Reg. No----------------

(15MMU604A)

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Under Section 3 of UGC Act 1956)

COIMBATORE – 641021

DEPARTMENT OF MATHEMATICS

SIXTH SEMESTER

II INTERNAL TEST- Feb'18

ADVANCED GRAPH THEORY

Date: .02.18( ) **Time: 2 Hours**

Class: III B.Sc Mathematics-B **Maximum: 50 Marks**

PART – A(20X1=20 Marks)

ANSWER ALL THE QUESTIONS

1. Any edge which is not in spanning tree is ___

a)Branch b) chord c) tree d) rank

2.Each of the largest non separable subgraphs is ______

a)link b) block c) connected d) disconnected

3.In adjacency matrix of graph all the entries along the leading diagonal are ___ iff the graph has no self-loops.

a) 1 b) 2 c) 3 d) 0

4.The determinant of every square sub matrix of an incidence matrix is_______

a) 1 , -1 or 0 b) 1, 2 or -1 c) 1 or -1 d) 0 or 2

5.The incidence matrix two elements 0 and 1 is ----------

a) binary matrix b) path matrix

c) adjacency matrix d)sub matrix

6. The incidence matrix A(G) every column has--------two 1's.

a) at most b) at least c) exactly d) more than

7. Every _____ edge in a graph is included in every covering of the graph.

a) pendant b) parallel c) adjacent d) finite

8.In a connected graph , any minimal set of edges containing atleast one branch of every spanning tree is _____

a) cut-set b) cut-vertex

c) fundamental cut-set d) chord

9.Every cut-set in a ____graph with more than two vertices contains atleast two edges.

a) Spanning tree b) separable

c) nonseparable d) vertex connectivity

10. A graph is ___ if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.

a) planar b) non planar

c) spanning tree d) cut-set

11. A connected planar graph with n vertices and e edges has ____ regions.

a) e-n+2 b) e-n+1 c) n-e+1 d) e-n+3

12. The rank of incident matrix must be atleast ______

a) n-1 b) n+1 c) n+2 d) n-2

13.In cut-set matrix, a column with all 0's corresponds to an ______ forming a self-loop.

a)edge b) vertex c) rank d) row

14.The spherical embedding of every ____ 3-connected graph is unique.

a)planar b) euler c) cut-set d) non planar

15.Kuratowski's second graph is _____

a) nonplanar b) planar c) cut- set d) separable

16. The number of branches in any spanning tree of G is ____

a) rank b) nullity

c) tree d) fundamental circuit

17.The number of 1's in a row is equal to the number of edges is

a) circuit b) path c) incidence d) adjacency

18.Complete graph with more than ______vertices is nonplanar

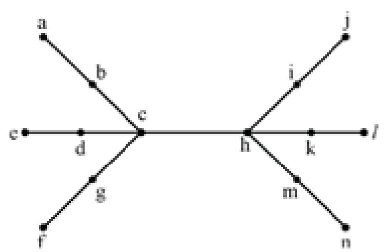
a) 1 b) 2 c) 3 d) 4

19. A ____ in a graph is a subset of edges in which no two edges are adjacent.

a) matching b) covering

c) chromatic d) chromatic partition

20. Centers of given tree are



a) d,h b) c,h c) c,k d)g,b, h, i

PART –B(3X10=30 MARKS)

ANSWER ALL THE QUESTIONS

21.a) Every connected graph has at least one spanning tree.

(OR)

b) Define with example.

i) planar ii)non planar iii)region iv)infinite region

22.a) Show that if G is connected simple planar graph with n(≥3) vertices and e is edge then e ≤ 3n-6.

(OR)

b) Prove that an n vertex graph is a tree iff $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$.

23.a) Prove that Kuratowski's second graph is also non planar.

(OR)

b) Prove that a connected graph with n vertices and e edges has e-n+2 regions.

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PART – A(20X1=20 Marks)
ANSWER ALL THE QUESTIONS

1. Any edge which is not in spanning tree is ___
   a)Branch **b) chord** c) tree d) rank
2. Each of the largest non separable subgraphs is ______
   a)link **b) block** c) connected d) disconnected
3. In adjacency matrix of graph all the entries along the leading diagonal are ___ iff the graph has no self-loops.
   a) 1 b) 2 c) 3 **d) 0**
4. The determinant of every square sub matrix of an incidence matrix is_______
   a) **1 , -1 or 0** b) 1, 2 or -1 c) 1 or -1 d) 0 or 2
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   a) **binary matrix** b) path matrix
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   a) at most b) at least **c) exactly** d) more than
7. Every _____ edge in a graph is included in every covering of the graph.
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8. In a connected graph , any minimal set of edges containing atleast one branch of every spanning tree is _____
   a) **cut-set** b) cut-vertex
   c) fundamental cut-set d) chord
9. Every cut-set in a ____graph with more than two vertices contains atleast two edges.
   a) Spanning tree b) separable
   c**) nonseparable** d) vertex connectivity
10. A graph is ___ if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.
    a) **planar** b) non planar
    c) spanning tree d) cut-set
11. A connected planar graph with n vertices and e edges has ____ regions.
    a) **e-n+2** b) e-n+1 c) n-e+1 d) e-n+3
12. The rank of incident matrix must be atleast _____
    a) **n-1** b) n+1 c) n+2 d) n-2
13. In cut-set matrix, a column with all 0's corresponds to an _____ forming a self-loop.
    a)edge b) vertex c) rank d) row
14. The spherical embedding of every ____ 3-connected graph is unique.
    a**)planar** b) euler c) cut-set d) non planar

15.Kuratowski’s second graph is _____

a) **nonplanar** b) planar c) cut- set d) separable

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a) **rank** b) nullity

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18.Complete graph with more than ______vertices is nonplanar

a) 1 **b) 2** c) 3 d) 4

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a) **matching** b) covering

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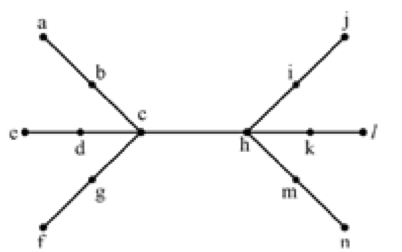
20. Centers of given tree are



a) d,h **b) c,h** c) c,k d)g,b, h, i

PART –B(3X10=30 MARKS)

ANSWER ALL THE QUESTIONS

21.a) Every connected graph has at least one spanning tree.

Proof: The fact that T is a spanning tree of G follows immediately from Theorem 9.1(ii). It remains only to show that the total weight of T is a minimum. In order to do this, suppose that S is a spanning tree of G such that W(S) < W(T). If ek is the first edge in the above sequence that does not lie in S, then the subgraph of T formed by adding ek to S contains a unique cycle C containing the edge ek. Since C contains an edge e lying in S but not in T, the subgraph obtained from S on replacing e by ek is a spanning tree 5". But by the construction, w(ek) < w(e), and so W(S') < W(S), and S' has one more edge in common with T than S. It follows on repeating this procedure that we can change S into 7, one step at a time, with the total weight decreasing at each stage. Hence W(T) < W(S), giving the required contradiction.

(OR)

b) Define with example.

i) planar

A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect':,only at their ends. Such a drawing of a planar graph G is called a planar embedding of G.

ii)non planar

iii)region iv)infinite region

22.a) Show that if G is connected simple planar graph with n(≥3) vertices and e is edge then e ≤ 3n-6.

Proof : We can assume that we have a plane drawing of G. Since each face is bounded by at least three edges, it follows on counting up the edges around each face that 3f< 2m; the factor 2 appears since each edge bounds two faces. We obtain the required result by combining this inequality with Euler's formula, (ii) This part follows in a similar way, except that the inequality 3f< 2m is replaced by 4/< 2m

(OR)

b) Prove that an n vertex graph is a tree iff $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.

Proof. We use induction on the number of edges of G, and prove that if all but one of the edges have been coloured with at most A colours, then there is a A-colouring of the edges of G. So suppose that each edge of G has been coloured, except for the edge vw. Then there is at least one colour missing at the vertex v, and at least one colour missing at the vertex w. If some colour is missing from both v and w, then we colour the edge vw with this colour. If this is not the case, then let a be a colour missing at v, and p be a colour missing at w, and let //ap be the connected subgraph of G consisting of the vertex v and those edges and vertices of G that can be reached from v by a path consisting entirely of edges coloured a or p

23.a) Prove that Kuratowski's second graph is also non planar.

Proof => We may assume that G is a simple connected plane graph. Then its geometric dual G* is a map, and the 4-colourability(v) of G follows immediately from the fact that this map is 4-colourable(f), by Theorem 19.2. <= Conversely, let G be a map and let G* be its geometric dual. Then G* is a simple planar graph and is therefore 4-colourable(v). It follows immediately that G is 4-colourable(f). // Duality can also be used to prove the following theorem.

(OR)

b) Prove that a connected graph with n vertices and e edges has e-n+2 regions.

Proof. The result is trivial if n = 2. We therefore assume that n > 3. If n is odd, then we can ^-colour the edges of Kn by placing the vertices of Kn in the form of a regular n-gon, colouring the edges around the boundary with a different colour for each edge, and then colouring each remaining edge with the colour used for the boundary edge parallel to it The fact that Kn is not (n - 1)- colourable(e) follows by observing that the largest possible number of edges of the same colour is (n - l)/2, and so Kn has at most (n - l)/2 x %'(Kn) edges.

Karpagam Academy Of Higher Education
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B.Sc Degree Examination, April 2018
Sixth Semester

Part-A (20×1=20 marks) online

Part-B (5×8=40 marks)

21. a) Prove that the number of vertices of odd degree in a graph is always even.

Proof;
Let G be a graph with n vertices. Among these some may have odd degree and some may have even degree.

$$\therefore \sum_{i=1}^{n} d(v_i) = \sum_{odd} d(v_j) + \sum_{even} d(v_k)$$

$$\Rightarrow \sum_{odd} d(v_j) = \sum_{i=1}^{n} d(v_i) - \sum_{even} d(v_k) \quad \text{[by using ⊛]}$$

$$= 2q - \text{even number}$$

(where q is the no. of edges of G)

$$= \text{even}$$

Hence the theorem

⊛ ⇒ Theorem: The sum of the degrees of all the vertices in a graph G is equal to twice the no. of edges.

Proof: Since each edge contributes two to the degree corresponding to each terminal vertex, the sum of the degrees of all vertices in G is equal to twice the no. of edges of G.

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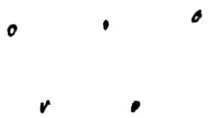
Q1(b) Define Graph. Explain the various types of graph with an example.

Graph: A Graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2 \ldots\}$ called vertices and another set $E = \{e_1, e_2 \ldots e_n\}$ whose elements are called edges such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

Various types of Graph:

1) Null graph: Any graph with edge set empty is called Null graph

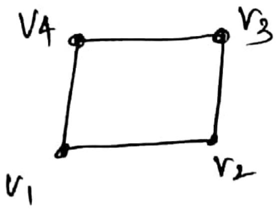
Example:



Null graph.

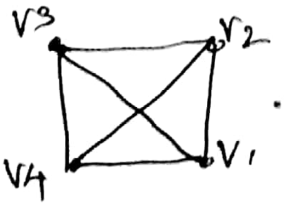
2) Simple graph: A graph in which every vertex has the same degree is called a regular graph

Example:



3) complete graph: A simple graph G with n vertices is said to be a complete graph if the degree of every vertex is n-1

Example



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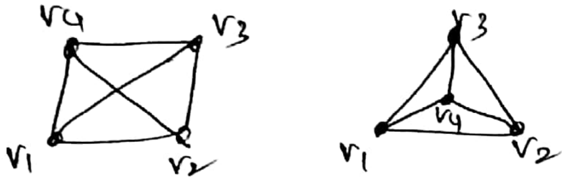
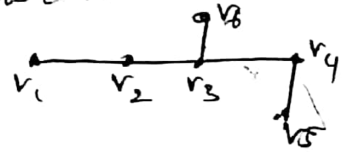
4. Isomorphic graphs:

Two graphs G & G' are said to be isomorphic if there is 1-1 correspondence between their vertices and between their edges such that the incidence relationship is preserved.

example: 

5. connected graph:

A graph G is said to be connected if there is atleast one path between every pair of vertices in G otherwise G is said to be disconnected.

Example: 

22. (a)

Proof: Let p be the no. of pendant vertices in a binary tree T. Then n-p-1 is the no. of vertices of degree three.

$$\therefore \text{no. of edges in } T = \frac{1}{2}\left[p + 3(n-p-1) + 2\right] = n-1$$

$$\Rightarrow \quad p = \frac{n+1}{2}$$

Hence the theorem

(or)

(b) (i) edge connectivity:

Each cut set of a connected graph G consists of a certain no. of edges. The no. of edges in the smallest cut-set (cut-set with fewest no. of edges) is defined as the edge connectivity of G.

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ii) Vertex connectivity : The vertex connectivity of a connected graph G is defined as the minimum no. of vertices whose removal from G leaves the remaining graph disconnected

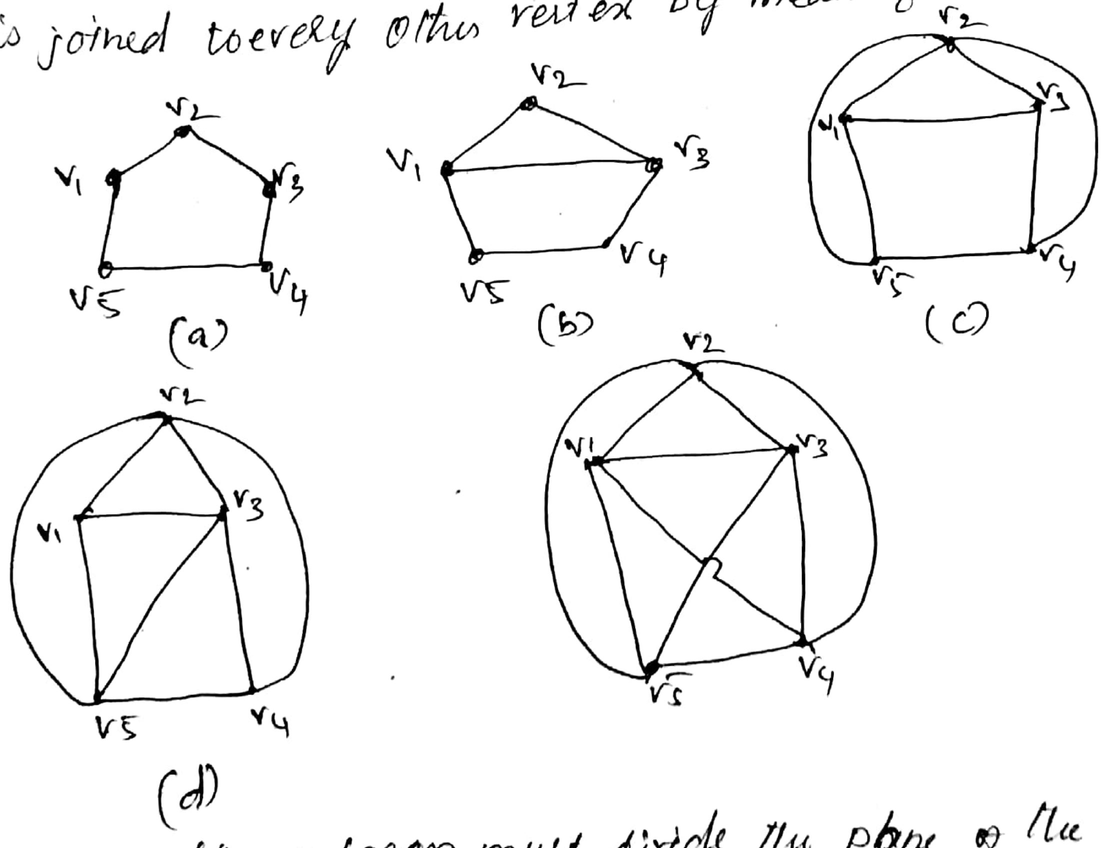
(iii) Minimally connected :

23. (a)

Proof :

Let the five vertices in the complete graph is named as v_1, v_2, v_3, v_4 and v_5.

A complete graph, is a simple graph in which every vertex is joined to every other vertex by means of an edge.



In (a) this pentagon must divide the plane of the paper into 2 regions one inside & the other outside (Jordan curve theorem)

Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon.

Suppose that we choose to draw a line from v_1 to v_3

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inside the pentagon (in b). Now we have to draw an edge ③
from v_2 to v_4 & another one from v_2 to v_5.

Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn we draw both these edges outside the pentagon. The edge connecting v_3 & v_5 cannot be drawn outside the pentagon without crossing the edge b/w v_2 & v_4.

$\therefore$ v_3 & v_5 have to be connected with an edge inside the pentagon. Thus the graph cannot be embedded in the plane.

Hence the theorem.

(or)

(23) (b)

Proof: Since any simple planar graph can have plane representation such that each edge is a straight line.

Any planar graph can be drawn such that each region is a polygon.

Let polygon net representing the given graph consist of f regions or faces and let k_p be the no. of p-side regions.

$\because$ each edge is on the boundary of exactly two regions

$$3 \cdot k_3 + 4 \cdot k_4 + 5 \cdot k_5 + \dots + r \cdot k_r = 2e \quad -①$$

when k_r is the no. of polygons, with maximum edges.

$$\text{Also, } k_3 + k_4 + k_5 + \dots + k_r = f \quad -②$$

Sum of all angles subtended at each vertex in the polygonal net is $2\pi n$ -③

Recalling the sum of all interior angles of p-sided polygon is $\pi(p-2)$
& sum of the exterior angle is $\pi(p+2)$.

$\therefore$ sum is $\pi(3-2) \cdot k_3 + \pi(4-2) k_4 + \dots + \pi(r-2) k_r + 4\pi$

$$= \pi(2e - 2f) + 4\pi \quad -④$$

Equation ③ & ④ we get

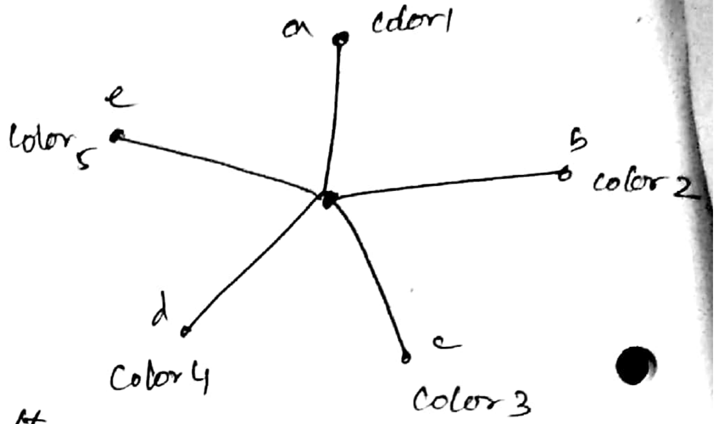
$2\pi(e-f)+4\pi = 2\pi n$

$\Rightarrow e-f+2 = n$

$\therefore$ no. of regions is $f = e-n+2$

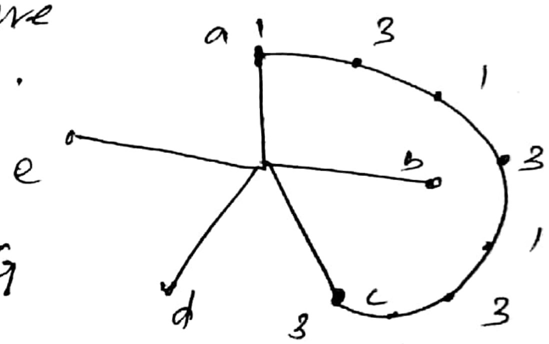
(24) (a)

Proof:



we prove this theorem by induction since all the vertices are properly colored with the five colors.

consider the planar graph G with n vertices. Since G is planar, it must have at least one vertex with degree 5 or less. Let this vertex be v.



Let G' be a graph obtained from G by deleting vertex v. graph G' requires no more than 5 colours, according to induction hypothesis. Suppose that the vertices in G' have been properly colored & now we add to it v and all the edges incident on v. If the degree of v is 1, 2, 3, or 4 we have no difficulty in assigning a proper color to v.

This leaves only the case in which the degree of v is 5 & all the five colors have been used in coloring the vertices adjacent to v.

Suppose that there is a path G' b/w vertices a & c coloured alternately with colors 1 & 3 as shown in fig. Then a similar path b/w b and d, colored alternately with colors 2 & 4 cannot exist otherwise two paths will intersect & G to be non-planar. Hence the thm.

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(or)

Proof:

(b) (c) (d)

Suppose that covering g contains a path of length 3 & it is $v_1 e_1 v_2 e_2 v_3 e_3 v_4$

edge e_2 can be removed without leaving its end vertices v_2 & v_3 uncovered. $\therefore$ g is minimal covering.

Conversely, if a covering g contains no paths of length 3 or more all its components must be a star graph. From a star graph no edge can be removed without leaving a vertex uncovered

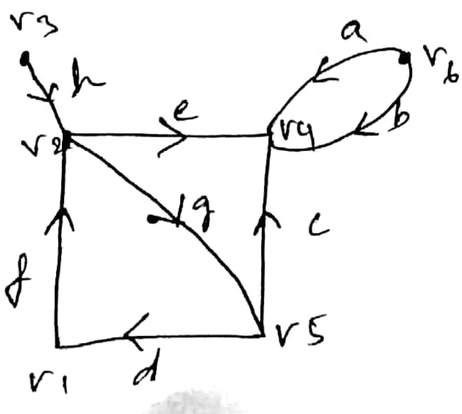
(i.e) g must be minimal covering.

(25)(a)

Circuit matrix of a Digraph: Let G be a digraph with e edges & q circuits. An arbitrary orientation is assigned to each of the q circuits. Then a circuit matrix $B = [b_{ij}]$ of a digraph G

$$b_{ij} = \begin{cases} 1, & \text{if } i\text{th circuit includes } j\text{th edge \& orientation of the edge \& circuit coincide} \\ -1, & \text{if } i\text{th circuit includes } j\text{th edge, but the orientations of the two are opposite} \\ 0, & \text{if } i\text{th circuit does not include the } j\text{th edge.} \end{cases}$$

Example:



circuit matrix

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

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(or)

25(b)

Proof:

Deleting any one row from A, we get A_f, the (n-1) by e reduced incidence matrix. The vertex corresponding to the

The argument leading can be extended to rank of A(G) is n-k if G is a disconnected graph with n vertices & k components.

If we remove any one row from the Incidence matrix of a connected graph, the remaining (n-1) by e submatrix is of rank n-1. In other words, the remaining (n-1) row vectors are linearly independent.

Thus we need only n-1 rows of an incidence matrix to specify the corresponding the graph completely for n-1 rows contain the same amount of information as entire matrix.

Such an (n-1) by e submatrix A_f of A is called a reduced incidence matrix.

∴ a tree is connected graph with n vertices & n-1 edges its reduced incidence matrix is a square matrix of order & rank n-1

Hence the theorem.

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Date: 20 .01.18(FN) **Time: 2 Hours**
Class: III B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)
ANSWER ALL THE QUESTIONS

1. An edge having the same vertex as both its end vertices is called _____________.
   a. graph b. tree **c. self-loop** d. trival
2. A graph with no edges is ___________.
   a. null graph b. trival **c. empty** d. parallel
3. The maximum number of edges in a simple graph with n vertices is __________.
   a. n b. (n-2)/2 c. n-1 **d. (n-1)/2**
4. A vertex of degree zero is called an -------------------------
   a. null vertex b. isolated vertex
   c. pendant vertex d. null graph
5. Vertices with which a walk begins or ends are called its___________.
   a. end vertex b. isolated vertex
   c. pendant vertex **d. terminal vertices**
6. -Edges that have the same end vertices are _______________.
   a. same **b. parallel** c. null graph d.connected
7. A graph with no vertices is a ___________.
   a. null graph b. trival c. empty d. parallel
8. A tree with ________ vertices has at least two vertices of degree 1.
   a. 2 b. 3 **c. n** d. 0
9. A ____________ is connected graph without circuit.
   a. graph b. directed graph
   c. undirected graph **d. tree**
10. The sum of the degrees of all vertices of a graph is equal to ___________ the number of edges.
    a. twice b. thrice c. same d. any
11. A graph is ____________ if it has no parallel edges or self-loops.
    a. simple b. directed c. adjacent d. self-loop
12. A graph in which some edges are directed and some are undirected is called________________.
    a. mixed graph b. regular graph
    c. complete graph d. simple graph
13. Every graph is its own _______________.
    a . mixed graph **b. sub graph**
    c . complete graph d. simple graph
14. A tree in which one vertex is distinguished from all others is called a ___________.
    a. tree **b. rooted tree**
    c. connected d. pendant vertex
15. ____________ is also called cycle.
    a. walk b. closed walk
    c. circuit d. path

16. If no vertex appears more than once in an open walk then it is called a ___________.

a. walk b. closed walk c. circuit **d. path**

17. The number of edges in a path is called the ___________ of the path.

a. length b. same c. walk d. circuit

18. A graph with only one vertex is __________.

a. null graph **b. trivial** c.empty d.parallel

19. A simple graph G with n vertices is said to be a___________ if the degree of every vertex is n-1.

a. null graph b. regular graph

c. complete graph d. simple graph

20.A walk is also called____________.

a. chain b. edge c. vertex d. graph

PART –B(3X10=30 MARKS)

ANSWER ALL THE QUESTIONS

(OR)

21.a) A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

Proof. => Suppose that P is an Eulerian trail of G. Whenever P passes through a vertex, there is a contribution of 2 towards the degree of that vertex. Since each edge occurs exactly once in P, each vertex must have even degree. <= The proof is by induction on the number of edges of G. Suppose that the degree of each vertex is even. Since G is connected, each vertex has degree at least 2 and so, by Lemma 6.1, G contains a cycle C. If C contains every edge of G, the proof is complete. If not, we remove from G the edges of C to form a new, possibly disconnected, graph H with fewer edges than G and in which each vertex still has even degree. By the induction hypothesis, each component of H has an Eulerian trail. Since each component of H has at least one vertex in common with C, by connectedness, we obtain the required Eulerian trail of G by following the edges of C until a non-isolated vertex of H is reached, tracing the Eulerian trail of the component ofH that contains that vertex, and then continuing along the edges of C until we reach a vertex belonging to another component of H, and so on. The whole process terminates when we return to the initial vertex

(OR)

b) Show that a simple graph with n vertices and k-components can have at most $\frac{(n-k)(n-k+1)}{2}$

Proof : We prove the lower bound m>n-kby induction on the number of edges of G, the result being trivial if G is a null graph. If G contains as few edges as possible (say mo), then the removal of any edge of G must increase the number of components by 1, and the graph that remains has n vertices, k + 1 components, and m$ - 1 edges. It follows from the induction hypothesis that m0 - 1 > n - (k + 1), giving m$> n - k, as required.

To prove the upper bound, we can assume that each component of G is a complete graph. Suppose, then, that there are two components Q and Cj with nt and rij vertices, respectively, where rij > rij^ > 1. If we replace C/ and Cj by complete graphs on Uj + 1 and rij - 1 vertices, then the total number of vertices remains unchanged, and the number of edges is changed by {{n^^n^n^-l^ll-injinj^^-inj-l^nj-l^ll^n-nj+l, which is positive. It follows that, in order to attain the maximum number of edges, G must consist of a complete graph onn-k+ 1 vertices and k-\ isolated vertices. The result now follows.

22.a) State and prove the Handshaking theorem.

Proof Consider the incidence matrix M. The sum of the entries in the row corresponding to vertex v is precisely d(v), and therefore L d(v) is just . veV the sum of all entries in M. But this sum is also 2£, since each of the e column sums of M is 2 0 Corollary 1.1 In any graph, the number of vertices of odd degree is even. Proof Let VI and Vz be the sets of vertices of odd and even degree iIi G, respectively. Then L d(v)+ L d(v) = L d(v) veVl veV2 veV is even, by theorem 1.1. Since L d (v) is also even, it follows that L d(v) is veV2 veVt even. Thus IVII is even

(OR)

b) Define graph. Explain the various types of graph with an example.

A graph is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices. E is a multiset, in other words, its elements can occur more than once so that every element has a multiplicity. Often, we label the vertices with letters (for example: a, b, c, . . . or v1, v2, . . .) or numbers 1, 2, . . .

1. The two vertices u and v are end vertices of the edge (u, v).
2. Edges that have the same end vertices are parallel.
3. An edge of the form (v, v) is a loop.
4. A graph is simple if it has no parallel edges or loops.
5. A graph with no edges (i.e. E is empty) is empty.
6. A graph with no vertices (i.e. V and E are empty) is a null graph.
7. A graph with only one vertex is trivial.
8. Edges are adjacent if they share a common end vertex.
9. Two vertices u and v are adjacent if they are connected by an edge, in other words, (u, v) is an edge.
10. The degree of the vertex v, written as d(v), is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately.
11. A pendant vertex is a vertex whose degree is 1.
12. An edge that has a pendant vertex as an end vertex is a pendant edge.
13. An isolated vertex is a vertex whose degree is 0.

23.a) Prove that the number of vertices of odd degree in a graph is always even.

Proof: The graph G = (V, E), where V = {v1, . . . , vn} and E = {e1, . . . , em}, satisfies Xn i=1 d(vi) = 2m.

Corollary. Every graph has an even number of vertices of odd degree.

Proof. If the vertices v1, . . . , vk have odd degrees and the vertices vk+1, . . . , vn have even degrees, then (Theorem 1.1) d(v1) + · · · + d(vk) = 2m − d(vk+1) − · · · − d(vn) is even. Therefore, k is even.

(OR)

b) Prove that the number of pendent vertices of a tree is equal to $\frac{n+1}{2}$

Proof. If a tree has n(≥ 2) vertices, then the sum of the degrees is 2(n − 1). If every vertex has a degree ≥ 2, then the sum will be ≥ 2n (√). On the other hand, if all but one vertex have degree ≥ 2, then the sum would be ≥ 1 + 2(n − 1) = 2n − 1

Reg. No----------------
(15MMU604A)

KARPAGAM UNIVERSITY
(Under Section 3 of UGC Act 1956)
COIMBATORE – 641021
DEPARTMENT OF MATHEMATICS
SIXTH SEMESTER
I INTERNAL TEST- JAN'18
ADVANCED GRAPH THEORY

Date: .01.18( ) **Time: 2 Hours**
Class: III B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)
ANSWER ALL THE QUESTIONS

1. An edge having the same vertex as both its end vertices is called _____________.
   a. graph b. tree **c. self-loop** d. trival
2. A graph with no edges is ___________.
   a. null graph b. trival **c. empty** d. parallel
3. The maximum number of edges in a simple graph with n vertices is __________.
   a. n b. (n-2)/2 c. n-1 **d. (n-1)/2**
4. A vertex of degree zero is called an -------------------------
   a. null vertex b. isolated vertex
   c. pendant vertex d. null graph
5. Vertices with which a walk begins or ends are called its___________.
   a. end vertex b. isolated vertex
   c. pendant vertex **d. terminal vertices**

-

6. Edges that have the same end vertices are _______________.
   a. same **b. parallel** c. null graph d.connected
7. A graph with no vertices is a ___________.
   a. null graph b. trival c. empty d. parallel
8. A tree with _________ vertices has at least two vertices of degree 1.
   a. 2 b. 3 **c. n** d. 0
9. A ____________ is connected graph without circuit.
   a. graph b. directed graph
   c. undirected graph **d. tree**
10. The sum of the degrees of all vertices of a graph is equal to ___________ the number of edges.
   a. twice b. thrice c. same d. any
11. A graph is ____________ if it has no parallel edges or self-loops.
   a. simple b. directed c. adjacent d. self-loop
12. A graph in which some edges are directed and some are undirected is called________________.
   a. mixed graph b. regular graph
   c. complete graph d. simple graph
13. Every graph is its own _______________.
   a . mixed graph **b. sub graph**
   c . complete graph d. simple graph
14. A tree in which one vertex is distinguished from all others is called a ___________.
   a. tree **b. rooted tree**
   c. connected d. pendant vertex
15. ____________ is also called cycle.
   a. walk b. closed walk
   c. circuit d. path

16. If no vertex appears more than once in an open walk then it is called a __________.

a. walk b. closed walk c. circuit **d. path**

17. The number of edges in a path is called the __________ of the path.

a. length b. same c. walk d. circuit

18. A graph with only one vertex is _________.

a. null graph **b. trivial** c.empty d.parallel

19. A simple graph G with n vertices is said to be a__________ if the degree of every vertex is n-1.

a. null graph b. regular graph

c. complete graph d. simple graph

20.A walk is also called___________.

a. chain b. edge c. vertex d. graph

PART –B(3X10=30 MARKS)

ANSWER ALL THE QUESTIONS

21.a) Show that a connected graph G is an Euler graph if and only if the degree of every vertex in G is even.

(OR)

b)Show that a simple graph with n vertices and k-components can have at most $\frac{(n-k)(n-k+1)}{2}$

22.a) State and prove the Handshaking theorem.

(OR)

b) Define graph. Explain the various types of graph with an example.

23.a) Prove that the number of vertices of odd degree in a graph is always even.

(OR)

b) Prove that the number of pendent vertices of a tree is equal to $\frac{n+1}{2}$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: A.NEERAJAH
SUBJECT NAME: Advanced Graph Theory SUB.CODE:15MMU604A
SEMESTER: VI CLASS: III B.SC MATHEMATICS-B

S.No	Lecture Hours (Hr)	Topics to be covered	Support Materials
		Unit-I	
1	1	Undirected graph: Basic concepts	T1: Chapter 1; Pg.no:1-2
2	1	Introduction and Definitions on Graphs	T1: Chapter 1; Pg.no:2-6
3	1	Incidence and Degree of vertices	T1: Chapter 1; Pg.no:7-8
4	1	Isolated vertex pendant vertex	T1: Chapter 1; Pg.no:8-9
5	1	Path and Circuits: Isomorphism	T1: Chapter 2; Pg.no:14-16
6	1	Sub graphs	T1: Chapter 2; Pg.no:16-17
7	1	Walks, Paths and Circuits	T1: Chapter 2; Pg.no:19-21
8	1	Continuation on Walks, Paths and Circuits	R3: Chapter 2, Pg.no:28
9	1	Connected graphs and concepts	T1: Chapter 2; Pg.no:21-22
10	1	Continuation on Connected graphs and concepts	T1: Chapter 2; Pg.no:22-23
11	1	Euler graphs	T1: Chapter 2; Pg.no:23-24
12	1	Continuation on Euler graphs	T1: Chapter 2; Pg.no:24-26
13	1	Hamilton graph	T1: Chapter 2; Pg.no:30-32
14	1	Complete graph	T1: Chapter 2; Pg.no:32-34
15	1	Traveling Salesman problem.	T1: Chapter 2; Pg.no:34-35
16	1	Recapitulation and discussion of Important questions	
Total	**16 Hrs**		
		Unit-II	
1	1	Trees:Introduction and Definitions	T1: Chapter 3; Pg.no:39-41
2	1	Theorems on some properties of trees	T1: Chapter 3; Pg.no:41-45

3	1	Continuation on Theorems on some properties of trees	T1: Chapter 3; Pg.no:45-48
4	1	Rooted and Binary trees	T1: Chapter 3; Pg.no:48-51
5	1	Continuation on Rooted and Binary trees	T1: Chapter 3; Pg.no:52-55
6	1	Spanning trees	T1: Chapter 3; Pg.no:55-58
7	1	Continuation on Spanning trees	T1: Chapter 3; Pg.no:58-60
8	1	Continuation on Spanning trees	T1: Chapter 3; Pg.no:61-63
9	1	Cut set and cut vertices: Introduction and definitions	T1: Chapter 4; Pg.no:68-69
10	1	some properties of a cut set	T1: Chapter 4; Pg.no:69-71
11	1	All cut sets in a graph- Theorems	T1: Chapter 4; Pg.no:71-73
12	1	Fundamental circuits and cut sets	T1: Chapter 4; Pg.no:73-74
13	1	Continuation on Fundamental circuits and cut sets	T1: Chapter 4; Pg.no:74-75
14	1	Connectivity and Separability – Theorems	T1: Chapter 4; Pg.no:75-77
15	1	Continuation on Theorems on Connectivity and Separability	T1: Chapter 4; Pg.no:77-79
16	1	Recapitulation and discussion of Important questions	
Total	**16 Hrs**		

Unit-III			
1	1	Planar graphs: Introduction and Definitions	T1: Chapter5; Pg.no:90
2	1	Theorems on Planar graphs	R1: Chapter11;Pg.no:102-106
3	1	Theorems on Kuratowski’ two graphs	T1: Chapter5; Pg.no:90-93
4	1	Continuation on Theorems on Kuratowski’ two graphs	R1: Chapter11;Pg.no:108-112
5	1	Different representation of a planar graph	T1: Chapter5; Pg.no:93-96
6	1	Continuation on Different representation of a planar graph	T1: Chapter5; Pg.no:97-99
7	1	Detection of planarity	T1: Chapter5; Pg.no:99-103
8	1	Continuation on Detection of planarity	T1: Chapter5; Pg.no:104-108
9	1	Thickness and crossings	T1: Chapter5; Pg.no:108-109
10	1	Recapitulation and discussion of important questions	
Total	**10 Hrs**		
Unit-IV			
1	1	Colorings ,Covering and	T1: Chapter8; Pg.no:165

		partitioning: Introduction and Definitions	R1: Chapter9;Pg.no:244-245
2	1	Chromatic number Theorems	T1: Chapter8; Pg.no:165-168
3	1	Chromatic partitioning	T1: Chapter8; Pg.no:169
4	1	Independent set	T1: Chapter8; Pg.no:170
5	1	Finding a maximal independent set	T1: Chapter8; Pg.no:170-171
6	1	Dominating set	T1: Chapter8; Pg.no:171
7	1	Finding minimal dominating set	T1: Chapter8; Pg.no:171-173
8	1	Chromatic polynomial	T1: Chapter8; Pg.no:174-177
9	1	Coverings: Introduction and Definitions	T1: Chapter8; Pg.no:182-183
10	1	Theorems on Coverings	T1: Chapter8; Pg.no:184-186
11	1	Four colour problem	R2: Chapter11;Pg.no:287-289
12	1	Five colour Theorem.	R1: Chapter12;Pg.no:131
13	1	Recapitulation and discussion of important questions	
Total	**13Hrs**		
		Unit-V	
1	1	Directed graph: Introduction and Definitions	T1: Chapter9; Pg.no:194-197
2	1	Some types of di-graphs	T1: Chapter9; Pg.no:197-199
3	1	Continuation on types of di-graphs	T1: Chapter9; Pg.no:199-201
4	1	Directed path and connectedness	T1: Chapter9; Pg.no:201-203
5	1	Euler di-graphs	T1: Chapter9; Pg.no:203-204
6	1	Continuation on Euler di-graphs	T1: Chapter9; Pg.no:204-206
7	1	Trees with direct edges	T1: Chapter9; Pg.no:206-207
8	1	Continuation on Trees with direct edges	T1: Chapter9; Pg.no:207-209
9	1	Ordered trees	T1: Chapter9; Pg.no:209-211
10	1	Matrix representation	T1: Chapter9; Pg.no:213 R1: Chapter13;Pg.no:150
11	1	Incidence matrix	T1: Chapter9; Pg.no:214-216
12	1	Circuit matrix	T1: Chapter9; Pg.no:216-220
13	1	Adjacency matrix	T1: Chapter9; Pg.no:220-222
14	1	Continuation on Adjacency matrix	T1: Chapter9; Pg.no:223-227
15	1	Tournaments.	T1: Chapter9; Pg.no:227-228
16	1	Continuation on Tournaments.	T1: Chapter9; Pg.no:228-230
17	1	Recapitulation and discussion of important questions	
18	1	Discussion of previous ESE question papers	
19	1	Discussion of previous ESE	

		question papers	
20	1	Discussion of previous ESE question papers	
Total	**20 Hrs**		

TEXT BOOK
T1.Narsingh Deo., 2007. Graph Theory with Applications to Engineering and Computer Science, Prentice Hall of India Pvt. Ltd, New Delhi.
REFERENCES
R1. Harary F., 1969. Graph Theory, Addision-Wesley publishing company,Inc., Amsterdam.
R2. Bondy.J.A., and U.S.R.Murty., 2008. Graph theory and applications, Springer.
R3. Balakrishnan, 2011, Graph theory, Springer publications.
R4. West D.B., 2011. Introduction to Graph Theory, Prentice Hall, New Delhi.

Hall of India, New Delhi.

Reg No ------------

[15MMU604A]

KARPAGAM ACADEMY OF HIGHER EDUCTION
COIMBATORE – 641021
DEPARTMENT OF MATHEMATICS
MODEL EXAMINATION- MARCH 2018
Sixth Semester
Elective-II:Advanced Graph Theory

Date: -03-18( ) **Time: 3 Hours**
Class:III B.ScMathematics-B **Maximum: 60 Marks**

PART – A (20 x 1 = 20 marks)
ANSWER ALL THE QUESTIONS

1. A vertex of degree one is ____
   a) pendant vertex b) isolated vertex
   c) null graph d) regular graph
2. A graph with n vertices and deg(v) = n - 1 equal for all vertices is_________
   a) regular b) null
   c) complete d) disconnected
3. A graph must have atleast ____ vertex.
   a) 1 b) 2 c) 0 d) 3
4. Every vertex in a null graph is an _______
   a) isolated vertex b) pendant vertex
   c) complete graph d) null graph
5. An edge in a spanning tree is called
   a) pendant b) branch
   c) chord d) root
6. Rank + nullity = number of ______in a graph
   a) edges b) vertices
   c) cycles d) odd vertices
7. A vertex connectivity of a tree is____
   a) 1 b) 0 c) 2 d) 4
8. In a degree constrained shortest spanning tree, deg(G) ≤
   a) 3 b) 1 c) 4 d) 2
9. Every circuit has an _____number of edges in common with any cut set
   a) Eve n b) odd c) 3 d) zero
10. Cover of a graph is a subset of _______
    a) vertices b) edges
    c) both vertices and edges d) neither edge nor vertex
11. Parallel edges produce identical columns in the ____ matrix.
    a) cut set b) path
    c) incidence d) adjacency
12. A graph consisting of one circuit with $n \geq 3$ vertices is 2-chromatic if n is _____
    a) even b) odd c) degree d) link
13. A ____ in a graph is a subset of edges in which no two edges are adjacent.
    a) matching b) covering
    b) chromatic d) chromatic partition
14. A column of all zeros corresponds to a non circuit edge is ____
    a) circuit matrix b) column matrix
    c) path matrix d) adjacency matrix
15. Every bipartite graph is ---------chromatic
    a) 2 b)3 c) 1 d) 4
16. A digraph that has no self loop or parallel edges is _____

a) simple b) symmetric
c) complete d) asymmetric

17. A balanced digraph is _____
a) isograph b) simple graph
c) complete digraph d) anti symmetric

18. The number of vertices in the largest ______ set of a graph
a) Independent b) dominating set
c) number d) digraph

19. The minimum cardinality of a total dominating set is ____
a) domination number b) dominating set
c) independent set d) independent number

20. A set of vertices in a graph is independent set if no two vertices in the set are ____
a) adjacent b) independent
c) dominate d) tree

PART – B (5 x 8 = 40 marks)

ALL THE QUESTIONS CARRY EQUAL MARKS

21. a) Define (i) Bipartite Graph

(ii) Regular Graph

(iii) Complete Graph.

Give an example for each.

(OR)

b) Prove that in a complete graph with n vertices there are $\frac{n-1}{2}$ edge disjoint Hamiltonian circuits if n is an odd number$\geq$ 3.

22. a) Define i) distance between two spanning trees
ii) cyclic interchange
iii) rank
iv) nullity

(OR)

b) Prove that the ring sum of any two cut sets in a graph is either a third cut set or an edge disjoint union of cut sets.

23. a) Show that if G is connected simple planar graph with n($\geq$3) vertices and e is edge then $e \leq 3n-6$.

(OR)

b) Prove that the vertices of every planar graph can be properly colored with five colors

24. a) Define chromatic number. Find the chromatic polynomial for the cycle of length 4. Hence find its chromatic number.

(OR)

b) Show that every tree with two or more vertices is 2-chromatic.

25. a) Discuss about the digraph.

(OR)

b) Discuss about the binary relations in a digraph.

Reg No ------------

[15MMU604A]

KARPAGAM ACADEMY OF HIGHER EDUCTION
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DEPARTMENT OF MATHEMATICS
MODEL EXAMINATION- MARCH 2018
Sixth Semester
Elective-II:Advanced Graph Theory

Date: -03-18( ) **Time: 3 Hours**
Class:III B.ScMathematics-B **Maximum: 60 Marks**

PART – A (20 x 1 = 20 marks)
ANSWER ALL THE QUESTIONS

1. A vertex of degree one is ____
   a) **pendant vertex** b) isolated vertex
   c) null graph d) regular graph
2. A graph with n vertices and deg(v) = n - 1 equal for all vertices is_________
   a) **regular** b) null
   c) complete d) disconnected
3. A graph must have atleast ____ vertex.
   a) **1** b) 2 c) 0 d) 3
4. Every vertex in a null graph is an _______
   a) **isolated vertex** b) pendant vertex
   c) complete graph d) null graph
5. An edge in a spanning tree is called
   a) pendant b**) branch**
   c) chord d) root
6. Rank + nullity = number of ______in a graph
   a) **edges** b) vertices
   c) cycles d) odd vertices
7. A vertex connectivity of a tree is____
   a) **1** b) 0 c) 2 d) 4
8. In a degree constrained shortest spanning tree, deg(G) ≤
   a) **3** b) 1 c) 4 d) 2
9. Every circuit has an _____number of edges in common with any cut set
   a) **Eve n** b) odd c) 3 d) zero
10. Cover of a graph is a subset of _______
    a) **vertices** b) edges
    c) both vertices and edges d) neither edge nor vertex
11. Parallel edges produce identical columns in the ____ matrix.
    a) **cut set** b) path
    c) incidence d) adjacency
12. A graph consisting of one circuit with $n \geq 3$ vertices is 2-chromatic if n is _____
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13. A ____ in a graph is a subset of edges in which no two edges are adjacent.
    a) **matching** b) covering
    b) chromatic d) chromatic partition
14. A column of all zeros corresponds to a non circuit edge is ____
    a) **circuit matrix** b) column matrix
    c) path matrix d) adjacency matrix
15. Every bipartite graph is ---------chromatic
    a) **2** b)3 c) 1 d) 4

16. A digraph that has no self loop or parallel edges is _____

a) **simple** b) symmetric

c) complete d) asymmetric

17. A balanced digraph is _____

a) **isograph** b) simple graph

c) complete digraph d) anti symmetric

18. The number of vertices in the largest ______ set of a graph

a) **Independent** b) dominating set

c) number d) digraph

19. The minimum cardinality of a total dominating set is ____

a) **domination number** b) dominating set

c) independent set d) independent number

20. A set of vertices in a graph is independent set if no two vertices in the set are ____

a) **adjacent** b) independent

c) dominate d) tree

PART – B (5 x 8 = 40 marks)

ALL THE QUESTIONS CARRY EQUAL MARKS

21. a) Define (i) Bipartite Graph

A graph can be divided

(ii) Regular Graph

A graph in which all vertices are of equal degree, is called a regular graph. If the degree of each vertex is r, then the graph is called a regular graph of degree r

(iii) Complete Graph.

A simple graph G is said to be complete if every vertex in G is connected with every other vertex. i.e., if G contains exactly one edge between each pair of distinct vertices.

Give an example for each.

(OR)

b) Prove that in a complete graph with n vertices there are $\frac{n-1}{2}$ edge disjoint Hamiltonian circuits if n is an odd number$\geq$ 3.

Proof : A complete graph with n vertices has (1) 2 n n − edges, and a hamiltonian circuit consists of n edges. Therefore, the number of edge-disjoint hamiltonian circuits in G cannot exceed (1) 2 n − . This implies there are 1 2 n − edge-disjoint hamiltonian circuits, when n is odd it can be shown asby keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by360 (1 n −) , 2.360 (1 n −) , 3.360 (1 n −) , ..., 3 2 n − . 360 (1 n −) degrees. At each rotation we get a hamiltonian circuit that has no edge in common with any of the previous ones. Thus we have 3 2 n − new hamiltonian circuits, all edges disjoint from one and also edge disjoint among themselves. Hence the proof.

22. a) Define i) distance between two spanning trees

In a connected graph G, the distance between the vertices u and v, denoted by d(u, v) is the length of the shortest path.

ii) cyclic interchange

iii) rank

iv) nullity

For a graph G with n vertices, m edges and k components we define the rank of G and is denoted by $\rho(G)$ and the nullity of G is denoted by $\mu(G)$ as follows. $\rho(G)$ = Rank of G = $n - k$ $\mu(G)$ = Nullity of G = $m - \rho(G) = m - n + k$ If G is connected, then we have $\rho(G) = n - 1$ and $\mu(G) = m - n + 1$.

(OR)

b) Prove that the ring sum of any two cut sets in a graph is either a third cut set or an edge disjoint union of cut sets.

23. a) Show that if G is connected simple planar graph with $n(\geq 3)$ vertices and e is edge then $e \leq 3n-6$.

Proof. Each region is bounded by atleast three edges (since the graphs discussed here are simple graphs, no multiple edges that could produce regions of degree 2 or loops that could produce regions of degree 1, are permitted) and edges belong to exactly two regions. $2e \geq 3r$ If we combine this with Euler's formula, $n - e + r = 2$, we get $3r = 6 - 3n + 3e \leq 2e$ which is equivalent to $e \leq 3n - 6$.

(OR)

b) Prove that the vertices of every planar graph can be properly colored with five colors

Proof. We proceed by induction on the number P of points. For any planar graph having $P \leq 5$ points, the result follows trivially since the graph is P-colorable. As the inductive hypothesis we assume that all planar graphs with P points, $P \geq 5$, are 5-colourable. Let G be a plane graph with $P + 1$ vertices, G contains a vertex v of degree 5 or less. By hypothesis, the plane graph $G - v$ is 5-colourable. Consider an assignment of colours to the vertices of $G - v$ so that a 5-colouring results, when the colours are denoted by C_i, $1 \leq i \leq 5$. Certainly, if some colour, say C_j, is not used in the colouring of the vertices adjacent with v, then by assigning the colour C_j to v, a 5-colouring of G results. This leaves only the case to consider in which deg $v = 5$ and five colours are used for the vertices of G adjacent with v. Permute the colours, if necessary, so that the vertices coloured C1, C2, C3, C4 and C5 are arranged cyclically about v, Now label the vertex adjacent with v and coloured C_i by v_i, $1 \leq i \leq 5$ (see Figure 2.100) Fig. 2.100. A step in the proof of the five colour theorem. Let G13 denote the subgraph of $G - v$ induced by those vertices coloured C1 or C3. If v1 and v3 belong to different components of G13, then a 5-coloring of $G - v$ may be accomplished by interchanging the colors of the vertices in the component of G13 containing v1. In this 5-coloring however, no vertex adjacent with v is colored C1, so by coloring v with the color C1, a 5-coloring of G results. If, on the other hand, v1 and v3 belong to the same component of G13, then there exists in G a path between v1 and v3 all of whose vertices are colored C1 or C3. This path together with the path v1 vv3 produces a cycle which necessarily encloses the vertex v2 or both the vertices v4 and v5. In any case, there exists no path joining v2 and v4, all of whose vertices are coloured C2 or C4. Hence, if we let G24 denote the subgraph of $G - v$ induced by the vertices coloured C2 or C4, then v2 and v4 belong to different components of G24. Thus if we interchange colors of the vertices in the component of G24 containing v2, a 5-colouring of G – v is produced in which no vertex adjacent with v is coloured C2. We may then obtain a 5-coloring of G by assigning to v the colour C2.

24. a) Define chromatic number. Find the chromatic polynomial for the cycle of length 4. Hence find its chromatic number.

Find the chromatic polynomial and chromatic number for the graph K3, 3. Solution. Chromatic polynomial for K3, 3 is given by $\lambda(\lambda - 1)5$. Thus chromatic number of this graph is 2. Since $\lambda(\lambda - 1)5 > 0$ first when $\lambda = 2$. Here, only two distinct colours are required to colour K3, 3. The vertices a, b and c may have one colours, as they are not adjacent. Similarly, vertices d, e and f can be coloured in proper way using one colour. But a vertex from {a, b, c} and a vertex from {d, e, f} both cannot have the same colour. In fact every chromatic number of any bipartite graph is always 2

(OR)

b) Show that every tree with two or more vertices is 2-chromatic.

Proof. Since Tree T is a bipartite graph. The vertex set V of G can be partitioned into two subsets V1 and V2 such that no two vertices of the set V1 are adjacent and two vertices of the set V2 are adjacent. Now colour the vertices of the set V1 by the colour 1 and the vertices of the set V2 by the colour 2. This colouring is a proper colouring. Hence, chromatic number of G ≤ 2, and since T contains atleast one edge chromatic number of G ≥ 2. Thus, chromatic number of G is 2.

25. a) Discuss about the digraph.

A digraph D consists of a finite set V of points and a collection of ordered pairs of distinct points. Any such pair (u, v) is called an arc or directed line and will usually be denoted uv. The arc uv goes from u to v and is incident with u and v. We also say that u is adjacent to v and v is adjacent from u. The outdegree od(r) of a point v is the number ofpoints adjacent from it, and the indegree id(t') is the number adjacent to it. A (directed) walk in a digraph is an alternating sequence of points and arcs, VQ, .Y,, r,, • • •, x,,, v,, in which each arc x, is r,_ ,t,. The length of such a walk is n, the number of occurrences of arcs in it. A closed walk has the same first and last points, and a spanning walk contains all the points. A path is a walk in which all points are distinct; a cycle is a nontrivial closed walk with all points distinct (except the first and last). If there is a path fromM to v. then v is said to be reachablefrom u, ano the distance, d{u, r), from u to v is the length of any shortest such path. Each walk is directed from the first point p0 to the last v,,. We also need a concept which does not have this property of direction and is analogous to a walk in a graph. A semiwalk is again an alternating sequence r0, x,, r,. • • •, xm v,, of points and arcs, but each arc x, may be either Oj_1»i or r,t', _,. A semipath, semicycle, and so forth, are defined as expected. Whereas a graph is either connected or it is not, there are three different ways in which a digraph may be connected, and each has its own idiosyncrasies. A digraph is strongly connected, or strong, if every two points are mutually reachable; it is unila'erally connected, or unilateral, if for any two points at least one is reachable from the other; and it is weakly connected, or weak, if every two points are joined by a semipath. Clearly, every strong digraph is unilateral and every unilateral digraph is weak, but the converse statements are not true. A digraph is disconnected if it is not even weak. We note that the trivial digraph, consisting of exactly one point, is (vacuously) strong since it does not contain two distinct points. We may now state necessary and sufficient conditions for a digraph to satisfy each of the three kinds of connectedness.

(OR)

b) Discuss about the binary relations in a digraph.

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Subject: Advanced Graph Theory **Subject Code: 15MMU604A**

Class : III - B.Sc. Mathematics-B **Semester : VI**

UNIT -I Undirected graph

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The degree of the leaf is	1	2	n	n-1	1
A graph in which all vertices are of equal degree is	complete graph	regular graph	null graph	both complete and	complete graph
A graph is a finite number of vertices and finite number of edges	finite graph	star graph	isolated graph	infinite graph	finite graph
A isolated vertex having no ____	incident edges	edges	series	adjancent edges	incident edge
Every edge of a ___ is cutset	tree	graph	incident edges	adjacent edge	tree
The degree of every vertex n-1 is	complete graph	regular graph	null graph	subgraph	complete graph
A pendent vertex of degree is	1	2	3		1

A regular graph with n vertices and their degree is	n-1	n-2	n+1	n+2	n-1
Atleast one vertex is	graph	incident vertex	degree	pendent	graph
Isolated vertex is	null graph	pendent graph	complete graph	regular graph	null graph
A null graph containing only	isolated vertex	regular graph	complete graph	simple graph	isolated vertex
All the edges of a graph is	euler line	euler edge	euler graph	euler trail	euler line
G is a subgraph of G then	G-g	G∩g	G+g	G/g	G-g
A connected graph G is	Hamiltonian circuit	hamiltonian graph	hamiltonian path	circuit	hamiltonian circuit
The number of edges incident on a vertex with self-loop counted	degree	adjacent	link	block	degree
In any tree there __ two pendent vertices	atleast	atmost	some	sum of	atleast
The length of a hamiltonian path of a ____ with n vertices n-1	connected graph	star graph	simple graph	complete graph	connected graph
A valency is degree of	vertex	edges	series	link	vertex
Two adjacent edges are series if their common vertex is of degree	one	two	three	zero	two
A single vertex in a graph G is	subgraph	regular graph	component	series	subgraph

A walk is ___ alternating sequence of vertices and edges beginnir	finite	infinte	atmost	some of	finite
Each connected subgraph is ____	component	star graph	series	link	component
A complete graph G is an Euler graph only if the number of vertices is	even	odd	2	6	odd
Euler line contains all the −−−−−−−−of a graph	vertices	edges	isolated vertices	pendant vertices	edges
Euler graphs do not have	even vertices	odd vertices	isolated vertices	pendant vertices	isolated vertices
$If\ G\ is\ a\ star\ with\ n\ vertices\ then\ \Delta(G) =$	n	n-1	$\frac{n}{2}$	$\frac{n-1}{2}$	n-1
$If\ G\ is\ a\ star\ with\ n\ vertices\ then\ \delta(G) =$	n	n-1	$\frac{n}{2}$	1	1
$If\ G\ is\ a\ star\ with\ n\ vertices\ then\ number\ of\ vertices\ with\ degree\ 1 =$	n	n-1	$\frac{n}{2}$	1	n-1
$G \oplus G =$	null graph	star graph	K_2	K_3	null graph
$If\ g\ is\ a\ subgraph\ of\ G,\ G \oplus g =$	$G \cup g$	$G \cap g$	$G - g$	G	$G - g$
$A\ graph\ G\ is\ said\ to\ decomposed\ into\ two\ subgraphs\ g_1\ and\ g_2\ then$	$g_1 \cup g_2 = g_1$	$g_1 \cup g_2 = g_2$	$g_1 \cup g_2 = G$	$g_1 \cap g_2 = G$	$g_1 \cup g_2 = G$
$A\ graph\ G\ is\ said\ to\ decomposed\ into\ two\ subgraphs\ g_1\ and\ g_2\ then$	$g_1 \cap g_2 = a\ null\ graph$	$g_1 \cup g_2 = g_1$	$g_1 \cup g_2 = g_2$	$g_1 \cap g_2 = G$	$g_1 \cap g_2 = a\ null\ graph$
$A\ graph\ containing\ m\ edges\ can\ be\ decomposed\ into ----$ $--\ different\ ways\ into\ pairs\ of\ subgraphs\ g_1\ and\ g_2$	2^m	2^{m-1}	2^{m+1}	2^{m+2}	2^{m-1}

If e is edge of an graph G then $G - e =$	$G \oplus e$	$G \cup e$	$G \cap e$	G	$G \oplus e$
A Hamiltonian circuit in a graph of n vertices consists of	n edges	$n-1$ edges	$n-2$ edges	$n-3$ edges	n edges
If G is an Euler graph then G	is connected	is not connected	with 2 componen	with pendeant vert	is connected
If G has an Hamiltonian circuit then G	is connected	is not connected	with 2 componen	with pendeant vert	is connected
Length of a Hamiltonian path of a connected graph with n vertices is	n	n-1	$\frac{n}{2}$	$\frac{n-1}{2}$	n-1
A graph is a infinte number of vertices and infinite number of edg	infinite graph	finite graph	link	regular graph	infinite graph
A graph with n vertices is a tree if	G is connected	G has $n-1$ edges	G is not connected	G is circuitless and has $n-1$ edges	G is circuitless and has $n-1$ edges
In any tree there are -------- two pendant vertices	0	1	2	3	1
Distance between any two vertices is	pendant vertex	isolated vertex	centre	odd vertex	centre
Number of circuits in a tree is	0	1	2	3	1
Distance between any two vertices in a complete graph is	n	n-1	n-2	n-3	n

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Class : III - B.Sc. Mathematics-B **Semester : VI**

UNIT -II Trees

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A tree T is said to be a spanning tree of G if T contains	all vertices of G	all edges of G	some vertices of G	some edges of G	all vertices of G
Spanning tree defined only for a	complete graph	connected graph	disconnected graph	star graph	connected graph
A disconnected graph with kcomponents has ------- spanning trees	k-3	k-1	k-2	k	k
A circuit free graph which contains all the vertices of G is a	tree	spanning tree	star graph	complete graph	spanning tree
A skeleton of a graph is called as	tree	spanning tree	star graph	complete graph	spanning tree
Suppose G is a graph with n vertices and T is a spanning tree of G. Then number of branches in T is	n	n-1	n-2	n-3	n-1
Number of chords for a complete graph is	4851	4850	4852	4853	4851
Suppose k is denoted as the number of components of G. Then G is connected if	k=0	k=1	k=2	k=3	k=1

Suppose G is a graph with n vertices and k is denoted as the number of components of G. Then rank of G =	n-k	n+k	n/k	n	n-k	
Suppose G is a graph with n vertices, e edges and k is denoted as the number of components of G. Then the nullity of G =	e-n+k	e+n+k	e+n	e-k	e-n+k	
$Rank\ of\ G =$	number of branches	number of chords	number of edges	number of vertices	number of branches	
$Nullity\ of\ G =$	number of branches	number of chords	number of edges	number of vertices	number of chords	
$Rank\ of\ G + nullity\ of\ G =$	number of branches	number of chords	number of edges	number of vertices	number of edges	
A connected graph is a tree if adding an edge between any 2 vertices in G creates ---- circuit	exactly one	atmost one	atleast one	no	exactly one	
Creating a circuit by adding any one chord to T is called	cycle	fundamental circuit	elementary circuit	circuit	fundamental circuit	
Distance between two spanning trees T_i and T_j is the number of edges present in	T_i	T_j	T_i not in T_j	T_i and T_j	T_i not in T_j	
Distance between two spanning trees T_i and T_j =	$\frac{1}{2}N(T_i \oplus T_j)$	$\frac{1}{2}N(T_i \cup T_j)$	$\frac{1}{2}N(T_i \cap T_j)$	$\frac{1}{2}N(T_i - T_j)$	$\frac{1}{2}N(T_i \oplus T_j)$	
If δ(G) and Δ(G) are the minimum and maximum degree in a graph G then the edge connectivity of G is	δ(G)	Δ(G)	2		δ(G)	
The number of branches in any ____ of G is rank	spanning tree	tree	shortest spanning tree	minimal spanning tree	spanning tree	
Weight of a spanning tree T is	sum of weights of all branches of G	sum of weights of all branches of T	sum of weights of all edges of G	sum of weights of all edges of T	sum of weights of all branches of T	
In a graph of n vertices in which every edge has unit weight, then spanning tree T has ------ weight	n	n-1	n-2	n-3	n-1	
In a graph of n vertices in which every edge has 3 unit weight, then spanning tree T has ------ weight	3n	3(n-1)	3(n-2)	3(n-3)	3(n-1)	

A graph in which all nodes are of equal degree is called	complete graph	regular graph	null graph	multi graph	regular graph	
Two ismoephic graphs must have	Equal number of vertices	equal number of edges	an equal number of vertices with a given degree	all of the above	all of the above	
In a separable graph, a vertex whose removal disconnects the graph _____	cut vertex	cut edge	*odd vertex*	*even vertex*	*every edge*	
– – – – – – of a star is a cut set	*every vertex*	*every edge*	*odd vertex*	*even vertex*	cut- vertex	
Edge connectivity of K_2 *is*	1	2	3	4	1	
Each of the largest ___ subgraph is block	nonseparable1	separable	tree	cut-set	nonseparable	
Edge connectivity of a tree is	1	2	3	4	1	
Edge connectivity of a star graph is	1	2	3	4	1	
A separable graph consists of two or more non separable _____	subgraph	tree	spanning tree	complete graph	subgraph	
The ring sum of two cut set is	cut set	not cut set	may cut set	empty set	cut set	
The edge connectivity of a connected graph is minimum number of edges removal reduces the rank of by -----	4	3	2	1	1	
The vertex connectivity of a tree is	4	3	2	1	1	
A graph is planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its ____ intersect	edges	vertices	link	block	edges	
The vertex connectivity of a star graph is	1	2	3	4	1	

Every cut-set in a nonseparable graph with more than two vertices contains ___ two edges.	atleast	atmost	exactly	graph	atleast	
Any edge which is not spanning tree is ______	branch	chord	tree	rank	chord	
$In\ a\ tree, v\ is\ \ cut\ vertex\ if\ \deg(v)$	≥ 1		< 1	> 1		
A tree in which ___ vertex is distinguished from all others is called rooted tree	1	3	2	4	1	
A connected ___ graph with n vertices and e edges has e - n + 2 regions	planar	non planar	complete graph	cut-set	planar	
The distance between____ of a connected graph is eccentricity	edges	vertices	self loop	loop	vertices	
in a ___ graph, any minimal set of edges containing atleast one branch of every spanning tree is cut-set	connected graph	disconnected graph	complete graph	tree	connected graph	
$K_n\ \ is\ planar\ \ for\ n$	4	5	6	7	4	
diameter is length of the longest path in the _____	tree	spanning tree	shortest spanning tree	euler graph	tree	
In a degree constrained shortest spanning tree deg(G)≤	3	4	5	2	3	
Every circuit has an ___ number of edges in common with any cut set	even	odd	zero	three	even	
A ___ is separable if its vertex connectivity is one.	connected graph	simple graph	planar graph	non planar graph	connected graph	
A ______is a connected graph without any circuit.	tree	spanning tree	weighted spanning tree	hamiltonian circuit	tree	
Any connected graph with n vertices and n-1 edges is _____	tree	spanning tree	fundamental circuit	fundamental circuit	tree	

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UNIT -III Planar graphs

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Every ___ edge in a graph G is included in every covering of G.	pendant	isolated	link	block	pendant
The complete graph of 5 vertices is _______	planar	nonplanar	embedding	complete	nonplanar
The number of ----- in each row is the degree of the corresponding vertex	1	0	2	3	1
A row o's in the incidence matrix represents	an isolated vertex	pendeant vertex	odd vetrex	even vertex	an isolated vertex
Every degree of a vertex v equals the number of ____ in the corresponding	circuit	vertex	edge	singular	circuit
If G is a graph with n vertices then rank of $A(G) =$	n	n-1	n-2	n-3	n-1
____ produce identical columns in the cut set matrix	parallel edges	parallel vertex	vertex	edge	parallel edge
Cover of a graph is a _____ of vertices	subset	set	matrix	singular	subset
The incidence matrix of a graph G is	square matrix	rectangular matrix	column matrix	row matrix	rectangular matrix

If G is a tree then A(G) is	square matrix	rectangular matrix	column matrix	row matrix	square matrix
If G is a tree with n vertices then order of A(G) =	n	n-1	n-2	n-3	n-1
The reduced incidence matrix of a tree is	singular	nonsingular	cannot be determi	of 1 determinant	nonsingular
A matching in a graph is a subset of edges in which no ___ edges are adjac	2	4	3	1	2
Every ____ is 2 - chromatic	bipartite graph	null graph	simple graph	complete graphs	bipartite graph
If A(G) is the adjacency matrix of a graph with 0's in ____ then G is comp	diagonal	non diagonal	matrix	tree	diagonal
A column of all ___ corresponds to a non circuit edge is circuit matrix	0's	1's	n	n+1	0's
Every degree of a vertex v equals the number of ____ in the corresponding	1's	0's	diagonal	matrix	1's
Suppose $A(G) = I_n$*, the identity matrix with order n. Then G is*	connected	disconnected	simple graph	complete	disconnected
X(G) =In, identity matrix if G has _____ and disconnected with k=n	self loop	connected	loop	link	self loop
Suppose G is complete graph wiht n vertices.Then number of rows in A(G) with exaclty one 0 is	n	n-1	0	1	n
Suppose G is complete graph wiht n vertices. Then the mnain diagonal element of A(G) is	1	0	0 or 1	2	0
A column of B(G) of all zeros corresponds to a noncircuit − − − − − − −	edge	vertex	both vertex and ed	neither edge nor ve	edge
The incidence matrix A(G) every column has ____ two 1's	atmost	atleast	exactly	more than	exactly
The number of 1's in a row of B(G) =	number of vertices i	number of edges i	number of odd ver	number of even ve	number of edges

The ______ matrix two elements 0 and 1 is binary matirx	incidence	adjacence	cut set	circuit	incidence
$If\ B(G)\ is\ a\ circuit\ matrix\ of\ a\ connected\ graph\ with\ n\ vertices\ and\ e\ edges\ then\ rank\ of\ B(G) =$	n	e	1	e-n+1	e-n+1
$If\ G\ is\ a\ tree\ with\ n\ vertices\ then\ rank\ of\ B(G) =$	1	0	2	3	0
In A(G), the _______ matrix, a row with all 0's represents isolated vertex	adjacent	path	circuit	incident	incident
$A\ column\ of\ P(x,y)\ all\ 0's\ corresponds\ to\ an\ edge\ that\ does\ not\ lie\ in\ ----path\ between\ x\ and\ y$	any	some	no	exactly one	any
$A\ column\ of\ P(x,y)\ all\ 1's\ corresponds\ to\ an\ edge\ that\ lies\ in\ ----path\ between\ x\ and\ y$	any	some	no	exactly one	any
$Number\ of\ rows\ in\ P(x,y)\ with\ all\ 0's\ is$	0	1	2	3	0
$If\ the\ entries\ along\ the\ principal\ diagonal\ of\ an\ adjacency\ matrix\ are\ all\ of\ 0's\ then\ G\ has$	self loop	no self loop	parallel edges	isolated vertex	self loop
$The\ degree\ of\ a\ vertex\ equals\ the\ number\ of\ 1's\ in\ the\ corresponding\ ----\ -\ of\ adjacency\ matrix$	row only	column only	both row and colu	either row or column	
$A\ graph\ consisting\ of\ only\ isolated\ vertices\ is$	1-chromatic	2-chromatic	3 -chromatic	4-chromatic	1-chromatic
$A\ graph\ with\ one\ edge\ is\ atleast$	1-chromatic	2-chromatic	3 -chromatic	4-chromatic	2-chromatic
$A\ graph\ with\ one\ edge\ is\ -----\ 2\ chromatic$	exactly one	atmost	atleast	not	atleast
The number of edges in a largest maximal matching is ____	matching	matching number	maximal matching	minimal matching	matching number
A graph that cannot be drawn on a plane without a cross over between its e	planar	nonplanar	embedding	graph	planar
Complete graph with more than one vertices is _____	planar	nonplanar	embedding	graph	nonplanar

The determinant of every square submatrix of an ___ matrix is 1,-1 or 0	incidence	adjacence	circuit	cut set	incidence
_______ discovered nonplanar graph unique property	Kasimir Kuratoasws	Rowan Hamilton	Euler	Fermat	Kasimir Kuratoaswski
The complete graph of ________ vertices is nonplanar	four	six	seven	five	five
A pentagon divide the plane of the paper into two regions is called ______	Jordan curve	Kuratowski	Euler	Konigsberg bridge	Jordan curve
In adjacency matrix of graph all the entries along the leading diagonal are (	self loop	loop	block	link	self loop
The number of ___ in a minimal covering of the smallest size is covering n	edges	vertices	loop	block	edges
In ____ matrix, a colum with all 0's corresponds to an edge forming a self -	cut-set	circuit	path	adjacency	cut-set
The rank of ____ matrix must be atleast n-1	incident	path	circuit	cut-set	incident
A _____ in which every vertex is of degree one is dimer covering	covering	minimal covering	maximal covering	matching	covering
A hamiltonian ____ in a graph is covering	circuit	path	vertex	edge	circuit
A graph with ___ or more edges is atleast 2 - chromatic	1	2	3	4	1

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UNIT -IV Colourings

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A ____ that has no self loop or parallel edges is simple	digraph	graph	tree	spanning tree	digraph
A balanced digraph is__________	isograph	simple graph	complete digraph	digraph	isograph
A ____ oriented graph.	digraph	complete graph	simple graph	Euler graph	digraph
In any graph, we have	$\alpha(G)=\beta(G)$	$\alpha(G)\leq\beta(G)$	$\alpha(G)<\beta(G)$	$\alpha(G)\geq\beta(G)$	$\alpha(G)\leq\beta(G)$
A vertex v ia called pendant vertex if d+(v)+d-(v)=	1	2	3	4	1
A graph G is an Euler graph if d+(v) is odd then d-(v)=	odd	even	3	5	even
A graph with one or more edges is atleast	4-chromatic	3-chromatic	2-chromatic	1-chromatic	2-chromatic
A complete graph with n vertices is--------------	4-chromatic	3-chromatic	2-chromatic	n-chromatic	n-chromatic

Every graph having ----------- is atleast 3-chromatic	triangle	square	odd vertices	even vertices	triangle	
Every graph having triangle is atleast ------------------	4-chromatic	3-chromatic	2-chromatic	n-chromatic	3-chromatic	
A complete graph with 5 vertices is--------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	5-chromatic	
Every tree with two or more vertices is------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic	
Every --------with 2 or more vertics is 2-chromatic	tree	complete	connected	disconnected	tree	
A graph consisting of simply one circuit with greater than or equa	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic	
A graph consisting of simply one circuit with greater than or equa	even	odd	3	0	even	
A graph consisting of simply one circuit with greater than or equa	4-chromatic	3-chromatic	2-chromatic	5-chromatic	3-chromatic	
A graph consisting of simply one circuit with greater than or equa	even	odd	3	0	odd	
A graph with ------- one edge is 2-chromatic if it has no circuits of	atleast	atmost	exactly	3	atleast	
A graph with atleast ---------- edge is 2-chromatic if it has no cir	1	2	3	4	1	
A graph with atleast one edge is 2-chromatic if it has no circui	odd	even	0	4	odd	
A graph with atleast one edge is 2-chromatic if it has ---------	0	1	2	3	0	
A graph with atleast one edge is ------------- if it has no circuits (	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic	

A star graph is ----------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic	
Every tree with -------------- vertics is 2-chromatic	greater than 2	less than 2	equal to 2	greater than or equa	greater than or equal to 2	
Every ------------ graph is 2-chromatic	bipartiate	complete	regular	connected	bipartiate	
Every biparitate graph is ------------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic	
Two regions are said to be adjacent if they have a common --------	edge	vertex	edge and vertex	neither edge nor ve	edge	
Two ---------------------- are said to be adjacent if they have a comm	faces	regions	egdes	vertices	regions	
Proper coloring of ---------------- is called map coloring	faces	regions	egdes	vertices	regions	
A covering exists for a graph if the graph has no --------------------	isolated vertex	odd vertex	even vertex	pendant vertex	isolated vertex	
Every ------------------ in a graph included in every covering of th	pendant edge	odd vertex	even vertex	pendant vertex	pendant edge	
Every pendant edge in a graph included in ---------- covering (	no	some	all	finite number of	all	
Cover of a graph is a sub set of ----------------------	2^m	edges	both vertices and e	neither edge nor ve	vertices	
A complete graph with _______ vertices is one of the 2 graphs o	2	3	5	1	5	
The second graph of Kuratowski is a regular connected graph wit	six,seven	six,nine	six,five	five,six	six,nine	
The two common geometric representations in Kuratowski graph	homeomorphics	planar representat	infinite region	isomorphic	isomorphic	

A graph in which all vertices are of equal degree is called a _____	complete graph	regular graph	planar graph	nonplanar graph	regular graph	
Removal of one edge or a vertex makes each a _______ graph.	complete	planar	$\frac{n}{2}$ nonplanar	Euler $\frac{n-1}{2}$	planar	
The complete graph of 5 vertices is _______	planar	nonplanar	embedding	complete	nonplanar	
The rank of an ____ of a digraph with n vertices is n-1	incidence matrix	cutset matrix	path matrix	circuit matrix	incidence matrix	
A _______ in which there is exactly one edge directed from every	simple digraph	complete digraph	regular digraph	symmetric digraph	simple digraph	
A graph with n vertices is a tree if	G is connected	G has $n-1$ edges	G is not connected	there is exactly one path between every pair of vertices in G	there is exactly one path between every pair of vertices in G	
A graph with n vertices is a tree if	G is connected	G has $n-1$ edges	G is not connected	G is minimally connected graph	G is minimally connected graph	
In any tree there are ––––––––two pendant vertices	atleast two	atmost two	exactly	no	atleast two	
Distance between any two vertices is	< 0	> 0	≤ 0	≥ 0	≥ 0	
Number of circuits in a tree is	0	1	2	3	0	
Distance between any two vertices in a complete graph is	0	1	2	3	1	
A vertex with minimum eccentricity is called	pendant vertex	isolated vertex	centre	odd vertex	centre	
If $G = (V, E)$ *is a complete graph and* $v \in V$ *then* $E(v) =$	0	1	2	3	1	
If G is a complete graph with n vertices then number of centre of G is	n	n-1	n-2	n-3	n	

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Advanced Graph Theory **Subject Code: 15MMU604A**

Class : III - B.Sc. Mathematics-B **Semester : VI**

UNIT -V Directed graph

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The isolated vertex in degree and out degree are equal to	0	1	2	3	0
The minimum cardinality of a _____ is equal to domination number	set	graph	cutset	vertex	set
The dominating set N[S] is _____	v	1	0	2	v
Suppose G is a complete graph wiyh n vertices. Then number of independent set of vertices is	n	n-1	n+1	n+2	n
Every dominating set contain ___ one minmal dominating set	atleast	atmost	equal	every	atleast
The number of ____ in the largest independent set of a graph	vertices	edges	links	blocks	vertices
The minimum cardinality of a total dominating set is dominating set is _______	domination number	independent set	dominating set	independent numb	domination number
A set of vertices in a graph is ______ if no two vertices in the set are adjacent	independent set	independent num	dominating set	dominating numbe	independent set
The number of ___ incident out of a vertex is out degree	edges	vertices	links	blocks	edges
The minimum cardinality of an independent dominating set G is _______	domination number	independent set	dominating set	independent domi	independent domination number
A _____ dominating set from which no vertex can be removed without destroying its dominance p	minimal	maximal	independent	independent numb	minimal
A ______ dominating set may or may not be independent	minimal	maximal	independent	independent numb	minimal

A_______ contains atleast one minimal dominating set.	domination number	independent set	dominating set	independent numb	dominating set
The set of all _____ is trivially a dominating set in graph	vertices	edges	cutset	blocks	vertices
An _____ has the dominance property only if it is a maximal independent set	domination number	independent set	dominating set	independent numb	independent set
A graph may have many ______ and of different sizes.	minimal dominating	independent set	dominating set	independent numb	minimal dominating set
The number of ___ in a minimal covering of the smallest size is covering number of the graph	edges	vertices	loop	block	edges
In ____ matrix, a colum with all 0's corresponds to an edge forming a self -loop	cut-set	circuit	path	adjacency	cut-set
The rank of ____ matrix must be atleast n-1	incident	path	circuit	cut-set	incident
A _____ in which every vertex is of degree one is dimer covering	covering	minimal covering	maximal covering	matching	covering
A hamiltonian ____ in a graph is covering	circuit	path	vertex	edge	circuit
A graph with ___ or more edges is atleast 2 - chromatic	1	2	3	$n - 3\ edges$ 4	1
A pendent vertex of degree is	1	2	3		1
A regular graph with n vertices and their degree is	n-1	n-2	n+1	n+2	n-1
Atleast one vertex is	graph	incident vertex	degree	pendent	graph
Isolated vertex is	null graph	pendent graph	complete graph	regular graph	null graph
A graph is a infinte number of vertices and infinite number of edges is	infinite graph	finite graph	link	regular graph	infinite graph
The number of edges in a largest maximal matching is ____	matching	matching number	maximal matching	minimal matching	matching number
A graph that cannot be drawn on a plane without a cross over between its edges is called _______	planar	nonplanar	embedding	graph	planar
Complete graph with more than one vertices is _____	planar	nonplanar	embedding	graph	nonplanar

The determinant of every square submatrix of an ___ matrix is 1,-1 or 0	incidence	adjacence	circuit	cut set	incidence
_______ discovered nonplanar graph unique property	Kasimir Kuratoasws	Rowan Hamilton	Euler	Fermat	Kasimir Kuratoaswski
The complete graph of ________ vertices is nonplanar	four	six	seven	five	five
A pentagon divide the plane of the paper into two regions is called _______	Jordan curve	Kuratowski	Euler	Konigsberg bridge	Jordan curve
In adjacency matrix of graph all the entries along the leading diagonal are 0 if and only if the graph	self loop	loop	block	link	self loop
A star graph is ----------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Every tree with -------------- vertics is 2-chromatic	greater than 2	less than 2	equal to 2	greater than or equ	greater than or equal to 2
Every ------------ graph is 2-chromatic	bipartiate	complete	regular	connected	bipartiate
Every biparitate graph is ------------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Two regions are said to be adjacent if they have a common -----------between them	edge	vertex	edge and vertex	neither edge nor ve	edge
Two --------------------- are said to be adjacent if they have a common egde between them	faces	regions	egdes	vertices	regions
Proper coloring of ---------------- is called map coloring	faces	regions	egdes	vertices	regions
A covering exists for a graph if the graph has no ------------------------	isolated vertex	odd vertex	even vertex	pendant vertex	isolated vertex
Every ------------------ in a graph included in every covering of the graph	pendant edge	odd vertex	even vertex	pendant vertex	pendant edge
Every pendant edge in a graph included in ---------- covering of the graph	no	some	all	finite number of	all
Cover of a graph is a sub set of ----------------------	vertices	edges	both vertices and e	neither edge nor ve	vertices
In a degree constrained shortest spanning tree deg(G)≤	3	4	5	2	3
Every circuit has an ___ number of edges in common with any cut set	even	odd	zero	three	even

A ___ is separable if its vertex connectivity is one.	connected graph	simple graph	planar graph	non planar graph	connected graph
A ______is a connected graph without any circuit.	tree	spanning tree	weighted spanning	hamiltonian circuit	tree
Any connected graph with n vertices and n-1 edges is ____	tree	spanning tree	fundamental circu	fundamental circui	tree
The number of edges incident on a vertex with self-loop counted twice is	degree	adjacent	link	block	degree
In any tree there __ two pendent vertices	atleast	atmost	some	sum of	atleast
The length of a hamiltonian path of a ____ with n vertices n-1	connected graph	star graph	simple graph	complete graph	connected graph

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Coimbatore – 641 021.

SYLLABUS

Semester -VI

		L	T	P	C
15MMU604A	**ELECTIVE-II**	**5**	**0**	**0**	**5**

ADVANCED GRAPH THEORY

Scope: This course is intended to introduce the area of structural graph theory to the learners. Basic principles underlying this theory and algorithmic applications are also surveyed.

Objectives: To enable the students to understand the basic concepts of Graph Theory and its applications.

UNIT I
Undirected graph- Basic concepts- incidence and Degree of vertices- isolated vertex – pendant vertex – Path and Circuits: Isomorphism – Sub graphs – Walks, Paths and Circuits – Connected graphs and concepts – Euler graphs – Hamilton graph – Complete graph – Traveling Salesman problem.

UNIT II
Trees – Definition – some properties of trees – Theorems – Rooted and Binary trees – Spanning trees. Cut set and cut vertices – some properties of a cut set – sets in a graph – Theorems – Fundamental circuits and cut sets – Connectivity and Separability – Theorems.

UNIT III
Planar graphs – Kuratowski‟s two graphs – Theorems – Different representation of a planar graph – Detection of planarity – Thickness and crossings.

UNIT IV
Colourings – Covering partitioning – Chromatic number Theorems –Chromatic partitioning – Independent set – Finding a maximal independent set – Dominating set – Finding minimal dominating set – Chromatic polynomial – Theorems. Coverings – Theorems – Four colour problem - Five colour Theorem.

UNIT V
Directed graph – Definition – Some types of di-graphs – Directed path and connectedness – Euler di-graphs – Theorems – Trees with direct edges - Theorems – odded trees – Matrix representation – incidence matrix – Theorems – Circuit matrix – Adjacency matrix – Tournaments.

TEXT BOOK

1.Narsingh Deo., 2007. Graph Theory with Applications to Engineering and Computer Science, Prentice Hall of India Pvt. Ltd, New Delhi.

REFERENCES

1. Harary F., 1969. Graph Theory, Addision-Wesley publishing company,Inc., Amsterdam.
2. Bondy.J.A., and U.S.R.Murty., 2008. Graph theory and applications, Springer.
3. Balakrishnan, 2011, Graph theory, Springer publications.
4. West D.B., 2011. Introduction to Graph Theory, Prentice Hall, New Delhi.

____.

UNIT-I

SYLLABUS

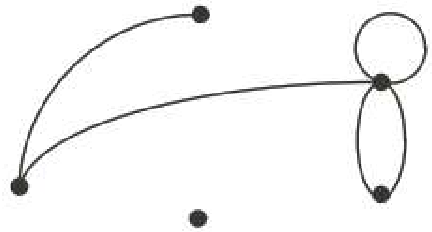
Basic concepts- incidence and Degree of vertices- isolated vertex – pendant vertex – Path and Circuits: Isomorphism – Sub graphs – Walks, Paths and Circuits – Connected graphs and concepts – Euler graphs – Hamilton graph – Complete graph – Traveling Salesman problem.

KAHE

Definitions and Fundamental Concepts

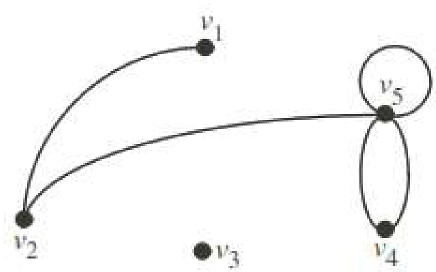
Conceptually, a *graph* is formed by *vertices* and *edges* connecting the vertices.

Example.



Formally, a graph is a pair of sets (V, E), where V is the *set of vertices* and E is the *set of edges*, formed by pairs of vertices. E is a *multiset*, in other words, its elements can occur more than once so that every element has a *multiplicity*. Often, we label the vertices with letters (for example: $a, b, c, \ldots$ or $v_1, v_2, \ldots$) or numbers $1, 2, \ldots$ Throughout this lecture material, we will label the elements of V in this way.

Example. *(Continuing from the previous example) We label the vertices as follows:*

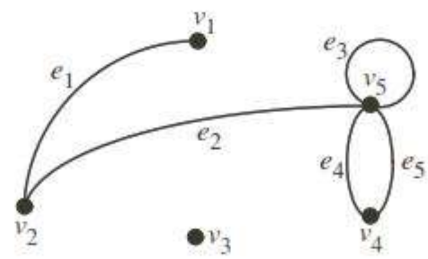


We have $V = \{v_1, \ldots, v_5\}$ *for the vertices and* $E = \{(v_1, v_2), (v_2, v_5), (v_5, v_5), (v_5, v_4), (v_5, v_4)\}$ *for the edges.*

Similarly, we often label the edges with letters (for example: $a, b, c, \ldots$ or $e_1, e_2, \ldots$) or numbers $1, 2, \ldots$ for simplicity.

Remark. *The two edges (u, v) and (v, u) are the same. In other words, the pair is not* ordered.

Example. *(Continuing from the previous example) We label the edges as follows:*



So $E = \{e_1, \ldots, e_5\}$.

We have the following terminologies:

1. The two vertices u and v are *end vertices* of the edge (u, v).
2. Edges that have the same end vertices are *parallel*.
3. An edge of the form (v, v) is a *loop*.
4. A graph is *simple* if it has no parallel edges or loops.
5. A graph with no edges (i.e. E is empty) is *empty*.
6. A graph with no vertices (i.e. V and E are empty) is a *null graph*.
7. A graph with only one vertex is *trivial*.
8. Edges are *adjacent* if they share a common end vertex.
9. Two vertices u and v are *adjacent* if they are connected by an edge, in other words, (u, v) is an edge.
10. The *degree* of the vertex v, written as $d(v)$, is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately.
11. A *pendant vertex* is a vertex whose degree is 1.
12. An edge that has a pendant vertex as an end vertex is a *pendant edge*.
13. An *isolated vertex* is a vertex whose degree is 0.

Example. *(Continuing from the previous example)*

- v_4 *and* v_5 *are end vertices of* e_5.
- e_4 *and* e_5 *are parallel.*
- e_3 *is a loop.*
- *The graph is not simple.*
- e_1 *and* e_2 *are adjacent.*

- v_1 *and* v_2 *are adjacent.*
- *The degree of* v_1 *is* 1 *so it is a pendant vertex.*
- e_1 *is a pendant edge.*
- *The degree of* v_5 *is* 5.
- *The degree of* v_4 *is* 2.
- *The degree of* v_3 *is* 0 *so it is an isolated vertex.*

In the future, we will label graphs with letters, for example:

$$G = (V, E).$$

The *minimum degree* of the vertices in a graph G is denoted $\delta(G)$ ($= 0$ if there is an isolated vertex in G). Similarly, we write $\Delta(G)$ as the *maximum degree* of vertices in G.

Example. *(Continuing from the previous example)* $\delta(G) = 0$ *and* $\Delta(G) = 5$.

Remark. *In this course, we only consider* finite *graphs, i.e.* V *and* E *are finite sets.*

Since every edge has two end vertices, we get

The bridges of Konigsberg: The graph Theory began in 1736 when Leonhard Euler solved the problem of seven bridges on Pregel river in the town of Konigsberg in Prussia (now Kaliningrad in Russia). The two islands and seven bridges are shown below:

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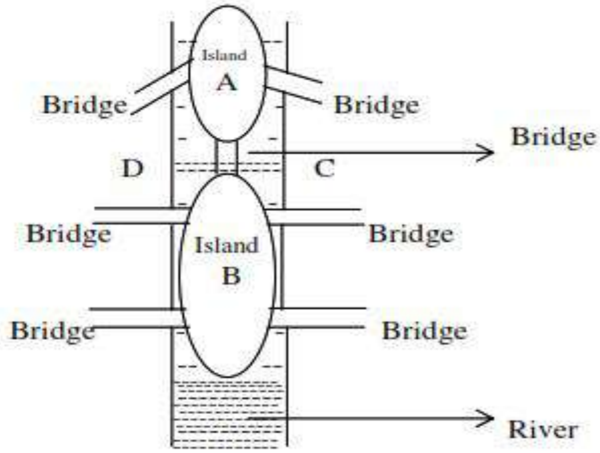
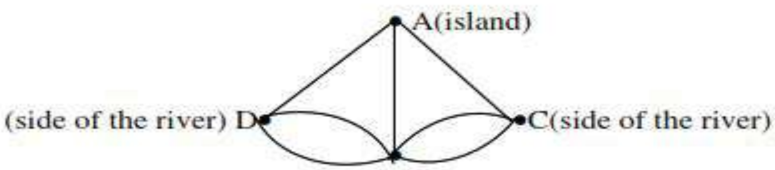


The people of Konigsgerg posed the following question to famous Swiss Mathematician Leonhard Euler:
"Beginning anywhere and ending any where, can a person walk through the town of Konigsberg crossing all the seven bridges exactly once?

Euler showed that such a walk is impossible. He replaced the islands A, B and the two sides (banks) C and D of the river by vertices and the bridges as edges of a graph. We note then that

$$\deg(A) = 3$$
$$\deg(B) = 5$$
$$\deg(C) = 3$$
$$\deg(D) = 3$$

Thus the graph of the problem is



(Euler's graphical representation of seven bridge problem)

The problem then reduces to

"Is there any Euler's path in the above diagram?".

To find the answer, we note that there are more than two vertices having odd degree. Hence there exist no Euler path for this graph.

Definition: An edge in a connected graph is called a **Bridge** or a **Cut Edge** if deleting that edge creates a disconnected graph.

Theorem 1.1. *The graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, satisfies*

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

Corollary. *Every graph has an even number of vertices of odd degree.*

Proof. If the vertices $v_1, \ldots, v_k$ have odd degrees and the vertices $v_{k+1}, \ldots, v_n$ have even degrees, then (Theorem 1.1)

$$d(v_1) + \cdots + d(v_k) = 2m - d(v_{k+1}) - \cdots - d(v_n)$$

is even. Therefore, k is even. □

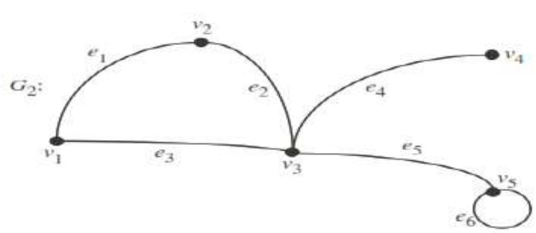
Example. *(Continuing from the previous example) Now the sum of the degrees is $1 + 2 + 0 + 2 + 5 = 10 = 2 \cdot 5$. There are two vertices of odd degree, namely v_1 and v_5.*

A simple graph that contains every possible edge between all the vertices is called a *complete graph*. A complete graph with n vertices is denoted as K_n. The first four complete graphs are given as examples:

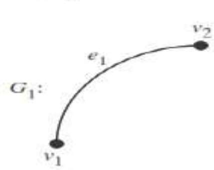
The graph $G_1 = (V_1, E_1)$ is a *subgraph* of $G_2 = (V_2, E_2)$ if

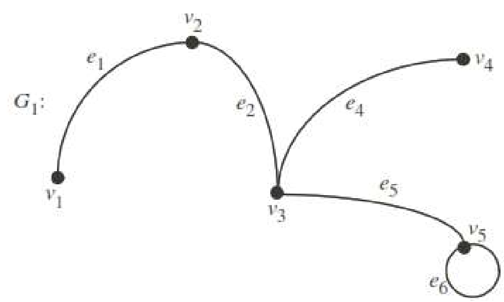
1. $V_1 \subseteq V_2$ and
2. Every edge of G_1 is also an edge of G_2.

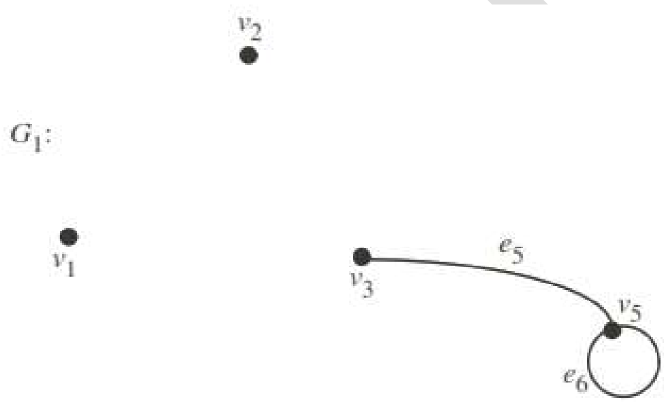
Example. *We have the graph*



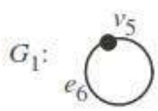
and some of its subgraphs are




G_1:
v_1
v_2
v_3
v_4
v_5
e_1
e_2
e_4
e_5
e_6


G_1:
v_1
v_2
v_3
v_5
e_5
e_6

and

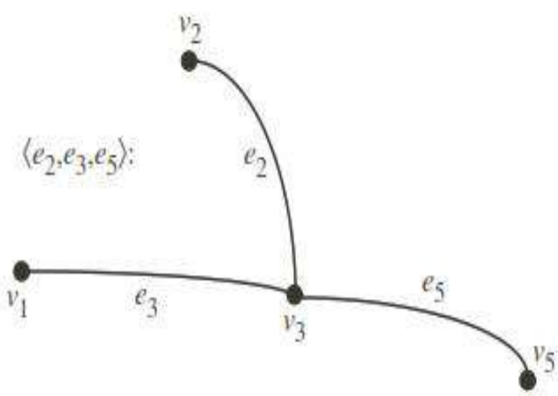


The *subgraph of* $G = (V, E)$ *induced by the edge set* $E_1 \subseteq E$ is:

$$G_1 = (V_1, E_1) =_{\text{def.}} \langle E_1 \rangle,$$

where V_1 consists of every end vertex of the edges in E_1.

Example. *(Continuing from above) From the original graph* G, *the edges* e_2, e_3 *and* e_5 *induce the subgraph*



The *subgraph of* $G = (V, E)$ *induced by the vertex set* $V_1 \subseteq V$ is:

$$G_1 = (V_1, E_1) =_{\text{def.}} \langle V_1 \rangle,$$

where E_1 consists of every edge between the vertices in V_1.

Example. *(Continuing from the previous example) From the original graph G, the vertices v_1, v_3 and v_5 induce the subgraph*

A complete subgraph of G is called a *clique* of G.

Walks, Trails, Paths, Circuits, Connectivity, Components

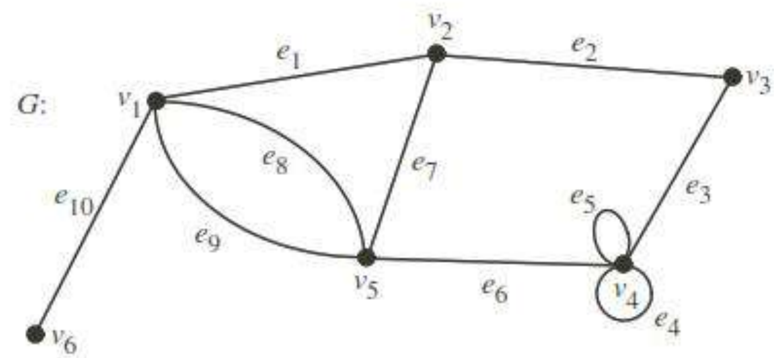
Remark. *There are many different variations of the following terminologies. We will adhere to the definitions given here.*

A *walk* in the graph $G = (V, E)$ is a finite sequence of the form

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k},$$

which consists of alternating vertices and edges of G. The walk starts at a vertex. Vertices $v_{i_{t-1}}$ and v_{i_t} are end vertices of e_{j_t} $(t = 1, \ldots, k)$. v_{i_0} is the *initial vertex* and v_{i_k} is the *terminal vertex*. k is the *length* of the walk. A zero length walk is just a single vertex v_{i_0}. It is allowed to visit a vertex or go through an edge more than once. A walk is *open* if $v_{i_0} \neq v_{i_k}$. Otherwise it is *closed*.

Example. *In the graph*



the walk

$$v_2, e_7, v_5, e_8, v_1, e_8, v_5, e_6, v_4, e_5, v_4, e_5, v_4$$

is open. On the other hand, the walk

$$v_4, e_5, v_4, e_3, v_3, e_2, v_2, e_7, v_5, e_6, v_4$$

is closed.

A walk is a *trail* if any edge is traversed at most once. Then, the number of times that the vertex pair u, v can appear as consecutive vertices in a trail is at most the number of parallel edges connecting u and v.

Example. *(Continuing from the previous example) The walk in the graph*

$$v_1, e_8, v_5, e_9, v_1, e_1, v_2, e_7, v_5, e_6, v_4, e_5, v_4, e_4, v_4$$

is a trail.

A trail is a *path* if any vertex is visited at most once except possibly the initial and terminal vertices when they are the same. A closed path is a *circuit*. For simplicity, we will assume in the future that a circuit is not empty, i.e. its length ≥ 1. We identify the paths and circuits with the subgraphs induced by their edges.

Example. *(Continuing from the previous example) The walk*

$$v_2, e_7, v_5, e_6, v_4, e_3, v_3$$

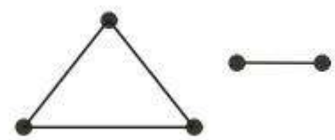
is a path and the walk

$$v_2, e_7, v_5, e_6, v_4, e_3, v_3, e_2, v_2$$

is a circuit.

The walk starting at u and ending at v is called an *u–v walk*. u and v are *connected* if there is a u–v walk in the graph (then there is also a u–v path!). If u and v are connected and v and w are connected, then u and w are also connected, i.e. if there is a u–v walk and a v–w walk, then there is also a u–w walk. A graph is *connected* if all the vertices are connected to each other. (A trivial graph is connected by convention.)

Example. *The graph*



is not connected.

The subgraph G_1 (not a null graph) of the graph G is a *component* of G if

1. G_1 is connected and
2. Either G_1 is trivial (one single isolated vertex of G) or G_1 is not trivial and G_1 is the subgraph induced by those edges of G that have one end vertex in G_1.

Different components of the same graph do not have any common vertices because of the following theorem.

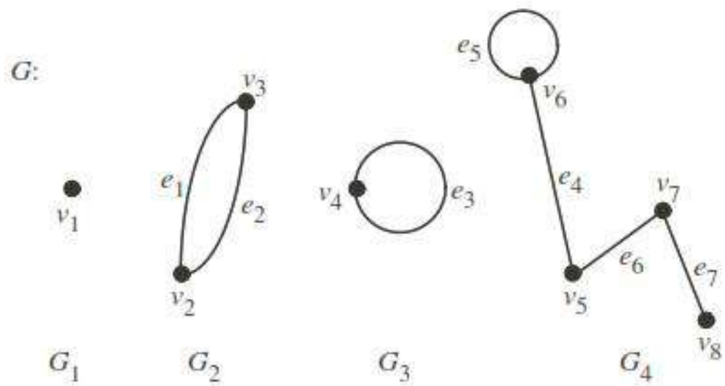
Theorem 1.2. *If the graph G has a vertex v that is connected to a vertex of the component G_1 of G, then v is also a vertex of G_1.*

Proof. If v is connected to vertex v' of G_1, then there is a walk in G

$$v = v_{i_0}, e_{j_1}, v_{i_1}, \ldots, v_{i_{k-1}}, e_{j_k}, v_{i_k} = v'.$$

Since v' is a vertex of G_1, then (condition #2 above) e_{j_k} is an edge of G_1 and $v_{i_{k-1}}$ is a vertex of G_1. We continue this process and see that v is a vertex of G_1. □

Example.



The components of G are G_1, G_2, G_3 and G_4.

Theorem 1.3. *Every vertex of G belongs to exactly one component of G. Similarly, every edge of G belongs to exactly one component of G.*

Proof. We choose a vertex v in G. We do the following as many times as possible starting with $V_1 = \{v\}$:

(∗) If v' is a vertex of G such that $v' \notin V_1$ and v' is connected to some vertex of V_1, then $V_1 \leftarrow V_1 \cup \{v'\}$.

Since there is a finite number of vertices in G, the process stops eventually. The last V_1 induces a subgraph G_1 of G that is the component of G containing v. G_1 is connected because its vertices are connected to v so they are also connected to each other. Condition #2 holds because we can not repeat (∗). By Theorem 1.2, v does not belong to any other component.

The edges of the graph are incident to the end vertices of the components. □

Theorem 1.3 divides a graph into distinct components. The proof of the theorem gives an algorithm to do that. We have to repeat what we did in the proof as long as we have free vertices that do not belong to any component. Every isolated vertex forms its own component. A connected graph has only one component, namely, itself.

A graph G with n vertices, m edges and k components has the *rank*

$$\rho(G) = n - k.$$

The *nullity* of the graph is

$$\mu(G) = m - n + k.$$

We see that $\rho(G) \geq 0$ and $\rho(G) + \mu(G) = m$. In addition, $\mu(G) \geq 0$ because

Theorem 1.4. $\rho(G) \leq m$

Proof. We will use the second principle of induction (strong induction) for m.

Induction Basis: $m = 0$. The components are trivial and $n = k$.

Induction Hypothesis: The theorem is true for $m < p$. $(p \geq 1)$

Induction Statement: The theorem is true for $m = p$.

Induction Statement Proof: We choose a component G_1 of G which has at least one edge. We label that edge e and the end vertices u and v. We also label G_2 as the subgraph of G and G_1, obtained by removing the edge e from G_1 (but not the vertices u and v). We label G' as the graph obtained by removing the edge e from G (but not the vertices u and v) and let k' be the number of components of G'. We have two cases:

1. G_2 is connected. Then, $k' = k$. We use the Induction Hypothesis on G':

$$n - k = n - k' = \rho(G') \leq m - 1 < m.$$

2. G_2 is not connected. Then there is only one path between u and v:

$$u, e, v$$

and no other path. Thus, there are two components in G_2 and $k' = k + 1$. We use the Induction Hypothesis on G':

$$\rho(G') = n - k' = n - k - 1 \leq m - 1.$$

Hence $n - k \leq m$. □

These kind of combinatorial results have many consequences. For example:

Theorem 1.5. *If G is a connected graph and $k \geq 2$ is the maximum path length, then any two paths in G with length k share at least one common vertex.*

Proof. We only consider the case where the paths are not circuits (Other cases can be proven in a similar way.). Consider two paths of G with length k:

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k} \quad \text{(path } p_1\text{)}$$

and

$$v_{i'_0}, e_{j'_1}, v_{i'_1}, e_{j'_2}, \ldots, e_{j'_k}, v_{i'_k} \quad \text{(path } p_2\text{)}.$$

Let us consider the counter hypothesis: The paths p_1 and p_2 do not share a common vertex. Since G is connected, there exists an v_{i_0}–$v_{i'_k}$ path. We then find the last vertex on this path which is also on p_1 (at least v_{i_0} is on p_1) and we label that vertex v_{i_t}. We find the first vertex of the v_{i_t}–$v_{i'_k}$ path which is also on p_2 (at least $v_{i'_k}$ is on p_2) and we label that vertex $v_{i'_s}$. So we get a v_{i_t}–$v_{i'_s}$ path

$$v_{i_t}, e_{j''_1}, \ldots, e_{j''_\ell}, v_{i'_s}.$$

The situation is as follows:

$$
\begin{array}{c}
v_{i_0}, e_{j_1}, v_{i_1}, \ldots, v_{i_t}, e_{j_{t+1}}, \ldots, e_{j_k}, v_{i_k} \\
e_{j''_1} \\
\vdots \\
e_{j''_\ell} \\
v_{i'_0}, e_{j'_1}, v_{i'_1}, \ldots, v_{i'_s}, e_{j'_{s+1}}, \ldots, e_{j'_k}, v_{i'_k}
\end{array}
$$

From here we get two paths: v_{i_0}–$v_{i'_k}$ path and $v_{i'_0}$–v_{i_k} path. The two cases are:

- $t \geq s$: Now the length of the v_{i_0}–$v_{i'_k}$ path is $\geq k+1$. $\checkmark$[1]
- $t < s$: Now the length of the $v_{i'_0}$–v_{i_k} path is $\geq k+1$. $\checkmark$ □

A graph is *circuitless* if it does not have any circuit in it.

Theorem 1.6. *A graph is circuitless exactly when there are no loops and there is at most one path between any two given vertices.*

Proof. First let us assume G is circuitless. Then, there are no loops in G. Let us assume the counter hypothesis: There are two different paths between distinct vertices u and v in G:

$$u = v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k} = v \quad \text{(path } p_1\text{)}$$

and

$$u = v_{i'_0}, e_{j'_1}, v_{i'_1}, e_{j'_2}, \ldots, e_{j'_\ell}, v_{i'_\ell} = v \quad \text{(path } p_2\text{)}$$

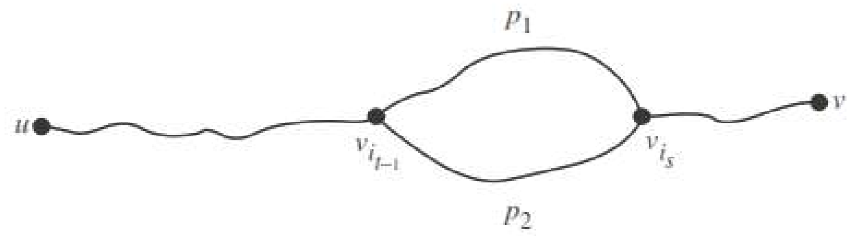
(here we have $i_0 = i'_0$ and $i_k = i'_\ell$), where $k \geq \ell$. We choose the smallest index t such that

$$v_{i_t} \neq v_{i'_t}.$$

There is such a t because otherwise

[1]From now on, the symbol $\checkmark$ means contradiction. If we get a contradiction by proceeding from the assumptions, the hypothesis must be wrong.

1. $k > \ell$ and $v_{i_k} = v = v_{i'_\ell} = v_{i_\ell}$ ($\checkmark$) or
2. $k = \ell$ and $v_{i_0} = v_{i'_0}, \ldots, v_{i_\ell} = v_{i'_\ell}$. Then, there would be two parallel edges between two consecutive vertices in the path. That would imply the existence of a circuit between two vertices in G. $\checkmark$



We choose the smallest index s such that $s \geq t$ and v_{i_s} is in the path p_2 (at least v_{i_k} is in p_2). We choose an index r such that $r \geq t$ and $v_{i'_r} = v_{i_s}$ (it exists because p_1 is a path). Then,

$$v_{i_{t-1}}, e_{j_t}, \ldots, e_{j_s}, v_{i_s}(= v_{i'_r}), e_{j'_r}, \ldots, e_{j'_t}, v_{i'_{t-1}}(= v_{i_{t-1}})$$

is a circuit. $\surd$ (Verify the case $t = s = r$.)

Let us prove the reverse implication. If the graph does not have any loops and no two distinct vertices have two different paths between them, then there is no circuit. For example, if

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k} = v_{i_0}$$

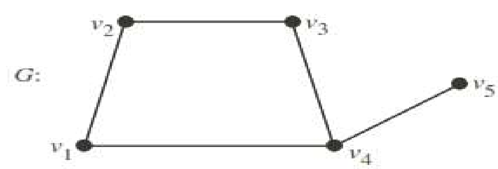
is a circuit, then either $k = 1$ and e_{j_1} is a loop ($\surd$), or $k \geq 2$ and the two vertices v_{i_0} and v_{i_1} are connected by two distinct paths

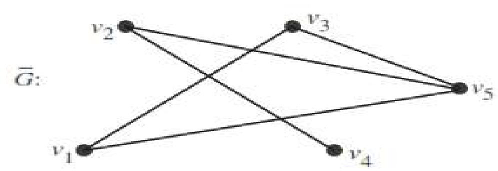
$$v_{i_0}, e_{j_1}, v_{i_1} \quad \text{and} \quad v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k} = v_{i_0} \quad (\surd). \qquad \square$$

Graph Operations

The *complement* of the simple graph $G = (V, E)$ is the simple graph $\overline{G} = (V, \overline{E})$, where the edges in $\overline{E}$ are exactly the edges not in G.

Example.

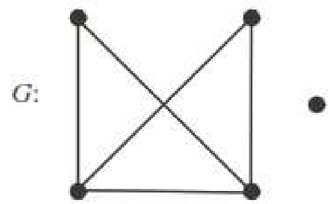


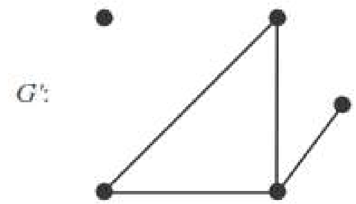


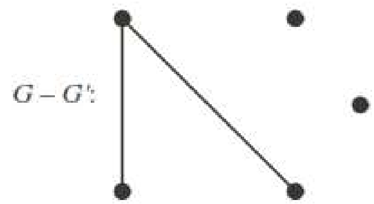
Example. *The complement of the complete graph K_n is the empty graph with n vertices.*

Obviously, $\overline{\overline{G}} = G$. If the graphs $G = (V, E)$ and $G' = (V', E')$ are simple and $V' \subseteq V$, then the *difference* graph is $G - G' = (V, E'')$, where E'' contains those edges from G that are not in G' (simple graph).

Example.







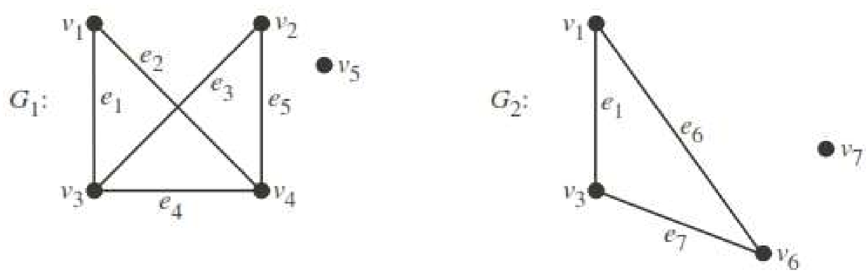
Here are some binary operations between two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$:

- The *union* is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ (simple graph).
- The *intersection* is $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ (simple graph).
- The *ring sum* $G_1 \oplus G_2$ is the subgraph of $G_1 \cup G_2$ induced by the edge set $E_1 \oplus E_2$ (simple graph). *Note!* The set operation $\oplus$ is the *symmetric difference*, i.e.

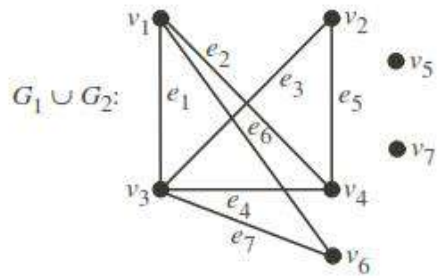
$$E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

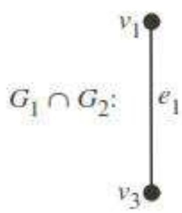
Since the ring sum is a subgraph induced by an edge set, there are no isolated vertices. All three operations are commutative and associative.

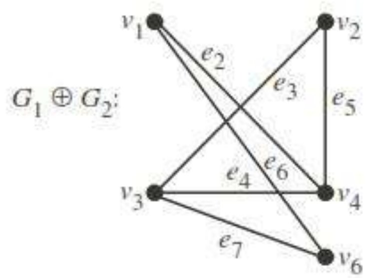
Example. *For the graphs*



we have







Remark. *The operations $\cup$, $\cap$ and $\oplus$ can also be defined for more general graphs other than simple graphs. Naturally, we have to "keep track" of the multiplicity of the edges:*

$\cup$: *The multiplicity of an edge in $G_1 \cup G_2$ is the larger of its multiplicities in G_1 and G_2.*

$\cap$: *The multiplicity of an edge in $G_1 \cap G_2$ is the smaller of its multiplicities in G_1 and G_2.*

$\oplus$: *The multiplicity of an edge in $G_1 \oplus G_2$ is $|m_1 - m_2|$, where m_1 is its multiplicity in G_1 and m_2 is its multiplicity in G_2.*

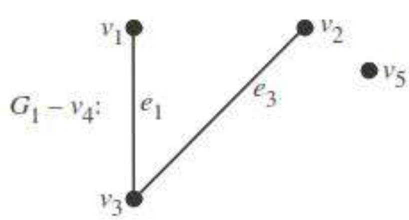
(We assume zero multiplicity for the absence of an edge.) In addition, we can generalize the difference operation for all kinds of graphs if we take account of the multiplicity. The multiplicity of the edge e in the difference $G - G'$ is

$$m_1 \dot{-} m_2 = \begin{cases} m_1 - m_2, \text{ if } m_1 \geq m_2 \\ 0, \text{ if } m_1 < m_2 \end{cases} \qquad \textit{(also known as the } \text{proper difference}\textit{).}$$

where m_1 and m_2 are the multiplicities of e in G_1 and G_2, respectively.

If v is a vertex of the graph $G = (V, E)$, then $G - v$ is the subgraph of G induced by the vertex set $V - \{v\}$. We call this operation the *removal of a vertex*.

Example. *(Continuing from the previous example)*



Similarly, if e is an edge of the graph $G = (V, E)$, then $G - e$ is graph (V, E'), where E' is obtained by removing e from E. This operation is known as *removal of an edge*. We remark that we are not talking about removing an edge as in Set Theory, because the edge can have nonunit multiplicity and we only remove the edge once.

Eulerian Paths And Circuits

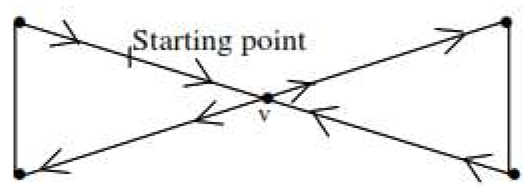
Definition: A path in a graph G is called an **Euler Path** if it includes **every edge exactly once**.

Definition: A circuit in a graph G is called an **Euler Circuit** if it includes every edge exactly once. Thus, an Euler circuit (Eulerian trail) for a graph G is a sequence of adjacent vertices and edges in G that starts and ends at the same vertex, uses every vertex of G at least once, and uses **every edge of G exactly once**.

Definition: A graph is called **Eulerian graph** if there exists a Euler circuit for that graph.

Theorem 1. If a graph has an Euler circuit, then every vertex of the graph has even degree.

Proof: Let G be a graph which has an Euler circuit. Let v be a vertex of G. We shall show that degree of v is even. By definition, Euler circuit contains every edge of graph G. Therefore the Euler circuit contains all edges incident on v. We start a journey beginning in the middle of one of the edges adjacent to the start of Euler circuit and continue around the Euler circuit to end in the middle of the starting edge. Since Euler circuit uses every edge exactly once, the edges incident on v occur



in entry / exist pair and hence the degree of v is a multiple of 2. Therefore the degree of v is even. This completes the proof of the theorem.

We know that contrapositive of a conditional statement is logically equivalent to statement. Thus the above theorem is equivalent to:

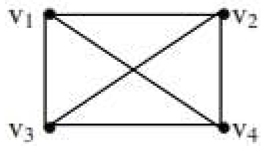
Theorem:2. If a vertex of a graph is not of even degree, then it does not have an Euler circuit.

or

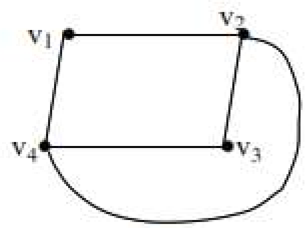
"If some vertex of a graph has odd degree, then that graph does not have an Euler circuit".

Example: Show that the graphs below do not have Euler circuits.

(a)



(b)



Solution: In graph (a), degree of each vertex is 3. Hence this **does not** have a Euler circuit.

In graph (b), we have

$$\deg(v_2) = 3$$

$$\deg(v_4) = 3$$

Since there are vertices of odd degree in the given graph, therefore it **does not** have an Euler circuit.

Theorem 3. If G is a connected graph and every vertex of G has even degree, then G has an Euler circuit.

Proof: Let every vertex of a connected graph G has even degree. If G consists of a single vertex, the trivial walk from v to v is an Euler circuit. So suppose G consists of more than one vertices. We start from any verted v of G. Since the degree of each vertex of G is even, if we reach each vertex other than v by travelling on one edge, the same vertex can be reached by travelling on another previously unused edge. Thus a sequence of distinct adjacent edges can be produced indefinitely as long as v is not reached. Since number of edges of the graph is finite (by definition of graph), the sequence of distinct edges will terminate. Thus the sequence must return to the starting vertex. We thus obtain a sequence of adjacent vertices and edges starting and ending at v without repeating any edge. Thus we get a circuit C.

If C contains every edge and vertex of G, then C is an Eular circuit.

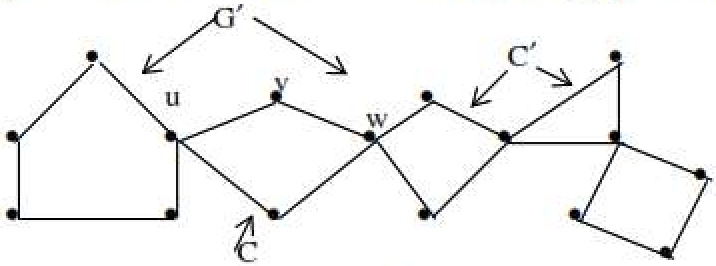
If C does not contain every edge and vertex of G, remove all edges of C from G and also any vertices that become isolated when the edges of C are removed. Let the resulting subgraph be G'. We note that when we removed edges of C, an even number of edges from each vertex have been removed. Thus degree of each remaining vertex remains even.

Further since G is connected, there must be at least one vertex common to both C and G'. Let it be w(in fact there are two such vertices). Pick any sequence of adjacent vertices and edges of G' starting and ending at w without repeating an edge. Let the resulting circuit be C'.

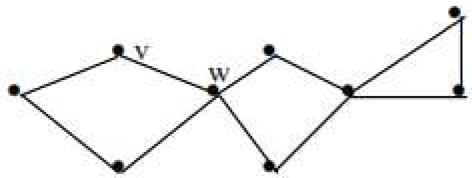
Join C and C' together to create a new circuit C''. Now, we observe that if we start from v and follow C all the way to reach w and then follow C' all the way to reach back to w. Then continuing travelling along the untravelled edges of C, we reach v.

If C'' contains every edge an vertex of C, then C'' is an Euler circuit. If not, then we again repeat our process. Since the graph is finite, the process must terminate.

The process followed has been described in the graph G shown below:



G



C''

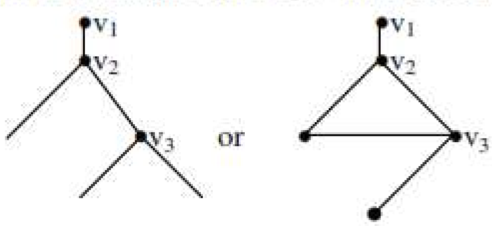
Theorems 1 and 3 taken together imply :

Theorem 4. (Euler Theorem) A finite connected graph G has an Euler circuit if and only if every vertex of G has even degree.

Thus finite connected graph is Eulerian if and only if each vertex has even degree.

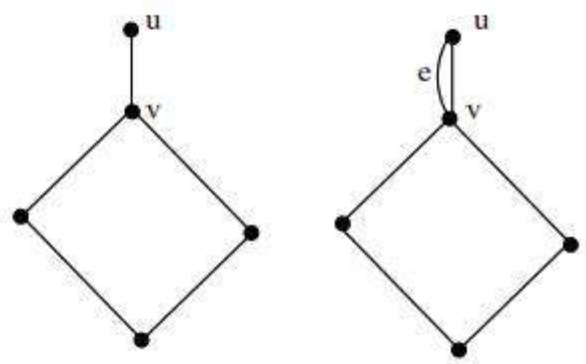
Theorem 5. If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.

Proof : Let v_1, v_2 and v_3 be vertices of odd degree. Since each of these vertices had odd degree, any possible Euler path must leave (arrive at) each of v_1, v_2, v_3 with no way to return (or leave). One vertex of these three vertices may be the beginning of Euler path and another the end but this leaves the third vertex at one end of an untravelled edge. Thus there is no Euler path.



Theorem 6. If G is a connected graph and has exactly two vertices of odd degree, then there is an Euler path in G. Further, any Euler path in G must begin at one vertex of odd degree and end at the other.

Proof: Let u and v be two vertices of odd degree in the given connected graph G.



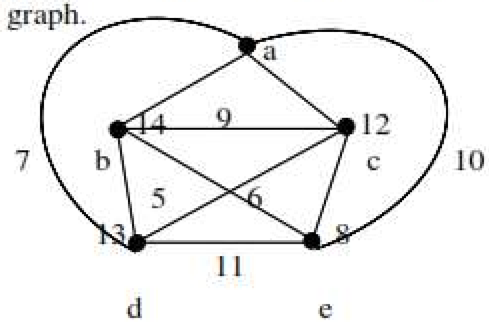
G G′

If we add the edge e to G, we get a connected graph G′ all of whose vertices have even degree. Hence there will be an Euler circuit in G′. If we omit e from Euler circuit, we get an Euler path beginning at u(or v) and edning at v(or u).

TRAVELLING SALESPERSON PROBLEM

This problem requires the determination of a **shortest Hamiltonian circuit** in a given graph of cities and lines of transportation to minimize the total fare for a travelling person who wants to made a tour of n cities visiting each city exactly once before returning home.
The weighted graph model for this problem consists of vertices representing cities and edges with weight as distances (fares) between the cities. The salesman starts and end his journey at the same city and visits each of $n - 1$ cities once and only once. We want to find minimum total distance.

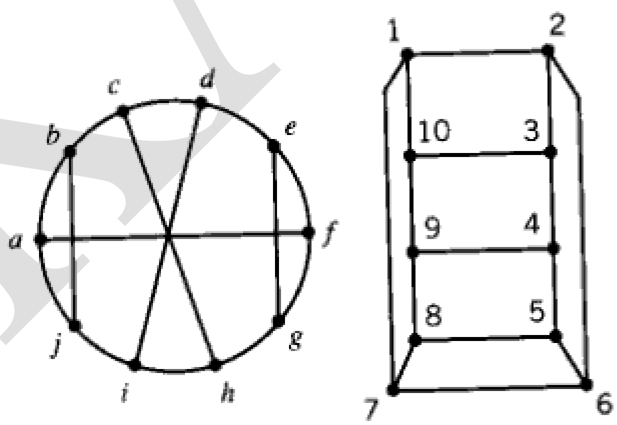
We discuss the case of five cities and so consider the following weighted graph.



POSSIBLE QUESTIONS

Answer All The Questions(5 X 8=40 Marks)

1) Define isomorphism and isomorphic graphs. Determine the following graphs are isomorphic are not?



2) State and prove the Handshaking theorem.

3) Prove that a connected graph G is an Euler graph iff it can be decomposed into circuits.

4) Prove that in a complete graph with n vertices there are $\frac{n-1}{2}$ edge disjoint Hamiltonian circuits if n is an odd number$\geq$ 3.

5) Prove that the number of vertices of odd degree in a graph is always even.

6) Define graph. Explain the various types of graph with an example.

7) Show that the sum of the degree of all vertices in a graph equal to twice in a number of edges incidence in G.

8) Show that if a graph G has exactly two vertices of odd degree there is a path joining these two vertices.

9) Show that a simple graph with n vertices and k-components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

10) Define (i) Bipartite Graph

(ii) Regular Graph

(iii) Complete Graph.

Give an example for each.

KAHE

UNIT-II

SYLLABUS

Definition – some properties of trees – Theorems – Rooted and Binary trees – Spanning trees. Cut set and cut vertices – some properties of a cut set – sets in a graph – Theorems – Fundamental circuits and cut sets – Connectivity and Separability – Theorems.

KAHE

Definition: A graph is said to be a **Tree** if it is a connected acyclic graph.

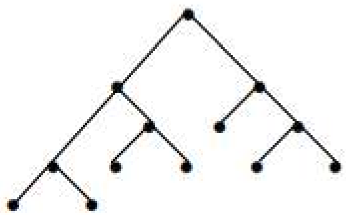
A **trivial tree** is a graph that consists of a single vertex. An **empty tree** is a tree that does not have any vertices or edges.

For example, the graphs shown below are all trees.

(i)
trivial tree

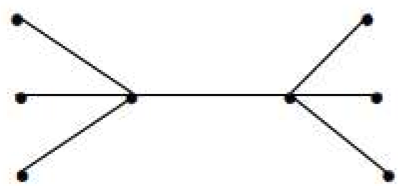
(ii)
Tree of 3 vertices

(iii)
Tree of 4 vertices



(iv)

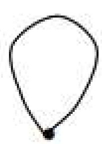
Tree of 13 vertices



(v)

Tree of 8 vertices

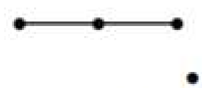
But the graphs shown below are not trees:



(i)

Has a cycle
and so is not a tree
so is not a tree

(ii)
has a cycle
and so is not a tree



(iii)
Disconnected graph
and so is not a tree

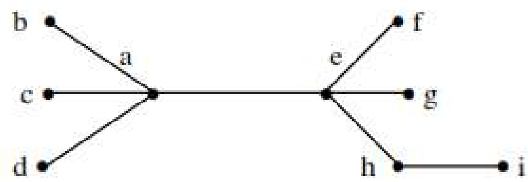
Definition: A collection of disjoint trees is called a **forest.**

Thus a graph is a forest if and only if it is circuit free.

Definition: A vertex of degree 1 in a tree is called a **leaf** or a **terminal node** or a **terminal vertex**.

Definition: A vertex of degree greater than 1 in a tree is called a **Branch node** or **Internal node** or **Internal vertex**.

Consider the tree shown below:



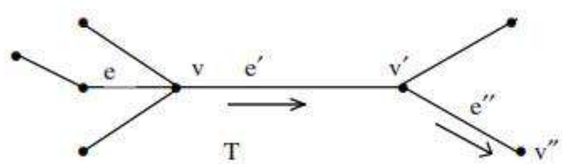
In this tree the vertices b, c, d, f, g, and i are leaves whereas the vertices a, e, h are branch nodes.

CHARACTERIZATION OF TREES

We have the following interesting characterization of trees:

Lemma 1: A tree that has more than one vertex has at least one vertex of degree 1.

Proof: Let T be a particular but arbitrary chosen tree having more than one vertex.



1. Choose a vertex v of T. Since T is connected and has at least two vertices, v is not isolated and there is an edge e incident on v.

2. If deg (v) > 1, there is an edge $e' \neq e$ because there are, in such a case, at least two edges incident on v. Let v' be the vertex at the other end of e'. This is possible because e' is not a loop by the definition of a tree.

3. If $\deg(v') > 1$, then there are at least two edges incident on v'. Let e'' be the other edge different from e' and v'' be the vertex at other end of e''. This is again possible because T is acyclic.

4. If $\deg(v'') > 1$, repeat the above process. Since the number of vertices of a tree is finite and T is circuit free, the process must terminate and we shall arrive at a vertex of degree 1.

Remark: In the proof of the above lemma, after finding a vertex of degree 1, if we return to v and move along a path outward from v starting with e, we shall reach to a vertex of degree 1 again. Thus it follows that **"Any tree that has more than one vertex has at least two vertices of degree 1".**

Lemma 2: There is a unique path between every two vertices in a tree.

Proof: Suppose on the contrary that there are more than one path between any two vertices in a given tree T. Then T has a cycle which contradicts the definition of a tree because T is acyclic. Hence the lemma is proved.

Lemma 3: The number of vertices is one more than the number of edges in a tree.

Or

For any positive integer n, a tree with n vertices has n-1 edges.

Proof: We shall prove the lemma by mathematical induction.

Let T be a tree with **one** vertex. Then T has no edges, that is, T has 0 edge. But $0 = 1 - 1$. Hence the lemma is true for $n = 1$.
Suppose that the lemma is true for $k > 1$. We shall show that it is then true for $k + 1$ also. Since the lemma is true for k, the tree has k vertices and k-1 edges. Let T be a tree with $k + 1$ vertices. Since k is +ve, $k+1 \geq 2$ and so T has more than one vertex. Hence, by Lemma 1, T has a vertex v of degree 1. Also there is another vertex w and so there is an edge e connecting v and w. Define a subgraph T' of T so that

$$V(T') = V(T) - \{v\}$$

$$E(T') = E(T) - \{e\}$$

Then number of vertices in $T' = (k+1) - 1 = k$ and since T is circuit free and T' has been obtained on removing one edge and one vertex, it follows that T' is acyclic. Also T' is connected. Hence T' is a tree having k vertices and therefore by induction hypothesis, the number of edges in T' is k-1. But then

No. of edges in T = number of edges in T′ + 1

$$= k - 1 + 1 = k$$

Thus the Lemma is true for tree having k + 1 vertices. Hence the lemma is true by mathematical induction.

Corollary 1. Let C(G) denote the number of components of a graph. Then a forest G on n vertices has n – C(G) edges.

Proof: Apply Lemma 3 to each component of the forest G.

Corollary 2. Any graph G on n vertices has at least n – C(G) edges.

Proof: If G has cycle-edges, remove them one at a time until the resulting graph G* is acyclic. Then G* has n – C(G*) edges by corollary 1. Since we have removed only circuit, C(G*) = C(G). Thus G* has n – C(G) edges. Hence G has at least n – C(G) edges.

Lemma 4: A graph in which there is a unique path between every pair of vertices is a tree

(This lemma is converse of Lemma 2).

between pair of vertices. Thus the graph is connected and acyclic and so is a tree.

Lemma 5. (converse of Lemma 3) A connected graph G with e = v – 1 is a tree

Proof: The given graph is connected and

$$e = v - 1.$$

To prove that G is a tree, it is sufficient to show that G is acyclic. Suppose on the contrary that G has a cycle. Let m be the number of vertices in this cycle. Also, we know that **number of edges in a cycle is equal to number of vertices in that cycle**. Therefore number of edges in the present case is m. Since the graph is connected, every vertex of the graph which is not in cycle must be connected to the vertices in the cycle.

Now each edge of the graph that is not in the cycle can connect only one vertex to the vertices in the cycle. There are v-m vertices that are not in the cycle. So the graph must contain at least v − m edges that are not in the cycle. Thus we have

$$e \geq v - m + m = v,$$

which is a contradiction to our hypothesis. Hence there is no cycle and so the graph in a tree.

Second proof of Lemma 5: We shall show that a connected graph with v vertices and v – 1 edges is a tree. It is sufficient to show that G is acyclic. Suppose on the contrary that G is not circuit free and has a non trivial circuit C. If we remove one edge of C from the graph G, we obtain a graph G′ which is connected.

If G′ still has a nontrivial circuit, we repeat the above process and remove one edge of that circuit obtaining a new connected graph. Continuing this process, we obtain a connected graph G* which is circuit free. Hence G* is a tree. Since no vertex has been removed, the tree G* has v vertices. Therefore, by Lemma 3, G* has v-1 edges. But at least one edge of G has been removed to form G*. This means that G* has not more than $v - 1 - 1 = v - 2$ edges. Thus we arrive at a contradiction. Hence our supposition is wrong and G has no cycle. Therefore G is connected and cycle free and so is a tree.

Lemma 6: A graph G with $e = v - 1$, that has no circuit is a tree.

Proof: It is sufficient to show that G is connected. Suppose G is not connected and let G′, G″….. be connected component of G. Since each of G′, G″,…. is connected and has no cycle, they all are tree. Therefore, by Lemma 3,

$$e' = v' - 1$$

$$e'' = v'' - 1$$

------------- ,

where e', e'', … are the number of edges and v', v'',… are the number of vertices in G', G'', …respectively. We have, on adding

$$e' + e'' + \ldots\ldots = (v' - 1) + (v'' - 1) + \ldots\ldots$$

Since

$$e = e' + e'' + \ldots..$$
$$v = v' + v'' + \ldots. \quad ,$$

we have

$$e < v - 1 \quad ,$$

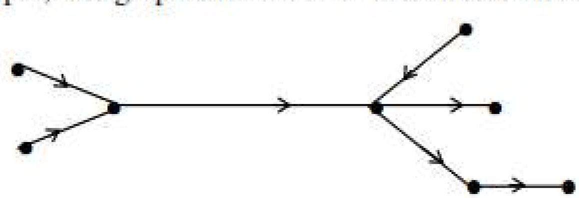
which contradicts our hypotheses. Hence G is connected. So G is connected and acyclic and is therefore a tree.

Example: Construct a graph that has 6 vertices and 5 edges but is not a tree.

Solution: We have, No. of vertices = 6, No. of edges = 5 . So the condition $e = v - 1$ is satisfied. Therefore, to construct graph with six vertices and 5 edges that is not a tree, we should keep in mind that the graph should not be connected. The graph shown below has 6 vertices and 5 edges but is not connected.

.

Definition: A directed graph is said to be a directed tree if it becomes a tree when the direction of edges are ignored.
For example, the graph shown below is a directed tree.



Definition: A directed tree is called a **rooted tree** if there is exactly one vertex whose incoming degree is 0 and the incoming degrees of all other vertices are 1.
The vertex with incoming degree 0 is called the **root** of the rooted tree.
A tree T with root v_0 will be denoted by (T, v_0).

Definition: In a rooted tree, a vertex, whose outgoing degree is 0 is called a **leaf** or **terminal node**, whereas a vertex whose outgoing degree is non - zero is called a **branch node** or an **internal node**.

Definition: Let u be a branch node in a rooted tree. Then a vertex v is said to be **child** (**son** or **offspring**) of u if there is an edge from u to v. In this case u is called **parent** (**father**) of v.

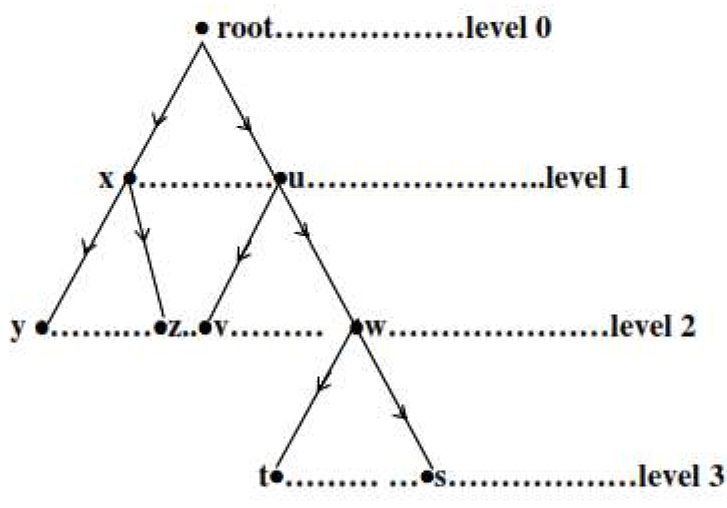
Definition: Two vertices in a rooted tree are said to be **siblings** (**brothers**) if they are both children of same parent.

Definition: A vertex v is said to be a **descendent** of a vertex u if there is a unique directed path from u to v.
In this case u is called the **ancestor** of v.

Definition: The **level** (or **path length**) of a vertex u in a rooted tree is the number of edges along the unique path between u and the root.

Definition: The **height** of a rooted tree is the maximum level to any vertex of the tree.

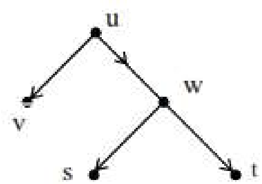
As an example of these terms consider the rooted tree shown below:



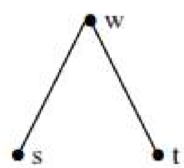
Here y is a child of x; x is the parent of y and z. Thus y and z are siblings. The descendents of u are v, w, t and s. Levels of vertices are shown in the figure. The height of this rooted tree is 3.

Definition: Let u be a branch node in the tree T = (V, E). Then the subgraph T' = (V', E') of T such that the vertices set V' contains u and all of its descendents and E' contains all the edges in all directed paths emerging from u is called a **subtree** with u as the root.

Definition: Let u be a branch node. By a subtree of u, we mean a subtree that has child of u as root.
In the above example, we note that the figure shown below is a subtree of T,



where as the figure shown below is a subtree of the branch node u .



is a subtree of the branch node u.

Example. Let

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

and let

$$E = (\{v_2, v_1), (v_2, v_3), (v_4, v_2), (v_4, v_5), (v_4, v_6), (v_6, v_7), (v_5, v_8)\}.$$

Show that (V, E) is rooted tree. Identify the root of this tree.

Solution: We note that

Incoming degree of $v_1 = 1$

Incoming degree of $v_2 = 1$

Incoming degree of $v_3 = 1$

Incoming degree of $v_4 = 0$

Incoming degree of $v_5 = 1$

Incoming degree of $v_6 = 1$

Incoming degree of $v_7 = 1$

Incoming degree of $v_8 = 1$

Since incoming degree of the vertex v_4 is 0, it follows that v_4 is root.

Further,

Outgoing degree of $v_1 = 0$

Outgoing degree of $v_3 = 0$

Outgoing degree of $v_7 = 0$

Outgoing degree of $v_8 = 0$

Therefore v_1, v_2, v_7, v_8 are leaves. Also ,

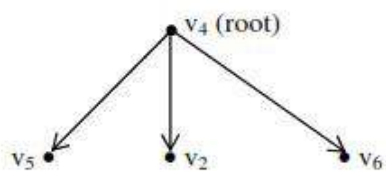
Outgoing degree of $v_2 = 2$

Outgoing degree of $v_4 = 3$

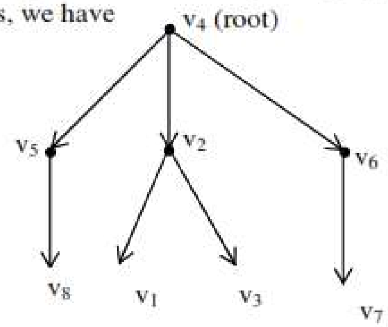
Outgoing degree of $v_5 = 1$

Outgoing degree of $v_6 = 1$

Now the root v_4 is connected to v_2, v_5 and v_6. So, we have

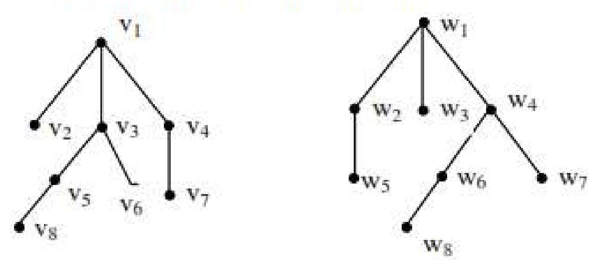


Now v_2 is connected to v_1 and v_3, v_5 is connected to v_8, v_6 is connected to v_7.
Thus, we have



We thus have a connected acyclic graph and so (V, E) is a rooted tree with root v_4.

Definition: Let T_1 and T_2 be rooted tree with roots r_1 and r_2 respectively. Then T_1 and T_2 are **isomorphic** if there exists a one-to-one, onto function f from the vertex set of T_1 to the vertex set of T_2 such that
(i) Vertices v_i and v_j are adjacent in T_1 if and only if the vertices $f(v_i)$ and $f(v_j)$ are adjacent in T_2.
(ii) $f(r_1) = r_2$
The function is then called an isomorphism.
Example: Show that the tree T_1 and T_2 are isomorphic.



Solution: We observe that T_1 and T_2 are rooted tree.
Define f: (Vertex set of T_1) $\rightarrow$ (Vertex set of T_2) by

$$f(v_1) = w_1$$

$$f(v_2) = w_3$$

$$f(v_3) = w_4$$

$$f(v_4) = w_2$$

$$f(v_5) = w_6$$

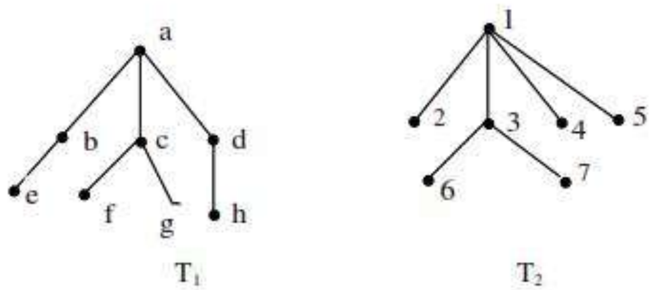
$$f(v_6) = w_7$$

$$f(v_7) = w_5$$

$$f(v_8) = w_8$$

Then f is one-to- one and adjacency relation is preserved. Hence f is an isomorphism and so the rooted tree T_1 and T_2 are isomorphic

Example: Show that the rooted tree shown below are not isomorphic:



Solution: We observe that the degree of root in T_1 is 3, whereas the degree of root in T_2 is 4. Hence T_1 is not isomorphic to T_2.

Definition: An ordered tree in which every branch node has atmost n offspring's is called a **n-ary tree** (or **n-tree**).

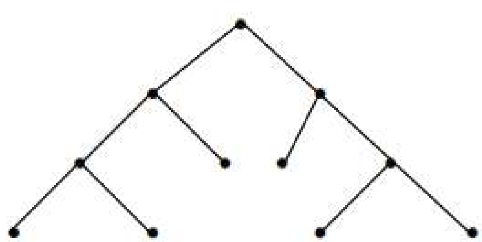
Definition: An n-ary tree is said to be **fully n-ary tree** (**complete n-ary tree** or **regular n ary tree**) if every branch node has exactly n offspring.

Definition: An ordered tree in which every branch node has almost 2 offsprings is called a **binary tree** (or **2 - tree**).

Definition: A binary tree in which every branch node (internal vertex) has exactly two offspring's is called a **fully binary tree**.

For example, the tree given below is a binary tree,

whereas the tree shown below is a fully binary tree.



Definition: Let T_1 and T_2 be binary trees roots r_1 and r_2 respectively. Then T_1 and T_2 are **isomorphic** if there is a one to one, onto function f from the vertex set of T_1 to the vertex set of T_2 satisfying

(i) Vertices v_i and v_j are adjacent in T_1 if and only if the vertices $f(v_i)$ and $f(v_j)$ are adjacent in T_2.

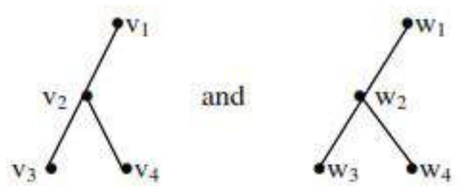
(ii) $f(r_1) = r_2$

(iii) v is a left child of w in T_1 if and only if f(v) is a left child of f(w) in T_2

(iv) v is a right child of w in T_1 if and only if f(v) is a right child of f(w) in T_2.

The function f is then called an **isomorphism** between binary tree T_1 and T_2

Example: Show that the trees given below are isomorphic.



Solution: Define f by $f(v_i) = w_i$, i = 1, 2, 3, 4. Then f satisfies all the properties for isomorphism. Hence T_1 and T_2 are isomorphic.

Example: Show that the trees given below are not isomorphic.

Solution: Since the root v_1 in T_1 has a left child but the root w_1 in T_2 has no left child, the binary trees are not isomorphic.

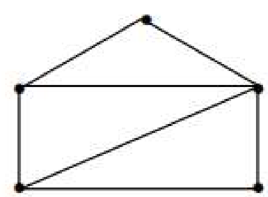
Definition: Let G be a graph, then a subgraph of G which is a tree is called **tree of the graph.**

Definition: A **spanning tree** for a graph G is a subgraph of G that contains every vertex of G and is a tree.

Or

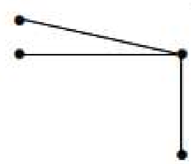
"A **spanning tree** for a graph G is a spanning subgroup of G which is a tree".

Example: Determine a tree and a spanning tree for the connected graph given below:

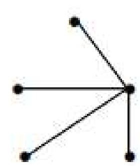


G

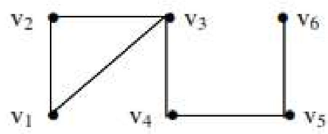
Solution: The given graph G contains circuits and we know that removal of the circuits gives a tree. So, we note that the figure below is a tree.



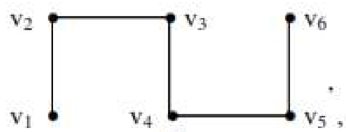
And the figure below is a spanning tree of the graph G.



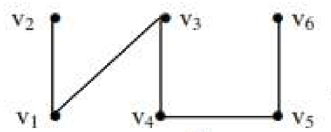
Example: Find all spanning trees for the graph G shown below:



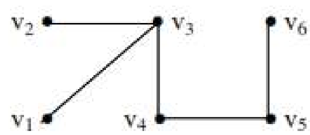
Solution: The given graph G has a circuit $v_1\, v_2\, v_3\, v_1$. We know that removal of any edge of the circuit gives a tree. So the spanning trees of G are



T_1



T_2



T_3

Remark: We know that a tree with n vertices has exactly n – 1 edges. Therefore if G is a connected graph with n vertices and m edges, a spanning tree of G must have n – 1 edges. Hence the number of edges that must be removed before a spanning tree is obtained must be

$$m - (n - 1) = m - n + 1.$$

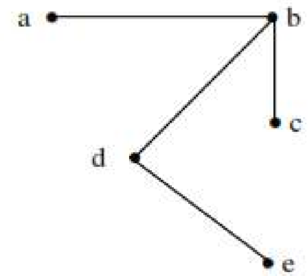
For Illustration, in the above example, n = 6, m = 6, so, we had to remove one edge to obtain a spanning tree.

Cut Sets

Let G be a connected graph. We know that the distance between two vertices v_1 and v_2, denoted by $d(v_1, v_2)$, is the **length of the shortest path.**

Definition: The **diameter** of a connected graph G, denoted by diam (G), is the maximum distance between any two vertices in G.

For example, in graph G shown below, we have



G

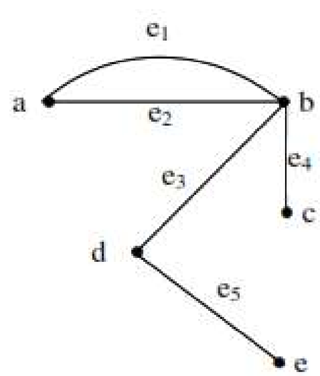
$d(a, e) = 3$, $d(a, c) = 2$, $d(b, e) = 2$ and diam $(G) = 3$.

Definition: A vertex in a connected graph G is called a **cut point** if $G - v$ is disconnected, where $G - v$ is the graph obtained from G by deleting v and all edges containing v.

For example, in the above graph, d is a cut point.

Definition: An edge e of a connected graph G is called a **bridge** (or cut edge) if $G - e$ is disconnected, where $G - e$ is the graph obtained by deleting the edge e.

For example, consider the graph G shown below :

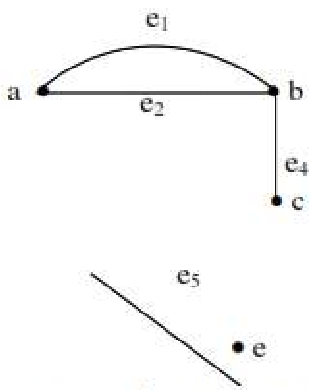


G

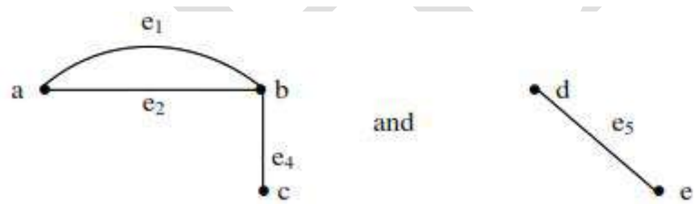
We observe that $G - e_3$ is disconnected. Hence the edge e_3 is a bridge.

Definition: A minimal set C of edges in a connected graph G is said to be a **cut set** (or **minimal edge – cut**) if the subgraph G – C has more connected components than G has.

For example, in the above graph, if we delete the edge (b, d) = e_3, the resulting subgraph G – e_3 is as shown below :

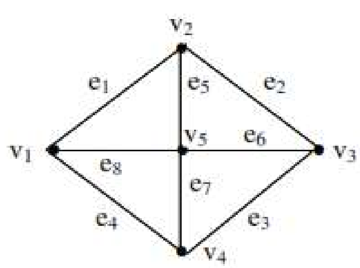


Thus G – e_3 has two connected components

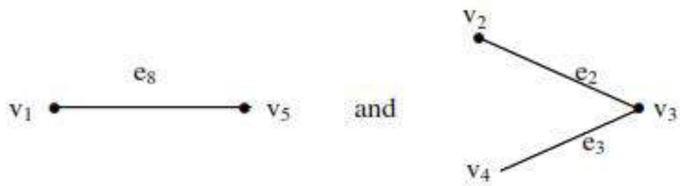


So, in this example, the cut set consists of single edge (b, d) = e_3, which is called edge or bridge.

Example: Find a cut set for the graph given below:



Solution : The given graph is connected. It is sufficient to reduce the graph into two connected components. To do so we have to remove the edges e_1, e_4, e_5, e_6, e_7. The two connected components are

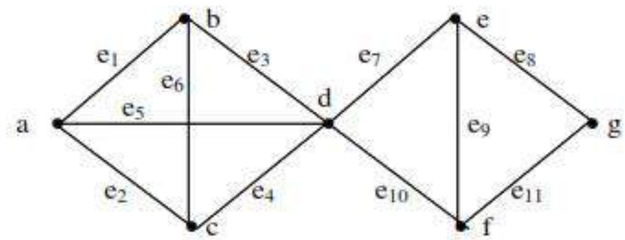


But, if we remove any proper subset of $\{e_1, e_4, e_5, e_6, e_7\}$, then there is no increase in connected components of G.
Hence

$$\{e_1, e_4, e_5, e_6, e_7\}$$

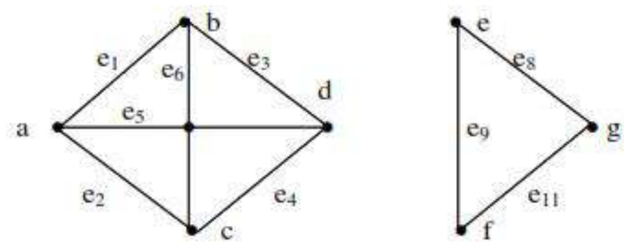
is a cut set.

Example: Find a cut set for the graph



G

Solution: The given graph is a connected graph. We note that removal of the edges e_7 and e_{10} creates two connected components of G shown below:



Hence the set $\{e_7, e_{10}\}$ is a cut set for the given graph G.

Theorem: Let G be a connected graph with n vertices. Then G is a tree if and only if every edge of G is a bridge (cut edge).

(This theorem asserts that every edge in a tree is a bridge).

Proof: Let G be a tree. Then it is connected and has $n - 1$ edges (proved already). Let e be an arbitrary edge of G. Since $G - e$ has $n - 2$ edges, and also we know that a graph G with n vertices has at least $n - c(G)$ edges, it follows that $n - 2 \geq n - c(G - e)$. Thus $G - e$ has at least two components. Thus removal of the edge e created more components than in the graph G. Hence e is a cut edge. This proves that every edge in a tree is a bridge.

Conversely, suppose that G is connected and every edge of G is a bridge. We have to show that G is a tree. To prove it, we have only to show that G is circuit – free. Suppose on the contrary that there exists a cycle between two points x and y in G. Then any edge on this cycle is

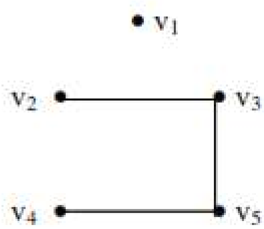
not a cut edge which contradicts the fact that every edge of G is a cut edge. Hence G has no cycle. Thus G is connected and acyclic and so is a tree.

Definition: Let v be the number of vertices and e be the number of edges in a graph G. Then the set of e – v + 1 circuits obtained by adding e – v + 1 chords to a spanning tree of G is called the **fundamental system of circuits relative to the spanning tree.**

A circuit in the fundamental system is called a **fundamental circuit.**

For example, $\{v_1, v_2, v_3, v_1\}$ is the fundamental circuit corresponding to the chord (v_1, v_2).

On the other hand, since each branch of a tree is cut edge, removal of any branch from a spanning tree breaks the spanning tree into two trees. For example, if we remove (v_1, v_3) from the above figured spanning tree, the resulting components are shown in the figure below :



Thus, **to every branch in a spanning tree, there is a corresponding cut set.** But, in a spanning tree, there are v – 1 branches. Therefore, **there are v – 1 cut sets corresponding to v – 1 branches.**

Definition: The set of v – 1 cut sets corresponding to v – 1 branches in a spanning tree of a graph with v vertices is called the **fundamental system of cut sets relative to the spanning tree.**

A cut – set in the fundamental system of cut – sets is called a **fundamental cut set.**

For example, the fundamental cut – sets in the spanning tree (figured above) is

$\{(v_1, v_2), (v_1, v_3)\}, \{(v_1, v_3), (v_2, v_3), (v_3, v_4)\},$

$\{(v_3, v_5), (v_4, v_5)\}, \{v_2, v_4), (v_4, v_5)\}.$

Theorem: A circuit and the complement of any spanning tree must have at least one edge in common.

Proof: We recall that the set of all chords of a tree is called the complement of the tree. Suppose on the contrary that a circuit has no common edge with the complement of a spanning tree. This means the circuit is wholly contained in the spanning tree. This contradicts the fact that a tree is acyclic (circuit – free). Hence a circuit has at least one edge in common with complement of a spanning tree.

Theorem: A cut – set and any spanning tree must have at least one edge in common.

Proof: Suppose on the contrary that there is a cut set which does not have a common edge with a spanning tree. Then removal of cut set has not effect on the tree, that is, the cut set will not separate the graph into two components. But this contradicts the definition of a cut set. Hence the result.

Theorem: Every circuit has an even number of edges in common with every cut – set.

Proof: We know that a cut – set divides the vertices of the graph into two subsets each being set of vertices in one of the two components. Therefore a path connecting two vertices in one subset must traverse the edges in the cut set an even number of times. Since a circuit is a path from some vertex to itself, it has an even number of edges in common with every cut – set.

POSSIBLE QUESTIONS

Answer All The Questions(5 X 8=40 Marks)

1) Define i) distance between two spanning trees
   ii) cyclic interchange
   iii) rank
   iv) nullity
2) Prove that the ring sum of any two cut sets in a graph is either a third cut set or an edge disjoint union of cut sets.
3) Every connected graph has at least one spanning tree.
4) Prove that every circuit has even number of edges in common with any cut-set.
5) Prove that the number of pendent vertices of a tree is equal to $\frac{n+1}{2}$
6)Define i) edge connectivity ii) vertex connectivity and iii) minimally connected. Give an example for each.
7) State and prove necessary and sufficient condition for a shortest spanning tree
8)Show that a graph G is a tree if and only if there is one and only one path between any two vertices of G
9) If G is a tree with n vertices then prove that G has n-1 edges.
10) Prove that every cut-set in a connected graph G must contain atleast one branch of every spanning tree of G.

UNIT-III

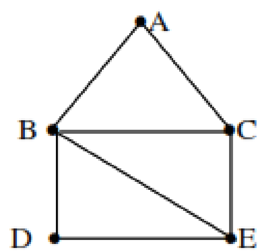
SYLLABUS

Kuratowski‟s two graphs – Theorems – Different representation of a planar graph – Detection of planarity – Thickness and crossings.

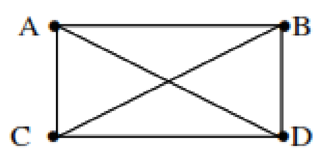
Planar Graphs

Definition: A graph which can be drawn in the plane so that its edges do not cross is said to be **planar.**

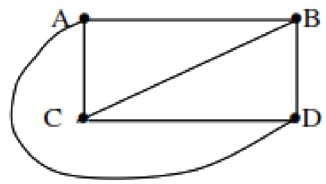
For example, the graph shown below is planar :



Also the complete graph K_4 shown below is planar.



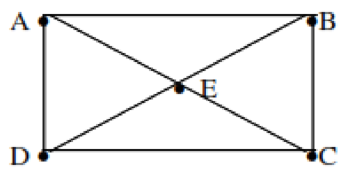
In fact, it can be redrawn as



,

so that no edges cross.

But the complete map K_5 is not planar because in this case, the edges cross each others.

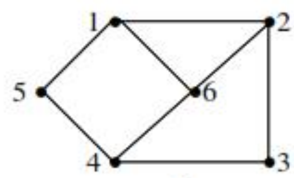


K_5

Definition: An area of the plane that is bounded by edges of the planar graph is not further subdivided into subareas is called a **region** or **face** of a planar graph.
A face is characterised by the cycle that forms its boundary.

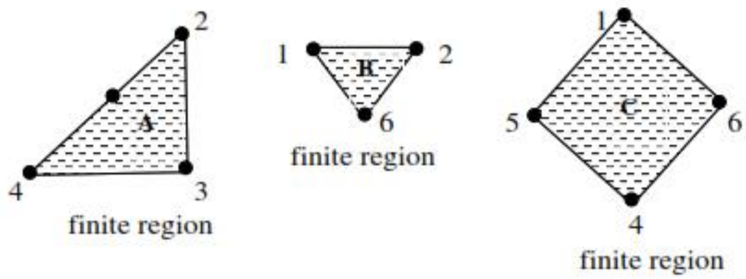
Definition: A region is said to be finite if its area is finite and infinite if its area is infinite. Clearly a planar graph has exactly one infinite region.

For example, consider the graph :

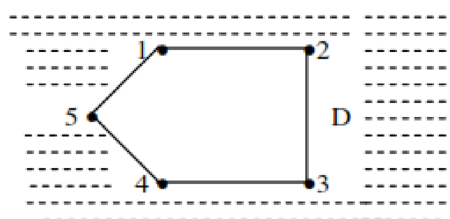


G_2

In graph G_2, there are four region **A, B, C, D**



and



infinite region

Definition: Let f be a face (region) in a planar graph. The length of the cycle (or closed walk) which borders f is called the **degree of the region f**. It is denoted by deg(f).

In a planar graph we note that **each edge either borders two regions or is contained in a region and will occur twice in any walk along the border of the region.** Thus we have

Theorem: The sum of the degrees of the regions of a map is equal to twice the number of edges.
For example, in the graph G_2, discussed above, we have

$$\deg(A) = 4, \deg(B) = 3, \deg(C) = 4, \deg(d) = 5$$

The sum of degrees of all regions = 4 + 3 + 4 + 5 = 16

No. of edges in G_2 = 8

Hence

"sum of degrees of region is twice the number of edges".

Theorem (Euler's formula for connected planar graphs): If G is a connected planar graph with e edges, v vertices and r regions, then

$$\mathbf{v - e + r = 2}$$

Proof: We shall use induction on the number of edges. Suppose that e = 0. Then the graph G consists of a single vertex, say P. Thus G is as shown below:

•P

and we have

$$e = 0, v = 1, r = 1$$

Thus

$$1 - 0 + 1 = 2$$

and the formula holds in this case.

Suppose that e = 1. Then the graph G is one of the two graphs shown below:

e = 1, v = 2, r = 1

e = 1, v = 1, r = 2

We see that, in either case, the formula holds.

Suppose that the formula holds for connected planar graph with n edges. We shall prove that this holds for graph with n + 1 edges. So, let G be the graph with n + 1 edges. Suppose first that G contains no cycles. Choose "a" vertex v_1 and trace a path starting at v_1. Ultimately, we will reach a vertex a with degree 1, that we cannot leave.

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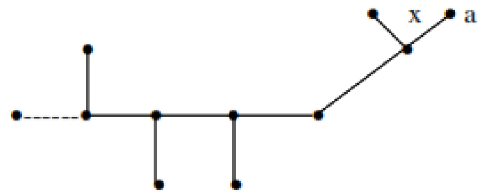
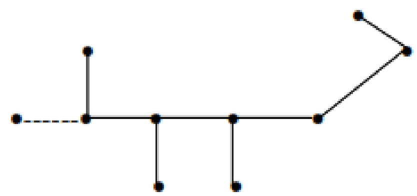


G

We delete "a" and the edge x incident on "a" from the graph G. The resulting graph G′ has n edges and so by induction hypothesis, the formula holds for G′. Since G has one more edge than G′,one more vertex than G′ and the same number of faces as G′, it follows that the formula $v - e + r = 2$ holds also for G.



G′

Now suppose that G contains a cycle. Let x be an edge in a cycle.

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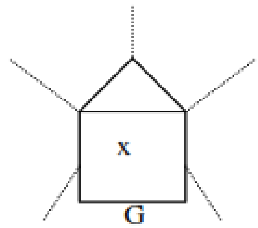
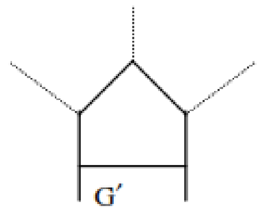


Now the edge x is part of a boundary for two faces. We delete the edge x but no vertices to obtain the graph G′

.



Thus G′ has n edges and so by induction hypothesis the formula holds. Since G has one more face (region) than G′, one more edge than G′ and the same number of vertices as G′, it follows that the formula $v - e + r = 2$ also holds for G. Hence, by Mathematical Induction, the theorem is true.

Theorem 6.2. (The Linear Bound) *If a simple connected planar graph G has $n \geq 3$ vertices and m edges, then*

$$m \leq 3n - 6.$$

Proof. If the regions of a planar embedding of G are $s_1, \ldots, s_f$, then we denote the number of boundary edges of s_i by r_i ($i = 1, \ldots, f$). The case $f = 1$ is obvious because G is then a tree and $m = n - 1 \leq 3n - 6$. Thus, we assume that $f \geq 2$. Since G is simple, every region has at least 3 boundary edges and thus

$$\sum_{i=1}^{f} r_i \geq 3f.$$

Every edge is a boundary edge of one or two regions in the planar embedding, so

$$\sum_{i=1}^{f} r_i \leq 2m.$$

The result now follows directly from Euler's Polyhedron Formula. □

Theorem 6.3. (The Minimum Degree Bound) *For a simple planar graph G, $\delta(G) \leq 5$.*

Proof. Let us prove by contradiction and consider the counter hypothesis: G is a simple planar graph and $\delta(G) \geq 6$. Then, (by Theorem 1.1) $m \geq 3n$, where n is the number of vertices and m is the number of edges in G. ($\surd$ Theorem 6.2) □

A characterization of planar graphs is obtained by examining certain forbidden subgraphs.

Theorem 6.4. (Kuratowski's Theorem) *A graph is planar if and only if none of its subgraphs can be transformed to K_5 or $K_{3,3}$ by contracting edges.*

The proof is quite complicated (but elegant!), refer e.g. to SWAMY & THULASIRAMAN for more information. K_5 and $K_{3,3}$ are not planar, which can be verified easily.

There are many fast but complicated algorithms for testing planarity and drawing planar embeddings. For example, the *Hopcroft–Tarjan Algorithm*[2] is one. We present a slower classical polynomial time algorithm, the *Demoucron–Malgrange–Pertuiset Algorithm*[3] (usually just called *Demoucron's Algorithm*). The idea of the algorithm is to try to draw a graph on a plane piece by piece. If this fails, then the graph is not planar.

If G is a graph and R is a planar embedding of a planar subgraph S of G, then an *R-piece* P of G is

- either an edge of $G - S$ whose end vertices are in S, or
- a component of the subgraph induced by vertices not in S which contains the edges (if any) that connect S to the component, known as *pending edges*, and their end vertices.

Those vertices of an R-piece of G that are end vertices of pending edges connecting them to S are called *contact vertices*. We say that a planar embedding R of the planar subgraph S is *planar extendable* to G if R can be extended to a planar embedding of the whole G by drawing more vertices and/or edges. Such an extended embedding is called a *planar extension* of R to G. We say further that an R-piece P of G is *drawable* in a region s of R if there is a planar extension of R to G where P is inside s. Obviously all contact vertices of P must then be boundary vertices of s, but this is of course not sufficient to guarantee planar extendability of R to G. Therefore we say that a P is *potentially drawable* in s if its contact vertices are boundary vertices of s. In particular, a piece with no contact vertices is potentially drawable in any region of R.

Theorem 6.5. (The Four-Color Theorem) *Every simple planar graph is* 4*-colorable.*

Proof. The only known proofs require extensive computer runs. The first such proof was obtained by Kenneth Appel ja Wolfgang Haken in 1976. It takes a whole book to present the proof: APPEL, K. & HAKEN, W.: *Every Planar Map is Four Colorable.* American Mathematical Society (1989). □

If we require a bit less, i.e. 5-colorability, then there is much more easily provable result, and an algorithm.

Theorem 6.6. (Heawood's Theorem or The Five-Color Theorem) *Every simple planar graph is* 5*-colorable.*

Proof. We may think of G as a planar embedding. We use induction on the number n of vertices of G.

Induction Basis: $n = 1$. Our graph is now 1-colorable since there are no edges.

Induction Hypothesis: The theorem is true for $n \leq \ell$. ($\ell \geq 1$)

Induction Statement: The theorem is true for $n = \ell + 1$.

Induction Statement Proof: According to the Minimum Degree Bound, there is a vertex v in G of degree at most 5. On the other hand, according to the Induction Hypothesis the graph $G - v$ is 5-colorable. If, in this coloring, the vertices adjacent to v are colored using at most four colors, then clearly we can 5-color G.

So we are left with the case where the vertices v_1, v_2, v_3, v_4, v_5 adjacent to v are colored using different colors. We may assume that the indexing of the vertices proceeds clockwise, and we label the colors with the numbers $1, 2, 3, 4, 5$ (in this order). We show that the coloring of $G - v$ can be changed so that (at most) four colors suffice for coloring v_1, v_2, v_3, v_4, v_5.

We denote by $H_{i,j}$ the subgraph of $G - v$ induced by the vertices colored with i and j. We have two cases:

- v_1 and v_3 are in different components H_1 and H_3 of $H_{1,3}$. We then interchange the colors 1 and 3 in the vertices of H_3 leaving the other colors untouched. In the resulting 5-coloring of $G - v$ the vertices v_1 and v_3 both have the color 1. We can then give the color 3 to v.

- v_1 and v_3 are connected in $H_{1,3}$. Then there is a v_1–v_3 path in $H_{1,3}$. Including the vertex v we get from this path a circuit C. Now, since we indexed the vertices v_1, v_2, v_3, v_4, v_5 clockwise, exactly one of the vertices v_2 and v_4 is inside C. We deduce that v_2 and v_4 are in different components of $H_{2,4}$, and we have a case similar to the previous one. □

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POSSIBLE QUESTIONS

Answer All The Questions(5 X 8=40 Marks)

1) Prove that a graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.
2) Define with example.
   i) planar ii)non planar iii)region iv)infinite region
3) Prove that the spherical embedding of every planar 3- connected graph is unique.
4) Show that if G is connected simple planar graph with n($\geq$3) vertices and e is edge then e $\leq$ 3n-6.
5) Prove that Kuratowski's second graph is also non planar.
6) Prove that a connected graph with n vertices and e edges has e-n+2 regions.
7) Show that a planar graph can be embedded in a plane such that any specified region can be made the infinite region.
8) Show that if G is connected simple planar graph with n($\geq$3) vertices and e is edge then

   e $\leq$ 3n-6.
9) Prove that the vertices of every planar graph can be properly colored with five colors.

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UNIT-IV

SYLLABUS

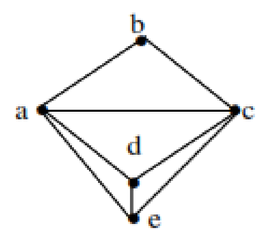
Covering partitioning – Chromatic number Theorems –Chromatic partitioning – Independent set – Finding a maximal independent set – Dominating set – Finding minimal dominating set – Chromatic polynomial – Theorems. Coverings – Theorems – Four colour problem - Five colour Theorem.

Colouring of Graph

Definition: Let G be a graph. The assignment of colours to the vertices of G, one colour to each vertex, so that the adjacent vertices are assigned different colours is called **vertex colouring** or **colouring of the graph G.**
Definition: A graph G is **n-colourable** if there exists a colouring of G which uses n colours.

Definition: The minimum number of colours required to paint (colour) a graph G is called the **chromatic number of G** and is denoted by **χ (G).**

Example: Find the chromatic number for the graph shown in the figure below:



G

Solution: The triangle a b c needs three colours. Suppose that we assign colours c_1, c_2, c_3 to a, b and c respectively. Since d is adjacent to a and c, d will have different colour than c_1 and c_3. So we paint d by c_2. Then e must be painted with a colour different from those of a, d and c, that is, we cannot colour e with c_1, c_2 or c_3. Hence, we have to give e a fourth colour c_4. Hence

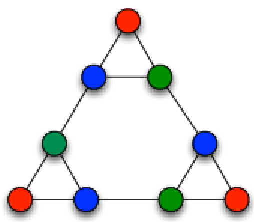
$$\chi(G) = 4.$$

Vertex Coloring of Graphs

(Vertex Coloring). Let $G = (V, E)$ be a graph and let $C = \{c_1, \ldots, c_k\}$

be a finite set of *colors* (labels). A *vertex coloring* is a mapping $c : V \to C$ with the property that if $\{v_1, v_2\} \in E$, then $c(v_1) \neq c(v_2)$.

We show an example of a graph coloring



DEFINITION 10.3 (k-Colorable). A graph $G = (V, E)$ is a k-colorable if there is a vertex coloring with k colors.

REMARK 10.4. Clearly, every graph $G = (V, E)$ is $|V|$ colorable, since we can assign a different color to each vertex. We are usually interested in the minimum number of colors we can get away with and still color a graph.

DEFINITION 10.5 (Chromatic Number). Let $G = (V, E)$ be a graph. The *chromatic number* of G, written $\chi(G)$ is the minimum integer k such that G is k-colorable.

PROPOSITION 10.6. *Every bipartite graph is* 2*-colorable.*

EXERCISE 85. Prove Proposition 10.6.

PROPOSITION 10.7. *If $G = (V, E)$ and $|V| = n$. Then:*

$$\chi(G) \geq \frac{n}{\alpha(G)} \tag{10.1}$$

where $\alpha(G)$ is the independence number of G.

PROOF. Suppose $\chi(G) = k$ and consider the set of vertices $V_i = \{v \in V : c(v) = c_i\}$. Then this set of vertices is an independent set and contains at most $\alpha(G)$ elements. Thus:

$$n = |V_1| + |V_2| + \cdots + |V_k| \leq \alpha(G) + \alpha(G) + \cdots + \alpha(G) \tag{10.2}$$

Thus:

$$n \leq k \cdot \alpha(G) \implies \frac{n}{\alpha(G)} \leq k \tag{10.3}$$

PROPOSITION 10.8. *The chromatic number of K_n is n.*

PROOF. From the previous proposition, we know that:

$$\chi(K_n) \geq \frac{n}{\alpha(K_n)} \tag{10.4}$$

But $\alpha(K_n) = 1$ and thus $\chi(K_n) \geq n$. From Remark 10.4, it is clear that $\chi(K_n) \leq n$. Thus, $\chi(K_n) = n$. □

Theorem 6.5. (The Four-Color Theorem) *Every simple planar graph is 4-colorable.*

Proof. The only known proofs require extensive computer runs. The first such proof was obtained by Kenneth Appel ja Wolfgang Haken in 1976. It takes a whole book to present the proof: APPEL, K. & HAKEN, W.: *Every Planar Map is Four Colorable.* American Mathematical Society (1989). □

If we require a bit less, i.e. 5-colorability, then there is much more easily provable result, and an algorithm.

Theorem 6.6. (Heawood's Theorem or The Five-Color Theorem) *Every simple planar graph is 5-colorable.*

Proof. We may think of G as a planar embedding. We use induction on the number n of vertices of G.

Induction Basis: $n = 1$. Our graph is now 1-colorable since there are no edges.

Induction Hypothesis: The theorem is true for $n \leq \ell$. ($\ell \geq 1$)

Induction Statement: The theorem is true for $n = \ell + 1$.

Induction Statement Proof: According to the Minimum Degree Bound, there is a vertex v in G of degree at most 5. On the other hand, according to the Induction Hypothesis the graph $G - v$ is 5-colorable. If, in this coloring, the vertices adjacent to v are colored using at most four colors, then clearly we can 5-color G.

So we are left with the case where the vertices v_1, v_2, v_3, v_4, v_5 adjacent to v are colored using different colors. We may assume that the indexing of the vertices proceeds clockwise, and we label the colors with the numbers $1, 2, 3, 4, 5$ (in this order). We show that the coloring of $G - v$ can be changed so that (at most) four colors suffice for coloring v_1, v_2, v_3, v_4, v_5.

We denote by $H_{i,j}$ the subgraph of $G - v$ induced by the vertices colored with i and j. We have two cases:

- v_1 and v_3 are in different components H_1 and H_3 of $H_{1,3}$. We then interchange the colors 1 and 3 in the vertices of H_3 leaving the other colors untouched. In the resulting 5-coloring of $G - v$ the vertices v_1 and v_3 both have the color 1. We can then give the color 3 to v.

- v_1 and v_3 are connected in $H_{1,3}$. Then there is a v_1–v_3 path in $H_{1,3}$. Including the vertex v we get from this path a circuit C. Now, since we indexed the vertices v_1, v_2, v_3, v_4, v_5 clockwise, exactly one of the vertices v_2 and v_4 is inside C. We deduce that v_2 and v_4 are in different components of $H_{2,4}$, and we have a case similar to the previous one. □

POSSIBLE QUESTIONS

Answer All The Questions(5 X 8=40 Marks)

1)Prove that the complete graph of five vertices is non planar

2) Prove that a covering g of a graph is minimal if and only if g contains no paths of length three or more.

3) State and prove five color theorem

4) Prove that an n vertex graph is a tree iff $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$.

5) State and prove four color problem.

6) Prove that the vertices of every planar graph can be properly colored with five colors.

7) Prove that a covering of a graph is minimal iff graph contains no paths of length three or more

8) Prove that a graph of n vertices is a complete graph iff its chromatic polynomial $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$

9) Define chromatic number. Find the chromatic polynomial for the cycle of length 4. Hence find its chromatic number.

10) Show that every tree with two or more vertices is 2-chromatic.

UNIT-V

SYLLABUS

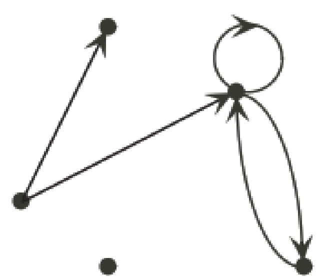
Definition – Some types of di-graphs – Directed path and connectedness – Euler di-graphs – Theorems – Trees with direct edges - Theorems – odded trees – Matrix representation – incidence matrix – Theorems – Circuit matrix – Adjacency matrix – Tournaments.

Directed Graphs

Definition

Intuitively, a *directed graph* or *digraph* is formed by vertices connected by *directed edges* or *arcs*.[1]

Example.



Formally, a digraph is a pair (V, E), where V is the vertex set and E is the set of vertex pairs as in "usual" graphs. The difference is that now the elements of E are ordered pairs: the arc from vertex u to vertex v is written as (u, v) and the other pair (v, u) is the opposite direction arc. We also have to keep track of the multiplicity of the arc (direction of a loop is irrelevant). We can pretty much use the same notions and results for digraphs from Chapter 1. However:

1. Vertex u is the *initial vertex* and vertex v is the *terminal vertex* of the arc (u, v). We also say that the arc is *incident out* of u and *incident into* v.
2. The *out-degree* of the vertex v is the number of arcs out of it (denoted $d^+(v)$) and the *in-degree* of v is the number of arcs going into it (denoted $d^-(v)$).
3. In the *directed walk* (*trail, path* or *circuit*),

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k}$$

   v_{i_ℓ} is the initial vertex and $v_{i_{\ell-1}}$ is the terminal vertex of the arc e_{j_ℓ}. .
4. When we treat the graph (V, E) as a usual undirected graph, it is the *underlying undirected graph* of the digraph $G = (V, E)$, denoted G_{u}.

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5. Digraph G is *connected* if G_u is connected. The *components* of G are the directed subgraphs of G that correspond to the components of G_u. The vertices of G are connected if they are connected in G_u. Other notions for undirected graphs can be used for digraphs as well by dealing with the underlying undirected graph.

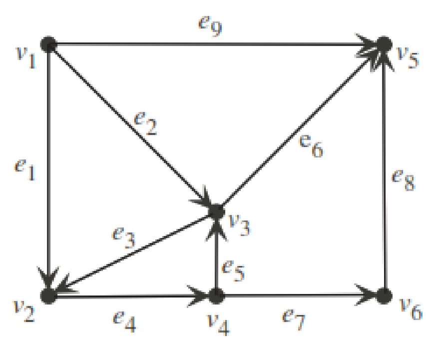
6. Vertices u and v are *strongly connected* if there is a directed u–v path and also a directed v–u path in G.

7. Digraph G is *strongly connected* if every pair of vertices is strongly connected. By convention, the trivial graph is strongly connected.

8. A *strongly connected component* H of the digraph G is a directed subgraph of G (not a null graph) such that H is strongly connected, but if we add any vertices or arcs to it, then it is not strongly connected anymore.

Every vertex of the digraph G belongs to one strongly connected component of G (compare to Theorem 1.3). However, an arc does not necessarily belong to any strongly connected component of G.

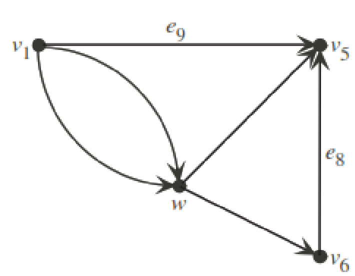
Example. *For the digraph G*



the strongly connected components are $(\{v_1\}, \emptyset)$, $(\{v_2, v_3, v_4\}, \{e_3, e_4, e_5\})$, $(\{v_5\}, \emptyset)$ *and* $(\{v_6\}, \emptyset)$.

The *condensed graph* G_c of the digraph G is obtained by contracting all the arcs in every strongly connected component.

Example. *(Continuing from the previous example) The condensed graph is*



Directed Trees

A directed graph is *quasi-strongly connected* if one of the following conditions holds for every pair of vertices u and v:

(i) $u = v$ or

(ii) there is a directed u–v path in the digraph or

(iii) there is a directed v–u path in the digraph or

(iv) there is a vertex w so that there is a directed w–u path and a directed w–v path.

Example. *(Continuing from the previous example) The digraph G is quasi-strongly connected.*

Quasi-strongly connected digraphs are connected but not necessarily strongly connected.

The vertex v of the digraph G is a *root* if there is a directed path from v to every other vertex of G.

Example. *(Continuing from the previous example) The digraph G only has one root, v_1.*

Theorem 3.1. *A digraph has at least one root if and only if it is quasi-strongly connected.*

Proof. If there is a root in the digraph, it follows from the definition that the digraph is quasi-strongly connected.

Let us consider a quasi-strongly connected digraph G and show that it must have at least one root. If G is trivial, then it is obvious. Otherwise, consider the vertex set $V = \{v_1, \ldots, v_n\}$ of G where $n \geq 2$. The following process shows that there must be a root:

1. Set $P \leftarrow V$.

2. If there is a directed u–v path between two distinct vertices u and v in P, then we remove v from P. Equivalently, we set $P \leftarrow P - \{v\}$. We repeat this step as many times as possible.

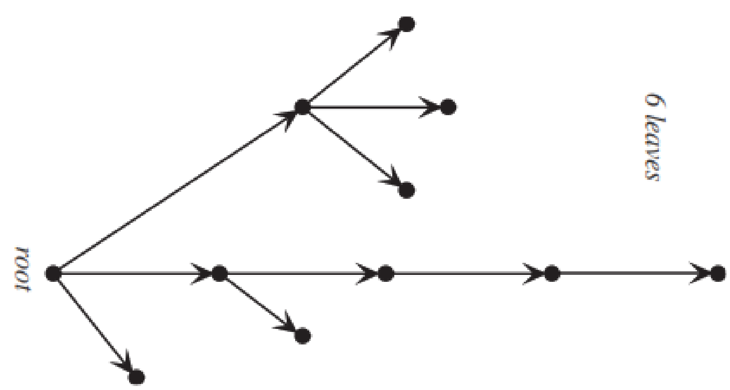
3. If there is only one vertex left in P, then it is the root. For other cases, there are at least two distinct vertices u and v in P and there is no directed path between them in either direction. Since G is quasi-strongly connected, from condition (iv) it follows that there is a vertex w and a directed w–u path as well as a directed w–v path. Since u is in P, w can not be in P. We remove u and v from P and add w, i.e. we set $P \leftarrow P - \{u, v\}$ and $P \leftarrow P \cup \{w\}$. Go back to step #2.

4. Repeat as many times as possible.

Every time we do this, there are fewer and fewer vertices in P. Eventually, we will get a root because there is a directed path from some vertex in P to every vertex we removed from P. □

The digraph G is a *tree* if G_u is a tree. It is a *directed tree* if G_u is a tree and G is quasi-strongly connected, i.e. it has a root. A *leaf* of a directed tree is a vertex whose out-degree is zero.

Example.



Theorem 3.2. *For the digraph G with $n > 1$ vertices, the following are equivalent:*

(i) *G is a directed tree.*

(ii) *G is a tree with a vertex from which there is exactly one directed path to every other vertex of G.*

(iii) *G is quasi-strongly connected but $G - e$ is not quasi-strongly connected for any arc e in G.*

(iv) *G is quasi-strongly connected and every vertex of G has an in-degree of 1 except one vertex whose in-degree is zero.*

(v) *There are no circuits in G (i.e. not in G_u) and every vertex of G has an in-degree of 1 except one vertex whose in-degree is zero.*

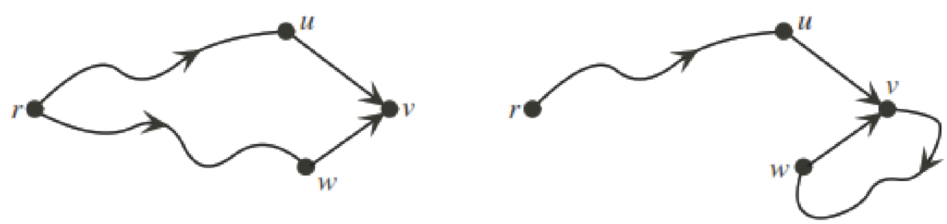
(vi) *G is quasi-strongly connected and there are no circuits in G (i.e. not in G_u).*

Proof. (i)⇒(ii): If G is a directed tree, then there is a root. This implies that there is a directed path from the root to every other vertex in G (but not more than one path since G_u is a tree).

(ii)⇒(iii): If (ii) is true, then G obviously is quasi-strongly connected. We will prove by contradiction by considering the counter hypothesis: There is an arc e in G such that $G - e$ is quasi-strongly connected. The arc e is not a loop because G is a directed tree. Let u and v be the two different end vertices of e. There does not exist a directed u–v path or a directed v–u

path in $G - e$ (otherwise G_u would have a circuit). Therefore, there is a vertex w and a directed w–u path as well as a directed w–v path. However, this leads to the existence of two directed w–u paths or two directed w–v paths in G depending on the direction of the arc e. Then, there is a circuit in the tree G_u. ($\checkmark$ by Theorem 1.6).

(iii)⇒(iv): If G quasi-strongly connected, then it has a root r (Theorem 3.1) so that the in-degrees of other vertices are ≥ 1. We start by considering the counter hypothesis: There exists a vertex $v \neq r$ and $d^-(v) > 1$. Then, v is the terminal vertex of two distinct arcs (u, v) and (w, v). If there were a loop e in G, then $G - e$ would be quasi-strongly connected ($\checkmark$). Thus, $u \neq v$ with $w \neq v$. Now, there are two distinct directed trails from r to v. The first one includes (u, v) and the second one includes (w, v). We have two possible cases:



In the digraph on the left, the paths r–u and r–w do not include the arcs (u, v) and (w, v). Both $G - (u, v)$ and $G - (w, v)$ are quasi-strongly connected. In the digraph on the right, the r–u path includes the arc (w, v) or (as in the figure) the r–w path includes the arc (u, v). In either case, only one of $G - (u, v)$ and $G - (w, v)$ is quasi-strongly connected because the root is r (Theorem 3.1). ($\checkmark$) We still have to show that $d^-(r) = 0$. Let us consider the counter hypothesis: $d^-(r) \geq 1$. Then, r is the terminal vertex of some arc e. However, the tree $G - e$ is then quasi-strongly connected since r is its root (Theorem 3.1). ($\checkmark$)

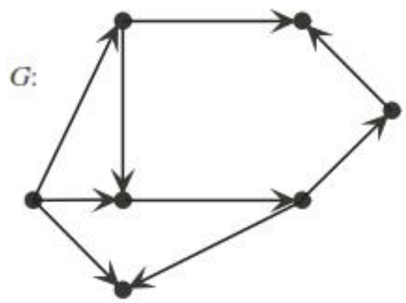
(iv)⇒(v): If (iv) is true, then it is enough to show that there are no circuits in G_u. The sum of in-degrees of all the vertices in G is $n - 1$ and the sum of out-degrees of all the vertices in G is also $n - 1$, i.e. there are $n - 1$ arcs in G. Since G is quasi-strongly connected, it is connected and it is a tree (Theorem 2.1). Therefore, there are no circuits in G_u.

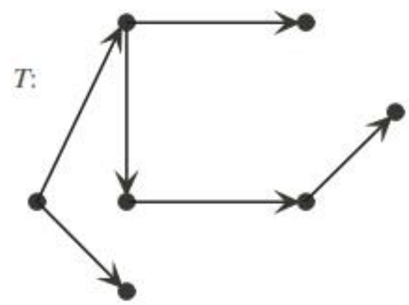
(v)⇒(vi): If we assume that (v) is true, then there are $n - 1$ arcs in G (compare to the previous proof). By Theorem 2.1, G is a tree. We denote by r the vertex satisfying condition (v). By Theorem 2.1, we see that there is exactly one path to any other vertex of G from r. These paths are also directed. Otherwise, $d^-(r) \geq 1$ or the in-degree of some vertex on that path is > 1 or the in-degree of some other vertex other than r on that path is zero. Hence, r is a root and G is quasi-strongly connected (Theorem 3.1).

(vi)⇒(i): If G is quasi-strongly connected, then it has a root (Theorem 3.1). Since G is connected and there are no circuits in G, it is a tree. □

A directed subgraph T of the digraph G is a *directed spanning tree* if T is a directed tree and T includes every vertex of G.

Example.





Theorem 3.3. *A digraph has a directed spanning tree if and only if it is quasi-strongly connected.*

Proof. If the digraph G has a directed spanning tree T, then the root of T is also a root for G and it is quasi-strongly connected (Theorem 3.1).

We now assume that G is quasi-strongly connected and show that it has a directed spanning tree. If G is a directed tree, then it is obvious. Otherwise, from Theorem 3.2, we know that there is an arc e in G so that if we remove e, G remains quasi-strongly connected. We systematically remove these kind of arcs until we get a directed tree. (Compare to the proof for Theorem 2.2) □

Matrix Representation of Graphs

A graph can be represented inside a computer by using the adjacency matrix or the incidence matrix of the graph.

Definition: Let G be a graph with n ordered vertices $v_1, v_2, \ldots, v_n$. Then the **adjacency matrix of G** is the $n \times n$ matrix $A(G) = (a_{ij})$ over the set of non-negative integers such that

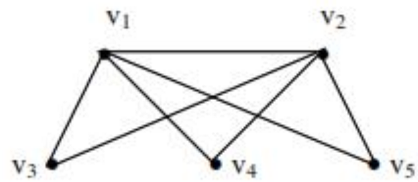
a_{ij} = the number of edges connecting v_i and v_j for all i, j = 1, 2,…,n.

We note that if G has no loop, then there is no edge joining v_i to v_i, i = 1, 2,…,n. Therefore, in this case, all the entries on the main diagonal will be 0.

Further, if G has no parallel edge, then the entries of A(G) are either 0 or 1.
It may be noted that adjacent matrix of a graph is symmetric.

Conversely, given a $n \times n$ symmetric matrix $A(G) = (a_{ij})$ over the set of non-negative integers, we can associate with it a graph G, whose adjacency matrix is A(G), by letting G have n vertices and joining v_i to vertex v_j by a_{ij} edges.

Example 1: Find the adjacency matrix of the graph shown below:



Solution: The adjacency matrix $A(G) = (a_{ij})$ is the matrix such that

a_{iJ} = No. of edges connecting v_i and v_j.

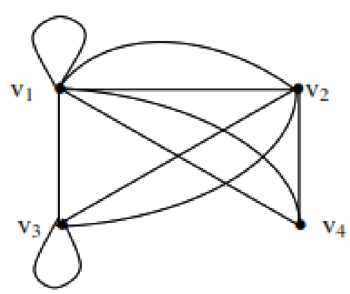
So we have for the given graph

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Example 2 : Find the graph that have the following adjacency matrix

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Solution: We note that there is a loop at v_1 and a loop at v_3. There are parallel edges between v_1, v_2; v_1, v_4; v_2, v_1; v_2, v_3, v_3, v_2 **;** v_4, v_1. Thus the graph is



The following theorem is stated without proof.

Circuit Matrix

We consider a loopless graph $G = (V, E)$ which contains circuits. We enumerate the circuits of G: $C_1, \ldots, C_\ell$. The *circuit matrix* of G is an $\ell \times m$ matrix $\mathbf{B} = (b_{ij})$ where

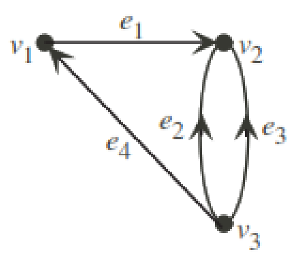
$$b_{ij} = \begin{cases} 1 \text{ if the arc } e_j \text{ is in the circuit } C_i \\ 0 \text{ otherwise} \end{cases}$$

(as usual, $E = \{e_1, \ldots, e_m\}$).

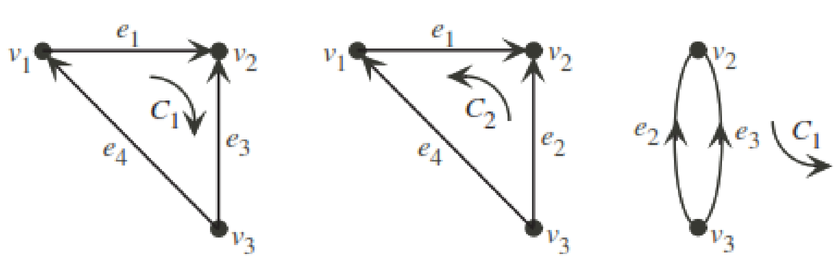
The circuits in the digraph G are *oriented*, i.e. every circuit is given an arbitrary *direction* for the sake of defining the circuit matrix. After choosing the orientations, the circuit matrix of G is $\mathbf{B} = (b_{ij})$ where

$$b_{ij} = \begin{cases} 1 \text{ if the arc } e_j \text{ is in the circuit } C_i \text{ and they in the same direction} \\ -1 \text{ if the arc } e_j \text{ is in the circuit } C_i \text{ and they are in the opposite direction} \\ 0 \text{ otherwise.} \end{cases}$$

Example. *For the directed graph*



the circuits are



and the circuit matrix is

$$\mathbf{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} e_1 \quad e_2 \quad e_3 \quad e_4 \\ \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ C_1 \\ C_2 \\ C_3 \end{array}$$

If the graph G is connected and contains at least one circuit, then it has a cospanning tree T^* and the corresponding fundamental circuits. By choosing the corresponding rows of the circuit matrix $\mathbf{B}$, we get an $(m-n+1) \times m$ matrix $\mathbf{B}_\mathrm{f}$, called the *fundamental circuit matrix*. Similarly, a connected digraph G with at least one circuit has a fundamental circuit matrix: the direction of a fundamental circuit is the same as the direction of the corresponding link in T^*.

When we rearrange the edges of G so that the links of T^* come last and sort the fundamental circuits in the same order, the fundamental circuit matrix takes the form

$$\mathbf{B}_\mathrm{f} = \left(\mathbf{B}_\mathrm{ft} \,\middle|\, \mathbf{I}_{m-n+1} \right),$$

where $\mathbf{I}_{m-n+1}$ is the identity matrix with $m-n+1$ rows. The rank of $\mathbf{B}_\mathrm{f}$ is thus $m-n+1 = \mu(G)$ and the rank of $\mathbf{B}$ is $\geq m-n+1$.

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POSSIBLE QUESTIONS

Answer All The Questions(5 X 8=40 Marks)

1)Discuss about the digraph.

2)Discuss about the binary relations in a digraph.

3)Prove that the determinant of every square submatrix of A , the incidence matrix of a digraph is 1, -1 or 0.

4)Prove that an arboresence is a tree in which every vertex other than the root has an in-degree of exactly one

5) Explain some types of digraphs with example.

6) Explain in detail of incidence matrix.

7) Explain circuit matrix of a digraph.

8) If A(G) is an incidence matrix of a connected graph G with n vertices then prove that the rank of A(G) is (n-1).

9)Prove that an arboresence is a tree in which every vertex other than the root has an in-degree of exactly one

10) Explain in detail: i) number of arboresence ii)connectedness and adjacency matrix

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