KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

		Semester – I
18MMP102	**REAL ANALYSIS**	**L T P C**
		4 0 0 4

Course Objectives

To understand basic principles of real analysis and to identify sets with various properties such as finiteness, countability, infiniteness, uncountability.

Course Outcomes

On successful completion of this course, students will be able to:

- Get specific skill in Riemann Stieltjes integral and Lebesgue integral.
- Enrich their knowledge of measure theory and extremum problems.
- Solve given problems at a high level of abstraction based on logical and structured reasoning.
- Attain knowledge in infinite series.

UNIT I

The Riemann – Stieltjes Integral: Introduction – Basic Definitions – Linear Properties – Integration by parts – Change of variable in a Riemann – Stieltjes Integral – Reduction to a Riemann Integral – Step functions as integrators – Reduction of a Riemann – Stieltjes Integral to a finite sum – Monotonically increasing – Additive and linear properties – Riemann condition – Comparison theorems – Integrators of bounded variation – Sufficient condition for Riemann Stieltjes integral.

UNIT II

Infinite series and infinite products: Introduction – Basic definitions – Ratio test and root test – Dirichlet test and Able's test –Rearrangement of series – Riemann's theorem on conditionally convergent series – Sub series - Double sequences – Double series – Multiplication of series – Cesaro summability.

UNIT III

Sequences of functions: Basic definitions – Uniform convergence and continuity - Uniform convergence of infinite series of functions – Uniform convergence and Riemann – Stieltjes integration – Non uniformly convergent sequence – Uniform convergence and differentiation – Sufficient condition for uniform convergence of a series.

UNIT IV

The Lebesgue integral: Introduction- The class of Lebesgue – integrable functions on a general interval- Basic properties of the Lebesgue integral- Lebesgue integration and sets of measure zero- The Levi monotone convergence theorem- The Lebesgue dominated convergence theorem- Applications of Lebesgue dominated convergence theorem- Lebesgue integrals on unbounded intervals as limit of integrals on bounded intervals- Improper Riemann integrals- Measurable functions.

UNIT V
Implicit functions and extremum problems: Introduction – Functions with non zero Jacobian determinant – Inverse function theorem – Implicit function theorem – Extrema of real valued functions of one variable and several variables

SUGGESTED READINGS

TEXT BOOK
1. Rudin. W., (1976) .Principles of Mathematical Analysis, Mcgraw Hill, New york .

REFERENCES
1. Tom .M. Apostol., (2002). Mathematical Analysis, Second edition, Narosa Publishing House, New Delhi.
2. Balli. N.P., (1981). Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.
3. Gupta.S.L. and Gupta.N.R.,(2003).Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd, Singapore.
4. Royden .H.L., (2002). Real Analysis, Third edition, Prentice hall of India,New Delhi.
5. Sterling. K. Berberian., (2004).A First Course in Real Analysis, Springer Pvt Ltd, New Delhi.

KARPAGAM ACADEMY OF HIGHER EDUCATION

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: K.AARTHIYA
SUBJECT NAME:REAL ANALYSIS SUB.CODE:18MMP102
SEMESTER: I CLASS: I M.SC MATHEMATICS

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1	1	Introduction,basic definitions on Reimann Steiltjes integral.	T1:Chap 6,P.No.120-128
2	1	Linear properties of Reimann Steiltjes integral	T1:Chap 6,P.No.128-131
3	1	Integration by parts	T1:Chap 6,P.No.134
4	1	Change of variables in Reimann Steiltjes integral	T1:Chap 6,P.No.132-133
5	1	Reduction to Reimann integral	R1:Chap 7,P.No.145-146
6	1	Step function as integrals	R1:Chap 7,P.No.147-148
7	1	Reduction of Reimann integral to finite sum	R1:Chap 7,P.No.148-149
8	1	Monotonically increasing and Reimann integral to finite sum	R1:Chap 7,P.No.150-154
9	1	Comparison Theorems	R1:Chap 7,P.No.155-156
10	1	Integrators of Bounded Variation	R1:Chap 7,P.No.156-158
11	1	Sufficient condition for Reimann Steiltjes integral	R1:Chap 7,P.No.159

12	1	Recapitulation and discussion on possible questions	-
	Total No of Hours Planned For Unit I=12		
		UNIT-II	
1	1	Introduction to infinte series and infinite products,Basic definitions	R1:Chap 8,P.No.183-191
2	1	Ratio test and root test	R1:Chap 8,P.No.193-194
3	1	Dirichlet test and Abel test	R3:Chap 6,P.No.6.1-6.4
4	1	Rearrangement of series	R1:Chap 8,P.No.196
5	1	Reimann's theorem on conditionally convergent	R1:Chap 8,P.No.197
6	1	Subseries	R1:Chap 8,P.No.197-199
7	1	Double sequences on conditionally convergent	R1:Chap 8,P.No.199-200
8	1	Double series	R1:Chap 8,P.No.200-202
9	1	Multiplication of series and Caesero Summability	R1:Chap 8,P.No.203-209
10	1	Recapitulation and discussion on possible questions	
	Total No of Hours Planned For Unit II=10		
		UNIT-III	
1	1	Basic definitions on Sequences of functions	R5:Chap 3,P.No.33-36
2	1	Uniform convergence of sequences of functions	R5:Chap 3,P.No.39-41
3	1	Uniform convergence of series of functions	T1:Chap 7,P.No.143-146
4	1	Uniform convergence and Reimann Steiljes integrals	T1:Chap 7,P.No.147-148 R2:Chap 9,P.No.533-534

5	1	Non uniformly convergent	T1:Chap 7,P.No.152-153
6	1	Uniform convergence and differentiation	T1:Chap 7,P.No.153-154 R1:Chap 9,P.No.228-229
7	1	Sufficient condition for uniform convergence	R1:Chap 9,P.No.230-231
8	1	Recapitulation and discussion on possible questions	
	Total No of Hours Planned For Unit III=8		
		UNIT-IV	
1	1	Introduction on Lebesgue integrable functions	R4:Chap 4,P.No.75-77
2	1	The class of Lebesgue integrable functions on a general interval	R1:Chap 10,P.No.254-256
3	1	Basic properties of Lebesgue integrals	R4:Chap 4,P.No.85-88
4	1	Lebesgue integration and sets of measure zero	R1:Chap 10,P.No.264-265
5	1	Levi Monotone theorem	R1:Chap 10,P.No.266-268
6	1	Lebesgue Dominated Convergence theorem and its applications	R1:Chap 10,P.No.268-273
7	1	Lebesgue integrals on unbounded intervals as limit of integrals on bounded intervals	R1:Chap 10,P.No.274-275
8	1	Improper Reimann integrals	R1:Chap 10,P.No.276-278
9	1	Measurable functions	R1:Chap 10,P.No.279-280
10	1	Recapitulation and discussion on possible questions	
	Total No of Hours Planned For Unit IV=10		
		UNIT-V	
1	1	Functions with non-zero Jacobian determinant and basic concepts	R1:Chap 13,P.No.367-371

2	1	Inverse Function theorem	R1:Chap 13,P.No.372-373
3	1	Implicit function theorem	R1:Chap 13,P.No.373-375
4	1	Extrema of real valued functions of one variable and several variables	R1:Chap 13,P.No.376-379
5	1	Recapitulation and discussion on possible questions	
6	1	Discussion on previous old ESE question papers	
7	1	Discussion on previous old ESE question papers	
8	1	Discussion on previous old ESE question papers	
	Total No of Hours Planned for unit V=8		
Total Planned Hours	**48**		

TEXTBOOKS:

T1:Rudin. W., (1976) .Principles of mathematical Analysis, Mcgraw hill, New York

REFERENCES:

R1: Tom .M. Apostol., (2002). Mathematical Analysis, Second edition, Narosa Publishing House,
New Delhi.

R2:Balli. N.P., (1981). Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

R3.Gupta.S.L and Gupta.N.R,(2003).Principles of Real Analysis, Second edition, Pearson Education ,Pvt.Ltd Singapore.

R4.Royden .H.L., (2002). Real Analysis, Third edition, Prentice hall of India,New Delhi.

R5:Sterling. K. Berberian., (2004).A First Course in Real Analysis, Springer Pvt Ltd, New Delhi

UNIT I
THE RIEMANN – STIELTJES INTEGRAL
SYLLABUS

Introduction – Basic Definitions – Linear Properties – Integration by parts – Change of variable in a Riemann – Stieltjes Integral – Reduction to a Riemann Integral – Step functions as integrators – Reduction of a Riemann – Stieltjes Integral to a finite sum – Monotonically increasing – Additive and linear properties – Riemann condition – Comparison theorems – Integrators of bounded variation – Sufficient condition for Riemann Stieltjes integral.

7.1 INTRODUCTION

Calculus deals principally with two geometric problems: finding the tangent line to a curve, and finding the area of a region under a curve. The first is studied by a limit process known as *differentiation*; the second by another limit process—*integration*—to which we turn now.

The reader will recall from elementary calculus that to find the area of the region under the graph of a positive function f defined on $[a, b]$, we subdivide the interval $[a, b]$ into a finite number of subintervals, say n, the kth subinterval having length Δx_k, and we consider sums of the form $\sum_{k=1}^{n} f(t_k)\,\Delta x_k$, where t_k is some point in the kth subinterval. Such a sum is an approximation to the area by means of rectangles. If f is sufficiently well behaved in $[a, b]$—continuous, for example—then there is some hope that these sums will tend to a limit as we let $n \to \infty$, making the successive subdivisions finer and finer. This, roughly speaking, is what is involved in Riemann's definition of the definite integral $\int_a^b f(x)\,dx$. (A precise definition is given below.)

The two concepts, derivative and integral, arise in entirely different ways and it is a remarkable fact indeed that the two are intimately connected. If we consider the definite integral of a continuous function f as a function of its upper limit, say we write

$$F(x) = \int_a^x f(t)\,dt,$$

then F has a derivative and $F'(x) = f(x)$. This important result shows that differentiation and integration are, in a sense, inverse operations.

7.2 NOTATION

For brevity we make certain stipulations concerning notation and terminology to be used in this chapter. We shall be working with a compact interval $[a, b]$ and, unless otherwise stated, all functions denoted by f, g, α, β, etc., will be assumed to be real-valued functions defined and *bounded* on $[a, b]$. Complex-valued functions are dealt with in Section 7.27, and extensions to unbounded functions and infinite intervals will be discussed in Chapter 10.

As in Chapter 6, a partition P of $[a, b]$ is a finite set of points, say

$$P = \{x_0, x_1, \ldots, x_n\},$$

such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. A partition P' of $[a, b]$ is said to be *finer* than P (or a *refinement* of P) if $P \subseteq P'$, which we also write $P' \supseteq P$. The symbol $\Delta\alpha_k$ denotes the difference $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$, so that

$$\sum_{k=1}^{n} \Delta\alpha_k = \alpha(b) - \alpha(a).$$

The set of all possible partitions of $[a, b]$ is denoted by $\mathscr{P}[a, b]$.

The norm of a partition P is the length of the largest subinterval of P and is denoted by $\|P\|$. Note that

$$P' \supseteq P \quad \text{implies} \quad \|P'\| \le \|P\|.$$

That is, refinement of a partition decreases its norm, but the converse does not necessarily hold.

7.3 THE DEFINITION OF THE RIEMANN–STIELTJES INTEGRAL

Definition 7.1. *Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and let t_k be a point in the subinterval $[x_{k-1}, x_k]$. A sum of the form*

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k$$

is called a Riemann–Stieltjes sum of f with respect to α. We say f is Riemann-integrable with respect to α on $[a, b]$, and we write "$f \in R(\alpha)$ on $[a, b]$" if there exists a number A having the following property: For every $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that for every partition P finer than P_ε and for every choice of the points t_k in $[x_{k-1}, x_k]$, we have $|S(P, f, \alpha) - A| < \varepsilon$.

7.4 LINEAR PROPERTIES

It is an easy matter to prove that the integral operates in a linear fashion on both the integrand and the integrator. This is the context of the next two theorems.

Theorem 7.2. *If $f \in R(\alpha)$ and if $g \in R(\alpha)$ on $[a, b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.$$

Proof. Let $h = c_1 f + c_2 g$. Given a partition P of $[a, b]$, we can write

$$S(P, h, \alpha) = \sum_{k=1}^{n} h(t_k)\, \Delta\alpha_k = c_1 \sum_{k=1}^{n} f(t_k)\, \Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\, \Delta\alpha_k$$
$$= c_1 S(P, f, \alpha) + c_2 S(P, g, \alpha).$$

Given $\varepsilon > 0$, choose P_ε' so that $P \supseteq P_\varepsilon'$ implies $|S(P, f, \alpha) - \int_a^b f\, d\alpha| < \varepsilon$, and choose P_ε'' so that $P \supseteq P_\varepsilon''$ implies $|S(P, g, \alpha) - \int_a^b g\, d\alpha| < \varepsilon$. If we take $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$, then, for P finer than P_ε, we have

$$\left| S(P, h, \alpha) - c_1 \int_a^b f\, d\alpha - c_2 \int_a^b g\, d\alpha \right| \le |c_1|\varepsilon + |c_2|\varepsilon,$$

and this proves the theorem.

Theorem 7.3. *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a, b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b f\, d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta.$$

The proof is similar to that of Theorem 7.2 and is left as an exercise.

A result somewhat analogous to the previous two theorems tells us that the integral is also additive with respect to the interval of integration.

Theorem 7.4. *Assume that $c \in (a, b)$. If two of the three integrals in* (1) *exist, then the third also exists and we have*

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha. \tag{1}$$

Proof. If P is a partition of $[a, b]$ such that $c \in P$, let

$$P' = P \cap [a, c] \quad \text{and} \quad P'' = P \cap [c, b],$$

denote the corresponding partitions of $[a, c]$ and $[c, b]$, respectively. The Riemann–Stieltjes sums for these partitions are connected by the equation

$$S(P, f, \alpha) = S(P', f, \alpha) + S(P'', f, \alpha).$$

Assume that $\int_a^c f\,d\alpha$ and $\int_c^b f\,d\alpha$ exist. Then, given $\varepsilon > 0$, there is a partition P'_ε of $[a, c]$ such that

$$\left|S(P', f, \alpha) - \int_a^c f\,d\alpha\right| < \frac{\varepsilon}{2} \qquad \text{whenever } P' \text{ is finer than } P'_\varepsilon,$$

and a partition P''_ε of $[c, b]$ such that

$$\left|S(P'', f, \alpha) - \int_c^b f\,d\alpha\right| < \frac{\varepsilon}{2} \qquad \text{whenever } P'' \text{ is finer than } P''_\varepsilon.$$

Then $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$ is a partition of $[a, b]$ such that P finer than P_ε implies $P' \supseteq P'_\varepsilon$ and $P'' \supseteq P''_\varepsilon$. Hence, if P is finer than P_ε, we can combine the foregoing results to obtain the inequality

$$\left|S(P, f, \alpha) - \int_a^c f\,d\alpha - \int_c^b f\,d\alpha\right| < \varepsilon.$$

This proves that $\int_a^b f\,d\alpha$ exists and equals $\int_a^c f\,d\alpha + \int_c^b f\,d\alpha$. The reader can easily verify that a similar argument proves the theorem in the remaining cases.

Using mathematical induction, we can prove a similar result for a decomposition of $[a, b]$ into a finite number of subintervals.

Definition 7.5. *If $a < b$, we define $\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha$ whenever $\int_a^b f\,d\alpha$ exists. We also define $\int_a^a f\,d\alpha = 0$.*

The equation in Theorem 7.4 can now be written as follows:

$$\int_a^b f\,d\alpha + \int_b^c f\,d\alpha + \int_c^a f\,d\alpha = 0.$$

7.5 INTEGRATION BY PARTS

A remarkable connection exists between the integrand and the integrator in a Riemann–Stieltjes integral. The existence of $\int_a^b f\,d\alpha$ implies the existence of $\int_a^b \alpha\,df$, and the converse is also true. Moreover, a very simple relation holds between the two integrals.

Theorem 7.6. *If $f \in R(\alpha)$ on $[a, b]$, then $\alpha \in R(f)$ on $[a, b]$ and we have*

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

NOTE. This equation, which provides a kind of reciprocity law for the integral, is known as the *formula for integration by parts.*

Proof. Let $\varepsilon > 0$ be given. Since $\int_a^b f\,d\alpha$ exists, there is a partition P_ε of $[a, b]$ such that for every P' finer than P_ε, we have

$$\left|S(P', f, \alpha) - \int_a^b f\,d\alpha\right| < \varepsilon. \tag{2}$$

Consider an arbitrary Riemann–Stieltjes sum for the integral $\int_a^b \alpha\,df$, say

$$S(P, \alpha, f) = \sum_{k=1}^n \alpha(t_k)\,\Delta f_k = \sum_{k=1}^n \alpha(t_k)f(x_k) - \sum_{k=1}^n \alpha(t_k)f(x_{k-1}),$$

where P is finer than P_ε. Writing $A = f(b)\alpha(b) - f(a)\alpha(a)$, we have the identity

$$A = \sum_{k=1}^n f(x_k)\alpha(x_k) - \sum_{k=1}^n f(x_{k-1})\alpha(x_{k-1}).$$

Subtracting the last two displayed equations, we find

$$A - S(P, \alpha, f) = \sum_{k=1}^n f(x_k)[\alpha(x_k) - \alpha(t_k)] + \sum_{k=1}^n f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})].$$

The two sums on the right can be combined into a single sum of the form $S(P', f, \alpha)$, where P' is that partition of $[a, b]$ obtained by taking the points x_k and t_k together. Then P' is finer than P and hence finer than P_ε. Therefore the inequality (2) is valid and this means that we have

$$\left|A - S(P, \alpha, f) - \int_a^b f\,d\alpha\right| < \varepsilon,$$

7.6 CHANGE OF VARIABLE IN A RIEMANN-STIELTJES INTEGRAL

Theorem 7.7. *Let $f \in R(\alpha)$ on $[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having endpoints c and d. Assume that $a = g(c)$,*

$b = g(d)$. Let h and β be the composite functions defined as follows:

$$h(x) = f[g(x)], \qquad \beta(x) = \alpha[g(x)], \qquad \text{if } x \in S.$$

Then $h \in R(\beta)$ on S and we have $\int_a^b f\,d\alpha = \int_c^d h\,d\beta$. That is,

$$\int_{g(c)}^{g(d)} f(t)\,d\alpha(t) = \int_c^d f[g(x)]\,d\{\alpha[g(x)]\}.$$

Proof. For definiteness, assume that g is strictly increasing on S. (This implies $c < d$.) Then g is one-to-one and has a strictly increasing, continuous inverse g^{-1} defined on $[a, b]$. Therefore, for every partition $P = \{y_0, \ldots, y_n\}$ of $[c, d]$, there corresponds one and only one partition $P' = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $x_k = g(y_k)$. In fact, we can write

$$P' = g(P) \qquad \text{and} \qquad P = g^{-1}(P').$$

Furthermore, a refinement of P produces a corresponding refinement of P', and the converse also holds.

If $\varepsilon > 0$ is given, there is a partition P'_ε of $[a, b]$ such that P' finer than P'_ε implies $|S(P', f, \alpha) - \int_a^b f\,d\alpha| < \varepsilon$. Let $P_\varepsilon = g^{-1}(P'_\varepsilon)$ be the corresponding partition of $[c, d]$, and let $P = \{y_0, \ldots, y_n\}$ be a partition of $[c, d]$ finer than P_ε. Form a Riemann–Stieltjes sum

$$S(P, h, \beta) = \sum_{k=1}^{n} h(u_k)\,\Delta\beta_k,$$

where $u_k \in [y_{k-1}, y_k]$ and $\Delta\beta_k = \beta(y_k) - \beta(y_{k-1})$. If we put $t_k = g(u_k)$ and $x_k = g(y_k)$, then $P' = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$ finer than P'_ε. Moreover, we then have

$$\begin{aligned} S(P, h, \beta) &= \sum_{k=1}^{n} f[g(u_k)]\{\alpha[g(y_k)] - \alpha[g(y_{k-1})]\} \\ &= \sum_{k=1}^{n} f(t_k)\{\alpha(x_k) - \alpha(x_{k-1})\} = S(P', f, \alpha), \end{aligned}$$

since $t_k \in [x_{k-1}, x_k]$. Therefore, $|S(P, h, \beta) - \int_a^b f\,d\alpha| < \varepsilon$ and the theorem is proved.

NOTE. This theorem applies, in particular, to Riemann integrals, that is, when $\alpha(x) = x$. Another theorem of this type, in which g is not required to be monotonic, will later be proved for Riemann integrals. (See Theorem 7.36.)

7.7 REDUCTION TO A RIEMANN INTEGRAL

The next theorem tells us that we are permitted to replace the symbol $d\alpha(x)$ by $\alpha'(x)\,dx$ in the integral $\int_a^b f(x)\,d\alpha(x)$ whenever α has a continuous derivative α'.

Theorem 7.8. *Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\,dx$ exists and we have*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

Proof. Let $g(x) = f(x)\alpha'(x)$ and consider a Riemann sum

$$S(P, g) = \sum_{k=1}^{n} g(t_k)\,\Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\,\Delta x_k.$$

The same partition P and the same choice of the t_k can be used to form the Riemann–Stieltjes sum

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k.$$

Applying the Mean-Value Theorem, we can write

$$\Delta\alpha_k = \alpha'(v_k)\,\Delta x_k, \qquad \text{where } v_k \in (x_{k-1}, x_k),$$

and hence

$$S(P, f, \alpha) - S(P, g) = \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\,\Delta x_k.$$

Since f is bounded, we have $|f(x)| \le M$ for all x in $[a, b]$, where $M > 0$. Continuity of α' on $[a, b]$ implies uniform continuity on $[a, b]$. Hence, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ (depending only on ε) such that

$$0 \le |x - y| < \delta \qquad \text{implies} \qquad |\alpha'(x) - \alpha'(y)| < \frac{\varepsilon}{2M(b - a)}.$$

If we take a partition P_ε' with norm $\|P_\varepsilon'\| < \delta$, then for any finer partition P we will have $|\alpha'(v_k) - \alpha'(t_k)| < \varepsilon/[2M(b - a)]$ in the preceding equation. For such P we therefore have

$$|S(P, f, \alpha) - S(P, g)| < \frac{\varepsilon}{2}.$$

On the other hand, since $f \in R(\alpha)$ on $[a, b]$, there exists a partition P_ε'' such that P finer than P_ε'' implies

$$\left|S(P, f, \alpha) - \int_a^b f\,d\alpha\right| < \frac{\varepsilon}{2}.$$

Combining the last two inequalities, we see that when P is finer than $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$, we will have $|S(P, g) - \int_a^b f\,d\alpha| < \varepsilon$, and this proves the theorem.

7.8 STEP FUNCTIONS AS INTEGRATORS

If α is constant throughout $[a, b]$, the integral $\int_a^b f\,d\alpha$ exists and has the value 0, since each sum $S(P, f, \alpha) = 0$. However, if α is constant except for a jump discontinuity at one point, the integral $\int_a^b f\,d\alpha$ need not exist and, if it does exist, its value need not be zero. The situation is described more fully in the following theorem:

Theorem 7.9. *Given $a < c < b$. Define α on $[a, b]$ as follows: The values $\alpha(a)$, $\alpha(c)$, $\alpha(b)$ are arbitrary;*

$$\alpha(x) = \alpha(a) \qquad \text{if } a \le x < c,$$

and

$$\alpha(x) = \alpha(b) \qquad \text{if } c < x \le b.$$

Let f be defined on $[a, b]$ in such a way that at least one of the functions f or α is continuous from the left at c and at least one is continuous from the right at c. Then $f \in R(\alpha)$ on $[a, b]$ and we have

$$\int_a^b f\,d\alpha = f(c)[\alpha(c+) - \alpha(c-)].$$

NOTE. The result also holds if $c = a$, provided that we write $\alpha(c)$ for $\alpha(c-)$, and it holds for $c = b$ if we write $\alpha(c)$ for $\alpha(c+)$. We will prove later (Theorem 7.29) that the integral does not exist if both f and α are discontinuous from the right or from the left at c.

Proof. If $c \in P$, every term in the sum $S(P, f, \alpha)$ is zero except the two terms arising from the subinterval separated by c, say

$$S(P, f, \alpha) = f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)],$$

where $t_{k-1} \le c \le t_k$. This equation can also be written as follows:

$$\Delta = [f(t_{k-1}) - f(c)][\alpha(c) - \alpha(c-)] + [f(t_k) - f(c)][\alpha(c+) - \alpha(c)],$$

where $\Delta = S(P, f, \alpha) - f(c)[\alpha(c+) - \alpha(c-)]$. Hence we have

$$|\Delta| \le |f(t_{k-1}) - f(c)|\,|\alpha(c) - \alpha(c-)| + |f(t_k) - f(c)|\,|\alpha(c+) - \alpha(c)|.$$

If f is continuous at c, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|P\| < \delta$ implies

$$|f(t_{k-1}) - f(c)| < \varepsilon \qquad \text{and} \qquad |f(t_k) - f(c)| < \varepsilon.$$

In this case, we obtain the inequality

$$|\Delta| \le \varepsilon|\alpha(c) - \alpha(c-)| + \varepsilon|\alpha(c+) - \alpha(c)|.$$

But this inequality holds whether or not f is continuous at c. For example, if f is discontinuous both from the right and from the left at c, then $\alpha(c) = \alpha(c-)$ and $\alpha(c) = \alpha(c+)$ and we get $\Delta = 0$. On the other hand, if f is continuous from the left and discontinuous from the right at c, we must have $\alpha(c) = \alpha(c+)$ and we get

$|\Delta| \le \varepsilon|\alpha(c) - \alpha(c-)|$. Similarly, if f is continuous from the right and discontinuous from the left at c, we have $\alpha(c) = \alpha(c-)$ and $|\Delta| \le \varepsilon|\alpha(c+) - \alpha(c)|$. Hence the last displayed inequality holds in every case. This proves the theorem.

7.9 REDUCTION OF A RIEMANN–STIELTJES INTEGRAL TO A FINITE SUM

The integrator α in Theorem 7.9 is a special case of an important class of functions known as *step functions*. These are functions which are constant throughout an interval except for a finite number of jump discontinuities.

Definition 7.10 (Step function). *A function α defined on $[a, b]$ is called a step function if there is a partition*

$$a = x_1 < x_2 < \cdots < x_n = b$$

such that α is constant on each open subinterval (x_{k-1}, x_k). The number $\alpha(x_k+) - \alpha(x_k-)$ is called the jump at x_k if $1 < k < n$. The jump at x_1 is $\alpha(x_1+) - \alpha(x_1)$, and the jump at x_n is $\alpha(x_n) - \alpha(x_n-)$.

Step functions provide the connecting link between Riemann–Stieltjes integrals and finite sums:

Theorem 7.11 (Reduction of a Riemann–Stieltjes integral to a finite sum). *Let α be a step function defined on $[a, b]$ with jump α_k at x_k, where $x_1, \ldots, x_n$ are as described in Definition 7.10. Let f be defined on $[a, b]$ in such a way that not both f and α are discontinuous from the right or from the left at each x_k. Then $\int_a^b f\,d\alpha$ exists and we have*

$$\int_a^b f(x)\,d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha_k.$$

Proof. By Theorem 7.4, $\int_a^b f\,d\alpha$ can be written as a sum of integrals of the type considered in Theorem 7.9.

MONOTONICALLY INCREASING INTEGRATORS. UPPER AND LOWER INTEGRALS

Definition 7.14. *Let P be a partition of $[a, b]$ and let*

$$M_k(f) = \sup \{f(x) : x \in [x_{k-1}, x_k]\},$$
$$m_k(f) = \inf \{f(x) : x \in [x_{k-1}, x_k]\}.$$

The numbers

$$U(P, f, \alpha) = \sum_{k=1}^{n} M_k(f)\, \Delta\alpha_k \qquad \textit{and} \qquad L(P, f, \alpha) = \sum_{k=1}^{n} m_k(f)\, \Delta\alpha_k,$$

are called, respectively, the upper and lower Stieltjes sums of f with respect to α for the partition P.

NOTE. We always have $m_k(f) \le M_k(f)$. If $\alpha \nearrow$ on $[a, b]$, then $\Delta\alpha_k \ge 0$ and we can also write $m_k(f)\, \Delta\alpha_k \le M_k(f)\, \Delta\alpha_k$, from which it follows that the lower sums do not exceed the upper sums. Furthermore, if $t_k \in [x_{k-1}, x_k]$, then

$$m_k(f) \le f(t_k) \le M_k(f).$$

Therefore, when $\alpha \nearrow$, we have the inequalities

$$L(P, f, \alpha) \le S(P, f, \alpha) \le U(P, f, \alpha)$$

relating the upper and lower sums to the Riemann–Stieltjes sums. These inequalities, which are frequently used in the material that follows, do not necessarily hold when α is not an increasing function.

The next theorem shows that, for increasing α, refinement of the partition increases the lower sums and decreases the upper sums.

Theorem 7.15. *Assume that $\alpha \nearrow$ on $[a, b]$. Then:*

i) *If P' is finer than P, we have*

$$U(P', f, \alpha) \le U(P, f, \alpha) \qquad \textit{and} \qquad L(P', f, \alpha) \ge L(P, f, \alpha).$$

ii) *For any two partitions P_1 and P_2, we have*

$$L(P_1, f, \alpha) \le U(P_2, f, \alpha).$$

Proof. It suffices to prove (i) when P' contains exactly one more point than P, say the point c. If c is in the ith subinterval of P, we can write

$$U(P', f, \alpha) = \sum_{\substack{k=1 \\ k \neq i}}^{n} M_k(f)\,\Delta\alpha_k + M'[\alpha(c) - \alpha(x_{i-1})] + M''[\alpha(x_i) - \alpha(c)],$$

where M' and M'' denote the sup of f in $[x_{i-1}, c]$ and $[c, x_i]$. But, since

$$M' \le M_i(f) \qquad \text{and} \qquad M'' \le M_i(f),$$

we have $U(P', f, \alpha) \le U(P, f, \alpha)$. (The inequality for lower sums is proved in a similar fashion.)

To prove (ii), let $P = P_1 \cup P_2$. Then we have

$$L(P_1, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_2, f, \alpha).$$

Definition 7.16. *Assume that $\alpha \nearrow$ on $[a, b]$. The upper Stieltjes integral of f with respect to α is defined as follows:*

$$\overline{\int_a^b} f\,d\alpha = \inf\{U(P, f, \alpha) : P \in \mathscr{P}[a, b]\}.$$

The lower Stieltjes integral is similarly defined:

$$\underline{\int_a^b} f\,d\alpha = \sup\{L(P, f, \alpha) : P \in \mathscr{P}[a, b]\}.$$

NOTE. We sometimes write $\bar{I}(f, \alpha)$ and $\underline{I}(f, \alpha)$ for the upper and lower integrals. In the special case where $\alpha(x) = x$, the upper and lower sums are denoted by $U(P, f)$ and $L(P, f)$ and are called upper and lower Riemann sums. The corresponding integrals, denoted by $\overline{\int_a^b} f(x)\,dx$ and by $\underline{\int_a^b} f(x)\,dx$, are called upper and lower Riemann integrals. They were first introduced by J. G. Darboux (1875).

Theorem 7.17. *Assume that $\alpha \nearrow$ on $[a, b]$. Then $\underline{I}(f, \alpha) \le \bar{I}(f, \alpha)$.*

Proof. If $\varepsilon > 0$ is given, there exists a partition P_1 such that

$$U(P_1, f, \alpha) < \bar{I}(f, \alpha) + \varepsilon.$$

By Theorem 7.15, it follows that $\bar{I}(f, \alpha) + \varepsilon$ is an upper bound to all lower sums $L(P, f, \alpha)$. Hence, $\underline{I}(f, \alpha) \le \bar{I}(f, \alpha) + \varepsilon$, and, since ε is arbitrary, this implies $\underline{I}(f, \alpha) \le \bar{I}(f, \alpha)$.

7.13 RIEMANN'S CONDITION

If we are to expect equality of the upper and lower integrals, then we must also expect the upper sums to become arbitrarily close to the lower sums. Hence it seems reasonable to seek those functions f for which the difference $U(P, f, \alpha) - L(P, f, \alpha)$ can be made arbitrarily small.

Definition 7.18. *We say that f satisfies Riemann's condition with respect to α on $[a, b]$ if, for every $\varepsilon > 0$, there exists a partition P_ε such that P finer than P_ε implies*

$$0 \le U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Theorem 7.19. *Assume that $\alpha \nearrow$ on $[a, b]$. Then the following three statements are equivalent:*

i) $f \in R(\alpha)$ *on* $[a, b]$.

ii) *f satisfies Riemann's condition with respect to α on $[a, b]$.*

iii) $\underline{I}(f, \alpha) = \overline{I}(f, \alpha)$.

Proof. We will prove that part (i) implies (ii), part (ii) implies (iii), and part (iii) implies (i). Assume that (i) holds. If $\alpha(b) = \alpha(a)$, then (ii) holds trivially, so we can assume that $\alpha(a) < \alpha(b)$. Given $\varepsilon > 0$, choose P_ε so that for any finer P and all choices of t_k and t'_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - A\right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left|\sum_{k=1}^{n} f(t'_k)\,\Delta\alpha_k - A\right| < \frac{\varepsilon}{3},$$

where $A = \int_a^b f\,d\alpha$. Combining these inequalities, we find

$$\left|\sum_{k=1}^{n} [f(t_k) - f(t_k')]\,\Delta\alpha_k\right| < \frac{2}{3}\varepsilon.$$

Since $M_k(f) - m_k(f) = \sup\{f(x) - f(x') : x, x' \text{ in } [x_{k-1}, x_k]\}$, it follows that for every $h > 0$ we can choose t_k and t_k' so that

$$f(t_k) - f(t_k') > M_k(f) - m_k(f) - h.$$

Making a choice corresponding to $h = \frac{1}{3}\varepsilon/[\alpha(b) - \alpha(a)]$, we can write

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{k=1}^{n} [M_k(f) - m_k(f)]\,\Delta\alpha_k$$

$$< \sum_{k=1}^{n} [f(t_k) - f(t_k')]\,\Delta\alpha_k + h\sum_{k=1}^{n} \Delta\alpha_k < \varepsilon.$$

Hence, (i) implies (ii).

Next, assume that (ii) holds. If $\varepsilon > 0$ is given, there exists a partition P_ε such that P finer than P_ε implies $U(P, f, \alpha) < L(P, f, \alpha) + \varepsilon$. Hence, for such P we have

$$\bar{I}(f, \alpha) \le U(P, f, \alpha) < L(P, f, \alpha) + \varepsilon \le \underline{I}(f, \alpha) + \varepsilon.$$

That is, $\bar{I}(f, \alpha) \le \underline{I}(f, \alpha) + \varepsilon$ for every $\varepsilon > 0$. Therefore, $\bar{I}(f, \alpha) \le \underline{I}(f, \alpha)$. But, by Theorem 7.17, we also have the opposite inequality. Hence (ii) implies (iii).

Finally, assume that $\bar{I}(f, \alpha) = \underline{I}(f, \alpha)$ and let A denote their common value. We will prove that $\int_a^b f\,d\alpha$ exists and equals A. Given $\varepsilon > 0$, choose P_ε' so that $U(P, f, \alpha) < \bar{I}(f, \alpha) + \varepsilon$ for all P finer than P_ε'. Also choose P_ε'' such that

$$L(P, f, \alpha) > \underline{I}(f, \alpha) - \varepsilon$$

for all P finer than P_ε''. If $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$, we can write

$$\underline{I}(f, \alpha) - \varepsilon < L(P, f, \alpha) \le S(P, f, \alpha) \le U(P, f, \alpha) < \bar{I}(f, \alpha) + \varepsilon$$

for every P finer than P_ε. But, since $\underline{I}(f, \alpha) = \bar{I}(f, \alpha) = A$, this means that $|S(P, f, \alpha) - A| < \varepsilon$ whenever P is finer than P_ε. This proves that $\int_a^b f\,d\alpha$ exists and equals A, and the proof of the theorem is now complete.

7.14 COMPARISON THEOREMS

Theorem 7.20. *Assume that $\alpha \nearrow$ on $[a, b]$. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ and if $f(x) \le g(x)$ for all x in $[a, b]$, then we have*

$$\int_a^b f(x)\,d\alpha(x) \le \int_a^b g(x)\,d\alpha(x).$$

Proof. For every partition P, the corresponding Riemann–Stieltjes sums satisfy

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k \le \sum_{k=1}^{n} g(t_k)\,\Delta\alpha_k = S(P, g, \alpha),$$

since $\alpha \nearrow$ on $[a, b]$. From this the theorem follows easily.

In particular, this theorem implies that $\int_a^b g(x)\,d\alpha(x) \ge 0$ whenever $g(x) \ge 0$ and $\alpha \nearrow$ on $[a, b]$.

Theorem 7.21. *Assume that $\alpha \nearrow$ on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$, then $|f| \in R(\alpha)$ on $[a, b]$ and we have the inequality*

$$\left|\int_a^b f(x)\, d\alpha(x)\right| \le \int_a^b |f(x)|\, d\alpha(x).$$

Proof. Using the notation of Definition 7.14, we can write

$$M_k(f) - m_k(f) = \sup\{f(x) - f(y) : x, y \text{ in } [x_{k-1}, x_k]\}.$$

Since the inequality $||f(x)| - |f(y)|| \le |f(x) - f(y)|$ always holds, it follows that we have

$$M_k(|f|) - m_k(|f|) \le M_k(f) - m_k(f).$$

Multiplying by $\Delta\alpha_k$ and summing on k, we obtain

$$U(P, |f|, \alpha) - L(P, |f|, \alpha) \le U(P, f, \alpha) - L(P, f, \alpha),$$

for every partition P of $[a, b]$. By applying Riemann's condition, we find that $|f| \in R(\alpha)$ on $[a, b]$. The inequality in the theorem follows by taking $g = |f|$ in Theorem 7.20.

NOTE. The converse of Theorem 7.21 is not true. (See Exercise 7.12.)

Theorem 7.22. *Assume that $\alpha \nearrow$ on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$, then $f^2 \in R(\alpha)$ on $[a, b]$.*

Proof. Using the notation of Definition 7.14, we have

$$M_k(f^2) = [M_k(|f|)]^2 \quad \text{and} \quad m_k(f^2) = [m_k(|f|)]^2.$$

Hence we can write

$$\begin{aligned} M_k(f^2) - m_k(f^2) &= [M_k(|f|) + m_k(|f|)][M_k(|f|) - m_k(|f|)] \\ &\le 2M[M_k(|f|) - m_k(|f|)]. \end{aligned}$$

POSSIBLE QUESTIONS

1. For any f ∈ R (α) on [a,b] and g∈ R (α) on [a,b] then $c_1 f + c_2 g \in R(\alpha)$ on [a,b] and
   we have $\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b (f\, d\alpha) + c_2 \int_a^b (g\, d\alpha)$
2. Let f be of bounded variation on [a,b] . if x ∈ [a,b] , let V (x) = V_r(a,x) and put V (a) =0. Then show that every point of continuity of f is also a point of continuity of V and Converse is also true.
3. Assume that α is increasing on [a,b] then prove that the following are equivalent
   (i) f ∈ R (α) on [a,b]

(ii) f satisfies Riemann condition w.r.to α on [a,b]

(iii) I_{-} (f ,α) = I^{-} (f ,α)

4. Assume that c ∈ (a,b). If two of the three integrals in are exist, then prove that the third also exists and we have $\int_a^c f\, d\,\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$
5. State and prove change of variable in Riemann – Stieltjes integral .
6. If f ∈ R (α) on [a,b] then prove that for α ∈ R (f) on [a,b,] we have
   $\int_a^b f(x) d\,\alpha(x) + \int_a^b \alpha(x) d\, f(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a)$.
7. State and prove Riemann – Stieltjes condition
8. Let f : [a,b] → R^n and g: [c,d] → R^n be two paths in R^n each of which is one –to –one on its domain, then prove that f and g are equivalent iff they have the same graph
9. State and Prove a Reduction to a Riemann integral.
10. If f is continuous on [a,b] and if f ' exists and is bounded in the interior ,say | f ' (x) | ≤ A for all in (a, b) then prove that f is of bounded variation on [a,b]
11. Assume that α is increasing on [a,b] ,then I_{-} (f ,α) ≤ I^{-} (f ,α) .

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Real Analysis **Subject**

Class : I - M.Sc. Mathematics

Unit I

Part A (20x1=20 Marks) **(Question Nos. 1 to**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	
x(y + z) = xy + xz is --------------- law	commutative	associative	distributive	closure	distributive
If x < y, then for every z we have -----------	(x + z) < (y + z)	(x + z) > (y + z)	(x + z) = (y + z)	x + z = 0	(x + z) < (y + z)
If x > 0 and y > 0, then ----------------	xy less than equal to 0	xy > 0	xy greater than equal to 0	xy < 0	xy > 0
If x > y and y > z, then -----------	x < y	x = z	x > z	x < z	x > z
If a less than equal to b + $ for every $ > 0, then -----------	a < b	a > b	a greater than equal to b	a less than equal to b	a less than equal to b
The set of all points between a and b is called ------------	integer	interval	elements	set	interval
The set {x: a < x < b} is --------------	(a, b)	[a, b]	(a, b]	[a, b)	(a, b)
A real number is called a positive integer if it belongs to -----------	interval	open interval	closed interval	inductive set	inductive set
If d is a divisor of n, then ---------------	n = c	n < cd	n > cd	n = cd	n = cd
If $a \mid bc$ and (a, b) =1, then ------------------	$a \mid c$	$a \mid b$	$b \mid a$	$c \mid a$	$a \mid c$
If $a \mid bc$ and (a, b) =1, then $a \mid c$ is ---------------	Unique factorisation theorem	additive property	approximation property	Euclid's lemma	Euclid's lemma
Rational numbers is of the form -----------------	pq	p + q	p/q	p - q	p/q
e is ---------------	rational	irrational	prime	composite	irrational
An integer n is called ----------- if the only possible divisors of n are 1 and n	rational	irrational	prime	composite	prime
If $d \mid a$ and $d \mid b$, then d is called ----------	LCM	common divisor	prime	function	common divisor
If (a, b) = 1, then a and b are called ---------------	twin prime	common factor	LCM	relatively prime	relatively prime
If an upper bound 'b' of a set S is also a member of S then 'b' is called ---------------	rational	irrational	maximum element	minimum element	maximum element
If an lower bound 'b' of a set S is also a member of S then 'b' is called ---------------	rational	irrational	maximum element	minimum element	minimum element
A set with no upper bound is called ------------	bounded above	bounded below	prime	function	bounded above
A set with no lower bound is called ------------	bounded above	bounded below	prime	function	bounded below
The least upper bound is called -----------	bounded above	bounded above	supremum	supremum	supremum
The greatest lower bound is called -----------	bounded above	bounded below	supremum	infimum	infimum
The supremum of {3, 4} is ----------	3	4	(3, 4)	[3, 4]	4
Every finite set of numbers is ----------	bounded	unbounded	unbounded	bounded above	bounded
A set S of real numbers which is bounded above and bounded below is called --------	bounded set	inductive set	super set	subset	bounded set
The set N of natural numbers is ----------	bounded	not bounded	irrational	rational	not bounded
The completeness axiom is ------------	b = sup S	S = sup b	b = inf S	S = inf b	b = sup S
The infimum of {3, 4} is ------------	3	4	(3, 4)	[3, 4]	3
Sup C = Sup A + Sup B is called -------------- property	approximation	additive	archimedean	comparison	additive
For any real x, there is a positive integer n such that -----------	n > x	n < x	n = x	n = 0	n > x
If x > 0 and if y is an arbitrary real number, there is a positive number n such that nx > y is -------------- property	approximation	additive	archimedean	comparison	archimedean
The set of positive integers is -------------	bounded above	bounded below	unbounded above	unbounded below	unbounded above
The absolute value of x is denoted by ---------------	$\lvert x\rvert$	$\lVert x\rVert$	x < 0	x > 0	$\lvert x\rvert$
If x < 0 then ---------------	$\lvert x\rvert$ = x	$\lVert x\rVert = \lvert x\rvert$	$\lVert x\rVert$ = -x	$\lvert x\rvert$ = -x	$\lvert x\rvert$ = -x
If S = [0, 1) then sup S = ----------------	0	1	(0, 1)	[0,1]	1
Triangle inequality is -------------------	$\lvert a\rvert + \lvert b\rvert$ greater equal to $\lvert a + b\rvert$	$\lvert a\rvert > \lvert a + b\rvert$	$\lvert b\rvert > \lvert a + b\rvert$	$\lvert a + b\rvert$ less than equal to $\lvert a \rvert + \lvert b\rvert$	$\lvert a + b\rvert$ less than equal to $\lvert a \rvert + \lvert b\rvert$
$\lvert x + y\rvert$ greater than equal to -----------------	$\lvert x\rvert + \lvert y\rvert$	$\lvert x\rvert \lvert y\rvert$	$\lvert x\rvert - \lvert y\rvert$	$\lvert \lvert x\rvert - \lvert y\rvert \rvert$	$\lvert \lvert x\rvert - \lvert y\rvert \rvert$

Set of real numbers S is bounded above implies S has a ----------	supremum	infimum	additive property	comparison property	supremum
In { (3n +2) / (2n + 1) such that n is in N}, the greatest lower bound is --------------	5 divided by 3	8 divided by 5	11 divided by 47	3 divided by 2	3 divided by 2
In Cauchy-Schwarz inequality, the equality holds iff ----------	akx = 0	akx + bkx = 0	akx + bk = 0	bk = 0	akx + bk = 0
If a set consists of a finite number of elements is called	infinite set	finite set	cantor set	null set	finite set
If A,B,C are three sets then what is A -(B -C) =	(A -B) U (A ∩ C)	A -(B ∩ C)	(A -B) U C	(A -B) U (A - C)	(A -B) U (A ∩ C)
If P (A) denotes the power set of A and A is the void set then P {P {P{P(A)}}}} =	0	1	4	16	16
If X R then	X/∞ = ∞	X/∞ = 0	X/∞ = X	X/∞ = - ∞	X/∞ = 0
If x < 0 then ---------------	X (- ∞) = -∞	X (- ∞) = ∞	X (- ∞) = 0	X (- ∞) = X	X (- ∞) = -∞
If R * is an extended real number system then the least upper bound is	∞	negative infinity	0	no least upper bound	∞
Let f : R → R be a functiondefined as f (x) = x | x| then	f is 1-1 but not onto	neither f is 1-1 nor onto	f is 1-1 both onto	f is onto but not one-one	f is 1-1 both onto
The value of (0, ∞) is	∞	0	not defined	can not be determined	0

UNIT II
INFINITE SERIES AND INFINITE PRODUCTS
SYLLABU

Introduction – Basic definitions – Ratio test and root test – Dirichlet test and Able's test –Rearrangement of series – Riemann's theorem on conditionally convergent series – Sub series - Double sequences – Double series – Multiplication of series – Cesaro summability.

8.14 THE RATIO TEST AND ROOT TEST

Theorem 8.25 (Ratio test). *Given a series $\sum a_n$ of nonzero complex terms, let*

$$r = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \qquad R = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

a) *The series $\sum a_n$ converges absolutely if $R < 1$.*
b) *The series $\sum a_n$ diverges if $r > 1$.*
c) *The test is inconclusive if $r \le 1 \le R$.*

Proof. Assume that $R < 1$ and choose x so that $R < x < 1$. The definition of R implies the existence of N such that $|a_{n+1}/a_n| < x$ if $n \ge N$. Since $x = x^{n+1}/x^n$, this means that

$$\frac{|a_{n+1}|}{x^{n+1}} < \frac{|a_n|}{x^n} \le \frac{|a_N|}{x^N}, \qquad \text{if } n \ge N,$$

and hence $|a_n| \le cx^n$ if $n \ge N$, where $c = |a_N|x^{-N}$. Statement (a) now follows by applying the comparison test.

To prove (b), we simply observe that $r > 1$ implies $|a_{n+1}| > |a_n|$ for all $n \ge N$ for some N and hence we cannot have $\lim_{n\to\infty} a_n = 0$.

To prove (c), consider the two examples $\sum n^{-1}$ and $\sum n^{-2}$. In both cases, $r = R = 1$ but $\sum n^{-1}$ diverges, whereas $\sum n^{-2}$ converges.

Theorem 8.26 (Root test). *Given a series $\sum a_n$ of complex terms, let*

$$\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

a) *The series $\sum a_n$ converges absolutely if $\rho < 1$.*
b) *The series $\sum a_n$ diverges if $\rho > 1$.*
c) *The test is inconclusive if $\rho = 1$.*

Proof. Assume that $\rho < 1$ and choose x so that $\rho < x < 1$. The definition of ρ implies the existence of N such that $|a_n| < x^n$ for $n \geq N$. Hence, $\sum |a_n|$ converges by the comparison test. This proves (a).

To prove (b), we observe that $\rho > 1$ implies $|a_n| > 1$ infinitely often and hence we cannot have $\lim_{n\to\infty} a_n = 0$.

Finally, (c) is proved by using the same examples as in Theorem 8.25.

NOTE. The root test is more "powerful" than the ratio test. That is, whenever the root test is inconclusive, so is the ratio test. But there are examples where the ratio test fails and the root test *is* conclusive. (See Exercise 8.4.)

8.15 DIRICHLET'S TEST AND ABEL'S TEST

All the tests in the previous section help us to determine *absolute* convergence of a series with complex terms. It is also important to have tests for determining convergence when the series might not converge absolutely. The tests in this section are particularly useful for this purpose. They all depend on the *partial summation formula* of Abel (equation (9) in the next theorem).

Theorem 8.27. *If $\{a_n\}$ and $\{b_n\}$ are two sequences of complex numbers, define*

$$A_n = a_1 + \cdots + a_n.$$

Then we have the identity

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k). \tag{9}$$

Therefore, $\sum_{k=1}^{\infty} a_k b_k$ converges if both the series $\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k)$ and the sequence $\{A_n b_{n+1}\}$ converge.

Proof. Writing $A_0 = 0$, we have

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1}.$$

The second assertion follows at once from this identity.

NOTE. Formula (9) is analogous to the formula for integration by parts in a Riemann–Stieltjes integral.

Theorem 8.28 (Dirichlet's test). *Let $\sum a_n$ be a series of complex terms whose partial sums form a bounded sequence. Let $\{b_n\}$ be a decreasing sequence which converges to 0. Then $\sum a_n b_n$ converges.*

Proof. Let $A_n = a_1 + \cdots + a_n$ and assume that $|A_n| \le M$ for all n. Then

$$\lim_{n\to\infty} A_n b_{n+1} = 0.$$

Therefore, to establish convergence of $\sum a_n b_n$ we need only show that $\sum A_k(b_{k+1} - b_k)$ is convergent. Since $b_n \searrow$, we have

$$|A_k(b_{k+1} - b_k)| \le M(b_k - b_{k+1}).$$

But the series $\sum(b_{k+1} - b_k)$ is a convergent telescoping series. Hence the comparison test implies *absolute* convergence of $\sum A_k(b_{k+1} - b_k)$.

Theorem 8.29 (Abel's test). *The series $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\{b_n\}$ is a monotonic convergent sequence.*

Proof. Convergence of $\sum a_n$ and of $\{b_n\}$ establishes the existence of the limit $\lim_{n\to\infty} A_n b_{n+1}$, where $A_n = a_1 + \cdots + a_n$. Also, $\{A_n\}$ is a bounded sequence. The remainder of the proof is similar to that of Theorem 8.28. (Two further tests, similar to the above, are given in Exercise 8.27.)

8.16 PARTIAL SUMS OF THE GEOMETRIC SERIES $\sum z^n$ ON THE UNIT CIRCLE $|z| = 1$

To use Dirichlet's test effectively, we must be acquainted with a few series having bounded partial sums. Of course, all *convergent* series have this property. The next theorem gives an example of a divergent series whose partial sums are bounded. This is the geometric series $\sum z^n$ with $|z| = 1$, that is, with $z = e^{ix}$ where x is real. The formula for the partial sums of this series is of fundamental importance in the theory of Fourier series.

Theorem 8.30. *For every real $x \ne 2m\pi$ (m is an integer), we have*

$$\sum_{k=1}^{n} e^{ikx} = e^{ix}\frac{1 - e^{inx}}{1 - e^{ix}} = \frac{\sin(nx/2)}{\sin(x/2)} e^{i(n+1)x/2}. \tag{10}$$

NOTE. This identity yields the following estimate:

$$\left|\sum_{k=1}^{n} e^{ikx}\right| \le \frac{1}{|\sin(x/2)|}. \tag{11}$$

Proof. $(1 - e^{ix}) \sum_{k=1}^{n} e^{ikx} = \sum_{k=1}^{n} (e^{ikx} - e^{i(k+1)x}) = e^{ix} - e^{i(n+1)x}$. This establishes the first equality in (10). The second follows from the identity

$$e^{ix}\frac{1 - e^{inx}}{1 - e^{ix}} = \frac{e^{inx/2} - e^{-inx/2}}{e^{ix/2} - e^{-ix/2}}\, e^{i(n+1)x/2}.$$

NOTE. By considering the real and imaginary parts of (10), we obtain

$$\sum_{k=1}^{n} \cos kx = \sin\frac{nx}{2}\cos(n+1)\frac{x}{2}\Big/\sin\frac{x}{2}$$

$$= -\frac{1}{2} + \frac{1}{2}\sin(2n+1)\frac{x}{2}\Big/\sin\frac{x}{2}, \tag{12}$$

$$\sum_{k=1}^{n} \sin kx = \sin\frac{nx}{2}\sin(n+1)\frac{x}{2}\Big/\sin\frac{x}{2}. \tag{13}$$

Using (10), we can also write

$$\sum_{k=1}^{n} e^{i(2k-1)x} = e^{-ix}\sum_{k=1}^{n} e^{ik(2x)} = \frac{\sin nx}{\sin x}\, e^{inx}, \tag{14}$$

an identity valid for every $x \neq m\pi$ (m is an integer). Taking real and imaginary parts of (14) we obtain

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2\sin x}, \tag{15}$$

$$\sum_{k=1}^{n} \sin(2k-1)x = \frac{\sin^2 nx}{\sin x}. \tag{16}$$

Formulas (12) and (16) occur in the theory of Fourier series.

8.17 REARRANGEMENTS OF SERIES

We recall that $\mathbf{Z}^+$ denotes the set of positive integers, $\mathbf{Z}^+ = \{1, 2, 3, \dots\}$.

Definition 8.31. *Let f be a function whose domain is $\mathbf{Z}^+$ and whose range is $\mathbf{Z}^+$, and assume that f is one-to-one on $\mathbf{Z}^+$. Let $\sum a_n$ and $\sum b_n$ be two series such that*

$$b_n = a_{f(n)} \quad \text{for } n = 1, 2, \dots \tag{17}$$

Then $\sum b_n$ is said to be a rearrangement of $\sum a_n$.

NOTE. Equation (17) implies $a_n = b_{f^{-1}(n)}$ and hence $\sum a_n$ is also a rearrangement of $\sum b_n$.

Theorem 8.32. *Let $\sum a_n$ be an absolutely convergent series having sum s. Then every rearrangement of $\sum a_n$ also converges absolutely and has sum s.*

Proof. Let $\{b_n\}$ be defined by (17). Then

$$|b_1| + \cdots + |b_n| = |a_{f(1)}| + \cdots + |a_{f(n)}| \le \sum_{k=1}^{\infty} |a_k|,$$

so $\sum |b_n|$ has bounded partial sums. Hence $\sum b_n$ converges absolutely.

To show that $\sum b_n = s$, let $t_n = b_1 + \cdots + b_n$, $s_n = a_1 + \cdots + a_n$. Given $\varepsilon > 0$, choose N so that $|s_N - s| < \varepsilon/2$ and so that $\sum_{k=1}^{\infty} |a_{N+k}| \le \varepsilon/2$. Then

$$|t_n - s| \le |t_n - s_N| + |s_N - s| < |t_n - s_N| + \frac{\varepsilon}{2}.$$

Choose M so that $\{1, 2, \dots, N\} \subseteq \{f(1), f(2), \dots, f(M)\}$. Then $n > M$ implies $f(n) > N$, and for such n we have

$$\begin{aligned} |t_n - s_N| &= |b_1 + \cdots + b_n - (a_1 + \cdots + a_N)| \\ &= |a_{f(1)} + \cdots + a_{f(n)} - (a_1 + \cdots + a_N)| \\ &\le |a_{N+1}| + |a_{N+2}| + \cdots \le \frac{\varepsilon}{2}, \end{aligned}$$

since all the terms $a_1, \dots, a_N$ cancel out in the subtraction. Hence, $n > M$ implies $|t_n - s| < \varepsilon$ and this means $\sum b_n = s$.

8.18 RIEMANN'S THEOREM ON CONDITIONALLY CONVERGENT SERIES

The hypothesis of absolute convergence is essential in Theorem 8.32. Riemann discovered that any *conditionally* convergent series of real terms can be rearranged to yield a series which converges to any prescribed sum. This remarkable fact is a consequence of the following theorem:

Theorem 8.33. *Let $\sum a_n$ be a conditionally convergent series with real-valued terms. Let x and y be given numbers in the closed interval $[-\infty, +\infty]$, with $x \le y$. Then there exists a rearrangement $\sum b_n$ of $\sum a_n$ such that*

$$\liminf_{n\to\infty} t_n = x \qquad \textit{and} \qquad \limsup_{n\to\infty} t_n = y,$$

where $t_n = b_1 + \cdots + b_n$.

Proof. Discarding those terms of a series which are zero does not affect its convergence or divergence. Hence we might as well assume that no terms of $\sum a_n$ are zero. Let p_n denote the nth positive term of $\sum a_n$ and let $-q_n$ denote its nth negative term. Then $\sum p_n$ and $\sum q_n$ are both divergent series of positive terms. [Why?] Next, construct two sequences of real numbers, say $\{x_n\}$ and $\{y_n\}$, such that

$$\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} y_n = y, \qquad \text{with } x_n < y_n, \quad y_1 > 0.$$

The idea of the proof is now quite simple. We take just enough (say k_1) positive terms so that

$$p_1 + \cdots + p_{k_1} > y_1,$$

followed by just enough (say r_1) negative terms so that

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} < x_1.$$

Next, we take just enough *further* positive terms so that

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k_1+1} + \cdots + p_{k_2} > y_2,$$

followed by just enough further negative terms to satisfy the inequality

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k_1+1} + \cdots + p_{k_2} - q_{r_1+1} - \cdots - q_{r_2} < x_2.$$

These steps are possible since $\sum p_n$ and $\sum q_n$ are both divergent series of positive terms. If the process is continued in this way, we obviously obtain a rearrangement of $\sum a_n$. We leave it to the reader to show that the partial sums of this rearrangement have limit superior y and limit inferior x.

8.19 SUBSERIES

Definition 8.34. *Let f be a function whose domain is $\mathbf{Z}^+$ and whose range is an infinite subset of $\mathbf{Z}^+$, and assume that f is one-to-one on $\mathbf{Z}^+$. Let $\sum a_n$ and $\sum b_n$ be two series such that*

$$b_n = a_{f(n)}, \qquad \text{if } n \in \mathbf{Z}^+.$$

Then $\sum b_n$ is said to be a subseries of $\sum a_n$.

Theorem 8.35. *If $\sum a_n$ converges absolutely, every subseries $\sum b_n$ also converges absolutely. Moreover, we have*

$$\left|\sum_{n=1}^{\infty} b_n\right| \le \sum_{n=1}^{\infty} |b_n| \le \sum_{n=1}^{\infty} |a_n|.$$

Proof. Given n, let N be the largest integer in the set $\{f(1), \ldots, f(n)\}$. Then

$$\left|\sum_{k=1}^{n} b_k\right| \le \sum_{k=1}^{n} |b_k| \le \sum_{k=1}^{N} |a_k| \le \sum_{k=1}^{\infty} |a_k|.$$

The inequality $\sum_{k=1}^{n} |b_k| \le \sum_{k=1}^{\infty} |a_k|$ implies absolute convergence of $\sum b_n$.

Theorem 8.36. *Let $\{f_1, f_2, \ldots\}$ be a countable collection of functions, each defined on $\mathbf{Z}^+$, having the following properties:*

a) *Each f_n is one-to-one on $\mathbf{Z}^+$.*
b) *The range $f_n(\mathbf{Z}^+)$ is a subset Q_n of $\mathbf{Z}^+$.*
c) *$\{Q_1, Q_2, \ldots\}$ is a collection of disjoint sets whose union is $\mathbf{Z}^+$.*

Let $\sum a_n$ be an absolutely convergent series and define

$$b_k(n) = a_{f_k(n)}, \qquad \text{if } n \in \mathbf{Z}^+, \quad k \in \mathbf{Z}^+.$$

Then:

i) *For each k, $\sum_{n=1}^{\infty} b_k(n)$ is an absolutely convergent subseries of $\sum a_n$.*
ii) *If $s_k = \sum_{n=1}^{\infty} b_k(n)$, the series $\sum_{k=1}^{\infty} s_k$ converges absolutely and has the same sum as $\sum_{k=1}^{\infty} a_k$.*

Proof. Theorem 8.35 implies (i). To prove (ii), let $t_k = |s_1| + \cdots + |s_k|$. Then

$$t_k \le \sum_{n=1}^{\infty} |b_1(n)| + \cdots + \sum_{n=1}^{\infty} |b_k(n)| = \sum_{n=1}^{\infty} (|b_1(n)| + \cdots + |b_k(n)|)$$
$$= \sum_{n=1}^{\infty} (|a_{f_1(n)}| + \cdots + |a_{f_k(n)}|).$$

But $\sum_{n=1}^{\infty} (|a_{f_1(n)}| + \cdots + |a_{f_k(n)}|) \le \sum_{n=1}^{\infty} |a_n|$. This proves that $\sum |s_k|$ has bounded partial sums and hence $\sum s_k$ converges absolutely.

To find the sum of $\sum s_k$, we proceed as follows: Let $\varepsilon > 0$ be given and choose N so that $n \ge N$ implies

Choose enough functions $f_1, \ldots, f_r$ so that each term $a_1, a_2, \ldots, a_N$ will appear somewhere in the sum

$$\sum_{n=1}^{\infty} a_{f_1(n)} + \cdots + \sum_{n=1}^{\infty} a_{f_r(n)}.$$

The number r depends on N and hence on ε. If $n > r$ and $n > N$, we have

$$\left|s_1 + s_2 + \cdots + s_n - \sum_{k=1}^{n} a_k\right| \le |a_{N+1}| + |a_{N+2}| + \cdots < \frac{\varepsilon}{2}, \qquad (19)$$

because the terms $a_1, a_2, \ldots, a_N$ cancel in the subtraction. Now (18) implies

$$\left|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k\right| < \frac{\varepsilon}{2}.$$

When this is combined with (19) we find

$$\left|s_1 + \cdots + s_n - \sum_{k=1}^{\infty} a_k\right| < \varepsilon,$$

if $n > r$, $n > N$. This completes the proof of (ii).

8.20 DOUBLE SEQUENCES

Definition 8.37. *A function f whose domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$ is called a double sequence.*

NOTE. We shall be interested only in real- or complex-valued double sequences.

Definition 8.38. *If $a \in \mathbf{C}$, we write $\lim_{p,q\to\infty} f(p, q) = a$ and we say that the double sequence f converges to a, provided that the following condition is satisfied: For every $\varepsilon > 0$, there exists an N such that $|f(p, q) - a| < \varepsilon$ whenever both $p > N$ and $q > N$.*

Theorem 8.39. *Assume that $\lim_{p,q\to\infty} f(p, q) = a$. For each fixed p, assume that the limit $\lim_{q\to\infty} f(p, q)$ exists. Then the limit $\lim_{p\to\infty} (\lim_{q\to\infty} f(p, q))$ also exists and has the value a.*

NOTE. To distinguish $\lim_{p,q\to\infty} f(p, q)$ from $\lim_{p\to\infty} (\lim_{q\to\infty} f(p, q))$, the first is called a *double limit*, the second an *iterated limit*.

Proof. Let $F(p) = \lim_{q\to\infty} f(p, q)$. Given $\varepsilon > 0$, choose N_1 so that

$$|f(p, q) - a| < \frac{\varepsilon}{2}, \qquad \text{if } p > N_1 \text{ and } q > N_1. \qquad (20)$$

For each p we can choose N_2 so that

$$|F(p) - f(p, q)| < \frac{\varepsilon}{2}, \qquad \text{if } q > N_2. \qquad (21)$$

(Note that N_2 depends on p as well as on ε.) For each $p > N_1$ choose N_2, and then choose a fixed q greater than both N_1 and N_2. Then both (20) and (21) hold and hence

$$|F(p) - a| < \varepsilon, \qquad \text{if } p > N_1.$$

Therefore, $\lim_{p\to\infty} F(p) = a$.

NOTE. A similar result holds if we interchange the roles of p and q.

Thus the existence of the double limit $\lim_{p,q\to\infty} f(p, q)$ and of $\lim_{q\to\infty} f(p, q)$ implies the existence of the iterated limit

$$\lim_{p\to\infty}\left(\lim_{q\to\infty} f(p, q)\right).$$

The following example shows that the converse is not true.

8.21 DOUBLE SERIES

Definition 8.40. *Let f be a double sequence and let s be the double sequence defined by the equation*

$$s(p, q) = \sum_{m=1}^{p} \sum_{n=1}^{q} f(m, n).$$

The pair (f, s) is called a double series and is denoted by the symbol $\sum_{m,n} f(m, n)$ or, more briefly, by $\sum f(m, n)$. The double series is said to converge to the sum a if

$$\lim_{p,q\to\infty} s(p, q) = a.$$

Each number $f(m, n)$ is called a *term* of the double series and each $s(p, q)$ is a partial sum. If $\sum f(m, n)$ has only positive terms, it is easy to show that it converges if, and only if, the set of partial sums is bounded. (See Exercise 8.29.) We say $\sum f(m, n)$ converges *absolutely* if $\sum |f(m, n)|$ converges. Theorem 8.18 is valid for double series. (See Exercise 8.29.)

8.24 MULTIPLICATION OF SERIES

Given two series $\sum a_n$ and $\sum b_n$, we can always form the double series $\sum f(m, n)$, where $f(m, n) = a_m b_n$. For every arrangement g of f into a sequence G, we are led to a further series $\sum G(n)$. By analogy with finite sums, it seems natural to refer to $\sum f(m, n)$ or to $\sum G(n)$ as the "product" of $\sum a_n$ and $\sum b_n$, and Theorem 8.44 justifies this terminology when the two given series $\sum a_n$ and $\sum b_n$ are absolutely convergent. However, if either $\sum a_n$ or $\sum b_n$ is *conditionally* convergent, we have no guarantee that either $\sum f(m, n)$ or $\sum G(n)$ will converge. Moreover, if one of them does converge, its sum need not be AB. The convergence and the sum will depend on the arrangement g. Different choices of g may yield different values of the product. There is one very important case in which the terms $f(m, n)$ are arranged "diagonally" to produce $\sum G(n)$, and then parentheses are inserted by grouping together those terms $a_m b_n$ for which $m + n$ has a fixed value. This product is called the *Cauchy product* and is defined as follows:

Definition 8.45. *Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, define*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \qquad \text{if } n = 0, 1, 2, \ldots \tag{22}$$

The series $\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of $\sum a_n$ and $\sum b_n$.

NOTE. The Cauchy product arises in a natural way when we multiply two power series. (See Exercise 8.33.)

Because of Theorems 8.44 and 8.13, absolute convergence of *both* $\sum a_n$ and $\sum b_n$ implies convergence of the Cauchy product to the value

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right). \tag{23}$$

This equation may fail to hold if both $\sum a_n$ and $\sum b_n$ are *conditionally* convergent. (See Exercise 8.32.) However, we can prove that (23) is valid if at least *one* of $\sum a_n$, $\sum b_n$ is absolutely convergent.

Theorem 8.46 (Mertens). *Assume that $\sum_{n=0}^{\infty} a_n$ converges absolutely and has sum A, and suppose $\sum_{n=0}^{\infty} b_n$ converges with sum B. Then the Cauchy product of these two series converges and has sum AB.*

Proof. Define $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$, where c_k is given by (22). Let $d_n = B - B_n$ and $e_n = \sum_{k=0}^{n} a_k d_{n-k}$. Then

$$C_p = \sum_{n=0}^{p} \sum_{k=0}^{n} a_k b_{n-k} = \sum_{n=0}^{p} \sum_{k=0}^{p} f_n(k), \tag{24}$$

where

$$f_n(k) = \begin{cases} a_k b_{n-k}, & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$

Then (24) becomes

$$C_p = \sum_{k=0}^{p} \sum_{n=0}^{p} f_n(k) = \sum_{k=0}^{p} \sum_{n=k}^{p} a_k b_{n-k} = \sum_{k=0}^{p} a_k \sum_{m=0}^{p-k} b_m = \sum_{k=0}^{p} a_k B_{p-k}$$

$$= \sum_{k=0}^{p} a_k(B - d_{p-k}) = A_p B - e_p.$$

To complete the proof, it suffices to show that $e_p \to 0$ as $p \to \infty$. The sequence $\{d_n\}$ converges to 0, since $B = \sum b_n$. Choose $M > 0$ so that $|d_n| \leq M$ for all n, and let $K = \sum_{n=0}^{\infty} |a_n|$. Given $\varepsilon > 0$, choose N so that $n > N$ implies $|d_n| < \varepsilon/(2K)$ and also so that

$$\sum_{n=N+1}^{\infty} |a_n| < \frac{\varepsilon}{2M}.$$

Then, for $p > 2N$, we can write

$$|e_p| \leq \sum_{k=0}^{N} |a_k d_{p-k}| + \sum_{k=N+1}^{p} |a_k d_{p-k}| \leq \frac{\varepsilon}{2K} \sum_{k=0}^{N} |a_k| + M \sum_{k=N+1}^{p} |a_k|$$

$$\leq \frac{\varepsilon}{2K} \sum_{k=0}^{\infty} |a_k| + M \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that $e_p \to 0$ as $p \to \infty$, and hence $C_p \to AB$ as $p \to \infty$.

A related theorem (due to Abel), in which no absolute convergence is assumed, will be proved in the next chapter. (See Theorem 9.32.)

Another product, known as the *Dirichlet product*, is of particular importance in the Theory of Numbers. We take $a_0 = b_0 = 0$ and, instead of defining c_n by (22), we use the formula

$$c_n = \sum_{d|n} a_d b_{n/d}, \qquad (n = 1, 2, \ldots), \tag{25}$$

where $\sum_{d|n}$ means a sum extended over all *positive divisors* of n (including 1 and n). For example, $c_6 = a_1 b_6 + a_2 b_3 + a_3 b_2 + a_6 b_1$, and $c_7 = a_1 b_7 + a_7 b_1$. The analog of Mertens' theorem holds also for this product. The Dirichlet product arises in a natural way when we multiply Dirichlet series. (See Exercise 8.34.)

8.25 CESARO SUMMABILITY

Definition 8.47. *Let s_n denote the nth partial sum of the series $\sum a_n$, and let $\{\sigma_n\}$ be the sequence of arithmetic means defined by*

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}, \qquad \text{if } n = 1, 2, \ldots \tag{26}$$

The series $\sum a_n$ is said to be Cesàro summable (or (C, 1) summable) if $\{\sigma_n\}$ converges. If $\lim_{n\to\infty} \sigma_n = S$, then S is called the Cesàro sum (or (C, 1) sum) of $\sum a_n$, and we write

$$\sum a_n = S \qquad (C, 1).$$

Example 1. Let $a_n = z^n$, $|z| = 1$, $z \neq 1$. Then

$$s_n = \frac{1}{1-z} - \frac{z^n}{1-z} \qquad \text{and} \qquad \sigma_n = \frac{1}{1-z} - \frac{1}{n}\frac{z(1-z^n)}{(1-z)^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z} \qquad (C, 1).$$

In particular,

$$\sum_{n=1}^{\infty} (-1)^{n-1} = \frac{1}{2} \qquad (C, 1).$$

Example 2. Let $a_n = (-1)^{n+1}n$. In this case,

$$\limsup_{n\to\infty} \sigma_n = \tfrac{1}{2}, \qquad \liminf_{n\to\infty} \sigma_n = 0,$$

and hence $\sum(-1)^{n+1}n$ is not (C, 1) summable.

Theorem 8.48. *If a series is convergent with sum S, then it is also* (C, 1) *summable with Cesàro sum S.*

Proof. Let s_n denote the nth partial sum of the series, define σ_n by (26), and introduce $t_n = s_n - S$, $\tau_n = \sigma_n - S$. Then we have

$$\tau_n = \frac{t_1 + \cdots + t_n}{n}, \tag{27}$$

and we must prove that $\tau_n \to 0$ as $n \to \infty$. Choose $A > 0$ so that each $|t_n| \le A$. Given $\varepsilon > 0$, choose N so that $n > N$ implies $|t_n| < \varepsilon$. Taking $n > N$ in (27), we obtain

$$|\tau_n| \le \frac{|t_1| + \cdots + |t_N|}{n} + \frac{|t_{N+1}| + \cdots + |t_n|}{n} < \frac{NA}{n} + \varepsilon.$$

Hence, $\limsup_{n\to\infty} |\tau_n| \le \varepsilon$. Since ε is arbitrary, it follows that $\lim_{n\to\infty} |\tau_n| = 0$.

NOTE. We have really proved that if a sequence $\{s_n\}$ converges, then the sequence $\{\sigma_n\}$ of arithmetic means also converges and, in fact, to the same limit.

Cesàro summability is just one of a large class of "summability methods" which can be used to assign a "sum" to an infinite series. Theorem 8.48 and Example 1 (following Definition 8.47) show us that Cesàro's method has a wider scope than ordinary convergence. The theory of summability methods is an important and fascinating subject, but one which we cannot enter into here. For an excellent account of the subject the reader is referred to Hardy's *Divergent Series* (Reference 8.1). We shall see later that (C, 1) summability plays an important role in the theory of Fourier series. (See Theorem 11.15.)

POSSIBLE QUESTIONS

1. Prove that if $\sum a_n$ converges to s, every series $\sum b_n$ obtained from $\sum a_n$ by inserting parentheses also converges to s.
2. State and prove Ratio Test Theorem.
3. Let $\sum a_n$ and b_n be two absolute converges series with sum a and b respectively. Let f be a double sequence defined by f (m,n) = $a_m\, b_n$ if f (m,n) $\in z^+ \times z^+$ then prove that $\sum_{m,n}$ f (m, n) converges absolutely and has sum $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ a_m b_n.
4. State and prove Dirchlet's test.
5. State and prove Rearrangement Theorem for double sequence .
6. Let $\sum a_n$ be an absolutely convergent series f having sum S then prove that every rearrangement of $\sum a_n$ also converges absolutely f has sum S.
7. State and Prove iterated limit Theorem.
8. Let $\sum a_n$ be a given series with real valued forms and define
   $P_n = \frac{|a_n| + a_n}{2}$; $q_n = \frac{|a_n| - a_n}{2}$ where n= 1,2,…… n then prove that
   (i) If $\sum a_n$ is conditionally convergent both $\sum P_n$ & $\sum q_n$ diverges.
   (ii) If $\sum |a_n|$ convergent both $\sum P_n$ & $\sum q_n$ converges and we have
   $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} P_n - \sum_{n=1}^{\infty} q_n$.
9. State and Prove Riemann theorem on conditionally convergent.
10. State and prove Merten's Theorem.
11. Let ($f_1, f_2, \ldots\ldots f_n$ } be a countable collection of function each defined on
    Z^+ having following properties
    (i) Each function f_n is one- one Z^+.
    (ii) the range fn(Z^+) is a subset Q_n of Z^+ .
    (iii){$Q_1, Q_2, \ldots\ldots Q_n$ } is the collection of disjoint sets whose union is Z^+.
    Let $\sum_{n=1}^{\infty} a_n$ be an absolute convergent series and define $b_k(n) = a_{f_k(n)}$ for k, n $\in Z^+$,then (i) for each fixed k the series $\sum_{n=1}^{\infty} b_k(n)$ is an absolutely convergent subseries of $\sum a_n$. (ii) if $s_k = \sum_{n=1}^{\infty} b_k(n)$, the series $\sum_{k=1}^{\infty} s_k$ k converges absolutely & has the sum same as $\sum_{k=1}^{\infty} a_k$.

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Subject: Real Analysis
Class : I - M.Sc. Mathematics

Unit II

Part A (20x1=20 Marks) **(Question**

Possible Questions

Question	**Choice 1**	**Choice 2**	**Choice 3**	**Choice 4**	
The coordinates (x,y) of a point represent an ------------ of numbers	function	relation	ordered pair	set	ordered pair
(a, b) = ----------------	{{a},{b},{a,b}}	{{a},{b}}	{{a},{a,b}}	{{a},{b},{}}	{{a},{a,b}}
(a, b) = (c, d) if and only if -------------	a = c & b = d	a = b & c = d	a = d & c = b	ab = cd	a = c & b = d
AxB denotes the ----------- of the sets A & B	product	cartesian product	polar form	complement	cartesian product
Any set of ordered pairs is called -------------	function	relation	ordered pair	set	relation
If S is a relation, the set of all elements that occur as first members in S is called the -----------	function	codomain	domain	range	domain
If S is a relation, the set of all elements that occur as second members in S is called the -----------	function	codomain	domain	range	range
If (x, y) belongs to F and (x, z) belongs to F, then -------------	x = z	x = y	xy = z	y = z	y = z
A mapping S into itself is called ------------	function	relation	domain	transformation	transformation
If F(x) = F(y) implies x =y is a --------------- function	one-one	onto	into	inverse	one-one
One-one function is also called -----------	injective	bijective	transformation	codomain	injective
S̆ **=** {(a,b) : (b,a) is in S} is called ---------------	inverse	domain	codomain	converse	converse
The composite functions are denoted by ---------------	GxF	GoF	GF	G + F	GoF
GoF(x) = ------------------	G[F(x)]	F[G(x)]	G(x)	F(x)	G[F(x)]
In general the composite function GoF is ------------	GoF = FoG	GoF is not equal to FoG	GoF < FoG	GoF > FoG	GoF is not equal to FoG
If m < n, then K(m) < K(n) implies that K is -------------	sequence	subsequence	order preserving	equinumerous	order preserving
Similar sets are also called as ------------ set	denumerable	uncountable	finite	equinumerous	equinumerous
If A and B are two sets andif there exists a one-one correspondence between them,then it is called ------------- set	denumerable	uncountable	finite	equinumerous	equinumerous
A set which is equinumerous with the set of all positive integers is called ---------------- set	finite	infinite	countably infinite	countably finite	countably infinite
A set which is either finite or countably infinite is called ------------ set	countable	uncountable	similar	equal	countable
Uncountable sets are also called ------------- set	denumerable	non-denumerable	similar	equal	non-denumerable
Countable sets are also called --------------- set	denumerable	non-denumerable	similar	equal	denumerable
Every subset of a countable set is ------------	countable	uncountable	rational	irrational	countable
The set of all real numbers is ----------------	countable	uncountable	rational	irrational	uncountable
The cartesian product of the set of all positive integers is ----------	countable	uncountable	rational	irrational	countable
The set of those elements which belong either to A or to B or to both is called ---------	complement	intersection	union	disjoint	union
The set of those elements which belong to both A and B is called ------------	complement	intersection	union	disjoint	intersection
Union of sets is --------------	commutative	not commutative	not associative	disjoint	commutative
The complement of A relative to B is denoted by --------------	B - A	B	A	A - B	B - A

If A intersection B is the empty set, then A and B are called --------	commutative	not commutative	not associative	disjoint	disjoint
B - (union A) = -----------------	union (B -A)	B - (intersection A)	intersection (B - A)	{}	intersection (B - A)
B - (intersection A) = -----------------------	union (B -A)	B - (union A)	intersection (B - A)	{}	union (B -A)
Union of countable sets is -----------------	uncountable	infinite	countable	disjoint	countable
The set of all rational numbers is ---------------	uncountable	infinite	countable	disjoint	countable
The set S of intervals with rational end points is ---------- set	uncountable	infinite	countable	disjoint	countable
A relation which is symmetric, reflexive and transitive is called ------------------ relation	equivalence	component	composite	countable	equivalence
Any collection of disjoint intervals of positive length is ------------	equivalence relation	countable set	composite function	uncountable set	countable set
If A similar to B and B similar to C, then --------------	C similar to A	A similar to C	A < C	A = C	A similar to C
If the root of an algebraic equation f(x) = 0, then the real number is called ----------	prime	positive	algebraic	composite	algebraic
For all subsets A and B of S with B contained in A, we have -------	f(A + B) = f(A)	f(A + B) = f(B)	f(A - B) = f(A) - f(B)	f(A - B) = f(A)	f(A - B) = f(A) - f(B)
If f(A U B) = f(A) + f(B), then the function f is called -------------	additive	multiplicative	disjoint	equinumerous	additive
f(A U B) = ----------------------	f(A) + f(B)	f(A) - f(B)	f(A) + f(B) - f(B - A)	f(A) + f(B - A)	f(A) + f(B - A)
The sequence < (-1) n > is	monotonically increasing	monotonically decreasing	either increasing or decreasing	neither monotonically increasing nor monotonically decreasing	neither monotonically increasing nor monotonically decreasing
An unbounded sequence	a limit point	does not have a limit point	may or may not have a limit point	unique limit point	may or may not have a limit point
Every absolutely convergent series is	convergent	conditionally convergent	absolutely divergent	need not convergent	convergent
The sequence { 1/n } is	convergent & bounded	divergent &unbounded	divergent & bounded	convergent & Unbounded	convergent & bounded
If a sequence {an}n=1 to ∞ converges to a real number then the given sequence is	unbounded sequence	convergent	divergent & bounded	bounded	unbounded sequence
Every subsequence has a ----------------	limit pount	convergent	monotonic subsequence	non monotoni sequence	monotonic subsequence
The series 1+ r+ r ^2 +r^3 +………… is oscillatory if	r=1	r= -1	r >1	r < 1	r= -1

UNIT III

SEQUENCES OF FUNCTIONS

SYLLABU

Basic definitions – Uniform convergence and continuity - Uniform convergence of infinite series of functions – Uniform convergence and Riemann – Stieltjes integration – Non uniformly convergent sequence – Uniform convergence and differentiation – Sufficient condition for uniform convergence of a series

9.4 UNIFORM CONVERGENCE AND CONTINUITY

Theorem 9.2. *Assume that $f_n \to f$ uniformly on S. If each f_n is continuous at a point c of S, then the limit function f is also continuous at c.*

NOTE. If c is an accumulation point of S, the conclusion implies that

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x).$$

Proof. If c is an isolated point of S, then f is automatically continuous at c. Suppose, then, that c is an accumulation point of S. By hypothesis, for every $\varepsilon > 0$ there is an M such that $n \geq M$ implies

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \text{for every } x \text{ in } S.$$

Since f_M is continuous at c, there is a neighborhood $B(c)$ such that $x \in B(c) \cap S$ implies

$$|f_M(x) - f_M(c)| < \frac{\varepsilon}{3}.$$

But

$$|f(x) - f(c)| \leq |f(x) - f_M(x)| + |f_M(x) - f_M(c)| + |f_M(c) - f(c)|.$$

If $x \in B(c) \cap S$, each term on the right is less than $\varepsilon/3$ and hence $|f(x) - f(c)| < \varepsilon$. This proves the theorem.

NOTE. Uniform convergence of $\{f_n\}$ is sufficient but not necessary to transmit continuity from the individual terms to the limit function. In Example 2 (Section 9.2), we have a nonuniformly convergent sequence of continuous functions with a continuous limit function.

9.6 UNIFORM CONVERGENCE OF INFINITE SERIES OF FUNCTIONS

Definition 9.4. *Given a sequence $\{f_n\}$ of functions defined on a set S. For each x in S, let*

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \qquad (n = 1, 2, \dots). \tag{4}$$

If there exists a function f such that $s_n \to f$ uniformly on S, we say the series $\sum f_n(x)$ converges uniformly on S and we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \qquad (\text{uniformly on } S).$$

Theorem 9.5 (Cauchy condition for uniform convergence of series). *The infinite series $\sum f_n(x)$ converges uniformly on S if, and only if, for every $\varepsilon > 0$ there is an N such that $n > N$ implies*

$$\left|\sum_{k=n+1}^{n+p} f_k(x)\right| < \varepsilon, \qquad \text{for each } p = 1, 2, \dots, \text{ and every } x \text{ in } S.$$

Proof. Define s_n by (4) and apply Theorem 9.3.

Theorem 9.6 (Weierstrass M-test). *Let $\{M_n\}$ be a sequence of nonnegative numbers such that*

$$0 \le |f_n(x)| \le M_n, \qquad \text{for } n = 1, 2, \dots, \text{ and for every } x \text{ in } S.$$

Then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

Proof. Apply Theorems 8.11 and 9.5 in conjunction with the inequality

$$\left|\sum_{k=n+1}^{n+p} f_k(x)\right| \le \sum_{k=n+1}^{n+p} M_k.$$

Theorem 9.7. *Assume that $\sum f_n(x) = f(x)$ (uniformly on S). If each f_n is continuous at a point x_0 of S, then f is also continuous at x_0.*

Proof. Define s_n by (4). Continuity of each f_n at x_0 implies continuity of s_n at x_0, and the conclusion follows at once from Theorem 9.2.

NOTE. If x_0 is an accumulation point of S, this theorem permits us to interchange limits and infinite sums, as follows:

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x).$$

9.8 UNIFORM CONVERGENCE AND RIEMANN–STIELTJES INTEGRATION

Theorem 9.8. *Let α be of bounded variation on $[a, b]$. Assume that each term of the sequence $\{f_n\}$ is a real-valued function such that $f_n \in R(\alpha)$ on $[a, b]$ for each $n = 1, 2, \ldots$ Assume that $f_n \to f$ uniformly on $[a, b]$ and define $g_n(x) = \int_a^x f_n(t)\, d\alpha(t)$ if $x \in [a, b]$, $n = 1, 2, \ldots$ Then we have:*

a) *$f \in R(\alpha)$ on $[a, b]$.*

b) *$g_n \to g$ uniformly on $[a, b]$, where $g(x) = \int_a^x f(t)\, d\alpha(t)$.*

NOTE. The conclusion implies that, for each x in $[a, b]$, we can write

$$\lim_{n\to\infty} \int_a^x f_n(t)\, d\alpha(t) = \int_a^x \lim_{n\to\infty} f_n(t)\, d\alpha(t).$$

This property is often described by saying that a uniformly convergent sequence can be integrated term by term.

Proof. We can assume that α is increasing with $\alpha(a) < \alpha(b)$. To prove (a), we will show that f satisfies Riemann's condition with respect to α on $[a, b]$. (See Theorem 7.19.)

Given $\varepsilon > 0$, choose N so that

$$|f(x) - f_N(x)| < \frac{\varepsilon}{3[\alpha(b) - \alpha(a)]}, \qquad \text{for all } x \text{ in } [a, b].$$

Then, for every partition P of $[a, b]$, we have

$$|U(P, f - f_N, \alpha)| \le \frac{\varepsilon}{3} \qquad \text{and} \qquad |L(P, f - f_N, \alpha)| \le \frac{\varepsilon}{3},$$

(using the notation of Definition 7.14). For this N, choose P_ε so that P finer than P_ε implies $U(P, f_N, \alpha) - L(P, f_N, \alpha) < \varepsilon/3$. Then for such P we have

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &\le U(P, f - f_N, \alpha) - L(P, f - f_N, \alpha) \\ &\quad + U(P, f_N, \alpha) - L(P, f_N, \alpha) \\ &< |U(P, f - f_N, \alpha)| + |L(P, f - f_N, \alpha)| + \frac{\varepsilon}{3} \le \varepsilon. \end{aligned}$$

This proves (a). To prove (b), let $\varepsilon > 0$ be given and choose N so that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]},$$

for all $n > N$ and every t in $[a, b]$. If $x \in [a, b]$, we have

$$|g_n(x) - g(x)| \le \int_a^x |f_n(t) - f(t)|\, d\alpha(t) \le \frac{\alpha(x) - \alpha(a)}{\alpha(b) - \alpha(a)} \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} < \varepsilon.$$

This proves that $g_n \to g$ uniformly on $[a, b]$.

Theorem 9.9. *Let α be of bounded variation on $[a, b]$ and assume that $\sum f_n(x) = f(x)$ (uniformly on $[a, b]$), where each f_n is a real-valued function such that $f_n \in R(\alpha)$ on $[a, b]$. Then we have:*

a) *$f \in R(\alpha)$ on $[a, b]$.*

b) $\int_a^x \sum_{n=1}^{\infty} f_n(t)\,d\alpha(t) = \sum_{n=1}^{\infty} \int_a^x f_n(t)\,d\alpha(t)$ (*uniformly on* $[a, b]$).

Proof. Apply Theorem 9.8 to the sequence of partial sums.

NOTE. This theorem is described by saying that a uniformly convergent series can be integrated term by term.

9.9 NONUNIFORMLY CONVERGENT SEQUENCES THAT CAN BE INTEGRATED TERM BY TERM

Uniform convergence is a sufficient but not a necessary condition for term-by-term integration, as is seen by the following example.

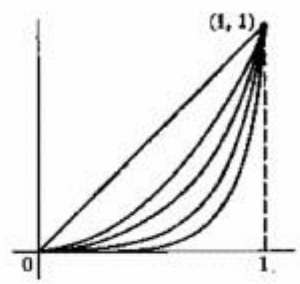

Figure 9.6

Example. Let $f_n(x) = x^n$ if $0 \le x \le 1$. (See Fig. 9.6.) The limit function f has the value 0 in $[0, 1)$ and $f(1) = 1$. Since this is a sequence of continuous functions with discontinuous limit, the convergence is not uniform on $[0, 1]$. Nevertheless, term-by-term integration on $[0, 1]$ leads to a correct result in this case. In fact, we have

$$\int_0^1 f_n(x)\,dx = \int_0^1 x^n\,dx = \frac{1}{n+1} \to 0 \text{ as } n \to \infty,$$

so $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx = 0$.

The sequence in the foregoing example, although not uniformly convergent on $[0, 1]$, is uniformly convergent on every closed subinterval of $[0, 1]$ not containing 1. The next theorem is a general result which permits term-by-term integration in examples of this type. The added ingredient is that we assume that $\{f_n\}$ is uniformly bounded on $[a, b]$ and that the limit function f is integrable.

Definition 9.10. *A sequence of functions* $\{f_n\}$ *is said to be boundedly convergent on* T *if* $\{f_n\}$ *is pointwise convergent and uniformly bounded on* T.

Theorem 9.11. *Let* $\{f_n\}$ *be a boundedly convergent sequence on* $[a, b]$. *Assume that each* $f_n \in R$ *on* $[a, b]$, *and that the limit function* $f \in R$ *on* $[a, b]$. *Assume also that there is a partition* P *of* $[a, b]$, *say*

$$P = \{x_0, x_1, \ldots, x_m\},$$

such that, on every subinterval $[c, d]$ *not containing any of the points* x_k, *the sequence* $\{f_n\}$ *converges uniformly to* f. *Then we have*

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt = \int_a^b f(t)\,dt. \tag{6}$$

Proof. Since f is bounded and $\{f_n\}$ is uniformly bounded, there is a positive number M such that $|f(x)| \le M$ and $|f_n(x)| \le M$ for all x in $[a, b]$ and all $n \ge 1$. Given $\varepsilon > 0$ such that $2\varepsilon < \|P\|$, let $h = \varepsilon/(2m)$, where m is the number of subintervals of P, and consider a new partition P' of $[a, b]$ given by

$$P' = \{x_0, x_0 + h, x_1 - h, x_1 + h, \ldots, x_{m-1} - h, x_{m-1} + h, x_m - h, x_m\}.$$

Since $|f - f_n|$ is integrable on $[a, b]$ and bounded by $2M$, the sum of the integrals of $|f - f_n|$ taken over the intervals

$$[x_0, x_0 + h],\ [x_1 - h, x_1 + h],\ \ldots,\ [x_{m-1} - h, x_{m-1} + h],\ [x_m - h, x_m],$$

is at most $2M(2mh) = 2M\varepsilon$. The remaining portion of $[a, b]$ (call it S) is the union of a finite number of closed intervals, in each of which $\{f_n\}$ is uniformly convergent to f. Therefore, there is an integer N (depending only on ε) such that for all x in S we have

$$|f(x) - f_n(x)| < \varepsilon \qquad \text{whenever } n \ge N.$$

Hence the sum of the integrals of $|f - f_n|$ over the intervals of S is at most $\varepsilon(b - a)$, so

$$\int_a^b |f(x) - f_n(x)|\, dx \le (2M + b - a)\varepsilon \qquad \text{whenever } n \ge N.$$

This proves that $\int_a^b f_n(x)\, dx \to \int_a^b f(x)\, dx$ as $n \to \infty$.

There is a stronger theorem due to Arzelà which makes no reference whatever to uniform convergence.

Theorem 9.12 (Arzelà). *Assume that $\{f_n\}$ is boundedly convergent on $[a,b]$ and suppose each f_n is Riemann-integrable on $[a, b]$. Assume also that the limit function f is Riemann-integrable on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx = \int_a^b f(x)\, dx. \tag{7}$$

The proof of Arzelà's theorem is considerably more difficult than that of Theorem 9.11 and will not be given here. In the next chapter we shall prove a theorem on Lebesgue integrals which includes Arzelà's theorem as a special case.

9.10 UNIFORM CONVERGENCE AND DIFFERENTIATION

By analogy with Theorems 9.2 and 9.8, one might expect the following result to hold: If $f_n \to f$ uniformly on $[a, b]$ and if f_n' exists for each n, then f' exists and $f_n' \to f'$ uniformly on $[a, b]$. However, Example 3 of Section 9.2 shows that this cannot be true. Although the sequence $\{f_n\}$ of Example 3 converges uniformly on $\mathbf{R}$, the sequence $\{f_n'\}$ does not even converge pointwise on $\mathbf{R}$. For example, $\{f_n'(0)\}$ diverges since $f_n'(0) = \sqrt{n}$. Therefore the analog of Theorems 9.2 and 9.8 for differentiation must take a different form.

Theorem 9.13. *Assume that each term of $\{f_n\}$ is a real-valued function having a finite derivative at each point of an open interval (a, b). Assume that for at least one point x_0 in (a, b) the sequence $\{f_n(x_0)\}$ converges. Assume further that there exists a function g such that $f_n' \to g$ uniformly on (a, b). Then:*

a) *There exists a function f such that $f_n \to f$ uniformly on (a, b).*

b) *For each x in (a, b) the derivative $f'(x)$ exists and equals $g(x)$.*

Proof. Assume that $c \in (a, b)$ and define a new sequence $\{g_n\}$ as follows:

$$g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c} & \text{if } x \neq c, \\ f_n'(c) & \text{if } x = c. \end{cases} \tag{8}$$

The sequence $\{g_n\}$ so formed depends on the choice of c. Convergence of $\{g_n(c)\}$ follows from the hypothesis, since $g_n(c) = f_n'(c)$. We will prove next that $\{g_n\}$ converges uniformly on (a, b). If $x \neq c$, we have

$$g_n(x) - g_m(x) = \frac{h(x) - h(c)}{x - c}, \tag{9}$$

where $h(x) = f_n(x) - f_m(x)$. Now $h'(x)$ exists for each x in (a, b) and has the value $f_n'(x) - f_m'(x)$. Applying the Mean-Value Theorem in (9), we get

$$g_n(x) - g_m(x) = f_n'(x_1) - f_m'(x_1), \tag{10}$$

where x_1 lies between x and c. Since $\{f_n'\}$ converges uniformly on (a, b) (by hypothesis), we can use (10), together with the Cauchy condition (Theorem 9.3), to deduce that $\{g_n\}$ converges uniformly on (a, b).

Now we can show that $\{f_n\}$ converges uniformly on (a, b). Let us form the particular sequence $\{g_n\}$ corresponding to the special point $c = x_0$ for which $\{f_n(x_0)\}$ is assumed to converge. From (8) we can write

$$f_n(x) = f_n(x_0) + (x - x_0)g_n(x),$$

an equation which holds for every x in (a, b). Hence we have

$$f_n(x) - f_m(x) = f_n(x_0) - f_m(x_0) + (x - x_0)[g_n(x) - g_m(x_0)].$$

This equation, with the help of the Cauchy condition, establishes the uniform convergence of $\{f_n\}$ on (a, b). This proves (a).

To prove (b), return to the sequence $\{g_n\}$ defined by (8) for an *arbitrary* point c in (a, b) and let $G(x) = \lim_{n\to\infty} g_n(x)$. The hypothesis that f_n' exists means that $\lim_{x\to c} g_n(x) = g_n(c)$. In other words, each g_n is continuous at c. Since $g_n \to G$ uniformly on (a, b), the limit function G is also continuous at c. This means that

$$G(c) = \lim_{x\to c} G(x), \tag{11}$$

the existence of the limit being part of the conclusion. But, for $x \neq c$, we have

$$G(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(c)}{x - c} = \frac{f(x) - f(c)}{x - c}.$$

Hence, (11) states that the derivative $f'(c)$ exists and equals $G(c)$. But

$$G(c) = \lim_{n\to\infty} g_n(c) = \lim_{n\to\infty} f_n'(c) = g(c);$$

hence $f'(c) = g(c)$. Since c is an arbitrary point of (a, b), this proves (b).

When we reformulate Theorem 9.13 in terms of series, we obtain

Theorem 9.14. *Assume that each f_n is a real-valued function defined on (a, b) such that the derivative $f_n'(x)$ exists for each x in (a, b). Assume that, for at least one point x_0 in (a, b), the series $\sum f_n(x_0)$ converges. Assume further that there exists a function g such that $\sum f_n'(x) = g(x)$ (uniformly on (a, b)). Then:*

a) *There exists a function f such that $\sum f_n(x) = f(x)$ (uniformly on (a, b)).*
b) *If $x \in (a, b)$, the derivative $f'(x)$ exists and equals $\sum f_n'(x)$.*

9.11 SUFFICIENT CONDITIONS FOR UNIFORM CONVERGENCE OF A SERIES

The importance of uniformly convergent series has been amply illustrated in some of the preceding theorems. Therefore it seems natural to seek some simple ways of testing a series for uniform convergence without resorting to the definition in each case. One such test, the *Weierstrass M-test*, was described in Theorem 9.6. There are other tests that may be useful when the M-test is not applicable. One of these is the analog of Theorem 8.28.

Theorem 9.15 (Dirichlet's test for uniform convergence). *Let $F_n(x)$ denote the nth partial sum of the series $\sum f_n(x)$, where each f_n is a complex-valued function defined on a set S. Assume that $\{F_n\}$ is uniformly bounded on S. Let $\{g_n\}$ be a sequence of real-valued functions such that $g_{n+1}(x) \le g_n(x)$ for each x in S and for every $n = 1, 2, \ldots$, and assume that $g_n \to 0$ uniformly on S. Then the series $\sum f_n(x)g_n(x)$ converges uniformly on S.*

Proof. Let $s_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$. By partial summation we have

$$s_n(x) = \sum_{k=1}^{n} F_k(x)(g_k(x) - g_{k+1}(x)) + g_{n+1}(x)F_n(x),$$

and hence if $n > m$, we can write

$$s_n(x) - s_m(x) = \sum_{k=m+1}^{n} F_k(x)(g_k(x) - g_{k+1}(x)) + g_{n+1}(x)F_n(x) - g_{m+1}(x)F_m(x).$$

Therefore, if M is a uniform bound for $\{F_n\}$, we have

$$\begin{aligned}|s_n(x) - s_m(x)| &\le M \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) + Mg_{n+1}(x) + Mg_{m+1}(x)\\ &= M(g_{m+1}(x) - g_{n+1}(x)) + Mg_{n+1}(x) + Mg_{m+1}(x)\\ &= 2Mg_{m+1}(x).\end{aligned}$$

Since $g_n \to 0$ uniformly on S, this inequality (together with the Cauchy condition) implies that $\sum f_n(x)g_n(x)$ converges uniformly on S.

The reader should have no difficulty in extending Theorem 8.29 (Abel's test) in a similar way so that it yields a test for uniform convergence. (Exercise 9.13.)

Example. Let $F_n(x) = \sum_{k=1}^{n} e^{ikx}$. In the last chapter (see Theorem 8.30), we derived the inequality $|F_n(x)| \le 1/|\sin(x/2)|$, valid for every real $x \ne 2m\pi$ (m is an integer). Therefore, if $0 < \delta < \pi$, we have the estimate

$$|F_n(x)| \le 1/\sin(\delta/2) \qquad \text{if } \delta \le x \le 2\pi - \delta.$$

Hence, $\{F_n\}$ is uniformly bounded on the interval $[\delta, 2\pi - \delta]$. If $\{g_n\}$ satisfies the conditions of Theorem 9.15, we can conclude that the series $\sum g_n(x)e^{inx}$ converges uniformly on $[\delta, 2\pi - \delta]$. In particular, if we take $g_n(x) = 1/n$, this establishes the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n}$$

on $[\delta, 2\pi - \delta]$ if $0 < \delta < \pi$. Note that the Weierstrass M-test cannot be used to establish uniform convergence in this case, since $|e^{inx}| = 1$.

POSSIBLE QUESTIONS

1. Assume that $\lim_{n\to\infty} f_n = f$ on [a,b].if $g \in R$ on [a,b]. define $h(x) = \int_a^b f(t)g(t)dt$ and $h_n(x) = \int_a^b f_n(t)g(t)dt$ if $x \in [a,b]$ then prove that $h_n \to h$ uniformly on [a,b].

2. State and prove uniform converges and double sequences.

3. State and prove Cauchy's condition for Uniform convergence .

4. Assume that $\sum f_n(x) = f(x)$ (Uniformly continuous on S) if each f_n is continuous at a point x_0 of S then prove that f is also continuous at x_0.

5. State and Prove Cauchys condition for uniform converges theorem .

6. Assume that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ on [a,b];define $h(x) = \int_a^x f(t)g(t)dt$ & $h_n(x) = \int_a^b f_n(t)\, g_n(t)\, dt$ if $x \in [a,b]$ then prove that $h_n \to h$ uniformly on [a,b].

7. Assume that each term of a sequence $\{ f_n \}$ is a real valued function having a finite derivative at each point of a $_n$ on [a,b].Assume that for atleast one point x_0 in [a,b].the sequence $\{ f_n(x_0) \}$ converges , assume further that ∃ a function G such that $f_n' \to G$ uniformly on [a,b]
   (i) ∃ a function f such that $f_n \to f$ uniformly on [a,b] .
   (ii) for each x in [a,b] the derivative $f'(x)$ exists and equals G (x)

8. Let α be bounded variation on [a,b]. Assume that each term of the sequence $\{ f_n \}$ is a real valued function such that $f_n \in R(\alpha)$ on [a,b] for each n = 1,2,… Assume that $f_n \to f$ uniformly on [a,b] and define $g_n(x) = \int_a^x f_n(t)d\alpha(t)$ n= 1,2,…. Then prove that (i) $f \in R(\alpha)$ on [a,b] (ii) $g_n \to g$ uniformly on [a,b]. where $g(x) = \int_a^x f(t)d(\alpha)(t)$.

9. Let α be of bounded variation on [a,b] & assume that $\sum f_n(x) = f(x)$ where each f_n is a real valued function such that $f_n \in R(\alpha)$ on [a,b] then prove that
   (i) $f \in R(\alpha)$ on[a,b] (ii) $\int_a^x \sum_{n=1}^{\infty} f_n(t)d\alpha(t) = \sum_{n=1}^{\infty} \int_a^x f_n(t)d\alpha(t)$

10. Let $f_n \to f$ uniformly on S. If each f_n is continuous at a point c of S, then prove that the limit function f is also continuous at c
11. State and prove Arzela Theorem.

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Subject: Real Analysis **Subject**

Class : I - M.Sc. Mathematics

Unit III

Part A (20x1=20 Marks) **(Question Nos. 1 to 20**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A collection of well defined object is called	set	uniformly bounded	convergent	point wise convergent and uniformly bounded	point wise convergent and uniformly bounded
A Sequence of functions is said to boundadly convergent on T is seq	point wise convergent	it contains no limit points	it contains some limit points	it contains infinite limit points	it contains all of its limit points
A set Fis closed if	it contains all of its limit points	uncountable collection of	countable collection of	uncountable collection of	countable collection of
Every open set of real numbers is the union of	countable collection of	open intervals	closed intervals	closed intervals	open intervals
	open intervals	a non-prime number and n >1	a non-prime number and n <1	a prime number and n <1	countable
composite number n is	a prime number	Uncountable	infinite	finite	
The union of a finite or collection of countable sets is	countable	a does not belongs to S	a is not lower bound of S	a is not upper bound of S	
An element a is an minimal element of set S,then	a belongs to S	n <x	n=x	n ≠ x	
For every real number x,there is a positive integer n such that	n>x	uncountable subset	proper subset	improper subset	countable subset
Every infinite set has a	countable subset	finite	countable	uncountable	finite
Set of real numbers is bounded above is Sup S	infinite	minimal element only	maximal and minimal	no maximal no minimal	minimal element only
The half interval [0,1) have	maximal element	finite	countable	uncountable	infinite
Set of real numbers is unbounded above is Sup S	infinite	closed	semi open	semi closed	closed
The arbitrary intersection of closed set is	open	a singleton set	a finite set	not a well defined set	not a well defined set
The set of intelligent student in a class is	a null set	sum of prime numbers	product of prime numbers	prime numbers or a product of prime numbers	prime numbers or a product of prime numbers
Every integer n>1 is	prime numbers	non ordered set	set of irrational numbers	does not satisfies principle induction	ordered set
The set of integer is	ordered set	unbounded below	unbounded above	no maximal	bounded above
The closed interval S= [0,1] is	bounded above	1	2	empty	1
If S is a set of real numbers which is bounded below then inf S is	a point of closure to S	closed set	uncountable set	countable set	closed set
If E is a nonemptyset then	inf E< sup E	∞	∞ (negative)	no infimum	∞ (negative)
If R is a extended real number system then inf R is	0	1	0	2	-1
The set of negative integers having least upper bound is	-1	closure of E contains non empty opensets	closure of E contains empty opensets	closure of E contains no non empty closedsets	closure of E contains no non empty opensets
Let S =[0,1) the maximal elementof S is	ϕ	its complement is closed set	its complement is null set	its complement is semiclosed set	its complement is closed set
the intersection finite collection of open set is	open set	finite set	unbounded set	unbounded set	open set
The set of real numbers is	unbounded	closed set	empty set	non empty set	closed set
The intersection of any collection of closed set is	open set	limit point	infinite limit point	finite limit point	limit point
An infinite set must possess a	does not have a	open intervals	open	closed intervals	open
single ton set { x } is	closed	open set	{ 0}	ϕ	open set
Every bounded infinite set has	smallest limit	countable	finite	infinite	countable
The set of all integers is	uncountable	countable	finite	infinite	countable
The cartesian product of two countable set is	uncountable	E " is null	E " is open	E " is closed	E " is closed
Let E " is the set of point of closure of E	E " is closed	semi open	closed intervals	open intervals	open set
Null set	open set	bounded above by 0 & minimal element is 0	bounded below by 1 & no maximal element	bounded above by 1 & maximal element is 1	bounded above by 1 & maximal element is 1
S=(0 , 1] is	bounded above by 1 & maximal	A - B is non empty set	A - B is closed set	A - B is empty set	A - B is open set

If A is open set and B is closed set then	A - B is open set	may be closed set	may not always be a closed set	open set	may not always be a closed set
The union of an arbitrary family of closed set	closed set	equal sequence	range set of a	null sequence	range set of a
The set of all distict element of a sequence is called	constant sequence	bounded below	bounded	neither bounded above nor bounded below	neither bounded above nor bounded below
A bounded sequence	converges	one limit	many limit	no limit point	more than one
A sequence can not converge to	more than one limit	must a member of the sequence	need not be a member of the	not a member of the sequence	need not be a member of the
limit point of a sequence	member of the sequence	no limit point	a limit point	more than two limit point	a limit point
Everu bounded real sequence has	many limit point				

UNIT IV
The Lebesgue integral: SYLLABUS

Introduction- The class of Lebesgue – integrable functions on a general interval- Basic properties of the Lebesgue integral- Lebesgue integration and sets of measure zero- The Levi monotone convergence theorem- The Lebesgue dominated convergence theorem-Applications of Lebesgue dominated convergence theorem- Lebesgue integrals on unbounded intervals as limit of integrals on bounded intervals- Improper Riemann integrals- Measurable functions.

10.6 THE CLASS OF LEBESGUE-INTEGRABLE FUNCTIONS ON A GENERAL INTERVAL

If u and v are upper functions, the difference $u - v$ is not necessarily an upper function. We eliminate this undesirable property by enlarging the class of integrable functions.

Definition 10.12. *We denote by $L(I)$ the set of all functions f of the form $f = u - v$, where $u \in U(I)$ and $v \in U(I)$. Each function f in $L(I)$ is said to be Lebesgue-integrable on I, and its integral is defined by the equation*

$$\int_I f = \int_I u - \int_I v. \tag{7}$$

If $f \in L(I)$ it is possible to write f as a difference of two upper functions $u - v$ in more than one way. The next theorem shows that the integral of f is independent of the choice of u and v.

Theorem 10.13. *Let u, v, u_1, and v_1 be functions in $U(I)$ such that $u - v = u_1 - v_1$. Then*

$$\int_I u - \int_I v = \int_I u_1 - \int_I v_1. \tag{8}$$

Proof. The functions $u + v_1$ and $u_1 + v$ are in $U(I)$ and $u + v_1 = u_1 + v$. Hence, by Theorem 10.6(a), we have $\int_I u + \int_I v_1 = \int_I u_1 + \int_I v$, which proves (8).

NOTE. If the interval I has endpoints a and b in the extended real number system $\mathbf{R}^*$, where $a \le b$, we also write

$$\int_a^b f \qquad \text{or} \qquad \int_a^b f(x)\,dx$$

for the Lebesgue integral $\int_I f$. We also define $\int_b^a f = -\int_a^b f$.

If $[a, b]$ is a compact interval, every function which is Riemann-integrable on $[a, b]$ is in $U([a, b])$ and therefore also in $L([a, b])$.

10.7 BASIC PROPERTIES OF THE LEBESGUE INTEGRAL

Theorem 10.14. *Assume $f \in L(I)$ and $g \in L(I)$. Then we have:*

a) $(af + bg) \in L(I)$ *for every real a and b, and*

$$\int_I (af + bg) = a \int_I f + b \int_I g.$$

b) $\int_I f \ge 0$ *if $f(x) \ge 0$ a.e. on I.*

c) $\int_I f \ge \int_I g$ *if $f(x) \ge g(x)$ a.e. on I.*

d) $\int_I f = \int_I g$ *if $f(x) = g(x)$ a.e. on I.*

Proof. Part (a) follows easily from Theorem 10.6. To prove (b) we write $f = u - v$, where $u \in U(I)$ and $v \in U(I)$. Then $u(x) \ge v(x)$ almost everywhere on I so, by Theorem 10.6(c), we have $\int_I u \ge \int_I v$ and hence

$$\int_I f = \int_I u - \int_I v \ge 0.$$

Part (c) follows by applying (b) to $f - g$, and part (d) follows by applying (c) twice.

Definition 10.15. *If f is a real-valued function, its positive part, denoted by f^+, and its negative part, denoted by f^-, are defined by the equations*

$$f^+ = \max(f, 0), \qquad f^- = \max(-f, 0).$$

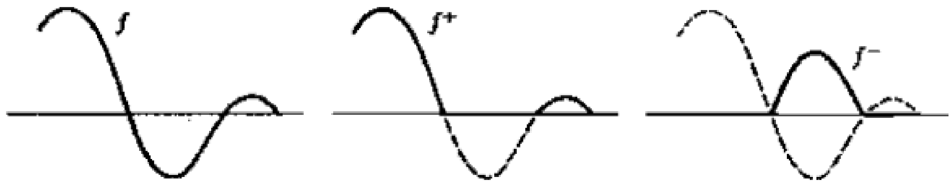

Figure 10.1

Note that f^+ and f^- are nonnegative functions and that

$$f = f^+ - f^-, \qquad |f| = f^+ + f^-.$$

Theorem 10.16. *If f and g are in $L(I)$, then so are the functions f^+, f^-, $|f|$,* max (f, g) *and* min (f, g). *Moreover, we have*

$$\left|\int_I f\right| \le \int_I |f|. \tag{9}$$

Proof. Write $f = u - v$, where $u \in U(I)$ and $v \in U(I)$. Then

$$f^+ = \max(u - v, 0) = \max(u, v) - v.$$

But $\max(u, v) \in U(I)$, by Theorem 10.9, and $v \in U(I)$, so $f^+ \in L(I)$. Since $f^- = f^+ - f$, we see that $f^- \in L(I)$. Finally, $|f| = f^+ + f^-$, so $|f| \in L(I)$.

Since $-|f(x)| \le f(x) \le |f(x)|$ for all x in I we have

$$-\int_I |f| \le \int_I f \le \int_I |f|,$$

which proves (9). To complete the proof we use the relations

$$\max(f, g) = \tfrac{1}{2}(f + g + |f - g|), \qquad \min(f, g) = \tfrac{1}{2}(f + g - |f - g|).$$

The next theorem describes the behavior of a Lebesgue integral when the interval of integration is translated, expanded or contracted, or reflected through the origin. We use the following notation, where c denotes any real number:

$$I + c = \{x + c : x \in I\}, \qquad cI = \{cx : x \in I\}.$$

Theorem 10.17. *Assume $f \in L(I)$. Then we have:*

a) *Invariance under translation. If $g(x) = f(x - c)$ for x in $I + c$, then $g \in L(I + c)$, and*

$$\int_{I+c} g = \int_I f.$$

b) *Behavior under expansion or contraction. If $g(x) = f(x/c)$ for x in cI, where $c > 0$, then $g \in L(cI)$ and*

$$\int_{cI} g = c \int_I f.$$

c) *Invariance under reflection. If $g(x) = f(-x)$ for x in $-I$, then $g \in L(-I)$ and*

$$\int_{-I} g = \int_I f.$$

NOTE. If I has endpoints $a < b$, where a and b are in the extended real number system $\mathbf{R}^*$, the formula in (a) can also be written as follows:

$$\int_{a+c}^{b+c} f(x - c)\, dx = \int_a^b f(x)\, dx.$$

Properties (b) and (c) can be combined into a single formula which includes both positive and negative values of c:

$$\int_{ca}^{cb} f(x/c)\, dx = |c| \int_a^b f(x)\, dx \qquad \text{if } c \neq 0.$$

Proof. In proving a theorem of this type, the procedure is always the same. First, we verify the theorem for step functions, then for upper functions, and finally for Lebesgue-integrable functions. At each step the argument is straightforward, so we omit the details.

Theorem 10.18. *Let I be an interval which is the union of two subintervals, say $I = I_1 \cup I_2$, where I_1 and I_2 have no interior points in common.*

a) *If $f \in L(I)$, then $f \in L(I_1)$, $f \in L(I_2)$, and*

$$\int_I f = \int_{I_1} f + \int_{I_2} f.$$

b) *Assume $f_1 \in L(I_1)$, $f_2 \in L(I_2)$, and let f be defined on I as follows:*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in I_1, \\ f_2(x) & \text{if } x \in I - I_1. \end{cases}$$

Then $f \in L(I)$ and $\int_I f = \int_{I_1} f_1 + \int_{I_2} f_2$.

Proof. Write $f = u - v$ where $u \in U(I)$ and $v \in U(I)$. Then $u = u^+ - u^-$ and $v = v^+ - v^-$, so $f = u^+ + v^- - (u^- + v^+)$. Now apply Theorem 10.10 to each of the nonnegative functions $u^+ + v^-$ and $u^- + v^+$ to deduce part (a). The proof of part (b) is left to the reader.

NOTE. There is an extension of Theorem 10.18 for an interval which can be expressed as the union of a finite number of subintervals, no two of which have interior points in common. The reader can formulate this for himself.

We conclude this section with two approximation properties that will be needed later. The first tells us that every Lebesgue-integrable function f is equal to an upper function u minus a nonnegative upper function v with a small integral. The second tells us that f is equal to a step function s plus an integrable function

g with a small integral. More precisely, we have:

Theorem 10.19. *Assume $f \in L(I)$ and let $\varepsilon > 0$ be given. Then:*

a) *There exist functions u and v in $U(I)$ such that $f = u - v$, where v is nonnegative a.e. on I and $\int_I v < \varepsilon$.*

b) *There exists a step function s and a function g in $L(I)$ such that $f = s + g$, where $\int_I |g| < \varepsilon$.*

Proof. Since $f \in L(I)$, we can write $f = u_1 - v_1$ where u_1 and v_1 are in $U(I)$. Let $\{t_n\}$ be a sequence which generates v_1. Since $\int_I t_n \to \int_I v_1$, we can choose N so that $0 \leq \int_I (v_1 - t_N) < \varepsilon$. Now let $v = v_1 - t_N$ and $u = u_1 - t_N$. Then both u and v are in $U(I)$ and $u - v = u_1 - v_1 = f$. Also, v is nonnegative *a.e.* on I and $\int_I v < \varepsilon$. This proves (a).

To prove (b) we use (a) to choose u and v in $U(I)$ so that $v \geq 0$ *a.e.* on I,

$$f = u - v \qquad \text{and} \qquad 0 \leq \int_I v < \frac{\varepsilon}{2}.$$

Now choose a step function s such that $0 \leq \int_I (u - s) < \varepsilon/2$. Then

$$f = u - v = s + (u - s) - v = s + g,$$

where $g = (u - s) - v$. Hence $g \in L(I)$ and

$$\int_I |g| \leq \int_I |u - s| + \int_I |v| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

10.8 LEBESGUE INTEGRATION AND SETS OF MEASURE ZERO

The theorems in this section show that the behavior of a Lebesgue-integrable function on a set of measure zero does not affect its integral.

Theorem 10.20. *Let f be defined on I. If $f = 0$ almost everywhere on I, then* $f \in L(I)$ and $\int_I f = 0$.

Proof. Let $s_n(x) = 0$ for all x in I. Then $\{s_n\}$ is an increasing sequence of step functions which converges to 0 everywhere on I. Hence $\{s_n\}$ converges to f almost everywhere on I. Since $\int_I s_n = 0$ the sequence $\{\int_I s_n\}$ converges. Therefore f is an upper function, so $f \in L(I)$ and $\int_I f = \lim_{n\to\infty} \int_I s_n = 0$.

Theorem 10.21. *Let f and g be defined on I. If $f \in L(I)$ and if $f = g$ almost everywhere on I, then $g \in L(I)$ and $\int_I f = \int_I g$.*

Proof. Apply Theorem 10.20 to $f - g$. Then $f - g \in L(I)$ and $\int_I (f - g) = 0$. Hence $g = f - (f - g) \in L(I)$ and $\int_I g = \int_I f - \int_I (f - g) = \int_I f$.

Example. Define f on the interval $[0, 1]$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then $f = 0$ almost everywhere on $[0, 1]$ so f is Lebesgue-integrable on $[0, 1]$ and its Lebesgue integral is 0. As noted in Chapter 7, this function is not Riemann-integrable on $[0, 1]$.

NOTE. Theorem 10.21 suggests a definition of the integral for functions that are defined almost everywhere on I. If g is such a function and if $g(x) = f(x)$ almost everywhere on I, where $f \in L(I)$, we say that $g \in L(I)$ and that

$$\int_I g = \int_I f.$$

10.9 THE LEVI MONOTONE CONVERGENCE THEOREMS

We turn next to convergence theorems concerning term-by-term integration of monotonic sequences of functions. We begin with three versions of a famous theorem of Beppo Levi. The first concerns sequences of step functions, the second sequences of upper functions, and the third sequences of Lebesgue-integrable functions. Although the theorems are stated for increasing sequences, there are corresponding results for decreasing sequences.

Theorem 10.22 (Levi theorem for step functions). *Let $\{s_n\}$ be a sequence of step functions such that*

a) *$\{s_n\}$ increases on an interval I, and*

b) *$\lim_{n\to\infty} \int_I s_n$ exists.*

Then $\{s_n\}$ converges almost everywhere on I to a limit function f in $U(I)$, and

$$\int_I f = \lim_{n\to\infty} \int_I s_n.$$

Proof. We can assume, without loss of generality, that the step functions s_n are nonnegative. (If not, consider instead the sequence $\{s_n - s_1\}$. If the theorem is true for $\{s_n - s_1\}$, then it is also true for $\{s_n\}$.) Let D be the set of x in I for which $\{s_n(x)\}$ diverges, and let $\varepsilon > 0$ be given. We will prove that D has measure 0 by showing that D can be covered by a countable collection of intervals, the sum of whose lengths is $< \varepsilon$.

Since the sequence $\{\int_I s_n\}$ converges it is bounded by some positive constant M. Let

$$t_n(x) = \left[\frac{\varepsilon}{2M} s_n(x)\right] \quad \text{if } x \in I,$$

where $[y]$ denotes the greatest integer $\le y$. Then $\{t_n\}$ is an increasing sequence of step functions and each function value $t_n(x)$ is a nonnegative integer.

If $\{s_n(x)\}$ converges, then $\{s_n(x)\}$ is bounded so $\{t_n(x)\}$ is bounded and hence $t_{n+1}(x) = t_n(x)$ for all sufficiently large n, since each $t_n(x)$ is an integer.

If $\{s_n(x)\}$ diverges, then $\{t_n(x)\}$ also diverges and $t_{n+1}(x) - t_n(x) \ge 1$ for infinitely many values of n. Let

$$D_n = \{x : x \in I \quad \text{and} \quad t_{n+1}(x) - t_n(x) \ge 1\}.$$

Then D_n is the union of a finite number of intervals, the sum of whose lengths we denote by $|D_n|$. Now

$$D \subseteq \bigcup_{n=1}^{\infty} D_n,$$

so if we prove that $\sum_{n=1}^{\infty} |D_n| < \varepsilon$, this will show that D has measure 0.

To do this we integrate the nonnegative step function $t_{n+1} - t_n$ over I and obtain the inequalities

$$\int_I (t_{n+1} - t_n) \geq \int_{D_n} (t_{n+1} - t_n) \geq \int_{D_n} 1 = |D_n|.$$

Hence for every $m \geq 1$ we have

$$\sum_{n=1}^{m} |D_n| \leq \sum_{n=1}^{m} \int_I (t_{n+1} - t_n) = \int_I t_{m+1} - \int_I t_1 \leq \int_I t_{m+1} \leq \frac{\varepsilon}{2M} \int_I s_{m+1} \leq \frac{\varepsilon}{2}.$$

Therefore $\sum_{n=1}^{\infty} |D_n| \leq \varepsilon/2 < \varepsilon$, so D has measure 0.

This proves that $\{s_n\}$ converges almost everywhere on I. Let

$$f(x) = \begin{cases} \lim_{n\to\infty} s_n(x) & \text{if } x \in I - D, \\ 0 & \text{if } x \in D. \end{cases}$$

Then f is defined everywhere on I and $s_n \to f$ almost everywhere on I. Therefore, $f \in U(I)$ and $\int_I f = \lim_{n\to\infty} \int_I s_n$.

Theorem 10.23 (Levi theorem for upper functions). *Let $\{f_n\}$ be a sequence of upper functions such that*

a) *$\{f_n\}$ increases almost everywhere on an interval I,*

and

b) $\lim_{n\to\infty} \int_I f_n$ *exists.*

Then $\{f_n\}$ converges almost everywhere on I to a limit function f in $U(I)$, and

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Proof. For each k there is an increasing sequence of step functions $\{s_{n,k}\}$ which generates f_k. Define a new step function t_n on I by the equation

$$t_n(x) = \max\{s_{n,1}(x), s_{n,2}(x), \ldots, s_{n,n}(x)\}.$$

Then $\{t_n\}$ is increasing on I because

$$\begin{aligned} t_{n+1}(x) &= \max\{s_{n+1,1}(x), \ldots, s_{n+1,n+1}(x)\} \geq \max\{s_{n,1}(x), \ldots, s_{n,n+1}(x)\} \\ &\geq \max\{s_{n,1}(x), \ldots, s_{n,n}(x)\} = t_n(x). \end{aligned}$$

But $s_{n,k}(x) \leq f_k(x)$ and $\{f_k\}$ increases almost everywhere on I, so we have

$$t_n(x) \leq \max\{f_1(x), \ldots, f_n(x)\} = f_n(x) \tag{10}$$

almost everywhere on I. Therefore, by Theorem 10.6(c) we obtain

$$\int_I t_n \leq \int_I f_n. \tag{11}$$

But, by (b), $\{\int_I f_n\}$ is bounded above so the increasing sequence $\{\int_I t_n\}$ is also bounded above and hence converges. By the Levi theorem for step functions, $\{t_n\}$ converges almost everywhere on I to a limit function f in $U(I)$, and $\int_I f = \lim_{n\to\infty} \int_I t_n$. We prove next that $f_n \to f$ almost everywhere on I.

The definition of $t_n(x)$ implies $s_{n,k}(x) \leq t_n(x)$ for all $k \leq n$ and all x in I. Letting $n \to \infty$ we find

$$f_k(x) \leq f(x) \quad \text{almost everywhere on } I. \tag{12}$$

Therefore the increasing sequence $\{f_k(x)\}$ is bounded above by $f(x)$ almost everywhere on I, so it converges almost everywhere on I to a limit function g satisfying $g(x) \leq f(x)$ almost everywhere on I. But (10) states that $t_n(x) \leq f_n(x)$ almost everywhere on I so, letting $n \to \infty$, we find $f(x) \leq g(x)$ almost everywhere on I. In other words, we have

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{almost everywhere on } I.$$

Finally, we show that $\int_I f = \lim_{n\to\infty} \int_I f_n$. Letting $n \to \infty$ in (11) we obtain

$$\int_I f \leq \lim_{n \to \infty} \int_I f_n. \tag{13}$$

Now integrate (12), using Theorem 10.6(c) again, to get $\int_I f_k \leq \int_I f$. Letting $k \to \infty$ we obtain $\lim_{k\to\infty} \int_I f_k \leq \int_I f$ which, together with (13), completes the proof.

NOTE. The class $U(I)$ of upper functions was constructed from the class $S(I)$ of step functions by a certain process which we can call P. Beppo Levi's theorem shows that when process P is applied to $U(I)$ it again gives functions in $U(I)$. The next theorem shows that when P is applied to $L(I)$ it again gives functions in $L(I)$.

Theorem 10.24 (Levi theorem for sequences of Lebesgue-integrable functions). *Let $\{f_n\}$ be a sequence of functions in $L(I)$ such that*

a) *$\{f_n\}$ increases almost everywhere on I,*

and

b) *$\lim_{n\to\infty} \int_I f_n$ exists.*

Then $\{f_n\}$ converges almost everywhere on I to a limit function f in $L(I)$, and

$$\int_I f = \lim_{n \to \infty} \int_I f_n.$$

We shall deduce this theorem from an equivalent result stated for *series* of functions.

Theorem 10.25 (Levi theorem for series of Lebesgue-integrable functions). *Let $\{g_n\}$ be a sequence of functions in $L(I)$ such that*

a) *each g_n is nonnegative almost everywhere on I,*

and

b) *the series $\sum_{n=1}^{\infty} \int_I g_n$ converges.*

Then the series $\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a sum function g in $L(I)$, and we have

$$\int_I g = \int_I \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int_I g_n. \tag{14}$$

Proof. Since $g_n \in L(I)$, Theorem 10.19 tells us that for every $\varepsilon > 0$ we can write

$$g_n = u_n - v_n,$$

where $u_n \in U(I)$, $v_n \in U(I)$, $v_n \geq 0$ *a.e.* on I, and $\int_I v_n < \varepsilon$. Choose u_n and v_n corresponding to $\varepsilon = (\frac{1}{2})^n$. Then

$$u_n = g_n + v_n, \qquad \text{where } \int_I v_n < (\tfrac{1}{2})^n.$$

The inequality on $\int_I v_n$ assures us that the series $\sum_{n=1}^{\infty} \int_I v_n$ converges. Now $u_n \geq 0$ almost everywhere on I, so the partial sums

$$U_n(x) = \sum_{k=1}^{n} u_k(x)$$

form a sequence of upper functions $\{U_n\}$ which increases almost everywhere on I. Since

$$\int_I U_n = \int_I \sum_{k=1}^{n} u_k = \sum_{k=1}^{n} \int_I u_k = \sum_{k=1}^{n} \int_I g_k + \sum_{k=1}^{n} \int_I v_k,$$

the sequence of integrals $\{\int_I U_n\}$ converges because both series $\sum_{k=1}^{\infty} \int_I g_k$ and $\sum_{k=1}^{\infty} \int_I v_k$ converge. Therefore, by the Levi theorem for upper functions, the sequence $\{U_n\}$ converges almost everywhere on I to a limit function U in $U(I)$, and $\int_I U = \lim_{n\to\infty} \int_I U_n$. But

$$\int_I U_n = \sum_{k=1}^{n} \int_I u_k,$$

so

$$\int_I U = \sum_{k=1}^{\infty} \int_I u_k.$$

Similarly, the sequence of partial sums $\{V_n\}$ given by

$$V_n(x) = \sum_{k=1}^{n} v_k(x)$$

converges almost everywhere on I to a limit function V in $U(I)$ and

$$\int_I V = \sum_{k=1}^{\infty} \int_I v_k.$$

Therefore $U - V \in L(I)$ and the sequence $\{\sum_{k=1}^{n} g_k\} = \{U_n - V_n\}$ converges almost everywhere on I to $U - V$. Let $g = U - V$. Then $g \in L(I)$ and

$$\int_I g = \int_I U - \int_I V = \sum_{k=1}^{\infty} \int_I (u_k - v_k) = \sum_{k=1}^{\infty} \int_I g_k.$$

This completes the proof of Theorem 10.25.

Proof of Theorem 10.24. Assume $\{f_n\}$ satisfies the hypotheses of Theorem 10.24. Let $g_1 = f_1$ and let $g_n = f_n - f_{n-1}$ for $n \geq 2$, so that

$$f_n = \sum_{k=1}^{n} g_k.$$

Applying Theorem 10.25 to $\{g_n\}$, we find that $\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a sum function g in $L(I)$, and Equation (14) holds. Therefore $f_n \to g$ almost everywhere on I and $\int_I g = \lim_{n\to\infty} \int_I f_n$.

In the following version of the Levi theorem for series, the terms of the series are not assumed to be nonnegative.

Theorem 10.26. *Let $\{g_n\}$ be a sequence of functions in $L(I)$ such that the series*

$$\sum_{n=1}^{\infty} \int_I |g_n|$$

is convergent. Then the series $\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a sum function g in $L(I)$ and we have

$$\int_I \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int_I g_n.$$

Proof. Write $g_n = g_n^+ - g_n^-$ and apply Theorem 10.25 to the sequences $\{g_n^+\}$ and $\{g_n^-\}$ separately.

The following examples illustrate the use of the Levi theorem for sequences.

10.10 THE LEBESGUE DOMINATED CONVERGENCE THEOREM

Levi's theorems have many important consequences. The first is Lebesgue's dominated convergence theorem, the cornerstone of Lebesgue's theory of integration.

Theorem 10.27 (Lebesgue dominated convergence theorem). *Let $\{f_n\}$ be a sequence of Lebesgue-integrable functions on an interval I. Assume that*

a) *$\{f_n\}$ converges almost everywhere on I to a limit function f,*
   and
b) *there is a nonnegative function g in $L(I)$ such that, for all $n \geq 1$,*

$$|f_n(x)| \leq g(x) \quad a.e. \text{ on } I.$$

Then the limit function $f \in L(I)$, the sequence $\{\int_I f_n\}$ converges and

$$\int_I f = \lim_{n \to \infty} \int_I f_n. \tag{15}$$

Proof. The idea of the proof is to obtain upper and lower bounds of the form

$$g_n(x) \leq f_n(x) \leq G_n(x) \tag{16}$$

where $\{g_n\}$ increases and $\{G_n\}$ decreases almost everywhere on I to the limit function f. Then we use the Levi theorem to show that $f \in L(I)$ and that $\int_I f = \lim_{n\to\infty} \int_I g_n = \lim_{n\to\infty} \int_I G_n$, from which we obtain (15).

To construct $\{g_n\}$ and $\{G_n\}$, we make repeated use of the Levi theorem for sequences in $L(I)$. First we define a sequence $\{G_{n,1}\}$ as follows:

$$G_{n,1}(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}.$$

Each function $G_{n,1} \in L(I)$, by Theorem 10.16, and the sequence $\{G_{n,1}\}$ is increasing on I. Since $|G_{n,1}(x)| \leq g(x)$ almost everywhere on I, we have

$$\left|\int_I G_{n,1}\right| \leq \int_I |G_{n,1}| \leq \int_I g. \tag{17}$$

Because of (17) we also have the inequality $-\int_I g \leq \int_I G_1$. Note that if x is a point in I for which $G_{n,1}(x) \to G_1(x)$, then we also have

$$G_1(x) = \sup\{f_1(x), f_2(x), \ldots\}.$$

In the same way, for each fixed $r \geq 1$ we let

$$G_{n,r}(x) = \max\{f_r(x), f_{r+1}(x), \ldots, f_n(x)\}$$

for $n \geq r$. Then the sequence $\{G_{n,r}\}$ increases and converges almost everywhere on I to a limit function G_r in $L(I)$ with

$$-\int_I g \leq \int_I G_r \leq \int_I g.$$

Also, at those points for which $G_{n,r}(x) \to G_r(x)$ we have

$$G_r(x) = \sup\{f_r(x), f_{r+1}(x), \ldots\},$$

so

$$f_r(x) \leq G_r(x) \quad a.e. \text{ on } I.$$

Now we examine properties of the sequence $\{G_n(x)\}$. Since $A \subseteq B$ implies $\sup A \leq \sup B$, the sequence $\{G_r(x)\}$ decreases almost everywhere and hence converges almost everywhere on I. We show next that $G_n(x) \to f(x)$ whenever

$$\lim_{n\to\infty} f_n(x) = f(x). \tag{18}$$

If (18) holds, then for every $\varepsilon > 0$ there is an integer N such that

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \qquad \text{for all } n \geq N.$$

Hence, if $m \geq N$ we have

$$f(x) - \varepsilon \leq \sup \{f_m(x), f_{m+1}(x), \ldots\} \leq f(x) + \varepsilon.$$

In other words,

$$m \geq N \quad \text{implies} \quad f(x) - \varepsilon \leq G_m(x) \leq f(x) + \varepsilon,$$

and this implies that

$$\lim_{m \to \infty} G_m(x) = f(x) \quad \text{almost everywhere on } I. \tag{19}$$

On the other hand, the decreasing sequence of numbers $\{\int_I G_n\}$ is bounded below by $-\int_I g$, so it converges. By (19) and the Levi theorem, we see that $f \in L(I)$ and

$$\lim_{n \to \infty} \int_I G_n = \int_I f.$$

By applying the same type of argument to the sequence

$$g_{n,r}(x) = \min \{f_r(x), f_{r+1}(x), \ldots, f_n(x)\},$$

for $n \geq r$, we find that $\{g_{n,r}\}$ decreases and converges almost everywhere to a limit function g_r in $L(I)$, where

$$g_r(x) = \inf \{f_r(x), f_{r+1}(x), \ldots\} \quad a.e. \text{ on } I.$$

Also, almost everywhere on I we have $g_r(x) \leq f_r(x)$, $\{g_r\}$ increases, $\lim_{n \to \infty} g_n(x) = f(x)$, and

$$\lim_{n \to \infty} \int_I g_n = \int_I f.$$

Since (16) holds almost everywhere on I we have $\int_I g_n \leq \int_I f_n \leq \int_I G_n$. Letting $n \to \infty$ we find that $\{\int_I f_n\}$ converges and that

$$\lim_{n \to \infty} \int_I f_n = \int_I f.$$

10.11 APPLICATIONS OF LEBESGUE'S DOMINATED CONVERGENCE THEOREM

The first application concerns term-by-term integration of series and is a companion result to Levi's theorem on series.

Theorem 10.28. *Let $\{g_n\}$ be a sequence of functions in $L(I)$ such that:*

a) *each g_n is nonnegative almost everywhere on I,*
and

b) *the series $\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a function g which is bounded above by a function in $L(I)$.*

Then $g \in L(I)$, the series $\sum_{n=1}^{\infty} \int_I g_n$ converges, and we have

$$\int_I \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int_I g_n.$$

Proof. Let

$$f_n(x) = \sum_{k=1}^{n} g_k(x) \qquad \text{if } x \in I.$$

Then $f_n \to g$ almost everywhere on I, and $\{f_n\}$ is dominated almost everywhere on I by the function in $L(I)$ which bounds g from above. Therefore, by the Lebesgue dominated convergence theorem, $g \in L(I)$, the sequence $\{\int_I f_n\}$ converges, and $\int_I g = \lim_{n\to\infty} \int_I f_n$. This proves the theorem.

The next application, sometimes called the *Lebesgue bounded convergence theorem*, refers to a bounded interval.

Theorem 10.29. *Let I be a bounded interval. Assume $\{f_n\}$ is a sequence of functions in $L(I)$ which is boundedly convergent almost everywhere on I. That is, assume there is a limit function f and a positive constant M such that*

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \textit{and} \qquad |f_n(x)| \le M, \quad \textit{almost everywhere on } I.$$

Then $f \in L(I)$ and $\lim_{n\to\infty} \int_I f_n = \int_I f$.

Theorem 10.30. *Let $\{f_n\}$ be a sequence of functions in $L(I)$ which converges almost everywhere on I to a limit function f. Assume that there is a nonnegative function g in $L(I)$ such that*

$$|f(x)| \leq g(x) \quad a.e. \text{ on } I.$$

Then $f \in L(I)$.

Proof. Define a new sequence of functions $\{g_n\}$ on I as follows:

$$g_n = \max \{\min (f_n, g), -g\}.$$

10.12 LEBESGUE INTEGRALS ON UNBOUNDED INTERVALS AS LIMITS OF INTEGRALS ON BOUNDED INTERVALS

Theorem 10.31. *Let f be defined on the half-infinite interval $I = [a, +\infty)$. Assume that f is Lebesgue-integrable on the compact interval $[a, b]$ for each $b \geq a$, and that there is a positive constant M such that*

$$\int_a^b |f| \leq M \qquad \text{for all } b \geq a. \tag{20}$$

Then $f \in L(I)$, the limit $\lim_{b \to +\infty} \int_a^b f$ exists, and

$$\int_a^{+\infty} f = \lim_{b \to +\infty} \int_a^b f. \tag{21}$$

Proof. Let $\{b_n\}$ be any increasing sequence of real numbers with $b_n \geq a$ such that $\lim_{n \to \infty} b_n = +\infty$. Define a sequence $\{f_n\}$ on I as follows:

$$f_n(x) = \begin{cases} f(x) & \text{if } a \leq x \leq b_n, \\ 0 & \text{otherwise.} \end{cases}$$

Each $f_n \in L(I)$ (by Theorem 10.18) and $f_n \to f$ on I. Hence, $|f_n| \to |f|$ on I. But $|f_n|$ is increasing and, by (20), the sequence $\{\int_I |f_n|\}$ is bounded above by M. Therefore $\lim_{n \to \infty} \int_I |f_n|$ exists. By the Levi theorem, the limit function $|f| \in L(I)$. Now each $|f_n| \leq |f|$ and $f_n \to f$ on I, so by the Lebesgue dominated convergence theorem, $f \in L(I)$ and $\lim_{n \to \infty} \int_I f_n = \int_I f$. Therefore

$$\lim_{n \to \infty} \int_a^{b_n} f = \int_a^{+\infty} f$$

for all sequences $\{b_n\}$ which increase to $+\infty$. This completes the proof.

There is, of course, a corresponding theorem for the interval $(-\infty, a]$ which concludes that

$$\int_{-\infty}^{a} f = \lim_{c \to -\infty} \int_{c}^{a} f,$$

provided that $\int_c^a |f| \le M$ for all $c \le a$. If $\int_c^b |f| \le M$ for all real c and b with $c \le b$, the two theorems together show that $f \in L(\mathbf{R})$ and that

$$\int_{-\infty}^{+\infty} f = \lim_{c \to -\infty} \int_{c}^{a} f + \lim_{b \to +\infty} \int_{a}^{b} f.$$

10.13 IMPROPER RIEMANN INTEGRALS

Definition 10.32. *If f is Riemann-integrable on $[a, b]$ for every $b \ge a$, and if the limit*

$$\lim_{b \to +\infty} \int_{a}^{b} f(x)\, dx \quad \textit{exists},$$

then f is said to be improper Riemann-integrable on $[a, +\infty)$ and the improper Riemann integral of f, denoted by $\int_a^{+\infty} f(x)\, dx$ or $\int_a^{\infty} f(x)\, dx$, is defined by the equation

$$\int_{a}^{+\infty} f(x)\, dx = \lim_{b \to +\infty} \int_{a}^{b} f(x)\, dx.$$

In Example 2 of the foregoing section the improper Riemann integral $\int_0^{+\infty} f(x)\, dx$ exists but f is not Lebesgue-integrable on $[0, +\infty)$. That example should be contrasted with the following theorem.

Theorem 10.33. *Assume f is Riemann-integrable on $[a, b]$ for every $b \ge a$, and assume there is a positive constant M such that*

$$\int_{a}^{b} |f(x)|\, dx \le M \qquad \textit{for every } b \ge a. \tag{22}$$

Then both f and $|f|$ are improper Riemann-integrable on $[a, +\infty)$. Also, f is Lebesgue-integrable on $[a, +\infty)$ and the Lebesgue integral of f is equal to the improper Riemann integral of f.

Proof. Let $F(b) = \int_a^b |f(x)|\, dx$. Then F is an increasing function which is bounded above by M, so $\lim_{b \to +\infty} F(b)$ exists. Therefore $|f|$ is improper Riemann-integrable on $[a, +\infty)$. Since

$$0 \le |f(x)| - f(x) \le 2|f(x)|,$$

the limit

$$\lim_{b \to +\infty} \int_a^b \{|f(x)| - f(x)\}\, dx$$

also exists; hence the limit $\lim_{b\to+\infty} \int_a^b f(x)\, dx$ exists. This proves that f is improper Riemann-integrable on $[a, +\infty)$. Now we use inequality (22), along with Theorem 10.31, to deduce that f is Lebesgue-integrable on $[a, +\infty)$ and that the Lebesgue integral of f is equal to the improper Riemann integral of f.

NOTE. There are corresponding results for improper Riemann integrals of the form

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx,$$

$$\int_a^c f(x)\, dx = \lim_{b \to c-} \int_a^b f(x)\, dx,$$

and

$$\int_c^b f(x)\, dx = \lim_{a \to c+} \int_a^b f(x)\, dx,$$

which the reader can formulate for himself.

If both integrals $\int_{-\infty}^a f(x)\, dx$ and $\int_a^{+\infty} f(x)\, dx$ exist, we say that the integral $\int_{-\infty}^{+\infty} f(x)\, dx$ exists, and its value is defined to be their sum,

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^a f(x)\, dx + \int_a^{+\infty} f(x)\, dx.$$

If the integral $\int_{-\infty}^{+\infty} f(x)\, dx$ exists, its value is also equal to the symmetric limit

$$\lim_{b \to +\infty} \int_{-b}^b f(x)\, dx.$$

However, it is important to realize that the symmetric limit might exist even when $\int_{-\infty}^{+\infty} f(x)\, dx$ does not exist (for example, take $f(x) = x$ for all x). In this case the symmetric limit is called the *Cauchy principal value* of $\int_{-\infty}^{+\infty} f(x)\, dx$. Thus $\int_{-\infty}^{+\infty} x\, dx$ has Cauchy principal value 0, but the integral does not exist.

10.14 MEASURABLE FUNCTIONS

Every function f which is Lebesgue-integrable on an interval I is the limit, almost everywhere on I, of a certain sequence of step functions. However, the converse is not true. For example, the constant function $f = 1$ is a limit of step functions on the real line $\mathbf{R}$, but this function is not in $L(\mathbf{R})$. Therefore, the class of functions which are limits of step functions is larger than the class of Lebesgue-integrable functions. The functions in this larger class are called *measurable functions.*

Definition 10.34. *A function f defined on I is called measurable on I, and we write $f \in M(I)$, if there exists a sequence of step functions $\{s_n\}$ on I such that*

$$\lim_{n\to\infty} s_n(x) = f(x) \quad \textit{almost everywhere on } I.$$

NOTE. If f is measurable on I then f is measurable on every subinterval of I.

As already noted, every function in $L(I)$ is measurable on I, but the converse is not true. The next theorem provides a partial converse.

Theorem 10.35. *If $f \in M(I)$ and if $|f(x)| \le g(x)$ almost everywhere on I for some nonnegative g in $L(I)$, then $f \in L(I)$.*

Proof. There is a sequence of step functions $\{s_n\}$ such that $s_n(x) \to f(x)$ almost everywhere on I. Now apply Theorem 10.30 to deduce that $f \in L(I)$.

Corollary 1. *If $f \in M(I)$ and $|f| \in L(I)$, then $f \in L(I)$.*

Corollary 2. *If f is measurable and bounded on a bounded interval I, then $f \in L(I)$.*

Further properties of measurable functions are given in the next theorem.

Theorem 10.36. *Let φ be a real-valued function continuous on $\mathbf{R}^2$. If $f \in M(I)$ and $g \in M(I)$, define h on I by the equation*

$$h(x) = \varphi[f(x), g(x)].$$

Then $h \in M(I)$. *In particular,* $f+g$, $f \cdot g$, $|f|$, $\max(f, g)$, *and* $\min(f, g)$ *are in* $M(I)$. *Also,* $1/f \in M(I)$ *if* $f(x) \neq 0$ *almost everywhere on* I.

Proof. Let $\{s_n\}$ and $\{t_n\}$ denote sequences of step functions such that $s_n \to f$ and $t_n \to g$ almost everywhere on I. Then the function $u_n = \varphi(s_n, t_n)$ is a step function such that $u_n \to h$ almost everywhere on I. Hence $h \in M(I)$.

The next theorem shows that the class $M(I)$ cannot be enlarged by taking limits of functions in $M(I)$.

POSSIBLE QUESTIONS

1. Let u,v,u_1 and v_1 be functions in U (I) such that u-v = u_1 - v_1 then prove that $\int_I u - \int_I v = \int_I u_1 - \int_I v_1$.

2. Let { f_n } be a sequence of functions in L (I) which converges a.e on I to a limit function f.Assume that there is a non-negative function g in L (I) such that | f (x) | ≤ g (x) a.e on I. Then prove that f ∈ L (I) .
3. State and prove Lesgue dominated convergence Theorem.

4. Prove that, let f is Riemann integrable on [a,b] ∀ b ≥ a and assume that there is a positive constant M such that $\int_a^b |f(x)|\, dx \leq M$ ∀ b ≥ a.
5. Let f be defined on I . If f =0 a.e on I,then prove that f ∈L (I) & $\int_I f = 0$.

6. Assume f ∈ L (I) and Let ∈ > 0 be given then prove that
   (i) there exist a function in u and v in U (I) such that f = u – v_1 where V is Non –negative integer on I and $\int_I v < \in$.
   (ii) there exist a step function S and a function g in L (I) such that f = S + g, where $\int_I |g| < \in$.
7. State and P rove Lebesgue integrals on unbounded intervals as limits of integrals on bounded intervals .

8. State and P rove Levi theorem for series of Lebesgue integrals functions.

9. State and prove Levi Monotone converges Theorem.
10. State and prove Levi theorem for upper functions .

11. Let f be defined on I & assume that {f_n } is a sequence of measurable functions on I such that $f_n \to f(x)$ a.e on I.Then prove that f is measurable on I.

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Subject: Real Analysis **Subject**

Class : I - M.Sc. Mathematics

Unit IV

Part A (20x1=20 Marks) **(Question Nos. 1 to 20**

Possible Questions

Question	**Choice 1**	**Choice 2**	**Choice 3**	**Choice 4**	
If a sequence of real number has cluster points then	convergent	bounded & convergent	bounded & divergent	unbounded & divergent	bounded & convergent
Sequence [1/n] is	unbounded &	decreasing sequence	monotonic sequence	oscillating sequence	bounded
The sequence { 1,0,1,0,1,…… } is	increasing	unbounded	bounded below	bounded above	divergent
Every convergent sequence is	bounded	convergent	unbounded	both converges and diverges	bounded
The series 1+3+5+7+…..	divergent	convergent	bounded	divergent	comparison test
Cauchy sequence is	unbounded &	root test	ratio test	leibnitz test	
Which one of the following test does not give absolute convergence series	comparison test	lim Sup $(x_n + y_n)$ > lim Sup x_n + lim Sup y_n	lim Sup $(x_n + y_n)$ = lim Sup x_n + lim Sup y_n	lim Sup $(x_n + y_n)$ < lim Sup x_n + lim Sup y_n	lim Sup $(x_n + y_n)$ ≤ lim Sup x_n + lim Sup y_n
If < x n > and < y n > sequence of real number	$_n$) ≤ lim Sup x_n + lim Sup y_n	exactly two constant sub sequence	constant sub sequence	exactly four constant sub sequence	exactly two constant sub sequence
The sequence < 1+ (-1) n > has	exactly one constant sub	bounded	having a subsequence	convergent	bounded
Let < a n > = least power of 2 that divides n.then < a n > is	divergent to infinity	convergent but not absolutely convergent	absolutely divergent	divergent	convergent but not absolutely convergent
A conditionally converges series is a series which is	absolutely convergent	bounded	not necessarily bounded	neither bounded nor unbounded	bounded
The set of limit points of a bounded sequence is	unbounded	sequence of rational numbers	sequence of irrational numbers	bounded sequence of rational numbers	sequence of real numbers
Cauchy sequence is convergent if it is a	sequence of real numbers	convergent sequencee	bounded sequencee	unboundedt sequencee	divergent sequencee
If a sequence is not a cauchy sequence then it is	divergent sequencee	bounded	not necessarily bounded	neither bounded nor unbounded	bounded
The set of limit points of a bounded sequence is	unbounded	2	3	4	2
If { x n } and { x n+1 }=√(2+x n) then the sequence { x n } converges to	1	need not be convergent	may be convergent	divergent subsequence	convergent subsequence
Every cauchy sequence contains	convergent	divergent	convergent	unbounded	bounded
The series ∑_(n=1)^∞▒ (-1) n n	bounded	infinite limit	unique limit	no limit	unique limit
Every convergent sequence is bounded and it has	finite limit	f is not Riemann integrable on [a,b]	f is Riemann integrable on R	f is integrable on R	f is Riemann integrable on [a,b]
If f : [a,b] →R is continuous and monotonic functions then	f is Riemann integrable on	Q	[S]	{S n }	{S n }
The notation of a sequence is	S	divergent	Uniformly Convergent	does not Uniformly Convergent	Uniformly Convergent
Union of two measurable sets is	not measurable	1	infinity	2	0
Cantor ternary set is measurable and its measure	0	uncountable	bounded	un bounded	uncountable
A set without measure different from zero is	countable	only one subcover	finite subcover	no subcover	finite subcover
A set A is said to be Compact if it has a	many subcover	closed	open as well as closed	neither open nor closed	open as well as closed
The empty set ϕ and whole set X	open	one limit	many limit	no limit point	no limit point
Finite sets in a metric space have	more than one	it ia set	It is a limit point	it is empty	it is an interval
A subset A of R is connnected if and only if	it is an interval	closed	semi open	semi closed	closed
In a metric space every singleton set { p } is	open	B - A is semi open set	B - A is closed set	B - A is empty set	B - A is closed set
If A is open set and B is closed set then	B - A is open set	unbounded	totally bounded	bounded below	totally bounded
A sequentially compact metric space is	bounded	non negative finite number	extended real number	extended rational number	non negative finite number
The total Variation on [a,b] is	non positive finite number	it is not of bounded variation of [a,b]	it is not of bounded variation of R	it is of bounded variation of R	it is of bounded variation of [a,b]

If f is absolutely continuous on [a,b]	it is of bounded variation of [a,b]	it is always a bounded variation	its never a function of bounded variation	may be a function of bounded variation	may or may not be a function of bounded variation
A continuous function is	be a function of bounded variation	atmost one cluster point	atleast one cluster point	unique cluster point	atleast one cluster point
Every infinite sequence { x n } in X has	more cluster point	A is complete metric space	undefined	need not be a complete metric space	A is incomplete metric space
If A is closed subset of a complete metric space	A is incomplete metric space	any infinite subcollection of F has empty intersection	any finite subcollection of F has empty set	any finite subcollection of F has non-empty intersection	any finite subcollection of F has empty intersection
A collection F of sets have finite intersection property if	any finite subcollection of	A is complete	completment of A is closed	completment of A is open	completment of A is closed
If A is ao open subset of complete metric space X then	A is incomplete	every sequence in X is divergent	every cauchy sequence in X is convergent	every cauchy sequence in X is divergent	every cauchy sequence in X is convergant
A metric space (X , p) is complete , if	every sequence in X is	closed set	empty set	non empty set	closed set
The union of any finite collection of non empty closed set is	open set	closed	open and closed	does not exist	open and closed
The empty set ϕ of a metric space is	open	not countable	may be countable	need not be a countable	not countable
The set [0,1] is	countable	C is of measure zero	C is uncountable and of measure zero	C is uncountable and of positive measure	C is uncountable and of measure zero
Let C be Cantor's middle third set then	C is not	1	2	3	0
The set of rational numbers lebesque outer measure is	0	infinite subcovering of F	no finite subcovering of F	no infinite subcovering of F	finite subcovering of F
If F is a closed and bounded aet of real numbers then each open covering is	finite subcovering of F	it is uncountable	it ia dense	it is perfect set	it ia dense
what is not correct about cantor ternary set	it is closed	not measurable	neither measurable nor not measurable	need not be measurable	measurable
If f is a measurable function and f =g almost everywhere,then g is	measurable	sum of an end points of the interval	Product of an end points of the interval	division of an end points of the interval	difference of an end points of the interval
The length of an interval I is	difference of an end points of the	every interval is measurable	every open set in R is measurable	every closed set in R is measurable	every closed set in R is measurable

UNIT V
IMPLICIT FUNCTIONS AND EXTREMUM PROBLEMS
SYLLABUS

Introduction – Functions with non zero Jacobian determinant – Inverse function theorem – Implicit function theorem – Extrema of real valued functions of one variable and several variables

13.1 INTRODUCTION

This chapter consists of two principal parts. The first part discusses an important theorem of analysis called the *implicit function theorem;* the second part treats extremum problems. Both parts use the theorems developed in Chapter 12.

The implicit function theorem in its simplest form deals with an equation of the form

$$f(x, t) = 0. \tag{1}$$

The problem is to decide whether this equation determines x as a function of t. If so, we have

$$x = g(t),$$

for some function g. We say that g is defined "implicitly" by (1).

The problem assumes a more general form when we have a system of several equations involving several variables and we ask whether we can solve these equations for some of the variables in terms of the remaining variables. This is the same type of problem as above, except that x and t are replaced by vectors, and f and g are replaced by vector-valued functions. Under rather general conditions, a solution always exists. The implicit function theorem gives a description of these conditions and some conclusions about the solution.

An important special case is the familiar problem in algebra of solving n linear equations of the form

$$\sum_{j=1}^{n} a_{ij}x_j = t_i \qquad (i = 1, 2, \ldots, n). \tag{2}$$

where the a_{ij} and t_i are considered as given numbers and $x_1, \ldots, x_n$ represent unknowns. In linear algebra it is shown that such a system has a unique solution if, and only if, the determinant of the coefficient matrix $A = [a_{ij}]$ is nonzero.

NOTE. The determinant of a square matrix $A = [a_{ij}]$ is denoted by det A or det $[a_{ij}]$. If det $[a_{ij}] \neq 0$, the solution of (2) can be obtained by Cramer's rule which expresses each x_k as a quotient of two determinants, say $x_k = A_k/D$, where $D = \det [a_{ij}]$ and A_k is the determinant of the matrix obtained by replacing the kth column of $[a_{ij}]$ by $t_1, \ldots, t_n$. (For a proof of Cramer's rule, see Reference 13.1, Theorem 3.14.) In particular, if each $t_i = 0$, then each $x_k = 0$.

Next we show that the system (2) can be written in the form (1). Each equation in (2) has the form

$$f_i(\mathbf{x}, \mathbf{t}) = 0 \qquad \text{where } \mathbf{x} = (x_1, \ldots, x_n), \quad \mathbf{t} = (t_1, \ldots, t_n),$$

and

$$f_i(\mathbf{x}, \mathbf{t}) = \sum_{j=1}^{n} a_{ij}x_j - t_i.$$

Therefore the system in (2) can be expressed as one vector equation $\mathbf{f}(\mathbf{x}, \mathbf{t}) = \mathbf{0}$, where $\mathbf{f} = (f_1, \ldots, f_n)$. If $D_j f_i$ denotes the partial derivative of f_i with respect to the jth coordinate x_j, then $D_j f_i(\mathbf{x}, \mathbf{t}) = a_{ij}$. Thus the coefficient matrix $A = [a_{ij}]$ in (2) is a Jacobian matrix. Linear algebra tells us that (2) has a unique solution if the determinant of this Jacobian matrix is nonzero.

Theorem 13.1. *If $f = u + iv$ is a complex-valued function with a derivative at a point z in $\mathbf{C}$, then $J_f(z) = |f'(z)|^2$.*

Proof. We have $f'(z) = D_1u + iD_1v$, so $|f'(z)|^2 = (D_1u)^2 + (D_1v)^2$. Also,

$$J_f(z) = \det \begin{bmatrix} D_1u & D_2u \\ D_1v & D_2v \end{bmatrix} = D_1u\, D_2v - D_1v\, D_2u = (D_1u)^2 + (D_1v)^2,$$

by the Cauchy–Riemann equations.

13.2 FUNCTIONS WITH NONZERO JACOBIAN DETERMINANT

This section gives some properties of functions with nonzero Jacobian determinant at certain points. These results will be used later in the proof of the implicit function theorem.

Theorem 13.2. *Let $B = B(\mathbf{a}; r)$ be an n-ball in $\mathbf{R}^n$, let ∂B denote its boundary,*

$$\partial B = \{\mathbf{x} : \|\mathbf{x} - \mathbf{a}\| = r\},$$

and let $\bar{B} = B \cup \partial B$ denote its closure. Let $\mathbf{f} = (f_1, \ldots, f_n)$ be continuous on $\bar{B}$, and assume that all the partial derivatives $D_j f_i(\mathbf{x})$ exist if $\mathbf{x} \in B$. Assume further that $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{a})$ if $\mathbf{x} \in \partial B$ and that the Jacobian determinant $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for each $\mathbf{x}$ in B. Then $\mathbf{f}(B)$, the image of B under $\mathbf{f}$, contains an n-ball with center at $\mathbf{f}(\mathbf{a})$.

Proof. Define a real-valued function g on ∂B as follows:

$$g(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| \qquad \text{if } \mathbf{x} \in \partial B.$$

Then $g(\mathbf{x}) > 0$ for each $\mathbf{x}$ in ∂B because $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{a})$ if $\mathbf{x} \in \partial B$. Also, g is continuous on ∂B since $\mathbf{f}$ is continuous on $\bar{B}$. Since ∂B is compact, g takes on its absolute minimum (call it m) somewhere on ∂B. Note that $m > 0$ since g is positive on ∂B. Let T denote the n-ball

$$T = B\left(\mathbf{f}(\mathbf{a}); \frac{m}{2}\right).$$

We will prove that $T \subseteq \mathbf{f}(B)$ and this will prove the theorem. (See Fig. 13.1.)

To do this we show that $\mathbf{y} \in T$ implies $\mathbf{y} \in \mathbf{f}(B)$. Choose a point $\mathbf{y}$ in T, keep $\mathbf{y}$ fixed, and define a new real-valued function h on $\bar{B}$ as follows:

$$h(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\| \qquad \text{if } \mathbf{x} \in \bar{B}.$$

Then h is continuous on the compact set $\bar{B}$ and hence attains its absolute minimum on $\bar{B}$. We will show that h attains its minimum somewhere in the open n-ball B. At the center we have $h(\mathbf{a}) = \|\mathbf{f}(\mathbf{a}) - \mathbf{y}\| < m/2$ since $\mathbf{y} \in T$. Hence the minimum value of h in $\bar{B}$ must also be $< m/2$. But at each point $\mathbf{x}$ on the boundary ∂B we have

$$h(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\| = \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - (\mathbf{y} - \mathbf{f}(\mathbf{a}))\|$$

$$\geq \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| - \|\mathbf{f}(\mathbf{a}) - \mathbf{y}\| > g(\mathbf{x}) - \frac{m}{2} \geq \frac{m}{2},$$

so the minimum of h cannot occur on the boundary ∂B. Hence there is an interior point $\mathbf{c}$ in B at which h attains its minimum. At this point the square of h also has a minimum. Since

$$h^2(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\|^2 = \sum_{r=1}^{n} [f_r(\mathbf{x}) - y_r]^2,$$

and since each partial derivative $D_k(h^2)$ must be zero at $\mathbf{c}$, we must have

$$\sum_{r=1}^{n} [f_r(\mathbf{c}) - y_r] D_k f_r(\mathbf{c}) = 0 \qquad \text{for } k = 1, 2, \ldots, n.$$

But this is a system of linear equations whose determinant $J_{\mathbf{f}}(\mathbf{c})$ is not zero, since $\mathbf{c} \in B$. Therefore $f_r(\mathbf{c}) = y_r$ for each r, or $\mathbf{f}(\mathbf{c}) = \mathbf{y}$. That is, $\mathbf{y} \in f(B)$. Hence $T \subseteq \mathbf{f}(B)$ and the proof is complete.

A function $f: S \to T$ from one metric space (S, d_S) to another (T, d_T) is called an *open mapping* if, for every open set A in S, the image $f(A)$ is open in T.

The next theorem gives a sufficient condition for a mapping to carry open sets onto open sets. (See also Theorem 13.5.)

Theorem 13.3. *Let A be an open subset of $\mathbf{R}^n$ and assume that $\mathbf{f} : A \to \mathbf{R}^n$ is continuous and has finite partial derivatives $D_j f_i$ on A. If $\mathbf{f}$ is one-to-one on A and if $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for each $\mathbf{x}$ in A, then $\mathbf{f}(A)$ is open.*

Proof. If $\mathbf{b} \in \mathbf{f}(A)$, then $\mathbf{b} = \mathbf{f}(\mathbf{a})$ for some $\mathbf{a}$ in A. There is an n-ball $B(\mathbf{a}; r) \subseteq A$ on which $\mathbf{f}$ satisfies the hypotheses of Theorem 13.2, so $\mathbf{f}(B)$ contains an n-ball with center at $\mathbf{b}$. Therefore, $\mathbf{b}$ is an interior point of $\mathbf{f}(A)$, so $\mathbf{f}(A)$ is open.

The next theorem shows that a function with continuous partial derivatives is locally one-to-one near a point where the Jacobian determinant does not vanish.

Theorem 13.4. *Assume that $\mathbf{f} = (f_1, \ldots, f_n)$ has continuous partial derivatives $D_j f_i$ on an open set S in $\mathbf{R}^n$, and that the Jacobian determinant $J_{\mathbf{f}}(\mathbf{a}) \neq 0$ for some point $\mathbf{a}$ in S. Then there is an n-ball $B(\mathbf{a})$ on which $\mathbf{f}$ is one-to-one.*

Proof. Let $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ be n points in S and let $\mathbf{Z} = (\mathbf{Z}_1; \ldots; \mathbf{Z}_n)$ denote that point in $\mathbf{R}^{n^2}$ whose first n components are the components of $\mathbf{Z}_1$, whose next n components are the components of $\mathbf{Z}_2$, and so on. Define a real-valued function h as follows:

$$h(\mathbf{Z}) = \det [D_j f_i(\mathbf{Z}_i)].$$

This function is continuous at those points $\mathbf{Z}$ in $\mathbf{R}^{n^2}$ where $h(\mathbf{Z})$ is defined because each $D_j f_i$ is continuous on S and a determinant is a polynomial in its n^2 entries. Let $\mathbf{Z}$ be the special point in $\mathbf{R}^{n^2}$ obtained by putting

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \cdots = \mathbf{Z}_n = \mathbf{a}.$$

Then $h(\mathbf{Z}) = J_{\mathbf{f}}(\mathbf{a}) \neq 0$ and hence, by continuity, there is some n-ball $B(\mathbf{a})$ such that $\det [D_j f_i(\mathbf{Z}_i)] \neq 0$ if each $\mathbf{Z}_i \in B(\mathbf{a})$. We will prove that $\mathbf{f}$ is one-to-one on $B(\mathbf{a})$.

Assume the contrary. That is, assume that $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y})$ for some pair of points $\mathbf{x} \neq \mathbf{y}$ in $B(\mathbf{a})$. Since $B(\mathbf{a})$ is convex, the line segment $L(\mathbf{x}, \mathbf{y}) \subseteq B(\mathbf{a})$ and we can apply the Mean-Value Theorem to each component of $\mathbf{f}$ to write

$$0 = f_i(\mathbf{y}) - f_i(\mathbf{x}) = \nabla f_i(\mathbf{Z}_i) \cdot (\mathbf{y} - \mathbf{x}) \qquad \text{for } i = 1, 2, \ldots, n,$$

where each $\mathbf{Z}_i \in L(\mathbf{x}, \mathbf{y})$ and hence $\mathbf{Z}_i \in B(\mathbf{a})$. (The Mean-Value Theorem is applicable because $\mathbf{f}$ is differentiable on S.) But this is a system of linear equations of the form

$$\sum_{k=1}^{n} (y_k - x_k) a_{ik} = 0 \qquad \text{with } a_{ik} = D_k f_i(\mathbf{Z}_i).$$

The determinant of this system is not zero, since $\mathbf{Z}_i \in B(\mathbf{a})$. Hence $y_k - x_k = 0$ for each k, and this contradicts the assumption that $\mathbf{x} \neq \mathbf{y}$. We have shown, therefore, that $\mathbf{x} \neq \mathbf{y}$ implies $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{y})$ and hence that $\mathbf{f}$ is one-to-one on $B(\mathbf{a})$.

Theorem 13.5. *Let A be an open subset of $\mathbf{R}^n$ and assume that $\mathbf{f} : A \to \mathbf{R}^n$ has continuous partial derivatives $D_j f_i$ on A. If $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in A, then $\mathbf{f}$ is an open mapping.*

Proof. Let S be any open subset of A. If $\mathbf{x} \in S$ there is an n-ball $B(\mathbf{x})$ in which $\mathbf{f}$ is one-to-one (by Theorem 13.4). Therefore, by Theorem 13.3, the image $\mathbf{f}(B(\mathbf{x}))$ is open in $\mathbf{R}^n$. But we can write $S = \bigcup_{\mathbf{x} \in S} B(\mathbf{x})$. Applying $\mathbf{f}$ we find $\mathbf{f}(S) = \bigcup_{\mathbf{x} \in S} \mathbf{f}(B(\mathbf{x}))$, so $\mathbf{f}(S)$ is open.

13.3 THE INVERSE FUNCTION THEOREM

Theorem 13.6. *Assume* $\mathbf{f} = (f_1, \ldots, f_n) \in C'$ *on an open set* S *in* $\mathbf{R}^n$, *and let* $T = \mathbf{f}(S)$. *If the Jacobian determinant* $J_{\mathbf{f}}(\mathbf{a}) \neq 0$ *for some point* $\mathbf{a}$ *in* S, *then there are two open sets* $X \subseteq S$ *and* $Y \subseteq T$ *and a uniquely determined function* $\mathbf{g}$ *such that*

a) $\mathbf{a} \in X$ *and* $\mathbf{f}(\mathbf{a}) \in Y$,

b) $Y = \mathbf{f}(X)$,

c) $\mathbf{f}$ *is one-to-one on* X,

d) $\mathbf{g}$ *is defined on* Y, $\mathbf{g}(Y) = X$, *and* $\mathbf{g}[\mathbf{f}(\mathbf{x})] = \mathbf{x}$ *for every* $\mathbf{x}$ *in* X,

e) $\mathbf{g} \in C'$ *on* Y.

Proof. The function $J_{\mathbf{f}}$ is continuous on S and, since $J_{\mathbf{f}}(\mathbf{a}) \neq 0$, there is an n-ball $B_1(\mathbf{a})$ such that $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in $B_1(\mathbf{a})$. By Theorem 13.4, there is an n-ball $B(\mathbf{a}) \subseteq B_1(\mathbf{a})$ on which $\mathbf{f}$ is one-to-one. Let B be an n-ball with center at $\mathbf{a}$ and radius smaller than that of $B(\mathbf{a})$. Then, by Theorem 13.2, $\mathbf{f}(B)$ contains an n-ball with center at $\mathbf{f}(\mathbf{a})$. Denote this by Y and let $X = \mathbf{f}^{-1}(Y) \cap B$. Then X is open since both $\mathbf{f}^{-1}(Y)$ and B are open. (See Fig. 13.2.)

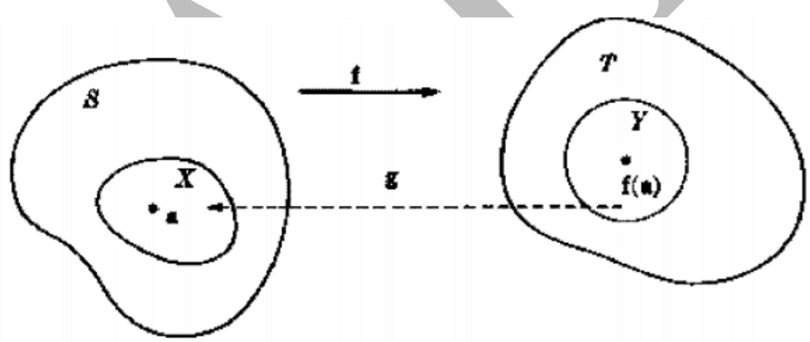

Figure 13.2

The set $\bar{B}$ (the closure of B) is compact and $\mathbf{f}$ is one-to-one and continuous on $\bar{B}$. Hence, by Theorem 4.29, there exists a function $\mathbf{g}$ (the inverse function $\mathbf{f}^{-1}$ of Theorem 4.29) defined on $\mathbf{f}(\bar{B})$ such that $\mathbf{g}[\mathbf{f}(\mathbf{x})] = \mathbf{x}$ for all $\mathbf{x}$ in $\bar{B}$. Moreover, $\mathbf{g}$ is continuous on $\mathbf{f}(\bar{B})$. Since $X \subseteq \bar{B}$ and $Y \subseteq \mathbf{f}(\bar{B})$, this proves parts (a), (b), (c) and (d). The uniqueness of $\mathbf{g}$ follows from (d).

open, $\mathbf{y} + t\mathbf{u}_r \in Y$ if t is sufficiently small.) Let $\mathbf{x} = \mathbf{g}(\mathbf{y})$ and let $\mathbf{x}' = \mathbf{g}(\mathbf{y} + t\mathbf{u}_r)$. Then both $\mathbf{x}$ and $\mathbf{x}'$ are in X and $\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x}) = t\mathbf{u}_r$. Hence $f_i(\mathbf{x}') - f_i(\mathbf{x})$ is 0 if $i \neq r$, and is t if $i = r$. By the Mean-Value Theorem we have

$$\frac{f_i(\mathbf{x}') - f_i(\mathbf{x})}{t} = \nabla f_i(\mathbf{Z}_i) \cdot \frac{\mathbf{x}' - \mathbf{x}}{t} \qquad \text{for } i = 1, 2, \ldots, n,$$

where each $\mathbf{Z}_i$ is on the line segment joining $\mathbf{x}$ and $\mathbf{x}'$; hence $\mathbf{Z}_i \in B$. The expression on the left is 1 or 0, according to whether $i = r$ or $i \neq r$. This is a system of n linear equations in n unknowns $(x_j' - x_j)/t$ and has a unique solution, since

$$\det [D_j f_i(\mathbf{Z}_i)] = h(\mathbf{Z}) \neq 0.$$

Solving for the kth unknown by Cramer's rule, we obtain an expression for $[g_k(\mathbf{y} + t\mathbf{u}_r) - g_k(\mathbf{y})]/t$ as a quotient of determinants. As $t \to 0$, the point $\mathbf{x}' \to \mathbf{x}$, since $\mathbf{g}$ is continuous, and hence each $\mathbf{Z}_i \to \mathbf{x}$, since $\mathbf{Z}_i$ is on the segment joining $\mathbf{x}$ to $\mathbf{x}'$. The determinant which appears in the denominator has for its limit the number $\det [D_j f_i(x)] = J_f(\mathbf{x})$, and this is nonzero, since $\mathbf{x} \in X$. Therefore, the following limit exists:

$$\lim_{t \to 0} \frac{g_k(\mathbf{y} + t\mathbf{u}_r) - g_k(\mathbf{y})}{t} = D_r g_k(\mathbf{y}).$$

This establishes the existence of $D_r g_k(\mathbf{y})$ for each $\mathbf{y}$ in Y and each $r = 1, 2, \ldots, n$. Moreover, this limit is a quotient of two determinants involving the derivatives $D_j f_i(\mathbf{x})$. Continuity of the $D_j f_i$ implies continuity of each partial $D_r g_k$. This completes the proof of (e).

NOTE. The foregoing proof also provides a method for computing $D_r g_k(\mathbf{y})$. In practice, the derivatives $D_r g_k$ can be obtained more easily (without recourse to a limiting process) by using the fact that, if $\mathbf{y} = \mathbf{f}(\mathbf{x})$, the product of the two Jacobian matrices $\mathbf{Df}(\mathbf{x})$ and $\mathbf{Dg}(\mathbf{y})$ is the identity matrix. When this is written out in detail it gives the following system of n^2 equations:

$$\sum_{k=1}^{n} D_k g_i(\mathbf{y}) D_j f_k(\mathbf{x}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For each fixed i, we obtain n linear equations as j runs through the values $1, 2, \ldots, n$. These can then be solved for the n unknowns, $D_1g_i(\mathbf{y}), \ldots, D_ng_i(\mathbf{y})$, by Cramer's rule, or by some other method.

13.4 THE IMPLICIT FUNCTION THEOREM

The reader knows that the equation of a curve in the xy-plane can be expressed either in an "explicit" form, such as $y = f(x)$, or in an "implicit" form, such as $F(x, y) = 0$. However, if we are given an equation of the form $F(x, y) = 0$, this does not necessarily represent a function. (Take, for example, $x^2 + y^2 - 5 = 0$.) The equation $F(x, y) = 0$ *does* always represent a *relation*, namely, that set of all

Theorem 13.7 (Implicit function theorem). *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a vector-valued function defined on an open set S in $\mathbf{R}^{n+k}$ with values in $\mathbf{R}^n$. Suppose $\mathbf{f} \in C'$ on S. Let $(\mathbf{x}_0; \mathbf{t}_0)$ be a point in S for which $\mathbf{f}(\mathbf{x}_0; \mathbf{t}_0) = \mathbf{0}$ and for which the $n \times n$ determinant* $\det [D_jf_i(\mathbf{x}_0; \mathbf{t}_0)] \neq 0$. *Then there exists a k-dimensional open set T_0 containing $\mathbf{t}_0$ and one, and only one, vector-valued function $\mathbf{g}$, defined on T_0 and having values in $\mathbf{R}^n$, such that*

a) $\mathbf{g} \in C'$ *on* T_0,

b) $\mathbf{g}(\mathbf{t}_0) = \mathbf{x}_0$,

c) $\mathbf{f}(\mathbf{g}(\mathbf{t}); \mathbf{t}) = \mathbf{0}$ *for every* $\mathbf{t}$ *in* T_0.

Proof. We shall apply the inverse function theorem to a certain vector-valued function $\mathbf{F} = (F_1, \ldots, F_n; F_{n+1}, \ldots, F_{n+k})$ defined on S and having values in $\mathbf{R}^{n+k}$. The function $\mathbf{F}$ is defined as follows: For $1 \leq m \leq n$, let $F_m(\mathbf{x}; \mathbf{t}) = f_m(\mathbf{x}; \mathbf{t})$, and for $1 \leq m \leq k$, let $F_{n+m}(\mathbf{x}; \mathbf{t}) = t_m$. We can then write $\mathbf{F} = (\mathbf{f}; \mathbf{I})$, where $\mathbf{f} = (f_1, \ldots, f_n)$ and where $\mathbf{I}$ is the identity function defined by $\mathbf{I}(\mathbf{t}) = \mathbf{t}$ for each $\mathbf{t}$ in $\mathbf{R}^k$. The Jacobian $J_{\mathbf{F}}(\mathbf{x}; \mathbf{t})$ then has the same value as the $n \times n$ determinant $\det [D_jf_i(\mathbf{x}; \mathbf{t})]$ because the terms which appear in the last k rows and also in the last k columns of $J_{\mathbf{F}}(\mathbf{x}; \mathbf{t})$ form a $k \times k$ determinant with ones along the main diagonal and zeros elsewhere; the intersection of the first n rows and n columns consists of the determinant $\det [D_jf_i(\mathbf{x}; \mathbf{t})]$, and

$$D_iF_{n+j}(\mathbf{x}; \mathbf{t}) = 0 \qquad \text{for } 1 \leq i \leq n, \quad 1 \leq j \leq k.$$

Hence the Jacobian $J_{\mathbf{F}}(\mathbf{x}_0; \mathbf{t}_0) \neq 0$. Also, $\mathbf{F}(\mathbf{x}_0; \mathbf{t}_0) = (\mathbf{0}; \mathbf{t}_0)$. Therefore, by Theorem 13.6, there exist open sets X and Y containing $(\mathbf{x}_0; \mathbf{t}_0)$ and $(\mathbf{0}; \mathbf{t}_0)$, respectively, such that $\mathbf{F}$ is one-to-one on X, and $X = \mathbf{F}^{-1}(Y)$. Also, there exists a local inverse function $\mathbf{G}$, defined on Y and having values in X, such that

$$\mathbf{G}[\mathbf{F}(\mathbf{x};\mathbf{t})] = (\mathbf{x};\mathbf{t}),$$

and such that $\mathbf{G} \in C'$ on Y.

Now $\mathbf{G}$ can be reduced to components as follows: $\mathbf{G} = (\mathbf{v};\mathbf{w})$ where $\mathbf{v} = (v_1, \ldots, v_n)$ is a vector-valued function defined on Y with values in $\mathbf{R}^n$ and $\mathbf{w} = (w_1, \ldots, w_k)$ is also defined on Y but has values in $\mathbf{R}^k$. We can now determine $\mathbf{v}$ and $\mathbf{w}$ explicitly. The equation $\mathbf{G}[\mathbf{F}(\mathbf{x};\mathbf{t})] = (\mathbf{x};\mathbf{t})$, when written in terms of the components $\mathbf{v}$ and $\mathbf{w}$, gives us the two equations

$$\mathbf{v}[\mathbf{F}(\mathbf{x};\mathbf{t})] = \mathbf{x} \quad \text{and} \quad \mathbf{w}[\mathbf{F}(\mathbf{x};\mathbf{t})] = \mathbf{t}.$$

Hence the function $\mathbf{G}$ can be described as follows: Given a point $(\mathbf{x};\mathbf{t})$ in Y, we have $\mathbf{G}(\mathbf{x};\mathbf{t}) = (\mathbf{x}';\mathbf{t})$, where $\mathbf{x}'$ is that point in $\mathbf{R}^n$ such that $(\mathbf{x};\mathbf{t}) = \mathbf{F}(\mathbf{x}';\mathbf{t})$. This statement implies that

$$\mathbf{F}[\mathbf{v}(\mathbf{x};\mathbf{t});\mathbf{t}] = (\mathbf{x};\mathbf{t}) \quad \text{for every } (\mathbf{x};\mathbf{t}) \text{ in } Y.$$

Now we are ready to define the set T_0 and the function $\mathbf{g}$ in the theorem. Let

$$T_0 = \{\mathbf{t} : \mathbf{t} \in \mathbf{R}^k, \quad (\mathbf{0};\mathbf{t}) \in Y\},$$

and for each $\mathbf{t}$ in T_0 define $\mathbf{g}(\mathbf{t}) = \mathbf{v}(\mathbf{0};\mathbf{t})$. The set T_0 is open in $\mathbf{R}^k$. Moreover, $\mathbf{g} \in C'$ on T_0 because $\mathbf{G} \in C'$ on Y and the components of $\mathbf{g}$ are taken from the components of $\mathbf{G}$. Also,

$$\mathbf{g}(\mathbf{t}_0) = \mathbf{v}(\mathbf{0};\mathbf{t}_0) = \mathbf{x}_0$$

because $(\mathbf{0};\mathbf{t}_0) = \mathbf{F}(\mathbf{x}_0;\mathbf{t}_0)$. Finally, the equation $\mathbf{F}[\mathbf{v}(\mathbf{x};\mathbf{t});\mathbf{t}] = (\mathbf{x};\mathbf{t})$, which holds for every $(\mathbf{x};\mathbf{t})$ in Y, yields (by considering the components in $\mathbf{R}^n$) the equation $\mathbf{f}[\mathbf{v}(\mathbf{x};\mathbf{t});\mathbf{t}] = \mathbf{x}$. Taking $\mathbf{x} = \mathbf{0}$, we see that for every $\mathbf{t}$ in T_0, we have $\mathbf{f}[\mathbf{g}(\mathbf{t});\mathbf{t}] = \mathbf{0}$, and this completes the proof of statements (a), (b), and (c). It remains to prove that there is only one such function $\mathbf{g}$. But this follows at once from the one-to-one character of $\mathbf{f}$. If there were another function, say $\mathbf{h}$, which satisfied (c), then we would have $\mathbf{f}[\mathbf{g}(\mathbf{t});\mathbf{t}] = \mathbf{f}[\mathbf{h}(\mathbf{t});\mathbf{t}]$, and this would imply $(\mathbf{g}(\mathbf{t});\mathbf{t}) = (\mathbf{h}(\mathbf{t});\mathbf{t})$, or $\mathbf{g}(\mathbf{t}) = \mathbf{h}(\mathbf{t})$ for every $\mathbf{t}$ in T_0.

13.5 EXTREMA OF REAL-VALUED FUNCTIONS OF ONE VARIABLE

In the remainder of this chapter we shall consider real-valued functions f with a view toward determining those points (if any) at which f has a local extremum, that is, either a local maximum or a local minimum.

We have already obtained one result in this connection for functions of one variable (Theorem 5.9). In that theorem we found that a necessary condition for a function f to have a local extremum at an interior point c of an interval is that $f'(c) = 0$, provided that $f'(c)$ exists. This condition, however, is not sufficient, as we can see by taking $f(x) = x^3$, $c = 0$. We now derive a sufficient condition.

Theorem 13.8. *For some integer $n \geq 1$, let f have a continuous* nth *derivative in the open interval (a, b). Suppose also that for some interior point c in (a, b) we have*

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \qquad but \qquad f^{(n)}(c) \neq 0.$$

Then for n even, f has a local minimum at c if $f^{(n)}(c) > 0$, and a local maximum at c if $f^{(n)}(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum at c.

Proof. Since $f^{(n)}(c) \neq 0$, there exists an interval $B(c)$ such that for every x in $B(c)$, the derivative $f^{(n)}(x)$ will have the same sign as $f^{(n)}(c)$. Now by Taylor's formula (Theorem 5.19), for every x in $B(c)$ we have

$$f(x) - f(c) = \frac{f^{(n)}(x_1)}{n!}(x - c)^n, \qquad \text{where } x_1 \in B(c).$$

If n is even, this equation implies $f(x) \geq f(c)$ when $f^{(n)}(c) > 0$, and $f(x) \leq f(c)$ when $f^{(n)}(c) \leq 0$. If n is odd and $f^{(n)}(c) > 0$, then $f(x) > f(c)$ when $x > c$, but $f(x) < f(c)$ when $x < c$, and there can be no extremum at c. A similar statement holds if n is odd and $f^{(n)}(c) < 0$. This proves the theorem.

13.6 EXTREMA OF REAL-VALUED FUNCTIONS OF SEVERAL VARIABLES

We turn now to functions of several variables. Exercise 12.1 gives a necessary condition for a function to have a local maximum or a local minimum at an interior point $\mathbf{a}$ of an open set. The condition is that each partial derivative $D_k f(\mathbf{a})$ must be zero at that point. We can also state this in terms of directional derivatives by saying that $f'(\mathbf{a}; \mathbf{u})$ must be zero for every direction $\mathbf{u}$.

The converse of this statement is not true, however. Consider the following example of a function of two real variables:

Here we have $D_1 f(0, 0) = D_2 f(0, 0) = 0$. Now $f(0, 0) = 0$, but the function assumes both positive and negative values in every neighborhood of (0, 0), so there is neither a local maximum nor a local minimum at (0, 0). (See Fig. 13.3.)

This example illustrates another interesting phenomenon. If we take a fixed straight line through the origin and restrict the point (x, y) to move along this line toward (0, 0), then the point will finally enter the region above the parabola $y = 2x^2$ (or below the parabola $y = x^2$) in which $f(x, y)$ becomes and stays positive for every $(x, y) \neq (0, 0)$. Therefore, along every such line, f has a minimum at (0, 0), but the origin is not a local minimum in any two-dimensional neighborhood of (0, 0).

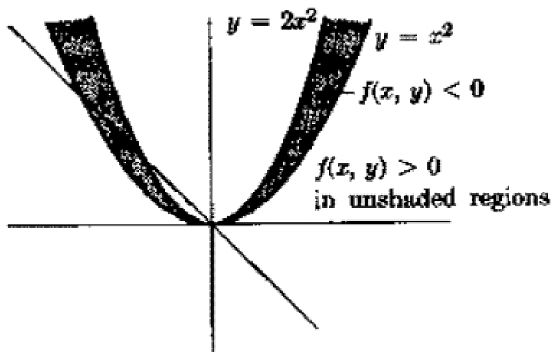

Figure 13.3

Definition 13.9. *If f is differentiable at $\mathbf{a}$ and if $\nabla f(\mathbf{a}) = \mathbf{0}$, the point $\mathbf{a}$ is called a stationary point of* **f**. *A stationary point is called a saddle point if every n-ball $B(\mathbf{a})$ contains points $\mathbf{x}$ such that $f(\mathbf{x}) > f(\mathbf{a})$ and other points such that $f(\mathbf{x}) < f(\mathbf{a})$.*

In the foregoing example, the origin is a saddle point of the function.

To determine whether a function of n variables has a local maximum, a local minimum, or a saddle point at a stationary point $\mathbf{a}$, we must determine the algebraic sign of $f(\mathbf{x}) - f(\mathbf{a})$ for all $\mathbf{x}$ in a neighborhood of $\mathbf{a}$. As in the one-dimensional case, this is done with the help of Taylor's formula (Theorem 12.14). Take $m = 2$ and $\mathbf{y} = \mathbf{a} + \mathbf{t}$ in Theorem 12.14. If the partial derivatives of f are differentiable on an n-ball $B(\mathbf{a})$ then

$$f(\mathbf{a} + \mathbf{t}) - f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{t} + \tfrac{1}{2} f''(\mathbf{z}; \mathbf{t}), \tag{3}$$

where $\mathbf{z}$ lies on the line segment joining $\mathbf{a}$ and $\mathbf{a} + \mathbf{t}$, and

$$f''(\mathbf{z}; \mathbf{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(\mathbf{z}) t_i t_j.$$

At a stationary point we have $\nabla f(\mathbf{a}) = \mathbf{0}$ so (3) becomes

$$f(\mathbf{a} + \mathbf{t}) - f(\mathbf{a}) = \tfrac{1}{2} f''(\mathbf{z}; \mathbf{t}).$$

Therefore, as $\mathbf{a} + \mathbf{t}$ ranges over $B(\mathbf{a})$, the algebraic sign of $f(\mathbf{a} + \mathbf{t}) - f(\mathbf{a})$ is determined by that of $f''(\mathbf{z}; \mathbf{t})$. We can write (3) in the form

$$f(\mathbf{a} + \mathbf{t}) - f(\mathbf{a}) = \tfrac{1}{2} f''(\mathbf{a}; \mathbf{t}) + \|\mathbf{t}\|^2 E(\mathbf{t}), \tag{4}$$

where

$$\|\mathbf{t}\|^2 E(\mathbf{t}) = \tfrac{1}{2} f''(\mathbf{z}; \mathbf{t}) - \tfrac{1}{2} f''(\mathbf{a}; \mathbf{t}).$$

The inequality

$$\|\mathbf{t}\|^2 \, |E(\mathbf{t})| \le \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |D_{i,j} f(\mathbf{z}) - D_{i,j} f(\mathbf{a})| \, \|\mathbf{t}\|^2,$$

shows that $E(\mathbf{t}) \to 0$ as $\mathbf{t} \to \mathbf{0}$ if the second-order partial derivatives of f are continuous at $\mathbf{a}$. Since $\|\mathbf{t}\|^2 E(\mathbf{t})$ tends to zero faster than $\|\mathbf{t}\|^2$, it seems reasonable to expect that the algebraic sign of $f(\mathbf{a} + \mathbf{t}) - f(\mathbf{a})$ should be determined by that of $f''(\mathbf{a}; \mathbf{t})$. This is what is proved in the next theorem.

Theorem 13.10 (Second-derivative test for extrema). *Assume that the second-order partial derivatives $D_{i,j} f$ exist in an n-ball $B(\mathbf{a})$ and are continuous at $\mathbf{a}$, where $\mathbf{a}$ is a stationary point of f. Let*

$$Q(\mathbf{t}) = \tfrac{1}{2} f''(\mathbf{a}; \mathbf{t}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(\mathbf{a}) t_i t_j. \tag{5}$$

a) *If $Q(\mathbf{t}) > 0$ for all $\mathbf{t} \ne \mathbf{0}$, f has a relative minimum at $\mathbf{a}$.*

b) *If $Q(\mathbf{t}) < 0$ for all $\mathbf{t} \ne \mathbf{0}$, f has a relative maximum at $\mathbf{a}$.*

c) *If $Q(\mathbf{t})$ takes both positive and negative values, then f has a saddle point at $\mathbf{a}$.*

Proof. The function Q is continuous at each point $\mathbf{t}$ in $\mathbf{R}^n$. Let $S = \{\mathbf{t} : \|\mathbf{t}\| = 1\}$ denote the boundary of the n-ball $B(\mathbf{0}; 1)$. If $Q(\mathbf{t}) > 0$ for all $\mathbf{t} \neq \mathbf{0}$, then $Q(\mathbf{t})$ is positive on S. Since S is compact, Q has a minimum on S (call it m), and $m > 0$. Now $Q(c\mathbf{t}) = c^2 Q(\mathbf{t})$ for every real c. Taking $c = 1/\|\mathbf{t}\|$ where $\mathbf{t} \neq \mathbf{0}$ we see that $c\mathbf{t} \in S$ and hence $c^2 Q(\mathbf{t}) \geq m$, so $Q(\mathbf{t}) \geq m\|\mathbf{t}\|^2$. Using this in (4) we find

$$f(\mathbf{a} + \mathbf{t}) - f(\mathbf{a}) = Q(\mathbf{t}) + \|\mathbf{t}\|^2 E(\mathbf{t}) \geq m\,\|\mathbf{t}\|^2 + \|\mathbf{t}\|^2 E(\mathbf{t}).$$

Since $E(\mathbf{t}) \to 0$ as $\mathbf{t} \to \mathbf{0}$, there is a positive number r such that $|E(\mathbf{t})| < \frac{1}{2}m$ whenever $0 < \|\mathbf{t}\| < r$. For such $\mathbf{t}$ we have $0 \leq \|\mathbf{t}\|^2\,|E(\mathbf{t})| < \frac{1}{2}m\|\mathbf{t}\|^2$, so

$$f(\mathbf{a} + \mathbf{t}) - f(\mathbf{a}) > m\|\mathbf{t}\|^2 - \tfrac{1}{2}m\|\mathbf{t}\|^2 = \tfrac{1}{2}m\|\mathbf{t}\|^2 > 0.$$

Therefore f has a relative minimum at $\mathbf{a}$, which proves (a). To prove (b) we use a similar argument, or simply apply part (a) to $-f$.

Finally, we prove (c). For each $\lambda > 0$ we have, from (4),

$$f(\mathbf{a} + \lambda\mathbf{t}) - f(\mathbf{a}) = Q(\lambda\mathbf{t}) + \lambda^2\|\mathbf{t}\|^2 E(\lambda\mathbf{t}) = \lambda^2\{Q(\mathbf{t}) + \|\mathbf{t}\|^2 E(\lambda\mathbf{t})\}.$$

Suppose $Q(\mathbf{t}) \neq 0$ for some $\mathbf{t}$. Since $E(\mathbf{y}) \to 0$ as $\mathbf{y} \to \mathbf{0}$, there is a positive r such that

$$\|\mathbf{t}\|^2 E(\lambda\mathbf{t}) < \tfrac{1}{2}|Q(\mathbf{t})| \qquad \text{if } 0 < \lambda < r.$$

Therefore, for each such λ the quantity $\lambda^2\{Q(\mathbf{t}) + \|\mathbf{t}\|^2 E(\lambda\mathbf{t})\}$ has the same sign as $Q(\mathbf{t})$. Therefore, if $0 < \lambda < r$, the difference $f(\mathbf{a} + \lambda\mathbf{t}) - f(\mathbf{a})$ has the same sign as $Q(\mathbf{t})$. Hence, if $Q(\mathbf{t})$ takes both positive and negative values, it follows that f has a saddle point at $\mathbf{a}$.

NOTE. A real-valued function Q defined on $\mathbf{R}^n$ by an equation of the type

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and the a_{ij} are real is called a *quadratic form*. The form is called *symmetric* if $a_{ij} = a_{ji}$ for all i and j, *positive definite* if $\mathbf{x} \neq \mathbf{0}$ implies $Q(\mathbf{x}) > 0$, and *negative definite* if $\mathbf{x} \neq \mathbf{0}$ implies $Q(\mathbf{x}) < 0$.

In general, it is not easy to determine whether a quadratic form is positive or negative definite. One criterion, involving eigenvalues, is described in Reference

13.1, Theorem 9.5. Another, involving determinants, can be described as follows. Let $\Delta = \det[a_{ij}]$ and let Δ_k denote the determinant of the $k \times k$ matrix obtained by deleting the last $(n - k)$ rows and columns of $[a_{ij}]$. Also, put $\Delta_0 = 1$. From the theory of quadratic forms it is known that a necessary and sufficient condition for a symmetric form to be positive definite is that the $n + 1$ numbers $\Delta_0, \Delta_1, \ldots, \Delta_n$ be positive. The form is negative definite if, and only if, the same $n + 1$ numbers are alternately positive and negative. (See Reference 13.2, pp. 304–308.) The quadratic form which appears in (5) is symmetric because the mixed partials $D_{i,j}f(\mathbf{a})$ and $D_{j,i}f(\mathbf{a})$ are equal. Therefore, under the conditions of Theorem 13.10, we see that f has a local minimum at $\mathbf{a}$ if the $(n + 1)$ numbers $\Delta_0, \Delta_1, \ldots, \Delta_n$ are all positive, and a local maximum if these numbers are alternately positive and negative. The case $n = 2$ can be handled directly and gives the following criterion.

Theorem 13.11. *Let f be a real-valued function with continuous second-order partial derivatives at a stationary point* $\mathbf{a}$ *in* $\mathbf{R}^2$. *Let*

$$A = D_{1,1}f(\mathbf{a}), \qquad B = D_{1,2}f(\mathbf{a}), \qquad C = D_{2,2}f(\mathbf{a}),$$

and let

$$\Delta = \det\begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2.$$

Then we have:

a) *If* $\Delta > 0$ *and* $A > 0$, *f has a relative minimum at* $\mathbf{a}$.

b) *If* $\Delta > 0$ *and* $A < 0$, *f has a relative maximum at* $\mathbf{a}$.

c) *If* $\Delta < 0$, *f has a saddle point at* $\mathbf{a}$.

Proof. In the two-dimensional case we can write the quadratic form in (5) as follows:

$$Q(x, y) = \tfrac{1}{2}\{Ax^2 + 2Bxy + Cy^2\}.$$

If $A \neq 0$, this can also be written as

$$Q(x, y) = \frac{1}{2A}\{(Ax + By)^2 + \Delta y^2\}.$$

If $\Delta > 0$, the expression in brackets is the sum of two squares, so $Q(x, y)$ has the same sign as A. Therefore, statements (a) and (b) follow at once from parts (a) and (b) of Theorem 13.10.

If $\Delta < 0$, the quadratic form is the product of two linear factors. Therefore, the set of points (x, y) such that $Q(x, y) = 0$ consists of two lines in the xy-plane intersecting at $(0, 0)$. These lines divide the plane into four regions; $Q(x, y)$ is positive in two of these regions and negative in the other two. Therefore f has a saddle point at $\mathbf{a}$.

POSSIBLE QUESTIONS

1. Let A be an open subset of R^n and assume that $f : A \to R^n$ has continuous partial derivatives $D_j f_j$ on A . If $J_f(x) \neq 0\ \forall x$ in A ,then prove that f is an open mapping.

2. Prove that , let A be an open subset of R^n and assume that $f : A \to R^n$ has continuous partial derivatives $D_j f_j$ on A.If f is 1- 1 on A and if $J_f(x) \neq 0\ \forall x$ in A , then f (A) is an open.

3. Let $\{ f_n \}$ be boundadly convergent sequence [a,b] Assume that each $f_n \in$. R on [a,b] and that the limit function $f \in$. R on [a,b].assume also that there is a partition P of [a,b] say $\{ x_0, x_1, x_2, \ldots\ldots x_m \}$ such that on every sub interval [c,d] not containing any of the points x_k the sequence $\{ f_n \}$ uniformly converges to f. then prove that we have $\int_a^b f_n(t)d\,t = \int_a^b \lim_{n\to\infty} f_n(t)d\,t = \int_a^b f(t)d\,t$.

4. State and prove functions with non Zero Jacobian determinant.

5. Assume that $f = \{ f_1, f_2, \ldots\ldots\ldots, f_n \}$ has continuous partial derivatives $D_j\ f_j$ on an open set in R^n & that prove that the Jacobian determinant $J_f(a) \neq 0$ for some point a in S ,then there is an n- ball B (a) on which f is 1 –1.

6. State and prove Implicit function theorem.

7. Define saddle point with example and Define Jacobian determinant .

8. (i)State and prove Cauchy condition for uniform convergence
(ii) Assume that $f_n \to f$ uniformly on S.If each f_n is continuous at a point C of S, then show that the limit function f is also continuous at c

9. State and Prove Second derivative for Exreme

10. For some integer $n \geq 1$, Let f have continuous n^{th} derivative in the open interval (a,b). Suppose for some interior point c in (a,b) we have $f'(c), f''(c) \ldots f^{(n+1)}(c) = 0$ but $f^{n}(c) \neq 0$.Then prove that for n even,f has a local minimum at c if $f^{n}(c) > 0$ & a local maximum at c if $f^{n}(c) < 0$. If n is odd, there is neither a local minimum nor a local maximum at c.

11. State and prove Inverse function Theorem.

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Subject: Real Analysis **Subject**
Class : I - M.Sc. Mathematics **Semester**

Unit V

Part A (20x1=20 Marks) **(Question Nos. 1 to 20**

Possible Questions

Question	**Choice 1**	**Choice 2**	**Choice 3**	**Choice 4**	
The outer measure of its interval is its	value	length	size of the interval	value	length
Every increasing function of its	bounded variation	variational value	bounded	variation	bounded variation
v (x) is monotonic---------------- function	increasing	decreasing	either increasing or decreasing	neither increasing nor decreasing	increasing
A function f of bounded variation is the expressible as a	monotonic increasing functions	monotonic increasing functions	monotonic increasing functions	monotonic increasing functions	monotonic increasing functions
Outer Lebesgue measure is also known as	Lebesgue exterior measure	measure	Lebesgue measure	Lebesgue interior measure	
If set A is Lebesgue measurable and m *(A ∆ B) =---------------	0	1	2	3	0
The sets S 1,S 2,……S n are called the --------------------- of the partition P	components	parts	partition	combined	components
The Refinement of P is denoted by	p1	p2	p '	P *	
Every singleton set is ----------------- set	disconnected set	connected	measurable	unmeasurable	connected
Intersection of finite number of open set is	closed	open	neither open nor not closed	semiopen	open
Union of finite number of closed set is	closed	open	neither open nor not closed	semi closed	closed
Every closed interval is	closed	compact	closed	not compact	compact
In a metric space (X , d) a non-empty X is	closed	compact	closed	not compact	compact
Every infinite set A has a	no limit point	neibourhood	limit point	need not be a limit point	limit point
If f is a continuous mapping of a compact metric space X into M.space y	f is uniformly continuous	not uniformly continuous	continuous	discotinuous	f is uniformly continuous
If E1 and E 2 are lebesgue measurable set then	E1 union E2 is also lebesgue measurable sets	E1 = E 2	E1 >E 2	E1 <E 2	E1 union E2 is also lebesgue measurable sets
Let f be an open covering of A , then	there exist countable collection of F which covers A	there exist Uncountable collection of F which covers A	there exist collection of F which covers A	there exist Uncountable collection of F which not covers A	there exist countable collection of F which covers A
If f is continuous Real valued function of Compact metric space then	f is bounded	f is unbounded	f is constant	f is a function	f is bounded
If f is an open covering of a closed and bounded set A then	there exist a finite subcollection of F which covers A	a finite subcollection of F which covers A	infinite subcollection of F which covers A	there exist a finite collection of F which covers A	there exist a finite subcollection of F which covers A
Any Countable set of points on the real axis line	no measure	measure is 1	measure zero	not measurable	measure zero
A real valued function ϕ is --------------- function if it is lebesgue measurable	large	simple	minor	major	simple
Every open and closed set is ------------------------- measurable	zero	lebesgue	not measurable	need not be measurable	lebesgue
Every ------------------- set is lebesgue measurable	cantor set	empty set	heine borel	non -empty set	Heine borel
Which one of the following is true ?	Family M of Lebesgue measurable sets is an algebra of sets	Family M of Lebesgue measurable sets is not algebra of sets	Family M of Lebesgue measurable sets	Lebesgue measurable sets	Family M of Lebesgue measurable sets is an algebra of sets
If Lebesgue outer measure of a set E, m * E= 0 then	E is measurable	E is not measurable	E is need not be measurable	neither measurable nor not measurable	E ist measurable
Which one of the following is true ?	Lebesgue exterior measure , m * is translation is invariant	Lebesgue exterior measure , m * is not translation is invariant	need not be measurable	Lebesgue interior measure , m * is not translation is invariant	Lebesgue exterior measure , m * is translation is invariant
If A is countable , then	m * A =0	m * A =infinity	m * A =1	m * A ≠ 0	m * A =0

If f : [a,b] →R monotonic then	f is of bounded variation	f is unbounded	discontinuity of f are uncountable	of f are uncountable	f is of bounded variation
Every sequence x n in a metric space X is convergent then every cauchys sequence	convergent	divergent	constant	convergent as well as divergent	convergent
If function f is the difference of two monotonic real valued functions on [a,b] then	f is of bounded variation	f is of not bounded variation	f is of need not bounded variation	f is of may be bounded variation	f is of bounded variation
If x is an accumulation point of S ⊆ R	every open ball B (x) contains X	every open ball B (x) contains finitely many points	(x) does not contains many of itspoints	every open ball B (x) contains few points	every open ball B (x) contains X
If a set S R^ n contains all its adherent points then S is	closed	open	neither open nor not closed	not closed	closed
If R ^ n -S is open ,then S R ^n is	open	closed	not open	not closed	closed
Let X is metric Space If X is sequentially compact then X is	unbounded	not compact	compact	bounded	compact
A metric space X has the BolZana weierstrass property , if X is	sequentially compact	unbounded	not compact	compact	sequentially compact
If we take g(x)=x and h(x) =1 in general mean value theorem we obtain	Lagrange's mean value theorem	Cauchy's mean value theorem	Rolle's theorem	Taylor's theorem	Cauchy's mean value theorem
Any closed interval with usual metric is	compact	not compact	need not be compact	sequentially compact	not compact
The Euclidean Space R^n is	not separable	separable	connected	disconnected	seperable
Every dense subset is ----------------------- in I_∞	countable	uncountable	bounded	unbounded	countable
The usual metric space (R ,d) is	seperable	not separable	not compact	compact	compact
The set of rational numbers lebesque outer measure is =	1	0	3	4	0
Every measurable set is nearly a finite union of	set	open sat	closed	intervals	intervals
Everuy convergent sequence of measurable functions is nearly	convergence	divergence	uniformly convergence	absolutely convergence	uniformly convergence

Reg. No................................

[17MMP102]

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M.Sc., DEGREE EXAMINATION, NOVEMBER 2017

First Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a. If $f \in R(\alpha)$ on $[a,b]$ and $f\, f \in R(\beta)$ on $[a,b]$ then $f \in R(c_1\ \alpha + c_2\,\beta)$ on $[a,b]$ we have $\int_a^b f d(c_1\alpha + c_2\beta) = c_1\int_a^b f d\alpha + c_2\int_a^b f d\beta$

Or

b. State and prove change of variable in Riemann – Stieltjes integral .

22. a. State and prove Merten's Theorem

Or

b. State and prove Ratio Test Theorem.

23. a. State and prove Cauchy's condition for Uniform convergence .

Or

b. Assume that $\Sigma f_n(x) = f(x)$ (Uniformly continuous on S) if each f_n is continuous at a point x_0 of S then f is also continuous at x_0.

24. a. State and prove Lesgue dominated convergence Theorem

Or

b. Assume f is Riemann integrable on [a,b] $\forall\, b \geq a$ and assume that there is a positive constant M such that $\int_a^b |f(x)|\,dx \leq M\ \forall\, b \geq a$.

25. a. Let A be an open subset of R^n and assume that $f : A \to R^n$ has continuous partial Derivatives $D_j f_j$ on A.If f is 1- 1 on A and if $J_f(x) \neq 0\ \forall\, x$ in A , then f (A) is an open.

Or

b. State and prove functions with non Zero Jacobian determinant.

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. State and prove Rearrangement Theorem for double sequence .

Reg. No................................

[15MMP102]

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M.Sc., DEGREE EXAMINATION, NOVEMBER 2015

First Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a) If $f \in R(\alpha)$ on $[a,b]$ and $f\, f \in R(\beta)$ on $[a,b]$ then $f \in R(c_1 \alpha + c_2 \beta)$ on $[a,b]$ we have $\int_a^b f d(c_1 \alpha + c_2 \beta) = c_1 \int_a^b f d\alpha + c_2 \int_a^b f d\beta$.

Or

b) State and prove change of variable in Riemann – Stieltjes integral .

22 . a) State and prove Merten's Theorem

Or

b) State and prove Ratio Test Theorem.

23. a) State and prove Cauchy's condition for Uniform convergence .

Or

b) Assume that $\Sigma f_n(x) = f(x)$ (uniformly on S) if each f_n is continuous at a point x_0 of S then f is also continuous at x_0.

24. a) State and prove Lesgue dominated convergence Theorem

Or

b) Assume f is Riemann integrable on $[a,b]$ $\forall\, b \geq a$ and assume there is a positive constant M such that $\int_a^b |f(x)|\, dx \leq M \;\; \forall\, b \geq a$.

25. a) Let A be an open subset of R^n and assume that $f : A \to R^n$ has continuous partial Derivatives $D_j f_j$ on A.If f is 1- 1 on A and if $J_f(x) \neq 0 \;\forall\, x$ in A , then f(A) is an open.

Or

b) State and prove functions with non Zero Jacobian determinant.

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. State and prove Rearrangement Theorem for double sequence.
