Key answer

Internal - I

complex analysis - II

Sub code : 15mmu602

class : III B.Sc mathematics.

Part - A

1. zero
2. entire function
3. constant
4. one
5. Isolated
6. boundary
7. m
8. 3
9. $|z| < \infty$
10. constant
11. zero
12. 2
13. polynomial of degree n
14. (zeros
15. order of vanishing
16. constant
17. simple zero
18. absolutely convergent
19. centre
20. 1

Theorem:

Cauchy inequality:

$|z-a|$

Suppose,

i) D is a simply-connected region.

ii) $f(z)$ is analytic in D.

iii) Γ is any circle $|z-a| = r$ in D, then $|f^{n}(a)| \leq \dfrac{n!\, M(f; r, a)}{r^{n}}$

where $M(f, r, a)$ is the maximum of $f(z)$ on Γ.

Proof:

From cauchy's Formula for n^{th} derivatives,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz \right|$$

$$= \frac{n!}{2\pi} \left| \int_{\Gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz \right|$$

2.1(b) Let the harmonic conjugate of $u(x,y)$ be $v(x,y)$.

Then $f(z)=u+iv$ is analytic in D and by the previous theorem then mean value of $f(z)$ on Γ. $|z-a|=r$ in D

is

$$f(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt.$$

If $a=\alpha+i\beta$ then

$f(a)=u(\alpha,\beta)+iv(\alpha,\beta)$ and

$f(a+re^{it})=u(\alpha+r\cos t,\beta+r\sin t)+iv(\alpha+r\cos t,\beta+r\sin t)$

$$u(\alpha,\beta)+iv(\alpha,\beta)=\frac{1}{2\pi}\int_0^{2\pi} u(\alpha+r\cos t,\beta+r\sin t)\,dt+\frac{i}{2\pi}\int_0^{2\pi} v(\alpha+r\cos t,\beta+r\sin t)\,dt$$

Equating the real parts.

$$u(\alpha,\beta)=\frac{1}{2\pi}\int_0^{2\pi} u(\alpha+r\cos t,\beta+r\sin t)\,dt$$

which is the mean value of the values of $u(x,y)$ on Γ.

Fundamental theorem of Algebra.

Every polynomial in z of degree equal to or greater than one, has atleast one zero.

Proof:

Let $P(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n$ be the given polynomial and suppose it has no zero. Then the function

$$f(z) = \frac{1}{P(z)}$$

is an entire function. Further $f(z)$ is bounded in the finite plane because, for no value of z, $p(z) = 0$.

Here by Liouville's Theorem, $f(z)$ is a constant. This means that $P(z)$ is constant. which is a contradiction. So the assumption that $P(z)$ has no zero, does not hold

$\therefore P(z)$ has atleast one zero.

Laurent Series

Suppose

1) D is a multiply-connected region.
2) $f(z)$ is analytic in D and.
3) $\Gamma_1 = |t-a| = R_1$, $\Gamma_2 : |t-a| = R_2$ are the smallest and largest circles. Such that, the annulus region D, b/w Γ_1 and Γ_2 lies in D.

Then for all z in D, the power series $\sum_{-\infty}^{+\infty} a_n(z-a)^n$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt$ and C is any sero curve in D is absolutely convergent and its sum is $f(z)$.

Proof

Let $\Gamma_1 : |t-a| = r_1$ and $\Gamma_2 : |t-a| = r_2$ be 2 positively oriented circles in D of which Γ_1' is smaller. Now by extension to Cauchy Integral formula for any z in D_2

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_2'} \frac{f(t)}{t-z} dt - \frac{1}{2\pi i} \int_{\Gamma_1'} \frac{f(t)}{t-z} dt$$

Case (i) Equation along Integral Γ_2'

$|z-a| < |t-a|$ (or) $\left|\frac{z-a}{t-a}\right| < 1$

Hence the geometric series

$$1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \cdots$$

is absolutely convergent and is sum is

$$\frac{1}{1-\frac{z-a}{t-a}} = \frac{t-a}{z-a}$$

multiplying both sides by $\frac{1}{2\pi i} \frac{f(t)}{t-a}$

$$\frac{1}{2\pi i} \frac{f(t)}{t-a} + \frac{1}{2\pi i} \frac{f(t)}{(t-a)^2}(z-a) + \frac{1}{2\pi i} \frac{f(t)}{(t-a)^3}(z-a)^2 + \cdots$$

$$\cdots = \frac{1}{2\pi i} \frac{f(t)}{t-z}$$

By term by term Integrating along Γ_2'

$$\sum_{n=0}^{\infty} (z-a)^n \frac{1}{2\pi i} \int_{\Gamma_2'} \frac{f(t)}{(t-a)^{n+1}} dt = \frac{1}{2\pi i} \int_{\Gamma_2'} \frac{f(t)}{t-z} dt$$

case (ii) Equation along Integral Γ_1'.

$$|t-a| < |z-a| \text{ (or) } \left|\frac{t-a}{z-a}\right| < 1$$

Hence the geometric series

$$1 + \left(\frac{t-a}{z-a}\right) + \left(\frac{t-a}{z-a}\right)^2 + \cdots$$

absolutely convergent and its sum function n is

$$\frac{1}{1-\frac{t-a}{z-a}} = \frac{z-a}{z-t} = -\frac{z-a}{t-z}$$

$$1 + \frac{t-a}{z-a} + \left(\frac{t-a}{z-a}\right)^2 + \cdots = -\frac{z-a}{t-z}$$

multiplying bothsides by $\frac{1}{2\pi i} \frac{f(t)}{z-a}$

$$\sum_{n=-1}^{-\infty} \frac{1}{2\pi i} \frac{f(t)}{(t-a)^{n+1}} (z-a)^n = \frac{-1}{2\pi i} \frac{f(t)}{t-z}$$

By term by term Integration along Γ_1'

$$\sum_{n=-1}^{-\infty} (z-a)^n \frac{1}{2\pi i} \int_{\Gamma_1'} \frac{f(t)}{(t-a)^{n+1}} dt = \frac{-1}{2\pi i} \int_{\Gamma_1'} \frac{f(t)}{t-z} dt$$

The series on the left is absolutely convergent

$$f(z) = \sum_{n=0}^{\infty} (z-a)^n \frac{1}{2\pi i} \int_{\Gamma_2'} \frac{f(t)}{(t-a)^{n+1}} dt + \sum_{n=-1}^{-\infty} (z-a)^n \frac{1}{2\pi i} \int_{\Gamma_1'} \frac{f(t)}{(t-a)^{n+1}} dt$$

Now C is a scro curve in b/w Γ_1' and Γ_2'

By extension to Cauchy ~~fundamental~~

$$\int_{\Gamma_2'} = \int_C \text{ and } \int_C = \int_{\Gamma_1'}$$

Hence become

$$f(z) = \sum_{n=-\infty}^{+\infty} (z-a)^n \cdot \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt$$

$$= \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt$

Uniqueness Theorem:

If $f(z)$ is analytic in an open circular disc with centre at $z=a$, then the Taylor's series about $z=a$ is the only power series in $z-a$ which converges at $f(z)$ in that disc.

The Taylor's series about $z=a$ for $f(z)$ in the disc is,

$$f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\dots$$

Suppose there is another series, say

$$a_0+a_1(z-a)+a_2(z-a)^2+a_3(z-a)^3 \to (1).$$

which converges in that disc to the same function $f(z)$. Then differentiating (1) term by term and successively.

$$a_1+2a_2(z-a)+3a_3(z-a)^2+4a_4(z-a)^3+\dots=f'(z)$$

$$2!\,a_2+3.2.a_3(z-a)+4.3.a_4(z-a)^2+\dots=f''(z)$$

$$3!\,a_3+4.3.2a_4(z-a)+\dots=f'''(z)$$

Setting $z=a$ in these series and in (1).

$$a_0=f(a),\ 1!\,a_1=f'(a),\ 2!\,a_2=f''(a),\dots$$

ie) $a_0=f(a),\ a_1=\frac{f'(a)}{1!},\ a_2=\frac{f''(a)}{2!},\dots$

∴ The series (1) becomes the Taylor's series itself.

Hence Taylor's series is the only power series in $z-a$ which converges to $f(z)$ in the disc.

Different Taylor's series for the same function

If $f(z)$ about $z=a$ and $z=b$, then

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\dots \text{ for all } z \text{ in } N(a)$$

$$f(z) = f(b) + \frac{f'(b)}{1!}(z-b) + \frac{f''(b)}{2!}(z-b)^2 + \cdots \text{ for all } z \text{ in } N(b)$$

These series are Taylor's series of the same function $f(z)$ about $z = a$ and $z = b$.

Find the Taylor's expansion about $z=0$ of $f(z) = \frac{z}{(z+1)(z-3)}$

Sol:

$$\frac{z}{(z+1)(z-3)} = \frac{A}{z+1} + \frac{B}{z-3}$$

$$\frac{z}{(z+1)(z-3)} = \frac{A(z-3)+B(z+1)}{(z+1)(z-3)}$$

$$z = A(z-3) + B(z+1) \rightarrow (A).$$

Put $z=3$ in (A)

$$3 = B(3+1)$$

$$\boxed{\frac{3}{4} = B}$$

Put $z=-1$ in (A).

$$-1 = A(-1-3)$$

$$\frac{-1}{-4} = A$$

$$\therefore \boxed{\frac{1}{4} = A}$$

$$\frac{z}{(z+1)(z-3)} = \frac{1/4}{z+1} + \frac{3/4}{z-3}$$

$$= 1/4 \frac{1}{z+1} + \frac{3}{4} \cdot \frac{1}{z-3}$$

Here, $z=-1$ $\quad$ $z=3$.

$\frac{z}{-1} = 1$ $\quad$ $\frac{z}{3} = 1$

$\left|\frac{z}{-1}\right| < 1$ $\quad$ | $\quad$ $|z/3| < 1$

$|z| < 1$ $\quad$ | $\quad$ $|z/3| < 1$

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$$\frac{z}{(z+1)\,(z-3)} = \frac{1}{4}\,\frac{1}{1+z} + \frac{3}{4}\,\frac{1}{3\left[\frac{z}{3}-1\right]}$$

$$= \frac{1}{4}\cdot\frac{1}{1+z} - \frac{1}{4}\cdot\frac{1}{1-z/3}$$

$$= \frac{1}{4}(1+z) - \frac{1}{4}\left(1 - z/3\right)$$

$$= \frac{1}{4}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{4}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 + \cdots\right]$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n (z)^n - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

Key answer

Internal - II

Complex analysis - II

Sub code : 15MMU602

Class : III B.Sc mathematics

Part-A

1. $|z|<R$
2. vanishes Identically
3. Singular part
4. Isolated
5. essential
6. not bounded
7. logarithmic functions
8. Laurent's series
9. negative power
10. three
11. at the origin
12. analytic
13. $z=a$
14. a pole
15. simple
16. Singular
17. $z=\infty$
18. multiple
19. $\cos z_1 \cos z_2 - \sin z_1 \sin z_2$
20. 1

Theorem:

If $z=a$ is a pole of order m of the function $f(z)$, then $z=a$ is a zero of order m of the function $\frac{1}{f(z)}$

Proof:

Since $z=a$ is a pole of order m of the function $f(z)$. there exists a deleted neighbourhood of $z=a$. In which the Laurent expansion for the $f(z)$ about $z=a$ is,

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \cdots \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots$$

Let $\phi(z) = (z-a)^m f(z)$

$$= a_{-m} + a_{-(m-1)}(z-a) + \cdots + a_{-(m-2)}(z-a)^2 + \cdots$$

Then $\phi(a) = a_{-m}$ and $\frac{1}{\phi(a)} \neq 0$.

Now $\frac{1}{f(z)} = (z-a)^m \left[\frac{1}{\phi(z)}\right], \frac{1}{\phi(a)} \neq 0$.

Therefore $\frac{1}{f(z)}$ has a zero of order m.

$n=0$

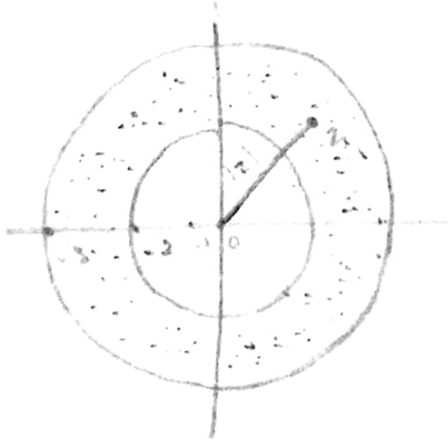
(2) Find the Laurent's series expansion for $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

(i) $2 < |z| < 3$ (ii) $|z| > 3$.

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+3)+B(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = A(z+3)+B(z+2) \rightarrow (A)$$



Put $z=-3$ in (A).

$$(-3)^2-1 = A(0)+B(-3+2)$$

$$9-1 = -B$$

$$8 = -B$$

$$\boxed{-8 = B}$$

Put $z=-2$ in (A).

$$(-2)^2-1 = A(-2+3)+B(-2+2)$$

$$\boxed{3 = A}$$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = \frac{3}{z+2} + \frac{-8}{z+3}$$

$$2 < |z| < 3$$

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$2 < |z|$ | $|z| < 3$

$\frac{2}{|z|} < 1$ | $\frac{|z|}{3} < 1$

$\left|\frac{2}{z}\right| < 1$ | $\left|\frac{z}{3}\right| < 1$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{3}{z+2} + \frac{8}{z+3}$$

$$= \frac{3}{z\left[1+\frac{2}{z}\right]} + \frac{8}{3\left[\frac{z}{3}+1\right]}$$

$$= \frac{3}{z\left[1+\frac{2}{z}\right]} + \frac{8}{3\left[1+\frac{z}{3}\right]}$$

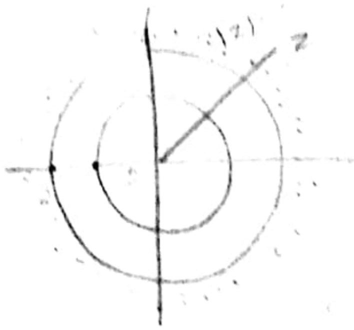
$$= \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1} - \frac{8}{3}\left[1+\frac{z}{3}\right]^{-1}$$

$$= 3/z\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\dots\right] - \frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\dots\right]$$

$$= \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

Case (ii)

$|z| > 3$



$$\frac{z^2-1}{(z+2)(z+3)} = \frac{3}{z+2} + \frac{-8}{z+3}$$

$|z| > 3$

$1 > \frac{3}{|z|}$

$1 > \left|\frac{3}{z}\right|$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{3}{z\left[1+\frac{2}{z}\right]} + \frac{-8}{z\left[1+\frac{3}{z}\right]}$$

$$= \frac{3}{z}\left[1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left[1+\frac{3}{z}\right]^{-1}$$

$$= \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1} - \frac{8}{z}\left[1+\frac{3}{z}\right]^{-1}$$

$$= 3/z\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\dots\right]$$

$$- 8/z\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\dots\right]$$

Residue Theorem:

STATEMENT: $\int_C f(z)\,dz = 2\pi i$

Suppose,

i) D is a simply-connected region

ii) F(z) is analytic in D except at its singularities which are finite in number.

iii) C is a SCSO curve in D not passing through any singularity

Then $\int_C f(z)\,dz = 2\pi i$ (sum of all residue of F(z) in C_i)

PROOF:

Let the singular points of F(z) in C_i be $a_1, a_2, a_3, \ldots a_n$

Let $\Gamma_1, \Gamma_2, \ldots \Gamma_n$ be positively oriented small circles.

$|z-a_1| = r_1$, $|z-a_2| = r_2$, ... $|z-a_n| = r_n$ in C_i

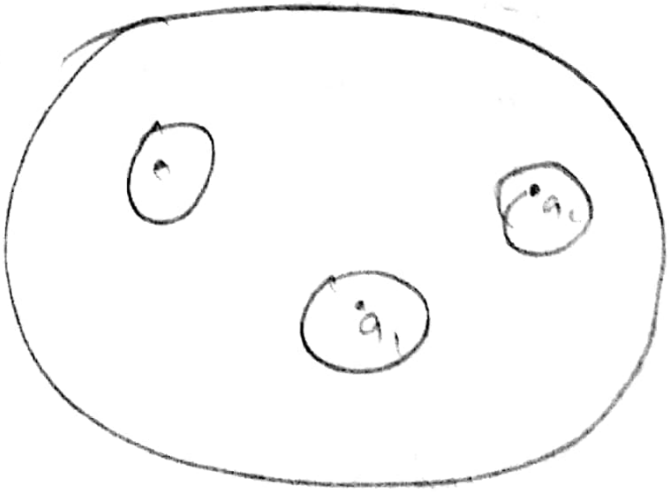
Such that no one intersects the other (Fig: 1)

Then by extension to cauchy's Fundamental theorem

$$\int_C F(z)\,dz = \int_{\Gamma_1} F(z)\,dz + \int_{\Gamma_2} F(z)\,dz + \dots + \int_{\Gamma_n} F(z)\,dz$$

$= 2\pi i$ (residue of $F(z)$ at $z = a_1$) $+ 2\pi i$ (resa of $F(z)$ at $z = a_2$) $+ 2\pi i$ (residue of $F(z)$ at z

$= 2\pi i$ (sum of the residues of $F(z)$ in C



(4) $f(z) = \dfrac{z^4}{(z-1)^4(z-2)(z-3)}$

i) $\{\text{Res } f(z)\}_{z=2} = \lim_{z \to 2} (z-2) \dfrac{z^4}{(z-1)^4(z-2)(z-3)}$

$= \lim_{z \to 2} \dfrac{z^4}{(z-1)^4(z-3)}$

$= \dfrac{(2)^4}{(2-1)^4(2-3)} = \dfrac{16}{(1)(-1)} = -16.$

ii) $\{\text{Res } f(z)\}_{z=3} = \lim_{z \to 3} (z-3) \dfrac{z^4}{(z-1)^4(z-2)(z-3)}$

$= \lim_{z \to 3} \dfrac{z^4}{(z-1)^4(z-2)}$

$= \dfrac{(3)^4}{(3-1)^4(3-2)} = \dfrac{81}{(2)^4(1)}$

$= \dfrac{81}{16}.$

$\{\text{Res } z\}_{z=2} = \lim_{z \to 2} (z-2) \dfrac{z^4}{(z-1)^4(z-2)(z-3)}$

$\{\text{Res } z\}_{z=a} = \lim_{z \to a} \dfrac{1}{(m-1)!} \dfrac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$

iii) $z=1$ is a pole of order 4.

$$\{Res\, f(z)\}_{z=a} = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$\{Res\, f(z)\}_{z=1} = \lim_{z\to 1} \frac{1}{(4-1)!} \frac{d^{4-1}}{dz^{4-1}} \left[(z-1)^4 \frac{z^4}{(z-1)^4(z-2)(z-3)}\right]$$

$$= \lim_{z\to 1} \frac{d^3}{dz^3} \frac{z^4}{(z-2)(z-3)}$$

$$= \lim_{z\to 1} \frac{d^3}{dz^3} \frac{z^4}{z^2-5z+6}$$

$$
\begin{array}{r l}
 & z^2+5z+19 \rightarrow Q \\
z^2-5z+6 \,\big) & z^4 \\
 & z^4-5z^3+6z \\
\hline
 & 5z^3-6z^2 \\
 & 5z^3-25z^2+30z \\
\hline
 & 19z^2-30z \\
 & 19z^2-95z+114 \\
\hline
 & 65z-114 \rightarrow R \\
\hline
\end{array}
$$

Dividing z^4 by z^2-5z+6, we get the quotient as $z^2+5z+19$ and the remainder as $65z-114$

$$\therefore \frac{z^4}{(z-1)^4(z-2)(z-3)} = z^2+5z+19+\frac{65z-114}{(z-2)(z-3)}$$

By Partial fractions

$$\frac{65z-114}{(z-2)(z-3)} = \frac{A}{(z-2)} + \frac{B}{(z-3)}$$

$$\frac{65z-114}{(z-2)(z-3)} = \frac{A(z-3)+B(z-2)}{(z-2)(z-3)}$$

$65z - 114 = A(z-3) + B(z-2)$

Put $z=3$ in ①

$65(3) - 114 = B(3-2)$

$195 - 114 = B$

$\therefore \boxed{81 = B}$

Put $z = 2$ in ①.

$65(2) - 114 = A(2-3)$

$130 - 114 = -A$

$16 = -A$

$\therefore \boxed{A = -16}$

$$\therefore \frac{65z - 114}{(z-2)(z-3)} = \frac{-16}{(z-2)} + \frac{81}{(z-3)}$$

By Ⓐ $\Rightarrow$ $$\frac{z^4}{(z-1)^4 (z-2)(z-3)} = z^2 + 5z + 19 - \frac{16}{(z-2)} + \frac{81}{(z-3)}$$

$$\text{Res}\{f(z)\}_{z=1} = \lim_{z\to 1} \frac{1}{(m-1)!} \frac{d^3}{dz^3}\left(z^2 + 5z + 19 - \frac{16}{(z-2)} + \frac{81}{(z-3)}\right)$$

$$= \lim_{z\to 1} \frac{1}{3!} \frac{d^3}{z^3}\left(z^2 + 5z + 19 - 16(z-2)^{-1} + 81(z-3)^{-1}\right)$$

$$= \lim_{z\to 1} \frac{1}{3!} \frac{d^2}{dz^2}\left(2z + 5 + 16(z-2)^{-2} - 81(z-3)^{-2}\right)$$

$$= \lim_{z\to 1} \frac{1}{3!} \frac{d}{dz}\left(2 - 32(z-2)^{-3} + 162(z-3)^{-3}\right)$$

$$= \lim_{z\to 1} \frac{1}{6}\left(96(z-2)^{-4} - 486(z-3)^{-4}\right)$$

$$= \frac{1}{6}\left[96(1-2)^{-4} - 486(1-3)^{-4}\right]$$

$$= \frac{1}{6}\left[96(-1)^{-4} - 486(-2)^{-4}\right]$$

$$= \frac{1}{6}\left[96 - \frac{486}{(-2)^4}\right]$$

$$\Rightarrow \frac{1}{6}\left[96 - \frac{486}{16}\right]$$

$$\Rightarrow \frac{1}{6}\left[\frac{1050}{16}\right]$$

$$\Rightarrow \frac{175}{16}$$

Weierstrass Theorem:

If $f(z)$ has an isolated essential singularity at $z=a$ and if c is any complex constant then for any positive ϵ, however small, there exist an z in every deleted neighbourhood of $z=a$ such that

$$|f(z)-c| < \epsilon$$

PROOF:

Let $D: 0 < |z-a| < \rho$

be an arbitarily chosen small deleted neighbourhood of $z=a$. Given an $\epsilon > 0$, Suppose it is not possible to find an z in D such that $|f(z)-c| < \epsilon$

Then, for all z in D,

$$|f(z)-c| \geq \epsilon \text{ (or)}$$

$$\left| \frac{1}{f(z)-c} \right| \leq \frac{1}{\epsilon}$$

Here $1/\{f(z)-c\}$ is bounded & analytic in D

And it is not analytic at $z=a$ because $f(z)$ is not analytic there. So, by Theorem

$F(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$

$z=a$ is a removable singularity of $\frac{1}{\{f(z)-c\}}$

$\therefore$ Its Laurent's expansion about $z=a$ is of the form, either

$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots, \quad a_0 \neq 0$

(or)

$a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \cdots, \quad a_m \neq 0.$

In other words,

$\frac{1}{\{f(z)-c\}}$ has a removable singularity at $z=a$ & either $z=a$ is not a zero of $\frac{1}{\{f(z)-c\}}$ (or) $z=a$ is a zero of order m of $\frac{1}{\{F(z)-c\}}$

Consequently,

either $F(z)-c$ has a removable singularity at $z=a$ (or) $F(z)-c$ has a pole $z=a$ of order m.

Either $F(z)$ has a removable singularity at $z=a$ (or) $F(z)$ has pole $z=a$ of order m.

This contradicts the hypothesis that $z=a$ is an essential singularity of $F(z)$.

Hence the Theorem as stated.

Theorem:

If $z=a$ is a pole of a function $f(z)$, then

$$\lim_{z \to a} f(z) = \infty$$

Proof:

Let the order of the pole be m and the Laurent's expansion about $z=a$ in the deleted neighbourhood $0<|z-a|<\rho$ be

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \dots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \dots$$

$$= \frac{a_{-m} + \dots + a_{-1}(z-a)^{m-1}}{(z-a)^m} + a_0 + a_1(z-a) + \dots$$

$$= \frac{\phi(z)}{(z-a)^m} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

Now as $z \to a$ taking any path

i) $\phi(z)$, a polynomial in $z-a$, tends to a_{-m}

ii) $\frac{1}{(z-a)^m}$ tends to ∞

iii) $a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$ tends to a_0.

Hence $f(z) \to \infty$ as $z \to a$.

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(i) Find the orders of the poles of $F(z) = \frac{1}{z(e^z - 1)}$

Sol: It is evident that the denominator vanishes when $z = 0$, $z = \pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \ldots$

So they are the poles of $F(z)$. Here we kn that,

$$\lim_{z \to 0} \frac{z}{e^z - 1} = 1$$

So, multiplying $F(x)$ by z^2 & taking limi as $z \to 0$

$$\lim_{z\to 0} z^2 F(z) = \lim_{z\to 0} \frac{z}{e^z - 1} = 1$$

which is finite, not equal too. So $z=0$ is a pole of order 2.

To find the order of $z = 2\pi i$, consider

$$\lim_{z\to 2\pi i} (z - 2\pi i) F(z) = \lim_{z\to 2\pi i} \frac{z - 2\pi i}{z(e^z - 1)}$$

$$= \lim_{z\to 2\pi i} \frac{1}{(e^z - 1) + z e^z} \quad \text{(by L' hospital's Rule)}$$

$$= \frac{1}{(1-1) + 2\pi i(1)} \Rightarrow \frac{1}{2\pi i}$$

which is finite & not equal to 0. So $z = 2\pi i$ and similarly the other poles are simple poles.

Key answer

model examination

complex Analysis - II

Sub code: 15mmu602

class: III B.SC mathematics

part-A

1. constant
2. continuous
3. Isolated
4. entire
5. <1
6. polynomial of degree n
7. logarithmic functions
8. Laurent's series
9. negative power
10. three
11. at the origin
12. analytic
13. simple pole
14. 0
15. $\cos z \neq 0$
16. $\cosh z$
17. 0
18. second and third
19. rational function
20. orders.

2.1(a) Let the harmonic conjugate of $u(x,y)$ be $v(x,y)$.

Then $f(z) = u + iv$ is analytic in D and by the previous theorem then mean value of $f(z)$ on $\Gamma: |z-a| = r$ in D

is

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt.$$

If $a = \alpha + i\beta$ then

$f(a) = u(\alpha,\beta) + iv(\alpha,\beta)$ and

$f(a+re^{it}) = u(\alpha + r\cos t, \beta + r\sin t) + iv(\alpha + r\cos t, \beta + r\sin t)$

$$u(\alpha,\beta) + iv(\alpha,\beta) = \frac{1}{2\pi}\int_0^{2\pi} u(\alpha + r\cos t, \beta + r\sin t)\,dt + \frac{i}{2\pi}\int_0^{2\pi} v(\alpha + r\cos t, \beta + r\sin t)\,dt$$

Equating the real parts

$$u(\alpha,\beta) = \frac{1}{2\pi}\int_0^{2\pi} u(\alpha + r\cos t, \beta + r\sin t)\,dt$$

which is the mean value of the values of $u(x,y)$ on Γ.

21 b) Poisson Integral

Statement

Suppose the function $u(x,y)$ is harmonic inside and on a circle $\Gamma: |z| = R$. If $re^{i\theta}$ is a point inside this circle then

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(R,\phi)\,(R^2-r^2)}{R^2+r^2-2Rr\cos(\theta-\phi)}\,d\phi$$

where $u(r,\theta)$ stands for $u(r\cos\theta, r\sin\theta)$ of

Proof

Suppose $f(z)$ is analytic inside and on Γ. and has $u(x,y)$ as its real part. Then by Cauchy Integral formula

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int \frac{f(z)}{z-re^{i\theta}}\,dz$$

$$0 = \frac{1}{2\pi i}\int \frac{f(z)}{z-(R^2/r)e^{i\theta}}\,dz$$

Subtracting the second Integral formula from the list

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_\Gamma f(z)\left[\frac{1}{z-re^{i\theta}} - \frac{r}{rz-R^2e^{i\theta}}\right]dz$$

$$= \frac{1}{2\pi i}\int_\Gamma f(z)\,\frac{-R^2e^{i\theta}+r^2e^{i\theta}}{(z-re^{i\theta})(rz-R^2e^{i\theta})}\,dz$$

$$= \frac{1}{2\pi i}\int_\Gamma f(z)\,\frac{-(R^2-r^2)\,e^{i\theta}}{(z-re^{i\theta})\left(re^{-i\theta}-\frac{R^2}{z}\right)e^{i\theta}z}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\theta})\,\frac{-(R^2-r^2)}{(Re^{i\theta}-re^{i\theta})(re^{-i\theta}-Re^{-i\theta})}\,i\,d\phi$$

Since on Γ, $z=Re^{i\theta}$ and $dz = iRe^{i\theta}d\phi = iz\,d\phi$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\,\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\phi)}\,d\phi$$

Equating the real parts we get the results as started.

Uniqueness Theorem:

If $f(z)$ is analytic in an open circular disc with centre at $z=a$, then the Taylor's series about $z=a$ is the only power series in $z-a$ which converges at $f(z)$ in that disc. $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$

The Taylor's series about $z=a$ for $f(z)$ in the disc is,

$$f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

Suppose there is another series, say

$$a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 \rightarrow (1).$$

which converges in that disc to the same function $f(z)$. Then differentiating (1) term by term and successively.

$$a_1 + 2a_2(z-a) + 3a_3(z-a)^2 + 4a_4(z-a)^3 + \cdots = f'(z)$$

$$2!\,a_2 + 3.2.a_3(z-a) + 4.3.a_4(z-a)^2 + \cdots = f''(z)$$

$$3!\,a_3 + 4.3.2a_4(z-a) + \cdots = f'''(z)$$

Setting $z=a$ in these series and in (1).

$$a_0 = f(a),\ 1!\,a_1 = f'(a),\ 2!\,a_2 = f''(a), \ldots$$

ie) $a_0 = f(a),\ a_1 = \frac{f'(a)}{1!},\ a_2 = \frac{f''(a)}{2!}, \ldots$

∴ The series (1) becomes the Taylor's series itself. Hence Taylor's series is the only power series in $z-a$ which converges to $f(z)$ in the disc.

Different Taylor's series for the same function

If $f(z)$ about $z=a$ and $z=b$, then

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \text{ for all } z \text{ in } N(a)$$

$$f(z) = f(b) + \frac{f'(b)}{1!}(z-b) + \frac{f''(b)}{2!}(z-b)^2 + \cdots \text{ for all } z \text{ in } N(b)$$

These series are Taylor's series of the same function $f(z)$ about $z=a$ and $z=b$.

32) b) $\cos z$ is an entire function that is a function analytic everywhere at $z = \infty$

we shall denote $\cos z$ by $f(z)$ then by successive differentiation, we have

$$f^n(z) = \cos\left(z + \frac{n\pi}{2}\right)$$

$$f^n(0) = \cos\left(0 + \frac{n\pi}{2}\right) = \cos\frac{n\pi}{2}$$

$$f^n\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$$

$$f^n\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \frac{n\pi}{2}\right)$$

(i) Here the expansion is about $z = 0$ so the expansion is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(z-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{\cos\left(0 + \frac{n\pi}{2}\right)}{n!}(z-0)^n$$

$$= \sum_{n=0}^{\infty} \frac{\cos\frac{n\pi}{2}}{n!} z^n$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

(ii) Here the expansion is about $z = \frac{\pi}{4}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{\pi}{4}\right)}{n!}\left(z - \frac{\pi}{4}\right)^n$$

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)}{n!} \left(z - \frac{\pi}{4}\right)^n$$

(iii) Here the expansion is about $z = \frac{\pi}{3}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{\pi}{3}\right)}{n!} \left(z - \frac{\pi}{3}\right)^n$$

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi}{3} + \frac{n\pi}{2}\right)}{n!} \left(z - \frac{\pi}{3}\right)^n$$

23) a

(i) $$\{\text{Res } f(z)\}_{z=2} = \underset{z\to 2}{Lt}\, (z-2) \frac{24}{(z-1)^4 (z-2)(z-3)}$$

$$= \underset{z\to 2}{Lt} \frac{24}{(z-1)^4 (z-3)}$$

$$= -16$$

(ii) $$\{\text{Res } f(z)\}_{z=3} = \underset{z\to 3}{Lt}\, (z-3) \frac{24}{(z-1)^4 (z-2)(z-3)}$$

$$= \underset{z\to 3}{Lt} \frac{24}{(z-1)^4 (z-2)}$$

$$= \frac{81}{16}$$

(iii) $$\{\text{Res } f(z)\}_{z=a} = \underset{z\to a}{Lt} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

$$= \underset{z\to 1}{Lt} \frac{d^3}{dz^3} \frac{24}{(z-2)(z-3)}$$

$$= \underset{z\to 1}{Lt} \frac{d^3}{dz^3} \frac{24}{z^2-5z+6}$$

$$
\begin{array}{r|l}
z^2-5z+6 & z^4 \quad (z^2+5z+19 \\
& z^4-5z^3+6z^2 \\
\hline
& 5z^3-6z^2 \\
& 5z^3-25z^2+30z \\
\hline
& 19z^2-30z \\
& 19z^2-95z+114 \\
\hline
& 65z-114
\end{array}
$$

$$\frac{z^4}{(z-1)^4(z-2)(z-3)} = z^2+5z+19+\frac{65z-114}{(z-2)(z-3)}$$

$$\frac{65z-114}{(z-2)(z-3)} = \frac{A}{z-2}+\frac{B}{z-3}$$

Solved by partial fraction

$A = -16$
$B = 81$

$$\frac{z^4}{(z-1)^4(z-2)(z-3)} = z^2+5z-119 \frac{-16}{z-2}+\frac{81}{z-3}$$

$$\{\text{Res } f(z)\}_{z=1} = \lim_{z\to 1} \frac{1}{3!}\frac{d^3}{dz^3}\left(z^2+5z+19-16(z-2)^{-1}+81(z-3)^{-1}\right)$$

$$= \frac{175}{16}$$

23) b **weierstrass theorem**

If $f(z)$ has an isolated essential singularity at $z=a$ and if C is any complex constant then for any positive ϵ, however small, there exist an z in every deleted neighbourhood of $z=a$ such that

$$|f(z)-c| < \epsilon.$$

Proof

Let $D: 0 < |z-a| < \rho$ be an arbitrarily chosen small deleted neighbourhood of $z=a$.

Given $\varepsilon > 0$ suppose It is not possible to find an z in D such that $|f(z) - c| < \varepsilon$.

Then for all z in D

$$|f(z) - c| \geq \varepsilon \text{ (or)}$$

$$\left| \frac{1}{f(z)-c} \right| \leq \frac{1}{\varepsilon}$$

Here $\frac{1}{\{f(z)-c\}}$ is bdd and analytic in D.

And It is not analytic at $z=a$ became $f(z)$ is not analytic there, so, by thm

$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$

$z=a$ is a removable singularity of $\frac{1}{\{f(z)-c\}}$

Its Laurent expansion about $z=a$ is of the form

either

$a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots \quad a_0 \neq 0$

$\frac{1}{\{f(z)-c\}}$ has removable singularity at $z=a$ &

either $z=a$ is not a zero of $\frac{1}{\{f(z)-c\}}$

Consequently

either $f(z)-c$ has a removable singularity at $z=a$

(or) $f(z)-c$ has a pole $z=a$ of order m.

Either $f(z)$ has pole $z=a$ of order m (or)

$f(z)$ has a removable singularity at $z=a$.

$\Rightarrow \Leftarrow$ the hypothesis that $z=a$ is an essential singularity of $f(z)$.

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$$

Put $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{(a+b\cos\theta)^2}$$

$$= \int_C \frac{\frac{dz}{iz}}{\left(a+b\left(\frac{z^2+1}{2z}\right)\right)^2} = \frac{4}{i}\int_C \frac{z\,dz}{(bz^2+2az+b)^2}$$

$$= \frac{4}{i}\int_C f(z)\,dz \quad \text{— (1)}$$

where $f(z) = \dfrac{z}{bz^2+2az+b}$

$$z = \frac{-a+\sqrt{a^2-b^2}}{b} \;;\; z = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$z = \alpha \;;\; z = \beta$$

$$f(z) = \frac{z}{b^2(z-\alpha)^2(z-\beta)^2}$$

$z=\alpha$ is a pole of order 2 lies inside $|z|=1$

$z=\beta$ is a pole of order 2 lies outside $|z|=1$.

$$\{\text{Res } f(z)\}_{z=\alpha} = \lim_{z\to\alpha} \frac{1}{1!}\frac{d}{dz}(z-\alpha)^2 \frac{z}{b^2(z-\alpha)^2(z-\beta)^2}$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left(\frac{z}{b^2(z-\beta)^2}\right)$$

$$= \frac{1}{b^2}\,\frac{-\alpha-\beta}{(\alpha-\beta)^3}$$

$$= \frac{1}{b^2} \times \frac{b^3}{8(a^2-b^2)^{3/2}} \left[\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b} + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} \right]$$

$$= \frac{b}{8(a^2-b^2)^{3/2}} \left(\frac{2a}{b} \right)$$

$$= \frac{a}{4(a^2-b^2)^{3/2}}$$

$\int_C f(z)dz = 2\pi i$ (sum of the residue)

$$= 2\pi i \times \frac{a}{4(a^2-b^2)^{3/2}}$$

$$= \frac{a\pi i}{2(a^2-b^2)^{3/2}}$$

To find the value of Given Integral

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{4}{i} \times \frac{a\pi i}{2(a^2-b^2)^{3/2}}$$

$$= \frac{2a\pi}{(a^2-b^2)^{3/2}}$$

/ Using contour integration Evaluate $I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a+b\cos\theta}$

$\sqrt{a^2-b^2}$

Sol:

Step 1: Transforming the variable θ in terms of 0.

Put $z = e^{i\theta}$ $\quad \frac{dz}{d\theta} = e^{i\theta} \cdot i$.

... with a theorem.

(OR)

b) State and prove Rouche's ...

$$dz = i \cdot e^{i\theta}\, d\theta$$

$$d\theta = \frac{dz}{i \cdot e^{i\theta}} = \frac{dz}{iz}$$

$$\therefore \cos\theta = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right)$$

$$= \tfrac{1}{2}\left(\frac{z^2+1}{z}\right)$$

Step 2:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \text{real part of } \left[\frac{1-z^2}{2}\right]$$

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a+b\cos\theta} = \int_C \frac{\text{R.P. of } \left[\frac{1-z^2}{2}\right] dz/iz}{a+b\left[\frac{z^2+1}{2z}\right]}$$

$$= \tfrac{1}{2} \text{ R.P of } \int_C (1-z^2) \times \frac{dz}{iz} \times \frac{2z}{bz^2+2az+b}$$

$$= \tfrac{1}{i} \text{ R.P of } \int_C \frac{(1-z)^2 dz}{bz^2+2az+b}$$

$$= \frac{1}{i} \text{ R.P. of } \int_C f(z)dz \rightarrow (i)$$

where $f(z) = \dfrac{1-z^2}{bz^2+2az+b}$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$\therefore z = \frac{-a+\sqrt{a^2-b^2}}{b} \;;\quad z = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$z = \alpha \;;\quad z = \beta$$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \;;\quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

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Since $a > b > 0$, $|\beta| > 1$.

This is clear by taking $a = -3$, $b = 2$.

$\alpha = \frac{-3+\sqrt{9-4}}{2} \Rightarrow \frac{-3+2.236}{2}$

$= -0.381 \Rightarrow |\alpha| = 0.381$

$\beta = \frac{-3-\sqrt{9-4}}{2} \Rightarrow \frac{-3-2.236}{2}$

$= -2.618 \Rightarrow |\beta| = 2.618$

$= |\beta| > 1.$

clearly $|\alpha| < 1$ and $|\beta| > 1$.

Step 3:

$Z = \alpha$ is order of simple pole lies inside on $|z| = 1$

$z = \beta$ is order of simple pole lies outside on $|z| = 1$

$$\therefore f(z) = \frac{1-z^2}{bz^2+2az+b} \Rightarrow \frac{1-z^2}{b(z-\alpha)(z-\beta)}$$

Step 4:

To find residue $\int_c f(z)\,dz$

$$\{\operatorname{Res} f(z)\}_{z=\alpha} = \lim_{z\to\alpha} (z-\alpha)\frac{1-z^2}{b(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1-z^2}{b(z-\alpha)(z-\beta)}(z-\alpha)$$

$$= \lim_{z\to\alpha} \frac{1-z^2}{b(z-\beta)}$$

$$= \frac{1-\alpha^2}{b(\alpha-\beta)}$$

$$= \frac{1-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)^2}{b\left[\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right]}$$

$$= \frac{1-\left(\frac{-a+\sqrt{a^2-b^2}}{b^2}\right)^2}{b\left[\frac{-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}}{}\right]}$$

$$= \frac{b^2 - (-a - \sqrt{a^2-b^2})^2}{b^2} \times \frac{1}{2\sqrt{a^2-b^2}}$$

$$= \frac{1}{b^2} \frac{b^2 - (-a + \sqrt{a^2-b^2})^2}{2\sqrt{a^2-b^2}}$$

$$= \frac{1}{b^2} \frac{b^2 - (a^2 + a^2 - b^2 - 2a\sqrt{a^2-b^2})}{2\sqrt{a^2-b^2}}$$

$$= \frac{1}{b^2} \frac{2b^2 - 2a^2 + 2a\sqrt{a^2-b^2}}{2\sqrt{a^2-b^2}}$$

$$= \frac{1}{b^2} \frac{2(b^2 - a^2 + a\sqrt{a^2-b^2})}{2\sqrt{a^2-b^2}}$$

$$= \frac{b^2 - a^2 + a\sqrt{a^2-b^2}}{b^2\sqrt{a^2-b^2}}$$

Step 5:

$\Rightarrow 2\pi i$ (Sum of the residues)

$$= 2\pi i \left[\frac{b^2 - a^2 + a\sqrt{a^2-b^2}}{b^2\sqrt{a^2-b^2}} \right]$$

Step 6:

(1) $\Rightarrow$

$$\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{1}{i} \text{ R.P of } \int_C f(z)\,dz$$

$$= \frac{1}{i} \text{ R.P of } \left[2\pi i \left(\frac{b^2 - a^2 + a\sqrt{a^2-b^2}}{b^2\sqrt{a^2-b^2}} \right) \right]$$

$$= \frac{2\pi}{b^2} \text{ R.P of } \left[\frac{b^2 - a^2 + a\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}} \right]$$

$$= \frac{-2\pi}{b^2} \text{ R.P of } \left[\frac{a^2 - b^2 - a\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}} \right]$$

$$= \frac{-2\pi}{b^2} \text{ R.P of } \left[\frac{a^2-b^2}{\sqrt{a^2-b^2}} - \frac{a\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\right]$$

$$= \frac{-2\pi}{b^2} \text{ R.P of } \left(\sqrt{a^2-b^2} - a\right)$$

$$= \frac{2\pi}{b^2} \text{ R.P of } \left(a - \sqrt{a^2-b^2}\right)$$

$$= \frac{2\pi}{b^2} \left(a - \sqrt{a^2-b^2}\right)$$

Hurwitz theorem

Let $\{f_K\}$ be a sequence of holomorphic functions on a connected open set G that coverage uniformly on compact subsets of G to a holomorphic function f which is not constantly zero on G. If f has a zero of order m at z_0 then for every small enough $\rho > 0$ and for sufficiently large $K \in N$ (depending on ρ), f_K has precisely m zeros In the disk defined by $|z - z_0| < \rho$ Including multiplicity.

Proof :-

Let f be an analytic function on open subset of the complex plane with a zero of order m at z_0. $\{f_n\}$ is a sequence of functions converging uniformly on compact subset.

fix some $\rho > 0$ s.t $f(z) \neq 0$ in $0 < |z - z_0| < \rho$.

Choose δ such that $|f(z)| > \delta$ for z on the circle $|z - z_0| = \rho$. since $f_K(z)$ converges uniformly on the disc we have chosen n such that $|f_K(z_0)| \geq \frac{\delta}{2}$ for every $K \geq n$ and every z on the circle, ensuring that the quotient $f_K'(z) / f_K(z)$ is well defined for all z on the circle $|z - z_0| = \rho$.

By morera's theorem

$$\frac{f_K'(z)}{f_K(z)} \to \frac{f'(z)}{f(z)}$$

Denoting the number of zeros of $f_K(z)$ in the disk by N_K, we may apply the argument principle to find

$$m = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)} dz = \lim_{K\to\infty} \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f_K'(z)}{f_K(z)} dz = \lim_{K\to\infty} N_K$$

we have shown that $N_K \to m$ as $K \to \infty$.
Since the N_K are Integer valued, N_K must equal to
m for lagre enough K.

25) b State and Prove Rouche's theorem.

Statement

Suppose

1) D is a simply connected region
2) f(z) is meromorphic in D.
3) C is any scro curve not passing through the poles (or) zeros of f(z).
4) f(z) is defined as the sum of two functions g(z) and h(z).

(ie) $g(z)+h(z) = f(z)$.

where g(z) is meromorphic function in D.

and also

$|g(z)| < |f(z)|$

Then in ci,

$n(z,f) - n(P,f) = n(z,g) - n(P,g)$.

Proof:-

from the Principles of argument, we have

$\Delta_C \arg f(z) = 2\pi [n(z,f) - n(P,f)]$

$2\pi [n(z,f) - n(P,f)] = \Delta_C \arg f(z)$

$= \Delta_C \arg [g(z)+h(z)]$

$= \Delta_C \arg [g(z)[1+\frac{h(z)}{g(z)}]]$.

$-\frac{2\pi}{b^2}$ RP of

$= \frac{-2\pi}{b^2}$

$= \frac{2\pi}{b^2}$

$= \frac{9\pi}{b}$

inte

$\frac{|z}{z}$

$) = \frac{z^2+}{2}$

$= \Delta_C \arg g(z) + \Delta_C \arg\left(1 + \frac{h(z)}{g(z)}\right)$

$= 2\pi\,[n(z, g) - n(P, g)] + \Delta_C \arg\left[1 + \frac{h(z)}{g(z)}\right]$

But $|g(z)| < |h(z)|$. Therefore $\left|\frac{h(z)}{g(z)}\right| < 1$

Then $1 + \frac{h(z)}{g(z)}$ lies in the interior of the circle $|z-1| < 1$. So, for any z in c,

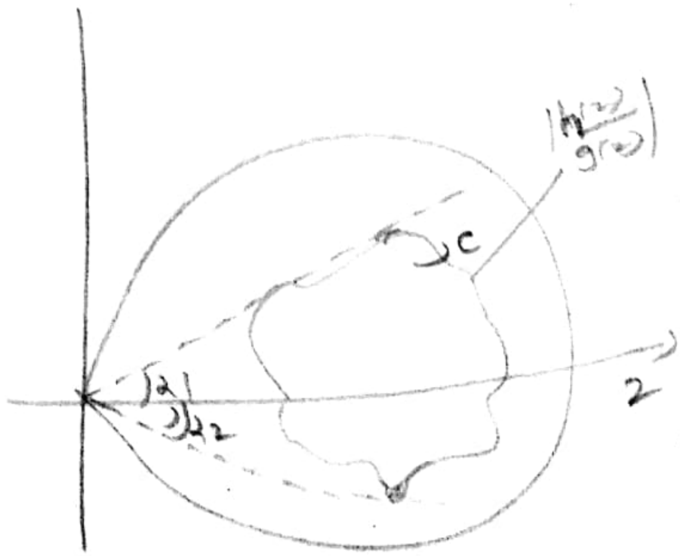
$-\frac{\pi}{2} < \alpha_2 \leq \arg\left[1 + \frac{h(z)}{g(z)}\right] \leq \alpha_2 < \frac{\pi}{2}$

$\therefore \left[1 + \frac{h(z)}{g(z)}\right] = 0$

$\therefore 2\pi\,[n(z, f) - n(P, f)] = 2\pi\,[n(z, g) - n(P, g)] + 0$

$n(z, f) - n(P, f) = n(z, g) - n(P, g)$

Hence the proof.



UNIT-I

SYLLABUS

Zero‟s of a function-Cauchy‟s inequality-Liouville‟s theorem-Fundamental theorem of Algebra- Maximum modulus theorem- Gauss mean value theorem-Mean value of the value- of a harmonic function on a circle-Term by term differentiation and integration of uniformly convergent series .

Zeros of Analytic Functions

Suppose that a function f is analytic at a point z_0. Since $f(z)$ is analytic at z_0, all of the derivatives $f^{(n)}(z)$ $(n = 1, 2, ...)$ exist at z_0. If $f(z_0) = 0$ and if there is a positive integer m such that $f^{(m)}(z+0) \neq 0$ and each derivative of lower order vanishes at z_0 , then f is said to have a ***zero of order*** m at z_0. The following theorem provides an alternative characterization of zeros of order m.

1. Zeros of Analytic function

A zero of an analytic function $f(z)$ is the value of z such that $f(z) = 0$.

Suppose $f(z)$ is analytic in a domain D and a is any point in D. Then, by Taylor's theorem,

$f(z)$ can be expanded about z = a in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad a_n = \frac{f^n(a)}{\lfloor n} \qquad (1)$$

Suppose $a_0 = a_1 = a_2 = \ldots\ldots = a_{m-1} = 0,\ a_m \neq 0$ (2)

so that $f(a) = f'(a) = \ldots\ldots = f^{m-1}(a) = 0, f^m(a) \neq 0$

In this case, we say that f(z) has a zero of order m at z = a and thus (1) takes the form

$$f(z) = \sum_{n=m}^{\infty} a_n (z-a)^n$$
$$= \sum_{n=0}^{\infty} a_{n+m}(z-a)^{n+m}$$
$$= (z-a)^m \sum_{n=0}^{\infty} a_{n+m}(z-a)^n$$

Taking $\sum_{n=0}^{\infty} a_{n+m} (z-a)^n = \phi(z)$ (3)

we get

$$f(z) = (z-a)^m \phi(z) \quad (4)$$

Now $\phi(a) = \left[\sum_{n=0}^{\infty} a_{n+m}(z-a)^n \right]_{z=a}$

$$= \left[a_m + \sum_{n=1}^{\infty} a_{n+m}(z-a)^n \right]_{z=a} = a_m$$

Since $a_m \neq 0$, so $\phi(a) \neq 0$

Thus, an analytic function f(z) is said to have a zero of order m at z = a if f(z) is expressible as

$$f(z) = (z-a)^m \phi(z)$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$.

Also, $f(z)$ is said to have a simple zero at z = a if z = a is a zero of order one.

1.1. Theorem. Zeros are isolated points.

Proof. Let us take the analytic function $f(z)$ which has a zero of order m at z = a. Then, by definition, $f'(z)$ can be expressed as

$f(z) = (z-a)^m \phi(z)$, where $\phi(z)$ is analytic and $\phi(a) \neq 0$.

Let $\phi(a) = 2K$. Since $\phi(z)$ is analytic in sufficiently small neighbourhood of a, if follows from the continuity of $\phi(z)$ in this neighbourhood that we can choose δ so small that, for $|z-a| < \delta$,

$$|\phi(z) - \phi(a)| < |K|$$

Hence
$$\begin{aligned} |\phi(z)| &= |\phi(a) + \phi(z) - \phi(a)| \\ &\geq |\phi(a)| - |\phi(z) - \phi(a)| \\ &> |2K| - |K| \\ &= |K|, \text{ for } |z-a| < \delta \end{aligned}$$

and thus, since $K \neq 0$, $\phi(z)$ does not vanish in the region $|z-a| < \delta$.

Since $f(z) = (z-a)^m \phi(z)$, it follows at once that $f(z)$ has no zero other than a in the same region. Thus we conclude that there exists a nbd of a in which the only zero of $f(z)$ is the point a itself i.e. a is an isolated zero.

The above theorem can also be stated as "Let $f(z)$ be analytic in a domain D, then unless $f(z)$ is identically zero, there exists a neighbourhood of each point in D throughout which the function has no zero except possibly at the point itself."

From the isolated nature of zeros of an analytic function, we are able to deduce the following remarkable result.

1.2. Theorem. If $f(z)$ is an analytic function, regular in a domain D and if $z_1, z_2, \ldots, z_n, \ldots$ is a sequence of zeros of $f(z)$, having a limiting point in the interior of D, then $f(z)$ vanishes identically in D.

Proof. Let a be the limiting point of the sequence of zeros $z_1, z_2, \ldots, z_n, \ldots$ of f(z). Then virtue of continuity of $f(z)$, $f(a) = 0$. Again, since $f(z)$ is regular in the domain D and a is an interior point of D, we can expand $f(z)$ as a power series in powers of z – a as

$$f(z) = \sum_{n=1}^{\infty} a_n (z-a)^n \quad (1)$$

which converges in the neighbourhood of a. Now, either $f(z)$ is identically zero in this region on account of the vanishing of all co-efficient a_n, or else there exists a first co-efficient a_m (say) which does not vanish. But if the latter is the case, we have already seen that there is a neighbourhood of which does not contain any zero of $f(z)$ other than a itself. This contradicts the hypothesis that a is the limiting point of the sequence of zeros $z_1, z_2,\ldots,z_n$. We are thus led to the conclusion that $f(z)$ is identically zero in the circle of convergence of the series (1) .

We are now free to repeat the same reasoning, starting with any point inside this circle, as the hypothesis now holds for any such point. In this manner by repeated employment of the same reasoning, it can be shown that $f(z)$ is identically zero throughout the interior of D.

Example

Consider the function $f(z) = cotz = \dfrac{cos\ z}{sin\ z}$, which is a quotient of the entire functions $p(z) = cos\ z$ and $q(z) = sin\ z$. Its singularities occur at the zeros of q. i.e., at the points $z = n\pi$ $(n = 0, \pm1, \pm2, \ldots)$. Since $p(n\pi) = (-1)^n \neq 0$, $q(n\pi) = 0$, and $q'(n\pi) = (-1)^n \neq 0$, each singular point $z = n\pi$ of f is a simple pole, with residue $= \dfrac{p(n\pi)}{q'(n\pi)} = \dfrac{(-1)^n}{(-1)^n} = 1$.

2. Complex Integration

Let [a, b] be a closed interval, where a, b are real numbers. Divide [a, b] into subintervals

$$[a = t_0, t_1], [t_1, t_2],\ldots, [t_{n-1}, t_n = b] \qquad (1)$$

by inserting n−1 points $t_1, t_2,\ldots, t_{n-1}$ satisfying the inequalities

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$$

Then the set $P = \{t_0, t_1,\ldots, t_n\}$ is called the partition of the interval [a, b] and the greatest of the numbers $t_1\ t_0$, $t_2 - t_1,\ldots, t_n - t_{n-1}$ is called the norm of the partition P. Thus the norm of the partition P is the maximum length of the subintervals in (1).

1.3. Remarks. The following two results are direct consequences of the above theorem

(i) If a function is regular in a region and vanishes at all points of a subregion of the given region, or along any arc of a continuous curve in the region, then it must be identically zero throughout the interior of the given region.

(ii) If two functions are regular in a region, and have identical values at an infinite number of points which have a limiting point in the region, they must be equal to each other throughout the interior of the given region.

i.e. If two functions, which are analytic in a domain, coincide in a part of that domain, then they coincide in the whole domain.

For this, we take $f(z) = f_1(z) - f_2(z)$.

2.29. Cauchy's Inequality (Cauchy's Estimate). If $f(z)$ is analytic within and on a circle C given by $|z - z_0| = R$ and if $|f(z)| \le M$ for every z on C, then

$$|f^n(z_0)| \le \frac{M\lfloor n}{R^n}$$

Proof. Since $f(z)$ is analytic inside C, we have by Cauchy's integral formula for nth derivative of an analytic function

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Since on the circle $|z - z_0| = R$,

$$z - z_0 = Re^{i\theta},\ dz = Re^{i\theta}\, i d\theta$$

and the length of the circle is $2\pi R$, therefore

$$|f^n(z_0)| = \frac{\lfloor n}{2\pi}\left|\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}\right|$$

$$\le \frac{\lfloor n}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}}$$

$$\le \frac{\lfloor n}{2\pi}\int_0^{2\pi} \frac{M|Re^{i\theta}id\theta|}{|Re^{i\theta}|^{n+1}} = \frac{\lfloor n}{2\pi}\int_0^{2\pi}\frac{M}{R^n}d\theta$$

$$= \frac{\lfloor n}{2\pi}\frac{M}{R^n}2\pi = \frac{M\lfloor n}{R^n}$$

Hence $$|f^n(z_0)| \le \frac{M\lfloor n}{R^n}$$

2.30. Liouville's Theorem. A function which is analytic in all finite regions of the complex plane, and is bounded, is identically equal to a constant.

or

If an integral function $f(z)$ is bounded for all values of z, then it is constant

or

The only bounded entire functions are the constant functions.

Proof. Let z_1, z_2 be arbitrary distinct points in z-plane and let C be a large circle with centre at origin and radius R such that C encloses z_1 and z_2 i.e. $|z_1| < R$, $|z_2| < R$.

Since $f(z)$ is bounded, there exists a positive number M such that $|f(z)| \le M\ \forall\ z$.
By Cauchy's integral formula,

$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_1}$$

$$f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_2}$$

$$\therefore \qquad f(z_2) - f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)(z_2-z_1)}{(z-z_2)(z-z_1)}dz$$

Thus

$$|f(z_2) - f(z_1)| \le \frac{|z_2-z_1|}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_1||z-z_2|}$$

$$\le \frac{M|z_2-z_1|}{2\pi}\int_C \frac{|dz|}{|z-z_1||z-z_2|}$$

$$\le \frac{M|z_2-z_1|}{2\pi}\int_C \frac{|dz|}{(|z|-|z_1|)(|z|-|z_2|)} \quad |\because |z-z_1| \ge |z|-|z_1|$$

Now, on the circle C, $z = R\,e^{i\theta}$, $|z| = R$,

$dz = Re^{i\theta}\, i d\theta$

Therefore,

$$|f(z_2) - f(z_1)| \le \frac{M|z_2-z_1|}{2\pi}\int_0^{2\pi} \frac{|Re^{i\theta}\, i d\theta|}{(R-|z_1|)(R-|z_2|)}$$

$$= \frac{M|z_2-z_1|}{2\pi}\frac{R}{(R-|z_1|(R-|z_2|)}2\pi$$

$$= \frac{M|z_2-z_1|}{\left(1-\frac{|z_1|}{R}\right)\left(1-\frac{|z_2|}{R}\right)}.\frac{1}{R}$$

which tends to zero as $R\to\infty$.

Hence $f(z_2) - f(z_1) = 0$ i.e. $f(z_1) = f(z_2)$

But z_1, z_2 are arbitrary, this holds for all couples of points z_1, z_2 in the z-plane, therefore $f(z)$ = constant.

2.31. The Fundamental Theorem of Algebra. Any polynomial

$P(z) = a_0 + a_1 z + \ldots + z_n z^n$, $a_n \ne 0$, $n \ge 1$ has at least one point $z = z_0$ such that $P(z_0) = 0$ i.e. P(z) has at least one zero.

Proof. We establish the proof by contradiction.

If P(z) does not vanish, then the function $f(z) = \dfrac{1}{P(z)}$ is analytic in the finite z-plane. Also when $|z|\to\infty$, $P(z)\to\infty$ and hence $f(z)$ is bounded in entire complex plane, including infinity. Liouville's theorem then implies that $f(z)$ and hence P(z) is a constant which violates $n \ge 1$ and thus contradicts the assumption that P(z) does not vanish. Hence it is concluded that P(z) vanishes at some point $z = z_0$

2.32. Remark. The above form of fundamental theorem of algebra does not tell about the number of zeros of P(z). Another form which tells that P(z) has exactly n zeros, will be discussed later on. Of course, here we can prove this result by using the process of algebra as follows :

(Gauss's mean value theorem) *If f is analytic in a simply connected domain D that contains the circle $C_R(z_0) = \{z : |z - z_0| = R\}$, then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right) d\theta.$$

Proof We parametrize the circle $C_R(z_0)$ by

$$C_R(z_0) : z(\theta) = z_0 + Re^{i\theta} \quad \text{and} \quad dz = iRe^{i\theta} d\theta, \qquad \text{for } 0 \le \theta \le 2\pi,$$

and use this parametrization along with Cauchy's integral formula to obtain

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + Re^{i\theta}\right) iRe^{i\theta}\, d\theta}{Re^{i\theta}} = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right) d\theta.$$

Harmonic functions

In this section we return to one aspect of the theory that concerns the analysis of harmonic functions, subject often called *potential theory.*

Recall that a C^2 function u on an open set $A \subseteq \mathbf{R}^2$ is said to be harmonic on A if $\Delta u = 0$ on A, where $\Delta = \partial_x^2 + \partial_y^2$ is the *Laplacian*. The next lemma collects the first elementary but fundamental facts about the relation between harmonic and holomorphic functions.

Lemma

If $f = u + iv$ is holomorphic on an open set $A \subseteq \mathbf{C}$ then its real and imaginary parts u and v are harmonic on A.

If u is a real harmonic function on a simply connected *open set $\mathcal{D}$, then there exists a real harmonic function v on $\mathcal{D}$ such that $u + iv$ is holomorphic on $\mathcal{D}$. In this case, we will say that v is the harmonic* conjugate *of u on $\mathcal{D}$.*

Proof. The first part follows from Subsection 1.2 .

Suppose now u is a real harmonic function on a simply connected open set $\mathcal{D}$. We wish to $v \in C^2(\mathcal{D})$ satisfying the CR-equations on $\mathcal{D}$, that is, such that

$$dv = (-\partial_y u)dx + (\partial_x u)dy\,.$$

The one on the right hand side is a closed differential since u is harmonic. Since $\mathcal{D}$ is simply connected, it is an exact differential, so such a v exists. It immediately follows that $u + iv$ is holomorphic. □

We remark that the hypothesis of $\mathcal{D}$ being simply connected cannot be relaxed. As an example, consider $A = \mathbf{C} \setminus \{0\}$ and $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$. Then u is real and harmonic. On $A \cap \{x + iy : x > 0\}$ is the real part of $\log z$, that cannot be extended to all of A. Hence, there exists no function holomorphic on A whose real part is u.

7.1. **Maximum principle.** We now prove the maximum principle for (real) harmonic functions.

Theorem 7.2. *Let $\Omega \subseteq \mathbf{C} \equiv \mathbf{R}^2$ be a domain (connected open set), $u : \Omega \to \mathbf{R}$ be harmonic. If there exists $z_0 \in \Omega$ and $r_0 > 0$ such that $D(z_0, r_0) \subseteq \Omega$ and $u(z_0) = \sup\{u(z) : z \in D(z_0, r_0)\}$, then u is constant on Ω.*

Proof. Let

$$\Omega' = \{z \in \Omega : \text{ there exists } r_z > 0 \text{ such that for } w \in D(z, r_z),\ u(w) = u(z_0)\}.$$

We wish to show that Ω' is open, closed in Ω and non-empty, thus showing that $\Omega' = \Omega$.

On $D(z_0, r_0)$ we can find h holomorphic such that $\operatorname{Re} h = u$. Take $f = e^h$. Since $|f| = e^{\operatorname{Re} h} = e^u$, $|f|$ attains its maximum at z_0. Hence f is constant on $D(z_0, r_0)$, so is u. Thus, $\Omega' \neq \emptyset$. Moreover, Ω' is open by construction.

Finally, let $z \in \overline{\Omega'}$. Let $D(z, r_z) \subseteq \Omega$. Since $z \in \overline{\Omega'}$, there exists some open disk on which u is constant. Let h_z be the holomorphic function on $D(z, r_z)$ whose real part is u. Then, h_z is constant on an open disk, hence on all of $D(z, r_z)$, so is u. Thus, $z \in \Omega'$, Ω' is closed, that is, $\Omega' = \Omega$. □

Theorem 7.5. (The mean value property) *Let $A \subseteq \mathbf{C}$ be open, $\overline{D(z_0, r)} \subseteq A$, u be harmonic on A. Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Proof. Let $s > r$ be such that $D(z_0, s) \subseteq A$ and let h be holomorphic on $D(z_0, s)$ and such that $u = \operatorname{Re} h$, $h = u + iv$. We can apply Cauchy's formula to h on $\gamma = \partial D(z_0, r)$, $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$. We have

$$\begin{aligned} h(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{h(\zeta)}{\zeta - z_0}\, d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{h(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta + i \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta. \end{aligned}$$

By passing to the real and imaginary part we obtain the conclusion. □

Although worth be to stated separately, the mean value property can be obtained as a particular case of the next result.

Term by term integration and differentiation

Sometimes the calculus one needs to do involves functions which cannot be defined in a traditional way by a formula, but only in terms of convergent series of 'elementary' functions. This then poses a question:
When is the formal term by term integration or differentiation of a series of functions valid (i.e. will give the same result as the integration or differentiation applied directly to the sum of the series)?

Earlier we proved the following.

Theorem 1. *Suppose that $f_n : [a,b] \to \mathbb{R}$, for each $k = 1, 2, \ldots$, is integrable on $[a,b]$ and that $f_n(x) \to f(x)$ uniformly on $[a,b]$ as $n \to \infty$.*

Then the limit function $f(x)$ is integrable on $[a,b]$ and $\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$.

This can be easily converted into a version for the series.

Theorem 1′ (Term by term integration). *Suppose that $u_k : [a,b] \to \mathbb{R}$, for each $k = 1, 2, \ldots$, is integrable on $[a,b]$ and $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a,b]$.*

Then the sum $f(x) = \sum_{k=1}^{\infty} u_k(x)$ is integrable on $[a,b]$ and $\int_a^b f(x)dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)$.

Proof. Put $s_n(x) = \sum_{k=1}^{n} u_k(x)$ and apply Theorem 1 to the sequence (s_n). □

Thus uniformly convergent series can be integrated term by term.

What about term by term differentiation? Here the situation is somewhat less elegant than with integration and there is a good reason for that.

Very *informally*, the integration is a 'bounded' operation whereas the differentiation is not. If $u(x)$ is 'small', say $|u(x)| < \varepsilon$ for each x, then the integral $|\int_0^1 u(x)dx| < \varepsilon$ is also 'small', but the derivative $|u'(x)|$ may be arbitrary 'large', consider e.g. $u(x) = \varepsilon \sin(x/\varepsilon^2)$ and let $\varepsilon \to 0$.

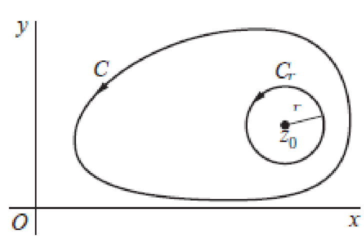
The next theorem is essentially a result for integration in disguise (as you will see from the proof). We can assume very little on the initial series, but the term by term differentiated series must satisfy a strong condition of being uniformly convergent.

Cauchy's Integral Formula

Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then $f(z_0) = \dfrac{1}{2\pi i}\int_C \dfrac{f(z)dz}{z - z_0}$.

Proof.

Let C_r denote a positively oriented circle $|z - z_0| = r$, where r is small enough that C_r is interior to C. Then, the quotient $\dfrac{f(z)}{(z - z_0)}$ is analytic between and on the contours C_r and C.



$$\int_C \frac{f(z)}{(z-z_0)}dz = \int_{C_r} \frac{f(z)}{(z-z_0)}dz.$$

This implies that,

$$\int_C \frac{f(z)}{(z-z_0)}dz - f(z_0)\int_{C_r} \frac{1}{(z-z_0)}dz. = \int_{C_r} \frac{f(z)-f(z_0)}{(z-z_0)}dz.$$

As in Problem 17, we obtain $\int_{C_r} \frac{1}{(z-z_0)}dz = 2\pi i$, so that

$$\int_C \frac{f(z)}{(z-z_0)}dz - 2\pi i\ f(z_0) = \int_{C_r} \frac{f(z)-f(z_0)}{(z-z_0)}dz.$$

Since f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, there is a positive number δ such that $|f(z)-f(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta$. Let the radius r of the circle C_r be smaller than the number δ. Then $|z-z_0| = r < \delta$ when z is on C_r, so that $|f(z)-f(z_0)| < \varepsilon$ when z is such a point. Therefore, by Theorem 2.1.4, we obtain

$$\int_C \frac{f(z)-f(z_0)}{z-z_0}dz < \frac{\varepsilon}{r}2\pi r = 2\pi\varepsilon.$$

Thus,

$$|\int_C \frac{f(z)}{(z-z_0)}dz - 2\pi i f(z_0)| < 2\pi\varepsilon.$$

Since, $\varepsilon > 0$ is arbitrary, it follows that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0}.$$

Cauchy's integral formula for Derivatives

The Cauchy's integral formula can be extended to provide an integral representation for derivatives of f at z_0. We assume that the function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sense and z_0 is any point interior to C. Then, for $n = 1, 2, 3, ...,$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}.$$

This is known as ***Cauchy's integral formula for derivatives.***

Problem

Evaluate $\int_c \frac{z^2-z+1}{z-1}dz$, where C is (a). $|z| = \frac{1}{2}$, and (b). $|z| = 2$, oriented in the positive sense.

Solution.

(a). Here, $f(z) = \dfrac{z^2-z+1}{z-1}$ is analytic at all points except the point $z = 1$ and $z = 1$ lies outside C.

Therefore by Cauchy – Goursat theorem, $\int_C \frac{z^2-z+1}{z-1}dz = 0.$

(b). Here $f(z) = z^2 - z + 1$ is analytic everywhere and C encloses the point $z = 1$. Therefore, by Cauchy's integral formula, we get

$$\int_C \frac{z^2-z+1}{z-1}dz = 2\pi i f(1) = 2\pi i.$$

Problem

Evaluate $\int_c \frac{z}{(9-z^2)(z+i)}dz$, where C is the positively oriented circle $|z| = 2$.

Solution.

Here, $f(z) = \frac{z}{(9-z^2)}$ is analytic within and on C, and $z_0 = -i$ lies inside C.
Therefore by Cauchy's integral formula, we get

$$\int_c \frac{z}{(9-z^2)(z+i)}dz = \int_c \frac{\frac{z}{(9-z^2)}}{(z-(-i))}dz = 2\pi i f(-i) = 2\pi i(\frac{-i}{10}) = \frac{\pi}{5}.$$

Problem

Evaluate $\int_c \frac{e^{2z}}{z^4}dz$, where C is the positively oriented unit circle.

Solution.

Here, $f(z) = e^{2z}$ is analytic within and on C, and $z_0 = 0$ lies inside C.
Therefore by Cauchy's integral formula for derivatives, we get

$$\int_c \frac{e^{2z}}{z^4}dz = \int_c \frac{e^{2z}}{(z-0)^{3+1}}dz = \frac{2\pi i}{3!}f^{(3)}(0) = \frac{8\pi i}{3}.$$

Problem

Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$, where C is the positively oriented circle $|z| = 3$.

Solution.

Here, $\frac{e^{2z}}{(z-1)(z-2)}$ is analytic everywhere, except the points $z = 1$ and $z = 2$, and both of these points lies inside C. Using partial fractions, we have

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Therefore by Cauchy's integral formula, we get

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}dz = \int_C \frac{e^{2z}}{z-2}dz - \int_C \frac{e^{2z}}{z-1}dz = 2\pi i(e^4 - e^2).$$

Theorem

Let f be continuous on a domain D. If $\int_C f(z)dz = 0$ for every closed contour C in D, then f is analytic throughout D.

Proof.

By Theorem 2.2.1, f has an antiderivative in D, i.e., there exists an analytic function F such that $F'(z) = f(z)$ at each point in D. Since f is the derivative of an analytic function F, it follows that (*See* the Remark above) f is analytic in D. □

4. Maximum Modulus Principle. Here, we continue the study of properties of analytic functions. Contrary to the case of real functions, we cannot speak of maxima and minima of a complex function $f(z)$, since ∀ is not an ordered field. However, it is meaningful to consider maximum and minimum values of the modulus $|f(z)|$ of the complex function $f(z)$, real part of

$f(z)$ and imaginary part of $f(z)$. The following theorem known as maximum modulus principle, is also true if $f(z)$ is not one-valued, provided $|f(z)|$ is one-valued.

4.3. Theorem. Let $f(z)$ be an analytic function, regular for $|z| < R$ and let $M(r)$ denote the maximum of $|f(z)|$ on $|z| = r$, then $M(r)$ is a steadily increasing function of r for $r < R$.

Proof. By maximum modulus principle, for two circles

$|z| = r_1$ and $|z| = r_2$, we have

$|f(z)| \leq M(r)$, where $r_1 < r_2$

which implies $M(r_1) \leq M(r_2)$, $r_1 < r_2$

and $M(r_1) = M(r_2)$ if $f(z)$ is constant.

Also M(r) cannot be bounded because if it were so, then $f(z)$ is a constant (by Lioureille's theorem). Hence M(r) is a steadily increasing function of r.

PART –B (EIGHT MARKS)

1.State and prove Cauchy's inequality.

2.State and prove weierstrass theorem.

3. If u(x,y) is a function harmonic in a simply connected region D, then prove that the mean value of u(x,y) taken along a circle in D is always equal to its value at the centre.

4. State and prove poisson's integral.

5. State and prove Liouville's theorem.

6. State and prove fundamental theorem of algebra.

7. State and prove Maximum modulus theorem.

8.State and prove Gauss mean value theorem.

9.Prove that Zero's of an analytic function are isolated.

10. Show that the values of the following integrals are zero, where c is the circle $|z| = 3$

i) $\int \frac{1}{z^2-4}\, dz$ ii) $\int \frac{1}{z^2+4}\, dz$

UNIT-III

SYLLABUS

Singularities–IsolatedSingularities-RemovableSingularity-Pole-Essential Singularity-Behaviour of a function at an isolated Singularity-Determination of the nature of Singularity -Problems-Residues- Residues Residues theorem(statement only)-problems

Singular Points and Residues

Recall that a point z_0 is called a ***singular point*** of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.

A singular point z_0 is said to be ***isolated*** if there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic.

If there is a positive number R_1 such that f is analytic for $R_1 < |z| < \infty$, then f is said to have an isolated singular point at $z_0 = \infty$.

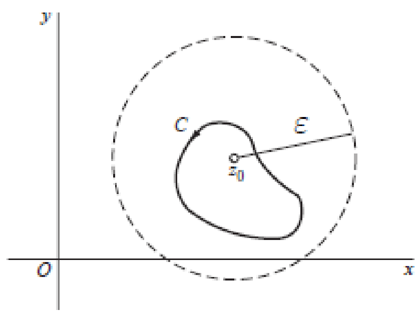
For example, the function $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has the three isolated singular points $z = 0$ and $z = \pm i$.

The function $f(z) = \dfrac{1}{sin(\pi/z)}$ has the singular points $z = 0$ and $z = 1/n$, $(n = \pm 1, \pm 2, ...)$, all lying on the segment of the real axis from $z = -1$ to $z = 1$. Each singular point except $z = 0$ is isolated. The singular point $z = 0$ is not isolated because every deleted ε-neighborhood of the origin contains other singular points of the function (since $1/n \to 0$ as $n \to \infty$).

Now, suppose that z_0 is an isolated singular point of a function f. Then there exists $\varepsilon > 0$ such that f is analytic in the annulus $0 < |z - z_0| < \varepsilon$. Hence $f(z)$ has a Laurent series representation

$$f(z) = \Sigma_{n=0}^{\infty} a_n (z - z_0)^n + \Sigma_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (0 < |z - z_0| < \varepsilon).$$

Let C is any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |z - z_0| < \varepsilon$.



Since $\int_C (z-z_0)^n dz = 0$ when $z \neq -1$, and $\int_C \frac{1}{(z - z_0)} dz = 2\pi i$, by integrating the above Laurent series, term by term around C, we obtain:

$$\int_C f(z)dz = 2\pi i \ b_1.$$

The complex number b_1, which is the coefficient of $1/(z-z_0)$ in the above Laurent series expansion of $f(z)$, is called the ***residue*** of f at the isolated singular point z_0, and we denote it as $\text{Res}_{z=z_0} f(z)$.

Therefore, we have $\int_C f(z)dz = 2\pi i \ \text{Res}_{z=z_0} f(z)$.

This provides a powerful method for evaluating certain integrals around simple closed contours.

Example

Consider the integral $\int_C z^2\ sin\ \frac{1}{z}\ dz$ where C is the positively oriented unit circle $|z| = 1$. Since the integrand is analytic everywhere in the finite complex plane except at $z = 0$, it has a Laurent series representation that is valid in the region $0 < |z| < \infty$. Therefore by the equation $\int_C f(z)dz = 2\pi i\ \text{Res}_{z=z_0} f(z)$, the value of integral $\int_C z^2\ sin\ \frac{1}{z}\ dz$ is $2\pi\ i$ times the residue of its integrand at $z = 0$.

Note that

$$z^2\ sin\ \frac{1}{z} = z^2(\frac{1}{z} - \frac{1}{3!}\cdot\frac{1}{z^3} + \frac{1}{5!}\cdot\frac{1}{z^5} - ....) = z - \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{5!}\cdot\frac{1}{z^3} - ....\quad 0 < |z| < \infty.$$

Here, the coefficient of $\frac{1}{z}$ is $\frac{-1}{3!}$. $\Rightarrow \text{Res}_{z=z_0} z^2\ sin\ \frac{1}{z} = \frac{-1}{3!}$. Therefore,

$$\int_C z^2\ sin\ \frac{1}{z}\ dz = 2\pi i\cdot\frac{-1}{3!} = \frac{-\pi\ i}{3}.$$

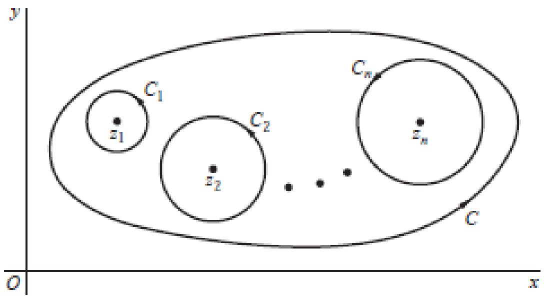
(Cauchy's Residue Theorem)

Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, ..., n)$ inside C, then

$$\int_C f(z)dz = 2\pi i\ \Sigma_{k=1}^{n} Res_{z=z_k} f(z)$$

Proof.

Let the points z_k $(k = 1, 2, ..., n)$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. The circles C_k, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain consisting of the points inside C and exterior to each C_k.



Hence, by the Cauchy–Goursat theorem for multiply connected domains,

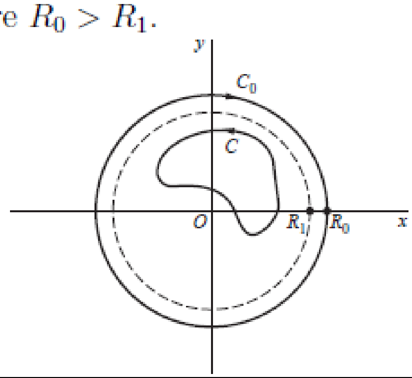
$$\int_C f(z)dz - \Sigma_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

. But, $\int_{C_k} f(z)dz = 2\pi i \text{ Res}_{z=z_k} f(z) \ (k = 1, 2, ..., n)$.

Therefore, $\int_C f(z)dz = 2\pi i \ \Sigma_{k=1}^{n} \text{Res}_{z=z_k} f(z)$. □

Residue at Infinity

Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C. Let R_1 denote a positive number which is large enough that C lies inside the circle $|z| = R_1$. Then, the function f is clearly analytic throughout the domain $R_1 < |z| < \infty$ and in this case, the point at infinity is said to be an isolated singular point of f. Now, let C_0 denote a circle $|z| = R_0$, oriented in the clockwise direction, where $R_0 > R_1$.



The residue of f at infinity is defined by means of the equation

$$\int_{C_0} f(z)dz = 2\pi\ i\ Res_{z=\infty} f(z) \quad ------(1)$$

Since f is analytic throughout the closed region bounded by C and C_0, the principle of deformation of paths implies that

$$\int_C f(z)dz = \int_{-C_0} f(z)dz = -\int_{C_0} f(z)dz.$$

Therefore, $\int_C f(z)dz = -2\pi\ i\ Res_{z=\infty} f(z) \quad ------(2)$.

Now to find this residue,we write the Laurent series

$$f(z) = \Sigma_{n=-\infty}^{\infty} c_n\ z^n \quad (R_1 < |z| < \infty),$$

where

$$c_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)dz}{z^{n+1}} \quad (n = 0, \pm1, \pm2, ...).$$

Replacing z by $1/z$ in the above Laurent series and then multiplying by $1/z^2$, we see that

$$\frac{1}{z^2} f(\frac{1}{z}) = \Sigma_{n=-\infty}^{\infty} c_n\ z^{n+2} = \Sigma_{n=-\infty}^{\infty} c_{n-2}\ z^n \quad (0 < |z| < \frac{1}{R_1}).$$

Therefore, by definition of residues, $c_{-1} = Res_{z=0}[\frac{1}{z^2} f(\frac{1}{z})]$. But, by the above formula to compute the coefficients of Laurent series,

$$c_{-1} = \frac{1}{2\pi i}\int_{-C_0} f(z)dz$$

$$\Rightarrow \int_{C_0} f(z)dz = -2\pi\ i\ Res_{z=0}[\frac{1}{z^2} f(\frac{1}{z})] \quad ------(3).$$

Now from equations (1) and (3), it follows that

$$Res_{z=\infty}f(z) = -Res_{z=0}[\frac{1}{z^2}f(\frac{1}{z})] \quad ------(4)$$

From equations (2) and (4), we obtain the following theorem, which is sometimes more efficient to use than Cauchy's residue theorem since it involves only one residue.

Problem

Evaluate the integral $\int_C \frac{5z-2}{z(z-1)}dz$, where C is the positively oriented circle $|z| = 2$.

Solution.

Here, the integrand $f(z) = \frac{5z-2}{z(z-1)}$ has the two isolated singularities $z = 0$ and $z = 1$, both of which are interior to C. We first expand $f(z) = \frac{5z-2}{z(z-1)}$ as a Laurent series about $z = 0$ as follows:

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \cdot \frac{-1}{1-z} = (5-\frac{2}{z})(-1-z-z^2-....) \quad (0 < |z| < 1).$$

Therefore, the $\text{Res}_{z=0}f(z)$ is the coefficient of $1/z$ in this Laurent series expansion, i.e., $\text{Res}_{z=0}f(z) = 2$.

Now we expand $f(z) = \dfrac{5z-2}{z(z-1)}$ as a Laurent series about $z = 1$ as follows:

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{(z-1)} \cdot \frac{1}{1+(z-1)} = (5+\frac{3}{(z-1)})[1-(z-1)+(z-1)^2-....],$$

when $0 < |z-1| < 1$. From this expansion, we get $\text{Res}_{z=1}f(z) = 3$, the coefficient of $i/(z-1)$. Therefore, by Cauchy's residue theorem,

$$\int_C \frac{5z-2}{z(z-1)}dz = 2\pi i\ [Res_{z=0}f(z) + Res_{z=1}f(z)] = 2\pi i\ [2+3] = 10\pi\ i.$$

Remark.

The above problem can also be solved by using Theorem 4.1.2. Here,

$$\frac{1}{z^2}f(\frac{1}{z}) = \frac{5-2z}{z(1-z)} = \frac{5-2z}{z} \cdot \frac{1}{1-z} = (\frac{5}{z}-2)\ (1+z+z^2+....)\ \ (0 < |z| < 1).$$

Therefore, $Res_{z=0}[\frac{1}{z^2}f(\frac{1}{z})]$ is the coefficient of $1/z$ in the above Laurent series expansion. i.e., $Res_{z=0}[\frac{1}{z^2}f(\frac{1}{z})] = 5$. Therefore,

$$\int_C f(z)dz = 2\pi\ i\ Res_{z=0}[\frac{1}{z^2}f(\frac{1}{z})] = 2\pi\ i\ \cdot 5 = 10\pi\ i\ .$$

EXAMPLE 7.2.1 Let us find $\int_\Gamma f(z)dz$, where $f(z) = \frac{1}{z(z-1)}$ and Γ is the positively oriented circle with centre 0 and radius 2.

By residue theorem,

$$\int_\Gamma \frac{dz}{z(z-1)} = 2\pi i[Res(f, z_1) + Res(f, z_2)], \qquad z_1 = 0,\ z_2 = 1.$$

Since $z_1 = 0$ and $z_2 = 1$ are simple poles of the function

$$Res(f, z_1) = \lim_{z\to 0} zf(z) = -1,$$

$$Res(f, z_2) = \lim_{z\to 1} (z-1)f(z) = 1.$$

Hence, $\int_\Gamma \frac{dz}{z(z-1)} = 0.$ □

EXAMPLE 7.2.2 Let us find $\int_\Gamma f(z)dz$, where $f(z) = \frac{1}{z(z-1)^2}$ and Γ is the positively oriented circle with centre 0 and radius 2. By residue theorem,

$$\int_\Gamma \frac{dz}{z(z-1)^2} = 2\pi i[Res(f, z_1) + Res(f, z_2)], \qquad z_1 = 0,\ z_2 = 1.$$

Since $z_1 = 0$ is a simple pole and $z_2 = 1$ is a pole of order 2,

$$Res(f, z_1) = \lim_{z\to 0} zf(z) = 1, \qquad Res(f, z_2) = \varphi'(1)$$

where $\varphi(z) = \frac{1}{z}$ so that $\varphi'(1) = -1$. Thus, $\int_\Gamma \frac{dz}{z(z-1)^2} = 0.$ □

Example

(Simple Pole)

Consider the function $f(z) = \dfrac{5z-2}{z-1}$.

Note that $f(z) = \dfrac{5z-2}{z-1} = \dfrac{5(z-1)+3}{(z-1)} = 5 + \dfrac{3}{(z-1)}$, which is the Laurent series expansion of $f(z) = \dfrac{5z-2}{z-1}$ about the isolated singular point $z = 1$. Here, the principal part contains only one nonzero term namely $\dfrac{3}{(z-1)}$, and hence $z = 1$ is a simple pole of $f(z)$ and $\text{Res}_{z=1} \dfrac{5z-2}{z-1} = 3$.

Example

(Pole of Order 2)

Consider the function $f(z) = \dfrac{1}{z^2(z+1)}$. Note that

$$f(z) = \frac{1}{z^2(z+1)} = \frac{1}{z^2}\,\frac{1}{1-(-z)} = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \ldots.. \ (0 < |z| < 1).$$

Thus, the principal part of the Laurent series expansion of $f(z) = \dfrac{1}{z^2(z+1)}$ about the isolated singular point $z = 0$ shows that $z = 0$ is a pole of order 2, and $\text{Res}_{z=0} \dfrac{1}{z^2(z+1)} = -1$.

Example

(Removable Singular Point)

Consider the function $f(z) = \dfrac{sin\ z}{z}$.

Note that $\dfrac{sin\ z}{z} = \dfrac{1}{z}\left[z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots.\right] = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{3!} - \ldots. \ (0 < |z| < \infty)$.

Thus, the principal part of the Laurent series expansion of $f(z) = \dfrac{sin\ z}{z}$ has no terms. $\Rightarrow z = 0$ is a removable singular point of $f(z)$, $\text{Res}_{z=0} \dfrac{sin\ z}{z} = 0$, and if we set $f(0) = 1$, $f(z)$ becomes an entire function.

Example

(Essential Singular Point)

We have

$$e^{\frac{1}{z}} = \Sigma_{n=0}^{\infty} \frac{1}{n!\ z^n} = 1 + \frac{1}{1!\ z} + \frac{1}{2!\ z^2} + \frac{1}{3!\ z^3} + ... \ (0 < |z| < \infty).$$

Thus, the principal part of the Laurent series expansion of $f(z) = e^{\frac{1}{z}}$ contains infinitely many terms. $\Rightarrow z = 0$ is an essential singular point of $f(z)$ and $\text{Res}_{z=0}\ e^{\frac{1}{z}} = 1$.

Problem 31.

Show that $z = 0$ is a removable singularity of the function $f(z) = \dfrac{1 - cos\ z}{z^2}$.

Solution.

We have the Macalurin' series expansion

$$cos\ z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...\ (|z| < \infty).$$

Therefore, for $0 < |z| < \infty$, we have

$$f(z) = \frac{1 - cos\ z}{z^2} = \frac{1}{z^2}[1 - (1 + \frac{1}{1!\ z} + \frac{1}{2!\ z^2} + \frac{1}{3!\ z^3} + ...)] = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + ...$$

The principal part of the Laurent series expansion has no terms. $\Rightarrow z = 0$ is a removable singular point of $\frac{1 - cos\ z}{z^2}$. If we set $f(0) = 1/2$, $f(z)$ becomes an entire function. ■

Problem 32.

Evaluate $\int_C e^{-1/z} sin\ (\frac{1}{z})\ dz$ where C is the positively oriented unit circle.

Solution.

We have the series expansions

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} - \frac{1}{3!} \cdot \frac{1}{z^3} + ...\ (0 < |z| < \infty)$$

and

$$sin\ (\frac{1}{z}) = \frac{1}{z} - \frac{1}{3!} \cdot \frac{1}{z^3} + \frac{1}{5!} \cdot \frac{1}{z^5} - ....\ (0 < |z| < \infty)$$

Therefore, for $0 < |z| < \infty$, we have

$$e^{-1/z} sin\ (\frac{1}{z}) = \frac{1}{z} - \frac{1}{z^2} + ...$$

$\Rightarrow$ The principal part of the Laurent series expansion of $e^{-1/z} sin\ (\frac{1}{z})$ has infinitely many terms. $\Rightarrow z = 0$ is an essential singular point with residue 1. Hence $\int_C e^{-1/z} sin\ (\frac{1}{z})\ dz = 2\pi\ i \cdot 1 = 2\pi\ i$. ■

Problem 34.

Find the value of the integral $\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$, taken counterclockwise around the circle $|z-2| = 2$.

Solution.

Here, $f(z) = \frac{3z^3+2}{(z-1)(z^2+9)}$ has the singular points $z = 1$ and $z = \pm 3\ i$ and all these are simple poles.

Here, C is $|z-2| = 2$. The simple poles $z = 1$ lies inside C , whereas $z = -3\imath$ and $z = 3\ i$ lies out side C.

As in above problem, we find that $\text{Res}_{z=1} f(z) = 1/2$. By Cauchy's residue theorem,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i\ Res_{z=1} f(z) = \pi\ i.$$

Problem 35.

Evaluate $\int_C \frac{e^z}{(z+1)^2}dz$, along the circle $|z-1| = 3$ taken in counterclockwise direction.

Solution.

Note that $f(z) = \frac{e^z}{(z+1)^2}$ is analytic at all points except $z = -1$.

Here, C is $|z-1| = 3$. $\Rightarrow$ The singular point $z = -1$ lies inside C .

Also, we can write $f(z) = \frac{\phi(z)}{(z+1)^2}$ where $\phi(z) = e^z$.

Since $\phi(z)$ is analytic and non zero at $z = -1$, it is a double pole of f, and $\text{Res}_{z=-1} f(z) = \frac{\phi'(-1)}{1!} = e^{-1}$.

Hence, $\int_C \frac{e^z}{(z+1)^2} dz = 2\pi i \; Res_{z=-1} f(z) = 2\pi \; i \cdot e^{-1} = \frac{2\pi \; i}{e}$.

Theorem

Let two functions p and q be analytic at a point z_0 . If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$, then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$Res_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Example

Consider the function $f(z) = cotz = \frac{cos \; z}{sin \; z}$, which is a quotient of the entire functions $p(z) = cos \; z$ and $q(z) = sin \; z$. Its singularities occur at the zeros of q. i.e., at the points $z = n\pi$ $(n = 0, \pm 1, \pm 2, ...)$. Since $p(n\pi) = (-1)^n \neq 0$, $q(n\pi) = 0$, and $q'(n\pi) = (-1)^n \neq 0$, each singular point $z = n\pi$ of f is a simple pole, with residue $= \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1$.

Problems:

Find the residues of $f(z) = \dfrac{z+1}{z^2+9}$ at singular points.

Solution.

Here, $f(z) = \dfrac{z+1}{z^2+9}$ is analytic at all points except $z = \pm 3\,i$. We can write $f(z) = \dfrac{\phi(z)}{z - 3\imath}$ where $\phi(z) = \dfrac{z+1}{z+3\,i}$. Since $\phi(z)$ is analytic at $3\,i$ and $\phi(3\,i) \neq 0$, $z = 3\,i$ is a simple pole of f, and $\text{Res}_{z=3\,i} f(z) = \phi(3\,i) = \dfrac{3-i}{6}$.

Similarly, $z = -3\,i$ is also a simple pole of f, and $\text{Res}_{z=-3\,i} f(z) = \dfrac{3+i}{6}$.

PART-B(EIGHT MARKS)

1.prove that If z = a is a pole of order m of a function f(z), then z = a is a zero of order m of the function 1/f(z).

2.A function has an isolated singularity at z=a but is analytic in the deleted neighbourhood $0<|z-a|<$R. If the function is bounded in the deleted neighbourhood, then prove that the singularity is a removable singularity.

3.Find the residues of $f(z) = \frac{z^4}{(z-2)(z-3)(z-1)^4}$ at its singularities.

4. State and prove Weierstrass theorem.

5. If z = a is a pole of a function f(z), then prove that $\lim_{z\to a} f(z) = \infty$,

6.Find the orders of poles of $f(z) = \frac{1}{z(e^z-1)}$.

7.State and prove residue theorem.

8.If z = a is a removable singularity of a function f(z), then prove that there exist a deleted neighbourhood of z = a in which f(z) is bounded.

9. If a function f(z) is analytic in a deleted neighbourhood of z = a and if $\lim_{z \to a} f(z) = \infty$, then prove that z=a is a pole of f(z).

10. If z=a is a zero of order m of an analytic function f(z), then z=a is a pole of order m of the function $\frac{1}{f(z)}$.

11.If a function f(z) is analytic in the extended plane except at a finite number of singularities including z = ∞, then the sum of the residues of f(z) is zero.

12.State the nature of singularity of f (z) in the following cases:

i) $\frac{e^z-1}{z}$ ii) $\frac{\sin z}{z}$ iii) $cosec\, z - \frac{1}{z}$

UNIT-V

SYLLABUS

Meromorphic functions: Theorem on number of zeros minus number of poles Principle of argument: Rouche‟s theorem- Theorem that a function which is meromorphic in the extended plane is a rational function.

5. Meromorphic Function. A function $f(z)$ is said to be meromorphic in a region D if it is analytic in D except at a finite number of poles. In other words, a function $f(z)$ whose only singularities in the entire complex plane are poles, is called a meromorphic function. The word meromorphic is used for the phrase "analytic except for poles". The concept of meromorphic is used in contrast to holomorphic. A meromorphic function is a ratio of entire functions.

Rational functions are meromorphic functions.

e.g.
$$f(z) = \frac{z^2-1}{z^5+2z^3+z} = \frac{(z+1)(z-1)}{z(z^4+2z^2+1)} = \frac{(z+1)(z-1)}{z(z^2+1)^2}$$

$$= \frac{(z+1)(z-1)}{z(z+i)^2(z-i)^2}$$

has poles at $z = 0$ (simple), at $z = \underline{+}\, i$ (both double) and zeros at $z = \underline{+}\, 1$ (both simple)

Since only singularities of $f(z)$ are poles, therefore $f(z)$ is a meromorphic function.

Similarly, tan z, cot z, sec z are all meromorphic functions.

A meromorphic function does not have essential singularity. The following theorem tells about the number of zeros and poles of a meromorphic function.

5.1. Theorem. Let $f(z)$ be analytic inside and on a simple closed contour C except for a finite number of poles inside C and let $f(z) \neq 0$ on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz = N - P$

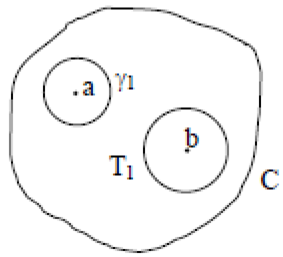
where N and P are respectively the total number of zeros and poles of $f(z)$ inside C, a zero (pole) of order m being counted m times.

Proof. Suppose that $f(z)$ is analytic within and on a simple closed contour C except at a pole $z = a$ of order p inside C and also suppose that $f(z)$ has a zero of order n at $z = b$ inside C.

Then, we have to prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = n-p$$

Let γ_1 and T_1 be the circles inside C with centre at z = a and z = b respectively.



Then, by cor. to Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\int_{T_1} \frac{f'(z)}{f(z)}\,dz \qquad (1)$$

Now, $f(z)$ has pole of order p at z = a, so

$$f(z) = \frac{g(z)}{(z-a)^p} \qquad (2)$$

where g(z) is analytic and non-zero within and on γ_1. Taking logarithm of (2) and differentiating, we get

$$\log f(z) = \log g(z) - p\log(z-a)$$

i.e., $$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{p}{z-a}$$

Therefore,

$$\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz = \int_{\gamma_1} \frac{g'(z)}{g(z)}\,dz - p\int_{\gamma_1} \frac{dz}{z-a} \qquad (3)$$

Since $\frac{g'(z)}{g(z)}$ is analytic within and on γ_1, by Cauchy theorem,

$$\int_{\gamma_1} \frac{g'(z)}{g(z)}\,dz = 0$$

Thus (3) gives $\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz = -2\pi ip \qquad (4)$

Again, $f(z)$ has a zero of order n at z = b, so we can write

$$f(z) = (z-b)^n\,\phi(z) \qquad (5)$$

where $\phi(z)$ is analytic and non-zero within and on T_1

Taking logarithm, then differentiating, we get

$$\frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{\phi'(z)}{\phi(z)}$$

or

$$\int_{T_1} \frac{f'(z)}{f(z)}\,dz = n\int_{T_1} \frac{dz}{z-b} + \int_{T_1} \frac{\phi'(z)}{\phi(z)}dz \qquad (6)$$

Since $\frac{\phi'(z)}{\phi(z)}$ is analytic within and on T_1, therefore

$$\int_{T_1} \frac{\phi'(z)}{\phi(z)}dz = 0 \text{ and thus (6) becomes}$$

$$\int_{T_1} \frac{f'(z)}{f(z)}dz = 2\pi in \qquad (7)$$

Writing (1) with the help of (4) and (7), we get

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = -p + n = n-p \qquad (8)$$

Now, suppose that $f(z)$ has poles of order p_m at $z = a_m$ for $m = 1, 2,\ldots,r$ and zeros of order n_m at $z = b_m$ for $m = 1, 2,\ldots,s$ within C. We enclose each pole and zero by circles $\gamma_1, \gamma_2,\ldots, \gamma_r$ and $T_1, T_2,\ldots, T_s$. Thus (8) becomes

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{m=1}^{s} n_m - \sum_{m=1}^{r} p_m$$

Taking $\sum_{m=1}^{s} n_m = N, \sum_{m=1}^{r} p_m = P$, we obtain

$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$ which proves the theorem. This theorem is also known as the argument principle which can be put in a more explicit manner as follows :

5.2. Theorem (The Argument Principle). Let $f(z)$ be meromorphic inside a closed contour C and analytic on C where $f(z) \neq 0$. When $f(z)$ describes C, the argument of $f(z)$ increases by a multiple of 2π, namely

$$\Delta_C \arg f(z) = 2\pi\,(N-P)$$

where N and P are respectively the total number of zeros and poles of $f(z)$ inside C, a zero (pole) of order m being counted m times.

Proof. Let $\arg f(z) = \phi$

So, we can write

$$f(z) = |f(z)|\, e^{i\phi}$$

i.e. $\log f(z) = \log |f(z)| + i\phi \qquad (1)$

Then as proved in the above theorem 4.1,

$$N - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$

$$= \frac{1}{2\pi i}\int_C d\,(\log f(z))$$

$$= \frac{1}{2\pi i}\int_C d\,(\log |f(z)| + i\,\phi)$$

$$= \frac{1}{2\pi i}\int_C d(\log|f(z)| + \frac{1}{2\pi}\int_C d\phi \qquad (2)$$

The first integral in (2) vanishes, since $\log|f(z)|$ is single valued, i.e., it returns to its original value at z goes round C. Now, $\int_C d\phi$ is the variation in the argument of $f(z)$ in describing the contour C,

Therefore $\int d\phi = \Delta_C \arg f(z)$
Thus, (2) becomes

$$\Delta_C \arg f(z) = 2\pi(N - P)$$

This formula makes it possible to compute the number N–P from the variation of the argument of $f(z)$ along the boundary of the closed contour C and is known as argument principle.

In particular, if $f(z)$ is analytic inside and on C, then P = 0

and $N = \frac{1}{2\pi}\Delta_C \arg f(z)$.

5.3. Rouche's Theorem. If $f(z)$ and g(z) are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Proof. First we prove that neither $f(z)$ nor $f(z) + g(z)$ has a zero on C.

If $f(z)$ has a zero at z = a on C, then $f(a) = 0$

Thus $|g(z) < |f(z)| \Rightarrow |g(a)| < f(a) = 0$

$\Rightarrow g(a) = 0 \Rightarrow |f(a)| = |g(a)|$

i.e. $|f(z)| = |g(z)|$ at z = a

which is contrary to the assumption that

$|g(z)| < |f(z)|$ on C.

Again, if $f(z) + g(z)$ has a zero at z = b on C,

then $f(b) + g(b) = 0 \Rightarrow f(b) = -g(b)$

i.e. $|f(b)| = |g(b)|$

again a contradiction.

Thus, neither $f(z)$ nor $f(z) + g(z)$ has a zero on C.

Now, let N and N′ be the number of zeros of $f(z)$ and $f(z) + g(z)$ respectively inside C. We are to prove that N = N′.

Since $f(z)$ and $f(z) + g(z)$ both are analytic within and on C and have no pole inside C, therefore the argument principle

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = N - P, \text{ with } P = 0, \text{ gives}$$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = N, \ \frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}dz = N'$$

Subtracting these two results, we get

$$\frac{1}{2\pi i}\int_C \left[\frac{f'+g'}{f+g} - \frac{f'}{f}\right]dz = N' - N$$

Let us take $\phi(z) = \dfrac{g(z)}{f(z)}$ so that $g = f\phi$

Now, $|g| < |f| \Rightarrow |g/f| < 1$ i.e. $|\phi| < |$

Therefore,

$$\frac{f'+g'}{f+g} = \frac{f'+f'\phi+f\phi'}{f+f\phi} = \frac{f'(1+\phi)+f\phi'}{f(1+\phi)}$$

$$= \frac{f'}{f} + \frac{\phi'}{1+\phi}$$

i.e. $$\frac{f'+g'}{f+g} - \frac{f'}{f} + \frac{\phi'}{1+\phi} \qquad (2)$$

Using (2) in (1), we get

$$N' - N = \frac{1}{2\pi i}\int_C \frac{\phi'}{1+\phi}dz = \frac{1}{2\pi i}\int_C \phi'(1+\phi)^{-1}\, dz \qquad (3)$$

Since we have observed that $|\phi| < 1$, so binomial expansion of $(1 + \phi)^{-1}$ is possible and this expansion in powers of ϕ is uniformly convergent and hence term by term integration is possible.

Thus, $\int_C \phi'(1+\phi)^{-1}dz = \int_C \phi'(1-\phi+\phi^2-\phi^3+\ldots)\, dz$

$$= \int_C \phi'\, dz - \int_C \phi\phi'\, dz + \int_C \phi^2\phi'\, dz \qquad (4)$$

Now, the functions f and g both are analytic within and on C and $f \neq 0$ $g \neq 0$ for any point on C, therefore $\phi = g/f$ is analytic and non-zero for any point on C. Thus ϕ and its all derivatives are analytic and so by Cauchy's theorem, each integral on R.H.S. of (4) vanishes. Thus

$$\int_C \phi'(1+\phi)^{-1}\, dz = 0$$

and therefore from (3), we conclude $N' - N = 0$

i.e. $N = N'$

5.5. Example. Determine the number of roots of the equation

$$z^8 - 4z^5 + z^2 - 1 = 0$$

that lie inside the circle $|z| = 1$

Solution. Let C be the circle defined by $|z| = 1$

Let us take $f(z) = z^8 - 4z^5$, $g(z) = z^2-1$.

On the circle C,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^2-1}{z^8-4z^5}\right| \leq \frac{|z|^2+1}{|z|^5|4-z^3|}$$

$$\leq \frac{1+1}{4-|z|^3} = \frac{2}{4-1} = \frac{2}{3} < 1$$

Thus $|g(z)| < |f(z)|$ and both $f(z)$ and $g(z)$ are analytic within and on C, Rouche's theorem implies that the required number of roots is the same as the number of roots of the equation $z^8 - 4z^5 = 0$ in the region $|z| < 1$. Since $z^3 - 4 \neq 0$ for $|z| < 1$, therefore the required number of roots is found to be 5.

5.6. Inverse Function. If $f(z) = w$ has a solution $z = F(w)$, then we may write

$f\{F(w)\} = w$, $F\{f(z)\} = z$. The function F defined in this way, is called inverse function of f.

5.7. Theorem. (Inverse Function Theorem). Let a function $w = f(z)$ be analytic at a point $z = z_0$ where $f'(z_0) \neq 0$ and $w_0 = f(z_0)$.

Then there exists a neighbourhood of w_0 in the w-plane in which the function $w = f(z)$ has a unique inverse $z = F(w)$ in the sense that the function F is single-valued and analytic in that neighbourhood such that $F(w_0) = z_0$ and

$$F'(w) = \frac{1}{f'(z)}.$$

Proof. Consider the function $f(z) - w_0$. By hypothesis, $f(z_0) - w_0 = 0$. Since $f'(z_0) \neq 0$, f is not a constant function and therefore, neither $f(z) - w_0$ not $f'(z)$ is identically zero. Also $f(z) - w_0$ is analytic at $z = z_0$ and so it is analytic in some neighbourhood of z_0. Again, since zeros are isolated, neither $f(z) - w_0$ nor $f'(z)$ has any zero in some deleted neighbourhood of z_0. Hence there exists $\in > 0$ such that $f(z) - w_0$ is analytic for $|z - z_0| \leq \in$ and $f(z) - w_0 \neq 0$, $f'(z) \neq 0$ for $0 < |z-z_0| \leq \in$. Let D denote the open disc

$$\{z : |z - z_0| < \in\}$$

and C denotes its boundary

$$\{z : |z-z_0| = \in\}.$$

Since $f(z) - w_0$ for $|z - z_0| \leq \in$, we conclude that $|f(z) - w_0|$ has a positive minimum on the circle C. Let

$$\min_{z \in C} |f(z) - w_0| = m$$

and choose δ such that $0 < \delta < m$.

We now show that the function $f(z)$ assumes exactly once in D every value w_1 in the open disc

$T = \{ w : |w - w_0| < \delta\}$. We apply Rouche's theorem to the functions $w_0 - w_1$ and $f(z) - w_0$. The condition of the theorem are satisfied, since

$$|w_0 - w_1| < \delta < m = \min_{z \in C} |f(z) - w_0| \leq |f(z) - w_0| \text{ on C.}$$

Thus we conclude that the functions.

$$f(z) - w_0 \text{ and } (f(z) - w_0) + (w_0 - w_1) = f(z) - w_1$$

have the same number of zeros in D. But the function $f(z) - w_0$ has only one zero in D i.e. a simple zeros at z_0, since $(f(z) - w_0)' = f'(z) \neq 0$ at z_0.

Hence $f(z)-w_1$ must also have only one zero, say z_1 in D. This means that the function $f(z)$ assumes the value w, exactly once in D. It follows that the function $w = f(z)$ has a unique inverse, say $z = F(w)$ in D such that F is single-valued and $w = f\{F(w)\}$. We now show that the function F is analytic in D. For fix w_1 in D, we have $f(z) = w_1$ for a unique z_1 in D. If w is in T and $F(w) = z$, then

$$\frac{F(w)-F(w_1)}{w-w_1} = \frac{z-z_1}{f(z)-f(z_1)} \qquad (1)$$

It is noted that T is continuous. Hence $z \to z_1$ whenever $w \to w_1$. Since $z_1 \in D$, as shown above $f'(z_1)$ exists and is zero. If we let $w \to w$, then (1) shows that

$$F'(w_1) = \frac{1}{f'(z_1)}.$$

Thus $F'(w)$ exists in the neighbourhood T of w_0 so that the function F is analytic there.

Lemma 0.1. *A rational function has a pole or removable singularity at infinity. It has a removable singularity if and only if* $\deg Q \geq \deg P$.

Theorem 0.1. *The only meromorphic functions on $\hat{\mathbb{C}}$ are rational functions.*

Proof. Let $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a meromorphic function.

Claim-1. F has only finitely many poles $\{p_1, \cdots, p_n\}$ in the complex plane C.

To see this, note that $F(1/z)$ has either a pole or zero at $z = 0$. In either case there is a small neighborhood $|z| < \varepsilon$ which has no other pole. Which is the same as saying that F has no finite pole in $|z| > 1/\varepsilon$. But $|z| \leq 1/\varepsilon$ is compact, and since all poles are isolated, this shows that there are only finitely many poles. Now, corresponding to each of the poles $p_k \in \mathbb{C}$ there exists a polynomial P_k (see Remark 0.2 in Lecture-20) such that

$$F(z) = P_k\Big(\frac{1}{z-p_k}\Big) + G_k(z),$$

where G_k is holomorphic on a whole neighborhood around p_k (including at the point p_k). Similarly, we can write

$$F\Big(\frac{1}{z}\Big) = P_\infty\Big(\frac{1}{z}\Big) + G_\infty(z),$$

where as before, $G(z)$ is holomorphic in a neighborhood of $z = 0$.

Claim-2. The function

$$H(z) = F(z) - P_\infty(z) - \sum_{k=1}^{n} P_k\Big(\frac{1}{z-p_k}\Big)$$

is an entire and bounded function.

Assuming the claim, by Liouville's theorem, $H(z$ is a constant, and hence $F(z)$ must be rational, and the theorem is proved. To prove the claim, first note that clearly, $H(z)$ is holomorphic away from $\{p_1, \cdots, p_n\}$. At some $z = p_k$, $P_j(1/z - p_j)$ is holomorphic for all $j \neq k$. On the other hand, near p_k,

$$F(z) - P_k\Big(\frac{1}{z-p_k}\Big) = G_k(z)$$

which is holomorphic. This shows that $H(z)$ is entire. As a consequence, to show boundedness, we only need to show boundedness on $|z| > R$ for some large R. To see, first observe that since P_k are polynomials,

$$\lim_{z\to\infty} P_k\Big(\frac{1}{z-p_k}\Big) = 0.$$

Hence it is enough to show that $F(z) - P_\infty(z)$ is bounded near infinity. But this follows immediately from noting that

$$G_\infty(z) = F\Big(\frac{1}{z}\Big) - P_\infty\Big(\frac{1}{z}\Big)$$

is holomorphic near $z = 0$ and hence is bounded on $|z| < \varepsilon$ for some $\varepsilon > 0$. In particular $F(z) - P_\infty(z)$ is bounded on $|z| > 1/\varepsilon$. This proves the claim, and hence completes the proof of the theorem.

□

A simple consequence of the proof is the following theorem on partial fraction decomposition that we take for granted as an important tool in integration theory, but never see the proof of.

Corollary 0.1. *For any rational function $R(z) = P(z)/Q(z)$ has a partial fraction decomposition of the form*

$$R(z) = P_\infty(z) + \sum_{k=1}^{n} P_k\Big(\frac{1}{z-p_k}\Big),$$

where p_k is a root of $Q(z)$ of order m_k, P_k is a polynomial of degree m_k, and $\deg P_\infty = \deg P - \deg Q$.

PART – B (EIGHT MARKS)

1.State and prove the principle of argument in meromorphic function.

2.State and prove fundamental theorem of algebra in meromorphic function.

3. Prove that

i) D is a simply connected region,

ii) f(z) is a meromorphic function in D

iii) C is a scro curve in D, not passing through a pole or zero of f(z).

Then prove that $\int_C \frac{f'(z)}{f(z)} dz = 2\pi i[n(Z,f) - n(P,f)]$, where n(Z,f) and n(P,f) denote respectively the number of zeros and number of poles of f(z) in C_i, the zeros and poles being counted as many times as their orders.

4. Show that one root of $z^4 + z^3 + 1 = 0$ lies in the first quadrant.

5. State and prove Rouche's theorem**.**

6. Show that the number of zeros of the function $f(z) = z^4 - 5z + 1$ which lie in the annulus region $1 < |z| < 2$.

7. Prove that a function which is meromorphic in the extended plane is a rational function.

8. State and prove Hurwitz's theorem.

9. State and prove Rouche's theorem.

Reg. No..................................

[13MMU602]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2013 onwards)

B.Sc., DEGREE EXAMINATION, APRIL 2016

Sixth Semester

MATHEMATICS

COMPLEX ANALYSIS - II

Time: 3 hours   Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 x 8 = 40 Marks) (2 ½ Hours)
Answer ALL the Questions

21. a) State and prove Cauchy's inequality.

Or

b) If u(x,y) is a function harmonic in a simply connected region D, then prove that the mean value of u(x,y) taken along a circle in D is always equal to its value at the centre.

22. a) State and prove Laurent's theorem.

Or

b) Prove that $f_1(z)$ and $f_2(z)$ are two functions analytic in a region D such that, for all z_n, $f_1(z_n) = f_2(z_n)$, where $\{ z_n \}$ is a sequence of points in D coverging to a point in D. then $f_1(z) \equiv f_2(z)$ in D.

23. a) If z = a is a pole of order m of a function f(z), then z = a is a zero of order m of the function 1/f(z).

Or

b) Find the residues of $f(z) = \frac{z^4}{(z-2)(z-3)(z-1)^4}$ at its singularities.

24. a) Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)} dx$

Or

b) Show that $\int_0^{2\pi} \frac{1}{a+b\cos\theta} d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$ $a > |b| > 0$.

25. a) Prove that i) D is a simply connected region, ii) f(z) is a meromorphic function in D iii) C is a scro curve in D, not passing through a pole or zero of f(z). Then prove that $\int_C \frac{f'(z)}{f(z)} dz = 2\pi i\,[n(Z,f) - n(P,f)]$, where n(Z,f) and n(P,f) denote respectively the number of zeros and number of poles of f(z) in C,, the zeros and poles being counted as many times as their orders.

Or

b) Show that the number of zeros of the function $f(z) = z^4 - 5z + 1$ which lie in the annulus region $1 < |z| < 2$.

Reg. No.................................

[11MCU601]

KARPAGAM UNIVERSITY
(Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2011 onwards)

B.Sc. DEGREE EXAMINATION, APRIL 2014

Sixth Semester

MATHEMATICS

COMPLEX ANALYSIS

3 hours — Maximum : 100 marks

PART – A (15 x 2 = 30 Marks)
Answer ALL the Questions

1. Z_1 and Z_2 are two complex numbers, then prove that $\arg\left(\frac{Z_1}{Z_2}\right) = \arg Z_1 - \arg Z_2$.
2. Show that the equation $\arg\left(\frac{Z-1}{Z+1}\right) = \mu$ represent orthogonal circle.
3. Find the locus of Z if $\text{Im}\left(\frac{Z-Z_1}{Z-Z_2}\right) = 0$
4. Show that $f(Z) = \overline{Z} = x - iy$ is not differentiable at Zero
5. Prove that $f'(Z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$.
6. Show that the function $f(z) = e^x(\cos y + i\sin y)$ is analytic.
7. Prove that under a bilinear transformation no two points in the Z plane go to same point in the W plane.
8. Prove that cross ratio is preserved by a bilinear transformation.
9. Find the bilinear transformation which maps $Z = 2,1,0$ to $W = 1,0,i$.
10. Prove that $\int_C \frac{1}{z-a}dz = 2\pi i$ where C the positively oriented circle is whose radius is r and centre is $z = a$.
11. State Cauchy's integral formula for n^{th} derivative.
12. Define Zero's of a function.
13. Find the taylor's series for the function $\log(1-z)$.
14. If $z = a$ is a zero of order m of an analytic function f(z) then prove that z=a is a pole of order m of $\frac{1}{f(z)}$.

15. Find the residue of $f(z) = \frac{2z+1}{z^2-z-2}$.

PART B (5 X 14 = 70 Marks)
Answer ALL the Questions

16. (a) Explain stereographic projection.

Or

(b) State and prove Heine-Borel Theorem

17. (a) State and prove sufficient condition for differentiability.

Or

(b) If $f(z)$ is a function analytic in a region D, then prove that $f(z)$ is constant in D, if in D, either (i) it's real part is constant (or) (ii) it's imaginary part is constant (or) (iii) $|f(z)|$ is constant (iv) $\arg f(z)$ is constant.

18 (a) If p,q are fixed point's under a bilinear transformation, then prove that the transformation can be expressed in the form $\frac{w-p}{w-q} = k\frac{z-p}{z-q}$ (ii) If p is the coincident fixed point then prove that $\frac{1}{w-p} = k + \frac{1}{z-p}$

Or

(b) Discuss the mapping of $w = z^2$.

19. (a) State and prove Cauchy's theorem using Goursat lemma.

Or

(b) State and prove Cauchy's integral formula.

20. (a) State and prove Taylor's series.

Or

(b) State and prove Weierstrass theorem.

2

Reg No----------------------

[15MMU602]

KARPAGAM ACADEMY OF HIGHER EDUCATION
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
SIXTH SEMESTER
I INTERNAL TEST-Jan'18
COMPLEX ANALYSIS-II

Date : .01.2018 **Time: 2 Hours**
Class : III B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)

Answer all the questions

1. If a is one zero, then P(a) equal to ……………
   a)Singular b)analytic c)non zero d)zero
2. If a/an ………………… is bounded in the finite plane, then it is a constant.
   a) simple function b)analytic function
   c)complex function d)entire function
3. Every function analytic in the extended plane is a ……………….
   a) constant b)zero c)non zero d)analytic
4. Every polynomial in z of degree equal to or greater than 1, has at least…………… zero
   a)two b)three c)one d)four
5. Zeros of an analytic functions are…………….
   a)constant b) isolated c)non zero d)singular
6. If f(z) is analytic inside and on an scr curve C,then the maximum always occurs at a ...….point.
   a)interior b)boundary
   c)singular d)analytic
7. If $f(z) = (z – a)^m[a_0 + a_1(z-a)………]$,$a_0 \neq 0$, then z =a is a zero of order …..
   a)m b)1 c)2 d)0
8. The function $(z – i)^2 (z+1)^3 e^z$ has a zero i of order 2 and a zero -1 of order ………….
   a)1 b)2 c)3 d)0
9. Maclaurin's series expansion of the function coshz is valid in ………………………
   a)$|z| < 0$ b)$|z| < \infty$ c)$|z|= 0$ d)$|z| < 1$
10. In maximum modulus theorem, the maximum of modulus f(z) in the whole of C_i and C occurs only on C unless f(z) is a ……………..
    a)analytic b)function
    c)constant d)maximum
11. If a function f(z) is such that $f(z) = (z - a)^k g(z)$, where k is a positive integer and g(a) not equal to zero, then a is said to be a …………of f(z) of order k.
    a)Constant b)zero
    c)function d)analytic
12. The function $(z-i)^2$ have a zero i of order…..
    a)2 b)1 c)0 d)3
13. The functions of the form, $P_n(Z)=a_0+a_1z+a_2z^2+……+a_nz^n$, $a_n\neq 0$ is called a…………………
    a)polynomial of degree n b) polynomial of degree 5
    c)polynomial of degree 2n d)polynomial of degree n-1

14. The power series of the form $a_0 + a_1(z - a) + a_2(z - a)^2$ +…. is said to be a series about ………………….
a)zeros b)poles c)residues d)points

15. The multiplicity of a zero *a* is also known as the …………….. of the function at *a*.
a)order of analytic b)order of vanishing
c)order of pole d)order of singularity

16. A bounded entire function is ………………………..
a)analytic b)function
c)Constant d)zero

17. A zero of order 1 is also called a …………..
a)simple analytic b)simple pole
c)simple zero d)simple function

18. If D is a simply-connected region, f(z) is analytic and a is a point in D and C is the largest circle whose interior lies in D then the power series is… and its sum is f(z).
a)Convergent b)uniformly convergent
c)divergent d)absolutely convergent

19. In Gauss' mean value theorem, mean of the values of the function on C is the value of the function at its…………
a)radius b)centre c)diameter d)points

20. f(z) = sin z has a zero of order ……………………….. at z = 0.
a)0 b)1 c)2 d)∞

PART – B (3X10=30 Marks)

Answer all the questions

21. a) State and prove Cauchy's inequality.

(OR)

b) If u(x,y) is a function harmonic in a simply connected region D, then prove that the mean value of u(x,y) taken along a circle in D is always equal to its value at the centre.

22. a) State and prove fundamental theorem of algebra.

(OR)

b) State and prove Laurent's theorem.

23. a) State and prove uniqueness theorem.

(OR)

b) Find the Taylor's expansion about z = 0 of $f(z) = \frac{z}{(z+1)(z-3)}$.

Key answer

Semester examination – 2018

Complex Analysis – II

Class: III B.Sc (mathematics)

Sub code: 15MMU602

Handled staff: M. Sangeetha.

2.1a) Let the harmonic conjugate of $u(x,y)$ be $v(x,y)$.

Then $f(z) = u+iv$ is analytic in D and by the previous theorem then mean value of $f(z)$ on Γ: $|z-a|=r$ in D

is

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt.$$

If $a = \alpha + i\beta$ then

$f(a) = u(\alpha,\beta) + iv(\alpha,\beta)$ and

$f(a+re^{it}) = u(\alpha+r\cos t, \beta + r\sin t) + iv(\alpha + r\cos t, \beta + r\sin t)$

$$u(\alpha,\beta) + iv(\alpha,\beta) = \frac{1}{2\pi}\int_0^{2\pi} u(\alpha+r\cos t, \beta+r\sin t)\,dt + \frac{i}{2\pi}\int_0^{2\pi} v(\alpha+r\cos t, \beta+r\sin t)\,dt$$

Equating the real parts

$$u(\alpha,\beta) = \frac{1}{2\pi}\int_0^{2\pi} u(\alpha+r\cos t, \beta+r\sin t)\,dt$$

which is the mean value of the values of $u(x,y)$ on Γ.

2.1b) Poisson Integral

Statement

Suppose the function $u(x,y)$ is harmonic inside and on a circle $\Gamma: |z| = R$. If $re^{i\theta}$ is a point inside this circle then

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(R,\phi)(R^2-r^2)}{R^2+r^2-2Rr\cos(\theta-\phi)}\,d\phi$$

where $u(r,\theta)$ stands for $u(r\cos\theta, r\sin\theta)$

Proof

Suppose $f(z)$ is analytic inside and on Γ. and has $u(x,y)$ as its real part. Then by Cauchy Integral formula

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int \frac{f(z)}{z - re^{i\theta}}\,dz$$

$$0 = \frac{1}{2\pi i}\int \frac{f(z)}{z - (R^2/r)e^{i\theta}}\,dz$$

Subtracting the second Integral formula from the list

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_\Gamma f(z)\left[\frac{1}{z - re^{i\theta}} - \frac{r}{rz - R^2 e^{i\theta}}\right] dz$$

$$= \frac{1}{2\pi i}\int_\Gamma f(z)\,\frac{-R^2 e^{i\theta} + r^2 e^{i\theta}}{(z - re^{i\theta})(rz - R^2 e^{i\theta})}\,dz$$

$$= \frac{1}{2\pi i}\int_\Gamma f(z)\,\frac{-(R^2 - r^2)\,e^{i\theta}}{(z - re^{i\theta})\left(re^{-i\theta} - \frac{R^2}{z}\right)e^{i\theta}z}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\theta})\,\frac{-(R^2 - r^2)}{(Re^{i\theta} - re^{i\theta})(re^{-i\theta} - Re^{-i\theta})}\,i\,d\phi$$

Since on Γ, $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\phi = iz\,d\phi$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\,\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \phi)}\,d\phi$$

Equating the real parts we get the results as started.

Laurent Series

Suppose

1) D is a multiply-connected region.
2) $f(z)$ is analytic in D and .
3) $\Gamma_1 = |t-a| = R_1$ $\Gamma_2 : |t-a| = R_2$ are the smallest and largest circles. Such that, the annulus region D, b/w Γ_1 and Γ_2 lies in D.

Then for all z in D, the power series $\sum_{-\infty}^{+\infty} a_n(z-a)^n$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}\,dt$ and C is any sero curve in D is absolutely convergent and its sum is $f(z)$.

Proof

Let $\Gamma_1 : |t-a| = r_1$ and $\Gamma_2 : |t-a| = r_2$ be 2 positively oriented circles in D of which Γ_1' is smaller. Now by extension to Cauchy Integral formula for any z in D_2

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_2'} \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_{\Gamma_1'} \frac{f(t)}{t-z}\,dt$$

Case (i) Equation along Integral Γ_2'

$|z-a| < |t-a|$ (or) $\left|\frac{z-a}{t-a}\right| < 1$

Hence the geometric series

$$1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \ldots$$

is absolutely convergent and is sum is

$$\frac{1}{1-\frac{z-a}{t-a}} = \frac{t-a}{z-a}$$

multiplying both sides by $\frac{1}{2\pi i}\frac{f(t)}{t-a}$

$$\frac{1}{2\pi i}\frac{f(t)}{t-a} + \frac{1}{2\pi i}\frac{f(t)}{(t-a)^2}(z-a) + \frac{1}{2\pi i}\frac{f(t)}{(t-a)^3}(z-a)^3 + \ldots$$

$$\ldots = \frac{1}{2\pi i}\frac{f(t)}{t-z}$$

By term by term Integrating along Γ_2'

$$\sum_{n=0}^{\infty} (z-a)^n \frac{1}{2\pi i} \int_{\Gamma_2'} \frac{f(t)}{(t-a)^{n+1}} dt = \frac{1}{2\pi i} \int_{\Gamma_2'} \frac{f(t)}{t-z} dt$$

<u>case (ii)</u> Equation along Integral Γ_1'.

$$|t-a| < |z-a| \text{ (or) } \left|\frac{t-a}{z-a}\right| < 1$$

Hence the geometric series

$$1 + \left(\frac{t-a}{z-a}\right) + \left(\frac{t-a}{z-a}\right)^2 + \cdots$$

absolutely convergent and its sum function n is

$$\frac{1}{1-\frac{t-a}{z-a}} = \frac{z-a}{z-t} = -\frac{z-a}{t-z}$$

$$1 + \frac{t-a}{z-a} + \left(\frac{t-a}{z-a}\right)^2 + \cdots = -\frac{z-a}{t-z}$$

multiplying bothsides by $\frac{1}{2\pi i} \frac{f(t)}{z-a}$

$$\sum_{n=-1}^{-\infty} \frac{1}{2\pi i} \frac{f(t)}{(t-a)^{n+1}} (z-a)^n = \frac{-1}{2\pi i} \frac{f(t)}{t-z}$$

By term by term Integration along Γ_1'

$$\sum_{n=-1}^{-\infty} (z-a)^n \frac{1}{2\pi i} \int_{\Gamma_1'} \frac{f(t)}{(t-a)^{n+1}} dt = \frac{-1}{2\pi i} \int_{\Gamma_1'} \frac{f(t)}{t-z} dt$$

The series on the left is absolutely convergent

$$f(z) = \sum_{n=0}^{\infty} (z-a)^n \frac{1}{2\pi i} \int_{\Gamma_2'} \frac{f(t)}{(t-a)^{n+1}} dt + \sum_{n=-1}^{-\infty} (z-a)^n$$

$$\frac{1}{2\pi i} \int_{\Gamma_1'} \frac{f(t)}{(t-a)^{n+1}} dt$$

Now c is a scro curve. in b/w Γ_1' and Γ_2'

By extension. to Cauchy fundamental

$$\int_{\Gamma_2'} = \int_C \quad \text{and} \quad \int_C = \int_{\Gamma_1'}$$

Hence become

$$f(z) = \sum_{n=-\infty}^{+\infty} (z-a)^n \cdot \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt$$

$$= \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt$

22) b) cos z is an entire function that is a function analytic everywhere at $z = \infty$

we shall denote cos z by f(z) then by successive differentiation, we have

$$f^n(z) = \cos\left(z + \frac{n\pi}{2}\right)$$

$$f^n(0) = \cos\left(0 + \frac{n\pi}{2}\right) = \cos\frac{n\pi}{2}$$

$$f^n\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$$

$$f^n\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \frac{n\pi}{2}\right)$$

(i) Here the expansion is about z=0 so the expansion is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} (z-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{\cos\left(0 + \frac{n\pi}{2}\right)}{n!} (z-0)^n$$

$$= \sum_{n=0}^{\infty} \frac{\cos\frac{n\pi}{2}}{n!} z^n$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

(ii) Here the expansion is about $z = \frac{\pi}{4}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{\pi}{4}\right)}{n!} \left(z - \frac{\pi}{4}\right)^n$$

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)}{n!}\left(z-\frac{\pi}{4}\right)^n$$

(iii) Here the expansion is about $z = \frac{\pi}{3}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{\pi}{3}\right)}{n!}\left(z-\frac{\pi}{3}\right)^n$$

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi}{3}+\frac{n\pi}{2}\right)}{n!}\left(z-\frac{\pi}{3}\right)^n$$

23) a
(i) $$\{\operatorname{Res} f(z)\}_{z=2} = \underset{z\to 2}{Lt}\ (z-2)\frac{24}{(z-1)^4(z-2)(z-3)}$$

$$= \underset{z\to 2}{Lt}\ \frac{24}{(z-1)^4(z-3)}$$

$$= -16$$

(ii) $$\{\operatorname{Res} f(z)\}_{z=3} = \underset{z\to 3}{Lt}\ (z-3)\frac{24}{(z-1)^4(z-2)(z-3)}$$

$$= \underset{z\to 3}{Lt}\ \frac{24}{(z-1)^4(z-2)}$$

$$= \frac{81}{16}$$

(iii) $$\{\operatorname{Res} f(z)\}_{z=a} = \underset{z\to a}{Lt}\ \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

$$= \underset{z\to 1}{Lt}\ \frac{d^3}{dz^3}\frac{24}{(z-2)(z-3)}$$

$$= \underset{z\to 1}{Lt}\ \frac{d^3}{dz^3}\frac{24}{z^2-5z+6}$$

$$
\begin{array}{r|l}
z^2-5z+6 & \overline{z^2+5z+19} \\
 & z^4 \\
 & z^4-5z^3+6z \\
 & \overline{\quad 5z^3-6z^2} \\
 & \quad 5z^3-25z^2+30z \\
 & \overline{\quad\quad 19z^2-30z} \\
 & \quad\quad 19z^2-95z+114 \\
 & \overline{\quad\quad\quad 65z-114}
\end{array}
$$

$$\frac{z^4}{(z-1)^4(z-2)(z-3)} = z^2+5z+19+\frac{65z-114}{(z-2)(z-3)}$$

$$\frac{65z-114}{(z-2)(z-3)} = \frac{A}{z-2}+\frac{B}{z-3}$$

Solved by partial fraction

$A = -16$
$B = 81$

$$\frac{z^4}{(z-1)^4(z-2)(z-3)} = z^2+5z-19 \; \frac{-16}{z-2}+\frac{81}{z-3}$$

$$\{\text{Res } f(z)\}_{z=1} = \lim_{z\to 1} \frac{1}{3!}\frac{d^3}{dz^3}\left[z^2+5z+19-16(z-2)^{-1}+81(z-3)^{-1}\right]$$

$$= \frac{175}{16}$$

23) b) **Weierstrass theorem**

If $f(z)$ has an isolated essential singularity at $z=a$ and if C is any complex constant then for any positive ϵ, however small, there exist an z in every deleted neighbourhood of $z=a$ such that

$$|f(z)-C| < \epsilon.$$

Proof

Let $D: 0 < |z-a| < \rho$ be an arbitrarily chosen small deleted neighbourhood of $z=a$.

Given $\varepsilon > 0$ suppose it is not possible to find an z in D such that $|f(z) - c| < \varepsilon$.

Then for all z in D

$$|f(z) - c| \geq \varepsilon \text{ (or)}$$

$$\left|\frac{1}{f(z)-c}\right| \leq \frac{1}{\varepsilon}$$

Here $\frac{1}{\{f(z)-c\}}$ is bdd and analytic in D.

And it is not analytic at $z=a$ because $f(z)$ is not analytic there, so, by thm

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

$z=a$ is a removable singularity of $\frac{1}{\{f(z)-c\}}$

Its Laurent expansion about $z=a$ is of the form

either

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \quad a_0 \neq 0$$

$\frac{1}{\{f(z)-c\}}$ has removable singularity at $z=a$ &

either $z=a$ is not a zero of $\frac{1}{\{f(z)-c\}}$

Consequently

either $f(z)-c$ has a removable singularity at $z=a$

(or) $f(z)-c$ has a pole $z=a$ of order m.

Either $f(z)$ has pole $z=a$ of order m (or)

$f(z)$ has a removable singularity at $z=a$.

$\Rightarrow \Leftarrow$ the hypothesis that $z=a$ is an essential singularity of $f(z)$.

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$$

Put $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{(a+b\cos\theta)^2}$$

$$= \int_C \frac{\frac{dz}{iz}}{\left(a+b\left(\frac{z^2+1}{2z}\right)\right)^2} = \frac{4}{i}\int_C \frac{z\,dz}{(bz^2+2az+b)^2}$$

$$= \frac{4}{i}\int_C f(z)\,dz \quad \text{— (1)}$$

where $f(z) = \frac{z}{bz^2+2az+b}$

$$z = \frac{-a+\sqrt{a^2-b^2}}{b} \quad ; \quad z = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$z = \alpha \quad ; \quad z = \beta$$

$$f(z) = \frac{z}{b^2(z-\alpha)^2(z-\beta)^2}$$

$z=\alpha$ is a pole of order 2 lies Inside $|z|=1$

$z=\beta$ is a pole of order 2 lies outside $|z|=1$.

$$\{\text{Res } f(z)\}_{z=\alpha} = \lim_{z\to\alpha} \frac{1}{1!}\frac{d}{dz}(z-\alpha)^2 \frac{z}{b^2(z-\alpha)^2(z-\beta)^2}$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left(\frac{z}{b^2(z-\beta)^2}\right)$$

$$= \frac{1}{b^2}\frac{-\alpha-\beta}{(\alpha-\beta)^3}$$

$$= \frac{1}{b^2} \times \frac{b^3}{8(a^2-b^2)^{3/2}} \left[\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b} + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} \right]$$

$$= \frac{b}{8(a^2-b^2)^{3/2}} \left(\frac{2a}{b} \right)$$

$$= \frac{a}{4(a^2-b^2)^{3/2}}$$

$\int_C f(z)dz = 2\pi i$ (sum of the residue)

$$= 2\pi i \times \frac{a}{4(a^2-b^2)^{3/2}}$$

$$= \frac{a\pi i}{2(a^2-b^2)^{3/2}}$$

To find the value of Given Integral

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{4}{i} \times \frac{a\pi i}{2(a^2-b^2)^{3/2}}$$

$$= \frac{2a\pi}{(a^2-b^2)^{3/2}}$$

24) b $\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)} dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}, m>0, a>0.$

Sol consider $\int_C \frac{e^{imz}}{(z^2+a^2)^2} dz$

let $\phi(z) = \frac{e^{imz}}{(z^2+a^2)^2}$

$(z^2+a^2)^2 = 0$

$[(z+ai)(z-ai)]^2 = 0$

$z = ai, -ai$

$z = ai$ lies In the upper half Plane at Pole of order 2.

$z=-ai$ lies in the lower half plane of pole of order 2.

$$\{Res\,\phi(z)\}_{z=ai} = \lim_{z\to ai} \frac{1}{(2-1)!}\frac{d}{dz}\left[(z-ai)^2 \frac{e^{imz}}{(z+ai)^2(z-ai)^2}\right]$$

$$= e^{im(ai)}\left[\frac{im(ai+ai)-2}{(ai+ai)^3}\right]$$

$$= e^{-am}\left(\frac{Im(2ai)-2}{(2ai)^3}\right)$$

$$= e^{-ma}\left[\frac{-2am-2}{-8a^3 i}\right]$$

$$= \frac{-2e^{-ma}[am+1]}{-8a^3 i}$$

$$= \frac{-ie^{-ma}[1+ma]}{4a^3}$$

$$\int_{-R}^{+R} \phi(x)\,dx + \int_{\Gamma} \phi(z)\,dz = \int_C \phi(z)\,dz \quad - (1)$$

$$\int_{-R}^{+R} \frac{e^{imx}}{(x^2+a^2)^2}\,dx + \int_{\Gamma} \frac{e^{imz}}{(z^2+a^2)^2}\,dz = 2\pi i\,(\text{Sum of the residue})$$

$$= \frac{2\pi i \times -ie^{-ma}(1+ma)}{4a^3}$$

$$= \frac{\pi e^{-ma}(1+ma)}{2a^3} \quad - (2)$$

Now $\left|\frac{1}{(z^2+a^2)^2}\right| \le \frac{1}{(R^2-a^2)^2}$

Since RHS $\to 0$ as $R\to\infty$

LHS $\to 0$ as $R\to\infty$ on $|z|=R$.

(12)

$$\int_{\Gamma} \frac{e^{imz}}{(z^2+a^2)^2}\,dz \to 0 \text{ as } R \to \infty \quad -(3)$$

Letting $R \to \infty$ in (2) and using (3)

$$\int_{-\infty}^{+\infty} \frac{e^{imx}}{(a^2+x^2)^2}\,dx = \frac{\pi(1+ma)\,e^{-ma}}{2a^3}$$

$$2\int_{0}^{\infty} \frac{e^{imx}}{(a^2+x^2)^2}\,dx = \frac{\pi(1+ma)\,e^{-ma}}{4a^3}$$

$$\int_{0}^{\infty} \frac{\cos mx}{(a^2+x^2)}\,dx = \frac{\pi(1+ma)\,e^{-ma}}{4a^3}.$$

Hurwitz theorem

...ment

Let $\{f_K\}$ be a sequence of holomorphic functions on a connected open set G that coverage uniformly on compact subsets of G to a holomorphic function f which is not constantly zero on G. If f has a zero of order m at z_0 then for every small enough $\rho > 0$ and for sufficiently large $K \in N$ (depending on ρ), f_K has precisely m zeros in the disk defined by $|z - z_0| < \rho$ Including multiplicity.

Proof:-

Let f be an analytic function on open subset of the complex plane with a zero of order m at z_0. $\{f_n\}$ is a sequence of functions converging uniformly on compact subset.

fixe some $\rho > 0$ s.t $f(z) \neq 0$ in $0 < |z - z_0| < \rho$.

on the circle $|z - z_0| = \rho$. Choose δ such that $|f(z)| > \delta$ for z on the disc we have chosen since $f_K(z)$ converges uniformly on the disc we have chosen N such that $|f_K(z_0)| \geq \frac{\delta}{2}$ for every $K \geq N$ and every z on the circle, ensuring that the quotient $f_K'(z)/f_K(z)$ is well defined for all z on the circle $|z - z_0| = \rho$.

By morera's theorem

$$\frac{f_K'(z)}{f_K(z)} \rightarrow \frac{f'(z)}{f(z)}$$

Denoting the number of zeros of $f_K(z)$ in the disk by N_K, we may apply the argument principle to find

$$m = \frac{1}{2\pi i} \int_{|z-z_0| = \rho} \frac{f'(z)}{f(z)} dz = \lim_{K \to \infty} \frac{1}{2\pi i} \int_{|z-z_0| = \rho} \frac{f_K'(z)}{f_K(z)} dz = \lim_{K \to \infty} N_K$$

we have shown that $N_K \to m$ as $K \to \infty$. Since the N_K are integer valued, N_K must equal to m for lagre enough K.

25) b State and Prove Rouche's theorem.

Statement

Suppose

1) D is a simply connected region
2) f(z) is meromorphic in D
3) C is any scro curve not passing through the poles (or) zeros of f(z).
4) f(z) is defined as the sum of two functions g(z) and h(z).

(ie) $g(z)+h(z) = f(z)$.

where g(z) is meromorphic function in D.

and also

$|g(z)| < |h(z)|$

Then in ci,

$$n(z,f) - n(p,f) = n(z,g) - n(p,g).$$

Proof:-

from the Principles of argument, we have

$$\Delta_C \arg f(z) = 2\pi [n(z,f)] - n(p,f)]$$

$$2\pi [n(z,f) - n(p,f)] = \Delta_C \arg f(z)$$

$$= \Delta_C \arg [g(z) + h(z)]$$

$$= \Delta_C \arg \left[g(z)\left[1+\frac{h(z)}{g(z)}\right]\right].$$

$= \Delta_C \arg g(z) + \Delta_C \arg \left(1 + \frac{h(z)}{g(z)}\right]$

$= 2\pi \left[n(z,g) - n(P,g)\right] + \Delta_C \arg \left[1 + \frac{h(z)}{g(z)}\right]$

But $|g(z)| < |h(z)|$. Therefore $\left|\frac{h(z)}{g(z)}\right| < 1$

Then $1 + \frac{h(z)}{g(z)}$ lies in the interior of the circle $|z-1| < 1$. So, for any z in C,

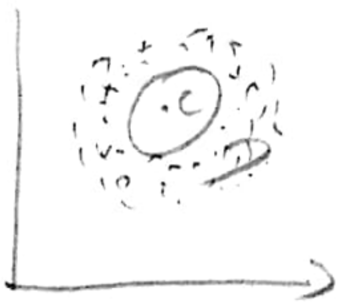
$-\frac{\pi}{2} < \alpha_2 \leq \arg\left[1 + \frac{h(z)}{g(z)}\right] \leq \alpha_2 < \frac{\pi}{2}$

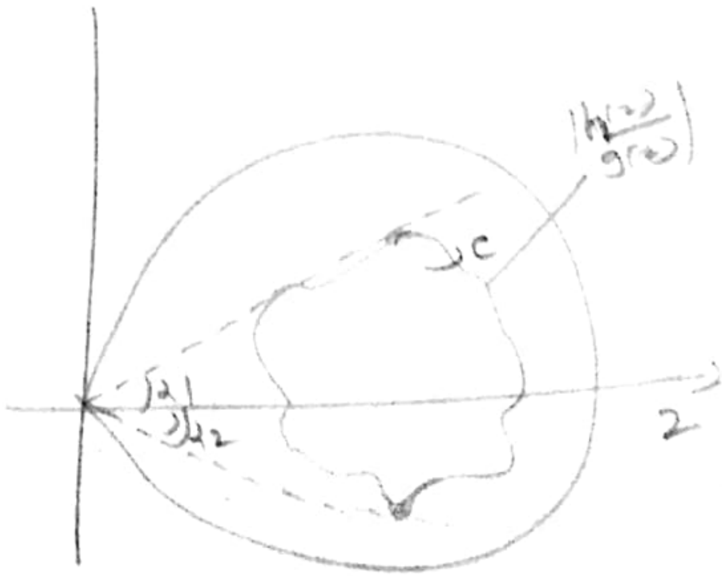
$\therefore \left[1 + \frac{h(z)}{g(z)}\right] = 0$

$\therefore 2\pi\left[n(z,f) - n(P,f)\right] = 2\pi\left[n(z,g) - n(P,g)\right] + 0$

$n(z,f) - n(P,f) = n(z,g) - n(P,g)$

Hence the proof.





Reg. No.................................

[15MMU602]

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari Post, Coimbatore – 641 021.
(For the candidates admitted from 2015 onwards)

B.Sc., DEGREE EXAMINATION, APRIL 2018

Sixth Semester

MATHEMATICS

COMPLEX ANALYSIS - II

Time: 3 hours — Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 x 8 = 40 Marks) (2 ½ Hours)
Answer ALL the Questions

21.a. If u(x,y) is a function harmonic in a simply connected region D, then prove that the mean value of u(x,y) taken along a circle in D is always equal to its value at the centre.

Or

b. State and prove poisson's integral.

22.a. State and prove Laurent's theorem.

Or

b. Expand cos z as Taylor's about the points given below.

i) $z=0$ ii) $z=\frac{\pi}{4}$ iii) $z=\frac{\pi}{3}$

23.a. Find the residues of $f(z)=\frac{z^4}{(z-2)(z-3)(z-1)^4}$ at its singularities.

Or

b. State and prove Weierstrass theorem.

24.a. Evaluate $\int_0^{2\pi}\frac{1}{(a+b\cos\theta)^2}d\theta$, $a>|b|>0$.

Or

b. Using contour integration show that

$$\int_0^{\infty}\frac{\cos mx}{(x^2+a^2)^2}dx=\frac{\pi}{4a^3}(1+ma)e^{-ma},m>0,a>0.$$

25.a. State and prove Hurwitz's theorem.

Or

b. State and prove Rouche's theorem.

Reg No ....

(15MMU602)

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE - 21

DEPARTMENT OF MATHEMATICS

SIXTH SEMESTER

II INTERNAL TEST

COMPLEX ANALYSIS -II

Date :27.02.2018 (FN) **Time: 2 Hours**

Class: III B.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 x 1 = 20 Marks)

Answer all the questions

1)The sum f(z) of a powerseries is analytic in ....

a) $|z|>R$ b) $|z|<R$ c) $|z|\le R$ d) $|z|=R$

2) suppose f(z) is analytic in a region D and z_n , n= 1,2,3,...., in D are the zeros of f(z), where the sequence $\{z_n\}$ converges to a limit z = a in D then f(z).... in D.

a) constant b) vanishes identically

c) analytic d) bounded

3) A .... of a function is a point at which the function ceases to be analytic.

a) singular point b) analytic point

c)pole point d)essential point

4) A singular point z_0 of f is said to be......................... if there is a neighborhood of z_0 which contains no singular points of f (z_0).

a) isolated b) Constant

c) bounded d) analytic

5) A singularity that is neither a pole or removable is called an ....singularity.

a) pole b)analytic c)essential d)singular

6) If z = a is a pole of f(z), then f(z) is .... in every deleted neighbourhood of z=a

a) zero b)not bounded c)constant d)bounded

7) The inverse function of the exponential function is the ......

a) trigonometric functions b) hyperbolic functions

c) harmonic functions d) logarithmic functions

8) The expansion with positive and negative powers of z – a is called ....series about z = a.

a) Laurent's series b) Taylor's series

c) Convergent series d) Power series

9) Classification of isolated singularities is done with reference to the .... of the Laurent's expansion of the function about the singular point.

a) positive power b) zero

c) constant d) negative power

10) Isolated singularities are classified into .....different groups

a) two b) three c) many d) five

11) The function f(z) = |z| is differentiable -----

a) on real part b) on imaginary part

c) at the origin d) at the point 2

12) If C is the largest circle with z=a as its center such that f(z) is analytic in C_i but is not .... somewhere on C.

a) analytic b) singular c) zero d) constant

13. The point z=a is a singular point of f(z) if f(z) is not defined at ....

a) y=a b) x=a c) u=a d) z=a

14. If z = a is an isolated singular point of a function f(z), then the singularity is called..... according as the Laurent's series about z=a of f(z), valid in a deleted neighbourhood of z=a has a finite number of negative powers

a) a removable singularity b) an essential singularity
c) a pole d) isolated singularity

15. When the order of a pole is 1, the pole is said to be a .... Pole.

a) zero b) double c) simple d) finite

16. The entire function $f(z) = e^z$ is not defined at $z=\infty$ and $z=\infty$ is the only .... Point

a) singular b) analytic c) pole d) essential

17. The point .... is a singular point of all the trigonometric function and hyperbolic functions because they are function of e^z.

a) z=0 b) z=1 c) $z=\infty$ d) z=a

18. The logarithmic function is a .... valued function

a) single b) multiple c) two d) zero

19. $\cos(z_1 + z_2)$ = ...

a) $\cos z_1 \cos z_2 - \sin z_1 \sin z_2$ b) $\cos z_1 \sin z_2 - \sin z_1 \cos z_2$
c) $\cos z_1 \cos z_2 + \sin z_1 \sin z_2$ d) $\sin z_1 \cos z_2 - \cos z_1 \sin z_2$

20. The residue of f(z) = cosz/z at its pole is ....

a) 2 b) 3 c) 0 d) 1

PART-B (3x 10=30Marks)

Answer all the questions

21.(a) If z = a is a pole of order m of a function f(z), then prove that z = a is a zero of order m of the function 1/f(z).

(OR)

(b) Find the Laurent's expansion for $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in

i) $2 < |z| < 3$ ii) $|z| > 3$.

22.(a) State and prove residue theorem.

(OR)

(b) Find the residues of $f(z) = \frac{z^4}{(z-2)(z-3)(z-1)^4}$ at its singularities.

23.(a) State and prove Weierstrass theorem.

(OR)

(b) (i) If z = a is a pole of a function f(z), then prove that $\lim_{z \to a} f(z) = \infty$,

ii) Find the orders of poles of $f(z) = \frac{1}{z(e^z-1)}$.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: M.Sangeetha
SUBJECT NAME: Complex Analysis-II SUB.CODE:15MMU602
SEMESTER: VI CLASS: III B.Sc (MATHS)

Lecture plan

UNIT-I

S.No	Lecture Hour	Topics to be Covered	Support Materials
1	1	Zero's of a function	T1:Ch:8:P.No:157-158
2	1	Cauchy's inequality	T1:Ch:8:P.No:158-159
3	1	Liouville's theorem	T1:Ch:8:P.No:159-161
4	1	Fundamental theorem of Algebra	R4:Ch:6:P.No:366-367
5	1	Continuation of fundamental theorem of Algebra	R2:Ch:8:P.No:220-221
6	1	Maximum modulus theorem	T1:Ch:8:P.No:161-162
7	1	Gauss mean value theorem	T1:Ch:8:P.No: 161-162
8	1	Mean value of the values of a harmonic function on a circle	T1:Ch:8:P.No: 162-163
9	1	Poisson's integral	T1:Ch:8:P.No: 163-164
10	1	Term by term differentiation and integration of uniformly convergent series	T1:Ch:8:P.No: 164-166
11	1	Problem on related integral theorem	T1:Ch:8:P.No:177-178

		integral theorem	
12	1	Continuation of problem on related Integral theorem	T1:Ch:8:P.No:178-179
13	1	Recapitulation and discussion of possible questions	
		Total Hours	**13**

UNIT-II

S.No	Lecture Hour	Topics to be Covered	Support Materials
1	1	Taylor's series:Introduction	T1:Ch:9:P.No:179-181
2	1	Theorem on Taylor's series	R2:Ch:7:P.No:173-175
3	1	Uniqueness Theorem	T1:Ch:9:P.No:181-182
4	1	Continuation of Uniqueness Theorem	T1:Ch:9:P.No:183-184
5	1	Zero's of an analytic function	R2:Ch:7:P.No:197-198
6	1	Continuation of Zero's of an analytic function	R2:Ch:9:P.No:199
7	1	Problems on Taylor's series	T1:Ch:9:P.No:199-202
8	1	Continuation on Problems on Taylor's series	R2:Ch:9:P.No:176-180
9	1	Laurent's series:Introduction	R1:Ch:5:P.No:184-186 R2:Ch:7:P.No:181-182
10	1	Theorem on Laurent's series	T1:Ch:9:P.No:184-186 R2:Ch:7:P.No:182-184
11	1	Problems on Laurent's series	T1:Ch:9:P.No:187-188
12	1	Continuation of problems on Laurent's series series	T1:Ch:9:P.No:202-205
13	1	Continuation of problems on Laurent's series series	R2:Ch:7:P.No:185-189

14	1	Continuation of problems on Laurent's series series	R2:Ch:7:P.No:190-194
15	1	Continuation of problems on Laurent's series series	R2:Ch:7:P.No:195-197
16	1	Cauchy product and division	T1:Ch:9:P.No:188-189
17	1	Problems on Cauchy product and division	T1:Ch:9:P.No:211-212
18	1	Recapitulation and discussion of possible questions.	
		Total Hours	**18**

UNIT-III

S.No	Lecture Hour	Topics to be Covered	Support Materials
1	1	Singularity:Introduction	R2:Ch:7:P.No:200-201
2	1	Isolated singularities:Definition and examples	T1:Ch:9:P.No:190-191 R3:Ch:7:P.No:105-107
3	1	Removable singularity:Definition and examples	T1:Ch:9:P.No:191-192
4	1	Theorems on Removable Singularities	R1:Ch:4:P.No:124-126
5	1	Pole:Definition and examples	T1:Ch:9:P.No:192
6	1	Theorems on pole ans its problems	R1:Ch:4:P.No:126-128
7	1	Essential singularity: Definition and examples	T1:Ch:9:P.No:192-193
8	1	Behaviour of a function at an isolated singularity	T1:Ch:9:P.No:193-195
9	1	Theorem on Behaviour of a function at an isolated singularity	T1:Ch:9:P.No:196-197
10	1	Determination of the nature of singularities	T1:Ch:9:P.No:197-199
11	1	Residues:Definition and examples	R3:Ch:8:P.No:112-114
12	1	Continuation of problems on Residues	R2:Ch:8:P.No:209-210
13	1	Continuation of problems on Residues	R2:Ch:8:P.No:211-217

14	1	Residue theorem	T1:Ch:10:P.No:217-220
15	1	Problem on Residue theorem	T1:Ch:10:P.No:236-237
16		Continuation of problem on Residues	R1:Ch:4:P.No:152-154
17	1	Recapitulation and discussion of possible questions.	
		Total Hours	17

UNIT-IV

S.No	Lecture Hour	Topics to be Covered	Support Materials
1	1	Real definite integrals:Introduction and types of integrals	T1:Ch:10:P.No:220-221
2	1	Type 1:Integral with 0 and 2π as lower and upper limit of f(cos θ,sinθ)	T1:Ch:10:P.No:221-222
3	1	Problems on Integral with 0 and 2π as lower and upper limit of f(cos θ,sinθ)	T1:Ch:10:P.No:222-223
4	1	Type 2:Integral with -∞ and ∞ as lower and upper limit of P(x)/Q(x)	T1:Ch:10:P.No:223-224
5	1	Problems on Integral with -∞ and ∞ as lower and upper limit of P(x)/Q(x)	T1:Ch:10:P.No:224-225
6	1	Type 3:Integral of the form with -∞ and ∞ as lower and upper limit of sin ax f(x) dx and cos ax f(x) dx	T1:Ch:10:P.No:225-226
7	1	Problems on Integral of the form with -∞ and ∞ as lower and upper limit of sin ax f(x) dx and cos ax f(x) dx	T1:Ch:10:P.No:226-227
8	1	Type 4:Integrals of the form with -∞ and ∞ as lower and upper limit of f(x) dx	T1:Ch:10:P.No:227-228
9	1	Problems on Integrals of the form with -∞ and ∞ as lower and upper limit of f(x) dx	T1:Ch:10:P.No:228-229
10	1	Integrals of the type x ∫ x^(a-1)/(1+x) dx	T1:Ch:10:P.No:230-231
11	1	Problems on Integrals of the type x ∫ x^(a-1)/(1+x) dx	T1:Ch:10:P.No:231-232
12	1	Recapitulation and discussion of possible questions.	
		Total Hours	**12**

UNIT-V

S. No	Lecture Hour	Topics to be Covered	Support Materials
1	1	Meromorphic functions:Introduction	T1:Ch:11:P.No:249
2	1	Continuation of problems on meromorphic functions	R3:Ch:7:P.No:108-109
3	1	Theorem on number of zeros minus number of poles	T1:Ch:11:P.No:249-250
4	1	Continuation of number on zeros minus number of poles	R3:Ch:7:P.No:110-112
5	1	Principle of argument	T1:Ch:11:P.No:250-251
6	1	Rouche's theorem	T1:Ch:11:P.No:251-252
7	1	Continuation of Rouche's theorem	T1:Ch:11:P.No:252-253
8	1	Fundamental theorem of algebra	T1:Ch:11:P.No:253-254
9	1	Hurwitz's theorem	T1:Ch:11:P.No:254-255
10	1	Theorem on function which is meromorphic in the extended plane	T1:Ch:11:P.No:255-256
11	1	Continuation of theorem on function which is meromorphic in the extended plane	T1:Ch:11:P.No:256-257
12	1	Recapitulation and discussion of possible questions	
13	1	Discussion of previous ESE question papers.	
14	1	Discussion of previous ESE question papers.	
15	1	Discussion of previous ESE question papers.	
		Total Hours	15

TEXT BOOK

T1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex Analysis, Emerald publishers, Chennai-2

REFERENCES

R1. Lars V.Ahlfors.,1979. Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi.

R2. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.

R3. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd , New Delhi.

R4. Vasishtha A.R ., 2005. Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.

Reg. No ------------
(15MMU602)

KARPAGAM ACADEMY OF HIGHER EDUCATION
COIMBATORE-21
Department of Mathematics
SIXTH SEMESTER
Model Examination-Mar'18
COMPLEX ANALYSIS-II

Date: .03.18( ) **Time: 3 hours**
Class: III B.Sc Mathematics **Maximum Marks: 60**

PART - A (20X1 = 20 Marks)
Answer all the Questions

1) In maximum modulus theorem, the maximum of modulus f(z) in the whole of C_i and C occurs only on C unless f(z) is a ……………..
   a) analytic b) function c) constant d) maximum
2) The derivative of an analytic function is also …
   a) analytic b) continuous c) derivative d) bounded
3) Zeros of an analytic functions are…………….
   a) constant b) isolated c) singular d) non zero
4) A function which is analytic everywhere in the finite plane is called an ----------------- function.
   a) single b) multi c) continuous d) entire
5) The binomial series is valid when $|z|$ ……………
   a) < 1 b) equal to 1 c) > 1 d) > 0
6) The functions of the form, $P_n(Z)= a_0+a_1z+a_2z^2+\ldots\ldots+a_nz^n$, $a_n\neq0$ is called a………..
   a) polynomial of degree n b) polynomial of degree 5
   c) polynomial of degree 2n d) polynomial of degree n-1
7) The inverse function of the exponential function is the ……
   a) trigonometric functions b) hyperbolic functions
   c) harmonic functions d) logarithmic functions
8) The expansion with positive and negative powers of z – a is called ………series about z = a.
   a) laurent's series b) taylor's series
   c) convergent series d) Power series
9) Classification of isolated singularities is done with reference to the ……………………. of the Laurent's expansion of the function about the singular point.
   a) positive power b) zero
   c) constant d) negative power
10) Isolated singularities are classified into ………..different groups.
   a) two b) three c) many d) five
11) The function f(z) = |z| is differentiable -----
   a) on real part b) on imaginary part
   c) at the origin d) at the point 2
12) If C is the largest circle with z=a as its center such that f(z) is analytic in C_i but is not ……… somewhere on C.
   a) analytic b) singular c) zero d) constant
13) If a function h(z) is analytic at z=a and h(a) not equal to zero then z=a is a …………… of the function f(z)=h(z)/(z-a).
   a) simple zero b) double pole
   c) simple pole d) finite pole

14) Principle value of log z is obtained when n = ...

a) 0 b) -1 c) 1 d) 2

15) Cot z and cosec z are analytic in a bounded region in which ....

a) cot z $\neq$ 0 b) cosecz $\neq$ 0
c) sinz $\neq$ 0 d) cosz $\neq$0

16) The Value of cosiz is = ...

a) cosz b) icosz c) icoshz d) coshz

17) Find the number of zeros of the function f(z) = z^6+z^3 -6z +9 which lie inside the unit circle C:|z| = 1 ....

a) 1 b) 2 c) 0 d) 4

18) $z^4+z^3+4z^2+2z+3$ has two zeros each in the ...... quadrants.

a) first and fourth b) first and second
c) second and third d) third and fourth

19) A function which is meromorphic in the extended plane is a --

a) real function b) irrational function
c) rational function d) complex function

20) The zeros and poles being counted as many times as their ....

a) zeros b) orders c) poles d) ones

PART -B(5X8= 40 Marks)

ANSWER ALL THE QUESTIONS:

21. a) If u(x,y) is a function harmonic in a simply connected region D, then prove that the mean value of u(x,y) taken along a circle in D is always equal to its value at the centre.

(OR)

b) State and prove poisson's integral.

22 .a) State and prove uniqueness theorem.

(OR)

b) Expand cos z as Taylor's about the points given below.

i) z=0 ii) z=$\frac{\pi}{4}$ iii) z= $\frac{\pi}{3}$

23. a) Find the residues of $f(z) = \frac{z^4}{(z-2)(z-3)(z-1)^4}$ at its singularities.

(OR)

b) State and prove Weierstrass theorem.

24. a) Evaluate $\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2} d\theta$, $a > |b| > 0$.

(OR)

b) Using contour integration evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta$

25. a) State and prove Hurwitz's theorem.

(OR)

b) State and prove Rouche's theorem.

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Coimbatore – 641 021.

Semester IV

		L	T	P	C
15MMU602	**COMPLEX ANALYSIS-II**	**5**	**0**	**0**	**5**

Scope: This course will enhance the learner to understand the extended concepts of analytic function of complex variables and the application of residues etc which plays a crucial role in the field of applied mathematics.

Objectives: To enable the students to learn complex number system, complex function and complex integration, Singularitie, real definite integrals, Meromorphic functions etc.

UNIT I

Zero's of a function-Cauchy‟s inequality-Liouville‟s theorem-Fundamental theorem of Algebra-
Maximum modulus theorem- Gauss mean value theorem- Mean value of the value of a harmonic function on a circle- Term by term differentiation and integration of uniformly convergent series.

UNIT II

Taylor‟s series and Laurent‟s series : Taylor‟s series-Theorems and some related problems- Zero‟s of an analytic function- Laurent‟s series – Theorems and some related problems- Cauchy product and division.

UNIT III

Singularities – Isolated Singularities- Removable Singularity- Pole-Essential Singularity-Behaviour of a function at an isolated Singularity-Determination of the nature of Singularity-Problems-Residues- Residues theorem(statement only)-Problems.

UNIT IV

Real definite integrals: Evaluation using the calculus of residues – Integration on the unit circle –Integral with - ∞ and + ∞ as lower and upper limits with the following integrals:
i) P(x) /Q(x) where the degree of Q(x) exceeds that of P(x) at least 2.

ii) (sin ax).f(x), (cos ax).f(x), where a>0 and f(z) →0 as z→∞ and f(z) does not have a pole
on the real axis.
iii) f(x) where f(z) has a finite number of poles on the real axis.

UNIT V
Meromorphic functions: Theorem on number of zeros minus number of poles – Principle of argument: Rouche‟s theorem – Theorem that a function which is meromorphic in the extended plane is a rational function.

TEXT BOOK

1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex Analysis, Emerald publishers, Chennai-2

REFERENCES
1. Lars V.Ahlfors.,1979. Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi.

2. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.

3. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd , New Delhi.

4. Vasishtha A.R ., 2005. Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.

5. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan (printers & publishers) pvt Ltd, Madras.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: COMPLEX ANALYSIS-II **Subject Code: 15MMU602**

Class : III B.Sc Mathematics **Semester : VI**

UNIT -I

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If a is one zero, then P (a) equal to ……………	Singular	analytic	non zero	zero	zero
If a/an ………………… is bounded in the finite plane, then it is a constant	simple function	analytic function	complex function	entire function	entire function
Every function analytic in the extended plane is a ………………	Constant	zero	non zero	analytic	Constant
Every polynomial in z of degree equal to or greater than 1, has at least…………… zero	two	three	one	four	one
Zeros of an analytic functions are……………	constant	isolated	non zero	singular	isolated
If f(z) is analytic inside and on an scr curve C, then the maximum always occurs at a …………………point	interior	boundary	singular	analytic	boundary
If $f(z) = (z – a)^m[a_0 + a_1(z-a)………]$,$a_0 \neq 0$, then z =a is a zero of order …..	m	1	2	0	m
The function $(z – i)^2 (z+1)^3 e^z$ has a zero i of order 2 and a zero -1 of order …………	1	2	3	0	3
In maximum modulus theorem, the maximum of modulus f(z) in the whole of C_i and C occurs only on C unless f(z) is a ……………..	analytic	function	constant	maximum	constant
If a function f(z) is such that $f(z) = (z - a)^k g(z)$, where k is a positive integer and g(a) not equal to zero, then a is said to be a …………of f(z) of order k.	Constant	zero	function	analytic	zero

The function $(z-i)^2$ have a zero i of order…..	2	1	0	3	2
If $f(z) = (z - a)^m[a_0 + a_1(z-a)………]$, $a_0 \neq 0$, then z =a is a zero of order …..	m	1	2	0	m
………… of an analytic function are isolated	zeros	poles	residues	points	zeros
The multiplicity of a zero *a* is also known as the ………………….. of the function at *a*	order of analytic	order of vanishing	order of pole	order of singularity	order of vanishing
A bounded entire function is ………………………..	analytic	function	Constant	zero	Constant
A zero of order 1 is also called a ………….	simple analytic	simple pole	simple zero	simple function	simple zero
In Gauss' mean value theorem, mean of the values of the function on C is the value of the function at its…………	radius	centre	diameter	points	centre
f(z) = sin z has a zero of order ……………………….. at z = 0.	0	1	2	∞	1
A function analytic in D has …………. of all orders in D	derivatives	points	curves	zeros	derivatives
The square of real number is ---------	Non negative	Non positive	Negative	absolute value	absolute value
A function which is analytic everywhere in the finite plane is called an ----------------- function.	single	multi	entire	continuous	entire
$f(z) = \|z\|^2$ is --------------- everywhere	analytic	not analytic	continuous	exist	not analytic
The quotient of two polynomials is called a	Exponential function	logarithmic function	Continuous function	rational function	rational function
It the region lies to the left of a person when he travels along C, then the curve C is called a ……………………	postively oriented simple closed curve	negatively oriented simple closed curve	open curve	simple closed curve	postively oriented simple closed curve
The simple closed rectifible curve is abbreviated as…………………	curve	scr curve	scro curve	arc	scr curve
The set of complex points is called ……………………	arc	simple arc	closed arc	open arc	simple arc
In cauchy's fundamental theorem, ò f(z) dz=………………..	1	2	0	4	0
The simple closed rectifiable positively oriented curve is abbreviated as ………………..	curve	scr curve	scro curve	arc	scro curve
The bounded region of C is called ……………	interior	exterior	interior nor exterior	interior and exterior	interior

A region D is said to be …………… for every closed curve in D , Ci is contained in D	connected	simply - connected	disconnected	disjoint	simply - connected
In cauchy’s fundamental theorem, ò f(z) dz=…	1	2	0	4	0
If f(z) is analytic in a simply connected domain , then the values of the integrals of f(z) along all paths in the region joining ------ fixed points are the same	one	two	three	multiple	two
The bounded region of C is called ……………	interior	exterior	interior nor exterior	interior and exterior	interior
A region D is said to be …………… for every closed curve in D , Ci is contained in D	connected	simply - connected	disconnected	disjoint	simply - connected
The set of complex points is called ….	arc	simple arc	closed arc	open arc	simple arc
If a curve intersects itself at a point then the point is said to be a……	single	multiple points	double valued	trile	multiple points
The equation $z = \cos t+i\sin t, 0\leq t\leq\pi$ represents a ……………	arc	simple arc	closed arc	curve	simple arc
The unit circle $z=\cos t+i\sin t, 0\leq t\leq 2\pi$ are	positively oriented circle	negatively oriented circle	circle	unit circle	positively oriented circle
The unit circle $z = \cos(-t) +i\sin(-t), 0\leq t\leq 2\pi$ are ……..	positively oriented circle	negatively oriented circle	circle	unit circle	negatively oriented circle
The simple closed rectifible curve is abbreviated as……	curve	scr curve	scro curve	arc	scr curve
The simple arc is also known as ….	multiple	Jordan	double	multiple	Jordan
The derivative of an analytic function is also …	analytic	continuous	derivative	bounded	continuous
The function $(z-i)^2$ have a zero i of order…..	2	1	0	3	2

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Subject: COMPLEX ANALYSIS-II **Subject Code: 15MMU602**

Class : III B.Sc Mathematics **Semester : VI**

UNIT -II

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If f(z) can be expanded as a series of non- negative integral powers which is convergent for all z in C_i then the series is called ……….for f(z) about z=a.	Power series	Taylor’s series	Laurent’s series	convergent series	Taylor’s series
If C is the largest circle with z=a as its center such that f(z) is analytic in C_i but is not …………… somewhere on C.	analytic	singular	zero	constant	analytic
………………….. is the expansion of f(z) about z = 0.	convergent series	taylor’s series	Power series	maclaurin’s series	maclaurin’s series
The binomial series is valid when $\lvert z \rvert$ ……………	< 1	equal to 1	> 1	> 0	< 1
Maclaurin's series expansion of the function cosz is valid in ………………………	$\lvert z\rvert > 1$	$\lvert z\rvert = 0$	$\lvert z\rvert < \infty$	$\lvert z\rvert < 1$	$\lvert z\rvert < \infty$
Suppose f(z) is not identically zero and analytic in a region D. In any closed bounded region D, f(z) has ……………….. number of zeros	infinite	only a finite	many	finite and many	only a finite
suppose f(z) is analytic in a region D and z_n , n= 1,2,3,…., in D are the zeros of f(z), where the sequence $\{z_n\}$ converges to a limit z = a in D then f(z)…. in D.	Constant	vanishes identically	bounded	analytic	vanishes identically
Maclaurin's series expansion of the function e^(-z) is valid in ………………………	$\lvert z\rvert = 0$	$\lvert z\rvert < 1$	$\lvert z\rvert < \infty$	$\lvert z\rvert > 1$	$\lvert z\rvert < \infty$

If D is a simply-connected region, f(z) is analytic and a is a point in D and C is the largest circle whose interior lies in D then the power series is… and its sum is f(z)	Convergent	uniformly convergent	divergent	absolutely convergent	absolutely convergent
The function $e^z - 1$ has a zero z = 0 of order………..	two	one	three	zero	one
Maclaurin's series expansion of the function log(1+z) and log(1-z) is valid in ……………………..					
The whole series is absolutely convergent if both the positive and negative series are………………..	absolutely convergent	uniformly convergent	Convergent	divergent	absolutely convergent
The logarithmic series is valid when $\|z\|$ ……………	< 1	equal to 1	> 1	> 0	< 1
The functions of the form, $P_n(Z)= a_0+a_1z+a_2z^2+\ldots\ldots+a_nz^n$, $a_n\neq 0$ is called a………………………………	polynomial of degree n	polynomial of degree 5	polynomial of degree 2n	polynomial of degree n-1	polynomial of degree n
The power series of the form $a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$ is said to be a series about …………………………	z = 0	z = -a	z = a	z = ∞	z = a
If R = 0 the series is divergent in the extended plane except at ……………………..	z = 0	z =1	z = ∞	z = -1	z = 0
The inverse function of the exponential function is the ……	Trignometric functions	hyberbolic functions	harmonic functions	Logarithmic functions	Logarithmic functions
Maclaurin's series expansion of the function coshz is valid in ……………………..	$\|z\| < 0$	$\|z\| < \infty$	$\|z\|= 0$	$\|z\| < 1$	$\|z\| < \infty$
. The function f(z) = |z| is differentiable -----	on real part	on imaginary part	at the origin	at the point 2	at the origin
The expansion of f(z) = e^z is valid in the ……………………….plane	partial complex	entire real	entire complex	partial real	entire complex
The region of validity for Taylor's series about z = 0 of the function e^z is ……………..	$\|z\|= 0$	$\|z\| < 1$	$\|z\| < \infty$	$\|z\| > 1$	$\|z\| < 1$
Maclaurin's series expansion of the function sinz is valid in ……………………..	$\|z\| < \infty$	$\|z\| < 1$	$\|z\| < 0$	$\|z\| > 1$	$\|z\| < \infty$
The sum f(z) of a powerseries is analytic in ………..	$\|z\| > R$	$\|z\| < R$	$\|z\| \leq R$	$\|z\| = R$	$\|z\| < R$
If f(z) is analytic in an open circular disc with center at z = a, then the …… about z = a is the only power series in z – a which converges to f(z) in that disc.	Taylor's series	Power series	Convergent series	Maclaurin's series	Taylor's series

Maclaurin's series expansion of the function sinhz is valid in ………………………	$\|z\| < \infty$	$\|z\| < 0$	$\|z\| < 1$	$\|z\| > 1$	$\|z\| < \infty$
The expansion with positive and negative powers of z – a is called ……………series about z = a.	Laurent’s series	Taylor’s series	convergent series	Power series	Laurent’s series

UNIT-II

SYLLABUS

Taylor's series and Laurent's series : Taylor's series- Theorems and some related problems- Zero's of an analytic function- Laurent's series – Theorems and some related problems- Cauchy product and division.

Taylor and Laurent Series

Taylor series. Suppose f is analytic on the open disk $|z - z_0| < r$. Let z be any point in this disk and choose C to be the positively oriented circle of radius ρ, where $|z - z_0| < \rho < r$. Then for $s \epsilon C$ we have

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0}\left[\frac{1}{1-\frac{z-z_0}{s-z_0}}\right] = \sum_{j=0}^{\infty} \frac{(z-z_0)^j}{(s-z_0)^{j+1}}$$

since $|\frac{z-z_0}{s-z_0}| < 1$. The convergence is uniform, so we may integrate

$$\int_C \frac{f(s)}{s-z} ds = \sum_{j=0}^{\infty}\left(\int_C \frac{f(s)}{(s-z_0)^{j+1}} ds\right)(z-z_0)^j, \text{ or}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z} ds = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}} ds\right)(z-z_0)^j.$$

We have thus produced a power series having the given analytic function as a limit:

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j, \ |z - z_0| < r,$$

where

$$c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds.$$

This is the celebrated **Taylor Series** for f at $z = z_0$.

We know we may differentiate the series to get

$$f'(z) = \sum_{j=1}^{\infty} j c_j (z - z_0)^{j-1}$$

and this one converges uniformly where the series for f does. We can thus differentiate again and again to obtain

$$f^{(n)}(z) = \sum_{j=n}^{\infty} j(j-1)(j-2)\ldots(j-n+1)c_j(z-z_0)^{j-n}.$$

Hence,

$$f^{(n)}(z_0) = n!c_n, \text{ or}$$
$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

But we also know that

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}ds.$$

This gives us

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}ds, \text{ for } n = 0, 1, 2, \ldots.$$

This is the famous **Generalized Cauchy Integral Formula.** Recall that we previously derived this formula for $n = 0$ and 1.

What does all this tell us about the radius of convergence of a power series? Suppose we have

$$f(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j,$$

and the radius of convergence is R. Then we know, of course, that the limit function f is analytic for $|z - z_0| < R$. We showed that if f is analytic in $|z - z_0| < r$, then the series converges for $|z - z_0| < r$. Thus $r \leq R$, and so f cannot be analytic at any point z for which $|z - z_0| > R$. In other words, the circle of convergence is the largest circle centered at z_0 inside of which the limit f is analytic.

Example

Let $f(z) = \exp(z) = e^z$. Then $f(0) = f'(0) = \ldots = f^{(n)}(0) = \ldots = 1$, and the Taylor series for f at $z_0 = 0$ is

$$e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$$

and this is valid for all values of z since f is entire. (We also showed earlier that this particular series has an infinite radius of convergence.)

Exercises

1. Show that for all z,

$$e^z = e\sum_{j=0}^{\infty}\frac{1}{j!}(z-1)^j.$$

2. What is the radius of convergence of the Taylor series $\left(\sum_{j=0}^{n} c_j z^j\right)$ for $\tanh z$?

3. Show that

$$\frac{1}{1-z} = \sum_{j=0}^{\infty}\frac{(z-i)^j}{(1-i)^{j+1}}$$

for $|z-i| < \sqrt{2}$.

4. If $f(z) = \frac{1}{1-z}$, what is $f^{(10)}(i)$?

5. Suppose f is analytic at $z = 0$ and $f(0) = f'(0) = f''(0) = 0$. Prove there is a function g analytic at 0 such that $f(z) = z^3 g(z)$ in a neighborhood of 0.

6. Find the Taylor series for $f(z) = \sin z$ at $z_0 = 0$.

7. Show that the function f defined by

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

is analytic at $z = 0$, and find $f'(0)$.

Laurent series. Suppose f is analytic in the region $R_1 < |z - z_0| < R_2$, and let C be a positively oriented simple closed curve around z_0 in this region. (Note: we include the possiblites that R_1 can be 0, and $R_2 = \infty$.) We shall show that for $z \notin C$ in this region

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j},$$

where

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, \text{ for } j = 0, 1, 2, \ldots$$

and

$$b_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{-j+1}} ds, \text{ for } j = 1, 2, \ldots.$$

The sum of the limits of these two series is frequently written

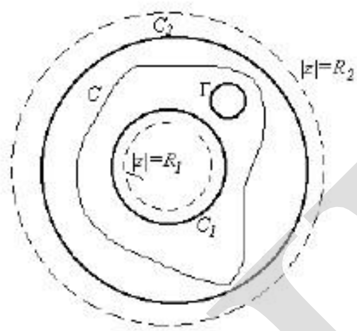
$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j,$$

where

$$c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}}, j = 0, \pm 1, \pm 2, \ldots.$$

This recipe for $f(z)$ is called a **Laurent series**, although it is important to keep in mind that it is really two series.

Okay, now let's derive the above formula. First, let r_1 and r_2 be so that $R_1 < r_1 \le |z - z_0| \le r_2 < R_2$ and so that the point z and the curve C are included in the region $r_1 \le |z - z_0| \le r_2$. Also, let Γ be a circle centered at z and such that Γ is included in this region.



Then $\frac{f(s)}{s-z}$ is an analytic function (of s) on the region bounded by C_1, C_2, and Γ, where C_1 is the circle $|z| = r_1$ and C_2 is the circle $|z| = r_2$. Thus,

$$\int_{C_2} \frac{f(s)}{s-z}\,ds = \int_{C_1} \frac{f(s)}{s-z}\,ds + \int_{\Gamma} \frac{f(s)}{s-z}\,ds.$$

(All three circles are positively oriented, of course.) But $\int_{\Gamma} \frac{f(s)}{s-z} ds = 2\pi i f(z)$, and so we have

$$2\pi i f(z) = \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds.$$

Look at the first of the two integrals on the right-hand side of this equation. For $s \epsilon C_2$, we have $|z - z_0| < |s - z_0|$, and so

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{(s-z_0)-(z-z_0)} \\ &= \frac{1}{s-z_0}\left[\frac{1}{1-(\frac{z-z_0}{s-z_0})}\right] \\ &= \frac{1}{s-z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^j \\ &= \sum_{j=0}^{\infty}\frac{1}{(s-z_0)^{j+1}}(z-z_0)^j. \end{aligned}$$

Hence,

$$\int_{C_2} \frac{f(s)}{s-z}\,ds = \sum_{j=0}^{\infty}\left(\int_{C_2} \frac{f(s)}{(s-z_0)^{j+1}}\,ds\right)(z-z_0)^j$$
$$= \sum_{j=0}^{\infty}\left(\int_{C} \frac{f(s)}{(s-z_0)^{j+1}}\,ds\right)(z-z_0)^j$$

For the second of these two integrals, note that for $s \in C_1$ we have $|s-z_0| < |z-z_0|$, and so

$$\frac{1}{s-z} = \frac{-1}{(z-z_0)-(s-z_0)} = \frac{-1}{z-z_0}\left[\frac{1}{1-\left(\frac{s-z_0}{z-z_0}\right)}\right]$$
$$= \frac{-1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^j = -\sum_{j=0}^{\infty}(s-z_0)^j\frac{1}{(z-z_0)^{j+1}}$$
$$= -\sum^{\infty}(s-z_0)^{j-1}\frac{1}{(z-z_0)^j} = -\sum^{\infty}\left(\frac{1}{(s-z_0)^{-j+1}}\right)\frac{1}{(z-z_0)^j}$$

As before,

$$\int_{C_1} \frac{f(s)}{s-z}\,ds = -\sum_{j=1}^{\infty}\left(\int_{C_2} \frac{f(s)}{(s-z_0)^{-j+1}}\,ds\right)\frac{1}{(z-z_0)^j}$$
$$= -\sum_{j=1}^{\infty}\left(\int_{C} \frac{f(s)}{(s-z_0)^{-j+1}}\,ds\right)\frac{1}{(z-z_0)^j}$$

Putting this altogether, we have the Laurent series:

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s-z}\,ds$$
$$= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{(s-z_0)^{j+1}}\,ds\right)(z-z_0)^j + \sum_{j=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{(s-z_0)^{-j+1}}\,ds\right)\frac{1}{(z-z_0)^j}.$$

Let f be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

First, observe that f is analytic in the region $0 < |z| < 1$. Let's find the Laurent series for f valid in this region. First,

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

From our vast knowledge of the Geometric series, we have

$$f(z) = -\frac{1}{z} - \sum_{j=0}^{\infty} z^j.$$

Now let's find another Laurent series for f, the one valid for the region $1 < |z| < \infty$. First,

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right].$$

Now since $|\frac{1}{z}| < 1$, we have

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] = \frac{1}{z}\sum_{j=0}^{\infty} z^{-j} = \sum_{j=1}^{\infty} z^{-j},$$

and so

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} + \sum_{j=1}^{\infty} z^{-j}$$

$$f(z) = \sum_{j=2}^{\infty} z^{-j}.$$

Exercises

8. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which the series converge to $f(z)$.

9. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z(1+z^2)}$$

and specify the regions in which the series converge to $f(z)$.

10. Find the Laurent series in powers of $z-1$ for $f(z) = \frac{1}{z}$ in the region $1 < |z-1| < \infty$.

POSSIBLE QUESTIONS

1.State and prove Taylor's theorem**.**

2.Prove that if $f_1(z)$ and $f_2(z)$ are two functions analytic in a region D such that, for all z_n, $f_1(z_n) = f_2(z_n)$, where $\{ z_n \}$ is a sequence of points in D converging to a point in D. then $f_1(z) \equiv f_2(z)$ in D.

3.State and prove Laurent's theorem.

4. Expand cos z as Taylor's about the points given below.
   i) z=0 ii) $z=\frac{\pi}{4}$ iii) $z=\frac{\pi}{3}$

5. State and prove uniqueness theorem.

6.Expand log(1+z) as a taylor's series about z = 0.

7.If f(z) is analytic in a region D and z_n, n = 1,2,3…… in D are the zeros of f(z), where the sequence $\{z_n\}$ converges to a limit z=a in D then prove that f(z) vanishes identically in D.

8. Find the Taylor's expansion about z = 0 of $f(z) = \frac{z}{(z+1)(z-3)}$.

9. Expand $\frac{1}{\{(z-1)(z-2)\}}$ as a power series in z valid in i)$|z| < 1$ $ii) 1 < |z| < 2$ $iii)\ |z| > 2$.

10.Find the Laurent's expansion for $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in i) $2 < |z| < 3$ $ii)\ |z| > 3$.

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UNIT -III

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A ……………………………. of a function is a point at which the function ceases to be analytic	singular point	analytic point	pole point	essential point	singular point
If z = a is a pole of f(z), then f(z) is ………………….. in every deleted neighbourhood of z=a	zero	not bounded	Constant	bounded	not bounded
The point z=a is a singular point of f(z) if f(z) = ……………………………..	one	infinity	zero	finite	infinity
The point z=a is a singular point of f(z) if f(z) is not defined at ……………………………..	y=a	x=a	u=a	z=a	z=a
If z = a is ………………………….. of f(z), then f(z) is not bounded in every deleted neighbourhood of z=a	a pole	a removable singularity	an essential singularity	isolated singularity	a pole
The entire function f(z) = e^z is not defined at z=∞ and z=∞ is the only ……………………………. Point	singular	analytic	pole	essential	singular
The point ………………………… is a singular point of all the trigonometric function and hyperbolic functions because they are function of e^z	z=0	z =1	z=∞	z=a	z=∞
If z = a is an isolated singular point of a function f(z), then the singularity is called…….. according as the Laurent's series about z=a of f(z), valid in a deleted neighbourhood of z=a has an infinite number of negative powers	a pole	an essential singularity	isolated singularity	a removable singularity	an essential singularity

A singular point z0 of f is said to be……………………. if there is a neighborhood of z0 which contains no singular points of f save z0.	isolated	Constant	bounded	analytic	isolated
If z = a is an isolated singular point of a function f(z), then the singularity is called….. according as the Laurent's series about z=a of f(z), valid in a deleted neighbourhood of z=a has a finite number of negative powers	a removable singularity	an essential singularity	a pole	isolated singularity	a pole
A singularity that is neither a pole or removable is called an ………………...singularity.	pole	analytic	essential	singular	essential
If z = a is a singularity of f(z), then it is ………………………………. If limit z tends to a point a of f(z) exists and is finite	a pole	a removable singularity	isolated singularity	an essential singularity	a removable singularity
Classification of isolated singularities is done with reference to the ………………………………. of the Laurent's expansion of the function about the singular point	positive power	zero	constant	negative power	negative power
If z = a is an isolated singular point of a function f(z), then the singularity is called….. according as the Laurent's series about z=a of f(z), valid in a deleted neighbourhood of z=a has no negative powers	an essential singularity	a removable singularity	isolated singularity	a pole	a removable singularity
A point z0 is a singular point of a function f if f not ……………………….. at z0	singular	analytic	pole	essential	analytic
An isolated singular point z0 of f such that the Laurent series at z0 includes only a finite number of terms involving negative powers of z - z0 is called a …………………………	essential	singular	analytic	pole	pole
If z = a is a singularity of f(z), then it is ………………………………. If limit z tends to a point a of f(z) does not exists	an essential singularity	a pole	isolated singularity	a removable singularity	an essential singularity
Isolated singularities are classified into ……………………………different groups	two	three	many	five	three
If z=a is a singular point of a function and if the limit of the function as z tends to a exists and is finite, then z=a is …………………………. of the function	a removable singularity	a pole	isolated singularity	an essential singularity	a removable singularity

The removable singularities and the poles are ……………………………………	singularity	non-isolated singularity	isolated singularity	an essential singularity	isolated singularity
If z = a is ………………………….. of f(z), then f(z) is bounded in every deleted neighbourhood of z=a	an essential singularity	isolated singularity	a removable singularity	a pole	a removable singularity
…………………………….. are the singular points of the function 1/z(z-1)	2 and 0	2 and 3	0 and 1	1 and 2	0 and 1
If z=a is …………………… of f(z) , then the behaviour of f(z), close to z=a is complicated and at z=a the function is not defined	an essential singularity	a pole	isolated singularity	a removable singularity	an essential singularity
When the order of a pole is 1, the pole is said to be a …………………………….. Pole	zero	double	simple	finite	simple
If z=a is a zero of order m of an analytic function f(z), then z=a is a…………………………….. of order m of the function 1/f(z)	a pole	a removable singularity	an essential singularity	isolated singularity	a pole
When the order of a pole is 2, the pole is said to be a …………………………….. Pole	double	finite	simple	zero	double
If z=a is a …………………….. of order m of an analytic function f(z), then z=a is a pole of order m of the function 1/f(z)	a pole	zero	analytic	singular	zero
The point z=a is a singular point of f(z) if f(z) is not ……………………… at z= a	continuous	not continuous	differentiable	bounded	differentiable
If z = a is a singularity of f(z), then it is ………………………………. if limit z tends to a point a of f(z) exists and equals infinity	a removable singularity	an essential singularity	a pole	isolated singularity	a pole
If z = a is a removable singularity of f(z), then f(z) is ………………….. in every deleted neighbourhood of z=a	bounded	Constant	zero	not bounded	bounded
If z is a ………………….. |f(z)| tends ∞ as z tends to a	Constant	singular	double	pole	pole
singular points of logz are……………………..	z = 0 and z = ∞	z = 1 and z =0	z =0 and z = -1	z =1 and z = ∞	z = 0 and z = ∞
The limit point of zero's of an analytic function is a …………….point of the function	singular	nonsingular	poles	zeros	singular
A region which has only one hole is an …………………..region	origin	set	annular	moment	annular

tanz and secz are analytic in a bounded region in which ……………………..	tanz $\neq 0$	sec z $\neq 0$	sinz $\neq 0$	cos z $\neq 0$	sinz $\neq 0$
cot z and cosecz are analytic in a bounded region in which ……………………..	cot z $\neq 0$	cosecz $\neq 0$	sinz $\neq 0$	cosz $\neq$0	cosz $\neq$0
The poles of an analytic function are ……………………	essential	removable	pole	isolated	isolated

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UNIT -IV

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Principle value of logz is obtained when n = …………………………….	0	-1	1	2	0
If a function h(z) is analytic at z=a and h(a) not equal to zero then the residue of function f(z)=h(z)/(z-a) at z=a is ……………..	h(z)	h(a)	f(a)	z(a)	h(a)
From x= rcosθ and y = rsinθ weget θ = ………………………	$\sin^{-1} y/x$	$\cos^{-1} y/x$	$\tan^{-1} y/x$	$\cot^{-1} y/x$	$\tan^{-1} y/x$
The complex integration along the scro curve used in evaluating the definite integral is called……………………………	differentiation	contour differentiation	contour integration	integration	contour integration
cosiz = ………………………..	cosz	icosz	icoshz	coshz	coshz
If a function h(z) is analytic at z=a and h(a) not equal to zero then z=a is a …………………. of the function f(z)=h(z)/(z-a)	simple zero	double pole	simple pole	finite pole	simple pole
In a compact set every continuous function is …………………………	bounded in s	uniformly continuous in s	unique	does not exist	
siniz = ………………………..	sinz	sinhz	isinz	isinhz	isinhz
Principle value of logz is obtained when n = ……………………….	0	-1	1	2	0

To show that the integral along gamma vanishes as R tends to infinity, a result called ……………………………………….	origin lemma	jacobi's lemma	annular lemma	jordan's lemma	jordan's lemma
$\cosh^2 z - \sinh^2 z$ = …………………………	0	1	-1	∞	1
The logarithmic function is a ………………………….. valued function	single	multiple	two	zero	multiple
In a complex field z = x + iy then θ = …………..	$\sin^{-1}(y/x)$	$\cos^{-1}(y/x)$	$\tan^{-1}(y/x)$	$\cot^{-1}(y/x)$	$\tan^{-1}(y/x)$
The function $f(z) = z^{1/2}$ is ………… valued function	single	multi	double	triple	double
The sum f(z) of a powerseries is analytic in …………..	$\vert z\vert > R$	$\vert z\vert < R$	$\vert z\vert \le R$	$\vert z\vert = R$	$\vert z\vert < R$
$\cos(z_1 + z_2)$ = ……………………………	$\cos z_1 \cos z_2 - \sin z_1 \sin z_2$	$\cos z_1 \sin z_2 - \sin z_1 \cos z_2$	$\cos z_1 \cos z_2 + \sin z_1 \sin z_2$	$\sin z_1 \cos z_2 - \cos z_1 \sin z_2$	$\cos z_1 \cos z_2 - \sin z_1 \sin z_2$
The residue of f(z) = cosz/z at its pole is ………………..	2	3	0	1	1
The residues at z=0 of the function f(z)=e^(1/z) is ……………………	0	2	1	3	1
The residue of f(z) = 1/(z^2+a^2)^2 at its singularities is ………………	i/a^3	$3i/a^4$	$i/4a^3$	$5i/a^2$	$i/4a^3$

UNIT-IV

SYLLABUS

Real definite integrals: Evaluation using the calculus of residues – Integration on the unit circle –Integral with - ∞ and + ∞ as lower and upper limits with the following integrals:
i) P(x) /Q(x) where the degree of Q(x) exceeds that of P(x) at least 2.
ii) (sin ax).f(x), (cos ax).f(x), where a>0 and f(z) →0 as z→∞ and f(z) does not have a pole
on the real axis.
iii) f(x) where f(z) has a finite number of poles on the real axis

Residues and Real Integrals

Residue theorem

Definition

Suppose f is holomorphic in a deleted neigbourhood D_0 of $z_0 \in \mathbb{C}$ and $\Gamma_r := \{z \in \mathbb{C} : |z - z_0| = r\} \subseteq D_0$. Then **residue** of f at z_0 is defined by

$$Res(f, z_0) = \frac{1}{2\pi i} \int_{\Gamma_r} f(z) dz.$$

$\Diamond$

Recall that if f is holomorphic in a deleted neigbourhood D_0 of $z_0 \in \mathbb{C}$, then f has Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad z \in D_0,$$

and we know that

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz, \qquad n \in \mathbb{Z}.$$

Thus,

$$Res(f, z_0) = a_{-1}.$$

The following theorem, known as **residue theorem** follows from Cauchy's theorem.

Theorem (Residue theorem) *Suppose Γ is a simple closed contour and $z_1, \ldots, z_k$ are points in Γ_Γ which are the only singular points of f in $\Gamma \cup \Omega_\Gamma$. Then*

$$\int_\Gamma f(z) dz = 2\pi i \sum_{j=1}^{k} Res(f, z_j).$$

Calculation of Residues

Suppose z_0 is a pole of order m of a holomorphic function f. Then we know that there exists a holomorphic function φ in a neigbourhood D of z_0 such that

$$f(z) = (z - z_0)^{-m}\varphi(z) \qquad \forall\, z \in D_0 := D \setminus \{z_0\}.$$

Let

$$\varphi(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n, \qquad z \in D.$$

Then we have

$$\alpha_n = \frac{\varphi^{(n)}(z_0)}{n!}, \qquad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

so that

$$f(z) = (z - z_0)^{-m}\varphi(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^{n-m} = \sum_{n=-m}^{\infty} a_n (z - z_0)^n.$$

Hence,

$$Res(f, z_0) = a_{-1} = \alpha_{m-1} = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

Thus, we have proved the following theorem.

Theorem *Suppose z_0 is a pole of order m of a holomorphic function f. Then the function*

$$z \mapsto \varphi(z) := (z - z_0)^m f(z)$$

defined in a deleted neigbourhood of z_0 has a holomorphic extension to a neigbourhood D of z_0, again denoted by φ, and

$$Res(f, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

In particular, if z_0 is a simple pole of f, then

$$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Corollary 7.2.2 *Suppose g and h are holomorphic in a neigbourhood D of z_0 and z_0 is a zero of h of order m. If $h(z) = (z - z_0)^m h_0(z)$ with $h_0(z_0) \neq 0$, then*

$$Res(\frac{g}{h}, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!},$$

where $\varphi(z) = g(z)/h_0(z)$. In particular, if $m = 1$, then

$$Res(\frac{g}{h}, z_0) = \frac{g(z_0)}{h_0(z_0)} = \frac{g(z_0)}{h'(z_0)}.$$

EXAMPLE Let us find $\int_\Gamma f(z)dz$, where $f(z) = \frac{1}{z(z-1)}$ and Γ is the positively oriented circle with centre 0 and radius 2.

By residue theorem,

$$\int_\Gamma \frac{dz}{z(z-1)} = 2\pi i[Res(f, z_1) + Res(f, z_2)], \qquad z_1 = 0,\ z_2 = 1.$$

Since $z_1 = 0$ and $z_2 = 1$ are simple poles of the function

$$Res(f, z_1) = \lim_{z \to 0} z f(z) = -1,$$

$$Res(f, z_2) = \lim_{z \to 1} (z-1) f(z) = 1.$$

Hence, $\int_\Gamma \frac{dz}{z(z-1)} = 0.$ □

EXAMPLE Let us find $\int_\Gamma f(z)dz$, where $f(z) = \frac{1}{z(z-1)^2}$

and Γ is the positively oriented circle with centre 0 and radius 2. By residue theorem,

$$\int_\Gamma \frac{dz}{z(z-1)^2} = 2\pi i[Res(f,z_1) + Res(f,z_2)], \qquad z_1 = 0,\ z_2 = 1.$$

Since $z_1 = 0$ is a simple pole and $z_2 = 1$ is a pole of order 2,

$$Res(f,z_1) = \lim_{z\to 0} zf(z) = 1, \qquad Res(f,z_2) = \varphi'(1)$$

where $\varphi(z) = \frac{1}{z}$ so that $\varphi'(1) = -1$. Thus, $\int_\Gamma \frac{dz}{z(z-1)^2} = 0$. □

Exercise 1. Find $Res(f, z_0)$, where

(a) $f(z) = ze^{1/z}, \quad z_0 = 0.$

(b) $f(z) = \dfrac{z+2}{z(z+1)}, \quad$ (i) $z_0 = 0$, (ii) $z_0 = -1$.

2. Evaluate $\int_\Gamma f(z)dz$, where

(a) $\dfrac{3z+1}{z(z-1)^3}$ and $\Gamma = \{z : |z| = 2\}$.

(b) $\dfrac{z+1}{2z^3 - 3z^2 - 2z}$ and $\Gamma = \{z : |z| = 1\}$.

(c) $\dfrac{z+1/z}{z(2z-1/(2z))}$ and $\Gamma = \{z : |z| = 1\}$.

(d) $\dfrac{\log(z+2)}{2z+1}$ and $\Gamma = \{z : |z| = 1\}$.

(e) $\dfrac{\cosh(1/z)}{z}$ and $\Gamma = \{z : |z| = 1\}$.

Evaluation of Improper Integrals

In this section we shall evaluate integrals of the form

$$\int_0^\infty f(x)dx \quad \text{and} \quad \int_{-\infty}^\infty f(x)dx,$$

where f is a continuous function.

EXAMPLE 7.3.1 Let us evaluate $\int_{-\infty}^\infty \frac{dx}{1+x^2}$. For this consider the function $f(z) = \frac{dx}{1+z^2}$ for $z \neq 0$. Note that $z = i$ is the only singularity of f in the upper half plane and it is a simple pole. Consider the positively oriented curve Γ_R consisting of the semicircle with centre 0 and radius R, i.e., $S_R := \{z : |z| = R, \text{Im}(z) > 0\}$ and the line segment $L_R := [-R, R]$. Then, by Cauchy's theorem,

$$\int_{\Gamma_R} f(z)dz = \int_{C_r} f(z)dz,$$

where $C_r = \{z : |z - i| = r\}$ with $0 < r < R$. But,

$$\int_{C_r} f(z)dz = 2\pi i \, Res(f, i) = 2\pi i \lim_{z\to i}(z - i)f(z) = \pi.$$

Thus,

$$\int_{\Gamma_R} f(z)dz = \pi.$$

Also, we have

$$\int_{\Gamma_R} f(z)dz = \int_{S_R} f(z)dz + \int_{L_R} f(z)dz = \int_{S_R} f(z)dz + \int_{-R}^{R} f(x)dx.$$

But, for $z \in S_R$,

$$|f(z)| \leq \frac{1}{R^2 - 1}.$$

Hence,

$$\Big|\int_{S_R} f(z)dz\Big| \leq \frac{\ell(S_R)}{R^2 - 1} = \frac{\pi R}{R^2 - 1}.$$

Hence,

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = \lim_{R\to\infty}\int_{S_R} f(z)dz + \lim_{R\to\infty}\int_{L_R} f(z)dz$$
$$= 0 + \int_{-\infty}^{\infty} f(x)dx.$$

Thus,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = \pi.$$

□

EXAMPLE Let us evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx$. Since

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx\right)$$

we consider the function

$$f(z) = \frac{e^{iz}}{1+z^2}, \qquad z \notin \{i, -i\}.$$

Following the arguments as in the previous example, one arrive at

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx = \frac{\pi}{e}.$$

But,

$$\text{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx\right) = \int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}dx = 0.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}.$$

EXAMPLE Let us evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x}dx$. Since

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx\right)$$

we consider the function

$$f(z) = \frac{e^{iz}}{z}, \qquad z \neq 0.$$

For $r > 0$, let $S_r := \{z : |z| = r\}$ with positive orientation. Then, taking $0 < \varepsilon < R$ and Γ as the curve consisting of S_R, $[-R, -\varepsilon]$, $\widetilde{S}_\varepsilon$, $[\varepsilon, R]$, using Cauchy's theorem,

$$\begin{aligned} 0 &= \int_\Gamma f(z)dz \\ &= \int_{S_R} f(z)dz + \int_{-R}^{-\varepsilon} f(x)dx + \int_{\widetilde{S}_\varepsilon} f(z)dz + \int_\varepsilon^R f(x)dx \\ &= \int_{S_R} f(z)dz + \int_{-R}^{-\varepsilon} f(x)dx - \int_{S_\varepsilon} f(z)dz + \int_\varepsilon^R f(x)dx \end{aligned}$$

But,

$$\int_{-R}^{-\varepsilon} \frac{\cos x}{x} dx + \int_\varepsilon^R \frac{\cos x}{x} dx = 0$$

and

$$\int_{-R}^{-\varepsilon} \frac{\sin x}{x} dx + \int_\varepsilon^R \frac{\sin x}{x} dx = 2\int_\varepsilon^R \frac{\sin x}{x} dx.$$

Hence,

$$\int_\varepsilon^R f(x)dx + \int_{-R}^{-\varepsilon} f(x)dx = 2i\int_\varepsilon^R \frac{\sin x}{x} dx.$$

Thus,

$$2i\int_\varepsilon^R \frac{\sin x}{x} dx = \int_{S_\varepsilon} f(z)dz - \int_{S_R} f(z)dz$$

Now, we note that with the parametrization $\gamma(t) = Re^{it}$, $0 \le t \le \pi$ of S_R,

$$\int_{S_R} f(z)dz = \int_0^{\pi} \frac{e^{iR(\cos t + i\sin t)}}{Re^{iRt}} iRe^{iRt} dt = i\int_0^{\pi} e^{iR(\cos t + i\sin t)} dt.$$

Hence,

$$\left|\int_{S_R} f(z)dz\right| \le \int_0^{\pi} e^{-R\sin t} dt = 2\int_0^{\pi/2} e^{-R\sin t} dt.$$

Since $\dfrac{\sin t}{t}$ is decreasing in $[0, \pi/2]$, we have $\dfrac{\sin t}{t} \ge \dfrac{\sin \pi/2}{\pi/2}$ so that $\sin t \ge 2t/\pi$.Thus,

$$\left|\int_{S_R} f(z)dz\right| \le 2\int_0^{\pi/2} e^{-2Rt/\pi} dt = \frac{\pi}{2R}(1 - e^{-2R}) \to 0 \quad \text{as} \quad R \to \infty.$$

Next, we observe that

$$\frac{e^{iz}}{z} = \frac{1}{z} + \varphi(z)$$

where φ is an entire function. Hence, there exists $M > 0$ such that $|\varphi(z)| \le M$ for all z with $|z| \le 1$. Thus,

$$\int_{S_\varepsilon} f(z)dz = \int_{S_\varepsilon} \frac{dz}{z} + \int_{S_\varepsilon} \varphi(z)dz,$$

where

$$\left|\int_{S_\varepsilon} \varphi(z)dz\right| \le M\pi\varepsilon, \qquad 0 < \varepsilon \le 1.$$

Hence,

$$\begin{aligned} 2i\int_{\varepsilon}^{R} \frac{\sin x}{x} dx &= \int_{S_\varepsilon} \frac{dz}{z} + \int_{S_\varepsilon} \varphi(z)dz - \int_{S_R} f(z)dz \\ &= \pi i + \int_{S_\varepsilon} \varphi(z)dz - \int_{S_R} f(z)dz, \end{aligned}$$

where

$$\int_{S_\varepsilon} \varphi(z)dz \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad \int_{S_R} f(z)dz \to 0 \quad \text{as} \quad R \to \infty.$$

Thus,

$$\int_\varepsilon^R \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

□

Problems

1. Find the residues of the following functions:

   (i) $\frac{z^3}{z-1}$ (ii) $\frac{z^3}{(z-1)^2}$

2. If f and g are holomorphic in a neigbourhood of z_0, and z_0 is a simple a pole of g, then prove that $Res(f/g, z_0) = f(z_0)/g'(z_0)$.

3. Determine the residues of each of the following functions at each of their singularities:

$$\text{(i)} \frac{z^3}{1-z^4}, \quad \text{(ii)} \frac{z^5}{(z^2-1)^2}, \quad \text{(iii)} \frac{\cos z}{1+z+z^2}.$$

4. If f is holomorphic in a neigbourhood of z_0, and z_0 is a zero of f order m, then prove that $Res(f'/f, z_0) = m$.

5. Evaluate the following using complex integrals:

$$\text{(i)} \int_0^\infty \frac{e^{ix}}{x} dx, \quad \text{(ii)} \int_0^\infty \frac{dx}{1+x^2},$$

$$\text{(iii)} \int_0^\infty \frac{\sin^2 x}{x} dx, \quad \text{(iii)} \int_0^\infty \frac{\cos ax}{x^2+b^2} dx, \quad a \geq 0,\, b > 0.$$

POSSIBLE QUESTIONS

1. Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$

2. Show that $\int_0^{2\pi} \frac{1}{a+b\cos\theta}d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a > |b| > 0.$

3. Evaluate $\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}d\theta$, $a > |b| > 0.$

4. Using contour integration show that $\int_0^\infty \frac{\cos mx}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}, m > 0, a > 0$.

5. Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$

6. Using contour integration evaluate $\int_0^{2\pi} \frac{sin^2\theta}{a+b\ cos\theta}d\theta$

7. Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$

8. Using contour integration evaluate $\int_0^{2\pi} \frac{sin^2\theta}{a+b\ cos\theta}d\theta$

9. Evaluate: $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$

10. Evaluate $\int_0^{2\pi} \frac{1}{(a+b\sin\theta)^2}d\theta$, $a > |b| > 0.$

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Subject: COMPLEX ANALYSIS-II **Subject Code: 15MMU602**

Class : III B.Sc Mathematics **Semester : VI**

UNIT -V

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A function which has no singularities in a region other than a finite number of poles is said to be ………………………… in that region	analytic	meromorphic	isomorphic	morphic	meromorphic
Find the number of roots of the function f(z) = z^8 - 5z^5 - 2z+1 which lie inside the unit circle C:|z| = 1 …………………………………	2	4	6	5	5
n(P,f) denotes the number of …………………… of f(z) in Ci	ones	constants	zeros	poles	poles
Find the number of zeros of the function f(z) = z^6+z^3 - 6z +9 which lie inside the unit circle C:|z| = 1 …………………………………	1	2	0	4	0
A polynomial of degree n ≥ 1 has ……………………………. Zeros	n-2	n-1	n+1	n	n
z^8+3z^3+7z+5 has two zeros in the …………………….. quadrant	zeros	first	third	second	first
By Hurwitz's theorem, f(z) has …………………………. in D.	no zeros	poles	orders	zeros	no zeros
z^3+iz+1=0 has one root in the ……………… quadrant.	first	fourth	third	second	first

Find the number of zeros of the function f(z) = z^7 - 4z^5 +z^2 -1 which lie inside the unit circle C:|z| = 1 ………………………………….	5	2	4	3	5
z^4+z^3+4z^2+2z+3 has two Zeros each in the ………………………... quadrants	first and fourth	first and second	second and third	third and fourth	second and third
A function which is meromorphic in the extended plane is a …………………………………….	real function	irrational function	rational function	complex function	rational function
Find the number of zeros of the function f(z) = 2z^9 - 5z^5 +z^2 -1 which lie inside the unit circle C:|z| = 1 ………………………………….	2	1	5	3	5
By ……………………………….. theorem, f(z) and g(z) have the same number of zeros inside C.	Fundamental	Hurwitz's	Rouche's	principle of argument	Rouche's
The zeros and poles being counted as many times as their ………………..	zeros	orders	poles	ones	orders
one root of z^4+z^3+1 = 0 lies in the ……………………quadrant	zeros	second	third	first	first
The number of zeros of the function f(z)= z^4-5z+1 which lies in the ………………………….. region is 3	inner	annular	domain	outer	annular
The equation e^(z-a) - z=0, a> 1 has just one root in the ……………………..of the circle C: |z|=1	exterior	poles	interior	orders	interior
The fundamental theorem of algebra can be proved by applying ………………………………….. Theorem	principle of argument	isomorphic	Rouche's	Hurwitz's	Rouche's
The equation e^(z-a) - z=0, a> 1 has ……………………….. root in the interior of the circle C: |z|=1	4	2	1	0	1
The equation e^(z-a) - z=0, a> 1 has just one root and it is ………………………………………….	real and negative	imaginary and positive	imaginary and negative	real and positive	real and positive
z^8+3z^3+7z+5 has ……………………….zeros in the first quadrant	2	1	5	3	2
n(Z,f) denotes the number of ……………………. of f(z) in Ci	poles	zeros	ones	constants	zeros
z^4-z^2+1 =0 has ……………………. root in first quadrant	one	two	zeros	three	one

Find the number of zeros of the function $f(z) = z^8 - 4z^4 + z^2 - 1$ which lie inside the unit circle $C:\|z\| = 1$ …………………………………	6	2	4	5	4
$z^4+z^3+4z^2+2z+3$ has …………………… zeros each in the second and third quadrants	two	three	zeros	one	two
Find the number of zeros of the function $f(z) = z^4 - 5z + 1$ which lie inside the unit circle $C:\|z\| = 1$ …………………………………	0	4	1	2	1