KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

Semester – I

		L	T	P	C
18MMU103	**LOGIC AND SETS**	**6**	**2**	**0**	**6**

Scope: On successful completion of course the learners gain about propositional equivalence, relation and its applications.
Objectives: To enable the students to learn and gain knowledge about propositions, negation, conjunction, disjunction, logical equivalences and counting principle.
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UNIT I
Introduction, propositions, truth table, negation, conjunction and disjunction. Implications, biconditional propositions, converse, contra positive and inverse propositions and precedence of logical operators.

UNIT II
Propositional equivalence: Logical equivalences.
Predicates and quantifiers: Introduction, Quantifiers, Binding variables and Negations.

UNIT III
Sets: Subsets, Set operations and the laws of set theory and Venn diagrams. Examples of finite and infinite sets.

UNIT IV
Finite sets and counting principle. Empty set, properties of empty set. Standard set operations. Classes of sets. Power set of a set. Difference and Symmetric difference of two sets. Set identities, Generalized union and intersections.

UNIT V
Relation: Product set, Composition of relations, Types of relations, Partitions. Equivalence Relations with example of congruence modulo relation, Partial ordering relations, n-ary relations.

SUGGESTED READINGS

TEXT BOOK

1. Grimaldi R.P.,(2004). Discrete Mathematics and Combinatorial Mathematics, Pearson Education, Pvt.Ltd, Singapore.

REFERENCES

1. Bourbaki .N(2004),Theory of sets, Springer Pvt Ltd, Paris.
2. Halmos P.R.,(2011). Naive Set Theory, Springer Pvt Ltd, New Delhi.
3. Kamke E., (2010).Theory of Sets, Dover Publishers, New York.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: U. R. Ramakrishnan
Subject Name: Logic and Sets **Sub.Code:18MMU103**
Semester: I **Class: I B.Sc Mathematics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
UNIT-I			
1.	1	Introduction to logic and sets	R4:Ch: 12; Pg.No:333
2.	1	Propositions-Definition with examples	R4:Ch: 12; Pg.No:334,335
3.	1	Truth table	T1:Ch:2; Pg.No:47-49
4.	1	Tutorial-1	
5.	1	Continuation of problems on truth table	T1:Ch:2;Pg.No:50-53
6.	1	Problems on Negation and Conjunction	R4:Ch:12; Pg.No:335-336
7.	1	Problems on disjunction	R4:Ch:12;Pg.No:336-338
8.	1	Tutorial-2	
9.	1	Implications-Definition and problems	R4:Ch:12; Pg.No:362-364
10.	1	Biconditional propositions	R4:Ch:12;Pg.No:349-350
11.	1	Continuation of Biconditional propositions	R4:Ch:12;Pg.No:351-352
12.	1	Tutorial-3	
13.	1	Converse and contra positive propositions	R4:Ch: 12; Pg.No:344-348
14.	1	Precedence of logical operators	R4: Ch: 12; Pg.No:342-343
15.	1	Problems on logical operators	R4: Ch:12; Pg.No:346-350

16.			
17.	1	Tutorial-4	
18.	1	Continuation of problems on logical operators	R4: Ch:12; Pg.No:350-351
19.	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit I=18			
T1. Grimaldi R.P.,(2004). Discrete Mathematics and Combinatorial Mathematics, Pearson Education, Pvt.Ltd, Singapore. R4.Sharma.J.K.,(2015).Discrete mathematics,Tata Mc Graw-Hill publishing company ltd, New Delhi.			
UNIT-II			
1.	1	Propositional equivalence	T_1:Chap 5 P.No:97-98
2.	1	Logical Equivalence	T1: Ch: 2; Pg. No :55-56
3.	1	Properties on logical equivalence	T1: Ch: 2; Pg. No :55-56
4.	1	Tutorial-1	
5.	1	Predicates :Introduction with example	R7: Ch: 2; Pg. No :2.1-2.2
6.	1	Quantifiers:Introduction with example	R7: Ch: 2; Pg. No :2.1-2.2
7.	1	Problems on Predicates	R7: Ch: 2; Pg. No :2.2-2.3
8.	1	Tutorial-2	
9.	1	Universal Quantifiers-Definition with examples	R7: Ch: 2; Pg. No :2.2-2.3
10.	1	Existential Quantifiers-Definition with examples	R7: Ch:2; Pg.No:2.3-2.4
11.	1	Problems on Quantifiers	R7: Ch:2; Pg.No:2.3-2.4
12.	1	Tutorial-3	
13.	1	Continuation of problems on Quantifiers	R1: Ch: 4; Pg. No :38-41
14.	1	Binding Variables:Definition with example	R7: Ch: 2; Pg. No :2.4-2.5
15.	1	Problems on binding variables	R7: Ch: 2; Pg. No :2.4-2.5
16.	1	Tutorial-4	
17.	1	Negations of a quantified expressions	R4: Ch:12; Pg. No :336-337

18.	1	Negations – problems	R7: Ch:2;Pg.No:2.7-2.8
19.	1	Recapitulation and Discussion of possible questions	

Total No of Hours Planned For Unit II=19

T1. Grimaldi R.P.,(2004). Discrete Mathematics and Combinatorial Mathematics, Pearson
R1. Bourbaki .N(2004),Theory of sets, Springer Pvt Ltd, Paris
R4.Sharma.J.K.,(2015).Discrete mathematics,Tata Mc Graw-Hill publishing company ltd, New Delhi
R7.Sundaresan,V.,Ganapathy Subramaniam,K.S and Ganesan.K.(2009).Discrete mathematics,AR Publications,India.

UNIT-III

1.	1	Set-Definitions with examples	T1: Ch: 3; Pg. No:123-124
2.	1	Subsets: Definitions and examples	R3: Ch: 1; Pg. No:5-8
3.	1	Tutorial-1	
4.	1	Theorems on subsets	T1: Ch: 3; Pg. No:125-133
5.	1	Set operations: Definitions and examples	T1: Ch: 3; Pg. No :136-139
6.	1	Laws of set theory:Definitions and example	T1: Ch:3;Pg.No:139-140
7.	1	Tutorial-2	
8.	1	Theorems on laws of set theory	T1:Ch:3;Pg.No:140-141
9.	1	Venn diagrams:Definitions	T1: Ch:3, Pg. No:140-141
10.	1	Problems on venn diagrams	T1: Ch: 3; Pg. No:142-150
11.	1	Tutorial-3	
12.	1	Problems on finite sets	R7: Ch: 2; Pg. No :3.7-3.8
13.	1	Theorems on finite sets	R7:Ch:2:Pg.No:3.8-3.9
14.	1	Infinite sets-Definition with example	R7:Ch:2;Pg.No:3.10-3.11
15.	1	Tutorial-4	
16.	1	Problems on infinite sets	R7:Ch:2;Pg.No:3.10-3.11
17.	1	Theorems on Infinite sets	R7:Ch:2;Pg.No:3.11-3.12
18.	1	Tutorial-5	

19.	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit III=19			
T1. Grimaldi R.P.,(2004). Discrete Mathematics and Combinatorial Mathematics, Pearson R3. Kamke E., (2010).Theory of Sets, Dover Publishers, New York. R7.Sundaresan,V.,Ganapathy Subramaniam,K.S and Ganesan.K.(2009). Discrete mathematics,AR Publications,India.			
UNIT-IV			
1.	1	Finite set and counting principle	R6:Ch:1; Pg,No:9-17
2.	1	Empty set and Property on empty set	R5:Ch:1; Pg,No:6-7
3.	1	Standard set operations	R5:Ch:1; Pg,No:7-8
4.	1	Tutorial-1	
5.	1	Classes of sets	R5:Ch:1; Pg.No:8-9
6.	1	Power set of a set	R2:Ch:5; Pg,No:19-21
7.	1	Problems on power set	R2:Ch:5; Pg,No:19-21
8.	1	Tutorial-2	
9.	1	Difference of two sets	R5:Ch:1; Pg,No:9-10
10.	1	Symmetric difference of two sets	R5:Ch:1; Pg,No:10-11
11.	1	Set identities	R5:Ch:1;Pg.No:11-12
12.	1	Tutorial-3	
13.	1	Generalized union	R2:Ch:4;Pg.No:12-16
14.	1	Problems on generalized union	R2:Ch:4;Pg.No:12-16
15.	1	Theorems on union	R4:Ch:4;Pg.No:10-12
16.	1	Tutorial-4	
17.	1	Theorem on intersection	R4:Ch:4;Pg.No:12-13
18.	1	Continuation of theorem on intersection	R4:Ch:4;Pg.No:13
19.	1	Tutorial-5	
20.	1	Recapitulation and Discussion of	

		possible questions	
Total No of Hours Planned For Unit IV=20			
R2. Halmos P.R.,(2011). Naive Set Theory, Springer Pvt Ltd, New Delhi. R5.Chowdhary.K.R.,(2012). Fundamentals of Discrete mathematical structures,second edition, phi learning pvt ltd,New Delhi. R6.Seymour Lipschutz,Marc Lars Lipson.,(2001).Theory and problems of discrete mathematics,Tata Mc Graw-Hill publishing company ltd,New Delhi.			
UNIT-V			
1.	1	Relation definition with example	R4:Ch:3.1; Pg.No:72-73
2.	1	Product set	R4:Ch:3.1; Pg.No:73-74
3.	1	Composition of relation and types of relations	R4:Ch:3.1; Pg.No:79-80
4.	1	Tutorial-1	
5.	1	Types of relations	R4:Pg.No:92-93
6.	1	Partial order relations	R1:Ch:3; Pg.No:78-79
7.	1	Equivalence relations: Definitions and problems	R4:Ch:3;Pg.No:82-83
8.	1	Tutorial-2	
9.	1	Congruence modulo	R4:Ch:3; Pg.No:83-84
10.	1	Theorem on reduced groups	R4:Ch:3; Pg.No:84-85
11.	1	Partial ordering relations: problems	R4:Ch:3; Pg.No:80-81
12.	1	Tutorial-3	
13.	1	Partial ordering relations: Theorems	R4:Ch:3;Pg.No:81-82
14.	1	n-ary relations	R7:Ch:1:Pg.No:20-21
15.	1	Continuation of n-ary operation	R7:Ch:1:Pg.No:21-22
16.	1	Tutorial-4	
17.	1	Recapitulation and Discussion of possible questions	
18.	1	Discuss on Previous ESE Question Papers	
19.	1	Discuss on Previous ESE Question Papers	
20.	1	Discuss on Previous ESE Question Papers	

Total No of Hours Planned for unit V=20	
R1. Bourbaki .N(2004),Theory of sets, Springer Pvt Ltd, Paris. R4.Sharma.J.K.,(2015).Discrete mathematics,Tata Mc Graw-Hill publishing company ltd, New Delhi. R7.Sundaresan,V.,Ganapathy Subramaniam,K.S and Ganesan.K.(2009).Discrete mathematics,AR Publications,India.	
Total Planned Hours	**96**

SUGGESTED READINGS

TEXT BOOK
T1. Grimaldi R.P.,(2004). Discrete Mathematics and Combinatorial Mathematics, Pearson

REFERENCES
R1. Bourbaki .N(2004),Theory of sets, Springer Pvt Ltd, Paris.
R2. Halmos P.R.,(2011). Naive Set Theory, Springer Pvt Ltd, New Delhi.
R3. Kamke E., (2010).Theory of Sets, Dover Publishers, New York.
R4.Sharma.J.K.,(2015).Discrete mathematics,Tata Mc Graw-Hill publishing company ltd, New Delhi.
R5.Chowdhary.K.R.,(2012). Fundamentals of Discrete mathematical structures,second edition, phi learning pvt ltd,New Delhi.
R6.Seymour Lipschutz,Marc Lars Lipson.,(2001).Theory and problems of discrete mathematics,Tata Mc Graw-Hill publishing company ltd,New Delhi.
R7.Sundaresan,V.,Ganapathy Subramaniam,K.S and Ganesan.K.(2009).Discrete mathematics,AR Publications,India.

UNIT – I

Introduction, propositions, truth table, negation, conjunction and disjunction. Implications, biconditional propositions, converse, contra positive and inverse propositions and precedence of logical operators.

UNIT – I

Statements (Propositions): Sentences that claim certain things, either true or false

Notation: A, B, …P, Q, R, …., p, q, r, etc.

Examples of statements: Today is Monday. This book is expensive
If a number is smaller than 0 then it is positive.

Examples of sentences that **are not statements**: Close the door! What is the time?

Propositional variables: A, B, C, …, P., Q, R, … Stand for statements. May have true or false value.

Propositional constants:

T – true
F - false

Basic logical connectives: NOT, AND, OR
Other logical connectives can be represented by means of the basic connectives

Logical connectives	**pronounced**	**Symbol in Logic**
Negation	NOT	¬, ~, ‘
Conjunction	AND	$\wedge$
Disjunction	OR	$\vee$
Conditional	if then	$\rightarrow$
Biconditional	if and only if	$\leftrightarrow$
Exclusive or	Exclusive or	$\oplus$

Propositions. Compound Statements. Truth Tables

Truth tables - Define formally the meaning of the logical operators.
The abbreviation **iff** means **if and only if**

a. Negation (NOT, ~, ¬, ')

P	~P	
T	F	~P is true if and only if P is false
F	T	

b. Conjunction (AND, Λ, &&)

P	Q	P Λ Q	
T	T	T	P Λ Q is true iff both P and Q are true. In all other cases P Λ Q is false
T	F	F	
F	T	F	
F	F	F	

c. Disjunction / Inclusive OR (OR, V, ||)

P	Q	P V Q	
T	T	T	P V Q is true iff P is true or Q is true or both are true.
T	F	T	P V Q is false iff both P and Q are false
F	T	T	
F	F	F	

d. Conditional , known also as implication (→)

P	Q	P→ Q	
T	T	T	The implication P→ Q is false iff P is true however Q is false.
T	F	F	In all other cases the implication is true
F	T	T	
F	F	T	

e. Biconditional (↔)

P	Q	P ↔ Q	
T	T	T	P↔ Q is true iff P and Q have same values - both are true or both are false.
T	F	F	If P and Q have different values, the biconditional is false.
F	T	F	
F	F	T	

f. Exclusive OR (⊕)

P	Q	P ⊕ Q	
T	T	F	P⊕ Q is true iff P and Q have different values
T	F	T	We say: "P or Q but not both"
F	T	T	
F	F	F	

Precedence of the logical connectives:

Connectives within parentheses, innermost parentheses first

$\neg$	**negation**
$\wedge$	**conjunction**
$\vee$	**disjunction**
$\rightarrow$	**conditional**
$\leftrightarrow$, $\oplus$	**biconditional, exclusive OR**

Compound Statements: Logical expressions that consist of propositional variables and logical connectives. They may contain also propositional constants.

Evaluating compound statements : by building their truth tables

Example: $\neg P \vee Q$

P	Q	$\neg$P	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$(P \vee Q) \wedge \neg(P \wedge Q)$

P	Q	$P \vee Q$ A	$P \wedge Q$ B	$\neg(P \wedge Q)$ $\neg$B	$(P \vee Q) \wedge \neg(P \wedge Q)$ $A \wedge \neg B$ (the letters A and B are used as shortcuts)
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

1. Tautologies and Contradictions

A propositional expression is a **tautology** if and only if for all possible assignments of truth values to its variables its truth value is **T**

Example: $P \vee \neg P$ is a tautology

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

A propositional expression is a **contradiction** if and only if for all possible assignments of truth values to its variables its truth value is **F**

Example: $P \wedge \neg P$ is a contradiction

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Usage of tautologies and contradictions - in proving the validity of arguments; for rewriting expressions using only the basic connectives.

Definition: Two propositional expressions P and Q are logically equivalent, if and only if $P \leftrightarrow Q$ is a tautology. We write $P \equiv Q$ or $P \Leftrightarrow Q$.

Note that the symbols $\equiv$ and $\Leftrightarrow$ are **not logical connectives**

Exercise:

a) Show that $P \rightarrow Q \leftrightarrow \neg P \vee Q$ is a tautology, i.e. $P \rightarrow Q \equiv \neg P \vee Q$

P	Q	$\neg P$	$\neg P \vee Q$	$P \rightarrow Q$	$P \rightarrow Q \leftrightarrow \neg P \vee Q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

2. Logical equivalences

Similarly to standard algebra, there are **laws** to manipulate logical expressions, given as logical equivalences.

1. Commutative laws	$P \vee Q \equiv Q \vee P$	
	$P \wedge Q \equiv Q \wedge P$	
2. Associative laws	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	
	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
3. Distributive laws:	$(P \vee Q) \wedge (P \vee R) \equiv P \vee (Q \wedge R)$	
	$(P \wedge Q) \vee (P \wedge R) \equiv P \wedge (Q \vee R)$	
4. Identity	$P \vee F \equiv P$	
	$P \wedge T \equiv P$	
5. Complement properties	$P \vee \neg P \equiv T$	(excluded middle)
	$P \wedge \neg P \equiv F$	(contradiction)
6. Double negation	$\neg(\neg P) \equiv P$	
7. Idempotency (consumption)	$P \vee P \equiv P$	
	$P \wedge P \equiv P$	
8. De Morgan's Laws	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	
	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	
9. Universal bound laws (Domination)	$P \vee T \equiv T$	
	$P \wedge F \equiv F$	
10. Absorption Laws	$P \vee (P \wedge Q) \equiv P$	
	$P \wedge (P \vee Q) \equiv P$	
11. Negation of T and F:	$\neg T \equiv F$	
	$\neg F \equiv T$	

1. Truth table of the conditional statement

P	Q	$P \to Q$
T	T	T
T	**F**	**F**
F	T	T
F	F	T

P is called **antecedent**
Q is called **consequent**
Meaning of the conditional statement: The truth of P implies (leads to) the truth of Q

Note that when P is false the conditional statement is true no matter what the value of Q is. We say that in this case the conditional statement is **true by default or vacuously true.**

2. Representing the implication by means of disjunction

$$\mathbf{P \to Q \equiv \neg P \vee Q}$$

P	Q	$\neg$P	**P→ Q**	**¬P V Q**
T	T	F	T	T
T	F	F	**F**	**F**
F	T	T	T	T
F	F	T	T	T

Same truth tables

Usage:

1. To rewrite "OR" statements as conditional statements and vice versa (for better understanding)
2. To find the negation of a conditional statement using De Morgan's Laws

3. Rephrasing "or" sentences as "if-then" sentences and vice versa

Consider the sentence:
(1) "The book can be found in the library or in the bookstore".

Let

A = The book can be found in the library
B = The book can be found in the bookstore

Logical form of (1): **A V B**

Rewrite A V B **as a conditional statement**

In order to do this we need to use the commutative laws, the equivalence $\neg(\neg P) \equiv P$, and the equivalence $P \rightarrow Q \equiv \neg P \vee Q$

Thus we have:

$A \vee B \equiv \neg(\neg A) \vee B \equiv \neg A \rightarrow B$

The last expression $\neg A \rightarrow B$ is translated into English as
**"If the book cannot be found in the library,
it can be found in the bookstore"**.

Here the statement "The book cannot be found in the library" is represented by $\neg A$

There is still one more conditional statement to consider.
$A \vee B \equiv B \vee A$ (commutative laws)

Then, following the same pattern we have:

$B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

The English sentence is: **"If the book cannot be found in the bookstore, it can be found in the library.**

We have shown that:

$A \vee B \equiv \quad \neg(\neg A) \vee B \equiv \neg A \rightarrow B$
$A \vee B \equiv B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

Thus the sentence **"The book can be found in the library or in the bookstore"** can be rephrased as:
"If the book cannot be found in the library, it can be found in the bookstore".
"If the book cannot be found in the bookstore, it can be found in the library.

4. Negation of conditional statements

Positive: The sun shines
Negative: The sun does not shine

Positive: " If the temperature is 250°F then the compound is boiling "
Negative: ?
In order to find the negation, we use De Morgan's Laws.

Let
P = the temperature is 250°F
Q = the compound is boiling

Positive: $P \rightarrow Q \equiv \neg P \vee Q$
Negative: $\neg (P \rightarrow Q) \equiv \neg(\neg P \vee Q) \equiv \neg(\neg P) \wedge \neg Q \equiv P \wedge \neg Q$

Negative: The temperature is 250°F however the compound is not boiling

IMPORTANT TO KNOW:

The negation of a disjunction is a conjunction.
The negation of a conjunction is a disjunction

The **negation of a conditional statement is a conjunction**, not another if-then statement

Question: Which logical connective when negated will result in a conditional statement?

5. Necessary and sufficient conditions

Definition:

"P is a **sufficient condition** for Q" means : **if P then Q, $P \rightarrow Q$**
"P is a **necessary condition** for Q" means: **if not P then not Q, $\sim P \rightarrow \sim Q$**
The statement **$\sim P \rightarrow \sim Q$ is equivalent to $Q \rightarrow P$**

Hence given the statement **$P \rightarrow Q$,**
P is a sufficient condition for Q, and Q is a necessary condition for P.

Examples:

If n is divisible by 6 then n is divisible by 2.
The sufficient condition to be divisible by 2 is to be divisible by 6.
The necessary condition to be divisible by 6 is to be divisible by 2

If n is odd then n is an integer.
The sufficient condition to be an integer to be odd.
The necessary condition to be odd is to be an integer.

If and only if - the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \leftrightarrow Q$ is true whenever P and Q have same values. Otherwise it is false.

This means that **both $P \to Q$ and $Q \to P$ have to be true**

P	Q	**$P \to Q$**	**$Q \to P$**	$P \leftrightarrow Q$
T	**T**	**T**	**T**	**T**
T	F	F	T	F
F	T	T	F	F
F	**F**	**T**	**T**	**T**

Contrapositive

Definition: The expression $\sim Q \to \sim P$ is called **contrapositive** of $P \to Q$

The conditional statement $P \to Q$ and its contrapositive $\sim Q \to \sim P$ **are equivalent**.
The proof is done by comparing the truth tables

The truth table for $P \to Q$ and $\neg Q \to \neg P$ is:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We can also prove the equivalence by using the disjunctive representation:

$P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \to \neg P$

Converse and inverse

Definition: The converse of $P \to Q$ is the expression $Q \to P$

Definition: The inverse of $P \to Q$ is the expression $\sim P \to \sim Q$

Neither the converse nor the inverse are equivalent to the original implication.
Compare the truth tables and you will see the difference.

P	Q	¬ P	¬ Q	P→ Q	Q→ P	**¬ P → ¬ Q**
T	T	F	F	T	T	T
T	F	F	T	**F**	**T**	**T**
F	T	T	F	**T**	**F**	**F**
F	F	T	T	T	T	T

Valid and Invalid Arguments.

Definition: An argument is a sequence of statements, ending in a conclusion. All the statements but the final one (the conclusion) are called premises(or assumptions, hypotheses)

Verbal form of an argument:

(1) If Socrates is a human being then Socrates is mortal.
(2) Socrates is a human being

Therefore (3) Socrates is mortal

Another way to write the above argument:

$$P \rightarrow Q$$
$$P$$
$$\therefore \quad Q$$

2. Testing an argument for its validity

Three ways to test an argument for validity:

A. Critical rows

1. Identify the assumptions and the conclusion and assign variables to them.
2. Construct a truth table showing all possible truth values of the assumptions and the conclusion.
3. Find the **critical rows - rows in which all assumptions are true**
4. For each critical row determine whether the conclusion is also true.
   a. If the conclusion is **true in all critical rows**, then **the argument is valid**
   b. If there is at least one row **where the assumptions are true, but the conclusion is false**, then the argument is **invalid**

B. Using tautologies

The argument is true if the conclusion is true whenever the assumptions are true.
This means: If all assumptions are true, then the conclusion is true.
"All assumptions" means the conjunction of all the assumptions.

Thus, let A1, A2, … An be the assumptions, and B - the conclusion.

For the argument to be valid, the statement
If (A1 Λ A2 Λ… Λ An) then B must be **a tautology -** true for all assignments of values to its variables, i.e. its column in the truth table must contain only **T**

i.e.

$$(A1 \wedge A2 \wedge \ldots \wedge An) \rightarrow B \equiv \mathbf{T}$$

C. Using contradictions

If the argument is valid, then we have $(A1 \wedge A2 \wedge \ldots \wedge An) \rightarrow B \equiv \mathbf{T}$
This means that the negation of $(A1 \wedge A2 \wedge \ldots \wedge An) \rightarrow B$ should be a contradiction - containing only **F** in its truth table

In order to find the negation we have first to represent the conditional statement as a disjunction and then to apply the laws of De Morgan

$(A1 \wedge A2 \wedge \ldots \wedge An) \rightarrow B \equiv \sim(A1 \wedge A2 \wedge \ldots \wedge An) \vee B \equiv$

$\sim A1 \vee \sim A2 \vee \ldots . \vee \sim An \vee B$.

The negation is:

$\sim((A1 \wedge A2 \wedge \ldots \wedge An) \rightarrow B) \equiv \sim(\sim A1 \vee \sim A2 \vee \ldots . \vee \sim An \vee B)$

$\equiv A1 \wedge A2 \wedge \ldots . \wedge An \wedge \sim B$

The argument is valid if $A1 \wedge A2 \wedge \ldots . \wedge An \wedge \sim B \equiv F$

There are two ways to show that a logical form is a tautology or a contradiction:

a. by constructing the truth table
b. by logical transformations applying the logical equivalences (logical identities)

Examples:

1. Consider the argument:

$$P \rightarrow Q$$
$$P$$
$$\therefore \quad Q$$

Testing its validity:

a. by examining the truth table:

P	Q	P→ Q
T	**T**	**T**
T	F	F
F	T	T
F	F	T

b. By showing that the statement 'If all premises then the conclusion" is a tautology:
The premises are P and P→ Q. The statement to be considered is:

$(P \wedge (P\rightarrow Q)) \rightarrow Q$

We shall show that it is a tautology by using the following identity laws:

(1) $P\rightarrow Q \equiv \sim P \vee Q$

(2) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ commutative laws

$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

(3) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$ distributive law

(4) $P \wedge \sim P \equiv F$

(5) $P \vee \sim P \equiv T$

(6) $P \vee F \equiv P$

(7) $P \vee T \equiv T$

(8) $P \wedge T \equiv P$

(9) $P \wedge F \equiv F$

(10) $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ De Morgan's Laws

		$(P \wedge (P \rightarrow Q)) \rightarrow Q$
by (1)	$\equiv$	$\sim(P \wedge (P \rightarrow Q)) \vee Q$
by (10)	$\equiv$	$(\sim P \vee \sim(P \rightarrow Q)) \vee Q$
by (1)	$\equiv$	$(\sim P \vee \sim(\sim P \vee Q)) \vee Q$
by (10)	$\equiv$	$(\sim P \vee (P \wedge \sim Q)) \vee Q$
by (3)	$\equiv$	$((\sim P \vee P) \wedge (\sim P \vee \sim Q)) \vee Q$
by (5)	$\equiv$	$(\mathbf{T} \wedge (\sim P \vee \sim Q)) \vee Q$
by (8)	$\equiv$	$(\sim P \vee \sim Q) \vee Q$
by (2)	$\equiv$	$\sim P \vee (\sim Q \vee Q)$
by (5)	$\equiv$	$\sim P \vee \mathbf{T}$
by(7)	$\equiv$	$\mathbf{T}$

2. Consider the argument

$$P \rightarrow Q$$
$$Q$$
$$\therefore \quad P$$

We shall show that this argument is invalid by examining the truth tables of the assumptions and the conclusion. The critical rows are in boldface.

P	Q	P→ Q	
T	**T**	**T**	
T	F	F	
F	**T**	**T**	**here the assumptions are true, however the conclusion is false**
F	F	T	

Exercise:

Show the validity of the argument:

1. P V Q (premise)
2. ~Q (premise)

Therefore P (conclusion)

a. by using critical rows
b. by contradiction using logical identities

Solution:

a. by critical rows

conclusion		Premises		
P	**Q**	**P V Q**	**~Q**	
T	T	T	F	
T	F	**T**	**T**	**Critical row**
F	T	T	F	
F	F	F	T	

b. By contradiction using identities

$((P \vee Q) \wedge \sim Q) \wedge \sim P \equiv$

$((P \wedge \sim Q) \vee (Q \wedge \sim Q)) \wedge \sim P \equiv$

$((P \wedge \sim Q) \vee \mathbf{F}) \wedge \sim P \equiv$

$(P \wedge \sim Q) \wedge \sim P \equiv$

$P \wedge \sim P \wedge \sim Q \equiv \mathbf{F} \wedge \sim Q \equiv \mathbf{F}$

Possible Questions

2 Marks Questions:

1. Define proposition with example.
2. Define tautology with example
3. Define modular statement with example
4. Define truth value.
5. Define conjunction with truth table formula.

6 Mark Questions:

6. Construct the truth table for $(P\to(Q\to R)) \to ((P\to Q) \to (P\to R))$
7. State the converse, contrapositive and inverse of "A positive integer is a prime if it has no divisors other than 1 and itself "
8. Write the following statement in symbolic form
   i) You can access the internet from campus only if you are a computer science major or you are not a freshman,
   ii) You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old
9. Construct the truth table for $(P \to(Q\to S)) \wedge (\rceil R \vee P) \wedge Q$
10. Construct the truth table for $((P \to Q) \to R) \to S$
11. State the converse, contrapositive and inverse of the following
    i)If you watch television your mind will decay.
    ii) School is closed if more than 2 feet of snow falls.
12. Construct the truth table for $\rceil(Q\to R) \wedge R \wedge (P\to Q)$
13. State the converse, contrapositive and inverse of the following
    i) If today is Thursday, then I have a test today.
    ii) I come to class whenever there is going to be a quiz.
14. Construct the truth table for $(P \leftrightarrow Q) \leftrightarrow (R \leftrightarrow S)$.
15. State the converse, contrapositive and inverse of the following
    i) If it snows today, I will ski tomorrow.
    ii) A positive integer is a prime only if it has no divisors other than 1 and itself.

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Subject: Liogic and Sets **Subject Code: 18MMU103**
Class : I B.Sc Mathematics **Semester : I**

UNIT -I
PART A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The equivalent statement for P and not P	F	T	F and T	none	F
The implications of P	P	not P	P or Q	P and Q	P or Q
The implications of P and Q is	P	Q	P or Q	not P	P
P or P "equivalent to" P is called as	idempotent	associative	closure	identity	idempotent
not(not P) "equivalent to" P is called as	Involution	Absorption	Associative	none	Involution
If P then Q is "equivalent to"	not P or Q	not P and Q	P and Q	P or Q	not P or Q
A statement which has true as the truth value for all the assignments is called	contradiction	tautology	either tautology or contradiction	none	tautology
A statement which has false as the truth value for all the assignments is called	contradiction	tautology	either tautology or contradiction	none	contradiction
If P has T and Q has F as their truth value, then P or Q has ----- as truth value	T	F	0	none	T
A biconditional statement P if and only if Q is " equivalent to "	(Not P or Q) and (not Q or P)	(Not P or Q) or (not Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(Not P or Q) and (not Q or P)
A biconditional statement notP if and only if Q is " equivalent to "	(Not P or Q) and (not Q or P)	(Not P or Q) or (not Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(P or Q) and (not Q or P)
A biconditional statement P if and only if not Q is " equivalent to "	(Not P or Q) and (not Q or P)	(Not P or Q) or (not Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(Not P or Q) and (Q or P)
A biconditional statement notP if and only if not Q is " equivalent to "	(Not P or Q) and (not Q or P)	(P or Q) and (Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(P or Q) and (Q or P)
if R: Mark is rich and H: Mark is happy , then Mark is poor or he is both rich and unhappy can be symbolically written as	not R or (R and not H)	not R or (R or not H)	not R and (R and not H)	R or (R and not H)	not R or (R and not H)
In the statement If P then Q the antecedent is	P	Q	notP	not Q	P
In the statement If P then Q the consequent is	P	Q	notP	not Q	Q
Out of the following which is the well formed formula	P and Q	(P or Q	if P then Q)	if (if P then Q) then Q)	P and Q
Elementary products are	P and not P	P	P andQ	not P	all of these
Elementary sum are	P	Not Q	P or Q	not P or P	all of these
pcnf contains	product of maxterms	sum of max terms	sum of minterms	product of min terms	product of maxterms
pdnf contains	product of maxterms	sum of max terms	sum of minterms	product of min terms	sum of minterms
dual of a statement is obtained by replacing "and" , "or" , "not" by	"or", "and", "not"	"or", "and", "and"	"and", "or", "not"	"or", "or", "not"	"or", "and", "not"
dual of the statement Pand Q is	P or Q	Q and P	Q and not P	none	P or Q
dual of "if P then Q" is	not P and Q	P and Q	P or Q	Not P or Q	not P and Q
P "exclusive or" Q is the negation of	if P then Q	if Q then P	P if and only if Q	Q if and only if P	P if and only if Q
The other name of tautology is	identically true	identically false	universally false	false	identically true
The other name of contradiction is	identically true	identically false	universally true	true	identically false
The converse of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	" If Q then P"
The contra positive of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	"if not Q then not P"
The inverse of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	" if not P then not Q"
A statement A is said to tautologically imply a statement B if an donly if " if A then B "is a	tautology	contradiction	false	none	tautology
P and (P or Q) is	P	Q	P or Q	P and Q	P
P " exclusive or" Q is true if both P, Q has ---- truth values	same	different	none	all of these	different
A conditional statement and its contrapositive are …………………	A tautulogy	a contradiction	Logically equivalent	an assumption	Logically equivalent
A rule of inference is a form of argument that is …………….	valid	a contradiction	an assumption	A tautulogy	valid
An or statement is false if, and only if, both components are ………………..	TRUE	FALSE	not true	neither true nor false	FALSE
Two statement forms are logically equivalent if, and only if they always have………………………….	not same truth values	the same truth values	the different truth values	the same false values	the same truth values
P " exclusive or" Q is false if both P, Q has ---- truth values	same	different	none	all of these	same

UNIT-II

SYLLABUS

Propositional equivalence: Logical equivalences.
Predicates and quantifiers: Introduction, Quantifiers, Binding variables and Negations.

Logical Equivalences as Tautologies

The Idea, and Definition, of Logical Equivalence

In lay terms, two statements are logically equivalent when they say the same thing, albeit perhaps in different ways. To a mathematician, two statements are called logically equivalent when they will always be simultaneously true or simultaneously false. To see that these notions are compatible, consider an example of a man named John N. Smith who lives alone at 12345 North Fictional Avenue in Miami, Florida, and has a United States Social Security number 987-65-4325.[14] Of course there should be exactly one person with a given Social Security number. Hence, when we ask any person the questions, "are you John N. Smith of 12345 North Fictional Avenue in Miami, Florida?" and "is your U.S. Social Security number 987-65-4325?" we would be in essence asking the same question in both cases. Indeed, the answers to these two questions would always be both yes, or both no, so the statements "you are John N. Smith of 12345 North Fictional Avenue in Miami, Florida," and "your U.S. Social Security number is 987-65-4325," are logically equivalent. The notation we would use is the following:

you are John N. Smith of 12345 North Fictional Avenue in Miami, Florida
$\Longleftrightarrow$ your U.S. Social Security number is 987-65-4325.

The motivation for the notation " $\Longleftrightarrow$ " will be explained shortly.

On a more abstract note, consider the statements $\sim (P \vee Q)$ and $(\sim P) \wedge (\sim Q)$. Below we compute both of these compound statements' truth values in one table:

P	Q	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

the same

We see that these two statements are both true or both false, under any of the $2^2 = 4$ possible circumstances, those being the possible truth value combinations of the underlying, independent component statements P and Q. Thus the statements $\sim (P \vee Q)$ and $(\sim P) \wedge (\sim Q)$ are indeed logically equivalent in the sense of always having the same truth value. Having established this, we would write

$$\sim (P \vee Q) \iff (\sim P) \wedge (\sim Q).$$

Note that in logic, this symbol " $\Longleftrightarrow$ " is similar to the symbol "=" in algebra and elsewhere.[15] There are a couple of ways it is read out loud, which we will consider momentarily. For now we take the occasion to list the formal definition of logical equivalence:

Definition: Given n independent statements P , $\cdots, P_n$, and two statements R, S which are compound statements of the $P_1, \cdots, P_n$, we say that R and S are **logically equivalent**, *which we then denote $R \iff S$, if and only if their truth table columns have the same entries for each of the 2^n distinct combinations of truth values for the $P_1, \cdots, P_n$. When R and S are logically equivalent, we will also call $R \iff S$ a* **valid equivalence**.

Again, this is consistent with the idea that to say statements R and S are logically equivalent is to say that, under any circumstances, they are both true or both false, so that asking if R is true is—functionally—exactly the same as asking if S is true. (Recall our example of John N. Smith's Social Security number.)

Note that if two statements' truth values always match, then connecting them with $\longleftrightarrow$ yields a tautology. Indeed, the bi-implication yields T if the connected statements have the same truth value, and F otherwise. Since two logically equivalent statements will have matching truth values in all cases, connecting with $\longleftrightarrow$ will always yield T, and we will have a tautology. On the other hand, if connecting two statements with $\longleftrightarrow$ forms a tautology, then the connected statements must have always-matching truth values, and thus be equivalent. This argument yields our first theorem:[16]

Theorem : Suppose R and S are compound statements of $P \cdots, P_n$. *Then R and S are logically equivalent if and only if $R \longleftrightarrow S$ is a tautology.*

The theorem above gives us the motivation behind the notation $\iff$. Assuming R and S are compound statements built upon component statements $P_1 \cdots, P_n$, then

$$R \iff S \qquad \text{means that} \qquad R \longleftrightarrow S \text{ is a tautology.} \tag{1.1}$$

To be clear, when we write $R \longleftrightarrow S$ we understand that this might have truth value T or F, i.e., it might be true or false. However, when we write $R \iff S$, we mean that $R \longleftrightarrow S$ is always true (i.e., a tautology), which partially explains why we call $R \iff S$ a *valid equivalence*.[17]

To prove $R \iff S$, we could (but usually will not) construct $R \longleftrightarrow S$, and show that it is a tautology. We do so below to prove

$$\underbrace{\sim (P \vee Q)}_{\text{"}R\text{"}} \iff \underbrace{(\sim P) \wedge (\sim Q)}_{\text{"}S\text{"}}.$$

P	Q	$P \vee Q$	$\overbrace{\sim (P \vee Q)}^{R}$	$\sim P$	$\sim Q$	$\overbrace{(\sim P) \wedge (\sim Q)}^{S}$	$\overbrace{[\sim (P \vee Q)] \longleftrightarrow [(\sim P) \wedge (\sim Q)]}^{R \longleftrightarrow S}$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

However, our preferred method will be as in the previous truth table, where we simply show that the truth table columns for R and S have the same entries at each horizontal level, i.e., for each truth value combination of the component statements. That approach saves space and reinforces our original notion of equivalence (matching truth values). However it is still important to understand the connection between $\longleftrightarrow$ and $\iff$, as given in (1.1).

Equivalences for Negations

Much of the intuition achieved from studying symbolic logic comes from examining various logical equivalences. Indeed we will make much use of these, for the theorems we use throughout the text are often stated in one form, and then used in a different, but logically equivalent form. When we prove a theorem, we may prove even a third, logically equivalent form.

The first logical equivalences we will look at here are the negations of the our basic operations. We already looked at the negations of $\sim P$ and $P \vee Q$. Below we also look at negations of $P \wedge Q$, $P \longrightarrow Q$ and $P \longleftrightarrow Q$. Historically, (1.3) and (1.4) below are called *De Morgan's Laws*, but each basic negation is important. We now list these negations.

$$\sim(\sim P) \iff P \tag{1.2}$$

$$\sim(P \vee Q) \iff (\sim P) \wedge (\sim Q) \tag{1.3}$$

$$\sim(P \wedge Q) \iff (\sim P) \vee (\sim Q) \tag{1.4}$$

$$\sim(P \longrightarrow Q) \iff P \wedge (\sim Q) \tag{1.5}$$

$$\sim(P \longleftrightarrow Q) \iff [P \wedge (\sim Q)] \vee [Q \wedge (\sim P)]. \tag{1.6}$$

Fortunately, with a well chosen perspective these are intuitive. Recall that any statement R can also be read "R is true," while the negation asserts the original statement is false. For example $\sim R$ can be read as the statement "R is false," or a similar wording (such as "it is not the case that R"). Similarly the statement $\sim(P \vee Q)$ is the same as "'P or Q' is false." With that it is not difficult to see that for $\sim(P \vee Q)$ to be true requires both that P be false *and* Q be false. For a specific example, consider our earlier P and Q:

P: I will eat pizza
Q: I will drink soda
$P \vee Q$: I will eat pizza or I will drink soda
$\sim(P \vee Q)$: It is not the case that (either) I will eat pizza or I will drink soda
$(\sim P) \wedge (\sim Q)$: It is not the case that I will eat pizza, and it is not the case that I will drink soda

That these last two statements essentially have the same content, as stated in (1.3), should be intuitive. An actual *proof* of (1.3) is best given by truth tables, and can be found on page 15.

Next we consider (1.5). This states that $\sim(P \longrightarrow Q) \iff P \wedge (\sim Q)$. Now we can read $\sim(P \longrightarrow Q)$ as "it is not the case that $P \longrightarrow Q$," or "$P \longrightarrow Q$ is false." Recall that there was only one case for which we considered $P \longrightarrow Q$ to be false, which was the case that P was true but Q was false, which itself can be translated to $P \wedge (\sim Q)$. For our earlier example, the negation of the statement "if I eat pizza then I will drink soda" is the statement "I will eat pizza but (and) I will not drink soda." While this discussion is correct and may be intuitive, the actual proof (1.5) is by truth table:

P	Q	$P \to Q$	$\sim(P \to Q)$	P	$\sim Q$	$P \wedge (\sim Q)$
T	T	T	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	F	F
F	F	T	F	F	T	F

the same

We leave the proof of (1.6) by truth tables to the exercises. Recall that $P \longleftrightarrow Q$ states that we have P true if and only if we also have Q true, which we further translated as the idea that we cannot have P true without Q true, and cannot have Q true without P true. Now $\sim (P \longleftrightarrow Q)$ is the statement that $P \longleftrightarrow Q$ is false, which means that P is true and Q false, or Q is true and P false, which taken together form the statement $[P \wedge (\sim Q)] \vee [Q \wedge (\sim P)]$, as reflected in (1.6) above. For our example P and Q from before, $P \longleftrightarrow Q$ is the statement "I will at pizza if and only if I will drink soda," the negation of which is "I will eat pizza and not drink soda, or I will drink soda and not eat pizza."

Another intuitive way to look at these negations is to consider the question of exactly when is someone uttering the original statement lying? For instance, if someone states $P \wedge Q$ (or some English equivalent), when are they lying? Since they stated "P and Q," it is not difficult to see they are lying exactly when at least one of the statements P, Q is false, i.e., when P is false or Q is false,[18] i.e., when we can truthfully state $(\sim P) \vee (\sim Q)$. That is the kind of thinking one should employ when examining (1.4), that is $\sim (P \wedge Q) \iff (\sim P) \vee (\sim Q)$, intuitively.

Equivalent Forms of the Implication

In this subsection we examine two statements which are equivalent to $P \longrightarrow Q$. The first is more important conceptually, and the second is more important computationally. We list them both now before contemplating them further:

$$P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P) \tag{1.7}$$
$$P \longrightarrow Q \iff (\sim P) \vee Q. \tag{1.8}$$

We will combine the proofs into one truth table, where we compute $P \longrightarrow Q$, followed in turn by $(\sim Q) \longrightarrow (\sim P)$ and $(\sim P) \vee Q$.

P	Q	$P \to Q$	$\sim Q$	$\sim P$	$(\sim Q) \to (\sim P)$	$\sim P$	Q	$(\sim P) \vee Q$
T	T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F	F
F	T	T	F	T	T	T	T	T
F	F	T	T	T	T	T	F	T

the same

The form (1.7) is important enough that it warrants a name:

Definition *Given any implication $P \longrightarrow Q$, we call the (logically equivalent) statement $(\sim Q) \longrightarrow (\sim P)$ its* **contrapositive** *(and vice-versa, see below).*

In fact, note that the contrapositive of $(\sim Q) \longrightarrow (\sim P)$ would be $[\sim (\sim P)] \longrightarrow [\sim (\sim Q)]$, i.e., $P \longrightarrow Q$, so $P \longrightarrow Q$ and $(\sim Q) \longrightarrow (\sim P)$ are contrapositives *of each other.*

We have proved that $P \longrightarrow Q$, its contrapositive $(\sim Q) \longrightarrow (\sim P)$, and the other form $(\sim P) \vee Q$ are equivalent using the truth table above, but developing the intuition that these *should* be equivalent can require some effort. Some examples can help to clarify this.

$$\begin{aligned} P &: \text{I will eat pizza} \\ Q &: \text{I will drink soda} \\ P \longrightarrow Q &: \text{If I eat pizza, then I will drink soda} \\ (\sim Q) \longrightarrow (\sim P) &: \text{If I do not drink soda, then I will not eat pizza} \\ (\sim P) \vee Q &: \text{I will not eat pizza, or I will drink soda.} \end{aligned}$$

Perhaps more intuition can be found when Q is a more natural consequence of P. Consider the following P, Q combination which might be used by parents communicating to their children.

$$\begin{aligned} P &: \text{you leave your room messy} \\ Q &: \text{you get spanked} \\ P \longrightarrow Q &: \text{if you leave your room messy, then you get spanked} \\ (\sim Q) \longrightarrow (\sim P) &: \text{if you do not get spanked, then you do (did) not leave your room messy} \\ (\sim P) \vee Q &: \text{you do not leave your room messy, or you get spanked.} \end{aligned}$$

A mathematical example could look like the following (assuming x is a "real number," as discussed later in this text):

$$\begin{aligned} P &: x = 10 \\ Q &: x^2 = 100 \\ P \longrightarrow Q &: \text{if } x = 10, \text{ then } x^2 = 100 \\ (\sim Q) \longrightarrow (\sim P) &: \text{if } x^2 \neq 100, \text{ then } x \neq 10 \\ (\sim P) \vee Q &: x \neq 10 \text{ or } x^2 = 100. \end{aligned}$$

The contrapositive is very important because many theorems are given as implications, but are often used in their logically equivalent, contrapositive forms. However, it is equally important to avoid confusing $P \longrightarrow Q$ with either of the statements $P \longleftrightarrow Q$ or $Q \longrightarrow P$. For instance, in the second example above, the child may get spanked without leaving the room messy, as there are quite possibly other infractions which would result in a spanking. Thus leaving the room messy does not follow from being spanked, and leaving the room messy is not necessarily connected with the spanking by an "if and only if." In the last, algebraic example above, all the forms of the statement are true, but $x^2 = 100$ does not imply $x = 10$. Indeed, it is possible that $x = -10$. In fact, the correct bi-implication is $x^2 = 100 \longleftrightarrow [(x = 10) \vee (x = -10)]$.

Other Valid Equivalences

While negations and equivalent alternatives to the implication are arguably the most important of our valid logical equivalences, there are several others. Some are rather trivial, such as

$$P \wedge P \iff P \iff P \vee P. \tag{1.9}$$

Also rather easy to see are the "commutativities" of $\wedge, \vee$ and $\longleftrightarrow$:

$$P \wedge Q \iff Q \wedge P, \qquad P \vee Q \iff Q \vee P, \qquad P \longleftrightarrow Q \iff Q \longleftrightarrow P. \tag{1.10}$$

There are also associative rules. The latter was in fact a topic in the previous exercises:

$$P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R \tag{1.11}$$
$$P \vee (Q \vee R) \iff (P \vee Q) \vee R. \tag{1.12}$$

However, it is not so clear when we mix together $\vee$ and $\wedge$. In fact, these "distribute over each other" in the following ways:

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R), \tag{1.13}$$
$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R). \tag{1.14}$$

We prove the first of these distributive rules below, and leave the other for the exercises.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

the same

To show that this is reasonable, consider the following:

P : I will eat pizza;
Q : I will drink cola;
R : I will drink lemon-lime soda.

Then our logically equivalent statements become

$P \wedge (Q \vee R)$: I will eat pizza, and drink cola or lemon-lime soda;
$(P \wedge Q) \vee (P \wedge R)$: I will eat pizza and drink cola, or
I will eat pizza and drink lemon-lime soda.

Table 1.3, page 22 gives these and some further valid equivalences. It is important to be able to read these and, through reflection and the exercises, to be able to see the reasonableness of each of these. Each can be proved using truth tables.

For instance we can prove that $P \longleftrightarrow Q \iff (P \longrightarrow Q) \wedge (Q \longrightarrow P)$, justifying the choice of the double-arrow symbol $\longleftrightarrow$:

P	Q	$P \longleftrightarrow Q$	$P \longrightarrow Q$	$Q \longrightarrow P$	$(P \longrightarrow Q) \wedge (Q \longrightarrow P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

the same

This was discussed in Example 1.1.4 on page 7.

For another example of such a proof, we next demonstrate the following interesting equivalence:

$$P \longrightarrow (Q \wedge R) \iff (P \longrightarrow Q) \wedge (P \longrightarrow R)$$

P	Q	R	$Q \wedge R$	$P \longrightarrow (Q \wedge R)$	$P \longrightarrow Q$	$P \longrightarrow R$	$(P \longrightarrow Q) \wedge (P \longrightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

the same

This should be somewhat intuitive: if P is to imply $Q \wedge R$, that should be the same as P implying Q *and* P implying R. This equivalence will be (1.33), page 22. According to (1.34) below it, we can replace $\wedge$ with $\vee$ and get another valid equivalence.

Still one must be careful about declaring two statements to be equivalent. These are all ultimately intuitive, but intuition must be informed.[19] For instance, left to the exercises are some valid equivalences which may seem counter-intuitive. These are in fact left off of our Table 1.3 because they are somewhat obscure, but we include them here to illustrate that not all equivalences are transparent. Consider

$$(P \vee Q) \longrightarrow R \iff (P \longrightarrow R) \wedge (Q \longrightarrow R), \tag{1.15}$$
$$(P \wedge Q) \longrightarrow R \iff (P \longrightarrow R) \vee (Q \longrightarrow R). \tag{1.16}$$

Upon reflection one can see how these are reasonable. For instance, we can look more closely at (1.15) with the following P, Q and R:

P : I eat pizza,
Q : I eat chicken,
R : I drink cola.

Then the left and right sides of (1.15) become

$(P \vee Q) \longrightarrow R$: If I eat pizza or chicken, then I drink cola
$(P \longrightarrow R) \wedge (Q \longrightarrow R)$: If I eat pizza then I drink cola, and if I eat chicken then I drink cola.

In fact (1.16) is perhaps more difficult to see.

At the end of the chapter there will be an optional section for the reader interested in achieving a higher level of symbolic logic sophistication. That section is devoted to finding and proving valid equivalences (and implications as seen in the next section) without relying on truth tables. The technique centers on using a small number of established equivalences to rewrite compound statements into alternative, equivalent forms. With those techniques one can quickly prove (1.15) and (1.16), again without truth tables. It is akin to proving trigonometric identities, or the leap from memorizing single-digit multiplication tables and applying them to several-digit problems.

$$
\begin{aligned}
P \wedge P &\iff P \iff P \vee P && (1.17)\\
\sim(\sim P) &\iff P && (1.18)\\
\sim(P \vee Q) &\iff (\sim P) \wedge (\sim Q) && (1.19)\\
\sim(P \wedge Q) &\iff (\sim P) \vee (\sim Q) && (1.20)\\
\sim(P \longrightarrow Q) &\iff P \wedge (\sim Q) && (1.21)\\
\sim(P \longleftrightarrow Q) &\iff [P \wedge (\sim Q)] \vee [Q \wedge (\sim P)] && (1.22)\\
P \vee Q &\iff Q \vee P && (1.23)\\
P \wedge Q &\iff Q \wedge P && (1.24)\\
P \vee (Q \vee R) &\iff (P \vee Q) \vee R && (1.25)\\
P \wedge (Q \wedge R) &\iff (P \wedge Q) \wedge R && (1.26)\\
P \wedge (Q \vee R) &\iff (P \wedge Q) \vee (P \wedge R) && (1.27)\\
P \vee (Q \wedge R) &\iff (P \vee Q) \wedge (P \vee R) && (1.28)\\
P \longrightarrow Q &\iff (\sim P) \vee Q && (1.29)\\
P \longrightarrow Q &\iff (\sim Q) \longrightarrow (\sim P) && (1.30)\\
P \longrightarrow Q &\iff \sim [P \wedge (\sim Q)] && (1.31)\\
P \longleftrightarrow Q &\iff (\sim P) \longleftrightarrow (\sim Q) && (1.32)\\
P \longrightarrow (Q \wedge R) &\iff (P \longrightarrow Q) \wedge (P \longrightarrow R) && (1.33)\\
P \longrightarrow (Q \vee R) &\iff (P \longrightarrow Q) \vee (P \longrightarrow R) && (1.34)\\
(P \longrightarrow Q) \wedge (Q \longrightarrow P) &\iff P \longleftrightarrow Q && (1.35)\\
(P \longrightarrow Q) \wedge (Q \longrightarrow R) \wedge (R \longrightarrow P) &\iff (P \longleftrightarrow Q) \wedge (Q \longleftrightarrow R)\\
&\qquad \wedge (P \longleftrightarrow R) && (1.36)
\end{aligned}
$$

Table 1.3: Table of common valid logical equivalence.

For a glance at the process, we can look at such a proof of the equivalence of the contrapositive: $P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P)$. To do so, we require (1.29), that $P \longrightarrow Q \iff (\sim P) \vee Q$. The proof runs as follows:

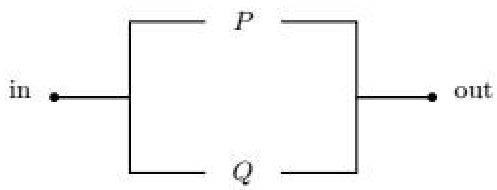
$$
\begin{aligned}
P \longrightarrow Q &\iff (\sim P) \vee Q\\
&\iff Q \vee (\sim P)\\
&\iff [\sim(\sim Q)] \vee (\sim P)\\
&\iff (\sim Q) \longrightarrow (\sim P).
\end{aligned}
$$

The first line used (1.29), the second commutativity (1.23), the third that $Q \iff \sim(\sim Q)$ (1.18), and the fourth used (1.29) again but with the part of "P" played by $(\sim Q)$ and the part of "Q" played by $(\sim P)$. This proof is not much more efficient than a truth table proof, but for (1.15) and (1.16) this technique of proofs without truth tables is much faster. However that technique assumes that the more primitive equivalences used in the proof are valid, and those are ultimately proved using truth tables. The extra section which develops such techniques, namely Section 1.6, is supplemental and not required reading for understanding sufficient symbolic logic to aid in developing the calculus. For that we need only up through Section 1.4.

Circuits and Logic

While we will not develop this next theory deeply, it is worthwhile to consider a short introduction. The idea is that we can model compound logic statements with electrical switching circuits.[20] When current is allowed to flow across a switch, the switch is considered "on" when the statement it represents has truth value T and current can flow through the switch, and "off" and not allowing current to flow through when the truth value is F. We can decide if the compound circuit is "on" or "off" based upon whether or not current could flow from one end to the other, based on whether the compound statement has truth value T or F. The analysis can be complicated if the switches are not necessarily independent (P is "on" when $\sim P$ is "off" for instance), but this approach is interesting nonetheless.

For example, the statement $P \vee Q$ is represented by a parallel circuit:



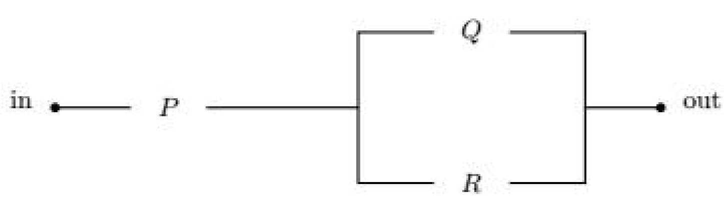
If either P or Q is on (T), then the current can flow from the "in" side to the "out" side of the circuit. On the other hand, we can represent $P \wedge Q$ by a series circuit:

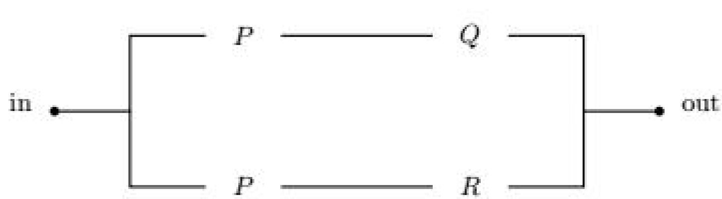
Of course $P \wedge Q$ is only true when both P and Q are true, and the circuit reflects this: current can flow exactly when both "switches" P and Q are "on."

It is interesting to see diagrams of some equivalent compound statements, illustrated as circuits. For instance, (1.27), i.e., the distributive-type equivalence

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$$

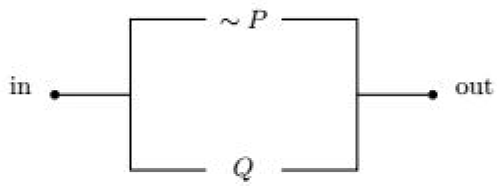
can be seen as the equivalence of the two cicruits below:





In both circuits, we must have P "on," and also either Q or R for current to flow. Note that in the second circuit, P is represented in two places, so it is either "on" in both places, or "off" in both places. Situations such as these can complicate analyses of switching circuits but this one is relatively simple.

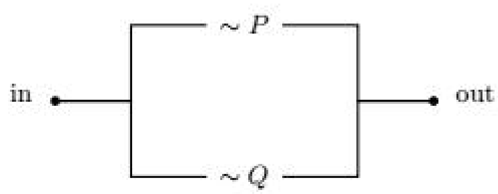
We can also represent negations of simple statements. To represent $\sim P$ we simply put "$\sim P$" into the circuit, where it is "on" if $\sim P$ is true, i.e., if P is false. This allows us to construct circuits for the implication by using (1.29), i.e., that $P \longrightarrow Q \iff (\sim P) \vee Q$:



We see that the only time the circuit does *not* flow is when P is true ($\sim P$ is false) and Q is false, so this matches what we know of when $P \longrightarrow Q$ is false. From another perspective, if P is true, then the top part of the circuit won't flow so Q must be true, for the whole circuit to be "on," or "true."

When negating a whole circuit it gets even more complicated. In fact, it is arguably easier to look at the original circuit and simply note when current will *not* flow. For instance, we know $\sim (P \wedge Q) \iff (\sim P) \vee (\sim Q)$, so we can construct $P \wedge Q$:

and note that it is *off* exactly when either P is off or Q is off. We then note that that is exactly when the circuit for $(\sim P) \vee (\sim Q)$ is *on*.

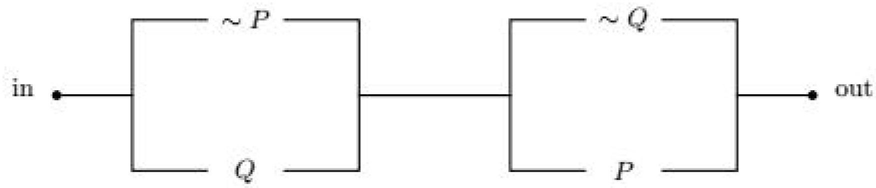


There are, in fact, electrical/mechanical means by which one can take a circuit and "negate" its truth value, for instance with relays or reverse-position switch levers, but that subject is more complicated than we wish to pursue here.

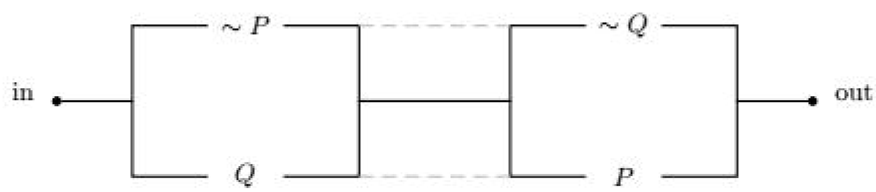
It is interesting to consider $P \longleftrightarrow Q$ as a circuit. It will be "on" if P and Q are both "on" or both "off," and the circuit will be "off" if P and Q do not match. Such a circuit is actually used commonly, such as for a room with two light switches for the same light. To construct such a circuit we note that

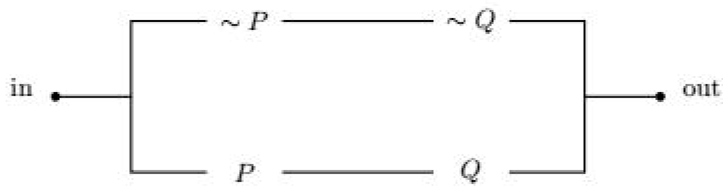
$$\begin{aligned} P \longleftrightarrow Q &\iff (P \longrightarrow Q) \wedge (Q \longrightarrow P) \\ &\iff [(\sim P) \vee Q] \wedge [(\sim Q) \vee P] \end{aligned}$$

We will use the last form to draw our diagram:



The reader is invited to study the above diagram to be convinced it represents $P \longleftrightarrow Q$, perhaps most easily in the sense that, "you can not have one (P or Q) without the other, but you can have neither." While the above diagram does represent $P \longleftrightarrow Q$ by the more easily diagrammed $[(\sim P) \vee Q] \wedge [(\sim Q) \vee P]$, it also suggests another equivalence, since the circuits below seems to be functionally equvialent. In the first, we can add two more wires to replace the "center" wire, and also switch the $\sim Q$ and P, since $(\sim Q) \vee P$ is the same as $P \vee (\sim Q)$:



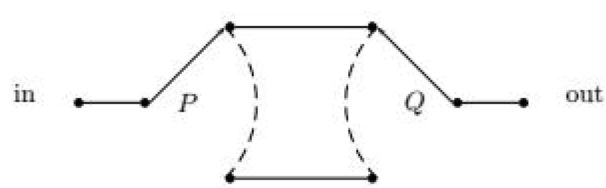


This circuit represents $[(\sim P) \wedge (\sim Q)] \vee [P \wedge Q]$, and so we have (as the reader can check)

$$P \longleftrightarrow Q \iff [(\sim P) \wedge (\sim Q)] \vee [P \wedge Q], \tag{1.37}$$

which could be added to our previous Table 1.3, page 22 of valid equivalences. It is also consistent with a more colloquial way of expressing $P \longleftrightarrow Q$, such as "neither or both."

Incidentally, the circuit above is used in applications where we wish to have two switches within a room which can both change a light (or other device) from on to off or vice versa. When switch P is "on," switch Q can turn the circuit on or off by matching P or being its negation. Similarly when P is "off." Mechanically this is accomplished with "single pole, double throw (SPDT)" switches.



In the above, the switch P is in the "up" position when P is 'true, and "down" when P is false. Similarly with Q.

Because there are many possible "mechanical" diagrams for switching circuits, reading and writing such circuits is its own skill. However, for many simpler cases there is a relatively easy connection to our symbolic logic.

The Statements $\mathcal{T}$ and $\mathcal{F}$

Just as there is a need for zero in addition, we have use for a symbol representing a statement which is always true, and for another symbol representing a statement which is always false. For convenience, we will make the following definitions:

Definition *Let $\mathcal{T}$ represent any compound statement which is a* **tautology**, *i.e., whose truth value is always* T. *Similarly, let $\mathcal{F}$ represent any compound statement which is a* **contradiction**, *i.e., whose truth value is always* F.

We will assume there is a universal $\mathcal{T}$ and a universal $\mathcal{F}$, i.e., statements which are respectively true regardless of any other statements' truth values, and false regardless of any other statements' truth values. In doing so, we consider any tautology to be logically equivalent to $\mathcal{T}$, and any contradiction similarly equivalent to $\mathcal{F}$.[21]

So, for any given $P_1 \cdots, P_n$, we have that $\mathcal{T}$ is exactly that statement whose column in the truth table consists entirely of T's, and $\mathcal{F}$ is exactly that statement whose column in the truth table consists entirely of F's. For example, we can write

$$P \vee (\sim P) \iff \mathcal{T}; \tag{1.38}$$
$$P \wedge (\sim P) \iff \mathcal{F}. \tag{1.39}$$

These are easily seen by observing the truth tables.

P	$\sim P$	$P \vee (\sim P)$	$P \wedge (\sim P)$
T	F	T	F
F	T	T	F

We see that $P \vee (\sim P)$ is always true, and $P \wedge (\sim P)$ is always false. Anything which is always true we will dub $\mathcal{T}$, and anything which is always false we will call $\mathcal{F}$. In the table above, the third column represents $\mathcal{T}$, and the last column represents $\mathcal{F}$.

From the definitions we can also eventually get the following.

$$P \vee \mathcal{T} \iff \mathcal{T} \tag{1.40}$$
$$P \wedge \mathcal{T} \iff P \tag{1.41}$$
$$P \vee \mathcal{F} \iff P \tag{1.42}$$
$$P \wedge \mathcal{F} \iff \mathcal{F}. \tag{1.43}$$

[1]In fact it is not difficult to see that all tautologies are logically equivalent. Consider the tautologies $P \vee (\sim P)$, $(P \longrightarrow Q) \longleftrightarrow [(\sim Q) \longrightarrow (\sim P)]$, and $R \longrightarrow R$. A truth table for all three must contain independent component statements P, Q, R, and the abridged version of the table would look like

P	Q	R	$P \vee (\sim P)$	$(P \longrightarrow Q) \longleftrightarrow [(\sim Q) \longrightarrow (\sim P)]$	$R \longrightarrow R$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

So when all possible underlying independent component statements are included, we see the truth table columns of these tautologies are indeed the same (all T's!). Similarly all contradictions are equivalent.

To demonstrate how one would prove these, we prove here the first two, (1.40) and (1.41), using a truth table. Notice that all entries for $\mathcal{T}$ are simply T:

P	$\mathcal{T}$	$P \vee \mathcal{T}$	$P \wedge \mathcal{T}$
T	T	T	T
F	T	T	F

Equivalence (1.40) is demonstrated by the equivalence of the second and third columns, while (1.41) is shown by the equivalence of the first and fourth columns. The others are left as exercises.

These are also worth reflecting upon. Consider the equivalence $P \wedge \mathcal{T} \iff P$. When we use $\wedge$ to connect P to a statement which is always true, then the truth of the compound statement only depends upon the truth of P. There are similar explanations for the rest of (1.40)–(1.43).

Some other interesting equivalences involving these are the following:

$$\mathcal{T} \longrightarrow P \iff P \tag{1.44}$$

$$P \longrightarrow \mathcal{F} \iff \sim P. \tag{1.45}$$

We leave the proofs of these for the exercises. These are in fact interesting to interpret. The first says that if a true statement implies P, that is the same as in fact having P. The second says that if P implies a false statement, that is the same as having $\sim P$, i.e., as having P false. Both types of reasoning are useful in mathematics and other disciplines.

If a statement contains *only* $\mathcal{T}$ or $\mathcal{F}$, then in fact that statement itself must be a tautology ($\mathcal{T}$) or a contradiction ($\mathcal{F}$). This is because there is only one possible combination of truth values. For instance, consider the statement $\mathcal{T} \longrightarrow \mathcal{F}$, which is a contradiction. One proof is in the table:

$\mathcal{T}$	$\mathcal{F}$	$\mathcal{T} \longrightarrow \mathcal{F}$
T	F	F

Since the component statement $\mathcal{T} \longrightarrow \mathcal{F}$ always has truth value F, it is a contradiction. Thus $\mathcal{T} \longrightarrow \mathcal{F} \iff \mathcal{F}$.

Quantifiers

In this section we introduce quantifiers, which form the last class of logic symbols we will consider in this text. To use quantifiers, we also need some notions and notation from set theory. This section introduces sets and quantifiers to the extent required for our study of calculus here. For the interested reader, Section 1.5 will extend this introduction, though even with that section we would be only just beginng to delve into these topics if studying them for their own sakes. Fortunately what we need of these topics for our study of calculus is contained in this section.

Sets

Put simply, a set is a collection of objects, which are then called *elements* or *members* of the set. We give sets names just as we do variables and statements. For an example of the notation, consider a set A defined by

$$A = \{2, 3, 5, 7, 11, 13, 17\}.$$

We usually define a particular set by describing or listing the elements between "curly braces" $\{\ \}$ (so the reader understands it is indeed a *set* we are discussing). The defining of A above was accomplished by a complete listing, but some sets are too large for that to be possible, let alone practical. As an alternative, the set A above can also be written

$$A = \{x \mid x \text{ is a prime number less than } 18\}.$$

The above equation is usually read, "A is the set of all x such that x is a prime number less than 18." Here x is a "dummy variable," used only briefly to describe the set.[45] Sometimes it is convenient to simply write

$$A = \{\text{prime numbers between 2 and 17, inclusive}\}.$$

(Usually "inclusive" is meant by default, so here we would include 2 and 17 as possible elements, if they also fit the rest of the description.) Of course there are often several ways of describing a list of items. For instance, we can replace "between 2 and 17, inclusive" with "less than 18," as before.

Often an ellipsis "$\cdots$" is used when a pattern should be understood from a partial listing. This is particularly useful if a complete listing is either impractical or impossible. For instance, the set B of integers from 1 to 100 could be written

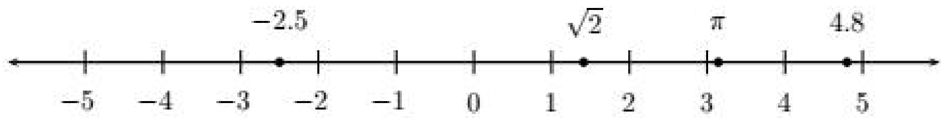
$$B = \{1, 2, 3, \cdots, 100\}.$$

To note that an object is in a set, we use the symbol $\in$. For instance we may write $5 \in B$, read "5 is an element of B." To indicate concisely that 5, 6, 7 and 8 are in B, we can write $5, 6, 7, 8 \in B$.

Just as we have use for zero in addition, we also define the *empty set*, or *null set* as the set which has no elements. We denote that set $\varnothing$. Note that $x \in \varnothing$ is always false, i.e.,

$$x \in \varnothing \iff \mathcal{F},$$

because it is impossible to find any element of any kind inside $\varnothing$. We will revisit this set repeatedly in the optional, more advanced Section 1.5.



The number line representing the set $\mathbb{R}$ of real numbers, with a few points plotted. On this graph, the hash marks fall at the integers.

Of course for calculus we are mostly interested in sets of numbers. While not the most important, the following three sets will occur from time to time in this text:

Natural Numbers[46]: $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, (1.67)

Integers: $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, (1.68)

Rational Numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; (p, q \in \mathbb{Z}) \wedge (q \neq 0) \right\}$. (1.69)

Here we again use the ellipsis to show that the established pattern continues forever in each of the cases $\mathbb{N}$ and $\mathbb{Z}$. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are examples of *infinite* sets, i.e., sets that do not have a finite number of elements. The rational numbers are those which are *ratios* of integers, except that division by zero is not allowed, for reasons we will consider later.[47]

For calculus the most important set is the set $\mathbb{R}$ of *real numbers*, which cannot be defined by a simple listing or by a simple reference to $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$. One intuitive way to describe the real numbers is to consider the horizontal *number line*, where geometric points on the line are represented by their *displacements* (meaning distances, but counted as positive if to the right and negative if to the left) from a fixed point, called the *origin* in this context. That fixed point is represented by the number 0, since the fixed point is a displacement of zero units from itself. In Figure 1.2 the number line representation of $\mathbb{R}$ is shown. Hash marks at convenient intervals are often included. In this case, they are at the integers. The arrowheads indicate the number line is an actual line and thus infinite in both directions. The points -2.5 and 4.8 on the graph are not integers, but are rational numbers, since they can be written $-25/10 = -5/2$, and $48/10 = 24/5$, respectively. The points $\sqrt{2}$ and π are real, but not rational, and so are called *irrational*. To summarize,

Definition *The set of all* **real** *numbers is the set $\mathbb{R}$ of all possible displacements, to the right or left, of a fixed point 0 on a line. If the displacement is to the right, the number is the positive distance from 0. If to the left, the number is the negative of the distance from 0.*[48]

Thus

$$\mathbb{R} = \{\text{displacements from 0 on the number line}\}. \qquad (1.70)$$

This is not a rigorous definition, not least because "right" and "left" require a fixed perspective. Even worse, the definition is really a kind of "circular reasoning," since we are effectively defining the number line in terms of $\mathbb{R}$, and then defining $\mathbb{R}$ in terms of (displacements on) the number line. We will give a more rigorous definition in Chapter 2 for the interested reader. For now this should do, since the number line is a simple and intuitive image.

Quantifiers

The three quantifiers used by nearly every professional mathematician are as follow:

universal quantifier: $\forall$, *read, "for all," or "for every;"*
existential quantifier: $\exists$, *read, "there exists;"*
uniqueness quantifier: $!$, *read, "unique."*

The first two are of equal importance, and far more important than the third which is usually only found after the second. Quantified statements are usually found in forms such as:

$(\forall x \in S)P(x)$, i.e., for all $x \in S$, $P(x)$ is true;
$(\exists x \in S)P(x)$, i.e., there exists an $x \in S$ such that $P(x)$ is true;
$(\exists! x \in S)P(x)$, i.e., there exists a unique (exactly one) $x \in S$ such that $P(x)$ is true.

Here S is a set and $P(x)$ is some statement about x. The meanings of these quickly become straightforward. For instance, consider

$$(\forall x \in \mathbb{R})(x + x = 2x): \text{ for all } x \in \mathbb{R},\ x + x = 2x;$$
$$(\exists x \in \mathbb{R})(x + 2 = 2): \text{ there exists (an) } x \in \mathbb{R} \text{ such that } x + 2 = 2;$$
$$(\exists! x \in \mathbb{R})(x + 2 = 2): \text{ there exists a unique } x \in \mathbb{R} \text{ such that } x + 2 = 2.$$

All three quantified statements above are true. In fact they are true under any circumstances, and can thus be considered tautologies. Unlike unquantified statements P, Q, R, etc., from our first three sections, a quantified statement is either true always or false always, and is thus, for our purposes, equivalent to either $\mathcal{T}$ or $\mathcal{F}$. Each has to be analyzed on its face, based upon known mathematical principles; we do not have a brute-force mechanism analogous to truth tables to analyze these systematically.[49] For a couple more short examples, consider the following cases from algebra which should be clear enough:

$$(\forall x \in \mathbb{R})(0 \cdot x = 0) \iff \mathcal{T};$$
$$(\exists x \in \mathbb{R})(x^2 = -1) \iff \mathcal{F}.$$

The optional advanced section shows how we can still find equivalent or implied statements from quantified statements in many circumstances.

Statements with Multiple Quantifiers

Many of the interesting statements in mathematics contain more than one quantifier. To illustrate the mechanics of multiply quantified statements, we will first turn to a more worldly setting. Consider the following sets:

$$M = \{\text{men}\},$$
$$W = \{\text{women}\}.$$

In other words, M is the set of all men, and W the set of all women. Consider the statement[50]

$$(\forall m \in M)(\exists w \in W)[w \text{ loves } m]. \tag{1.71}$$

Set to English, (1.71) could be written, "for every man there exists a woman who loves him."[51] So if (1.71) is true, we can in principle arbitrarily choose a man m, and then know that there is a woman w who loves him. It is important that the man m was quantified *first*. A common syntax that would be used by a logician or mathematician would be to say here that, once our choice of a man is *fixed*, we can in principle find a woman who loves him. Note that (1.71) allows that different men may need different women to love them, and also that a given man may be loved by more than (but not less than) one woman.

Alternatively, consider the statement

$$(\exists w \in W)(\forall m \in M)[w \text{ loves } m]. \tag{1.72}$$

A reasonable English interpretation would be, "there exists a woman who loves every man." Granted that is a summary, for the word-for-word English would read more like, "there exists a woman such that, for every man, she loves him." This says something very different from (1.71), because that earlier statement does not assert that we can find a woman who, herself, loves every man, but that for each man there is a woman who loves him.[52]

We can also consider the statement

$$(\forall m \in M)(\forall w \in W)[w \text{ loves } m]. \tag{1.73}$$

This can be read, "for every man and every woman, the woman loves the man." In other words, every man is loved by every woman. In this case we can reverse the order of quantification:

$$(\forall w \in W)(\forall m \in M)[w \text{ loves } m]. \tag{1.74}$$

In fact, if the two quantifiers are the same type—both universal or both existential—then the order does not matter. Thus

$$(\forall m \in M)(\forall w \in W)[w \text{ loves } m] \iff (\forall w \in W)(\forall m \in M)[w \text{ loves } m],$$
$$(\exists m \in M)(\exists w \in W)[w \text{ loves } m] \iff (\exists w \in W)(\exists m \in M)[w \text{ loves } m].$$

In both representations in the existential statements, we are stating that there is at least one man and one woman such that she loves him. In fact that above equivalence is also valid if we replace $\exists$ with $\exists!$, though it would mean then that there is exactly one man and exactly one woman such that the woman loves the man, but we will not delve too deeply into uniqueness here.

Note that in cases where the sets are the same, we can combine two similar quantifications into one, as in

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = y + x] \iff (\forall x, y \in \mathbb{R})[x + y = y + x]. \tag{1.75}$$

Similarly with existence.

However, we repeat the point at the beginning of the subsection, which is that the order does matter if the types of quantification are different.

For another, short example which is algebraic in nature, consider

$$(\forall x \in \mathbb{R})(\exists K \in \mathbb{R})(x = 2K). \qquad \text{(True.)} \tag{1.76}$$

This is read, "for every $x \in \mathbb{R}$, there exists $K \in \mathbb{R}$ such that $x = 2K$." That $K = x/2$ exists (and is actually unique) makes this true, while it would be false if we were to reverse the order of quantification:

$$(\exists K \in \mathbb{R})(\forall x \in \mathbb{R})(x = 2K). \qquad \text{(False.)} \tag{1.77}$$

Statement (1.77) claims (erroneously) that there exists $K \in \mathbb{R}$ so that, for every $x \in \mathbb{R}$, $x = 2K$. That is impossible, because no value of K is half of *every* real number x. For example the value of K which works for $x = 4$ is not the same as the value of K which works for $x = 100$.

Detour: Uniqueness as an Independent Concept

We will have occasional statements in the text which include uniqueness. However, most of those will not require us to rewrite the statements in ways which require actual manipulation of the uniqueness quantifier. Still, it is worth noting a couple of interesting points about this quantifier.

First we note that uniqueness can be formulated as a separate concept from existence, interestingly instead requiring the universal quantifier.

Definition **Uniqueness** *is the notion that if $x_1, x_2 \in S$ satisfy the same particular statement $P(\ )$, then they must in fact be the same object. That is, if $x_1, x_2 \in S$ and $P(x_1)$ and $P(x_2)$ are true, then $x_1 = x_2$. This may or may not be true, depending upon the set S and the statement $P(\ )$.*

Note that there is the vacuous case where nothing satisfies the statement $P(\ )$, in which case the uniqueness of any such hypothetical object is proved but there is actually no existence. Consider the following, symbolic representation of the uniqueness of an object x which satisfies $P(x)$:[53]

$$(\forall x, y \in S)[(P(x) \wedge P(y)) \longrightarrow x = y]. \tag{1.78}$$

Finally we note that a proof of a statement such as $(\exists! x \in S)P(x)$ is thus usually divided into two separate proofs:

(1) *Existence:* $(\exists x \in S)P(x)$;

(2) *Uniqueness:* $(\forall x, y \in S)[(P(x) \wedge P(y)) \longrightarrow x = y]$.

For example, in the next chapter we rigorously, axiomatically define the set of real numbers $\mathbb{R}$. One of the axioms[54] defining the real numbers is the existence of an additive identity:

$$(\exists z \in \mathbb{R})(\forall x \in \mathbb{R})(z + x = x). \tag{1.79}$$

The above statement indeed says that any two elements $x, y \in S$ which both satisfy P must be the same. Note that we use a single arrow here, because the statement between the brackets [ ] is not likely to be a tautology, but may be true for enough cases for the entire quantified statement to be true. Indeed, the symbols $\Longrightarrow$ and $\Longleftrightarrow$ belong *between* quantified statements, not *inside* them.

Recall that an *axiom* is an assumption, usually self-evident, from which we can logically argue towards theorems. Axioms are also known as *postulates*. If we attempt to argue only using "pure logic" (as a mathematician does when developing theorems, for instance), it eventually becomes clear that we still need to make some assumptions because one can not argue "from nothing." Indeed, some "starting points" from which to argue towards the conclusions are required. These are then called axioms.

The word "axiomatic" is often used colloquially to mean clearly evident and therefore not requiring proof. In

In fact it follows quickly that such a "z" must be unique, so we have

$$(\exists! z \in \mathbb{R})(\forall x \in \mathbb{R})(z + x = x). \tag{1.80}$$

To prove (1.80), we need to prove (1) existence, and (2) uniqueness. In this setting, the existence is an axiom so there is nothing to prove. We turn then to the uniqueness. A proof is best written in prose, but it is based upon proving that the following is true:

$$(\forall z_1, z_2 \in \mathbb{R})[(z_1 \text{ an additive identity}) \wedge (z_2 \text{ an additive identity}) \longrightarrow z_1 = z_2].$$

Now we prove this. Suppose z_1 and z_2 are additive identities, i.e., they can stand in for z in (1.79), which could also read $(\exists z \in \mathbb{R})(\forall x \in \mathbb{R})(x = z + x)$. Note the order there, where the identity z (think "zero") is placed on the left of x in the equation $x = z + x$. So, assuming z_1, z_2 are additive identities, we have:

$$\begin{aligned} z_1 &= z_2 + z_1 && \text{(since } z_2 \text{ is an additive identity)} \\ &= z_1 + z_2 && \text{(since addition is commutative—order is irrelevant)} \\ &= z_2 && \text{(since } z_1 \text{ is an additive identity).} \end{aligned}$$

This argument showed that if z_1 and z_2 are any real numbers which act as additive identities, then $z_1 = z_2$. In other words, if there are any additive identities, there must be only one. Of course, assuming its existence we call that unique real number *zero*. (It should be noted that the commutativity used above is another axiom of the real numbers. We will list fourteen in all.)

The distinction between existence and uniqueness of an object with some property P is often summarized as follows:

(1) Existence asserts that there is at least one such object.

(2) Uniqueness asserts that there is at most one such object.

If both hold, then there is exactly one such object.

Negating Universally and Existentially Quantified Statements

For statements with a single universal or existential quantifier, we have the following negations.

$$\sim [(\forall x \in S)P(x)] \iff (\exists x \in S)[\sim P(x)], \tag{1.81}$$
$$\sim [(\exists x \in S)P(x)] \iff (\forall x \in S)[\sim P(x)], \tag{1.82}$$

The left side of (1.81) states that it is not the case that $P(x)$ is true for all $x \in S$; the right side states that there is an $x \in S$ for which $P(x)$ is false. We could ask when is it a lie that for all x, $P(x)$ is true? The answer is when there is an x for which $P(x)$ is false, i.e., $\sim P(x)$ is true.

The left side of (1.82) states that it is not the case that there exists an $x \in S$ so that $P(x)$ is true; the right side says that $P(x)$ is false for all $x \in S$. When is it a lie that there is an x making $P(x)$ true? When $P(x)$ is false for all x.

Thus when we negate such a statement as $(\forall x)P(x)$ or $(\exists x)P(x)$, we change $\forall$ to $\exists$ or vice-versa, and negate the statement after the quantifiers.

Example *Negate* $(\forall x \in S)[P(x) \longrightarrow Q(x)]$.

Solution: *We will need* (1.21), page 22, *namely* $\sim (P \longrightarrow Q) \iff P \wedge (\sim Q)$.

$$\begin{aligned} \sim [(\forall x \in S)(P(x) \longrightarrow Q(x))] &\iff (\exists x \in S)[\sim (P(x) \longrightarrow Q(x))] \\ &\iff (\exists x \in S)[P(x) \wedge (\sim (Q(x)))]. \end{aligned}$$

The above example should also be intuitive. To say that it is not the case that, for all $x \in S$, $P(x) \longrightarrow Q(x)$ is to say there exists an x so that we do have $P(x)$, but not the consequent $Q(x)$.

Example *Negate* $(\exists x \in S)[P(x) \wedge Q(x)]$.

<u>*Solution*</u>: *Here we use* $\sim (P \wedge Q) \iff (\sim P) \vee (\sim Q)$, *so we can write*

$$\sim [(\exists x \in S)(P(x) \wedge Q(x))] \iff (\forall x)[(\sim P(x)) \vee (\sim (Q(x)))].$$

This last example shows that if it is not the case that there exists an $x \in S$ so that $P(x)$ and $Q(x)$ are both true, that is the same as saying that for all x, either $P(x)$ is false or $Q(x)$ is false.

Negating Statements Containing Mixed Quantifiers

Here we simply apply (1.81) and (1.82) two or more times, as appropriate. For a typical case of a statement first quantified by $\forall$, and then be $\exists$, we note that we can group these as follows:[55]

$$(\forall x \in R)(\exists y \in S)P(x, y) \iff (\forall x \in R)[(\exists y \in S)P(x, y)].$$

(Here "R" is another set, not to be confused with the set of real numbers $\mathbb{R}$.) Thus

$$\begin{aligned}\sim [(\forall x \in R)(\exists y \in S)P(x, y)] &\iff \sim \{(\forall x \in R)[(\exists y \in S)P(x, y)]\} \\ &\iff (\exists x \in R)\{\sim [(\exists y \in S)P(x, y)]\} \\ &\iff (\exists x \in R)(\forall y \in S)[\sim P(x, y)].\end{aligned}$$

Ultimately we have, in turn, the $\forall$'s become $\exists$'s, the $\exists$'s become $\forall$'s, the variables are quantified in the same order as before, and finally the statement P is replaced by its negation $\sim P$. The pattern would continue no matter how many universal and existential quantifiers arise. (The uniqueness quantifier is left for the exercises.) To summarize for the case of two quantifiers,

$$\sim [(\forall x \in R)(\exists y \in S)P(x, y)] \iff (\exists x \in R)(\forall y \in S)[\sim P(x, y)] \tag{1.83}$$
$$\sim [(\exists x \in R)(\forall y \in S)P(x, y)] \iff (\forall x \in R)(\exists y \in S)[\sim P(x, y)]. \tag{1.84}$$

Example *Consider the following statement, which is false:*

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[xy = 1].$$

One could say that the statement says every real number x has a real number reciprocal y. This is false, but before that is explained, we compute the negation which must be true:

$$\sim [(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)] \iff (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(xy \neq 1).$$

Indeed, there exists such an x, namely $x = 0$, such that $xy \neq 1$ for all y.

In the above, we borrowed one of the many convenient mathematical notations for the negations of various symbols. Some common negations follow:

$$\begin{aligned}\sim (x = y) &\iff x \neq y, \\ \sim (x < y) &\iff x \geq y, \\ \sim (x \leq y) &\iff x > y, \\ \sim (x \in S) &\iff x \notin S.\end{aligned}$$

Of course we can negate both sides of any one of these and get, for example, $x \in S \iff \sim (x \notin S)$. Reading one of these backwards, we can have $\sim (x \geq y) \iff x < y$.

Possible Questions

2 Mark Questions:

1. Give the symbolic form of the statement "every book with blue cover is a mathematics book"
2. Define subject with example.
3. Define contingency
4. Define Essential quantifier with example.
5. Define Predicates with examples.

6 Mark Questions:

1. Show that the following is a tautology implication $P\to(Q\to R) \Rightarrow (P\to Q)\to(P\to R)$
2. Let $Q(x,y,z)$ be the set "$x+y=z$".what are the truth values of the set

   (i) $\forall x\forall y\exists z\, Q(x,y,z)$ (ii) $\exists z\forall x\forall y\, Q(x,y,z)$ (iii) $\forall x\forall y\forall z\, Q(x,y,z)$
3. Show that $P\to(Q\to P) \Leftrightarrow \neg P\to(P\to Q)$
4. Let $Q(x, y)$ denote "$x + y=0$". what are the truth value of the quantification

   (i) $\exists y\forall x\, Q(x,y)$ (ii) $\forall x\exists y\, Q(x,y)$
5. Show that $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$ (use only the laws)
6. Prove that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H\to(A \wedge \neg B)$ and $(A \wedge \neg B) \to (R\vee S)$.
7. Show that the following are implication.
   i) $P \Rightarrow (Q \to P)$.
   ii) $(P \to (Q \to R)) \Rightarrow (P \to Q) \to (P \to R)$
8. Use quantifiers to express each of the following:
   (i) All humming birds are richly colored
   (ii) No large birds line on honey
   (iii) Birds that do not line honey are dull in color
   (iv) Humming birds are small

9. Show that the following premises are Inconsistent.

   i) If Jack misses many classes through illness, he fails in school.

   ii) If jack fails in school, then he is uneducated.

   iii) If jack reads a lot of books, then he is not uneducated.

   iv) Jack misses many classes through illness and reads a lot of books.

10. Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

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Subject: Liogic and Sets **Subject Code: 18MMU103**
Class : I B.Sc Mathematics **Semester : I**

UNIT -II
PART A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
{ "and", "not"} is called a ---- set	functionally complete	minimal functionally complete	maximal functionally complete	complete	minimal functionally complete
{"and", "or", "not"} is called a ---- set	functionally complete	functionally incomplete	complete	functional	functionally complete
For two variables the number of possible assignment of truth values is	2	2^n	n	2n	2^n
The substitution instance of a tautology is a	tautology	contradiction	identically false	all of these	tautology
Equivalence is a ----- relation	reflexive	symmetric	transitive	asymmetric	symmetric
A statement "A" is said to imply another statement "B" if ---- is a tautology	if A then B	if B then A	if (not A) then B	if (not B) then A	if A then B
The dual of "and" is	"and"	"or"	"not and"	"not or"	"or"
The dual of " or" is	"and"	"or"	"not and"	"not or"	"and"
The dual of NANDis	NAND	NOR	"or "	"and"	NOR
The dual of NOR is	NAND	NOR	"or "	"and"	NAND
The other name for pcnf is	product of sums canonical form	sum of products canonical form	product of products canonical form	sum of sums canonical form	product of sums canonical form
The other name for pdnf is	product of sums canonical form	sum of products canonical form	product of products canonical form	sum of sums canonical form	sum of products canonical form
The minterms are	P and Q	not P and Q	P and Q, not P and Q	none of these	P and Q, not P and Q
The max terms are	P or Q	P or not Q	not P or P	P or Q , P or not Q	P or Q , P or not Q
The statement B follows logically from the statement A if only if	if A then B is a tautology	if A then B is a contradition	if B then A is a tautology	if B then A is a contradiction	if A then B is a tautology
The Rule P in the inference is used to indicate the introduction of the	Premise	conclusion	contradiction	none	Premise
Symbolize the expression "Every student in this class has studied logic" by taking p(x) : x studied logic, q(x) : x is in this class	(ɏx)(if q(x) then p(x))	(ɏx)(if p(x) then q(x))	(ɏx)(if not q(x) then p(x))	(ɏx)(if q(x) then not p(x))	(ɏx)(if q(x) then p(x))
Symbolize the statement "This cricket ball is white"	W(b)	B(w)	W(b.c)	C(b,w)	W(b)
Symbolize the statement "Jack is taller than Smith"	T(j,s)	T(s,j)	J(s,t)	J(t,s)	T(j,s)
Symbolize the statement " Canada is to the north of United States"	N(c,s)	N(s,c)	S(n,c)	S(c,n)	N(c,s)
Universal Quantifier is	For all x	For some x	there exists x	there exists no x	For all x
Essential Quantifier is	For all x	For some x	there exists x	there exists no x	there exists x
In the statement "The cricket ball is white", the predicate is	white	ball	cricket ball	none	white
In the statement "Every mammal is warm blooded", the predicate is	warm blooded	mammal	warm	none	warm blooded
In the statement "Every mammal is warm blooded", the object is	warm blooded	mammal	warm	none	mammal
Use quantifiers to say that √3 is not a rational number	negation (there exists x a rational number)(x^2=3)	(there exists x a rational number)(x^2=3)	negation (there exists x a rational number)(x^2≠3)	none	negation (there exists x a rational number)(x^2=3)
Existential Specification is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	(there exists x)(A(x)) implies A(y)
Existential Generalisation is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	A(x) implies (there exists y)(A(y))
Universal Specification is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	(For all x) (A(x)) implies A(y)
Universal Generalisation is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	A(x) implies (For all y)(A(y))
Symbolize the statement" Every mammal is warm blooded"	(For all x) (M(x))→ W(x))	(there exists x) (M(x))→ W(x))	(For all x) (W(x))→ M(x))	(there exists x) (W(x))→ M(x))	(For all x) (M(x))→ W(x))
"x is shorter than y" can be symbolized as	G(x,y)	X(g)	Y(g)	G(y,x)	G(x,y)
The painting is red can be symbolized as	R(p)	P(r)	S(p,r)	R and P	R(p)
"Zaheer is a bowler and the ball is white" can be symbolized as	B(z) and W(b)	B(z) or W(b)	not B(z) and W(b)	not B(z) or W(b)	B(z) and W(b)
The rules used to check the validity of the premises is	US,UG	ES,EG	both	none	both
The statement form pv(~p) is a……………………..	Satisfiable	Unsatisfiable	Tautology	Invalid	Tautology
Let p and q be statements given by "p →q". Then q is called	hypothesis	conclusion	TRUE	FALSE	conclusion
The statement form p^(~p) is a…………	contradiction	Unsatisfiable	Tautology	Invalid	contradiction
If p and q are statement variables, the conditional of q by p is given by ……	~p → ~ q	p → ~ q	~p → q	p → q	p → q
Let p and q be statements given by "p →q". Then p is called………	hypothesis	conclusion	TRUE	FALSE	hypothesis
The statement (p → r) ∧(q → r) is equivalent to………..	p ∨q →~ r	p ∨q → r	p ∨~ q → r	~p ∨q → r	p ∨q → r
The Negation of a Conditional Statement p →q is given by ……	p ∧ ~ q	~p ∧ ~ q	p ∨ ~ q	p ∧q	p ∧ ~ q
Given statement variables p and q, the biconditional of p and q is given by …	p«~q	p→q	~p«q	p«q	p«q
The inverse of "if p then q" is …………	if ~ p then ~ q	if ~ p then ~ q	if ~ p then ~ q	if ~ p then ~ q	if ~ p then ~ q
The Some men are clever can be symbolized as	(there exists x)(M(x)→C(x))	(for all x)(M(x)→C(x))	(there exists x)(M(x) or C(x))	(for all x)(M(x) or C(x))	(there exists x)(M(x)→C(x))

UNIT-III

SYLLABUS

Sets: Subsets, Set operations and the laws of set theory and Venn diagrams. Examples of finite and infinite sets.

Sets

In this section we introduce set theory in its own right. We also apply the earlier symbolic logic to the theory of sets (rather than vice-versa). We also approach set theory visually and intuitively, while simultaneously introducing all the set-theoretic notation we will use throughout the text. To begin we make the following definition:

Definition *A* **set** *is a well-defined collection of objects.*

By well-defined, we mean that once we define the set, the objects contained in the set are totally determined, and so any given object is either in the set or not in the set. We might also note that in a sense a set is defined (or *determined*) by its elements; sets which are different collections of elements are different sets, while sets with exactly the same elements are the same set. We can also define equality by means of quantifiers:

Definition *Given two sets A and B, we defined the statement $A = B$ as being equivalent to the statement $(\forall x)[(x \in A) \longleftrightarrow (x \in B)]$:*

$$A = B \iff (\forall x)[(x \in A) \longleftrightarrow (x \in B)]. \tag{1.85}$$

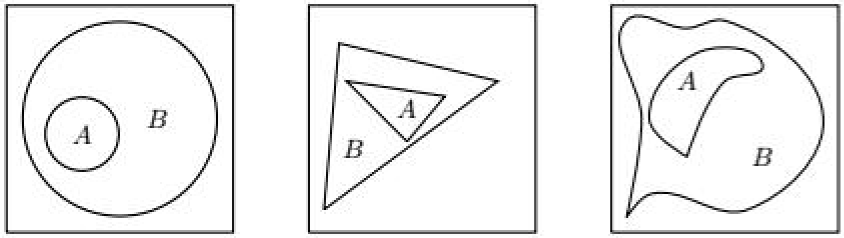
If we allow ourselves to understand that x is quantified universally (that is, we assume "$(\forall x)$" is understood) unless otherwise stated, we can write, instead of $A = B$, that $x \in A \iff x \in B$.

When we say a set is well-defined we also mean that once defined the set is *fixed*, and does not change. If elements can be listed in a table (finite or otherwise),[57] then the order we list the elements is not relevant; sets are defined by exactly which objects are elements, and which are not. Moreover, it is also irrelevant if objects are listed more than once in the set, such as when we list $\mathbb{Q} = \{x \mid x = p/q,\ p, q \in \mathbb{Z},\ q \neq 0\}$. In that definition $2 = 2/1 = 4/2 = 6/3$ is "listed" infinitely many times, but it is simply one element of the set of rational numbers $\mathbb{Q}$. While it actually is possible to "list" the elements of $\mathbb{Q}$ if we allow for the elipsis $(\cdots)$, it is more practical to describe the set, as we did, using some *defining property* of its elements (here they were ratios of integers, without dividing by zero), as long as it is exactly those elements in the set—no more and no fewer—which share that property. One usually uses a "dummy variable" such as x and then describes what properties all such x in the set should have. We could have just as easily used z or any other variable.[58]

Subsets and Set Equality

When all the elements of a set A are also elements of another set B, we say A is a *subset* of B. To express this in set notation, we write $A \subseteq B$. In this case we can also take another perspective, and say B is a *superset* of A, written $B \supseteq A$. Both symbols represent types of *set inclusions*, i.e., they show one set is contained in another.

A useful graphical device which can illustrate the notion that $A \subseteq B$ and other set relations is the *Venn Diagram*, as in Figure 1.3. There we see a visual representation of what it means for $A \subseteq B$. The sets are represented by enclosed areas in which we imagine the elements reside. In each representation given in Figure 1.3, all the elements inside A are also inside B.



Three possible Venn Diagrams illustrating $A \subseteq B$. (Note that in the first figure, for example, B is the set of all elements within the interior of the larger circle.) What is important is that all elements of A are necessarily contained in B as well. We do not necessarily know "where" in A are the elements of A, except that they are in the area which is marked by A. Since the area in A is also in B, we know the elements of A must also be contained in B in the illustrations above.

Using symbolic logic, we can *define* subsets, and the notation, as follows:

$$A \subseteq B \iff (\forall x)(x \in A \longrightarrow x \in B). \tag{1.86}$$

The role of the implication which is the main feature of (1.86) should seem intuitive. Perhaps less intuitive are some of the statements which are therefore logically equivalent to (1.86):

$$\begin{aligned} A \subseteq B &\iff (\forall x)(x \in A \longrightarrow x \in B) \\ &\iff (\forall x)\left[(\sim (x \in A)) \vee (x \in B)\right] \\ &\iff (\forall x)\left[(x \notin A) \vee (x \in B)\right], \end{aligned}$$

which uses the fact that $P \longrightarrow Q \iff (\sim P) \vee Q$, and

$$\begin{aligned} A \subseteq B &\iff (\forall x)\left[(\sim (x \in B)) \longrightarrow (\sim (x \in A))\right] \\ &\iff (\forall x)\left[(x \notin B) \longrightarrow (x \notin A)\right] \end{aligned}$$

which uses the contrapositive $P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P)$. Note that we used the shorthand notation $\sim (x \in A) \iff x \notin A$. With the definition (1.86) we can quickly see two more, technically interesting facts about subsets:

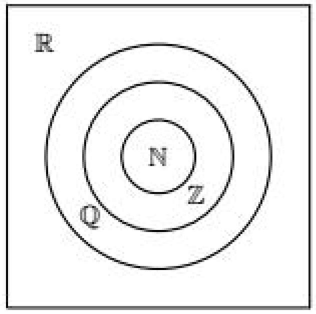
Theorem *For any sets A and B, the following hold true:*

$$A \subseteq A, \qquad \textit{and} \tag{1.87}$$
$$A = B \iff (A \subseteq B) \wedge (B \subseteq A). \tag{1.88}$$

Now we take a moment to remind ourselves of what is meant by *theorem:*

Definition : *A* **theorem** *is a statement which we know to be true because we have a proof of it. We can therefore accept it as a tautology.*

A theorem's scope may be very limited (the above theorem only applies to sets and subsets as we have defined them.) Furthermore, a theorem's scope and "truth" depends upon the axiomatic system upon which it rests, such the definitions we gave our symbolic logic symbols (which might not have always been completely obvious to the novice, as in our definitions of "$\vee$" and "*longrightarrow*"). For another example there is Euclidean geometry, the theroems of which



Venn Diagram illustrating $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

rest upon Euclid's Postulates (or axioms, or original assumptions), while other geometric systems begin with different postulates.

Nonetheless once we have the definitions and postulates one can say that a theorem is a statement which is always true (demonstrated by some form of proof), and in fact therefore equivalent to $\mathcal{T}$ (introduced on page 26). We will use that fact in the proof of (1.87), but for (1.88) we will instead demonstrate the validity of the equivalence ($\Longleftrightarrow$). For the first statement's proof, we have

$$A \subseteq A \iff (\forall x)[(x \in A) \longrightarrow (x \in A)] \iff \mathcal{T}.$$

Note that the above proof is based upon the fact that $P \longrightarrow P$ is a tautology (i.e., equivalent to $\mathcal{T}$). A glance at a Venn Diagram with a set A can also convince one of this fact, that any set is a subset of itself. For the proof of (1.88) we offer the following:

$$\begin{aligned} A = B &\iff (\forall x)[(x \in A) \longleftrightarrow (x \in B)] \\ &\iff (\forall x)[((x \in A) \longrightarrow (x \in B)) \wedge ((x \in B) \longrightarrow (x \in A))] \\ &\iff \big[(\forall x)[(x \in A) \longrightarrow (x \in B)\big] \wedge \big[(\forall x)[(x \in B) \longrightarrow (x \in A)\big] \\ &\iff (A \subseteq B) \wedge (B \subseteq A), \text{ q.e.d.}^{59} \end{aligned}$$

A consideration of Venn diagrams also leads one to believe that for all the area in A to be contained in B and vice versa, it must be the case that $A = B$. That $A = B$ implies they are mutual subsets is perhaps easier to see.

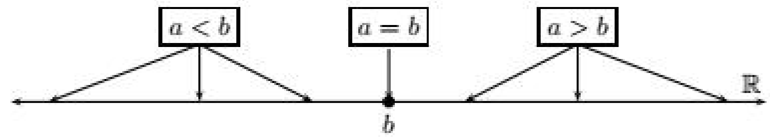
Note that the above arguments can also be made with supersets instead of subsets, with $\supseteq$ replacing $\subseteq$ and $\longleftarrow$ replacing $\longrightarrow$.

One needs to be careful with quantifiers and symbolic logic, as is discussed later in Section **??**, but in what we did above the $(\forall x)$ effectively went along for the ride.

Of course, Venn Diagrams can accommodate more than two sets. For example, we can illustrate the chain of set inclusions

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \tag{1.89}$$

using a Venn Diagram, as in Figure 1.4. Note that this is a compact way of writing six different set inclusions: $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{N} \subseteq \mathbb{Q}$, $\mathbb{N} \subseteq \mathbb{R}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Z} \subseteq \mathbb{R}$, and $\mathbb{Q} \subseteq \mathbb{R}$.



For any two real numbers a and b, we have the three cases concerning their relative positions on the real line: $a < b$, $a = b$, $a > b$. Arrows indicate the possible positions of a for the three cases.

Intervals and Inequalities in $\mathbb{R}$

The number line, which we will henceforth dub the *real line*, has an inherent order in which the numbers are arranged. Suppose we have two numbers $a, b \in \mathbb{R}$. Then the order relation between a and b has three possibilities, each with its own notation:

1. a is to the left of b, written $a < b$ and spoken "a is *less than* b."

2. a is to the right of b, written $a > b$ and spoken "a is *greater than* b."

3. a is at the same location as b, written $a = b$ and spoken "a equals b."

Figure 1.5 shows these three possibilities. Note that "less than" and "greater than" refer to relative positions on the real line, not how "large" or "small" the numbers are. For instance, $4 < 5$ but $-5 < -4$, though it is natural to consider -5 to be a "larger" number than -4. Similarly $-1000 < 1$.[60] Of course if $a < b \iff b > a$. We have further notation which describes when a is left of or at b, and when a is right of or at b:

4. a is at or left of b, written $a \leq b$ and spoken "a is less than or equal to b."

5. a is at or right of b, written $a \geq b$ and spoken "a is greater than or equal to b."

Using inequalities, we can describe *intervals* in $\mathbb{R}$, which are exactly the *connected* subsets of $\mathbb{R}$, meaning those sets which can be represented by darkening the real line at only those points which are in the subset, and where doing so can be theoretically accomplished without lifting our pencils as we darken. In other words, these are "unbroken" subsets of $\mathbb{R}$. Later we will see that intervals are subsets of particular interest in calculus.

Intervals can be classified as finite or infinite (referring to their lengths), and open, closed or half-open (referring to their "endpoints"). The finite intervals are of three types: closed, open and half-open. Intervals of these types, with real *endpoints* a and b, where $a < b$ (though the idea extends to work with $a \leq b$) are shown below respectively by graphical illustration, in *interval notation*, and using earlier set-theoretic notation:

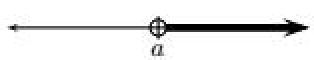
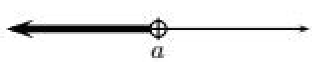
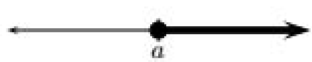
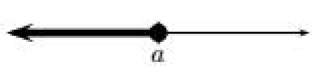
open:		(a, b)	$\{x \in \mathbb{R} \mid a < x < b\}$
closed:		$[a, b]$	$\{x \in \mathbb{R} \mid a \leq x \leq b\}$
half-open:		$[a, b)$	$\{x \in \mathbb{R} \mid a \leq x < b\}$
half-open:		$(a, b]$	$\{x \in \mathbb{R} \mid a < x \leq b\}$

Note that $a < x < b$ is short for $(a < x) \wedge (x < b)$, i.e., $(x > a) \wedge (x < b)$. The others are similar.

We will concentrate on the open and closed intervals in calculus. For the finite open interval above, we see that we do not include the endpoints a and b in the set, denoting this fact with parentheses in the interval notation and an "open" circle at each endpoint on the graph. What is crucial to calculus is that immediately surrounding any point $x \in (a, b)$ are only other points still inside the interval; if we pick a point x *anywhere* in the interval (a, b), we see that just left and just right of x are only points in the interval. Indeed, we have to travel some distance—albeit possibly short—to leave the interval from a point $x \in (a, b)$. Thus no point inside of (a, b) is on the boundary, and so each point in (a, b) is "safely" on the interior of the interval. This will be crucial to the concepts of continuity, limits and (especially) derivatives later in the text.

For a closed interval $[a, b]$, we *do* include the endpoints a and b, which are not surrounded by other points in the interval. For instance, immediately left of a is outside the interval $[a, b]$, though immediately right of a is on the interior.[61] We denote this fact with brackets in the interval notation, and a "closed" circle at each endpoint when we sketch the graph. Half-open (or half-closed) intervals are simple extensions of these ideas, as illustrated above.

For infinite intervals, we have either one or no endpoints. If there is an endpoint it is either not included in the interval or it is, the former giving an open interval and the latter a closed interval. An open interval which is infinite in one direction will be written (a, ∞) or $(-\infty, a)$, depending upon the direction in which it is infinite. Here ∞ (infinity) means that we can move along the interval to the right "forever," and $-\infty$ means we can move left without end. For infinite closed intervals the notation is similar: $[a, \infty)$ and $(-\infty, a]$. The whole real line is also considered an interval, which we denote $\mathbb{R} = (-\infty, \infty)$.[62] When an interval continues *without bound* in a direction, we also darken the arrow in that direction. Thus we have the following:

open:		(a, ∞)	$\{x \in \mathbb{R} \mid x > a\}$
open:		$(-\infty, a)$	$\{x \in \mathbb{R} \mid x < a\}$
closed:		$[a, \infty)$	$\{x \in \mathbb{R} \mid x \geq a\}$
closed:		$(-\infty, a]$	$\{x \in \mathbb{R} \mid x \leq a\}$

Note that we never use brackets to enclose an infinite "endpoint," since $-\infty, \infty$ are not actual boundaries but rather are concepts of unending continuance. Indeed, $-\infty, \infty \notin \mathbb{R}$, i.e., they are not points on the real line, so they can not be boundaries of subsets of $\mathbb{R}$; there are no elements "beyond" them.

Most General Venn Diagrams

Before we get to the title of this subsection, we will introduce a notion which we will have occasional use for, which is the concept of *proper subset.*

Definition *If* $(A \subseteq B) \wedge (A \neq B)$, *we call* A *a* **proper** *subset of* B, *and write* $A \subset B$[63].

Thus $A \subset B$ means A is contained in B, but A is not all of B. Note that $A \subset B \implies A \subseteq B$ (just as $P \wedge Q \implies P$). When we have that A is a subset of B and are not interested in emphasizing whether or not $A \neq B$ (or are not sure if this is true), we will use the "inclusive" notation $\subseteq$. In fact, the inclusive case is less complicated logically (just as $P \vee Q$ is easier than P XOR Q) and so we will usually opt for it even when we do know that $A \neq B$. We mention the exclusive case here mainly because it is useful in explaining the most general Venn Diagram for two sets A and B.

Of course it is possible to have two sets, A and B, where neither is a subset of the other. Then A and B may share some elements, or no elements. In fact, for any given sets A and B, exactly one of the following will be true:

case 1: $A = B$;

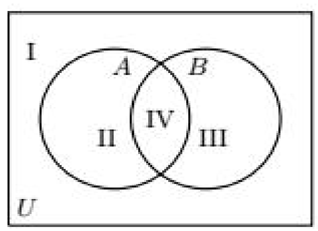
case 2: $A \subset B$, i.e., A is a proper subset of B;

case 3: $B \subset A$, i.e., B is a proper subset of A;

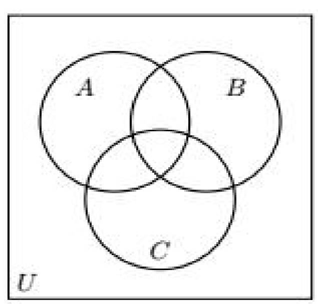
case 4: A and B share common elements, but neither is a subset of the other;

case 5: A and B have no common elements. In such a case the two sets are said to be *disjoint.*

Even if we do not know which of the five cases is correct, we can use a single illustration which covers all of these. That illustration is given in Figure 1.6, with the various regions labeled. (We will explain the meaning of U in the next subsection.) To see that this covers all cases, we take them in turn:



Most general Venn diagram for two arbitrary sets A and B. Here U is some superset of both A and B.



case 1: $A = B$: all elements of A and B are in Region IV; there are no elements in Regions II and III.

case 2: $A \subset B$: there are elements in Regions III and IV, and no elements in Region II.

case 3: $B \subset A$: there are elements in Regions II and IV, and no elements in Region III.

case 4: A and B share common elements, but neither is a subset of the other: there are elements in Region II, III and IV.

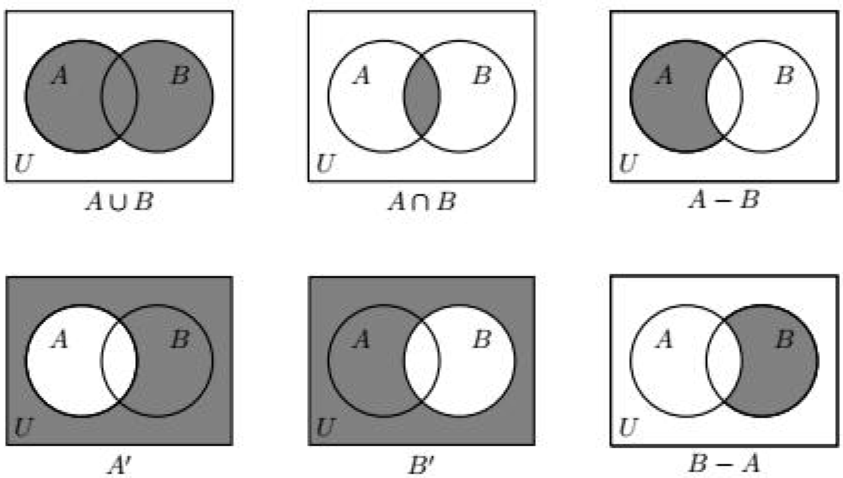
case 5: A and B have no common elements: there are no elements in Region IV.

Note that whether or not Region I has elements is irrelevant in the discussion above, though it will become important shortly.

The most general Venn diagram for three sets is given in Figure 1.7, though we will not exhaustively show this to be the most general. It is not important that the sets are represented by circles, but only that there are sufficiently many separate regions and that every case of an element being, or not being, in A, B and C is represented. Note that there are three sets for an element to be or not to be a member of, and so there are $2^3 = 8$ subregions needed.

Set Operations

When we are given two sets A and B, it is natural to combine or compare their memberships with each other and the universe of all elements of interest. In particular, we form new sets called the union and intersection of A and B, the difference of A and B (and of B and A), and the complement of A (and of B). The first three are straightforward, but the fourth requires



Some Venn Diagrams involving two sets A and B inside a universal set U, which is represented by the whole "box."

some clarification. Usually A and B contain only objects of a certain class like numbers, colors, etc. Thus we take elements of A and B from a specific *universal set* U of objects rather than an all-encompassing universe of all objects. It is unlikely in mathematics that we would need, for instance, to mix numbers with persons and planets and verbs, so we find it convenient to limit our universe U of considered objects. With that in mind (but without presently defining U), the notations for these new sets are as follow:

Definition

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\} \tag{1.90}$$
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\} \tag{1.91}$$
$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\} \tag{1.92}$$
$$A' = \{x \in U \mid (x \notin A)\}. \tag{1.93}$$

These are read "A union B," "A intersect B," "A minus B," and "A complement," respectively. Note that in the first three, we could have also written $\{x \in U | \cdots\}$, but since $A, B \subseteq U$, there it is unnecessary. Also note that one could define the complement in the following way, though (1.93) is more convenient for symbolic logic computations:

$$A' = \{x \mid (x \in U) \wedge (x \notin A)\} = U - A. \tag{1.94}$$

These operations are illustrated by the Venn diagrams of Figure 1.8, where we also construct B' and $B - A$. Note the connection between the logical $\vee$ and $\wedge$, and the set-theoretical $\cup$ and $\cap$.[64]

Example *Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$ if*

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$
$$B = \{5, 6, 7, 8, 9, 10\}.$$

Solution: Though not necessary (and often impossible), we will list these set elements in a table from which we can easily compare the membership.

$$\begin{array}{lllllllllllll} A & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & & & \} \;, \\ B & = & \{ & & & & & 5, & 6, & 7, & 8, & 9, & 10 \;\} \;. \end{array}$$

Now we can compare the memberships using the operations defined earlier.

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$
$$A \cap B = \{5, 6, 7\},$$
$$A - B = \{1, 2, 3, 4\},$$
$$B - A = \{8, 9, 10\}.$$

The complements depend upon the identity of the assumed universal set. If in the above example we had $U = \mathbb{N}$, then $A' = \{8, 9, 10, 11, \cdots\}$ and $B' = \{1, 2, 3, 4, 11, 12, 13, 14, 15 \cdots\}$. If instead we took $U = \mathbb{Z}$ we have $A' = \{\cdots, -3, -2, -1, 0, 8, 9, 10, 11, \cdots\}$, for instance. (We leave B' to the interested reader.)

Just as it is important to have a zero element in $\mathbb{R}$ for arithmetic and other purposes, it is also useful in set theory to define a set which contains no elements:

Definition *The set with no elements is called the* **empty set**,[65] *denoted $\varnothing$.*

One reason we need such a device is for cases of intersections of disjoint sets. If $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7, 8, 9, 10\}$, then $A \cup B = \{1, 2, 3, \cdots, 10\}$, while $A \cap B = \varnothing$. Notice that regardless of the set A, we will always have $A - A = \varnothing$, $A - \varnothing = A$, $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$, and $\varnothing \subseteq A$. The last statement is true because, after all, every element of $\varnothing$ is also an element of A.[66] Note also that $\varnothing' = U$ and $U' = \varnothing$.

The set operations for two sets A and B can only give us finitely many combinations of the areas enumerated in Figure 1.6. In fact, since each such area is either included or not, there are $2^4 = 16$ different diagram shadings possible for the general case as in Figure 1.6. The situation is more interesting if we have three sets A, B and C. Using Figure 1.7, we can prove several interesting set equalities. First we have some fairly obvious commutative laws (1.95), (1.96) and associative laws (1.97), (1.98):

$$A \cup B = B \cup A \tag{1.95}$$
$$A \cap B = B \cap A \tag{1.96}$$
$$A \cup (B \cup C) = (A \cup B) \cup C \tag{1.97}$$
$$A \cap (B \cap C) = (A \cap B) \cap C \tag{1.98}$$

Next are the following two *distributive laws*, which are the set-theory analogs to the logical equivalences (1.27) and (1.28), found on page 22.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \tag{1.99}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{1.100}$$

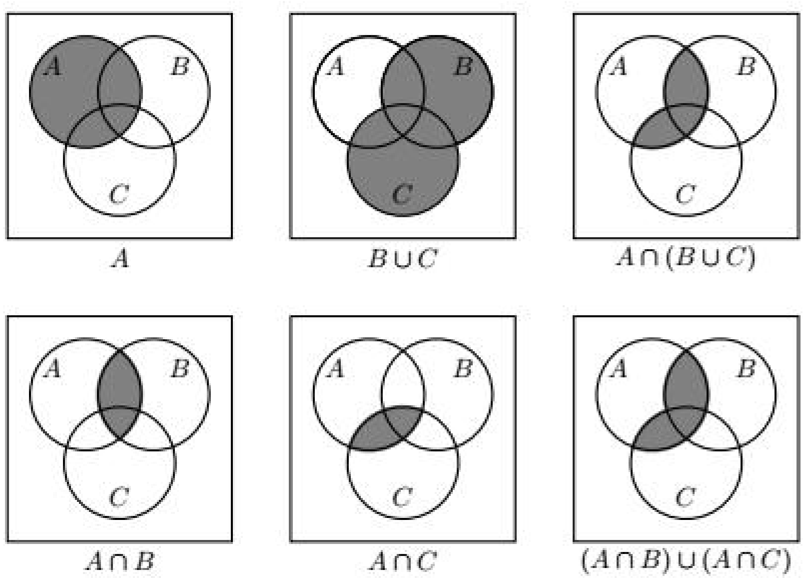


Figure Venn Diagrams for Example 1.5.2 verifying one of the distributive laws, specifically $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. It is especially important to note how one constructs the third box in each line from the first two.

Example : *We will show how to prove* (1.99) *using our previous symbolic logic, and then give a visual proof using Venn diagrams. Similar techniques can be used to prove* (1.100). *For the proof that* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, *we use definitions, and* (1.27) *from page 22 to get the following:*

$$\begin{aligned} x \in A \cap (B \cup C) &\iff (x \in A) \wedge (x \in B \cup C) \\ &\iff (x \in A) \wedge [(x \in B) \vee (x \in C)] \\ &\iff [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)] \\ &\iff [x \in (A \cap B)] \vee [x \in (A \cap C)] \\ &\iff x \in [(A \cap B) \cup (A \cap C)], \text{ q.e.d.} \end{aligned}$$

We proved that $(\forall x)[(x \in A \cap (B \cup C)) \longleftrightarrow (x \in (A \cap B) \cup (A \cap C))]$, *which is the definition for the sets in question to be equal. The visual demonstration of* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *is given in* Figure 1.9, *where we construct both sets of the equality in stages.*

To construct the left-hand side of the equation, in the first box we color A, *then* $B \cup C$ *in the second, and finally we take the area from the first, remove the area from the second, and are left with the difference* $A - (B \cup C)$. *To construct the right-hand side of the equation, we color* $A - B$ *and* $A - C$ *in separate boxes. Then we color the intersection of these, which is the area colored in the previous two boxes. This gives us our Venn Diagram for* $(A - B) \cap (A - C)$. *We see that the left- and right-hand sides are the same, and conclude the equality is valid.*

The next two are distributive in nature also:

$$A - (B \cup C) = (A - B) \cap (A - C) \tag{1.101}$$
$$A - (B \cap C) = (A - B) \cup (A - C). \tag{1.102}$$

Finally, if we replace A with U, we get the set-theoretic version of *de Morgan's Laws:*

$$(B \cup C)' = B' \cap C' \tag{1.103}$$
$$(B \cap C)' = B' \cup C'. \tag{1.104}$$

Note that these are very much like our earlier de Morgan's laws, and indeed use the previous versions (1.3) and (1.4), page 17 (also see page 22) in their proofs. For instance, assuming $x \in U$ where U is fixed, we have

$$\begin{aligned} x \in (B \cup C)' &\iff \sim (x \in B \cup C) \\ &\iff \sim ((x \in B) \vee (x \in C)) \\ &\iff [\sim (x \in B)] \wedge [\sim (x \in C)] \\ &\iff [x \in B'] \wedge [x \in C'] \\ &\iff x \in B' \cap C', \text{ q.e.d.} \end{aligned}$$

That proves (1.103), and (1.104) has a similar proof. It is interesting to prove these using Venn Diagrams as well (see exercises).

Example *Another example of how to prove these using logic and Venn diagrams is in order. We will prove (1.101) using both methods. First, with symbolic logic:*

$$\begin{aligned} x \in A - (B \cup C) &\iff (x \in A) \wedge [\sim (x \in B \cup C)] \\ &\iff (x \in A) \wedge [\sim ((x \in B) \vee (x \in C))] \\ &\iff (x \in A) \wedge [(\sim (x \in B)) \wedge (\sim (x \in C))] \\ &\iff (x \in A) \wedge (\sim (x \in B)) \wedge (\sim (x \in C)) \\ &\iff (x \in A) \wedge (\sim (x \in B)) \wedge (x \in A) \wedge (\sim (x \in C)) \\ &\iff [(x \in A) \wedge (\sim (x \in B)] \wedge [(x \in A) \wedge (\sim (x \in C))] \\ &\iff (x \in A - B) \wedge (x \in A - C) \\ &\iff x \in (A - B) \cap (A - C), \text{ q.e.d.} \end{aligned}$$

If we took the steps above in turn, we used the definition of set subtraction, the definition of union, (1.19), associative property of $\wedge$, added a redundant $(x \in A)$, regrouped, used the definition of set subtraction, and finally the definition of intersection.

Now we will see how we can use Venn diagrams to prove (1.101). As before, we will do this by constructing Venn Diagrams for the sets $A-(B\cup C)$ and $(A-B)\cap(A-C)$ separately, and verify that we get the same sets. We do this in Figure 1.10. *(If it is not visually clear how we proceed from one diagram to the next "all at once," a careful look at each of the $2^3=8$ distinct regions can verify the constructions.)*

More on Subsets

Before closing this section, a few more remarks should be included on the subject of subsets. Consider for instance the following:

Example *Let $A=\{1,2\}$. List all subsets of A.*

Solution: *As $A=\{1,2\}$ has two elements, it can have subsets which contain zero elements, one element, or two elements. The subsets are thus $\varnothing$, $\{1\}$, $\{2\}$ and $\{1,2\}=A$.*

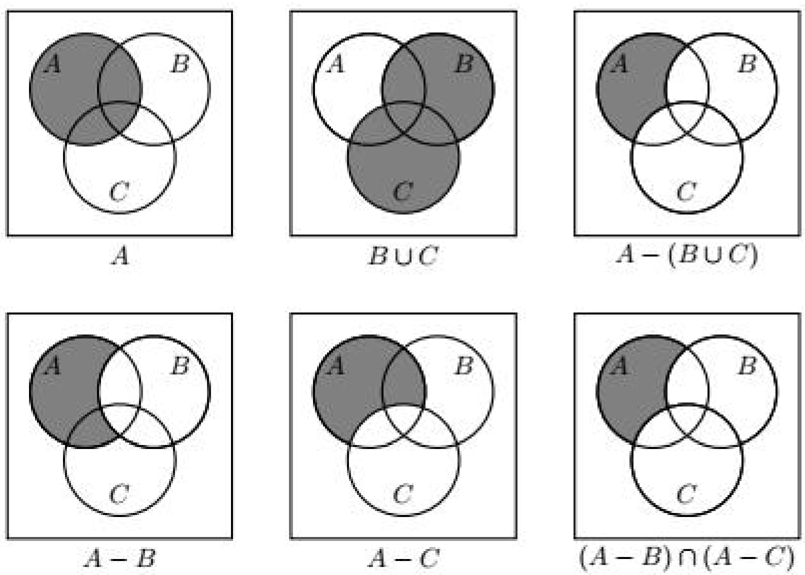


Figure Venn Diagrams for Example 1.5.3 verifying that $A-(B\cup C)=(A-B)\cap(A-C)$.

It is common for novices studying sets to forget that $\varnothing \subseteq A$, and $A\subseteq A$, though by definition,

$$x\in\varnothing \implies x\in A \qquad \text{(vacuously)},$$
$$x\in A \implies x\in A \qquad \text{(trivially)}.$$

If one wanted only *proper* subsets of A, those would be $\varnothing, \{1\}, \{2\}$ (we omit the set A).

Note that with our set $A = \{1, 2\}$, we can reduce rephrase the question of which subset we might refer to, instead into a question of exactly which elements are in it, from the choices 1 and 2. In other words, given a subset $B \subseteq A$, which (if any) of the following are true: $1 \in B$, $2 \in B$. From these statements we can construct a truth table-like structure to describe every possible subset of A:

$A = \{1, 2\}$

$1 \in B$	$2 \in B$	subset B
T	T	$\{1, 2\} = A$
T	F	$\{1\}$
F	T	$\{2\}$
F	F	$\varnothing$

Similarly, a question about subsets B of $A = \{a, b, c\}$ can be placed in context of a truth table-like construct:

$A = \{a, b, c\}$

$a \in B$	$b \in B$	$c \in B$	subset B
T	T	T	$\{a, b, c\} = A$
T	T	F	$\{a, b\}$
T	F	T	$\{a, c\}$
T	F	F	$\{a\}$
F	T	T	$\{b, c\}$
F	T	F	$\{b\}$
F	F	T	$\{c\}$
F	F	F	$\varnothing$

It would not be too difficult to list the elements of $A = \{1, 2, 3\}$ by listing subsets with zero, one, two and three elements separately, i.e., $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$, but if we were to need to list subsets of a set with significantly more elements, it might be easier to use the lexicographical order embedded in the truth table format to exhaust all the possibilities. The only disadvantage is that the order in which subsets are listed might not be quite as natural as the order we would likely find if we listed subsets with zero, one, two elements and so on.

Possible Questions

2 Mark Questions:

1. Define null set and singleton set.
2. Define subset
3. When two sets are said to be equal?
4. Define finite set with example
5. Define Poset.

6 Mark Questions:

1. If A={3,4,2},B={3,4,5,6} and C={2,4,6,8} then prove that
   $A \cap (B - C) = (A \cap B) - (A \cap C)$
2. Let U={x : x in N, $1 \le x \le 12$} be the universal set and A = {1, 9, 10},
   B = {3, 4, 6, 11} and C = {2, 5, 6} are subsets of U. Find the sets
   (i) $(A \cup B) \cap (A \cap C)$ (ii) $A \cup (B \cap C)$
3. If A, B, C are any three sets then prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. Use venn diagram to find the sets A and B if
   i) $A - B = \{1, 3, 7, 11\}$, $B - A = \{2, 6, 8\}$ and $A \cap B = \{1, 9\}$
   ii) $A - B = \{1, 2, 4\}$, $B - A= \{7, 8\}$ and $A \cup B =\{1, 2, 4, 5, 7, 8, 9\}$
5. Prove that $(A-C) \cap (C-B) = \Phi$
6. If A, B, C are the sets then prove that $A - (B \cap C) = (A - B) \cup (A - C)$
7. If A,B,C are sets prove that $A\cup (B\cap C) = (\overline{C} \cup \overline{B}) \cap \overline{A}$ using set identities
8. If A, B, C are any three sets then prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. Use venn diagram to prove that A∩(B–C) = (A∩B)–(A©
10. Simplify the following set using set identities $A\cup B \cup (A\cap B\cap C)$

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18MMU103

Class : I B.Sc Mathematics **Semester : I**

UNIT -III

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
____ is a collection of well-defined objects.	element	member	set	none of these	set
a,b,c} then cardinality of the set is ________	nullset	one	two	three	three
The two sets A and B are called as _________ if n(a) = n(B)	equal set	equalent set	null set	Subset	equalent set
The two sets A and B are called as _________ if the sets have the same elements.	equal set	equalent set	null set	Subset	equal set
If every element of the set A is an element of the another set B then A is ________ of B	subset	superset	empty set	universal set	subset
If every element of the set A is an element of the another set B then B is ________ of A	subset	superset	empty set	universal set	superset
if the cardinality of the set is zero then the set is ______________	subset	superset	empty set	universal set	empty set
Empty set is a __________ of every set.	subset	superset	empty set	universal set	subset
Universal set is the _____________ of all the sets.	subset	superset	empty set	universal set	superset
If A = {1,2,3,4} and B = {2,4} then A intersection B =	{2,4}	{1,2,3,4}	{1,2}	{}	{2,4}
If A = {1,2,3,4} and B = {2,4} then A union B =	{2,4}	{1,2,3,4}	{1,2}	{}	{1,2,3,4}
Two sets are said to be disjoint if A intersection B =	A		B	A union B	{}
If n subsets of a set are given, then the number of __________ is 2 power n	min terms	minimax terms	sets	none of these	min terms
If n subsets of a set are given, then the number of __________ is 2 power n	max terms	minimax terms	sets	none of these	max terms
Every singleton subset constitutes a ___________	set	partition	min term	max term	partition
The least uper bound of any element in a poset are __________	unique	dual	zero	one	unique
The greatest uper bound of any element in a poset are __________	unique	dual	zero	one	unique
An element m in a poset L is called the greatest element if __________ for all a in L	a less than or equal to m	a greater than or equal to m	a = m	a = 0	a less than or equal to m
An element m in a poset L is called the least element if __________ for all a in L	a less than or equal to m	a greater than or equal to m	a = m	a = 0	a greater than or equal to m
A set is a well defined collection of object called , the ________ of the set.	object	languages	element	letters	element
A ______ is a well defined collection of objects called members of the set.	object	set	element	letters	set
A set is represented in __________ ways	one	two	three	four	two
In ________ notation , all the elements of the set are listed	member	roster	element	object	roster
In _________ notation we specify elements of the set by specifying a property.	builder	roaster	object	number	builder
the set which contains all the objects under consideration is called ______ set.	singular	universal	null	empty	universal
A set which contains no elements at all is called_________ set.	singular	null	universal	finite	null
A set which contains ________ elements at all is called empty set.	all	no	two	one	no
Any subset A of the set B is called proper subset of B if there is atleast one element of B which _____ A	belongs to	does not belongs to	is contained in	is not contained in	is contained in
Two sets are said to be ______ if A and B are contained in both the sets	equal set	not equal	empty set	power	equal set
If A is a subset of B then B is called _______of A	super set	subset	proper set	power set	super set
Every set is a __ of itself	singleton set	subset	universal	empty set	subset
A set is said to be a_____ of B iff every element of A is also an element of B	subset	power set	universal set	empty set	subset
A set is said to be a subset of B iff _______of A is also an element of B	one element	every element	two element	three element	every element
A set which contains only ____ element is called a singleton set	2	1	3	5	1
A set which contains only one element is called a ________	universal set	singleton set	null set	empty set	singleton set
A set which contains ______ number of elements is called finite sets	more	single	finite	infinite	finite
A set which contains finite number of elements is called ______	infinite set	finite sets	subset	super set	finite sets
A set which contains ______ number of elements is called infinite sets	more	single	finite	infinite	infinite
A set which contains infinite number of elements is called ______	infinite set	finite sets	subset	super set	infinite sets
If a set A is finite set then the number of elements in A is called the __of A	member	degree	cardinality	order	cardinality
We can find a size of A if it is ___	infinite set	finite sets	subset	super set	finite sets
We can find a size of A if it is ___	infinite set	finite sets	subset	super set	finite sets

UNIT-IV

SYLLABUS

Finite sets and counting principle. Empty set, properties of empty set. Standard set operations. Classes of sets. Power set of a set. Difference and Symmetric difference of two sets. Set identities, Generalized union and intersections.

Sets

"A set is a Many that allows itself to be thought of as a One."
(Georg Cantor)

In the previous chapters, we have often encountered "sets", for example, prime numbers form a set, domains in predicate logic form sets as well. Defining a set formally is a pretty delicate matter, for now, we will be happy to consider an intuitive definition, namely:

Definition 24. A set is a collection of abstract objects.

A set is typically determined by its distinct elements, or members, by which we mean that the order does not matter, and if an element is repeated several times, we only care about one instance of the element. We typically use the bracket notation $\{\}$ to refer to a set.

Example 42. The sets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same, because the ordering does not matter. The set $\{1, 1, 1, 2, 3, 3, 3\}$ is also the same set as $\{1, 2, 3\}$, because we are not interested in repetition: either an element is in the set, or it is not, but we do not count how many times it appears.

One may specify a set *explicitly*, that is by listing all the elements the set contains, or *implicitly*, using a predicate description as seen in predicate logic, of the form $\{x,\ P(x)\}$. Implicit descriptions tend to be preferred for infinite sets.

Example 43. The set A given by $A = \{1, 2\}$ is an explicit description. The set $\{x,\ x \text{ is a prime number }\}$ is implicit.

Given a set S, one may be interested in elements belonging to S, or in subset of S. The two concepts are related, but different.

Definition 25. A set A is a subset of a set B, denoted by $A \subseteq B$, if and only if every element of A is also an element of B. Formally

$$A \subseteq B \iff \forall x(x \in A \to x \in B).$$

Note the two notations $A \subset B$ and $A \subseteq B$: the first one says that A is a subset of B, while the second emphasizes that A is a subset of B, possibly equal to B. The second notation is typically preferred if one wants to emphasize that one set is possibly equal to the other.

To say that A is not a subset of S, we use the negation of $\forall x(x \in A \to x \in B)$, which is (using the rules we have studied in predicate logic! namely negation of universal quantifier, conversion theorem, and De Morgan's law) $\exists x(x \in A \wedge x \notin B)$. The notation is $A \not\subseteq B$.

For an element x to be an element of a set S, we write $x \in S$. This is a notation that we used already in predicate logic. Note the difference between $x \in S$ and $\{x\} \subseteq S$: in the first expression, x is in element of S, while in the second, we consider the subset $\{x\}$, which is emphasized by the bracket notation.

Example 44. Consider the set $S = \{$ rock, paper, scissors $\}$, then $R = \{$ rock $\}$ is a subset of S, while rock $\in S$, it is an element of S.

Definition 26. The empty set is a set that contains no element. We denote it $\varnothing$ or $\{\}$.

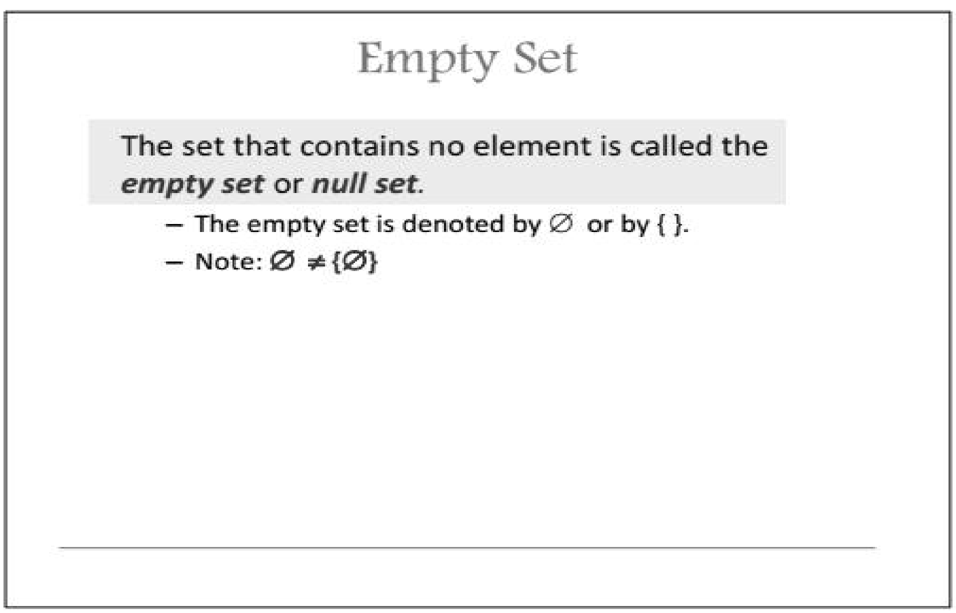
There is a difference between $\varnothing$ and $\{\varnothing\}$: the first one is an empty set, the second one is a set, which is not empty since it contains one element: the empty set!

Definition 27. The empty set is a set that contains no element. We denote it $\varnothing$ or $\{\}$.

Example 45. We say that two sets A and B are equal, denoted by $A = B$, if and only if $\forall x, (x \in A \leftrightarrow x \in B)$.

To say that two sets A and B are not equal, we use the negation from predicate logic, which is:

$$\neg(\forall x, (x \in A \leftrightarrow x \in B)) \equiv \exists x((x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)).$$



Set Equality

$A=B \triangleq \forall x(x\in A \leftrightarrow x\in B)$

– Two sets A, B are equal *iff* they have the same elements.

$A\neq B \triangleq \neg\forall x(x\in A \leftrightarrow x\in B)$

$\equiv \exists x\,[(x\in A\wedge x\notin B) \vee (x\in B\wedge x\notin A)]$

– Two sets are not equal if they do not have identical members, i.e., there is some element in one of the sets which is absent in the other.

- Example:

{1, 2, 3} = {3, 1, 2} = {1, 3, 2}= {1, 1, 1, 2, 3, 3, 3}

This makes our earlier example $\{1,2,3\} = \{1,1,1,2,3,3,3\}$ easier to justify than what we had intuitively before: both sets are equal because whenever a number belongs to one, it belongs to the other.

Definition 28. The cardinality of a set S is the number of distinct elements of S. If $|S|$ is finite, the set is said to be finite. It is said to be infinite otherwise.

We could say the number of elements of S, but then this may be confusing when elements are repeated as in $\{1,2,3\} = \{1,1,1,2,3,3,3\}$, while there is no ambiguity for distinct elements. There $|S| = |\{1,2,3\}| = 3$. The set of prime numbers is infinite, while the set of even prime numbers is finite, because it contains only 2.

Definition 29. The power set $P(S)$ of a set S is the set of all subsets of S:

$$P(S) = \{A,\ A \subseteq S\}.$$

If $S = \{1,2,3\}$, then $P(S)$ contains S and the empty set $\varnothing$, and all subsets of size 1, namely $\{1\}$, $\{2\}$, and $\{3\}$, and all subsets of size 2, namely $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.

The cardinality of $P(S)$ is 2^n when $|S| = n$. This is not such an obvious result, it may be derived in several ways, one of them being the so-called binomial theorem, which says that

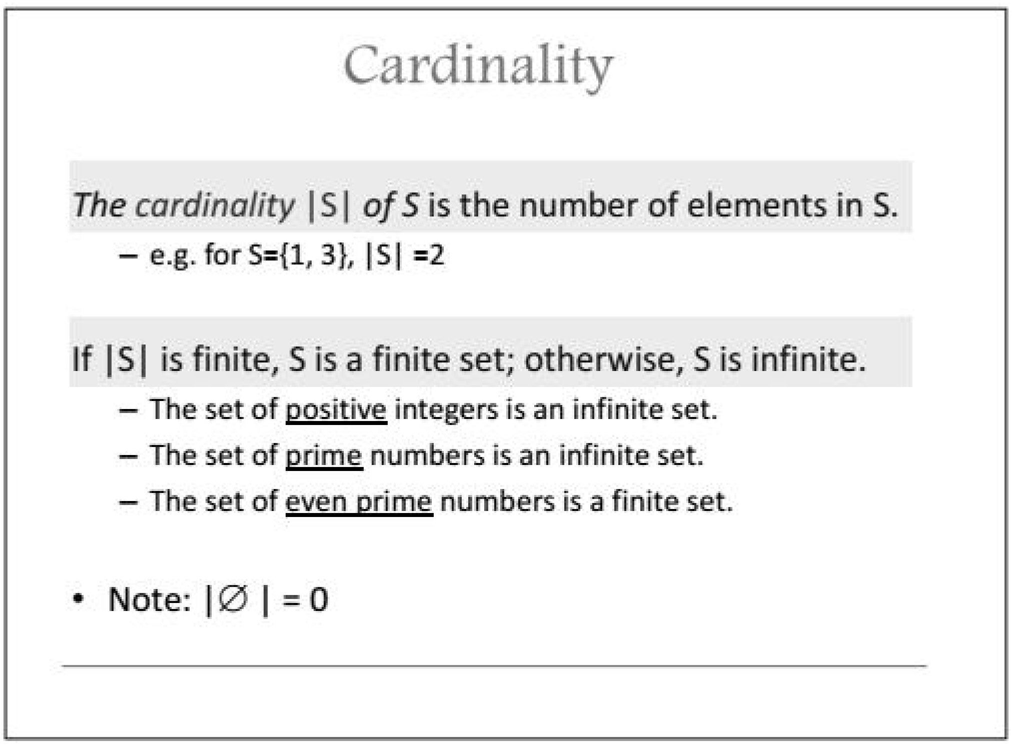
$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j},$$

where $\binom{n}{j}$ counts the number of ways to choose j elements out of n. The notation $\sum_{j=0}^{n}$ means that we sum for the values of j going from 0 to n. See Exercise 33 for a proof of the binomial theorem. When $n = 3$, evaluating in $x = y = 1$, we have

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

and we see that $\binom{3}{0}$ says we pick no element from 3, there is one way, and it corresponds to the empty set, then $\binom{3}{1}$ is telling us that we have 3 ways to choose a single subset, this is for $\{1\}$, $\{2\}$, and $\{3\}$, $\binom{3}{2}$ counts $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\binom{3}{3}$ counts the whole set $\{1, 2, 3\}$.

When dealing with sets, it is often useful to draw Venn diagrams to show how sets are interacting. They are useful to visualize "unions" and "intersections".



Power Set

The power set P(S) of a given set S is the set of all subsets of S: $P(S) = \{ A \mid A \subseteq S\}$.

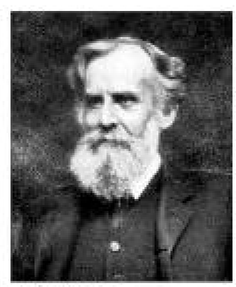
- Example
  - For $S= \{1,2,3\}$
    $P(S)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

- If a set A has n elements, then P(s) has 2^n elements.
  - Hint: Try to leverage *the Binomial theorem*

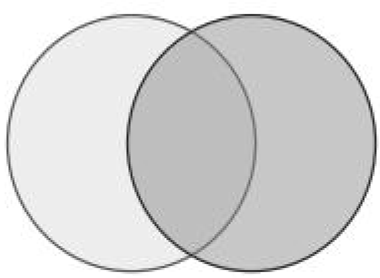
$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

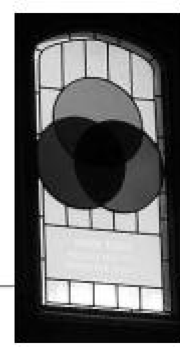
Venn Diagram

A Venn diagram is used to show/visualize the possible relations among a collection of sets.



John Venn (1834-1923)



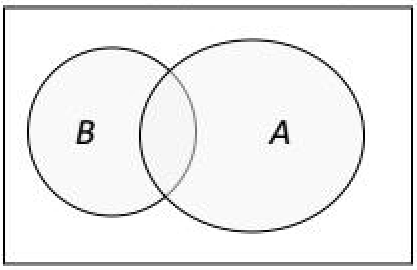


Pictures from wikipedia

Union and Intersection

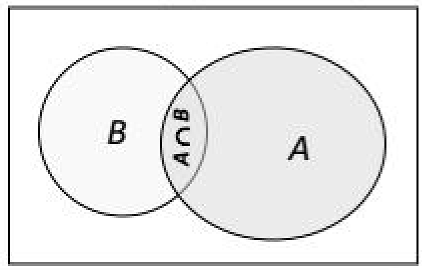
The *union of sets A and B* is the set of those *elements that are either in A or in B, or in both.*

$A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$



The *intersection of the sets A and B* is the set of *all elements that are in both A and B.*

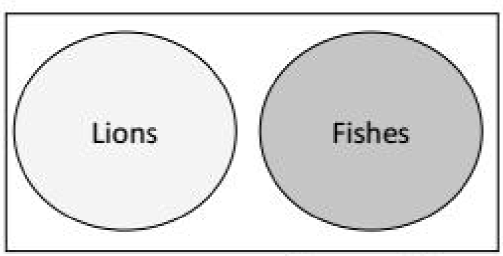
$A \cap B \triangleq \{x \mid x \in A \wedge x \in B\}$



Disjoint Sets

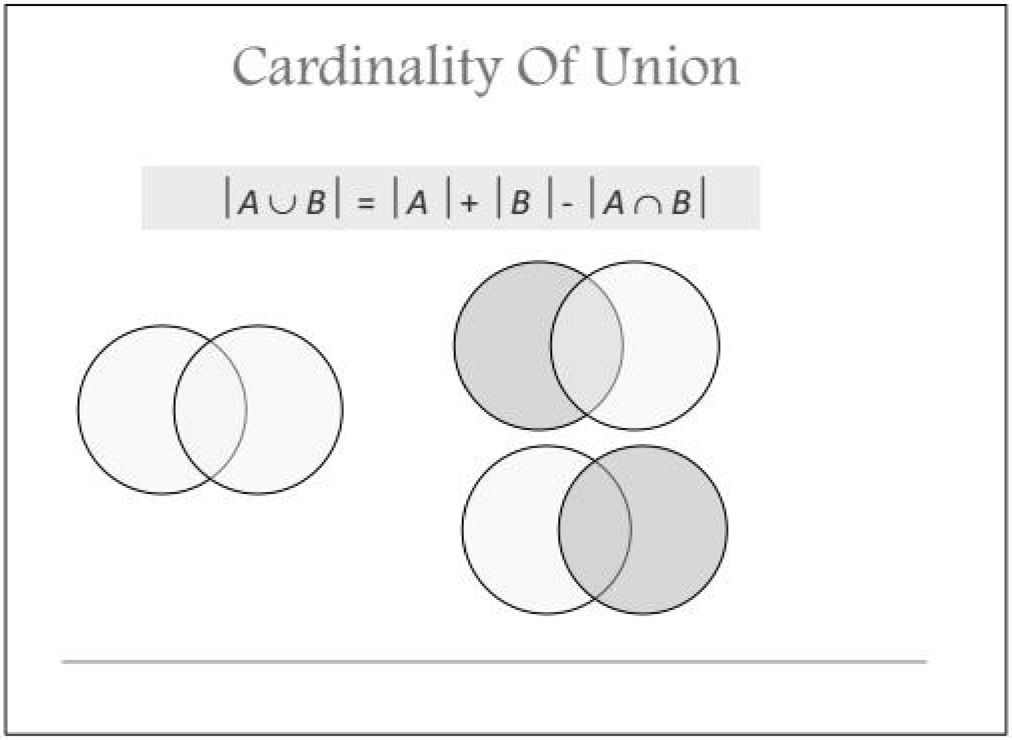
Sets A and B are *disjoint* iff $A \cap B = \varnothing$

– $|A \cap B| = 0$



Lions $\cap$ Fishes = $\varnothing$

© photographer



Definition 30. The union of the sets A and B is by definition

$$A \cup B = \{x,\ x \in A \vee x \in B\}.$$

The intersection of the sets A and B is by definition

$$A \cap B = \{x,\ x \in A \wedge x \in B\}.$$

When the intersection of A and B is empty, we say that A and B are disjoint.

The cardinality of the union and intersection of the sets A and B are related by:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is true, because to count the number of elements in $A \cup B$, we start by counting those in A, and then add those in B. If A and B were disjoint, then we are done, otherwise, we have double counted those in both sets, so we must subtract those in $A \cap B$.

Definition 31. The difference of A and B, also called complement of B with respect to A is the set containing elements that are in B but not in B:

$$A - B = \{x,\ x \in A \wedge x \notin B\}.$$

The complement of A is the complement of A with respect to the universe U:

$$\bar{A} = U - A = \{x,\ x \notin A\}.$$

The universe U is the set that serves as a framework for all our set computations, the biggest set in which all the other sets we are interested in lie. Note that $\overline{\overline{A}} = A$.

Definition 32. The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (a, b), where $a \in A$, $b \in B$:

$$A \times B = \{(a, b),\ a \in A \wedge b \in B\}.$$

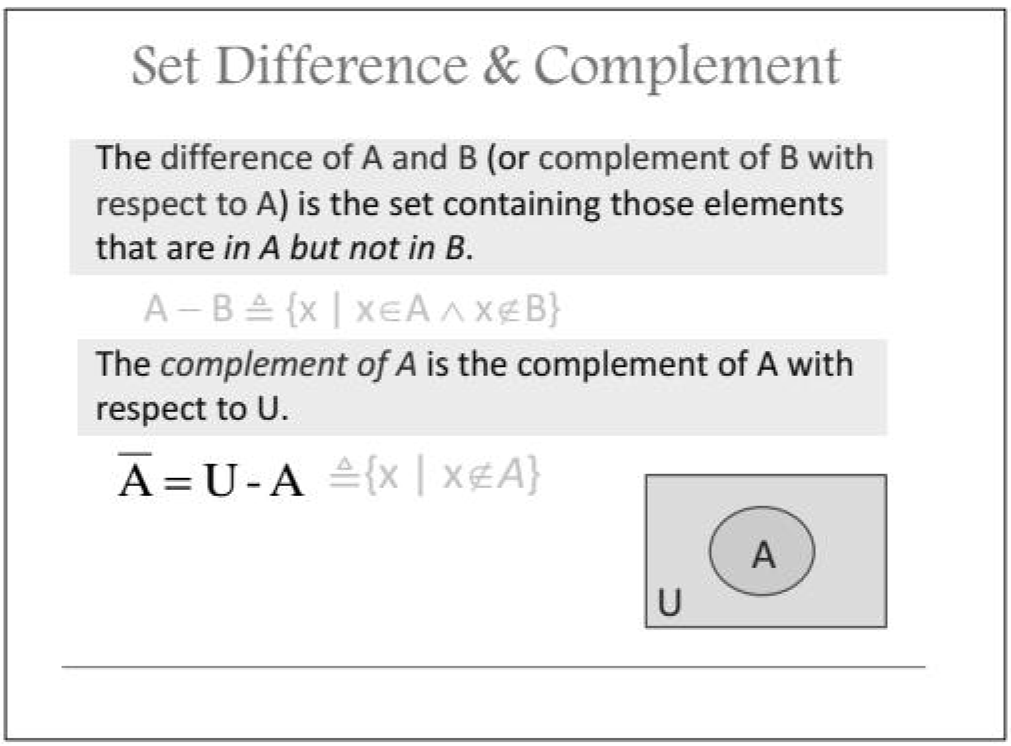
Example 46. Take $A = \{1, 2\}$, $B = \{x, y, z\}$. Then

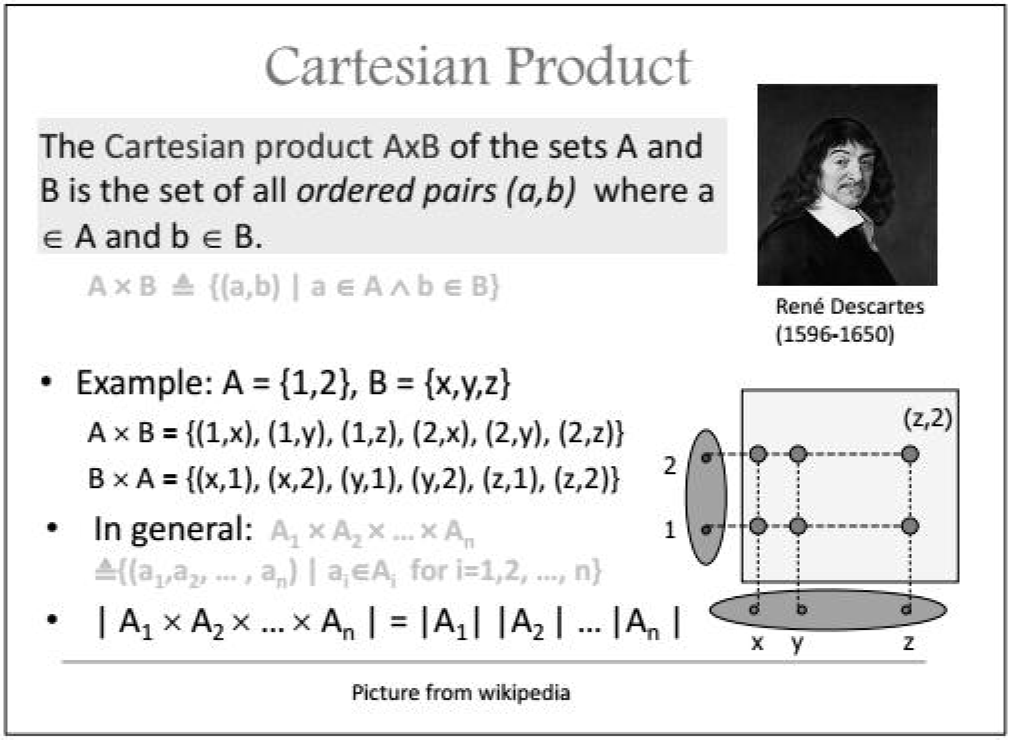
$$A \times B = \{(a, b),\ a \in \{1, 2\} \wedge b \in \{x, y, z\}\}$$

thus a can be either 1 or 2, and for each of these 2 values, b can be either x, y or z:

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}.$$

Note that $A \times B \neq B \times A$, and that a Cartesian product can be formed from n sets $A_1, \ldots, A_n$, which is denoted by $A_1 \times A_2 \times \cdots \times A_n$.





Definition 33. A collection of nonempty sets $\{A_1, \ldots, A_n\}$ is a partition of a set A if and only if

1. $A = A_1 \cup A_2 \cup \ldots A_n$
2. and $A_1, \ldots, A_n$ are mutually disjoint: $A_i \cap A_j = \varnothing$, $i \neq j$, $i, j = 1, 2, \ldots, n$.

Example 47. Consider $A = \mathbb{Z}$, $A_1 = \{$ even numbers $\}$, $A_2 = \{$ odd numbers $\}$. Then A_1, A_2 form a partition of A.

We next derive a series of set identities:

$$\boxed{A \cap \bar{B} = A - B.}$$

By Definition 31, $A - B = \{x,\ x \in A \wedge x \notin B\}$. Then $A \cap \bar{B} = \{x,\ x \in A \wedge x \in \bar{B}\}$, but by the definition of $\bar{B}$, $A \cap \bar{B} = \{x,\ x \in A \wedge x \notin B\}$, which completes the proof.

We have the set theoretic version of De Morgan's law:

$$\boxed{\overline{A \cap B} = \bar{A} \cup \bar{B}.}$$

We have $\overline{A \cap B} = \{x,\ x \notin A \cap B\} = \{x,\ \neg(x \in A \wedge x \in B)\}$, and using the usual De Morgan's law, we get $\overline{A \cap B} = \{x,\ x \notin A \vee x \notin B\}$ as desired.

Applying de Morgan's law on $\overline{A \cap \bar{B}}$, and $\bar{\bar{B}} = B$ we get:

$$\boxed{\overline{A \cap \bar{B}} = \bar{A} \cup B.}$$

Recall that U denotes the universe set, the one to which belongs all the sets that we are manipulating. In particular, $A \subset U$. We have

$$\boxed{A \cup \varnothing = A,\ A \cap U = A,\ A \cup U = U,\ A \cap \varnothing = \varnothing,\ A \cup A = A,\ A \cap A = A.}$$

Furthermore, the order in which $\cup$ or $\cap$ is done does not matter:

$$\boxed{A \cup B = B \cup A,\ A \cap B = B \cap A,\ A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C.}$$

Distributive laws hold as well:

$$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C),\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C).}$$

For example, $A \cap (B \cup C) = \{x,\ x \in A \wedge (x \in B \vee x \in C)\}$ and we can apply the distribute law from propositional logic to get the desired result. And finally

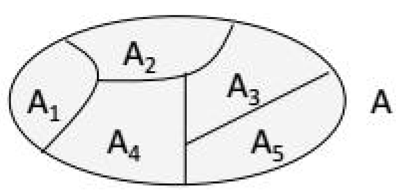
$$\boxed{A \cup (A \cap B) = A,\ A \cap (A \cup B) = A.}$$

This follows from the fact that $A \cap B$ is a subset of A, while A is a subset of $A \cup B$.

Partition

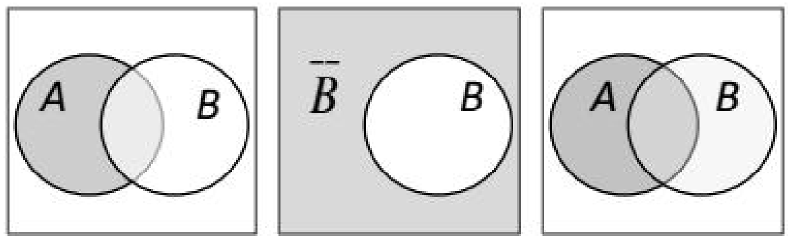
A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A, *iff* $A = A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1, A_2, \ldots, A_n$ are *mutually disjoint*, i.e.

$A_i \cap A_j = \varnothing$ for all $i, j = 1, 2, \ldots n$, and $i \neq j$.



Set Identities

$$A \cap \overline{B} = A - B$$

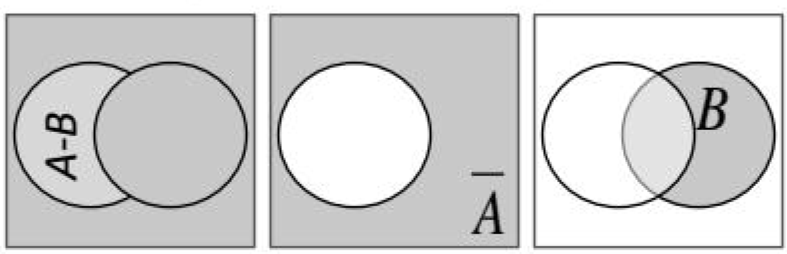


Compare $A \cap \overline{B}$ with $A\text{-}B = \{x \mid x \in A \wedge x \notin B\}$

(not a formal proof)

Set Identities

$$\overline{A \cap \overline{B}} = \overline{A} \cup B$$



- Consider $\overline{A - B} = \overline{A \cap \overline{B}}$
- Apply DeMorgan's Law $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ with X=A and Y= $\overline{B}$

Set Identities

Identity	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Double Complement laws

Set Identities

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws

Set Identities

Identity	Name
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A - B = A \cap \overline{B}$	Alternate Representation for set difference

Suppose that you want to prove that two sets A and B are equal. We will discuss 3 possible methods to do so:

1. Double inclusion: $A \subseteq B$ and $B \subseteq A$.
2. Set identities.
3. Membership tables.

Example 48. To show that $(B - A) \cup (C - A) = (B \cup C) - A$, we show the double inclusion.

- Take an element $x \in (B - A) \cup (C - A)$, then either $x \in (B - A)$, or $x \in (C - A)$. Then $x \in B \wedge x \notin A$, or $x \in C \wedge x \notin B$. Then either way, $x \in B \cup C \wedge x \notin A$, that is $x \in (B \cup C) - A$, and $(B - A) \cup (C - A) \subseteq (B \cup C) - A$ is shown.
- Now take an element $x \in (B \cup C) - A$, that is $x \in B \cup C$ but $x \notin A$. Then $x \in B$ and not in A, or $x \in C$ and not in A. Then $x \in B - A$ or $x \in C - A$. Thus either way, $x \in (B - A) \cup (C - A)$, which shows that $(B - A) \cup (C - A) \supseteq (B \cup C) - A$

Example 49. We show that $(A-B)-(B-C) = A-B$ using set identities.

$$\begin{aligned}(A-B)-(B-C) &= (A-B)\cap\overline{(B-C)}\\ &= (A\cap\bar{B})\cap\overline{(B\cap\bar{C})}\\ &= (A\cap\bar{B})\cap(\bar{B}\cup C)\\ &= [(A\cap\bar{B})\cap\bar{B}]\cup[(A\cap\bar{B})\cap C]\end{aligned}$$

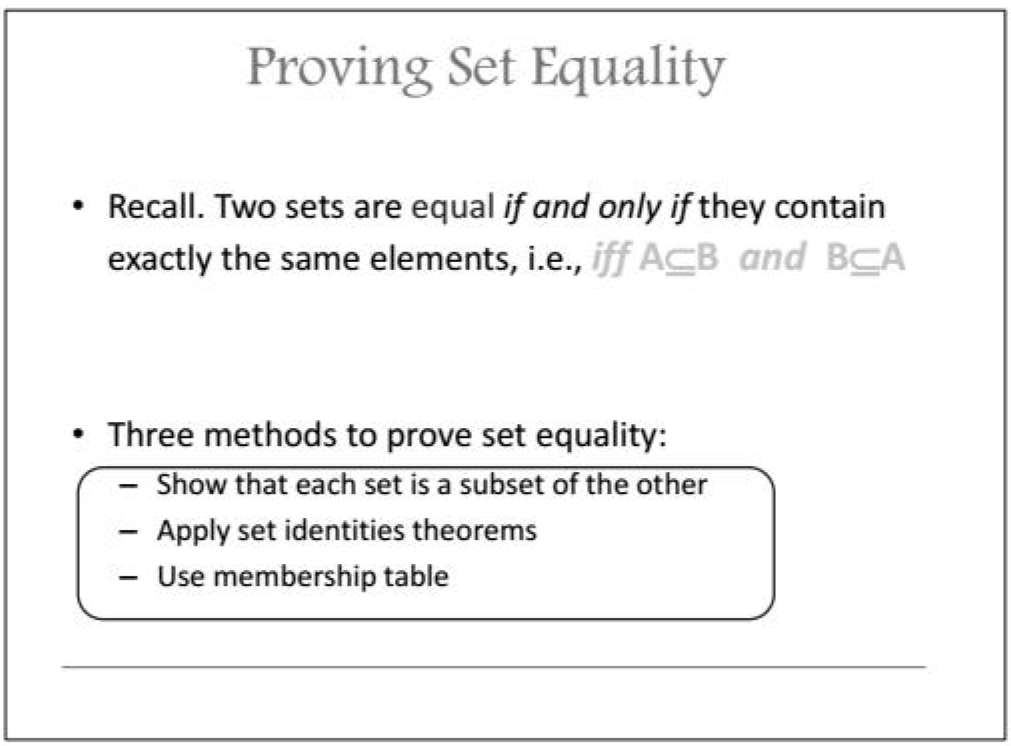
where the third equality is De Morgan's law, and the 4rth one is distributivity. We also notice that the first term can be simplified to get $(A\cap\bar{B})$. We then apply distributivity again:

$$(A\cap\bar{B})\cup[(A\cap\bar{B})\cap C] = [A\cup[(A\cap\bar{B})\cap C]]\cap[\bar{B}\cup[(A\cap\bar{B})\cap C]].$$

Since $(A\cap\bar{B})\cap C$ is a subset of A, then the first term is A. Similarly, since $(A\cap\bar{B})\cap C$ is a subset of $\bar{B}$, the second term is $\bar{B}$. Therefore

$$(A-B)-(B-C) = A\cap\bar{B} = A-B.$$

The third method is a membership table, where columns of the table represent different set expressions, and rows take combinations of memberships in constituent sets: 1 means membership, and 0 non-membership. For two sets to be equal, they need to have identical columns.



Each Others' Subset

Show that $(B\text{-}A) \cup (C\text{-}A) = (B \cup C)\text{-}A$.

For any $x \in$ LHS, $x \in (B\text{-}A)$ or $x \in (C\text{-}A)$ [or both].

when $x \in B - A$ $\Rightarrow (x \in B) \wedge (x \notin A)$

$\Rightarrow (x \in B \cup C) \wedge (x \notin A)$

$\Rightarrow x \in (B \cup C) - A$

when $x \in C - A$ $\Rightarrow (x \in C) \wedge (x \notin A)$

$\Rightarrow (x \in B \cup C) \wedge (x \notin A)$

$\Rightarrow x \in (B \cup C) - A$

Therefore, LHS $\subseteq$ RHS

Each Others' Subset

Show that $(B\text{-}A) \cup (C\text{-}A) = (B \cup C)\text{-}A$.

For any $x \in$ RHS, $x \in (B \cup C)$ and $x \notin A$.

when $x \in B$ and $x \notin A$

$(x \in B) \wedge (x \notin A) \Rightarrow x \in B - A$

$\Rightarrow x \in (B - A) \cup (C - A)$

when $x \in C$ and $x \notin A$,

$(x \in C) \wedge (x \notin A) \Rightarrow x \in C - A$

$\Rightarrow x \in (B - A) \cup (C - A)$

Therefore, RHS $\subseteq$ LHS

With LHS $\subseteq$ RHS and RHS $\subseteq$ LHS, we can conclude that **LHS = RHS**

Using Set Identities

Show that (A-B)-(B-C)=A-B

$(A-B)-(B-C) = (A\cap\overline{B})\cap\overline{(B\cap\overline{C})}$ (By alternate representation for set difference)

$= (A\cap\overline{B})\cap(\overline{B}\cup C)$ (By De Morgan's laws)

$= [(A\cap\overline{B})\cap\overline{B}]\cup[(A\cap\overline{B})\cap C]$ (By Distributive laws)

$= [A\cap(\overline{B}\cap\overline{B})]\cup[A\cap(\overline{B}\cap C)]$ (By Associative laws)

$= (A\cap\overline{B})\cup[A\cap(\overline{B}\cap C)]$ (By Idempotent laws)

$= A\cap[\overline{B}\cup(\overline{B}\cap C)]$ (By Distributive laws)

$= A\cap\overline{B}$ (By Absorption laws)

$= A-B$ (By the alternate representation for set difference)

Using Membership Tables

Similar to truth table (in propositional logic)

- Columns for different set expressions
- Rows for all combinations of memberships in constituent sets
- "1" = membership , "0" =non-membership
- Two sets are equal, *iff* they have identical columns

Prove that $(A\cup B)-B = A-B$

A	B	$A\cup B$	$(A\cup B)-B$	$A-B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Example 50. To prove $(A \cup B) - B = A - B$, we create a table

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0			
0	1			
1	0			
1	1			

The first row, if x is not in A and not in B, it will not be in any of the sets, therefore the first row contains only zeroes. If x is only in B, then it belongs to $A \cup B$, but not in the others, since B is removed. So the second row has only a 1 in $A \cup B$. Then if x is only in A, it belongs to all the three sets. Finally, if x is in both A and B, it is in their intersection, therefore it belongs to $A \cup B$, but not in the 2 others, since B is removed.

Exercises for Chapter 4

Exercise 33. 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \; n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Exercise 34. Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B), \; P(A), \; P(A \cup B), \; P(A \times B).$$

Exercise 35. Prove by contradiction that for two sets A and B

$$(A - B) \cap (B - A) = \varnothing.$$

Exercise 36. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B.$$

Exercise 37. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Exercise 38. Show that the empty set is a subset of all non-null sets.

Exercise 39. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

Exercise 40. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

Exercise 41. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

Exercise 42. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

Exercise 43. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Exercise 44. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.

2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

Exercise 45. In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

Possible Questions

2 Mark Questions:

1. Define generalized union of two sets.
2. Define partial ordering with an example
3. Define the symmetric difference of two sets
4. Define the difference of two sets
5. What is a power set?

6 Mark Questions:

1. Consider U={1, 2, ….., 9} and the sets A={1, 2, 3, 4, 5}, B={4, 5, 6, 7}, C={5, 6, 7, 8, 9}, D={1, 3, 5, 7, 9}, E={2, 4, 6, 8} and F={1, 5, 9}. Find
   (i) A^C, B^C, D^C, E^C, (ii) A\B, B\A, D\E, F\D, (iii) A+B, C+D, E+F.
2. Prove the following identity $(A \cup B) \cap (A \cup B^c) = A$
3. Prove that $(A\cup B)\setminus(A\cap B) = (A\setminus B) \cup (B\setminus A)$.
4. Consider U = {1, 2, ….., 9} and the sets A = {1, 2, 3, 4, 5}, B = {4, 5, 6, 7}, C = {5, 6, 7, 8, 9}, D={1,3,5,7,9}, E={2,4,6,8} and F={1,5,9}. Find
   i) $A\cap(B\cup E)$ ii) $(A\setminus E)^c$ iii) $(A\cap D)\setminus B$ iv) $(B\cap F) \cup (C\cap E)$.
5. Consider U = {1, 2, ….., 9} and the sets A = {1, 2, 3, 4, 5}, B = {4, 5, 6, 7}, C = {5, 6, 7, 8, 9}, D = {1, 3, 5, 7, 9}, E = {2, 4, 6, 8} and F = {1, 5, 9}. Find
   i) $A \cup B$ and $A \cap B$ ii) $B \cup D$ and $B \cap D$ iii) $A \cup C$ and $A \cap C$
   iv) $D \cup E$ and $D \cap E$ v) $E \cup F$ and $E \cap F$ vi) $D \cup F$ and $D \cap F$.
6. Consider the class A of sets A={{1, 2, 3}, {4, 5}, {6, 7, 8}}. Determine whether each of the following is true or false :
   i) $1\in A$ ii) $\{1, 2, 3\} \subseteq A$ iii) $\{6, 7, 8\}\in A$ iv) $\{4, 5\} \subseteq A$ v) $\phi\in A$ vi) $\phi \subseteq A$
7. In a survey of 60 people,it was found that 25 read news week magazine,26 read time,26 read fortune,9 read both news week and fortune,11 read both news week and time, 8 read both time and fortune and 3 read all three magazines. Find

i) The number of people who read atleast one of the 3 magazines,

ii) The number of people who read exactly one magazine.

8. Find the power set for i) A = {1, 2, 3, 4, 5}, ii) B = {a, b, c} iii) C = {}

9. If A and B are finite sets,then $A \cup B$ and $A \cap B$ are finite and

$\cap (A \cup B) = \cap (A) + \cap (B) - \cap (A \cap B)$

10. i) Let S = {red, blue, green, yellow}. Determine which of the following is a partition of S.

P1={{red},{blue, green}}

P2={{red, blue, green, yellow}}

P3={ϕ,{red, blue},{green, yellow}}

P4={{blue},{red, yellow, green}}

ii) Find all partitions of S={1, 2, 3}

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Subject: Liogic and Sets **Subject Code: 18MMU103**
Class : I B.Sc Mathematics **Semester : I**

UNIT -IV
PART A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A pair of objects whose components occur in a specific order is called an ----------	ordered pair	binary	pair	order	ordered pair
The ordered pairs (a,b) and (b,a) are -----unless a=b.	equal	not equal	parallel	not parallel	not equal
Two or more sets can be combined using ---	set identities	identities	operations	set operations	set operations
The --- of two sets A and B is the set of elements that belongs to A or to B	empty set	universal set	intersection	union	union
The set of all subsets of the set S is called the ---- of S.	power set	proper set	super set	subset	power set
Number of subsets of S having no element is called ------	null set	proper set	super set	subset	null set
| P(S)| =----	2n	2	n	n2	2n
The ordered pairs (a,b) and (c,d) are ---- iff a=c and b=d	equal	unequal	parallel	not parallel	parallel
The --- is finite	subset	superset	empty set	universal set	empty set
The set N,Z,Q and R are ----	finite sets	infinite sets	singleton sets	universal sets	infinite sets
The set of all lines in a given --- is an infinite set	line	point	plane	set	plane
Every ---- number is an integer	Real	rational	natural	irrational	natural
A set having only one element is called ---	singleton set	superset	empty set	universal set	singleton set
A set is said to be --- if the number of its elements is a positive integer	infinite	finite	empty	singleton	finite
A set having an unlimited number of elements is called an ---- sets	infinite	finite	empty	singleton	infinite
The set of ---less than 100 is a finite set.	even numbers	odd numbers	both even and odd	either even or odd	even numbers
N is a ---- subset of Z	regular	improper	proper	regular	proper
Every integer need not be a ---- number	whole	real	rational	natural	natural
The --- is unique	empty set	singleton set	universal set	finite set	empty set
If ---- then A and B are comparable sets	A=B	A<B	A>B	A^{1}B	A=B
If a set S has n elements, then its power set has ------	2	n	m	2^n	2^n
If A and B are sets, the set of all ordered pairs whose first component belongs to A and second component belongs to B is called the------	Binary product	Cartesian product	Ordered product	Binary relation	Cartesian product
The Cartesian product of more than n sets is the set of ordered of -----	s-tubles	n-tubles	2-tubles	4-tubles	n-tubles
The -------of two sets Aand B is the set of elements that belongs to both A and B.	Universal set	Empty set	intersection	Union	intersection
If A and B do not have any element in common then the sets A and B are said to be ----------	Disjoint	intersection	Union	Complement	Disjoint
The sets of elements which belongs to union but not belongs to A is called the of ------ A	Union	complement	Disjoint	intersection	complement
If A and B are any two sets, then the set of elements that belongs to A but do not belongs to B is called -------- of A and B	difference	Union	Intersection	Sum	difference
If A and B are any two sets, then the set of elements that belongs to A or B ,but not to both is called the --------- difference of A and B	symmetric	Antisymmetric	Reflexive	Irreflexive	symmetric
The principal of duality states that wherever S ,a statement of ------- of two expressions is valid then its dual is also valid	Intersection	equality	Unequal	Union	equality
All the sets identities of various laws one simply the --- of the corresponding set identities	Equal	duals	Non dual	Non equal	duals
The set of all points on a given ------- is an infinite set	Plane	Point	Set	Line	Line
{x/xϵR AND 0<X<1} is an	Finite set	Infinite set	Singleton set	Universal set	Infinite set
Q C R ,since every rational number is a -----number	whole	real	Rational	Irrational	real
If A={3,4,5} and B={x/xϵN and 2<x<6} then .	A=B	A<B	A>B	A$\neq$B	A=B
An -------is a subset of every set	Finite set	Singleton set	Empty set	Universal set	Empty set
If A is the set of odd integers and B is the set of even integers, then A and B are --------	Comparable set	Disjoint set	Equivalent set	Power set	Disjoint set
Union and intersection of sets are ----------	Empty	Component	Idempotent	Singleton	Idempotent
A set whose elements are also sets is called a-------- of sets	Degree	family	Order	Member	family
Complement of the intersection of two sets is the--------- of their complements	Union	complement	Disjoint	intersection	Union
Sets can also be represented graphically by means of------	Chart diagram	venn diagrams	Pie chart	Bar diagram	venn diagrams
The complement of union of two sets is the--------- of their complements	Idempotent	Complement	intersection	Union	intersection

UNIT-V

SYLLABUS

Relation: Product set, Composition of relations, Types of relations, Partitions. Equivalence Relations with example of congruence modulo relation, Partial ordering relations, n-ary relations.

Relations

Relations: Assume that we hav e a set of men M and a set of women W, some of whom are married. We want to express which men in M are married to which women in W. One way to do that is by listing the set of pairs (m, w) such that m is a man, w is a woman, and m is married to w. So, the relation "married to" can be represented by a subset of the Cartesian product $M \times W$. In general, a *relation* $\mathcal{R}$ from a set A to a set B will be understood as a subset of the Cartesian product $A \times B$, i.e., $\mathcal{R} \subseteq A \times B$. If an element $a \in A$ is related to an element $b \in B$, we often write $a \,\mathcal{R}\, b$ instead of $(a, b) \in \mathcal{R}$.

The set

$$\{a \in A \mid a \,\mathcal{R}\, b \text{ for some } b \in B\}$$

is called the *domain* of $\mathcal{R}$. The set

$$\{b \in B \mid a \,\mathcal{R}\, b \text{ for some } a \in A\}$$

is called the *range* of $\mathcal{R}$. For instance, in the relation "married to" above, the domain is the set of married men, and the range is the set of married women.

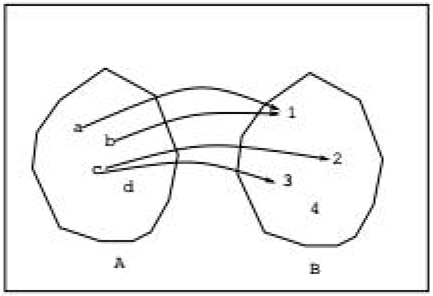
If A and B are the same set, then any subset of $A \times A$ will be a *binary relation* in A. For instance, assume $A = \{1, 2, 3, 4\}$. Then the binary relation "less than" in A will be:

$$\begin{aligned} <_A = \{(x, y) \in A \times A \mid x < y\} \\ = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}. \end{aligned}$$

Notation: A set A with a binary relation $\mathcal{R}$ is sometimes represented by the pair $(A, \mathcal{R})$. So, for instance, $(\mathbb{Z}, \leq)$ means the set of integers together with the relation of non-strict inequality.

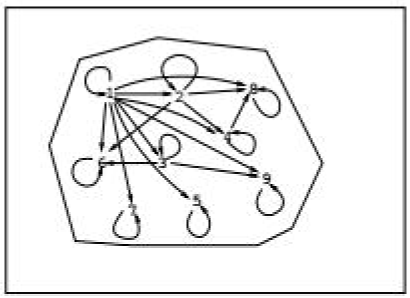
Representations of Relations.

Arrow diagrams. Venn diagrams and arrows can be used for representing relations between given sets. As an example, figure 2.14 represents the relation from $A = \{a, b, c, d\}$ to $B = \{1, 2, 3, 4\}$ given by $\mathcal{R} = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$. In the diagram an arrow from x to y means that x is related to y. This kind of graph is called *directed graph* or *digraph*.



Relation.

Another example is given in diagram 2.15, which represents the divisibility relation on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.



Binary relation of divisibility.

Matrix of a Relation. Another way of representing a relation $\mathcal{R}$ from A to B is with a matrix. Its rows are labeled with the elements of A, and its columns are labeled with the elements of B. If $a \in A$ and $b \in B$ then we write 1 in row a column b if $a \mathcal{R} b$, otherwise we write 0. For instance the relation $\mathcal{R} = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$ from $A = \{a, b, c, d\}$ to $B = \{1, 2, 3, 4\}$ has the following matrix:

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Inverse Relation. Given a relation $\mathcal{R}$ from A to B, the inverse of $\mathcal{R}$, denoted $\mathcal{R}^{-1}$, is the relation from B to A defined as

$$b \, \mathcal{R}^{-1} \, a \Leftrightarrow a \, \mathcal{R} \, b \, .$$

For instance, if $\mathcal{R}$ is the relation "being a son or daughter of", then $\mathcal{R}^{-1}$ is the relation "being a parent of".

Composition of Relations. Let A, B and C be three sets. Given a relation $\mathcal{R}$ from A to B and a relation $\mathcal{S}$ from B to C, then the composition $\mathcal{S} \circ \mathcal{R}$ of relations $\mathcal{R}$ and $\mathcal{S}$ is a relation from A to C defined by:

$$a\,(\mathcal{S} \circ \mathcal{R})\,c \Leftrightarrow \text{there exists some } b \in B \text{ such that } a\,\mathcal{R}\,b \text{ and } b\,\mathcal{S}\,c\,.$$

For instance, if $\mathcal{R}$ is the relation "to be the father of", and $\mathcal{S}$ is the relation "to be married to", then $\mathcal{S} \circ \mathcal{R}$ is the relation "to be the father in law of".

Properties of Binary Relations. A binary relation $\mathcal{R}$ on A is called:

1. *Reflexive* if for all $x \in A$, $x\,\mathcal{R}\,x$. For instance on $\mathbb{Z}$ the relation "equal to" $(=)$ is reflexive.
2. *Transitive* if for all $x, y, z \in A$, $x\,\mathcal{R}\,y$ and $y\,\mathcal{R}\,z$ implies $x\,\mathcal{R}\,z$. For instance equality $(=)$ and inequality $(<)$ on $\mathbb{Z}$ are transitive relations.
3. *Symmetric* if for all $x, y \in A$, $x\,\mathcal{R}\,y \Rightarrow y\,\mathcal{R}\,x$. For instance on $\mathbb{Z}$, equality $(=)$ is symmetric, but strict inequality $(<)$ is not.
4. *Antisymmetric* if for all $x, y \in A$, $x\,\mathcal{R}\,y$ and $y\,\mathcal{R}\,x$ implies $x = y$. For instance, non-strict inequality $(\leq)$ on $\mathbb{Z}$ is antisymmetric.

Partial Orders. A *partial order*, or simply, an *order* on a set A is a binary relation "$\preccurlyeq$" on A with the following properties:

1. *Reflexive*: for all $x \in A$, $x \preccurlyeq x$.
2. *Antisymmetric*: $(x \preccurlyeq y) \land (y \preccurlyeq x) \Rightarrow x = y$.
3. *Transitive*: $(x \preccurlyeq y) \land (y \preccurlyeq z) \Rightarrow x \preccurlyeq z$.

Examples:

1. The non-strict inequality $(\leq)$ in $\mathbb{Z}$.
2. Relation of divisibility on $\mathbb{Z}^+$: $a|b \Leftrightarrow \exists t,\ b = at$.

3. Set inclusion ($\subseteq$) on $\mathcal{P}(A)$ (the collection of subsets of a given set A).

Exercise: prove that the aforementioned relations are in fact partial orders. As an example we prove that integer divisibility is a partial order:

1. Reflexive: $a = a\,1 \Rightarrow a|a$.
2. Antisymmetric: $a|b \Rightarrow b = at$ for some t and $b|a \Rightarrow a = bt'$ for some t'. Hence $a = att'$, which implies $tt' = 1 \Rightarrow t' = t^{-1}$. The only invertible positive integer is 1, so $t = t' = 1 \Rightarrow a = b$.
3. Transitive: $a|b$ and $b|c$ implies $b = at$ for some t and $c = bt'$ for some t', hence $c = att'$, i.e., $a|c$.

Question: is the strict inequality ($<$) a partial order on $\mathbb{Z}$?

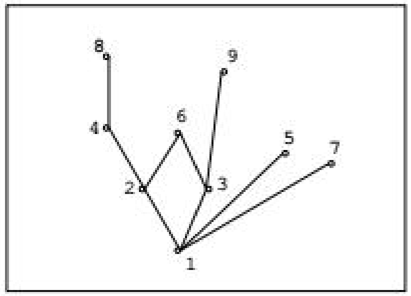
Two elements $a, b \in A$ are said to be *comparable* if either $x \preccurlyeq y$ or $y \preccurlyeq x$, otherwise they are said to be *non comparable*. The order is called *total* or *linear* when every pair of elements $x, y \in A$ are comparable. For instance $(\mathbb{Z}, \leq)$ is totally ordered, but $(\mathbb{Z}^+, |)$, where "$|$" represents integer divisibility, is not. A totally ordered subset of a partially ordered set is called a *chain*; for instance the set $\{1, 2, 4, 8, 16, \dots\}$ is a chain in $(\mathbb{Z}^+, |)$.

Hasse diagrams. A Hasse diagram is a graphical representation of a partially ordered set in which each element is represented by a dot (node or vertex of the diagram). Its immediate successors are placed above the node and connected to it by straight line segments. As an example, figure 2.16 represents the Hasse diagram for the relation of divisibility on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Question: How does the Hasse diagram look for a totally ordered set?

Equivalence Relations. An *equivalence relation* on a set A is a binary relation "$\sim$" on A with the following properties:

1. *Reflexive*: for all $x \in A$, $x \sim x$.
2. *Symmetric*: $x \sim y \Rightarrow y \sim x$.
3. *Transitive*: $(x \sim y) \wedge (y \sim z) \Rightarrow x \sim z$.



Hasse diagram for divisibility.

For instance, on $\mathbb{Z}$, the equality $(=)$ is an equivalence relation.

Another example, also on $\mathbb{Z}$, is the following: $x \equiv y \pmod 2$ ("x is congruent to y modulo 2") iff $x-y$ is even. For instance, $6 \equiv 2 \pmod 2$ because $6-2=4$ is even, but $7 \not\equiv 4 \pmod 2$, because $7-4=3$ is not even. Congruence modulo 2 is in fact an equivalence relation:

1. Reflexive: for every integer x, $x-x=0$ is indeed even, so $x \equiv x \pmod 2$.
2. Symmetric: if $x \equiv y \pmod 2$ then $x-y=t$ is even, but $y-x=-t$ is also even, hence $y \equiv x \pmod 2$.
3. Transitive: assume $x \equiv y \pmod 2$ and $y \equiv z \pmod 2$. Then $x-y=t$ and $y-z=u$ are even. From here, $x-z=(x-y)+(y-z)=t+u$ is also even, hence $x \equiv z \pmod 2$.

Equivalence Classes, Quotient Set, Partitions. Given an equivalence relation $\sim$ on a set A, and an element $x \in A$, the set of elements of A related to x are called the *equivalence class* of x, represented $[x] = \{y \in A \mid y \sim x\}$. Element x is said to be a *representative* of class

$[x]$. The collection of equivalence classes, represented $A/\!\sim\; = \{[x] \mid x \in A\}$, is called *quotient set* of A by $\sim$.

Congruence Modulo Relation:

Congruences are an important and useful tool for the study of divisibility. As we shall see, they are also critical in the art of cryptography.

Definition: *If a and b are integers and* $n>0$, *we write*

$$a \equiv b \bmod n$$

to mean $n|(b-a)$. *We read this as "a is congruent to b modulo (or mod) n.*

For example, $29 \equiv 8 \bmod 7$, and $60 \equiv 0 \bmod 15$.

The notation is used because the properties of congruence "$\equiv$" are very similar to the properties of equality "$=$". The next few result make this clear.

Theorem *For any integers a and b, and positive integer n, we have:*
1. $a \equiv a \bmod n$.
2. If $a \equiv b \bmod n$ *then* $b \equiv a \bmod n$.
3. If $a \equiv b \bmod n$ *and* $b \equiv c \bmod n$ *then* $a \equiv c \bmod n$

These results are classically called: 1. Reflexivity; 2. Symmetry; and 3. Transitivity. The proof is as follows:
1. $n|(a-a)$ since 0 is divisible by any integer. Therefore $a \equiv a \bmod n$.
2. If $a \equiv b \bmod n$ then $n|(b-a)$. Therefore, $n|(-1)(b-a)$ or $n|(a-b)$. Therefore, $b \equiv a \bmod n$.
3. If $a \equiv b \bmod n$ and $b \equiv c \bmod n$ then $n|(b-a)$ and $n|(c-b)$. Using the linear combination theorem, we have $n|(b-a+c-b)$ or $n|(c-a)$. Thus, $a \equiv c \bmod n$.

The following result gives an equivalent way of looking at congruence. It replaces the congruence sign with an equality.

Theorem *If* $a \equiv b \bmod n$ *then* $b = a + nq$ *for some integer q, and conversely.*

Proof: If $a \equiv b \bmod n$ then by definition $n|(b-a)$. Therefore, $b-a = nq$ for some q. Thus $b = a+nq$. Conversely if $b = a+nq$, then $b-a = nq$ and so $n|(b-a)$ and hence $a \equiv b \bmod n$ then $b = a + nq$.

We will use often this theorem for calculations. Thus, we can write $15 \equiv -2 \bmod 17$ by subtracting 17 from 15: $-2 = 15 + (-1) \cdot 17$. Similarly, $52 \equiv 12 \bmod 20$. Just subtract 40 (2 times 20) from 52.

A simple consequence is this: Any number is congruent mod n to its remainder when divided by n. For if $a = nq + r$, the above result shows that $a \equiv r \bmod n$. Thus for example, $23 \equiv 2 \bmod 7$ and $103 \equiv 3 \bmod 10$. For this reason, the remainder of a number a when divided by n is called $a \bmod n$. In EXCEL, as in many spreadsheets, this is written "MOD(a,n)." If you put the expression =MOD(23,7) in a cell, the readout will be simply 2. Try it!

Another way of relating congruence to remainders is as follows.

Theorem *If $a \equiv b \bmod n$ then a and b leave the same remainder when divided by n. Conversely if a and b leave the same remainder when divided by n, then $a \equiv b \bmod n$.*

Proof: Suppose $a \equiv b \bmod n$. Then by Theorem 3.3, $b = a + nq$. If a leaves the remainder r when divided by n, we have $a = nQ + r$ with $0 \leq r < n$. Therefore, $b = a + nq = nQ + r + nq = n(Q + r) + r$, and so b leaves the same remainder when divided by n. The converse is straightforward and we omit the proof.

We can now show some useful algebraic properties of congruences. Briefly, congruences can be added and multiplied.

Theorem *If $a \equiv b \bmod n$ and $c \equiv d \bmod n$ then*
1. $a + c \equiv b + d \bmod n$.
2. $ac \equiv bd \bmod n$.

Proof: Write $b = a + nq_1$ and $d = c + nq_2$, using Theorem 3.3. Then adding equalities, we get $b + d = a + c + nq_1 + nq_2 = a + c + n(q_1 + q_2)$. This shows that $a + c \equiv b + d \bmod n$ by Theorem 3.3.

Similarly, multiplying, we get $bd = (a + nq_1)(c + nq_2) = ac + naq_2 + ncq_1 + n^2q_1q_2$. Thus, $bd = ac + n(aq_2 + cq_1 + nq_1q_2$, and so $ac \equiv bd \bmod n$, again by Theorem 3.3.

Some Examples.
We have noted that $23 \equiv 2 \bmod 7$. We can square this (i.e. multiply this congruence by itself) to get $23^2 \equiv 4 \bmod 7$. What a nice way to find the remainder of 23^2 when it is divided by 7! Multiply again by $23 \equiv 2 \bmod 7$, to get

$$23^3 \equiv 8 \equiv 1 \bmod 7$$

(This string of congruences is similar to a string of inequalities. It is read 23^3 is congruent to 8 which is congruent to 1 mod 7. By transitivity (Theorem 3.2) this implies that 23^3 is congruent to 1 mod 7.) Once we know that $23^3 \equiv 1 \bmod 7$, we can raise to the 5th power (i.e. multiply this by itself 5 times) to get $23^{15} \equiv 1 \bmod 7$. The application of a few theorems and we have found remainders of huge numbers rather easily!

Example *Find* 17^{341} mod 5. As explained on page 26, this is the remainder when 17^{341} is divided by 5.

Method. We have

$$17 \equiv 2 \bmod 5$$

Squaring, we have

$$17^2 \equiv 4 \equiv -1 \bmod 5$$

Squaring again, we find

$$17^4 \equiv 1 \bmod 5$$

Now, 1 to any power is 1, so we raise this last congruence to the 85th power. Why 85? Just wait a moment to find out. We then find

$$17^{340} \equiv 1 \bmod 5$$

Finally, multiply by the first congruence to obtain

$$17^{341} \equiv 2 \bmod 5$$

So the required remainder is 2.

The strategy is to find some power of 17 to be 1 mod 5. Here, the power 4 worked. The we divided 4 into 341 to get a quotient 85, and this is the power we used on the congruence $17^4 \equiv 1 \bmod 5$. Note also the little trick of replacing 4 by -1 mod 5. This gives an easier number to square.

Example *Solve for* x : $5x \equiv 1 \bmod 12$.

One method is as follows. We know that $\gcd(5, 12) = 1$, so some linear combination of 5 and 12 is equal to 1. In Section 1 we had a general method for doing this, and we also had a spreadsheet approach. However, we can simply note by observation that

$$1 = 5 \cdot 5 + (-2) \cdot 12$$

So both sides of this equality are congruent to each other mod 12. Hence

$$1 \equiv 5 \cdot 5 + (-2) \cdot 12 \equiv 5 \cdot 5 \bmod 12$$

So one solution is $x = 5$. More generally, if $x \equiv 5 \bmod 12$ then

$$5x \equiv 25 \equiv 1 \bmod 12$$

Here is another approach: Start with the equation $5x \equiv 1 \bmod 12$. If this were an equality, we would simply divide by 5 to get $x = 1/5$. But we are in the realm of integers so this won't work. Instead we *multiply* by 5 to get $25x \equiv 5 \bmod 12$ or $x \equiv 5 \bmod 12$. Note that we multiplied by 5 to get a coefficient of 1: $5 \cdot 5 \equiv 1 \bmod 12$.

The algebra of congruences is sometime referred to as "clock arithmetic." This example illustrates this. Imagine you are a mouse and that each day you travel clockwise around a clock, passing through 25 minutes on the clock. You start at 12 o'clock. Here is what you journey will look like:

Start	Day 1	Day 2	Day 3	Day 4	Day 5
12 Midnight	5 o'clock	10 o'clock	3 o'clock	8 o'clock	1 o'clock

Note that the transition from 10 o'clock was not to 15 o'clock, but (working mod 12) to 15 mod 12 or 3 o'clock. In terms of clocks, we asked when the mouse would land at the 1 o'clock spot on the clock.

We can quickly find when the mouse will land at 4 o'clock. The equation is

$$5x \equiv 4 \bmod 12$$

Multiply by 5 to get $25x \equiv 20 \bmod 12$ or simply $x \equiv 8 \bmod 12$. It take 8 days.

Example *Same clock, different mouse. This mouse goes 23 minutes a day and starts at 12 o'clock. How many days before she reaches 9 minutes before 12?*

The appropriate congruence is $23x \equiv -9 \bmod 60$. We'll use the gcd method and find 1 as a linear combination of 23 and 60. A spreadsheet calculation gives

$$1 = -13 \cdot 23 + 5 \cdot 60$$

Taking this mod 60, we find

$$23(-13) \equiv 1 \bmod 60.$$

Multiply by -9 to get

$$23(117) \equiv -9 \bmod 60.$$

But $117 \equiv 57 \bmod 60$. And so the mouse must travel 57 days to reach 9 minutes before the hour. Note that $57 \equiv -3 \bmod 60$ so the mouse will take 3 days if she goes in the other direction.

Up to now, all of our congruences have been modulo one fixed n. The following results show how to change the modulus in certain situations.

Theorem *If $a \equiv b \bmod n$, and c is a positive integer, then $ca \equiv cb \bmod cn$*

Proof: This is little more than a divisibility theorem. Since $n|(b-a)$, we have $cn|c(b-a)$ or $cn|(cb-ca)$, and this is the result.

The converse is also valid. Thus, if $ca \equiv cb \bmod cn$ with $c > 0$ then $a \equiv b \bmod n$.

These results can be stated: A congruence can by multiplied through (including the modulus) and similarly, it can be divided by a common divisor.

Finally, we can mention that if $a \equiv b \bmod n$ and if $d|n$, then $a \equiv b \bmod d$. We leave the proof to the reader.

We can now tackle the general question of solving a linear congruence $ax \equiv b \bmod n$. We will find when this congruence has a solution, and how many solutions it has. We first consider the case $\gcd(a, n) = 1$. (In the examples above, this was the situation.) The following theorem answers this question and also shows how to find the solution.

Theorem *If $\gcd(a, n) = 1$, then the congruence $ax \equiv b \bmod n$ has a solution $x = c$. In this case, the general solution of the congruence is given by $x \equiv c \bmod n$.*

Proof: Since a and n are relative prime, we can express 1 as a linear combination of them:

$$ar + ns = 1$$

Multiply this by b to get $abr + nbs = b$. Take this mod n to get

$$abr + nbs \equiv b \bmod n \text{ or } abr \equiv b \bmod n$$

Thus $c = br$ is a solution of the congruence $ax \equiv b \bmod n$. In general, if $x \equiv c \bmod n$ we have $ax \equiv ac \equiv b \bmod n$.

We now claim that *any* solution of $ax \equiv b \bmod n$ is necessarily congruent to $c \bmod n$. For suppose $ax \equiv b \bmod n$. We already know that $ac \equiv b \bmod n$. Subtract to get

$$ax - ac \equiv 0 \bmod n \text{ or } a(x - c) \equiv 0 \bmod n$$

But this means that $n|a(x - br)$. But since a and n are relatively prime, this implies that $n|(x - c)$ and $x \equiv c \bmod n$. This completes the proof.
An important special case occurs when n is a prime p.

Corollary . *If p is a prime, the congruence $ax \equiv b \bmod p$ has a unique solution $x \bmod p$ provided $a \not\equiv 0 \bmod p$.*

The reason we single this case out is that this result is almost exactly like the similar result in high school algebra: The equation $ax = b$ has a unique solution provided $a \neq 0$. We shall soon delve further into this analogy. The reason this is true is that if an integer a is not divisible by p, it is relatively prime to p. Thus, if $a \not\equiv 0 \bmod p$, then a and p are relatively prime.

Theorem *If* $ab \equiv ac \bmod n$ *and if* $\gcd(a, n) = 1$*, then we have* $b \equiv c \bmod n$.

In short, we can cancel the factor a from both sides of the congruence so long as $\gcd(a, n) = 1$. In algebra, we learn that we "can divide an equation $ax = ay$ by a" if $a \neq 0$. Here we can "cancel the factor a from both sides of the congruence $ax \equiv ay \bmod n$" if a and n are relatively prime. This theorem is sometimes called the cancelation law for congruences.

Now suppose that we wish to solve the congruence $ax \equiv b \bmod n$ where $d = \gcd(a, n) > 1$. For example, consider the congruence $18x \equiv 12 \bmod 24$. Here $d = \gcd(18, 24) = 6$. We can divide this congruence by 6 to get the equivalent[15] congruence $3x \equiv 2 \bmod 4$. So we end up with the congruence $3x \equiv 2 \bmod 4$, in which $\gcd(3, 4) = 1$ and which has general solution $x \equiv 2 \bmod 4$. So this is the solution of the original congruence $18x \equiv 12 \bmod 24$. This worked because the gcd also divided the constant term 12. If it didn't there would be no solution. This is the content of the following theorem which generalizes this problem.

Theorem *Given the congruence* $ax \equiv b \bmod n$*. Let* $d = gcd(a, n)$*. Then*
1. If d *does not divide* b*, the congruence has no solution.*
2. If $d|b$ *then the congruence is equivalent to the congruence* $(a/d)x \equiv (b/d) \bmod (n/d)$ *which has a unique solution mod* n/d*.*

Proof: Suppose there were a solution of $ax \equiv b \bmod n$. Then we would have $ax \equiv b \bmod d$. But $a \equiv 0 \bmod d$ since $d|a$. So we would have $0 \equiv b \bmod d$ or $d|b$. So a necessary condition for a solution is that $d|b$. This prove part 1. As for part 2, divide the entire congruence by d as in the above example. The reduced congruence has a unique solution mod n/d since a/d and n/d are relatively prime.

Algebra on a Small Scale.

Corollary 3.11 has an interesting interpretation–if p is a prime and we work mod p, the integers mod p behave algebraically like the real numbers. In the real number system the equation $ax = b$ has a solution $x = b/a = ba^{-1}$ where $a^{-1} = 1/a$ is the reciprocal of a and is the solution of the equation $ax = 1$. What is the situation if we try to do this mod p?

Example *What is the value of* $5^{-1} \bmod 7$*?*

Method. It is required to find the solution of $5x \equiv 1 \bmod 7$. We can do this using the method of Example 3. Since

$$3 \cdot 5 + (-2)7 = 1$$

be observation, we have

$$3 \cdot 5 \equiv 1 \bmod 7$$

So $5^{-1} \equiv 3 \bmod 7$, or simply $5^{-1} = 3 \bmod 7$, where equality if used because it is understood that we are working mod 7.

Since we are working mod 7, there are only 7 different numbers mod 7, namely the remainders 0 through 6 when a number is divided by 7. So the algebra of numbers mod 7 is a strictly finite algebra. Here is the multiplication table for these numbers mod 7. We omit 0.

×	1	2	3	4	5	6
1	$\underline{1}$	2	3	4	5	6
2	2	4	6	$\underline{1}$	3	5
3	3	6	2	5	$\underline{1}$	4
4	4	$\underline{1}$	5	2	6	3
5	5	3	$\underline{1}$	6	4	2
6	6	5	4	3	2	$\underline{1}$

Multiplication Table mod 7

The number 1 is underlined in the body of the table. The row and column where a 1 appears are inverses, because the product is 1. By observation, we can see that 2 and 4 are inverses mod 7, as are 3 and 5. Both 1 and 6 are self inverses. (Note that $6 = -1$ mod 7, and so it is not surprising that 6 is its own inverse: $(-1)^{-1} = -1$.

Example *Solve the congruence* $8x \equiv 13 \bmod 29$.

First method. In analogy with algebra we expect the solution $x \equiv 13 \cdot 8^{-1} \bmod 29$. So we first compute $8^{-1} \bmod 29$. We express 1 as a linear combination of 8 and 29 by the method given in section 1, or using a spreadsheet. A possible result is

$$1 = 11 \cdot 8 - 3 \cdot 29$$

Taking this mod 29, we find $8^{-1} \equiv 11 \bmod 29$. So, solving for x, we find

$$x \equiv 13 \cdot 8^{-1} \equiv 13 \cdot 11 = 143 \equiv 27 \bmod 29$$

Second method. Using fractions, we write

$$x \equiv \frac{13}{8} \bmod 29$$

Ordinarily, we cancel factors in the numerator and denominator. We can't do this here, but we can *multiply* numerator and denominator by the same (non-zero) number. We choose 4, because this gets the denominator close to the modulus 29, making the quotient simpler. Thus

$$x \equiv \frac{13}{8} \equiv \frac{52}{32} \equiv \frac{23}{3} \bmod 29$$

Now do it again, using a factor 10:

$$\frac{23}{3} \equiv \frac{230}{30} \equiv \frac{27}{1} \equiv 27 \bmod 29$$

This is the same answer, of course. Here's the way the full solution works in one line:

$$x \equiv \frac{13}{8} \equiv \frac{52}{32} \equiv \frac{23}{3} \equiv \frac{230}{30} \equiv \frac{27}{1} \equiv 27 \bmod 29$$

Third method. When we write $x \equiv \frac{13}{8} \bmod 29$, we can cancel at least one factor 2, if we *add* 29 to the numerator. Thus,

$$x \equiv \frac{13}{8} \equiv \frac{42}{8} \equiv \frac{21}{4} \equiv \frac{50}{4} \equiv \frac{25}{2} \equiv \frac{54}{2} \equiv \frac{27}{1} \equiv 27 \bmod 29$$

We don't necessarily recommend this method, but we use it to illustrate that there are often many ways to attack a problem and to show the inner consistency of our small scale arithmetic.

Divisibility Tricks. The number 345,546,711 is divisible by 3. In fact it is divisible by 9. We can discover this easily using the following trick, which we shall prove.

A number is congruent mod 9 to the sum of the digits in that number.

Here we have

$$345,546,711 \equiv 3+4+5+5+4+6+7+1+1 = 36 \equiv 3+6 = 9 \equiv 0 \bmod 9$$

In fact, using this result, it is not even necessary to find the sum. There are short cuts. For example $3+4+5 = 12$ which is congruent to *its* digit sum $1+2 = 3 \bmod 9$. Continuing, add $5+5 = 10 \equiv 1$, so we add 1 to 3 to get 4. And so on. This is a lot easier to do than to explain. Briefly, any time you get a two digit answer, replace it by its digit sum.

The proof of this trick depends on the knowledge that the digits in an expansion of a number represent coefficient of powers of 10. Thus,

$$3,412 = 3 \times 10^3 + 4 \times 10^2 + 1 \times 10^1 + 2 \times 1$$

Since $10 \equiv 1 \bmod 9$, we can square to get $10^2 \equiv 1 \bmod 9$. Similarly, by cubing we get $10^3 \equiv 1 \bmod 9$, and so on. Thus,

$$3412 = 3 \times 10^3 + 4 \times 10^2 + 1 \times 10^1 + 2 \times 1 \equiv 3+4+1+2 \bmod 9$$

where the latter sum is simply the sum of the digits of 3412. This generalizes to give the result. It follows that a number is congruent to its digit sum mod 3, because if $a \equiv b \bmod n$ and $d|n$ then $a \equiv b \bmod n$. (Here $n = 9$ and $d = 3$.)

This simple trick has a useful application. It is a check on possible calculation errors. For example, suppose you are given the multiplication $341 \times 167 = 56847$ and you are suspicious of this result. (Perhaps someone was sloppy or didn't copy it down correctly.) Now if this multiplication were true, it would also be true mod 9. But $341 \equiv 8 \bmod 9$ (just add the digits!) and $167 \equiv 14 \equiv 5 \bmod 9$ so $341 \times 167 \equiv 8 \times 5 = 40 \equiv 4 \bmod 9$. But the answer given us was $56847 \equiv 30 \equiv 3 \bmod 9$, and so it was in error. This method is not failsafe, but it is a quick check.[16] Incidentally, you know that the multiplication $1234567 \times 245678 = 303305951435$ is wrong. (Hint: look at the last digits.) You know it's wrong by checking the answer mod 10.

There is another simple trick to find a number mod 11 using its digits. In this case, we find the alternating sum starting with the units column. For example, to find 56744 mod 11, we compute $56743 \equiv 3-4+7-6+5 = 5 \bmod 11$. The proof is similar to the proof above, and is based on the simple congruence $10 =\equiv -1 \bmod 11$. Squaring, we get $100 =\equiv 1 \bmod 11$. Cubing, we get $1000 \equiv 1 \bmod 11$, etc. Thus,

$$56743 = 3 + 4 \times 10 + 7 \times 10^2 + 6 \times 10^3 + 5 \times 10^4 \equiv 3 - 4 + 7 - 6 + 5 = 5 \bmod 11$$

The general proof is the same.

For example, the alleged calculation $345 \times 3456 = 1129320$ can be check mod 11. We have

$$345 \times 3456 \equiv (5-4+3)(6-5+4-3) = 4 \times 2 = 8 \bmod 11$$

The alleged answer is $1129320 \equiv 0-2+3-9+2-1+1 = -6 \equiv 5 \not\equiv 8 \bmod 11$. The actual answer for this multiplication is $11\underline{92}320$, so the error was a simple transposition of digits, a common error. The alternating sum will catch such an error.

Possible Questions

2 Mark Questions:

1. Define a relation on a set with examples
2. Define composition of relations with an example
3. Define equivalence class
4. Define partition of a set
5. Define generalized intersection of two sets.

6 Mark Questions:

1. State and prove equivalence class theorem on relations.
2. R and S are "congruent modulo 3" and "congruent modulo 4" relations respectively on the set of integers. Find

   (i) $R \cup S$ (ii) $R \cap S$ (iii) $R - S$ (iv) $S - R$ (v) $R \oplus S$.

3. If R is the relation on the set of integers such that (a, b) in R, iff $3a+4b = 7n$ for some integer n, prove that R is an equivalence relation.
4. Determine whether the relation R on the set off all integers is reflexive, symmetric, antisymmetric and /or transitive, where a R b iff (i) $a \neq b$ (ii) $ab \geq 0$ (iii) $ab \geq 1$ (iv) a is multiple of b
5. If R is the relation on A = {1, 2, 3} such that (a, b) in R, iff a + b = even. Find the relational matrix M_R.Find also the relational matrices R^{-1} ,R, R^2.

6. If R and S be relations on a set A represented by the matrices $M_R=\begin{bmatrix}0&1&0\\1&1&1\\1&0&0\end{bmatrix}$ and $M_S=\begin{bmatrix}0&1&0\\0&1&1\\1&1&1\end{bmatrix}$.

   Find the matrices that represent i) $R\cup S$ ii) $R \cap S$ iii) $R \bullet S$ iv) $S \bullet R$ v) $R \oplus S$

7. If the relation $R_1, R_2, ...R_6$ are defined on the set of real numbers as given below

   $R_1 = \{(a, b) / a > b\}$, $R_2 = \{(a, b) / a \geq b\}$, $R_3 = \{(a, b) / a < b\}$, $R_4 = \{(a, b) / a \leq b\}$, $R_5=\{(a, b) / a = b\}$, $R_6 = \{(a, b) / a \neq b\}$. Find the following composite relations $R_1 \bullet R_2$, $R_2 \bullet R_2$, $R_1 \bullet R_4$, $R_3 \bullet R_5$, $R_5 \bullet R_3$, $R_6 \bullet R_3$, $R_6 \bullet R_4$, $R_6 \bullet R_6$.

8. Let R={(1, 1), (1, 3), (3, 2), (3, 4), (4, 2)}and S={(2, 1), (3, 3), (3, 4), (4, 1)}. Find the following composite relations $R \bullet S$, $S \bullet R$, $R \bullet R$,$S \bullet S$, $(R \bullet S) \bullet R$, $R \bullet (S \bullet R)$, $R \bullet R \bullet R$.
9. The relation R on the set A={1, 2, 3, 4, 5} is defined by the rule (a, b) in R, if 3 divides a–b.

(i) List the elements of R and R^{-1}

(ii) Find the domain and range of R

(iii) Find the domain and range of R^{-1}

10. If R = {(1, 2), (2, 4), (3, 3)} and S={(1, 3), (2, 4), (4, 2)}. Find (i) $R \cup S$
(i) $R \cap S$ (iii) $R - S$ (iv) $S - R$ (v) $R \oplus S$. Also verify that
dom $(R \cup S)$ = dom (R) $\cup$ dom (S) and range $(R \cap S) \subseteq$ range (R) $\cap$ range(S)

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Class : I B.Sc Mathematics	**Semester : I**

UNIT -V

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A __________ R from a set A to a set B is a subset R of the cartesian product A´B	Relation	Binary relation	duality principle	partition of a set	Binary relation
Let R be a relation on a set A then if aRa for all a in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	reflexive
Let R be a relation on a set A then if aRb then bRa for all a,b in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	symmetric
Let R be a relation on a set A then if aRb and bRc then aRc for all a,b,c in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	transitive
A relation R on a set A is called an equivalence relation if R is	reflexive , symmetric and	reflexive , antisymmetric and	irreflexive , symmetric and	irreflexive , antisymmetric and	reflexive , symmetric and transitive
A relation R on a set A is called an partial order relation if R is	reflexive , symmetric and	reflexive , antisymmetric and	irreflexive , symmetric and	irreflexive , antisymmetric and	reflexive , antisymmetric and transitive
A poset in which every pair of elements have both a least upper bound and a greatest lower bound is called a ________	hasse diagram	maximal element	minimal element	lattice	lattice
If a relation is reflexive, symmetric and transitive then the relation is ___________	Relation	Binary relation	equivalence relation	partitial ordered relation	equivalence relation
If a relation is reflexive, anti symmetric and transitive then the relation is ___________	Relation	Binary relation	equivalence relation	partitial ordered relation	partitial ordered relation
The two relations symmetric and antisymmetric are __________	unique	equal	not equal	none of these	not equal
A binary relation R in a set X is said to be symmetric if ------	aRa	aRb implies bRa	aRb,bRc implies aRc	aRb,bRa implies a=b	aRb implies bRa
A binary relation R in a set X is said to be reflexive if ------	aRa	aRb implies bRa	aRb,bRc implies aRc	aRb,bRa implies a=b	aRa
A binary relation R in a set X is said to be antisymmetric if ------	aRa	aRb implies bRa	aRb,bRc implies aRc	aRb,bRa implies a=b	aRb,bRa implies a=b
A binary relation R in a set X is said to be transitive if ----	aRa	aRb implies bRa	aRb,bRc implies aRc	aRb,bRa implies a=b	aRb,bRc implies aRc
If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)} are relations then SoS = ------	{(4,2),(3,2),(1,4)}	{(1,5),(3,2),(2,5)}	{(1,2),(2,2)}	{(4,5),(3,3),(1,1)}	{(4,5),(3,3),(1,1)}
Let x = {1,2,3,4}, R = {(2,3),(4,1)} then the domain of R = ---------	{1,3}	{2,3}	{2,4}	{1,4}	{2,4}
Let x = {1,2,3,4}, R = {(2,3),(4,1)} then the range of R = ---------	{1,3}	{3,1}	{2,4}	{1,4}	{3,1}
In a relation matrix all the diagonal elements are one then it satisfies -------	symmetric	antisymmetric	transitive	reflexive	reflexive
In a relation matrix A=(aij) a_{ij} =a_{ji} then it satisfies ------- relation	symmetric	reflexive	transitive	antisymmetric	symmetric
A relation R on a set is said to be an equivalence relation if it is ------	Reflexive	Symmetric	Reflexive,Symmetric, Transitive	Transitive	Reflexive,Symmetric, Transitive
If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)} are relations then RoS = ------	{(4,2),(3,2),(1,4)}	{(1,5),(3,2),(2,5)}	{(1,2),(2,2)}	{(4,5),(3,3),(1,1)}	{(1,5),(3,2),(2,5)}
A relation R in a set X is ---------- if for every x in X, (x,x) in R	transitive	symmetric	irreflexive	reflexive	irreflexive
In N, define aRb if a+b = 7. This is symmetric when --------	b+a =7	a+a =7	b+c =7	a + c = 7	b+a =7
If the relation is ---------- relation if aRb,bRa implies a = b.	symmetric	reflexive	Antisymmetric	not reflexive	Antisymmetric
f from R to R, g from R to R defined by f(x) = 4x-1 and g(x) = cos x..The value of fog is ------	4cosx –1	4cosx	4cosx +1	1/4cosx	4cosx –1
The subsets in a partition are also called --- of partition	blocks	members	order	degree	blocks
The equivalence classes of A form a ----of A	member	partition	degree	order	partition
The --------A/R is a partition of A	quotient set	subset	super set	power set	quotient set
Symmetry and --- are not negative of each other	symmetry	not symmetry	anti symmetry	not anti symmetry	anti symmetry
The relation of similarity of --- is reflexive,symmetric and transitive	triangle	square	rectangle	cube	triangle
A relation R on a set A is aid to be --- if (a,a) in R	symmetric	reflexive	antisymmetric	irreflexive	reflexive
A relation R on a set A is ---- if there is no a in A	symmetry	not symmetry	reflexive	irreflexive	irreflexive
The quotient set A/R is ---of A	member	partition	degree	order	partition
The --- of A form a partition of A	equivalence logic	equivalence classes	class	logic	equivalence classes
Any element bÎ [a] is called --- of equivalence class [a]	member	order	degree	representative	representative
The collection of all equivalence classes of elements of A under an equivalence relation R is called --- of A by R	subset	super set	quotient set	universal set	quotient set

Reg no-------------

(18MMU103)

KARPAGAM ACADEMY OF HIGHER EDUCATION
Coimbatore-21
DEPARTMENT OF MATHEMATICS
First Semester
I Internal Test - AUG'2018
Logic and Sets

Date: 01-08-2018 **Time: 2 Hours**
Class: I-B.Sc Mathematics **Maximum Marks:50**

PART-A(20X1=20 Marks)

Answer all the Questions:

1. The equivalent statement for P and not P is ----------
   (a) T **(b) F** (c) F and T (d) T and F
2. The implications of P is --------------
   (a) P (b) not P (c) **P or Q** (d) P and Q
3. A statement which has true as the truth value for all the assignments is called -------
   (a) Contradiction (b) **tautology**
   (c) Either tautology or contradiction (d) true
4. P or P "equivalent to" P is called as -------
   (a) idempotent (b)associative (c)closure (d)identity
5. not(not P) "equivalent to" P is called as--------
   (a)Involution (b)Absorption (c)Associative (d)none
6. If P then Q is "equivalent to"----------
   (a) **not P or Q** (b) not P and Q (c)P and Q (d)P or Q
7. The substitution instance of a tautology is a-----
   (a) **tautology** (b)contradiction (c)identically false (d)false
8. If A = {1,2,3,4} and B = {2,4} then A intersection B = --------
   (a){2,4}(b){1,2,3,4} (c){1,2} d){}
9. A biconditional statement P if and only if Q is "equivalent to " --
   (a)**(Not P or Q) and (not Q or P)**
   (b)(Not P or Q) or (not Q or P)
   (c)(P or Q) and (not Q or P)
   (d)(Not P or Q) and (Q or P)
10. The other name of tautology is_______
    (a) **identically true** (b)identically false
    (c)universally false (d)false
11. The notation of "x+1>x" for all real value x, is ________
    (a) $\forall x P(x)$ (b) P(x) (c) $\exists x P(x)$ (d) $\sim P(x)$
12. In this statement $\forall x P(x)$, P(x) denotes ________
    (a) Essential quantifier (b) Universal quantifier
    (c) Predicates (d) subject
13. The statement ________is tautology.
    (a) $\mathbf{P \vee \neg P}$ (b) $P \rightarrow Q$ (c) $P \rightarrow (Q \rightarrow R)$ (d) P
14. Proposition is called________.
    (a) Statement (b) tautology (c) true (d) false
15. Tautologically implication means________
    (a) $P \rightarrow Q$ (b) $P \rightarrow Q$ is fallacy
    (c) **$P \rightarrow Q$ is tautology** (d) $P \rightarrow Q$ is Contradiction
16. The operation________ is called unary operator.
    (a) $\neg$ (b) $\wedge$ (c) $\vee$ (d) $\otimes$
17. The disjunction of two statement is denote the symbol________
    (a) $\neg$ (b) $\wedge$ **(c)** $\vee$ (d) $\otimes$
18. The equivalence $P \vee T \Leftrightarrow T$ is called ________
    (a) identity law (b) Negation law
    (c) Domination law (d) Absorption law
19. The Precedence of the operator $\leftrightarrow$is ________
    (a) 1 (b) 2 (c) 3 **(d) 5**
20. Which one is not Distributive?
    (a) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
    (b) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
    (c) $(P \vee Q) \wedge R \Leftrightarrow (P \wedge R) \vee (QC \wedge R)$
    (d) $\mathbf{(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)}$

PART-B (3X2=6 Marks)

Answer all the Questions:

21. Give the symbolic form of the statement "every book with a blue cover is a math6ematics book.

22. Construct the truth table for $(P \rightarrow Q) \rightarrow P$
23. Define bound variable.

PART-C (3X8=24 Marks)

Answer all the Questions:

24. (a) Without construct the truth table show that

$$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$$

(OR)

(b) Construct the truth table for $(P \Leftrightarrow Q) \Leftrightarrow (R \Leftrightarrow S)$

25. (a) Show that the following is an implication.

$$P \rightarrow (Q \rightarrow R) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$$

(OR)

(b) Define Tautology, Contraction and Fallacy with example.

26. (a) Write about predicate calculus.

(OR)

(b) Discuss about the types of quantifiers with example.