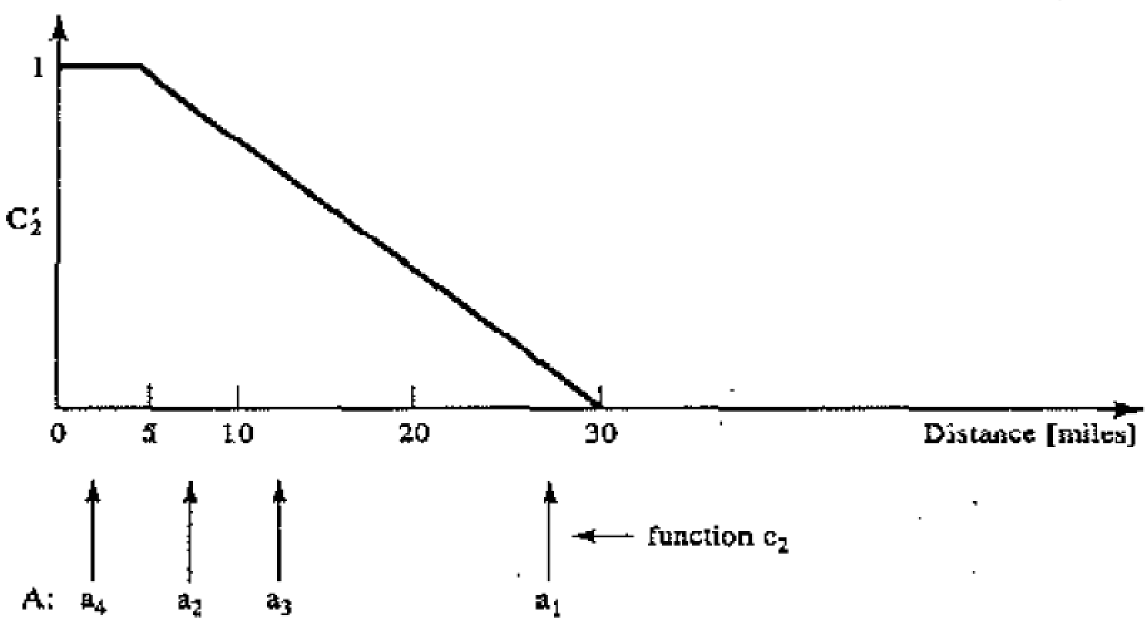
## Example

Suppose that an individual needs to decide which of four possible jobs, $a_1, a_2, a_3, a_4$, to choose. His or her goal is to choose a job that offers a high salary under the constraints that the job is interesting and within close driving distance. In this case, $A = \{a_1, a_2, a_3, a_4\}$, and the fuzzy sets involved represent the concepts of *high salary*, *interesting job*, and *close driving distance*. These concepts are highly subjective and context-dependent, and must be defined by the individual in a given context. The goal is expressed in monetary terms, independent of the jobs available. Hence, according to our notation, we denote the fuzzy set expressing the goal by $G'$. A possible definition of $G'$ is given in Fig. 15.1a, where we assume, for convenience, that the underlying universal set is $\mathbb{R}^+$. To express the goal in terms of set $A$, we need a function $g: A \to \mathbb{R}^+$, which assigns to each job the respective salary. Assume the following assignments:

Individual Decision Making




(a) Goal G′: High salary.




(b) Constraint $C_2'$: Close driving distance.

# FUZZY RANKING METHODS

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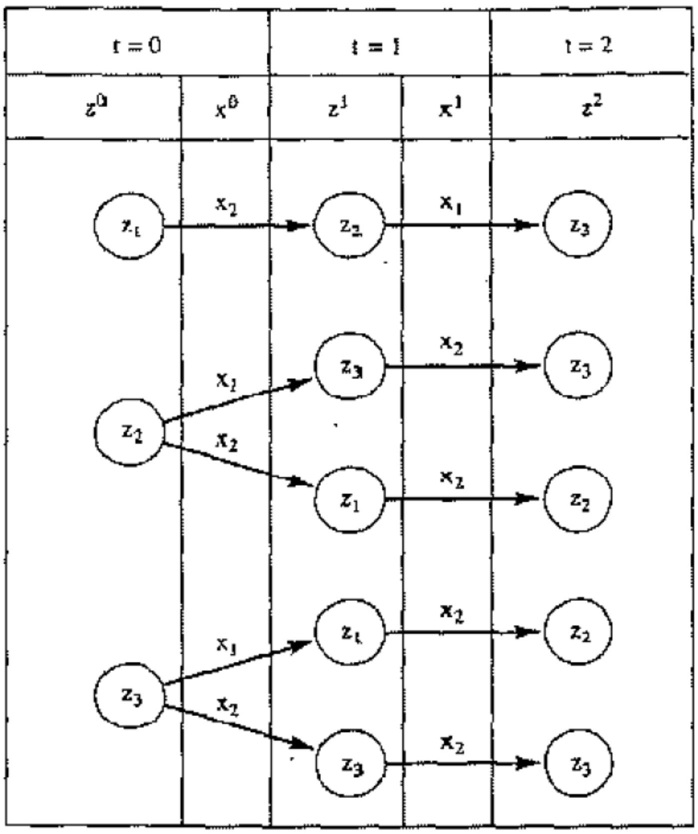


**Figure 15.4** Maximizing decisions in Example 15.5 for different initial states $z^0$.

Numerous methods for total ordering of fuzzy numbers have been suggested in the literature. Each method appears to have some advantages as well as disadvantages. In the context of each application, some methods seem more appropriate than others. However, the issue of choosing a proper ordering method in a given context is still a subject of active research. To illustrate the problem of total ordering of fuzzy numbers, we describe three simple methods and illustrate them by examples.

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have been suggested in the literature. A simple variation of this methods proceeds as follows. Given fuzzy numbers $A$ and $B$ to be compared, we select a particular value of $\alpha \in [0, 1]$ and determine the $\alpha$-cuts ${}^{\alpha}A = [a_1, a_2]$ and ${}^{\alpha}B = [b_1, b_2]$. Then, we define

$$A \le B \text{ if } a_2 \le b_2.$$

This definition is, of course, dependent on the chosen value of $\alpha$. It is usually required that $\alpha > 0.5$. More sophisticated methods based on $\alpha$-cuts, such as the one developed by Mabuchi [1988], aggregate appropriately defined degrees expressing the dominance of one fuzzy number over the other one for all $\alpha$-cuts.

The third method is based on the extension principle. This method can be employed for ordering several fuzzy numbers, say $A_1, A_2, \ldots, A_n$. The basic idea is to construct a fuzzy set $P$ on $\{A_1, A_2, \ldots, A_n\}$, called a *priority set*, such as $P(A_i)$ is the degree to which $A_i$ is ranked as the greatest fuzzy number. Using the extension principle, $P$ is defined for each $i \in \mathbb{N}_n$ by the formula

$$P(A_i) = \sup \min_{k \in \mathbb{N}_n} A_k(r_k), \tag{15.17}$$

where the supremum is taken over all vectors $\langle r_1, r_2, \ldots, r_n \rangle \in \mathbb{R}^n$ such that $r_i \ge r_j$ for all $j \in \mathbb{N}_n$.

**Example 15.6**

In this example, we illustrate and compare the three fuzzy ranking methods. Let $A$ and $B$ be fuzzy numbers whose triangular-type membership functions are given in Fig. 15.5a. Then, $\text{MAX}(A, B)$ is the fuzzy number whose membership function is indicated in the figure in bold. We can see that the Hamming distances $d(\text{MAX}(A, B), A)$ and $d(\text{MAX}(A, B), B)$ are expressed by the areas in the figure that are hatched horizontally and vertically, respectively. Using (15.16), we obtain

$$\begin{aligned} d(\text{MAX}(A, B), A) &= \int_{1.5}^{2} [x - 1 - \frac{x}{3}] dx + \int_{2}^{2.25} [-x + 3 - \frac{x}{3}] dx \\ &\quad + \int_{2.25}^{3} [\frac{x}{3} + x - 3] dx + \int_{3}^{4} [4 - x] dx \\ &= \frac{1}{12} + \frac{1}{24} + \frac{3}{8} + \frac{1}{2} = 1 \end{aligned}$$

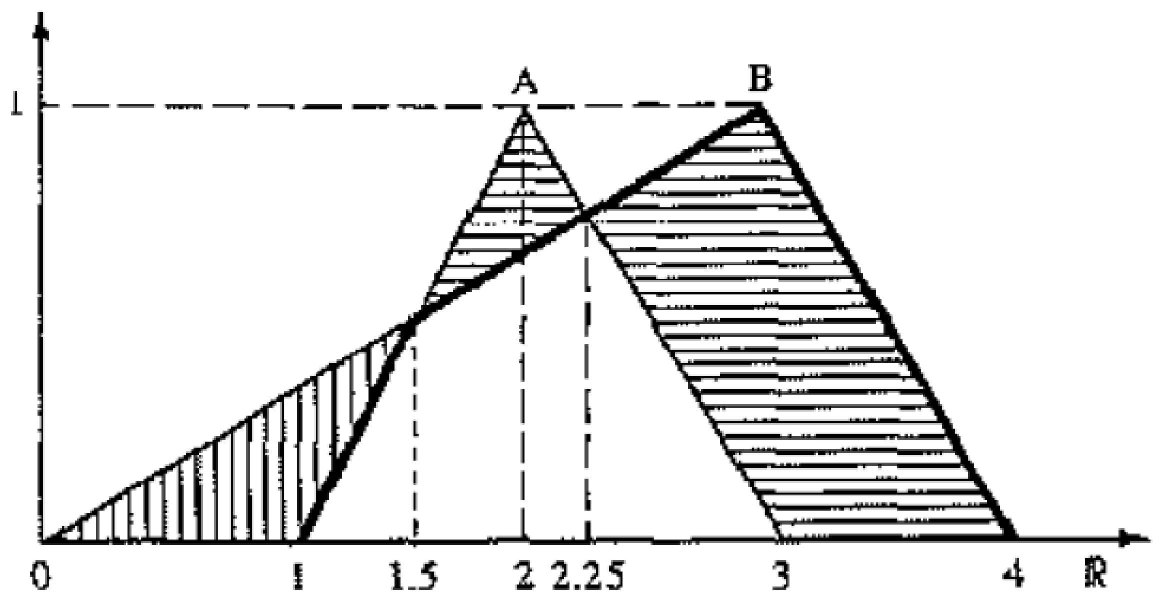
$$d(\mathrm{MAX}(A, B), B) = \int_0^{1.5} \frac{x}{3} dx - \int_1^{1.5} [x-1] dx$$

$$= \frac{3}{8} - \frac{1}{8} = 0.25.$$

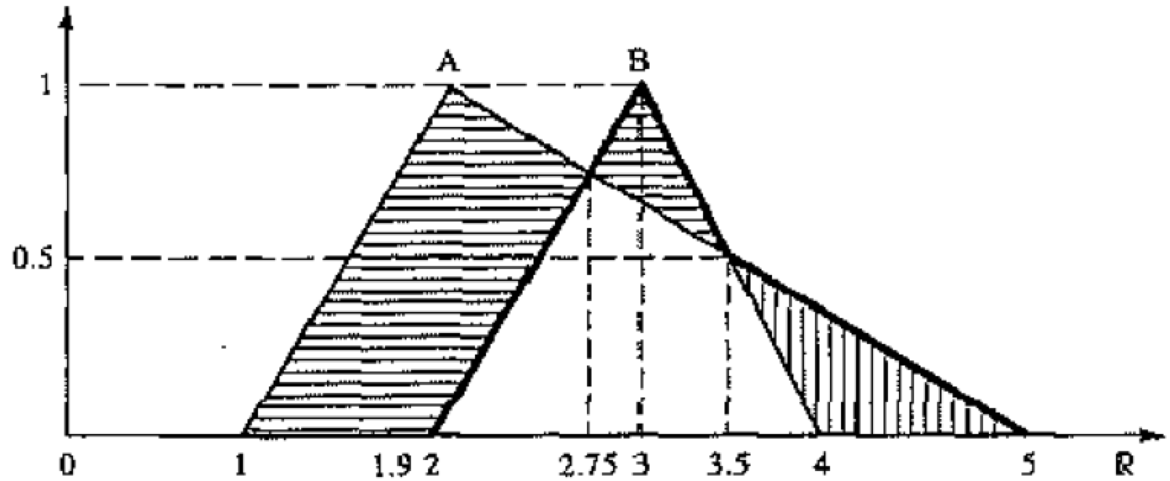
Since $d(\mathrm{MAX}(A, B), A) > d(\mathrm{MAX}(A, B), B)$, we may conclude that, according to the first ranking method, $A \leq B$. When applying the second method to the same example, we can easily find, from Fig. 15.5a, that $A \leq B$ for any $\alpha \in [0, 1]$. According to the third method, we construct the priority fuzzy set $P$ on $\{A, B\}$ as follows:

$$P(A) = \sup_{r_1 \geq r_2} \min[A(r_1), B(r_2)] = 0.75,$$

$$P(B) = \sup_{r_2 \geq r_1} \min[A(r_1), B(r_2)] = 1.$$




(a)




(b)

**Figure 15.5** Ranking of fuzzy members (Example 15.6).

Hence, again, we conclude that $A \leq B$.

Consider now the fuzzy numbers $A$ and $B$ whose membership functions are given in Fig. 15.5b. The horizontally and vertically hatched areas have the same meaning as before. We can easily find that

$$d(\mathrm{MAX}(A, B), A) = 1, d(\mathrm{MAX}(A, B), B) = 0.25.$$

Hence, $A \leq B$ *according to the first method. The second method gives the same result only for* $\alpha > 0.5$. This shows that the method is inconsistent. According to the third method, we again obtain $P(A) = 0.75$ and $P(B) = 1$; hence, $A \leq B$.

# FUZZY LINEAR PROGRAMMING

The *classical linear programming problem* is to find the minimum or maximum values of a linear function under constraints represented by linear inequalities or equations. The most typical linear programming problem is:

Minimize (or maximize) $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$
$$x_1, x_2, \ldots, x_n \geq 0.$$

The function to be minimized (or maximized) is called an *objective function*; let us denote it by $z$. The numbers $c_i$ ($i \in \mathbb{N}_n$) are called cost coefficients, and the vector $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ is called a *cost vector*. The matrix $\mathbf{A} = [a_{ij}]$, where $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$, is called a *constraint matrix*, and the vector $\mathbf{b} = \langle b_1, b_2, \ldots, b_m \rangle^T$ is called a *right-hand-side vector*. Using this notation, the formulation of the problem can be simplified as

$$\begin{aligned} \text{Min} \quad & z = \mathbf{cx} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b} \qquad (15.18) \\ & \mathbf{x} \geq 0, \end{aligned}$$

where $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle^T$ is a *vector of variables*, and s.t. stands for "subject to." The set of vectors $\mathbf{x}$ that satisfy all given constraints is called a *feasible set*. As is well known, many practical problems can be formulated as linear programming problems.

**Example 15.9**

Consider the following fuzzy linear programming problem:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & \langle 4, 2, 1 \rangle x_1 + \langle 5, 3, 1 \rangle x_2 \leq \langle 24, 5, 8 \rangle \\ & \langle 4, 1, 2 \rangle x_1 + \langle 1, .5, 1 \rangle x_2 \leq \langle 12, 6, 3 \rangle \\ & x_1, x_2 \geq 0. \end{aligned}$$

We can rewrite it as

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 4x_1 + 5x_2 \leq 24 \end{aligned}$$

$$4x_1 + x_2 \leq 12$$
$$2x_1 + 2x_2 \leq 19$$
$$3x_1 + 0.5x_2 \leq 6$$
$$5x_1 + 6x_2 \leq 32$$
$$6x_1 + 2x_2 \leq 15$$
$$x_1, x_2 \geq 0.$$

Solving this problem, we obtain $\hat{x}_1 = 1.5$, $\hat{x}_2 = 3$, $\hat{z} = 19.5$.

Notice that if we defuzzified the fuzzy numbers in the constraints of the original problem by the maximum method, we would obtain another classical linear programming problem:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 4x_1 + 5x_2 \leq 24 \\ & 4x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

We can see that this is a classical linear programming problem with a smaller number of constraints than the one converted from a fuzzy linear programming problem. Therefore, fuzziness in (15.21) results in stronger constraints, while fuzziness in (15.20) results in weaker constraints.

**POSSIBLE QUESTIONS**

**PART-B (5 x 8 =40 Marks)**

**Answer all the questions**

1. Describe about fuzzy decision making.
2. Describe about individual decision making with example.
3. Describe about fuzzy ranking methods with example.
4. Discuss the types of fuzzy Decision making.
5. Discuss about fuzzy linear programming.
6. Write short notes on multi criteria fuzzy decision making.
7. Write about decision making in fuzzy environment.
8. Describe about fuzzy multiple objective decision making.
9. Discuss about fuzzy multiple attribute decision making.
10. How to rank the fuzzy numbers by distance minimization method.

# UNIT-I

## SYLLABUS

Crisp sets and fuzzy sets – basic concept of fuzzy set – fuzzy logic-operations on fuzzy sets-general discussion fuzzy complents.

Research on the theory of fuzzy sets has been growing steadily since the inception of the theory in the mid-1960s. The body of concepts and results pertaining to the theory is now quite impressive. Research on a broad variety of applications has also been very active and has produced results that are perhaps even more impressive. In this book, we present an introduction to the major developments of the theory as well as to some of the most successful applications of the theory.

## Crisp Sets:

The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those that certainly belong in the set) and nonmembers (those that certainly do not). A sharp, unambiguous distinction exists between the members and nonmembers of the set. However, many classification concepts we commonly employ and express in natural language describe sets that do not exhibit this characteristic. Examples are the set of tall people, expensive cars, highly contagious diseases, close driving distances, modest profits, numbers much greater than one, or sunny days. We perceive these sets as having imprecise boundaries that facilitate gradual transitions from membership to nonmembership and vice versa.

## Fuzzy Sets:

As defined in the previous section, the characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and nonmembers of the crisp set under consideration. This function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote higher degrees of set membership. Such a function is called a *membership function*, and the set defined by it a *fuzzy set*.

The most commonly used range of values of membership functions is the unit interval [0, 1]. In this case, each membership function maps elements of a given *universal set X*, which is *always a crisp set*, into real numbers in [0, 1].

Two distinct notations are most commonly employed in the literature to denote membership functions. In one of them, the membership function of a fuzzy set $A$ is denoted by $\mu_A$; that is,

$$\mu_A : X \to [0, 1].$$

In the other one, the function is denoted by $A$ and has, of course, the same form:

$$A : X \to [0, 1].$$

## FUZZY SETS: BASIC CONCEPTS

In this section, we introduce some basic concepts and terminology of fuzzy sets. To illustrate the concepts, we consider three fuzzy sets that represent the concepts of a young,. middle-aged, and old person. A reasonable expression of these concepts by trapezoidal membership functions $A_1$, $A_2$, and $A_3$ is shown in Fig. 1.7. These functions are defined on the interval [0, 80] as follows:

$$A_1(x) = \begin{cases} 1 & \text{when } x \le 20 \\ (35 - x)/15 & \text{when } 20 < x < 35 \\ 0 & \text{when } x \ge 35 \end{cases}$$

$$A_2(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or } \ge 60 \\ (x - 20)/15 & \text{when } 20 < x < 35 \\ (60 - x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}$$

$$A_3(x) = \begin{cases} 0 & \text{when } x \le 45 \\ (x - 45)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } x \ge 60 \end{cases}$$

A possible discrete approximation, $D_2$, of function $A_2$, is also shown in Fig. 1.7; its explicit definition is given in Table 1.2. Such approximations are important because they are typical in computer representations of fuzzy sets.

One of the most important concepts of fuzzy sets is the concept of an *α-cut* and its variant, a *strong α-cut*. Given a fuzzy set $A$ defined on $X$ and any number $\alpha \in [0, 1]$, the $\alpha$-cut, ${}^{\alpha}A$, and the strong $\alpha$-cut, ${}^{\alpha+}A$, are the crisp sets

$$^{\alpha}A = \{x \mid A(x) \geq \alpha\}$$

That is, the $\alpha$-cut (or the strong $\alpha$-cut) of a fuzzy set $A$ is the crisp set $^{\alpha}A$ (or the crisp set $^{\alpha+}A$) that contains all the elements of the universal set $X$ whose membership grades in $A$ are greater than or equal to (or only greater than) the specified value of $\alpha$.

As an example, the following is a complete characterization of all $\alpha$-cuts and all strong $\alpha$-cuts for the fuzzy sets $A_1, A_2, A_3$ given in Fig. 1.7:

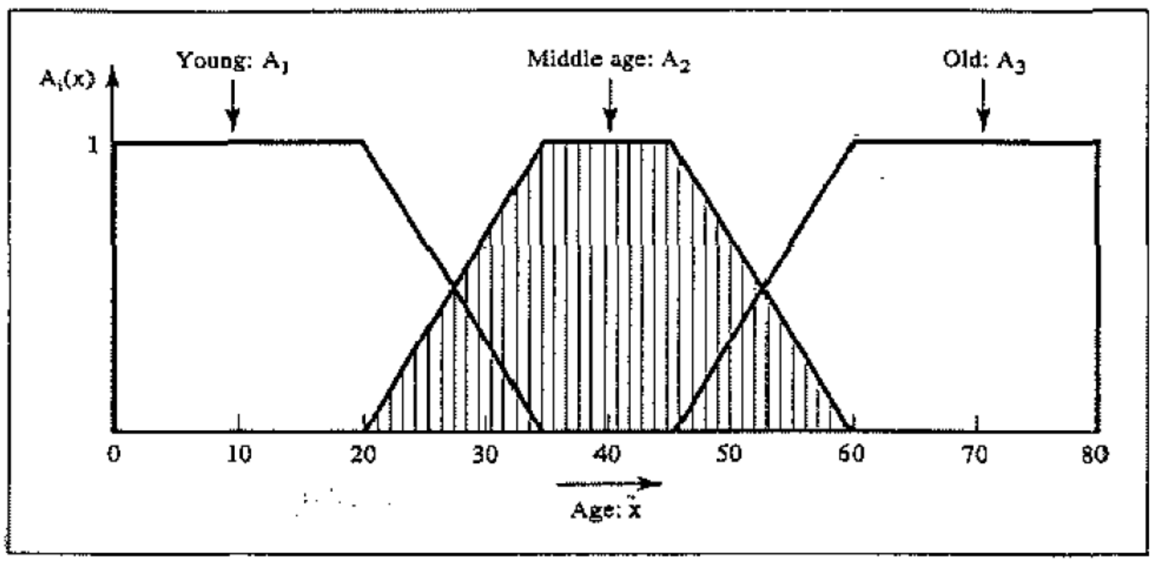



**Figure 1.7** Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation $D_2$ of $A_2$ is defined numerically in Table 1.2.

| $x$ | $D_2(x)$ |
|---|---|
| $x \notin \{22, 24, \ldots, 58\}$ | 0.00 |
| $x \in \{22, 58\}$ | 0.13 |
| $x \in \{24, 56\}$ | 0.27 |
| $x \in \{26, 54\}$ | 0.40 |
| $x \in \{28, 52\}$ | 0.53 |
| $x \in \{30, 50\}$ | 0.67 |
| $x \in \{32, 48\}$ | 0.80 |
| $x \in \{34, 46\}$ | 0.93 |
| $x \in \{36, 38, \ldots, 44\}$ | 1.00 |

**TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION $A_2$ (FIG. 1.7) BY FUNCTION $D_2$ OF THE FORM: $D_2 : \{0, 2, 4, \ldots, 80\} \to [0, 1]$**

$^{0}A_1 = {}^{0}A_2 = {}^{0}A_3 = [0, 80] = X;$

$^{\alpha}A_1 = [0, 35 - 15\alpha], {}^{\alpha}A_2 = [15\alpha + 20, 60 - 15\alpha], {}^{\alpha}A_3 = [15\alpha + 45, 80]$ for all $\alpha \in (0, 1];$

$^{\alpha+}A_1 = (0, 35 - 15\alpha), {}^{\alpha+}A_2 = (15\alpha + 20, 60 - 15\alpha), {}^{\alpha+}A_3 = (15\alpha + 45, 80)$ for all $\alpha \in [0, 1);$

$^{1+}A_1 = {}^{1+}A_2 = {}^{1+}A_3 = \varnothing.$

The set of all levels $\alpha \in [0, 1]$ that represent distinct $\alpha$-cuts of a given fuzzy set $A$ is called a *level set* of $A$. Formally,

$$\Lambda(A) = \{\alpha | A(x) = \alpha \text{ for some } x \in X\},$$

where $\Lambda$ denotes the level set of fuzzy set $A$ defined on $X$. For our examples, we have:

$\Lambda(A_1) = \Lambda(A_2) = \Lambda(A_3) = [0, 1]$, and

$\Lambda(D_2) = \{0, 0.13, 0.27, 0.4, 0.53, 0.67, 0.8, 0.93, 1\}.$

# Fuzzy Logic

## CLASSICAL LOGIC:

*Logic* is the study of the methods and principles of *reasoning* in all its possible forms. Classical logic deals with *propositions* that are required to be either *true* or *false*. Each proposition has its opposite, which is usually called a *negation* of the proposition. A proposition and its negation are required to assume opposite truth values.

### Logic formulas are then defined recursively as follows:

1. if $v$ denotes a logic variable, then $v$ and $\overline{v}$ are logic formulas;
2. if $a$ and $b$ denote logic formulas, then $a \wedge b$ and $a \vee b$ are also logic formulas;
3. the only logic formulas are those defined by the previous two rules.

Various forms of tautologies can be used for making deductive inferences. They are referred to as *inference rules*. Examples of some tautologies frequently used as inference rules are

$$(a \wedge (a \Rightarrow b)) \Rightarrow b \ (\textit{modus ponens}),$$

$$(\overline{b} \wedge (a \Rightarrow b)) \Rightarrow \overline{a} \ (\textit{modus tollens}),$$

$$((a \Rightarrow b) \wedge (b \Rightarrow c)) \Rightarrow (a \Rightarrow c) \ (\textit{hypothetical syllogism}).$$

### Operations on Classical Sets

Let A and B be two sets on the universe X. The union between the two sets, denoted A ∪ B, represents all those elements in the universe that reside in (or belong to) the set A, the set B, or both sets A and B. (This operation is also called the *logical or*; another form of the union is the *exclusive or* operation. The *exclusive or* will be described in Chapter 5.) The intersection of the two sets, denoted A ∩ B, represents all those elements in the universe X that simultaneously reside in (or belong to) both sets A and B. The complement of a set A, denoted $\overline{A}$, is defined as the collection of all elements in the universe that do not reside in the set A. The difference of a set A with respect to B, denoted A | B, is defined as the collection of all elements in the universe that reside in A and that do not reside in B simultaneously. These operations are shown below in set-theoretic terms.

*Union* $\mu_{\underset{\sim}{A}\cup\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x)$

*Intersection* $\mu_{\underset{\sim}{A}\cap\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x)$

*Complement* $\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x)$

Any fuzzy set $\underset{\sim}{A}$ defined on a universe X is a subset of that universe. Also by definition just as with classical sets, the membership value of any element $x$ in the null set Ø is 0

# FUZZY COMPLEMENTS

Let $A$ be a fuzzy set on $X$. Then, by definition, $A(x)$ is interpreted as *the degree to which $x$ belongs to $A$*. Let $cA$ denote a fuzzy *complement* of $A$ of type $c$. Then, $cA(x)$ may be interpreted not only as the degree to which $x$ belongs to $cA$, but also as *the degree to which $x$ does not belong to $A$*. Similarly, $A(x)$ may also be interpreted as the degree to which $x$ does not belong to $cA$.

As a notational convention, let a complement $cA$ be defined by a function

$$c : [0, 1] \to [0, 1],$$

which assigns a value $c(A(x))$ to each membership grade $A(x)$ of any given fuzzy set $A$. The value $c(A(x))$ is interpreted as the value of $cA(x)$. That is,

$$c(A(x)) = cA(x)$$

To produce meaningful fuzzy complements, function $c$ must satisfy at least the following two axiomatic requirements:

**Axiom c1.** $c(0) = 1$ and $c(1) = 0$ (*boundary conditions*).

**Axiom c2.** For all $a, b \in [0, 1]$, if $a \le b$, then $c(a) \ge c(b)$ (*monotonicity*).

In most cases of practical significance, it is desirable to consider various additional requirements for fuzzy complements. Each of them reduces the general class of fuzzy complements to a special subclass. Two of the most desirable requirements, which are usually listed in the literature among axioms of fuzzy complements, are the following:

**Axiom c3.** $c$ is a continuous function.

**Axiom c4.** $c$ is *involutive*, which means that $c(c(a)) = a$ for each $a \in [0, 1]$.

**Theorem 3.2.** Every fuzzy complement has at most one equilibrium.

***Proof:*** Let $c$ be an arbitrary fuzzy complement. An equilibrium of $c$ is a solution of the equation

$$c(a) - a = 0,$$

where $a \in [0, 1]$. We can demonstrate that any equation $c(a) - a = b$, where $b$ is a real constant, must have at most one solution, thus proving the theorem. In order to do so, we assume that $a_1$ and $a_2$ are two different solutions of the equation $c(a) - a = b$ such that $a_1 < a_2$. Then, since $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$, we get

$$c(a_1) - a_1 = c(a_2) - a_2. \tag{3.7}$$

However, because $c$ is monotonic nonincreasing (by Axiom c2), $c(a_1) \geq c(a_2)$ and, since $a_1 < a_2$,

$$c(a_1) - a_1 > c(a_2) - a_2.$$

This inequality contradicts (3.7), thus demonstrating that the equation must have at most one solution. ■

**POSSIBLE QUESTIONS**

**PART-B (5 x 8 =40 Marks)**

**Answer all the questions**

1. Explain operations of fuzzy sets
2. Prove that every fuzzy complement has at most one equilibrium
3. Prove that for each a ∈ $[0,1], d_a = c(a)$ if and only if $c(c(a)) = a$ that is which the complement is involutive.
4. Explain briefly about fuzzy complement.
5. If ‘c’ is a continuous fuzzy complement then prove that ‘c’ has a unique equilibrium.
6. Write briefly about Boolean algebra.
7. The fuzzy complement c has an equilibrium $e_c$, which is unique, then prove that $a \leq c(a)$ if and only if $a \leq e_c$ , $a \geq c(a)$and a ≥ $e_c$.
8. Explain extension principle of fuzzy sets.
9. Let the interval $X = [0,10]$ by the memebership rate function $A(x) = \frac{x}{x+2}$
   Find (i)$\bar{A}$
   (ii) $\alpha$ – cuts and strong $\alpha -$ cuts for $\alpha$ = 0.2, 0.5, 0.8, 1
10. If a complement ‘c’ has an equilibrium $e_c$, then prove that $d_{e_c'} = e_c$

# UNIT-II

## SYLLABUS

| |
|---|
| Fuzzy union, fuzzy intersection, combinations operations. |

## Fuzzy unions

### Definition

A **union** of two fuzzy sets $A$ and $B$ is given by a function of the form

$$u : [0,1] \times [0,1] \to [0,1].$$

A value is assigned to a pair of membership values $A(x)$ and $B(x)$ of an element $x$ of the universal set $X$. It represents membership of $x$ to the union of $A$ and $B$:

$$(A \cup B)(x) = u(A(x), B(x)), \qquad \text{for } x \in X.$$

Note:

- Intuitive requirements to be fulfilled by a function $u$ to qualify as a union of fuzzy sets are those of well known and extensively studied $t$-conorms (triangular conorms); the names *fuzzy union* and *t-conorm* are therefore used interchangeably in the literature.
- The value $(A \cup B)(x)$ does not depend on $x$, but only on $A(x)$ and $B(x)$.

## Fuzzy unions

### Axiomatic requirements

For all $a, b, d \in [0,1]$,

**Ax u1.** $u(a, 0) = a$. **boundary condition**

**Ax u2.** $b \leq d$ implies $u(a, b) \leq u(a, d)$. **monotonicity**

**Ax u3.** $u(a, b) = u(b, a)$. **commutativity**

**Ax u4.** $u(a, u(b, d)) = u(u(a, b), d)$. **associativity**

Axioms **u1** - **u4** are called **axiomatic skeleton for fuzzy unions**. They differ from the axiomatic skeleton of fuzzy intersections only in boundary condition.

For crisp sets, $u$ behaves like a classical (crisp) union.

## Fuzzy unions

### Additional (optional) requirements

For all $a, b, d \in [0, 1]$,

**Ax u5.** $u$ is a continuous function. **continuity**

**Ax u6.** $u(a, a) \geq a$. **superidempotency**

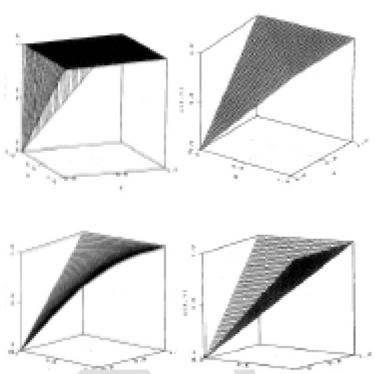
**Ax u7.** $a_1 < a_2$ and $b_1 < b_2$ implies $u(a_1, b_1) < u(a_2, b_2)$. **strict monotonicity**

Note:

Requirements **u5** - **u7** are analogous to Axioms **i5** - **i7**.
Superidempotency is a weaker requirement than idempotency.
A continuous superidempotent $t$-conorm is called **Archimedean $t$-conorm**.
The standard fuzzy union, $u(a, b) = \max[a, b]$, is the only idempotent $t$-conorm.

## Fuzzy unions

### Examples of $t$-conorms frequently used



- **Drastic union**
  $u(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$
- **Bounded sum**
  $u(a, b) = \min[1, a + b]$
- **Algebraic sum**
  $u(a, b) = a + b - ab$
- **Standard intersection**
  $u(a, b) = \max[a, b]$

## Fuzzy unions

### Properties

- $\max[a, b] \leq a + b - ab \leq \min(1, a + b) \leq u_{\max}(a, b)$.
- For all $a, b \in [0, 1]$, $\max[a, b] \leq u(a, b) \leq u_{\max}(a, b)$.

## Fuzzy unions

How to generate $t$-conorms

### Theorem

**(Characterization Theorem of $t$-conorms)** *Let $u$ be a binary operation on the unit interval. Then, $u$ is an Archimedean $t$-conorm iff there exists an increasing generator $g$ such that*

$$u(a,b) = g^{(-1)}(g(a)+g(b)), \qquad \text{for } a,b \in [0,1].$$

**Example:** A class of increasing generators $f_\omega(a) = a^\omega$, $\omega > 0$ generates a Yager class of $t$-conorms

$$u_\omega(a,b) = \min[1, (a^\omega + b^\omega)^{\frac{1}{\omega}}], \quad \omega > 0.$$

It can be proved that $\max[a,b] \le u_\omega(a,b) \le u_{max}(a,b)$.

## Fuzzy intersections

Definition

An **intersection** of two fuzzy sets $A$ and $B$ is given by a function of the form

$$i : [0,1] \times [0,1] \to [0,1].$$

A value is assigned to a pair of membership values $A(x)$ and $B(x)$ of an element $x$ of the universal set $X$. It represents membership of $x$ to the intersection of $A$ and $B$:

$$(A \cap B)(x) = i(A(x), B(x)), \qquad \text{for } x \in X.$$

Note:

- Intuitive requirements to be fulfilled by a function $i$ to qualify as an intersection of fuzzy sets are those of well known and extensively studied $t$-norms (triangular norms); the names *fuzzy intersection* and *$t$-norm* are therefore used interchangeably in the literature.
- The value $(A \cap B)(x)$ does not depend on $x$, but only on $A(x)$ and $B(x)$.

## Fuzzy intersections

Axiomatic requirements

For all $a,b,d \in [0,1]$,

**Ax i1.** $i(a,1) = a$. **boundary condition**

**Ax i2.** $b \le d$ implies $i(a,b) \le i(a,d)$. **monotonicity**

**Ax i3.** $i(a,b) = i(b,a)$. **commutativity**

**Ax i4.** $i(a, i(b,d)) = i(i(a,b),d)$. **associativity**

Axioms **i1** - **i4** are called **axiomatic skeleton for fuzzy intersections**.

If the sets are crisp, $i$ becomes the classical (crisp) intersection.

## Fuzzy intersections

Additional (optional) requirements

For all $a, b, d \in [0, 1]$,

**Ax i5.** $i$ is a continuous function. **continuity**

**Ax i6.** $i(a, a) \leq a$. **subidempotency**

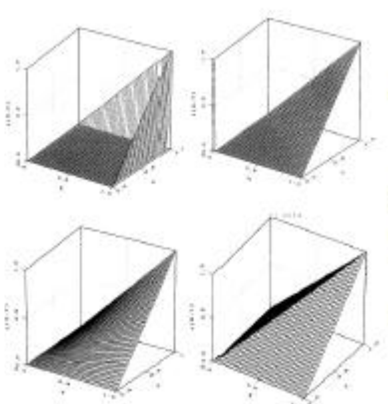
**Ax i7.** $a_1 < a_2$ and $b_1 < b_2$ implies $i(a_1, b_1) < i(a_2, b_2)$.
**strict monotonicity**

Note:

Subidempotency is a weaker requirement than idempotency, $i(a, a) = a$.
A continuous subidempotent $t$-norm is called **Archimedean $t$-norm**
The standard fuzzy intersection, $i(a, b) = \min[a, b]$, is the only idempotent $t$-norm.

## Fuzzy intersections

Examples of $t$-norms frequently used



- **Drastic intersection**
$$i(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$
- **Bounded difference**
$i(a, b) = \max[0, a + b - 1]$
- **Algebraic product**
$i(a, b) = ab$
- **Standard intersection**
$i(a, b) = \min[a, b]$

## Fuzzy intersections

Properties

- $i_{\min}(a, b) \leq \max(0, a + b - 1) \leq ab \leq \min(a, b)$.
- For all $a, b \in [0, 1]$, $i_{\min}(a, b) \leq i(a, b) \leq \min[a, b]$.

## Fuzzy intersections

How to generate $t$-norms

**Theorem**

**(Characterization Theorem of $t$-norms)** *Let $i$ be a binary operation on the unit interval. Then, $i$ is an Archimedean $t$-norm iff there exists a decreasing generator $f$ such that*

$$i(a, b) = f^{(-1)}(f(a) + f(b)), \qquad \text{for } a, b \in [0, 1].$$

**Example:** A class of decreasing generators $f_\omega(a) = (1-a)^\omega$, $\omega > 0$ generates a Yager class of $t$-norms

$$i_\omega(a,b) = 1 - \min[1, ((1-a)^\omega + (1-b)^\omega)^{\frac{1}{\omega}}], \quad \omega > 0.$$

It can be proved that $i_{\min}(a,b) \le i_\omega(a,b) \le \min[a,b]$.

## Decision as a fuzzy set

**Intersection**: No positive compensation (trade-off) between the memberships of the fuzzy sets observed.

**Union**: Full compensation of lower degrees of membership by the maximal membership.

In reality of decision making, rarely either happens.

(non-verbal) "merging connectives" $\rightarrow$ (language) connectives {'and', 'or',...,}.

Aggregation operations called **compensatory and** are needed to model fuzzy sets representing to, e.g., managerial decisions.

**Definitions:**

The membership function $\mu_{\tilde{C}}(x)$ of the *intersection* $\tilde{C} = \tilde{A} \cap \tilde{B}$ is pointwise defined by

$$\mu_{\tilde{C}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad x \in X$$

The membership function $\mu_{\tilde{D}}(x)$ of the *union* $\tilde{D} = \tilde{A} \cup \tilde{B}$ is pointwise defined by

$$\mu_{\tilde{D}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad x \in X$$

The membership function of the *complement* of a normalized fuzzy set $A$, $\mu_{\complement\tilde{A}}(x)$ is defined by

$$\mu_{\complement\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x), \quad x \in X$$

**Exmple:**

Let $\tilde{A}$ be the fuzzy set "comfortable type of house for a four-person family" from example 2–1a and $\tilde{B}$ be the fuzzy set "large type of house" defined as

$$\tilde{B} = \{(3, .2), (4, .4), (5, .6), (6, .8), (7, 1), (8, 1)\}$$

The intersection $\tilde{C} = \tilde{A} \cap \tilde{B}$ is then

$$\tilde{C} = \{(3, .2), (4, .4), (5, .6), (6, .3)\}$$

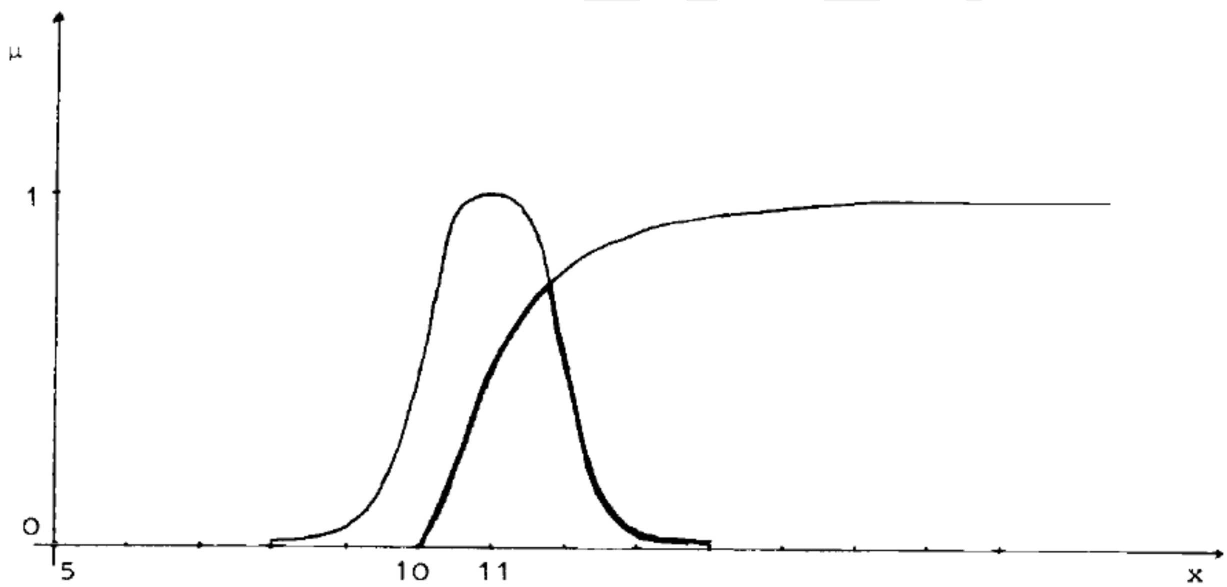
The union $\tilde{D} = \tilde{A} \cup \tilde{B}$ is

$$\tilde{D} = \{(1, .2), (2, .5), (3, .8), (4, 1)\ (5, .7)\ (6, .8), (7, 1), (8, 1)\}$$

The complement $\complement\tilde{B}$, which might be interpreted as "not large type of hous is

$$\complement\tilde{B} = \{(1, 1), (2, 1), (3, .8), (4, .6), (5, .4), (6, .2), (9, 1), (10, 1)\}$$

**Fig: Union and intersection on fuzzy sets**




Combinations of set operations

De Morgan laws and duality of fuzzy operations

De Morgan laws in classical set theory:

$$\overline{A \cap B} = \bar{A} \cup \bar{B} \quad \text{and} \quad \overline{A \cup B} = \bar{A} \cap \bar{B}.$$

The union and intersection operation are **dual** with respect to the complement.

De Morgan laws for fuzzy sets:

$$c(i(A, B)) = u(c(A), c(B)) \quad \text{and} \quad c(u(A, B)) = i(c(A), c(B))$$

for a $t$-norm $i$, a $t$-conorm $u$, and fuzzy complement $c$.

Notation: $\langle i, u, c\rangle$ denotes a **dual triple**.

## Duality of fuzzy set operations

Examples of dual triples

Dual triples with respect to the **standard** fuzzy complement

$$\langle \min(a,b), \max(a,b), c_s \rangle$$
$$\langle ab, a+b-ab, c_s \rangle$$
$$\langle \max(0, a+b-1), \min(1, a+b), c_s \rangle$$
$$\langle i_{\min}(a,b), u_{\max}(a,b), c_s \rangle$$

## Standard fuzzy operations

Recall:
for fuzzy sets $A, B$ on a reference set $X$, given by the corresponding membership functions $A(x)$ and $B(x)$:

$\bar{A}(x) = 1 - A(x)$ – fuzzy complement
$(A \cap B)(x) = \min[A(x), B(x)]$ – fuzzy intersection
$(A \cup B)(x) = \max[A(x), B(x)]$ – fuzzy union

for all $x \in X$.

## Properties of the standard operations

- They are **generalizations** of the corresponding (uniquely defined!) classical set operations.
- They satisfy the **cutworthy and strong cutworthy** properties. They are the only ones that do.
- The standard fuzzy **intersection** of two sets **contains** (is bigger than) all other fuzzy intersections of those sets.
- The standard fuzzy **union** of two sets **is contained in** (is smaller than) all other fuzzy unions of those sets.
- They inherently **prevent the compound of errors** of the operands.

**POSSIBLE QUESTIONS**

**PART-B (5 x 8 =40 Marks)**

**Answer all the questions**

1. Show that $u(a,b) \geq \max(a,b)$ if for all $a,b \in [0,1]$
2. Prove that $u(a,b) = \max(a,b)$ is the only continuous and idempotent fuzzy set union.
3. Prove that $\lim_{w\to\infty} i_w = \lim_{w\to\infty}\left[1 - \min\left(1, [(1-a)^w + (1-b)^w]^{\frac{1}{w}}\right)\right]$
4. If for all $a,b \in [0,1]$ then prove that $i(a,b) \geq i_{min}(a,b)$
5. Prove that $\lim_{w\to\infty} \min\left[1, (a^w + b^w)^{\frac{1}{w}}\right] = \max(a,b)$
6. prove that the standard fuzzy intersection is the only idempotent t – norms.
7. Show that for all $a,b \in [0,1]$ then $i(a,b) \leq \min(a,b)$
8. Prove that fuzzy set operation of union intersection and continuous complement that satisfies the law of excluded middle and the law of contradiction are not idempotent or distributive
9. Show that $u(a,b) \leq u_{max}(a,b)$ if for all $a,b \in [0,1]$
10. explain about fuzzy union and intersection.

**UNIT-III**

**SYLLABUS**

Fuzzy relations and fuzzy graphs , fuzzy relation on sets and fuzzy sets, composition of fuzzy relations, properties of the min-max composition, fuzzy graphs, special fuzzy relations.

Fuzzy relations are fuzzy subsets of $X \times Y$, that is, mappings from $X \to Y$. They have been studied by a number of authors, in particular by Zadeh [1965, 1971], Kaufmann [1975], and Rosenfeld [1975]. Applications of fuzzy relations are widespread and important. We shall consider some of them and point to more possible uses at the end of this chapter. We shall exemplarily consider only binary relations. A generalization to $n$-ary relations is straightforward.

# Fuzzy Relations on Sets and Fuzzy Sets

Fuzzy relations are fuzzy subsets of $X \times Y$, that is, mappings from $X \to Y$. They have been studied by a number of authors, in particular by Zadeh [1965, 1971], Kaufmann [1975], and Rosenfeld [1975]. Applications of fuzzy relations are widespread and important. We shall consider some of them and point to more possible uses at the end of this chapter. We shall exemplarily consider only binary relations. A generalization to $n$-ary relations is straightforward.

***Definition***

Let $X$, $Y \subseteq \mathbb{R}$ be universal sets; then

$$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$$

is called a *fuzzy relation* on $X \times Y$.

***Example***

Let $X = Y = \mathbb{R}$ and $\tilde{R}$: = "considerably larger than." The membership function of the fuzzy relation, which is, of course, a fuzzy set on $X \times Y$, can then be

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ \dfrac{(x-y)}{10y} & \text{for } y < x \leq 11y \\ 1 & \text{for } x > 11y \end{cases}$$

A different membership function for this relation could be

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ (1 + (y-x)^{-2})^{-1} & \text{for } x > y \end{cases}$$

For discrete supports, fuzzy relations can also be defined by matrixes.

## FUZZY RELATIONS AND FUZZY GRAPHS

### *Definition 6–2*

Let $X, Y \subseteq \mathbb{R}$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\},$$

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y \in Y\} \quad \text{two fuzzy sets.}$$

Then $\tilde{R} = \{[(x, y), \mu_{\tilde{R}}(x, y)] \mid (x, y) \in X \times Y\}$ is a *fuzzy relation* on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{A}}(x), \forall (x, y) \in X \times Y$$

and

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{B}}(y), \forall (x, y) \in X \times Y.$$

Let $\tilde{R}$ and $\tilde{Z}$ be two fuzzy relations in the same product space. The *union/intersection* of $\tilde{R}$ with $\tilde{Z}$ is then defined by

$$\mu_{\tilde{R}\cup\tilde{Z}}(x, y) = \max\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, \quad (x, y) \in X \times Y$$

$$\mu_{\tilde{R}\cap\tilde{Z}}(x, y) = \min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, \quad (x, y) \in X \times Y$$

## Example

Let $\tilde{R}$ and $\tilde{Z}$ be the two fuzzy relations defined in example 6–2. The union of $\tilde{R}$ and $\tilde{Z}$, which can be interpreted as "$x$ considerably larger or very close to $y$," is then given by

$\tilde{R} \cup \tilde{Z}$:

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | .8 | 1 | .9 | .7 |
| $x_2$ | .9 | .8 | .5 | .7 |
| $x_3$ | .9 | 1 | .8 | .8 |

The intersection of $\tilde{R}$ and $\tilde{Z}$ is represented by

$\tilde{R} \cap \tilde{Z}$:

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | .4 | 0 | .1 | .6 |
| $x_2$ | 0 | .4 | 0 | 0 |
| $x_3$ | .3 | 0 | .7 | .5 |

## Definition

Let $\tilde{R} = \{ [(x, y), \mu_{\tilde{R}}(x, y)] \mid (x, y) \in X \times Y\}$ be a fuzzy binary relation. The *first projection* of $\tilde{R}$ is then defined as

$$\tilde{R}^{(1)} = \{(x, \max_y \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$$

The *second projection* is defined as

$$\tilde{R}^{(2)} = \{(y, \max_x \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$$

and the *total projection* as

$$\tilde{R}^{(T)} = \max_{x} \max_{y} \{\mu_{\tilde{R}}(x, y) \mid (x, y) \in X \times Y\}$$

## Example

Let $\tilde{R}$ be a fuzzy relation defined by the following relational matrix. The first, second, and total projections are then shown at the appropriate places below.

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | First projection $[\mu_{\tilde{R}^{(1)}}(x)]$ |
|---|---|---|---|---|---|---|---|
| $x_1$ | .1 | .2 | .4 | .8 | 1 | .8 | 1 |
| $\tilde{R}$: $x_2$ | .2 | .4 | .8 | 1 | .8 | .6 | 1 |
| $x_3$ | .4 | .8 | 1 | .8 | .4 | .2 | 1 |
| Second projection: $[\mu_{\tilde{R}^{(2)}}(x)]$ | .4 | .8 | 1 | 1 | 1 | .8 | 1 Total projection |

## Compositions of Fuzzy Relations

Fuzzy relations in different product spaces can be combined with each other by the operation "composition." Different versions of "composition" have been suggested, which differ in their results and also with respect to their mathematical properties. The max-min composition has become the best known and the most frequently used one. However, often the so-called max-product or max-average compositions lead to results that are more appealing.

## Definition

*Max-min composition*: Let $\tilde{R}_1(x, y)$, $(x, y) \in X \times Y$ and $\tilde{R}_2(y, z)$, $(y, z) \in Y \times Z$ be two fuzzy relations. The max-min composition $\tilde{R}_1$ max-min $\tilde{R}_2$ is then the fuzzy set

$$\tilde{R}_1 \circ \tilde{R}_2 = \{[(x, z), \max_y \{\min \{\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z)\}\}] \mid x \in X, y \in Y, z \in Z\}$$

$\mu_{\tilde{R}_1 \circ \tilde{R}_2}$ is again the membership function of a fuzzy relation on fuzzy sets

## Definition

Let $\tilde{R}_1$ and $\tilde{R}_2$ be defined as in definition 6–7. The *max-* composition* of $\tilde{R}_1$ and $\tilde{R}_2$ is then defined as

$$\tilde{R}_1 \overset{\circ}{*} \tilde{R}_2 = \{[(x, z), \max_y (\mu_{\tilde{R}_1}(x, y) * \mu_{\tilde{R}_2}(y, z))] \mid x \in X, y \in Y, z \in Z\}$$

If * is an associative operation that is monotonically nondecreasing in each argument, then the max-* composition corresponds essentially to the max-min composition. Two special cases of the max-* composition are proposed in the next definition.

## Properties of the Min-Max Composition

**Associativity.** The max-min composition is *associative*, that is,

$$(\tilde{R}_3 \circ \tilde{R}_2) \circ \tilde{R}_1 = \tilde{R}_3 \circ (\tilde{R}_2 \circ \tilde{R}_1).$$

Hence $\tilde{R}_1 \circ \tilde{R}_1 \circ \tilde{R}_1 = \tilde{R}_1^3$, and the third power of a fuzzy relation is defined.

## Reflexivity

## Definition

Let $\tilde{R}$ be a fuzzy relation in $X \times X$.

1. $\tilde{R}$ is called *reflexive* [Zadeh 1971] if

$$\mu_{\tilde{R}}(x, x) = 1 \quad \forall x \in X$$

2. $\tilde{R}$ is called *ε-reflective* [Yeh 1975] if

$$\mu_{\tilde{R}}(x, x) \geqslant \varepsilon \ \forall x \in X$$

3. $\tilde{R}$ is called *weakly reflexive* [Yeh 1975] if

$$\left.\begin{array}{l}\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x) \\ \mu_{\tilde{R}}(y, x) \leq \mu_{\tilde{R}}(x, x)\end{array}\right\} \forall x, y \in X.$$

***Example***

Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.
The following relation "$y$ is close to $x$" is reflexive:

| $\tilde{R}$: | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | 1 | 0 | .2 | .3 |
| $x_2$ | 0 | 1 | .1 | 1 |
| $x_3$ | .2 | .7 | 1 | .4 |
| $x_3$ | 0 | 1 | .4 | 1 |

If $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive fuzzy relations, then the max-min composition $\tilde{R}_1 \circ \tilde{R}_2$ is also reflexive.

## Symmetry

***Definition***

A fuzzy relation $\tilde{R}$ is called *symmetric* if $\tilde{R}(x, y) = \tilde{R}(y, x)\ \forall x, y \in X$.

A relation is called *antisymmetric* if for

$$\left.\begin{array}{lll} x \neq y & \text{either} & \mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x) \\ & \text{or} & \mu_{\tilde{R}}(x, y) = \mu_{\tilde{R}}(y, x) = 0 \end{array}\right\} \forall x, y \in X$$

A relation is called *perfectly antisymmetric* if for $x \neq y$ whenever

$$\mu_{\tilde{R}}(x, y) > 0 \quad \text{then} \quad \mu_{\tilde{R}}(y, x) = 0 \ \forall x, y \in X$$

***Example***

$\tilde{R}_1$:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | .4 | 0 | .1 | .8 |
| $x_2$ | .8 | 1 | 0 | 0 |
| $x_3$ | 0 | .6 | .7 | 0 |
| $x_4$ | 0 | .2 | 0 | 0 |

$\tilde{R}_2$:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | .4 | 0 | .7 | 0 |
| $x_2$ | 0 | 1 | .9 | .6 |
| $x_3$ | .8 | .4 | .7 | .4 |
| $x_4$ | 0 | .1 | 0 | 0 |

$\tilde{R}_3$:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | .4 | .8 | .1 | .8 |
| $x_2$ | .8 | 1 | .0 | .2 |
| $x_3$ | .1 | .6 | .7 | .1 |
| $x_4$ | 0 | .2 | 0 | 0 |

$\tilde{R}_1$ is a perfectly antisymmetric relation, while $\tilde{R}_2$ is an antisymmetric, but not perfectly antisymmetric relation. $\tilde{R}_3$ is a nonsymmetric relation, that is, there exist $x, y \in X$ with $\mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x)$, which is not antisymmetric and therefore also not perfectly antisymmetric.

## *Remark*

For max-min compositions, the following properties hold:

1. If $\tilde{R}_1$ is reflexive and $\tilde{R}_2$ is an arbitrary fuzzy relation, then $\tilde{R}_1 \circ \tilde{R}_2 \supseteq \tilde{R}_2$ and $\tilde{R}_2 \circ \tilde{R}_1 \supseteq \tilde{R}_2$.
2. If $\tilde{R}$ is reflexive, then $\tilde{R} \subseteq \tilde{R} \circ \tilde{R}$.
3. If $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive relations, so is $\tilde{R}_1 \circ \tilde{R}_2$.
4. If $\tilde{R}_1$ and $\tilde{R}_2$ are symmetric, then $\tilde{R}_1 \circ \tilde{R}_2$ is symmetric if $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$.
5. If $\tilde{R}$ is symmetric, so is each power of $\tilde{R}$.

## Transitivity

## *Definition*

A fuzzy relation $\tilde{R}$ is called (max-min) *transitive* if

$$\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$$

## *Remark*

Combinations of the above properties give some interesting results for max-min compositions:

1. If $\tilde{R}$ is symmetric and transitive, then $\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x)$ for all $x, y \in X$.
2. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R} \circ \tilde{R} = \tilde{R}$.
3. If $\tilde{R}_1$ and $\tilde{R}_2$ are transitive and $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$, then $\tilde{R}_1 \circ \tilde{R}_2$ is transitive.

# Fuzzy Graphs

## Definition

Let $E$ be the (crisp) set of nodes. A *fuzzy graph* is then defined by

$$\tilde{G}(x_i, x_j) = \{((x_i, x_j), \mu_{\tilde{G}}(x_i, x_j)) \mid (x_i, x_j) \in E \times E\}$$

## Definition

$\tilde{H}(x_i, x_j)$ is a *fuzzy subgraph* of $\tilde{G}(x_i, x_j)$ if

$$\mu_{\tilde{H}}(x_i, x_j) \leq \mu_{\tilde{G}}(x_i, x_j) \; \forall (x_i, x_j) \in E \times E$$

$\tilde{H}(x_i, x_j)$ *spans* graph $\tilde{G}(x_i, x_j)$ if the node sets of $\tilde{H}(x_i, x_j)$ and $\tilde{G}(x_i, x_j)$ are equal, that is, if they differ only in their arc weights.

## Example

Let $\tilde{G}(x_i, x_j)$ be defined as in example 6–11b. A spanning subgraph of $\tilde{G}(x_i, x_j)$ is then

$$\tilde{H}(x_i, x_j) = \{[(x_1, x_2), .2], [(x_1, x_3), .4], [(x_3, x_2), .4], [(x_4, x_3), .7]\}$$

# Special Fuzzy Relations

## Definition

A *similarity relation* is a fuzzy relation $\mu_s(\cdot)$ that is reflexive, symmetrical, and max-min transitive.

A fuzzy relation that is (max-min) transitive and reflexive is called a *fuzzy preorder relation.*

A *total fuzzy order relation* [Kaufmann 1975, p. 112] or a *fuzzy linear ordering* [Dubois and Prade 1980a, p. 82; Zadeh 1971] is a fuzzy order relation such that $\forall x, y \in X$; $x \neq y$ either $\mu_{\tilde{R}}(x, y) > 0$ or $\mu_{\tilde{R}}(y, x) > 0$.

Any $\alpha$-cut of a fuzzy linear order is a *crisp linear order*.

## Example

$\tilde{R}$:

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | .7 | .4 | .8 | .8 |
| $x_2$ | 0 | 1 | 0 | .2 |
| $x_3$ | 0 | .6 | 0 | .4 |
| $x_4$ | 0 | 0 | 0 | .7 |

$\tilde{R}$ is a total fuzzy order relation.

**POSSIBLE QUESTIONS**

**PART-B 5x8=40**

**Answer all the questions**

1. Find i) first projection ii) second projection and iii) Total projection if $\tilde{R}$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | Y5 | Y6 |
|---|---|---|---|---|---|---|
| $X_1$ | 0.1 | 0.2 | 0.4 | 1 | 1 | 0.8 |
| $X_2$ | 0.2 | 0.4 | 0.8 | 0.8 | 0.8 | 0.6 |
| $X_3$ | 0.4 | 0.8 | 1 | 0.4 | 0.4 | 0.2 |

2. Compute i) $\tilde{R} \cup \tilde{Z}$ ii) $\tilde{R} \cap \tilde{Z}$

If $\tilde{R}$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ |
|---|---|---|---|---|
| $X_1$ | 0.8 | 1 | 0.1 | 0.7 |
| $X_2$ | 0 | 0.8 | 0 | 0 |
| $X_3$ | 0.9 | 1 | 0.7 | 0.8 |

And

$\tilde{Z}$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | Y4 |
|---|---|---|---|---|
| $X_1$ | 0.4 | 0 | 0.9 | 0.6 |
| X2 | 0.9 | 0.4 | 0.5 | 0.7 |
| X3 | 0.3 | 0 | 0.8 | 0.5 |

3. Compute $\tilde{R}\circ\tilde{R}$ for the fuzzy relation $\tilde{R}$ is defined as

| | $X_1$ | $X_2$ | $X_3$ | X4 |
|---|---|---|---|---|
| $X_1$ | 0.2 | 1 | 0.4 | 0.4 |
| $X_2$ | 0 | 0.6 | 0.3 | 0 |
| $X_3$ | 0 | 1 | 0.3 | 0 |
| $X_4$ | 0.1 | 1 | 1 | 0.1 |

4. Compute i) min-max composition, ii) max-prod composition iii) max-av composition of

if $\widetilde{R}_1$ and $\widetilde{R}_2$

$\widetilde{R}_1$ :

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ |
|---|---|---|---|---|---|
| $X_1$ | 0.1 | 0.2 | 0 | 1 | 0.7 |
| $X_2$ | 0.3 | 0.5 | 0 | 0.2 | 1 |
| X3 | 0.8 | 0 | 1 | 0.4 | 0.3 |

$\widetilde{R_2}$ :

| | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ |
|---|---|---|---|---|
| $Y_1$ | 0.9 | 0 | 0.3 | 0.4 |
| $Y_2$ | 0.2 | 1 | 0.8 | 0 |
| $Y_3$ | 0.8 | 0 | 0.7 | 1 |
| $Y_4$ | 0.4 | 0.2 | 0.3 | 0 |
| $Y_5$ | 0 | 1 | 0 | 0.8 |

5. Discuss the reflexivity

properties of the following fuzzy relation:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $x_1$ | 1 | 0.7 | 0.3 |
| $x_2$ | 0.4 | 0.5 | 0.8 |
| x3 | 0.7 | 0.5 | 1 |

6. Discuss the following relation is a similarity relation

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ |
|---|---|---|---|---|---|---|
| $X_1$ | 1 | 0.2 | 1 | 0.6 | 0.2 | 0.6 |
| $X_2$ | 0.2 | 1 | 0.2 | 0.2 | 0.8 | 0.2 |
| $X_3$ | 1 | 0.2 | 1 | 0.6 | 0.2 | 0.6 |
| $X_4$ | 0.6 | 0.2 | 0.6 | 1 | 0.2 | 0.8 |
| $X_5$ | 0.2 | 0.8 | 0.2 | 0.2 | 1 | 0.2 |
| $X_6$ | 0.6 | 0.2 | 0.6 | 0.8 | 0.2 | 1 |

7. Explain the graphs that are forests and give examples of forest and not forest.
8. Explain about properties of the min-max composition.