KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University)

(Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

		Semester – V
		L T P C
16MMU504B	**LINEAR PROGRAMMING**	4 2 0 4

Scope: On successful completion of this course student will gain knowledge about fundamental concepts of duality, economic interpretation of dual constraints and game theory**.**

Objectives: This course has been intended to provide the knowledge in understanding the need and origin of the optimization methods which plays an essential role in present future in the application of Mathematics.

UNIT I

Introduction to Linear Programming Problem – Graphical Linear Programming Solution- Theory of Simplex Method-Optimality and unboundedness-the Simplex algorithm –Simplex method in tableau format- Introduction to artificial variables – two –phase method – Big –M method and their comparison.

UNIT II

Duality – Definition of the dual Problems-Formulation of the dual Problem-Primal Dual relationship: Review of simplex matrix Operations –Simplex tableau Layout-Optimal Dual Solution-Simplex Tableau computations. Economic interpretation of the dual: Economic Interpretation of Dual Variables-Economic Interpretation of Dual Constraints.

UNIT III

Transportation Problem: Definition of the Transportation model – Nontraditional Transportation model – The Transportation Algorithm: Determination of the Starting Solution-Northwest – corner method, Least – corner method, Vogel approximation method- Iterative Computations of the Transportation Algorithm.

UNIT IV

The Assignment Model: Introduction to Assignment model- Mathematical Formulation of Assignment model- Hungarian method for solving assignment problem –Simplex Explanation of the Hungarian method.

UNIT V

Game theory: Formulation of two person zero games – Solving two person zero sum games, games with mixed strategies, graphical solution procedure, linear programming solution of games.

SUGGESTED READINGS

TEXT BOOK

1. Handy .A. Taha., (2007). Operations Research, Seventh edition, Prentice Hall of India Pvt Ltd, New Delhi .

REFERENCES

1. Hillier F.S., and Lieberman G.J., (2009). Introduction to Operations Research, Ninth Edition, Tata McGraw Hill, Singapore.

2. Mokhtar S. Bazaraa, John J. Jarvis and Hanif D. Sherali, (2004). Linear Programming and Network Flows, Second Edition, John Wiley and Sons, India.

3. Hadley G,(2002). Linear Programming, Narosa Publishing House, New Delhi.

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Department of Mathematics

LECTURE PLAN

Subject: Linear Programming **Subject Code: 16MMU504B**

S.No	Lecture Duration	Topic to be covered	Support Material
Unit – I			
1.	1	Introduction to Linear Programming problem	R1:Ch3:Pg:24-26
2.	1	Problems on the Graphical Linear Programming solution	T1:Ch2:Pg:15-19
3.	1	Tutorial-I	
4.	1	Theory of Simplex Method	R1:Ch5:Pg:190-192
5.	1	Optimality and Unboundedness	R2:Ch3:Pg:114-120
6.	1	Tutorial-II	
7.	1	The Simplex Algorithm	R2:Ch3:Pg:120-125
8.	1	Problems on the Simplex method in tableau format	R2:Ch3:Pg:125-131
9.	1	Tutorial-III	
10.	1	Introduction to artificial variables	R2:Ch3:Pg:153-154
11.	1	Problems on the Two Phase Method	R2:Ch3:Pg:154-160
12.	1	Tutorial-IV	
13.	1	Problems on the Big M method and their comparison	R2:Ch3:Pg:165-173
14.	1	Tutorial-V	
15.	1	Recapitulation and discussion of possible question	
Total No. of Lecture hours planned – 15 hours			
T1. Handy .A. Taha., (2007). Operations Research, Seventh edition, Prentice Hall of India Pvt Ltd, New Delhi . **R1.** Hillier F.S., and Lieberman G.J., (2009). Introduction to Operations Research, Ninth Edition, Tata McGraw Hill, Singapore. **R2.** Mokhtar S. Bazaraa, John J. Jarvis and Hanif D. Sherali, (2004). Linear Programming and Network Flows, Second Edition, John Wiley and Sons, India.			
Unit – II			
1.	1	Definition of dual problems	T1:Ch4:Pg:151-155
2.	1	Problems on Formulation of dual problem	R3:Ch8:Pg:233-234
3.	1	Tutorial-I	
4.	1	Primal Dual relationship: Review of Simplex Matrix Operations	T1:Ch4:Pg:156-158
5.	1	Tutorial-VII	

6.	1	Simplex tableau Layout	T1:Ch4:Pg:158-159
7.	1	Problems on Optimal dual Solution	T1:Ch4:Pg:159-161
8.	1	Tutorial-VIII	
9.	1	Simplex tableau computations	T1:Ch4:Pg:165-166
10.	1	Economic interpretation of the dual: Economic interpretation of dual Variables	T1:Ch4:Pg:169-171
11.	1	Tutorial-IX	
12.	1	Economic interpretation of Dual Constraints	T1:Ch4:Pg:172-173
13.	1	Recapitulation and discussion of possible question	
Total No. of Lecture hours planned – 13 hours			
T1. Handy .A. Taha., (2007). Operations Research, Seventh edition, Prentice Hall of India Pvt Ltd, New Delhi . **R3.** Hadley G.,(2002). Linear Programming, Narosa Publishing House, New Delhi.			
Unit – III			
1.	1	Definition of Transportation Model	T1:Ch5:Pg:194-195
2.	1	Tutorial-X	
3.	1	Nontraditional Transportation model	T1:Ch5:Pg:201-204
4.	1	Tutorial-XI	
5.	1	Terminology for Transportation model	R1:Ch8:Pg:354-356
6.	1	The Transportation Algorithm	T1:Ch5:Pg:206-207
7.	1	Tutorial-XII	
8.	1	Problems on Determination of Starting Solution using Northwest corner method	T1:Ch5:Pg:207-208
9.	1	Problems on Determination of Starting Solution using Least cost method	T1:Ch5:Pg:208-209
10.	1	Tutorial-XIII	
11.	1	Problems on Determination of Starting Solution using Vogel approximation method	T1:Ch5:Pg:209-211
12.	1	Problems on the Iterative computations of the Transportation Algorithm	T1:Ch5:Pg:211-215
13.	1	Tutorial-XIV	
14.	1	Recapitulation and discussion of possible question	
Total No. of Lecture hours planned – 14 hours			
T1. Handy .A. Taha., (2007). Operations Research, Seventh edition, Prentice Hall of India Pvt Ltd, New Delhi . **R1.** Hillier F.S., and Lieberman G.J., (2009). Introduction to Operations Research, Ninth Edition, Tata McGraw Hill, Singapore.			
Unit – IV			
1.	1	Introduction to Assignment Model	R1:Ch8:Pg:381-382
2.	1	Tutorial-XV	
3.	1	Mathematical Formulation of Assignment Model	R1:Ch8:Pg:383-386
4.	1	Terminology for Hungarian Method	R2:Ch10:Pg:535-538
5.	1	Tutorial-XVI	
6.	1	Continuation of terminology of Hungarian Method	R2:Ch10:Pg:538-539

7.	1	Algorithm for Hungarian Method for solving Assignment Problem	T1:Ch5:Pg:222-223
8.	1	Tutorial-XVII	
9.	1	Problems on Hungarian Method for solving Assignment Problem	T1:Ch5:Pg:223-225
10.	1	Continuation of Problems on Hungarian Method for solving Assignment Problem	T1:Ch5:Pg:225-227
11.	1	Tutorial-XVIII	
12.	1	Simplex Explanation of the Hungarian Method	T1:Ch5:Pg:228-229
13.	1	Tutorial-XIX	
14.	1	Recapitulation and discussion of possible question	
Total No. of Lecture hours planned – 14 hours			
T1. Handy .A. Taha., (2007). Operations Research, Seventh edition, Prentice Hall of India Pvt Ltd, New Delhi . **R1.** Hillier F.S., and Lieberman G.J., (2009). Introduction to Operations Research, Ninth Edition, Tata McGraw Hill, Singapore. **R2.** Mokhtar S. Bazaraa, John J. Jarvis and Hanif D. Sherali, (2004). Linear Programming and Network Flows, Second Edition, John Wiley and Sons, India.			
Unit – V			
1.	1	Introduction to Game Theory	R1:Ch14:Pg:726
2.	1	Formulation of two person zero games	R1:Ch14:Pg:767-728
3.	1	Tutorial-XX	
4.	1	Solving simple games	R1:Ch14:Pg:728-729
5.	1	Problems on Solving two person zero sum games	T1:Ch13:Pg:521-522
6.	1	Tutorial-XXI	
7.	1	Problems on Games with mixed strategies	R1:Ch14:Pg:733-735
8.	1	Tutorial-XXII	
9.	1	Graphical solution Procedure	R1:Ch14:Pg:735-738
10.	1	Tutorial-XXXIII	
11.	1	Problems on linear Programming solution of games	R1:Ch14:Pg:738-741
12.	1	Tutorial-XXIV	
13.	1	Recapitulation and discussion of possible question	
14.	1	Discussion of pervious ESE question papers	
15.	1	Discussion of pervious ESE question papers	
16.	1	Discussion of pervious ESE question papers	
Total No. of Lecture hours planned -16 hours			
T1. Handy .A. Taha., (2007). Operations Research, Seventh edition, Prentice Hall of India Pvt Ltd, New Delhi . **R1.** Hillier F.S., and Lieberman G.J., (2009). Introduction to Operations Research, Ninth Edition, Tata McGraw Hill, Singapore.			

TEXT BOOK

T1. Handy .A. Taha., (2007). Operations Research, Seventh edition, Prentice Hall of India Pvt Ltd, New Delhi .

REFERENCES

R1. Hillier F.S., and Lieberman G.J., (2009). Introduction to Operations Research, Ninth Edition, Tata McGraw Hill, Singapore.
R2. Mokhtar S. Bazaraa, John J. Jarvis and Hanif D. Sherali, (2004). Linear Programming and Network Flows, Second Edition, John Wiley and Sons, India.
R3. Hadley G.,(2002). Linear Programming, Narosa Publishing House, New Delhi.

UNIT-I

SYLLABUS

Introduction to Linear Programming Problem – Formulation of LPP – Graphical Linear Programming Solution- Theory of Simplex Method-Optimality and unboundedness-the Simplex algorithm –Simplex method in tableau format- Introduction to artificial variables – two –phase method – Big –M method and their comparison.

Introduction to Linear Programming Problem

The idea of Linear Programing is conceived by George B. Bantzing in 1947 and the work named " Programing in Liner Structure" done by Kantorovich (1939) was published in 1959. Koopmans coined the term linear programing in 1948.

Linear Programing is a versatile technique which can be applied to a variety of problems of management such as production, refinery operation, advertising, transportation, distribution and investment analysis. Over the years linear programing has been found useful not only in business and industry but also in non-profit organizations such as government, hospitals, libraries and education.

Terminology:

- The problem variable X and Y are called decision variables and they represent the solution or output decision from the problem.
- The profit function that the manufacture wishes to increase, represents the objective of making the decisions on the production quantities and it is called ***objective function.***
- The conditions matching the resource requirements are called ***constraints.***
- The decision variables should take non negative values. This is called ***non-negativity restriction.***
- The problem written in algebraic form represents the mathematical model of the given system and is called ***Problem Formulation.***

Formulation:

The problem formulation has the following steps:

- Identifying the decision variables.
- Writing the objective function.
- Writing the constraints.
- Writing the non-negativity restrictions.

In the above formulations, the objective function and the constraints are linear therefore the

formulated model is known as *Linear Programming Problem.*
The formulation of a linear programming problem can be illustrated through what is known as the product mix problem. Typically, it occurs in a manufacturing industry where it is possible to manufacture a variety of products. Each of the products has a certain margin of profit per unit. These products use a common pool of resources whose availability is limited. The linear programming techniques identify the combination of the products which will maximize the profit without violating the resource constraints.

STANDARD and CANONICAL form of the model: sometimes referred to as the *canonical* form:

	MINIMIZATION PROBLEM	MAXIMIZATION PROBLEM
STANDARD FORM	Minimize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1,\dots,m$; $x_j \geq 0, j = 1,\dots,n$	Maximize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1,\dots,m$; $x_j \geq 0, j = 1,\dots,n$

CANONICAL FORM	Maximize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$, $i = 1,\dots,m$; $x_j \geq 0, j = 1,\dots,n$	Maximize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1,\dots,m$; $x_j \geq 0, j = 1,\dots,n$

1. All RHS parameters $b_i \geq 0$ and $n > m$.
2. Use the *n*-dimensional vector x to represent the decision variables; i.e., x $= (x_1,\dots,x_n)$.
3. Might have simple upper bounds, say, $x_j \leq u_j$.
4. Convert inequalities to equalities in (2).
5. Vector form of constraint: $a_{i1}x_1 + \bullet\bullet\bullet + a_{in}x_n = b_i$; $a_i x = b_i$; $Ax = b$ Maximize $\{z = cx : Ax = b, x \geq 0\}$.

Feasible Solution and Feasible Region:

Any no-negative value of (X_1,X_2) i.e. $X_1 \geq 0$, $X_2 \geq 0$ is a feasible solution of the linear programing problem if it satisfies all the constraints. The collection of all **feasible solutions** is known as the **feasible region.**

Formulation as a Linear Programming Problem

To formulate the mathematical (linear programming) model for this problem, let

x_1 = number of batches of product 1 produced per week
x_2 = number of batches of product 2 produced per week
Z = total profit per week (in thousands of dollars) from producing these two products

Thus, x_1 and x_2 are the *decision variables* for the model. Using the bottom row of Table

, we obtain

$Z = 3x_1 + 5x_2.$

The objective is to choose the values of x_1 and x_2 so as to *maximize* $Z = 3x_1 + 5x_2$, subject to the restrictions imposed on their values by the limited production capacities available in the three plants. Table 3.1 indicates that each batch of product 1 produced per week uses 1 hour of production time per week in Plant 1, whereas only 4 hours per week are available. This restriction is expressed mathematically by the inequality $x_1 \leq 4$. Similarly, Plant 2 imposes the restriction that $2x_2 \leq 12$. The number of hours of production

Example Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

time used per week in Plant 3 by choosing x_1 and x_2 as the new products' production rates would be $3x_1 + 2x_2$. Therefore, the mathematical statement of the Plant 3 restriction is $3x_1 + 2x_2 \leq 18$. Finally, since production rates cannot be negative, it is necessary to restrict the decision variables to be nonnegative: $x_1 \geq 0$ and $x_2 \geq 0$.

To summarize, in the mathematical language of linear programming, the problem is to choose values of x_1 and x_2 so as to

$$\text{Maximize} \quad Z = 3x_1 + 5x_2,$$

subject to the restrictions

$$\begin{aligned} x_1 \quad\quad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

Example

Reddy Mikks produces both interior and exterior paints from two raw materials, *M*1 and *M*2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, *M*1	6	4	24
Raw material, *M*2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.
Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.
The LP model, as in any OR model, has three basic components.

1. Decision **variables** that we seek to determine.
2. **Objective** (goal) that we need to optimize (maximize or minimize).
3. **Constraints** that the solution must satisfy.

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

x_1 = Tons produced daily of exterior paint

x_2 = Tons produced daily of interior paint

To construct the objective function, note that the company wants to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

$$\text{Total profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Total profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting z represent the total daily profit (in thousands of dollars), the objective of the company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus

$$\text{Usage of raw material } M1 \text{ by exterior paint} = 6x_1 \text{ tons/day}$$

$$\text{Usage of raw material } M1 \text{ by interior paint} = 4x_2 \text{ tons/day}$$

Hence

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \le 24 \quad \text{(Raw material } M1\text{)}$$
$$x_1 + 2x_2 \le 6 \quad \text{(Raw material } M2\text{)}$$

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$, should not exceed 1 ton, which translates to

$$x_2 - x_1 \le 1 \quad \text{(Market limit)}$$

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$$x_2 \le 2 \quad \text{(Demand limit)}$$

An implicit (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values. The **nonnegativity restrictions,** $x_1 \ge 0$, $x_2 \ge 0$, account for this requirement.

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$6x_1 + 4x_2 \le 24 \qquad (1)$$
$$x_1 + 2x_2 \le 6 \qquad (2)$$
$$-x_1 + x_2 \le 1 \qquad (3)$$
$$x_2 \le 2 \qquad (4)$$
$$x_1, x_2 \ge 0 \qquad (5)$$

Any values of x_1 and x_2 that satisfy *all* five constraints constitute a **feasible solution**. Otherwise, the solution is **infeasible**. For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate *any* of the constraints, including the nonnegativity restrictions. To verify this result, substitute $(x_1 = 3, x_2 = 1)$ in the left-hand side of each constraint. In

constraint (1) we have $6x_1 + 4x_2 = 6 \times 3 + 4 \times 1 = 22$, which is less than the right-hand side of the constraint (= 24). Constraints 2 through 5 will yield similar conclusions (verify!). On the other hand, the solution $x_1 = 4$ and $x_2 = 1$ is infeasible because it does not satisfy constraint (1)—namely, $6 \times 4 + 4 \times 1 = 28$, which is larger than the right-hand side (= 24).

Graphical Linear Programming Solution

The graphical procedure includes two steps:

1. Determination of the feasible solution space.
2. Determination of the optimum solution from among all the feasible points in the solution space.

Example

This example solves the Reddy Mikks model

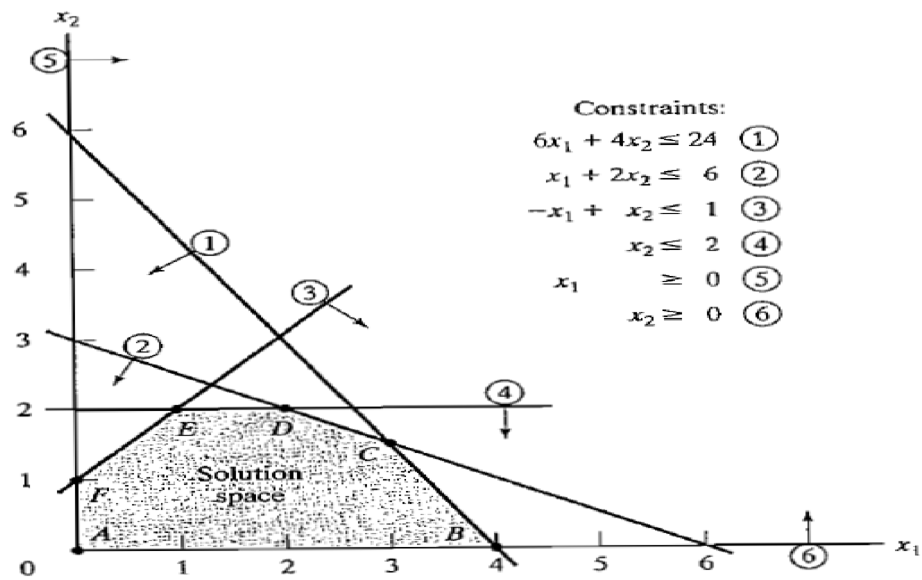
Step 1. *Determination of the Feasible Solution Space:*
First, we account for the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. the horizontal axis x_1 and the vertical axis x_2 represent the exterior- and interior-paint variables, respectively. Thus, the nonnegativity of the variables restricts the solution-space area to the first quadrant that lies above the x_1-axis and to the right of the x_2-axis.

To account for the remaining four constraints, first replace each inequality with an equation and then graph the resulting straight line by locating two distinct points on it. For example, after replacing $6x_1 + 4x_2 \leq 24$ with the straight line $6x_1 + 4x_2 = 24$, we can determine two distinct points by first setting $x_1 = 0$ to obtain $x_2 = \frac{24}{4} = 6$ and then setting $x_2 = 0$ to obtain $x_1 = \frac{24}{6} = 4$. Thus, the line passes through the two points $(0, 6)$ and $(4, 0)$, as shown by line (1)

Next, consider the effect of the inequality. All it does is divide the (x_1, x_2)-plane into two half-spaces, one on each side of the graphed line. Only one of these two halves satisfies the inequality. To determine the correct side, choose $(0, 0)$ as a *reference point*. If it satisfies the inequality, then the side in which it lies is the

FIGURE

Feasible space of the Reddy Mikks model



feasible half-space, otherwise the other side is. The use of the reference point $(0, 0)$ is illustrated with the constraint $6x_1 + 4x_2 \leq 24$. Because $6 \times 0 + 4 \times 0 = 0$ is less than 24, the half-space representing the inequality includes the origin

It is convenient computationally to select $(0, 0)$ as the reference point, unless the line happens to pass through the origin, in which case any other point can be used. For example, if we use the reference point $(6, 0)$, the left-hand side of the first constraint is $6 \times 6 + 4 \times 0 = 36$, which is larger than its right-hand side $(= 24)$, which means that the side in which $(6, 0)$ lies is not feasible for the inequality $6x_1 + 4x_2 \leq 24$. The conclusion is consistent with the one based on the reference point $(0, 0)$.

Application of the reference-point procedure to all the constraints of the model produces the constraints The **feasible solution space** of the problem represents the area in the first quadrant in which all the constraints are satisfied simultaneously. In any point in or on the boundary of the area *ABCDEF* is part of the feasible solution space. All points outside this area are infeasible.

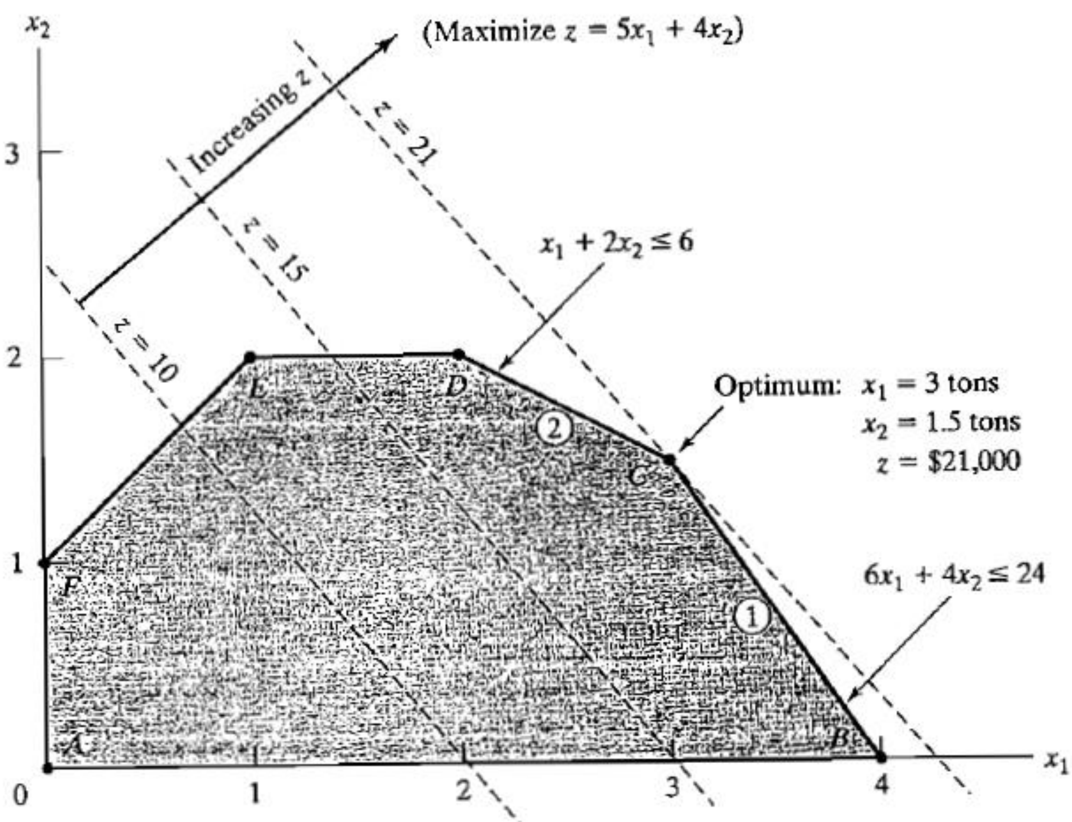
Step 2. *Determination of the Optimum Solution:*
The feasible space in Figure 2.1 is delineated by the line segments joining the points *A, B, C, D, E,* and *F*. Any point within or on the boundary of the space *ABCDEF* is feasible. Because the feasible space *ABCDEF* consists of an *infinite* number of points, we need a systematic procedure to identify the optimum solution.

The determination of the optimum solution requires identifying the direction in which the profit function $z = 5x_1 + 4x_2$ increases (recall that we are *maximizing* z). We can do so by assigning *arbitrary* increasing values to z. For example, using $z = 10$ and $z = 15$ would be equivalent to graphing the two lines $5x_1 + 4x_2 = 10$ and $5x_1 + 4x_2 = 15$. Thus, the direction of increase in z is as shown Figure 2.2. The optimum solution occurs at C, which is the point in the solution space beyond which any further increase will put z outside the boundaries of $ABCDEF$.

The values of x_1 and x_2 associated with the optimum point C are determined by solving the equations associated with lines (1) and (2)—that is,

$$6x_1 + 4x_2 = 24$$
$$x_1 + 2x_2 = 6$$

The solution is $x_1 = 3$ and $x_2 = 1.5$ with $z = 5 \times 3 + 4 \times 1.5 = 21$. This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is \$21,000.



Optimum solution of the Reddy Mikks model

An important characteristic of the optimum LP solution is that it is *always* associated with a **corner point** of the solution space (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or corner point C. Actually any point on the line segment BC will be an *alternative* optimum (see also Example 3.5-2), but the important observation here is that the line segment BC is totally defined by the *corner points* B and C.

THE SIMPLEX METHOD

Simplex method also called Simplex Technique was developed by G. B. Dantzig, an American mathematician. It has the advantage of being universal i.e. any linear model for which the solution exist can be solved by it. In principle, it consists of starting with a certain solution of which all that we know is that, it is feasible i.e. it satisfies non-negatively conditions. We improve this solution at consecutive stages until after a certain finite number of stages we arrive at optimal solution.

Basic Terminology Involved in Simplex Method

Standard Form—A linear program in which all of the constraints are written as equalities. The optimal solution of the standard form of a linear programming is the same optimal solution of the original formulation of the linear programme.

Slack Variable—Variables added to convert less than or equal to (≤) type constraints into equality are known as slack variable.

e.g. Let any constraint $a_1x_1 + a_2 x_2 \le b$

Some values have to be added to the L.H.S. of the constraints. Let this amount is s_1

Then $a_1x_1 + a_2 x_2 + s_1 = b_1$

s_1 is known as slack variable.

Surplus Variable—Variables subtracted to convert greater than or equal to (≥) type constraints into equality are known as surplus variable.

e.g. Let any constraint

$a_3x_3 + a_4 x_4 \ge b_2$

Some amount of values has to be subtracted to the R.H.S. of the constraints.

Let this amount is s_2

Then $a_3x_3 + a_4 x_4 - s_2 = b_2$

Artificial Variable—Sometimes to avoid negativity of the variables and to make an identity in the simplex table variable has to be introduce known as artificial variable.

e.g. Let any constraint

$a_3x_3 + a_4 x_4 \ge b_2$

To change into equality $a_3x_3 + a_4x_4 - s_2 + A_1 = b_2$

A_1 is an artificial variable.

Note an artificial variable has no physical meaning in theoretical problem.

TYPES OF SIMPLEX PROBLEM

Broadly, simplex problem either maximization or minimization types is solve by the method—

(a) Problem with only slack variables.
(b) Problem with artificial variables.
(c) Problem with degeneracy.
(d) Problem with unbounded solution.
(e) Primal dual problem.
(f) Problem with unrestricted problem.

Procedure to solve any LPP by Simplex Method—To solve any LPP by this method require to construct a ‗simplex tableau‖ which can be done by the following steps—

Step I—Formulate the given problem into Linear Programming problem as mathematical form—objective function

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to constraints

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$

..
..
..

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$

and Non-negative restrictions are decision variable.

$x_1, x_2, \ldots\ldots\ldots\ldots x_n \ge 0$

Step II—Now to express the model of LPP in the standard form as change the objective function as maximize if given problem minimize as

$Maximize(z) = Minimize(z)$

Objective function must be always maximize type. If, it does not so make it.
In the constraints, adding slack variables in the left hand side of the constraints and assign a zero coefficient to these in the objective function.

Thus, we can restate the problem as follows:—

$Maximize(z) = c_1x_1 + c_2x_2 + \ldots\ldots\ldots c_nx_n + 0.s_1 + \cdots + 0.s_m$

Subject to the linear constraints

$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n + s_1 = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots a_{2n}x_n + s_2 = b_2$

..
..

..
$a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n + s_m = b_m$

and $x_1, x_2, ..., x_n, s_1, s_2, \quad s_m \geq 0$

As, the contribution of slack variable is nothing due to zero coefficient in the objective function.

Step III—An initial basic feasible solution is obtained by setting $x_1 = x_2 = ... = x_n = 0$, we get $s_1 = s_2 = ... = s_m = b_m$

Step IV—To set up the initial simplex table

Basic Variables	Coefficient of B.V.	Solution	c_1	c_2	....	c_n	0	0	...	0	Min. Ratio (CB/KB)
			Coefficient Matrix				Identify Matrix				
			x_1	x_2	...	x_n	s_1	s_2	...	s_m	
s_1	0	b_1	a_{11}	a_{12}	...	a_{1n}	1	0	...	0	
s_2	0	b_2	a_{21}	a_{22}	...	a_{2n}	0	1	...	0	
...	...	...	...	...	...	...	...	...	...	...	
...	...	...	...	...	...	...	...	...	...	...	
s_n	0	b_n	a_{m1}	a_{m2}	...	a_{mn}	0	0	...	1	
			c_1	c_2	...	c_n	0	0	...	0	

Let Z_j represent the amount by which the value of objective function Z would be decreased or (increased) if one unit of given variable is added to the new solution.

$C_j - Z_j$ (Net Effect) = C_j (Incoming Unit Profit / Cost)

$-Z_j$ (Outgoing total profit / Cost)

Where Z_j =(Coefficient of basic Variables Column)×(Exchange Coefficient Column j)

Optimality Test—Calculate $\Delta_j == C_j - Z_j$ value for all non-basic variables. The

decision variable value corresponding to column of Basic variable coefficient. These may be following three cases:—

(i) If all $C_j - Z_j \leq 0$, then the basic feasible solution is optimal and alternative solution also exist if any non-variable with Δj is zero.

(ii) If at least one column of the coefficient matrix say key column leaving all the elements are negative, then there exists an unbounded solution to the given problem.

(iii) If at least one $\Delta j = (C_j - Z_j) < 0$ and each of these has at least one positive element (i.e. a_{ij}) for some row, then it indicates that an improvement in the value of objective function Z is possible.

(iv) Select the variable to enter the basis if (iii) case holds, then select the variable with largest $(C_j - Z_j)$ value to enter into the new solution. The column to be entered is known as key column.

- **Test for Feasibility—**Corresponding variable in the key column with minimum ratio of non-negative amount.

$$\frac{XBr}{arj} = \min\left\{\frac{XBi}{arj}; arj > 0\right\}$$

Step (V) —Finding the new Solution

(i) If the key element is 1, then the row remains the same in the new simplex table.

(ii) If the key element is other than 1, then divide each element in the key row (including elements in X_B column) by the key element, to find the new values for that now.

(iii) The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero.

For each row other than key row, we use the following formula:—

Number in new row= (Number in old row) ±

$$\left[\begin{pmatrix}\text{Number above or}\\ \text{below key element}\end{pmatrix} \times \begin{pmatrix}\text{Corresponding number in the new}\\ \text{row, that is row replaced in step (ii)}\end{pmatrix}\right]$$

The new entries in C_B (Coefficient of Basic Variables) and X_B (Value of basic variables) columns are updated in the new simplex table of the current solution.

Step (VI)—Repeat the procedure, until all the entries in the $C_j - Z_j$ row are either negative or zero.

Example-1—Use the simplex method to solve the following Linear Programming Problem—

$$Maximize(Z) = 2x_1 + 3x_2$$

Subject to the Constraints

$$x_1 + x_2 \le 1$$
$$3x_1 + x_2 \le 4$$

and $x_1, x_2 \ge 0$

Solution—Step I—Introducing non-negative slack variables s_1, s_2 to convert inequality constraints to equality. Then LPP becomes—

$$Maximize(Z) = 2x_1 + 3x_2 + 0.s_1 + 0.s_2$$

Subject to the constraints

$x_1 + x_2 + s_1 = 1$

$3x_1 + x_2 + s_2 = 4$

and $x_1, x_2, s_1, s_2 \geq 0$

Step II—Since all $b_i > 0$, (i=1, 2) we can choose initial basic feasible solution as $x_1 = x_2 = 0, s_1 = 1, s_2 = 4$ and maximum Z=0. This solution can be read from the simplex table by putting all the equations.

Step III—To test the optimality of the solution calculate $\Delta j = C_j - Z_j$ as

$$Z_i = (\text{Basic Variable Coefficients, } C_B) \times (j^{th} \text{ column of data matrix})$$

$Z_1 = (0 \times 1) + (0 \times 3) = 0$ $c_1 = 2$

$Z_2 = (0 \times 1) + (0 \times 1) = 0$ $c_2 = 3$

$\Delta j = C_j - Z_j;$ $\Delta_1 = C_1 - Z_1;$ $\Delta_2 = C_2 - Z_2$

= 2 – 0 = 2 = 3 – 0 = 3

C_j			2	3	0	0	
Profit per Unit CB	**Basic Variable BV**	**Solution b = X_B**	**x_1**	**x_2**	**s_1**	**s_2**	**Min Ratio X_B/X_2**
0	s_1	1	1	1	1	0	1/1
0	s_2	4	3	1	0	1	4/1
Z=0		Z_j	0	0	0	0	
		$C_j - Z_j$	2	3	0	0	

Since all $\Delta_j \geq 0$, so current solution is not optimal. Largest positive number of Δ_j is 3, so $x_2 \geq 0$ column is the key column.

Step 4—The variable to leave the basis is determined by /../ to leave the basis.

Step 5—The new row can be obtained by new operation ratio

$$R_1(new) \rightarrow R_1(old) \div 1(Key\ element)$$
$$\rightarrow 1/1, 1/1, 1/1, 1/1, 0/1$$

$\rightarrow 1, 1, 1, 1, 0$

$$R_2(New) \rightarrow R_2(Old) - 1R_2(New)$$

$4 - 1 \times 1 = 3$

$3 - 1 \times 1 = 2$

$1 - 1 \times 1 = 0$

$0 - 1 \times 1 = -1$

$1 - 1 \times 0 = 1$

Improved Table

C_j			2	3	0	0
CB	**BV**	**b = X_B**	**x_1**	**x_2**	**s_1**	**s_2**
3	x_2	1	1	1	1	0
0	s_2	3	2	0	−1	1
Z=3		$\Delta_j = C_j - Z_j$	−1	0	−3	0

All $\Delta_j \leq 0$ for non-basic variables, The optimal solution is $X_1 = 0, X_2 = 1$, optimal Z = 3.

Introduction to artificial variables

As demonstrated in Example 3.3-1, LPs in which all the constraints are ($\leq$) with non-negative right-hand sides offer a convenient all-slack starting basic feasible solution. Models involving ($=$) and/or ($\geq$) constraints do not.

The procedure for starting "ill-behaved" LPs with ($=$) and ($\geq$) constraints is to use **artificial variables** that play the role of slacks at the first iteration, and then dispose of them legitimately at a later iteration. Two closely related methods are introduced here: the *M*-method and the two-phase method.

M-Method

The *M*-method starts with the LP in equation form (Section 3.1). If equation *i* does not have a slack (or a variable that can play the role of a slack), an artificial variable, R_i, is added to form a starting solution similar to the convenient all-slack basic solution.

However, because the artificial variables are not part of the original LP model, they are assigned a very high **penalty** in the objective function, thus forcing them (eventually) to equal zero in the optimum solution. This will always be the case if the problem has a feasible solution. The following rule shows how the penalty is assigned in the cases of maximization and minimization:

Penalty Rule for Artificial Variables.

Given M, a sufficiently large positive value (mathematically, $M \to \infty$), the objective coefficient of an artificial variable represents an appropriate **penalty** if:

$$\text{Artificial variable objective coefficient} = \begin{cases} -M, & \text{in maximization problems} \\ M, & \text{in minimization problems} \end{cases}$$

Example

Minimize $z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Using x_3 as a surplus in the second constraint and x_4 as a slack in the third constraint, the equation form of the problem is given as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The third equation has its slack variable, x_4, but the first and second equations do not. Thus, we add the artificial variables R_1 and R_2 in the first two equations and penalize them in the objective function with $MR_1 + MR_2$ (because we are minimizing). The resulting LP is given as

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - x_3 + R_2 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

The associated starting basic solution is now given by $(R_1, R_2, x_4) = (3, 6, 4)$.

From the standpoint of solving the problem on the computer, M must assume a numeric value. Yet, in practically all textbooks, including the first seven editions of this book, M is manipulated algebraically in all the simplex tableaus. The result is an added, and unnecessary, layer of difficulty which can be avoided simply by substituting an appropriate numeric value for M (which is what we do anyway when we use the computer). In this edition, we will break away from the long tradition of manipulating M algebraically and use a numerical substitution instead. The intent, of course, is to simplify the presentation without losing substance.

Using $M = 100$, the starting simplex tableau is given as follows (for convenience, the z-column is eliminated because it does not change in all the iterations):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Before proceeding with the simplex method computations, we need to make the z-row consistent with the rest of the tableau. Specifically, in the tableau, $x_1 = x_2 = x_3 = 0$, which yields the starting basic solution $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$. This solution yields $z = 100 \times 3 + 100 \times 6 = 900$ (instead of 0, as the right-hand side of the z-row currently shows). This inconsistency stems from the fact that R_1 and R_2 have nonzero coefficients $(-100, -100)$ in the z-row (compare with the all-slack starting solution in Example 3.3-1, where the z-row coefficients of the slacks are zero).

We can eliminate this inconsistency by substituting out R_1 and R_2 in the z-row using the appropriate constraint equations. In particular, notice the highlighted elements $(= 1)$ in the R_1-row and the R_2-row. Multiplying *each* of R_1-row and R_2-row by 100 and adding the *sum* to the z-row will substitute out R_1 and R_2 in the objective row—that is,

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (100 \times R_1\text{-row} + 100 \times R_2\text{-row})$$

The modified tableau thus becomes (verify!)

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	696	399	−100	0	0	0	900
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Notice that $z = 900$, which is consistent now with the values of the starting basic feasible solution: $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$.

The last tableau is ready for us to apply the simplex method using the simplex optimality and the feasibility conditions, exactly as we did in Section 3.3.2. Because we are minimizing the objective function, the variable x_1 having the most *positive* coefficient in the z-row ($= 696$) enters the solution. The minimum ratio of the feasibility condition specifies R_1 as the leaving variable (verify!).

Once the entering and the leaving variables have been determined, the new tableau can be computed by using the familiar Gauss-Jordan operations.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	167	−100	−232	0	0	204
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	−1	$-\frac{4}{3}$	1	0	2
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

The last tableau shows that x_2 and R_2 are the entering and leaving variables, respectively. Continuing with the simplex computations, two more iterations are needed to reach the optimum: $x_1 = \frac{2}{5}$, $x_2 = \frac{9}{5}$, $z = \frac{17}{5}$ (verify with TORA!).

Note that the artificial variables R_1 and R_2 leave the basic solution in the first and second iterations, a result that is consistent with the concept of penalizing them in the objective function.

Two-Phase Method

In the M-method, the use of the penalty M, which by definition must be large relative to the actual objective coefficients of the model, can result in roundoff error that may impair the accuracy of the simplex calculations. The two-phase method alleviates this difficulty by eliminating the constant M altogether. As the name suggests, the method solves the LP in two phases: Phase I attempts to find a starting basic feasible solution, and, if one is found, Phase II is invoked to solve the original problem.

Summary of the Two-Phase Method

Phase I. Put the problem in equation form, and add the necessary artificial variables to the constraints (exactly as in the M-method) to secure a starting basic solution. Next, find a basic solution of the resulting equations that, regardless of whether the LP is maximization or minimization, *always* minimizes the sum of the artificial variables. If the minimum value of the sum is positive, the LP problem has no feasible solution, which ends the process (recall that a positive artificial variable signifies that an original constraint is not satisfied). Otherwise, proceed to Phase II.

Phase II. Use the feasible solution from Phase I as a starting basic feasible solution for the *original* problem.

Example

Minimize $z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Phase I

$$\text{Minimize } r = R_1 + R_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - x_3 + R_2 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

The associated tableau is given as

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

As in the M-method, R_1 and R_2 are substituted out in the r-row by using the following computations:

$$\text{New } r\text{-row} = \text{Old } r\text{-row} + (1 \times R_1\text{-row} + 1 \times R_2\text{-row})$$

The new r-row is used to solve Phase I of the problem, which yields the following optimum tableau (verify with TORA's Iterations ⇒ Two-phase Method):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	−1	1	1

Because minimum $r = 0$, Phase I produces the basic feasible solution $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, and $x_4 = 1$. At this point, the artificial variables have completed their mission, and we can eliminate their columns altogether from the tableau and move on to Phase II.

Phase II

After deleting the artificial columns, we write the *original* problem as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + \tfrac{1}{5}x_3 &= \tfrac{3}{5} \\ x_2 - \tfrac{3}{5}x_3 &= \tfrac{6}{5} \\ x_3 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Essentially, Phase I is a procedure that transforms the original constraint equations in a manner that provides a starting basic feasible solution for the problem, if one exists. The tableau associated with Phase II problem is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Again, because the basic variables x_1 and x_2 have nonzero coefficients in the z-row, they must be substituted out, using the following computations.

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row})$$

The initial tableau of Phase II is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Because we are minimizing, x_3 must enter the solution.

5. The Big-M Method

The following steps are involved in solving an LPP using the Big-M Method.

Step I—Express the given Linear Programming problem into Standard form.

Step II—Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type ≥ or =. However, addition of these artificial variable causes violation of the corresponding constraints. Therefore, we would like to get rid of these variables and would not allow them to appear in the final solution.

This is achieved by assigning a very large penalty (−M for maximization problems and M for minimization problems) in the objective function.

Step III—Solve the modified LPP by Simplex Method, until any one of the three causes may arises.

(i) If no artificial variable appears in the basis and optimal conditions are satisfied, then the current solution is an optimal basic feasible solution.

(ii) If at least one artificial variable in the basis at zero level and the optimality condition is satisfied then the current solutions is an optimal basic feasible solution.

(iii) If at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize that objective function, since it contains a very large penalty M and is called Pseudo optimal solution.

Example: Solve the following LPP—

$$Minimize(Z)=12x_1+20x_2$$

Subject to the constraints

$$6x_1+8x_2\geq 100$$
$$7x_1+12x_2\geq 120$$

and $x_1, x_2\geq 0$

Solution—Convert the given LPP into standard form

$$Maximize(Z) = -12x_1 - 20x_2 + 0.s_1 + 0.s_2 - MA_1 - MA_2$$

s.t. $6x_1 + 8x_2 - s_1 + A_1 = 100$

$7x_1 + 12x_2 - s_2 + A_2 = 120$

and $x_1, x_2, s_1, s_2 \geq 0$

Initial Simplex Table

C_j			−12	−20	0	0	−M	−M	Min Ratio
Basic Variable	**C_B**	**X_B**	**X_1**	**X_2**	**s_1**	**s_2**	**A_1**	**A_2**	
A_1	−M	100	6	8	−1	0	1	0	12.5
A_2	−M	120	7	12	0	−1	0	1	←10
		$\Delta_j \rightarrow$	−12 +13M	−20 +20M	M	M	0	0	

Key element = 12, Outgoing variable is A_2, Incoming Variable is X_2

C_j			−12	−20	0	0	−M	−M	Min Ratio
Basic Variable	**C_B**	**X_B**	**X_1**	**X_2**	**s_1**	**s_2**	**A_1**	**A_2**	
A_1	−M	20	4/3	0	−1	2/3	1	–	←15
X_2	−20	10	7/12	1	0	−1/12	0	–	120/7
		$\Delta_j \rightarrow$	$\frac{4M-1}{3}$ ↑	0	M	$\frac{2M-5}{3}$	0	–	=17.14

Key element is 4/3. Outgoing variable is A_1 and Incoming variable is X_1.

C_j			−12	−20	0	0	−M	−M	Min Ratio
Basic Variable	**C_B**	**X_B**	**X_1**	**X_2**	**s_1**	**s_2**	**A_1**	**A_2**	
X_1	−12	15	1	0	−3/4	1/2	–	–	
X_2	−20	5/4	0	1	7/16	−3/4	–	–	
		$\Delta_j \rightarrow$	0	0	−1/4	−9	–	–	

All $\Delta_j \leq 0$, so we have optimal solution.

$X_1 = 15$, $X_2 = 5/14$

Maximize Z = (−12 × 15)+(−20 × 5/4)

= – 205

Minimize Z = – Maximize (Z)

= 205

POSSIBLE QUESTIONS

PART - A (20 x 1 =20 Marks)
(Question Nos. 1 to 20 Online Examinations)

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Write the canonical form of LPP.
2. Write the standard form of LPP.
3. Define feasible solution.
4. Define degenerate basic solution.
5. Express the following LPP in standard form.
   Maximize $Z = 4x_1 + 2x_2 + 6x_3$
   Subject to $2x_1 + 3x_2 + 2x_3 \geq 6$
   $3x_1 + 4x_2 = 8$
   $6x_1 - 4x_2 + x_3 \leq 10$
   and $x_1, x_2, x_3 \geq 0$

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. A company manufactures 2 types of printed circuits. The requirements of transistors, resistors and capacitors for each type of printed circuits along with other data are given below.

	Circuit		Stock available
	A	B	
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
Profit	Rs.5	Rs.8	

How many circuits of each type should the company produce from the stock to earn maximum profit. Formulate this as a LPP and solve it graphically also.

2. A firm manufactures two types of product A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines $M_1$and M_2. Type A requires 1 minute of processing time on M_1 and 2 minutes on M_2. Type B requires 1 minute on M_1 and 1 minute on M_2. Machine M_1 is available for not more than 6 hours 40 minutes, while machines M_2 is a available for 10 hours during any work hours. Formulate the problem as LPP so as to maximize the profit.
3. Solve the following LPP by the graphical method.
   Maximize $Z = 3x_1 + 2x_2$
   Subject to the constrains
   $-2x_1 + x_2 \leq 1$
   $x_1 \leq 2$
   $x_1 + x_2 \leq 3$
   and $x_1, x_2 \geq 0$

4. Use graphical method to solve the following LPP

   Maximize $Z = 4x_1 + 10x_2$

   Subject to the constrains

   $2x_1 + x_2 \leq 50$

   $2x_1 + 5x_2 \leq 100$

   $2x_1 + 3x_2 \leq 90$

   and $x_1, x_2 \geq 0$

5. Use simplex method to solve the following LPP

   Maximize $Z = 5x_1 + 8x_2$

   Subject to the constrains

   $-2x_1 + x_2 \leq 1$

   $x_1 \leq 2$

   $x_1 + x_2 \leq 3$

   and $x_1, x_2 \geq 0$

6. Use simplex method to solve the following LPP

   Maximize $Z = 4x_1 + 10x_2$

   Subject to the constrains

   $2x_1 + x_2 \leq 50$

   $2x_1 + 5x_2 \leq 100$

   $2x_1 + 3x_2 \leq 90$

   and $x_1, x_2 \geq 0$

7. Solve the following LPP using Big M method.

   Maximize $Z = 3x_1 + 2x_2$

   Subject to $2x_1 + x_2 \leq 2$

   $3x_1 + 4x_2 \geq 12$

   And $x_1, x_2 \geq 0$

8. Use Penalty method to solve

   Minimize $Z = 4x_1 + 3x_2$

   Subject to $2x_1 + x_2 \geq 10$

   $-3x_1 + 2x_2 \leq 6$

   $x_1 + x_2 \geq 6$

   and $x_1, x_2 \geq 0$

9. Use Two – phase simplex method to solve

   Maximize $Z = 5x_1 + 8x_2$

   Subject to $3x_1 + 2x_2 \geq 3$

   $x_1 + 4x_2 \geq 4$

   $x_1 + x_2 \leq 5$

   and $x_1, x_2 \geq 0$

10. Solve the following using Two – phase simplex method.

Minimize $Z = -2x_1 - x_2$

Subject to $x_1 + x_2 \geq 2$

$x_1 + x_2 \geq 4$

and $x_1, x_2 \geq 0$

UNIT-II

SYLLABUS

Duality – Definition of the dual Problems-Formulation of the dual Problem-Primal Dual relationship: Review of simplex matrix Operations –Simplex tableau Layout-Optimal Dual Solution-Simplex Tableau computations. Economic interpretation of the dual: Economic Interpretation of Dual Variables-Economic Interpretation of Dual Constraints.

Duality

This chapter dealt with sensitivity of the optimal solution by determining the ranges for the model parameters that will keep the optimum basic solution unchanged. A natural sequel to sensitivity analysis is post-optimal analysis, where the goal is to determine the new optimum that results from makin targeted chanes in the model parameters. Although opst-optimal analysis can e carried out using the simplex tableau computations, this chapter is based entirely on the dual problem.

At a minimum we will need to study the dual problem and its economic interpretation. The mathematical definition of the dual problem is purely abstract. Yet, when we study, we will see that the dual problem leads to intriguing economic interpretations of the LP model, including dual prices and reduced costs. It also provides the foundation for the development of the new dual simplex algorithm, a prerequisite for post-optimal analysis.

DEFINITION OF THE DUAL PROBLEM

The **dual** problem is an LP defined directly and systematically from the **primal** (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints ($\leq$, $\geq$, or $=$), and orientation of the variables (nonnegative or unrestricted). This type of treatment is somewhat confusing, and for this reason we offer a *single* definition that automatically subsumes *all* forms of the primal.

Our definition of the dual problem requires expressing the primal problem in the *equation form* presented in Section 3.1 (all the constraints are equations with nonnegative right-hand side and all the variables are nonnegative). This requirement is consistent with the format of the simplex starting tableau. Hence, any results obtained from the primal optimal solution will apply directly to the associated dual problem.

To show how the dual problem is constructed, define the primal in *equation form* as follows:

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \ldots, m$$

$$x_j \geq 0, j = 1, 2, \ldots, n$$

The variables x_j, $j = 1, 2, \ldots, n$, include the surplus, slack, and artificial variables, if any.

Table 4.1 shows how the dual problem is constructed from the primal. Effectively, we have

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

Formulation of the dual problem:

TABLE 4.1 Construction of the Dual from the Primal

	Primal variables						
	x_1	x_2	$\ldots$	x_j	$\ldots$	x_n	
Dual variables	c_1	c_2	$\ldots$	c_j	$\ldots$	c_n	Right-hand side
y_1	a_{11}	a_{12}	$\ldots$	a_{1j}	$\ldots$	a_{1n}	b_1
y_2	a_{21}	a_{22}	$\ldots$	a_{2j}	$\ldots$	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\ldots$	a_{mj}	$\ldots$	a_{mn}	b_m
				↑ jth dual constraint			↑ Dual objective coefficients

TABLE 4.2 Rules for Constructing the Dual Problem

Primal problem objective[a]	Dual problem		
	Objective	*Constraints type*	*Variables sign*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

[a] All primal constraints are equations with nonnegative right-hand side and all the variables are nonnegative.

The rules for determining the sense of optimization (maximization or minimization), the type of the constraint ($\leq, \geq$, or $=$), and the sign of the dual variables are summarized in Table 4.2. Note that the sense of optimization in the dual is always opposite to that of the primal. An easy way to remember the constraint type in the dual (i.e., $\leq$ or $\geq$) is that if the dual objective is *minimization* (i.e., pointing *down*), then the constraints are all of the type $\geq$ (i.e., pointing *up*). The opposite is true when the dual objective is maximization.

The following examples demonstrate the use of the rules in Table 4.2 and also show that our definition incorporates all forms of the primal automatically.

Example

Primal	Primal in equation form	Dual variables
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 + x_3 \leq 10$	$x_1 + 2x_2 + x_3 + x_4 = 10$	y_1
$2x_1 - x_2 + 3x_3 = 8$	$2x_1 - x_2 + 3x_3 + 0x_4 = 8$	y_2
$x_1, x_2, x_3 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Minimize } w = 10y_1 + 8y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\geq 5 \\ 2y_1 - y_2 &\geq 12 \\ y_1 + 3y_2 &\geq 4 \\ \left.\begin{aligned} y_1 + 0y_2 &\geq 0 \\ y_1, y_2 &\text{ unrestricted} \end{aligned}\right\} &\Rightarrow (y_1 \geq 0,\ y_2 \text{ unrestricted}) \end{aligned}$$

PRIMAL-DUAL RELATIONSHIPS

Changes made in the original LP model will change the elements of the current optimal tableau, which in turn may affect the optimality and/or the feasibility of the current solution. This section introduces a number of primal-dual relationships that can be used to recompute the elements of the optimal simplex tableau. These relationships will form the basis for the economic interpretation of the LP model as well as for post-optimality analysis.

This section starts with a brief review of matrices, a convenient tool for carrying out the simplex tableau computations.

Review of Simple Matrix Operations

The simplex tableau computations use only three elementary matrix operations: (row vector) × (matrix), (matrix) × (column vector), and (scalar) × (matrix). These operations are summarized here for convenience. First, we introduce some matrix definitions:[1]

1. A *matrix*, **A**, of size $(m \times n)$ is a rectangular array of elements with m rows and n columns.
2. A *row vector*, **V**, of size m is a $(1 \times m)$ matrix.
3. A *column vector*, **P**, of size n is an $(n \times 1)$ matrix.

These definitions can be represented mathematically as

$$\mathbf{V} = (v_1, v_2, \ldots, v_m),\ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \vdots & a_{mn} \end{pmatrix},\ \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{pmatrix}$$

1. (Row vector × matrix, VA). The operation is defined only if the size of the row vector **V** equals the number of rows of **A**. In this case,

$$\mathbf{VA} = \left(\sum_{i=1}^{m} v_i a_{i1}, \sum_{i=1}^{m} v_i a_{i2}, \ldots, \sum_{i=1}^{m} v_i a_{in} \right)$$

For example,

$$(11, 22, 33)\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = (1 \times 11 + 3 \times 22 + 5 \times 33, 2 \times 11 + 4 \times 22 + 6 \times 33)$$

$$= (242, 308)$$

2. (Matrix × column vector, AP). The operation is defined only if the number of columns of **A** equals the size of column vector **P**. In this case,

$$\mathbf{AP} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}p_j \\ \sum_{j=1}^{n} a_{2j}p_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}p_j \end{pmatrix}$$

As an illustration, we have

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 11 \\ 22 \\ 33 \end{pmatrix} = \begin{pmatrix} 1 \times 11 + 3 \times 22 + 5 \times 33 \\ 2 \times 11 + 4 \times 22 + 6 \times 33 \end{pmatrix} = \begin{pmatrix} 242 \\ 308 \end{pmatrix}$$

3. (Scalar $\times$ matrix, αA). Given the scalar (or constant) quantity α, the multiplication operation $\alpha\mathbf{A}$ will result in a matrix of the same size as **A** whose (i, j)th element equals αa_{ij}. For example, given $\alpha = 10$,

$$(10)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix}$$

In general, $\alpha\mathbf{A} = \mathbf{A}\alpha$. The same operation is extended equally to the multiplication of vectors by scalars. For example, $\alpha\mathbf{V} = \mathbf{V}\alpha$ and $\alpha\mathbf{P} = \mathbf{P}\alpha$.

Simplex Tableau Layout

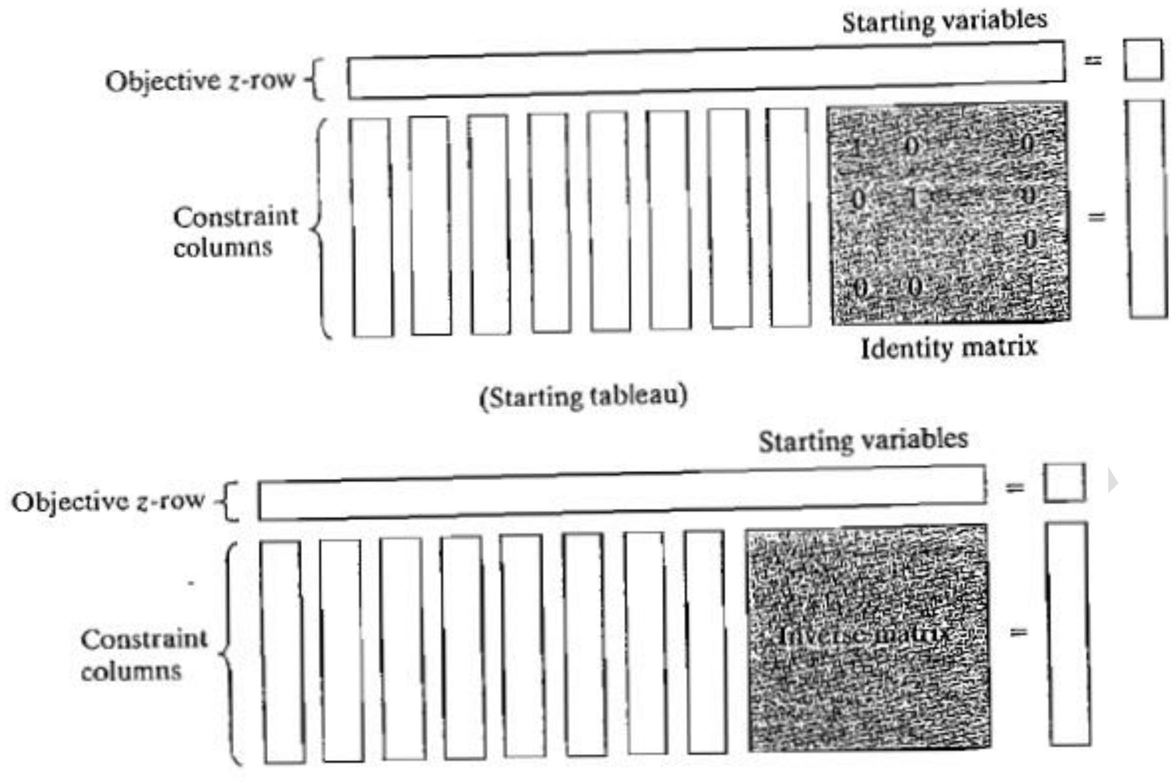
In Chapter

we followed a specific format for setting up the simplex tableau. This format is the basis for the development in this chapter.

Figure 4.1 gives a schematic representation of the *starting* and *general* simplex tableaus. In the starting tableau, the constraint coefficients under the starting variables form an **identity matrix** (all main-diagonal elements equal 1 and all off diagonal elements equal zero). With this arrangement, subsequent iterations of the simplex tableau generated by the Gauss-Jordan row operations (see Chapter 3) will modify the elements of the identity matrix to produce what is known as the **inverse matrix**. As we will see in the remainder of this chapter, the inverse matrix is key to computing all the elements of the associated simplex tableau.

FIGURE 4.1

Schematic representation of the starting and general simplex tableaus



Optimal Dual Solution

The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution to the other. Thus, in an LP model in which the number of variables is considerably smaller than the number of constraints, computational savings may be realized by solving the dual, from which the primal solution is determined automatically. This result follows because the amount of simplex computation depends largely (though not totally) on the number of constraints

This section provides two methods for determining the dual values. Note that the dual of the dual is itself the primal, which means that the dual solution can also be used to yield the optimal primal solution automatically.

Method 1.

$$\begin{pmatrix}\text{Optimal value of} \\ \textit{dual}\text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i \\ + \\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Method 2.

$$\begin{pmatrix}\text{Optimal values}\\ \text{of } dual \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original objective coefficients}\\ \text{of optimal } primal \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } primal\\ \text{inverse}\end{pmatrix}$$

The elements of the row vector must appear in the same order in which the basic variables are listed in the *Basic* column of the simplex tableau.

Example

Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$x_1 + 2x_2 + x_3 \le 10$$
$$2x_1 - x_2 + 3x_3 = 8$$
$$x_1, x_2, x_3 \ge 0$$

To prepare the problem for solution by the simplex method, we add a slack x_4 in the first constraint and an artificial R in the second. The resulting primal and the associated dual problems are thus defined as follows:

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$ subject to $x_1 + 2x_2 + x_3 + x_4 = 10$ $2x_1 - x_2 + 3x_3 + R = 8$ $x_1, x_2, x_3, x_4, R \ge 0$	Minimize $w = 10y_1 + 8y_2$ subject to $y_1 + 2y_2 \ge 5$ $2y_1 - y_2 \ge 12$ $y_1 + 3y_2 \ge 4$ $y_1 \ge 0$ $y_2 \ge -M$ ($\Rightarrow y_2$ unrestricted)

We now show how the optimal dual values are determined using the two methods described at the start of this section.

Method 1. In Table 4.4, the starting primal variables x_4 and R uniquely correspond to the dual variables y_1 and y_2, respectively. Thus, we determine the optimum dual solution as follows:

Starting primal basic variables	x_4	R
z-equation coefficients	$\frac{29}{5}$	$-\frac{2}{5} + M$
Original objective coefficient	0	$-M$
Dual variables	y_1	y_2
Optimal dual values	$\frac{29}{5} + 0 = \frac{29}{5}$	$-\frac{2}{5} + M + (-M) = -\frac{2}{5}$

Method 2. The optimal inverse matrix, highlighted under the starting variables x_4 and R, is given in Table 4.4 as

$$\text{Optimal inverse} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

First, we note that the optimal primal variables are listed in the tableau in *row order* as x_2 and then x_1. This means that the elements of the original objective coefficients for the two variables must appear in the same order—namely,

$$(\text{Original objective coefficients}) = (\text{Coefficient of } x_2, \text{coefficient of } x_1)$$
$$= (12, 5)$$

TABLE 4.4 Optimal Tableau of the Primal of Example 4.2-1

Basic	x_1	x_2	x_3	x_4	R	Solution
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5} + M$	$54\frac{4}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Thus, the optimal dual values are computed as

$$(y_1, y_2) = \begin{pmatrix} \text{Original objective} \\ \text{coefficients of } x_2, x_1 \end{pmatrix} \times (\text{Optimal inverse})$$
$$= (12, 5)\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$
$$= \left(\frac{29}{5}, -\frac{2}{5}\right)$$

Simplex Tableau Computations

This section shows how *any iteration* of the entire simplex tableau can be generated from the *original* data of the problem, the *inverse* associated with the iteration, and the dual problem. Using the layout of the simplex tableau in Figure we can divide the computations into two types:

1. Constraint columns (left- and right-hand sides).
2. Objective z-row.

Formula 1: Constraint Column Computations. In any simplex iteration, a left-hand or a right-hand side column is computed as follows:

$$\begin{pmatrix} \text{Constraint column} \\ \text{in iteration } i \end{pmatrix} = \begin{pmatrix} \text{Inverse in} \\ \text{iteration } i \end{pmatrix} \times \begin{pmatrix} \text{Original} \\ \text{constraint column} \end{pmatrix}$$

Formula 2: Objective z-row Computations. In any simplex iteration, the objective equation coefficient (reduced cost) of x_j is computed as follows:

$$\begin{pmatrix}\textit{Primal } z\text{-equation} \\ \text{coefficient of variable } x_j\end{pmatrix} = \begin{pmatrix}\text{Left-hand side of} \\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of} \\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix}$$

Example

We use the LP in Example 4.2-1 to illustrate the application of Formulas 1 and 2. From the optimal tableau in Table 4.4, we have

$$\text{Optimal inverse} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

The use of Formula 1 is illustrated by computing all the left- and right-hand side columns of the optimal tableau:

$$\begin{pmatrix} x_1\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \text{Inverse in} \\ \text{optimal iteration} \end{pmatrix} \times \begin{pmatrix} \text{original} \\ x_1\text{-column} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In a similar manner, we compute the remaining constraint columns; namely,

$$\begin{pmatrix} x_2\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_3\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{7}{5} \end{pmatrix}$$

$$\begin{pmatrix} x_4\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$$

$$\begin{pmatrix} R\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} \text{Right-hand side} \\ \text{column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 10 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{26}{5} \end{pmatrix}$$

Next, we demonstrate how the objective row computations are carried out using Formula 2. The optimal values of the dual variables, $(y_1, y_2) = \left(\frac{29}{5}, -\frac{2}{5}\right)$, were computed in Example 4.2-1 using two different methods. These values are used in Formula 2 to determine the associated z-coefficients; namely,

z-cofficient of $x_1 = y_1 + 2y_2 - 5 = \frac{29}{5} + 2 \times -\frac{2}{5} - 5 = 0$

z-cofficient of $x_2 = 2y_1 - y_2 - 12 = 2 \times \frac{29}{5} - \left(-\frac{2}{5}\right) - 12 = 0$

z-cofficient of $x_3 = y_1 + 3y_2 - 4 = \frac{29}{5} + 3 \times -\frac{2}{5} - 4 = \frac{3}{5}$

z-cofficient of $x_4 = y_1 - 0 = \frac{29}{5} - 0 = \frac{29}{5}$

z-cofficient of $R = y_2 - (-M) = -\frac{2}{5} - (-M) = -\frac{2}{5} + M$

Notice that Formula 1 and Formula 2 calculations can be applied at any iteration of either the primal or the dual problems. All we need is the inverse associated with the (primal or dual) iteration and the original LP data.

ECONOMIC INTERPRETATION OF DUALITY

The linear programming problem can be viewed as a resource allocation model in which the objective is to maximize revenue subject to the availability of limited resources. Looking at the problem from this standpoint, the associated dual problem offers interesting economic interpretations of the LP resource allocation model.

To formalize the discussion, we consider the following representation of the general primal and dual problems:

Primal	Dual
Maximize $z = \sum_{j=1}^{n} c_j x_j$	Minimize $w = \sum_{i=1}^{m} b_i y_i$
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ij} y_i \ge c_j,\ j = 1, 2, \ldots, n$
$x_j \ge 0,\ j=1, 2, \ldots, n$	$y_i \ge 0,\ i = 1, 2, \ldots, m$

Viewed as a resource allocation model, the primal problem has n economic activities and m resources. The coefficient c_j in the primal represents the revenue per unit of activity j. Resource i, whose maximum availability is b_i, is consumed at the rate a_{ij} units per unit of activity j.

Economic Interpretation of Dual Variables

It states that for any two primal and dual *feasible* solutions, the values of the objective functions, when finite, must satisfy the following inequality:

$$z = \sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i = w$$

The strict equality, $z = w$, holds when both the primal and dual solutions are optimal.

Let us consider the optimal condition $z = w$ first. Given that the primal problem represents a resource allocation model, we can think of z as representing revenue dollars. Because b_i represents the number of units available of resource i, the equation $z = w$ can be expressed dimensionally as

$$\$ = \sum_i (\text{units of resource } i) \times (\$ \text{ per unit of resource } i)$$

This means that the dual variable, y_i, represents the **worth per unit** of resource i. As stated in Section 3.6, the standard name **dual** (or **shadow**) **price** of resource i replaces the name *worth per unit* in all LP literature and software packages.

Using the same logic, the inequality $z < w$ associated with any two feasible primal and dual solutions is interpreted as

$$(\text{Revenue}) < (\text{Worth of resources})$$

This relationship says that so long as the total revenue from all the activities is less than the worth of the resources, the corresponding primal and dual solutions are not optimal. Optimality (maximum revenue) is reached only when the resources have been exploited completely, which can happen only when the input (worth of the resources) equals the output (revenue dollars). In economic terms, the system is said to be *unstable* (nonoptimal) when the input (worth of the resources) exceeds the output (revenue). Stability occurs only when the two quantities are equal.

Example

The Reddy Mikks model and its dual are given as:

Reddy Mikks primal	Reddy Mikks dual
Maximize $z = 5x_1 + 4x_2$	Minimize $w = 24y_1 + 6y_2 + y_3 + 2y_4$
subject to	subject to
$6x_1 + 4x_2 \leq 24$ (resource 1, $M1$)	$6y_1 + y_2 - y_3 \geq 5$
$x_1 + 2x_2 \leq 6$ (resource 2, $M2$)	$4y_1 + 2y_2 + y_3 + y_4 \geq 4$
$-x_1 + x_2 \leq 1$ (resource 3, market)	$y_1, y_2, y_3, y_4 \geq 0$
$x_2 \leq 2$ (resource 4, demand)	
$x_1, x_2 \geq 0$	
Optimal solution:	Optimal solution:
$x_1 = 3, x_2 = 1.5, z = 21$	$y_1 = .75, y_2 = 0.5, y_3 = y_4 = 0, w = 21$

Briefly, the Reddy Mikks model deals with the production of two types of paint (interior and exterior) using two raw materials $M1$ and $M2$ (resources 1 and 2) and subject to market and demand limits represented by the third and fourth constraints. The model determines the amounts (in tons/day) of interior and exterior paints that maximize the daily revenue (expressed in thousands of dollars).

The optimal dual solution shows that the dual price (worth per unit) of raw material $M1$ (resource 1) is $y_1 = .75$ (or \$750 per ton), and that of raw material $M2$ (resource 2) is $y_2 = .5$ (or \$500 per ton). These results hold true for specific *feasibility ranges* as we showed in Section 3.6. For resources 3 and 4, representing the market and demand limits, the dual prices are both zero, which indicates that their associated resources are abundant. Hence, their worth per unit is zero.

Economic Interpretation of Dual Constraints

The dual constraints can be interpreted by using Formula 2 in Section 4.2.4, which states that at any primal iteration,

$$\text{Objective coefficient of } x_j = \begin{pmatrix}\text{Left-hand side of}\\ \text{dual constraint } j\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ \text{dual constraint } j\end{pmatrix}$$

$$= \sum_{i=1}^{m} a_{ij}y_i - c_j$$

We use dimensional analysis once again to interpret this equation. The revenue per unit, c_j, of activity j is in dollars per unit. Hence, for consistency, the quantity $\sum_{i=1}^{m} a_{ij}y_i$ must also be in dollars per unit. Next, because c_j represents revenue, the quantity $\sum_{i=1}^{m} a_{ij}y_i$, which appears in the equation with an opposite sign, must represent cost. Thus we have

$$\$\text{ cost} = \sum_{i=1}^{m} a_{ij}y_i = \sum_{i=1}^{m} \begin{pmatrix}\text{usage of resource } i\\ \text{per unit of activity } j\end{pmatrix} \times \begin{pmatrix}\text{cost per unit}\\ \text{of resource } i\end{pmatrix}$$

The conclusion here is that the dual variable y_i represents the **imputed cost** per unit of resource i, and we can think of the quantity $\sum_{i=1}^{m} a_{ij}y_i$ as the imputed cost of all the resources needed to produce one unit of activity j.

In Section 3.6, we referred to the quantity $\left(\sum_{i=1}^{m} a_{ij}y_i - c_j\right)$ as the **reduced cost** of activity j. The maximization optimality condition of the simplex method says that an increase in the level of an unused (nonbasic) activity j can improve revenue only if its reduced cost is negative. In terms of the preceding interpretation, this condition states that

$$\begin{pmatrix}\text{Imputed cost of}\\ \text{resources used by}\\ \text{one unit of activity } j\end{pmatrix} < \begin{pmatrix}\text{Revenue per unit}\\ \text{of activity } j\end{pmatrix}$$

The maximization optimality condition thus says that it is economically advantageous to increase an activity to a positive level if its unit revenue exceeds its unit imputed cost.

POSSIBLE QUESTIONS

PART - A (20 x 1 =20 Marks)
(Question Nos. 1 to 20 Online Examinations)

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Write the statement of Fundamental theorem of Duality.
2. Write the statement of Existence theorem.
3. Write the statement of complementary slackness theorem.
4. Define unbounded solution.
5. What are the conditions to be followed to convert the primal problem which is of maximization type to dual problem?

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Explain the guidelines to construct the dual problem.
2. Write the dual of the following primal LPP.
   Maximize $F = x_1 + 2x_2 + x_3$
   Subject to $2x_1 + x_2 - x_3 \leq 2$
   $-2x_1 + x_2 - 5x_3 \geq -6$
   $4x_1 + x_2 + x_3 \leq 6$
   and $x_1, x_2, x_3 \geq 0$
3. Construct the dual of the LPP.
   Minimize $Z = 4x_1 + 6x_2 + 18x_3$
   Subject to $x_1 + 3x_2 \geq 3$
   $x_2 + 2x_3 \geq 5$
   and $x_1, x_2, x_3 \geq 0$
4. Write the dual of the following primal LPP.
   Minimize $Z = 4x_1 + 5x_2 - 3x_3$
   Subject to $x_1 + x_2 + x_3 = 22$
   $3x_1 + 5x_2 - 2x_3 \leq 65$
   $x_1 + 7x_2 + 4x_3 \geq 120$
   $x_1 \geq 0$, $x_2 \geq 0$ and x_3 unrestricted.
5. Express the dual of the following primal LPP.
   Maximize $Z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$
   Subject to $3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$
   $2x_1 + x_2 + 3x_4 + 9x_5 = 6$
   $x_1, x_2, x_3, x_4 \geq 0$ and x_5 unrestricted.
6. Write down the dual of the following LPP and solve it.
   Maximize $Z = 4x_1 + 2x_2$
   Subject to $-x_1 - x_2 \leq -3$
   $-x_1 + x_2 \geq -2$
   and $x_1, x_2 \geq 0$

7. Prove using duality theorem that the following linear program is feasible but has no optimal solution.
   Minimize $Z = x_1 - x_2 + x_3$
   Subject to $x_1 - x_3 \geq 4$
   $x_1 - x_2 + 2x_3 \geq 3$
   and $x_1, x_2, x_3 \geq 0$
8. Write the procedure for dual simplex method.
9. Using dual simplex method solve the LPP.
   Minimize $Z = 2x_1 + x_2$
   Subject to $3x_1 + x_2 \geq 3$
   $4x_1 + 3x_2 \geq 6$
   $x_1 + 2x_2 \geq 3$
   and $x_1, x_2 \geq 0$
10. Use dual simplex method to solve the LPP.
   Maximize $Z = -3x_1 - 2x_2$
   Subject to $x_1 + x_2 \geq 1$
   $x_1 + x_2 \leq 7$
   $x_1 + 2x_2 \geq 10$
   $x_2 \leq 3$
   and $x_1, x_2 \geq 0$

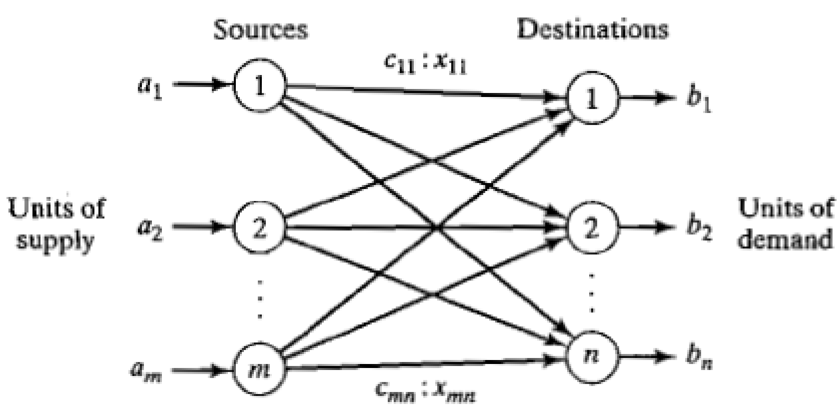
UNIT-III

SYLLABUS

Transportation Problem: Definition of the Transportation model – Nontraditional Transportation model – The Transportation Algorithm: Determination of the Starting Solution-Northwest – corner method, Least – corner method, Vogel approximation method- Iterative Computations of the Transportation Algorithm.

Transportation Problem

The transportation model is a special class of linear programs that deals with shipping a commodity from sources (e.g., factories) to destinations (e.., warehouses). The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits. The application of the transportation model can be extended to other areas of operation, including inventory control, employment scheduling and personnel assignment.



Representation of the transportation model with nodes and arcs

DEFINITION OF THE TRANSPORTATION MODEL

The general problem is represented by the network. There are m sources and n destinations, each represented by a **node**. The **arcs** represent the routes linking the sources and the destinations. Arc (i, j) joining source i to destination j carries two pieces of information: the transportation cost per unit, c_{ij}, and the amount shipped, x_{ij}. The amount of supply at source i is a_i and the amount of demand at destination j is b_j. The objective of the model is to determine the unknowns x_{ij} that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

Example

MG Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars. The mileage chart between the plants and the distribution centers is given in Table

The trucking company in charge of transporting the cars charges 8 cents per mile per car. The transportation costs per car on the different routes, rounded to the closest dollar, are given The LP model of the problem is given as

$$\text{Minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

TABLE 5.1 Mileage Chart

	Denver	Miami
Los Angeles	1000	2690
Detroit	1250	1350
New Orleans	1275	850

TABLE 5.2 Transportation Cost per Car

	Denver (1)	Miami (2)
Los Angeles (1)	\$80	\$215
Detroit (2)	\$100	\$108
New Orleans (3)	\$102	\$68

subject to

$$x_{11} + x_{12} = 1000 \quad \text{(Los Angeles)}$$
$$x_{21} + x_{22} = 1500 \quad \text{(Detroit)}$$
$$+ x_{31} + x_{32} = 1200 \quad \text{(New Oreleans)}$$
$$x_{11} + x_{21} + x_{31} = 2300 \quad \text{(Denver)}$$
$$x_{12} + x_{22} + x_{32} = 1400 \quad \text{(Miami)}$$
$$x_{ij} \geq 0, i = 1, 2, 3, j = 1, 2$$

These constraints are all equations because the total supply from the three sources (= 1000 + 1500 + 1200 = 3700 cars) equals the total demand at the two destinations (= 2300 + 1400 = 3700 cars).

The LP model can be solved by the simplex method. However, with the special structure of the constraints we can solve the problem more conveniently using the **transportation tableau** shown in Table 5.3.

TABLE 5.3 MG Transportation Model

	Denver	Miami	Supply
Los Angeles	80, x_{11}	215, x_{12}	**1000**
Detroit	100, x_{21}	108, x_{22}	**1500**
New Orleans	102, x_{31}	68, x_{32}	**1200**
Demand	**2300**	**1400**	

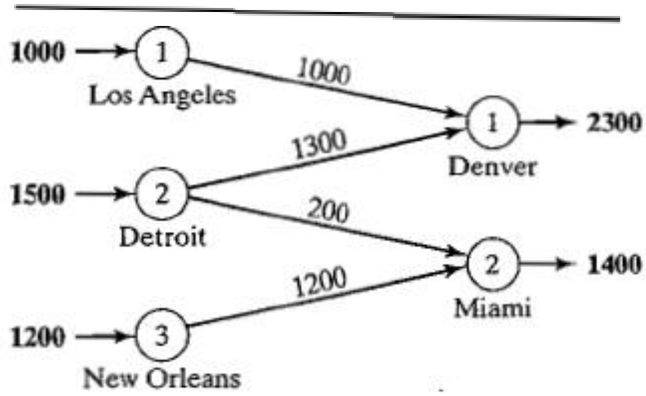


FIGURE 5.2

Optimal solution of MG Auto model

The optimal solution in Figure 5.2 calls for shipping 1000 cars from Los Angeles to Denver, 1300 from Detroit to Denver, 200 from Detroit to Miami, and 1200 from New Orleans to Miami. The associated minimum transportation cost is computed as $1000 \times \$80 + 1300 \times \$100 + 200 \times \$108 + 1200 \times \$68 = \$313{,}200$.

Balancing the Transportation Model. The transportation algorithm is based on the assumption that the model is balanced, meaning that the total demand equals the total supply. If the model is unbalanced, we can always add a dummy source or a dummy destination to restore balance.

Example

In the MG model, suppose that the Detroit plant capacity is 1300 cars (instead of 1500). The total supply (= 3500 cars) is less than the total demand (= 3700 cars), meaning that part of the demand at Denver and Miami will not be satisfied.

Because the demand exceeds the supply, a dummy source (plant) with a capacity of 200 cars (= 3700 − 3500) is added to balance the transportation model. The unit transportation costs from the dummy plant to the two destinations are zero because the plant does not exist.

Table 5.4 gives the balanced model together with its optimum solution. The solution shows that the dummy plant ships 200 cars to Miami, which means that Miami will be 200 cars short of satisfying its demand of 1400 cars.

We can make sure that a specific destination does not experience shortage by assigning a very high unit transportation cost from the dummy source to that destination. For example, a penalty of $1000 in the dummy-Miami cell will prevent shortage at Miami. Of course, we cannot use this "trick" with all the destinations, because shortage must occur somewhere in the system.

The case where the supply exceeds the demand can be demonstrated by assuming that the demand at Denver is 1900 cars only. In this case, we need to add a dummy distribution center to "receive" the surplus supply. Again, the unit transportation costs to the dummy distribution center are zero, unless we require a factory to "ship out" completely. In this case, we must assign a high unit transportation cost from the designated factory to the dummy destination.

TABLE 5.4 MG Model with Dummy Plant

	Denver	Miami	Supply
Los Angeles	(80) 1000	(215)	1000
Detroit	(100) 1300	(108)	1300
New Orleans	(102)	(68) 1200	1200
Dummy Plant	(0)	(0) 200	200
Demand	2300	1400	

TABLE 5.5 MG Model with Dummy Destination

	Denver	Miami	Dummy	Supply
Los Angeles	(80) 1000	(215)	(0)	1000
Detroit	(100) 900	(108) 200	(0) 400	1500
New Orleans	(102)	(68) 1200	(0)	1200
Demand	1900	1400	400	

NONTRADITIONAL TRANSPORTATION MODELS

The application of the transportation model is not limited to *transporting* commodities between geographical sources and destinations. This section presents two applications in the areas of production-inventory control and tool sharpening service.

Example (Production-Inventory Control)

Boralis manufactures backpacks for serious hikers. The demand for its product occurs during March to June of each year. Boralis estimates the demand for the four months to be 100, 200, 180, and 300 units, respectively. The company uses part-time labor to manufacture the backpacks and, accordingly, its production capacity varies monthly. It is estimated that Boralis can produce 50, 180, 280, and 270 units in March through June. Because the production capacity and demand for the different months do not match, a current month's demand may be satisfied in one of three ways.

1. Current month's production.
2. Surplus production in an earlier month.
3. Surplus production in a later month (backordering).

In the first case, the production cost per backpack is \$40. The second case incurs an additional holding cost of \$.50 per backpack per month. In the third case, an additional penalty cost of \$2.00 per backpack is incurred for each month delay. Boralis wishes to determine the optimal production schedule for the four months.

The situation can be modeled as a transportation model by recognizing the following parallels between the elements of the production-inventory problem and the transportation model:

Transportation	Production-inventory
1. Source i	1. Production period i
2. Destination j	2. Demand period j
3. Supply amount at source i	3. Production capacity of period i
4. Demand at destination j	4. Demand for period j
5. Unit transportation cost from source i to destination j	5. Unit cost (production + inventory + penalty) in period i for period j

The resulting transportation model is given in Table 5.12.

TABLE 5.12 Transportation Model for Example 5.2-1

	1	2	3	4	Capacity
1	\$40.00	\$40.50	\$41.00	\$41.50	**50**
2	\$42.00	\$40.00	\$40.50	\$41.00	**180**
3	\$44.00	\$42.00	\$40.00	\$40.50	**280**
4	\$46.00	\$44.00	\$42.00	\$40.00	**270**
Demand	**100**	**200**	**180**	**300**	

FIGURE 5.3
Optimal solution of the production–inventory model

The unit "transportation" cost from period i to period j is computed as

$$c_{ij} = \begin{cases} \text{Production cost in } i, i = j \\ \text{Production cost in } i + \text{ holding cost from } i \text{ to } j, i < j \\ \text{Production cost in } i + \text{ penaty cost from } i \text{ to } j, i > j \end{cases}$$

For example,

$$c_{11} = \$40.00$$

$$c_{24} = \$40.00 + (\$.50 + \$.50) = \$41.00$$

$$c_{41} = \$40.00 + (\$2.00 + \$2.00 + \$2.00) = \$46.00$$

The optimal solution is summarized in Figure 5.3. The dashed lines indicate back-ordering, the dotted lines indicate production for a future period, and the solid lines show production in a period for itself. The total cost is $31,455.

Example (Tool Sharpening)

Arkansas Pacific operates a medium-sized saw mill. The mill prepares different types of wood that range from soft pine to hard oak according to a weekly schedule. Depending on the type of wood being milled, the demand for sharp blades varies from day to day according to the following 1-week (7-day) data:

Day	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Sun.
Demand (blades)	24	12	14	20	18	14	22

The mill can satisfy the daily demand in the following manner:

1. Buy new blades at the cost of $12 a blade.
2. Use an overnight sharpening service at the cost of $6 a blade.
3. Use a slow 2-day sharpening service at the cost of $3 a blade.

The situation can be represented as a transportation model with eight sources and seven destinations. The destinations represent the 7 days of the week. The sources of the model are defined as follows: Source 1 corresponds to buying new blades, which, in the extreme case, can provide sufficient supply to cover the demand for all 7 days ($= 24 + 12 + 14 + 20 + 18 + 14 + 22 = 124$). Sources 2 to 8 correspond to the 7 days of the week. The amount of supply for each of these sources equals the number of used blades at the end of the associated day. For example, source 2 (i.e., Monday) will have a supply of used blades equal to the demand for Monday. The unit "transportation cost" for the model is $12, $6, or $3, depending on whether the blade is supplied from new blades, overnight sharpening, or 2-day sharpening. Notice that the overnight service means that used blades sent at the *end* of day i will be available for use at the *start* of day $i + 1$ or day $i + 2$, because the slow 2-day service will not be available until the *start* of day $i + 3$. The "disposal" column is a dummy destination needed to balance the model. The complete model and its solution are given in Table 5.13.

TABLE 5.13 Tool Sharpening Problem Expressed as a Transportation Model

	1 Mon.	2 Tue.	3 Wed.	4 Thu.	5 Fri.	6 Sat.	7 Sun.	8 Disposal	
1-New	$12 24	$12 2	$12	$12	$12	$12	$12	$0 98	124
2-Mon.	M	$6 10	$6 8	$3 6	$3	$3	$3	$0	24

3-Tue.	M	M	$6 6	$6	$3 6	$3	$3	$0	12
4-Wed.	M	M	M	$6 14	$6	$3	$3	$0	14
5-Thu.	M	M	M	M	$6 12	$6	$3 8	$0	20
6-Fri.	M	M	M	M	M	$6 14	$6	$0 4	18
7-Sat.	M	M	M	M	M	M	$6 14	$0	14
8-Sun.	M	M	M	M	M	M	M	$0 22	22
	24	12	14	20	18	14	22	124	

The problem has alternative optima at a cost of $840 (file toraEx5.2-2.txt). The following table summarizes one such solution.

Period	Number of sharp blades (Target day) New	Overnight	2-day	Disposal
Mon.	24 (Mon.)	10 (Tue.) + 8 (Wed.)	6 (Thu.)	0
Tues.	2 (Tue.)	6 (Wed.)	6 (Fri.)	0
Wed.	0	14 (Thu.)	0	0
Thu.	0	12 (Fri.)	8 (Sun.)	0
Fri.	0	14 (Sat.)	0	4
Sat.	0	14 (Sun.)	0	0
Sun.	0	0	0	22

Remarks. The model in Table 5.13 is suitable only for the first week of operation because it does not take into account the *rotational* nature of the days of the week, in the sense that this week's days can act as sources for next week's demand. One way to handle this situation is to assume that the very first week of operation starts with all new blades for each day. From then on, we use a model consisting of exactly 7 sources and 7 destinations corresponding to the days of the week. The new model will be similar to Table 5.13 less source "New" and destination "Dis- ... The remaining cells will have a posal." Also, only diagonal cells will be blocked (unit cost = M). The remaining cells have a unit cost of either $3.00 or $6.00. For example, the unit cost for cell (Sat., Mon.) is $6.00 and that for cells (Sat., Tue.), (Sat., Wed.), (Sat., Thu.), and (Sat., Fri.) is $3.00. The table below gives the solution costing $372. As expected, the optimum solution will always use the 2-day service only. The problem has alternative optima (see file toraEx5.2-2a.txt).

	Week $i+1$							
Week i	*Mon.*	*Tue.*	*Wed.*	*Thu.*	*Fri.*	*Sat.*	*Sun.*	Total
Mon.				6			18	24
Tue.					8		4	12
Wed.	12					2		14
Thu.	8	12						20
Fri.	4		14					18
Sat.				14				14
Sun.					10	12		22
Total	24	12	14	20	18	14	22	

THE TRANSPORTATION ALGORITHM

The transportation algorithm follows the *exact steps* of the simplex method (Chapter 3). However, instead of using the regular simplex tableau, we take advantage of the special structure of the transportation model to organize the computations in a more convenient form.

The special transportation algorithm was developed early on when hand computations were the norm and the shortcuts were warranted. Today, we have powerful computer codes that can solve a transportation model of any size as a regular LP.[4] Nevertheless, the transportation algorithm, aside from its historical significance, does provide insight into the use of the theoretical primal-dual relationships (introduced in Section 4.2) to achieve a practical end result, that of improving hand computations. The exercise is theoretically intriguing.

The details of the algorithm are explained using the following numeric example.

TABLE 5.16 SunRay Transportation Model

	Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo 1	10 x_{11}	2 x_{12}	20 x_{13}	11 x_{14}	15
Silo 2	12 x_{21}	7 x_{22}	9 x_{23}	20 x_{24}	25
Silo 3	4 x_{31}	14 x_{32}	16 x_{33}	18 x_{34}	10
Demand	5	15	15	15	

Example (SunRay Transport)

SunRay Transport Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (also in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the transportation model in Table 5.16. The unit transportation costs, c_{ij}, (shown in the northeast corner of each box) are in hundreds of dollars. The model seeks the minimum-cost shipping schedule x_{ij} between silo i and mill j $(i = 1, 2, 3; j = 1, 2, 3, 4)$.

Summary of the Transportation Algorithm. The steps of the transportation algorithm are exact parallels of the simplex algorithm.

Step 1. Determine a *starting* basic feasible solution, and go to step 2.

Step 2. Use the optimality condition of the simplex method to determine the *entering variable* from among all the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise, go to step 3.

Step 3. Use the feasibility condition of the simplex method to determine the *leaving variable* from among all the current basic variables, and find the new basic solution. Return to step 2.

Determination of the Starting Solution

A general transportation model with m sources and n destinations has $m + n$ constraint equations, one for each source and each destination. However, because the transportation model is always balanced (sum of the supply = sum of the demand), one of these equations is redundant. Thus, the model has $m + n - 1$ independent constraint equations, which means that the starting basic solution consists of $m + n - 1$ basic variables. Thus, in Example 5.3-1, the starting solution has $3 + 4 - 1 = 6$ basic variables.

The special structure of the transportation problem allows securing a nonartificial starting basic solution using one of three methods:[5]

1. Northwest-corner method
2. Least-cost method
3. Vogel approximation method

The three methods differ in the "quality" of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value. In general, though not always, the Vogel method yields the best starting basic solution, and the northwest-corner method yields the worst. The tradeoff is that the northwest-corner method involves the least amount of computations.

Northwest-Corner Method. The method starts at the northwest-corner cell (route) of the tableau (variable x_{11}).

Step 1. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step 2. Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both a row and a column net to zero simultaneously, *cross out one only*, and leave a zero supply (demand) in the uncrossed-out row (column).

Step 3. If *exactly one* row or column is left uncrossed out, stop. Otherwise, move to the cell to the right if a column has just been crossed out or below if a row has been crossed out. Go to step 1.

Example 5.3-2

The application of the procedure to the model of Example 5.3-1 gives the starting basic solution in Table 5.17. The arrows show the order in which the allocated amounts are generated.

The starting basic solution is

$$x_{11} = 5, x_{12} = 10$$

$$x_{22} = 5, x_{23} = 15, x_{24} = 5$$

$$x_{34} = 10$$

The associated cost of the schedule is

$$z = 5 \times 10 + 10 \times 2 + 5 \times 7 + 15 \times 9 + 5 \times 20 + 10 \times 18 = \$520$$

Least-Cost Method. The least-cost method finds a better starting solution by concentrating on the cheapest routes. The method assigns as much as possible to the cell with the smallest unit cost (ties are broken arbitrarily). Next, the satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.

TABLE 5.17 Northwest-Corner Starting Solution

	1	2	3	4	Supply
1	10 / **5** →	2 / **10** ↓	20	11	15
2	12	7 / **5** →	9 / **15** →	20 / **5** ↓	25
3	4	14	16	18 / **10**	10
Demand	5	15	15	15	

If both a row and a column are satisfied simultaneously, *only one is crossed out*, the same as in the northwest-corner method. Next, look for the uncrossed-out cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed out.

Example 5.3-3

The least-cost method is applied to Example 5.3-1 in the following manner:

1. Cell (1, 2) has the least unit cost in the tableau (= \$2). The most that can be shipped through (1, 2) is $x_{12} = 15$ truckloads, which happens to satisfy both row 1 and column 2 simultaneously. We arbitrarily cross out column 2 and adjust the supply in row 1 to 0.
2. Cell (3, 1) has the smallest uncrossed-out unit cost (= \$4). Assign $x_{31} = 5$, and cross out column 1 because it is satisfied, and adjust the demand of row 3 to $10 - 5 = 5$ truckloads.
3. Continuing in the same manner, we successively assign 15 truckloads to cell (2, 3), 0 truckloads to cell (1, 4), 5 truckloads to cell (3, 4), and 10 truckloads to cell (2, 4)

The resulting starting solution is summarized in Table 5.18. The arrows show the order in which the allocations are made. The starting solution (consisting of 6 basic variables) is $x_{12} = 15, x_{14} = 0, x_{23} = 15, x_{24} = 10, x_{31} = 5, x_{34} = 5$. The associated objective value is
$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$

The quality of the least-cost starting solution is better than that of the northwest-corner method (Example 5.3-2) because it yields a smaller value of z (\$475 versus \$520 in the northwest-corner method).

Vogel Approximation Method (VAM). VAM is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

Step 1. For each row (column), determine a penalty measure by subtracting the *smallest* unit cost element in the row (column) from the *next smallest* unit cost element in the same row (column).

TABLE 5.18 Least-Cost Starting Solution

	1	2	3	4	Supply
1	10	(start) 2 15	20	11 0	15
2	12	7	9 15	(end) 20 10	25
3	4 5	14	16	18 5	10
Demand	5	15	15	15	

Step 2. Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row *or* column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 3. **(a)** If exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) If one row (column) with *positive* supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least-cost method. Stop.

(c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the *zero* basic variables by the least-cost method. Stop.

(d) Otherwise, go to step 1.

Example 5.3-4

VAM is applied to Example 5.3-1. Table 5.19 computes the first set of penalties.

Because row 3 has the largest penalty (= 10) and cell (3, 1) has the smallest unit cost in that row, the amount 5 is assigned to x_{31}. Column 1 is now satisfied and must be crossed out. Next, new penalties are recomputed as in Table 5.20.

Table 5.20 shows that row 1 has the highest penalty (= 9). Hence, we assign the maximum amount possible to cell (1, 2), which yields $x_{12} = 15$ and simultaneously satisfies both row 1 and column 2. We arbitrarily cross out column 2 and adjust the supply in row 1 to zero.

Continuing in the same manner, row 2 will produce the highest penalty (= 11), and we assign $x_{23} = 15$, which crosses out column 3 and leaves 10 units in row 2. Only column 4 is left, and it has a positive supply of 15 units. Applying the least-cost method to that column, we successively assign $x_{14} = 0$, $x_{34} = 5$, and $x_{24} = 10$ (verify!). The associated objective value for this solution is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

This solution happens to have the same objective value as in the least-cost method.

TABLE 5.19 Row and Column Penalties in VAM

	1	2	3	4		Row penalty
1	10	2	20	11	**15**	10 − 2 = 8
2	12	7	9	20	**25**	9 − 7 = 2
3	4 **5**	14	16	18	**10**	14 − 4 = 10
	5	**15**	**15**	**15**		
Column penalty	10 − 4 = 6	7 − 2 = 5	16 − 9 = 7	18 − 11 = 7		

TABLE 5.20 First Assignment in VAM ($x_{31} = 5$)

	1	2	3	4		Row penalty
1	10	2	20	11	**15**	9
2	12	7	9	20	**25**	2
3	4 **5**	14	16	18	**10**	2
	5	**15**	**15**	**15**		
Column penalty	—	5	7	7		

Iterative Computations of the Transportation Algorithm

After determining the starting solution (using any of the three methods), we use the following algorithm to determine the optimum solution:

Step 1. Use the simplex *optimality condition* to determine the *entering variable* as the current nonbasic variable that can improve the solution. If the optimality condition is satisfied, stop. Otherwise, go to step 2.

Step 2. Determine the *leaving variable* using the simplex *feasibility condition*. Change the basis, and return to step 1.

The optimality and feasibility conditions do not involve the familiar row operations used in the simplex method. Instead, the special structure of the transportation model allows simpler computations.

Example 5.3-5

Solve the transportation model of Example 5.3-1, starting with the northwest-corner solution.

Table 5.21 gives the northwest-corner starting solution as determined in Table 5.17, Example 5.3-2.

The determination of the entering variable from among the current nonbasic variables (those that are not part of the starting basic solution) is done by computing the nonbasic coefficients in the z-row, using the **method of multipliers** (which, as we show in Section 5.3.4, is rooted in LP duality theory).

In the method of multipliers, we associate the multipliers u_i and v_j with row i and column j of the transportation tableau. For each current *basic* variable x_{ij}, these multipliers are shown in Section 5.3.4 to satisfy the following equations:

$$u_i + v_j = c_{ij}, \text{ for each } basic\ x_{ij}$$

As Table 5.21 shows, the starting solution has 6 basic variables, which leads to 6 equations in 7 unknowns. To solve these equations, the method of multipliers calls for arbitrarily setting any $u_i = 0$, and then solving for the remaining variables as shown below.

Basic variable	(u, v) Equation	Solution
x_{11}	$u_1 + v_1 = 10$	Set $u_1 = 0 \rightarrow v_1 = 10$
x_{12}	$u_1 + v_2 = 2$	$u_1 = 0 \rightarrow v_2 = 2$
x_{22}	$u_2 + v_2 = 7$	$v_2 = 2 \rightarrow u_2 = 5$
x_{23}	$u_2 + v_3 = 9$	$u_2 = 5 \rightarrow v_3 = 4$
x_{24}	$u_2 + v_4 = 20$	$u_2 = 5 \rightarrow v_4 = 15$
x_{34}	$u_3 + v_4 = 18$	$v_4 = 15 \rightarrow u_3 = 3$

To summarize, we have

$$u_1 = 0, u_2 = 5, u_3 = 3$$
$$v_1 = 10, v_2 = 2, v_3 = 4, v_4 = 15$$

Next, we use u_i and v_j to evaluate the nonbasic variables by computing

$$u_i + v_j - c_{ij}, \text{ for each } nonbasic\ x_{ij}$$

TABLE 5.21 Starting Iteration

	1	2	3	4	Supply
1	10 **5**	2 **10**	20	11	**15**
2	12	7 **5**	9 **15**	20 **5**	**25**
3	4	14	16	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

The results of these evaluations are shown in the following table:

Nonbasic variable	$u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 4 - 20 = -16$
x_{14}	$u_1 + v_4 - c_{14} = 0 + 15 - 11 = 4$
x_{21}	$u_2 + v_1 - c_{21} = 5 + 10 - 12 = 3$
x_{31}	$u_3 + v_1 - c_{31} = 3 + 10 - 4 = \mathbf{9}$
x_{32}	$u_3 + v_2 - c_{32} = 3 + 2 - 14 = -9$
x_{33}	$u_3 + v_3 - c_{33} = 3 + 4 - 16 = -9$

The preceding information, together with the fact that $u_i + v_j - c_{ij} = 0$ for each basic x_{ij}, is actually equivalent to computing the z-row of the simplex tableau, as the following summary shows.

Basic	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	↓ x_{31}	x_{32}	x_{33}	x_{34}
z	0	0	−16	4	3	0	0	0	9	−9	−9	0

Because the transportation model seeks to *minimize* cost, the entering variable is the one having the *most positive* coefficient in the z-row. Thus, x_{31} is the entering variable.

The preceding computations are usually done directly on the transportation tableau as shown in Table 5.22, meaning that it is not necessary really to write the (u, v)-equations explicitly. Instead, we start by setting $u_1 = 0$.[6] Then we can compute the v-values of all the columns that have *basic* variables in row 1—namely, v_1 and v_2. Next, we compute u_2 based on the (u, v)-equation of basic x_{22}. Now, given u_2, we can compute v_3 and v_4. Finally, we determine u_3 using the basic equation of x_{33}. Once all the u's and v's have been determined, we can evaluate the nonbasic variables by computing $u_i + v_j - c_{ij}$ for each nonbasic x_{ij}. These evaluations are shown in Table 5.22 in the boxed southeast corner of each cell.

Having identified x_{31} as the entering variable, we need to determine the leaving variable. Remember that if x_{31} enters the solution to become basic, one of the current basic variables must leave as nonbasic (at zero level).

TABLE 5.22 Iteration 1 Calculations

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	cost 10; **5**	cost 2; **10**	cost 20; [−16]	cost 11; [4]	**15**
$u_2 = 5$	cost 12; [3]	cost 7; **5**	cost 9; **15**	cost 20; **5**	**25**
$u_3 = 3$	cost 4; [9]	cost 14; [−9]	cost 16; [−9]	cost 18; **10**	**10**
Demand	**5**	**15**	**15**	**15**	

The selection of x_{31} as the entering variable means that we want to ship through this route because it reduces the total shipping cost. What is the most that we can ship through the new route? Observe in Table 5.22 that if route (3, 1) ships θ units (i.e., $x_{31} = \theta$), then the maximum value of θ is determined based on two conditions.

1. Supply limits and demand requirements remain satisfied.
2. Shipments through all routes remain nonnegative.

These two conditions determine the maximum value of θ and the leaving variable in the following manner. First, construct a *closed loop* that starts and ends at the entering variable cell, (3, 1). The loop consists of connected *horizontal* and *vertical* segments only (no diagonals are al-

1). The loop consists of connected horizontal and vertical [illegible] lowed).[7] Except for the entering variable cell, each corner of the closed loop must coincide with a basic variable. Table 5.23 shows the loop for x_{31}. Exactly one loop exists for a given entering variable.

Next, we assign the amount θ to the entering variable cell (3, 1). For the supply and demand limits to remain satisfied, we must alternate between subtracting and adding the amount θ at the successive *corners* of the loop as shown in Table 5.23 (it is immaterial whether the loop is traced in a clockwise or counterclockwise direction). For $\theta \geq 0$, the new values of the variables then remain nonnegative if

$$x_{11} = 5 - \theta \geq 0$$
$$x_{22} = 5 - \theta \geq 0$$
$$x_{34} = 10 - \theta \geq 0$$

The corresponding maximum value of θ is 5, which occurs when both x_{11} and x_{22} reach zero level. Because only one current basic variable must leave the basic solution, we can choose either x_{11} or x_{22} as the leaving variable. We arbitrarily choose x_{11} to leave the solution.

The selection of x_{31} (= 5) as the entering variable and x_{11} as the leaving variable requires adjusting the values of the basic variables at the corners of the closed loop as Table 5.24 shows. Because each unit shipped through route (3, 1) reduces the shipping cost by \$9 ($= u_3 + v_1 - c_{31}$), the total cost associated with the new schedule is \$9 × 5 = \$45 less than in the previous schedule. Thus, the new cost is \$520 − \$45 = \$475.

TABLE 5.23 Determination of Closed Loop for x_{31}

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10; $5 - \Theta$	2; $10 + \Theta$	20; −16	11; 4	15
$u_2 = 5$	12; 3	7; $5 - \Theta$	9; 15	20; $5 + \Theta$	25
$u_3 = 3$	4; Θ; 9	14; −9	16; −9	18; $10 - \Theta$	10
Demand	5	15	15	15	

TABLE 5.24 Iteration 2 Calculations

	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 −9	2 **15 − Θ** (−)	20 −16	11 **Θ** (+) 4	**15**
$u_2 = 5$	12 −6	7 **0 + Θ** (+)	9 **15**	20 **10 − Θ** (−)	**25**
$u_3 = 3$	4 **5**	14 −9	16 −9	18 **5**	**10**
Demand	**5**	**15**	**15**	**15**	

TABLE 5.25 Iteration 3 Calculations (Optimal)

	$v_1 = -3$	$v_2 = 2$	$v_3 = 4$	$v_4 = 11$	Supply
$u_1 \equiv 0$	10 −13	2 **5**	20 −16	11 **10**	**15**
$u_2 = 5$	12 −10	7 **10**	9 **15**	20 −4	**25**
$u_3 = 7$	4 **5**	14 −5	16 −5	18 5	**10**
Demand	**5**	**15**	**15**	**15**	

Given the new basic solution, we repeat the computation of the multipliers u and v, as Table 5.24 shows. The entering variable is x_{14}. The closed loop shows that $x_{14} = 10$ and that the leaving variable is x_{24}.

The new solution, shown in Table 5.25, costs $4 × 10 = $40 less than the preceding one, thus yielding the new cost $475 − $40 = $435. The new $u_i + v_j - c_{ij}$ are now negative for all nonbasic x_{ij}. Thus, the solution in Table 5.25 is optimal.

The following table summarizes the optimum solution.

From silo	To mill	Number of truckloads
1	2	5
1	4	10
2	2	10
2	3	15
3	1	5
3	4	5
Optimal cost = $435		

POSSIBLE QUESTIONS

PART - A (20 x 1 =20 Marks)
(Question Nos. 1 to 20 Online Examinations)

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define unbalanced transportation problem.
2. Define basic feasible solution.
3. Define balanced transportation problem.
4. Write the method for solving transportation problem using North west corner Rule.
5. Explain the algorithm to determine the optimum solution using iterative computations of the transportation algorithm.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Find the optimal solution to the following transportation problem.

	1	2	3	4	Supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	

2. Solve the transportation problem.

	To				Supply
From	1	2	3	4	6
	4	3	2	0	8
	0	2	2	1	10
Demand	4	6	8	6	

3. Solve the transportation problem.

		Distribution Centers				
		D1	D2	D3	D4	Available
Origin	A	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
Demand		200	225	275	250	

4. Find the initial basic feasible solution by using North-West Corner Rule and Least cost entry method

W→ F ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

5. Determine an initial basic feasible solution to the following transportation problem

using Vogel's approximation method.

		I	II	III	IV	Supply
	A	13	11	15	20	2000
From	B	17	14	12	13	6000
	C	18	18	15	12	7000
	Demand	3000	3000	4000	5000	

6. Determine basic feasible solution to the following transportation problem using North west corner rule.

Sink

		A	B	C	D	E	Supply
	P	2	11	10	3	7	4
Origin	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

7. Find the non-degenerate basic feasible solution for the following transportation problem using
   (i) North west corner rule
   (ii) Least cost method
   (iii) Vogel's approximation method

	To				Supply
	10	20	5	7	10
	13	9	12	8	20
From	4	5	7	9	30
	14	7	1	0	40
	3	12	5	19	50
Demand	60	60	20	10	

8. Solve the transportation problem with unit transportation costs in rupees, demands and supplies as given below:

		Destination			
		D_1	D_2	D_3	Supply (units)
	A	5	6	9	100
	B	3	5	10	75
Origin	C	6	7	6	50
	D	6	4	10	75
Demand (units)			70	80	120

9. Solve the following transportation problem to maximize profit.

	Profit (Rs) / Unit Destination				
	A	B	C	D	Supply
1	4	19	22	11	100
Source 2	0	9	14	14	30
3	6	6	16	14	70
Demand	40	20	60	30	

10. Solve the following transportation problem to maximize profit.

Destination

		A	B	C	D	Supply
	1	15	51	42	33	23
Source	2	80	42	26	81	44
	3	90	40	66	60	33
Demand		23	31	16	30	100

UNIT-IV

SYLLABUS

The Assignment Model: Introduction to Assignment model- Mathematical Formulation of Assignment model- Hungarian method for solving assignment problem –Simplex Explanation of the Hungarian method.

Introduction to Assignment model

The **assignment problem** is a special type of linear programming problem where **assignees** are being assigned to perform **tasks.** For example, the assignees might be employees who need to be given work assignments. Assigning people to jobs is a common application of the assignment problem. However, the assignees need not be people. They also could be machines, or vehicles, or plants, or even time slots to be assigned tasks. The first example below involves machines being assigned to locations, so the tasks in this case simply involve holding a machine. A subsequent example involves plants being assigned products to be produced.

To fit the definition of an assignment problem, these kinds of applications need to be formulated in a way that satisfies the following assumptions.

1. The number of assignees and the number of tasks are the same. (This number is denoted by n.)
2. Each assignee is to be assigned to exactly *one* task.
3. Each task is to be performed by exactly *one* assignee.
4. There is a cost c_{ij} associated with assignee i ($i = 1, 2, \ldots, n$) performing task j ($j = 1, 2, \ldots, n$).
5. The objective is to determine how all n assignments should be made to minimize the total cost.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithms designed specifically for assignment problems.

The first three assumptions are fairly restrictive. Many potential applications do not quite satisfy these assumptions. However, it often is possible to reformulate the problem to make it fit. For example, *dummy assignees* or *dummy tasks* frequently can be used for this purpose.

Mathematical Formulation of Assignment model

The general assignment model with n workers and n jobs is represented in Table 5.31.

The element c_{ij} represents the cost of assigning worker i to job j $(i, j = 1, 2, \ldots, n)$. There is no loss of generality in assuming that the number of workers always

TABLE 5.31 Assignment Model

Worker	Jobs 1	2	…	n	
1	c_{11}	c_{12}	…	c_{1n}	1
2	c_{21}	c_{22}	…	c_{2n}	1
⋮	⋮	⋮	⋮	⋮	⋮
n	c_{n1}	c_{n2}	…	c_{nn}	1
	1	1	…	1	

equals the number of jobs, because we can always add fictitious workers or fictitious jobs to satisfy this assumption.

The assignment model is actually a special case of the transportation model in which the workers represent the sources, and the jobs represent the destinations. The supply (demand) amount at each source (destination) exactly equals 1. The cost of "transporting" worker i to job j is c_{ij}. In effect, the assignment model can be solved directly as a regular transportation model. Nevertheless, the fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the **Hungarian method.** Although the new solution method appears totally unrelated to the transportation model, the algorithm is actually rooted in the simplex method, just as the transportation model is.

Hungarian method for solving assignment problem

We will use two examples to present the mechanics of the new algorithm. The next section provides a simplex-based explanation of the procedure.

Example 5.4-1

Joe Klyne's three children, John, Karen, and Terri, want to earn some money to take care of personal expenses during a school trip to the local zoo. Mr. Klyne has chosen three chores for his children: mowing the lawn, painting the garage door, and washing the family cars. To avoid anticipated sibling competition, he asks them to submit (secret) bids for what they feel is fair pay for each of the three chores. The understanding is that all three children will abide by their father's decision as to who gets which chore.

The assignment problem will be solved by the Hungarian method.

Step 1. For the original cost matrix, identify each row's minimum, and subtract it from all the entries of the row.

TABLE 5.32 Klyne's Assignment Problem

	Mow	Paint	Wash
John	\$15	\$10	\$9
Karen	\$9	\$15	\$10
Terri	\$10	\$12	\$8

Step 2. For the matrix resulting from step 1, identify each column's minimum, and subtract it from all the entries of the column.

Step 3. Identify the optimal solution as the feasible assignment associated with the zero elements of the matrix obtained in step 2.

Let p_i and q_j be the minimum costs associated with row i and column j as defined in steps 1 and 2, respectively. The row minimums of step 1 are computed from the original cost matrix as shown in Table 5.33.

Next, subtract the row minimum from each respective row to obtain the reduced matrix in Table 5.34.

The application of step 2 yields the column minimums in Table 5.34. Subtracting these values from the respective columns, we get the reduced matrix in Table 5.35.

TABLE 5.33 Step 1 of the Hungarian Method

	Mow	Paint	Wash	Row minimum
John	15	10	9	$p_1 = 9$
Karen	9	15	10	$p_2 = 9$
Terri	10	12	8	$p_3 = 8$

TABLE 5.34 Step 2 of the Hungarian Method

	Mow	Paint	Wash
John	6	1	0
Karen	0	6	1
Terri	2	4	0
Column minimum	$q_1 = 0$	$q_2 = 1$	$q_3 = 0$

TABLE 5.35 Step 3 of the Hungarian Method

	Mow	Paint	Wash
John	6	<u>0</u>	0
Karen	<u>0</u>	5	1
Terri	2	3	<u>0</u>

The cells with underscored zero entries provide the optimum solution. This means that John gets to paint the garage door, Karen gets to mow the lawn, and Terri gets to wash the family cars. The total cost to Mr. Klyne is 9 + 10 + 8 = \$27. This amount also will always equal $(p_1 + p_2 + p_3) + (q_1 + q_2 + q_3) = (9 + 9 + 8) + (0 + 1 + 0) = \$27$.

Example 5.4-2

Suppose that the situation discussed in Example 5.4-1 is extended to four children and four chores. Table 5.36 summarizes the cost elements of the problem.

The application of steps 1 and 2 to the matrix in Table 5.36 (using $p_1 = 1$, $p_2 = 7$, $p_3 = 4$, $p_4 = 5$, $q_1 = 0$, $q_2 = 0$, $q_3 = 3$, and $q_4 = 0$) yields the reduced matrix in Table 5.37 (verify!).

The locations of the zero entries do not allow assigning unique chores to all the children. For example, if we assign child 1 to chore 1, then column 1 will be eliminated, and child 3 will not have a zero entry in the remaining three columns. This obstacle can be accounted for by adding the following step to the procedure outlined in Example 5.4-1:

Step 2a. If no feasible assignment (with all zero entries) can be secured from steps 1 and 2,

(i) Draw the *minimum* number of horizontal and vertical lines in the last reduced matrix that will cover *all* the zero entries.

TABLE 5.36 Assignment Model

		Chore			
		1	2	3	4
Child	1	\$1	\$4	\$6	\$3
	2	\$9	\$7	\$10	\$9
	3	\$4	\$5	\$11	\$7
	4	\$8	\$7	\$8	\$5

TABLE 5.37 Reduced Assignment Matrix

Child \ Chore	1	2	3	4
1	**0**	3	2	2
2	2	**0**	**0**	2
3	**0**	1	4	3
4	3	2	**0**	**0**

TABLE 5.38 Application of Step 2a

Child \ Chore	1	2	3	4
1	0	3	2	2
2	**2**	0	0	2
3	0	*1*	4	3
4	**3**	2	0	0

TABLE 5.39 Optimal Assignment

Child \ Chore	1	2	3	4
1	**0**	2	1	1
2	3	0	**0**	2
3	0	**0**	3	2
4	4	2	0	**0**

(ii) Select the *smallest un*covered entry, subtract it from every uncovered entry, then add it to every entry at the intersection of two lines.

(iii) If no feasible assignment can be found among the resulting zero entries, repeat step 2a. Otherwise, go to step 3 to determine the optimal assignment.

The application of step 2a to the last matrix produces the shaded cells in Table 5.38. The smallest unshaded entry (shown in italics) equals 1. This entry is added to the bold intersection cells and subtracted from the remaining shaded cells to produce the matrix in Table 5.39.

The optimum solution (shown by the underscored zeros) calls for assigning child 1 to chore 1, child 2 to chore 3, child 3 to chore 2, and child 4 to chore 4. The associated optimal cost is $1 + 10 + 5 + 5 = \$21$. The same cost is also determined by summing the p_i's, the q_j's, and the entry that was subtracted after the shaded cells were determined—that is, $(1 + 7 + 4 + 5) + (0 + 0 + 3 + 0) + (1) = \$21$.

Simplex Explanation of the Hungarian Method

The assignment problem in which n workers are assigned to n jobs can be represented as an LP model in the following manner: Let c_{ij} be the cost of assigning worker i to job j, and define

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Then the LP model is given as

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1$$

The optimal solution of the preceding LP model remains unchanged if a constant is added to or subtracted from any row or column of the cost matrix (c_{ij}). To prove this point, let p_i and q_j be constants subtracted from row i and column j. Thus, the cost element c_{ij} is changed to

$$c'_{ij} = c_{ij} - p_i - q_j$$

Now

$$\sum_i \sum_j c'_{ij}x_{ij} = \sum_i \sum_j (c_{ij} - p_i - q_j)x_{ij} = \sum_i \sum_j c_{ij}x_{ij} - \sum_i p_i\Big(\sum_j x_{ij}\Big) - \sum_j q_j\Big(\sum_i x_{ij}\Big)$$

$$= \sum_i \sum_j c_{ij}x_{ij} - \sum_i p_i(1) - \sum_j q_j(1)$$

$$= \sum_i \sum_j c_{ij}x_{ij} - \text{constant}$$

Because the new objective function differs from the original one by a constant, the optimum values of x_{ij} must be the same in both cases. The development thus shows that steps 1 and 2 of the Hungarian method, which call for subtracting p_i from row i and then subtracting q_j from column j, produce an equivalent assignment model. In this regard, if a feasible solution can be found among the zero entries of the cost matrix created by steps 1 and 2, then it must be optimum because the cost in the modified matrix cannot be less than zero.

If the created zero entries cannot yield a feasible solution (as Example 5.4-2 demonstrates), then step 2a (dealing with the covering of the zero entries) must be applied. The validity of this procedure is again rooted in the simplex method of linear programming and can be explained by duality theory and the complementary slackness theorem . We will not present the details of the proof here because they are somewhat involved.

The reason $(p_1 + p_2 + \cdots + p_n) + (q_1 + q_2 + \cdots + q_n)$ gives the optimal objective value is that it represents the dual objective function of the assignment model. This result can be seen through comparison with the dual objective function of the transportation model .

POSSIBLE QUESTIONS

PART - A (20 x 1 =20 Marks)
(Question Nos. 1 to 20 Online Examinations)

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Write the general form of an assignment problem.
2. Define cost matrix.
3. What are the difference between the transportation problem and the assignment problem?
4. Define optimal solution.
5. Define bounded solution.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Write algorithm for assignment problem (Hungarian Method)
2. The assignment cost of assigning any one operator to any one machine is given in the following table

Operator

Machine	I	II	III	IV
1	10	5	13	15
2	3	9	18	3
3	10	7	3	2
4	5	11	9	7

3. A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

	A	B	C	D
I	18	24	28	32
II	8	13	17	19
III	10	15	19	22

What are job assignments which will minimize the cost?

4. Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:

Job

	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
From C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

5. Solve the assignment problem.

	A	B	C	D	
I	11	17	8	16	
II	9	7	12	6	
III	13	16	15	12	
IV		14	10	12	11

6. Explain the comparison of assignment problem with transportation model.

7. Solve the assignment problem.

Machines

	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

8. Assign four trucks 1, 2, 3 and 4 to vacant spaces A, B, C, D, E and F so that the distance travelled is minimized. The matrix below shows the distance.

	1	2	3	4
A	4	7	3	7
B	8	2	5	5
C	4	9	6	9
D	7	5	4	8
E	6	3	5	4
F	6	8	7	3

9. Find the assignment of salesmen to various districts which will yield maximum profit.

Districts

Salesmen	1	2	3	4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

10. Solve the assignment problem for maximization given the profit matrix (profit in rupees).

Machines

Job	P	Q	R	S
A	51	53	54	50
B	47	50	48	50
C	49	50	60	61
D	63	64	60	60

UNIT-V

SYLLABUS

Game theory: Formulation of two person zero games – Solving two person zero sum games, games with mixed strategies, graphical solution procedure, linear programming solution of games.

Game Theory

Life is full of conflict and competition. Numerous examples involving adversaries in conflict include parlor games, military battles, political campaigns, advertising and marketing campaigns by competing business firms, and so forth. A basic feature in many of these situations is that the final outcome depends primarily upon the combination of strategies selected by the adversaries. Game theory is a mathematical theory that deals with the general features of competitive situations like these in a formal, abstract way. It places particular emphasis on the decision-making processes of the adversaries.

As briefly surveyed research on game theory continues to delve into rather complicated types of competitive situations. However, the focus in this chapter is on the simplest case, called **two-person, zero-sum games.** As the name implies, these games involve only two adversaries or *players* (who may be armies, teams, firms, and so on). They are called *zero-sum* games because one player wins whatever the other one loses, so that the sum of their net winnings is zero.
introduces the basic model for two-person, zero-sum games, and the next four sections describe and illustrate different approaches to solving such games. The chapter concludes by mentioning some other kinds of competitive situations that are dealt with by other branches of game theory.

THE FORMULATION OF TWO-PERSON, ZERO-SUM GAMES

To illustrate the basic characteristics of two-person, zero-sum games, consider the game called *odds and evens.* This game consists simply of each player simultaneously showing either one finger or two fingers. If the number of fingers matches, so that the total number for both players is even, then the player taking evens (say, player 1) wins the bet (say, $1) from the player taking odds (player 2). If the number does not match, player 1 pays $1 to player 2. Thus, each player has two *strategies:* to show either one finger or two fingers. The resulting payoff to player 1 in dollars is shown in the *payoff table* given in Table 14.1.

In general, a two-person game is characterized by

1. The strategies of player 1
2. The strategies of player 2
3. The payoff table

TABLE 14.1 Payoff table for the odds and evens game

		Player 2	
Strategy		**1**	**2**
Player 1	1	1	−1
	2	−1	1

Before the game begins, each player knows the strategies she or he has available, the ones the opponent has available, and the payoff table. The actual play of the game consists of each player simultaneously choosing a strategy without knowing the opponent's choice.

A strategy may involve only a simple action, such as showing a certain number of fingers in the odds and evens game. On the other hand, in more complicated games involving a series of moves, a **strategy** is a predetermined rule that specifies completely how one intends to respond to each possible circumstance at each stage of the game. For example, a strategy for one side in chess would indicate how to make the next move for *every* possible position on the board, so the total number of possible strategies would be astronomical. Applications of game theory normally involve far less complicated competitive situations than chess does, but the strategies involved can be fairly complex.

The **payoff table** shows the gain (positive or negative) for player 1 that would result from each combination of strategies for the two players. It is given only for player 1 because the table for player 2 is just the negative of this one, due to the zero-sum nature of the game.

The entries in the payoff table may be in any units desired, such as dollars, provided that they accurately represent the *utility* to player 1 of the corresponding outcome. However, utility is not necessarily proportional to the amount of money (or any other commodity) when large quantities are involved. For example, $2 million (after taxes) is probably worth much less than twice as much as $1 million to a poor person. In other words, given the choice between (1) a 50 percent chance of receiving $2 million rather than nothing and (2) being sure of getting $1 million, a poor person probably would much prefer the latter. On the other hand, the outcome corresponding to an entry of 2 in a payoff table should be "worth twice as much" to player 1 as the outcome corresponding to an entry of 1. Thus, given the choice, he or she should be indifferent between a 50 percent chance of receiving the former outcome (rather than nothing) and definitely receiving the latter outcome instead.[1]

A primary objective of game theory is the development of *rational criteria* for selecting a strategy. Two key assumptions are made:

1. *Both* players are *rational*.
2. *Both* players choose their strategies solely to *promote their own welfare* (no compassion for the opponent).

Game theory contrasts with *decision analysis* where the assumption is that the decision maker is playing a game with a passive opponent—nature—which chooses its strategies in some random fashion.

We shall develop the standard game theory criteria for choosing strategies by means of illustrative examples. In particular, the next section presents a prototype example that illustrates the formulation of a two-person, zero-sum game and its solution in some simple situations.

Solving two person zero sum games

Two politicians are running against each other for the U.S. Senate. Campaign plans must now be made for the final 2 days, which are expected to be crucial because of the closeness of the race. Therefore, both politicians want to spend these days campaigning in two key cities, Bigtown and Megalopolis. To avoid wasting campaign time, they plan to travel at night and spend either 1 full day in each city or 2 full days in just one of the cities. However, since the necessary arrangements must be made in advance, neither politician will learn his (or her)[1] opponent's campaign schedule until after he has finalized his own. Therefore, each politician has asked his campaign manager in each of these cities to assess what the impact would be (in terms of votes won or lost) from the various possible combinations of days spent there by himself and by his opponent. He then wishes to use this information to choose his best strategy on how to use these 2 days.

Formulation as a Two-Person, Zero-Sum Game

To formulate this problem as a two-person, zero-sum game, we must identify the two *players* (obviously the two politicians), the *strategies* for each player, and the *payoff table*.

As the problem has been stated, each player has the following three strategies:

Strategy 1 = spend 1 day in each city.
Strategy 2 = spend both days in Bigtown.
Strategy 3 = spend both days in Megalopolis.

By contrast, the strategies would be more complicated in a different situation where each politician learns where his opponent will spend the first day before he finalizes his own plans for his second day. In that case, a typical strategy would be: Spend the first day in Bigtown; if the opponent also spends the first day in Bigtown, then spend the second day in Bigtown; however, if the opponent spends the first day in Megalopolis, then spend the second day in Megalopolis. There would be eight such strategies, one for each combination of the two first-day choices, the opponent's two first-day choices, and the two second-day choices.

Each entry in the payoff table for player 1 represents the *utility* to player 1 (or the negative utility to player 2) of the outcome resulting from the corresponding strategies used by the two players. From the politician's viewpoint, the objective is to *win votes,*

TABLE 14.2 Form of the payoff table for politician 1 for the political campaign problem

		Total Net Votes Won by Politician 1 (in Units of 1,000 Votes)		
		Politician 2		
Strategy		1	2	3
	1			
Politician 1	2			
	3			

and each additional vote (before he learns the outcome of the election) is of equal value to him. Therefore, the appropriate entries for the payoff table for politician 1 are the *total net votes won* from the opponent (i.e., the sum of the net vote changes in the two cities) resulting from these 2 days of campaigning. Using units of 1,000 votes, this formulation is summarized in Table 14.2. Game theory assumes that both players are using the same formulation (including the same payoffs for player 1) for choosing their strategies.

However, we should also point out that this payoff table would *not* be appropriate if additional information were available to the politicians. In particular, assume that they know exactly how the populace is planning to vote 2 days before the election, so that each politician knows exactly how many net votes (positive or negative) he needs to switch in his favor during the last 2 days of campaigning to win the election. Consequently, the only significance of the data prescribed by Table 14.2 would be to indicate which politician would win the election with each combination of strategies. Because the ultimate goal is to win the election and because the size of the plurality is relatively inconsequential, the utility entries in the table then should be some positive constant (say, +1) when politician 1 wins and −1 when he loses. Even if only a *probability* of winning can be determined for each combination of strategies, the appropriate entries would be the probability of winning minus the probability of losing because they then would represent *expected* utilities. However, sufficiently accurate data to make such determinations usually are not available, so this example uses the thousands of total net votes won by politician 1 as the entries in the payoff table.

Using the form given in Table 14.2, we give three alternative sets of data for the payoff table to illustrate how to solve three different kinds of games.

Variation 1 of the Example

Given that Table 14.3 is the payoff table for player 1 (politician 1), which strategy should each player select?

This situation is a rather special one, where the answer can be obtained just by applying the concept of **dominated strategies** to rule out a succession of inferior strategies until only one choice remains.

TABLE 14.3 Payoff table for player 1 for variation 1 of the political campaign problem

		Player 2		
Strategy		**1**	**2**	**3**
	1	1	2	4
Player 1	2	1	0	5
	3	0	1	−1

A strategy is **dominated** by a second strategy if the second strategy is *always at least as good* (and sometimes better) regardless of what the opponent does. A dominated strategy can be eliminated immediately from further consideration.

At the outset, Table 14.3 includes no dominated strategies for player 2. However, for player 1, strategy 3 is dominated by strategy 1 because the latter has larger payoffs ($1 > 0, 2 > 1, 4 > -1$) regardless of what player 2 does. Eliminating strategy 3 from further consideration yields the following reduced payoff table:

	1	2	3
1	1	2	4
2	1	0	5

Because both players are assumed to be rational, player 2 also can deduce that player 1 has only these two strategies remaining under consideration. Therefore, player 2 now *does* have a dominated strategy—strategy 3, which is dominated by both strategies 1 and 2 because they always have smaller losses for player 2 (payoffs to player 1) in this reduced payoff table (for strategy 1: $1 < 4$, $1 < 5$; for strategy 2: $2 < 4$, $0 < 5$). Eliminating this strategy yields

	1	2
1	1	2
2	1	0

At this point, strategy 2 for player 1 becomes dominated by strategy 1 because the latter is better in column 2 ($2 > 0$) and equally good in column 1 ($1 = 1$). Eliminating the dominated strategy leads to

	1	2
1	1	2

Strategy 2 for player 2 now is dominated by strategy 1 (1 < 2), so strategy 2 should be eliminated.

Consequently, both players should select their strategy 1. Player 1 then will receive a payoff of 1 from player 2 (that is, politician 1 will gain 1,000 votes from politician 2).

In general, the payoff to player 1 when both players play optimally is referred to as the **value of the game.** A game that has a value of 0 is said to be a **fair game.** Since this particular game has a value of 1, it is *not* a fair game.

The concept of a dominated strategy is a very useful one for reducing the size of the payoff table that needs to be considered and, in unusual cases like this one, actually identifying the optimal solution for the game. However, most games require another approach to at least finish solving, as illustrated by the next two variations of the example.

Variation 2 of the Example

Now suppose that the current data give Table 14.4 as the payoff table for player 1 (politician 1). This game does not have dominated strategies, so it is not obvious what the players should do. What line of reasoning does game theory say they should use?

Consider player 1. By selecting strategy 1, he could win 6 or could lose as much as 3. However, because player 2 is rational and thus will seek a strategy that will protect himself from large payoffs to player 1, it seems likely that player 1 would incur a loss by playing strategy 1. Similarly, by selecting strategy 3, player 1 could win 5, but more probably his rational opponent would avoid this loss and instead administer a loss to player 1 which could be as large as 4. On the other hand, if player 1 selects strategy 2, he is guaranteed not to lose anything and he could even win something. Therefore, because it provides the *best guarantee* (a payoff of 0), strategy 2 seems to be a "rational" choice for player 1 against his rational opponent. (This line of reasoning assumes that both players are averse to risking larger losses than necessary, in contrast to those individuals who enjoy gambling for a large payoff against long odds.)

Now consider player 2. He could lose as much as 5 or 6 by using strategy 1 or 3, but is guaranteed at least breaking even with strategy 2. Therefore, by the same reasoning of seeking the best guarantee against a rational opponent, his apparent choice is strategy 2.

If both players choose their strategy 2, the result is that both break even. Thus, in this case, neither player improves upon his best guarantee, but both also are forcing the opponent into the same position. Even when the opponent deduces a player's strategy, the opponent cannot exploit this information to improve his position. Stalemate.

TABLE 14.4 Payoff table for player 1 for variation 2 of the political campaign problem

		Player 2			
Strategy		1	2	3	Minimum
	1	−3	−2	6	−3
Player 1	2	2	0	2	0 ← Maximin value
	3	5	−2	−4	−4
	Maximum:	5	0 ↑ Minimax value	6	

The end product of this line of reasoning is that each player should play in such a way as to *minimize his maximum losses* whenever the resulting choice of strategy cannot be exploited by the opponent to then improve his position. This so-called **minimax criterion** is a standard criterion proposed by game theory for selecting a strategy. In effect, this criterion says to select a strategy that would be best even if the selection were being announced to the opponent before the opponent chooses a strategy. In terms of the payoff table, it implies that *player 1* should select the strategy whose *minimum payoff* is *largest,* whereas *player 2* should choose the one whose *maximum payoff to player 1* is the *smallest.* This criterion is illustrated in Table 14.4, where strategy 2 is identified as the *maximin strategy* for player 1 and strategy 2 is the *minimax strategy* for player 2. The resulting payoff of 0 is the value of the game, so this is a fair game.

Notice the interesting fact that the same entry in this payoff table yields both the maximin and minimax values. The reason is that this entry is both the minimum in its row and the maximum of its column. The position of any such entry is called a **saddle point.**

The fact that this game possesses a saddle point was actually crucial in determining how it should be played. Because of the saddle point, neither player can take advantage of the opponent's strategy to improve his own position. In particular, when player 2 predicts or learns that player 1 is using strategy 2, player 2 would incur a loss instead of breaking even if he were to change from his original plan of using his strategy 2. Similarly, player 1 would only worsen his position if he were to change his plan. Thus, neither player has any motive to consider changing strategies, either to take advantage of his opponent or to prevent the opponent from taking advantage of him. Therefore, since this is a **stable solution** (also called an *equilibrium solution*), players 1 and 2 should exclusively use their maximin and minimax strategies, respectively.

As the next variation illustrates, some games do not possess a saddle point, in which case a more complicated analysis is required.

Variation 3 of the Example

Late developments in the campaign result in the final payoff table for player 1 (politician 1) given by Table 14.5. How should this game be played?

Suppose that both players attempt to apply the minimax criterion in the same way as in variation 2. Player 1 can guarantee that he will lose no more than 2 by playing strategy 1. Similarly, player 2 can guarantee that he will lose no more than 2 by playing strategy 3.

TABLE 14.5 Payoff table for player 1 for variation 3 of the political campaign problem

		Player 2			
Strategy		**1**	**2**	**3**	**Minimum**
	1	0	−2	2	−2 ← Maximin value
Player 1	2	5	4	−3	−3
	3	2	3	−4	−4
	Maximum:	5	4	2 ↑ Minimax value	

However, notice that the maximin value (−2) and the minimax value (2) do not coincide in this case. The result is that there is *no saddle point*.

What are the resulting consequences if both players plan to use the strategies just derived? It can be seen that player 1 would win 2 from player 2, which would make player 2 unhappy. Because player 2 is rational and can therefore foresee this outcome, he would then conclude that he can do much better, actually winning 2 rather than losing 2, by playing strategy 2 instead. Because player 1 is also rational, he would anticipate this switch and conclude that he can improve considerably, from −2 to 4, by changing to strategy 2. Realizing this, player 2 would then consider switching back to strategy 3 to convert a loss of 4 to a gain of 3. This possibility of a switch would cause player 1 to consider again using strategy 1, after which the whole cycle would start over again. Therefore, even though this game is being played only once, *any* tentative choice of a strategy leaves that player with a motive to consider changing strategies, either to take advantage of his opponent or to prevent the opponent from taking advantage of him.

In short, the originally suggested solution (player 1 to play strategy 1 and player 2 to play strategy 3) is an **unstable solution,** so it is necessary to develop a more satisfactory solution. But what kind of solution should it be?

The key fact seems to be that whenever one player's strategy is predictable, the opponent can take advantage of this information to improve his position. Therefore, an essential feature of a rational plan for playing a game such as this one is that neither player should be able to deduce which strategy the other will use. Hence, in this case, rather than applying some known criterion for determining a single strategy that will definitely be used, it is necessary to choose among alternative acceptable strategies on some kind of random basis. By doing this, neither player knows in advance which of his own strategies will be used, let alone what his opponent will do.

This suggests, in very general terms, the kind of approach that is required for games lacking a saddle point. In the next section we discuss the approach more fully. Given this foundation, the following two sections will develop procedures for finding an optimal way of playing such games. This particular variation of the political campaign problem will continue to be used to illustrate these ideas as they are developed.

GAMES WITH MIXED STRATEGIES

Whenever a game does not possess a saddle point, game theory advises each player to assign a probability distribution over her set of strategies. To express this mathematically, let

x_i = probability that player 1 will use strategy i $(i = 1, 2, \ldots, m)$,

y_j = probability that player 2 will use strategy j $(j = 1, 2, \ldots, n)$,

where m and n are the respective numbers of available strategies. Thus, player 1 would specify her plan for playing the game by assigning values to $x_1, x_2, \ldots, x_m$. Because these values are probabilities, they would need to be nonnegative and add to 1. Similarly, the plan for player 2 would be described by the values she assigns to her decision variables $y_1, y_2, \ldots, y_n$. These plans $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ are usually referred to as **mixed strategies,** and the original strategies are then called **pure strategies.**

When the game is actually played, it is necessary for each player to use one of her pure strategies. However, this pure strategy would be chosen by using some random device to obtain a random observation from the probability distribution specified by the mixed strategy, where this observation would indicate which particular pure strategy to use.

To illustrate, suppose that players 1 and 2 in variation 3 of the political campaign problem (see Table 14.5) select the mixed strategies $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, 0)$ and $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{1}{2})$, respectively. This selection would say that player 1 is giving an equal chance (probability of $\frac{1}{2}$) of choosing either (pure) strategy 1 or 2, but he is discarding strategy 3 entirely. Similarly, player 2 is randomly choosing between his last two pure strategies. To play the game, each player could then flip a coin to determine which of his two acceptable pure strategies he will actually use.

Although no completely satisfactory measure of performance is available for evaluating mixed strategies, a very useful one is the *expected payoff.* By applying the probability theory definition of expected value, this quantity is

$$\text{Expected payoff for player 1} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j,$$

where p_{ij} is the payoff if player 1 uses pure strategy i and player 2 uses pure strategy j. In the example of mixed strategies just given, there are four possible payoffs $(-2, 2, 4, -3)$, each occurring with a probability of $\frac{1}{4}$, so the expected payoff is $\frac{1}{4}(-2 + 2 + 4 - 3) = \frac{1}{4}$.

Thus, this measure of performance does not disclose anything about the risks involved in playing the game, but it does indicate what the average payoff will tend to be if the game is played many times.

By using this measure, game theory extends the concept of the minimax criterion to games that lack a saddle point and thus need mixed strategies. In this context, the **minimax criterion** says that a given player should select the mixed strategy that *minimizes* the *maximum expected loss* to himself. Equivalently, when we focus on payoffs (player 1) rather than losses (player 2), this criterion says to *maximin* instead, i.e., *maximize* the *minimum expected payoff* to the player. By the *minimum expected payoff* we mean the smallest possible expected payoff that can result from any mixed strategy with which the opponent can counter. Thus, the mixed strategy for player 1 that is *optimal* according to this criterion is the one that provides the *guarantee* (minimum expected payoff) that is *best* (maximal). (The value of this best guarantee is the *maximin value,* denoted by $\underline{v}$.) Similarly, the *optimal* strategy for player 2 is the one that provides the *best guarantee,* where *best* now means *minimal* and *guarantee* refers to the *maximum expected loss* that can be administered by any of the opponent's mixed strategies. (This best guarantee is the *minimax value,* denoted by $\overline{v}$.)

Recall that when only pure strategies were used, games not having a saddle point turned out to be *unstable* (no stable solutions). The reason was essentially that $\underline{v} < \overline{v}$, so that the players would want to change their strategies to improve their positions. Similarly, for games with mixed strategies, it is necessary that $\underline{v} = \overline{v}$ for the optimal solution to be *stable.* Fortunately, according to the minimax theorem of game theory, this condition always holds for such games.

Minimax theorem: If mixed strategies are allowed, the pair of mixed strategies that is optimal according to the minimax criterion provides a *stable solution* with $\underline{v} = \overline{v} = v$ (the value of the game), so that neither player can do better by unilaterally changing her or his strategy.

Although the concept of mixed strategies becomes quite intuitive if the game is played *repeatedly,* it requires some interpretation when the game is to be played just *once.* In this case, using a mixed strategy still involves selecting and using *one* pure strategy (randomly selected from the specified probability distribution), so it might seem more sensible to ignore this randomization process and just choose the one "best" pure strategy to be used. However, we have already illustrated for variation 3 in the preceding section that a player must *not* allow the opponent to deduce what his strategy will be (i.e., the solution procedure under the rules of game theory must not *definitely* identify which pure strategy will be used when the game is unstable). Furthermore, even if the opponent is able to use only his knowledge of the tendencies of the first player to deduce probabilities (for the pure strategy chosen) that are different from those for the optimal mixed strategy, then the opponent still can take advantage of this knowledge to reduce the expected payoff to the first player. Therefore, the only way to guarantee attaining the optimal expected payoff v is to randomly select the pure strategy to be used from the probability distribution for the optimal mixed strategy.

Now we need to show how to find the optimal mixed strategy for each player. There are several methods of doing this. One is a graphical procedure that may be used whenever one of the players has only two (undominated) pure strategies; this approach is described in the next section.

GRAPHICAL SOLUTION PROCEDURE

Consider any game with mixed strategies such that, after dominated strategies are eliminated, one of the players has only two pure strategies. To be specific, let this player be player 1. Because her mixed strategies are (x_1, x_2) and $x_2 = 1 - x_1$, it is necessary for her to solve only for the optimal value of x_1. However, it is straightforward to plot the expected payoff as a function of x_1 for each of her opponent's pure strategies. This graph can then be used to identify the point that maximizes the minimum expected payoff. The opponent's minimax mixed strategy can also be identified from the graph.

To illustrate this procedure, consider variation 3 of the political campaign problem Notice that the third pure strategy for player 1 is dominated by her second, so the payoff table can be reduced to the form given in Table 14.6. Therefore, for

TABLE 14.6 Reduced payoff table for player 1 for variation 3 of the political campaign problem

			Player 2		
		Probability	y_1	y_2	y_3
	Probability	**Pure Strategy**	**1**	**2**	**3**
Player 1	x_1	1	0	−2	2
	$1 - x_1$	2	5	4	−3

each of the pure strategies available to player 2, the expected payoff for player 1 will be

(y_1, y_2, y_3)	Expected Payoff
(1, 0, 0)	$0x_1 + 5(1 - x_1) = 5 - 5x_1$
(0, 1, 0)	$-2x_1 + 4(1 - x_1) = 4 - 6x_1$
(0, 0, 1)	$2x_1 - 3(1 - x_1) = -3 + 5x_1$

Now plot these expected-payoff lines on a graph, as shown in Fig. 14.1. For any given values of x_1 and (y_1, y_2, y_3), the expected payoff will be the appropriate weighted average of the corresponding points on these three lines. In particular,

$$\text{Expected payoff for player 1} = y_1(5 - 5x_1) + y_2(4 - 6x_1) + y_3(-3 + 5x_1).$$

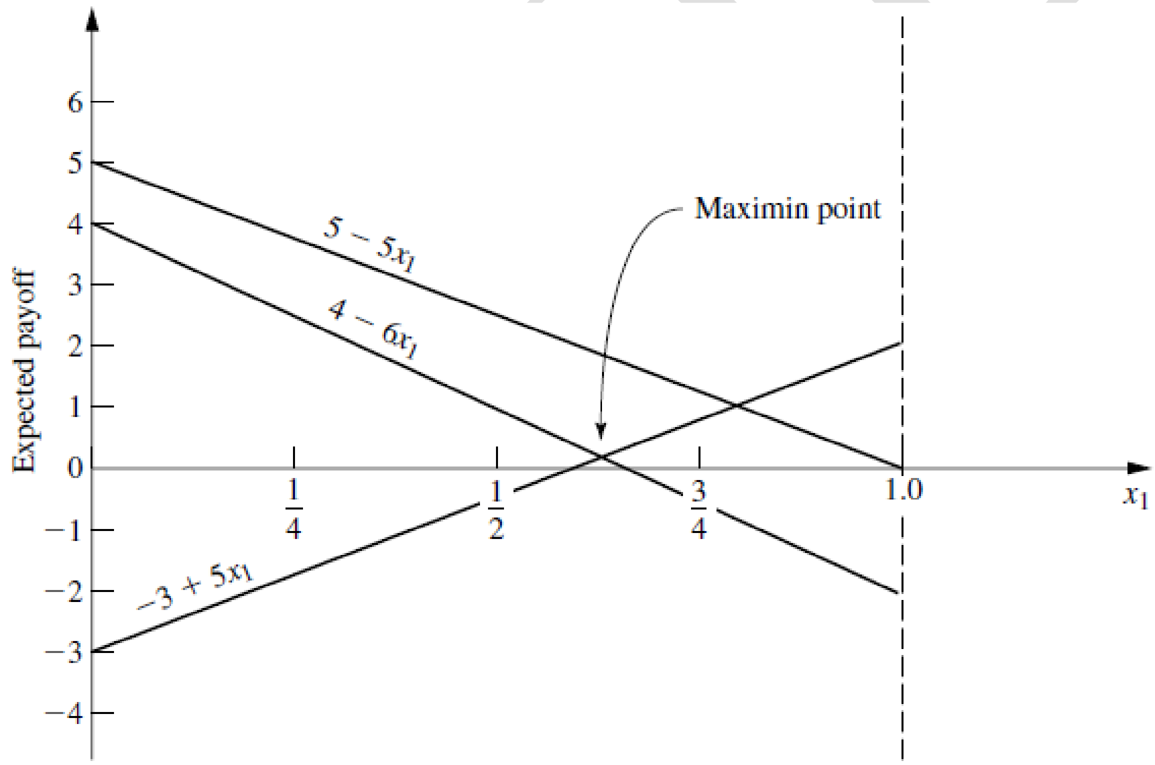
Remember that player 2 wants to minimize this expected payoff for player 1. Given x_1, player 2 can minimize this expected payoff by choosing the pure strategy that corresponds to the "bottom" line for that x_1 in Fig. 14.1 (either $-3 + 5x_1$ or $4 - 6x_1$, but never $5 - 5x_1$). According to the minimax (or maximin) criterion, player 1 wants to maximize this minimum expected payoff. Consequently, player 1 should select the value of x_1 where the bottom line peaks, i.e., where the $(-3 + 5x_1)$ and $(4 - 6x_1)$ lines intersect, which yields an expected payoff of

$$\underline{v} = v = \max_{0 \le x_1 \le 1} \{\min\{-3 + 5x_1, 4 - 6x_1\}\}.$$

To solve algebraically for this optimal value of x_1 at the intersection of the two lines $-3 + 5x_1$ and $4 - 6x_1$, we set

$$-3 + 5x_1 = 4 - 6x_1,$$

FIGURE 14.1
Graphical procedure for solving games



which yields $x_1 = \frac{7}{11}$. Thus, $(x_1, x_2) = (\frac{7}{11}, \frac{4}{11})$ is the *optimal mixed strategy* for player 1, and

$$\underline{v} = v = -3 + 5\left(\frac{7}{11}\right) = \frac{2}{11}$$

is the value of the game.

To find the corresponding optimal mixed strategy for player 2, we now reason as follows. According to the definition of the minimax value $\overline{v}$ and the minimax theorem, the expected payoff resulting from the optimal strategy $(y_1, y_2, y_3) = (y_1^*, y_2^*, y_3^*)$ will satisfy the condition

$$y_1^*(5 - 5x_1) + y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \le \bar{v} = v = \frac{2}{11}$$

for all values of x_1 ($0 \le x_1 \le 1$). Furthermore, when player 1 is playing optimally (that is, $x_1 = \frac{7}{11}$), this inequality will be an equality (by the minimax theorem), so that

$$\frac{20}{11}y_1^* + \frac{2}{11}y_2^* + \frac{2}{11}y_3^* = v = \frac{2}{11}.$$

Because (y_1, y_2, y_3) is a probability distribution, it is also known that

$$y_1^* + y_2^* + y_3^* = 1.$$

Therefore, $y_1^* = 0$ because $y_1^* > 0$ would violate the next-to-last equation; i.e., the expected payoff on the graph at $x_1 = \frac{7}{11}$ would be above the maximin point. (In general, any line that does not pass through the maximin point must be given a zero weight to avoid increasing the expected payoff above this point.)

Hence,

$$y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \begin{cases} \le \frac{2}{11} & \text{for } 0 \le x_1 \le 1, \\ = \frac{2}{11} & \text{for } x_1 = \frac{7}{11}. \end{cases}$$

But y_2^* and y_3^* are numbers, so the left-hand side is the equation of a straight line, which is a fixed weighted average of the two "bottom" lines on the graph. Because the ordinate of this line must equal $\frac{2}{11}$ at $x_1 = \frac{7}{11}$, and because it must never exceed $\frac{2}{11}$, the line necessarily is horizontal. (This conclusion is always true unless the optimal value of x_1 is either 0 or 1, in which case player 2 also should use a single pure strategy.) Therefore,

$$y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) = \frac{2}{11}, \qquad \text{for } 0 \le x_1 \le 1.$$

Hence, to solve for y_2^* and y_3^*, select two values of x_1 (say, 0 and 1), and solve the resulting two simultaneous equations. Thus,

$$4y_2^* - 3y_3^* = \frac{2}{11},$$

$$-2y_2^* + 2y_3^* = \frac{2}{11},$$

which has a simultaneous solution of $y_2^* = \frac{5}{11}$ and $y_3^* = \frac{6}{11}$. Therefore, the *optimal mixed strategy* for player 2 is $(y_1, y_2, y_3) = (0, \frac{5}{11}, \frac{6}{11})$.

If, in another problem, there should happen to be more than two lines passing through the maximin point, so that more than two of the y_j^* values can be greater than zero, this condition would imply that there are many ties for the optimal mixed strategy for player 2. One such strategy can then be identified by setting all but two of these y_j^* values equal to zero and solving for the remaining two in the manner just described. For the remaining two, the associated lines must have positive slope in one case and negative slope in the other.

Although this graphical procedure has been illustrated for only one particular problem, essentially the same reasoning can be used to solve any game with mixed strategies that has only two undominated pure strategies for one of the players.

SOLVING BY LINEAR PROGRAMMING

Any game with mixed strategies can be solved by transforming the problem to a linear programming problem. As you will see, this transformation requires little more than applying the minimax theorem and using the definitions of the maximin value $\underline{v}$ and minimax value $\bar{v}$.

First, consider how to find the optimal mixed strategy for player 1.

$$\text{Expected payoff for player 1} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j$$

and the strategy $(x_1, x_2, \ldots, x_m)$ is optimal if

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j \geq \underline{v} = v$$

for all opposing strategies $(y_1, y_2, \ldots, y_n)$. Thus, this inequality will need to hold, e.g., for each of the pure strategies of player 2, that is, for each of the strategies $(y_1, y_2, \ldots, y_n)$ where one $y_j = 1$ and the rest equal 0. Substituting these values into the inequality yields

$$\sum_{i=1}^{m} p_{ij} x_i \geq v \qquad \text{for } j = 1, 2, \ldots, n,$$

so that the inequality *implies* this set of n inequalities. Furthermore, this set of n inequalities *implies* the original inequality (rewritten)

$$\sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} p_{ij} x_i \right) \geq \sum_{j=1}^{n} y_j v = v,$$

since

$$\sum_{j=1}^{n} y_j = 1.$$

Because the implication goes in both directions, it follows that imposing this set of n linear inequalities is *equivalent* to requiring the original inequality to hold for all strategies $(y_1, y_2, \ldots, y_n)$. But these n inequalities are legitimate linear programming constraints, as are the additional constraints

$$x_1 + x_2 + \cdots + x_m = 1$$

$$x_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m$$

that are required to ensure that the x_i are probabilities. Therefore, any solution $(x_1, x_2, \ldots, x_m)$ that satisfies this entire set of linear programming constraints is the desired optimal mixed strategy.

Consequently, the problem of finding an optimal mixed strategy has been reduced to finding a feasible solution for a linear programming problem, which can be done as described in Chap. 4. The two remaining difficulties are that (1) v is unknown and (2) the linear programming problem has no objective function. Fortunately, both these difficulties can be resolved at one stroke by replacing the unknown constant v by the variable x_{m+1} and then *maximizing* x_{m+1}, so that x_{m+1} automatically will equal v (by definition) at the *optimal* solution for the linear programming problem!

To summarize, player 1 would find his optimal mixed strategy by using the simplex method to solve the linear programming problem:

Maximize x_{m+1},

subject to

$$\begin{aligned} p_{11}x_1 + p_{21}x_2 + \cdots + p_{m1}x_m - x_{m+1} &\geq 0 \\ p_{12}x_1 + p_{22}x_2 + \cdots + p_{m2}x_m - x_{m+1} &\geq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ p_{1n}x_1 + p_{2n}x_2 + \cdots + p_{mn}x_m - x_{m+1} &\geq 0 \\ x_1 + x_2 + \cdots + x_m &= 1 \end{aligned}$$

and

$$x_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

Note that x_{m+1} is not restricted to be nonnegative, whereas the simplex method can be applied only after *all* the variables have nonnegativity constraints. However, this matter can be easily rectified, as will be discussed shortly.

Now consider player 2. He could find his optimal mixed strategy by rewriting the payoff table as the payoff to himself rather than to player 1 and then by proceeding exactly as just described. However, it is enlightening to summarize his formulation in terms of the original payoff table. By proceeding in a way that is completely analogous to that just described, player 2 would conclude that his optimal mixed strategy is given by an optimal solution to the linear programming problem:

Minimize y_{n+1},

subject to

$$\begin{aligned} p_{11}y_1 + p_{12}y_2 + \cdots + p_{1n}y_n - y_{n+1} &\leq 0 \\ p_{21}y_1 + p_{22}y_2 + \cdots + p_{2n}y_n - y_{n+1} &\leq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ p_{m1}y_1 + p_{m2}y_2 + \cdots + p_{mn}y_n - y_{n+1} &\leq 0 \\ y_1 + y_2 + \cdots + y_n &= 1 \end{aligned}$$

and

$$y_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

It is easy to show (see Prob. 14.5-5 and its hint) that this linear programming problem and the one given for player 1 are *dual* to each other in the sense described in Secs. 6.1 and 6.4. This fact has several important implications. One implication is that the optimal mixed strategies for both players can be found by solving only one of the linear programming problems because the optimal dual solution is an automatic by-product of the simplex method calculations to find the optimal primal solution. A second implication is that this brings all *duality theory* (described in Chap. 6) to bear upon the interpretation and analysis of games.

A related implication is that this provides a simple proof of the minimax theorem. Let x^*_{m+1} and y^*_{n+1} denote the value of x_{m+1} and y_{n+1} in the optimal solution of the respective linear programming problems. It is known from the *strong duality property* given in Sec. 6.1 that $-x^*_{m+1} = -y^*_{n+1}$, so that $x^*_{m+1} = y^*_{n+1}$. However, it is evident from the definition of $\underline{v}$ and $\overline{v}$ that $\underline{v} = x^*_{m+1}$ and $\overline{v} = y^*_{n+1}$, so it follows that $\underline{v} = \overline{v}$, as claimed by the minimax theorem.

One remaining loose end needs to be tied up, namely, what to do about x_{m+1} and y_{n+1} being unrestricted in sign in the linear programming formulations. If it is clear that $v \geq 0$ so that the optimal values of x_{m+1} and y_{n+1} are nonnegative, then it is safe to introduce nonnegativity constraints for these variables for the purpose of applying the simplex method. However, if $v < 0$, then an adjustment needs to be made. One possibility is to use the approach described in Sec. 4.6 for replacing a variable without a nonnegativity constraint by the difference of two nonnegative variables. Another is to reverse players 1 and 2 so that the payoff table would be rewritten as the payoff to the original player 2, which would make the corresponding value of v positive. A third, and the most commonly used, procedure is to add a sufficiently large fixed constant to all the entries in the payoff table that the new value of the game will be positive. (For example, setting this constant equal to the absolute value of the largest negative entry will suffice.) Because this same constant is added to every entry, this adjustment cannot alter the optimal mixed strategies in any way, so they can now be obtained in the usual manner. The indicated value of the game would be increased by the amount of the constant, but this value can be readjusted after the solution has been obtained.

To illustrate this linear programming approach, consider again variation 3 of the political campaign problem after dominated strategy 3 for player 1 is eliminated (see Table 14.6). Because there are some negative entries in the reduced payoff table, it is unclear at the outset whether the *value* of the game v is *nonnegative* (it turns out to be). For the moment, let us assume that $v \geq 0$ and proceed without making any of the adjustments discussed in the preceding paragraph.

To write out the linear programming model for player 1 for this example, note that p_{ij} in the general model is the entry in row i and column j of Table 14.6, for $i = 1, 2$ and $j = 1, 2, 3$. The resulting model is

Maximize $\quad x_3,$

subject to

$$\begin{aligned} 5x_2 - x_3 &\geq 0 \\ -2x_1 + 4x_2 - x_3 &\geq 0 \\ 2x_1 - 3x_2 - x_3 &\geq 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

Applying the simplex method to this linear programming problem (after adding the constraint $x_3 \geq 0$) yields $x_1^* = \frac{7}{11}$, $x_2^* = \frac{4}{11}$, $x_3^* = \frac{2}{11}$ as the optimal solution. (See Probs. 14.5-7 and 14.5-8.) Consequently, just as was found by the graphical procedure in the preceding section, the optimal mixed strategy for player 1 according to the minimax criterion is $(x_1, x_2) = (\frac{7}{11}, \frac{4}{11})$, and the value of the game is $v = x_3^* = \frac{2}{11}$. The simplex method also yields the optimal solution for the dual (given next) of this problem, namely, $y_1^* = 0$, $y_2^* = \frac{5}{11}$, $y_3^* = \frac{6}{11}$, $y_4^* = \frac{2}{11}$, so the optimal mixed strategy for player 2 is $(y_1, y_2, y_3) = (0, \frac{5}{11}, \frac{6}{11})$.

The dual of the preceding problem is just the linear programming model for player 2 (the one with variables $y_1, y_2, \ldots, y_n, y_{n+1}$) shown earlier in this section. (See Prob. 14.5-6.) By plugging in the values of p_{ij} from Table 14.6, this model is

Minimize $\quad y_4,$

subject to

$$\begin{aligned} -2y_2 + 2y_3 - y_4 &\leq 0 \\ 5y_1 + 4y_2 - 3y_3 - y_4 &\leq 0 \\ y_1 + y_2 + y_3 &= 1 \end{aligned}$$

and

$$y_1 \geq 0, \qquad y_2 \geq 0, \qquad y_3 \geq 0.$$

Applying the simplex method directly to this model (after adding the constraint $y_4 \geq 0$) yields the optimal solution: $y_1^* = 0$, $y_2^* = \frac{5}{11}$, $y_3^* = \frac{6}{11}$, $y_4^* = \frac{2}{11}$ (as well as the optimal dual solution $x_1^* = \frac{7}{11}$, $x_2^* = \frac{4}{11}$, $x_3^* = \frac{2}{11}$). Thus, the optimal mixed strategy for player 2 is $(y_1, y_2, y_3) = (0, \frac{5}{11}, \frac{6}{11})$, and the value of the game is again seen to be $v = y_4^* = \frac{2}{11}$.

Because we already had found the optimal mixed strategy for player 2 while dealing with the first model, we did not have to solve the second one. In general, you always can find optimal mixed strategies for *both* players by choosing just one of the models (either one) and then using the simplex method to solve for both an optimal solution and an optimal dual solution.

When the simplex method was applied to both of these linear programming models, a nonnegativity constraint was added that assumed that $v \geq 0$. If this assumption were violated, both models would have no feasible solutions, so the simplex method would stop quickly with this message. To avoid this risk, we could have added a positive constant, say, 3 (the absolute value of the largest negative entry), to all the entries in Table 14.6. This then would increase by 3 all the coefficients of x_1, x_2, y_1, y_2, and y_3 in the inequality constraints of the two models.

POSSIBLE QUESTIONS

PART - A (20 x 1 =20 Marks)
(Question Nos. 1 to 20 Online Examinations)

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define n-player game.
2. Define Zero-Sum Game.
3. What are the main characteristics of game theory
4. Define pay-off matrix.
5. Write a note on saddle point.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Solve the game whose pay-off matrix is given by

Player B

		B_1	B_2	B_3
Player A	A_1	1	3	1
	A_2	0	-4	-3
	A_3	1	5	-1

2. Determine the range of value of p and q that will make the payoff element a_{22} a saddle point for the game whose payoff matrix (a_{ij}) is given below:

Player B

$$\text{Player A} \begin{pmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{pmatrix}$$

3. Solve the following 2 x 2 game.

B

$$A \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}$$

4. In a game of matching coins with two players, suppose A wins one unit value when there are two heads, wins nothing when there are two tails, and looses ½ unit value when there are one head and one tail. Determine the payoff matrix, the best strategy for each player, and the value of the game.

5. For the payoff matrix given below, decide optimum strategies for A and B.

		B: 1	B: 2
A	1	200	80
A	2	110	170

6. Solve the following game using dominance property.

		B: I	B: II	B: III
A	I	1	7	2
A	II	6	2	7
A	III	6	1	6

7. Use the notion of dominance to simplify the rectangular game with the following payoff, and solve it graphically.

		Player K: I	II	III	IV
Player L	1	18	4	6	4
Player L	2	6	2	13	7
Player L	3	11	5	17	3
Player L	4	7	6	12	2

8. Solve the following 2 x 4 game graphically.

Player B

$$\text{Player A} \begin{pmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & -2 & 5 \end{pmatrix}$$

9. Solve the following game by using simplex method.

Player B

$$\text{Player A} \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{pmatrix}$$

10. Two companies A and B competing for the same product. Their strategies are given in the following payoff matrix.

Company A

$$\text{Company B} = \begin{pmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{pmatrix}$$