UNIT-I

SYLLABUS

Limits of functions , sequential criterion for limits, divergence criteria. Limit theorems, one sided limits. Infinite limits and limits at infinity. Continuous functions, sequential criterion for continuity and discontinuity.

LIMITS

Limits of Functions

4.1.1 Definition Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

This definition is rephrased in the language of neighborhoods as follows: A point c is a cluster point of the set A if every δ-neighborhood $V_\delta(c) = (c - \delta, c + \delta)$ of c contains at least one point of A distinct from c.

Note The point c may or may not be a member of A, but even if it is in A, it is ignored when deciding whether it is a cluster point of A or not, since we explicitly require that there be points in $V_\delta(c) \cap A$ distinct from c in order for c to be a cluster point of A.

For example, if $A := \{1, 2\}$, then the point 1 is not a cluster point of A, since choosing $\delta := \frac{1}{2}$ gives a neighborhood of 1 that contains no points of A distinct from 1. The same is true for the point 2, so we see that A has no cluster points.

4.1.2 Theorem *A number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.*

Proof. If c is a cluster point of A, then for any $n \in \mathbb{N}$ the $(1/n)$-neighborhood $V_{1/n}(c)$ contains at least one point a_n in A distinct from c. Then $a_n \in A$, $a_n \neq c$, and $|a_n - c| < 1/n$ implies $\lim(a_n) = c$.

Conversely, if there exists a sequence (a_n) in $A \backslash \{c\}$ with $\lim(a_n) = c$, then for any $\delta > 0$ there exists K such that if $n \geq K$, then $a_n \in V_\delta(c)$. Therefore the δ-neighborhood $V_\delta(c)$ of c contains the points a_n, for $n \geq K$, which belong to A and are distinct from c. Q.E.D.

4.1.3 Examples **(a)** For the open interval $A_1 := (0, 1)$, every point of the closed interval [0,1] is a cluster point of A_1. Note that the points 0, 1 are cluster points of A_1, but do not belong to A_1. All the points of A_1 are cluster points of A_1.

(b) A finite set has no cluster points.

(c) The infinite set $\mathbb{N}$ has no cluster points.

(d) The set $A_4 := \{1/n : n \in \mathbb{N}\}$ has only the point 0 as a cluster point. None of the points in A_4 is a cluster point of A_4.

4.1.4 Definition Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Remarks **(a)** Since the value of δ usually depends on ε, we will sometimes write $\delta(\varepsilon)$ instead of δ to emphasize this dependence.

(b) The inequality $0 < |x - c|$ is equivalent to saying $x \neq c$.

If L is a limit of f at c, then we also say that f **converges to L at c**. We often write

$$L = \lim_{x \to c} f(x) \quad \text{or} \quad L = \lim_{x \to c} f.$$

4.1.5 Theorem *If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.*

Proof. Suppose that numbers L and L' satisfy Definition 4.1.4. For any $\varepsilon > 0$, there exists $\delta(\varepsilon/2) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta(\varepsilon/2)$, then $|f(x) - L| < \varepsilon/2$. Also there exists $\delta'(\varepsilon/2)$ such that if $x \in A$ and $0 < |x - c| < \delta'(\varepsilon/2)$, then $|f(x) - L'| < \varepsilon/2$. Now let $\delta := \inf\{\delta(\varepsilon/2), \delta'(\varepsilon/2)\}$. Then if $x \in A$ and $0 < |x - c| < \delta$, the Triangle Inequality implies that

$$|L - L'| \leq |L - f(x)| + |f(x) - L'| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L - L' = 0$, so that $L = L'$. Q.E.D.

The definition of limit can be very nicely described in terms of neighborhoods. (See Figure 4.1.1.) We observe that because

$$V_\delta(c) = (c - \delta,\ c + \delta) = \{x : |x - c| < \delta\},$$

the inequality $0 < |x - c| < \delta$ is equivalent to saying that $x \neq c$ and x belongs to the δ-neighborhood $V_\delta(c)$ of c. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to saying that $f(x)$ belongs to the ε-neighborhood $V_\varepsilon(L)$ of L. In this way, we obtain the following result. The reader should write out a detailed argument to establish the theorem.

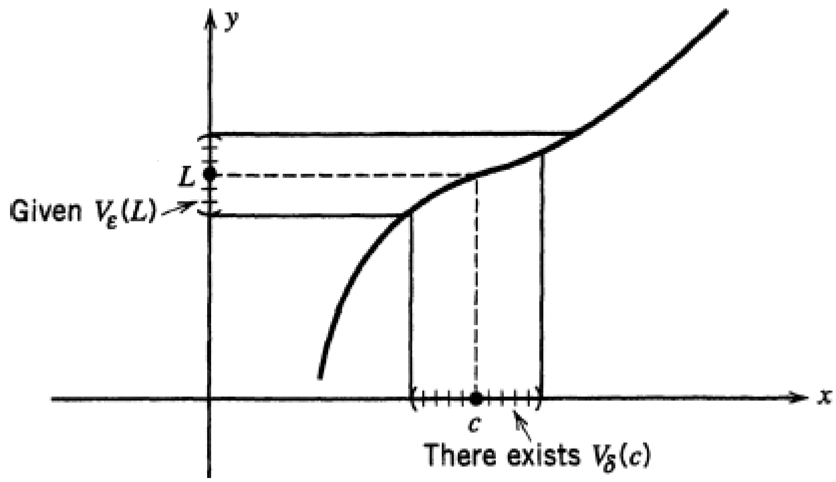


Figure 4.1.1 The limit of f at c is L

4.1.6 Theorem *Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.*

(i) $\lim\limits_{x\to c} f(x) = L$.

(ii) *Given any ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(L)$.*

4.1.7 Examples (a) $\lim\limits_{x\to c} b = b$.

To be more explicit, let $f(x) := b$ for all $x \in \mathbb{R}$. We want to show that $\lim\limits_{x\to c} f(x) = b$. If $\varepsilon > 0$ is given, we let $\delta := 1$. (In fact, any strictly positive δ will serve the purpose.) Then if

$0 < |x - c| < 1$, we have $|f(x) - b| = |b - b| = 0 < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, v conclude from Definition 4.1.4 that $\lim\limits_{x\to c} f(x) = b$.

(b) $\lim\limits_{x\to c} x = c$.

Let $g(x) := x$ for all $x \in \mathbb{R}$. If $\varepsilon > 0$, we choose $\delta(\varepsilon) := \varepsilon$. Then if $0 < |x - c| < \delta(\varepsilon$ we have $|g(x) - c| = |x - c| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim\limits_{x\to c} g = c$.

(c) $\lim\limits_{x\to c} x^2 = c^2$.

Let $h(x) := x^2$ for all $x \in \mathbb{R}$. We want to make the difference

$$\left|h(x) - c^2\right| = \left|x^2 - c^2\right|$$

less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to c. To do so, we note that $x^2 - c^2 = (x+c)(x-c)$. Moreover, if $|x-c| < 1$, then

$$|x| < |c| + 1 \qquad \text{so that} \qquad |x+c| \le |x| + |c| < 2\,|c| + 1.$$

Therefore, if $|x-c| < 1$, we have

$$(1) \qquad \left|x^2 - c^2\right| = |x+c||x-c| < (2\,|c|+1)\,|x-c|.$$

Moreover this last term will be less than ε provided we take $|x-c| < \varepsilon/(2|c|+1)$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\left\{1, \frac{\varepsilon}{2|c|+1}\right\},$$

then if $0 < |x-c| < \delta(\varepsilon)$, it will follow first that $|x-c| < 1$ so that (1) is valid, and therefore, since $|x-c| < \varepsilon/(2|c|+1)$ that

$$\left|x^2 - c^2\right| < (2|c|+1)|x-c| < \varepsilon.$$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim_{x\to c} h(x) = \lim_{x\to c} x^2 = c^2$.

(d) $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ if $c > 0$.

Let $\varphi(x) := 1/x$ for $x > 0$ and let $c > 0$. To show that $\lim_{x\to c} \varphi = 1/c$ we wish to make the difference

$$\left|\varphi(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right|$$

less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to $c > 0$. We first note that

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{1}{cx}(c-x)\right| = \frac{1}{cx}|x-c|$$

for $x > 0$. It is useful to get an upper bound for the term $1/(cx)$ that holds in some neighborhood of c. In particular, if $|x-c| < \frac{1}{2}c$, then $\frac{1}{2}c < x < \frac{3}{2}c$ (why?), so that

$$0 < \frac{1}{cx} < \frac{2}{c^2} \qquad \text{for} \qquad |x-c| < \frac{1}{2}c.$$

Therefore, for these values of x we have

$$(2) \qquad \left|\varphi(x) - \frac{1}{c}\right| \le \frac{2}{c^2}|x-c|.$$

In order to make this last term less than ε it suffices to take $|x-c| < \frac{1}{2}c^2\varepsilon$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\left\{\frac{1}{2}c,\ \frac{1}{2}c^2\varepsilon\right\},$$

then if $0 < |x-c| < \delta(\varepsilon)$, it will follow first that $|x-c| < \frac{1}{2}c$ so that (2) is valid, and therefore, since $|x-c| < \left(\frac{1}{2}c^2\right)\varepsilon$, that

$$\left|\varphi(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon.$$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim_{x\to c} \varphi = 1/c$.

(e) $\lim_{x\to 2} \dfrac{x^3-4}{x^2+1} = \dfrac{4}{5}$.

Let $\psi(x) := (x^3-4)/(x^2+1)$ for $x \in \mathbb{R}$. Then a little algebraic manipulation gives us

$$\left|\psi(x) - \frac{4}{5}\right| = \frac{|5x^3 - 4x^2 - 24|}{5(x^2+1)}$$
$$= \frac{|5x^3 + 6x + 12|}{5(x^2+1)} \cdot |x-2|.$$

To get a bound on the coefficient of $|x-2|$, we restrict x by the condition $1 < x < 3$. For x in this interval, we have $5x^2 + 6x + 12 \le 5 \cdot 3^2 + 6 \cdot 3 + 12 = 75$ and $5(x^2+1) \ge 5(1+1) = 10$, so that

$$\left|\psi(x) - \frac{4}{5}\right| \le \frac{75}{10}|x-2| = \frac{15}{2}|x-2|.$$

Now for given $\varepsilon > 0$, we choose

$$\delta(\varepsilon) := \inf\left\{1,\ \frac{2}{15}\varepsilon\right\}.$$

Then if $0 < |x-2| < \delta(\varepsilon)$, we have $|\psi(x) - (4/5)| \le (15/2)|x-2| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the assertion is proved. □

4.1.8 Theorem *(Sequential Criterion) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.*

(i) $\lim_{x \to c} f = L$.
(ii) *For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.*

Proof. (i) $\Rightarrow$ (ii). Assume f has limit L at c, and suppose (x_n) is a sequence in A with $\lim(x_n) = c$ and $x_n \neq c$ for all n. We must prove that the sequence $(f(x_n))$ converges to L. Let $\varepsilon > 0$ be given. Then by Definition 4.1.4, there exists $\delta > 0$ such that if $x \in A$ satisfies $0 < |x - c| < \delta$, then $f(x)$ satisfies $|f(x) - L| < \varepsilon$. We now apply the definition of convergent sequence for the given δ to obtain a natural number $K(\delta)$ such that if $n > K(\delta)$ then $|x_n - c| < \delta$. But for each such x_n we have $|f(x_n) - L| < \varepsilon$. Thus if $n > K(\delta)$, then $|f(x_n) - L| < \varepsilon$. Therefore, the sequence $(f(x_n))$ converges to L.

(ii) $\Rightarrow$ (i). [The proof is a contrapositive argument.] If (i) is not true, then there exists an ε_0-neighborhood $V_{\varepsilon_0}(L)$ such that no matter what δ-neighborhood of c we pick, there will be at least one number x_δ in $A \cap V_\delta(c)$ with $x_\delta \neq c$ such that $f(x_\delta) \notin V_{\varepsilon_0}(L)$. Hence for every $n \in \mathbb{N}$, the $(1/n)$-neighborhood of c contains a number x_n such that

$$0 < |x_n - c| < 1/n \quad \text{and} \quad x_n \in A,$$

but such that

$$|f(x_n) - L| \geq \varepsilon_0 \quad \text{for all} \quad n \in \mathbb{N}.$$

We conclude that the sequence (x_n) in $A \backslash \{c\}$ converges to c, but the sequence $(f(x_n))$ does not converge to L. Therefore we have shown that if (i) is not true, then (ii) is not true. We conclude that (ii) implies (i). Q.E.D.

4.1.9 Divergence Criteria *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.*

(a) *If $L \in \mathbb{R}$, then f does **not** have limit L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge to L.*

(b) *The function f does **not** have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge in $\mathbb{R}$.*

We now give some applications of this result to show how it can be used.

4.1.10 Examples **(a)** $\lim_{x\to 0}(1/x)$ does not exist in $\mathbb{R}$.

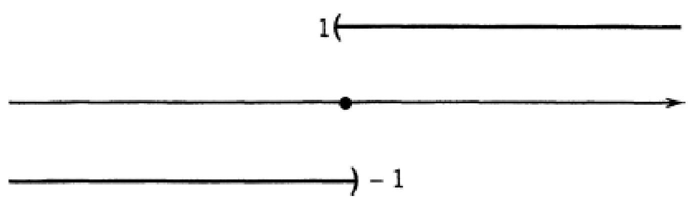
As in Example 4.1.7(d), let $\varphi(x) := 1/x$ for $x > 0$. However, here we consider $c = 0$. The argument given in Example 4.1.7(d) breaks down if $c = 0$ since we cannot obtain a bound such as that in (2) of that example. Indeed, if we take the sequence (x_n) with $x_n := 1/n$ for $n \in \mathbb{N}$, then $\lim(x_n) = 0$, but $\varphi(x_n) = 1/(1/n) = n$. As we know, the sequence $(\varphi(x_n)) = (n)$ is not convergent in $\mathbb{R}$, since it is not bounded. Hence, by Theorem 4.1.9(b), $\lim_{x\to 0}(1/x)$ does not exist in $\mathbb{R}$.

(b) $\lim_{x\to 0}\operatorname{sgn}(x)$ does not exist.

Let the **signum function** sgn be defined by

$$\operatorname{sgn}(x) := \begin{cases} +1 & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x = 0, \\ -1 & \text{for} \quad x < 0. \end{cases}$$

Note that $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$. (See Figure 4.1.2.) We shall show that sgn does not have a limit at $x = 0$. We shall do this by showing that there is a sequence (x_n) such that $\lim(x_n) = 0$, but such that $(\operatorname{sgn}(x_n))$ does not converge.



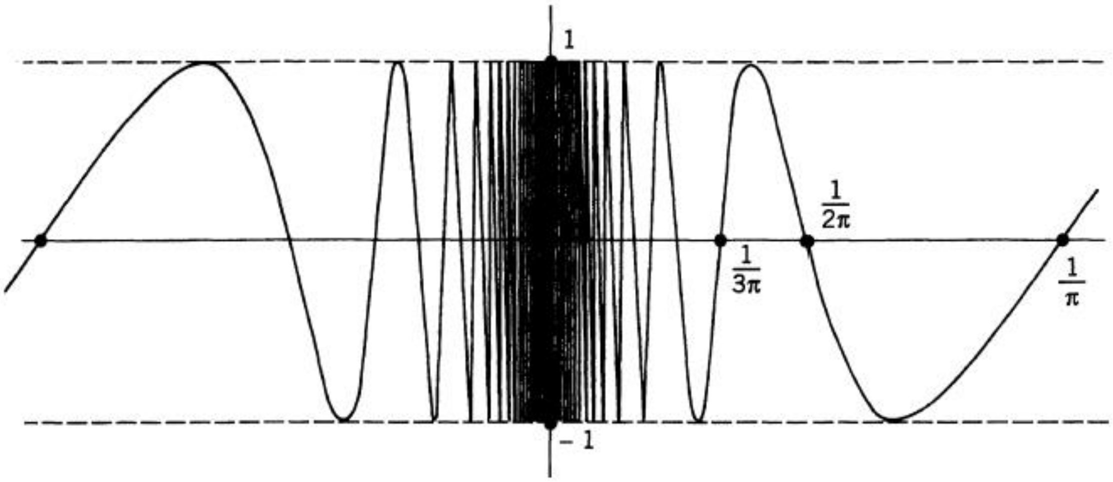
Indeed, let $x_n := (-1)^n/n$ for $n \in \mathbb{N}$ so that $\lim(x_n) = 0$. However, since

$$\operatorname{sgn}(x_n) = (-1)^n \quad \text{for} \quad n \in \mathbb{N},$$

it follows from Example 3.4.6(a) that $(\operatorname{sgn}(x_n))$ does not converge. Therefore $\lim_{x\to 0}\operatorname{sgn}(x)$ does not exist.

(c)[†] $\lim_{x\to 0}\sin(1/x)$ does not exist in $\mathbb{R}$.

Let $g(x) := \sin(1/x)$ for $x \neq 0$. (See Figure 4.1.3.) We shall show that g does not have a limit at $c = 0$, by exhibiting two sequences (x_n) and (y_n) with $x_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$ and such that $\lim(x_n) = 0$ and $\lim(y_n) = 0$, but such that $\lim(g(x_n)) \neq \lim(g(y_n))$. In view of Theorem 4.1.9 this implies that $\lim_{x\to 0} g$ cannot exist. (Explain why.)



Indeed, we recall from calculus that $\sin t = 0$ if $t = n\pi$ for $n \in \mathbb{Z}$, and that $\sin t = +1$ if $t = \frac{1}{2}\pi + 2\pi n$ for $n \in \mathbb{Z}$. Now let $x_n := 1/n\pi$ for $n \in \mathbb{N}$; then $\lim(x_n) = 0$ and $g(x_n) = \sin n\pi = 0$ for all $n \in \mathbb{N}$, so that $\lim(g(x_n)) = 0$. On the other hand, let $y_n := \left(\frac{1}{2}\pi + 2\pi n\right)^{-1}$ for $n \in \mathbb{N}$; then $\lim(y_n) = 0$ and $g(y_n) = \sin\left(\frac{1}{2}\pi + 2\pi n\right) = 1$ for all $n \in \mathbb{N}$, so that $\lim(g(y_n)) = 1$. We conclude that $\lim_{x \to 0} \sin(1/x)$ does not exist. □

Limit Theorems

4.2.1 Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. We say that f is **bounded on a neighborhood of** c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that we have $|f(x)| \le M$ for all $x \in A \cap V_\delta(c)$.

4.2.2 Theorem *If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.*

Proof. If $L := \lim_{x \to c} f$, then for $\varepsilon = 1$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < 1$; hence (by Corollary 2.2.4(a)),

$$|f(x)| - |L| \le |f(x) - L| < 1.$$

Therefore, if $x \in A \cap V_\delta(c),\ x \ne c$, then $|f(x)| \le |L| + 1$. If $c \notin A$, we take $M = |L| + 1$, while if $c \in A$ we take $M := \sup\{|f(c)|, |L| + 1\}$. It follows that if $x \in A \cap V_\delta(c)$, then $|f(x)| \le M$. This shows that f is bounded on the neighborhood $V_\delta(c)$ of c. Q.E.D.

4.2.3 Definition Let $A \subseteq \mathbb{R}$ and let f and g be functions defined on A to $\mathbb{R}$. We define the **sum** $f+g$, the **difference** $f-g$, and the **product** fg on A to $\mathbb{R}$ to be the functions given by

$$(f+g)(x) := f(x)+g(x), \qquad (f-g)(x) := f(x)-g(x),$$
$$(fg)(x) := f(x)g(x)$$

for all $x \in A$. Further, if $b \in \mathbb{R}$, we define the **multiple** bf to be the function given by

$$(bf)(x) := bf(x) \quad \text{for all} \quad x \in A.$$

Finally, if $h(x) \neq 0$ for $x \in A$, we define the **quotient** f/h to be the function given by

$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)} \qquad \text{for all} \qquad x \in A.$$

4.2.4 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Further, let $b \in \mathbb{R}$.*

(a) *If $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then:*

$$\lim_{x\to c}(f+g) = L+M, \qquad \lim_{x\to c}(f-g) = L-M,$$
$$\lim_{x\to c}(fg) = LM, \qquad \lim_{x\to c}(bf) = bL.$$

(b) *If $h : A \to \mathbb{R}$, if $h(x) \neq 0$ for all $x \in A$, and if $\lim_{x\to c} h = H \neq 0$, then*

$$\lim_{x\to c}\left(\frac{f}{h}\right) = \frac{L}{H}.$$

Proof. One proof of this theorem is exactly similar to that of Theorem 3.2.3. Alternatively, it can be proved by making use of Theorems 3.2.3 and 4.1.8. For example, let (x_n) be any sequence in A such that $x_n \neq c$ for $n \in \mathbb{N}$, and $c = \lim(x_n)$. It follows from Theorem 4.1.8 that

$$\lim(f(x_n)) = L, \qquad \lim(g(x_n)) = M.$$

On the other hand, Definition 4.2.3 implies that

$$(fg)(x_n) = f(x_n)g(x_n) \qquad \text{for} \quad n \in \mathbb{N}.$$

Therefore an application of Theorem 3.2.3 yields

$$\begin{aligned}\lim((fg)(x_n)) &= \lim(f(x_n)g(x_n)) \\ &= [\lim(f(x_n))]\,[\lim(g(x_n))] = LM.\end{aligned}$$

Consequently, it follows from Theorem 4.1.8 that

$$\lim_{x\to c}(fg) = \lim((fg)(x_n)) = LM.$$

Remark Let $A \subseteq \mathbb{R}$, and let $f_1, f_2, \ldots, f_n$ be functions on A to $\mathbb{R}$, and let c be a cluster point of A. If $L_k := \lim_{x \to c} f_k$ for $k = 1, \ldots, n$, then it follows from Theorem 4.2.4 by an Induction argument that

$$L_1 + L_2 + \cdots + L_n = \lim_{x \to c}(f_1 + f_2 + \cdots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim(f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if $L = \lim_{x \to c} f$ and $n \in \mathbb{N}$, then

$$L^n = \lim_{x \to c}(f(x))^n.$$

If we apply Theorem 4.2.4(b), we have

$$\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{\lim_{x \to 2}(x^3 - 4)}{\lim_{x \to 2}(x^2 + 1)} = \frac{4}{5}.$$

Note that since the limit in the denominator [i.e., $\lim_{x \to 2}(x^2 + 1) = 5$] is not equal to 0, then Theorem 4.2.4(b) is applicable.

(d) $\lim_{x \to 2} = \dfrac{x^2 - 4}{3x - 6} = \dfrac{4}{3}.$

If we let $f(x) := x^2 - 4$ and $h(x) := 3x - 6$ for $x \in \mathbb{R}$, then we *cannot* use Theorem 4.2.4(b) to evaluate $\lim (f(x)/h(x))$ because

4.2.5 Examples (a) Some of the limits that were established in Section 4.1 can be proved by using Theorem 4.2.4. For example, it follows from this result that since $\lim_{x \to c} x = c$, then $\lim_{x \to c} x^2 = c^2$, and that if $c > 0$, then

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{\lim_{x \to c} x} = \frac{1}{c}.$$

(b) $\lim_{x \to 2}(x^2 + 1)(x^3 - 4) = 20.$

It follows from Theorem 4.2.4 that

$$\lim_{x \to 2}(x^2 + 1)(x^3 - 4) = \left(\lim_{x \to 2}(x^2 + 1)\right)\left(\lim_{x \to 2}(x^3 - 4)\right) = 5 \cdot 4 = 20.$$

(c) $\lim_{x \to 2}\left(\dfrac{x^3 - 4}{x^2 + 1}\right) = \dfrac{4}{5}.$

If we apply Theorem 4.2.4(b), we have

$$\lim_{x\to 2}\frac{x^3-4}{x^2+1}=\frac{\lim\limits_{x\to 2}(x^3-4)}{\lim\limits_{x\to 2}(x^2+1)}=\frac{4}{5}.$$

Note that since the limit in the denominator [i.e., $\lim_{x\to 2}(x^2+1)=5$] is not equal to 0, then Theorem 4.2.4(b) is applicable.

(d) $\lim_{x\to 2} = \dfrac{x^2-4}{3x-6}=\dfrac{4}{3}.$

If we let $f(x):=x^2-4$ and $h(x):=3x-6$ for $x\in\mathbb{R}$, then we *cannot* use Theorem 4.2.4(b) to evaluate $\lim (f(x)/h(x))$ because

$$H=\lim_{x\to 2}h(x)=\lim_{x\to 2}(3x-6)=3\cdot 2-6=0.$$

However, if $x\neq 2$, then it follows that

$$\frac{x^2-4}{3x-6}=\frac{(x+2)(x-2)}{3(x-2)}=\frac{1}{3}(x+2).$$

Therefore we have

$$\lim_{x\to 2}\frac{x^2-4}{3x-6}=\lim_{x\to 2}\frac{1}{3}(x+2)=\frac{1}{3}\left(\lim_{x\to 2}x+2\right)=\frac{4}{3}.$$

Note that the function $g(x)=(x^2-4)/(3x-6)$ has a limit at $x=2$ *even though it is not defined there.*

(e) $\lim_{x\to 0}\dfrac{1}{x}$ does not exist in $\mathbb{R}$.

Of course $\lim_{x\to 0}1=1$ and $H:=\lim_{x\to 0}x=0$. However, since $H=0$, we *cannot* use Theorem 4.2.4(b) to evaluate $\lim_{x\to 0}(1/x)$. In fact, as was seen in Example 4.1.10(a), the function $\varphi(x)=1/x$ does not have a limit at $x=0$. This conclusion also follows from Theorem 4.2.2 since the function $\varphi(x)=1/x$ is not bounded on a neighborhood of $x=0$.

(f) If p is a polynomial function, then $\lim_{x\to c}p(x)=p(c)$.

Let p be a polynomial function on $\mathbb{R}$ so that $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ for all $x\in\mathbb{R}$. It follows from Theorem 4.2.4 and the fact that $\lim_{x\to c}x^k=c^k$ that

$$\begin{aligned}\lim_{x\to c}p(x)&=\lim_{x\to c}\left[a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right]\\&=\lim_{x\to c}(a_nx^n)+\lim_{x\to c}\left(a_{n-1}x^{n-1}\right)+\cdots+\lim_{x\to c}(a_1x)+\lim_{x\to c}a_0\\&=a_nc^n+a_{n-1}c^{n-1}+\cdots+a_1c+a_0\\&=p(c).\end{aligned}$$

Hence $\lim_{x\to c}p(x)=p(c)$ for any polynomial function p.

(g) If p and q are polynomial functions on $\mathbb{R}$ and if $q(c)\neq 0$, then

$$\lim_{x\to c}\frac{p(x)}{q(x)}=\frac{p(c)}{q(c)}.$$

Since $q(x)$ is a polynomial function, it follows from a theorem in algebra that there are at most a finite number of real numbers $\alpha_1, \dots, \alpha_m$ [the real zeroes of $q(x)$] such that $q(\alpha_j) = 0$ and such that if $x \notin \{\alpha_1, \dots, \alpha_m\}$, then $q(x) \neq 0$. Hence, if $x \notin \{\alpha_1, \dots, \alpha_m\}$, we can define

$$r(x) := \frac{p(x)}{q(x)}.$$

If c is not a zero of $q(x)$, then $q(c) \neq 0$, and it follows from part (f) that $\lim_{x\to c} q(x) = q(c) \neq 0$. Therefore we can apply Theorem 4.2.4(b) to conclude that

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)} = \frac{p(c)}{q(c)}. \qquad \square$$

4.2.7 Squeeze Theorem *Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$f(x) \leq g(x) \leq h(x) \quad \textit{for all} \quad x \in A, x \neq c,$$

and if $\lim_{x\to c} f = L = \lim_{x\to c} h$, then $\lim_{x\to c} g = L$.

4.2.8 Examples **(a)** $\lim_{x\to 0} x^{3/2} = 0 \; (x > 0)$.

Let $f(x) := x^{3/2}$ for $x > 0$. Since the inequality $x < x^{1/2} \leq 1$ holds for $0 < x \leq 1$ (why?), it follows that $x^2 \leq f(x) = x^{3/2} \leq x$ for $0 < x \leq 1$. Since

$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0} x = 0,$$

it follows from the Squeeze Theorem 4.2.7 that $\lim_{x\to 0} x^{3/2} = 0$.

(b) $\lim_{x\to 0} \sin x = 0$.

It will be proved later (see Theorem 8.4.8), that

$$-x \leq \sin x \leq x \quad \text{for all} \quad x \geq 0.$$

Since $\lim_{x\to 0} (\pm x) = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} \sin x = 0$.

(c) $\lim_{x\to 0} \cos x = 1$.

It will be proved later (see Theorem 8.4.8) that

(1) $$1 - \tfrac{1}{2}x^2 \leq \cos x \leq 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

Since $\lim_{x\to 0} \left(1 - \frac{1}{2}x^2\right) = 1$, it follows from the Squeeze Theorem that $\lim_{x\to 0} \cos x = 1$.

(d) $\lim_{x\to 0} \left(\dfrac{\cos x - 1}{x}\right) = 0$.

We cannot use Theorem 4.2.4(b) to evaluate this limit. (Why not?) However, it follows from the inequality (1) in part (c) that

$$-\tfrac{1}{2}x \le (\cos x - 1)/x \le 0 \quad \text{for} \quad x > 0$$

and that

$$0 \le (\cos x - 1)/x \le -\tfrac{1}{2}x \quad \text{for} \quad x < 0.$$

Now let $f(x) := -x/2$ for $x \ge 0$ and $f(x) := 0$ for $x < 0$, and let $h(x) := 0$ for $x \ge 0$ and $h(x) := -x/2$ for $x < 0$. Then we have

$$f(x) \le (\cos x - 1)/x \le h(x) \quad \text{for} \quad x \ne 0.$$

Since it is readily seen that $\lim_{x\to 0} f = 0 = \lim_{x\to 0} h$, it follows from the Squeeze Theorem that $\lim_{x\to 0} (\cos x - 1)/x = 0$.

(e) $\lim_{x\to 0} \left(\dfrac{\sin x}{x}\right) = 1.$

One-Sided Limits

There are times when a function f may not possess a limit at a point c, yet a limit does exist when the function is restricted to an interval on one side of the cluster point c.

For example, the signum function considered in Example 4.1.10(b), and illustrated in Figure 4.1.2, has no limit at $c = 0$. However, if we restrict the signum function to the interval $(0, \infty)$, the resulting function has a limit of 1 at $c = 0$. Similarly, if we restrict the signum function to the interval $(-\infty, 0)$, the resulting function has a limit of -1 at $c = 0$. These are elementary examples of right-hand and left-hand limits at $c = 0$.

4.3.1 Definition Let $A \in \mathbb{R}$ and let $f : A \to \mathbb{R}$.

(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A: x > c\}$, then we say that $L \in \mathbb{R}$ is a **right-hand limit of f at c** and we write

$$\lim_{x\to c+} f = L \quad \text{or} \quad \lim_{x\to c+} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

(ii) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, c) = \{x \in A: x < c\}$, then we say that $L \in \mathbb{R}$ is a **left-hand limit of f at c** and we write

$$\lim_{x\to c-} f = L \quad \text{or} \quad \lim_{x\to c-} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$.

Notes (1) The limits $\lim_{x \to c+} f$ and $\lim_{x \to c-} f$ are called **one-sided limits of f at c**. It is possible that neither one-sided limit may exist. Also, one of them may exist without the other existing. Similarly, as is the case for $f(x) := \text{sgn}(x)$ at $c = 0$, they may both exist and be different.

(2) If A is an interval with left endpoint c, then it is readily seen that $f : A \to \mathbb{R}$ has a limit at c if and only if it has a right-hand limit at c. Moreover, in this case the limit $\lim_{x \to c} f$ and the right-hand limit $\lim_{x \to c+} f$ are equal. (A similar situation occurs for the left-hand limit when A is an interval with right endpoint c.)

The reader can show that f can have only one right-hand (respectively, left-hand) limit at a point. There are results analogous to those established in Sections 4.1 and 4.2 for two-sided limits. In particular, the existence of one-sided limits can be reduced to sequential considerations.

4.3.2 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:*

(i) $\lim_{x \to c+} f = L$.

(ii) *For every sequence (x_n) that converges to c such that $x_n \in A$ and $x_n > c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.*

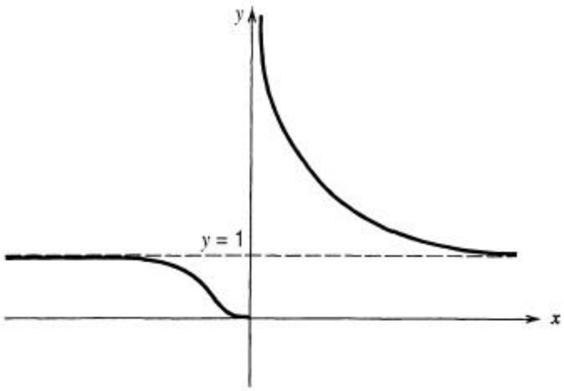


Figure 4.3.1 Graph of $g(x) = e^{1/x}$ $(x \neq 0)$

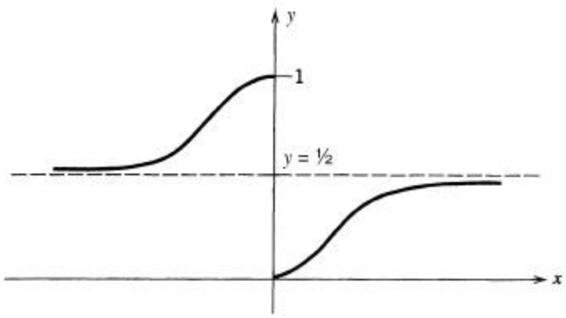


Figure 4.3.2 Graph of $h(x) = 1/(e^{1/x} + 1)$ $(x \neq 0)$

We leave the proof of this result (and the formulation and proof of the analogous result for left-hand limits) to the reader. We will not take the space to write out the formulations of the one-sided version of the other results in Sections 4.1 and 4.2.

The following result relates the notion of the limit of a function to one-sided limits. We leave its proof as an exercise.

4.3.3 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f = L$ if and only if $\lim_{x \to c+} f = L = \lim_{x \to c-} f$.*

4.3.4 Examples **(a)** Let $f(x) := \operatorname{sgn}(x)$.

We have seen in Example 4.1.10(b) that sgn does not have a limit at 0. It is clear that $\lim_{x \to 0+} \operatorname{sgn}(x) = +1$ and that $\lim_{x \to 0-} \operatorname{sgn}(x) = -1$. Since these one-sided limits are different, it also follows from Theorem 4.3.3 that $\operatorname{sgn}(x)$ does not have a limit at 0.

(b) Let $g(x) := e^{1/x}$ for $x \neq 0$. (See Figure 4.3.1.)

We first show that g does not have a finite right-hand limit at $c = 0$ since it is not bounded on any right-hand neighborhood $(0, \delta)$ of 0. We shall make use of the inequality

$$(1) \qquad 0 < t < e^t \quad \text{for} \quad t > 0,$$

which will be proved later (see Corollary 8.3.3). It follows from (1) that if $x > 0$, then $0 < 1/x < e^{1/x}$. Hence, if we take $x_n = 1/n$, then $g(x_n) > n$ for all $n \in \mathbb{N}$. Therefore $\lim_{x \to 0+} e^{1/x}$ does not exist in $\mathbb{R}$.

However, $\lim_{x \to 0-} e^{1/x} = 0$. Indeed, if $x < 0$ and we take $t = -1/x$ in (1) we obtain $0 < -1/x < e^{-1/x}$. Since $x < 0$, this implies that $0 < e^{1/x} < -x$ for all $x < 0$. It follows from this inequality that $\lim_{x \to 0-} e^{1/x} = 0$.

(c) Let $h(x) := 1/\left(e^{1/x} + 1\right)$ for $x \neq 0$. (See Figure 4.3.2.)

We have seen in part (b) that $0 < 1/x < e^{1/x}$ for $x > 0$, whence

$$0 < \frac{1}{e^{1/x} + 1} < \frac{1}{e^{1/x}} < x,$$

which implies that $\lim_{x \to 0+} h = 0$.

Since we have seen in part (b) that $\lim_{x \to 0-} e^{1/x} = 0$, it follows from the analogue of Theorem 4.2.4(b) for left-hand limits that

$$\lim_{x \to 0-} \left(\frac{1}{e^{1/x} + 1} \right) = \frac{1}{\lim_{x \to 0-} e^{1/x} + 1} = \frac{1}{0 + 1} = 1.$$

Note that for this function, both one-sided limits exist in $\mathbb{R}$, but they are unequal.

Infinite Limits

The function $f(x) := 1/x^2$ for $x \neq 0$ (see Figure 4.3.3) is not bounded on a neighborhood of 0, so it cannot have a limit in the sense of Definition 4.1.4. While the symbols $\infty(= +\infty)$ and $-\infty$ do not represent real numbers, it is sometimes useful to be able to say that "$f(x) = 1/x^2$ tends to ∞ as $x \to 0$." This use of $\pm\infty$ will not cause any difficulties, provided we exercise caution and *never* interpret ∞ or $-\infty$ as being real numbers.

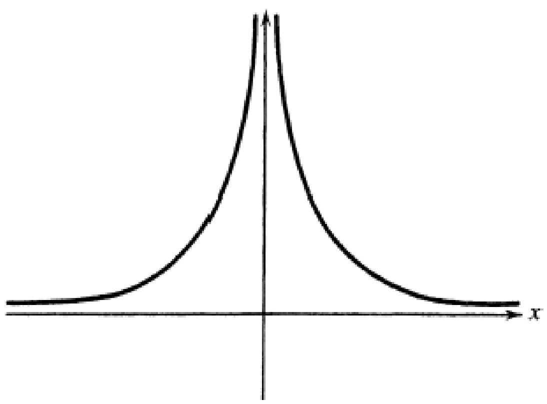


Figure 4.3.3 Graph of $f(x) = 1/x^2 \quad (x \neq 0)$

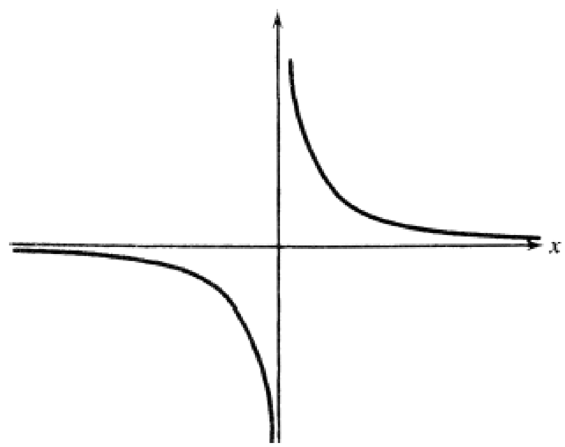


Figure 4.3.4 Graph of $g(x) = 1/x \quad (x \neq 0)$

4.3.5 Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A.

(i) We say that f **tends to ∞ as $x \to c$**, and write

$$\lim_{x \to c} f = \infty,$$

if for every $\alpha \in \mathbb{R}$ there exists $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.

(ii) We say that f **tends to $-\infty$ as $x \to c$**, and write

$$\lim_{x \to c} f = -\infty,$$

if for every $\beta \in \mathbb{R}$ there exists $\delta = \delta(\beta) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < \beta$.

4.3.6 Examples (a) $\lim_{x \to 0} (1/x^2) = \infty$.

For, if $\alpha > 0$ is given, let $\delta := 1/\sqrt{\alpha}$. It follows that if $0 < |x| < \delta$, then $x^2 < 1/\alpha$ so that $1/x^2 > \alpha$.

(b) Let $g(x) := 1/x$ for $x \neq 0$. (See Figure 4.3.4.)

The function g does *not* tend to either ∞ or $-\infty$ as $x \to 0$. For, if $\alpha > 0$ then $g(x) < \alpha$ for all $x < 0$, so that g does not tend to ∞ as $x \to 0$. Similarly, if $\beta < 0$ then $g(x) > \beta$ for all $x > 0$, so that g does not tend to $-\infty$ as $x \to 0$. □

While many of the results in Sections 4.1 and 4.2 have extensions to this limiting notion, not all of them do since $\pm\infty$ are not real numbers. The following result is an analogue of the Squeeze Theorem 4.2.7. (See also Theorem 3.6.4.)

4.3.7 Theorem *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Suppose that $f(x) \leq g(x)$ for all $x \in A, x \neq c$.*

(a) *If $\lim_{x \to c} f = \infty$, then $\lim_{x \to c} g = \infty$.*

(b) *If $\lim_{x \to c} g = -\infty$, then $\lim_{x \to c} f = -\infty$.*

Proof. (a) If $\lim_{x \to c} f = \infty$ and $\alpha \in \mathbb{R}$ is given, then there exists $\delta(\alpha) > 0$ such that if $0 < |x - c| < \delta(\alpha)$ and $x \in A$, then $f(x) > a$. But since $f(x) \leq g(x)$ for all $x \in A, x \neq c$, it follows that if $0 < |x - c| < \delta(\alpha)$ and $x \in A$, then $g(x) > \alpha$. Therefore $\lim_{x \to c} g = \infty$.

The proof of (b) is similar. Q.E.D.

4.3.8 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that f **tends to** ∞ [respectively, $-\infty$] as $x \to c+$, and we write

$$\lim_{x \to c+} f = \infty \quad \left[\text{respectively, } \lim_{x \to c+} f = -\infty\right],$$

if for every $\alpha \in \mathbb{R}$ there is $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

4.3.9 Examples **(a)** Let $g(x) := 1/x$ for $x \neq 0$. We have noted in Example 4.3.6(b) that $\lim_{x\to 0} g$ does not exist. However, it is an easy exercise to show that

$$\lim_{x\to 0+} (1/x) = \infty \quad \text{and} \quad \lim_{x\to 0-} (1/x) = -\infty.$$

(b) It was seen in Example 4.3.4(b) that the function $g(x) := e^{1/x}$ for $x \neq 0$ is not bounded on any interval $(0, \delta)$, $\delta > 0$. Hence the right-hand limit of $e^{1/x}$ as $x \to 0+$ does not exist in the sense of Definition 4.3.1(i). However, since

$$1/x < e^{1/x} \quad \text{for} \quad x > 0,$$

it is readily seen that $\lim_{x\to 0+} e^{1/x} = \infty$ in the sense of Definition 4.3.8. □

Limits at Infinity

It is also desirable to define the notion of the limit of a function as $x \to \infty$. The definition as $x \to -\infty$ is similar.

4.3.10 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that $L \in \mathbb{R}$ is a **limit of** f **as** $x \to \infty$, and write

$$\lim_{x\to\infty} f = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L,$$

if given any $\varepsilon > 0$ there exists $K = K(\varepsilon) > a$ such that for any $x > K$, then $|f(x) - L| < \varepsilon$.

The reader should note the close resemblance between 4.3.10 and the definition of a limit of a sequence.

We leave it to the reader to show that the limits of f as $x \to \pm\infty$ are unique whenever they exist. We also have sequential criteria for these limits; we shall only state the criterion as $x \to \infty$. This uses the notion of the limit of a properly divergent sequence (see Definition 3.6.1).

4.3.11 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $L = \lim_{x\to\infty} f$.

(ii) *For every sequence (x_n) in $A \cap (a, \infty)$ such that $\lim(x_n) = \infty$, the sequence $(f(x_n))$ converges to L.*

We leave it to the reader to prove this theorem and to formulate and prove the companion result concerning the limit as $x \to -\infty$.

4.3.12 Examples **(a)** Let $g(x) := 1/x$ for $x \neq 0$.

It is an elementary exercise to show that $\lim_{x\to\infty}(1/x) = 0 = \lim_{x\to-\infty}(1/x)$. (See Figure 4.3.4.)

(b) Let $f(x) := 1/x^2$ for $x \neq 0$.

The reader may show that $\lim_{x\to\infty}(1/x^2) = 0 = \lim_{x\to-\infty}(1/x^2)$. (See Figure 4.3.3.) One way to do this is to show that if $x \geq 1$ then $0 \leq 1/x^2 \leq 1/x$. In view of part (a), this implies that $\lim_{x\to\infty}(1/x^2) = 0$. □

Just as it is convenient to be able to say that $f(x) \to \pm\infty$ as $x \to c$ for $c \in \mathbb{R}$, it is convenient to have the corresponding notion as $x \to \pm\infty$. We will treat the case where $x \to \infty$.

4.3.13 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in A$. We say that f **tends to** ∞ [respectively, $-\infty$] **as** $x \to \infty$, and write

$$\lim_{x\to\infty} f = \infty \quad \left[\text{respectively, } \lim_{x\to\infty} f = -\infty\right]$$

if given any $\alpha \in \mathbb{R}$ there exists $K = K(\alpha) > a$ such that for any $x > K$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

As before there is a sequential criterion for this limit.

4.3.14 Theorem *Let $A \in \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $\lim_{x\to\infty} f = \infty$ [*respectively*, $\lim_{x\to\infty} f = -\infty$].

(ii) *For every sequence (x_n) in (a, ∞) such that* $\lim(x_n) = \infty$, *then* $\lim(f(x_n)) = \infty$ [*respectively*, $\lim(f(x_n)) = -\infty$].

The next result is an analogue of Theorem 3.6.5.

4.3.15 Theorem *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose further that $g(x) > 0$ for all $x > a$ and that for some $L \in \mathbb{R}, L \neq 0$, we have*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

(i) *If* $L > 0$, *then* $\lim_{x\to\infty} f = \infty$ *if and only if* $\lim_{x\to\infty} g = \infty$.

(ii) *If* $L < 0$, *then* $\lim_{x\to\infty} f = -\infty$ *if and only if* $\lim_{x\to\infty} g = \infty$.

Proof. (i) Since $L > 0$, the hypothesis implies that there exists $a_1 > a$ such that

$$0 < \frac{1}{2}L \le \frac{f(x)}{g(x)} < \frac{3}{2}L \qquad \text{for} \qquad x > a_1.$$

Therefore we have $\left(\frac{1}{2}L\right)g(x) < f(x) < \left(\frac{3}{2}L\right)g(x)$ for all $x > a_1$, from which the conclusion follows readily.

The proof of (ii) is similar. Q.E.D.

We leave it to the reader to formulate the analogous result as $x \to -\infty$.

4.3.16 Examples (a) $\lim_{x\to\infty} x^n = \infty$ for $n \in \mathbb{N}$.

Let $g(x) := x^n$ for $x \in (0, \infty)$. Given $\alpha \in \mathbb{R}$, let $K := \sup\{1, \alpha\}$. Then for all $x > K$, we have $g(x) = x^n \ge x > \alpha$. Since $\alpha \in \mathbb{R}$ is arbitrary, it follows that $\lim_{x\to\infty} g = \infty$.

(b) $\lim_{x\to-\infty} x^n = \infty$ for $n \in \mathbb{N}$, n even, and $\lim_{x\to-\infty} x^n = -\infty$ for $n \in \mathbb{N}$, n odd.

We will treat the case n odd, say $n = 2k + 1$ with $k = 0, 1, \ldots$. Given $\alpha \in \mathbb{R}$, let $K := \inf\{\alpha, -1\}$. For any $x < K$, then since $(x^2)^k \ge 1$, we have $x^n = (x^2)^k x \le x < \alpha$. Since $\alpha \in \mathbb{R}$ is arbitrary, it follows that $\lim_{x\to-\infty} x^n = -\infty$.

(c) Let $p : \mathbb{R} \to \mathbb{R}$ be the polynomial function

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then $\lim_{x\to\infty} p = \infty$ if $a_n > 0$, and $\lim_{x\to\infty} p = -\infty$ if $a_n < 0$.

Indeed, let $g(x) := x^n$ and apply Theorem 4.3.15. Since

$$\frac{p(x)}{g(x)} = a_n + a_{n-1}\left(\frac{1}{x}\right) + \cdots + a_1\left(\frac{1}{x^{n-1}}\right) + a_0\left(\frac{1}{x^n}\right),$$

it follows that $\lim_{x\to\infty}(p(x)/g(x)) = a_n$. Since $\lim_{x\to\infty} g = \infty$, the assertion follows from Theorem 4.3.15.

(d) Let p be the polynomial function in part (c). Then $\lim_{x\to-\infty} p = \infty$ [respectively, $-\infty$] if n is even [respectively, odd] and $a_n > 0$.

We leave the details to the reader. □

POSSIBLE QUESTIONS

2 Mark Questions:

1. Define cluster point.
2. Define limit of a function.
3. Define one sided limit.
4. State sequential criterion for limits.
5. State sequential criterion for continuity.
6. Define algebra of functions.
7. Define infinite limits.
8. Explain continuous function.

8 Mark Questions:

1. Prove that composition of two continuous function is also a continuous function.
2. If f is continuous at c, prove that $|f|$ is continuous at c.
3. If p is a polynomial function, prove that $\lim_{x\to c} p(x) = p(c)$.
4. Let $A \subset R$, let f and g be continuous functions on A to R. Suppose f and g are continuous at $c \in A$. Then prove that $f + g, fg$ and bf are continuous at c.
5. Prove that $\lim_{x\to 0}(\frac{1}{x})$ does not exist in R .
6. Discuss about the limit of signum function at 0.
7. State and prove sequential criterion for limits.
8. State and prove squeeze theorem.

UNIT-II

SYLLABUS

Algebra of continuous functions. Continuous functions on an interval, intermediate value theorem, location of roots theorem, preservation of intervals theorem. Uniform continuity, non-uniform continuity criteria, uniform continuity theorem.

CONTINUOUS FUNCTIONS

Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any number $\varepsilon > 0$, there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

If f fails to be continuous at c, then we say that f is **discontinuous at** c.

As with the definition of limit, the definition of continuity at a point can be formulated very nicely in terms of neighborhoods. This is done in the next result. We leave the verification as an important exercise for the reader. See Figure 5.1.1.

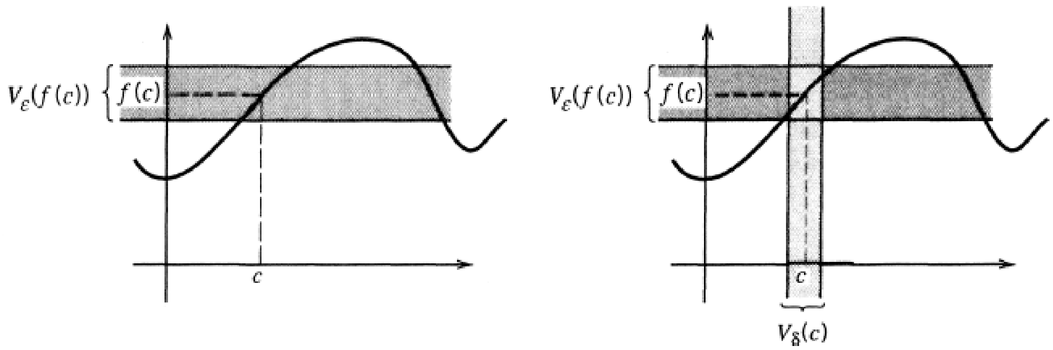


Figure 5.1.1 Given $V_\varepsilon(f(c))$, a neighborhood $V_\delta(c)$ is to be determined

5.1.2 Theorem *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if and only if given any ε-neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there exists a δ-neighborhood $V_\delta(c)$ of c such that if x is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\varepsilon(f(c))$, that is,*

$$f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c)).$$

Remarks **(1)** If $c \in A$ is a cluster point of A, then a comparison of Definitions 4.1.4 and 5.1.1 show that f is continuous at c if and only if

$$f(c) = \lim_{x \to c} f(x). \tag{1}$$

Thus, if c is a cluster point of A, then three conditions must hold for f to be continuous at c:

(i) f must be defined at c (so that $f(c)$ makes sense),
(ii) the limit of f at c must exist in $\mathbb{R}$ (so that $\lim_{x \to c} f(x)$ makes sense), and
(iii) these two values must be equal.

(2) If $c \in A$ is not a cluster point of A, then there exists a neighborhood $V_\delta(c)$ of c such that $A \cap V_\delta(c) = \{c\}$. Thus we conclude that a function f is automatically continuous at a point $c \in A$ that is not a cluster point of A. Such points are often called "isolated points" of A. They are of little practical interest to us, since they have no relation to a limiting process. Since continuity is automatic for such points, we generally test for continuity only at cluster points. Thus we regard condition (1) as being characteristic for continuity at c.

A slight modification of the proof of Theorem 4.1.8 for limits yields the following sequential version of continuity at a point.

5.1.3 Sequential Criterion for Continuity *A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

5.1.4 Discontinuity Criterion *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$.*

So far we have discussed continuity at a *point*. To talk about the continuity of a function on a *set*, we will simply require that the function be continuous at each point of the set. We state this formally in the next definition.

5.1.5 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If B is a subset of A, we say that f is **continuous on the set** B if f is continuous at every point of B.

5.1.6 Examples **(a)** The constant function $f(x) := b$ is continuous on $\mathbb{R}$.

It was seen in Example 4.1.7(a) that if $c \in \mathbb{R}$, then $\lim_{x \to c} f(x) = b$. Since $f(c) = b$, we have $\lim_{x \to c} f(x) = f(c)$, and thus f is continuous at every point $c \in \mathbb{R}$. Therefore f is continuous on $\mathbb{R}$.

(b) $g(x) := x$ is continuous on $\mathbb{R}$.

It was seen in Example 4.1.7(b) that if $c \in \mathbb{R}$, then we have $\lim_{x \to c} g = c$. Since $g(c) = c$, then g is continuous at every point $c \in \mathbb{R}$. Thus g is continuous on $\mathbb{R}$.

(c) $h(x) := x^2$ is continuous on $\mathbb{R}$.

It was seen in Example 4.1.7(c) that if $c \in \mathbb{R}$, then we have $\lim_{x \to c} h = c^2$. Since $h(c) = c^2$, then h is continuous at every point $c \in \mathbb{R}$. Thus h is continuous on $\mathbb{R}$.

(d) $\varphi(x) := 1/x$ is continuous on $A := \{x \in \mathbb{R} : x > 0\}$.

It was seen in Example 4.1.7(d) that if $c \in A$, then we have $\lim_{x \to c} \varphi = 1/c$. Since $\varphi(c) = 1/c$, this shows that φ is continuous at every point $c \in A$. Thus φ is continuous on A.

(e) $\varphi(x) := 1/x$ is not continuous at $x = 0$.

Indeed, if $\varphi(x) = 1/x$ for $x > 0$, then φ is not defined for $x = 0$, so it cannot be continuous there. Alternatively, it was seen in Example 4.1.10(a) that $\lim_{x \to 0} \varphi$ does not exist in $\mathbb{R}$, so φ cannot be continuous at $x = 0$.

(f) The signum function sgn is not continuous at 0.

The signum function was defined in Example 4.1.10(b), where it was also shown that $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist in $\mathbb{R}$. Therefore sgn is not continuous at $x = 0$ (even though sgn 0 is defined). It is an exercise to show that sgn is continuous at every point $c \neq 0$.

Note In the next two examples, we introduce functions that played a significant role in the development of real analysis. Discontinuities are emphasized and it is not possible to graph either of them satisfactorily. The intuitive idea of drawing a curve in the plane to represent a function simply does not apply, and plotting a handful of points gives only a hint of their character. In the nineteenth century, these functions clearly demonstrated the need for a precise and rigorous treatment of the basic concepts of analysis. They will reappear in later sections.

(g) Let $A := \mathbb{R}$ and let f be Dirichlet's "discontinuous function" defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

We claim that f is *not continuous at any point of* $\mathbb{R}$. (This function was introduced in 1829 by P. G. L. Dirichlet.)

Indeed, if c is a rational number, let (x_n) be a sequence of irrational numbers that converges to c. (Corollary 2.4.9 to the Density Theorem 2.4.8 assures us that such a sequence does exist.) Since $f(x_n) = 0$ for all $n \in \mathbb{N}$, we have $\lim(f(x_n)) = 0$, while $f(c) = 1$. Therefore f is not continuous at the rational number c.

On the other hand, if b is an irrational number, let (y_n) be a sequence of rational numbers that converge to b. (The Density Theorem 2.4.8 assures us that such a sequence does exist.) Since $f(y_n) = 1$ for all $n \in \mathbb{N}$, we have $\lim(f(y_n)) = 1$, while $f(b) = 0$. Therefore f is not continuous at the irrational number b.

Since every real number is either rational or irrational, we deduce that f is not continuous at any point in $\mathbb{R}$.

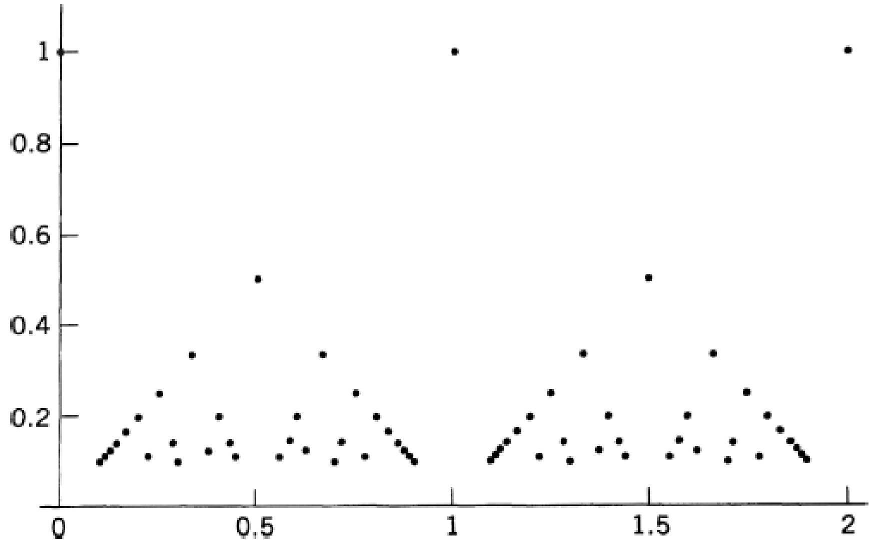


Figure 5.1.2 Thomae's function

(h) Let $A := \{x \in \mathbb{R}: x > 0\}$. For any irrational number $x > 0$ we define $h(x) := 0$. For a rational number in A of the form m/n, with natural numbers m, n having no common factors except 1, we define $h(m/n) := 1/n$. (We also define $h(0) := 1$.)

We claim that *h is continuous at every irrational number in A, and is discontinuous at every rational number in A*. (This function was introduced in 1875 by K. J. Thomae.)

Indeed, if $a > 0$ is rational, let (x_n) be a sequence of irrational numbers in A that converges to a. Then $\lim(h(x_n)) = 0$, while $h(a) > 0$. Hence h is discontinuous at a.

On the other hand, if b is an irrational number and $\varepsilon > 0$, then (by the Archimedean Property) there is a natural number n_0 such that $1/n_0 < \varepsilon$. There are only a finite number of rationals with denominator less than n_0 in the interval $(b-1, b+1)$. (Why?) Hence $\delta > 0$ can be chosen so small that the neighborhood $(b-\delta, b+\delta)$ contains no rational numbers with denominator less than n_0. It then follows that for $|x-b| < \delta, x \in A$, we have $|h(x) - h(b)| = |h(x)| \le 1/n_0 < \varepsilon$. Thus h is continuous at the irrational number b.

5.1.7 Remarks **(a)** Sometimes a function $f: A \to \mathbb{R}$ is not continuous at a point c because it is not defined at this point. However, if the function f has a limit L at the point c and if we define F on $A \cup \{c\} \to \mathbb{R}$ by

$$F(x) := \begin{cases} L & \text{for} \quad x = c, \\ f(x) & \text{for} \quad x \in A, \end{cases}$$

then F is continuous at c. To see this, one needs to check that $\lim_{x \to c} F = L$, but this follows (why?), since $\lim_{x \to c} f = L$.

(b) If a function $g : A \to \mathbb{R}$ does not have a limit at c, then there is no way that we can obtain a function $G : A \cup \{c\} \to \mathbb{R}$ that is continuous at c by defining

$$G(x) := \begin{cases} C & \text{for} \quad x = c, \\ g(x) & \text{for} \quad x \in A. \end{cases}$$

To see this, observe that if $\lim\limits_{x \to c} G$ exists and equals C, then $\lim\limits_{x \to c} g$ must also exist and equal C.

5.1.8 Examples **(a)** The function $g(x) := \sin(1/x)$ for $x \neq 0$ (see Figure 4.1.3) does not have a limit at $x = 0$ (see Example 4.1.10(c)). Thus there is no value that we can assign at $x = 0$ to obtain a continuous extension of g at $x = 0$.

(b) Let $f(x) := x \sin(1/x)$ for $x \neq 0$. (See Figure 5.1.3.) It was seen in Example 4.2.8(f) that $\lim\limits_{x \to 0} (x \sin(1/x)) = 0$. Therefore it follows from Remark 5.1.7(a) that if we define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) := \begin{cases} 0 & \text{for} \quad x = 0, \\ x \sin(1/x) & \text{for} \quad x \neq 0, \end{cases}$$

then F is continuous at $x = 0$. □

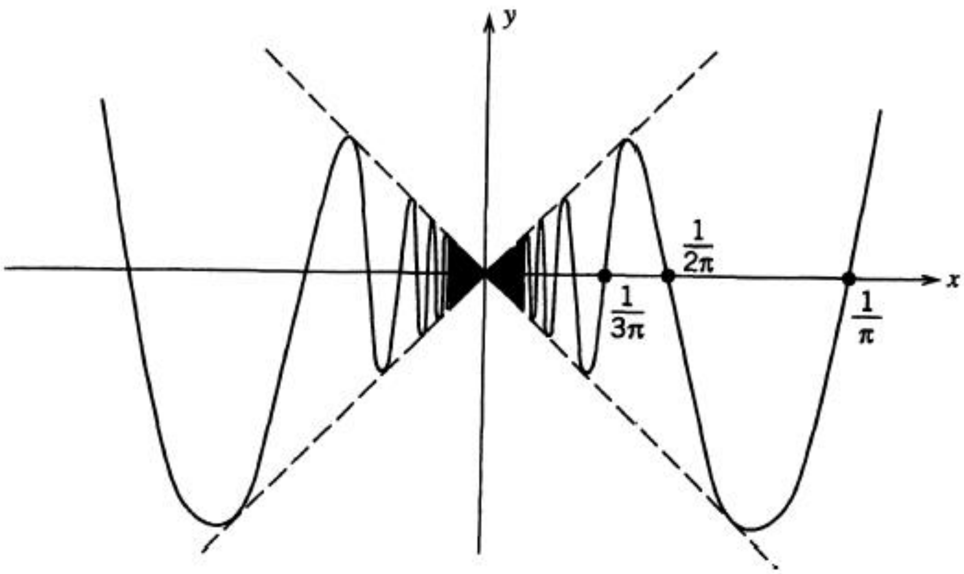


Figure 5.1.3 Graph of $f(x) = x \sin(1/x)$ $(x \neq 0)$

Continuous Functions on Intervals

Definition A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in A$.

In other words, a function is bounded on a set if its range is a bounded set in $\mathbb{R}$. To say that a function is *not* bounded on a given set is to say that no particular number can serve as a bound for its range. In exact language, a function f is not bounded on the set A if given any $M > 0$, there exists a point $x_M \in A$ such that $|f(x_M)| > M$. We often say that f is **unbounded on** A in this case.

For example, the function f defined on the interval $A := (0, \infty)$ by $f(x) := 1/x$ is not bounded on A because for any $M > 0$ we can take the point $x_M := 1/(M+1)$ in A to get $f(x_M) = 1/x_M = M + 1 > M$. This example shows that continuous functions need not be bounded. In the next theorem, however, we show that continuous functions on a certain type of interval are necessarily bounded.

5.3.2 Boundedness Theorem† *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.*

Proof. Suppose that f is not bounded on I. Then, for any $n \in \mathbb{N}$ there is a number $x_n \in I$ such that $|f(x_n)| > n$. Since I is bounded, the sequence $X := (x_n)$ is bounded. Therefore, the Bolzano-Weierstrass Theorem 3.4.8 implies that there is a subsequence $X' = (x_{n_r})$ of X that converges to a number x. Since I is closed and the elements of X' belong to I, it follows from Theorem 3.2.6 that $x \in I$. Then f is continuous at x, so that $(f(x_{n_r}))$ converges to $f(x)$. We then conclude from Theorem 3.2.2 that the convergent sequence $(f(x_{n_r}))$ must be bounded. But this is a contradiction since

$$|f(x_{n_r})| > n_r \ge r \quad \text{for} \quad r \in \mathbb{N}.$$

Therefore the supposition that the continuous function f is not bounded on the closed bounded interval I leads to a contradiction. Q.E.D.

To show that each hypothesis of the Boundedness Theorem is needed, we can construct examples that show the conclusion fails if any one of the hypotheses is relaxed.

(i) The interval must be bounded. The function $f(x) := x$ for x in the unbounded, closed interval $A := [0, \infty)$ is continuous but not bounded on A.

(ii) The interval must be closed. The function $g(x) := 1/x$ for x in the half-open interval $B := (0, 1]$ is continuous but not bounded on B.

(iii) The function must be continuous. The function h defined on the closed interval $C := [0, 1]$ by $h(x) := 1/x$ for $x \in (0, 1]$ and $h(0) := 1$ is discontinuous and unbounded on C.

The Maximum-Minimum Theorem

5.3.3 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f **has an absolute maximum** on A if there is a point $x^* \in A$ such that

$$f(x^*) \geq f(x) \quad \text{for all} \quad x \in A.$$

We say that f **has an absolute minimum** on A if there is a point $x_* \in A$ such that

$$f(x_*) \leq f(x) \quad \text{for all} \quad x \in A.$$

We say that x^* is an **absolute maximum point** for f on A, and that x_* is an **absolute minimum point** for f on A, if they exist.

We note that a continuous function on a set A does not necessarily have an absolute maximum or an absolute minimum on the set. For example, $f(x) := 1/x$ has neither an absolute maximum nor an absolute minimum on the set $A := (0, \infty)$. (See Figure 5.3.1.) There can be no absolute maximum for f on A since f is not bounded above on A, and there is no point at which f attains the value $0 = \inf\{f(x) : x \in A\}$. The same function has

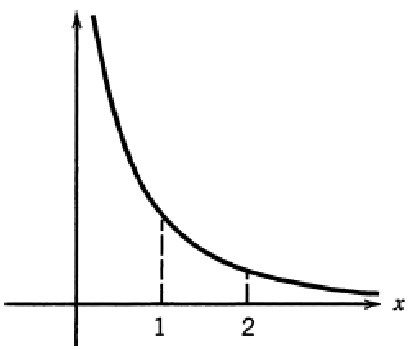


Figure 5.3.1 The function $f(x) = 1/x \quad (x > 0)$

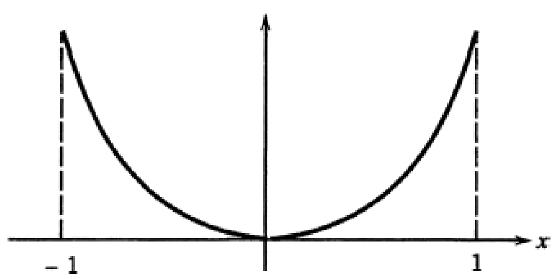


Figure 5.3.2 The function $g(x) = x^2 \quad (|x| \leq 1)$

neither an absolute maximum nor an absolute minimum when it is restricted to the set (0, 1), while it has *both* an absolute maximum and an absolute minimum when it is restricted to the set [1, 2]. In addition, $f(x) = 1/x$ has an absolute maximum but no absolute minimum when restricted to the set $[1, \infty)$, but no absolute maximum and no absolute minimum when restricted to the set $(1, \infty)$.

It is readily seen that if a function has an absolute maximum point, then this point is not necessarily uniquely determined. For example, the function $g(x) := x^2$ defined for $x \in A := [-1, +1]$ has the two points $x = \pm 1$ giving the absolute maximum on A, and the single point $x = 0$ yielding its absolute minimum on A. (See Figure 5.3.2.) To pick an extreme example, the constant function $h(x) := 1$ for $x \in \mathbb{R}$ is such that *every point* of $\mathbb{R}$ is both an absolute maximum and an absolute minimum point for h.

5.3.4 Maximum-Minimum Theorem *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.*

Proof. Consider the nonempty set $f(I) := \{f(x) : x \in I\}$ of values of f on I. In Theorem 5.3.2 it was established that $f(I)$ is a bounded subset of $\mathbb{R}$. Let $s^* := \sup f(I)$ and $s_* := \inf f(I)$. We claim that there exist points x^* and x_* in I such that $s^* = f(x^*)$ and $s_* = f(x_*)$. We will establish the existence of the point x^*, leaving the proof of the existence of x_* to the reader.

Since $s^* = \sup f(I)$, if $n \in \mathbb{N}$, then the number $s^* - 1/n$ is not an upper bound of the set $f(I)$. Consequently there exists a number $x_n \in I$ such that

$$(1) \qquad s^* - \frac{1}{n} < f(x_n) \le s^* \quad \text{for all} \quad n \in \mathbb{N}.$$

Since I is bounded, the sequence $X := (x_n)$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{n_r})$ of X that converges to some number x^*. Since the elements of X' belong to $I = [a, b]$, it follows from Theorem 3.2.6 that $x^* \in I$. Therefore f is continuous at x^* so that $\lim(f(x_{n_r})) = f(x^*)$. Since it follows from (1) that

$$s^* - \frac{1}{n_r} < f(x_{n_r}) \le s^* \quad \text{for all} \quad r \in \mathbb{N},$$

we conclude from the Squeeze Theorem 3.2.7 that $\lim(f(x_{n_r})) = s^*$. Therefore we have

$$f(x^*) = \lim(f(x_{n_r})) = s^* = \sup f(I).$$

We conclude that x^* is an absolute maximum point of f on I. Q.E.D.

The next result is the theoretical basis for locating roots of a continuous function by means of sign changes of the function. The proof also provides an algorithm, known as the **Bisection Method**, for the calculation of roots to a specified degree of accuracy and can be readily programmed for a computer. It is a standard tool for finding solutions of equations of the form $f(x) = 0$, where f is a continuous function. An alternative proof of the theorem is indicated in Exercise 5.3.11.

5.3.5 Location of Roots Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We assume that $f(a) < 0 < f(b)$. We will generate a sequence of intervals by successive bisections. Let $I_1 := [a_1, b_1]$, where $a_1 := a, b_1 := b$, and let p_1 be the midpoint $p_1 := \frac{1}{2}(a_1 + b_1)$. If $f(p_1) = 0$, we take $c := p_1$ and we are done. If $f(p_1) \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$. If $f(p_1) > 0$, then we set $a_2 := a_1, b_2 := p_1$, while if $f(p_1) < 0$, then we set $a_2 := p_1, b_2 := b_1$. In either case, we let $I_2 := [a_2, b_2]$; then we have $I_2 \subset I_1$ and $f(a_2) < 0, f(b_2) > 0$.

We continue the bisection process. Suppose that the intervals $I_1, I_2, \ldots, I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0$ and $f(b_k) > 0$, and we set $p_k := \frac{1}{2}(a_k + b_k)$. If $f(p_k) = 0$, we take $c := p_k$ and we are done. If $f(p_k) > 0$, we set $a_{k+1} := a_k, b_{k+1} := p_k$, while if $f(p_k) < 0$, we set $a_{k+1} := p_k, b_{k+1} := b_k$. In either case, we let $I_{k+1} := [a_{k+1}, b_{k+1}]$; then $I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0, f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n := [a_n, b_n]$ such that for every $n \in \mathbb{N}$ we have

$$f(a_n) < 0 \quad \text{and} \quad f(b_n) > 0.$$

Furthermore, since the intervals are obtained by repeated bisection, the length of I_n is equal to $b_n - a_n = (b - a)/2^{n-1}$. It follows from the Nested Intervals Property 2.5.2 that there exists a point c that belongs to I_n for all $n \in \mathbb{N}$. Since $a_n \le c \le b_n$ for all $n \in \mathbb{N}$ and $\lim(b_n - a_n) = 0$, it follows that $\lim(a_n) = c = \lim(b_n)$. Since f is continuous at c, we have

$$\lim(f(a_n)) = f(c) = \lim(f(b_n)).$$

The fact that $f(a_n) < 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(a_n)) \le 0$. Also, the fact that $f(b_n) > 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(b_n)) \ge 0$. Thus, we conclude that $f(c) = 0$. Consequently, c is a root of f. Q.E.D.

The following example illustrates how the Bisection Method for finding roots is applied in a systematic fashion.

5.3.6 Example The equation $f(x) = xe^x - 2 = 0$ has a root c in the interval [0, 1], because f is continuous on this interval and $f(0) = -2 < 0$ and $f(1) = e - 2 > 0$. Using a calculator we construct the following table, where the sign of $f(p_n)$ determines the interval at the next step. The far right column is an upper bound on the error when p_n is used to approximate the root c, because we have

$$|p_n - c| \le \tfrac{1}{2}(b_n - a_n) = 1/2^n.$$

We will find an approximation p_n with error less than 10^{-2}.

n	a_n	b_n	p_n	$f(p_n)$	$\frac{1}{2}(b_n - a_n)$
1	0	1	.5	−1.176	.5
2	.5	1	.75	−.412	.25
3	.75	1	.875	+.099	.125
4	.75	.875	.8125	−.169	.0625
5	.8125	.875	.84375	−.0382	.03125
6	.84375	.875	.859375	+.0296	.015625
7	.84375	.859375	.8515625	—	.0078125

We have stopped at $n = 7$, obtaining $c \approx p_7 = .8515625$ with error less than .0078125. This is the first step in which the error is less than 10^{-2}. The decimal place values of p_7 past the second place cannot be taken seriously, but we can conclude that $.843 < c < .860$. □

5.3.7 Bolzano's Intermediate Value Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.*

Proof. Suppose that $a < b$ and let $g(x) := f(x) - k$; then $g(a) < 0 < g(b)$. By the Location of Roots Theorem 5.3.5 there exists a point c with $a < c < b$ such that $0 = g(c) = f(c) - k$. Therefore $f(c) = k$.

If $b < a$, let $h(x) := k - f(x)$ so that $h(b) < 0 < h(a)$. Therefore there exists a point c with $b < c < a$ such that $0 = h(c) = k - f(c)$, whence $f(c) = k$. Q.E.D.

5.3.8 Corollary *Let $I = [a, b]$ be a closed, bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying*

$$\inf f(I) \le k \le \sup f(I),$$

then there exists a number $c \in I$ such that $f(c) = k$.

Proof. It follows from the Maximum-Minimum Theorem 5.3.4 that there are points c_* and c^* in I such that

$$\inf f(I) = f(c_*) \le k \le f(c^*) = \sup f(I).$$

The conclusion now follows from Bolzano's Theorem 5.3.7. Q.E.D.

The next theorem summarizes the main results of this section. It states that the image of a closed bounded interval under a continuous function is also a closed bounded interval.

5.3.9 Theorem *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) := \{f(x) : x \in I\}$ is a closed bounded interval.*

Proof. If we let $m := \inf f(I)$ and $M := \sup f(I)$, then we know from the Maximum-Minimum Theorem 5.3.4 that m and M belong to $f(I)$. Moreover, we have $f(I) \subseteq [m, M]$. If k is any element of $[m, M]$, then it follows from the preceding corollary that there exists a point $c \in I$ such that $k = f(c)$. Hence, $k \in f(I)$ and we conclude that $[m, M] \subseteq f(I)$. Therefore, $f(I)$ is the interval $[m, M]$. Q.E.D.

Warning If $I := [a, b]$ is an interval and $f : I \to \mathbb{R}$ is continuous on I, we have proved that $f(I)$ is the interval $[m, M]$. We have *not* proved (and it is not always true) that $f(I)$ is the interval $[f(a), f(b)]$. (See Figure 5.3.3.) □

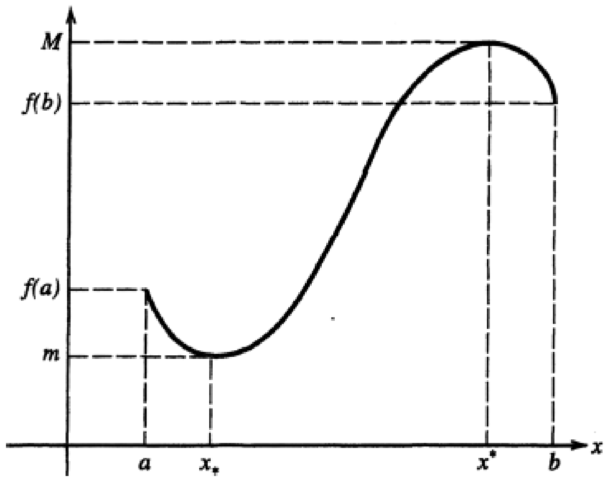


Figure 5.3.3 $f(I) = [m, M]$

The preceding theorem is a "preservation" theorem in the sense that it states that the continuous image of a closed bounded interval is a set of the same type. The next theorem extends this result to general intervals. However, it should be noted that although the continuous image of an interval is shown to be an interval, it is *not* true that the image interval necessarily has the *same form* as the domain interval. For example, the continuous image of an open interval need not be an open interval, and the continuous image of an unbounded closed interval need not be a closed interval. Indeed, if $f(x) := 1/(x^2 + 1)$ for $x \in \mathbb{R}$, then f is continuous on $\mathbb{R}$ [see Example 5.2.3(b)]. It is easy to see that if $I_1 := (-1, 1)$, then $f(I_1) = (\frac{1}{2}, 1]$, which is not an open interval. Also, if $I_2 := [0, \infty)$, then $f(I_2) = (0, 1]$, which is not a closed interval. (See Figure 5.3.4.)

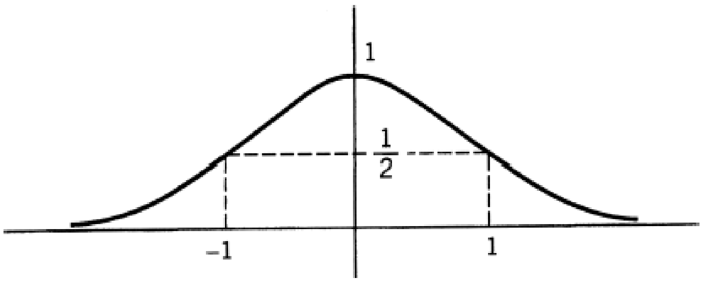


Figure 5.3.4 Graph of $f(x) = 1/(x^2+1)$ $(x \in \mathbb{R})$

Uniform Continuity

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Definition 5.1.1 states that the following statements are equivalent:

(i) f is continuous at every point $u \in A$;

(ii) given $\varepsilon > 0$ and $u \in A$, there is a $\delta(\varepsilon, u) > 0$ such that for all x such that $x \in A$ and $|x - u| < \delta(\varepsilon, u)$, then $|f(x) - f(u)| < \varepsilon$.

The point we wish to emphasize here is that δ depends, in general, on *both* $\varepsilon > 0$ and $u \in A$. The fact that δ depends on u is a reflection of the fact that the function f may change its values rapidly near certain points and slowly near other points. [For example, consider $f(x) := \sin(1/x)$ for $x > 0$; see Figure 4.1.3.]

Now it often happens that the function f is such that the number δ can be chosen to be independent of the point $u \in A$ and to depend only on ε. For example, if $f(x) := 2x$ for all $x \in \mathbb{R}$, then

$$|f(x) - f(u)| = 2|x - u|,$$

and so we can choose $\delta(\varepsilon, u) := \varepsilon/2$ for all $\varepsilon > 0$ and all $u \in \mathbb{R}$. (Why?)

On the other hand if $g(x) := 1/x$ for $x \in A := \{x \in \mathbb{R} : x > 0\}$, then

(1) $$g(x) - g(u) = \frac{u - x}{ux}.$$

If $u \in A$ is given and if we take

(2) $$\delta(\varepsilon, u) := \inf\{\tfrac{1}{2}u, \tfrac{1}{2}u^2\varepsilon\},$$

then if $|x - u| < \delta(\varepsilon, u)$, we have $|x - u| < \frac{1}{2}u$ so that $\frac{1}{2}u < x < \frac{3}{2}u$, whence it follows that $1/x < 2/u$. Thus, if $|x - u| < \frac{1}{2}u$, the equality (1) yields the inequality

(3) $$|g(x) - g(u)| \le (2/u^2)|x - u|.$$

Consequently, if $|x-u| < \delta(\varepsilon, u)$, then (2) and (3) imply that

$$|g(x)-g(u)| < (2/u^2)\left(\tfrac{1}{2}u^2\varepsilon\right) = \varepsilon.$$

We have seen that the selection of $\delta(\varepsilon, u)$ by the formula (2) "works" in the sense that it enables us to give a value of δ that will ensure that $|g(x)-g(u)| < \varepsilon$ when $|x-u| < \delta$ and $x, u \in A$. We note that the value of $\delta(\varepsilon, u)$ given in (2) certainly depends on the point $u \in A$. If we wish to consider *all* $u \in A$, formula (2) does not lead to one value $\delta(\varepsilon) > 0$ that will "work" simultaneously for all $u > 0$, since $\inf\{\delta(\varepsilon, u) : u > 0\} = 0$.

In fact, there is no way of choosing one value of δ that will "work" for all $u > 0$ for the function $g(x) = 1/x$. The situation is exhibited graphically in Figures 5.4.1 and 5.4.2 where, for a given ε-neighborhood $V_\varepsilon\left(\frac{1}{2}\right)$ about $\frac{1}{2} = f(2)$ and $V_\varepsilon(2)$ about $2 = f\left(\frac{1}{2}\right)$, the corresponding maximum values of δ are seen to be considerably different. As u tends to 0, the permissible values of δ tend to 0.

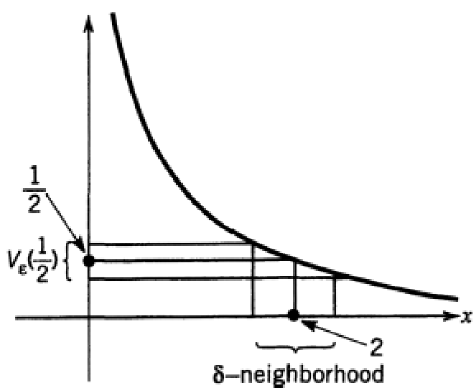


Figure 5.4.1 $g(x) = 1/x \quad (x > 0)$

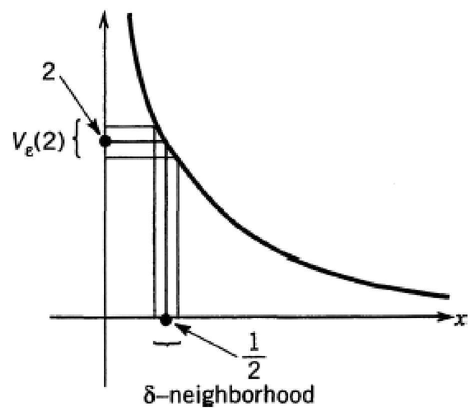


Figure 5.4.2 $g(x) = 1/x \quad (x > 0)$

5.4.1 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is **uniformly continuous** on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ are any numbers satisfying $|x-u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

It is clear that if f is uniformly continuous on A, then it is continuous at every point of A. In general, however, the converse does not hold, as is shown by the function $g(x) = 1/x$ on the set $A := \{x \in \mathbb{R} : x > 0\}$.

It is useful to formulate a condition equivalent to saying that f is *not* uniformly continuous on A. We give such criteria in the next result, leaving the proof to the reader as an exercise.

5.4.2 Nonuniform Continuity Criteria *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:*

(i) *f is not uniformly continuous on A.*

(ii) *There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.*

(iii) *There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.*

We can apply this result to show that $g(x) := 1/x$ is not uniformly continuous on $A := \{x \in \mathbb{R} : x > 0\}$. For, if $x_n := 1/n$ and $u_n := 1/(n+1)$, then we have $\lim(x_n - u_n) = 0$, but $|g(x_n) - g(u_n)| = 1$ for all $n \in \mathbb{N}$.

5.4.3 Uniform Continuity Theorem *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

Proof. If f is not uniformly continuous on I then, by the preceding result, there exists $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in I such that $|x_n - u_n| < 1/n$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Since I is bounded, the sequence (x_n) is bounded; by the Bolzano-Weierstrass Theorem 3.4.8 there is a subsequence (x_{n_k}) of (x_n) that converges to an element z. Since I is closed, the limit z belongs to I, by Theorem 3.2.6. It is clear that the corresponding subsequence (u_{n_k}) also converges to z, since

$$|u_{n_k} - z| \leq |u_{n_k} - x_{n_k}| + |x_{n_k} - z|.$$

Now if f is continuous at the point z, then both of the sequences $(f(x_{n_k}))$ and $(f(u_{n_k}))$ must converge to $f(z)$. But this is not possible since

$$|f(x_n) - f(u_n)| \geq \varepsilon_0$$

for all $n \in \mathbb{N}$. Thus the hypothesis that f is not uniformly continuous on the closed bounded interval I implies that f is not continuous at some point $z \in I$. Consequently, if f is continuous at every point of I, then f is uniformly continuous on I. Q.E.D.

Lipschitz Functions

5.4.4 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u| \tag{4}$$

for all $x, u \in A$, then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

The condition (4) that a function $f : I \to \mathbb{R}$ on an interval I is a Lipschitz function can be interpreted geometrically as follows. If we write the condition as

$$\left|\frac{f(x)-f(u)}{x-u}\right| \le K, \qquad x, u \in I, x \neq u,$$

then the quantity inside the absolute values is the slope of a line segment joining the points $(x, f(x))$ and $(u, f(u))$. Thus a function f satisfies a Lipschitz condition if and only if the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

5.4.5 Theorem *If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Proof. If condition (4) is satisfied, then given $\varepsilon > 0$, we can take $\delta := \varepsilon/K$. If $x, u \in A$ satisfy $|x-u| < \delta$, then

$$|f(x) - f(u)| < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Therefore f is uniformly continuous on A. Q.E.D.

5.4.6 Examples **(a)** If $f(x) := x^2$ on $A := [0, b]$, where $b > 0$, then

$$|f(x) - f(u)| = |x+u||x-u| \le 2b|x-u|$$

for all x, u in $[0, b]$. Thus f satisfies (4) with $K := 2b$ on A, and therefore f is uniformly continuous on A. Of course, since f is continuous and A is a closed bounded interval, this can also be deduced from the Uniform Continuity Theorem. (Note that f does *not* satisfy a Lipschitz condition on the interval $[0, \infty)$.)

(b) Not every uniformly continuous function is a Lipschitz function.

Let $g(x) := \sqrt{x}$ for x in the closed bounded interval $I := [0, 2]$. Since g is continuous on I, it follows from the Uniform Continuity Theorem 5.4.3 that g is uniformly continuous on I. However, there is no number $K > 0$ such that $|g(x)| \le K|x|$ for all $x \in I$. (Why not?) Therefore, g is not a Lipschitz function on I.

We consider $g(x) := \sqrt{x}$ on the set $A := [0, \infty)$. The uniform continuity of g on the interval $I := [0, 2]$ follows from the Uniform Continuity Theorem as noted in (b). If $J := [1, \infty)$, then if both x, u are in J, we have

$$|g(x) - g(u)| = \left|\sqrt{x} - \sqrt{u}\right| = \frac{|x-u|}{\sqrt{x}+\sqrt{u}} \le \tfrac{1}{2}|x-u|.$$

5.4.8 Continuous Extension Theorem *A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.*

Proof. $(\Leftarrow)$ This direction is trivial.

$(\Rightarrow)$ Suppose f is uniformly continuous on (a, b). We shall show how to extend f to a; the argument for b is similar. This is done by showing that $\lim_{x\to c} f(x) = L$ exists, and this is accomplished by using the sequential criterion for limits. If (x_n) is a sequence in (a, b) with $\lim(x_n) = a$, then it is a Cauchy sequence, and by the preceding theorem, the sequence $(f(x_n))$ is also a Cauchy sequence, and so is convergent by Theorem 3.5.5. Thus the limit $\lim(f(x_n)) = L$ exists. If (u_n) is any other sequence in (a, b) that converges to a, then $\lim(u_n - x_n) = a - a = 0$, so by the uniform continuity of f we have

$$\begin{aligned}\lim(f(u_n)) &= \lim(f(u_n) - f(x_n)) + \lim(f(x_n))\\ &= 0 + L = L.\end{aligned}$$

Since we get the same value L for every sequence converging to a, we infer from the sequential criterion for limits that f has limit L at a. If we define $f(a) := L$, then f is continuous at a. The same argument applies to b, so we conclude that f has a continuous extension to the interval $[a, b]$. Q.E.D.

Since the limit of $f(x) := \sin(1/x)$ at 0 does not exist, we infer from the Continuous Extension Theorem that the function is not uniformly continuous on $(0, b]$ for any $b > 0$. On the other hand, since $\lim_{x\to 0} x\sin(1/x) = 0$ exists, the function $g(x) := x\sin(1/x)$ is uniformly continuous on $(0, b]$ for all $b > 0$.

POSSIBLE QUESTIONS

2 Mark Questions:

1. Find the location of root of $xe^x - 2 = 0$
2. Define bounded function.
3. Define absolute minimum of a function.
4. Define absolute maximum of a function.
5. Find the location of root of$x^2 - 2\sin x + 3 = 0$.

8 Mark Questions:

1. State and prove boundedness theorem.
2. State and prove maximum –minimum theorem.
3. State and prove location of root theorem.
4. State and prove Bolzano’s intermediate value theorem.
5. State and prove preservation of intervals theorem.
6. Find the root of $f(x) = xe^x - 2 = 0$ in the interval [0,1].
7. If $f: [a, b] \to R$ is a continuous function, prove that f is bounded on [a,b].
8. If $f: [a, b] \to R$ is a continuous function, prove that f has an absolute maximum and absolute minimum on [a,b].
9. If $f: [a, b] \to R$ is a continuous function and if $f(a) < 0 < f(b)$ or if $f(b) < 0 < f(a)$, then prove that there exists a number $c \in (a, b)$ such that $f(c) = 0$.
10. State and prove uniform continuity theorem

UNIT-III

SYLLABUS

Differentiability of a function at a point and in an interval, Caratheodory's theorem, algebra of differentiable functions. Relative extrema, interior extremum theorem. Rolle's theorem.

The Derivative

6.1.1 Definition Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the **derivative of f at c** if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon. \tag{1}$$

In this case we say that f is **differentiable** at c, and we write $f'(c)$ for L.

In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{2}$$

provided this limit exists. (We allow the possibility that c may be the endpoint of the interval.)

Note It is possible to define the derivative of a function having a domain more general than an interval (since the point c need only be an element of the domain and also a cluster point of the domain) but the significance of the concept is most naturally apparent for functions defined on intervals. Consequently we shall limit our attention to such functions.

Whenever the derivative of $f : I \to \mathbb{R}$ exists at a point $c \in I$, its value is denoted by $f'(c)$. In this way we obtain a function f' whose domain is a subset of the domain of f. In working with the function f', it is convenient to regard it also as a function of x. For example, if $f(x) := x^2$ for $x \in \mathbb{R}$, then at any c in $\mathbb{R}$ we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Thus, in this case, the function f' is defined on all of $\mathbb{R}$ and $f'(x) = 2x$ for $x \in \mathbb{R}$.

We now show that continuity of f at a point c is a necessary (but not sufficient) condition for the existence of the derivative at c.

6.1.2 Theorem *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.*

Proof. For all $x \in I$, $x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c).$$

Since $f'(c)$ exists, we may apply Theorem 4.2.4 concerning the limit of a product to conclude that

$$\begin{aligned}\lim_{x\to c}(f(x) - f(c)) &= \lim_{x\to c}\left(\frac{f(x) - f(c)}{x - c}\right)\left(\lim_{x\to c}(x - c)\right)\\ &= f'(c)\cdot 0 = 0.\end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = f(c)$ so that f is continuous at c. Q.E.D.

The continuity of $f : I \to \mathbb{R}$ at a point does not assure the existence of the derivative at that point. For example, if $f(x) := |x|$ for $x \in \mathbb{R}$, then for $x \neq 0$ we have $(f(x) - f(0))/(x - 0) = |x|/x$, which is equal to 1 if $x > 0$, and equal to -1 if $x < 0$. Thus the limit at 0 does not exist [see Example 4.1.10(b)], and therefore the function is not differentiable at 0. Hence, continuity at a point c is *not* a sufficient condition for the derivative to exist at c.

6.1.3 Theorem *Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:*

(a) *If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c, and*

(3) $$(\alpha f)'(c) = \alpha f'(c).$$

(b) *The function $f + g$ is differentiable at c, and*

(4) $$(f + g)'(c) = f'(c) + g'(c).$$

(c) *(Product Rule) The function fg is differentiable at c, and*

(5) $$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

(d) *(Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and*

(6) $$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Proof. We shall prove (c) and (d), leaving (a) and (b) as exercises for the reader.

(c) Let $p := fg$; then for $x \in I$, $x \neq c$, we have

$$\begin{aligned}\frac{p(x)-p(c)}{x-c} &= \frac{f(x)g(x)-f(c)g(c)}{x-c} \\ &= \frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c} \\ &= \frac{f(x)-f(c)}{x-c}\cdot g(x)+f(c)\cdot\frac{g(x)-g(c)}{x-c}.\end{aligned}$$

Since g is continuous at c, by Theorem 6.1.2, then $\lim_{x\to c} g(x) = g(c)$. Since f and g are differentiable at c, we deduce from Theorem 4.2.4 on properties of limits that

$$\lim_{x\to c}\frac{p(x)-p(c)}{x-c} = f'(c)g(c)+f(c)g'(c).$$

Hence $p := fg$ is differentiable at c and (5) holds.

(d) Let $q := f/g$. Since g is differentiable at c, it is continuous at that point (by Theorem 6.1.2). Therefore, since $g(c) \neq 0$, we know from Theorem 4.2.9 that there exists an interval $J \subseteq I$ with $c \in J$ such that $g(x) \neq 0$ for all $x \in J$. For $x \in J$, $x \neq c$, we have

$$\begin{aligned}\frac{q(x)-q(c)}{x-c} &= \frac{f(x)/g(x)-f(c)/g(c)}{x-c} = \frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)(x-c)} \\ &= \frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)} \\ &= \frac{1}{g(x)g(c)}\left[\frac{f(x)-f(c)}{x-c}\cdot g(c)-f(c)\cdot\frac{g(x)-g(c)}{x-c}\right].\end{aligned}$$

Using the continuity of g at c and the differentiability of f and g at c, we get

$$q'(c) = \lim_{x\to c}\frac{q(x)-q(c)}{x-c} = \frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}.$$

Thus, $q = f/g$ is differentiable at c and equation (6) holds. Q.E.D.

Mathematical Induction may be used to obtain the following extensions of the differentiation rules.

6.1.4 Corollary *If $f_1, f_2, \ldots, f_n$ are functions on an interval I to $\mathbb{R}$ that are differentiable at $c \in I$, then:*

(a) *The function $f_1 + f_2 + \cdots + f_n$ is differentiable at c and*

(7) $$(f_1+f_2+\cdots+f_n)'(c) = f_1'(c)+f_2'(c)+\cdots+f_n'(c).$$

(b) *The function $f_1 f_2 \cdots f_n$ is differentiable at c, and*

(8) $$(f_1 f_2 \cdots f_n)'(c) = f_1'(c) f_2(c) \cdots f_n(c) + f_1(c) f_2'(c) \cdots f_n(c) + \cdots + f_1(c) f_2(c) \cdots f_n'(c).$$

An important special case of the extended product rule (8) occurs if the functions are equal, that is, $f_1 = f_2 = \cdots = f_n = f$. Then (8) becomes

(9) $$(f^n)'(c) = n(f(c))^{n-1} f'(c).$$

In particular, if we take $f(x) := x$, then we find the derivative of $g(x) := x^n$ to be $g'(x) = nx^{n-1}$, $n \in \mathbb{N}$. The formula is extended to include negative integers by applying the Quotient Rule 6.1.3(d).

Notation If $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$, we have introduced the notation f' to denote the function whose domain is a subset of I and whose value at a point c is the derivative $f'(c)$ of f at c. There are other notations that are sometimes used for f'; for example, one sometimes writes Df for f'. Thus one can write formulas (4) and (5) in the form:

$$D(f+g) = Df + Dg, \qquad D(fg) = (Df) \cdot g + f \cdot (Dg).$$

When x is the "independent variable," it is common practice in elementary courses to write df/dx for f'. Thus formula (5) is sometimes written in the form

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{df}{dx}(x)\right) g(x) + f(x)\left(\frac{dg}{dx}(x)\right).$$

This last notation, due to Leibniz, has certain advantages. However, it also has certain disadvantages and must be used with some care.

6.1.5 Carathéodory's Theorem *Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there exists a function φ on I that is continuous at c and satisfies*

(10) $$f(x) - f(c) = \varphi(x)(x - c) \quad \textit{for} \quad x \in I.$$

In this case, we have $\varphi(c) = f'(c)$.

Proof. ($\Rightarrow$) If $f'(c)$ exists, we can define φ by

$$\varphi(x) := \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{for} \quad x \neq c,\ x \in I, \\ f'(c) & \text{for} \quad x = c. \end{cases}$$

The continuity of φ follows from the fact that $\lim_{x \to c} \varphi(x) = f'(c)$. If $x = c$, then both sides of (10) equal 0, while if $x \neq c$, then multiplication of $\varphi(x)$ by $x - c$ gives (10) for all other $x \in I$.

($\Leftarrow$) Now assume that a function φ that is continuous at c and satisfying (10) exists. If we divide (10) by $x - c \neq 0$, then the continuity of φ implies that

$$\varphi(c) = \lim_{x \to c} \varphi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. Therefore f is differentiable at c and $f'(c) = \varphi(c)$. Q.E.D.

To illustrate Carathéodory's Theorem, we consider the function f defined by $f(x) := x^3$ for $x \in \mathbb{R}$. For $c \in \mathbb{R}$, we see from the factorization

$$x^3 - c^3 = \left(x^2 + cx + c^2\right)(x - c)$$

that $\varphi(x) := x^2 + cx + c^2$ satisfies the conditions of the theorem. Therefore, we conclude that f is differentiable at $c \in \mathbb{R}$ and that $f'(c) = \varphi(c) = 3c^2$.

We will now establish the Chain Rule. If f is differentiable at c and g is differentiable at $f(c)$, then the Chain Rule states that the derivative of the composite function $g \circ f$ at c is the product $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$. Note that this can be written as

$$(g \circ f)' = (g' \circ f) \cdot f'.$$

One approach to the Chain Rule is the observation that the difference quotient can be written, when $f(x) \neq f(c)$, as the product

$$\frac{g(f(x)) - g(f(c))}{x - c} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}.$$

This suggests the correct limiting value. Unfortunately, the first factor in the product on the right is undefined if the denominator $f(x) - f(c)$ equals 0 for values of x near c, and this presents a problem. However, the use of Carathéodory's Theorem neatly avoids this difficulty.

6.1.6 Chain Rule *Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and*

(11) $$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Proof. Since $f'(c)$ exists, Carathéodory's Theorem 6.1.5 implies that there exists a function φ on J such that φ is continuous at c and $f(x) - f(c) = \varphi(x)(x - c)$ for $x \in J$, and where $\varphi(c) = f'(c)$. Also, since $g'(f(c))$ exists, there is a function ψ defined on I such that ψ is continuous at $d := f(c)$ and $g(y) - g(d) = \psi(y)(y - d)$ for $y \in I$, where $\psi(d) = g'(d)$. Substitution of $y = f(x)$ and $d = f(c)$ then produces

$$g(f(x)) - g(f(c)) = \psi(f(x))(f(x) - f(c)) = [(\psi \circ f(x)) \cdot \varphi(x)](x - c)$$

for all $x \in J$ such that $f(x) \in I$. Since the function $(\psi \circ f) \cdot \varphi$ is continuous at c and its value at c is $g'(f(c)) \cdot f'(c)$, Carathéodory's Theorem gives (11). Q.E.D.

If g is differentiable on I, if f is differentiable on J, and if $f(J) \subseteq I$, then it follows from the Chain Rule that $(g \circ f)' = (g' \circ f) \cdot f'$, which can also be written in the form $D(g \circ f) = (Dg \circ f) \cdot Df$.

6.1.7 Examples **(a)** If $f : I \to \mathbb{R}$ is differentiable on I and $g(y) := y^n$ for $y \in \mathbb{R}$ and $n \in \mathbb{N}$, then since $g'(y) = ny^{n-1}$, it follows from the Chain Rule 6.1.6 that

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad \text{for} \quad x \in I.$$

Therefore we have $(f^n)'(x) = n(f(x))^{n-1} f'(x)$ for all $x \in I$ as was seen in (9).

(b) Suppose that $f : I \to \mathbb{R}$ is differentiable on I and that $f(x) \neq 0$ and $f'(x) \neq 0$ for $x \in I$. If $h(y) := 1/y$ for $y \neq 0$, then it is an exercise to show that $h'(y) = -1/y^2$ for $y \in \mathbb{R},\ y \neq 0$. Therefore we have

$$\left(\frac{1}{f}\right)'(x) = (h \circ f)'(x) = h'(f(x))f'(x) = -\frac{f'(x)}{(f(x))^2} \quad \text{for} \quad x \in I.$$

(c) The absolute value function $g(x) := |x|$ is differentiable at all $x \neq 0$ and has derivative $g'(x) = \operatorname{sgn}(x)$ for $x \neq 0$. (The signum function is defined in Example 4.1.10(b).) Though sgn is defined everywhere, it is not equal to g' at $x = 0$ since $g'(0)$ does not exist.

Now if f is a differentiable function, then the Chain Rule implies that the function $g \circ f = |f|$ is also differentiable at all points x where $f(x) \neq 0$, and its derivative is given by

$$|f|'(x) = \operatorname{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x) & \text{if} \quad f(x) > 0, \\ -f'(x) & \text{if} \quad f(x) < 0. \end{cases}$$

If f is differentiable at a point c with $f(c) = 0$, then it is an exercise to show that $|f|$ is differentiable at c if and only if $f'(c) = 0$. (See Exercise 7.)

For example, if $f(x) := x^2 - 1$ for $x \in \mathbb{R}$, then the derivative of its absolute value $|f|(x) = |x^2 - 1|$ is equal to $|f|'(x) = \operatorname{sgn}(x^2 - 1) \cdot (2x)$ for $x \neq 1, -1$. See Figure 6.1.1 for a graph of $|f|$.

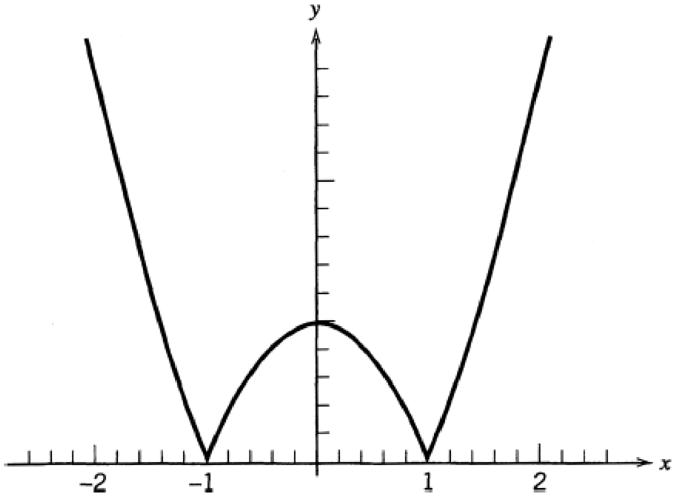


Figure 6.1.1 The function $|f|(x) = |x^2 - 1|$.

(d) It will be proved later that if $S(x) := \sin x$ and $C(x) := \cos x$ for all $x \in \mathbb{R}$, then

$$S'(x) = \cos x = C(x) \quad \text{and} \quad C'(x) = -\sin x = -S(x)$$

for all $x \in \mathbb{R}$. If we use these facts together with the definitions

$$\tan x := \frac{\sin x}{\cos x}, \quad \sec x := \frac{1}{\cos x},$$

for $x \neq (2k+1)\pi/2$, $k \in \mathbb{Z}$, and apply the Quotient Rule 6.1.3(d), we obtain

$$D\tan x = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = (\sec x)^2,$$

$$D\sec x = \frac{0 - 1(-\sin x)}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} = (\sec x)(\tan x)$$

for $x \neq (2k+1)\pi/2, k \in \mathbb{Z}$.

Similarly, since

$$\cot x := \frac{\cos x}{\sin x}, \qquad \csc x := \frac{1}{\sin x}$$

for $x \neq k\pi$, $k \in \mathbb{Z}$, then we obtain

$$D\cot x = -(\csc x)^2 \qquad \text{and} \qquad D\csc x = -(\csc x)(\cot x)$$

for $x \neq k\pi$, $k \in \mathbb{Z}$.

(e) Suppose that f is defined by

$$f(x) := \begin{cases} x^2 \sin(1/x) & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

If we use the fact that $D \sin x = \cos x$ for all $x \in \mathbb{R}$ and apply the Product Rule 6.1.3(c) and the Chain Rule 6.1.6, we obtain (why?)

$$f'(x) = 2x \sin(1/x) - \cos(1/x) \quad \text{for} \quad x \neq 0.$$

If $x = 0$, none of the calculational rules may be applied. (Why?) Consequently, the derivative of f at $x = 0$ must be found by applying the definition of derivative. We find that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin(1/x) = 0.$$

Hence, the derivative f' of f exists at all $x \in \mathbb{R}$. However, the function f' does not have a limit at $x = 0$ (why?), and consequently f' is discontinuous at $x = 0$. Thus, a function f that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative f'. □

6.1.8 Theorem *Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the strictly monotone and continuous function inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d := f(c)$ and*

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}. \qquad (12)$$

Proof. Given $c \in \mathbb{R}$, we obtain from Carathéodory's Theorem 6.1.5 a function φ on I with properties that φ is continuous at c, $f(x) - f(c) = \varphi(x)(x - c)$ for $x \in I$, and $\varphi(c) = f'(c)$. Since $\varphi(c) \neq 0$ by hypothesis, there exists a neighborhood $V := (c - \delta, c + \delta)$ such that $\varphi(x) \neq 0$ for all $x \in V \cap I$. (See Theorem 4.2.9.) If $U := f(V \cap I)$, then the inverse function g satisfies $f(g(y)) = y$ for all $y \in U$, so that

$$y - d = f(g(y)) - f(c) = \varphi(g(y)) \cdot (g(y) - g(d)).$$

Since $\varphi(g(y)) \neq 0$ for $y \in U$, we can divide to get

$$g(y) - g(d) = \frac{1}{\varphi(g(y))} \cdot (y - d).$$

Since the function $1/(\varphi \circ g)$ is continuous at d, we apply Theorem 6.1.5 to conclude that $g'(d)$ exists and $g'(d) = 1/\varphi(g(d)) = 1/\varphi(c) = 1/f'(c)$. Q.E.D.

6.1.10 Examples **(a)** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^5 + 4x + 3$ is continuous and strictly monotone increasing (since it is the sum of two strictly increasing functions). Moreover, $f'(x) = 5x^4 + 4$ is never zero. Therefore, by Theorem 6.1.8, the inverse function $g = f^{-1}$ is differentiable at every point. If we take $c = 1$, then since $f(1) = 8$, we obtain $g'(8) = g'(f(1)) = 1/f'(1) = 1/9$.

(b) Let $n \in \mathbb{N}$ be even, let $I := [0, \infty)$, and let $f(x) := x^n$ for $x \in I$. It was seen at the end of Section 5.6 that f is strictly increasing and continuous on I, so that its inverse function $g(y) := y^{1/n}$ for $y \in J := [0, \infty)$ is also strictly increasing and continuous on J. Moreover, we have $f'(x) = nx^{n-1}$ for all $x \in I$. Hence it follows that if $y > 0$, then $g'(y)$ exists and

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n(g(y))^{n-1}} = \frac{1}{ny^{(n-1)/n}}.$$

Hence we deduce that

$$g'(y) = \frac{1}{n} y^{(1/n)-1} \quad \text{for} \quad y > 0.$$

However, g is *not* differentiable at 0. (For a graph of f and g, see Figures 5.6.4 and 5.6.5.)

(c) Let $n \in \mathbb{N}, n \neq 1$, be odd, let $F(x) := x^n$ *for* $x \in \mathbb{R}$, and let $G(y) := y^{1/n}$ be its inverse function defined for all $y \in \mathbb{R}$. As in part (b) we find that G is differentiable for $y \neq 0$ and that $G'(y) = (1/n)y^{(1/n)-1}$ for $y \neq 0$. However, G is not differentiable at 0, even though G is differentiable for all $y \neq 0$. (For a graph of F and G, see Figures 5.6.6 and 5.6.7.)

(d) Let $r := m/n$ be a positive rational number, let $I := [0, \infty)$, and let $R(x) := x^r$ for $x \in I$. (Recall Definition 5.6.6.) Then R is the composition of the functions $f(x) := x^m$ and $g(x) := x^{1/n}$, $x \in I$. That is, $R(x) = f(g(x))$ for $x \in I$. If we apply the Chain Rule 6.1.6 and the results of (b) [or (c), depending on whether n is even or odd], then we obtain

$$\begin{aligned} R'(x) &= f'(g(x))g'(x) = m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n} x^{(1/n)-1} \\ &= \frac{m}{n} x^{(m/n)-1} = rx^{r-1} \end{aligned}$$

for all $x > 0$. If $r > 1$, then it is an exercise to show that the derivative also exists at $x = 0$ and $R'(0) = 0$. (For a graph of R see Figure 5.6.8.)

(e) The sine function is strictly increasing on the interval $I := [-\pi/2, \pi/2]$; therefore its inverse function, which we will denote by Arcsin, exists on $J := [-1, 1]$. That is, if $x \in [-\pi/2, \pi/2]$ and $y \in [-1, 1]$ then $y = \sin x$ if and only if $\text{Arcsin}\, y = x$. It was asserted (without proof) in Example 6.1.7(d) that sin is differentiable on I and that $D \sin x = \cos x$ for $x \in I$. Since $\cos x \neq 0$ for x in $(-\pi/2, \pi/2)$ it follows from Theorem 6.1.8 that

$$D\, \text{Arcsin}\, y = \frac{1}{D \sin x} = \frac{1}{\cos x}$$

$$= \frac{1}{\sqrt{1-(\sin x)^2}} = \frac{1}{\sqrt{1-y^2}}$$

for all $y \in (-1, 1)$. The derivative of Arcsin does *not* exist at the points -1 and 1. □

We begin by looking at the relationship between the relative extrema of a function and the values of its derivative. Recall that the function $f : I \to \mathbb{R}$ is said to have a **relative maximum** [respectively, **relative minimum**] at $c \in I$ if there exists a neighborhood $V := V_\delta(c)$ of c such that $f(x) \leq f(c)$ [respectively, $f(c) \leq f(x)$] for all x in $V \cap I$. We say that f has a **relative extremum** at $c \in I$ if it has either a relative maximum or a relative minimum at c.

The next result provides the theoretical justification for the familiar process of finding points at which f has relative extrema by examining the zeros of the derivative. However, it must be realized that this procedure applies only to *interior* points of the interval. For example, if $f(x) := x$ on the interval $I := [0, 1]$, then the endpoint $x = 0$ yields the unique relative minimum and the endpoint $x = 1$ yields the unique maximum of f on I, but neither point is a zero of the derivative of f.

6.2.1 Interior Extremum Theorem *Let c be an interior point of the interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$.*

Proof. We will prove the result only for the case that f has a relative maximum at c; the proof for the case of a relative minimum is similar.

If $f'(c) > 0$, then by Theorem 4.2.9 there exists a neighborhood $V \subseteq I$ of c such that

$$\frac{f(x) - f(c)}{x - c} > 0 \quad \text{for} \quad x \in V,\ x \neq c.$$

If $x \in V$ and $x > c$, then we have

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

But this contradicts the hypothesis that f has a relative maximum at c. Thus we cannot have $f'(c) > 0$. Similarly (how?), we cannot have $f'(c) < 0$. Therefore we must have $f'(c) = 0$. Q.E.D.

6.2.2 Corollary *Let $f : I \to \mathbb{R}$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Then either the derivative of f at c does not exist, or it is equal to zero.*

We note that if $f(x) := |x|$ on $I := [-1, 1]$, then f has an interior minimum at $x = 0$; however, the derivative of f fails to exist at $x = 0$.

6.2.3 Rolle's Theorem *Suppose that f is continuous on a closed interval $I := [a, b]$, that the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.*

Proof. If f vanishes identically on I, then any c in (a, b) will satisfy the conclusion of the theorem. Hence we suppose that f does not vanish identically; replacing f by $-f$ if necessary, we may suppose that f assumes some positive values. By the Maximum-Minimum Theorem 5.3.4, the function f attains the value $\sup\{f(x) : x \in I\} > 0$ at some point c in I. Since $f(a) = f(b) = 0$, the point c must lie in (a, b); therefore $f'(c)$ exists.

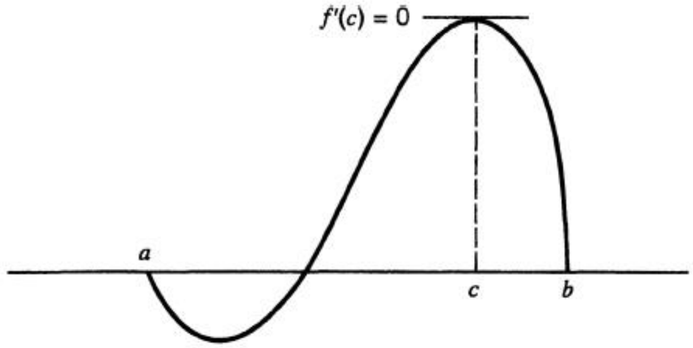


Figure 6.2.1 Rolle's Theorem

Since f has a relative maximum at c, we conclude from the Interior Extremum Theorem 6.2.1 that $f'(c) = 0$. (See Figure 6.2.1.) Q.E.D.

POSSIBLE QUESTIONS

2 Mark Questions:

1. If $f(x) = x^2$, prove that $f'(x) = 2x$.
2. Give an example for continuity is not sufficient for continuity.
3. State the product rule of derivative.
4. State the quotient rule of derivative.
5. Define relative extremum of a function.

8 Mark Questions:

1. If $f: [a, b] \to R$ has a derivative at $c \in [a, b]$, prove that f is continuous at c. Also prove that

   the converse need not be true.
2. State and prove Caratheoodory theorem.
3. State and prove the product rule and quotient rule for derivative.
4. State and prove chain rule.
5. State and prove interior extremum theorem.
6. State and prove Rolle 's Theorem.
7. If $f: I \to R$ has a relative extermum at an interior point c of I and if derivative of f exists at c, prove that $f'(c) = 0$.

UNIT-IV

SYLLABUS

Mean value theorem, intermediate value property of derivatives, Darboux's theorem. Applications of mean value theorem to inequalities and approximation of polynomials, Taylor's theorem to inequalities.

6.2.4 Mean Value Theorem *Suppose that f is continuous on a closed interval $I := [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Consider the function φ defined on I by

$$\varphi(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

[The function φ is simply the difference of f and the function whose graph is the line segment joining the points $(a, f(a))$ and $(b, f(b))$; see Figure 6.2.2.] The hypotheses of

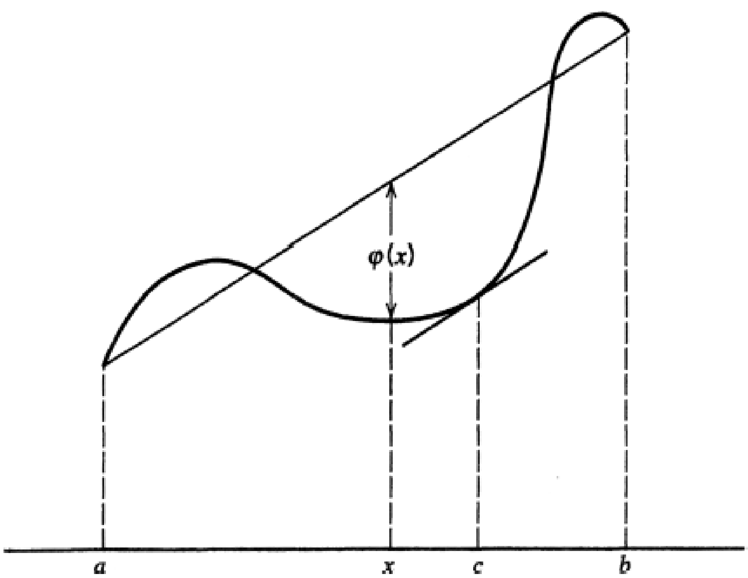


Figure 6.2.2 The Mean Value Theorem

Rolle's Theorem are satisfied by φ since φ is continuous on $[a, b]$, differentiable on (a, b), and $\varphi(a) = \varphi(b) = 0$. Therefore, there exists a point c in (a, b) such that

$$0 = \varphi'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Hence, $f(b) - f(a) = f'(c)(b - a)$. Q.E.D.

Remark The geometric view of the Mean Value Theorem is that there is some point on the curve $y = f(x)$ at which the tangent line is parallel to the line segment through the points $(a, f(a))$ and $(b, f(b))$. Thus it is easy to remember the statement of the Mean Value Theorem by drawing appropriate diagrams. While this should not be discouraged, it tends to suggest that its importance is geometrical in nature, which is quite misleading. In fact the Mean Value Theorem is a wolf in sheep's clothing and is *the* Fundamental Theorem of Differential Calculus. In the remainder of this section, we will present some of the consequences of this result. Other applications will be given later.

The Mean Value Theorem permits one to draw conclusions about the nature of a function f from information about its derivative f'. The following results are obtained in this manner.

6.2.5 Theorem *Suppose that f is continuous on the closed interval $I := [a, b]$, that f is differentiable on the open interval (a, b), and that $f'(x) = 0$ for $x \in (a, b)$. Then f is constant on I.*

Proof. We will show that $f(x) = f(a)$ for all $x \in I$. Indeed, if $x \in I$, $x > a$, is given, we apply the Mean Value Theorem to f on the closed interval $[a, x]$. We obtain a point c (depending on x) between a and x such that $f(x) - f(a) = f'(c)(x - a)$. Since $f'(c) = 0$ (by hypothesis), we deduce that $f(x) - f(a) = 0$. Hence, $f(x) = f(a)$ for any $x \in I$. Q.E.D.

6.2.6 Corollary *Suppose that f and g are continuous on $I := [a, b]$, that they are differentiable on (a, b), and that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant C such that $f = g + C$ on I.*

Recall that a function $f : I \to \mathbb{R}$ is said to be **increasing** on the interval I if whenever x_1, x_2 in I satisfy $x_1 < x_2$, then $f(x_1) \le f(x_2)$. Also recall that f is **decreasing** on I if the function $-f$ is increasing on I.

6.2.7 Theorem *Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then:*

(a) *f is increasing on I if and only if $f'(x) \ge 0$ for all $x \in I$.*
(b) *f is decreasing on I if and only if $f'(x) \le 0$ for all $x \in I$.*

Proof. (a) Suppose that $f'(x) \ge 0$ for all $x \in I$. If x_1, x_2 in I satisfy $x_1 < x_2$, then we apply the Mean Value Theorem to f on the closed interval $J := [x_1, x_2]$ to obtain a point c in (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) \geq 0$ and $x_2 - x_1 > 0$, it follows that $f(x_2) - f(x_1) \geq 0$. (Why?) Hence, $f(x_1) \leq f(x_2)$ and, since $x_1 < x_2$ are arbitrary points in I, we conclude that f is increasing on I.

For the converse assertion, we suppose that f is differentiable and increasing on I. Thus, for any point $x \neq c$ in I, we have $(f(x) - f(c))/(x - c) \geq 0$. (Why?) Hence, by Theorem 4.2.6 we conclude that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0.$$

(b) The proof of part (b) is similar and will be omitted. Q.E.D.

A function f is said to be **strictly increasing** on an interval I if for any points x_1, x_2 in I such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$. An argument along the same lines of the proof of Theorem 6.2.7 can be made to show that a function having a strictly positive derivative on an interval is strictly increasing there. (See Exercise 13.) However, the converse assertion is not true, since a strictly increasing differentiable function may have a derivative that vanishes at certain points. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^3$ is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$. The situation for strictly decreasing functions is similar.

Remark It is reasonable to define a function to be **increasing at a point** if there is a neighborhood of the point on which the function is increasing. One might suppose that, if the derivative is strictly positive at a point, then the function is increasing at this point. However, this supposition is false; indeed, the differentiable function defined by

$$g(x) := \begin{cases} x + 2x^2 \sin(1/x) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

is such that $g'(0) = 1$, yet it can be shown that g is not increasing in any neighborhood of $x = 0$. (See Exercise 10.)

We next obtain a sufficient condition for a function to have a relative extremum at an interior point of an interval.

6.2.8 First Derivative Test for Extrema *Let f be continuous on the interval $I := [a, b]$ and let c be an interior point of I. Assume that f is differentiable on (a, c) and (c, b). Then:*

(a) *If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \geq 0$ for $c - \delta < x < c$ and $f'(x) \leq 0$ for $c < x < c + \delta$, then f has a relative maximum at c.*

(b) *If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \leq 0$ for $c - \delta < x < c$ and $f'(x) \geq 0$ for $c < x < c + \delta$, then f has a relative minimum at c.*

Proof. (a) If $x \in (c-\delta, c)$, then it follows from the Mean Value Theorem that there exists a point $c_x \in (x, c)$ such that $f(c) - f(x) = (c-x)f'(c_x)$. Since $f'(c_x) \geq 0$ we infer that $f(x) \leq f(c)$ for $x \in (c-\delta, c)$. Similarly, it follows (how?) that $f(x) \leq f(c)$ for $x \in (c, c+\delta)$. Therefore $f(x) \leq f(c)$ for all $x \in (c-\delta, c+\delta)$ so that f has a relative maximum at c.

(b) The proof is similar. Q.E.D.

Remark The converse of the First Derivative Test 6.2.8 is *not* true. For example, there exists a differentiable function $f: \mathbb{R} \to \mathbb{R}$ with absolute minimum at $x = 0$ but such that f' takes on both positive and negative values on both sides of (and arbitrarily close to) $x = 0$. (See Exercise 9.)

Further Applications of the Mean Value Theorem

We will continue giving other types of applications of the Mean Value Theorem; in doing so we will draw more freely than before on the past experience of the reader and his or her knowledge concerning the derivatives of certain well-known functions.

6.2.9 Examples **(a)** Rolle's Theorem can be used for the location of roots of a function. For, if a function g can be identified as the derivative of a function f, then between any two roots of f there is at least one root of g. For example, let $g(x) := \cos x$, then g is known to be the derivative of $f(x) := \sin x$. Hence, between any two roots of $\sin x$ there is at least one root of $\cos x$. On the other hand, $g'(x) = -\sin x = -f(x)$, so another application of Rolle's Theorem tells us that between any two roots of cos there is at least one root of sin. Therefore, we conclude that the roots of sin and cos *interlace each other*. This conclusion is probably not news to the reader; however, the same type of argument can be applied to the *Bessel functions* J_n of order $n = 0, 1, 2, \ldots$ by using the relations

$$[x^n J_n(x)]' = x^n J_{n-1}(x), \quad [x^{n-1} J_n(x)]' = -x^{-n} J_{n+1}(x) \qquad \text{for} \quad x > 0.$$

(b) We can apply the Mean Value Theorem for approximate calculations and to obtain error estimates. For example, suppose it is desired to evaluate $\sqrt{105}$. We employ the Mean Value Theorem with $f(x) := \sqrt{x}$, $a = 100$, $b = 105$, to obtain

$$\sqrt{105} - \sqrt{100} = \frac{5}{2\sqrt{c}}$$

for some number c with $100 < c < 105$. Since $10 < \sqrt{c} < \sqrt{105} < \sqrt{121} = 11$, we can assert that

$$\frac{5}{2(11)} < \sqrt{105} - 10 < \frac{5}{2(10)},$$

whence it follows that $10.2272 < \sqrt{105} < 10.2500$. This estimate may not be as sharp as desired. It is clear that the estimate $\sqrt{c} < \sqrt{105} < \sqrt{121}$ was wasteful and can be improved by making use of our conclusion that $\sqrt{105} < 10.2500$. Thus, $\sqrt{c} < 10.2500$ and we easily determine that

$$0.2439 < \frac{5}{2(10.2500)} < \sqrt{105} - 10.$$

Our improved estimate is $10.2439 < \sqrt{105} < 10.2500$. □

6.2.10 Examples **(a)** The exponential function $f(x) := e^x$ has the derivative $f'(x) = e^x$ for all $x \in \mathbb{R}$. Thus $f'(x) > 1$ for $x > 0$, and $f'(x) < 1$ for $x < 0$. From these relationships, we will derive the inequality

(1) $$e^x \geq 1 + x \quad \text{for} \quad x \in \mathbb{R},$$

with equality occurring if and only if $x = 0$.

If $x = 0$, we have equality with both sides equal to 1. If $x > 0$, we apply the Mean Value Theorem to the function f on the interval $[0, x]$. Then for some c with $0 < c < x$ we have

$$e^x - e^0 = e^c(x - 0).$$

Since $e^0 = 1$ and $e^c > 1$, this becomes $e^x - 1 > x$ so that we have $e^x > 1 + x$ for $x > 0$. A similar argument establishes the same strict inequality for $x < 0$. Thus the inequality (1) holds for all x, and equality occurs only if $x = 0$.

(b) The function $g(x) := \sin x$ has the derivative $g'(x) = \cos x$ for all $x \in \mathbb{R}$. On the basis of the fact that $-1 \leq \cos x \leq 1$ for all $x \in \mathbb{R}$, we will show that

(2) $$-x \leq \sin x \leq x \quad \text{for all} \quad x \geq 0.$$

Indeed, if we apply the Mean Value Theorem to g on the interval $[0, x]$, where $x > 0$, we obtain

$$\sin x - \sin 0 = (\cos c)(x - 0)$$

for some c between 0 and x. Since $\sin 0 = 0$ and $-1 \leq \cos c \leq 1$, we have $-x \leq \sin x \leq x$. Since equality holds at $x = 0$, the inequality (2) is established.

(c) (Bernoulli's inequality) If $\alpha > 1$, then

(3) $$(1 + x)^\alpha \geq 1 + \alpha x \quad \text{for all} \quad x > -1,$$

with equality if and only if $x = 0$.

6.2.11 Lemma *Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, let $c \in I$, and assume that f has a derivative at c. Then:*

(a) *If $f'(c) > 0$, then there is a number $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ such that $c < x < c + \delta$.*

(b) *If $f'(c) < 0$, then there is a number $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ such that $c - \delta < x < c$.*

Proof. (a) Since

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0,$$

it follows from Theorem 4.2.9 that there is a number $\delta > 0$ such that if $x \in I$ and $0 < |x - c| < \delta$, then

$$\frac{f(x) - f(c)}{x - c} > 0.$$

If $x \in I$ also satisfies $x > c$, then we have

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

Hence, if $x \in I$ and $c < x < c + \delta$, then $f(x) > f(c)$.

The proof of (b) is similar. Q.E.D.

6.2.12 Darboux's Theorem *If f is differentiable on $I = [a, b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point c in (a, b) such that $f'(c) = k$.*

Proof. Suppose that $f'(a) < k < f'(b)$. We define g on I by $g(x) := kx - f(x)$ for $x \in I$. Since g is continuous, it attains a maximum value on I. Since $g'(a) = k - f'(a) > 0$, it follows from Lemma 6.2.11(a) that the maximum of g does not occur at $x = a$. Similarly, since $g'(b) = k - f'(b) < 0$, it follows from Lemma 6.2.11(b) that the maximum does not occur at $x = b$. Therefore, g attains its maximum at some c in (a, b). Then from Theorem 6.2.1 we have $0 = g'(c) = k - f'(c)$. Hence, $f'(c) = k$. Q.E.D.

6.2.13 Example The function $g: [-1, 1] \to \mathbb{R}$ defined by

$$g(x) := \begin{cases} 1 & \text{for} \quad 0 < x \le 1, \\ 0 & \text{for} \quad x = 0, \\ -1 & \text{for} \quad -1 \le x < 0, \end{cases}$$

(which is a restriction of the signum function) clearly fails to satisfy the intermediate value property on the interval $[-1, 1]$. Therefore, by Darboux's Theorem, there does not exist a function f such that $f'(x) = g(x)$ for all $x \in [-1, 1]$. In other words, g is *not* the derivative on $[-1, 1]$ of any function. □

Convex Functions

The notion of convexity plays an important role in a number of areas, particularly in the modern theory of optimization. We shall briefly look at convex functions of one real variable and their relation to differentiation. The basic results, when appropriately modified, can be extended to higher dimensional spaces.

6.4.5 Definition Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be **convex** on I if for any t satisfying $0 \le t \le 1$ and any points x_1, x_2 in I, we have

$$f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2).$$

Note that if $x_1 < x_2$, then as t ranges from 0 to 1, the point $(1-t)x_1 + tx_2$ traverses the interval from x_1 to x_2. Thus if f is convex on I and if $x_1, x_2 \in I$, then the chord joining any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f lies above the graph of f. (See Figure 6.4.1.)

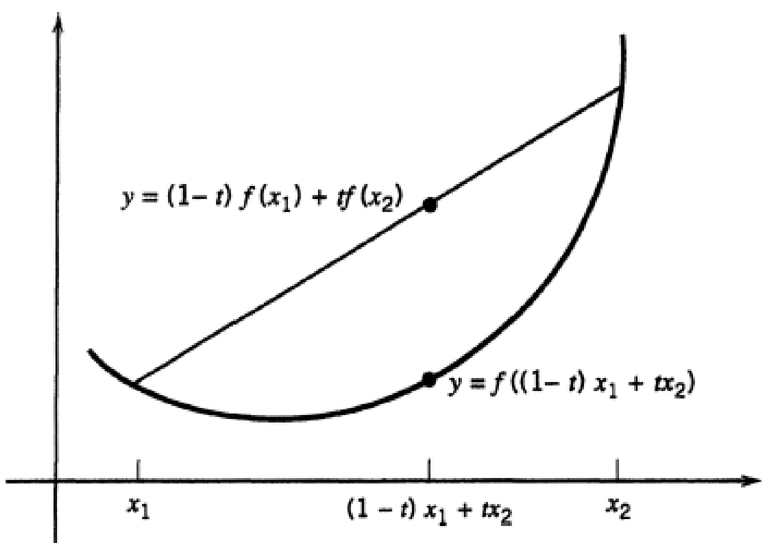


Figure 6.4.1 A convex function

6.4.6 Theorem *Let I be an open interval and let $f : I \to \mathbb{R}$ have a second derivative on I. Then f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$.*

Proof. ($\Rightarrow$) We will make use of the fact that the second derivative is given by the limit

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \tag{4}$$

for each $a \in I$. (See Exercise 16.) Given $a \in I$, let h be such that $a+h$ and $a-h$ belong to I. Then $a = \frac{1}{2}((a+h) + (a-h))$, and since f is convex on I, we have

$$f(a) = f\left(\tfrac{1}{2}(a+h) + \tfrac{1}{2}(a-h)\right) \leq \tfrac{1}{2}f(a+h) + \tfrac{1}{2}f(a-h).$$

Therefore, we have $f(a+h) - 2f(a) + f(a-h) \geq 0$. Since $h^2 > 0$ for all $h \neq 0$, we see that the limit in (4) must be nonnegative. Hence, we obtain $f''(a) \geq 0$ for any $a \in I$.

($\Leftarrow$) We will use Taylor's Theorem. Let x_1, x_2 be any two points of I, let $0 < t < 1$, and let $x_0 := (1-t)x_1 + tx_2$. Applying Taylor's Theorem to f at x_0 we obtain a point c_1 between x_0 and x_1 such that

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \tfrac{1}{2}f''(c_1)(x_1 - x_0)^2,$$

and a point c_2 between x_0 and x_2 such that

$$f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \tfrac{1}{2}f''(c_2)(x_2 - x_0)^2.$$

If f'' is nonnegative on I, then the term

$$R := \tfrac{1}{2}(1-t)f''(c_1)(x_1 - x_0)^2 + \tfrac{1}{2}tf''(c_2)(x_2 - x_0)^2$$

is also nonnegative. Thus we obtain

$$\begin{aligned}(1-t)f(x_1) + tf(x_2) &= f(x_0) + f'(x_0)((1-t)x_1 + tx_2 - x_0) \\ &\quad + \tfrac{1}{2}(1-t)f''(c_1)(x_1 - x_0)^2 + \tfrac{1}{2}\,tf''(c_2)(x_2 - x_0)^2 \\ &= f(x_0) + R \\ &\geq f(x_0) = f((1-t)x_1 + tx_2).\end{aligned}$$

Hence, f is a convex function on I.

POSSIBLE QUESTIONS

2 Mark Questions:

1. State the geometric view of mean value theorem.
2. Define an increasing function.
3. Define decreasing function.
4. Write the sufficient condition for a function to have a relative extermum.
5. State first derivative test for extrema.

8 Mark Questions:

1. State and prove mean value theorem.
2. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then prove that f is constant on $[a, b]$ if $f'(x) = 0$for $x \in (a, b)$.
3. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then prove that there exists at least one point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.
4. State and prove first derivative test for extrema.
5. State and prove intermediate value property of derivatives.
6. State and prove Darboux's theorem.
7. Apply the mean value theorem for approximate value of $\sqrt{105}$

UNIT-V

SYLLABUS

Cauchy's mean value theorem. Taylor's theorem with Lagrange's form of remainder, Taylor's theorem with Cauchy's form of remainder, application of Taylor's theorem to convex functions, relative extrema. Taylor's series and Maclaurin's series expansions of exponential and trigonometric functions, ln(1 + x), 1/ax+b and $(1 +x)^n$.

6.3.2 Cauchy Mean Value Theorem *Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all x in (a, b). Then there exists c in (a, b) such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. As in the proof of the Mean Value Theorem, we introduce a function to which Rolle's Theorem will apply. First we note that since $g'(x) \neq 0$ for all x in (a, b), it follows from Rolle's Theorem that $g(a) \neq g(b)$. For x in $[a, b]$, we now define

$$h(x) := \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)) - (f(x)-f(a)).$$

Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$. Therefore, it follows from Rolle's Theorem 6.2.3 that there exists a point c in (a, b) such that

$$0 = h'(c) = \frac{f(b)-f(a)}{g(b)-g(a)}g'(c) - f'(c).$$

Since $g'(c) \neq 0$, we obtain the desired result by dividing by $g'(c)$. Q.E.D.

Remarks The preceding theorem has a geometric interpretation that is similar to that of the Mean Value Theorem 6.2.4. The functions f and g can be viewed as determining a curve in the plane by means of the parametric equations $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$. Then the conclusion of the theorem is that there exists a point $(f(c), g(c))$ on the curve for some c in (a, b) such that the slope $g'(c)/f'(c)$ of the line tangent to the curve at that point is equal to the slope of the line segment joining the endpoints of the curve.

Note that if $g(x) = x$, then the Cauchy Mean Value Theorem reduces to the Mean Value Theorem 6.2.4.

$k = 1, 2, \ldots, n$. In fact, the polynomial

$$P_n(x) := f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{1}$$

has the property that it and its derivatives up to order n agree with the function f and its derivatives up to order n, at the specified point x_0. This polynomial P_n is called the n**th Taylor polynomial for** f **at** x_0. It is natural to expect this polynomial to provide a reasonable approximation to f for points near x_0, but to gauge the quality of the approximation, it is necessary to have information concerning the remainder $R_n := f - P_n$. The following fundamental result provides such information.

6.4.1 Taylor's Theorem *Let $n \in \mathbb{N}$, let $I := [a, b]$, and let $f : I \to \mathbb{R}$ be such that f and its derivatives $f', f'', \ldots, f^{(n)}$ are continuous on I and that $f^{(n+1)}$ exists on (a, b). If $x_0 \in I$, then for any x in I there exists a point c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{2}$$

Proof. Let x_0 and x be given and let J denote the closed interval with endpoints x_0 and x. We define the function F on J by

$$F(t) := f(x) - f(t) - (x-t)f'(t) - \cdots - \frac{(x-t)^n}{n!}f^{(n)}(t)$$

for $t \in J$. Then an easy calculation shows that we have

$$F'(t) = -\frac{(x-t)^n}{n!}f^{(n+1)}(t).$$

If we define G on J by

$$G(t) := F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0)$$

for $t \in J$, then $G(x_0) = G(x) = 0$. An application of Rolle's Theorem 6.2.3 yields a point c between x and x_0 such that

$$0 = G'(c) = F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}}F(x_0).$$

Hence, we obtain

$$\begin{aligned} F(x_0) &= -\frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}F'(c) \\ &= \frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}\frac{(x-c)^n}{n!}f^{(n+1)}(c) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}, \end{aligned}$$

which implies the stated result. Q.E.D.

We shall use the notation P_n for the nth Taylor polynomial (1) of f, and R_n for the remainder. Thus we may write the conclusion of Taylor's Theorem as $f(x) = P_n(x) + R_n(x)$ where R_n is given by

(3) $$R_n(x) := \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

for some point c between x and x_0. This formula for R_n is referred to as the **Lagrange form** (or the **derivative form**) of the remainder. Many other expressions for R_n are known; one is in terms of integration and will be given later. (See Theorem 7.3.18.)

6.4.2 Examples **(a)** Use Taylor's Theorem with $n = 2$ to approximate $\sqrt[3]{1+x},\ x > -1$.

We take the function $f(x) := (1+x)^{1/3}$, the point $x_0 = 0$, and $n = 2$. Since $f'(x) = \frac{1}{3}(1+x)^{-2/3}$ and $f''(x) = \frac{1}{3}(-\frac{2}{3})(1+x)^{-5/3}$, we have $f'(0) = \frac{1}{3}$ and $f''(0) = -2/9$. Thus we obtain

$$f(x) = P_2(x) + R_2(x) = 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + R_2(x),$$

where $R_2(x) = \frac{1}{3!}f'''(c)x^3 = \frac{5}{81}(1+c)^{-8/3}\,x^3$ for some point c between 0 and x.

For example, if we let $x = 0.3$, we get the approximation $P_2(0.3) = 1.09$ for $\sqrt[3]{1.3}$. Moreover, since $c > 0$ in this case, then $(1+c)^{-8/3} < 1$ and so the error is at most

$$R_2(0.3) \le \frac{5}{81}\left(\frac{3}{10}\right)^3 = \frac{1}{600} < 0.17 \times 10^{-2}.$$

Hence, we have $|\sqrt[3]{1.3} - 1.09| < 0.5 \times 10^{-2}$, so that two decimal place accuracy is assured.

(b) Approximate the number e with error less than 10^{-5}.

We shall consider the function $g(x) := e^x$ and take $x_0 = 0$ and $x = 1$ in Taylor's Theorem. We need to determine n so that $|R_n(1)| < 10^{-5}$. To do so, we shall use the fact that $g'(x) = e^x$ and the initial bound of $e^x \le 3$ for $0 \le x \le 1$.

Since $g'(x) = e^x$, it follows that $g^{(k)}(x) = e^x$ for all $k \in \mathbb{N}$, and therefore $g^{(k)}(0) = 1$ for all $k \in \mathbb{N}$. Consequently the nth Taylor polynomial is given by

$$P_n(x) := 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

and the remainder for $x = 1$ is given by $R_n(1) = e^c/(n+1)!$ for some c satisfying $0 < c < 1$. Since $e^c < 3$, we seek a value of n such that $3/(n+1)! < 10^{-5}$. A calculation reveals that $9! = 362,880 > 3 \times 10^5$ so that the value $n = 8$ will provide the desired accuracy; moreover, since $8! = 40,320$, no smaller value of n will be certain to suffice. Thus, we obtain

$$e \approx P_8(1) = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{8!} = 2.718\,28$$

with error less than 10^{-5}.

6.4.3 Examples (a) $1 - \frac{1}{2}x^2 \le \cos x$ for all $x \in \mathbb{R}$.

Use $f(x) := \cos x$ and $x_0 = 0$ in Taylor's Theorem, to obtain

$$\cos x = 1 - \frac{1}{2}x^2 + R_2(x),$$

where for some c between 0 and x we have

$$R_2(x) = \frac{f'''(c)}{3!}x^3 = \frac{\sin c}{6}\ x^3.$$

If $0 \le x \le \pi$, then $0 \le c < \pi$; since c and x^3 are both positive, we have $R_2(x) \ge 0$. Also, if $-\pi \le x \le 0$, then $-\pi \le c \le 0$; since $\sin c$ and x^3 are both negative, we again have $R_2(x) \ge 0$. Therefore, we see that $1 - \frac{1}{2}x^2 \le \cos x$ for $|x| \le \pi$. If $|x| \ge \pi$, then we have $1 - \frac{1}{2}x^2 < -3 \le \cos x$ and the inequality is trivially valid. Hence, the inequality holds for all $x \in \mathbb{R}$.

(b) For any $k \in \mathbb{N}$, and for all $x > 0$, we have

$$x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k}x^{2k} < \ln(1+x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Using the fact that the derivative of $\ln(1+x)$ is $1/(1+x)$ for $x > 0$, we see that the nth Taylor polynomial for $\ln(1+x)$ with $x_0 = 0$ is

$$P_n(x) = x - \frac{1}{2}x^2 + \cdots + (-1)^{n-1}\frac{1}{n}x^n$$

and the remainder is given by

$$R_n(x) = \frac{(-1)^n c^{n+1}}{n+1}x^{n+1}$$

for some c satisfying $0 < c < x$. Thus for any $x > 0$, if $n = 2k$ is even, then we have $R_{2k}(x) > 0$; and if $n = 2k+1$ is odd, then we have $R_{2k+1}(x) < 0$. The stated inequality then follows immediately.

(c) $e^\pi > \pi^e$

Taylor's Theorem gives us the inequality $e^x > 1 + x$ for $x > 0$, which the reader should verify. Then, since $\pi > e$, we have $x = \pi/e - 1 > 0$, so that

$$e^{(\pi/e-1)} > 1 + (\pi/e - 1) = \pi/e.$$

This implies $e^{\pi/e} > (\pi/e)e = \pi$, and thus we obtain the inequality $e^\pi > \pi^e$. □

6.4.4 Theorem *Let I be an interval, let x_0 be an interior point of I, and let $n \ge 2$. Suppose that the derivatives $f', f'', \ldots, f^{(n)}$ exist and are continuous in a neighborhood of x_0 and that $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \ne 0$.*

(i) *If n is even and $f^{(n)}(x_0) > 0$, then f has a relative minimum at x_0.*

(ii) *If n is even and $f^{(n)}(x_0) < 0$, then f has a relative maximum at x_0.*

(iii) *If n is odd, then f has neither a relative minimum nor relative maximum at x_0.*

Proof. Applying Taylor's Theorem at x_0, we find that for $x \in I$ we have

$$f(x) = P_{n-1}(x) + R_{n-1}(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n,$$

where c is some point between x_0 and x. Since $f^{(n)}$ is continuous, if $f^{(n)}(x_0) \neq 0$, then there exists an interval U containing x_0 such that $f^{(n)}(x)$ will have the same sign as $f^{(n)}(x_0)$ for $x \in U$. If $x \in U$, then the point c also belongs to U and consequently $f^{(n)}(c)$ and $f^{(n)}(x_0)$ will have the same sign.

(i) If n is even and $f^{(n)}(x_0) > 0$, then for $x \in U$ we have $f^{(n)}(c) > 0$ and $(x - x_0)^n \geq 0$ so that $R_{n-1}(x) \geq 0$. Hence, $f(x) \geq f(x_0)$ for $x \in U$, and therefore f has a relative minimum at x_0.

(ii) If n is even and $f^{(n)}(x_0) < 0$, then it follows that $R_{n-1}(x) \leq 0$ for $x \in U$, so that $f(x) \leq f(x_0)$ for $x \in U$. Therefore, f has a relative maximum at x_0.

(iii) If n is odd, then $(x - x_0)^n$ is positive if $x > x_0$ and negative if $x < x_0$. Consequently, if $x \in U$, then $R_{n-1}(x)$ will have opposite signs to the left and to the right of x_0. Therefore, f has neither a relative minimum nor a relative maximum at x_0. Q.E.D.

Theorem 1 (Taylor-Maclaurin series). *Suppose that $f(x)$ has a power series expansion at $x = a$ with radius of convergence $R > 0$, then the series expansion of $f(x)$ takes the form*

$$(2) \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots,$$

that is, the coefficient c_n in the expansion of $f(x)$ centered at $x = a$ is precisely $c_n = \dfrac{f^{(n)}(a)}{n!}$. The expansion (2) is called **Taylor series**. *If $a = 0$, the expansion*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \ldots,$$

is called **Maclaurin Series**.

Example 1. The function $f(x) = e^x$ satisfies $f^{(n)}(x) = e^x$ for any integer $n \geq 1$ and in particular $f^{(n)}(0) = 1$ for all n and then the Maclaurin series of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

observe that the radius of convergence of $f(x)$ is computed by noting that $c_n x^n = \dfrac{x^n}{n!}$ so that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0,$$

and the radius of convergence is $R = \infty$ since the above computation shows that the series converges absolutely for any x. Note that for any other center, say $x = a$ we have $f^{(n)}(a) = e^a$, so that the Taylor expansion of $f(x)$ is

$$e^x = \sum_{n=0}^{\infty} \frac{e^a (x-a)^n}{n!}.$$

and this series also has radius of convergence $R = \infty$.

Taylor Series Expansions of Exponential Functions

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad -\infty < x < \infty$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots \qquad -\infty < x < \infty$$

$$a^x = e^{x\ln a} = 1 + \frac{x\ln a}{1!} + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \cdots \qquad -\infty < x < \infty$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \cdots \qquad -\infty < x < \infty$$

$$e^{\cos x} = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31x^6}{720} + \cdots\right) \qquad -\infty < x < \infty$$

$$e^{\tan x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \cdots \qquad |x| < \frac{\pi}{2}$$

$$e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \cdots + \frac{\left(\sqrt{2}\right)^n \sin\left(\frac{n\pi}{4}\right) x^n}{n!} + \cdots \qquad -\infty < x < \infty$$

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots + \frac{\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) x^n}{n!} + \cdots \qquad -\infty < x < \infty$$

Taylor Series Expansions of Trigonometric Functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad -\infty < x < \infty$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} \cdots + \frac{2^{2n}\left(2^{2n}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots + \frac{2\left(2^{2n-1}-1\right)B_n x^{2n-1}}{(2n)!} + \cdots \quad 0 < |x| < \pi$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - - \cdots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} - \cdots \quad 0 < |x| < \pi$$

POSSIBLE QUESTIONS

2 Mark Questions:

1. State the geometric view of Cauchy's mean value theorem.
2. Write the condition to obtain mean value theorem form Cauchy's mean value theorem.
3. Approximate $\sqrt[3]{1+x}$ with $n = 2$.
4. Approximate the number e with error less than 10^{-5}.
5. Define convex function.

8 Mark Questions:

1. State and prove Cauchy's mean value theorem
2. State and prove Taylor's theorem
3. Prove that $1 - \frac{1}{2}x^2 \leq \cos x$ for all $x \in R$
4. Let $f: (a, b) \to R$ have a second derivative on (a, b) . Prove that f is a convex function on (a, b) iff $f''(x) \geq 0$ for all $x \in (a, b)$
5. For any $k \in N$ and for all $x > 0$, prove that $x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k}x^{2k} < \ln(1+x) < x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k+1}x^{2k+1}$
6. State and prove mean value theorem. Hence approximate $\sqrt{105}$
7. State and prove Bernoulli's inequality