Reg. No……………………………

[17MMP205A]

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari Post, Coimbatore – 641 021.
(For the candidates admitted from 2017 onwards)

M.Sc., DEGREE EXAMINATION, APRIL 2018

Second Semester

MATHEMATICS

GRAPH THEORY AND ITS APPLICATIONS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21.a. Define i. Bipartite Graph
ii. Regular Graph
iii .Complete Graph. Give an example for each.

Or

b. If G is a tree with n vertices then prove that G has n-1 edges.

22.a. Explain Kruskal algorithm and Prim's algorithm for shortest spanning tree with example.

Or

b. Prove that the maximum flow possible between two vertices a and b in a network is equal to the minimum of the capacities of all cut-sets with respect to a and b.

23.a. Explain about incidence matrix in a graph.

Or

b. Prove that a graph of n vertices is a complete graph if and only if its chromatic polynomial $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-n+1)$

24. a. Discuss about the digraph.

Or

b. Explain counting labeled tress.

25.a. Explain domination number of a graph with examples.

Or

b. Explain briefly about applications of radio stations.

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Prove that every circuit has an even number of edges in common with any cut-set.

Reg. No................................

[14MMP303]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2014 onwards)

M.Sc., DEGREE EXAMINATION, NOVEMBER 2015

Third Semester

MATHEMATICS

GRAPH THEORY

Time: 3 hours — Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 x 8 = 40 Marks) (2 ½ Hours)
Answer ALL the Questions

21. a) Show that a connected graph G is an Euler graph if and only if the degree of every vertex in G is even.

Or

b) Explain Travelling Salesman Problem with example.

22. a) i) Show that a graph G is a tree if and only if it is minimally connected.
ii) Show that every tree has one (or) two centres.

Or

b) i) Explain with example : (1) Kruskal's algorithm. (2) Prim's algorithm.
ii) Show that the minimum height of a n-vertex binary tree is equal to $[\log_2 (n+1)-1]$.

23. a) Prove that $K_{3,3}$ is a non-planar graph.

Or

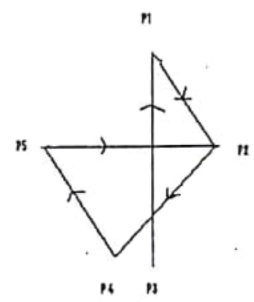
b) Prove that K_5 is a non-planar graph.

24. a) Show that every planar graph is 5-colourable.

Or

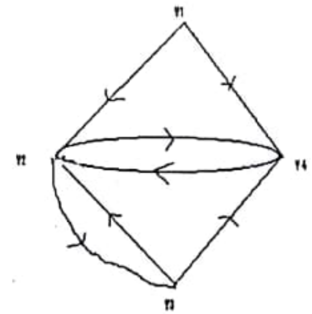
b) i) Show that a graph with at least one edge is 2-chromatic if and only if it has no circuit.
ii) Show that a graph of 5-vertices is a complete graph if and only if its chromatic polynomial is $P_5(\lambda) = \lambda(\lambda-1)(\lambda^2 -5\lambda+7)$

25. a) i) Show that every complete tournament has a directed Hamiltonian.
ii) Prove that $Q = A^+$ where Q is the path matrix and A is a adjacency matrix for the following diagraph.

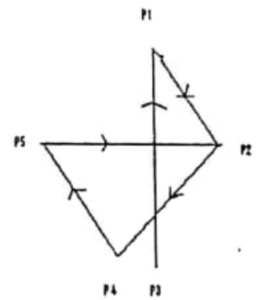


Or

b) Write the adjacency matrix, incident matrix and path matrix for the following graphs.



(a)



(b)

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester - II

		L	T	P	C
17MMP205A	**GRAPH THEORY AND ITS APPLICATIONS**	**4**	**0**	**0**	**4**

Scope: On successful completion of this course the learner gains knowledge about the concept of graphs, spanning trees, incidence matrix, graph colorings, domination in graphs which provides the basis for networks.

Objectives: To be familiar with different types of Graphs and their incidence matrices ,spanning trees and to be exposed with colourings & Domination in Graphs.

UNIT I

Graphs – Introduction – Isomorphism – Sub graphs – Walks, Paths, Circuits – Connectedness – Components – Euler Graphs – Hamiltonian Paths and Circuits – Trees – Properties of trees – Distance and Centers in Tree – Rooted and Binary Trees - Spanning trees – Fundamental Circuits.

UNIT II

Spanning Trees in a Weighted Graph – Cut Sets – Properties of Cut Set – All Cut Sets – Fundamental Circuits and Cut Sets – Connectivity and separability – Network flows – 1-Isomorphism – 2-Isomorphism – Combinational versus Geometric Graphs – Planer Graphs – Different Representation of a Planer Graph.

UNIT III

Incidence matrix – Sub matrices – Circuit Matrix – Path Matrix – Adjacency Matrix – Chromatic Number – Chromatic partitioning – Chromatic polynomial - Matching - Covering – Four Color Problem.

UNIT IV

Directed Graphs – Types of Directed Graphs - Types of enumeration, counting labeled trees, counting unlabelled trees, Polya's counting theorem, graph enumeration with Polya's theorem.

UNIT V

Domination in graphs: Introduction – Terminology and concepts – Applications – Dominating set and domination number – Independent set and independence number – History of domination in graphs.

TEXT BOOKS

1. Deo N, (2004). Graph Theory with Applications to Engineering and Computer Science, Prentice Hall Inc ,Upper Saddle River, NJ, USA. **(for Unit I to IV).**

2. Teresa W. Haynes, Stephen T. Hedetniemi and Peter J.Slater, (1998), Fundamentals of Domination in Graphs, Marcel Dekker, New York **(for Unit V)**

REFERENCES

1. Jonathan L Gross, Jay Yellen, (2014). Handbook of Graph Theory, CRC Press LLC. Taylor &Francis Group,Boca Rotan.

2. Diestel. R Springer-Verlag, (2012). Graph Theory. Springer-Verleg,New York.

3. Jensen.TR and Toft.B., (1995). Graph Coloring Problems. Wiley-Interscience , , New York.

4. Fred Buckley and Frank Harary, (1990). Distance in Graphs, Addison - Wesley Publications. Redwood City, California.

5. C. R. Flouds , (2009). Graph Theory Applications, Narosa Publishing House. New Delhi,India.

6. Arumugam. S, Ramachandran. S ,(2003). Invitation to graph theory, Scitech publications, Chennai.

7. Harary F, (1972).Graph Theory, Addison- Wesley publications, Massachusetts Menlo Park, California, London

Reg . No__________

[17MMP205A]

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE- 641 021

DEPARTMENT OF MATHEMATICS

SIXTH SEMESTER

I - INTERNAL TEST JAN-2018

GRAPH THEORY AND ITS APPLICATIONS

Class: I M.Sc Mathematics   Max. Marks : 50 Marks

Date: 2.2.2018 (FN)   Time : 2 Hours

PART – A(20 × 1 = 20 marks)

ANSWER ALL THE QUESTIONS

1. A vertex of degree one is________
   (a) Pendant vertex (b) isolated vertex
   (c) simple graph   (d) null graph
2. A graph in which all vertices are of equal degree is____
   (a) regular graph   (b)graph
   (c) isolated vertex   (d) Pendant vertex
3. A vertex with minimum eccentricity in graph G is ____
   (a) center   (b) diameter
   (c)radius   (d) bicenters
4. A tree with n vertices has____ edges
   (a) n-1 (b) n+1 (c) n (d) 1
5. A graph that has neither self loops nor parallel edges ____
   (a) null graph   (b) simple graph
   (c) regular graph   (d) complete graph
6. A connected graph is said to be ____ if its vertex connectivity is one
   (a) separable   (b) vertex connectvity
   (c) edge connectivity   (d) complete graph
7. Every edge of a tree is a ____
   (a) cut set   (b) cut vertices
   (c) euler graph   (d) graph
8. ______ of a connected graph can be defined as the minimum number of edges whose removal reduces the rank of the graph by one.
   (a) edge connectivity (b) euler graph
   (c) vertex connectvity   (d) simple graph
9. A spanning tree with the smallest weight in a weighted graph is called a ______
   (a) shortest spanning tree (b) spanning tree
   (c) tree   (d) cut set
10. Every binary tree is a ______ tree
    (a) Rooted tree   (b) tree
    (c) spanning tree (d) shortest spanning tree
11. A collection of trees is ________
    (a) forest   (b) spanning tree
    (c) shortest spanning tree (d) Rooted tree

12. A graph containing only isolated vertex is ____________
   (a) null graph (b) simple graph
   (c) complete graph (d) regular graph
13. A ______ is a connected graph without any circuit.
   (a) tree (b) spanning tree
   (c) shortest spanning tree (d) rooted tree
14. The eccentricity of a center of a tree is ______
   (a) radius (b) diameter (c) length (d) distance
15. The reduced incidence matrix of a tree is _________
   (a) singular (b) non singular
   (c) submatrix (d) circuit matrix
16. A vertices which a walk begins and ends are ________
   (a) terminal vertices (b) path
   (c) tree (d) graph
17. A closed walk in which no vertices appears more than once is ___
   (a) circuit (b) path
   (c) cut set (d) length
18. Each connected subgraph is_______
   (a) component (b) cycle (c) tree (d) path
19. A trail is trace every edge of G exactly once is
   (a) euler trail (b) eular graph
   (c) graph (d) circuit
20. The degree of leaf is______
   (a)1 (b)0 (c) 2 (d)3

PART–B (3× 2 = 6 *marks*)

Answer all the questions

21. Define incidence matrix with example
22. Definition of cut-set.
23. Define hamilitonian path

PART–C (3× 8 = 24 *marks*)

Answer all the questions

24. (a) A graph G is a tree iff it is minimally connected **(or)**
    (b) Prove that the number of vertices of odd degree in a graph is always even.
25. (a) prove that every cut-set in a connected graph G must contain atleast one branch of every spanning tree of G. **(or)**
    (b) Prove that every circuit has an even number of edges in common with any cut-set.
26. (a) Prove that any connected graph with n vertices and n-1 edges is a tree **(or)**
    (b) Discuss brifly about the circuit matrix

Reg no-------------

(17MMP205A)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore-21

DEPARTMENT OF MATHEMATICS

Second Semester

II Internal Test - March'2018

Graph theory and its applications

Date:23 -03-2018(FN) Time: 2 Hours

Class: I- M.Sc Maths Maximum Marks:50

PART – A (20 X 1 = 20 MARKS)

ANSWER ALL THE QUESTIONS

1. The number of edges in a largest maximal matching is ______
   a) matching b) matching number
   c) maximal matching d) minimal matching
2. A digraph is ______ graph
   a) oriented b) simple
   c) bipartite d) euler
3. The number of edges incident out of a vertex is _____
   a) out-degree b) in-degree
   c) link d) digraph
4. Every bipartite graph is ---------chromatic
   a) 2 b) 3 c) 1 d) 4
5. A digraph that has no self loop or parallel edges is ______
   a) simple b) symmetric
   c) complete d) asymmetric
6. A balanced digraph is ______
   a) isograph b) simple graph
   c) complete digraph d) anti symmetric

7. The number of vertices in the largest _______ set of a graph
   a) independent b) dominating set
   c) number d) digraph
8. The minimum cardinality of a total dominating set is _____
   a) domination number b) dominating set
   c) independent set d) independent number
9. A set of vertices in a graph is independent set if no two vertices in the set are _____
   a) adjacent b) independent
   c) dominate d) tree
10. A digraph is ______ graph
    a) oriented b) simple
    c) bipartite d) euler
11. The number of edges incident out of a vertex is _____
    a) out-degree b) in-degree
    c) link d) digraph
12. In any graph G, we have
    a) $\alpha(G) = \beta(G)$ b) $\alpha(G) \geq \beta(G)$
    c) $\alpha(G) \leq \beta(G)$ d) $\alpha(G) < \beta(G)$
13. A vertex v is called pendant vertex if $d^+(v) + d^-(v) =$
    a) 1 b) 2 c) 3 d) 4
14. The minimum cardinality of an independent dominating set of G is _____
    a) Independent domination number
    b) domination number
    c) independent number
    d) independent set
15. A graph G is an Euler graph if $d^+(v)$ is odd then $d^-(v) =$
    a) odd b) even c) 3 d) 5
16. The rank of an incidence matrix of a digraph with n vertices is ______
    a) n-1 b) n c) n+1 d) n+2

17. A simple digraph in which there is exactly one edge directed from every vertex to every other vertex is ______
a) complete symmetric digraph b) symmetric digraph
c) simple digraph d) balanced

18. Every dominating set contains atleast ____ minimal dominating set.
a) 1 b) 2 c) 3 d) 4

19. A____ dominating set from which no vertex can be removed without destroying its dominanace property.
a) minimal b) maximal
c) independent d) independent number

20. A minimal dominating set may or may not be ---------
a) dependent b) independent
c) empty d) zero

PART-B (3X2=6 Marks)

ANSWER ALL THE QUESTIONS

21. Define digraph with an example
22. Define of matching
23. Define dominating set

PART-C (3X8=24 Marks)

ANSWER ALL THE QUESTIONS

24. a) Prove that the vertices of every planar graph can be properly colored with five colors.

(OR)

b). Prove that a covering G of a graph is minimal if and only G contains no paths of length three or more.

25. a) Prove that there are n^{n-2} labeled trees with n vertices ($n \geq 2$).

(OR)

b) Discuss about some types of digraph with suitable example

26. a) Discuss dominating set of a graph with examples.

(OR)

b) Prove that if a connected graph with $n \geq 2$ vertices has a dominating set S then the complement of S is also a dominating set of G.

UNIT III
Syllabus

Incidence matrix – Sub matrices – Circuit Matrix – Path Matrix – Adjacency Matrix – Chromatic Number – Chromatic partitioning – Chromatic polynomial - Matching - Covering – Four Color Problem.

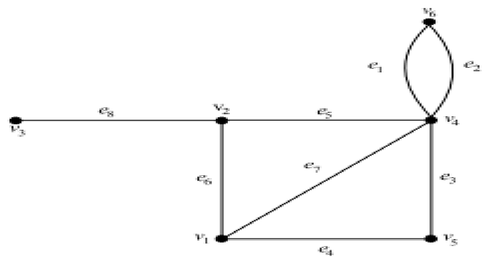
INCIDENCE MATRIX

Let G be a graph with n vertices, m edges and without self-loops. The incidence matrix A of G is an n×m matrix A = [ai j] whose n rows correspond to the n vertices and the m columns correspond to m edges such that ai j = 1 , i f jth edge mj is incident on the ith vertex 0 , otherwise. It is also called vertex-edge incidence matrix and is denoted by A(G).

Example Consider the graphs given in Figure 10.1. The incidence matrix of G_1 is

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The incidence matrix of G_2 is



G_1
(a)

Theorem 10.1 Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns.

Proof Let the graphs G_1 and G_2 be isomorphic. Then there is a one-one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved. Thus $A(G_1)$ and $A(G_2)$ are either same or differ only by permutation of rows and columns.

The converse follows, since permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph. ❑

SUB MATRICES

Let H be a subgraph of a graph G, and let $A(H)$ and $A(G)$ be the incidence matrices of H and G respectively. Clearly, $A(H)$ is a submatrix of $A(G)$, possibly with rows or columns permuted. We observe that there is a one-one correspondence between each $n \times k$ submatrix of $A(G)$ and a subgraph of G with k edges, k being a positive integer, $k < m$ and n being the number of vertices in G.

The following is a property of the submatrices of $A(G)$.

Theorem 10.4 Let $A(G)$ be the incidence matrix of a connected graph G with n vertices. An $(n-1) \times (n-1)$ submatrix of $A(G)$ is non-singular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitutes a spanning tree in G.

Proof Let G be a connected graph with n vertices and m edges. So, $m \geq n-1$.

Let $A(G)$ be the incidence matrix of G, so that $A(G)$ has n rows and m columns $(m \geq n-1)$.

We know every square submatrix of order $(n-1) \times (n-1)$ in $A(G)$ is the reduced incidence matrix of some subgraph H in G with $n-1$ edges, and vice versa. We also know that a square submatrix of $A(G)$ is non-singular if and only if the corresponding subgraph is a tree.

Obviously, the tree is a spanning tree because it contains $n-1$ edges of the n-vertex graph.

Hence $(n-1) \times (n-1)$ submatrix of $A(G)$ is non-singular if and only if $n-1$ edges corresponding to $n-1$ columns of this matrix forms a spanning tree. ❑

CIRCUIT MATRIX

We consider a loopless graph $G = (V, E)$ which contains circuits. We enumerate the circuits of G: $C_1, \ldots, C_\ell$. The *circuit matrix* of G is an $\ell \times m$ matrix $\mathbf{B} = (b_{ij})$ where

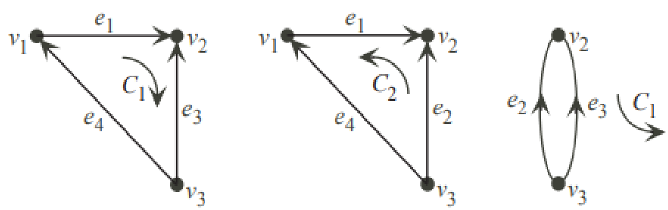
$$b_{ij} = \begin{cases} 1 \text{ if the arc } e_j \text{ is in the circuit } C_i \\ 0 \text{ otherwise} \end{cases}$$

(as usual, $E = \{e_1, \ldots, e_m\}$).

The circuits in the digraph G are *oriented*, i.e. every circuit is given an arbitrary *direction* for the sake of defining the circuit matrix. After choosing the orientations, the circuit matrix of G is $\mathbf{B} = (b_{ij})$ where

$$b_{ij} = \begin{cases} 1 \text{ if the arc } e_j \text{ is in the circuit } C_i \text{ and they in the same direction} \\ -1 \text{ if the arc } e_j \text{ is in the circuit } C_i \text{ and they are in the opposite direction} \\ 0 \text{ otherwise.} \end{cases}$$

EXAMPLE



he circuit matrix is

$$\mathbf{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array}$$

PATH MATRIX

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q\times m}$, where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & \text{if } jth \text{ edge lies in the } ith \text{ path}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u, v)$ correspond to different paths between u and v, and the columns correspond to different edges in G. For example, consider the graph in Figure 10.10.

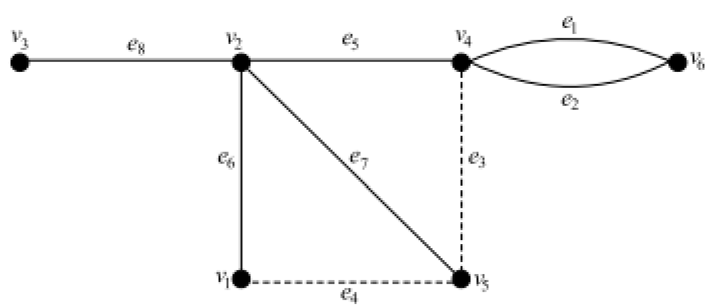


Fig. 10.10

The different paths between the vertices v_3 and v_4 are

$p_1 = \{e_8, e_5\}$, $p_2 = \{e_8, e_7, e_3\}$ and $p_3 = \{e_8, e_6, e_4, e_3\}$.

The path matrix for v_3, v_4 is given by

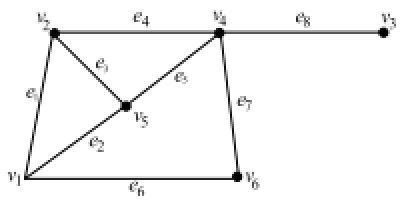
$$P(v_3, v_4) = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

ADJACENCY MATRIX

Let $V = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and without parallel edges. The adjacency matrix of G is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1, & if\ v_iv_j \in E, \\ 0, & otherwise. \end{cases}$$

Example Consider the graph G given in Figure 10.12.



The adjacency matrix of G is given by

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

CHROMATIC NUMBER

Lemma 6.1.1 **Let G be a connected graph that is not an odd cycle. Then**

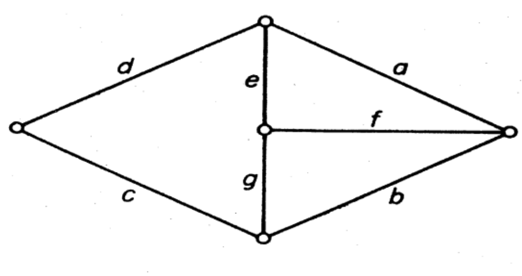


Figure 6.1

G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.

Proof We may clearly assume that G is nontrivial. Suppose, first, that G is eulerian. If G is an even cycle, the proper 2-edge colouring of G has the required property. Otherwise, G has a vertex v_0 of degree at least four. Let $v_0e_1v_1 \ldots e_\varepsilon v_0$ be an Euler tour of G, and set

$$E_1 = \{e_i \mid i \text{ odd}\} \quad \text{and} \quad E_2 = \{e_i \mid i \text{ even}\} \tag{6.2}$$

Then the 2-edge colouring (E_1, E_2) of G has the required property, since each vertex of G is an internal vertex of $v_0e_1v_1 \ldots e_\varepsilon v_0$.

If G is not eulerian, construct a new graph G^* by adding a new vertex v_0 and joining it to each vertex of odd degree in G. Clearly G^* is eulerian. Let $v_0e_1v_1 \ldots e_{\varepsilon^*} v_0$ be an Euler tour of G^* and define E_1 and E_2 as in (6.2). It is then easily verified that the 2-edge colouring $(E_1 \cap E, E_2 \cap E)$ of G has the required property □

Lemma 6.1.2 Let $\mathscr{C} = (E_1, E_2, \ldots, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Proof Let u be a vertex that satisfies the hypothesis of the lemma, and denote by H the component of $G[E_i \cup E_j]$ containing u. Suppose that H is not an odd cycle. Then, by lemma 6.1.1, H has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in H. When we recolour the edges of H with colours i and j in this way, we obtain a new k-edge colouring $\mathscr{C}' = (E_1', E_2', \ldots, E_k')$ of G. Denoting by $c'(v)$ the number of distinct colours at v in the colouring $\mathscr{C}'$, we have

$$c'(u) = c(u) + 1$$

since, now, both i and j are represented at u, and also

$$c'(v) \geq c(v) \quad \text{for} \quad v \neq u$$

Thus $\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$, contradicting the choice of $\mathscr{C}$. It follows that H is indeed an odd cycle □

CHROMATIC PARTITIONING

Theorem 6.3 If G is bipartite, and if $p \geq \Delta$, then there exist p disjoint matchings $M_1, M_2, \ldots, M_p$ of G such that

$$E = M_1 \cup M_2 \cup \ldots \cup M_p \tag{6.4}$$

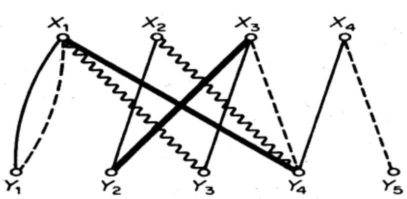
and, for $1 \leq i \leq p$

$$[\varepsilon/p] \leq |M_i| \leq \{\varepsilon/p\} \tag{6.5}$$

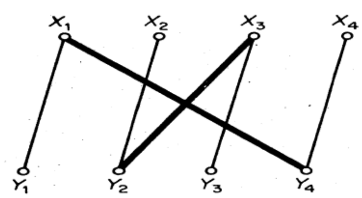
Proof Let G be a bipartite graph. By theorem 6.1, the edges of G can be partitioned into Δ matchings $M'_1, M'_2, \ldots, M'_\Delta$. Therefore, for any $p \geq \Delta$, there exist p disjoint matchings $M'_1, M'_2, \ldots, M'_p$ (with $M'_i = \emptyset$ for $i > \Delta$) such that

$$E = M'_1 \cup M'_2 \cup \ldots \cup M'_p$$

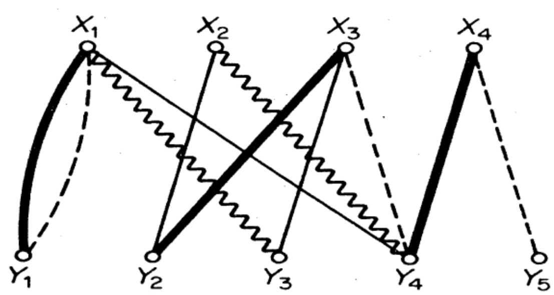
By repeatedly applying lemma 6.3 to pairs of these matchings that differ in size by more than one, we eventually obtain p disjoint matchings $M_1, M_2, \ldots, M_p$ of G satisfying (6.4) and (6.5), as required □



(a)



(b)



(a)

	Period 1	2	3	4
X_1	Y_4	Y_1	Y_3	Y_1
X_2	Y_2	–	Y_4	–
X_3	Y_3	Y_4	–	Y_2
X_4	–	Y_5	–	Y_4

(b)

As an example, suppose that there are four teachers and five classes, and that the teaching requirement matrix $\mathbf{P} = [p_{ij}]$ is as given in figure 6.4*a*. One possible 4-period timetable is shown in figure 6.4*b*.

CHROMATIC POLYNOMIAL

Theorem 1.1. $\mathrm{chr}(G,k)$ *is a polynomial of* k.

Proof. For any coloring of G the nonempty color classes constitute a partition of $V(G)$ where each part is a stable vertex set. We may count those colorings that give a certain partition and add them up for all such partitions to find the total number of colorings. Since $V(G)$ is a finite set, it has a finite number of partitions, so it is sufficient to show that the number of colorings for a single partition is a polynomial of k.

Fix a partition with p parts, each of them being a stable set. By assigning a different color to each part, we get all the colorings belonging to the partition. We may pick the first color in k possible ways, the second in $k-1$ ways, etc. so there are $k(k-1)\dots(k-p+1)$ colorings, which is obviously a polynomial. Note that this also works when $k < p$. □

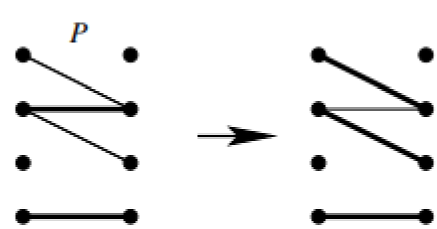
MATCHING

Let M be a matching. The vertices that are incident to an edge of M are *matched* or *covered* by M. If U is a set of vertices covered by M, then we say that M *saturates* U. The vertices which are not covered are said to be *exposed*.

Let $G = (V,E)$ be a graph and M a matching. An M-*alternating* path in G is a path whose edges are alternatively in $E \setminus M$ and in M. An M-alternating path whose two endvertices are exposed is M-*augmenting*. We can use an M-augmenting path P to transform M into a greater matching (see Figure 6.1). Indeed, if P is M-alternating, then the symmetric difference between M and $E(P)$

$$M' = M \triangle E(P) = (M \setminus (E(P) \cap M) \cup (E(P) \setminus M)$$

is also a matching. Its size $|M'|$ equals $|M| - 1 + x$ where x is the number of exposed ends of P.



Theorem 6.1 (Berge 1957). *Let M be a matching in a graph G. Then M is maximum if and only if there are no M-augmenting paths.*

Proof. Necessity was shown above so we just need to prove sufficiency. Let us assume that M is not maximum and let M' be a maximum matching. The symmetric difference $Q = M \triangle M'$ is a subgraph with maximum degree 2. Its connected components are cycles and paths where the edges of M and M' alternate. Hence, the cycles have even length and contain as many edges of M and of M'. Since M' is greater than M, Q contains at least one path P that contains more edges of M' than of M. Therefore, the first and the last edges of P belong to M', and so P is M-augmenting.

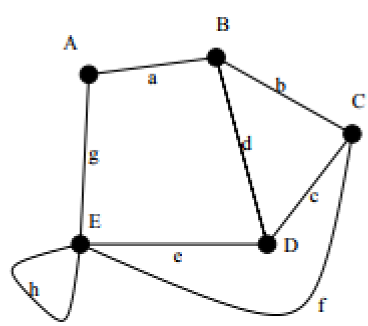
COVERING

Definition 2.1. Let $X = (D, V; I, \lambda)$ and $\tilde{X} = (\tilde{D}, \tilde{V}, \tilde{I}, \tilde{\lambda})$ be graphs. A graph epimorphism $p : \tilde{X} \to X$ is called a *covering projection* if, for every vertex $\tilde{u} \in \tilde{X}$, p maps the neighborhood $\tilde{D}_{\tilde{u}}$ of $\tilde{u}$ bijectively onto the neighborhood $D_{p\tilde{u}}$ of $p\tilde{u}$. The graph X is usually referred to as the *base graph* or a *quotient graph* and $\tilde{X}$ is called the *covering graph*. By $\mathrm{fib}_u = p^{-1}u$ and $\mathrm{fib}_x = p^{-1}x$ we denote the *fibre* over $u \in V$ and $x \in D$, respectively.

FOUR COLOR PROBLEM

2. Background. To understand the principles of the Four Color Theorem, we must know some basic graph theory.

A *graph* is a pair of sets, whose elements called *vertices* and *edges* respectively. Associated to each edge are two distinguished vertices called *ends*. The two ends are allowed to coincide; if they do, the edge is called a *loop*. Each vertex is represented by a point in the plane. Each edge is represented by a continuous curve between its two ends. We say an edge *connects* its two ends. Figure 2-1 shows a graph with vertices



A, B, C, D, E and edges a, b, c, d, e, f, g, h. The edge h connects the vertex E to itself; so h is a loop.

PART B (5x6=30)

1. If A(G) is an incidence matrix of a connected graph G with n vertices then prove that the rank of A(G) is (n-1).
2. Show that every tree with two or more vertices is 2-chromatic.
3. Explain about incidence matrix in a graph.
4. Prove that a graph of n vertices is a complete graph if and only if its chromatic polynomial $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-n+1)$
5. Explain about chromatic polynomial in a graph.
6. Prove that a covering G of a graph is minimal if and only G contains no paths of length three or more.
7. Prove that a graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd length.
8. Explain about coverings
9. Discuss about the chromatic partition.
10. Show that any tree T on n-vertices has chromatic polynomial $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.

PART – C (1x10=10)

1. Prove that the vertices of every planar graph can be properly colored with five colors.
2. Explain every tree with two or more vertices is 2-chromatic.
3. Discuss a graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd length

Karpagam Academy of Higher Education - Coimbatore - 21

Department of Mathematics

Second Semester

II internal Test Answer key - March '2018

Graph Theory and its Applications

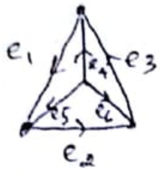
PART - A.

1) b) matching number
2) a) oriented
3) a) Out - degree
4) a) 2
5) a) Simple
6) a) isograph
7) a) independent
8) a) domination number
9) a) adjacent
10) a) Oriented
11) a) out - degree
12) c) $\alpha(G) \leq \beta(G)$
13) a) 1
14) a) independent domination number.
15) b) even
16) a) n-1
17) a) complete Symmetric digraph
18) a) 1
19) a) minimal
20) b) independent

PART - B.

21) A directed graph (or a digraph) G consists of a set of vertices $V = \{v_1, v_2, \ldots\}$, a set of edges $E = \{e_1, e_2 \ldots\}$ and a mapping ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).

Ex:-



22) A matching in a graph is a subset of edges in which no two edges are adjacent.

23). A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

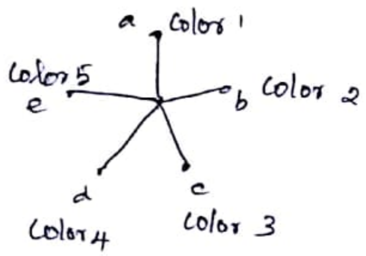
PART - C

24) a). The theorem will be proved by induction. Since the vertices of all graphs with 1, 2, 3, 4 or 5 vertices can be properly colored with 5 colors.

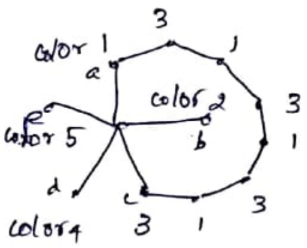
* consider the planar graph G with n vertices. Since G is planar, it must have atleast one vertex with degree 5 or less.

* The graph G' requires no more than 5 colors, according to the induction hypothesis.

* There is a path in G' b/w vertices a & c colored alternately with colors 1 & 3.



(a)



(b)

* Similar path b/w b & d colored alternately with colors 2 & 4 cannot exist.

* The interchange will paint vertex b with color 4 and yet keep G' properly colored.

* There is no path b/w a & c of vertices painted alternately with colors 1 & 3, we would have released color 1 (or) 3 instead of color 2.

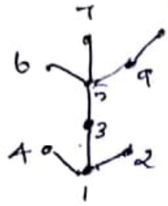
24) b). Suppose that a covering g contains a path of length three $v_1 e_1 v_2 e_2 v_3 e_3 v_4$

* e_2 can be removed without leaving its end vertices v_2 & v_3 uncovered.
$\therefore$ g is not a minimal covering.

* conversely, if a covering g contains no path of length 3 or more, all its components must be star graphs.
* A star gp no edge can be removed without leaving a vertex uncovered.
∴ g must be a minimal covering.

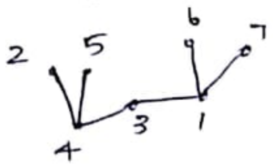
26) a). Let n vertices of a tree T be labeled 1, 2, 3 .. n Remove the pendant vertex having the smallest label a_1. Suppose that b_1 was the vertex adjacent to a_1. Remove the edge (a_2, b_2).

* The tree T defines the sequence $(b_1, b_2, \dots b_{n-2})$



The sequence is (1, 1, 3, 5, 5, 5, 9)

* conversely, determine the first number in the sequence 1, 2, 3 ... n

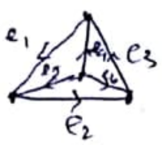


The sequence is (4, 4, 3, 1, 1)

* For each of $(n-2)$ elements in the form n^{n-2} sequences.
* Hence the proof.

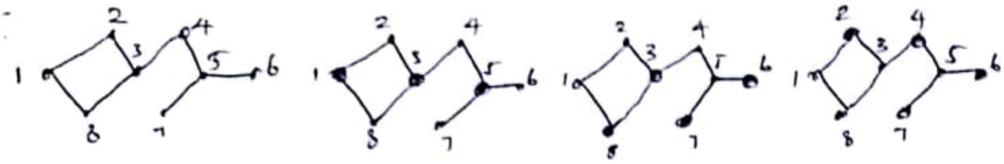
25 b) simple Digraphs:- No self-loop (or) parallel edges
* Asymmetric Digraphs:- Atmost one directed edge b/w a pair of vertices, but are allowed to have self-loop.
* Symmetric digraph :- $(a, b) \Rightarrow (b, a)$
* Simple symmetric digraph :- simple & symmetric
* Asymmetric digraph :- simple & asymmetric
* complete digraph:- Every vertex is joined to every other vertex exactly by one edge.

Ex:-



26) a) A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

* for every vertex $v \in V-S$, $\exists$ a vertex $u \in S$ such that v is adjacent to u.
* for every vertex $v \in V-S$, $d(v,S) \le 1$
* $N[S] = V$
* for every vertex $v \in V-S$, $|N(v) \cap S| \ge 1$, (ie) every vertex $u \in V-S$ is adjacent to atleast one vertex in S.
* for every vertex $v \in V$, $|N(v) \cap S| \ge 1$
* $V-S$ is enclaveless.
* Ex:-



2b) b). * Let T be any spanning tree of G, and let u be any vertex in V.

* Then the vertices in T fall into two disjoint sets S & S' consisting of the vertices with an even and odd distance from u in T.

* Both S and $S' = V-S$ are dominating sets of G.

* Hence the proof.

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Karpagam Academy of Higher Education - Coimbatore - 21
Department of Mathematics
Second Semester
I - Internal Test Answer Key - March' 2018
Graph Theory and its applications

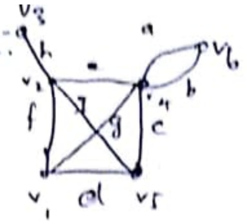
PART - A

1) a) Pendant vertex
2) a) Regular graph
3) a) center
4) a) n-1
5) a) null graph
6) a) Separable
7) a) cut set
8) a) edge connectivity
9) a) shortest spanning tree
10) a) Rooted tree
11) a) Forest
12) b) simple graph
13) a) tree
14) a) radius
15) b) non-singular
16) a) terminal vertices
17) a) circuit
18) a) Component
19) a) Euler trail
20) a) 1

PART - B.

21) Let G be a graph with n vertices, e edges and no self loops.
The matrix element $a_{ij} = 1$, if j^{th} edge e_j is incident on i^{th} vertex v_i;
$= 0$, otherwise

Ex:-



$$\begin{array}{c c} & \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

22) In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of those edges disconnects G.

23). If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called a Hamiltonian path.

PART - C

24) a). The minimum no. of line segments needed is $n-1$. It requires no background in electrical engineering to realize that to short n points together, one needs atleast $n-1$ pieces of wire. Hence the tree

* A connected graph with n vertices & without any circuits has $n-1$ edges. We can show that a graph with n vertices which has no circuit and has $n-1$ edges is always connected.

24) b) * Let V_1, V_2 be the set of all vertices of even degree and set of all vertices of odd degree in a graph $G = (V, E)$

$$\sum d(v) = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j) \quad \text{— ①}$$

$$2e = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j) \quad \text{— ②}$$

* L.H.S of eqn ② is even. and R.H.S of eqn ② is even.

$\sum_{v_j \in V_2} d(v_j)$ is even.

* Hence the no. of vertices of odd degree is even. Hence the proof.

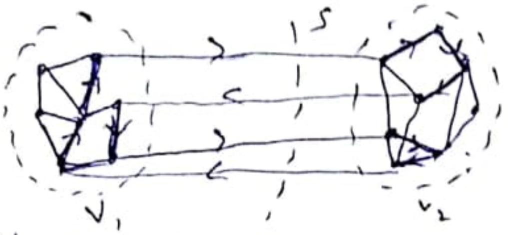
25) a) * In a connected graph G, let Q be a minimal set of edges containing atleast one branch of every spanning tree of G.

* The subgraph $G - Q$ contains no spanning tree of G, $G - Q$ is disconnected.

* The subgraph $G-Q+e$ is connected. ∴ Q is a minimal set of edge whose removal from G disconnects G. Hence the proof.

25) b). * Consider a cut-set S in graph G. Let the removal of S partition the vertices of G into two subsets V_1 & V_2.

* consider a circuit Γ in G (ie) $N(S \cap \Gamma) = 0$. an even number.



* If on the other hand, some vertices in Γ are in V_1 and some in V_2, we traverse back & forth b/w the sets V_1 & V_2 as we traverse the circuit and because of V_1 & V_2 are even.

* Every edge in S has one end in V_1 & the other in V_2, and no other edge in G has its property the no. of edges common to S & Γ is even.

Hence the proof.

26) a). A connected graph is said to be minimally connected if removal of any one edge from it disconnects the graph.

* A minimally connected graph cannot have a circuit.

* (ie) It is a tree.

* Hence any connected graph with n vertices & $n-1$ edges is a tree.

26) b). * Let G be a graph with n vertices, e edges, and no self loops.

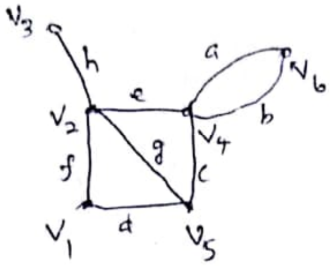
* The matrix $A = [a_{ij}]$,

$$a_{ij} = 1, \text{ if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex } v_i$$
$$= 0, \text{ Otherwise.}$$

* A matrix A is called the vertex-edge incidence matrix (or) incidence matrix.

* Matrix A for a graph G, $A(G)$, the incidence matrix contains only 2 elements 0 & 1. Such a matrix is called binary matrix or (0,1) matrix

Ex :-



$$
\begin{array}{c|c}
 & \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\
\begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{matrix} &
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

— x —

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: R.GAYATHRI
SUBJECT NAME: GRAPH THEORY AND ITS APPLICATIONS
SUB.CODE:17MMU205A
SEMESTER: II CLASS: I M.SC MATHEMATICS

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1.	1	Introduction and Definition of a Graphs	T1:Chap:1.1:Pg.No:1-2
2.	1	Isomorphism of graphs and sub graphs	T1:Chap:2.1:Pg.No:14-16
3.	1	Walks, Paths, Circuits	R2:Chap:1.3:Pg.No:6-9
4.	1	Connected , connectedness of graphs and components of graphs	T1:Chap:2.5:Pg.No:19-21
5.	1	Euler graphs and Euler graphs based on theorems	T1:Chap:2.6:Pg.No:21-23
6.	1	Hamiltonian paths and circuits	R1:Chap:4.5:Pg.No:314-316
7.	1	Introduction and definition of a trees	R2:Chap:1.5:Pg.No:9-12
8.	1	Theorems on some properties of trees	R4:Chap:3:Pg.No:39-41
9.	1	Distance and centers in tree	T1:Chap:3.4:Pg.No:43-45

10.	1	Rooted and binary trees and spanning trees	T1:Chap:3.5:Pg.No:45-55
11.	1	Fundamentals Circuits	T1:Chap:3.8:Pg.No:55-57
12.	1	Recapitulation & discussion of possible questions	
	Total No of Hours Planned For Unit 1= 12		
		UNIT-II	
1	1	Spanning trees in a Weights Graph	R6:Chap:3.10:Pg.No:58-61
2	1	Definition of a Cut Sets	T1:Chap:4.1:Pg.No:68-69
3	1	Theorems on some properties of Cut Sets and all Cut Sets	T1:Chap:4.2:Pg.No:69-71
4	1	Fundamental Circuits and Cut Sets	T1:Chap:4.5:Pg.No:73-75
5	1	Connectivity and separability	T1:Chap:4.5:Pg.No:73-75
6	1	Network flows	R1:Chap:11:Pg.No:1377-1380
7	1	Theorems on some 1-Isomorphism	T1:Chap:4.7:Pg.No:80-82
8	1	Theorems on some 2-Isomorphism	T1:Chap:4.5:Pg.No:73-75
9	1	Combinational versus Geometric Graphs	T1:Chap:5.1:Pg.No:88-89
10	1	Planar Graphs	T1:Chap:5.2:Pg.No:90-93
11	1	Different Representation of a	T1:Chap:5.4:Pg.No:93-

		Planar Graph	99
12	1	Recapitulation & discussion of possible questions	
	Total No of Hours Planned For Unit II = 12		
		UNIT-III	
1	1	Introduction and definition of a incidence matrix	T1:Chap:7.1:Pg.No:137-139
2	1	Sub matrices of incidence matrices	T1:Chap:7.2:Pg.No:140-141
3	1	Circuits matrix based on problems	T1:Chap:7.3:Pg.No:142-146
4	1	Path matrix and adjacency matrix based on problems	T1:Chap:7.8:Pg.No:156-161
5.	1	Chromatic Number theorems	R3:Chap:1.12:Pg.No:257-258
6.	1	Chromatic partitioning	R3:Chap:16.14:Pg.No:258-259
7	1	Chromatic polynomial	T1:Chap:8.3:Pg.No:174-177
8.	1	Matching	T1:Chap:8.4:Pg.No:177-182
9.	1	covering	T1:Chap:8.5:Pg.No:182-190
10.	1	Four color problem	R3:Chap:2.1:Pg.No:31-35
11.	1	Recapitulation & discussion of possible questions	
	Total No of Hours Planned For Unit III = 11		
		UNIT-IV	

1	1	Introduction and definition of Directed Graphs	R7:Chap:3.1:Pg.No:163-165
2	1	Some types of Directed Graphs	T1:Chap:9.2:Pg.No:197-198
3	1	Conutination on Some types of Directed Graphs	T1:Chap:9.2:Pg.No:197-198
4	1	Types of enumeration	T1:Chap:10.1:Pg.No:238 -240
5.	1	Counting labeled trees	T1:Chap:10.2:Pg.No:240 -241
6.	1	Continuation on theorem on Counting labeled trees	T1:Chap:10.2:Pg.No:240 -241
7	1	Counting unlabeled trees	T1:Chap:10.3:Pg.No:241 -250
8.	1	Continuation on theorem on Counting unlabeled trees	T1:Chap:10.3:Pg.No:241 -250
9.	1	Polya's counting theorem	T1:Chap:10.4:Pg.No:250 -260
10.	1	Graph enumeration with Polya's theorem	T1:Chap:10.5:Pg.No:260 -264
11.	1	Recapitulation & discussion of possible questions	
	Total No of Hours Planned For Unit IV=24		
		UNIT-V	
1	1	Introduction : Domination in Graphs	T1:Chap:1.1:Pg.No:15-16
2	1	Terminology and concepts	T1:Chap:1.1:Pg.No:15-16
3	1	Applications of Domination in	R5:Chap:5.1:Pg.No:71-

		graphs	73
4	1	Dominating set	T2:Chap:1.2:Pg.No:16-18
5	1	Continuation on Dominating set	T2:Chap:1.2:Pg.No:16-18
6	1	Domination number	T2:Chap:1.2:Pg.No:17-18
7	1	Continuation on Domination number	T2:Chap:1.2:Pg.No:17-18
8	1	Independent set	T2:Chap:1.3:Pg.No:19-20
9	1	Independent number	T2:Chap:1.3:Pg.No:19-20
10	1	History of domination in graphs	T2:Chap:1.13:Pg.No:36-37
11	1	Recapitulation & discussion of possible questions	
12	1	Disscussion of previous ESE question papers	
13	1	Disscussion of previous ESE question papers	
14	1	Disscussion of previous ESE question papers	
	Total No of Hours Planned for unit V = 14		
Total Planned Hours	**60**		

TEXT BOOKS

1. Deo N, (2004). Graph Theory with Applications to Engineering and Computer Science, Prentice Hall Inc ,Upper Saddle River, NJ, USA. **(for Unit I to IV).**

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KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Graph Theory and its Applications **Subject Code: 17MMP205A**
Class : I M.Sc Mathematics **Semester : II**

UNIT -I
PART A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The degree of the leaf is	1	2	n	n-1	1
A graph in which all vertices are of equal degree is	complete graph	regular graph	null graph	both complete and regular graph	complete graph
A graph is a finite number of vertices and finite number of edges are called ___	finite graph	star graph	isolated graph	infinite graph	finite graph
A isolated vertex having no ____	incident edges	edges	series	adjancent edges	incident edge
Every edge of a ___ is cutset	tree	graph	incident edges	adjacent edge	tree
The degree of every vertex n-1 is	complete graph	regular graph	null graph	subgraph	complete graph
A pendent vertex of degree is	1	2	3	4	1
A regular graph with n vertices and their degree is	n-1	n-2	n+1	n+2	n-1
Atleast one vertex is	graph	incident vertex	degree	pendent	graph
Isolated vertex is	null graph	pendent graph	complete graph	regular graph	null graph
A null graph containing only	isolated vertex	regular graph	complete graph	simple graph	isolated vertex
All the edges of a graph is	euler line	euler edge	euler graph	euler trail	euler line
G is a subgraph of G then	G-g	G∩g	G+g	G/g	G-g
A connected graph G is	Hamiltonian circuit	hamiltonian graph	hamiltonian path	circuit	hamiltonian circuit
The number of edges incident on a vertex with self-loop counted twice is	degree	adjacent	link	block	degree
In any tree there __ two pendent vertices	atleast	atmost	some	sum of	atleast
The length of a hamiltonian path of a ____ with n vertices n-1	connected graph	star graph	simple graph	complete graph	connected graph
A valency is degree of	vertex	edges	series	link	vertex
degree	one	two	three	zero	two
A single vertex in a graph G is	subgraph	regular graph	component	series	subgraph
A walk is ___ alternating sequence of vertices and edges beginning and ending vertices such that each edge is incident with the vertices	finite	infinte	atmost	some of	finite
Each connected subgraph is ____	component	star graph	series	link	component
A complete graph G is an Euler graph only if the number of vertices is	even	odd	2	6	odd
Euler line contains all the ____of a graph	vertices	edges	isolated vertices	pendant vertices	edges
Euler graphs do not have	even vertices	odd vertices	isolated vertices	pendant vertices	isolated vertices
If G is a star with n vertices then Δ(G) =	n	n-1	$\frac{n}{2}$	$\frac{n-1}{2}$	n-1
If G is a star with n vertices then δ(G) =	n	n-1	$\frac{n}{2}$	1	1
If G is a star with n vertices then number of vertices with degree 1=	n	n-1	$\frac{n}{2}$	1	n-1
A Hamiltonian circuit in a graph of n vertices consists of		n-1 edges	n-2 edges	n-3 edges	n edges

If G is an Euler graph then G	is connected	is not connected	with 2 components	with pendeant vertices	is connected			
If G has an Hamiltonian circuit then G	is connected	is not connected	with 2 components	with pendeant vertices	is connected $2m-1$			
Length of a Hamiltonian path of a connected graph with n vertice	n	n-1			n-1 $G\oplus e$			
Agraph with n vertices is a tree if	n $edges$	G is connected	G has n-1 edges	G is not connected	G is connected and has n-1 edges			
A graph is a infinte number of vertices and infinite number of edg	infinite graph	finite graph	link	regular graph	infinite graph			
A graph with n vertices is a tree if	G is connected	G has n-1 edges	G is not connected	G is circuitless and has n-1 edges	G is circuitless and has n-1 edges			
A graph with n vertices is a tree if	G is connected	G has n-1 edges	G is not connected	there is exactly one path between every pair of vertices in G	there is exactly one path between every pair of vertices in G			
A graph with n vertices is a tree if	G is connected	G has n-1 edges	G is not connected	G is minimally connected graph	G is minimally connected graph			
In any tree there are _____ two pendant vertices	atleast two	atmost two	exactly	no	atleast two			
Distance between any two vertices is	< 0							
Number of circuits in a tree is	0	1	2	3	0			
Distance between any two vertices in a complete graph is	0	1	2	3	1			
A vertex with minimum eccentricity is	pendant vertex	isolated vertex	centre	odd vertex	centre			
If G is a complete graph with n vertices then number of centre of	n	n-1	n-2	n-3	n			

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Class : I M.Sc Mathematics **Semester : II**

UNIT -II

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A tree T is said to be a spanning tree of G if T contains	all vertices of G	all edges of G	some vertices of G	some edges of G	all vertices of G
Spanning tree defined only for a	complete graph	connected graph	disconnected graph	star graph	connected graph
A disconnected graph with k components has _____ spanning tree	k-3	k-1	k-2	k	k
A circuit free graph which contains all the vertices of G is a	tree	spanning tree	star graph	complete graph	spanning tree
A skeleton of a graph is	tree	spanning tree	star graph	complete graph	spanning tree
Suppose G is a graph with n vertices and T is a spanning tree of G. Then number of branches in T is	n	n-1	n-2	n-3	n-1
Number of chords for a complete graph is	4851	4850	4852	4853	4851
Suppose k is denoted as the number of components of G. Then G is connected if	k=0	k=1	k=2	k=3	k=1
Suppose G is a graph with n vertices and k is denoted as the number of components of G. Then the rank of G	n-k	n+k	n/k	n	n-k
Suppose G is a graph with n vertices, e edges and k is denoted as the components of G. Then the nullity of G	e-n+k	e+n+k	e+n	e-k	e-n+k
Rank of G =	number of branches	number of chords	number of edges	number of vertices	number of branches
Nullity of G =	number of branches	number of chords	number of edges	number of vertices	number of chords
Rank of G+ nullity of G =	number of branches	number of chords	number of edges	number of vertices	number of edges
A connected graph is atree if adding an edge between any two vertices in G creates ____ circuit	exactly one	atmost one	atleast one	no	exactly one
Creating a circuit by adding anyone chord to T is	cycle	fundamental circuit	elementary circuit	circuit	fundamental circuit
Distance between two spanning trees T_i and T_j is the number of edges present in	T_i	T_j	T_i not in T_j	T_i and T_j	T_i not in T_j
Distance between two spanning trees T_i and T_j =	$\frac{1}{2}N(T_i \oplus T_j)$	$\frac{1}{2}N(T_i \cup T_j)$	$\frac{1}{2}N(T_i \cap T_j)$	$\frac{1}{2}N(T_i - T_j)$	$\frac{1}{2}N(T_i \oplus T_j)$
If δ(G) and Δ(G) are the minimum and maximum degree in a graph G then the edge connectivity of G is	δ(G)	Δ(G)	2		δ(G)
The number of branches in any ____ of G is rank	spanning tree	tree	shortest spanning tree	minimal spanning tree	spanning tree
Weight of a spanning tree T is	sum of weights of all branches of G	sum of weights of all branches of T	sum of weights of all edges of G	sum of weights of all edges of T	sum of weights of all branches of T
In a graph of n vertces in which every edge has unit weight, then spanning tree T has _____ weight	n	n-1	n-2	n-3	n-1
In a graph of n vertces in which every edge has 3 unit weight, then spanning tree T has _____ weight	3n	3(n-1)	3(n-2)	3(n-3)	3(n-1)
A graph in which all nodes are of equal degree is called	complete graph	regular graph	null graph	multi graph	regular graph
Two ismoephic graphs must have	Equal number of vertices	equal number of edges	an equal number of vertices with a given degree	all of the above	all of the above
In a separable graph, a vertex whose removal disconnects the graph ______	cut vertex	cut edge	odd vertex	even vertex	every edge
______ of a star is a cut set	every vertex	every edge	odd vertex	even vertex	cut- vertex
Edge connectivity of K_2 is	1	2	3	4	1
Each of the largest ___ subgraph is block	nonseparable1	separable	tree	cut-set	nonseparable
Edge connectivity of a tree is	1	2	3	4	1
Edge connectivity of a star graph is	1	2	3	4	1
A separable graph consists of two or more non separable _____	subgraph	tree	spanning tree	complete graph	subgraph
The ring sum of two cut set is	cut set	not cut set	may cut set	empty set	cut set
The edge connecetivity of a connected graph is minimum number of edges removal reduces the rank of by _______	4	3	2	1	1
The vertex connectivity of a tree is	4	3	2	1	1
A graph is planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its ____ intersect	edges	vertices	link	block	edges
The vertex connectivity of a star graph is	1	2	3	4	1
Every cut-set in a nonseparable graph with more than two vertices contains ___ two edges.	atleast	atmost	exactly	graph	atleast
Any edge which is not spanning tree is _____	branch	chord	tree	rank	chord
In a tree, v is a cut vertex if deg(v)	≥ 1	$=$	< 1	> 1	> 1
A tree in which ___ vertex is distinguished from all others is called rooted tree	1	3	2	4	1
A connected ___ graph with n vertices and e edges has e - n + 2 regions	planar	non planar	complete graph	cut-set	planar
The distance between____ of a connected graph is eccentricity	edges	vertices	self loop	loop	vertices
in a ___ graph, any minimal set of edges containing atleast one branch of every spanning tree is cut-set	connected graph	disconnected graph	complete graph	tree	connected graph
K_n is planar for n	4	5	6	7	4
diameter is length of the longest path in the ____	tree	spanning tree	shortest spanning tree	euler graph	tree
In a degree constrained shortest spanning tree deg(G)≤	3	4	5	2	3
Every circuit has an ___ number of edges in common with any cut set	even	odd	zero	three	even
A ___ is separable if its vertex connectivity is one.	connected graph	simple graph	planar graph	non planar graph	connected graph
A _____is a connected graph without any circuit.	tree	spanning tree	weighted spanning tree	hamiltonian circuit	tree
Any connected graph with n vertices and n-1 edges is ____	tree	spanning tree	fundamental circuit	fundamental circuit	tree

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Class : I M.Sc Mathematics **Semester : II**

UNIT - III
PART A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Every ___ edge in a graph G is included in every covering of G.	pendant	isolated	link	block	pendant
The complete graph of 5 vertices is _______	planar	nonplanar	embedding	complete	nonplanar
The number of _______ in each row is the degreee of the corresponding vertex	1	0	2	3	1
A row 0's in the incidence matrix represents	an isolated vertex	pendeant vertex	odd vetrex	even vertex	an isolated vertex
Every degree of a vertex v equals the number of ____ in the correspondingrow or column of x(G)	circuit	vertex	edge	singular	circuit
If G is a graph with n vertices then A(G) =	n	n-1	n-2	n-3	n-1
____ produce identical columns in the cut set matrix	parallel edges	parallel vertex	vertex	edge	parallel edge
Cover of a graph is a _____ of vertices	subset	set	matrix	singular	subset
The incidence matrix of a graph G is	square matrix	rectangular matrix	column matrix	row matrix	rectangular matrix
If G is a tree then A(G) is	square matrix	rectangular matrix	column matrix	row matrix	square matrix
If G is a tree with n vertices then order of A(G) =	n	n-1	n-2	n-3	n-1
The reduced incidence matrix of a tree is	singular	nonsingular	cannot be determined	of 1 determinant	nonsingular
A matching in a graph is a subset of edges in which no ___ edges are adjacent	2	4	3	1	2
Every ____ is 2 - chromatic	bipartite graph	null graph	simple graph	complete graphs	bipartite graph
If A(G) is the adjacency matrix of a graph with 0's in ____ then G is complete	diagonal	non diagonal	matrix	tree	diagonal
A column of all ___ corresponds to a non circuit edge is circuit matrix	0's	1's	n	n+1	0's
Every degree of a vertex v equals the number of ____ in the correspondingrow or column of x(G)	1's	0's	diagonal	matrix	1's
Suppose $A(G) = I_n$, *the identity matrix with order n. Then G is*	connected	disconnected	simple graph	complete	disconnected
X(G) = In, identity matrix if G has _____ and disconnected with k = n	self loop	connected	loop	link	self loop
Suppose G is complete graph with n vertices. Then number of rows in A(G) with exactly one 0 is	n	n-1	0	1	n
Suppose G is complete graph with n vertices. Then the main diagonal element of A(G) is	1	0	0 or 1	2	0
A column of B(G) of all zeros corresponds to a non circuit ____________	edge	vertex	both vertex and edge	neither edge nor vertex	edge
The incidence matrix A(G) every column has ____ two 1's	atmost	atleast	exactly	more than	exactly
The number of 1's in a row of B(G) =	number of vertices in G	number of edges in G	number of odd vertices	number of even vertices	number of edges
The ______ matrix two elements 0 and 1 is binary matirx	incidence	adjacence	cut set	circuit	incidence
If B(G) is a circuit matrix of a connected graph with n vertices and e edges then rank of B(G) is	n	e	1	e-n+1	e-n+1
If G is a tree with n vertices then rank of B(G) is	1	0	2	3	0
In A(G), the _______ matrix, a row with all 0's represents isolated vertex	adjacent	path	circuit	incident	incident
$G \oplus G =$	any	some	no	exactly one	any
A column of P(x,y) all 1's corresponds to an edge that lies in _______ path between x and y	any	some	no	exactly one	any
Number of rows in P(x,y) with all 0's is	0	1	2	3	0
If the entries along the principal diagonal of an adjacancy matrix are all of 0's then G has	self loop	no self loop	parallel edges	isolated vertex	self loop
The degree of a vertex equals the number of 1's in the corresponding _____ of adjacency matrix	row only	column only	both row and column	either row or column	2^{m-1}
A graph consisting of only isolated vertices is	1-chromatic	2-chromatic	3 -chromatic	4-chromatic	1-chromatic
A graph with one edge is atleast	1-chromatic	2-chromatic	3 -chromatic	4-chromatic	2-chromatic
If G is an Euler graph then G	exactly one	atmost	atleast	not	atleast
The number of edges in a largest maximal matching is ____	matching	matching number	maximal matching	minimal matching	matching number
A graph that cannot be drawn on a plane without a cross over between its edges is called _______	planar	nonplanar	embedding	graph	planar
Complete graph with more than one vertices is _____	planar	nonplanar	embedding	graph	nonplanar
The determinant of every square submatrix of an ___ matrix is 1,-1 or 0	incidence	adjacence	circuit	cut set	incidence
_______ discovered nonplanar graph unique property	Kasimir Kuratoaswski	Rowan Hamilton	Euler	Fermat	Kasimir Kuratoaswski
The complete graph of ________ vertices is nonplanar	four	six	seven	five	five
A pentagon divide the plane of the paper into two regions is called _______	Jordan curve	Kuratowski	Euler	Konigsberg bridges	Jordan curve
In adjacency matrix of graph all the entries along the leading diagonal are 0 if and only if the graph has no _____	self loop	loop	block	link	self loop
The number of ___ in a minimal covering of the smallest size is covering number of the graph	edges	vertices	loop	block	edges
In ____ matrix, a colum with all 0's corresponds to an edge forming a self -loop	cut-set	circuit	path	adjacency	cut-set
The rank of ____ matrix must be atleast n-1	incident	path	circuit	cut-set	incident
A _____ in which every vertex is of degree one is dimer covering	covering	minimal covering	maximal covering	matching	covering
A hamiltonian ____ in a graph is covering	circuit	path	vertex	edge	circuit
A graph with ___ or more edges is atleast 2 - chromatic	1	2	3	4	1

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UNIT - IV

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A ____ that has no self loop or parallel edges is simple	digraph	graph	tree	spanning tree	digraph
A balanced digraph is__________	isograph	simple graph	complete digraph	digraph	isograph
A ____ oriented graph.	digraph	complete graph	simple graph	Euler graph	digraph
In any graph, we have	α(G)=β(G)	α(G)≤β(G)	α(G)<β(G)	α(G)≥β(G)	α(G)≤β(G)
A vertex v ia called pendant vertex if d+(v)+d-(v)=	1	2	3	4	1
A graph G is an Euler graph if d+(v) is odd then d-(v)=	odd	even	3	5	even
A graph with one or more edges is atleast	4-chromatic	3-chromatic	2-chromatic	1-chromatic	2-chromatic
A complete graph with n vertices is--------------	4-chromatic	3-chromatic	2-chromatic	n-chromatic	n-chromatic
Every graph having ----------- is atleast 3-chromatic	triangle	square	odd vertices	even vertices	triangle
Every graph having triangle is atleast ------------------	4-chromatic	3-chromatic	2-chromatic	n-chromatic	3-chromatic
A complete graph with 5 vertices is--------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	5-chromatic
Every tree with two or more vertices is------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Every --------with 2 or more vertics is 2-chromatic	tree	complete	connected	disconnected	tree
A graph consisting of simply one circuit with greater than or equal to 3 vertices is ------------- if n is even	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
A graph consisting of simply one circuit with greater than or equal to 3 vertices is 2-chromatic if n is ----------	even	odd	3	0	even
A graph consisting of simply one circuit with greater than or equal to 3 vertices is ------------- if n is odd	4-chromatic	3-chromatic	2-chromatic	5-chromatic	3-chromatic
A graph consisting of simply one circuit with greater than or equal to 3 vertices is 3-chromatic if n is ---------	even	odd	3	0	odd
A graph with ------- one edge is 2-chromatic if it has no circuits of odd length	atleast	atmost	exactly	3	atleast
A graph with atleast ---------- edge is 2-chromatic if it has no circuits of odd length	1	2	3	4	1
A graph with atleast one edge is 2-chromatic if it has no circuits of -------- length	odd	even	0	4	odd
A graph with atleast one edge is 2-chromatic if it has ---------- circuits of odd length	0	1	2	3	0
A graph with atleast one edge is ------------- if it has no circuits of odd length	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
A star graph is ----------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Every tree with -------------- vertics is 2-chromatic	greater than 2	less than 2	equal to 2	greater than or equal to 2	greater than or equal to 2
Every ------------ graph is 2-chromatic	bipartiate	complete	regular	connected	bipartiate
Every biparitate graph is ------------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Two regions are said to be adjacent if they have a common -----------between them	edge	vertex	edge and vertex	neither edge nor vertex	edge
Two ---------------------- are said to be adjacent if they have a common egde between them	faces	regions	egdes	vertices	regions
Proper coloring of ---------------- is called map coloring	faces	regions	egdes	vertices K_3	regions
A covering exists for a graph if the graph has no ------------------------	isolated vertex	odd vertex	even vertex	pendant vertex	isolated vertex
Every ------------------ in a graph included in every covering of the graph	pendant edge	odd vertex	even vertex	pendant vertex	pendant edge
Every pendant edge in a graph included in ---------- covering of the graph	no	some	all	finite number of	all
Cover of a graph is a sub set of ----------------------	vertices	edges	both vertices and edges	neither edge nor vertex	vertices
A complete graph with _______ vertices is one of the 2 graphs of Kuratowski.	2	3	5	1	5
The second graph of Kuratowski is a regular connected graph with _______ vertices and _______ edges	six,seven	six,nine	six,five	five,six	six,nine
The two common geometric representations in Kuratowski graph it is fairly easy to see that the graphs are ________	homeomorphics	planar representation	infinite region	isomorphic	isomorphic
A graph in which all vertices are of equal degree is called a _______	complete graph	regular graph	planar graph	nonplanar graph	regular graph
Removal of one edge or a vertex makes each a _______ graph.	complete	planar	nonplanar	Euler	planar
The complete graph of 5 vertices is _______	planar	nonplanar	embedding	complete	nonplanar
The rank of an ____ of a digraph with n vertices is n-1	incidence matrix	cutset matrix	path matrix	circuit matrix	incidence matrix
A _______ in which there is exactly one edge directed from every vertex to every other vertex is complete symmetric digraph	simple digraph	complete digraph	regular digraph	symmetric digraph	simple digraph

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UNIT - V
PART A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The isolated vertex in degree and out degree are equal to	0	1	2	3	0
The minimum cardinality of a _____ is equal to domination number	set	graph	cutset	vertex	set
The dominating set N[S] is _____	v	1	0	2	v
Suppose G is a complete graph wiyh n vertices. Then number of independent set of vertices is	n	n-1	n+1	n+2	n
Every dominating set contain ___ one minmal dominating set	atleast	atmost	equal	every	atleast
The number of ____ in the largest independent set of a graph	vertices	edges	links	blocks	vertices
The minimum cardinality of a total dominating set is dominating set is _______	domination number	independent set	dominating set	independent number	domination number
A set of vertices in a graph is ______ if no two vertices in the set are adjacent	independent set	independent number	dominating set	dominating number	independent set
The number of ___ incident out of a vertex is out degree	edges	vertices	links	blocks	edges
The minimum cardinality of an independent dominating set G is _______	domination number	independent set	dominating set	independent domination number	independent domination number
A _____ dominating set from which no vertex can be removed without destroying its dominance property.	minimal	maximal	independent	independent number	minimal
A ______ dominating set may or may not be independent	minimal	maximal	independent	independent number	minimal
A_______ contains atleast one minimal dominating set.	domination number	independent set	dominating set	independent number	dominating set
The set of all _____ is trivially a dominating set in graph	vertices	edges	cutset	blocks	vertices
An _____ has the dominance property only if it is a maximal independent set	domination number	independent set	dominating set	independent number	independent set
A graph may have many ______ and of different sizes.	minimal dominating set	independent set	dominating set	independent number	minimal dominating set
The number of ___ in a minimal covering of the smallest size is covering number of the graph	edges	vertices	loop	block	edges
In ____ matrix, a colum with all 0's corresponds to an edge forming a self -loop	cut-set	circuit	path	adjacency	cut-set
The rank of ____ matrix must be atleast n-1	incident	path	circuit	cut-set	incident
A _____ in which every vertex is of degree one is dimer covering	covering	minimal covering	maximal covering	matching	covering
A hamiltonian ____ in a graph is covering	circuit	path	vertex	edge	circuit
A graph with ___ or more edges is atleast 2 - chromatic	1	2	3	4	1
A pendent vertex of degree is	1	2	3		1
A regular graph with n vertices and their degree is	n-1	n-2	n+1	n+2	n-1
Atleast one vertex is	graph	incident vertex	degree	pendent	graph
Isolated vertex is	null graph	pendent graph	complete graph	regular graph	null graph
A graph is a infinte number of vertices and infinite number of edges is	infinite graph	finite graph	link	regular graph	infinite graph
The number of edges in a largest maximal matching is ____	matching disconnected	matching number	maximal matching	minimal matching	matching number
A graph that cannot be drawn on a plane without a cross over between its edges is called _______	planar	nonplanar	embedding	graph	planar
Complete graph with more than one vertices is _____	planar	nonplanar	embedding	graph	nonplanar
The determinant of every square submatrix of an ___ matrix is 1,-1 or 0	incidence	adjacence	circuit	cut set	incidence
_______ discovered nonplanar graph unique property	Kasimir Kuratoaswski	Rowan Hamilton	Euler	Fermat	Kasimir Kuratoaswski
The complete graph of ________ vertices is nonplanar	four	six	seven	five	five
A pentagon divide the plane of the paper into two regions is called _______	Jordan curve	Kuratowski	Euler	Konigsberg bridges	Jordan curve
In adjacency matrix of graph all the entries along the leading diagonal are 0 if and only if the graph has no _____	self loop	loop	block	link	self loop
A star graph is ----------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Every tree with -------------- vertics is 2-chromatic	greater than 2	less than 2	equal to 2	greater than or equal to 2	greater than or equal to 2
Every ------------ graph is 2-chromatic	bipartiate	complete	regular	connected	bipartiate
Every biparitate graph is ------------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Two regions are said to be adjacent if they have a common -----------between them	edge	vertex	edge and vertex	neither edge nor vertex	edge
Two ---------------------- are said to be adjacent if they have a common egde between them	faces	regions	egdes	vertices	regions
Proper coloring of ---------------- is called map coloring	faces	regions	egdes	vertices	regions
A covering exists for a graph if the graph has no ------------------------	isolated vertex	odd vertex	even vertex	pendant vertex	isolated vertex
Every ------------------ in a graph included in every covering of the graph	pendant edge	odd vertex	even vertex	pendant vertex	pendant edge
Every pendant edge in a graph included in ---------- covering of the graph	no	some	all	finite number of	all
Cover of a graph is a sub set of ----------------------	vertices	edges	both vertices and edges	neither edge nor vertex	vertices
In a degree constrained shortest spanning tree deg(G)≤	3	4	5	2	3
Every circuit has an ___ number of edges in common with any cut set	even	odd	zero	three	even
A ___ is separable if its vertex connectivity is one.	connected graph	simple graph	planar graph	non planar graph	connected graph
A _____is a connected graph without any circuit.	tree	spanning tree	weighted spanning tree	hamiltonian circuit	tree
Any connected graph with n vertices and n-1 edges is ____	tree	spanning tree	fundamental circuit	fundamental circuit	tree
The number of edges incident on a vertex with self-loop counted twice is	degree	adjacent	link	block	degree
In any tree there __ two pendent vertices	atleast	atmost	some	sum of	atleast

The length of a hamiltonian path of a ____ with n vertices n-1	connected graph	star graph	simple graph	complete graph	connected graph	

Graph theory and its applications

I - M. Sc. Mathematics

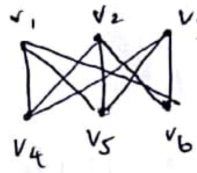
17MMP205A

PART - B

21.) a) (i) Bipartite Graph:-

A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y. So that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph.

Ex:-



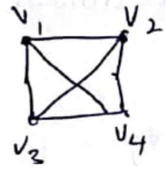
$X = v_1, v_2, v_3$

$Y = v_4, v_5, v_6$

(ii) Regular Graph:-

A graph G is said to be regular if all the vertices in G have the same degree.

Ex:-



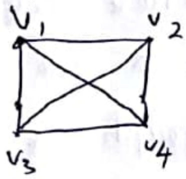
$d(v_1) = 3$ $d(v_3) = 3$

$d(v_2) = 3$ $d(v_4) = 3$

(iii) Complete Graph:-

A simple graph in which each pair of vertices is joined by an edge is called a complete graph.

Ex:-



1.b)

Proof:-

We prove the theorem by induction on the no. of vertices

If $n = 1$, the no. of edges is zero.

If $n = 2$, the no. of edges is one.

Assume that the thm. is true for all vertices with less than n vertices.

Let us consider a tree G with n vertices. Let e_k be any edge in G between the vertices v_i & v_j

Since G is a tree the edge e_k only path between v_i & v_j. delete an edge e_k then G will be disconnected graph.

(ii) $G - e_k$ is not connected. $G - e_k$ contain exactly two components. Let G_1 & G_2 be components & also a tree because there is no circuit in tree & it also a connected.

Let n_1 & n_2 be no. of vertices G_1 & G_2 respectively. $\therefore n_1 < n$ & $n_2 < n$

$\therefore$ no. of edges in $G_1 = n_1 - 1$

no. of edges in $G_2 = n_2 - 1$

The no. of edges in $G - e_k = n_1 - 1 + n_2 - 1 = n - 1$

$\therefore$ no. of edges in $G = n - 1$

HTP.

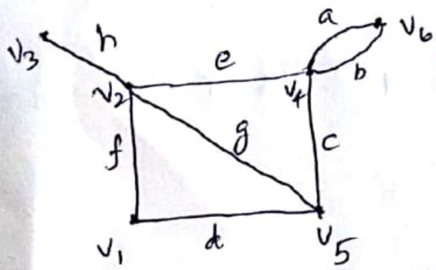
23). a). Explain about incidence matrix in a graph.

Let G be a graph with n vertices, e edges, & no self loops. Define an n by e matrix $A = [a_{ij}]$ whose n rows correspond to the n vertices & the e columns correspond to the e edges, as follows.

The matrix element $a_{ij} = 1$, if jth edge e_j is incident on ith vertex v_i, and

$= 0$, otherwise.

Such a matrix A is called the vertex-edge incidence matrix, or simply incidence matrix. Matrix A for a graph G is sometimes also written as $A(G)$. A graph & its incidence matrix is given below.



$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\left[\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

The incidence matrix contains only two elements 0 & 1. Such a matrix is called a binary matrix or a (0,1) - matrix

23) b). $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$

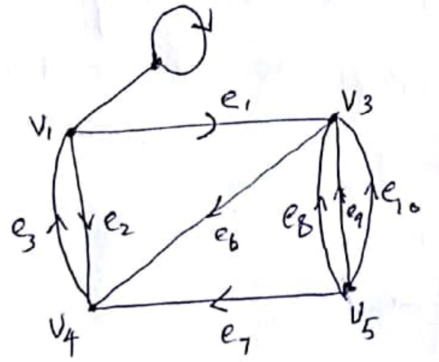
With λ colors, there are λ different ways of coloring any selected vertex of a graph. A second vertex can be colored properly in exactly $\lambda-1$ ways, the third in $(\lambda-2)$ ways, the fourth in $\lambda-3$ ways, ... and the nth in $\lambda-n+1$ ways. if and only if every vertex is adjacent to every other. (ie) if and only if the graph is complete.

24) a)

A directed graph (digraph) G consists of a set of vertices $V=\{v_1, v_2, \ldots\}$, a set of edges $E=\{e_1, e_2, \ldots\}$ and a mapping ψ that maps every edge onto some ordered pair of vertices (v_i, v_j)

A digraph is called an oriented graph.

In a digraph an edge is not only incident on a vertex, but is also incident out of a vertex and incident into a vertex. The vertex v_i, which edge e_k is incident out of, is called the initial vertex of e_k. The vertex v_j, which e_k is incident into is called the terminal vertex of e_k.



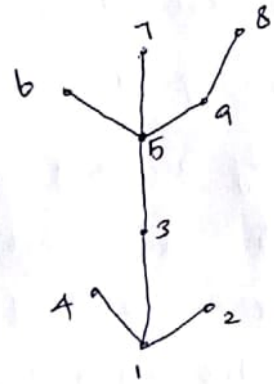
An edge for which the initial & terminal vertices are the same forms a self loop, e_5.

The no. of edges incident out of a vertex $v_i \Rightarrow$ out degree $d^+(v_i)$
The no. of edges incident into $v_i \Rightarrow$ in degree $d^-(v_i)$

$d^+(v_1) = 3$, $d^+(v_2) = 1$, $d^+(v_5) = 4$, $d^-(v_1) = 1$, $d^-(v_2) = 2$, $d^-(v_3) = 0$

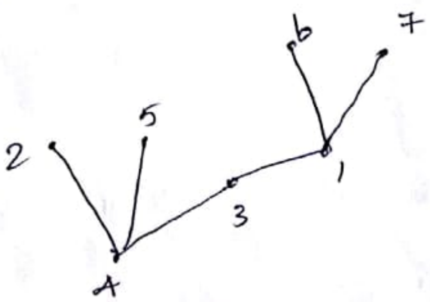
$$\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i)$$

24) b). Let the n vertices of a tree T be labeled $1, 2, 3, \ldots n$.
Remove the pendant vertex having the smallest label, a_1.
Suppose that b_1 was the vertex adjacent to a_1.
Among the remaining $n-1$ vertices let a_2 be the pendant vertex with the smallest label and b_2 be the vertex adjacent to a_2. Remove the edge (a_2, b_2).
This operation is repeated on the remaining $n-2$ vertices, & $n-3$ vertices and so on.
The process is terminated after $n-2$ steps, when only two vertices are left. the tree T defines the sequence $(b_1, b_2 \ldots b_{n-2})$ uniquely.



$(1, 1, 3, 5, 5, 5, 9)$

Determine the first no. in the sequence $1, 2, 3, \ldots n$ that does not appear in sequence. In the remaining sequence of the first no that does not appear. we can construct a seven - vertex tree as follows.

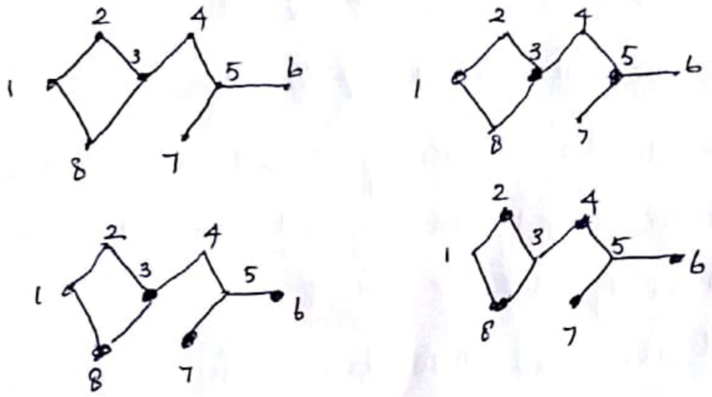


Since each tree defines one of those sequences, there is one - to - one correspondence between the trees & the n^{n-2} sequences.

25) a). A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $u \in V$ is either an element of S or adjacent to an element of S.

The domination number $\gamma(G)$ of a graph G equals the minimum cardinality of a set.

The upper domination number $\Gamma(G)$ equals the maximum cardinality of a set.



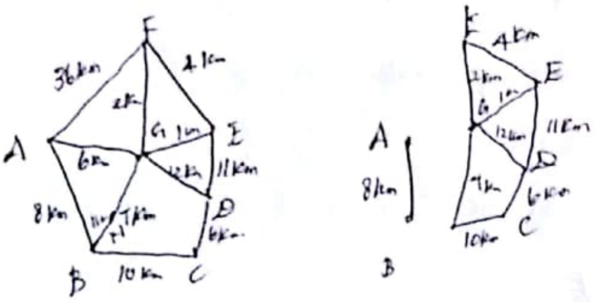
$\gamma(G) = 3$, $\Gamma(G) = 5$, the set $S = \{1, 3, 5\}$ is a dominating set of minimum cardinality. This is called independent dominating set of G. The minimum cardinality of an independent dominating set of G is the independent domination number $i(G)$.

25 b). Suppose that we have a collection of small villages in a remote part of the world. Ex:- Himalayas, Siberia, Andes, serengetti

To locate radio stations in some of these villages so that messages can be broadcast to all of the villages in the region.

radio station has a limited broadcasting range, we must use several stations to reach all villages.

village $\Rightarrow$ vertex , distance $\Rightarrow$ edge.



22a) Kruskal's Algorithm :-

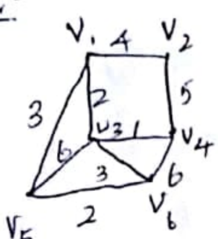
Step 1 :- List all edges of weighted graph G in the ascending order.

Step 2 :- Select the smallest weight of the edge in G.

Step 3 :- Select from all the remaining edges of G connected to the already selected edges. The smallest edge that makes no circuit with the previously selected edges.

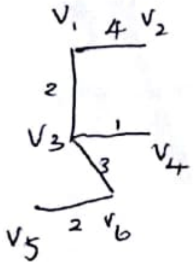
Step 4 :- Continue until n-1 edges have been selected & there n-1 edges will the shortest spanning tree.

Ex :-



weight	edge	weight	edge
1	(v_3, v_4)	3	(v_3, v_6)
2	(v_1, v_3)	4	(v_1, v_2)
2	(v_5, v_6)	5	(v_2, v_4)
3	(v_1, v_5)	6	(v_4, v_6)
		6	(v_3, v_5)

The shortest spanning tree is $v_3 \to v_4$, $v_1 \to v_3$, $v_3 \to v_6$, $v_5 \to v_6$, $v_1 \to v_2$



The weight of the minimal spanning tree = 12 //

Prim's Algorithm :-

Step 1 :- label the n vertices graph as $v_1, v_2 \ldots v_n$

Step 2 :- Tabulate the weights of the edges of G in $n \times n$ table.

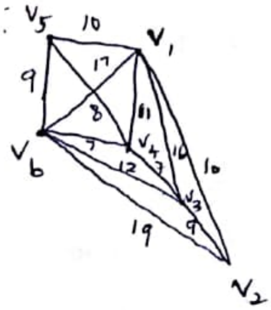
Step 3 :- Set the weights of non existen edges as infinity.

Step 4:- start with the vertex v_1 & connect it to the vertex which has smallest entry in the row of the table (v_k)

Step 5:- Consider the subgraph with vertices v_1 & v_k & an edge b/w them that has the smallest entry rows 1 & k. Let this new vertex v_j.

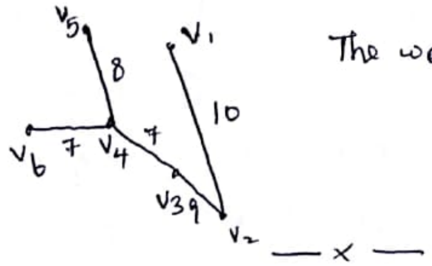
Step 6:- Consider the tree with vertices v_1, v_k & v_j as a subgraph & continue the process until all the n vertices have been connected by n-1 edges.

Ex:-



	v_1	v_2	v_3	v_4	v_5	v_6
v_1	∞	10	16	11	10	17
v_2	10	∞	9	∞	∞	19
v_3	16	9	∞	7	∞	12
v_4	11	∞	7	∞	8	7
v_5	10	∞	∞	8	∞	9
v_6	17	19	12	7	9	∞

The shortest spanning tree is $v_1 \to v_2$, $v_2 \to v_3$, $v_3 \to v_4$, $v_4 \to v_6$, $v_5 \to v_4$



The weight of the minimal spanning tree is 41.

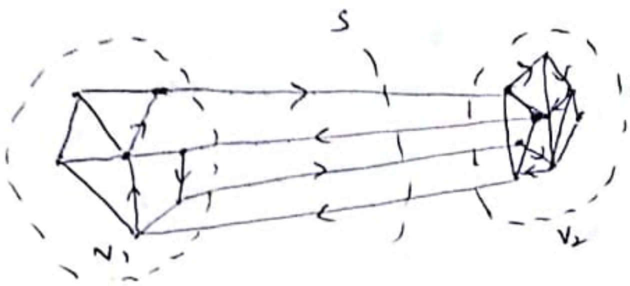
— x —

22 b). * Consider any cut-set S w.r.to vertices a & b in G. In the subgraph G-S there is no path b/w a & b.

* Every path in G b/w a & b must contain atleast one edge of S.
* Every flow from a to b must pass through one or more edges of S.
* The total flow rate b/w these two vertices cannot exceed the capacity of S.
* Since this holds for all cut-sets w.r.to a & b, the flow rate cannot exceed the minimum of their capacities.

PART - C

26).



* Let the removal of S partition the vertices of G into two subsets V_1 & V_2.

* If all the vertices in circuit Γ are entirely within vertex set V_1, the no. of edges common to S & Γ is zero. $N(S \cap \Gamma) = 0$, an even no.

* If some vertices in Γ are in V_1 & some in V_2, we traverse ~~because~~ back & forth b/w the sets V_1 & V_2 as we traverse the circuit because of the closed nature of a circuit, the no. of edges we traverse b/w V_1 & V_2 must be even.

* Every edge in S has one end in V_1 & the other in V_2, and no other edge in G of ~~sp~~ separating sets V_1 & V_2 the no. of edges common to S and Γ is even.

HTP.

UNIT – I

SYLLABUS

Graphs – introduction – isomorphism – sub graphs – walks, paths, circuits – connectedness – components – euler graphs – Hamiltonian path and circuits – trees – properties of trees – distance and centres in tree – rooted and binary trees – spanning trees – fundamental circuits

Graphs

Basic Concepts

Definition 8.1.1. [Pseudograph, Vertex set and Edge set] A **pseudograph** or a **general graph** G is a pair (V, E) where V is a nonempty set and E is a multiset of unordered pairs of points of V. The set V is called the **vertex set** and its elements are called **vertices**. The set E is called the **edge set** and its elements are called **edges**.

Example 8.1.2. $G = \Big([4], \big\{\{1,1\}, \{1,2\}, \{2,2\}, \{3,4\}, \{3,4\}\big\}\Big)$ is a pseudograph.

Discussion 8.1.3. A pseudograph can be represented in picture in the following way.

1. Put different points on the paper for vertices and label them.
2. If $\{u, v\}$ appears in E some k times, draw k distinct lines joining the points u and v.
3. A loop at u is drawn if $\{u, u\} \in E$.

Example 8.1.4. A picture for the pseudograph in Example 8.1.2 is given in Figure 8.1.

Definition 8.1.5. [Loop, End vertex and Incident vertex/edge]

1. An edge $\{u, v\}$ is sometimes denoted uv. An edge uu is called a **loop**. The vertices u and v are called the **end vertices** of the edge uv. Let e be an edge. We say 'e is **incident** on u' to mean that 'u is an end vertex of e'.

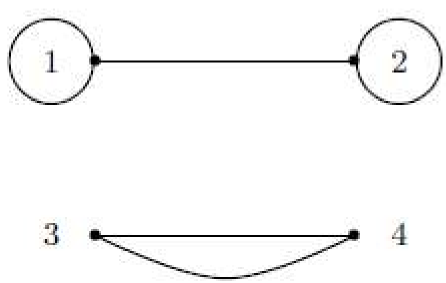


Figure 8.1: A pseudograph

2. [**Multigraph and simple graph**] A **multigraph** is a pseudograph without loops. A multigraph is a **simple graph** if no edge appears twice.[1]

3. Henceforth, all graphs in this book are simple with a finite vertex set, unless stated otherwise.

4. We use $\boldsymbol{V(G)}$ (or simply $\boldsymbol{V}$) and $\boldsymbol{E(G)}$ (or simply $\boldsymbol{E}$) to denote the vertex set and the edge set of G, respectively. The number $|V(G)|$ is the **order** of the graph G. Sometimes it is denoted $|\boldsymbol{G}|$. By $\|\boldsymbol{G}\|$ we denote the number of edges of G. A graph with n vertices and m edges is called a $\boldsymbol{(n,m)}$ **graph**. The $(1,0)$ graph is the **trivial graph**.

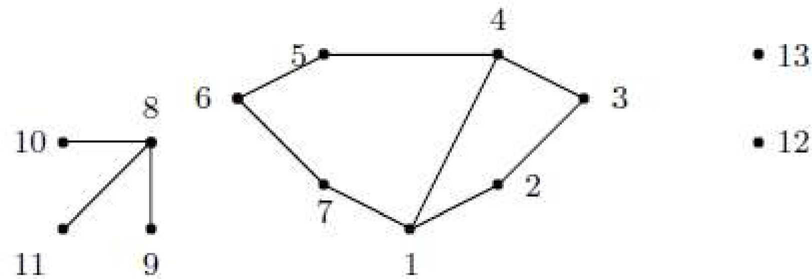
5. [**Neighbor and independent set**] If uv is an edge in G, then we say 'u and v are **adjacent** in G' or 'u is a **neighbor** of v'. We write $u \sim v$ to denote that 'u is adjacent to v'. Two edges e_1 and e_2 are **adjacent** if they have a common end vertex. A set of vertices or edges is **independent** if no two of them are adjacent.

6. [**Isolated and pendant vertex**] If $v \in V(G)$, by $\boldsymbol{N(v)}$ or $\boldsymbol{N_G(v)}$, we denote the set of neighbors of v in G and $|N(v)|$ is called the **degree** of v. It is usually denoted by $\boldsymbol{d_G(v)}$ or $\boldsymbol{d(v)}$. A vertex of degree 0 is called **isolated**. A vertex of degree one is called a **pendant** vertex.

Example 8.1.7. Consider the graph G in Figure 8.2. The vertex 12 is an isolated vertex. We have $N(1) = \{2,4,7\}$, $d(1) = 3$. The set $\{9,10,11,2,4,7\}$ is an independent vertex set. The set $\{\{1,2\},\{8,10\},\{4,5\}\}$ is an independent edge set. The vertices 1 and 6 are not adjacent.

Definition 8.1.8. [**Complete graph, path graph, cycle graph and bipartite graph**] Let $G = (V,E)$ be a graph on n vertices, say $V = \{v_1,\ldots,v_n\}$. Then, G is said to be a

1. **complete** graph, denoted K_n, if each pair of vertices in G are adjacent.
2. **path graph**, denoted $\boldsymbol{P_n}$, if $E = \{v_iv_{i+1} \mid 1 \le i \le n-1\}$.



3. **cycle graph**, denoted C_n, if $E = \{v_i v_{i+1} \mid 1 \le i \le n-1\} \cup \{v_n v_1\}$.

4. **complete bipartite graph**, denoted $K_{r,s}$ and $E = \{v_i v_j \mid 1 \le i \le r, r+1 \le j \le n\}$ with $r + s = n$.

Lemma 8.1.10. **[Hand shaking lemma]** *In any graph G, $\sum\limits_{v \in V} d(v) = 2|E|$. Thus, the number of vertices of odd degree is even.*

Proof. Each edge contributes 2 to the sum $\sum\limits_{v \in V} d(v)$. Hence, $\sum\limits_{v \in V} d(v) = 2|E|$. Note that

$$2|E| = \sum_{v \in V} d(v) = \sum_{d(v) \text{ is odd}} d(v) + \sum_{d(v) \text{ is even}} d(v)$$

is even. So, $\sum\limits_{d(v) \text{ is odd}} d(v)$ is even. Hence, the number of vertices of odd degree is even. ■

Proposition 8.1.12. *In a graph G with $n = |G| \geq 2$, there are two vertices of equal degree.*

Proof. If G has two or more isolated vertices, we are done. So, suppose G has exactly one isolated vertex. Then, the remaining $n-1$ vertices have degree between 1 and $n-2$ and hence by PHP, the result follows. If G has no isolated vertex then G has n vertices whose degree lie between 1 and $n-1$. Now, again apply PHP to get the required result. ∎

Example 8.1.13. The graph in Figure 8.5 is called the **Petersen graph**. We shall use it as an example in many places.

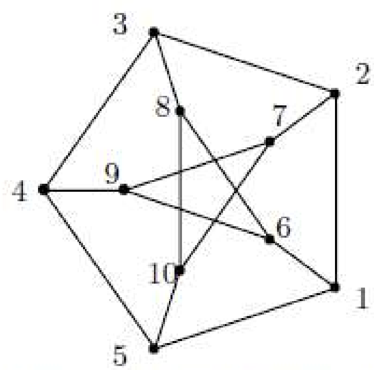


Figure 8.5: Petersen graphs

Definition 8.1.15. **[Regular graph, cubic graph]** The minimum degree of a vertex in G is denoted $\delta(G)$ and the maximum degree of a vertex in G is denoted $\boldsymbol{\Delta(G)}$. A graph G is called **k-regular** if $d(v) = k$ for all $v \in V(G)$. A 3-regular graph is called **cubic**.

Definition 8.1.18. **[Subgraph, induced subgraph, spanning subgraph and k-factor]** A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $U \subseteq V(G)$, then the subgraph **induced by** U is denoted by $\langle U \rangle = (U, E)$, where the edge set $E = \{uv \in E(G) \mid u, v \in U\}$. A subgraph H of G is a **spanning subgraph** if $V(G) = V(H)$. A k-regular spanning subgraph is called a **k-factor**.

Example 8.1.22. Consider the graph G in Figure 8.2. Let H_2 be the graph with $V(H_2) = \{6, 7, 8, 9, 10, 12\}$ and $E(H_2) = \{\{6, 7\}, \{8, 10\}\}$. Consider the edge $e = \{8, 9\}$. Then, $H_2 + e$ is the induced subgraph $\langle\{6, 7, 8, 9, 10, 12\}\rangle$ and $H_2 - 8 = \langle\{6, 7, 9, 10, 12\}\rangle$.

Definition 8.1.23. **[Complement graph]** The **complement** $\overline{G}$ of a graph G is defined as $(V(G), E)$, where $E = \{uv \mid u \neq v, uv \notin E(G)\}$.

Example 8.1.24. 1. See the graphs in Figure 8.6.

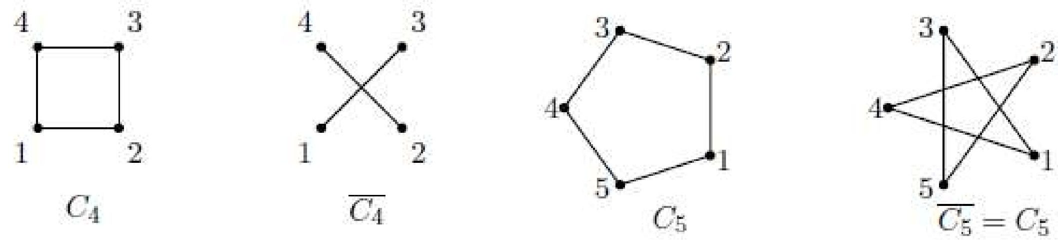
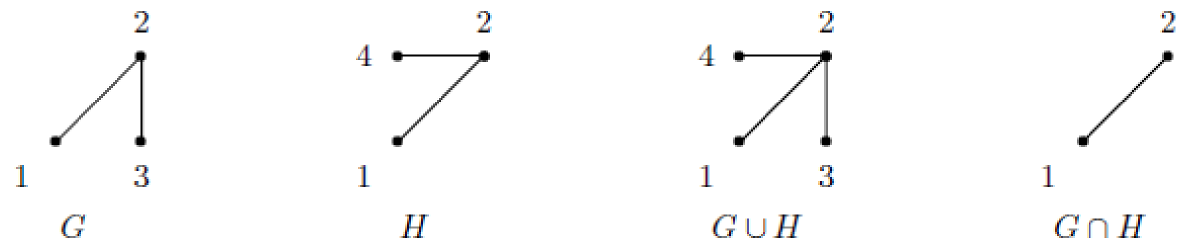


Figure 8.6: Complement graphs

2. The complement of K_3 contains 3 isolated points.
3. For any graph G, $\|G\| + \|\overline{G}\| = C(|G|, 2)$.
4. In any graph G of order n, $d_G(v) + d_{\overline{G}}(v) = n - 1$. Thus, $\Delta(G) + \Delta(\overline{G}) \geq n - 1$.

Definition 8.1.26. **[Intersection, union and disjoint union]** The **intersection** of two graphs G and H, denoted $G \cap H$, is defined as $(V(G) \cap V(H), E(G) \cap E(H))$. The **union** of two graphs G and H, denoted $G \cup H$, is defined as $(V(G) \cup V(H), E(G) \cup E(H))$. A **disjoint union** of two graphs is the union while treating the vertex sets as disjoint sets.

Example 8.1.27. Two graphs G and H are shown below. The graphs $G \cup H$ and $G \cap H$ are also shown below.



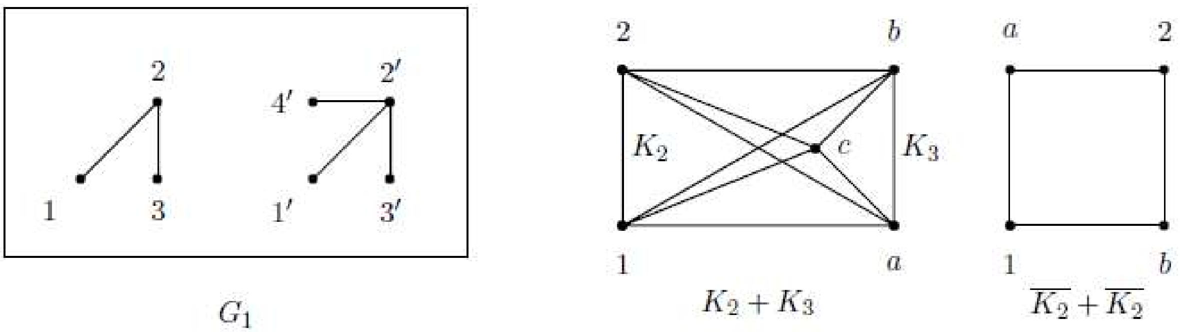


Figure 8.7: Disjoint union and join of graphs

Definition 8.1.28. [**Join of two graphs**] If $V(G) \cap V(G') = \emptyset$, then the **join** $G + G'$ is defined as $G \cup G' + \{vv' : v \in V, v' \in V'\}$. The first '+' means the join of two graphs and the second '+' means adding a set of edges to a given graph.

Connectedness

Definition 8.2.1. [**Walk, trail, path, cycle, circuit, length and internal vertex**] An u-v **walk** in G is a finite sequence of vertices $[u = v_1, v_2, \cdots, v_k = v]$ such that $v_i v_{i+1} \in E$, for all $i = 1, \cdots, k-1$. The **length** of a walk is the number of edges on it. A walk is called a **trail** if edges on the walk are not repeated. A v-u walk is a called a **path** if the vertices involved are all distinct, except that v and u may be the same. A path can have length 0. A walk (trail, path) is called **closed** if $u = v$. A closed path is called a **cycle/circuit**. Thus, in a simple graph a cycle has length at least 3. A cycle (walk, path) of length k is also written as a k-cycle (k-walk, k-path). If P is an u-v path with $u \neq v$, then we sometimes call u and v as the **end vertices of P** and the remaining vertices on P as the **internal vertices**.

Proposition 8.2.3 (Technique). *Let G be a graph and $u, v \in V(G)$, $u \neq v$. Let $W = [u = u_1, \ldots, u_k = v]$ be a walk. Then, W contains an u-v-path.*

Proof. If no vertex on W repeats, then W is itself a path. So, let $u_i = u_j$ for some $i < j$. Now, consider the walk $W_1 = [u_1, \ldots, u_{i-1}, u_j, u_{j+1}, \ldots u_k]$. This is also an u-v walk but of shorter length. Thus, using induction on the length of the walk, the desired result follows. ∎

Proposition 8.2.10. *Every graph G containing a cycle satisfies $g(G) \leq 2\,\mathsf{diam}(G) + 1$.*

Proof. Let $C = [v_1, v_2, \ldots, v_k, v_1]$ be the shortest cycle and $\mathsf{diam}(G) = r$. If $k \geq 2r + 2$, then consider the path $P = [v_1, v_2, \ldots, v_{r+2}]$. Since the length of P is $r + 1$ and $\mathsf{diam}(G) = r$, there is a v_{r+2}-v_1 path R of length at most r. Note that P and R are different v_1-v_{r+2} paths. By Proposition 8.2.9, the closed walk $P \cup R$ of length at most $2r + 1$ contains a cycle. Hence, the length of this cycle is at most $2r+1$, a contradiction to C having the smallest length $k \geq 2r+2$. ■

Definition 8.2.11. [Chord, chordal and acyclic graphs] Let $C = [v_1, \ldots, v_k = v_1]$ be a cycle. An edge v_iv_j is called a **chord** of C if it is not an edge of C. A graph is called **chordal** if each cycle of length at least 4 has a chord. A graph is **acyclic** if it has no cycles.

Definition 8.2.14. 1. **[Maximal and minimal graph]** A graph G is said to be **maximal** with respect to a property P if G has property P and no proper supergraph of G has the property P. We similarly define the term **minimal**.

Proposition 8.2.17. *If $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.*

Proof. Let $[v_1, \cdots, v_k]$ be a longest path in G. As $d(v_k) \geq 2$, v_k is adjacent to some vertex $v \neq v_{k-1}$. If v is not on the path, then we have a path that is longer than $[v_1, \cdots, v_k]$ path. A contradiction. Let i be the smallest positive integer such that v_i is adjacent to v_k. Thus,

$$\delta(G) \leq d(v_k) \leq |\{v_i, v_{i+1}, \cdots, v_{k-1}\}|.$$

Hence, the cycle $C = [v_i, v_{i+1}, \cdots, v_k, v_i]$ has length at least $\delta(G) + 1$ and the length of the path $P = [v_i, v_{i+1}, \cdots, v_k]$ is at least $\delta(G)$. ■

Definition 8.2.18. [Edge density] The **edge density**, denoted $\varepsilon(G)$, is defined to be the number $\frac{|E(G)|}{|V(G)|}$. Observe that $\varepsilon(G)$ is also a graph invariant.

Isomorphism in Graphs

Definition 8.3.1. **[Isomorphic graphs]** Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be **isomorphic** if there is a bijection $f : V \to V'$ such that $u \sim v$ is G if and only if $f(u) \sim f(v)$ in G', for each $u, v \in V$. In other words, an isomorphism is a bijection between the vertex sets which preserves adjacency. We write $G \cong G'$ to mean that G is isomorphic to G'.

Example 8.3.2. Consider the graphs in Figure 8.9. Then, note that

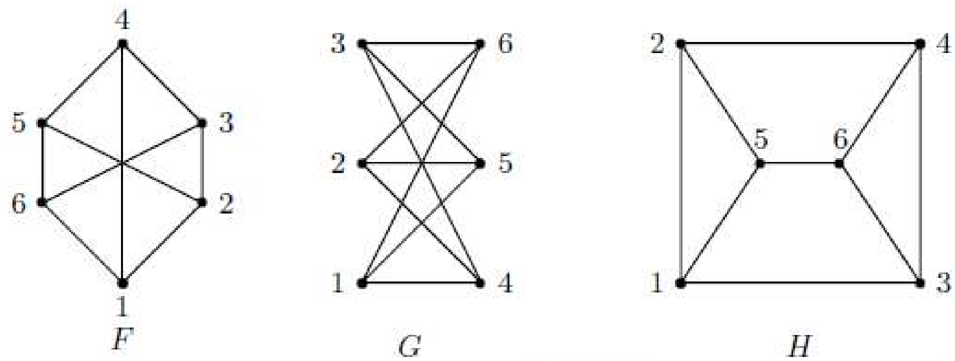


Figure 8.9: F is isomorphic to G but F is not isomorphic to H

1. the graph F is not isomorphic to H as the **independence number**, denoted $\alpha(F)$, of F (the maximum size of an independent vertex set) is 3 whereas $\alpha(H) = 2$. Alternately, H has a 3-cycle, whereas F does not.
2. the graph F is isomorphic to G as the map $f : V(F) \to V(G)$ defined by $f(1) = 1$, $f(2) = 5$, $f(3) = 3$, $f(4) = 4$, $f(5) = 2$ and $f(6) = 6$ gives an isomorphism.

Definition 8.3.5. **[Self-complementary]** A graph G is called **self-complementary** if $G \cong \overline{G}$.

Example 8.3.6. 1. Note that the cycle $C_5 = [0, 1, 2, 3, 4, 0]$ is self complimentary. An isomorphism from G to $\overline{G}$ is described by $f(i) = 2i \pmod 5$.

2. If $|G| = n$ and $G \cong \overline{G}$ then $\|G\| = n(n-1)/4$. Thus, $n = 4k$ or $n = 4k + 1$.

Definition 8.3.8. **A graph invariant** is a function which assigns the same value (output) to isomorphic graphs.

Example 8.3.9. Observe that some of the graph invariants are: $|G|$, $\|G\|$, $\Delta(G)$, $\delta(G)$, the multiset $\{d(v) : v \in V(G)\}$, $\omega(G)$ and $\alpha(G)$.

Definition 8.3.12. An isomorphism of G to G is called an **automorphism**.

Example 8.3.13. 1. Identity map is always an automorphism on any graph.

2. Any permutation in S_n is an automorphism of K_n.

3. There are only two automorphisms of a path P_8.

Trees

Definition 8.4.1. [**Tree and forest**] A connected acyclic graph is called a **tree**. A **forest** is a graph whose components are trees.

Proposition 8.4.2. *Let T be a tree and $u, v \in V(T)$. Then, there is a unique u-v-path in T.*

Proof. On the contrary, assume that there are two u-v-paths in T. Then, by Proposition 8.2.9, T has a cycle, a contradiction. ∎

Proposition 8.4.3. *Let G be a graph with the property that 'between each pair of vertices there is a unique path'. Then, G is a tree.*

Proof. Clearly, G is connected. If G has a cycle $[v_1, v_2, \cdots, v_k = v_1]$, then $[v_1, v_2, \ldots, v_{k-1}]$ and $[v_1, v_{k-1}]$ are two v_1-v_{k-1} paths. A contradiction. ∎

Definition 8.4.4. [**Cut vertex**] Let G be a connected graph. A vertex v of G is called a **cut vertex** if $G - v$ is disconnected. Thus, $G - v$ is connected if and only if v is not a cut vertex.

Proposition 8.4.5. *Let G be a connected graph with $|G| \geq 2$. If $v \in V(G)$ with $d(v) = 1$, then $G - v$ is connected. That is, a vertex of degree 1 is never a cut vertex.*

Proof. Let $u, w \in V(G - v)$, $u \neq w$. As G is connected, there is an u-w path P in G. The vertex v cannot be an internal vertex of P, as each internal vertex has degree at least 2. Hence, the path P is available in $G - v$. So, $G - v$ is connected. ∎

Proposition 8.4.6 (Technique). *Let G be a connected graph with $|G| \geq 2$ and let $v \in V(G)$. If $G - v$ is connected, then either $d(v) = 1$ or v is on a cycle.*

Proof. Assume that $G - v$ is connected. If $d_G(v) = 1$, then there is nothing to show. So, assume that $d(v) \geq 2$. We need to show that v is on a cycle in G.

Let u and w be two distinct neighbors of v in G. As $G - v$ is connected there is a path, say $[u = u_1, \ldots, u_k = w]$, in $G - v$. Then, $[u = u_1, \ldots, u_k = w, v, u]$ is a cycle in G containing v. ∎

Definition 8.4.8. [**Cut edge**] Let G be a graph. An edge e in G is called a **cut edge** or a **bridge** if $G - e$ has more connected components than that of G.

Proposition 8.4.9 (Technique). *Let G be connected and $e = [u, v]$ be a cut edge. Then, $G - e$ has two components, one containing u and the other containing v.*

Proof. If $G - e$ is not disconnected, then by definition, e cannot be a cut edge. So, $G - e$ has at least two components. Let G_u (respectively, G_v) be the component containing the vertex u (respectively, v). We claim that these are the only components.

Let $w \in V(G)$. Then, G is a connected graph and hence there is a path, say P, from w to u. Moreover, either P contains v as its internal vertex or P doesn't contain v. In the first case, $w \in V(G_v)$ and in the latter case, $w \in V(G_u)$. Thus, every vertex of G is either in $V(G_v)$ or in $V(G_u)$ and hence the required result follows. ■

Proposition 8.4.10 (Technique). *Let G be a graph and e be an edge. Then, e is a cut edge if and only if e is not on a cycle.*

Proof. Suppose that $e = [u, v]$ is a cut edge of G. Let F be the component of G that contains e. Then, by Proposition 8.4.9, $F - e$ has two components, namely, F_u that contains u and F_v that contains v.

Let if possible, $C = [u, v = v_1, \ldots, v_k = u]$ be a cycle containing $e = [u, v]$. Then, $[v = v_1, \ldots, v_k = u]$ is an u-v path in $F - e$. Hence, $F - e$ is still connected. A contradiction. Hence, e cannot be on any cycle.

Conversely, let $e = [u, v]$ be an edge which is not on any cycle. Now, suppose that F is the component of G that contains e. We need to show that $F - e$ is disconnected.

Let if possible, there is an u-v-path, say $[u = u_1, \ldots, u_k = v]$, in $F - e$. Then, $[v, u = u_1, \ldots, u_k = v]$ is a cycle containing e. A contradiction to e not lying on any cycle.

Hence, e is a cut edge of F. Consequently, e is a cut edge of G. ■

Theorem 8.4.12. *Let G be a graph with $V(G) = [n]$. Then, the following are equivalent.*

1. *G is a tree.*
2. *G is a minimal connected graph on n vertices.*
3. *G is a maximal acyclic graph on n vertices.*

Proof. (a)$\Rightarrow$(b). Suppose that G is a tree. If it is not a minimal connected graph on n vertices, then there is an edge $[u, v]$ such that $G - [u, v]$ is connected. But then, by Theorem 8.4.10, $[u, v]$ is on a cycle in G. A contradiction.

(b)$\Rightarrow$(c). Suppose G is a minimal connected graph on n vertices. If G has a cycle, say Γ, then select an edge $e \in \Gamma$. Thus, by Theorem 8.4.10, $G - e$ is still connected graph on n vertices, a contradiction to the fact that G is a minimal connected graph on n vertices. Hence, G is acyclic. Since G is connected, for any new edge e, the graph $G + e$ contains a cycle and hence, G is maximal acyclic graph.

(c)$\Rightarrow$(a). Suppose G is maximal acyclic graph on n vertices. If G is not connected, let G_1 and G_2 be two components of G. Select $v_1 \in G_1$ and $v_2 \in G_2$ and note that $G + [v_1, v_2]$ is acyclic graph on n vertices. This contradicts that G is a maximal acyclic graph on n vertices. Thus, G is connected and acyclic and hence is a tree. ■

Proposition 8.4.15. *Let T be a tree on n vertices. Then, T has $n-1$ edges.*

Proof. We proceed by induction. Take a tree on $n \geq 2$ vertices and delete an edge e. Then, we get two subtrees T_1, T_2 of order n_1, n_2, respectively, where $n_1 + n_2 = n$. So, $E(T) = E(T_1) \cup E(T_2) \cup \{e\}$. By induction hypothesis $\|T\| = \|T_1\| + \|T_2\| + 1 = n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1 = n - 1$. ■

Proposition 8.4.16. *Let G be a connected graph with n vertices and $n-1$ edges. Then, G is acyclic.*

Proof. On the contrary, assume that G has a cycle, say Γ. Now, select an edge $e \in \Gamma$ and note that $G - e$ is connected. We go on selecting edges from G that lie on cycles and keep removing them, until we get an acyclic graph H. Since the edges that are being removed lie on some cycle, the graph H is still connected. So, by definition, H is a tree on n vertices. Thus, by Proposition 8.4.15, $|E(H)| = n - 1$. But, in the above argument, we have deleted at least one edge and hence, $|E(G)| \geq n$. This gives a contradiction to $|E(G)| = n - 1$. ■

Proposition 8.4.17. *Let G be an acyclic graph with n vertices and $n-1$ edges. Then, G is connected.*

Proof. Let if possible, G be disconnected with components $G_1, \ldots, G_k, k \geq 2$. As G is acyclic, by definition, each G_i is a tree on, say $n_i \geq 1$ vertices, with $\sum i = 1^k n_i = n$. Thus, $\|G\| = \sum_{i=1}^{k} (n_i - 1) = n - k < n - 1 = \|G\|$, as $k \geq 2$. A contradiction. ■

Eulerian Graphs

Definition 8.6.1. [Eulerian graph] An **Eulerian tour** in a graph G is a closed walk $[v_0, v_1, \ldots, v_k, v_0]$ such that each edge of the graph appears exactly once in the walk. The graph G is said to be **Eulerian** if it has an Eulerian tour.

Theorem 8.6.2. **[Euler,** 1736] *A connected graph G is Eulerian if and only if $d(v)$ is even, for each $v \in V(G)$.*

Proof. Let G have an Eulerian tour, say $[v_0, v_1, \ldots, v_k, v_0]$. Then, $d(v) = 2r$, if $v \neq v_0$ and v appears r times in the tour. Also, $d(v_0) = 2(r-1)$, if v_0 appears r times in the tour. Hence, $d(v)$ is always even.

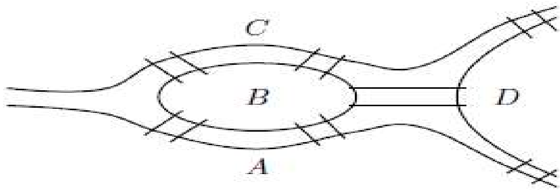


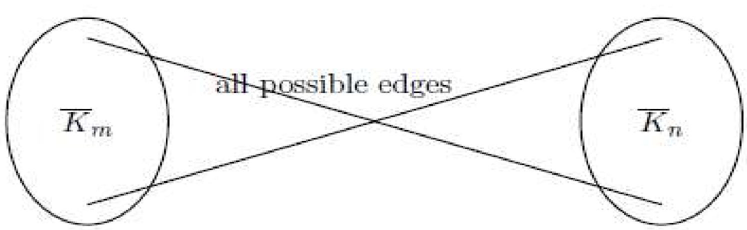
Figure 8.12: Königsberg bridge problem

Conversely, let G be a connected graph with each vertex having even degree. Let $W = v_0v_1 \cdots v_k$ be a longest walk in G without repeating any edge in it. As v_k has an even degree it follows that $v_k = v_0$, otherwise W can be extended. If W is not an Eulerian tour then there exists an edge, say $e' = v_iw$, with $w \neq v_{i-1}, v_{i+1}$. In this case, $wv_i \cdots v_k(= v_0)v_1 \cdots v_{i-1}v_i$ is a longer walk, a contradiction. Thus, there is no edge lying outside W and hence W is an Eulerian tour. ∎

Proposition 8.6.3. *Let G be a connected graph with exactly two vertices of odd degree. Then, there is an Eulerian walk starting at one of those vertices and ending at the other.*

Proof. Let x and y be the two vertices of odd degree and let v be a symbol such that $v \notin V(G)$. Then, the graph H with $V(H) = V(G) \cup \{v\}$ and $E(H) = E(G) \cup \{xv, yv\}$ has each vertex of even degree and hence by Theorem 8.6.2, H is Eulerian. Let $\Gamma = [v, v_1 = x, \ldots, v_k = y, v]$ be an Eulerian tour. Then, $\Gamma - v$ is an Eulerian walk with the required properties. ∎

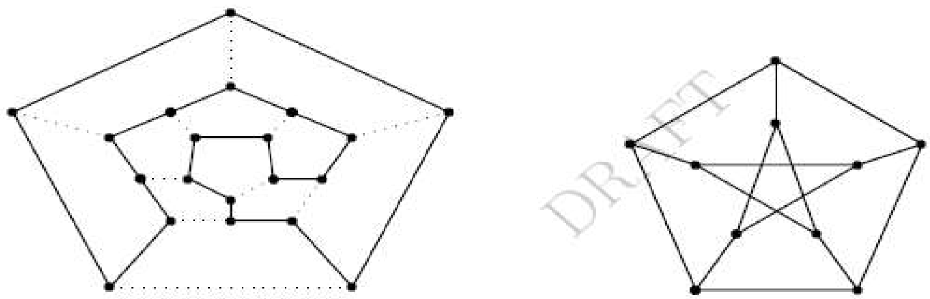
Definition 8.6.8. **[bipartite graph]** A graph $G = (V, E)$ is said to be **bipartite** if $V = V_1 \cup V_2$ such that $|V_1|, |V_2| \geq 1$, $V_1 \cap V_2 = \emptyset$ and $\|\langle V_1 \rangle\| = 0 = \|\langle V_2 \rangle\|$. The complete bipartite graph $K_{m,n}$ is shown below. Notice that $K_{m,n} = \overline{K}_m + \overline{K}_n$.



Hamiltonian Graphs

Definition 8.7.1. [**Hamiltonian**] A cycle in G is said to be **Hamiltonian** if it contains all vertices of G. If G has a Hamiltonian cycle, then G is called a **Hamiltonian** graph. Finding a nice characterization of a Hamiltonian graph is an unsolved problem.

Example 8.7.2. 1. For each positive integer $n \geq 3$, the cycle C_n is Hamiltonian.



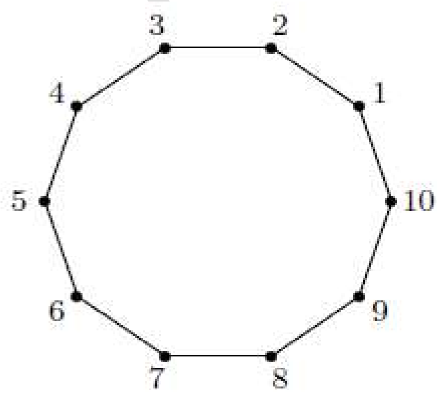
The dodecahedron graph The Petersen graph

Figure 8.13: A Hamiltonian and a non-Hamiltonian graph

2. The graphs corresponding to all platonic solids are Hamiltonian.
3. The Petersen graph is a non-Hamiltonian Graph (the proof appears below).

Proposition 8.7.3. *The Petersen graph is not Hamiltonian.*

Proof. Suppose that the Petersen graph, say G, is Hamiltonian. Also, each vertex of G has degree 3 and hence, $G = C_{10} + M$, where M is a set of 5 chords in which each vertex appears as an endpoint. We assume that $C_{10} = [1, 2, \ldots, 10, 1]$. Now, consider the vertices 1, 2 and 3.



Since, $g(G) = 5$, the vertex 1 can only be adjacent to one of the vertices $5, 6$ or 7. Hence, if 1 is adjacent to 5, then the third vertex that is adjacent to 10 creates cycles of length 3 or 4. Similarly, if 1 is adjacent to 7, then there is no choice for the third vertex that can be adjacent to 2. So, let 1 be adjacent to 6. Then, 2 must be adjacent to 8. In this case, note that there is no choice for the third vertex that can be adjacent to 3. Thus, the Petersen graph is non-Hamiltonian. ∎

Theorem 8.7.4. *Let G be a Hamiltonian graph. Then, for $S \subseteq V(G)$ with $S \neq \emptyset$, the graph $G - S$ has at most $|S|$ components.*

Proof. Note that by removing k vertices from a cycle, one can create at most k connected components. Hence, the required result follows. ∎

Theorem 8.7.5. [Dirac, 1952] *Let G be a graph with $|G| = n \geq 3$ and $d(v) \geq n/2$, for each $v \in V(G)$. Then, G is Hamiltonian.*

Proof. Let is possible, G be disconnected. Then, G has a component, say H, with $|V(H)| = k \leq n/2$. Hence, $d(v) \leq k - 1 < n/2$, for each $v \in V(H)$. A contradiction to $d(v) \geq n/2$, for each $v \in V(G)$. Now, let $P = [v_1, v_2, \cdots, v_k]$ be a longest path in G. Since P is the longest path, all neighbors of v_1 and v_k are in P.

We claim that there exists an i such that $v_1 \sim v_i$ and $v_{i-1} \sim v_k$. Otherwise, for each $v_i \sim v_1$, we must have $v_{i-1} \nsim v_k$. Then, $|N(v_k)| \leq k - 1 - |N(v_1)|$. Hence, $|N(v_1)| + |N(v_k)| \leq k - 1 < n$, a contradiction to $d(v) \geq n/2$, for each $v \in V(G)$. So, the claim is valid and hence, we have a cycle $\tilde{P} := v_1 v_i v_{i+1} \cdots v_k v_{i-1} \cdots v_1$ of length k.

We now prove that $\tilde{P}$ gives a Hamiltonian cycle. Suppose not. Then, there exists $v \in V(G)$ such that v is outside P and v is adjacent to some v_j. Note that in this case, P cannot be the path of longest length, a contradiction. Thus, the required result follows. ∎

Definition 8.7.8. [closure of a graph] The **closure** of a graph G, denoted $C(G)$, is obtained by repeatedly choosing pairs of nonadjacent vertices u, v such that $d(u) + d(v) \geq n$ and adding edges between them.

Proposition 8.7.9. *The closure of G is unique.*

Proof. Let K be a closure obtained by adding edges $e_1 = u_1 v_1, \ldots, e_k = u_k v_k$ sequentially and F be a closure obtained by adding edges $f_1 = x_1 y_1, \ldots, f_r = x_r y_r$ sequentially. Let e_i be the first edge in the e-sequence which does not appear in the f-sequence. Put $H = G + e_1 + \cdots + e_{i-1}$. Then, $e_i = u_i v_i$ implies that $e_i \notin E(H)$ and $d_H(u_i) + d_H(v_i) \geq n$. Also, H is a subgraph of F and hence, $d_F(u_i) + d_F(v_i) \geq n$. Moreover, $e_i = u_i v_i \notin F$ as e_i does not appear in the f-sequence. Thus, F cannot be a closure and therefore the required result follows. ■

EXERCISE **8.7.10.** *Let G be a graph on $n \geq 3$ vertices.*

1. *If G has a cut vertex, then prove that $C(G) \neq K_n$.*
2. *Then, prove a generalization of Dirac's theorem: If the closure $C(G) \cong K_n$, then G is Hamiltonian.*

Theorem 8.7.11. *Let $d_1 \leq \cdots \leq d_n$ be the vertex degrees of G. Suppose that, for each $k < n/2$ with $d_k \leq k$, the condition $d_{n-k} \geq n-k$ holds. Then, prove that G is Hamiltonian.*

Proof. We show that under the above condition $H = C(G) \cong K_n$. On the contrary, assume that there exist a pair of vertices $u, v \in V(G)$ such that $uv \notin E(G)$ and $d_H(u) + d_H(v) \leq n-1$. Among the above pairs, choose a pair $u, v \in V(G)$ such that $uv \notin E(H)$ and $d_H(u) + d_H(v)$ is maximum. Assume that $d_H(v) \geq d_H(u) = k$ (say). Clearly, $k < n/2$.

Now, let $S_v = \{x \in V(H) \mid xv \notin E(H), x \neq v\}$ and $S_u = \{w \in V(H) \mid wu \notin E(H), w \neq u\}$. Therefore, the assumption that $d_H(u) + d_H(v)$ is the maximum among each pair of vertices u, v with $uv \notin E(H)$ and $d_H(u) + d_H(v) \leq n-1$ implies that $|S_v| = n - 1 - d_H(v) \geq d_H(u) = k$ and for each $x \in S_v$, $d_H(x) \leq d_H(u) = k$. So, there are at least k vertices in H (elements of S_v) with degrees at most k.

Also, for any $w \in S_u$, note that the choice of the pair u, v implies that $d_H(w) \leq d_H(v) \leq n - 1 - d_H(u) = n - 1 - k < n - k$. Moreover, $|S_u| = n - 1 - k$. Further, the condition $d_H(u) + d_H(v) \leq n-1$, $d_H(v) \geq d_H(u) = k$ and $u \notin S_u$ implies that $d_H(u) \leq n - 1 - d_H(v) \leq n - 1 - k < n - k$. So, there are $n-k$ vertices in H with degrees less than $n-k$.

Therefore, if $d'_1 \leq \cdots \leq d'_n$ are the vertex degrees of H, then we observe that there exists a $k < n/2$ for which $d'_k \leq k$ and $d'_{n-k} < n-k$. As $k < n/2$ and $d_i \leq d'_i$, we get a contradiction. ■

Binary tree: A *binary tree* is a tree where each node has at most two children. A *full binary tree* is a tree where all nodes have exactly two children and all leaves are at the same depth.

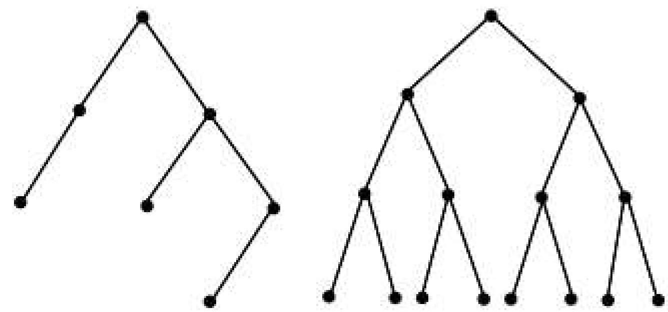


Figure 2: A binary tree on the left and a full binary tree of height 3 on the right.

A tree with n vertices has $n-1$ edges.

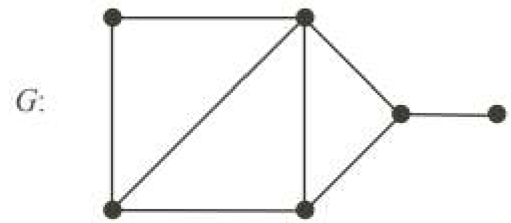
Algebraic expressions involving binary operations can be represented by labeled binary trees. The leaves are labeled as operands, and the internal nodes are labeled as binary operations. For any internal node, the binary operation of its label is performed on the expressions associated with its left and right subtrees. The binary tree below represents the algebraic expression $(2+x)-(3*y)$.

-

Fundamental circuits

If the branches of the spanning tree T of a connected graph G are $b_1, \ldots, b_{n-1}$ and the corresponding links of the cospanning tree T^* are $c_1, \ldots, c_{m-n+1}$, then there exists one and only one circuit C_i in $T + c_i$ (which is the subgraph of G induced by the branches of T and c_i) (Theorem 2.1). We call this circuit a *fundamental circuit*. Every spanning tree defines $m - n + 1$ fundamental circuits $C_1, \ldots, C_{m-n+1}$, which together form a *fundamental set of circuits*. Every fundamental circuit has exactly one link which is not in any other fundamental circuit in the fundamental set of circuits. Therefore, we can not write any fundamental circuit as a ring sum of other fundamental circuits in the same set. In other words, the fundamental set of circuits is linearly independent under the ring sum operation.

Example.



PART –B (5x6=30)

1. Show that the sum of the degree of all vertices in a graph equal to twice in a number of edges incidence in G.
2. i) Show that if a graph G has exactly two vertices of odd degree there is a path joining these two vertices.

ii) Show that a simple graph with n vertices and k-components can have at most $\frac{(n-k)(n-k+1)}{2}$

3. Define (i) Bipartite Graph
   (ii) Regular Graph
   (iii) Complete Graph.
   Give an example for each.
4. If G is a tree with n vertices then prove that G has n-1 edges.
5. Prove that a connected graph G is an Euler graph if and only if it can be decomposed into circuits.
6. Explain Hamiltonian graph.
7. Define walks and paths and Give an example.
8. Prove that a graph G is a tree if and only if it is minimally connected
9. Prove that every tree has either one or two centers.
10. Define (i) weighted graph
    (ii)isomorphic
    (iii) Euler trail. Give an example of each

PART – C (1 x 10 = 10 marks)

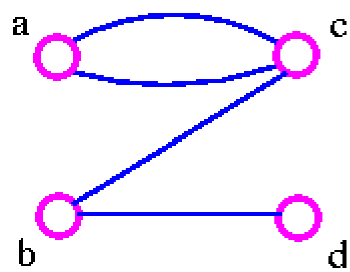
1. Prove that a graph G is a tree I and only if there is one and only one path between any two vertices of G.
2. Explain walks and paths and Give an example.
3. Discuss Hamiltonian graph.

UNIT – II

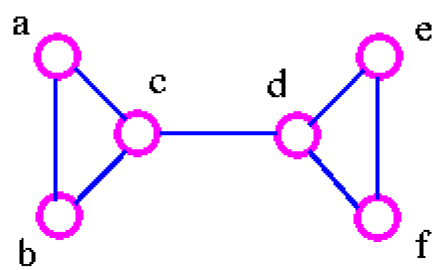
SYLLABUS

Spanning tree in weighted graph – cut sets – properties of cut sets – all cuts sets – fundamentals circuits and cut sets – connectivity and separability – network flows – I isomorphism – II isomorphism – combinational verus geometric graphs – planer graphs – different representation of planar graphs

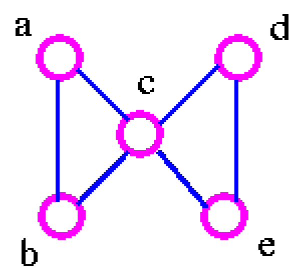
Cut Edge (Bridge) A bridge is a single edge whose removal disconnects a graph.



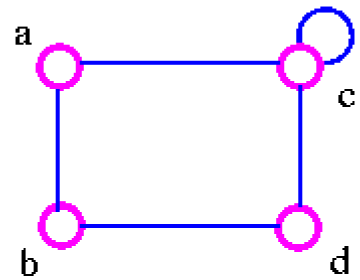
The above graph G1 can be split up into two components by removing one of the edges bc or bd. Therefore, edge bc or bd is a bridge.



The above graph G2 can be disconnected by removing a single edge, cd. Therefore, edge cdis a bridge.



The above graph G3 cannot be disconnected by removing a single edge, but the removal of two edges (such as ac and bc) disconnects it.



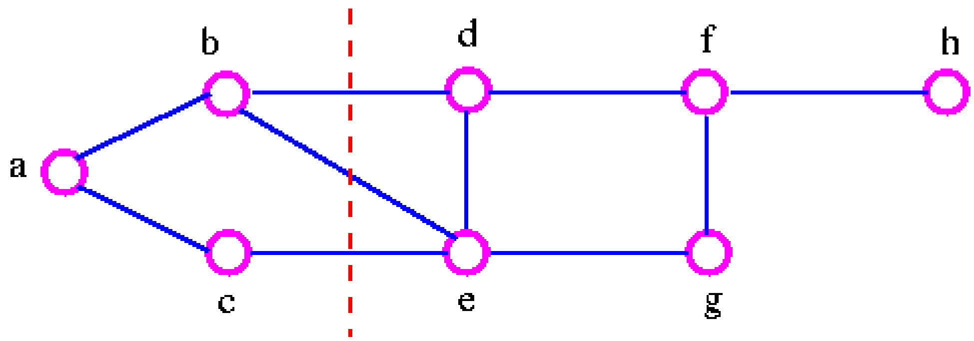
The above graph G4 can be disconnected by removing two edges such as ac and dc.

Cut Set

A cut set of a connected graph *G* is a set S of edges with the following properties

- The removal of all edges in S disconnects *G*.
- The removal of some (but not all) of edges in S does not disconnects *G*.

As an example consider the following graph



We can disconnect *G* by removing the three edges bd, bc, and ce, but we cannot disconnect it by removing just two of these edges. Note that a cut set is a set of edges in which no edge is redundant.

Cut-Vertex

A cut-vertex is a single vertex whose removal disconnects a graph.

It is important to note that the above definition breaks down if G is a complete graph, since we cannot then disconnects G by removing vertices. Therefore, we make the following definition.

Connectivity of Complete Graph

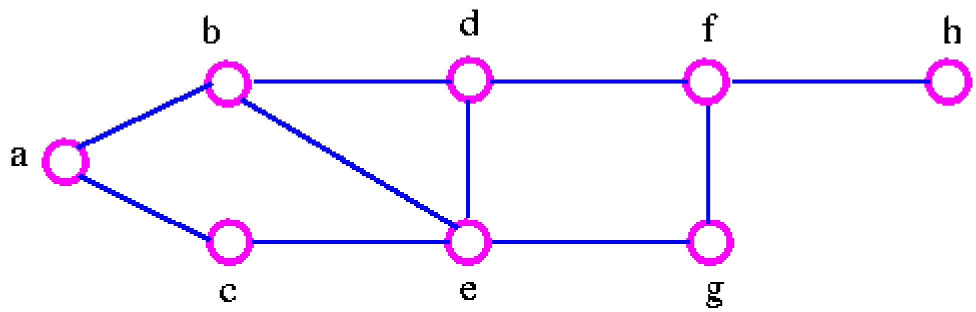
The connectivity $k(k_n)$ of the complete graph k_n is n-1. When $n-1 \geq k$, the graph k_n is said to be k-connected.

Vertex-Cut set

A vertex-cut set of a connected graph G is a set S of vertices with the following properties.

a. the removal of all the vertices in S disconnects G.
b. the removal of some (but not all) of vertices in S does not disconnects G.

Consider the following graph



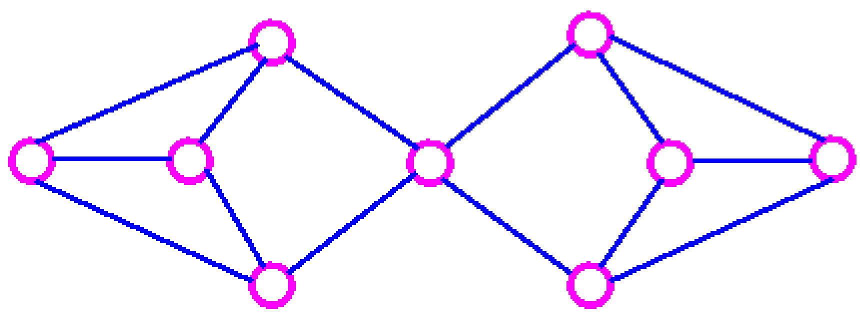
We can disconnects the graph by removing the two vertices b and e, but we cannot disconnect it by removing just one of these vertices. the vertex-cutset of G is {b, e}.

Note that the connectivity k(G) does not exceed the edge-connectivity λ(G). This inequality holds for all connected graph.

Formally, for any connected graph G we have

$$K(G) \leq \lambda(G) \leq \delta(G)$$

where δ(G) is the smallest vertex-degree in G. But it is certainly possible for both inequality in above theorem to be strict inequalities (that is, $k(G) < \lambda(G) < \delta(G)$) For example, in the following graph,



K(G)=1, λ(G) = 2, and δ(G) = 3.

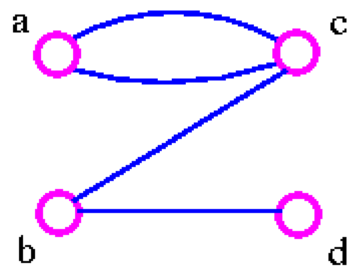
Edge Connectivity

The edge-connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G. When $\lambda(G) \geq k$, the graph G is said to be k-edge-connected.

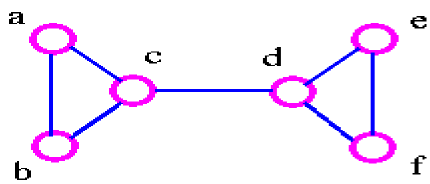
For example, the edge connectivity of the below four graphs G1, G2, G3, and G4 are as follows:

- *G1* has edge-connectivity 1.
- *G2* has edge connectivity 1.

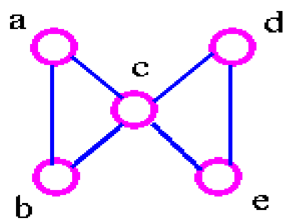
- *G3* has edge connectivity 2.
- *G4* has edge connectivity 2.



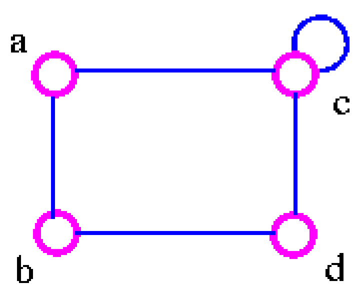
The above graph G1 can be split up into two components by removing one of the edges bcor bd. Therefore, edge bc or bd is a bridge.



The above graph G2 can be disconnected by removing a single edge, cd. Therefore, edge cdis a bridge.



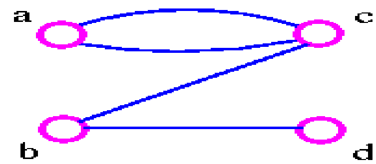
The above graph G3 cannot be disconnected by removing a single edge, but the removal of two edges (such as ac and bc) disconnects it.



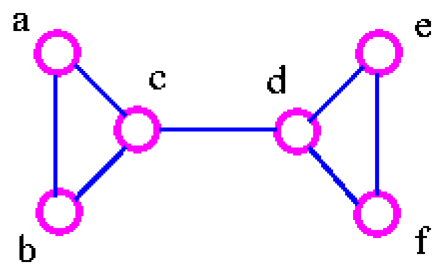
The above graph G4 can be disconnected by removing two edges such as ac and dc.

Vertex Connectivity

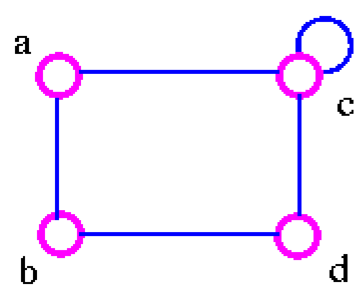
The connectivity (or vertex connectivity) $\mathbf{K}(G)$ of a connected graph G (other than a complete graph) is the minimum number of vertices whose removal disconnects G. When$\mathbf{K}(G) \geq k$, the graph is said to be k-connected (or k-vertex connected). When we remove a vertex, we must also remove the edges incident to it. As an example consider following graphs.



The above graph G can be disconnected by removal of single vertex (either b or c). The G has connectivity 1.



The above graph G can be disconnected by removal of single vertex (either c or d). The vertex c or d is a cut-vertex. The G has connectivity 1.



The above G cannot be disconnected by removing a single vertex, but the removal of two non-adjacent vertices (such as b and c) disconnects it. The G has connectivity 2.

Connectivity

Proposition 8.5.1. *Let G be a connected graph on vertex set $[n]$. Then, its vertices can be labeled in such a way that the induced subgraph on the set $[i]$ is connected for $1 \leq i \leq n$.*

Proof. If $n = 1$, there is nothing to prove. Assume that the statement is true if $n < k$ and let G be a connected graph on the vertex set $[k]$. If G is a tree, pick any pendant vertex and label it k. If G has a cycle, pick a vertex on a cycle and label it k. In both the case $G - k$ is connected. Now, use the induction hypothesis to get the required result. ■

Definition 8.5.2. **[Separating set]** Let G be a graph. Then, a set $X \subseteq V(G) \cup E(G)$ is called a **separating set** if $G - X$ has more connected components than that of G.

Definition 8.5.4. [**Vertex connectivity**] A graph G is said to be **k-connected** if $|G| > k$ and G is connected even after deletion of any $k-1$ vertices. The **vertex connectivity $\kappa(G)$** of a non trivial graph G is the largest k such that G is k-connected. Convention: $\kappa(K_1) = 0$.

Example 8.5.5. 1. Each connected graph of order more than one is 1-connected.

2. A 2-connected graph is also a 1-connected graph.

3. For a disconnected graph, $\kappa(G) = 0$ and for $n > 1$, $\kappa(K_n) = n - 1$.

4. The graph G in Figure 8.11 is 2-connected but not 3-connected. Thus, $\kappa(G) = 2$.

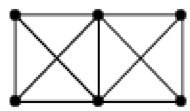


Figure 8.11: graph with vertex connectivity 2

5. The Petersen graph is 3-connected.

Definition 8.5.6. [**Edge connectivity**] A graph G is called **l-edge connected** if $|G| > 1$ and $G - F$ is connected for every $F \subseteq E(G)$ with $|F| < l$. The greatest integer l such that G is l-edge connected is the **edge connectivity** of G, denoted $\boldsymbol{\lambda(G)}$. Convention: $\lambda(K_1) = 0$.

Example 8.5.7. 1. Note that $\lambda(P_n) = 1, \lambda(C_n) = 2$ and $\lambda(K_n) = n - 1$, whenever $n > 1$.

2. Let T be a tree on n vertices. Then, $\lambda(T) = 1$.

3. For the graph G in Figure 8.11, $\lambda(G) = 3$.

4. For the Petersen graph G, $\lambda(G) = 3$.

Theorem 8.5.9. [**H. Whitney,** 1932] *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. If G is disconnected or $|G| = 1$, then we have nothing to prove. So, let G be connected graph and $|G| \geq 2$. Then, there is a vertex v with $d(v) = \delta(G)$. If we delete all edges incident on v, then the graph is disconnected. Thus, $\delta(G) \geq \lambda(G)$.

Suppose that $\lambda(G) = 1$ and $G - uv$ is disconnected with components C_u and C_v. If $|C_u| = |C_v| = 1$, then $G = K_2$ and $\kappa(G) = 1$. If $|C_u| > 1$, then we delete u to see that $\kappa(G) = 1$.

If $\lambda(G) = k \geq 2$, then there is a set of edges, say $e_1, \ldots, e_k$, whose removal disconnects G. Notice that $G - \{e_1, \ldots, e_{k-1}\}$ is a connected graph with a bridge, say $e_k = uv$. For each of $e_1, \ldots, e_{k-1}$ select an end vertex other than u or v. Deletion of these vertices from G results in a graph H with uv as a bridge of a connected component. Note that $\kappa(H) \leq 1$. Hence, $\kappa(G) \leq \lambda(G)$. ■

Separability

PROPOSITION 2.6. *For any positive integer p and any integer q such that* $0 \leqq q \leqq p(p-1)/2$ *there is a simple* (p, q) *graph.*

Proof. The proof is by induction on p. Since the trivial graph is simple, the result is true for $p = 1$. Assume the proposition is true for $p \leqq k$, and let q be an integer such that

$$0 \leqq q \leqq (k+1)k/2.$$

If $q \leqq k(k-1)/2$ then the result follows from the induction hypothesis since there is a simple (k, q) graph from which a simple $(k+1, q)$ graph is obtained by adding a point of degree zero (Corollary 2.5).

On the other hand if q satisfies

$$k(k-1)/2 \leqq q$$

then $q = l + k(k-1)/2$ where $0 \leqq l \leqq k$. To get the desired simple graph we adjoin a new point to any l points of K_k. The graph so obtained is simple because its complement is the union of $K_{1,k-l}$ and trivial graphs.

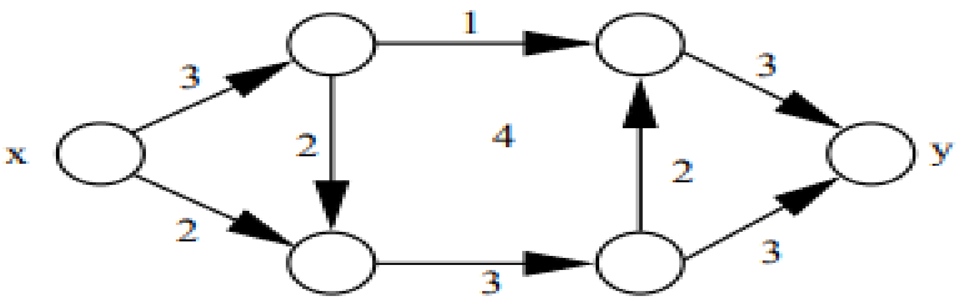
The next result goes in the opposite direction.

Network flow

Networks can be used to represent the transportation of some commodity through a system of delivery channels.

There are *sources* (x) and *sinks* (y).

The network is a directed graph, where each arc a is associated with a *capacity*, $c(a)$.



Isomorphism in Graphs

Definition 8.3.1. [**Isomorphic graphs**] Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be **isomorphic** if there is a bijection $f : V \to V'$ such that $u \sim v$ is G if and only if $f(u) \sim f(v)$ in G', for each $u, v \in V$. In other words, an isomorphism is a bijection between the vertex sets which preserves adjacency. We write $G \cong G'$ to mean that G is isomorphic to G'.

Example 8.3.2. Consider the graphs in Figure 8.9. Then, note that

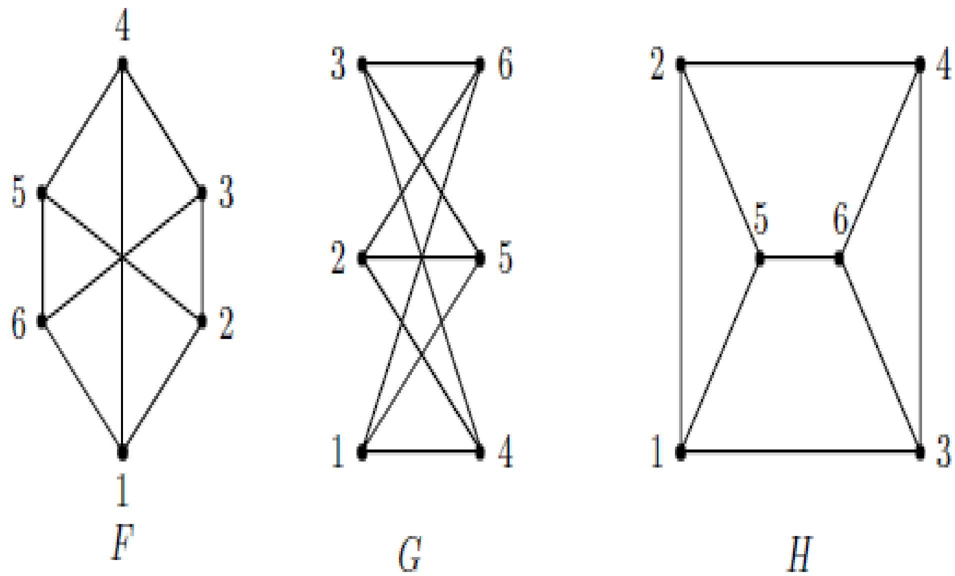


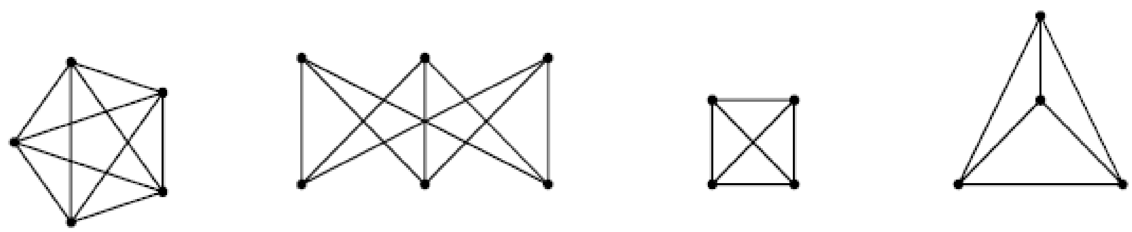
Figure 8.9: F is isomorphic to G but F is not isomorphic to H

1. the graph F is not isomorphic to H as the **independence number**, denoted $\alpha(F)$, of F (the maximum size of an independent vertex set) is 3 whereas $\alpha(H) = 2$. Alternately, H has a 3-cycle, whereas F does not.

2. the graph F is isomorphic to G as the map $f : V(F) \to V(G)$ defined by $f(1) = 1$, $f(2) = 5$, $f(3) = 3$, $f(4) = 4$, $f(5) = 2$ and $f(6) = 6$ gives an isomorphism.

Planar Graphs

Definition 8.12.1. [Embedding, Planar graph] A graph is said to be **embedded** on a surface S when it is drawn on S so that no two edges intersect. A graph is said to be **planar** if it can be embedded on the plane. A **plane graph** is a graph which is embedded on the plane.



K_5-Non-planar $K_{3,3}$-Non-planar K_4 K_4 - Planar embedding

Figure 8.15: Planar and non-planar graphs

Example 8.12.2.

1. A tree is embed-able on a plane and when it is embedded we have only one face, the exterior face.
2. Any cycle C_n, $n \geq 3$ is planar and any plane representation of C_n has two faces.
3. The planar embedding of K_4 is given in Figure 8.15.
4. Draw a planar embedding of $K_{2,3}$.
5. Draw a planar embedding of the three dimensional cube.

Definition 8.12.3. [Face of a planar embedding] Consider a planar embedding of a graph G. The regions on the plane defined by this embedding are called **faces/regions** of G. The unbounded face/region is called the **exterior face** (see Figure 8.16).

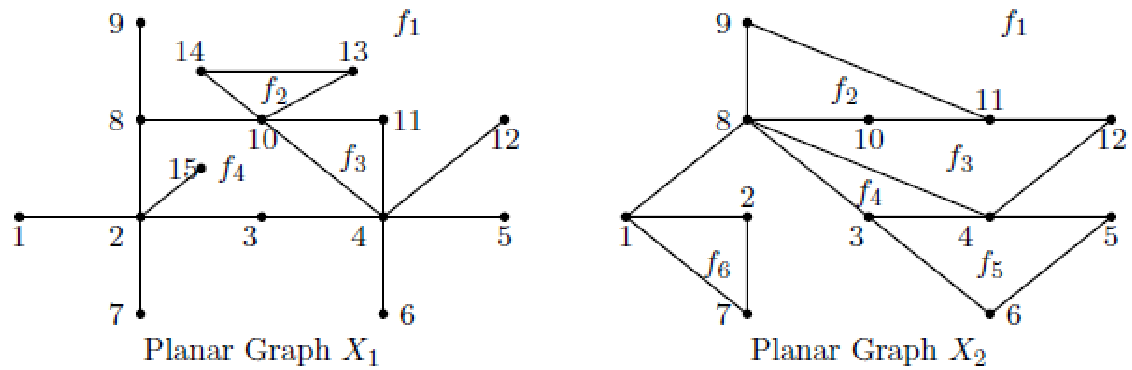


Figure 8.16: Planar graphs with labeled faces to understand the Euler's theorem

Theorem 8.12.4. [Euler formula] *Let G be a connected plane graph with f as the number of faces. Then,*

$$|G| - \|G\| + f = 2. \tag{8.3}$$

Proof. We use induction on f. Let $f = 1$. Then, G cannot have a subgraph isomorphic to a cycle. For if, G has a subgraph isomorphic to a cycle then in any planar embedding of G, $f \geq 2$. Therefore, G is a tree and hence $|G| - \|G\| + f = n - (n-1) + 1 = 2$.

So, assume that Equation (8.3) is true for all plane connected graphs having $2 \leq f < n$. Now, let G be a connected planar graph with $f = n$. Now, choose an edge that is not a cut-edge, say e. Then, $G - e$ is still a connected graph. Also, the edge e is incident with two separate faces and hence it's removal will combine the two faces and thus $G - e$ has only $n - 1$ faces. Thus,

$$|G| - \|G\| + f = |G - e| - (\|G - e\| + 1) + n = |G - e| - \|G - e\| + (n - 1) = 2$$

using the induction hypothesis. Hence, the required result follows. ∎

Lemma 8.12.5. *Let G be a plane bridgeless graph with $\|G\| \geq 2$. Then, $2\|G\| \geq 3f$. Further, if G has no cycle of length 3 then, $2\|G\| \geq 4f$.*

Proof. For each edge put two dots on either side of the edge. The total number of dots is $2\|G\|$. If G has a cycle then each face has at least three edges. So, the total number of dots is at least $3f$. Further, if G does not have a cycle of length 3, then $2\|G\| \geq 4f$. ∎

Theorem 8.12.6. *The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar.*

Proof. If K_5 is planar, then consider a plane representation of it. By Equation (8.3), $f = 7$. But, by Lemma 8.12.5, one has $20 = 2\|G\| \geq 3f = 21$, a contradiction.

If $K_{3,3}$ is planar, then consider a plane representation of it. Note that it does not have a C_3. Also, by Euler's formula, $f = 5$. Hence, by Lemma 8.12.5, one has $18 = 2\|G\| \geq 4f = 20$, a contradiction. ∎

Definition 8.12.7. [**Subdivision, homeomorphic**] Let G be a graph. Then, a **subdivision** of an edge uv in G is obtained by replacing the edge by two edges uw and wv, where w is a new vertex. Two graphs are said to be **homeomorphic** if they can be obtained from the same graph by a sequence of subdivisions.

Different representation of a planar graphs

Definition 8.12.12. [**Maximal planar**] A graph is called **maximal planar** if it is planar and addition of any more edges results in a non-planar graph. A maximal plane graph is necessarily connected.

Proposition 8.12.13. *If G is a maximal planar graph with m edges and $n \geq 3$ vertices, then every face is a triangle and $m = 3n - 6$.*

Proof. Suppose there is a face, say f, described by the cycle $[u_1, \ldots, u_k, u_1]$, $k \geq 4$. Then, we can take a curve joining the vertices u_1 and u_3 lying totally inside the region f, so that $G + u_1u_3$ is planar. This contradicts the fact that G is maximal planar. Thus, each face is a triangle. It follows that $2m = 3f$. As $n - m + f = 2$, we have $2m = 3f = 3(2 - n + m)$ or $m = 3n - 6$. ∎

PART B (5X6=30 Marks)

1.Prove that every cut-set in a connected graph G must contain atleast one branch of every spanning tree of G.

2. Explain about fundamental cut-sets and fundamental circuit in a graph

3. Explain Kruskal algorithm and Prim's algorithm for shortest spanning tree with example.
4. Prove that the maximum flow possible between two vertices a and b in a network is equal to the minimum of the capacities of all cut-sets with respect to a and b.
5. Explain about Network flows in a graph.

6. Prove that the complete graph of five vertices is nonplanar

7. Prove that the vertex connectivity of any graph G can never exceed the edge connectivity of G.
8. Expalin about planar graphs
9. Prove that the minimum height of a n vertex binary tree is equal to [log(n+1)-1]
10. Define (i) edge connectivity
(ii) vertex connectivity
(iii) separable graph

PART C (1X10=10 Marks)

1.Prove that every circuit has an even number of edges in common with any cut-set

2. Explain the complete graph of five vertices is nonplanar

3. Discuss about fundamental cut-sets and fundamental circuit in a graph

UNIT-IV

SYLLABUS

Directed Graphs- Types of Directed Graphs- Types of enumeration, counting labeled trees, counting unlabelled trees, Polya's counting theorem, graph enumeration with polya's theorem.

DIRECTED GRAPH

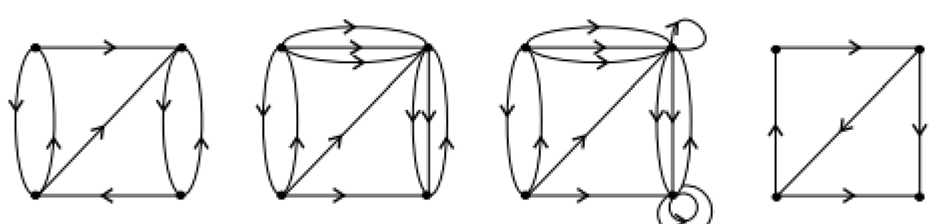
Digraphs (Directed graphs): A digraph D is a pair (V, A), where V is a nonempty set whose elements are called the *vertices* and A is the subset of the set of ordered pairs of distinct elements of V. The elements of A are called the *arcs* of D (Fig. 11.1(a)).

TYPES OF DIRECTED GRAPHS

Multidigraphs: A multidigraph D is a pair (V, A), where V is a nonempty set of vertices and A is a multiset of arcs, which is a multisubset of the set of ordered pairs of distinct elements of V. The number of times an arc occurs in D is called its *multiplicity* and arcs with multiplicity greater than one are called *multiple arcs* of D (Fig. 11.1(b)).

General digraphs: A general digraph D is a pair (V, A), where V is a nonempty set of vertices, and A is a multiset of arcs, which is a multisubset of the cartesian product of V with itself. An arc of the form uu is called a *loop* of D and arcs which are not loops are called *proper arcs* of D. The number of times an arc occurs is called its multiplicity. A loop with multiplicity greater than one is called a *multiple loop* (Fig. 11.1(c)).

Oriented graph: A digraph containing no symmetric pair of arcs is called an oriented graph (Fig. 11.1(d)).



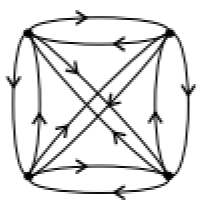
(a) (b) (c) (d)

A vertex v for which $d^+(v) = d^-(v) = 0$ is called an *isolate*. A vertex v is called a *transmitter* or a *receiver* according as $d^+(v) > 0$, $d^-(v) = 0$ or $d^+(v) = 0$, $d^-(v) > 0$. A vertex v is called a *carrier* if $d^+(v) = d^-(v) = 1$.

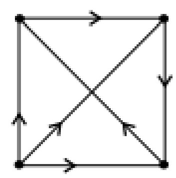
Underlying graph of a digraph: Let $D = (V, A)$ be a digraph. The graph $G = (V, E)$, where $uv \in E$ if and only if uv or vu or both are in A, is called the *underlying graph* of D. This is also called the *covering graph* $C(D)$ of D. Here we denote $C(D)$ by $G(D)$ or simply by G.

In case $G = (V, E)$ is a graph, the digraph with vertex set V and a symmetric uv whenever $uv \in E$, is called the *digraph corresponding* to G, and is denoted by $D(G)$, or D. Clearly, $D(G)$ is a symmetric digraph. An oriented graph obtained from the graph $G = (V, E)$ by replacing each edge $uv \in E$ by an arc uv or vu, but not both is called an *orientation* of G and is denoted by $O(G)$ or O.

Complete symmetric digraph: A digraph $D = (V, A)$ is said to be *complete* if both uv and $vu \in A$, for all $u, v \in V$. Obviously this corresponds to K_n, where $|V| = n$, and is denoted by K_n^*. A complete antisymmetric digraph, or a complete oriented graph is called a *tournament*. Clearly, a tournament is an orientation of K_n (Fig. 11.2).



Complete symmetric digraph



Tournament

We note that the number of arcs in K_n^* is $n\,(n-1)$ and the number of arcs in a tournament is $\frac{n(n-1)}{2}$.

TYPES OF ENUMERATION

A ***permutation of a set*** S is a one-to-one, onto function from S to itself.

Cor 1.6.4. Every permutation of a finite set S can be represented as the composition of disjoint cycles of elements of S.

DEF: A closed non-empty collection P of permutations on a set Y of objects that forms a group under the operation of composition is called a ***permutation group.*** The combined structure may be denoted $\mathcal{P} = [P : Y]$. It is often denoted P when the set Y of objects is understood from context.

DEF: A permutation on a set Y whose representation in disjoint cycle form has only one cycle containing more than one element of Y is called a ***cyclic permutation.***

The number of elements in that one cycle is called the ***length of that cycle.*** Also, a cycle of length k is called a k-cycle.

Proposition 9.1.1. *Let $n \in \mathbb{Z}^+$, and let*

$$\alpha = (1 \quad 2 \quad \cdots \quad n)$$

Then for $j = 1, \ldots, n-1$ and for $r = 1, \ldots, n$, we have

$$\alpha^j(r) = \begin{cases} r+j & \text{if } r+j \leq n \\ r+j \bmod n & \text{otherwise} \end{cases}$$

Proof: This is provable by induction on the power j. It is clearly true for $j = 1$. The inductive hypothesis is that

$$\alpha^{j-1}(r) = \begin{cases} r + j - 1 & \text{if } r + j - 1 \leq n \\ r + j - 1 \bmod n & \text{otherwise} \end{cases}$$

The inductive step is that

$$\alpha^{j}(r) = \alpha^1\alpha^{j-1}(r) = \begin{cases} \alpha^1(r + j - 1) & \text{if } r + j - 1 \leq n \\ \alpha^1(r + j - 1 \bmod n) & \text{otherwise} \end{cases}$$
$$= \begin{cases} r + j & \text{if } r + j \leq n \\ r + j \bmod n & \text{otherwise} \end{cases} \qquad \diamondsuit$$

Example 9.1.2: Let α be the 5-cycle $(1 \quad 2 \quad 3 \quad 4 \quad 5)$. Then we have

$\alpha^3 = (1 \quad 4 \quad 2 \quad 5 \quad 3)$ and $\alpha^4 = (1 \quad 5 \quad 4 \quad 3 \quad 2)$

DEF: The group of permutations in Table 9.1.2 is called a ***cyclic permutation group*** on the set $\{1, 2, \ldots, n\}$. It can be denoted $\left[\mathbb{Z}_n : [1 : n]\right]$, but is more usually denoted, simply, $\mathbb{Z}_n$.

Example 9.1.4: Let $\alpha = (1 \quad 2 \quad \cdots \quad 12)$ in the cyclic permutation group $\mathbb{Z}_{12}$. Since

$$\gcd(8, 12) = 4 \quad \text{and} \quad \frac{12}{4} = 3$$

Proposition 9.1.4 implies that the permutation α^8 has four 3-cycles. In fact,

$$\alpha^8 = (1 \quad 9 \quad 5)(2 \quad 10 \quad 6)(3 \quad 11 \quad 7)(4 \quad 12 \quad 8)$$

DEF: Let π be a permutation on a set of n objects. Then the ***cycle structure*** of π is the n-variable monomial

$$\zeta(\pi) = \prod_{i=j}^{n} t_j^{r_j} = t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$$

where t_j is a formal variable, and where r_j is the number of j-cycles in the disjoint cycle form of π.

COUNTING LABELED TREES

Let $G = (V,X)$ be a graph where $V = \{v_1, v_2, \cdots, v_p\}$ is the set of points, and X its set of lines; see [2]. A *partial labeling* of G is an injection f of $N = \{1, 2, \cdots, n\}$ into V for $n \leqslant p$. A graph G together with a partial labeling f will be called *partially labeled*. Two partially labeled graphs (G, f_1) and (G, f_2) are *identical* if there is an automorphism γ of G such that $f_2(i) = \gamma(f_1(i))$ for $1 \leqslant i \leqslant n$.

A partially labeled tree (T,f) will be called *end-labeled* if $f(N)$ is the set of endpoints of T. Let $t(p)$ and $T(p)$ denote the number of end-labeled trees and end-labeled rooted trees, respectively, having p points.

Theorem 1.

(1)
$$t(p) = B(p-2)$$
and
(2)
$$T(p) = B(p-1),$$
where
$$B(n) = \sum_{k=1}^{n} S(n,k)$$
is a Bell number, i.e., $S(n,k)$ is a Stirling number of the second kind.

Both (1) and (2) follow from the same line of argument so that only (1) will be proved. We will present two derivations of this simple result; the second illustrates a general principle for enumerating partially labeled graphs.

First Proof. Let (T,f) be a p-point end-labeled tree with $V - f(N) = \{v_{n+1}, \cdots, v_p\}$, so that T may be regarded as a labeled tree. Consider the Prufer sequence $(i_1, i_2, \cdots, i_{p-2})$ associated with T (see for example Moon [6] or Harary and Palmer [4]). Each i_j $(1 \leq j \leq p-2)$ satisfies $n+1 \leq i_j \leq p$, so that the sequence $(i_1, i_2, \cdots, i_{p-2})$ may be regarded as a distribution of $p-2$ distinct objects into $p-n$ identical cells with no cell empty. The number of such distributions is of course $S(p-2, p-n)$, and hence
$$t(p) = \sum_{n=2}^{p-1} S(p-2, p-n),$$
as asserted.

The second method requires several lemmas. Let U be the set of endpoints of a tree T, and let $\Gamma = \Gamma(T)$ denote its automorphism group. Furthermore, let us define $\Gamma^* = \Gamma^*(T)$ to be the restriction of Γ to U. Then Γ^* is well-defined since U is invariant under any automorphism of T.

Lemma 1. For any tree T, $\Gamma(T)$ is isomorphic to $\Gamma^*(T)$.

Proof. It is clear that the mapping h defined by $\gamma \to \gamma|_U$ for any $\gamma \in \Gamma(T)$ is a homomorphism of Γ onto Γ^*. Now let γ be an arbitrary nontrivial automorphism of T. It is easy to show (see for example Prins [5, p. 17]) that there exist endpoints u and v $(u \neq v)$ such that $\gamma(u) = v$. Hence, h has a trivial kernel.

Lemma 2. Let T be a tree with n endpoints. The number of distinct end-labeled copies of T is $n!/|\Gamma(T)|$.

Proof. Using Lemma 1, this follows from the argument which establishes the analogous result for labeled graphs (see for example Chao [1] or Harary and Palmer [4, p. 4]).

COUNTING UNLABELLED TREES

Lemma 5. *Let n be a positive integer and let W_n be the set of coding trees with black vertex set $[n]$. Let W_n° be the set of rooted coding trees obtained by rooting a tree in W_n at a colored vertex, let $W_n^{\bullet}$ be the set of rooted coding trees obtained by rooting a tree in W_n at a black vertex, and let $W_n^{\circ-\bullet}$ be the* *set of rooted coding trees obtained by rooting a tree in W_n at an edge. Then there is a bijection Θ from $W_n^{\circ} \cup W_n^{\bullet}$ to $W_n \cup W_n^{\circ\!-\!\bullet}$ that commutes with the actions of $\mathfrak{S}_n$ on vertex labels and of $\mathfrak{S}_{k+1}$ on colors.*

Proof. Every coding tree has a unique center vertex, either black or colored, which is the midpoint of every longest path in the tree, and the center is fixed by both group actions. Let T be a rooted tree in $W_n^{\circ} \cup W_n^{\bullet}$. If T is rooted at its center then we define $\Theta(T)$ to be the underlying unrooted tree of T. Otherwise, there is a unique path from the root r of T to the center, and we take $\Theta(T)$ to be the underlying tree of T rooted at the first edge on the path from r to the center. It is easily seen that Θ is a bijection that commutes with the actions of $\mathfrak{S}_n$ and $\mathfrak{S}_{k+1}$. $\square$

Theorem 7. *The generating function U for unlabeled k-trees is given by*

$$U = B + C - E,$$

where

$$B = \sum_{\lambda \vdash k+1} B_\lambda / z_\lambda, \tag{3a}$$

$$C = \sum_{\mu \vdash k} C_\mu / z_\mu, \tag{3b}$$

$$E = \sum_{\mu \vdash k} \overline{B}_\mu C_\mu / z_\mu, \tag{3c}$$

$$B_\lambda = x \prod_i C_{\lambda^i}(x^i), \tag{3d}$$

$$\overline{B}_\mu = x \prod_i C_{\mu^i}(x^i), \tag{3e}$$

$$C_\mu = \exp\left(\sum_{m=1}^{\infty} \frac{\overline{B}_{\mu^m}(x^m)}{m}\right). \tag{3f}$$

In (3d)*, λ is a partition of $k+1$ and in* (3e) *and* (3f)*, μ is a partition of k. In the products in* (3d) *and* (3e)*, i runs through the parts of λ and μ with multiplicities; i.e., if i occurs m times as a part then i is taken m times in the product.*

Proof. Formula (3a) follows directly from Lemma 1 (Burnside's lemma).

The generating function C for color-orbits under $\mathfrak{S}_{k+1}$ of colored-rooted trees is the same as the generating function for color-orbits under the action of $\mathfrak{S}_k$, permuting the colors 1 through k, of $k+1$-rooted trees, since every color-orbit of colored-rooted trees contains a $k+1$-rooted tree. Then (3b) follows from Lemma 1.

Similarly, the generating function E for color-orbits under $\mathfrak{S}_{k+1}$ of coding trees rooted at an edge is the same as the generating function for color-orbits under the action of $\mathfrak{S}_k$, permuting the colors 1 through k, of coding trees rooted at an edge incident with a vertex of color $k+1$. Removing the root edge from such a tree leaves a $k+1$-rooted tree together with a $k+1$-reduced black-rooted tree. Thus, if $\pi \in \mathfrak{S}_{k+1}$ fixes $k+1$, the generating function for such pairs fixed by π is $C_\pi \overline{B}_\pi$, so (3c) follows.

Next, for $\pi \in \mathfrak{S}_{k+1}$ we find an equation for B_π, which counts black-rooted trees fixed by π. The root of such a tree has $k+1$ children, one of each of the colors from 1 to $k+1$. If we delete the root, we are left with trees $T_1, \dots, T_{k+1}$, where tree T_j is rooted at a vertex of color j. Now suppose

that j is in a cycle of π of length i. Then the orbit of T_j under π consists of $T_j, T_{\pi(j)} = \pi(T_j), \dots, T_{\pi^{i-1}(j)} = \pi^{i-1}(T_j)$, and we must have $\pi^i(T_j) = T_j$. Thus to determine a black-rooted tree fixed by π, we choose from each cycle of π an arbitrary element j, and take T_j to be a j-rooted tree that is fixed by π^i, where i is the length of the cycle of π containing j. Then $T_{\pi(j)}, \dots, T_{\pi^{i-1}(j)}$ are determined and all have the same weight as T_j. The generating function for j-rooted trees fixed by π^i is $C_{\pi^i}(x)$ (independently of the choice of j), so the contribution to B_π from a cycle of π of length i is $C_{\pi^i}(x^i)$. Thus

$$B_\pi = x \prod_c C_{\pi^{|c|}}(x^{|c|}) \tag{4}$$

where c runs over the cycles of π and $|c|$ is the size of the cycle c. Thus (3d) follows, and a similar argument gives (3e).

Next we need to find a formula for C_π. Since $C_\pi = 0$ if π has no fixed points, we may assume without loss of generality that $k+1$ is a fixed point of π. Suppose that T is a $k+1$-rooted tree that is fixed by the permutation π. Removing the root from T leaves a multiset of $k+1$-reduced black-rooted trees that is fixed by π. Thus C_π is the generating function for these multisets, and applying Lemma 2 gives

$$C_\pi = \exp\left(\sum_{m=1}^{\infty} \frac{\overline{B}_{\pi^m}(x^m)}{m}\right),$$

and (3f) follows. □

POLYA'S COUNTING THEOREM

In this section, we shall prove Polya's enumeration theorem and Burnside's lemma.

Suppose G is a group of permutations of a set X, and let $\hat{G}$ be the induced group of permutations of the set Ψ of colorings of X. Now each permutation g in G induces a permutation $\hat{g}$ of Ψ in the following way. Given a coloring ω, we define $\hat{g}(\omega)$ to be the coloring in which the color assigned to x is the color ω assigns to $g(x)$; that is,

$$(\hat{g}(\omega))(x) = \omega(g(x)).$$

We require the generating function $K_E(c_1, c_2, ..., c_k)$, where E is a set of colorings containing one representative of each orbit of $\hat{G}$ on Ψ. The coefficient of $c_1^s c_2^t ....$ in K_E will be the number of distinguishable colorings in which color c_1 is used s times, color c_2 is used t times, and so on.

Polya's theorem state that K_E is obtained from the cycle index $Z_G(a_1, a_2, ..., a_n)$ by substituting

$$c_1^i + c_2^i + ... + c_k^i$$

for a_i $(1 \leq i \leq n)$. Before going to the proof, let us see how this works in the simple case of the red-and-white colorings of the corners of a square.

PART – B (5X6=30 Marks)

1.Prove that the number of simple, labeled graphs of n vertices is $2^{n(n-1)/2}$.

2. Define (i) Directed Graph
(ii) Euler Digraph

3.Discuss about the digraph.

4. Explain counting labeled tress.
5. Discuss about the binary relations in a digraph.

6. Explain about the counting unlabeled tress

7.Discuss about some types of digraph with suitable example.

8.Explain euler digraphs

9.Explain about adjacency matrix of a digraph.

10. Prove that the determinant of every square submatrix of A, the incidence matrix of a digraph is 1,-1,0.

PART – C (1X10= 10 Marks)

1. Prove that there are n^{n-2} labeled trees with n vertices ($n \geq 2$).
2. Discuss euler digraphs

3. Explain (i) Directed Graph
(ii) Euler Digraph

Domination in graphs: Introduction- Terminolgy and concepts- Applications – Dominating set and domination number – Independent set and independentnumber – History of domination in graphs.

Graph Theory: Terminology and Concepts

A *graph* $G = (V, E)$ consists of a (finite) set denoted by V, or by $V(G)$ if one needs to make clear which graph is under consideration, and a collection E, or $E(G)$, of unordered pairs $\{u, v\}$ of distinct elements from V. Each element of V is called a *vertex* (or a point, or a node), and each element of E is called an *edge* (or a line, or a link). The number of vertices, the cardinality of V, is called the *order* of G and is denoted by $|V|$, and $|E|$ is called the *size* of G. We usually use n to denote the order and m the size and typically have $V(G) = \{v_1, v_2, ..., v_n\}$. We write $v_iv_j \in E(G)$ to mean $\{v_i, v_j\} \in E(G)$, and if $e = v_iv_j \in E(G)$, we say that v_i and v_j are *adjacent* and that e and v_i are *incident*. For example, $V(R_4) = V(N_4) = \{a, b, , c, ..., p\}$, $fg \in E(R_4)$, $fg \notin E(N_4)$, and so f and g are adjacent in R_4 but not in N_4.

Theorem 1 *For a graph G of size $|E| = m$,*

$$\sum_{v \in V} deg(v) = 2m.$$

Proof. One can simply count the number of incidences in two ways. First, each vertex v is in $deg\ v$ incidences. Alternatively, each of the m edges has two incidences. □

TWO APPLICATIONS

Centrality And Domination For Faculities Location

Suppose that each vertex in a graph represents a site where customers are located, and we can choose one or more sites at which to locate facilities to serve these customers optimally. Measures of optimality typically involve centrality measures such as choosing centers, medians, or centroids. For example, suppose we have a fixed number p of facilities to locate. If we want to minimize the maximum distance a customer has to travel in order to get to the facility vertex closest to him/her, then we have the well-studied *p-center problem*. If, on the other hand, we wish to minimize the average distance of a customer to his/her nearest facility vertex, then we have the also well-studied *p-median problem*. There is also a *centroidal* measure of centrality that is used in competitive facility location problems. (See P. J. Slater, Maximin facility location, *J. Res. Nat. Bur. Standards 79B* (1975) 107–115.) For these three problems one does as well as possible, given the constraint that only p facility vertices can be chosen. For the tree T in Figure 5, a 2-center solution is $\{u, y\}$, and $\{u, w\}$ is a 2-median solution.

Independence and chromatic number for scheduling

For a graph $G = (V, E)$, a set $S \subset V$ is *independent* if no two vertices in S are adjacent. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in G. A maximum independent set is called a β_0-set. The minimum k such that we can partition $V = S_1 \cup S_2 \cup \ldots \cup S_k$, where each S_i is independent, is the *chromatic number* $\chi(G)$.

vertex $v \in V(G) - S$ that is not adjacent to any vertex in S, and replace S by $S \cup \{v\}$. When no more vertices can be added to S, we have a maximal independent set. For example, consider the cycle C_8 in Figure 10, and assume we examine the vertices in the order (5,8,1,6,4,3,7,2). Vertices 5 and 8 are added to S, and the next possible vertex one can add to S is the vertex 3. Then {5,8,3} is a *maximal independent set*, namely, an independent set S with the property that any vertex set properly containing S is not independent. However, $\beta_0(C_8) = 4$ with maximum independent sets {1,3,5,7} and {2,4,6,8}. The *lower independence number* $i(G)$ is the minimum cardinality of a maximal independent set of G. Clearly, $i(C_8) = 3$. Also, for the star, $\beta_0(K_{1,n-1}) = n - 1$ and $i(K_{i,n-1}) = 1$. For paths, $\beta_0(P_n) = \lceil n/2 \rceil$ and $i(P_n) = \lceil n/3 \rceil$. For grids, $\beta_0(P_j \times P_k) = \lceil jk/2 \rceil$ and, for j and k large, $i(P_j \times P_k)$ is approximately $jk/5$.

Dominating Queens

The following problem can be said to be the origin of the study of dominating sets in graphs. Figure 1.1 illustrates a standard 8×8 chessboard on which is placed a queen. According to the rules of chess a queen can, in one move, advance any number of squares horizontally, vertically, or diagonally (assuming that no other chess piece lies in its way). Thus, the queen in Figure 1.1 can move to (or attack, or dominate) all of the squares marked with an 'x'. In the 1850s, chess enthusiasts in Europe considered the problem of determining the minimum number of queens that can be placed on a chessboard so that all squares are either attacked by a queen or are occupied by a queen. Figure 1.1

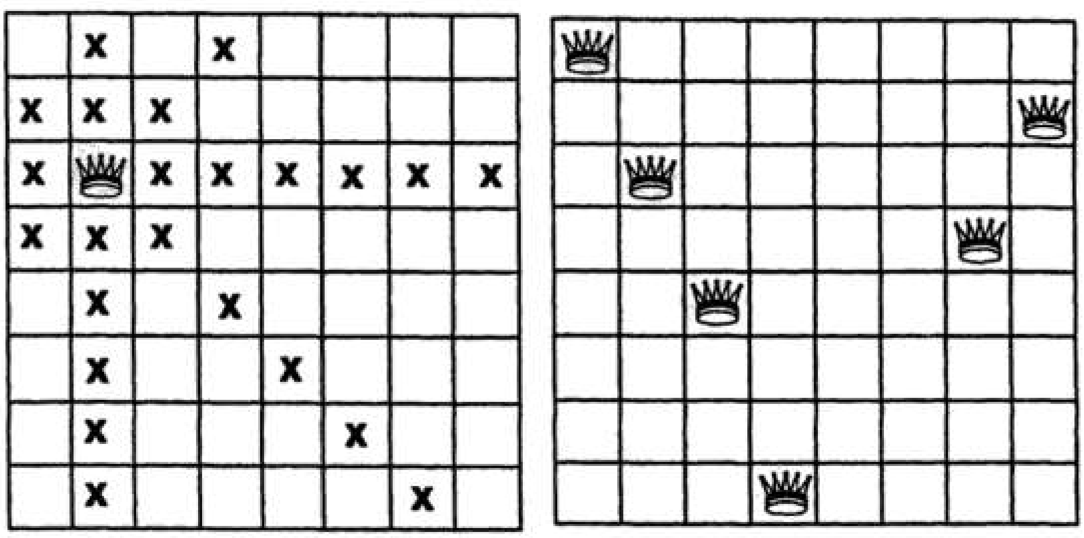


Figure: Queens

The problem of dominating the squares of a chessboard can be stated more generally as a problem of dominating the vertices of a graph.

Definition. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

Definitions. For $S \subseteq V$, a vertex $v \in S$ is called an *enclave* of S if $N[v] \subseteq S$, and $v \in S$ is an *isolate* of S if $N(v) \subseteq V - S$. A set is said to be *enclaveless* if it does not contain any enclaves.

There are several different ways to define a dominating set in a graph, each of which illustrates a different aspect of the concept of domination. Consider the following equivalent definitions. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a *dominating set* if and only if:

(i) for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that v is adjacent to u;

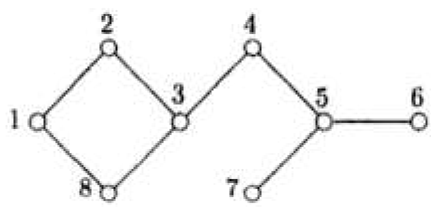
(ii) for every vertex $v \in V - S$, $d(v, S) \leq 1$;

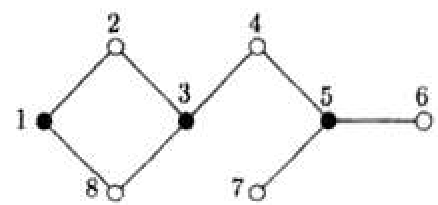
(iii) $N[S] = V$;

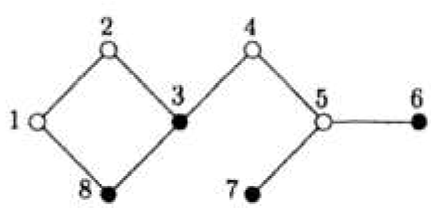
(iv) for every vertex $v \in V - S$, $|N(v) \cap S| \geq 1$, that is, every vertex $v \in V - S$ is adjacent to at least one vertex in S;

(v) for every vertex $v \in V$, $|N[v] \cap S| \geq 1$;

(vi) $V - S$ is enclaveless.







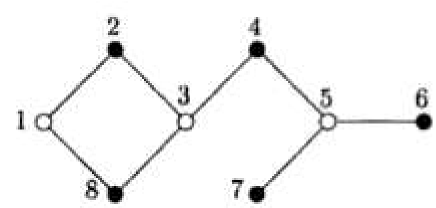


Figure: Minimal dominating sets

The first three theorems about dominating sets in graphs were given by Ore in his 1962 book, *Theory of Graphs*, as follows:

Theorem 1.1 [924] *A dominating set S is a minimal dominating set if and only if for each vertex $u \in S$, one of the following two conditions holds:*

(a) *u is an isolate of S,*

(b) *there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.*

Proof. Assume that S is a minimal dominating set of G. Then for every vertex $u \in S$, $S - \{u\}$ is not a dominating set. This means that some vertex v in $(V - S) \cup \{u\}$ is not dominated by any vertex in $S - \{u\}$. Now either $v = u$, in which case u is an isolate of S, or $v \in V - S$. If v is not dominated by $S - \{u\}$, but is dominated by S, then vertex v is adjacent only to vertex u in S, that is, $N(v) \cap S = \{u\}$.

Conversely, suppose that S is a dominating set and for each vertex $u \in S$, one of the two stated conditions holds. We show that S is a minimal dominating set. Suppose that S is not a minimal dominating set, that is, there exists a vertex $u \in S$ such that $S - \{u\}$ is a dominating set. Hence, u is adjacent to at least one vertex in $S - \{u\}$, that is, condition (a) does not hold. Also, if $S - \{u\}$ is a dominating set, then every vertex in $V - S$ is adjacent to at least one vertex in $S - \{u\}$, that is, condition (b) does not hold for u. Thus neither condition (a) nor (b) holds, which contradicts our assumption that at least one of these conditions holds. □

Theorem 1.2 [924] *Every connected graph G of order $n \geq 2$ has a dominating set S whose complement $V - S$ is also a dominating set.*

Proof. Let T be any spanning tree of G, and let u be any vertex in V. Then the vertices in T fall into two disjoint sets S and S' consisting, respectively, of the vertices with an even and odd distance from u in T. Clearly, both S and $S' = V - S$ are dominating sets for G. □

Theorem 1.3 [924] *If G is a graph with no isolated vertices, then the complement $V - S$ of every minimal dominating set S is a dominating set.*

Proof. Let S be any minimal dominating set of G. Assume vertex $u \in S$ is not dominated by any vertex in $V - S$. Since G has no isolated vertices, u must be dominated by at least one vertex in $S - \{u\}$, that is, $S - \{u\}$ is a dominating set, contradicting the minimality of S. Thus every vertex in S is dominated by at least one vertex in $V - S$, and $V - S$ is a dominating set. □

Definitions. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a set in $MDS(G)$, or equivalently, the minimum cardinality of a dominating set in G. The *upper domination number* $\Gamma(G)$ equals the maximum cardinality of a set in $MDS(G)$, or equivalently, the maximum cardinality of a

minimal dominating set of G. It is easy to see that for the graph G in Figure 1.2, $\gamma(G) = 3$, while $\Gamma(G) = 5$. Notice that the set $S = \{1, 3, 5\}$ is a dominating set of minimum cardinality; this is called a *γ-set* of G. Notice further that S is an independent set. This is also called an *independent dominating set* of G. The minimum cardinality of an independent dominating set of G is the *independent domination number* $i(G)$.

Theorem 1.4 [920] *For any graph G, $\gamma(G) + \varepsilon_F(G) = n$.*

Now that we have defined and illustrated dominating sets, minimum dominating sets, and the domination number of a graph, we describe, in Sections 1.3 through 1.10, a variety of situations in which dominating sets naturally occur.

School Bus Routing

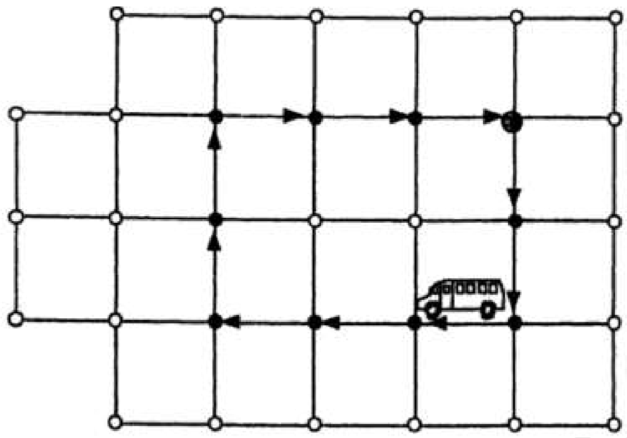


Figure:Distance 2-cycle domination

Consider the graph in Figure 1.5. Let us say that this represents a street map of part of a city, where each edge represents one city block. The school is located at the large vertex. Let us assume that the school district has decided that no child shall have to walk more than two blocks in order to be picked up by a school bus. Therefore, we must construct a route for a school bus that leaves the school, gets within two blocks of every child and returns to the school. One such simple route is indicated by the directed edges in Figure 1.5. Notice that some of the children live close enough to walk to school.

Computer Communication Networks

Consider a computer network modeled by a graph $G = (V, E)$, for which vertices represent computers and edges represent direct communication links between pairs of computers. Let the vertices in Figure 1.7 represent an array, or network, of 16 computers, or processors. Each processor can pass information to the processors to which it is directly connected. Assume that from time to time we need to collect information from all processors. We do this by having each processor route its information to one of a small set of collecting processors (a dominating set). Since this must be done relatively often and relatively fast, we cannot route this information over too long a path. Thus we need to identify a small set of processors which are close to all other processors. Let us say that we will tolerate at most a two-unit delay between the time a processor sends its

information and the time it arrives at a nearby collector. In this case we seek a *distance-2 dominating set* among the set of all processors. The two shaded vertices form a distance-2 dominating set in the hypercube network in Figure 1.7.

Radio Stations

Let each village be represented by a vertex. An edge between two villages is labelled with the distance, say in kilometers, between the two villages (see Figure 1.9(a)). Let us assume that a radio station has a broadcast range of fifty kilometers. What is the least number of stations in a set which dominates (within distance 50) all other vertices in this graph? A set $\{B, F, H, J\}$ of cardinality four is indicated in Figure 1.9(b). Notice in this case that since we have assumed that a radio station has a broadcast range of only fifty kilometers, we can essentially remove all edges in the graph in Figure 1.9(a) which represent a distance of more than fifty kilometers. This gives us the graph in Figure 1.9(b).

Theorem 1.5 [772] *If a γ-set S of a connected graph G of order $n \geq 2$ is a status of G, then S is an independent dominating set of cardinality two.*

Proof. Let S be a γ-set of G which is a status. Since G is connected and has no isolated vertices, there must be at least one vertex $v \in V - S$. Since S is a γ-set, v must be adjacent to at least one vertex in S. But since S is a status, every vertex of S must be adjacent to v. Furthermore, every vertex in S must be adjacent to every vertex in $V - S$. Now since S is a status, $|S| \geq 2$.

Assume that $|S| \geq 3$, and let $u \in S$ and $v \in V - S$. Since S is a status, u is adjacent to every vertex in $V - S$, and v is adjacent to every vertex in S. Thus, $\{u, v\}$ is a dominating set, contradicting the minimality of S. Therefore, $|S| = 2$. If u is adjacent to v, then clearly $\{u\}$ is a dominating set of G, again contradicting the minimality of S. Therefore, $|S| = 2$ and S is an independent set. □

Corollary 1.6 [772] *If a γ-set S of a connected graph G is also a structurally equivalent set, then S consists of two independent vertices each of which has degree $n - 1$.*

Land Surveying

In the field of surveying, a typical task is to compile a topographic map of a tract of land by determining the positions and elevations of a carefully selected set of control points. A grid method can be used in areas where the topography is fairly regular. The area is divided uniformly into squares or rectangles, by two sets of lines running in perpendicular directions and spaced uniformly apart, for example, 100 feet (as in Figure 1.10).

One way the surveyor identifies the grid points to be mapped is to set stakes at the intersections of these lines. It is then necessary to determine the elevations of all grid points. This is done with the use of a total station instrument which is an electronic theodolite (or transit) containing an integral EDM (electronic distance measuring instrument) and angle measuring capability. An EDM sends out light from one point on a line to another point. At the other point, a retroprism (a reflector or a transmitter-receiver), reflects the light back to the

NP-Completeness of the Domination Problem

Let us proceed to the basic question: how difficult is it to compute the domination number of an arbitrary graph? We will show that the domination problem is NP-complete for arbitrary graphs.

Stated in the now accepted format, as established by Garey and Johnson in their seminal book on NP-completeness [524], the basic complexity question concerning the decision problem for the domination number takes the following form:

DOMINATING SET

Theorem 1.7 [524] *DOMINATING SET is NP-complete.*

Proof. We must do two things. First, we must show that DOMINATING SET $\in$ **NP**. This is easy to do since it is easy to verify a 'yes' instance of DOMINATING SET in polynomial time, that is, for a graph $G = (V, E)$, a positive integer k and an arbitrary set $S \subseteq V$ with $|S| \leq k$, it is easy to verify in polynomial time whether S is a dominating set.

Mathematical History of Domination in Graphs

Although the mathematical study of dominating sets in graphs began around 1960, the subject has historical roots dating back to 1862 when de Jaenisch [347] studied the problem of determining the minimum number of queens which are necessary to cover (or dominate) an $n \times n$ chessboard. As reported by W. W. Rouse Ball in 1892 [72], chess enthusiasts in the late 1800s studied, among others, the following three basic types of problems:

1. Covering - what is the minimum number of chess pieces of a given type which are necessary to cover/attack/dominate every square of an $n \times n$ board? This is an example of the problem of finding a dominating set of minimum cardinality.

2. Independent Covering - what is the minimum number of mutually nonattacking chess pieces of a given type which are necessary to dominate every square of an $n \times n$ board? This is an example of the problem of finding a minimum cardinality independent dominating set.

3. Independence - what is the maximum number of chess pieces of a given type which can be placed on an $n \times n$ chessboard in such a way that no two of them attack/dominate each other? This is an example of the problem of finding the maximum cardinality of an independent set. When the chess piece is the queen, this problem is known as the *N-queens Problem*. It is known that for every positive integer $n \geq 4$, it is possible to place n nonattacking (independent) queens on an $n \times n$ board. For over a hundred years people have studied different ways of doing this.

These three problem-types were studied in detail by Yaglom and Yaglom around 1964 [1155]. These two brothers produced elegant solutions to some of these problems for the rooks, knights, kings and bishops chess pieces.

In 1958 Claude Berge [95] wrote a book on graph theory, in which he defined for the first time the concept of the domination number of a graph (although he called this number the 'coefficient of external stability'). In 1962 Oystein Ore [924] published his book on graph theory, in which he used, for the first time, the names 'dominating set' and 'domination number' (although he used the notation $d(G)$ for the domination number of a graph). In 1977 Cockayne and Hedetniemi [280] published a survey of the few results known at that time about dominating sets in graphs. In this survey paper, Cockayne and Hedetniemi were the first to use the notation $\gamma(G)$ for the domination number of a graph, which subsequently became the accepted notation.

This survey paper seems of have set in motion the modern study of domination in graphs. Some twenty years later more than 1,200 research papers have been published on this topic, and the number of papers is steadily growing. This book is inspired by the somewhat explosive growth of this field of study. It is also motivated by a desire to put some order into this huge collection of research papers, to organize the study of dominating sets in graphs into meaningful subareas, and to attempt to place the study of dominating sets in even broader mathematical and algorithmic contexts.

POSSIBLE QUESTIONS

PART B (5x6 = 30 Marks)

1. Explain about minimal dominating set.
2. Prove that if a connected graph with n ≥ 2vertices has a dominating set S then the complement of S is also a dominating set of G.
3. Explain domination number of a graph with examples
4. Explain briefly about applications of radio stations.
5. If G is a graph with no isolated vertices, then prove that the complement V-S of every minimal dominating set S is a dominating set.
6. Explain independent set of a graph with examples.

7. Prove that every connected graph G of order n ≥ 2 has a dominating set S whose complement V-S is also a dominating set.
8. Explain independence number of a graph with examples.
9. Explain dominating set of a graph with examples.
10. If a γ- set S of a connected graph G of order n ≥ 2 is a status of G, then prove that S is an independent dominating set of cardinality two.

PART C(1x10 =10 Marks)

1. Prove that a dominating set S is a minimal dominating set if and only if for each vertex u ε

S, one of the following two conditions holds.

(i) u is an isolate of S.

(ii) there exists a vertex v ε V-S for which N(v)∩S = {u}.

2. Discuss dominating set of a graph with examples.
3. Explain if a connected graph with $n \geq 2$ vertices has a dominating set S then the complement of S is also a dominating set of G.