KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

DEPARTMENT OF MATHEMATICS

17MMP306 INTEGRAL EQUATIONS AND TRANSFORMS

SEMESTER-III

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Course Objectives:

To enable the students

- To come to know volterra integral equations and Fredholm integral equations .
- how to reduce the differential equations to integral equations .

Course Outcomes:

On successful completion of this course the students will be able to ,

- Calculate the Laplace equation in half plane of standard functions both from the definition and by using tables.
- Equation with seperable kernel and Fredholm alternative approximation method
- Select and combine the necessary Laplace transform techniques to solve second-
- order ordinary differential equations .
  Calculate both real and complex forms of the Fourier series .
- Calculate the Fourier transform of elementary functions from the definition.

UNIT I

Fourier transforms: Fourier Transforms – Definition of Inversion theorem –Fourier cosine transforms - Fourier sine transforms – Fourier transforms of derivatives -Fourier transforms of some simple functions - Fourier transforms of rational function.

UNIT II

The convolution integral – convolution theorem – Parseval's relation for Fourier transforms – solution of PDE by Fourier transform – Laplace's Equation in Half plane – Laplace's Equation in an infinite strip - The Linear diffusion equation on a semi-infinite line - The two-dimensional diffusion equation.

UNIT III

Integral equations: Types of Integral equations–Equation with separable kernel- Fredholm Alternative Approximate method – Volterra integral equations–Classical Fredholm theory – Fredholm's First, Second, Third theorems.

UNIT IV

Application of Integral equation to ordinary differential equation – initial value problems – Boundary value problems – singular integral equations – Abel Integral equation .

UNIT V

Calculus of variations: Variation and its properties – Euler's equation – Functionals of the integral forms - Functional dependent on higher order derivatives – functionals dependent on the functions of several independent variables – variational problems in parametric form.

SUGGESTED READINGS

TEXT BOOKS

1. Sneedon. I. N, (1974). The Use of Integral Transforms, Tata Mc Graw Hill, New Delhi. **(For Unit –I & II)**
2. Kanwal, R. P, (2013). Linear integral Equations Theory and Technique, Academic press, New York. **(For Unit –III & IV**)
3. Elsgots, L., (2003). Differential Equations and Calculus of Variation, Mir Publication Moscow. **(For Unit –V)**

REFERENCES

1. Gelfand, I. M and Francis, S.V. (2000). Calculus of Variation, Prentice Hall, India.
2. Tricomi.F.G, (1985). Integral Equations, Dover, New York.
3. Larry C. Andrews and Bhimson K. Shivamoggi, (1999). The Integral transforms for Engineers ,Spie Press, Washington.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: M.Sangeetha
Subject Name: Integral Equations and Transforms **Subject Code:17MMU306**
Semester:III **Class: II M.Sc Mathematics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1.	1	Introduction to Fourier transforms & definition of Inversion Theorem	T1:chap 2.3:pg.No:36-41
2.	1	Fourier cosine transforms	T1:chap 2.4:pg.No:42-43
3.	1	Fourier sine transforms	T1:chap 2.5:pg.No:44-45
4.	1	Fourier transforms of derivatives	T1:chap 2.6:pg.No:46-48
5.	1	Fourier transforms of some simple functions	R3:chap 2.4:pg.No:49-52
6.	1	Problems on Fourier transforms of some simple functions	R3:chap 2.4:pg.No:53-56
7.	1	Fourier transforms of rational function	R3:chap 2.4:pg.No:57-61
8.	1	Continuation on Fourier transforms of rational function	R3:chap 2.4:pg.No:62-65
9.	1	Recapitulation and discussion of important questions	
		Total Hours	**9 Hours**
		UNIT-II	
1.	1	Introduction to the convolution integral	T1:chap 2.9:pg.No:58-59
2.	1	Convolution theorem	T1:chap 2.9:pg.No:59-60
3.	1	Parseval's relation for Fourier transforms	T1:chap 2.10:pg.No:61-62

4.	1	Solution of PDE by Fourier transform	T1:chap 2.16:pg.No:92-93
5.	1	Laplace's equation in Half plane	R3:chap 2.7:pg.No:72-74
6.	1	Continuation on Laplace's equation in Half plane	R3:chap 2.7:pg.No:75-77
7.	1	Laplace's equation in an infinite strip	R3:chap 2.7:pg.No:78-80
8.	1	The linear diffusion equation on a semi infinite line	R3:chap 2.7:pg.No:81-83
9.	1	The two dimensional diffusion equation	R3:chap 2.4:pg.No:84-85
10.	1	Recapitulation and discussion of important questions	
	Total Hours		**10 Hours**
		UNIT-III	
1.	1	Introduction to Integral equation and types of integral equations	T2:chap 2:pg.No:7-10
2.	1	Equations with separable kernel	T2:chap 2:pg.No:11-15
3.	1	Fredholm Alternative Approximate method	T2:chap 2:pg.No:16-20
4.	1	Continuation on Fredholm Alternative Approximate method	T2:chap 2:pg.No:21-25
5.	1	Volterra integral equations	R2:chap 1:pg.No:2-8
6.	1	Classical Fredholm theory	T2:chap 3:pg.No:31-35
7.	1	Fredholm's First,second,third theorems	T2:chap 3:pg.No:36-39
8.	1	Continuation on Fredholm's First,second,third theorems	R2:chap 2:pg.No:49-56
9.	1	Recapitulation and discussion of important questions	
	Total Hours		**9 Hours**
		UNIT-IV	
1.	1	Introduction for Application of integral equation to ordinary differential equation	T2:chap 5:pg.No:61-63
2.	1	Initial value problems	T2:chap 5:pg.No:64-65
3.	1	Continuation on initial value problems	T2:chap 5:pg.No:66-67
4.	1	Boundary value problems	R2:chap 4:pg.No:57-58

5.	1	Continuation on boundary value problems	R2:chap 4:pg.No:59-61
6.	1	Singular integral equations	T2:chap 8:pg.No:165-166
7.	1	Continuation on singular integral equations	T2:chap 8:pg.No:167-168
8.	1	Abel integral equation	T2:chap 8:pg.No:169-170
9.	1	Continuation on Abel integral equation	T2:chap 8:pg.No:171-172
10.	1	Recapitulation and discussion of important questions	
		Total Hours	**10 Hours**
		UNIT-V	
1.	1	Introduction to calculus transformations and its properties	T3:chap 6:pg.No:293-298
2.	1	Euler's equations and related examples	R1:chap 1:pg.No:5-9
3.	1	Functionals of the integral forms	T3:chap 6:pg.No:305-310
4.	1	Functional dependent on higher order derivatives	T3:chap 6:pg.No:311-313
5.	1	Functional dependent on the functions of several independent variables	T3:chap 6:pg.No:314-316
6.	1	Variational problems in parametric form	R1:chap 2:pg.No:36-40
7.	1	Recapitulation and discussion of important questions	
8.	1	Discuss on previous ESE question papers	
9.	1	Discuss on previous ESE question papers	
10.	1	Discuss on previous ESE question papers	
		Total Hours	**10 Hours**
		Total Planned Hours	**48 Hours**

SUGGESTED READINGS

TEXT BOOKS

1. Sneedon. I. N, (1974). The Use of Integral Transforms, Tata Mc Graw Hill, New Delhi. **(For Unit –I & II)**
2. Kanwal, R. P, (2013). Linear integral Equations Theory and Technique, Academic press, New York. **(For Unit –III & IV**)
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REFERENCES

1. Gelfand, I. M and Francis, S.V. (2000). Calculus of Variation, Prentice Hall, India.
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2. Larry C. Andrews and Bhimson K. Shivamoggi, (1999). The Integral transforms for Engineers , Spie Press, Washington.

UNIT – I
SYLLABUS

Introduction to Fourier transforms & definition of Inversion Theorem-Fourier cosine transforms Fourier sine transforms-Fourier transforms of derivatives- Fourier transforms of some simple functions- Fourier transforms of rational function.

The Fourier Transform

Fourier transforms as integrals

There are several ways to define the Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$. In this section, we define it using an integral representation and state some basic uniqueness and inversion properties, without proof. Thereafter, we will consider the transform as being defined as a suitable limit of Fourier series, and will prove the results stated here.

Definition 1 *Let $f : \mathbb{R} \to \mathbb{R}$. The Fourier transform of $f \in L^1(\mathbb{R})$, denoted by $\mathcal{F}[f](.)$, is given by the integral:*

$$\mathcal{F}[f](x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\exp(-ixt)dt$$

for $x \in \mathbb{R}$ for which the integral exists. *

We have the **Dirichlet condition** for inversion of Fourier integrals.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that (1) $\int_{-\infty}^{\infty} |f|\, dt$ converges and (2) in any finite interval, f, f' are piecewise continuous with at most finitely many maxima/minima/discontinuities. Let $F = \mathcal{F}[f]$. Then if f is continuous at $t \in \mathbb{R}$, we have*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\exp(itx)dx.$$

Moreover, if f is discontinuous at $t \in \mathbb{R}$ and $f(t+0)$ and $f(t-0)$ denote the right and left limits of f at t, then

$$\frac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} F(x)\exp(itx)dx.$$

From the above, we deduce a uniqueness result:

Theorem 2 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous, f', g' piecewise continuous. If*

$$\mathcal{F}[f](x) = \mathcal{F}[g](x),\ \forall x$$

then

$$f(t) = g(t),\ \forall t.$$

Proof: We have from inversion, easily that

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}[f](x)\exp(itx)dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}[g](x)\exp(itx)dx$$
$$= g(t).$$

□

Example 1 *Find the Fourier transform of $f(t) = \exp(-|t|)$ and hence using inversion, deduce that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ and $\int_0^\infty \frac{x\sin(xt)}{1+x^2}dx = \frac{\pi\exp(-t)}{2}$, $t > 0$.*

Solution We write

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\exp(-ixt)dt$$
$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} \exp(t(1-ix))dt + \int_0^\infty \exp(-t(1+ix))\right]$$
$$= \sqrt{\frac{2}{\pi}}\frac{1}{1+x^2}.$$

Now by the inversion formula,

$$\exp(-|t|) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\exp(ixt)dx$$
$$= \frac{1}{\pi}\left[\int_0^\infty \frac{\exp(ixt)+\exp(-ixt)}{1+x^2}dt\right]$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\cos(xt)}{1+x^2}dx.$$

Now this formula holds at $t = 0$, so substituting $t = 0$ into the above gives the first required identity. Differentiating with respect to t as we may for $t > 0$, gives the second required identity. □.

Proceeding in a similar way as the above example, we can easily show that

$$\mathcal{F}[\exp(-\frac{1}{2}t^2)](x) = \exp(-\frac{1}{2}x^2),\ x \in \mathbb{R}.$$

We will discuss this example in more detail later in this chapter.

We will also show that we can reinterpret Definition 1 to obtain the Fourier transform of any complex valued $f \in L^2(\mathbb{R})$, and that the Fourier transform is unitary on this space:

Theorem 3 *If $f, g \in L^2(\mathbb{R})$ then $\mathcal{F}[f], \mathcal{F}[g] \in L^2(\mathbb{R})$ and*

$$\int_{-\infty}^{\infty} f(t)\overline{g}(t)\ dt = \int_{-\infty}^{\infty} \mathcal{F}[f](x)\overline{\mathcal{F}[g](x)}\ dx.$$

This is a result of fundamental importance for applications in signal processing.

1.2 The transform as a limit of Fourier series

We start by constructing the Fourier series (complex form) for functions on an interval $[-\pi L, \pi L]$. The ON basis functions are

$$e_n(t) = \frac{1}{\sqrt{2\pi L}} e^{\frac{int}{L}}, \qquad n = 0, \pm 1, \cdots,$$

and a sufficiently smooth function f of period $2\pi L$ can be expanded as

$$f(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} f(x) e^{-\frac{inx}{L}} dx \right) e^{\frac{int}{L}}.$$

For purposes of motivation let us abandon periodicity and think of the functions f as differentiable everywhere, vanishing at $t = \pm\pi L$ and identically zero outside $[-\pi L, \pi L]$. We rewrite this as

$$f(t) = \sum_{n=-\infty}^{\infty} e^{\frac{int}{L}} \frac{1}{2\pi L} \hat{f}(\frac{n}{L})$$

which looks like a Riemann sum approximation to the integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda \qquad (1.2.1)$$

to which it would converge as $L \to \infty$. (Indeed, we are partitioning the λ interval $[-L, L]$ into $2L$ subintervals, each with partition width $1/L$.) Here,

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt. \quad (1.2.2)$$

Similarly the Parseval formula for f on $[-\pi L, \pi L]$,

$$\int_{-\pi L}^{\pi L} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi L}|\hat{f}(\frac{n}{L})|^2$$

goes in the limit as $L \to \infty$ to the *Plancherel identity*

$$2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda. \quad (1.2.3)$$

Expression (1.2.2) is called the *Fourier integral* or *Fourier transform* of f. Expression (1.2.1) is called the *inverse Fourier integral* for f. The Plancherel identity suggests that the Fourier transform is a one-to-one norm preserving map of the Hilbert space $L^2[-\infty, \infty]$ onto itself (or to another copy of itself). We shall show that this is the case. Furthermore we shall show that the pointwise convergence properties of the inverse Fourier transform are somewhat similar to those of the Fourier series. Although we could make a rigorous justification of the the steps in the Riemann sum approximation above, we will follow a different course and treat the convergence in the mean and pointwise convergence issues separately.

A second notation that we shall use is

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = \frac{1}{\sqrt{2\pi}}\hat{f}(\lambda) \quad (1.2.4)$$

$$\mathcal{F}^*[g](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\lambda)e^{i\lambda t}d\lambda \quad (1.2.5)$$

Note that, formally, $\mathcal{F}^*[\hat{f}](t) = \sqrt{2\pi}f(t)$. The first notation is used more often in the engineering literature. The second notation makes clear that $\mathcal{F}$ and $\mathcal{F}^*$ are linear operators mapping $L^2[-\infty, \infty]$ onto itself in one view, and $\mathcal{F}$ mapping the *signal space* onto the *frequency space* with $\mathcal{F}^*$ mapping the frequency space onto the signal space in the other view. In this notation the Plancherel theorem takes the more symmetric form

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}[f](\lambda)|^2 d\lambda.$$

Examples:

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = \int_{-\pi}^{\pi} \cos(3t)\cos(\lambda t)dt$$

$$= \frac{2\lambda \sin(\lambda\pi)}{9-\lambda^2}.$$

3. A truncated sine wave.

$$f(t) = \begin{cases} \sin 3t & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

Since the sine is an odd function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = -i\int_{-\pi}^{\pi} \sin(3t)\sin(\lambda t)dt$$

$$= \frac{-6i\sin(\lambda\pi)}{9-\lambda^2}.$$

4. A triangular wave.

$$f(t) = \begin{cases} 1+t & \text{if } -1 \le t \le 0 \\ -1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.8}$$

Then, since f is an even function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = 2\int_{0}^{1} (1-t)\cos(\lambda t)dt$$

$$= \frac{2-2\cos\lambda}{\lambda^2}.$$

NOTE: The Fourier transforms of the discontinuous functions above decay as $\frac{1}{\lambda}$ for $|\lambda| \to \infty$ whereas the Fourier transforms of the continuous functions decay as $\frac{1}{\lambda^2}$. The coefficients in the Fourier series of the analogous functions decay as $\frac{1}{n}$, $\frac{1}{n^2}$, respectively, as $|n| \to \infty$.

1.2.1 Properties of the Fourier transform

Recall that

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt = \frac{1}{\sqrt{2\pi}} \hat{f}(\lambda)$$

$$\mathcal{F}^*[g](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\lambda) e^{i\lambda t} d\lambda$$

We list some properties of the Fourier transform that will enable us to build a repertoire of transforms from a few basic examples. Suppose that f, g belong to $L^1[-\infty, \infty]$, i.e., $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ with a similar statement for g. We can state the following (whose straightforward proofs are left to the reader):

1. $\mathcal{F}$ and $\mathcal{F}^*$ are linear operators. For $a, b \in C$ we have

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g], \quad \mathcal{F}^*[af + bg] = a\mathcal{F}^*[f] + b\mathcal{F}^*[g].$$

2. Suppose $t^n f(t) \in L^1[-\infty, \infty]$ for some positive integer n. Then

$$\mathcal{F}[t^n f(t)](\lambda) = i^n \frac{d^n}{d\lambda^n} \{\mathcal{F}[f](\lambda)\}.$$

3. Suppose $\lambda^n f(\lambda) \in L^1[-\infty, \infty]$ for some positive integer n. Then

$$\mathcal{F}^*[\lambda^n f(\lambda)](t) = i^n \frac{d^n}{dt^n} \{\mathcal{F}^*[f](t)\}.$$

4. Suppose the nth derivative $f^{(n)}(t) \in L^1[-\infty, \infty]$ and piecewise continuous for some positive integer n, and f and the lower derivatives are all continuous in $(-\infty, \infty)$. Then

$$\mathcal{F}[f^{(n)}](\lambda) = (i\lambda)^n \mathcal{F}[f](\lambda)\}.$$

5. Suppose nth derivative $f^{(n)}(\lambda) \in L^1[-\infty, \infty]$ for some positive integer n and piecewise continuous for some positive integer n, and f and the lower derivatives are all continuous in $(-\infty, \infty)$. Then

$$\mathcal{F}^*[f^{(n)}](t) = (-it)^n \mathcal{F}^*[f](t).$$

6. The Fourier transform of a translation by real number a is given by

$$\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a}\mathcal{F}[f](\lambda).$$

7. The Fourier transform of a scaling by positive number b is given by

$$\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f](\frac{\lambda}{b}).$$

8. The Fourier transform of a translated and scaled function is given by

$$\mathcal{F}[f(bt-a)](\lambda) = \frac{1}{b}e^{-i\lambda a/b}\mathcal{F}[f](\frac{\lambda}{b}).$$

Examples

- We want to compute the Fourier transform of the rectangular box function with support on $[c, d]$:

$$R(t) = \begin{cases} 1 & \text{if } c < t < d \\ \frac{1}{2} & \text{if } t = c, d \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the box function

$$\Pi(t) = \begin{cases} 1 & \text{if } -\pi < t < \pi \\ \frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

has the Fourier transform $\hat{\Pi}(\lambda) = 2\pi \text{ sinc } \lambda$. but we can obtain R from Π by first translating $t \rightarrow s = t - \frac{(c+d)}{2}$ and then rescaling $s \rightarrow \frac{2\pi}{d-c}s$:

$$R(t) = \Pi(\frac{2\pi}{d-c}t - \pi\frac{c+d}{d-c}).$$

$$\hat{R}(\lambda) = \frac{4\pi^2}{d-c}e^{i\pi\lambda(c+d)/(d-c)}\text{sinc}(\frac{2\pi\lambda}{d-c}). \qquad (1.2.9)$$

Furthermore, from (??) we can check that the inverse Fourier transform of $\hat{R}$ is R, i.e., $\mathcal{F}^*(\mathcal{F})R(t) = R(t)$.

- Consider the truncated sine wave

$$f(t) = \begin{cases} \sin 3t & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$$

with

$$\hat{f}(\lambda) = \frac{-6i\sin(\lambda\pi)}{9-\lambda^2}.$$

Note that the derivative f' of $f(t)$ is just $3g(t)$ (except at 2 points) where $g(t)$ is the truncated cosine wave

$$g(t) = \begin{cases} \cos 3t & \text{if } -\pi < t < \pi \\ -\frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

We have computed

$$\hat{g}(\lambda) = \frac{2\lambda\sin(\lambda\pi)}{9-\lambda^2}.$$

so $3\hat{g}(\lambda) = (i\lambda)\hat{f}(\lambda)$, as predicted.

Lemma 1 *$f * g \in L^1[-\infty, \infty]$ and*

$$\int_{-\infty}^{\infty} |f * g(t)|dt = \int_{-\infty}^{\infty} |f(x)|dx \int_{-\infty}^{\infty} |g(t)|dt.$$

Sketch of proof:

$$\int_{-\infty}^{\infty} |f * g(t)|dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x)g(t-x)|dx\right) dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |g(t-x)|dt\right) |f(x)|dx = \int_{-\infty}^{\infty} |g(t)|dt \int_{-\infty}^{\infty} |f(x)|dx.$$

□

Theorem 4 *Let $h = f * g$. Then*

$$\hat{h}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda).$$

Sketch of proof:

$$\hat{h}(\lambda) = \int_{-\infty}^{\infty} f * g(t) e^{-i\lambda t} dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) g(t-x) dx \right) e^{-i\lambda t} dt$$

$$= \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} \left(\int_{-\infty}^{\infty} g(t-x) e^{-i\lambda(t-x)} dt \right) dx = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx \ \hat{g}(\lambda)$$

$$= \hat{f}(\lambda)\hat{g}(\lambda).$$

Exercise 20 *Prove the following: If f is even,*

$$\mathcal{F}[f](x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(xt) dt$$

and if f is odd,

$$\mathcal{F}[f](x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin(xt) dt.$$

Exercise 21 *The Fourier Cosine ($\mathcal{F}_c[f](.)$) and Fourier Sine ($\mathcal{F}_s[f](.)$) of $f : \mathbb{R} \to \mathbb{R}$ are defined as follows:*

$$\mathcal{F}_c[f](x) := \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(xt) dt.$$

$$\mathcal{F}_s[f](x) := \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin(xt) dt.$$

The Fourier Cosine Transform (FCT)

Definitions and Relations to the Exponential Fourier Transforms

$$F_c(\omega)=\int_0^\infty f(t)\cos\omega t\,dt,\quad \omega\geq 0, \tag{3.2.1}$$

subject to the existence of the integral. The definition is sometimes more compactly represented as an operator $\mathscr{F}_c$ applied to the function $f(t)$, so that

$$\mathscr{F}_c\left[f(t)\right]=F_c(\omega)=\int_0^\infty f(t)\cos\omega t\,dt. \tag{3.2.2}$$

The subscript c is used to denote the fact that the kernel of the transformation is a cosine function. The unit normalization constant used here provides for a definition for the inverse Fourier cosine transform, given by

$$\mathscr{F}_c^{-1}\left[F_c(\omega)\right]=\frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega t\,d\omega,\quad t\geq 0, \tag{3.2.3}$$

again subject to the existence of the integral used in the definition. The functions $f(t)$ and $F_c(\omega)$, if they exist, are said to form a Fourier cosine transform pair.

Because the cosine function is the real part of an exponential function of purely imaginary argument, that is,

$$\cos(\omega t)=\mathrm{Re}\left[e^{j\omega t}\right]=\frac{1}{2}\left[e^{j\omega t}+e^{-j\omega t}\right], \tag{3.2.4}$$

it is easy to understand that there exists a very close relationship between the Fourier transform and the cosine transform. To see this relation, consider an even extension of the function $f(t)$ defined over the entire real line so that

$$f_e(t)=f(|t|),\quad t\in R. \tag{3.2.5}$$

Its Fourier transform is defined as

$$\mathscr{F}\left[f_e(t)\right]=\int_{-\infty}^{\infty} f_e(t)e^{-j\omega t}dt,\quad \omega\in R. \tag{3.2.6}$$

The integral in (3.2.6) can be evaluated in two parts over $(-\infty, 0]$ and $[0, \infty)$. Then using (3.2.5) and changing the integrating variable in the $(-\infty, 0]$ integral from t to $-t$, we have

$$\mathscr{F}\left[f_e(t)\right]=\left[\int_0^\infty f(t)e^{-j\omega t}dt+\int_0^\infty f(t)e^{j\omega t}dt\right]=2\int_0^\infty f(t)\cos\omega t\,dt,$$

by (3.2.4), and thus

$$\mathscr{F}[f_e(t)] = 2\mathscr{F}_c[f(t)], \quad \text{if } f_e(t) = f(|t|). \tag{3.2.7}$$

Many of the properties of the Fourier cosine transforms can be derived from the properties of Fourier transforms of symmetric, or even, functions. Some of the basic properties and operational rules are discussed in Section 3.2.2.

3.2.2 Basic Properties and Operational Rules

1. *Inverse Transformation*: As stated in (3.2.3), the inverse transformation is exactly the same as the forward transformation except for the normalization constant. This leads to the so-called Fourier cosine integral formula, which states that

$$\begin{aligned} f(t) &= \frac{2}{\pi}\int_0^{\infty} F_c(\omega)\cos\omega t\, d\omega \\ &= \frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(\tau)\cos\omega\tau\, d\tau\right]\cos\omega t\, d\omega. \end{aligned} \tag{3.2.8}$$

The sufficient conditions for the inversion formula (3.2.3) are that $f(t)$ be absolutely integrable in $[0, \infty)$ and that $f'(t)$ be piece-wise continuous in each bounded subinterval of $[0, \infty)$. In the range where the function $f(t)$ is continuous, (3.2.8) represents f. At the point t_0 where $f(t)$ has a jump discontinuity, (3.2.8) converges to the mean of $f(t_0 + 0)$ and $f(t_0 - 0)$, that is,

$$\frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(\tau)\cos(\omega\tau)\,d\tau\right]\cos(\omega t_0)\,d\omega = \frac{1}{2}\left[f(t_0+0)+f(t_0-0)\right]. \tag{3.2.8'}$$

2. *Transforms of Derivatives*: It is easy to show, because of the Fourier cosine kernel, that the transforms of even-order derivatives are reduced to multiplication by even powers of the conjugate variable ω, much as in the case of the Laplace transforms. For the second-order derivative, using integration by parts, we can show that,

$$\begin{aligned} \mathscr{F}_c\left[f''(t)\right] &= \int_0^{\infty} f''(t)\cos(\omega t)\,dt \\ &= -f'(0) - \omega^2\int_0^{\infty} f(t)\cos\omega t\, dt \\ &= -\omega^2 F_c(\omega) - f'(0) \end{aligned} \tag{3.2.9}$$

where we have assumed that $f(t)$ and $f'(t)$ vanish as $t \to \infty$. These form the sufficient conditions for (3.2.9) to be valid. As the transform is applied to higher order derivatives, corresponding

4. *Shifting*:

(a) Shifting in the t-domain: The shift-in-t property for the cosine transform is somewhat less direct compared with the exponential Fourier transform for two reasons. First, a shift to the left will require extending the definition of the function $f(t)$ onto the negative real line. Secondly, a shift-in-t in the transform kernel does not result in a constant phase factor as in the case of the exponential kernel.

If $f_c(t)$ is defined as the even extension of the function $f(t)$ such that $f_c(t) = f(|t|)$, and if $f(t)$ is piece-wise continuous and absolutely integrable over $[0, \infty)$, then

$$\mathcal{F}_c\left[f_c(t+a)+f_c(t-a)\right]=\int_0^{\infty}\left[f_c(t+a)+f_c(t-a)\right]\cos\omega t\,dt$$
$$=\int_a^{\infty} f_c(\tau)\cos\omega(\tau+a)d\tau$$
$$+\int_{-a}^{\infty} f_c(\tau)\cos\omega(\tau-a)d\tau\,.$$

By expanding the compound cosine functions and using the fact that the function $f_c(\tau)$ is even, these combine to give:

$$\mathcal{F}_c[f_c(t+a)+f_c(t-a)] = 2F_c(\omega)\cos a\omega\,, \qquad a>0. \tag{3.2.17}$$

This is sometimes called the kernel-product property of the cosine transform. In terms of the function $f(t)$, it can be written as:

$$\mathcal{F}_c[f(t+a)+f(|t-a|)] = 2F_c(\omega)\cos a\omega\,. \tag{3.2.18}$$

Similarly, the kernel-product $2F_c(\omega)\sin(a\omega)$ is related to the Fourier sine transform:

$$\mathcal{F}_s[f(|t-a|)-f(t+a)] = 2F_c(\omega)\sin a\omega\,, \quad a>0. \tag{3.2.19}$$

(b) Shifting in the ω-domain:
To consider the effect of shifting in ω by the amount of $\beta(>0)$, we examine the following,

$$F_c(\omega+\beta)=\int_0^{\infty} f(t)\cos(\omega+\beta)t\,dt$$
$$=\int_0^{\infty} f(t)\cos\beta t\cos\omega t\,dt-\int_0^{\infty} f(t)\sin\beta t\sin\omega t\,dt \tag{3.2.20}$$
$$=\mathcal{F}_c\left[f(t)\cos\beta t\right]-\mathcal{F}_s\left[f(t)\sin\beta t\right].$$

Similarly,

$$F_c(\omega-\beta) = \mathcal{F}_c[f(t)\cos\beta t] + \mathcal{F}_s[f(t)\sin\beta t]. \tag{3.2.20'}$$

Combining (3.2.20) and (3.2.20′) produces a shift-in-ω operational rule involving only the Fourier cosine transform as

$$\mathcal{F}_c\left[f(t)\cos\beta t\right]=\frac{1}{2}\left[F_c(\omega+\beta)+F_c(\omega-\beta)\right]. \tag{3.2.21}$$

More generally, for $a, \beta > 0$, we have,

$$\mathcal{F}_c\left[f(at)\cos\beta t\right]=\frac{1}{2a}\left[F_c\left(\frac{\omega+\beta}{a}\right)+F_c\left(\frac{\omega-\beta}{a}\right)\right]. \qquad (3.2.22)$$

Similarly, we can easily derive:

$$\mathcal{F}_c\left[f(at)\sin\beta t\right]=\frac{1}{2a}\left[F_s\left(\frac{\omega+\beta}{a}\right)-F_s\left(\frac{\omega-\beta}{a}\right)\right]. \qquad (3.2.22')$$

5. *Differentiation in the ω domain*: Similar to differentiation in the t domain, the transform operation reduces a differentiation operation into multiplication by an appropriate power of the conjugate variable. In particular, even-order derivatives in the ω domain are transformed as:

$$F_c^{(2n)}(\omega)=\mathcal{F}_c\left[(-1)^n t^{2n} f(t)\right]. \qquad (3.2.23)$$

We show here briefly, the derivation for $n = 1$:

$$\begin{aligned}F_c^{(2)}(\omega)&=\frac{d^2}{d\omega^2}\int_0^\infty f(t)\cos\omega t\,dt\\&=\int_0^\infty f(t)\frac{d^2}{d\omega^2}\cos\omega t\,dt\\&=\int_0^\infty f(t)(-1)t^2\cos\omega t\,dt\\&=\mathcal{F}_c\left[(-1)t^2 f(t)\right].\end{aligned}$$

For odd orders, these are related to Fourier sine transforms

$$F_c^{(2n+1)}(\omega)=\mathcal{F}_s\left[(-1)^{n+1} t^{2n+1} f(t)\right]. \qquad (3.2.24)$$

In both (3.2.23) and (3.2.24), the existence of the integrals in question is assumed. This means that $f(t)$ should be piece-wise continuous and that $t^{2n}f(t)$ and $t^{2n+1}f(t)$ should be absolutely integrable over $[0, \infty)$.

6. *Asymptotic behavior*: When the function $f(t)$ is piece-wise continuous and absolutely integrable over the region $[0, \infty)$, the Reimann-Lebesque theorem for Fourier series* can be invoked to provide the following asymptotic behavior of its cosine transform:

7. *Integration:*
   (a) Integration in the t domain:
   Integration in the t domain is transformed to division by the conjugate variable, very similar to the cases of Laplace transforms and Fourier transforms, except the resulting transform is a Fourier sine transform. Thus,

$$\mathcal{F}_c\left[\int_t^\infty f(\tau)\,d\tau\right] = \int_0^\infty \int_t^\infty f(\tau)\,d\tau \cos\omega t\,dt$$

$$= \int_0^\infty \left[\int_0^\tau \cos\omega t\,dt\right] f(\tau)\,d\tau$$

   by reversing the order of integration. The inner integral results in a sine function and is the kernel for the Fourier sine transform. Therefore,

$$\mathcal{F}_c\left[\int_t^\infty f(\tau)d\tau\right] = \frac{1}{\omega}\mathcal{F}_s\left[f(\tau)\right] = \frac{1}{\omega}F_s(\omega). \tag{3.2.26}$$

   Here, again, $f(t)$ is subject to the usual sufficient conditions of being piece-wise continuous and absolutely integrable in $[0, \infty)$.
   (b) Integration in the ω domain:
   A similar and symmetric relation exists for integration in the ω-domain.

$$\mathcal{F}_s^{-1}\left[\int_\omega^\infty F_c(\beta)d\beta\right] = -\frac{1}{t}f(t). \tag{3.2.27}$$

   Note that the integral transform inversion is of the Fourier sine type instead of the cosine type. Also the aysmptotic behavior of $F_c(\omega)$ has been invoked.
8. *The convolution property:* Let $f(t)$ and $g(t)$ be defined over $[0, \infty)$ and satisfy the sufficiency condition for the existence of F_c and G_c. If $f_e(t) = f(|t|)$ and $g_e(t) = g(|t|)$ are the even extensions of f and g, respectively, over the entire real line, then the convolution of f_e and g_e is given by:

$$f_e * g_e = \int_{-\infty}^\infty f_e(\tau)g_e(t-\tau)d\tau \tag{3.2.28}$$

   where $*$ has been used to denote the convolution operation. It is easy to see that in terms of f and g, we have:

$$f_e * g_e = \int_0^\infty f(\tau)\left[g(t+\tau) + g(|t-\tau|)\right]d\tau \tag{3.2.29}$$

   which is an even function. Applying the exponential Fourier transform on both sides and using (3.2.7) and the convolution property of the exponential Fourier transform, we obtain the convolution property for the cosine transform:

$$2F_c(\omega)G_c(\omega)=\mathcal{F}_c\left\{\int_0^\infty f(\tau)\left[g(t+\tau)+g(|t-\tau|)\right]d\tau\right\}. \quad (3.2.30)$$

In a similar way, the cosine transform of the convolution of odd extended functions is related to the sine transforms. Thus,

$$2F_s(\omega)G_s(\omega)=\mathcal{F}_c\left\{\int_0^\infty f(\tau)\left[g(t+\tau)+g_o(t-\tau)\right]d\tau\right\}. \quad (3.2.31)$$

where

$$\begin{aligned} g_o(t) &= g(t) && \text{for } t>0, \\ &= -g(-t) && \text{for } t<0, \end{aligned} \quad (3.2.32)$$

is defined as the odd extension of the function $g(t)$.

The Fourier Sine Transform (FST)

Definitions and Relations to the Exponential Fourier Transforms

Similar to the Fourier cosine transform, the Fourier sine transform of a function $f(t)$, which is piecewise continuous and absolutely integrable over $[0, \infty)$, is defined by application of the operator $\mathcal{F}_s$ as:

$$F_s(\omega)=\mathcal{F}_s\left[f(t)\right]=\int_0^\infty f(t)\sin\omega t\,dt, \quad \omega>0. \quad (3.3.1)$$

The inverse operator $\mathcal{F}_s^{-1}$ is similarly defined:

$$f(t)=\mathcal{F}_s^{-1}\left[F_s(\omega)\right]=\frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega t\,d\omega, \quad t\geq 0, \quad (3.3.2)$$

subject to the existence of the integral. Functions $f(t)$ and $F_s(\omega)$ defined by (3.3.2) and (3.3.1), respectively, are said to form a Fourier sine transform pair. It is noted in (3.2.3) and (3.3.2) for the inverse FCT and inverse FST that both transform operators have symmetric kernels and that they are involuntary or unitary up to a factor of $\sqrt{(2/\pi)}$.

Fourier sine transforms are also very closely related to the exponential Fourier transform defined in (3.2.6). Using the property that

$$\sin\omega t = \mathrm{Im}\left[e^{j\omega t}\right] = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right], \tag{3.3.3}$$

one can consider the odd extension of the function $f(t)$ defined over $[0, \infty)$ as

$$\begin{aligned} f_o(t) &= f(t) \qquad t \geq 0, \\ &= -f(-t) \quad t < 0. \end{aligned}$$

Then the Fourier transform of $f_o(t)$ is

$$\begin{aligned} \mathscr{F}\left[f_o(t)\right] &= \int_{-\infty}^{\infty} f_o(t) e^{-j\omega t} dt = -\int_0^{\infty} f(t) e^{j\omega t} dt + \int_0^{\infty} f(t) e^{-j\omega t} dt \\ &= -2j\int_0^{\infty} f(t)\sin\omega t \, dt = -2j\mathscr{F}_s\left[f(t)\right], \end{aligned}$$

and therefore,

$$\mathscr{F}_s\left[f(t)\right] = -\frac{1}{2j}\mathscr{F}\left[f_o(t)\right]. \tag{3.3.4}$$

Equation (3.3.4) provides the relation between the FST and the exponential Fourier transform. As in the case for cosine transforms, many properties of the sine transform can be related to those for the Fourier transform through this equation. We shall present some properties and operational rules for FST in the next section.

3.3.2 Basic Properties and Operational Rules

1. *Inverse Transformation*: The inverse transformation is exactly the same as the forward transformation except for the normalization constant. Combining the forward and inverse transformations leads to the Fourier sine integral formula, which states that,

$$f(t)=\frac{2}{\pi}\int_0^{\infty}F_s(\omega)\sin\omega t\,d\omega=\frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty}f(\tau)\sin\omega\tau\,d\tau\right]\sin\omega t\,d\omega. \qquad (3.3.5)$$

The sufficient conditions for the inversion formula (3.3.2) are the same as for the cosine transform. Where $f(t)$ has a jump discontinuity at $t=t_0$, (3.3.5) converges to the mean of $f(t_0+0)$ and $f(t_0-0)$.

2. *Transforms of Derivatives*: Derivatives transform in a fashion similar to FCT, even orders involving sine transforms only and odd orders involving cosine transforms only. Thus, for example,

$$\mathcal{F}_s[f''(t)] = -\omega^2F_s(\omega) + \omega f(0) \qquad (3.3.6)$$

and

$$\mathcal{F}_s[f'(t)] = -\omega F_c(\omega), \qquad (3.3.7)$$

where $f(t)$ is assumed continuous to the first order.

For the fourth-order derivative, we apply (3.3.6) twice to obtain,

$$\mathcal{F}_s[f^{(iv)}(t)] = \omega^4F_s(\omega) - \omega^3f(0) + \omega f''(0), \qquad (3.3.8)$$

if $f(t)$ is continuous at least to order three. When the function $f(t)$ and its derivatives have jump discontinuities at $t=t_0$, (3.3.8) is modified to become,

$$\begin{aligned}\mathcal{F}_s[f^{(iv)}(t)] &= \omega^4F_s(\omega) - \omega^3f(0) + \omega f''(0) - \omega^3 d\cos\omega t_0\\ &\quad + \omega^2 d'\sin\omega t_0 + \omega d''\cos\omega t_0 - d'''\sin\omega t_0\end{aligned} \qquad (3.3.9)$$

where the jump discontinuities d, d', and d''' are as defined in (3.2.13). Similarly, for odd-order derivatives, when the function $f(t)$ has jump discontinuities, the operational rule must be modified. For example, (3.3.7) will become:

$$\mathcal{F}_s[f'(t)] = -\omega F_c(\omega) + d\sin\omega t_0. \qquad (3.3.7')$$

Generalization to other orders and to more than one location for the jump discontinuities is straightforward.

3. *Scaling*: Scaling in the t-domain for the FST has exactly the same effect as in the case of FCT, giving,

$$\mathcal{F}_s\left[f(at)\right]=\frac{1}{a}F_s(\omega/a) \quad a>0. \qquad (3.3.10)$$

4. *Shifting*:
   (a) Shift in the t-domain:
   As in the case of the Fourier cosine transform, we first define the even and odd extensions of the function $f(t)$ as,

$$f_e(t)=f(|t|), \quad \text{and } f_o(t)=\frac{t}{|t|}f(|t|). \tag{3.3.11}$$

   Then it can be shown that:

$$\mathcal{F}_s[f_o(t+a)+f_o(t-a)]=2F_s(\omega)\cos a\omega \tag{3.3.12}$$

   and

$$\mathcal{F}_c[f_o(t+a)+f_o(t-a)]=2F_s(\omega)\sin a\omega\,; \quad a>0. \tag{3.3.13}$$

   These, together with (3.2.18) and (3.2.19), form a complete set of kernel-product relations for the cosine and the sine transforms.
   (b) Shift in the ω-domain:
   For a positive β shift in the ω-domain, it is easily shown that

$$\mathcal{F}_s[\omega+\beta]=F_s[f(t)\cos\beta t]+F_c[f(t)\sin\beta t] \tag{3.3.14}$$

   and combining with the result for a negative shift, we get:

$$\mathcal{F}_s[f(t)\cos\beta t]=(1/2)[F_s(\omega+\beta)+F_s(\omega-\beta)]. \tag{3.3.15}$$

   More generally, for $a, \beta > 0$, we have,

$$\mathcal{F}_s\left[f(at)\cos\beta t\right]=(1/2a)\left[F_s\left(\frac{\omega+\beta}{a}\right)+F_s\left(\frac{\omega-\beta}{a}\right)\right]. \tag{3.3.16}$$

   Similarly, we can easily show that

$$\mathcal{F}_s\left[f(at)\sin\beta t\right]=-(1/2a)\left[F_c\left(\frac{\omega+\beta}{a}\right)-F_c\left(\frac{\omega-\beta}{a}\right)\right]. \tag{3.3.17}$$

   The shift-in-ω properties are useful in deriving some FCTs and FSTs. As well, because the quantities being transformed are modulated sinusoids, these are useful in applications to communication problems.
5. *Differentiation in the ω-domain*: The sine transform behaves in a fashion similar to the cosine transform when it comes to differentiation in the ω-domain. Even-order derivatives involve only sine transforms and odd-order derivatives involve only cosine transforms. Thus,

$$F_s^{(2n)}(\omega)=\mathcal{F}_s\left[(-1)^n t^{2n} f(t)\right],$$

and

$$F_s^{(2n+1)}(\omega)=\mathcal{F}_c\left[(-1)^n t^{2n+1} f(t)\right]. \tag{3.3.18}$$

It is again assumed that the integrals in (3.3.18) exist.

5. *Asymptotic behavior*: The Reimann-Lebesque theorem guarantees that any Fourier sine transform converges to zero as ω tends to infinity, that is,

$$\lim_{\omega\to\infty} F_s(\omega)=0. \tag{3.3.19}$$

7. *Integration*:

(a) Integration in the t-domain. In analogy to (3.2.26), we have

$$\mathcal{F}_s\left[\int_0^t f(\tau)d\tau\right]=(1/\omega)F_c(\omega) \tag{3.3.20}$$

provided $f(t)$ is piece-wise smooth and absolutely integrable over $[0, \infty)$.

(b) Integration in the ω-domain. As in the Fourier cosine transform, integration in the ω-domain results in division by t in the t-domain, giving,

$$\mathcal{F}_c^{-1}\left[\int_\omega^\infty F_s(\beta)d\beta\right]=(1/t)f(t) \tag{3.3.21}$$

in parallel with (3.2.27).

8. *The convolution property*: If functions $f(t)$ and $g(t)$ are piece-wise continuous and absolutely integrable over $[0, \infty)$, a convolution property involving $F_s(\omega)$ and $G_c(\omega)$ is

$$2F_s(\omega)G_c(\omega)=\mathcal{F}_s\left\{\int_0^\infty f(\tau)\left[g(|t-\tau|)-g(t+\tau)\right]d\tau\right\}. \tag{3.3.22}$$

Equivalently,

$$2F_s(\omega)G_c(\omega)=\mathcal{F}_s\left\{\int_0^\infty g(\tau)\left[f(t+\tau)+f_o(t-\tau)\right]d\tau\right\} \tag{3.3.23}$$

where $f_o(x)$ is the odd extension of the function $f(x)$ defined as in (3.3.11).

One can establish a convolution theorem involving only sine transforms. This is obtained by imposing an additional condition on one of the functions, say, $g(t)$. We define the function $h(t)$ by,

$$h(t)=\int_t^\infty g(\tau)d\tau. \tag{3.3.24}$$

Then $g(t)$ must satisfy the condition that its integral $h(t)$ is absolutely integrable over $[0, \infty)$, so that the Fourier cosine transform of $h(t)$ exists. We note from (3.2.26) that

$$H_c(\omega) = (1/\omega)G_s(\omega) \tag{3.3.25}$$

Applying (3.3.22) to $f(t)$ and $h(t)$ yields immediately,

$$(2/\omega)F_s(\omega)G_s(\omega)=\mathcal{F}_s\left[\int_0^\infty f(\tau)\int_{|t-\tau|}^{t+\tau} g(\eta)d\eta\, d\tau\right] \tag{3.3.26}$$

noting that $g(t) = -h'(t)$.

Because the FSTs have properties and operation rules very similar to those for the FCTs, we refer the reader to Section 3.2.24 for simple examples on the use of these rules for FCTs.

Possible Questions

PART-B (SIX MARKS)

UNIT I

1.Define self reciprocal and prove that $e^{-\frac{x^2}{2}}$ is self reciprocal under the Fourier transform.

2.Explain about Fourier transforms of some simple functions with examples.

3. show that the Fourier transform of f(x) = $\begin{cases} a^2 - x^2\ , |x| < a \\ 0\ , |x| > a > 0 \end{cases}$ is 2$\sqrt{\frac{2}{\pi}}$ ($\frac{sinas - ascosas}{s^3}$).Hence deduce that $\int_0^\infty \frac{sint - tcost}{t^3}$ dt = $\frac{\pi}{4}$.

4. Expalin the Fourier transforms of derivatives.

5.Prove that $\mathcal{F}_c[f'(t);\xi] = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} f(0) + \xi\mathcal{F}_s[f(t);\xi]$

6.Prove that$\mathcal{F}_s[f'(t);\xi] = -\xi\mathcal{F}_c[f(t);\xi]$

7.Obtain the Fourier transform of some simple functions.

8. Obtain Fourier sine transform.

9.Obtain the Fourier cosine transform

10.Find the Fourier Transform of f(x) = $\begin{cases} a - |x|, for\ |x| < a \\ 0, for\ |x| > a > 0\ . \end{cases}$ .Deduce that $\int_0^\infty \frac{sin^2}{t^2}$dt= $\frac{\pi}{2}$

PART-C (TEN MARKS)

11. Derive the Fourier transforms of Rational Functions

UNIT II

SYLLABUS

Introduction to The Convolution integral -Convolution theorem- Parseval's relation for Fourier transforms-Solution of PDE by Fourier transform-Laplace's equation in Half plane -Laplace's equation in an infinite strip-The linear diffusion equation on a semi-infinite line
The two dimensional diffusion equation.

The Fourier Transform

Fourier transformation is the most powerful technique for solving differential equations of different type arising in science and engineering. There are a variety of both analytical and numerical approaches rely on Fourier transforms. FFT (Fast Fourier Transform) is , e.g., the backbone of numerical approaches for problems in signal analysis. Besides all the traditional applications the modern technique of wavelet transform is based on (actually is an special version of) the Fourier transform.

Suppose that f is a function on $\mathbb{R}$. For any $L > 0$ we can expand f on the interval $[-L, L]$ in a Fourier series,

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{\infty} c_{n,L} e^{i\frac{\pi n}{L}x}, \quad \text{where} \quad C_{n,L} = \int_{-L}^{L} f(y) e^{-i\frac{\pi n}{L}y} dy. \qquad (4.1.1)$$

Let

$$\frac{\pi}{L} = \Delta\xi \quad \text{and define} \quad \xi_n := \frac{\pi n}{L} = n\Delta\xi.$$

Then the formulas in (4.1.1) become

$$f(x) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} c_{n,L} e^{i\xi_n x}\Delta\xi, \quad \text{where} \quad C_{n,L} = \int_{-L}^{L} f(y)e^{-i\xi_n y}dy. \tag{4.1.2}$$

Suppose that $f(x)$ vanishes rapidly as $x \to \pm\infty$, then for sufficiently large L we get

$$C_{n,L} = \int_{-L}^{L} f(y)e^{-i\xi_n y}dy \approx \int_{-\infty}^{\infty} f(y)e^{-i\xi_n y}dy. \tag{4.1.3}$$

Introducing the notation

$$\hat{f}(\xi_n) := \int_{-\infty}^{\infty} f(y)e^{-i\xi_n y}dy, \tag{4.1.4}$$

we have

$$f(x) \approx \frac{1}{2\pi}\sum_{-\infty}^{\infty} \hat{f}(\xi_n)e^{i\xi_n x}\Delta\xi, \quad \text{where} \quad |x| < L. \tag{4.1.5}$$

Let $L \to \infty$, so that $\Delta\xi \to 0$ and the sum in (4.1.5) should turn into an integral, thus:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}d\xi, \quad \text{where} \quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx. \tag{4.1.6}$$

$\hat{f}$ is called the *Fourier transform* of f and the formula (4.1.6) is the *Fourier inversion theorem.*

Definition 12. *If f is an integrable function on $\mathbb{R}$, i.e., $f \in L^1(\mathbb{R})$, its Fourier transform is the function $\hat{f}$ on $\mathbb{R}$, defined by*

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx := \mathcal{F}\Big[f(x)\Big](\xi) := \mathcal{F}\Big[f(x)\Big]. \tag{4.1.7}$$

Lemma 8. *The Fourier transform $\hat{f}(\xi)$ is (i) bounded, and (ii) continuous.*

Proof. (i) Since $\hat{f}(\xi)$ is defined for $f \in L^1(\mathbb{R})$, and $|e^{-i\xi x}| = 1$, the integral converges absolutely for all ξ,

$$|\hat{f}(\xi)| = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \le \int_{-\infty}^{\infty} f(x)dx < \infty \quad \text{where} \quad f \in L^1(\mathbb{R}).$$

(ii) Let $\xi \to \xi_0$. We want to show that $\hat{f}(\xi) \to \hat{f}(\xi_0)$. Since

$$|f(x)e^{i\xi x}| = |f(x)| \quad \forall\ \xi \quad \text{and} \quad f \in L_1(\mathbb{R}),\ i.e.,\ \int_{-\infty}^{\infty} f(x)dx < \infty,$$

the *dominating convergence theorem* give us

$$\lim_{\xi\to\xi_0} \hat{f}(\xi) = \int_{-\infty}^{\infty} \lim_{\xi\to\xi_0} f(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)e^{-i\xi_0 x}dx = \hat{f}(\xi_0),$$

and the proof is complete. □

Basic properties of the Fourier transform

Some of the basic properties of the Fourier transform are given in the following theorem.

Theorem 15. *Suppose $f \in L^1$, then*
(a) *For any $a \in \mathbb{R}$, we have*

$$(\mathbf{a1}) \quad \mathcal{F}\Big[(x-a)\Big] = e^{-ia\xi}\hat{f}(\xi) \quad and \quad (\mathbf{a2}) \quad \mathcal{F}\Big[e^{ia\xi}f(x)\Big] = \hat{f}(\xi - a).$$

(b) *If $\delta > 0$, then we have the scaling formula:*

$$\mathcal{F}\Big[f(\delta x)\Big](\xi) = \frac{1}{\delta}\hat{f}\Big(\frac{\xi}{\delta}\Big).$$

(c) *If f is continuous and piecewise smooth and $f' \in L^1$, then*

$$(\mathbf{c1}) \qquad \mathcal{F}[f'(x)](\xi) = i\xi\hat{f}(\xi).$$

On the other hand, if $xf(x)$ is integrable, then

$$(\mathbf{c2}) \qquad \mathcal{F}\Big[xf(x)\Big] = i\hat{f}'(\xi).$$

Proof. **(a1)** From the definition we have

$$\mathcal{F}\Big[(x-a)\Big] = \int_{-\infty}^{\infty} f(x-a)e^{-i\xi x}dx.$$

Convolutions

Definition 13. *If f and g are functions on $\mathbb{R}$, their convolution is the function $f * g$ defined by*

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \qquad \forall\, x \in \mathbb{R} \tag{4.3.1}$$

With a change of variables we have evidently

$$\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy. \tag{4.3.2}$$

We can think of the convolution integral as a limit of the Riemann sum:

$$\int_{-\infty}^{\infty} f(x-y)g(y)dy \approx \sum_{j=-\infty}^{\infty} f(x-y_j)g(y_j)\Delta y_j.$$

The function $f_j(x) := f(x-y_j)$ is a translation of f along the x-axis by the amount y_j, so the sum on the Right is a linear combination of translates of f with coefficients $g(y_j)\Delta y_j$. We can therefore think of $f * g$ as a continuous superposition of translates of f.

The weighted average of f on $[a, b]$ with respect to a nonnegative weight function g is

$$\frac{\int_a^b f(y)g(y)dy}{\int_a^b g(y)dy}.$$

Suppose now that $\int_a^b g(y)dy = 1$. If we now use the identity (4.3.2) and write $f * g(x)$ as $\int_{-\infty}^{\infty} f(y)g(x-y)dy$, we see that $f * g(x)$ is the weighted average of f with respect to the weight function $g(x-y)$.

In the next two theorems we state (without proof) some basic algebraic and analytic properties of convolutions.

Theorem 16. *Convolution obeys the same algebraic laws as ordinary multiplication:*
(i) The associative law: $f*(ag+bh) = a(f*g)+b(f*h)$, *for* a, b *constants.*
(ii) The commutative law: $f*g = g*f$.
(iii) The distributive law: $f*(g*h) = (f*g)*h$.

Theorem 17. *Suppose that* f *and* g *are differentiable and the convolutions* $f*g$, $f'*g$ *and* $f*g'$ *are well-defined. Then* $f*g$ *is differentiable and*

$$(f*g)'(x) = (f'*g)(x) = (f*g')(x).$$

Now we can give the proof for the convolution theorem:

Theorem 18 (The convolution theorem). *Suppose that* $f,\, g \in L^1$, *then*

$$\mathcal{F}[f*g] = (f*g)\hat{} = \hat{f}\hat{g}.$$

Proof. By the definition

$$(f*g)\hat{}(\xi) = \int_{-\infty}^{\infty}(f*g)(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)e^{-i\xi x}dydx.$$

Since $f,\, g \in L^1$ we can use Fubini's theorem to change the order of integration. Substituting also $x - y = z$, it follow that

$$\begin{aligned}
(f*g)\hat{}(\xi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)e^{-i\xi x}dxdy \\
&= \int_{-\infty}^{\infty} g(y)\Big\{\int_{-\infty}^{\infty} f(z)e^{-i\xi(y+z)}\,dz\Big\}dy \\
&= \Big(\int_{-\infty}^{\infty} g(y))e^{-i\xi y}dy\Big)\Big(\int_{-\infty}^{\infty} f(z)e^{-i\xi z}dz\Big) = \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

and thus we have

$$(f*g)\hat{}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

and the proof is complete.

Exempel 7. *Determine the Fourier transform for the function $f(x) = e^{-|x|}$.*

Solution: Using the definition of the Fourier transform it follows that

$$\mathcal{F}\Big[e^{-|x|}\Big](\xi) = \int_{-\infty}^{\infty} e^{-|x|}e^{-i\xi x}dx = \int_{-\infty}^{0} e^{(1-i\xi)x}dx + \int_{0}^{\infty} e^{-(1+i\xi)x}dx$$
$$= \Big[\frac{e^{(1-i\xi)x}}{1-i\xi}\Big]_{-\infty}^{0} + \Big[\frac{e^{-(1+i\xi)x}}{-(1+i\xi)}\Big]_{0}^{\infty} = \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}.$$

Lemma 9. *Let $f(x) = sign(x)\cdot e^{-a|x|}$, then $\hat{f}(\xi) = \frac{-2i\xi}{a^2+\xi^2}$.*

Proof. A straightforward calculation yields

$$\begin{aligned}\mathcal{F}\Big[sign(x)\cdot e^{-a|x|}\Big] &= \int_{-\infty}^{\infty} sign(x)\cdot e^{-a|x|}e^{-i\xi x}dx \\ &= \int_{-\infty}^{0} -e^{(a-i\xi)x}dx + \int_{0}^{\infty} e^{-(a+i\xi)x}dx \\ &= \Big[\frac{e^{(a-i\xi)x}}{a-i\xi}\Big]_{-\infty}^{0} + \Big[\frac{e^{-(a+i\xi)x}}{-(a+i\xi)}\Big]_{0}^{\infty} \\ &= \frac{-1}{a-i\xi} + \frac{1}{a+i\xi} = \frac{-2i\xi}{a^2+\xi^2}.\end{aligned} \tag{4.4.1}$$

Exempel 8. *Find the Fourier transform for the function $f(x) = e^{-x^2}$.*

Solution: By the definition we have that the Fourier transform for $f(x) = e^{-x^2}$ is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-x^2}e^{-i\xi x}dx.$$

It will be easier if we first compute $(\hat{f})'(\xi)$. Then $\hat{f}(\xi)$ will follow easily using theorem 15(c):

$$\begin{aligned}(\hat{f})'(\xi) &= \int_{-\infty}^{\infty}(-ix)e^{-x^2}e^{-i\xi x}dx \\ &= \Big[\frac{i}{2}e^{-x^2}e^{-i\xi x}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{i}{2}e^{-x^2}(-i\xi)e^{-i\xi x}dx \\ &= -\frac{\xi}{2}\int_{-\infty}^{\infty} e^{-x^2}e^{-i\xi x}dx = -\frac{\xi}{2}\hat{f}(\xi),\end{aligned} \tag{4.4.2}$$

where we used partial integration and the fact that $\left[\frac{i}{2}e^{-x^2}e^{-i\xi x}\right]_{-\infty}^{\infty} = 0$. *Consequently we have the differential equation* $\hat{f}'(\xi) + \frac{\xi}{2}\hat{f}(\xi) = 0$, *where solution is* $\hat{f}(\xi) = Ce^{-\frac{\xi^2}{4}}$, *with* $C = \hat{f}(0)$.

Note that for $\xi = 0$,

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \quad thus \quad C = \sqrt{\pi},$$

and hence

$$\hat{f}(\xi) = \mathcal{F}\left[e^{-x^2}\right](\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}. \qquad (4.4.3)$$

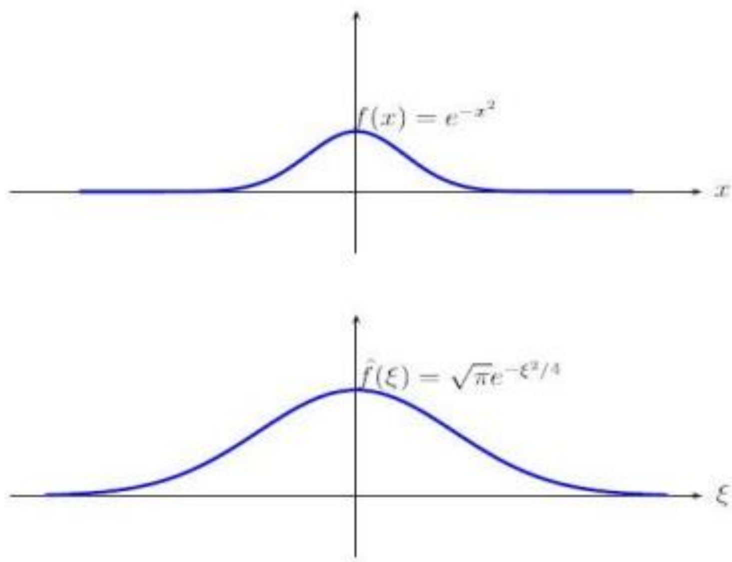


Figure 4.2: $f(x) = e^{-x^2}$ and its Fourier transform $\hat{f}(\xi) = \sqrt{\pi}e^{-\xi^2/4}$.

This means that for a Gaussian distribution f *its Fourier transform* $\hat{f}$ *is equivalent to a scaling of* f *preserving both its shape and regularity. In particular, as we shall see below, the Fourier transform of* $e^{-x^2/2}$ *is the same function multiplied by* $\sqrt{2\pi}$.

As a consequence of this example we have the following important formula for the Fourier transform of a general Gaussian function:

Lemma 10.

$$\mathcal{F}\Big[e^{-\frac{ax^2}{2}}\Big](\xi) = \sqrt{\frac{2\pi}{a}}e^{-\frac{\xi^2}{2a}}. \tag{4.4.4}$$

Proof. The proof is straightforward using the scaling formula with $\delta = \sqrt{\frac{a}{2}}$, viz,

$$\mathcal{F}\Big[e^{-\frac{ax^2}{2}}\Big](\xi) = \sqrt{\frac{2}{a}}\sqrt{\pi}e^{-\frac{\left(\xi\sqrt{\frac{2}{a}}\right)^2}{4}} = \sqrt{\frac{2\pi}{a}}e^{-\frac{\xi^2}{2a}}.$$

□

Later on we shall use the above formula with the substituting: $x = \xi$ and $\xi = (x - y)$:

$$\mathcal{F}\Big[e^{-\frac{a\xi^2}{2}}\Big](x-y) = \sqrt{\frac{2\pi}{a}}e^{-\frac{(x-y)^2}{2a}}. \tag{4.4.5}$$

Theorem 19 (The Fourier Inversion Theorem). *Suppose $f \in L^1(\mathbb{R})$, f, piecewise continuous, and defined at its points of discontinuity so as to satisfy $f(x) = \frac{1}{2}\Big[f(x-) + f(x+)\Big]$ for all $x \in \mathbb{R}$. Then*

$$f(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}e^{-\frac{\epsilon^2\xi^2}{2}}d\xi. \tag{4.5.1}$$

Moreover, since $\hat{f} \in L^1(\mathbb{R})$, the f is continuous and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}d\xi. \tag{4.5.2}$$

Proof. Note that the *cutoff function* $e^{-\epsilon^2\xi^2/2}$ in (4.5.1) is just to make the integrals converge, then passing to the limit the cutoff is removed. A straightforward calculation yields

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}d\xi &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(y)e^{-i\xi y}e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}dyd\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\Big\{\int_{-\infty}^{\infty}e^{-\frac{\varepsilon^2\xi^2}{2}}e^{-i\xi(y-x)}d\xi\Big\}dy \quad (4.5.3) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\mathcal{F}\Big[e^{-\frac{\varepsilon^2\xi^2}{2}}\Big](y-x)dy.\end{aligned}$$

Now we apply (4.4.5) above with $a = \varepsilon^2$ to get

$$\mathcal{F}\Big[e^{-\frac{\varepsilon^2\xi^2}{2}}\Big](y-x) = \frac{\sqrt{2\pi}}{\varepsilon}e^{-\frac{(x-y)^2}{2\varepsilon^2}}.$$

Replacing in (4.5.3) it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\frac{1}{\varepsilon}e^{-\left(\frac{y-x}{\sqrt{2}\varepsilon}\right)^2}dy. \qquad (4.5.4)$$

Substituting $\frac{y-x}{\sqrt{2}\varepsilon} = z$ gives $y = x + \sqrt{2}\varepsilon z$ and $dy = \sqrt{2}\varepsilon dz$. Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}f(x+\sqrt{2}\varepsilon z)e^{-z^2}dz. \qquad (4.5.5)$$

Now since f is bounded we have

$$\Big|f(x+\sqrt{2}\varepsilon z)e^{-z^2}\Big| \le Me^{-z^2} \quad \text{and} \quad \Big|\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}\Big| \le \Big|\hat{f}(\xi)\Big| \in L^1.$$

Taking limit in both sides of (4.5.5), by Lebesgue dominated convergence theorem, we can pass the limits inside integrals to get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}\Big\{\lim_{\varepsilon\to 0}e^{-\frac{\varepsilon^2\xi^2}{2}}\Big\}d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{\varepsilon\to 0}f(x+\sqrt{2}\varepsilon z)e^{-z^2}dz.$$

Hence by the continuity of f it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}f(x)e^{-z^2}dz = f(x)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}dz = f(x),$$

and the proof is complete.

The Fourier inversion formula can simply be interpreted as a improper integral if f is integrable and piecewise smooth on $\mathbb{R}$. Below, we state this as a theorem (without proof!):

Theorem 20. *If f is integrable and piecewise smooth on $\mathbb{R}$, then*

$$\lim_{r\to\infty}\int_{-r}^{r} e^{i\xi x}\hat{f}(\xi)d\xi = \frac{1}{2}\Big[f(x-)+f(x+)\Big], \qquad (4.5.6)$$

for every $x \in \mathbb{R}$.

Theorem 21. *Suppose that $f, \hat{f}, g$ and $\hat{g}$ are in L^1. Then*

$$2\pi\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle. \qquad (4.6.1)$$

Proof. Using the Fourier inversion theorem for g:

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(\xi)e^{i\xi x}d\xi,$$

and the definition of the inner product yields

$$2\pi\langle f, g\rangle = 2\pi\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty}\overline{\hat{g}(\xi)}e^{-i\xi x}d\xi$$
$$= \int_{-\infty}^{\infty}\overline{\hat{g}(\xi)}\Big\{\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\Big\}d\xi = \int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \langle \hat{f}, \hat{g}\rangle,$$

where we used the fact that since $f, \hat{g} \in L^1$, and the proof is complete.

Remark. The definition of the Fourier transform can be developed to arbitrary L^2-functions. If $f, g, \hat{f}$ and $\hat{g}$ are in L^1, then $f, g, \hat{f}$ and $\hat{g}$ are also in L^2.

Because of our interest in L_2 spaces we formulate the following result:

Theorem 22 (The Plancherel Theorem). *The Fourier transform, defined originally on $L^1 \cap L^2$, extends uniquely to a map on L^2 satisfying*

$$2\pi\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \quad \textit{for all} \quad f, g \in L^2.$$

As a consequence of the Plancherel theorem we have
The Parsevals formula: For $f = g \in L^2$ we have that

$$2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi,$$

or

$$2\pi \|f(x)\|_{L^2}^2 = \|\hat{f}(\xi)\|_{L^2}^2. \tag{4.6.2}$$

Exempel 10. *Recalling som of our key examples:*

$$\mathcal{F}\left[e^{-|x|}\right] = \frac{2}{1+\xi^2} \quad \textit{and} \quad \mathcal{F}\left[e^{-a|x|}\right] = \frac{2a}{\xi^2+a^2}.$$

The symmetry rule give us

$$\mathcal{F}\left[\frac{2}{1+x^2}\right] = 2\pi e^{-|-\xi|} = 2\pi e^{-|\xi|} \longrightarrow \mathcal{F}\left[\frac{1}{1+x^2}\right] = \pi e^{-|\xi|}. \tag{4.7.4}$$

Similarly, by the symmetry rule

$$\mathcal{F}\left[\frac{2a}{x^2+a^2}\right] = 2\pi e^{-a|\xi|} \Longrightarrow \mathcal{F}\left[\frac{1}{x^2+a^2}\right] = \left(\frac{\pi}{a}\right) e^{-a|\xi|}. \tag{4.7.5}$$

Exempel 11. *Since*

$$\mathcal{F}\left[e^{-\frac{ax^2}{2}}\right](\xi) = \sqrt{\frac{2\pi}{a}} e^{-\frac{\xi^2}{2a}},$$

by the symmetry rule

$$\mathcal{F}\left[\sqrt{\frac{2\pi}{a}} e^{-\frac{x^2}{2a}}\right](\xi) = 2\pi e^{-\frac{a\xi^2}{2}} \tag{4.7.6}$$

Applications of Fourier transform

Partial differential equations

We now use the Fourier transform to solve problems on unbounded regions. The Fourier transform converts differentiation into a simple algebraic operation and we can reduce partial differential equations to easily solvable ordinary differential equations.

Exempel 13. *Consider the heat flow in an infinitely long road, given the initial temperature $u(x, 0) = f(x)$:*

$$u_t = k u_{xx}, \quad t > 0, \quad -\infty < x < \infty. \tag{4.8.1}$$

Solution: To find the temperature $u(x,t)$, *let* $\hat{u}(\xi,t) = \mathcal{F}_x\Big[u(x,t)\Big](\xi)$. *Then*

$$\mathcal{F}[u_t](\xi) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\xi x} dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x,t) e^{-i\xi x} dx = \frac{\partial \hat{u}}{\partial t}.$$

Further $\mathcal{F}[u_x](\xi) = i\xi\hat{u}(\xi)$ *gives that* $\mathcal{F}[u_{xx}](\xi) = (i\xi)^2\hat{u}(\xi) = -\xi^2\hat{u}(\xi)$. *Hence the Fourier transform of* (4.8.1) *yields*

$$\frac{\partial \hat{u}}{\partial t} = -k\xi^2\hat{u}(\xi), \tag{4.8.2}$$

with the general solution

$$\hat{u}(\xi,t) = Ce^{-k\xi^2 t}. \tag{4.8.3}$$

Fourier transform of the the initial data $u(\xi,0) = f(\xi)$: $\hat{u}(\xi,0) = \hat{f}(\xi)$, *inserted in* (4.8.3) *give* $\hat{u}(\xi,0) = C = \hat{f}(\xi)$. *Thus we have*

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-k\xi^2 t}. \tag{4.8.4}$$

To recover the solution u *we recall that* $\mathcal{F}\Big[e^{-\frac{ax^2}{2}}\Big](\xi) = \sqrt{\frac{2\pi}{a}} e^{-\frac{\xi^2}{2a}}$. *Letting* $\frac{1}{2a} = kt$ *thus* $a = \frac{1}{2kt}$, *we then have*

$$\mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi) = \sqrt{4\pi kt}\cdot e^{-k\xi^2 t}, \quad \textit{hence} \quad e^{-k\xi^2 t} = \frac{1}{\sqrt{4\pi kt}}\mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi).$$

Inserting in (4.8.4) *we get*

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{4\pi kt}}\hat{f}(\xi)\mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi) := \frac{1}{\sqrt{4\pi kt}}\hat{g}(\xi)\hat{f}(\xi), \tag{4.8.5}$$

where $\hat{g}(\xi) := \mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi)$. *Using the convolution theorem:* $\hat{f}\hat{g} = (f*g)\hat{}$ *it follows that*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}(f*g)(x) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)dy. \tag{4.8.6}$$

Exempel 14. *Solve the Poisson's equation,*

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \tag{4.8.7}$$

where the boundary condition, $u(x,0) = f(x)$, is bounded.

Solution: As in the previous example the Fourier transform of the equation and the boundary, with respect to x, yields to the following ordinary differential equation in y;

$$-\xi^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} = 0 \quad and \quad \hat{u}(\xi, 0) = \hat{f}(\xi), \tag{4.8.8}$$

with the general solution given by

$$\hat{u}(\xi, y) = C_1(\xi) e^{|\xi| y} + C_2(\xi) e^{-|\xi| y}. \tag{4.8.9}$$

By the boundedness requirement we have that $C_1(\xi) = 0$. Moreover using the Fourier transform of the boundary data from (4.8.8) we get $\hat{u}(\xi, 0) = C_2(\xi) = \hat{f}(\xi)$. Thus

$$\hat{u}(\xi, y) = \hat{f}(\xi) e^{-|\xi| y}. \tag{4.8.10}$$

To take the inverse transform, in this case, the appropriate Fourier transform formula is:

$$\mathcal{F}\Big[\frac{1}{x^2 + a^2}\Big] = \frac{\pi}{a} e^{-a|\xi|}, \quad where \quad a > 0. \tag{4.8.11}$$

Choosing $a = y$ in (4.8.11) we get

$$\mathcal{F}\Big[\frac{y}{\pi} \cdot \frac{1}{x^2 + y^2}\Big] = \frac{\pi}{y} \cdot e^{-|\xi| y}. \tag{4.8.12}$$

Thus the inverse transform of (4.8.10) is

$$u(x, y) = f(x) * \frac{y}{\pi} \cdot \frac{1}{x^2 + y^2} = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-s)}{s^2 + y^2} \, ds, \tag{4.8.13}$$

which is the Poisson integral formula for the solution the given problem.

Remark. This solution make sense since the Poisson kernel $\frac{y}{\pi(x^2+y^2)} \in L^1$ and $f(x)$ is bounded, $|f(x)| \le M$. Thus we have

$$|u(x, y)| \le M \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{s^2 + y^2} ds = \frac{M}{\pi} \arctan\left(\frac{s}{y}\right)_{-\infty}^{\infty} = M.$$

Exempel 15. *Solve the Dirichlet problem*

$$u_{xx} + u_{yy} = 0, \quad x > 0 \qquad y > 0, \quad where \tag{4.8.14}$$

$$u(0, y) = 0, \quad u(x, 0) = \frac{x}{x^2 + 1} \quad and \quad u(x, y) \quad is\ bounded. \tag{4.8.15}$$

Solution: First we solve the following full range (in x) problem:

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \qquad y > 0, \quad where \tag{4.8.16}$$

$$u(x, 0) = \frac{x}{x^2 + 1} \quad and \quad u(x, y) \quad is\ bounded. \tag{4.8.17}$$

In this case since $\frac{x}{x^2+1}$ is odd then $u(x, y)$ is odd in x and we have automatically the condition $u(0, y) = 0$. Now we recall the formula

$$sign x \cdot e^{-a|x|} \supset^{\mathcal{F}} \frac{-2i\xi}{a^2 + \xi^2}.$$

By the symmetry rule we get

$$\frac{x}{a^2 + x^2} \supset^{\mathcal{F}} -\pi i \cdot sign(\xi) \cdot e^{-a|\xi|}. \tag{4.8.18}$$

Thus for $a = 1$ we have

$$\frac{x}{1 + x^2} \supset^{\mathcal{F}} -\pi i \cdot sign(\xi) \cdot e^{-|\xi|}.$$

hence

$$u(x, 0) := f(x) \supset^{\mathcal{F}} -\pi i \cdot sign(\xi) \cdot e^{-a|\xi|}.$$

Now the Fourier transform of the solution $\hat{u}(\xi, y) = \hat{f}(\xi)e^{-|\xi|y}$, (see previous example), can be written as

$$\hat{u}(\xi, y) = -\pi i \cdot sign(\xi) \cdot e^{-|\xi|}e^{-y|\xi|} = -\pi i \cdot sign(\xi) \cdot e^{-(1+y)|\xi|}.$$

Thus with $a = 1 + y$ in (4.8.18) we finally get

$$u(x, y) = \frac{x}{x^2 + (1 + y)^2}. \tag{4.8.19}$$

Definition 14. *Let $f \in L^1(0,\infty)$. Then the Fourier cosine transform and Fourier sine transform of f are the functions $\mathcal{F}_c[f](\xi)$ and $\mathcal{F}_s[f](\xi)$ on $[0,\infty)$ defined by*

$$\mathcal{F}_c[f](\xi) = \int_0^\infty f(x)\cos\xi x\,dx \quad \text{and} \quad \mathcal{F}_s[f](\xi) = \int_0^\infty f(x)\sin\xi x\,dx. \quad (4.9.5)$$

Thus, if f_{even} and f_{odd} are the even and odd extensions of f to $\mathbb{R}$, then $\mathcal{F}_c[f](\xi)$ and $\mathcal{F}_s[f](\xi)$ are restrictions to $[0,\infty)$ of $\frac{1}{2}\hat{f}_{even}$ and $\frac{i}{2}\hat{f}_{odd}$, since

$$\hat{f}_{even}(\xi) = 2\int_0^\infty f_{even}(x)\cos\xi x\,dx = 2\mathcal{F}_c[f](\xi),$$

$$\hat{f}_{odd}(\xi) = -2i\int_0^\infty f_{odd}(x)\sin\xi x\,dx = -2i\mathcal{F}_s[f](\xi) = \frac{2}{i}\mathcal{F}_s[f](\xi).$$

The *inversion formulas* therefore become

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c[f](\xi)\cos\xi x\,d\xi = \frac{2}{\pi}\int_0^\infty \mathcal{F}_s[f](\xi)\sin\xi x\,d\xi.$$

Plancherel Theorem for $\mathcal{F}_c[f]$ and $\mathcal{F}_s[f]$.

Using the above relations it follows that the norm of $\mathcal{F}_c[f](\xi)$ on $[0,\infty)$ is given by

$$\|\mathcal{F}_c[f]\|^2_{L^2([0,\infty))} = \int_0^\infty \left|\frac{1}{2}\hat{f}_{even}(\xi)\right|^2 d\xi = \frac{1}{4}\cdot\frac{1}{2}\int_{-\infty}^\infty |\hat{f}_{even}(\xi)|^2 d\xi,$$

i.e.,

$$\|\mathcal{F}_c[f]\|^2_{L^2[0,\infty)} = \frac{1}{8}\|\hat{f}_{even}(\xi)\|^2_{L^2(-\infty,\infty)}. \quad (4.9.6)$$

Recalling the Parsevals formula: $\|\hat{f}(\xi)\|^2_{L^2(-\infty,\infty)} = 2\pi\|f(x)\|^2_{L^2(-\infty,\infty)}$, the relation (4.9.6) is written as

$$\|\mathcal{F}_c[f]\|^2_{L^2[0,\infty)} = \frac{\pi}{4}\int_{-\infty}^\infty |f_{even}(x)|^2\,dx = \frac{\pi}{2}\|f_{even}\|^2. \quad (4.9.7)$$

Similarly,

$$\|\mathcal{F}_s[f]\|^2 = \frac{\pi}{2}\int_0^\infty |f_{odd}(x)|^2\,dx = \frac{\pi}{2}\|f_{odd}\|^2. \quad (4.9.8)$$

We summarize the relation (4.9.7) and (4.9.8) in the:

Theorem 25 (Plancherel Theorem for cos and sin transforms). *$\mathcal{F}_c[f]$ and $\mathcal{F}_s[f]$ extend to maps from $L^2(0,\infty)$ onto itself that satisfy*

$$\|\mathcal{F}_c[f]\|^2 = \|\mathcal{F}_s[f]\|^2 = \frac{\pi}{2}\|f\|^2.$$

Exempel 16. *Use the Fourier sine transform to find a bounded solution $u(x,y)$ for the problem:*

$$u_{xx} + u_{yy} = 0, \qquad x > 0, \qquad y > 0, \tag{4.9.9}$$

with the boundary conditions

$$u(0,y) = 0, \quad \textit{and} \quad u(x,0) = \frac{x}{x^2+1}.$$

Fourier Transforms of impulse functions

The Dirac's delta function is an even function defined by

$$\delta(x) = 0, \qquad \text{for} \quad x \neq 0, \tag{4.10.6}$$

and

$$\int_{-a}^{a} \delta(x)dx = 1 \qquad \text{for all} \quad a > 0. \tag{4.10.7}$$

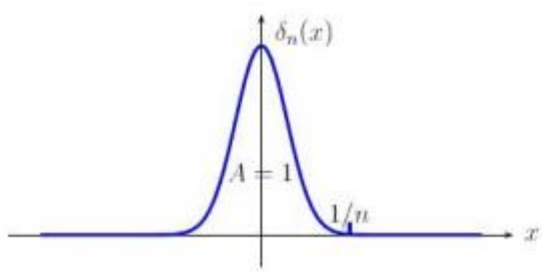


Figure 4.7: The Dirac function $\delta_n(x)$.

For $x = t - T$ this definition give

$$\delta(t-T) = \int_{-\infty}^{\infty} \delta(t-T)dx = 1. \tag{4.10.8}$$

To derive the Fourier transform of $\delta(t-T)$, we recall that by the evaluation formula:

$f(t)\delta(t-T) = f(T)\delta(t-T)$ we have $e^{-j\omega t}\delta(t-T) = e^{-j\omega T}\delta(t-T)$

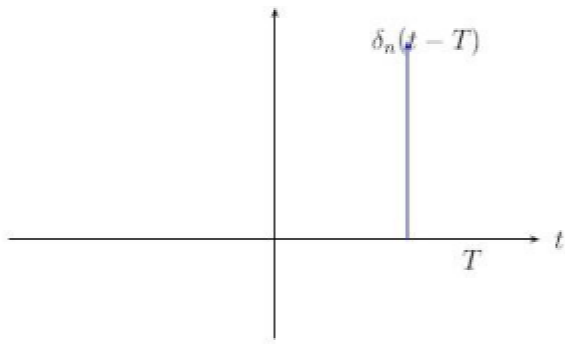


Figure 4.8: The Dirac function $\delta_{(t-T)}$.

and thus we have

$$\delta(t-T) \supset^{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t-T)e^{-j\omega t}dt = e^{-jwT}\int_{-\infty}^{\infty} \delta(t-T)dt = e^{-j\omega T}. \quad (4.10.9)$$

Then for $T = 0$: $\delta(t) \supset^{\mathcal{F}} = e^0 = 1$. Using symmetry rule and the fact that δ is an even function we have the following "formal relations": $1 \supset^{\mathcal{F}} 2\pi\delta(-\omega) = 2\pi\delta(\omega)$, i.e., we have

$$\delta(t) \supset^{\mathcal{F}} = 1, \qquad \text{and} \qquad 1 \supset^{\mathcal{F}} 2\pi\delta(\omega). \qquad (4.10.10)$$

The steady-state temperature distribution for y > 0 with the prescribed temperature u(x, 0) = f(x) on an infinite wall, y = 0, is described by the equation: PDE: uxx + uyy = 0, −∞ < x < ∞, y > 0 (1) BC: u(x, 0) = f(x), −∞ < x < ∞, (2) where u is bounded as y → ∞. Both u and ux → 0 as |x| → ∞. Solution. To solve this problem, we proceed as follows. Let F[u](ω) = ˆu(ω, y), F[f(x)] = ˆf(ω). Step 1. (Transforming the problem using FT) Taking FT of the PDE (1) in the variable x and using linearity property we have F[uxx] + F[uyy] = 0. (3) Since u and ux → 0 as |x| → ∞, it follows that −ω 2F[u](ω, y) + 1 √ 2π ∫ ∞ −∞ uyye −iωxdx = 0 =⇒ −ω 2uˆ(ω, y) + ∂ 2 ∂y2 [∫ ∞ −∞ u(x, y)e −iωxdx] = 0 =⇒ d 2 dy2 uˆ(ω, y) − ω 2uˆ(ω, y) = 0, (4) which is a second-order linear ODE in y. Taking FT of the BC yields uˆ(ω, 0) = F[f(x)] = ˆf(ω). (5) Step 2. (Solving the Transformed the problem) The general solution of (4) is given by uˆ(ω, y) = A(ω)e ωy + B(ω)e −ωy , (6) where A(ω) and B(ω) are to be determined. Since u is bounded as y → ∞, its FT ˆu(ω, y) must be bounded as y → ∞. This implies A(ω) = 0 for ω > 0. If ω < 0 then B(ω) = 0. Thus, uˆ(ω, y) = Ke−|ω|y , K = is a constant. (7)

Possible Questions

PART-B (SIX MARKS)

UNIT II

1. Using Fourier transforms evaluate $\int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$.

2.Derive the solution of two dimensional diffusion equations in an infinite region.

3.State and prove Parseval's theorem for Fourier transform.

4.Obtain Parseval's theorem for cosine and sine transform.

5.Derive the solutions of two dimensional diffusion equations.

6. Find the solution of a Laplace equation in a half plane.

7.Find the solutions of a Linear diffusion equation on a semi-strip

8.State and proof convolution theorem for Fourier transform.

9.Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

PART-B (TEN MARKS)

1.Derive the solution of Laplace's equation in an infinite strip.
2.Find the solutions of a Linear diffusion equation on a semi-strip

UNIT III
SYLLABUS

Introduction to Integral equations and types of integral equations -Equation with separable kernel-Fredholm Alternative Approximate method- Volterra integral equations-Classical Fredholm theory-Fredholm'sFirst, second, third theorems

An integral equation is an equation in which an unknown function appears under one or more integral signs. Naturally, in such an equation there can occur other terms as well. For example, for $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, the equations

$$f(s) = \int_a^b K(s,t)g(t)\,dt\,, \tag{1}$$

$$g(s) = f(s) + \int_a^b K(s,t)g(t)\,dt\,, \tag{2}$$

$$g(s) = \int_a^b K(s,t)[g(t)]^2\,dt\,, \tag{3}$$

where the function $g(s)$ is the unknown function while all the other functions are known, are integral equations. These functions may be complex-valued functions of the real variables s and t.

Integral equations occur naturally in many fields of mechanics and mathematical physics. They also arise as representation formulas for the solutions of differential equations. Indeed, a differential equation can be replaced by an integral equation which incorporates its boundary conditions. As such, each solution of the integral equation automatically satisfies these boundary conditions. Integral equations also form one of the most useful tools in many branches of pure analysis, such as the theories of functional analysis and stochastic processes.

One can also consider integral equations in which the unknown function is dependent not only on one variable but on several variables. Such, for example, is the equation

$$g(s) = f(s) + \int_\Omega K(s,t)g(t)\,dt\,, \tag{4}$$

where s and t are n-dimensional vectors and Ω is a region of an n-dimensional space. Similarly, one can also consider systems of integral equations with several unknown functions.

An integral equation is called linear if only linear operations are performed in it upon the unknown function. The equations (1) and (2) are linear, while (3) is nonlinear. In fact, the equations (1) and (2) can be written as

$$L[g(s)] = f(s) , \tag{5}$$

where L is the appropriate integral operator. Then, for any constants c_1 and c_2, we have

$$L[c_1 g_1(s) + c_2 g_2(s)] = c_1 L[g_1(s)] + c_2 L[g_2(s)] . \tag{6}$$

This is the general criterion for a linear operator. In this book, we shall deal only with linear integral equations.

The most general type of linear integral equation is of the form

$$h(s) g(s) = f(s) + \lambda \int_a K(s,t) g(t)\, dt , \tag{7}$$

where the upper limit may be either variable or fixed. The functions f, h, and K are known functions, while g is to be determined; λ is a nonzero, real or complex, parameter. The function $K(s, t)$ is called the kernel. The following special cases of equation (7) are of main interest.

(i) FREDHOLM INTEGRAL EQUATIONS. In all Fredholm integral equations, the upper limit of integration b, say, is fixed.

(i) In the Fredholm integral equation of the first kind, $h(s) = 0$.

Thus,

$$f(s) + \lambda \int_a^b K(s,t) g(t)\, dt = 0 . \tag{8}$$

(ii) In the Fredholm integral equation of the second kind, $h(s) = 1$;

$$g(s) = f(s) + \lambda \int_a^b K(s,t) g(t)\, dt . \tag{9}$$

(iii) The homogeneous Fredholm integral equation of the second kind is a special case of (ii) above. In this case, $f(s) = 0$;

$$g(s) = \lambda \int_a^b K(s,t)\, g(t)\, dt \,. \tag{10}$$

(ii) VOLTERRA EQUATIONS. Volterra equations of the first, homogeneous, and second kinds are defined precisely as above except that $b = s$ is the variable upper limit of integration.

Equation (7) itself is called an integral equation of the third kind.

(iii) SINGULAR INTEGRAL EQUATIONS. When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is called singular. For example, the integral equations

$$g(s) = f(s) + \lambda \int_{-\infty}^{\infty} \left(\exp - |s-t|\right) g(t)\, dt \tag{11}$$

and

$$f(s) = \int_0^s [1/(s-t)^\alpha]\, g(t)\, \mathrm{dt}\,, \qquad 0 < \alpha < 1 \tag{12}$$

are singular integral equations.

(i) SEPARABLE OR DEGENERATE KERNEL. A kernel $K(s,t)$ is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of s only and a function of t only, i.e.,

$$K(s,t) = \sum_{i=1}^{n} a_i(s)\, b_i(t)\,. \tag{1}$$

The functions $a_i(s)$ can be assumed to be linearly independent, otherwise the number of terms in relation (1) can be reduced (by linear independence it is meant that, if $c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0$, where c_i are arbitrary constants, then $c_1 = c_2 = \cdots = c_n = 0$).

(ii) SYMMETRIC KERNEL. A complex-valued function $K(s, t)$ is called symmetric (or Hermitian) if $K(s, t) = K^*(t, s)$, where the asterisk denotes the complex conjugate. For a real kernel, this coincides with definition $K(s, t) = K(t, s)$.

1.4. EIGENVALUES AND EIGENFUNCTIONS

If we write the homogeneous Fredholm equation as

$$\int_a^b K(s, t) g(t)\, dt = \mu g(s) , \qquad \mu = 1/\lambda ,$$

we have the classical eigenvalue or characteristic value problem; μ is the eigenvalue and $g(t)$ is the corresponding eigenfunction or characteristic function. Since the linear integral equations are studied in the form (1.1.10), it is λ and not $1/\lambda$ which is called the eigenvalue.

1.5. CONVOLUTION INTEGRAL

Many interesting problems of mechanics and physics lead to an integral equation in which the kernel $K(s, t)$ is a function of the difference $(s-t)$ only:

$$K(s, t) = k(s-t) , \tag{1}$$

where k is a certain function of one variable. The integral equation

$$g(s) = f(s) + \lambda \int_a^s k(s-t) g(t)\, dt , \tag{2}$$

and the corresponding Fredholm equation are called integral equations of the convolution type.

The function defined by the integral

$$\int_0^s k(s-t)\,g(t)\,dt = \int_0^s k(t)\,g(s-t)\,dt \tag{3}$$

is called the convolution or the Faltung of the two functions k and g. The integrals occurring in (3) are called the convolution integrals.

The convolution defined by relation (3) is a special case of the standard convolution

$$\int_{-\infty}^{\infty} k(s-t)\,g(t)\,dt = \int_{-\infty}^{\infty} k(t)\,g(s-t)\,dt\ . \tag{4}$$

The integrals in (3) are obtained from those in (4) by taking $k(t) = g(t) = 0$, for $t < 0$ and $t > s$.

2.1. REDUCTION TO A SYSTEM OF ALGEBRAIC EQUATIONS

In Chapter 1, we have defined a degenerate or a separable kernel $K(s,t)$ as

$$K(s,t) = \sum_{i=1}^{n} a_i(s)\,b_i(t)\ , \tag{1}$$

where the functions $a_1(s), \ldots, a_n(s)$ and the functions $b_1(t), \ldots, b_n(t)$ are linearly independent. With such a kernel, the Fredholm integral equation of the second kind,

$$g(s) = f(s) + \lambda \int K(s,t)\,g(t)\,dt \tag{2}$$

becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int b_i(t)\,g(t)\,dt\ . \tag{3}$$

It emerges that the technique of solving this equation is essentially dependent on the choice of the complex parameter λ and on the definition of

$$c_i = \int b_i(t)\,g(t)\,dt\ . \tag{4}$$

The quantities c_i are constants, although hitherto unknown.

Substituting (4) in (3) gives

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} c_i a_i(s) , \tag{5}$$

and the problem reduces to finding the quantities c_i. To this end, we put the value of $g(s)$ as given by (5) in (3) and get

$$\sum_{i=1}^{n} a_i(s) \{c_i - \int b_i(t) [f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t)] \, dt\} = 0 . \tag{6}$$

But the functions $a_i(s)$ are linearly independent; therefore,

$$c_i - \int b_i(t) [f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t)] \, dt = 0 , \qquad i = 1, \ldots, n . \tag{7}$$

Using the simplified notation

$$\int b_i(t) f(t) = f_i , \qquad \int b_i(t) a_k(t) \, dt = a_{ik} , \tag{8}$$

where f_i and a_{ik} are known constants, equation (7) becomes

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i , \qquad i = 1, \ldots, n ; \tag{9}$$

that is, a system of n algebraic equations for the unknowns c_i. The determinant $D(\lambda)$ of this system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \vdots & & & \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix} , \tag{10}$$

which is a polynomial in λ of degree at most n. Moreover, it is not identically zero, since, when $\lambda = 0$, it reduces to unity.

For all values of λ for which $D(\lambda) \neq 0$, the algebraic system (9), and thereby the integral equation (2), has a unique solution. On the other hand, for all values of λ for which $D(\lambda)$ becomes equal to zero, the algebraic system (9), and with it the integral equation (2), either is insoluble or has an infinite number of solutions. Setting $\lambda = 1/\mu$ in

equation (9), we have the eigenvalue problem of matrix theory. The eigenvalues are given by the polynomial $D(\lambda) = 0$. They are also the eigenvalues of our integral equation.

Note that we have considered only the integral equation of the second kind, where alone this method is applicable.

This method is illustrated with the following examples.

2.2. EXAMPLES

Example 1. Solve the Fredholm integral equation of the second kind

$$g(s) = s + \lambda \int_0^1 (st^2 + s^2 t) g(t)\, dt \,. \qquad (1)$$

The kernel $K(s,t) = st^2 + s^2 t$ is separable and we can set

$$c_1 = \int_0^1 t^2 g(t)\, dt \,, \qquad c_2 = \int_0^1 t g(t)\, dt \,.$$

Equation (1) becomes

$$g(s) = s + \lambda c_1 s + \lambda c_2 s^2 \,, \qquad (2)$$

which we substitute in (1) to obtain the algebraic equations

$$\begin{aligned} c_1 &= \tfrac{1}{4} + \tfrac{1}{4}\lambda c_1 + \tfrac{1}{5}\lambda c_2 \,, \\ c_2 &= \tfrac{1}{3} + \tfrac{1}{3}\lambda c_1 + \tfrac{1}{4}\lambda c_2 \,. \end{aligned} \qquad (3)$$

The solution of these equations is readily obtained as

$$c_1 = (60+\lambda)/(240 - 120\lambda - \lambda^2) \,, \qquad c_2 = 80/(240 - 120\lambda - \lambda^2) \,. \quad (4)$$

From (2) and (4), we have the solution

$$g(s) = [(240-60\lambda)s + 80\lambda s^2]/(240-120\lambda-\lambda^2) . \tag{5}$$

Example 2. Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^1 (s+t)\, g(t)\, dt \tag{6}$$

and find the eigenvalues.

Here, $a_1(s) = s$, $a_2(s) = 1$, $b_1(t) = 1$, $b_2(t) = t$,

$$a_{11} = \int_0^1 t\, dt = \tfrac{1}{2} , \qquad a_{12} = \int_0^1 dt = 1 ,$$

$$a_{21} = \int_0^1 t^2\, dt = \tfrac{1}{3} , \qquad a_{22} = \int_0^1 t\, dt = \tfrac{1}{2} ,$$

$$f_1 = \int_0^1 f(t)\, dt , \qquad f_2 = \int_0^1 tf(t)\, dt .$$

Substituting these values in (2.1.9), we have the algebraic system

$$(1-\tfrac{1}{2}\lambda)c_1 - \lambda c_2 = f_1 , \qquad -\tfrac{1}{3}\lambda c_1 + (1-\tfrac{1}{2}\lambda)c_2 = f_2 .$$

The determinant $D(\lambda) = 0$ gives $\lambda^2 + 12\lambda - 12 = 0$. Thus, the eigenvalues are

$$\lambda_1 = (-6 + 4\sqrt{3}) , \qquad \lambda_2 = (-6 - 4\sqrt{3}) .$$

For these two values of λ, the homogeneous equation has a nontrivial solution, while the integral equation (6) is, in general, not soluble. When λ differs from these values, the solution of the above algebraic system is

$$c_1 = [-12f_1 + \lambda(6f_1 - 12f_2)]/(\lambda^2 + 12\lambda - 12) ,$$

$$c_2 = [-12f_2 - \lambda(4f_1 - 6f_2)]/(\lambda^2 + 12\lambda - 12) .$$

Using the relation (2.1.5), there results the solution

$$g(s) = f(s) + \lambda \int_0^1 \frac{6(\lambda-2)(s+t) - 12\lambda st - 4\lambda}{\lambda^2 + 12\lambda - 12} f(t)\, dt \,. \qquad (7)$$

The function $\Gamma(s, t; \lambda)$,

$$\Gamma(s, t; \lambda) = [6(\lambda-2)(s+t) - 12\lambda st - 4\lambda]/(\lambda^2 + 12\lambda - 12) \,, \quad (8)$$

is called the resolvent kernel. We have therefore succeeded in inverting the integral equation because the right-hand side of the above formula is a known quantity.

2.3. FREDHOLM ALTERNATIVE

In the previous sections, we have found that, if the kernel is separable, the problem of solving an integral equation of the second kind reduces to that of solving an algebraic system of equations. Unfortunately, integral equations with degenerate kernels do not occur frequently in practice. But since they are easily treated and, furthermore, the results derived for such equations lead to a better understanding of integral equations of more general types, it is worthwhile to study them. Last, but not least, any reasonably well-behaved kernel can be written as an infinite series of degenerate kernels.

When an integral equation cannot be solved in closed form, then recourse has to be taken to approximate methods. But these approximate methods can be applied with confidence only if the existence of the solution is assured in advance. The Fredholm theorems explained in this chapter provide such an assurance. The basic theorems of the general theory of integral equations, which were first presented by Fredholm, correspond to the basic theorems of linear algebraic systems. Fredholm's classical theory shall be presented in Chapter 4 for general kernels. Here, we shall deal with degenerate kernels and borrow the results of linear algebra.

In Section 2.1, we have found that the solution of the present problem rests on the investigation of the determinant (2.1.10) of the coefficients of the algebraic system (2.1.9). If $D(\lambda) \neq 0$, then that system has only one solution, given by Cramer's rule

$$c_i = (D_{1i}f_1 + D_{2i}f_2 + \cdots + D_{ni}f_n)/D(\lambda) \,, \qquad i = 1, 2, \cdots, n \,, \quad (1)$$

where D_{hi} denotes the cofactor of the (h, i)th element of the determinant (2.1.10). Consequently, the integral equation (2.1.2) has the unique solution (2.1.5), which, in view of (1), becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} \frac{D_{1i}f_1 + D_{2i}f_2 + \cdots + D_{ni}f_n}{D(\lambda)} a_i(s) , \tag{2}$$

while the corresponding homogeneous equation

$$g(s) = \lambda \int K(s,t) g(t) \, dt \tag{3}$$

has only the trivial solution $g(s) = 0$.

Substituting for f_i from (2.1.8) in (2), we can write the solution $g(s)$ as

$$g(s) = f(s) + [\lambda / D(\lambda)] \times \int \{ \sum_{i=1}^{n} [D_{1i} b_1(t) + D_{2i} b_2(t) + \cdots + D_{ni} b_n(t)] a_i(s) \} f(t) \, dt. \tag{4}$$

Now consider the determinant of $(n+1)$th order

$$D(s,t;\lambda) = - \begin{vmatrix} 0 & a_1(s) & a_2(s) & \cdots & a_n(s) \\ b_1(t) & 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ b_2(t) & -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \vdots & & & & \\ b_n(t) & -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix} . \tag{5}$$

By developing it by the elements of the first row and the corresponding minors by the elements of the first column, we find that the expression in the brackets in equation (4) is $D(s,t;\lambda)$. With the definition

$$\Gamma(s,t;\lambda) = D(s,t;\lambda)/D(\lambda) , \tag{6}$$

equation (4) takes the simple form

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t) \, dt . \tag{7}$$

The function $\Gamma(s,t;\lambda)$ is the resolvent (or reciprocal) kernel we have already encountered in Examples 2 and 4 in the previous section. We shall see in Chapter 4 that the formula (6) has many important consequences. For the time being, we content ourselves with the observation that the only possible singular points of $\Gamma(s,t;\lambda)$ in the λ-plane are the roots of the equation $D(\lambda) = 0$, i.e., the eigenvalues of the kernel $K(s,t)$.

The above discussion leads to the following basic Fredholm theorem.

Fredholm Theorem. The inhomogeneous Fredholm integral equation (2.1.2) with a separable kernel has one and only one solution, given by formula (7). The resolvent kernel $\Gamma(s,t;\lambda)$ coincides with the quotient (6) of two polynomials.

If $D(\lambda)=0$, then the inhomogeneous equation (2.1.2) has no solution in general, because an algebraic system with vanishing determinant can be solved only for some particular values of the quantities f_i. To discuss this case, we write the algebraic system (2.1.9) as

$$(\mathbf{I}-\lambda\mathbf{A})\,\mathbf{c} = \mathbf{f}\,, \tag{8}$$

where $\mathbf{I}$ is the unit (or identity) matrix of order n and $\mathbf{A}$ is the matrix (a_{ij}). Now, when $D(\lambda)=0$, we observe that for each nontrivial solution of the homogeneous algebraic system

$$(\mathbf{I}-\lambda\mathbf{A})\,\mathbf{c} = 0 \tag{9}$$

there corresponds a nontrivial solution (an eigenfunction) of the homogeneous integral equation (3). Furthermore, if λ coincides with a certain eigenvalue λ_0 for which the determinant $D(\lambda_0)=|\mathbf{I}-\lambda_0\mathbf{A}|$ has the rank p, $1\leqslant p\leqslant n$, then there are $r=n-p$ linearly independent solutions of the algebraic system (9); r is called the index of the eigenvalue λ_0. The same holds for the homogeneous integral equation (3). Let us denote these r linearly independent solutions as $g_{01}(s), g_{02}(s), \cdots, g_{0r}(s)$, and let us also assume that they have been normalized. Then, to each eigenvalue λ_0 of index $r=n-p$, there corresponds a solution $g_0(s)$ of the homogeneous integral equation (3) of the form

$$g_0(s) = \sum_{k=1}^{r} \alpha_k\, g_{0k}(s)\,,$$

where α_k are arbitrary constants.

Let m be the multiplicity of the eigenvalue λ_0, i.e., $D(\lambda)=0$ has m equal roots λ_0. Then, we infer from the theory of linear algebra that, by using the elementary transformations on the determinant $|\mathbf{I}-\lambda\mathbf{A}|$, we shall have at most $m+1$ identical rows and this maximum is achieved only if $\mathbf{A}$ is symmetric. This means that the rank p of $D(\lambda_0)$ is greater than or equal to $n-m$. Thus,

$$r = n-p \leqslant n-(n-m) = m\,,$$

and the equality holds only when $a_{ij} = a_{ji}$.

Thus we have proved the theorem of Fredholm that, if $\lambda = \lambda_0$ is a root of multiplicity $m \geqslant 1$ of the equation $D(\lambda) = 0$, then the homogeneous integral equation (3) has r linearly independent solutions; r is the index of the eigenvalue such that $1 \leqslant r \leqslant m$.

The numbers r and m are also called the geometric multiplicity and algebraic multiplicity of λ_0, respectively. From the above result, it follows that the algebraic multiplicity of an eigenvalue must be greater than or equal to its geometric multiplicity.

To study the case when the inhomogeneous Fredholm integral equation (2.1.2) has solutions even when $D(\lambda) = 0$, we need to define and study the transpose of the equation (2.1.2). The integral equation[1]

$$\psi(s) = f(s) + \lambda \int K(t,s)\psi(t)\,dt \tag{10}$$

is called the transpose (or adjoint) of the equation (2.1.2). Observe that the relation between (2.1.2) and its transpose (10) is symmetric, since (2.1.2) is the transpose of (10).

If the separable kernel $K(s,t)$ has the expansion (2.1.1), then the kernel $K(t,s)$ of the transposed equation has the expansion

$$K(t,s) = \sum_{i=1}^{n} a_i(t)\,b_i(s)\ . \tag{11}$$

Proceeding as in Section 2.1, we end up with the algebraic system

$$(\mathbf{I} - \lambda \mathbf{A}^{\mathrm{T}})\,\mathbf{c} = \mathbf{f}\ , \tag{12}$$

where $\mathbf{A}^{\mathrm{T}}$ stands for the transpose of $\mathbf{A}$ and where c_i and f_i are now defined by the relations

$$c_i = \int a_i(t)\,g(t)\,dt\ , \qquad f_i = \int a_i(t) f(t)\,dt\ . \tag{13}$$

The interesting feature of the system (12) is that the determinant $D(\lambda)$ is the same function as (2.1.10) except that there has been an interchange of rows and columns in view of the interchange in the functions a_i and b_i. Thus, the eigenvalues of the transposed integral equation are the same as those of the original equation. This means that *the transposed equation* (10) *also possesses a unique solution whenever* (2.1.2) *does*.

As regards the eigenfunctions of the homogeneous system

$$|\mathbf{I}-\lambda\mathbf{A}^{\mathrm{T}}|\,\mathbf{c} = 0\,, \tag{14}$$

we know from linear algebra that these are different from the corresponding eigenfunctions of the system (9). The same applies to the eigenfunctions of the transposed integral equation. Since the index r of λ_0 is the same in both these systems, the number of linearly independent eigenfunctions is also r for the transposed system. Let us denote them by $\psi_{01}, \psi_{02}, \cdots, \psi_{0r}$ and let us assume that they have been normalized. Then, any solution $\psi_0(s)$ of the transposed homogeneous integral equation

$$\psi(s) = \lambda \int K(t,s)\psi(t)\,dt \tag{15}$$

corresponding to the eigenvalue λ_0 is of the form

$$\psi_0(s) = \sum \beta_i \psi_{0i}(s)\,,$$

where β_i are arbitrary constants.

We prove in passing that eigenfunctions $g(s)$ and $\psi(s)$ corresponding to distinct eigenvalues λ_1 and λ_2, respectively, of the homogeneous integral equation (3) and its transpose (15) are orthogonal. In fact, we have

$$g(s) = \lambda_1 \int K(s,t)\,g(t)\,dt\,, \qquad \psi(s) = \lambda_2 \int K(t,s)\,\psi(t)\,dt.$$

Multiplying both sides of the first equation by $\lambda_2\psi(s)$ and those of the second equation by $\lambda_1 g(s)$, integrating, and then subtracting the resulting equations, we obtain

$$(\lambda_2-\lambda_1)\int_a^b g(s)\psi(s)\,ds = 0\,.$$

But $\lambda_1 \neq \lambda_2$, and the result follows.

We are now ready to discuss the solution of the inhomogeneous Fredholm integral equation (2.1.2) for the case $D(\lambda)=0$. In fact, we can prove that the necessary and sufficient condition for this equation to have a solution for $\lambda=\lambda_0$, a root of $D(\lambda)=0$, is that $f(s)$ be orthogonal to the r eigenfunctions ψ_{0i} of the transposed equation (15).

The necessary part of the proof follows from the fact that, if equation (2.1.2) for $\lambda=\lambda_0$ admits a certain solution $g(s)$, then

$$\int f(s)\psi_{0i}(s)\,ds = \int g(s)\psi_{0i}(s)\,ds - \lambda_0 \int \psi_{0i}(s)\,ds \int K(s,t)g(t)\,dt$$
$$= \int g(s)\psi_{0i}(s)\,ds - \lambda_0 \int g(t)\,dt \int K(s,t)\psi_{0i}(s)\,ds = 0\,,$$

because λ_0 and $\psi_{0i}(s)$ are eigenvalues and corresponding eigenfunctions of the transposed equation.

To prove the sufficiency of this condition, we again appeal to linear algebra. In fact, the corresponding condition of orthogonality for the linear-algebraic system assures us that the inhomogeneous system (8) reduces to only $n-r$ independent equations. This means that the rank of the matrix $(\mathbf{I}-\lambda\mathbf{A})$ is exactly $p=n-r$, and therefore the system (8) or (2.1.9) is soluble. Substituting this solution in (2.1.5), we have the solution to our integral equation.

Finally, the difference of any two solutions of (2.1.2) is a solution of

the homogeneous equation (3). Hence, the most general solution of the inhomogeneous integral equation (2.1.2) has the form

$$g(s) = G(s) + \alpha_1 g_{01}(s) + \alpha_2 g_{02}(s) + \cdots + \alpha_r g_{0r}(s)\,, \tag{16}$$

where $G(s)$ is a suitable linear combination of the functions $a_1(s)$, $a_2(s), \cdots, a_n(s)$.

We have thus proved the theorem that, if $\lambda=\lambda_0$ is a root of multiplicity $m \geqslant 1$ of the equation $D(\lambda)=0$, then the inhomogeneous equation has a solution if and only if the given function $f(s)$ is orthogonal to all the eigenfunctions of the transposed equation.

The results of this section can be collected to establish the following theorem.

Fredholm Alternative Theorem. Either the integral equation

$$g(s) = f(s) + \lambda \int K(s,t)g(t)\,dt \tag{17}$$

with fixed λ possesses one and only one solution $g(s)$ for arbitrary $\mathscr{L}_2$-functions $f(s)$ and $K(s,t)$, in particular the solution $g=0$ for $f=0$; or the homogeneous equation

$$g(s) = \lambda \int K(s,t)g(t)\,dt \tag{18}$$

possesses a finite number r of linearly independent solutions g_{0i}, $i = 1, 2, \cdots, r$. In the first case, the transposed inhomogeneous equation

$$\psi(s) = f(s) + \lambda \int K(t, s) \psi(t)\, dt \qquad (19)$$

also possesses a unique solution. In the second case, the transposed homogeneous equation

$$\psi(s) = \lambda \int K(t, s) \psi(t)\, dt \qquad (20)$$

also has r linearly independent solutions ψ_{0i}, $i = 1, 2, \cdots, r$; the inhomogeneous integral equation (7) has a solution if and only if the given function $f(s)$ satisfies the r conditions

$$(f, \psi_{0i}) = \int f(s) \psi_{0i}(s)\, ds = 0\,, \qquad i = 1, 2, \cdots, r\,. \qquad (21)$$

In this case, the solution of (17) is determined only up to an additive linear combination $\sum_{i=1}^{r} c_i g_{0i}$.

The following examples illustrate the theorems of this section.

Example 1. Show that the integral equation

$$g(s) = f(s) + (1/\pi) \int_0^{2\pi} [\sin(s + t)]\, g(t)\, dt \qquad (1)$$

possesses no solution for $f(s) = s$, but that it possesses infinitely many solutions when $f(s) = 1$.

For this equation,

$$K(s, t) = \sin s \cos t + \cos s \sin t\,,$$

$$a_1(s) = \sin s\,, \qquad a_2(s) = \cos s\,, \qquad b_1(t) = \cos t\,, \qquad b_2(t) = \sin t\,.$$

Therefore,

$$a_{11} = \int_0^{2\pi} \sin t \cos t\, dt = 0 = a_{22}\,,$$

$$a_{12} = \int_0^{2\pi} \cos^2 t \, dt = \pi = a_{21} .$$

$$D(\lambda) = \begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = 1 - \lambda^2\pi^2 . \qquad (2)$$

The eigenvalues are $\lambda_1 = 1/\pi$, $\lambda_2 = -1/\pi$ and equation (1) contains $\lambda_1 = 1/\pi$. Therefore, we have to examine the eigenfunctions of the transposed equation (note that the kernel is symmetric)

$$g(s) = (1/\pi) \int_0^{2\pi} \sin(s+t) g(t) \, dt . \qquad (3)$$

The algebraic system corresponding to (3) is

$$c_1 - \lambda\pi c_2 = 0 , \qquad -\lambda\pi c_1 + c_2 = 0 ,$$

which gives

$$c_1 = c_2 \quad \text{for} \quad \lambda_1 = 1/\pi ; \qquad c_1 = -c_2 \quad \text{for} \quad \lambda_2 = -1/\pi .$$

Therefore, the eigenfunctions for $\lambda_1 = 1/\pi$ follow from the relation (2.1.5) and are given by

$$g(s) = c(\sin s + \cos s) . \qquad (4)$$

Since

$$\int_0^{2\pi} (s \sin s + s \cos s) \, ds = -2\pi \neq 0 ,$$

while

$$\int_0^{2\pi} (\sin s + \cos s) \, ds = 0 ,$$

we have proved the result.

Example 2. Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^1 (1 - 3st) g(t) \, dt . \qquad (5)$$

The algebraic system (2.1.9) for this equation is

$$(1-\lambda)c_1 + \tfrac{3}{2}\lambda c_2 = f_1\,, \qquad -\tfrac{1}{2}\lambda c_1 + (1+\lambda)c_2 = f_2\,, \tag{6}$$

while

$$D(\lambda) = \begin{vmatrix} 1-\lambda & \tfrac{3}{2}\lambda \\ -\tfrac{1}{2}\lambda & 1+\lambda \end{vmatrix} = \tfrac{1}{4}(4-\lambda^2)\,. \tag{7}$$

Therefore, the inhomogeneous equation (5) will have a unique solution if and only if $\lambda \neq \pm 2$. Then the homogeneous equation

$$g(s) = \lambda \int_0^1 (1-3st)\,g(t)\,dt \tag{8}$$

has only the trivial solution.

Let us now consider the case when λ is equal to one of the eigenvalues and examine the eigenfunctions of the transposed homogeneous equation

$$g(s) = \lambda \int_0^1 (1-3st)\,g(t)\,dt\,. \tag{9}$$

For $\lambda = +2$, the algebraic system (6) gives $c_1 = 3c_2$. Then, (2.1.5) gives the eigenfunction

$$g(s) = c(1-s)\,, \tag{10}$$

where c is an arbitrary constant. Similarly, for $\lambda = -2$, the corresponding eigenfunction is

$$g(s) = c(1-3s)\,. \tag{11}$$

It follows from the above analysis that the integral equation

$$g(s) = f(s) + 2\int_0^1 (1-3st)\,g(t)\,dt$$

will have a solution if $f(s)$ satisfies the condition

$$\int_0^1 (1-s)f(s)\,ds = 0\,,$$

while the integral equation

$$g(s) = f(s) - 2\int_0^1 (1-3st)\,g(t)\,dt$$

will have a solution if the following holds:

$$\int_0^1 (1-3s)f(s)\,ds = 0\,.$$

VOLTERRA INTEGRAL EQUATION

The same iterative scheme is applicable to the Volterra integral equation of the second kind. In fact, the formulas corresponding to (3.1.13) and (3.1.25) are, respectively,

$$g(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_a^s K_m(s,t) f(t)\, dt , \tag{1}$$

$$g(s) = f(s) + \lambda \int_a^s \Gamma(s,t;\lambda) f(t)\, dt , \tag{2}$$

where the iterated kernel $K_m(s,t)$ satisfies the recurrence formula

$$K_m(s,t) = \int_t^s K(s,x)\, K_{m-1}(x,t)\, dx \tag{3}$$

with $K_1(s,t) = K(s,t)$, as before. The resolvent kernel $\Gamma(s,t;\lambda)$ is given by the same formula as (3.1.26), and it is an entire function of λ for any given (s,t) (see Exercise 8).

We shall illustrate it by the following examples.

From the formula (3.3.3), we have

$$K_1(s,t) = (s-t) ,$$

$$K_2(s,t) = \int_t^s (s-x)(x-t)\, dx = \frac{(s-t)^3}{3!} ,$$

$$K_3(s,t) = \int_t^s \frac{(s-x)(x-t)^3}{3!}\, dx = \frac{(s-t)^5}{5!} ,$$

and so on. Thus,

$$g(s) = 1 + s + \lambda\left(\frac{s^2}{2!} + \frac{s^3}{3!}\right) + \lambda^2\left(\frac{s^4}{4!} + \frac{s^5}{5!}\right) + \cdots . \tag{2}$$

For $\lambda = 1$, $g(s) = e^s$.

Example 2. Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^s e^{s-t} g(t)\, dt \tag{3}$$

and evaluate the resolvant kernel.

For this case,

$$K_1(s,t) = e^{s-t},$$

$$K_2(s,t) = \int_t^s e^{s-x} e^{x-t}\, dx = (s-t)e^{s-t},$$

$$K_3(s,t) = \int_t^s (x-t) e^{s-x} e^{x-t}\, dx = \frac{(s-t)^2}{2!} e^{s-t},$$

$$K_m(s,t) = \frac{(s-t)^{m-1}}{(m-1)!} e^{s-t}.$$

The resolvent kernel is

$$\Gamma(s,t;\lambda) = \begin{cases} e^{s-t} \sum_{m=1}^{\infty} \frac{\lambda^{m-1}(s-t)^{m-1}}{(m-1)!} = e^{(\lambda+1)(s-t)}, & t \leqslant s, \\ 0, & t > s. \end{cases} \tag{4}$$

Hence, the solution is

$$g(s) = f(s) + \lambda \int_0^s e^{(\lambda+1)(s-t)} f(t)\, dt.$$

4.1. THE METHOD OF SOLUTION OF FREDHOLM

In the previous chapter, we have derived the solution of the Fredholm integral equation

$$g(s) = f(s) + \lambda \int K(s,t)\, g(t)\, dt \qquad (1)$$

as a uniformly convergent power series in the parameter λ for $|\lambda|$ suitably small. Fredholm gave the solution of equation (1) in general form for all values of the parameter λ. His results are contained in three theorems which bear his name. We have already studied them in Chapter 2 for the special case when the kernel is separable. In this chapter, we shall study equation (1) when the function $f(s)$ and the kernel $K(s,t)$ are any integrable functions. Furthermore, the present method enables us to get explicit formulas for the solution in terms of certain determinants.

The method used by Fredholm consists in viewing the integral equation (1) as the limiting case of a system of linear algebraic equations. This theory applies to two- or higher-dimensional integrals, although we shall confine our discussion to only one-dimensional integrals in the interval (a,b). Let us divide the interval (a,b) into n equal parts,

$$s_1 = t_1 = a\,, \quad s_2 = t_2 = a+h\,, \quad \ldots, \quad s_n = t_n = a+(n-1)h\,,$$

where $h = (b-a)/n$. Thereby, we have the approximate formula

$$\int K(s,t)\, g(t)\, dt \simeq h \sum_{j=1}^{n} K(s,s_j)\, g(s_j)\,. \qquad (2)$$

Equation (1) then takes the form

$$g(s) \simeq f(s) + \lambda h \sum_{j=1}^{n} K(s,s_j)\, g(s_j)\,, \qquad (3)$$

which must hold for all values of s in the interval (a,b). In particular, this equation is satisfied at the n points of division s_i, $i = 1, \ldots, n$. This leads to the system of equations

$$g(s_i) = f(s_i) + \lambda h \sum_{j=1}^{n} K(s_i,s_j)\, g(s_j)\,, \qquad i = 1, \ldots, n\,. \qquad (4)$$

Writing

$$f(s_i) = f_i\,, \qquad g(s_i) = g_i\,, \qquad K(s_i,s_j) = K_{ij}\,, \qquad (5)$$

equation (4) yields an approximation for the integral equation (1) in terms of the system of n linear equations

$$g_i - \lambda h \sum_{j=1}^{n} K_{ij} g_j = f_i \,, \qquad i = 1, \ldots, n \,, \tag{6}$$

in n unknown quantities $g_1, \ldots, g_n$. The values of g_i obtained by solving this algebraic system are approximate solutions of the integral equation (1) at the points $s_1, s_2, \ldots, s_n$. We can plot these solutions g_i as ordinates and by interpolation draw a curve $g(s)$ which we may expect to be an approximation to the actual solution. With the help of this algebraic system, we can also determine approximations for the eigenvalues of the kernel.

The resolvent determinant of the algebraic system (6) is

$$D_n(\lambda) = \begin{vmatrix} 1-\lambda h K_{11} & -\lambda h K_{12} & \cdots & -\lambda h K_{1n} \\ -\lambda h K_{21} & 1-\lambda h K_{22} & \cdots & -\lambda h K_{2n} \\ \vdots & & & \\ -\lambda h K_{n1} & -\lambda h K_{n2} & \cdots & 1-\lambda h K_{nn} \end{vmatrix} . \tag{7}$$

The approximate eigenvalues are obtained by setting this determinant equal to zero. We illustrate it by the following example.

Example.

$$g(s) - \lambda \int_0^{\pi} \sin(s+t)\, g(t)\, dt = 0 \,.$$

By taking $n = 3$, we have $h = \pi/3$ and therefore

$$s_1 = t_1 = 0 \,, \qquad s_2 = t_2 = \pi/3 \,, \qquad s_3 = t_3 = 2\pi/3 \,,$$

ınd the values of K_{ij} are readily calculated to give

$$(K_{ij}) = \begin{vmatrix} 0 & 0.866 & 0.866 \\ 0.866 & 0.866 & 0 \\ 0.866 & 0 & -0.866 \end{vmatrix} .$$

The homogeneous system corresponding to (6) will have a nontrivial solution if the determinant

$$D_n(\lambda) = \begin{vmatrix} 1 & -0.907\lambda & -0.907\lambda \\ -0.907\lambda & (1-0.907\lambda) & 0 \\ -0.907\lambda & 0 & (1+0.907\lambda) \end{vmatrix} = 0 ,$$

or when $1-3(0.0907)^2\lambda^2=0$. The roots of this equation are $\lambda = \pm 0.6365$. This gives a rather close agreement with the exact values (see Example 3, Section 3.2), which are $\pm\sqrt{2}/\pi = \pm 0.6366$.

In general, the practical applications of this method are limited because one has to take a rather large n to get a reasonable approximation.

4.2. FREDHOLM'S FIRST THEOREM

The solutions $g_1, g_2, \ldots, g_n$ of the system of equations (4.1.6) are obtained as ratios of certain determinants, with the determinant $D_n(\lambda)$ given by (4.1.7) as the denominator provided it does not vanish. Let us expand the determinant (4.1.7) in powers of the quantity $(-\lambda h)$. The constant term is obviously equal to unity. The term containing $(-\lambda h)$ in the first power is the sum of all the determinants containing only one column $-\lambda h K_{\mu\nu}$, $\mu = 1, \ldots, n$. Taking the contribution from all the columns $\nu = 1, \ldots, n$, we find that the total contribution is $-\lambda h \sum_{\nu=1}^{n} K_{\nu\nu}$.

The factor containing the factor $(-\lambda h)$ to the second power is the sum of all the determinants containing two columns with that factor. This results in the determinants of the form

$$(-\lambda h)^2 \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix} ,$$

where (p,q) is an arbitrary pair of integers taken from the sequence $1, \ldots, n$, with $p < q$. In the same way, it follows that the term containing the factor $(-\lambda h)^3$ is the sum of the determinants of the form

$$(-\lambda h)^3 \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix},$$

where (p,q,r) is an arbitrary triplet of integers selected from the sequence $1, \ldots, n$, with $p < q < r$.

The remaining terms are obtained in a similar manner. Therefore, we conclude that the required expansion of $D_n(\lambda)$ is

$$D_n(\lambda) = 1 - \lambda h \sum_{\nu=1}^{n} K_{\nu\nu} + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$$

$$+ \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix} + \cdots$$

$$+ \frac{(-\lambda h)^n}{n!} \sum_{p_1,p_2,\ldots,p_n=1}^{n} \begin{vmatrix} K_{p_1p_1} & K_{p_1p_2} & \cdots & K_{p_1p_n} \\ K_{p_2p_1} & K_{p_2p_2} & \cdots & K_{p_2p_n} \\ \vdots & & & \\ K_{p_np_1} & K_{p_np_2} & \cdots & K_{p_np_n} \end{vmatrix}, \qquad (1)$$

where we now stipulate that the sums are taken over all permutations of pairs (p,q), triplets (p,q,r), etc. This convention explains the reason for dividing each term of the above series by the corresponding number of permutations.

The analysis is simplified by introducing the following symbol for the determinant formed by the values of the kernel at all points (s_i, t_j)

$$\begin{vmatrix} K(s_1,t_1) & K(s_1,t_2) & \cdots & K(s_1,t_n) \\ K(s_2,t_1) & K(s_2,t_2) & \cdots & K(s_2,t_n) \\ \vdots & & & \\ K(s_n,t_1) & K(s_n,t_2) & \cdots & K(s_n,t_n) \end{vmatrix} = K\begin{pmatrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{pmatrix}, \qquad (2)$$

the so-called Fredholm determinant. We observe that, if any pair of arguments in the upper or lower sequence is transposed, the value of the determinant changes sign because the transposition of two arguments in the upper sequence corresponds to the transposition of two rows of the determinant and the transposition of two arguments in the lower sequence corresponds to the transposition of two columns.

In this notation, the series (1) takes the form

$$D_n(\lambda) = 1 - \lambda h \sum_{p=1}^{n} K(s_p, s_p) + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^{n} K\begin{pmatrix} s_p, s_q \\ s_p, s_q \end{pmatrix} + \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^{n} K\begin{pmatrix} s_p, s_q, s_r \\ s_p, s_q, s_r \end{pmatrix} + \cdots . \qquad (3)$$

If we now let n tend to infinity, then h will tend to zero, and each term of the sum (3) tends to some single, double, triple integral, etc. There results Fredholm's first series:

$$D(\lambda) = 1 - \lambda \int K(s,s)\, ds + \frac{\lambda^2}{2!} \int K\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} ds_1\, ds_2 - \frac{\lambda^3}{3!} \iiint K\begin{pmatrix} s_1, s_2, s_3 \\ s_1, s_2, s_3 \end{pmatrix} ds_1\, ds_2\, ds_3 + \cdots . \qquad (4)$$

Hilbert gave a rigorous proof of the fact that the sequence $D_n(\lambda) \to D(\lambda)$ in the limit, while the convergence of the series (4) for all values of λ was proved by Fredholm on the basis that the kernel $K(s, t)$ is a bounded and integrable function.[1] Thus, $D(\lambda)$ is an entire function of the complex variable λ.

We are now ready to solve the Fredholm equation (4.1.1) and express the solutions in the form of a quotient of two power series in the parameter λ, where the Fredholm function $D(\lambda)$ is to be the divisor. In this connection, recall the relations (2.3.6) and (2.3.7). Indeed, we seek solutions of the form

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t)\, dt \,, \tag{5}$$

and expect the resolvent kernel $\Gamma(s,t;\lambda)$ to be the quotient

$$\Gamma(s,t;\lambda) = D(s,t;\lambda)/D(\lambda)\,, \tag{6}$$

where $D(s,t;\lambda)$, still to be determined, is the sum of certain functional series.

Now, we have proved in Section 3.5 that the resolvent $\Gamma(s,t;\lambda)$ itself satisfies a Fredholm integral equation of the second kind (3.5.5):

$$\Gamma(s,t;\lambda) = K(s,t) + \lambda \int K(s,x)\,\Gamma(x,t;\lambda)\, dx \,. \tag{7}$$

From (6) and (7), it follows that

$$D(s,t;\lambda) = K(s,t)\, D(\lambda) + \lambda \int K(s,x)\, D(x,t;\lambda)\, dx \,. \tag{8}$$

The form of the series (4) for $D(\lambda)$ suggests that we seek the solution of equation (8) in the form of a power series in the parameter λ:

$$D(s,t;\lambda) = C_0(s,t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!}\, C_p(s,t) \,. \tag{9}$$

For this purpose, write the numerical series (4) as

$$D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!}\, c_p \,, \tag{10}$$

where

$$c_p = \int \cdots \int K\begin{pmatrix} s_1, s_2, \ldots, s_p \\ s_1, s_2, \ldots, s_p \end{pmatrix} ds_1 \cdots ds_p \,. \tag{11}$$

The next step is to substitute the series for $D(s,t;\lambda)$ and $D(\lambda)$ from (9) and (10) in (8) and compare the coefficients of equal powers of λ. The following relations result:

$$C_0(s,t) = K(s,t)\,, \tag{12}$$

$$C_p(s,t) = c_p K(s,t) - p\int K(s,x)\,C_{p-1}(x,t)\,dx\,. \tag{13}$$

Our contention is that we can write the function $C_p(s,t)$ in terms of the Fredholm determinant (2) in the following way:

$$C_p(s,t) = \int\cdots\int K\begin{pmatrix} s, x_1, x_2, \ldots, x_p \\ t, x_1, x_2, \ldots, x_p \end{pmatrix} dx_1 \cdots dx_p\,. \tag{14}$$

In fact, for $p = 1$, the relation (13) becomes

$$\begin{aligned} C_1(s,t) &= c_1 K(s\ t) - \int K(s,x)\,C_0(x,t)\,dx \\ &= K(s,t)\int K(x,x)\,dx - \int K(s,x)\,K(x,t)\,dx \\ &= \int K\begin{pmatrix} s & x \\ t & x \end{pmatrix} dx\,, \end{aligned} \tag{15}$$

where we have used (11) and (12).

To prove that (14) holds for general p, we expand the determinant under the integral sign in the relation:

$$K\begin{pmatrix} s, x_1, \ldots, x_p \\ t, x_1, \ldots, x_p \end{pmatrix} = \begin{vmatrix} K(s,t) & K(s,x_1) & \cdots & K(s,x_p) \\ K(x_1,t) & K(x_1,x_1) & \cdots & K(x_1,x_p) \\ \vdots & & & \\ K(x_p,t) & K(x_p,x_1) & \cdots & K(x_p,x_p) \end{vmatrix},$$

with respect to the elements of the given row, transposing in turn the first column one place to the right, integrating both sides, and using the definition of c_p as in (11); the required result then follows by induction.

From (9), (12), and (14) we derive Fredholm's second series:

$$D(s,t;\lambda) = K(s,t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!}\int\cdots\int K\begin{pmatrix} s, x_1, \ldots, x_p \\ t, x_1, \ldots, x_p \end{pmatrix} dx_1 \cdots dx_p\,. \tag{16}$$

This series also converges for all values of the parameter λ. It is interesting to observe the similarity between the series (4) and (16).

Having found both terms of the quotient (6), we have established the existence of a solution to the integral equation (4.1.1) for a bounded and integrable kernel $K(s,t)$, provided, of course, that $D(\lambda) \neq 0$. Since both terms of this quotient are entire functions of the parameter λ, it follows that the resolvent kernel $\Gamma(s,t;\lambda)$ is a meromorphic function of λ, i.e., an analytic function whose singularities may only be the poles, which in the present case are zeros of the divisor $D(\lambda)$.

Next, we prove that the solution in the form obtained by Fredholm is unique and is given by

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t)\, dt\,. \tag{17}$$

In this connection, we first observe that the integral equation (7) satisfied by $\Gamma(s,t;\lambda)$ is valid for all values of λ for which $D(\lambda) \neq 0$. Indeed, (7) is known to hold for $|\lambda| < B^{-1}$ from the analysis of Chapter 3, and since both sides of this equation are now proved to be meromorphic, the above contention follows. To prove the uniqueness of the solution, let us suppose that $g(s)$ is a solution of the equation (4.1.1) in the case $D(\lambda) \neq 0$. Multiply both sides of (4.1.1) by $\Gamma(s,t;\lambda)$, integrate, and get

$$\int \Gamma(s,x;\lambda) g(x)\, dx = \int \Gamma(s,x;\lambda) f(x)\, dx + \lambda \int \left[\int \Gamma(s,x;\lambda) K(x,t)\, dx\right] g(t)\, dt\,. \tag{18}$$

Substituting from (7) into left side of (18), this becomes

$$\int K(s,t) g(t)\, dt = \int \Gamma(s,x;\lambda) f(x)\, dx\,, \tag{19}$$

which, when joined by (4.1.1), yields

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t)\, dt\,, \tag{20}$$

but this form is unique.

In particular, the solution of the homogeneous equation

$$g(s) = \lambda \int K(s,t) g(t)\, dt \tag{21}$$

is identically zero.

The above analysis leads to the following theorem.

Fredholm's First Theorem. The inhomogeneous Fredholm equation

$$g(s) = f(s) + \lambda \int K(s,t)\, g(t)\, dt\,, \tag{22}$$

where the functions $f(s)$ and $g(t)$ are integrable, has a unique solution

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t)\, dt\,, \tag{23}$$

where the resolvent kernel $\Gamma(s,t;\lambda)$,

$$\Gamma(s,t;\lambda) = D(s,t;\lambda)/D(\lambda)\,, \tag{24}$$

with $D(\lambda) \neq 0$, is a meromorphic function of the complex variable λ, being the ratio of two entire functions defined by the series

$$D(s,t;\lambda) = K(s,t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int \cdots \int K\begin{pmatrix} s, x_1, \ldots, x_p \\ t, x_1, \ldots, x_p \end{pmatrix} dx_1 \cdots dx_p\,, \tag{25}$$

and

$$D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int \cdots \int K\begin{pmatrix} x_1, \ldots, x_p \\ x_1, \ldots, x_p \end{pmatrix} dx_1 \cdots dx_p\,, \tag{26}$$

both of which converge for all values of λ. In particular, the solution of the homogeneous equation

$$g(s) = \lambda \int K(s,t)\, g(t)\, dt \tag{27}$$

is identically zero.

4.3. EXAMPLES

Example 1. Evaluate the resolvent for the integral equation

$$g(s) = f(s) + \lambda \int_0^1 (s+t)\, g(t)\, dt\,. \tag{1}$$

The solution to this example is obtained by writing

$$\Gamma(s,t;\lambda) = \left[\sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} C_p(s,t)\right] \Big/ \sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} c_p\,, \tag{2}$$

where C_p and c_p are defined by the relations (4.2.11) and (4.2.13):

$$c_0 = 1 , \qquad C_0(s,t) = K(s,t) = (s+t) . \tag{3}$$

$$c_p = \int C_{p-1}(s,s)\, ds , \tag{4}$$

$$C_p = c_p K(s,t) - p \int_0^1 K(s,x)\, c_{p-1}(x,t)\, dx . \tag{5}$$

Thus,

$$c_1 = \int_0^1 2s\, ds = 1 ,$$

$$C_1(s,t) = (s+t) - \int_0^1 (s+x)(x+t)\, dx = \tfrac{1}{2}(s+t) - st - \tfrac{1}{3} ,$$

$$c_2 = \int_0^1 (s - s^2 - \tfrac{1}{3})\, ds = -\tfrac{1}{6} ,$$

$$C_2(s,t) = -\tfrac{1}{6}(s+t) - 2\int_0^1 (s+x)[\tfrac{1}{2}(x+t) - xt - \tfrac{1}{3})]\, dx = 0 .$$

Since $C_2(x,t)$ vanishes, it follows from (5) that the subsequent coefficients C_k and c_k also vanish. Therefore,

$$\Gamma(s,t;\lambda) = \frac{(s+t) - [\tfrac{1}{2}(s+t) - st - \tfrac{1}{3}]\,\lambda}{1 - \lambda - (\lambda^2/12)} , \tag{6}$$

which agrees with result (2.2.8) found by a different method.

Example 2. Solve the integral equation

$$g(s) = s + \lambda \int_0^1 [st + (st)^{1/2}]\, g(t)\, dt . \tag{7}$$

In this case,

$$c_0 = 1 , \qquad C_0(s,t) = st + (st)^{1/2} ,$$

$$c_1 = \int_0^1 (s^2+s)\,ds = \tfrac{5}{6}\,,$$

$$C_1(s,t) = \tfrac{5}{6}[st+(st)^{1/2}] - \int_0^1 [sx+(sx)^{1/2}][xt+(xt)^{1/2}]\,dt$$

$$= \tfrac{1}{2}st + \tfrac{1}{3}(st)^{1/2} - \tfrac{2}{5}(st^{1/2}+ts^{1/2})\,,$$

$$c_2 = \int_0^1 (\tfrac{1}{2}s^2+\tfrac{1}{3}s-\tfrac{4}{5}s^{3/2})\,ds = 1/75\,,$$

$$C_2(s,t) = 0\,,$$

and therefore all the subsequent coefficients vanish. The value of the resolvent is

$$\Gamma(s,t;\lambda) = \frac{st+(st)^{1/2} - \{\tfrac{1}{2}st + \tfrac{1}{3}(st)^{1/2} - \tfrac{2}{5}(st^{1/2}+s^{1/2}t)\}\lambda}{1-\tfrac{5}{6}\lambda + (1/150)\lambda^2}\,. \quad (8)$$

The solution $g(s)$ then follows by using the relation (4.2.23),

$$g(s) = \frac{150s + \lambda(60\sqrt{s}-75s) + 21\lambda^2 s}{\lambda^2 - 125\lambda + 150}\,. \quad (9)$$

4.4. FREDHOLM'S SECOND THEOREM

Fredholm's first theorem does not hold when λ is a root of the equation $D(\lambda) = 0$. We have found in Chapter 2 that, for a separable kernel, the homogeneous equation

$$g(s) = \lambda \int K(s,t)\,g(t)\,dt \quad (1)$$

has nontrivial solutions. It might be expected that same holds when the kernel is an arbitrary integrable function and we shall then have a spectrum of eigenvalues and corresponding eigenfunctions. The second theorem of Fredholm is devoted to the study of this problem.

We first prove that every zero of $D(\lambda)$ is a pole of the resolvent kernel (4.2.24); the order of this pole is at most equal to the order of the zero of $D(\lambda)$. In fact, differentiate the Fredholm's first series (4.2.26) and interchange the indices of the variables of integration to get

$$D'(\lambda) = -\int D(s,s;\lambda)\,ds\,. \quad (2)$$

From this relation, it follows that, if λ_0 is a zero of order k of $D(\lambda)$, then it is a zero of order $k-1$ of $D'(\lambda)$ and consequently λ_0 may be a zero of order at most $k-1$ of the entire function $D(s,t;\lambda)$. Thus, λ_0 is the pole of the quotient (4.2.24) of order at most k. In particular, if λ_0 is a simple zero of $D(\lambda)$, then $D(\lambda_0)=0$, $D'(\lambda_0)\neq 0$, and λ_0 is a simple pole of the resolvent kernel. Moreover, it follows from (2) that $D(s,t;\lambda)\neq 0$. For this particular case, we observe from equation (4.1.8) that, if $D(\lambda)=0$ and $D(s,t;\lambda)\neq 0$, then $D(s,t;\lambda)$, as a function of s, is a solution of the homogeneous equation (1). So is $\alpha D(s,t;\lambda)$, where α is an arbitrary constant.

Let us now consider the general case when λ is a zero of an arbitrary multiplicity m, that is, when

$$D(\lambda_0)=0\,, \qquad \ldots\,, \qquad D^{(r)}(\lambda_0)=0\,, \qquad D^{(m)}(\lambda_0)\neq 0\,, \tag{3}$$

where the superscript r stands for the differential of order r, $r=1$, ..., $m-1$. For this case, the analysis is simplified if one defines a determinant known as the Fredholm minor:

$$D_n\left(\begin{matrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle|\, \lambda\right) = K\begin{pmatrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \times \int \cdots \int K\begin{pmatrix} s_1, \ldots, s_n, x_1, \ldots, x_p \\ t_1, \ldots, t_n, x_1, \ldots, x_p \end{pmatrix} dx_1\, dx_2 \cdots dx_p\,, \tag{4}$$

where $\{s_i\}$ and $\{t_i\}$, $i=1,2,\ldots,n$, are two sequences of arbitrary variables. Just as do the Fredholm series (4.2.25) and (4.2.26), the series (4) also converges for all values of λ and consequently is an entire function of λ. Furthermore, by differentiating the series (4.2.26) n times and comparing it with the series (4), there follows the relation

$$\frac{d^n D(\lambda)}{d\lambda^n} = (-1)^n \int \cdots \int D_n\left(\begin{matrix} s_1, \ldots, s_n \\ s_1, \ldots, s_n \end{matrix}\middle|\, \lambda\right) ds_1 \cdots ds_n\,. \tag{5}$$

From this relation, we conclude that, if λ_0 is a zero of multiplicity m of the function $D(\lambda)$, then the following holds for the Fredholm minor of order m for that value of λ_0:

$$D_m\left(\begin{matrix} s_1, s_2, \ldots, s_m \\ t_1, t_2, \ldots, t_m \end{matrix}\middle|\, \lambda_0\right) \neq 0\,.$$

Of course, there might exist minors of order lower than m which also do not identically vanish (compare the discussion in Section 2.3).

Let us find the relation among the minors that corresponds to the resolvent formula (4.2.7). Expansion of the determinant under the integral sign in (4),

$$\begin{vmatrix} K(s_1,t_1) & K(s_1,t_2) & \cdots & K(s_1,t_n) & K(s_1,x_1) & \cdots & K(s_1,x_p) \\ K(s_2,t_1) & K(s_2,t_2) & \cdots & K(s_2,t_n) & K(s_2,x_1) & \cdots & K(s_2,x_p) \\ \vdots & & & & & & \\ K(s_n,t_1) & K(s_n,t_2) & \cdots & K(s_n,t_n) & K(s_n,x_1) & \cdots & K(s_n,x_p) \\ K(x_1,t_1) & K(x_1,t_2) & \cdots & K(x_1,t_n) & K(x_1,x_1) & \cdots & K(x_1,x_p) \\ \vdots & & & & & & \\ K(x_p,t_1) & K(x_p,t_2) & \cdots & K(x_p,t_n) & K(x_p,x_1) & \cdots & K(x_p,x_p) \end{vmatrix} \quad (6)$$

by elements of the first row and integrating p times with respect to $x_1, x_2, \ldots, x_p$ for $p \geqslant 1$, we have

$$\begin{aligned} &\int \cdots \int K\begin{pmatrix} s_1, \ldots, s_n, x_1, \ldots, x_p \\ t_1, \ldots, t_n, x_1, \ldots, x_p \end{pmatrix} dx_1 \cdots dx_p \\ &\quad = \sum_{h=1}^{n} (-1)^{h+1} K(s_1, t_h) \\ &\qquad \times \int \cdots \int K\begin{pmatrix} s_2, \ldots, s_h, \ldots, & s_n, x_1, \ldots, x_p \\ t_1, \ldots, t_{h-1}, t_{h+1}, \ldots, & t_n, x_1, \ldots, x_p \end{pmatrix} dx_1\, dx_2 \cdots dx_p \\ &\qquad + \sum_{h=1}^{p} (-1)^{h+n-1} \\ &\qquad \times \int \cdots \int K(s_1, x_h)\, K\begin{pmatrix} s_2, \ldots, & s_n, x_1, & x_2, & \ldots, & x_h, & \ldots, & x_p \\ t_1, \ldots, t_{n-1}, & t_n, x_1, & \ldots, & x_{h-1}, & x_{h+1}, & \ldots, & x_p \end{pmatrix} \\ &\qquad \times dx_1 \cdots dx_p\,. \end{aligned} \quad (7)$$

Note that the symbols for the determinant K on the right side of (7) do not contain the variables s_1 in the upper sequence and the variables t_h or x_h in the lower sequence. Furthermore, it follows by transposing the variable s_h in the upper sequence to the first place by means of $h+n-2$ transpositions that all the components of the second sum on the right side are equal. Therefore, we can write (7) as

$$\int \cdots \int K\begin{pmatrix} s_1, \ldots, s_n, x_1, \ldots, x_p \\ t_1, \ldots, t_n, x_1, \ldots, x_p \end{pmatrix}$$

$$= \sum_{h=1}^{n} (-1)^{h+1} K(s_1, t_h)$$

$$\times \int \cdots \int K\begin{pmatrix} s_2, \ldots, & s_n, x_1, \ldots, x_p \\ t_1, \ldots, t_{h-1}, t_{h+1}, \ldots, & t_n, x_1, \ldots, x_p \end{pmatrix} dx_1 \cdots dx_p$$

$$- p \int K(s_1, x)\left[\int \cdots \int K\begin{pmatrix} x, s_2, \ldots, s_n, x_1, \ldots, x_{p-1} \\ t_1, t_2, \ldots, t_n, x_1, \ldots, x_{p-1} \end{pmatrix} \times dx_1 \cdots dx_{p-1}\right] dx \qquad (8)$$

where we have omitted the subscript h from x. Substituting (8) in (7), we find that Fredholm minor satisfies the integral equation

$$D_n\left(\begin{matrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{matrix}\middle| \lambda\right) = \sum_{h=1}^{n} (-1)^{h+1} K(s_1, t_h)\, D_{n-1}\begin{pmatrix} s_2, \ldots & , s_n \\ t_1, \ldots, t_{h-1}, t_{h+1}, & t_n \end{pmatrix}$$
$$+ \lambda \int K(s_1, x)\, D_n\left(\begin{matrix} x, x_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle| \lambda\right) dx\,. \qquad (9)$$

Expansion by the elements of any other row leads to a similar identity, with x placed at the corresponding place. If we expand the determinant (6) with respect to the first column and proceed as above, we get the integral equation

$$D_n\left(\begin{matrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{matrix}\middle| \lambda\right) = \sum_{h=1}^{n} (-1)^{h+1} K(s_h, t_1)\, D_{n-1}\begin{pmatrix} s_1, \ldots, s_{h-1}, s_{h+1}, \ldots, & s_n \\ t_2, \ldots & , t_n \end{pmatrix}$$
$$+ \lambda \int K(x, t_1)\, D_n\begin{pmatrix} s_1, \ldots & , s_n \\ x, t_2, \ldots, & t_n \end{pmatrix} dx\,, \qquad (10)$$

and a similar result would follow if we were to expand by any other column. The formulas (9) and (10) will play the role of the Fredholm series of the previous section.

Note that the relations (9) and (10) hold for all values of λ. With the help of (9), we can find the solution of the homogeneous equation (1) for the special case when $\lambda = \lambda_0$ is an eigenvalue. To this end, let us suppose that $\lambda = \lambda_0$ is a zero of multiplicity m of the function $D(\lambda)$. Then, as remarked earlier, the minor D_m does not identically vanish and even the minors $D_1, D_2, \ldots, D_{m-1}$ may not identically vanish. Let D_r be the first minor in the sequence $D_1, D_2, \ldots, D_{m-1}$ that does not vanish identically. The number r lies between 1 and m and is the index of the eigenvalue λ_0 as defined in Section 2.3. Moreover, this means that $D_{r-1} = 0$. But then the integral equation (9) implies that

$$g_1(s) = D_r\left(\begin{matrix} s, s_2, \ldots, s_r \\ t_1, \ldots, \quad t_r \end{matrix}\middle| \lambda_0\right) \tag{11}$$

is a solution of the homogeneous equation (1). Substituting s at different points of the upper sequence in the minor D_r, we obtain r nontrivial solutions $g_i(s)$, $i = 1, \ldots, r$, of the homogeneous equation. These solutions are usually written as

$$\Phi_i(s) = \frac{D_r\left(\begin{matrix} s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_r \\ t_1, \ldots \qquad\qquad\qquad , t_r \end{matrix}\middle| \lambda_0\right)}{D_r\left(\begin{matrix} s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_r \\ t_1, \ldots \qquad\qquad\qquad , t_r \end{matrix}\middle| \lambda_0\right)}, \qquad i = 1, 2, \ldots, r. \tag{12}$$

Observe that we have already established that the denominator is not zero.

The solutions Φ_i as given by (12) are linearly independent for the following reason. In the determinant (6) above, if we put two of the arguments s_i equal, this amounts to putting two rows equal, and consequently the determinant vanishes. Thus, in (12), we see that $\Phi_k(s_i) = 0$ for $i \neq k$, whereas $\Phi_k(s_k) = 1$. Now, if there exists a relation $\sum_k C_k \Phi_k \equiv 0$, we may put $s = s_i$, and it follows that $C_i \equiv 0$; and this proves the linear independence of these solutions. This system of solutions Φ_i is called the fundamental system of the eigenfunctions of λ_0 and any linear combination of these functions gives a solution of (1).

Conversely, we can show that any solution of equation (1) must be

a linear combination of $\Phi_1(s), \Phi_2(s), \ldots, \Phi_v(s)$. We need to define a kernel $H(s,t;\lambda)$ which corresponds to the resolvent kernel $\Gamma(s,t;\lambda)$ of the previous section

$$H(s,t;\lambda) = D_{r+1}\left(\begin{matrix} s, s_1, \ldots, s_r \\ t, t_1, \ldots, t_r \end{matrix}\middle| \lambda_0\right) \Big/ D_r\left(\begin{matrix} s_1, \ldots, s_r \\ t_1, \ldots, t_r \end{matrix}\middle| \lambda_0\right). \qquad (13)$$

In (10), take n to be equal to r, and add extra arguments s and t to obtain

$$\begin{aligned} D_{r+1}\left(\begin{matrix} s, s_1, \ldots, s_r \\ t, t_1, \ldots, t_r \end{matrix}\middle| \lambda_0\right) &= K(s,t)\, D_r\left(\begin{matrix} s_1, \ldots, s_r \\ t_1, \ldots, t_r \end{matrix}\middle| \lambda_0\right) \\ &+ \sum_{h=1}^{r} (-1)^h K(s_h, y)\, D_r\left(\begin{matrix} s, s_1, \ldots, s_{h-1}, s_{h+1}, \ldots, s_r \\ t_1, t_2, \ldots \qquad\qquad , t_r \end{matrix}\middle| \lambda_0\right) \\ &+ \lambda_0 \int K(x,t)\, D_{r+1}\left(\begin{matrix} s, s_1, \ldots, s_r \\ x, t_1, \ldots, t_r \end{matrix}\middle| \lambda_0\right) dx\,. \end{aligned} \qquad (14)$$

In every minor D_r in the above equation, we transpose the variable s from the first place to the place between the variables s_{h-1} and s_{h+1} and divide both sides by the constant

$$D_r\left(\begin{matrix} s_1, \ldots, s_r \\ t_1, \ldots, t_r \end{matrix}\middle| \lambda_0\right) \neq 0 ,$$

to obtain

$$H(s,t;\lambda) - K(s,t) - \lambda_0 \int H(s,x;\lambda) K(x,t)\,dx$$
$$= -\sum_{h=1}^{r} K(s_h, t)\,\Phi_h(s) . \qquad (15)$$

If $g(s)$ is any solution to (1), we multiply (15) by $g(t)$ and integrate with respect to t,

$$\int g(t)\,H(s,t;\lambda)\,dt - \frac{g(s)}{\lambda_0} - \int g(x)\,\Gamma(s,x;\lambda)\,dx$$
$$= -\sum_{h=1}^{r} \frac{g(s_h)}{\lambda_0}\,\Phi_h(s) , \qquad (16)$$

where we have used (1) in all terms but the first; we have also taken $\lambda_0 \int K(s_h, t)\,g(t)\,dt = g(s_h)$. Cancelling the equal terms, we have

$$g(s) = \sum_{h=1}^{r} g(s_h)\,\Phi_h(s) . \qquad (17)$$

This proves our assertion. Thus we have established the following result.

Fredholm's Second Theorem. If λ_0 is a zero of multiplicity m of the function $D(\lambda)$, then the homogeneous equation

$$g(s) = \lambda_0 \int K(s,t)\,g(t)\,dt \qquad (18)$$

possesses at least one, and at most m, linearly independent solutions

$$g_i(s) = D_r\left(\begin{matrix} s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_r \\ t_1, \ldots \qquad\qquad , t_r \end{matrix}\middle| \lambda_0\right),$$
$$i = 1, \ldots, r ; \qquad 1 \leqslant r \leqslant m \qquad (19)$$

not identically zero. Any other solution of this equation is a linear combination of these solutions.

4.5. FREDHOLM'S THIRD THEOREM

In the analysis of Fredholm's first theorem, it has been shown that the inhomogeneous equation

$$g(s) = f(s) + \lambda \int K(s,t)\, g(t)\, dt \qquad (1)$$

possesses a unique solution provided $D(\lambda) \neq 0$. Fredholm's second theorem is concerned with the study of the homogeneous equation

$$g(s) = \lambda \int K(s,t)\, g(t)\, dt ,$$

when $D(\lambda) = 0$. In this section, we investigate the possibility of (1) having a solution when $D(\lambda) = 0$. The analysis of this section is not much different from the corresponding analysis for separable kernels as given in Section 2.3. In fact, the only difference is that we shall now give an explicit formula for the solution. Qualitatively, the discussion is the same.

Recall that the transpose (or adjoint) of equation (1) is (under the same assumption as in Section 2.3)

$$\psi(s) = f(s) + \lambda \int_a^b K(t,s)\, \psi(t)\, dt . \qquad (2)$$

It is clear that Fredholm's first series $D(\lambda)$ as given by (4.1.26) is the same for the transposed equation, while the second series is $D(t,s;\lambda)$ as obtained from (4.1.25) by interchanging the roles of s and t. This means that the kernels of equation (1) and its transpose (2) have the same eigenvalues. Furthermore, the resolvent kernel for (2) is

$$\Gamma(t,s;\lambda) = D(t,s;\lambda)/D(\lambda) , \qquad (3)$$

and therefore the solution of (2) is

$$\psi(s) = f(s) + \lambda \int [D(t,s;\lambda)/D(\lambda)] f(t)\, dt , \qquad (4)$$

provided λ is not an eigenvalue.

It is also clear that not only has the transposed kernel the same eigenvalues as the original kernel, but also the index r of each of the eigenvalues is equal. Moreover, corresponding to equation (4.4.12), the eigenfunctions of the transposed equation for an eigenvalue for λ_0 are given as

$$\Psi_i(t) = \frac{D_r\left(\begin{matrix} s_1, \dots & , s_r \\ t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_r \end{matrix} \middle| \lambda_0\right)}{D_r\left(\begin{matrix} s_1, \dots & , s_r \\ t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_r \end{matrix} \middle| \lambda_0\right)}, \tag{5}$$

where the values $(s_1, \dots, s_r)$ and $(t_1, \dots, t_r)$ are so chosen that the denominator does not vanish. Substituting r in different places in the lower sequence of this formula, we obtain a linearly independent system of r eigenfunctions. Also recall that each Φ_i is orthogonal to each Ψ_i with different eigenvalues.

If a solution $g(s)$ of (1) exists, then multiply (1) by each member $\Psi_k(s)$ of the above-mentioned system of functions and integrate to obtain

$$\int f(s)\Psi_k(s)\,ds = \begin{matrix} \int g(s)\Psi_k(s)\,ds - \lambda \int\int K(s,t)\,g(t)\,\Psi_k(s)\,ds\,dt \\ \int g(s)\,ds\,[\Psi_k(s) - \lambda \int K(t,s)\,\Psi_k(t)\,dt] = 0\,, \end{matrix} \tag{6}$$

where the term in the bracket vanishes because $\Psi_k(s)$ is an eigenfunction of the transposed equation. From (6), we see that a necessary condition for (1) to have a solution is that the inhomogeneous term $f(s)$ be orthogonal to each solution of the transposed homogeneous equation.

Conversely, we shall show that the condition (6) of orthogonality is sufficient for the existence of a solution. Indeed, we shall present an explicit solution in that case. With this purpose, we again appeal to the resolvent function $H(s, t; \lambda)$ as defined by (4.4.13) under the assumption that $D_r \neq 0$ and that r is the index of the eigenvalue λ_0.

Our contention is that if the orthogonality condition is satisfied, then the function

$$g_0(s) = f(s) + \lambda_0 \int H(s, t; \lambda) f(t)\, dt \tag{7}$$

is a solution. Indeed, substitute this value for $g(s)$ in (1), obtaining

$$f(s) + \lambda_0 \int H(s, t; \lambda) f(t)\, dt = f(s) + \lambda_0 \int K(s, t) \times \left[f(t) + \lambda_0 \int H(t, x; \lambda) f(x)\, dx \right] dt$$

or

$$\int f(t)\, dt \left[H(s, t; \lambda) - K(s, t) - \lambda_0 \int K(s, x)\, H(x, t; \lambda)\, dx \right] = 0 . \tag{8}$$

Now, just as we obtained equation (4.4.15), we can obtain its "transpose,"

$$H(s, t; \lambda) - K(s, t) - \lambda_0 \int K(s, x)\, H(x, t; \lambda)\, dx = -\sum_{h=1}^{r} K(s, t_h)\, \Psi_h(t) . \tag{9}$$

Substituting this in (8) and using the orthogonality condition, we have an identity, and thereby the assertion is proved.

The difference of any two solutions of (1) is a solution of the homogeneous equation. Hence, the most general solution of (1) is

$$g(s) = f(s) + \lambda_0 \int H(s, t; \lambda) f(t)\, dt + \sum_{h=1}^{r} C_h \Phi_h(s) . \tag{10}$$

The above analysis leads to the following theorem.

Fredholm's Third Theorem. For an inhomogeneous equation

$$g(s) = f(s) + \lambda_0 \int K(s,t)\, g(t)\, dt\,, \tag{11}$$

to possess a solution in the case $D(\lambda_0) = 0$, it is necessary and sufficient that the given function $f(s)$ be orthogonal to all the eigenfunctions $\Psi_i(s)$, $i = 1, 2, \ldots, v$, of the transposed homogeneous equation corresponding to the eigenvalue λ_0. The general solution has the form

$$g(s) = f(s) + \lambda_0 \int \left[D_{r+1}\left(\begin{matrix} s, s_1, s_2, \ldots, s_r \\ t, t_1, \ldots \quad , t_r \end{matrix} \middle| \lambda_0 \right) \right] \Big/ D_r\left(\begin{matrix} s_1, s_2, \ldots, s_r \\ t_1, \ldots \quad , t_r \end{matrix} \middle| \lambda_0 \right) \times f(t)\, dt + \sum_{h=1}^{r} C_h \Phi_h(s)\,. \tag{12}$$

POSSIBLE QUESTIONS
(SIX MARKS)

1.Solve $g(s) = 1 + \lambda \int_0^1 (1 - 3st) g(t) dt$
2.Obtain the reduction to a system of algebraic equation.
3.Show that the integral equation $g(s) = f(s) + \frac{1}{\pi} \int_0^{2\pi} \sin(s + t)\, g(t) dt$ passes number of solution for *f*(s) = s it passes many solution when *f*(s) = 1.
4. Find the resolvent kernel for the integral equation $g(s) = f(s) + \lambda \int_{-1}^1 (st + s^2 t^2) g(t) dt$
5. State and prove Fredholm theorem for First and Second Kind.
6. Solve $g(s) = f(s) + \lambda \int_0^s e^{(s-t)} g(t) dt$ and evaluate resolvent kernel.
7. State and prove Basic Fredholm theorem.
8. Show that the integral equation $g(s) = f(s) + 2 \int_0^1 (1 - 3t) g(t) dt$ will have a solution if f satisfies the condition $\int_0^1 (1 - s) f(s) ds = 0$
9.Solve the IE and find the Eigen value of $g(s) = f(s) + \lambda \int_0^1 (s + t) g(t) dt$.
10.State and prove Ferholm's First Theorem

(TEN MARKS)

11.Explain the Fredholm alternative approximate method.

UNIT – IV

Introduction for Application of Integral equation to ordinary differential equation-Initial value Problems-Boundary value problems-Singular integral equations-Abel integral equation.

APPLICATION OF INTEGRAL EQUATION TOORDINARY DIFFERENTIAL EQUATION

INITIAL VALUE PROBLEMS

There is a fundamental relationship between Volterra integral equations and ordinary differential equations with prescribed initial values. We begin our discussion by studying the simple initial value problem

$$y'' + A(s)y' + B(s)y = F(s)\,, \tag{1}$$

$$y(a) = q_0\,, \qquad y'(a) = q_1\,, \tag{2}$$

where a prime implies differentiation with respect to s, and the functions A, B, and F are defined and continuous in the closed interval $a \leqslant s \leqslant b$.

The result of integrating the differential equation (1) from a to s and using the initial values (2) is

$$y'(s) - q_1 = -A(s)y(s) - \int_a^s [B(s_1) - A'(s_1)]\,y(s_1)\,ds_1 + \int_a^s F(s_1)\,ds_1 + A(a)q_0\,.$$

Similarly, a second integration yields

$$y(s) - q_0 = -\int_a^s A(s_1)y(s_1)\,ds_1 - \int_a^s\int_a^{s_2} [B(s_1) - A'(s_1)]\,y(s_1)\,ds_1\,ds_2 + \int_a^s\int_a^{s_2} F(s_1)\,ds_1\,ds_2 + [A(a)q_0 + q_1](s-a)\,. \tag{3}$$

With the help of the identity (see Appendix, Section A.1)

$$\int_a^s\int_a^{s_2} F(s_1)\,ds_1\,ds_2 = \int_a^s (s-t)F(t)\,dt\,, \tag{4}$$

the two double integrals in (3) can be converted to single integrals. Hence, the relation (3) takes the form

$$y(s) = q_0 + [A(a)q_0 + q_1](s-a) + \int_a^s (s-t)F(t)\,dt - \int_a^s \{A(t) + (s-t)[B(t) - A'(t)]\}\,y(t)\,dt\,. \tag{5}$$

From relations (5)–(7), we have the Volterra integral equation of the second kind:

$$y(s) = f(s) + \int_a^s K(s,t)\, y(t)\, dt \, . \tag{8}$$

Conversely, any solution $g(s)$ of the integral equation (8) is, as can be verified by two differentiations, a solution of the initial value problem (1)–(2).

Note that the crucial step is the use of the identity (4). Since we have proved the corresponding identity for an arbitrary integer n in the Appendix, Section A.1, it follows that the above process of converting an initial value problem to a Volterra integral equation is applicable to a linear ordinary differential equation of order n when there are n prescribed initial conditions. An alternative approach is somewhat simpler for proving the above-mentioned equivalence for a general differential equation. Indeed, let us consider the linear differential equation of order n:

$$\frac{d^n y}{ds^n} + A_1(s)\frac{d^{n-1} y}{ds^{n-1}} + \cdots + A_{n-1}(s)\frac{dy}{ds} + A_n(s)\, y = F(s) \, , \tag{9}$$

with the initial conditions

$$y(a) = q_0 \, , \qquad y'(a) = q_1, \qquad \ldots, \qquad y^{(n-1)}(a) = q_{n-1} \, , \tag{10}$$

where the functions $A_1, A_2, \ldots, A_n$ and F are defined and continuous in $a \leqslant s \leqslant b$.

The reduction of the initial value problem (9)–(10) to the Volterra integral equation is accomplished by introducing an unknown function $g(s)$:

$$d^n y / ds^n = g(s) \, . \tag{11}$$

From (10) and (11), it follows that

$$\frac{d^{n-1}(y)}{ds^{n-1}} = \int_a^s g(t)\,dt + q_{n-1}\,,$$

$$\frac{d^{n-2}y}{ds^{n-2}} = \int_a^s (s-t)\,g(t)\,dt + (s-a)q_{n-1} + q_{n-2}\,, \qquad \text{(12) (continued)}$$

$$\frac{dy}{ds} = \int_a^s \frac{(s-t)^{n-2}}{(n-2)!}\,g(t)\,dt + \frac{(s-a)^{n-2}}{(n-2)!}\,q_{n-1} + \frac{(s-a)^{n-3}}{(n-3)!}\,q_{n-2} + \cdots + (s-a)q_2 + q_1\,,$$

$$y = \int_a^s \frac{(s-t)^{n-1}}{(n-1)!}\,g(t)\,dt + \frac{(s-a)^{n-1}}{(n-1)!}\,q_{n-1} + \frac{(s-a)^{n-2}}{(n-2)!}\,q_{n-2} + \cdots + (s-a)q_1 + q_0\,. \qquad (12)$$

Now, if we multiply relations (11) and (12) by 1, $A_1(s)$, $A_2(s)$, etc. and add, we find that the initial value problem defined by (9)–(10) is reduced to the Volterra integral equation of the second kind

$$g(s) = f(s) + \int_a^s K(s,t)\,g(t)\,dt\,, \qquad (13)$$

where

$$K(s,t) = -\sum_{k=1}^{n} A_k(s)\frac{(s-t)^{k-1}}{(k-1)!} \qquad (14)$$

and

$$f(s) = F(s) - q_{n-1} A_1(s) - [(s-a)q_{n-1} + q_{n-2}] A_2(s) - \cdots - \{[(s-a)^{n-1}/(n-1)!] q_{n-1} + \cdots + (s-a)q_1 + q_0\} \times A_n(s) . \tag{15}$$

Conversely, if we solve the integral equation (13) and substitute the value obtained for $g(s)$ in the last equation of the system (12), we derive the (unique) solution of the initial value problem (9)–(10).

5.2. BOUNDARY VALUE PROBLEMS

Just as initial value problems in ordinary differential equations lead to Volterra-type integral equations, boundary value problems in ordinary differential equations lead to Fredholm-type integral equations. Let us illustrate this equivalence by the problem

$$y''(s) + A(s)y' + B(s)y = F(s) , \tag{1}$$

$$y(a) = y_0 , \qquad y(b) = y_1 . \tag{2}$$

When we integrate equation (1) from a to s and use the boundary condition $y(a) = y_0$, we get

$$y'(s) = C + \int_a^s F(s)\, ds - A(s)y(s) + A(a)y_0$$

$$+ \int_a^s [A'(s) - B(s)]\, y(s)\, ds ,$$

where C is a constant of integration.

A second integration similarly yields

$$y(s) - y_0 = [C + A(a)\, y_0](s-a) + \int_a^s \int_a^{s_2} F(s_1)\, ds_1\, ds_2$$

$$- \int_a^s A(s_1)\, y(s_1)\, ds_1 + \int_a^s \int_a^{s_2} [A'(s_1) - B(s_1)]\, y(s_1)\, ds_1\, ds_2 . \quad (3)$$

Using the identity (5.1.4), the relation (3) becomes

$$y(s) - y_0 = [C + A(a)\, y_0](s-a) + \int_a^s (s-t)\, F(t)\, dt$$

$$- \int_a^s \{A(t) - (s-t)[A'(t) - B(t)]\}\, y(t)\, dt . \quad (4)$$

The constant C can be evaluated by setting $s = b$ in (4) and using the second boundary condition $y(b) = y_1$:

$$y_1 - y_0 = [C + A(a)\, y_0](b-a) + \int (b-t)\, F(t)\, dt$$

$$- \int \{A(t) - (b-t)[A'(t) - B(t)]\}\, y(t)\, dt ,$$

or

$$C + A(a)\, y_0 = [1/(b-a)]\, \{(y_1 - y_0) - \int (b-t)\, F(t)\, dt$$

$$+ \int \{A(t) - (b-t)[A'(t) - B(t)]\}\, y(t)\, dt\} . \quad (5)$$

From (4) and (5), we have the relation

$$\begin{aligned} y(s) = y_0 + \int_a^s (s-t)F(t)\,dt + [(s-a)/(b-a)] \\ \times [(y_1-y_0) - \int (b-t)F(t)\,dt] \\ - \int_a^s \{A(t) - (s-t)[A'(t) - B(t)]\}\,y(t)\,dt \\ + \int [(s-a)/(b-a)]\{A(t) - (b-t)[A'(t) - B(t)]\}\,y(t)\,dt\,. \qquad (6) \end{aligned}$$

Equation (6) can be written as the Fredholm integral equation

$$y(s) = f(s) + \int K(s,t)\,y(t)\,dt\,, \qquad (7)$$

provided we set

$$\begin{aligned} f(s) = y_0 + \int_a^s (s-t)F(t)\,dt \\ + [(s-a)/(b-a)][(y_1-y_0) - \int (b-t)F(t)\,dt] \qquad (8) \end{aligned}$$

and

$$K(s,t) = \begin{cases} [(s-a)/(b-a)]\{A(t) - (b-t)[A'(t) - B(t)]\}\,, & s < t\,, \\ A(t)\{[(s-a)/(b-a)] - 1\} - [A'(t) - B(t)] \times [(t-a)(b-s)/(b-a)]\,, & s > t\,. \end{cases} \qquad (9)$$

For the special case when A and B are constants, $a = 0$, $b = 1$, and $y(0) = y(1) = 0$, the above kernel simplifies to

$$K(s,t) = \begin{cases} Bs(1-t) + As\,, & s < t\,, \\ Bt(1-s) + As - A\,, & s > t\,. \end{cases} \tag{10}$$

Note that the kernel (10) is asymmetric and discontinuous at $t = s$, unless $A \equiv 0$. We shall elaborate on this point in Section 5.4.

Example 1. Reduce the initial value problem

$$y''(s) + \lambda y(s) = F(s)\,, \tag{1}$$

$$y(0) = 1\,, \qquad y'(0) = 0\,, \tag{2}$$

to a Volterra integral equation.

Comparing (1) and (2) with the notation of Section 5.1, we have $A(s) = 0$, $B(s) = \lambda$. Therefore, the relations (5.1.6)–(5.1.8) become

$$K(s,t) = \lambda(t-s)\,,$$

$$f(s) = 1 + \int_0^s (s-t)\,F(t)\,dt\,, \tag{3}$$

and

$$y(s) = 1 + \int_0^s (s-t)\,F(t)\,dt + \lambda \int_0^s (t-s)\,y(t)\,dt\,.$$

Example 2. Reduce the boundary value problem

$$y''(s) + \lambda P(s)\,y = Q(s)\,, \tag{4}$$

$$y(a) = 0\,, \qquad y(b) = 0 \tag{5}$$

to a Fredholm integral equation.

Comparing (4) and (5) with the notation of Section 5.2, we have $A = 0$, $B = \lambda P(s)$, $F(s) = Q(s)$, $y_0 = 0$, $y_1 = 0$. Substitution of these values in the relations (5.2.8) and (5.2.9) yields

$$f(s) = \int_a^s (s-t)\,Q(t)\,dt - [(s-a)/(b-a)] \int (b-t)\,Q(t)\,dt \qquad (6)$$

and

$$K(s,t) = \begin{cases} \lambda P(t)\,[(s-a)(b-t)/(b-a)]\,, & s < t\,, \\ \lambda P(t)\,[(t-a)(b-s)/(b-a)]\,, & s > t\,, \end{cases} \qquad (7)$$

which, when put in (5.2.7), gives the required integral equation. Note that the kernel is continuous at $s = t$.

As a special case of the above example, let us take the boundary value problem

$$y'' + \lambda y = 0 , \tag{8}$$

$$y(0) = 0 , \qquad y(\ell) = 0 . \tag{9}$$

Then, the relations (6) and (7) take the simple forms: $f(s) = 0$, and

$$K(s,t) = \begin{cases} (\lambda s/\ell)(\ell - t) , & s < t , \\ (\lambda t/\ell)(\ell - s) , & s > t . \end{cases} \tag{10}$$

Note that, although the kernels (7) and (10) are continuous at $s = t$, their derivatives are not continuous. For example, the derivative of the kernel (10) is

$$\partial K(s,t)/\partial s = \begin{cases} \lambda[1 - (t/\ell)] , & s < t , \\ -\lambda t/\ell , & s > t . \end{cases}$$

The value of the jump of this derivative at $s = t$ is

$$\left[\frac{dK(s,t)}{ds}\right]_{t+0} - \left[\frac{dK(s,t)}{ds}\right]_{t-0} = -\lambda .$$

Similarly, the value of the jump of the derivative of the kernel (7) at $s = t$ is

$$\left[\frac{dK(s,t)}{ds}\right]_{t+0} - \left[\frac{dK(s,t)}{ds}\right]_{t-0} = -\lambda P(t) .$$

An integral equation is called singular if either the range of integration is infinite or the kernel has singularities within the range of integration. Such equations occur rather frequently in mathematical physics and possess very unusual properties. For instance, one of the simplest singular integral equations is the Abel integral equation

$$f(s) = \int_0^s [g(t)/(s-t)^\alpha]\, dt\,, \qquad 0 < \alpha < 1\,, \tag{1}$$

which arises in the following problem in mechanics. A material point moving under the influence of gravity along a smooth curve in a vertical plane takes the time $f(s)$ to move from the vertical height s to a fixed point 0 on the curve. The problem is to find the equation of that curve. Equation (1) with $\alpha = 1/2$ is the integral-equation formulation of this problem.

The integral equation (1) is readily solved by multiplying both sides by the factor $ds/(u-s)^{1-\alpha}$ and integrating it with respect to s from 0 to u:

$$\int_0^u \frac{f(s)\, ds}{(u-s)^{1-\alpha}} = \int_0^u \frac{ds}{(u-s)^{1-\alpha}} \int_0^s \frac{g(t)\, dt}{(s-t)^\alpha}\,. \tag{2}$$

The double integration on the right side of the above equation is so written that first it is to be integrated in the t direction from 0 to s and then the resulting single integral is to be integrated in the s direction from 0 to u. The region of integration therefore is the triangle lying below the diagonal $s = t$. We change the order of integration so that we fir integrate from $s = t$ to $s = u$ and afterwards in the t direction from $t =$ to $t = u$. Equation (2) then becomes

$$\int_0^u \frac{f(s)\,ds}{(u-s)^{1-\alpha}} = \int_0^u g(t)\,dt \int_t^u \frac{ds}{(u-s)^{1-\alpha}(s-t)^\alpha}\,. \qquad ($$

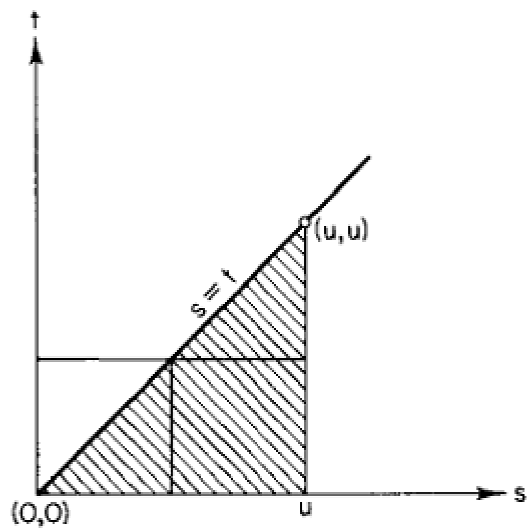


Figure 8.1

To evaluate the integral

$$\int_t^u \frac{ds}{(u-s)^{1-\alpha}(s-t)^\alpha}\,,$$

one sets $y=(u-s)/(u-t)$, and obtains

$$\int_t^u (u-s)^{\alpha-1}(s-t)^{-\alpha}\,ds = \int_0^1 y^{\alpha-1}(1-y)^{-\alpha}\,dy = \pi/\sin\alpha\pi\ ,$$

where we have used the value of the Eulerian beta function $B(\alpha,1-\alpha) = \pi/\sin\alpha\pi$. Substituting this result in (3), we have

$$\frac{\sin\alpha\pi}{\pi}\int_0^u \frac{f(s)\,ds}{(u-s)^{1-\alpha}} = \int_0^u g(t)\,dt\ ,$$

which, when differentiated with respect to u, and then changing u to t, gives the required solution:

$$g(t) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dt}\left[\int_0^t f(s)(t-s)^{\alpha-1}\,ds\right]. \tag{4}$$

The integral equation (1) is a special case of the singular integral equation [18]

$$f(s) = \int_a^s \frac{g(t)\,dt}{[h(s)-h(t)]^{\alpha}}\ ,\quad 0<\alpha<1\ , \tag{5}$$

where $h(t)$ is a strictly monotonically increasing and differentiable function in (a,b), and $h'(t)\neq 0$ in this interval. To solve this, we consider the integral

$$\int_a^s \frac{h'(u)f(u)\,du}{[h(s)-h(u)]^{1-\alpha}}\ ,$$

and substitute for $f(u)$ from (5). This gives

$$\int_a^s \int_a^u \frac{g(t)\, h'(u)\, dt\, du}{[h(u)-h(t)]^{\alpha}\,[h(s)-h(u)]^{1-\alpha}},$$

which, by change of the order of integration, becomes

$$\int_a^s g(t)\, dt \int_t^s \frac{h'(u)\, du}{[h(u)-h(t)]^{\alpha}\,[h(s)-h(u)]^{1-\alpha}}.$$

The inner integral is easily proved to be equal to the beta function $B(\alpha, 1-\alpha)$. We have thus proved that

$$\int_a^s \frac{h'(u) f(u)\, du}{[h(s)-h(u)]^{1-\alpha}} = \frac{\pi}{\sin \alpha\pi} \int_a^s g(t)\, dt, \qquad (6)$$

and by differentiating both sides of (6), we obtain the solution

$$g(t) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dt} \int_a^t \frac{h'(u) f(u)\, du}{[h(t)-h(u)]^{1-\alpha}}. \qquad (7)$$

Similarly, the integral equation

$$f(s) = \int_s^b \frac{g(t)\, dt}{[h(t)-h(s)]^{\alpha}}, \qquad 0 < \alpha < 1, \qquad (8)$$

and $a < s < b$, with $h(t)$ a monotonically increasing function, has the solution

$$g(t) = -\frac{\sin\alpha\pi}{\pi}\frac{d}{dt}\int_t^b \frac{h'(u)f(u)\,du}{[h(u)-h(t)]^{1-\alpha}}\,. \tag{9}$$

We close this section with the remark that a Fredholm integral equation with a kernel of the type

$$K(s,t) = H(s,t)/|t-s|^{\alpha}\,, \qquad 0 < \alpha < 1\,, \tag{10}$$

where $H(s,t)$ is a bounded function, can be transformed to a kernel which is bounded. It is done by the method of iterated kernels. Indeed, it can be shown [11, 15, 20] that, if the singular kernel has the form as given by the relation (10), then there always exists a positive integer p_0, dependent on α, such that, for $p > p_0$, the iterated kernel $K_p(s,t)$ is bounded. For this reason, the kernel (10) is called weakly singular.

Note that, for this hypothesis, the condition $\alpha < 1$ is essential. For the important case $\alpha = 1$, the integral equation differs radically from the equations considered in this section. Moreover, we need the notion of Cauchy principal value for this case. But, before considering the case $\alpha = 1$, let us give some examples for the case $\alpha < 1$.

Example 1. Solve the integral equation

$$s = \int_0^s \frac{g(t)\,dt}{(s-t)^{1/2}} . \tag{1}$$

Comparing this with integral equation (8.1.1), we find that $f(s) = s$, $\alpha = 1/2$. Substituting these values in (8.1.4), there results the solution:

$$\begin{aligned} g(t) &= \frac{1}{\pi}\frac{d}{dt}\left[\int_0^t \frac{s}{(t-s)^{1/2}}\,ds\right] \\ &= \frac{1}{\pi}\frac{d}{dt}\left[-\frac{2}{3}(s+2t)(t-s)^{1/2}\right]_0^t \\ &= \frac{1}{\pi}\frac{d}{dt}\left[\frac{4}{3}t^{3/2}\right] = \frac{2t^{1/2}}{\pi} . \end{aligned} \tag{2}$$

Example 2. Solve the integral equation

$$f(s) = \int_a^s \frac{g(t)\,dt}{(\cos t - \cos s)^{1/2}} , \qquad 0 \leqslant a < s < b \leqslant \pi . \tag{3}$$

Comparing (8.1.5) and (3), we see that $\alpha = 1/2$, and $h(t) = 1 - \cos t$, a strictly monotonically increasing function in $(0, \pi)$. Substituting this value for $h(u)$ in (8.1.7), we have the required solution

$$g(t) = \frac{1}{\pi}\frac{d}{dt}\left[\int_a^t \frac{(\sin u) f(u)\,du}{(\cos u - \cos t)^{1/2}}\right], \qquad a < t < b . \tag{4}$$

Similarly, the integral equation

$$f(s) = \int_s^b \frac{g(t)\,dt}{(\cos s - \cos t)^{1/2}}, \qquad 0 \leqslant a < s < b \leqslant \pi, \tag{5}$$

has the solution

$$g(t) = -\frac{1}{\pi}\frac{d}{dt}\left[\int_t^b \frac{(\sin u) f(u)\,du}{(\cos t - \cos u)^{1/2}}\right], \qquad a < t < b. \tag{6}$$

Example 3. Solve the integral equations

(a) $$f(s) = \int_a^s \frac{g(t)\,dt}{(s^2 - t^2)^\alpha}, \qquad 0 < \alpha < 1; \quad a < s < b, \tag{7}$$

and

(b) $$f(s) = \int_s^b \frac{g(t)\,dt}{(t^2 - s^2)^\alpha}, \qquad 0 < \alpha < 1; \quad a < s < b. \tag{8}$$

From (8.1.5) and (7), we find that $h(t) = t^2$, which is a strictly monotonic function. The solution, therefore, follows from (8.1.7):

$$g(t) = \frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_a^t \frac{u f(u)\,du}{(t^2 - u^2)^{1-\alpha}}, \qquad a < t < b. \tag{9}$$

Similarly, the solution of the integral equation (8) is

$$g(t) = -\frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_t^b \frac{u f(u)\,du}{(u^2 - t^2)^{1-\alpha}}, \qquad a < t < b. \tag{10}$$

The results (9) and (10) remain valid when a tends to 0 and b tends to $+\infty$. Hence, the solution of the integral equation

$$f(s) = \int_0^s \frac{g(t)\,dt}{(s^2 - t^2)^\alpha}, \qquad 0 < \alpha < 1, \tag{11}$$

is

$$g(t) = \frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_0^t \frac{uf(u)\,du}{(t^2 - u^2)^{1-\alpha}}. \tag{12}$$

Similarly, the solution of the integral equation

$$f(s) = \int_s^\infty \frac{g(t)\,dt}{(t^2 - s^2)^\alpha}, \qquad 0 < \alpha < 1, \tag{13}$$

is

$$g(t) = -\frac{2\sin\alpha\pi}{\pi}\frac{d}{dt}\int_t^\infty \frac{uf(u)\,du}{(u^2 - t^2)^{1-\alpha}}. \tag{14}$$

Possible Questions

PART-B (Six Marks)

1) Obtain the relationship between Voltera integral equation and initial value problem.

2) Reduce the BVP to a Fredholm integral equation $y"(s) + \lambda y(s) = 0$ with y(0) = 0 , y(l) = 0.

3) Solve the homogeneous fredholm integral equation $g(s) = \lambda \int_0^1 e^{st} g(t)dt$

4) Obtain the Abel integral equation.

5) Solve y"+sy =1 , y(0)= y(1) =0.

6) Solve the integral equation $s = \int_0^s \frac{g(t)}{(s-t)^{1/2}} dt$

7) Reduce the IVB $y"(s) + \lambda y(s) = F(s)$, $y(0) = 1, y'(0) = 0$ to a voltera integral equation.

8) Solve the integral equation $f(s) = \int_a^s \frac{g(t)}{(s^2-t^2)\alpha'} dt$, $0 < \alpha < 1, a < s < b$ and

$f(s) = \int_s^b \frac{g(t)}{(t^2-s^2)\alpha} dt$, $0 < \alpha < 1, a < s < b.$

9) Reduce the BVP to a Fredholm integral equation $y"(s) + \lambda p(s)y(s) = g(s)$ with y(a) = 0 and y(b) = 0.

10) Solve the BVP y"- y = F(s), y(0) = y(1) = 0 .

PART-C (Ten Marks)

1) Reduce the boundary value problem$y''(s) + \lambda P(s)y = Q(s)$, y(a)=0, y(b)=0 to a Fredholm integral equation.

UNIT – V

Calculus of Variations and its properties-Euler's equation- Functionals of the integral forms-Functional dependent on higher order derivatives-Functional dependent on the functions of several independent variables-Variational problems in parametric form.

The method of variations in problems with fixed boundaries

1. Variation and Its Properties

Methods of solving variational problems, i. e. problems involving the investigation of functionals for maxima and minima, are extremely similar to the methods of investigating functions for maxima and minima. It is therefore worth while recalling briefly the theory of maxima and minima of functions and in parallel introduce analogous concepts and prove similar theorems for functionals.

1. A variable z is a *function* of a variable quantity x [written $z=f(x)$] if to every value of x over a certain range of x there corresponds a value of z; i.e., we have a correspondence: to the number x there corresponds a number z.

Functions of several variables are defined in similar fashion.

1. A variable quantity v is a *functional* dependent on a function $y(x)$ [written $v=v[y(x)]$] if to each function $y(x)$ of a certain class of functions $y(x)$ there corresponds a value v, i.e. we have a correspondence: to the function $y(x)$ there corresponds a number v.

Functionals dependent on several functions, and functionals dependent on functions of several independent variables are similarly defined.

2. The *increment* Δx of the argument x of a function $f(x)$ is the difference between two values of the variable $\Delta x = x - x_1$. If x is the independent variable, then the differential x coincides with the increment, $dx = \Delta x$.

2. The *increment*, or *variation*, δy of the argument $y(x)$ of a functional $v[y(x)]$ is the difference between two functions $\delta y = y(x) - y_1(x)$. Here it is assumed that $y(x)$ varies in arbitrary fashion in some class of functions.

3. A function $f(x)$ is called *continuous* if to a small change of x there corresponds a small change in the function $f(x)$.

3. A functional $v[y(x)]$ is called *continuous* if to a small change of $y(x)$ there corresponds a small change in the functional $v[y(x)]$.

The latter definition requires some explanation, for the question immediately arises as to what changes of the function $y(x)$, which is the argument of the functional, are called small or, what is the same, what curves $y=y(x)$ and $y=y_1(x)$ are considered close or only slightly different.

It may be taken that the functions $y(x)$ and $y_1(x)$ are close if the absolute value of their difference $y(x)-y_1(x)$ is small for all values of x for which the functions $y(x)$ and $y_1(x)$ are prescribed; that is, we can consider as close such curves as have close-lying ordinates.

However, for such a definition of proximity of curves, the functionals of the kind

$$v[y(x)]=\int_{x_0}^{x_1} F(x,\ y,\ y')\,dx$$

that frequently occur in applications will be continuous only in exceptional cases due to the presence of the argument y' in the integrand function. For this reason, in many cases it is more natural to consider as close only those curves which have close-lying ordinates and are close as regards the directions of tangents at the respective points; that is, to require that, for close curves, not only should the absolute value of the difference $y(x)-y_1(x)$ be small, but also the absolute value of the difference $y'(x)-y_1'(x)$.

It is sometimes necessary to consider as close only those functions for which the absolute values of each of the following differences are small:

$$y(x)-y_1(x),\quad y'(x)-y_1'(x),$$
$$y''(x)-y_1''(x),\ \ldots,\ y^{(k)}(x)-y_1^{(k)}(x).$$

This compels us to introduce the following definitions of proximity of the curves $y=y(x)$ and $y=y_1(x)$.

The curves $y=y(x)$ and $y=y_1(x)$ are close in the sense of zero-order proximity if the absolute value of the difference $y(x)-y_1(x)$ is small.

The curves $y=y(x)$ and $y=y_1(x)$ are close in the sense of first-order proximity if the absolute values of the differences $y(x)-y_1(x)$ and $y'(x)-y_1'(x)$ are small.

The curves

$$y=y(x) \text{ and } y=y_1(x)$$

are close in the sense of kth order proximity if the absolute values of the differences

$$y(x) - y_1(x),$$
$$y'(x) - y_1'(x),$$
$$\cdots\cdots$$
$$y^{(k)}(x) - y_1^{(k)}(x)$$

are small.

Fig. 6.1 exhibits curves close in the sense of zero-order proximity but not close in the sense of first-order proximity, since the ordinates are close but the directions of the tangents are not. In Fig. 6.2. are depicted curves close in the sense of first-order proximity.

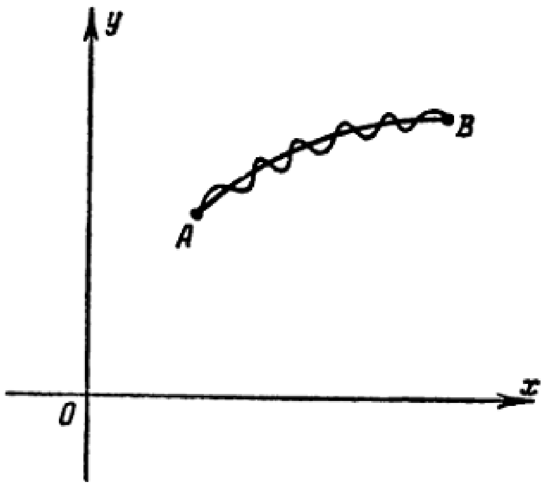


Fig. 6-1

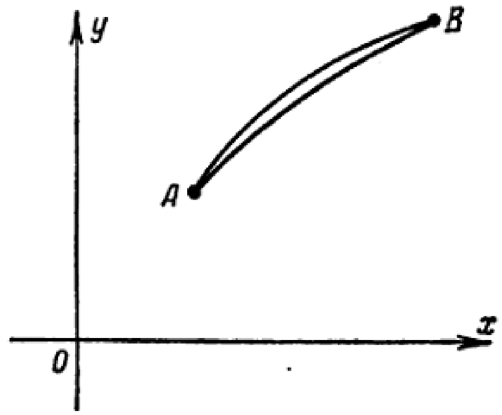


Fig. 6-2

From these definitions it follows that if the curves are close in the sense of kth order proximity, then they are definitely close in the sense of any lesser order of proximity.

We can now refine the concept of continuity of a functional.

3′. A function $f(x)$ is *continuous* at $x=x_0$ if for any positive ε there is a $\delta>0$ such that $|f(x)-f(x_0)|<\varepsilon$ for $|x-x_0|<\delta$.

It is assumed here that x takes on values at which the function $f(x)$ is defined.

3′. The functional $v[y(x)]$ is *continuous* at $y=y_0(x)$ in the sense of kth order proximity if for any positive ε there is a $\delta>0$ such that $|v[y(x)]-v[y_0(x)]|<\varepsilon$ for

$$|y(x)-y_0(x)|<\delta,$$
$$|y'(x)-y_0'(x)|<\delta,$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$|y^{(k)}(x)-y_0^{(k)}(x)|<\delta.$$

It is assumed here that the function $y(x)$ is taken from a class of functions on which the functional $v[y(x)]$ is defined.

One might also define the notion of distance $\rho(y_1, y_2)$ between the curves $y=y_1(x)$ and $y=y_2(x)$ $(x_0 \leqslant x \leqslant x_1)$ and then close-lying curves would be curves with small separation.

If we assume that

$$\rho(y_1, y_2) = \max_{x_0 \leqslant x \leqslant x_1} |y_1(x) - y_2(x)|,$$

that is if we introduce the space metric C_0 (see pages 54-55), we have the concept of zero-order proximity. If it is taken that

$$\rho(y_1, y_2) = \sum_{p=1}^{k} \max_{x_0 \leqslant x \leqslant x_1} |y_1^{(p)}(x) - y_2^{(p)}(x)|$$

(it is assumed that y_1 and y_2 have continuous derivatives up to order k inclusive), then the proximity of the curves is understood in the sense of kth order proximity.

4. A *linear function* is a function $l(x)$ that satisfies the following conditions:

$$l(cx) = cl(x),$$

where c is an arbitrary constant, and

$$l(x_1 + x_2) = l(x_1) + l(x_2).$$

A linear function of one variable is of the form

$$l(x) = kx,$$

4. A *linear functional* is a functional $L[y(x)]$ that satisfies the following conditions

$$L[cy(x)] = cL[y(x)],$$

where c is an arbitrary constant and

$$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)].$$

The following is an instance of a linear functional:

where k is constant.

$$L[y(x)] = \int_{x_0}^{x_1} (p(x)y + q(x)y')\,dx.$$

5. If the increment of a function

$$\Delta f = f(x+\Delta x) - f(x)$$

may be represented in the form

$$\Delta f = A(x)\Delta x + \beta(x, \Delta x)\cdot\Delta x,$$

where $A(x)$ does not depend on Δx, and $\beta(x, \Delta x) \to 0$ as $\Delta x \to 0$, then the function is called *differentiable*, while the part of the increment that is linear with respect to Δx — $A(x)\Delta x$ — is called the *differential* of the function and is denoted by df. Dividing

5. If the increment of a functional

$$\Delta v = v[y(x)+\delta y] - v[y(x)]$$

may be represented in the form

$$\Delta v = L[y(x), \delta y] + \beta(y(x), \delta y)\max|\delta y|,$$

where $L[y(x), \delta y]$ is a functional linear with respect to δy, and $\max|\delta y|$ is the maximum value of $|\delta y|$ and $\beta(y(x), \delta y) \to 0$ as $\max|\delta y| \to 0$, then the part of the increment of the functional that is linear with respect to

since by virtue of linearity

$$L(y, \alpha\delta y) = \alpha L(y, \delta y)$$

and

$$\lim_{\alpha\to 0} \frac{\beta[y(x), \alpha\delta y]\,|\alpha|\max|\delta y|}{\alpha} = \lim_{\alpha\to 0} \beta[y(x), \alpha\delta y]\max|\delta y| = 0$$

because $\beta[y(x), \alpha\delta y] \to 0$ as $\alpha \to 0$. Thus, if there exists a variation in the sense of the principal linear part of the increment of the functional, then there also exists a variation in the sense of the derivative with respect to the parameter for the initial value of the parameter, and both of these definitions are equivalent.

The latter definition of a variation is somewhat broader than the former, since there are instances of functionals, from the increments of which it is impossible to isolate the principal linear part, but the variation exists in the meaning of the second definition.

6. The *differential* of a function $f(x)$ is equal to

$$\frac{\partial}{\partial\alpha} f(x+\alpha\,\Delta x)\big|_{\alpha=0}.$$

6. The *variation* of a functional $v[y(x)]$ is equal to

$$\frac{\partial}{\partial\alpha} v[y(x)+\alpha\,\delta y]\big|_{\alpha=0}.$$

Definition. *A functional $v[y(x)]$ reaches a maximum on a curve $y=y_0(x)$ if the values of the functional $v[y(x)]$ on any curve close to $y=y_0(x)$ do not exceed $v[y_0(x)]$*; that is $\Delta v = v[y(x)] - v[y_0(x)] \leqslant 0$.

If $\Delta v \leqslant 0$, and $\Delta v = 0$ only for $y(x)=y_0(x)$, then it is said that a *strict maximum* is reached on the curve $y=y_0(x)$. The curve $y=y_0(x)$, on which a *minimum* is achieved, is defined in similar fashion. In this case, $\Delta v \geqslant 0$ for all curves close to the curve $y=y_0(x)$.

7. ***Theorem.*** *If a differentiable function $f(x)$ achieves a maximum or a minimum at an interior point $x=x_0$ of the domain of definition of the function, then at this point*

$$df=0.$$

7. ***Theorem.*** *If a functional $v[y(x)]$ having a variation achieves a maximum or a minimum at $y=y_0(x)$, where $y(x)$ is an interior point of the domain of definition of the functional, then at $y=y_0(x)$,*

$$\delta v=0.$$

Proof of the theorem for functionals. For fixed $y_0(x)$ and δy $v[y_0(x)+\alpha\,\delta y]=\varphi(\alpha)$ is a function of α, which for $\alpha=0$, by hypothesis, reaches a maximum or a minimum; hence, the derivative

$$\varphi'(0)=0^{*}, \quad \text{and} \quad \frac{\partial}{\partial\alpha} v[y_0(x)+\alpha\,\delta y]\Big|_{\alpha=0}=0,$$

* α can take on either positive or negative values in the neighbourhood of the point $\alpha=0$, since $y_0(x)$ is an interior point of the domain of definition of the functional.

since by virtue of linearity

$$L(y, \alpha \delta y) = \alpha L(y, \delta y)$$

and

$$\lim_{\alpha \to 0} \frac{\beta [y(x), \alpha \delta y] |\alpha| \max |\delta y|}{\alpha} = \lim_{\alpha \to 0} \beta [y(x), \alpha \delta y] \max |\delta y| = 0$$

because $\beta [y(x), \alpha \delta y] \longrightarrow 0$ as $\alpha \longrightarrow 0$. Thus, if there exists a variation in the sense of the principal linear part of the increment of the functional, then there also exists a variation in the sense of the derivative with respect to the parameter for the initial value of the parameter, and both of these definitions are equivalent.

The latter definition of a variation is somewhat broader than the former, since there are instances of functionals, from the increments of which it is impossible to isolate the principal linear part, but the variation exists in the meaning of the second definition.

6. The *differential* of a function $f(x)$ is equal to

$$\frac{\partial}{\partial \alpha} f(x + \alpha \Delta x)|_{\alpha = 0}.$$

6. The *variation* of a functional $v[y(x)]$ is equal to

$$\frac{\partial}{\partial \alpha} v[y(x) + \alpha \delta y]|_{\alpha = 0}.$$

Definition. *A functional $v[y(x)]$ reaches a maximum on a curve $y = y_0(x)$ if the values of the functional $v[y(x)]$ on any curve close to $y = y_0(x)$ do not exceed $v[y_0(x)]$*; that is $\Delta v = v[y(x)] - v[y_0(x)] \leqslant 0$.

If $\Delta v \leqslant 0$, and $\Delta v = 0$ only for $y(x) = y_0(x)$, then it is said that a *strict maximum* is reached on the curve $y = y_0(x)$. The curve $y = y_0(x)$, on which a *minimum* is achieved, is defined in similar fashion. In this case, $\Delta v \geqslant 0$ for all curves close to the curve $y = y_0(x)$.

7. ***Theorem.*** *If a differentiable function $f(x)$ achieves a maximum or a minimum at an interior point $x = x_0$ of the domain of definition of the function, then at this point*

$$df = 0.$$

7. ***Theorem.*** *If a functional $v[y(x)]$ having a variation achieves a maximum or a minimum at $y = y_0(x)$, where $y(x)$ is an interior point of the domain of definition of the functional, then at $y = y_0(x)$,*

$$\delta v = 0.$$

Proof of the theorem for functionals. For fixed $y_0(x)$ and δy $v[y_0(x) + \alpha \delta y] = \varphi(\alpha)$ is a function of α, which for $\alpha = 0$, by hypothesis, reaches a maximum or a minimum; hence, the derivative

$$\varphi'(0) = 0^*, \quad \text{and} \quad \frac{\partial}{\partial \alpha} v[y_0(x) + \alpha \delta y]\Big|_{\alpha = 0} = 0,$$

i. e. $\delta v = 0$. Thus, the variation of a functional is zero on curves on which an extremum of the functional is achieved.

The concept of the *extremum* of a functional must be made more specific. When speaking of a maximum or a minimum, more precisely, of a relative maximum or minimum, we had in view the largest or smallest value of the functional only relative to values of the functional on close-lying curves. But, as has already been pointed out, the proximity of curves may be understood in different ways, and for this reason it is necessary, in the definition of a maximum or minimum, to indicate the order of proximity.

If a functional $v[y(x)]$ reaches a maximum or a minimum on a curve $y = y_0(x)$ with respect to all curves for which the absolute value of the difference $y(x) - y_0(x)$ is small, i.e. with respect to curves close to $y = y_0(x)$ in the sense of zero-order proximity, then the maximum of minimum is called *strong*.

However, if a functional $v[y(x)]$ attains, on the curve $y = y_0(x)$, a maximum or minimum only with respect to curves $y = y(x)$ close to $y = y_0(x)$ in the sense of first-order proximity, i.e. with respect to curves close to $y = y_0(x)$ not only as regards ordinates but also as regards the tangent directions, then the maximum or the minimum is termed *weak*.

Quite obviously, if a strong maximum (or minimum) is attained on a curve $y = y_0(x)$, then most definitely a weak one has been attained as well, since if the curve is close to $y = y_0(x)$ in the sense of first-order proximity, then it is also close in the sense of zero-order proximity. It is possible, however, that on the curve $y = y_0(x)$ a weak maximum (minimum) has been attained, yet a strong maximum (minimum) is not achieved; in other words, among the curves $y = y(x)$ close to $y = y_1(x)$ both as to ordinates and as to the tangent directions, there may not be any curves for which $v[y(x)] > v[y_0(x)]$ (in the case of a minimum $v[y(x)] < v[y_0(x)]$), and among the curves $y = y(x)$ that are close as regards ordinates but not close as regards the tangent directions there may be those for which $v[y(x)] > v[y_0(x)]$ (in the case of a minimum $v[y(x)] < v[y_0(x)]$). The difference between a strong and weak extremum will not have essential meaning in the derivation of the basic necessary condition for an extremum, but it will be extremely essential in Chapter 8 in studying the sufficient conditions for an extremum.

Note also that if on a curve $y=y_0(x)$ an extremum is attained, then not only $\frac{\partial}{\partial\alpha} v[y_0(x)+\alpha\delta y]\Big|_{\alpha=0}=0$, but also $\frac{\partial}{\partial\alpha} v[y(x,\alpha)]\Big|_{\alpha=0}=0$, where $y(x, \alpha)$ is any family of admissible curves, and for $\alpha=0$ and $\alpha=1$ the function $y(x, \alpha)$ must, respectively, transform to $y_0(x)$ and $y_0(x)+\delta y$. Indeed, $v[y(x, \alpha)]$ is a function of α since

specifying α determines a curve of the family $y=y(x, \alpha)$, and this means that it also defines the value of the functional $v[y(x, \alpha)]$.

It is assumed that this function achieves an extremum at $\alpha=0$, hence, the derivative of this function vanishes at $\alpha=0$.*

Thus, $\frac{\partial}{\partial\alpha} v[y(x, \alpha)]\Big|_{\alpha=0}=0$, however, this derivative generally speaking will no longer coincide with the variation of the function but will, as has been shown above, vanish simultaneously with δv on curves that achieve an extremum of the functional.

All definitions of this section and the fundamental theorem (page 302) can be extended almost without any change to functionals dependent on several unknown functions:

$$v[y_1(x),\ y_2(x),\ \ldots,\ y_n(x)]$$

or dependent on one or several functions of many variables:

$$v[z(x_1,\ x_2,\ \ldots,\ x_n)],$$

$$v[z_1(x_1,\ x_2,\ \ldots,\ x_n),\ z_2(x_1,\ x_2,\ \ldots,\ x_n),\ \ldots,\ z_m(x_1,\ x_2,\ \ldots,\ x_n)].$$

For example, the variation δv of the functional $v[z(x, y)]$ may be defined either as the principal part of the increment

$$\Delta v = v[z(x,\ y)+\delta z] - v[z(x,\ y)],$$

linear in δz, or as a derivative with respect to the parameters for the initial value of the parameter

$$\frac{\partial}{\partial \alpha} v [z (x, y) + \alpha \delta z] \Big|_{\alpha = 0},$$

and if for $z = z(x, y)$ the functional v attains an extremum, then for $z = z(x, y)$ the variation $\delta v = 0$, since $v[z(x, y) + \alpha \delta z]$ is a function of α, which for $\alpha = 0$, by hypothesis, attains an extremum and, hence, the derivative of this function with respect to α for $\alpha = 0$ vanishes, $\frac{\partial}{\partial \alpha} v [z (x, y) + \alpha \delta z] \Big|_{\alpha=0} = 0$ or $\delta v = 0$.

2. Euler's Equation

Let us investigate the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \tag{6.1}$$

* It is assumed that α can take on any values close to $\alpha = 0$ and $\frac{\partial v [y (x, \alpha)]}{\partial \alpha} \Big|_{\alpha = 0}$ exists.

for an extreme value, the boundary points of the admissible curves being fixed: $y(x_0) = y_0$ and $y(x_1) = y_1$ (Fig. 6.3). We will consider the function $F(x, y, y')$ three times differentiable.

We already know that a necessary condition for an extremum is that the variation of the functional vanish. We will now show how this basic theorem is applied to the functional under consideration, and we will repeat the earlier argument as applied to the functional (6.1). Assume that the extremum is attained on a twice-differentiable curve $y = y(x)$ (by only requiring that admissible

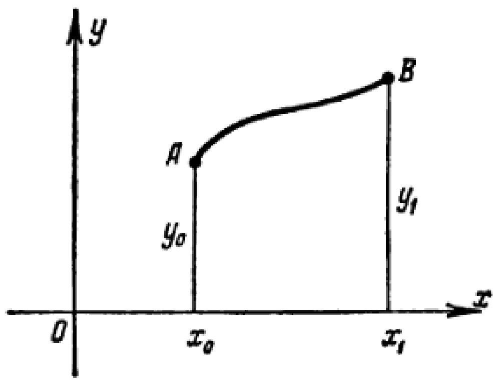


Fig. 6-3

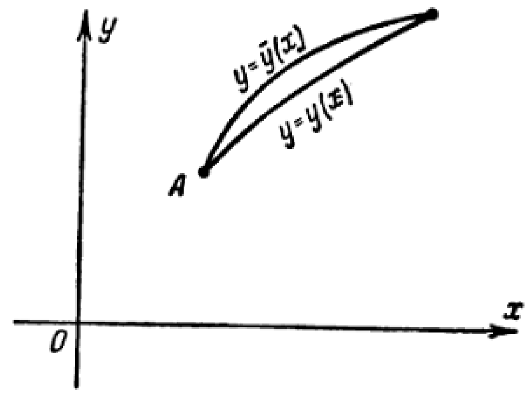


Fig. 6-4

curves have first-order derivatives, we can prove by a different method that the curve which achieves the extremum has a second derivative as well).

Take some admissible curve $y=\bar{y}(x)$ close to $y=y(x)$ and include the curves $y=y(x)$ and $y=\bar{y}(x)$ in a one-parameter family of curves

$$y(x, \alpha)=y(x)+\alpha(\bar{y}(x)-y(x));$$

for $\alpha=0$ we get the curve $y=y(x)$, for $\alpha=1$ we have $y=\bar{y}(x)$ (Fig. 6.4). As we already know, the difference $\bar{y}(x)-y(x)$ is called the variation of the function $y(x)$ and is symbolized as δy.

In variational problems, the variation δy plays a role similar to that of the increment of the independent variable Δx in problems involving investigating functions $f(x)$ for extreme values. The variation $\delta y=\bar{y}(x)-y(x)$ of the function is a function of x. This function may be differentiated once or several times; $(\delta y)'=\bar{y}'(x)-y'(x)=\delta y'$, that is, the derivative of the variation is equal to the variation of the derivative, and similarly

$$(\delta y)''=\bar{y}''(x)-y''(x)=\delta y'',$$

$$\cdots\cdots\cdots\cdots$$

$$(\delta y)^{(k)}=\bar{y}^{(k)}(x)-y^{(k)}(x)=\delta y^{(k)}.$$

We consider the family $y=y(x, \alpha)$, where $y(x, \alpha)=y(x)+\alpha\delta y$, which for $\alpha=0$ contains a curve on which an extreme value is achieved, and which for $\alpha=1$ contains a certain close-lying admissible curve, the so-called comparison curve.

If one considers the values of the functional

$$v[y(x)]=\int_{x_0}^{x_1} F(x, y, y')\,dx$$

only on curves of the family $y=y(x, \alpha)$, then the functional becomes a function of α:

$$v[y(x, \alpha)]=\varphi(\alpha),$$

since the value of the parameter α determines the curve of the family $y=y(x, \alpha)$ and thus determines also the value of the functional $v[y(x, \alpha)]$. This function $\varphi(\alpha)$ is extremized for $\alpha=0$ since for $\alpha=0$ we have $y=y(x)$, and the functional is assumed to have achieved an extremum in comparison with any neighbouring admissible curve and, in particular, with respect to curves of the family $y=y(x, \alpha)$ in the neighbourhood. A necessary condition for the extremum of the function $\varphi(\alpha)$ for $\alpha=0$ is, as we know, that its derivative for $\alpha=0$ vanish:

$$\varphi'(0)=0.$$

Since

$$\varphi(\alpha)=\int_{x_0}^{x_1} F(x, y(x, \alpha), y'_x(x, \alpha))\,dx,$$

it follows that

$$\varphi'(\alpha)=\int_{x_0}^{x_1}\left[F_y\frac{\partial}{\partial\alpha}y(x, \alpha)+F_{y'}\frac{\partial}{\partial\alpha}y'(x, \alpha)\right]dx,$$

where

$$F_y = \frac{\partial}{\partial y} F(x, y(x, \alpha), y'(x, \alpha)),$$

$$F_{y'} = \frac{\partial}{\partial y'} F(x, y(x, \alpha), y'(x, \alpha)),$$

or since

$$\frac{\partial}{\partial \alpha} y(x, \alpha) = \frac{\partial}{\partial \alpha}[y(x) + \alpha \delta y] = \delta y$$

and

$$\frac{\partial}{\partial \alpha} y'(x, \alpha) = \frac{\partial}{\partial \alpha}[y'(x) + \alpha \delta y'] = \delta y',$$

we get

$$\varphi'(\alpha) = \int_{x_0}^{x_1} [F_y(x, y(x, \alpha), y'(x, \alpha))\,\delta y + \\ + F_{y'}(x, y(x, \alpha), y'(x, \alpha))\,\delta y']\,dx;$$

$$\varphi'(0) = \int_{x_0}^{x_1} [F_y(x, y(x), y'(x))\,\delta y + F_{y'}(x, y(x), y'(x))\,\delta y']\,dx.$$

As we already know, $\varphi'(0)$ is called the variation of the functional and is denoted by δv. A necessary condition for the extremum of a functional v is that its variation vanish: $\delta v = 0$. For the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

this condition has the form

$$\int_{x_0}^{x_1} [F_y\,\delta y + F_{y'}\,\delta y']\,dx = 0.$$

We integrate the second term by parts and, taking into account that $\delta y' = (\delta y)'$, we get

$$\delta v = [F_{y'}\,\delta y]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx}F_{y'}\right)\delta y\,dx.$$

But

$$\delta y\,|_{x=x_0} = \bar{y}(x_0) - y(x_0) = 0 \quad \text{and} \quad \delta y\,|_{x=x_1} = \bar{y}(x_1) - y(x_1) = 0,$$

because all admissible curves in the elementary problem under consideration pass through fixed boundary points and, hence,

$$\delta v = \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx}F_{y'}\right)\delta y\,dx.$$

Thus, the necessary condition for an extremum takes the form

$$\int_{x_0}^{x_1} \left(F_y - \frac{d}{dx}F_{y'}\right)\delta y\,dx = 0, \tag{6.2}$$

the first factor $F_y - \frac{d}{dx}F_{y'}$ on the extremizing curve $y = y(x)$ is a given continuous function, while the second factor δy, because of the arbitrary choice of the comparison curve $y = \bar{y}(x)$, is an arbitrary function that satisfies only certain very general conditions, namely: at the boundary points $x = x_0$ and $x = x_1$ the function δy vanishes, it is continuous and differentiable once or several times; δy or δy and $\delta y'$ are small in absolute value.

To simplify the condition obtained, (6.2), let us take advantage of the following lemma.

The fundamental lemma of the calculus of variations. *If for every continuous function* $\eta(x)$

$$\int_{x_0}^{x_1} \Phi(x)\,\eta(x)\,dx = 0,$$

where the function $\Phi(x)$ *is continuous on the interval* $[x_0, x_1]$, *then*

$$\Phi(x) \equiv 0$$

on that interval.

Note. The statement of the lemma and its proof do not change if the following restrictions are imposed on the functions: $\eta(x_0) = \eta(x_1) = 0$; $\eta(x)$ has continuous derivatives to order p, $|\eta^{(s)}(x)| < \varepsilon$ $(s = 0, 1, \ldots, q;\ q \leqslant p)$.

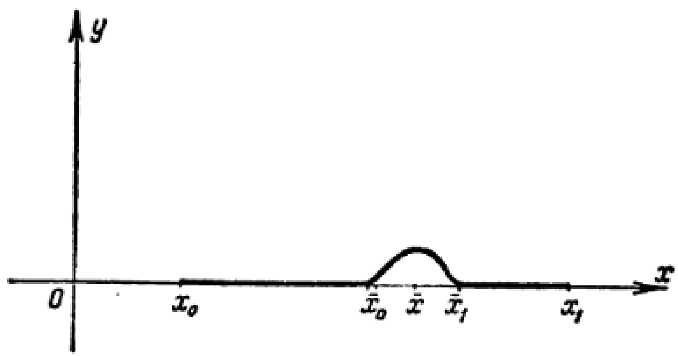


Fig. 6-5

Proof. Assuming that at the point $x = \bar{x}$ lying on the interval $x_0 \leqslant x \leqslant x_1$, $\Phi(x) \neq 0$, we arrive at a contradiction. Indeed, from the continuity of the function $\Phi(x)$ it follows that if $\Phi(\bar{x}) \neq 0$, then $\Phi(x)$ maintains its sign in a certain neighbourhood $(\bar{x}_0 \leqslant x \leqslant \bar{x}_1)$ of the point $\bar{x}$; but then, having chosen a function $\eta(x)$, which also maintains its sign in this neighbourhood and is equal to zero outside this neighbourhood (Fig. 6.5), we get

$$\int_{x_0}^{x_1} \Phi(x)\,\eta(x)\,dx = \int_{\bar{x}_0}^{\bar{x}_1} \Phi(x)\,\eta(x)\,dx \neq 0,$$

since the product $\Phi(x)\eta(x)$ does not change sign on the interval $(\bar{x}_0 \leqslant x \leqslant \bar{x}_1)$ and vanishes outside this interval. We have thus arrived at a contradiction; hence, $\Phi(x) \equiv 0$. The function $\eta(x)$ may for example be chosen thus: $\eta(x) \equiv 0$ outside the interval $(\bar{x}_0 \leqslant x \leqslant \bar{x}_1)$; $\eta(x) = k(x-\bar{x}_0)^{2n}(x-\bar{x}_1)^{2n}$ on the interval $(\bar{x}_0 \leqslant x \leqslant \bar{x}_1)$, where n is a positive integer and k is a constant factor. It is obvious that the function $\eta(x)$ satisfies the above conditions: it is continuous, has continuous derivatives up to order $2n-1$, vanishes at the points x_0 and x_1 and may be made arbitrarily small in absolute value together with its derivatives by reducing the absolute value of the constant k.

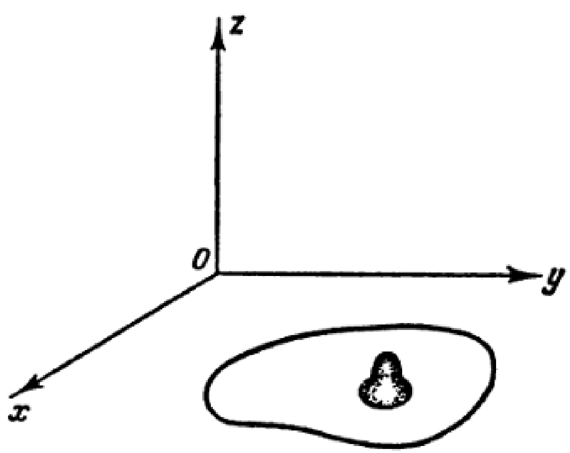


Fig. 6-6

Note. Repeating this argument word for word, one can prove that if the function $\Phi(x, y)$ is continuous in the region D on the plane (x, y) and $\iint_D \Phi(x, y)\,\eta(x, y)\,dx\,dy = 0$ for an arbitrary choice of the function $\eta(x, y)$ satisfying only certain general conditions (continuity, differentiability once or several times, and vanishing at the boundaries of the region D, $|\eta| < \varepsilon$, $|\eta'_x| < \varepsilon$, $|\eta'_y| < \varepsilon$), then $\Phi(x, y) \equiv 0$ in the region D. When proving the fundamental lemma, the function $\eta(x, y)$ may be chosen, for example, as follows: $\eta(x, y) \equiv 0$ outside a circular neighbourhood of sufficiently small radius ε_1 of the point $(\bar{x}, \bar{y})$ in which $\Phi(\bar{x}, \bar{y}) \neq 0$, and in this neighbourhood of the point $(\bar{x}, \bar{y})$ the function $\eta(x, y) = k[(x-\bar{x})^2 + (y-\bar{y})^2 - \varepsilon_1^2]^{2n}$ (Fig. 6.6). An analogous lemma holds true for n-fold multiple integrals.

Now let us use the fundamental lemma to simplify the above-obtained condition (6.2) for the extremum of the elementary functional (6.1)

$$\int_{x_0}^{x_1} \left(F_y - \frac{d}{dx} F_{y'}\right) \delta y\, dx = 0. \qquad (6.2)$$

All conditions of the lemma are fulfilled: on the extremizing curve the factor $\left(F_y - \frac{d}{dx} F_{y'}\right)$ is a continuous function, and the variation δy is an arbitrary function on which only restrictions of a general nature that are provided for by the fundamental lemma have

been imposed; hence, $F_y - \frac{d}{dx} F_{y'} \equiv 0$ on the curve $y = y(x)$ which extremizes the functional under consideration, i.e. $y = y(x)$ is a solution of the second-order differential equation

$$F_y - \frac{d}{dx} F_{y'} = 0,$$

or in expanded form

$$F_y - F_{xy'} - F_{yy'} y' - F_{y'y'} y'' = 0.$$

This equation is called *Euler's equation* (it was first published in 1744). The integral curves of Euler's equation $y = y(x, C_1, C_2)$ are called *extremals*. It is only on extremals that the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

can be extremized. To find the curve that extremizes the functional (6.1), integrate the Euler equation and determine both arbitrary constants that enter into the general solution of this equation, proceeding from the conditions on the boundary $y(x_0) = y_0$, $y(x_1) = y_1$. Only on extremals that satisfy these conditions can the functional be extremized. However, in order to establish whether indeed an extremum (and whether it is a maximum or a minimum) is achieved on them, one has to take advantage of the sufficient conditions for an extremum given in Chapter 8.

Recall that the boundary-value problem

$$F_y - \frac{d}{dx} F_{y'} = 0, \qquad y(x_0) = y_0, \qquad y(x_1) = y_1$$

does not always have a solution and if the solution exists, it may not be unique (see page 166).

Note that in many variational problems the existence of a solution is obvious from the physical or geometrical meaning of the problem and if the solution of Euler's equation satisfying the boundary conditions is unique, then this unique extremal will be the solution of the given variational problem.

Example 1. On what curves can the functional

$$v[y(x)] = \int_0^{\frac{\pi}{2}} [(y')^2 - y^2]\, dx; \; y(0) = 0, \; y\left(\frac{\pi}{2}\right) = 1$$

be extremized? The Euler equation is of the form $y'' + y = 0$; its general solution is $y = C_1 \cos x + C_2 \sin x$. Utilizing the boundary conditions we get $C_1 = 0$, $C_2 = 1$; hence, only on the curve $y = \sin x$ can an extremum be achieved.

3. Functionals of the Form

$$\int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx$$

In order to obtain the necessary conditions for the extremum of a functional v of a more general type

$$v[y_1, y_2, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx$$

for the given boundary conditions of all functions

$$y_1(x_0) = y_{10}, \quad y_2(x_0) = y_{20}, \; \ldots, y_n(x_0) = y_{n0},$$
$$y_1(x_1) = y_{11}, \quad y_2(x_1) = y_{21}, \; \ldots, y_n(x_1) = y_{n1},$$

we shall vary only one of the functions

$$y_j(x) \quad (j = 1, 2, \ldots, n),$$

holding the other functions unchanged. Then the functional $v[y_1, y_2, \ldots, y_n]$ will reduce to a functional dependent only on a single varied function, for example, on $y_i(x)$,

$$v[y_1, y_2, \ldots, y_n] = \tilde{v}[y_i]$$

of the form considered in Sec. 2, and, hence, the extremizing function must satisfy Euler's equation

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0.$$

Since this argument is applicable to any function y_i $(i = 1, 2, \ldots, n)$, we get a system of second-order differential equations

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0 \qquad (i = 1, 2, \ldots, n),$$

which, generally speaking, define a $2n$-parameter family of integral curves in the space $x, y_1, y_2, \ldots, y_n$—which is the family of extremals of the given variational problem.

If, for example, the functional depends only on two functions $y(x)$ and $z(x)$:

$$v[y(x), \quad z(x)] = \int_{x_0}^{x_1} F(x, y, z, y', z')\,dx;$$

$$y(x_0) = y_0, \qquad z(x_0) = z_0, \qquad y(x_1) = y_1, \qquad z(x_1) = z_1,$$

that is to say, it is defined by the choice of space curve $y = y(x)$, $z = z(x)$ (Fig. 6.11), then by varying $y(x)$ alone and holding $z(x)$ constant we can change our curve so that its projection on the xz-plane does not change, i.e. the curve all the time remains on the projecting cylinder $z = z(x)$ (Fig. 6.12).

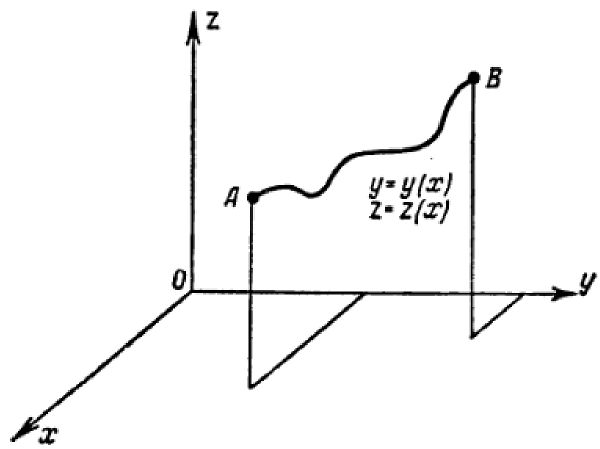
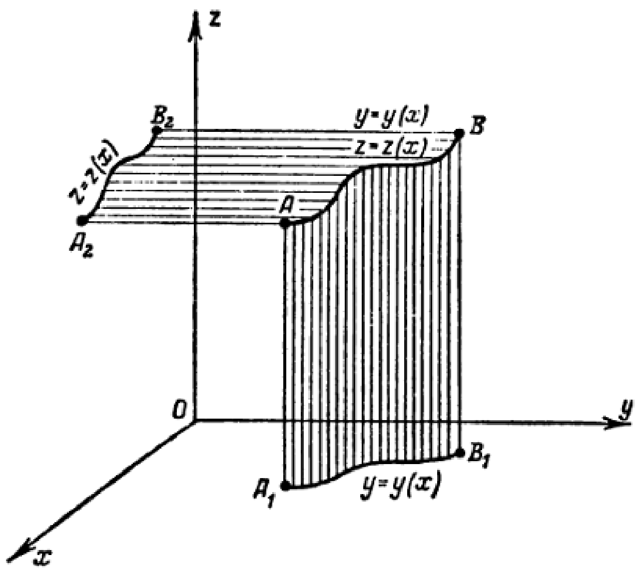


Fig. 6-11



· Fig. 6-12

Similarly, by fixing $y(x)$ and varying $z(x)$, we vary the curve so that all the time it lies on the projecting cylinder $y = y(x)$. We then obtain a system of two Euler's equations:

$$F_y - \frac{d}{dx} F_{y'} = 0 \quad \text{and} \quad F_z - \frac{d}{dx} F_{z'} = 0.$$

Example 1. Find the extremals of the functional

$$v[y(x),\ z(x)] = \int_0^{\frac{\pi}{2}} [y'^2 + z'^2 + 2yz]\, dx, \qquad y(0) = 0, \qquad y\left(\frac{\pi}{2}\right) = 1,$$

$$z(0) = 0, \qquad z\left(\frac{\pi}{2}\right) = -1.$$

The system of Euler's differential equations is of the form

$$y'' - z = 0,$$
$$z'' - y = 0.$$

Eliminating one of the unknown functions, say z, we get $y^{IV} - y = 0$. Integrating this linear equation with constant coefficients, we obtain

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x;$$
$$z = y''; \quad z = C_1 e^x + C_2 e^{-x} - C_3 \cos x - C_4 \sin x.$$

Using the boundary conditions, we find

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = 0, \quad C_4 = 1;$$

hence, $y = \sin x$, $z = -\sin x$.

Example 2. Find the extremals of the functional

$$v[y(x),\ z(x)] = \int_{x_0}^{x_1} F(y',\ z')\, dx.$$

The system of Euler's equations is of the form

$$F_{y'y'} y'' + F_{y'z'} z'' = 0; \quad F_{y'z'} y'' + F_{z'z'} z'' = 0,$$

whence, assuming $F_{y'y'} F_{z'z'} - (F_{y'z'})^2 \neq 0$, we get $y'' = 0$ and $z'' = 0$ or $y = C_1 x + C_2$, $z = C_3 x + C_4$ are a family of straight lines in space.

Example 3. Find the differential equations of the lines of propagation of light in an optically nonhomogeneous medium in which the speed of light is $v(x, y, z)$.

According to Fermat's principle, light is propagated from one point $A(x_0, y_0)$ to another $B(x_1, y_1)$ along a curve for which the time T of passage of light will be least. If the equation of the desired curve $y=y(x)$ and $z=z(x)$, then

$$T=\int_{x_0}^{x_1} \frac{\sqrt{1+y'^2+z'^2}}{v(x, y, z)}\, dx.$$

For this functional, the system of Euler's equations

$$\frac{\partial v}{\partial y}\frac{\sqrt{1+y'^2+z'^2}}{v^2}+\frac{d}{dx}\frac{y'}{v\sqrt{1+y'^2+z'^2}}=0,$$

$$\frac{\partial v}{\partial z}\frac{\sqrt{1+y'^2+z'^2}}{v^2}+\frac{d}{dx}\frac{z'}{v\sqrt{1+y'^2+z'^2}}=0$$

will be a system that defines the lines of light propagation.

4. Functionals Dependent on Higher-Order Derivatives

Let us investigate the extreme value of the functional

$$v[y(x)]=\int_{x_0}^{x_1} F(x, y(x), y'(x), \ldots, y^{(n)}(x))\, dx,$$

where we consider the function F differentiable $n+2$ times with respect to all arguments and we assume that the boundary conditions are of the form

$$y(x_0)=y_0,\ y'(x_0)=y'_0,\ \ldots,\ y^{(n-1)}(x_0)=y_0^{(n-1)};$$
$$y(x_1)=y_1,\ y'(x_1)=y'_1,\ \ldots,\ y^{(n-1)}(x_1)=y_1^{(n-1)},$$

i.e. at the boundary points are given the values not only of the function but also of its derivatives up to the order $n-1$ inclusive. Suppose that an extremum is attained on the curve $y=y(x)$, which is $2n$ times differentiable, and let $y=\bar{y}(x)$ be the equation of some comparison curve, which is also $2n$ times differentiable.

Consider the one-parameter family of functions

$$y(x, \alpha)=y(x)+\alpha[\bar{y}(x)-y(x)] \text{ or } y(x, \alpha)=y(x)+\alpha\delta y.$$

For $\alpha=0$, $y(x, \alpha)=y(x)$ and for $\alpha=1$, $y(x, \alpha)=\bar{y}(x)$. If one considers the value of the functional $v[y(x)]$ only on curves of the family $y=y(x, \alpha)$, then the functional reduces to a function of the parameter α, which is extremized for $\alpha=0$; hence, $\frac{d}{d\alpha}v[y(x,\alpha)]\Big|_{\alpha=0}=0$. According to Sec. 1, this derivative is called the *variation of the functional* v and is symbolized by δv:

$$\delta v=\left[\frac{d}{d\alpha}\int_{x_0}^{x_1}F(x, y(x, \alpha), y'(x, \alpha), \ldots, y^{(n)}(x, \alpha))\,dx\right]_{\alpha=0}=$$

$$=\int_{x_0}^{x_1}(F_y\,\delta y+F_{y'}\,\delta y'+F_{y''}\,\delta y''+\ldots+F_{y^{(n)}}\,\delta y^{(n)})\,dx.$$

Integrate the second summand on the right once term-by-term

$$\int_{x_0}^{x_1}F_{y'}\,\delta y'\,dx=[F_{y'}\,\delta y]_{x_0}^{x_1}-\int_{x_0}^{x_1}\frac{d}{dx}F_{y'}\,\delta y\,dx,$$

the third summand twice:

$$\int_{x_0}^{x_1}F_{y''}\,\delta y''\,dx=[F_{y''}\,\delta y']_{x_0}^{x_1}-\left[\frac{d}{dx}F_{y''}\,\delta y\right]_{x_0}^{x_1}+\int_{x_0}^{x_1}\frac{d^2}{dx^2}F_{y''}\,\delta y\,dx,$$

and so forth; the last summand, n times:

$$\int_{x_0}^{x_1} F_{y^{(n)}}\,\delta y^{(n)}\,dx = [F_{y^{(n)}}\,\delta y^{(n-1)}]_{x_0}^{x_1} - \left[\frac{d}{dx}F_{y^{(n)}}\,\delta y^{(n-2)}\right]_{x_0}^{x_1} + \dots$$

$$\dots + (-1)^n \int_{x_0}^{x_1} \frac{d^n}{dx^n} F_{y^{(n)}}\,\delta y\,dx.$$

Taking into account the boundary conditions, by virtue of which for $x = x_0$ and for $x = x_1$, the variations $\delta y = \delta y' = \delta y'' = \dots = \delta y^{(n-1)} = 0$, we finally get

$$\delta v = \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}}\right)\delta y\,dx.$$

Since on the extremizing curve we have

$$\delta v = \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}}\right)\delta y\,dx = 0$$

for an arbitrary choice of the function δy and since the first factor under the integral sign is a continuous function of x on the same curve $y = y(x)$, it follows that by virtue of the fundamental lemma the first factor is identically zero:

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} \equiv 0.$$

Thus, the function $y = y(x)$, which extremizes the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y, y', y'', \dots, y^{(n)})\,dx,$$

must be a solution of the equation

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} = 0.$$

This differential equation of order $2n$ is called the *Euler-Poisson equation*, and its integral curves are termed *extremals* of the variational problem under consideration. The general solution of this equation contains $2n$ arbitrary constants, which, generally speaking, may be determined from the $2n$ boundary conditions:

$$y(x_0)=y_0,\ y'(x_0)=y'_0,\ \ldots,\ y^{(n-1)}(x_0)=y_0^{(n-1)};$$
$$y(x_1)=y_1,\ y'(x_1)=y'_1,\ \ldots,\ y^{(n-1)}(x_1)=y_1^{(n-1)}.$$

Example 1. Find the extremal of the functional

$$v[y(x)]=\int_0^1 (1+y''^2)\,dx;$$
$$y(0)=0,\ y'(0)=1,\ y(1)=1,\ y'(1)=1.$$

The Euler-Poisson equation is of the form $\frac{d^2}{dx^2}(2y'')=0$ or $y^{IV}=0$; its general solution is $y=C_1x^3+C_2x^2+C_3x+C_4$. Using the boundary conditions, we get

$$C_1=0,\ C_2=0,\ C_3=1,\ C_4=0.$$

And so the extremum can be attained only on the straight line $y=x$.

Example 2. Determine the extremal of the functional

$$v[y(x)]=\int_0^{\frac{\pi}{2}} (y''^2-y^2+x^2)\,dx,$$

that satisfies the conditions

$$y(0)=1,\ y'(0)=0,\ y\left(\frac{\pi}{2}\right)=0,\ y'\left(\frac{\pi}{2}\right)=-1.$$

The Euler-Poisson equation is of the form $y^{IV}-y=0$; its general solution is $y=C_1e^x+C_2e^{-x}+C_3\cos x+C_4\sin x$. Using the boundary conditions, we get $C_1=0$, $C_2=0$, $C_3=1$, $C_4=0$. And so the extremum can be achieved only on the curve $y=\cos x$.

Example 3. Determine the extremal of the functional

$$v[y(x)]=\int_{-l}^{l}\left(\frac{1}{2}\mu y''^2+\rho y\right)dx,$$

that satisfies the boundary conditions

$$y(-l)=0, \quad y'(-l)=0, \quad y(l)=0, \quad y'(l)=0.$$

This is the variational problem to which is reduced the problem of finding the axis of a flexible bent cylindrical beam fixed at the

ends. If the beam is homogeneous, then ρ and μ are constants and the Euler-Poisson equation has the form

$$\rho+\frac{d^2}{dx^2}(\mu y'')=0 \text{ or } y^{IV}=-\frac{\rho}{\mu},$$

whence

$$y=-\frac{\rho x^4}{24\mu}+C_1x^3+C_2x^2+C_3x+C_4.$$

Using the boundary conditions, we finally get

$$y=-\frac{\rho}{24\mu}(x^4-2l^2x^2+l^4) \text{ or } y=-\frac{\rho}{24\mu}(x^2-l^2)^2.$$

If the functional v is of the form

$$v[y(x), \ z(x)]=\int_{x_0}^{x_1} F(x, \ y, \ y', \ \ldots, \ y^{(n)}, \ z, \ z' \ \ldots, \ z^{(m)})\,dx,$$

then by varying only $y(x)$ and assuming $z(x)$ to be fixed, we find that the extremizing functions $y(x)$ and $z(x)$ must satisfy the Euler-Poisson equation

$$F_y-\frac{d}{dx}F_{y'}+\ldots+(-1)^n\frac{d^n}{dx^n}F_{y^{(n)}}=0,$$

whereas by varying $z(x)$ and holding $y(x)$ fixed we find that the very same functions must satisfy the equation

$$F_z-\frac{d}{dx}F_{z'}+\ldots+(-1)^m\frac{d^m}{dx^m}F_{z^{(m)}}=0.$$

Thus, the functions $z(x)$ and $y(x)$ must satisfy a system of two equations:

$$F_y - \frac{d}{dx} F_{y'} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0,$$

$$F_z - \frac{d}{dx} F_{z'} + \ldots + (-1)^m \frac{d^m}{dx^m} F_{z^{(m)}} = 0.$$

We can argue in the same fashion when investigating for the extremum of a functional dependent on any number of functions:

$$v[y_1, y_2, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_1', \ldots, y_1^{(n_1)}, y_2, y_2', \ldots, y_2^{(n_2)}, \ldots, y_m, y_m', \ldots, y_m^{(n_m)})\, dx.$$

Varying some one function $y_i(x)$ and holding the others fixed, we get the basic necessary condition for an extremum in the form

$$F_{y_i} - \frac{d}{dx} F_{y_i'} + \ldots + (-1)^{n_i} \frac{d^{n_i}}{dx^{n_i}} F_{y_i^{(n_i)}} = 0 \qquad (i = 1, 2, \ldots, m).$$

5. Functionals Dependent on the Functions of Several Independent Variables

Let us investigate the following functional for an extremum:

$$v[z(x, y)] = \iint_D F\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right) dx\, dy;$$

the values of the function $z(x, y)$ are given on the boundary C of domain D, that is, a spatial path (or contour) $\tilde{C}$ is given, through

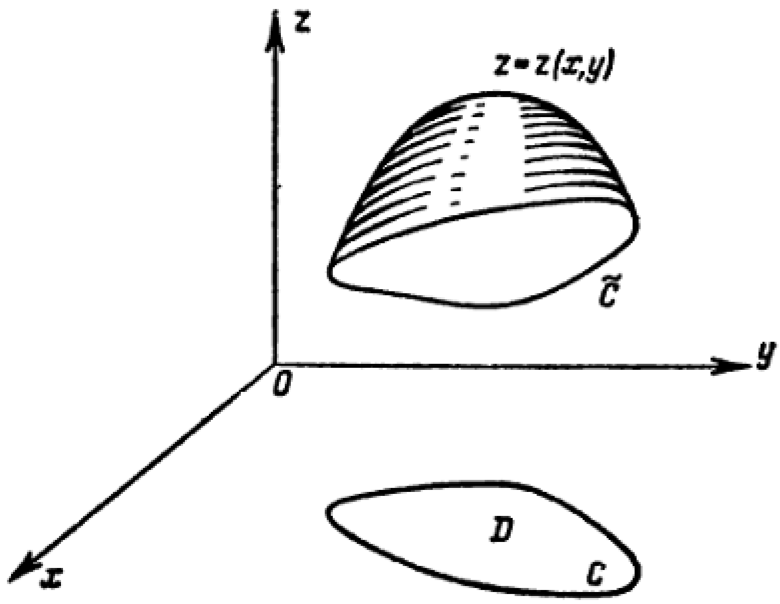


Fig. 6-13

which all permissible surfaces have to pass (Fig. 6.13). To abbreviate notation, put $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$. We will consider the function F as three times differentiable. We assume the extremizing surface $z = z(x, y)$ to be twice differentiable.

Let us consider a one-parameter family of surfaces $z = z(x, y, \alpha) = z(x, y) + \alpha\delta x$, where $\delta z = \bar{z}(x, y) - z(x, y)$, including for $\alpha = 0$ the surface $z = z(x, y)$ on which the extremum is achieved, and for $\alpha = 1$, a certain permissible surface $z = \bar{z}(x, y)$. On functions of the family $z(x, y, \alpha)$, the functional v reduces to the function α, which has to have an extremum for $\alpha = 0$; consequently, $\frac{\partial}{\partial \alpha} v[z(x, y, \alpha)]|_{\alpha=0} = 0$. If, in accordance with Sec. 1, we call the derivative of $v[z(x, y, \alpha)]$ with respect to α, for $\alpha = 0$, the *variation of the functional* and symbolize it by δv, we will have

$$\delta v = \left\{ \frac{\partial}{\partial \alpha} \iint_D F(x, y, z(x, y, \alpha), p(x, y, \alpha), q(x, y, \alpha))\, dx\, dy \right\}_{\alpha=0} = \iint_D [F_z \delta z + F_p \delta p + F_q \delta q]\, dx\, dy,$$

where

$$z(x, y, \alpha) = z(x, y) + \alpha\,\delta z,$$
$$p(x, y, \alpha) = \frac{\partial z(x, y, \alpha)}{\partial x} = p(x, y) + \alpha\,\delta p,$$
$$q(x, y, \alpha) = \frac{\partial z(x, y, \alpha)}{\partial y} = q(x, y) + \alpha\,\delta q.$$

Since

$$\frac{\partial}{\partial x}\{F_p\,\delta z\} = \frac{\partial}{\partial x}\{F_p\}\,\delta z + F_p\,\delta p,$$
$$\frac{\partial}{\partial y}\{F_q\,\delta z\} = \frac{\partial}{\partial y}\{F_q\}\,\delta z + F_q\,\delta q,$$

it follows that

$$\iint_D (F_p\,\delta p + F_q\,\delta q)\,dx\,dy =$$
$$= \iint_D \left[\frac{\partial}{\partial x}\{F_p\,\delta z\} + \frac{\partial}{\partial y}\{F_q\,\delta z\}\right] dx\,dy -$$
$$- \iint_D \left[\frac{\partial}{\partial x}\{F_p\} + \frac{\partial}{\partial y}\{F_q\}\right] \delta z\,dx\,dy,$$

where $\frac{\partial}{\partial x}\{F_p\}$ is the so-called total partial derivative with respect to x. When calculating it, y is assumed to be fixed, but the dependence of z, p and q upon x is taken into account:

$$\frac{\partial}{\partial x}\{F_p\} = F_{px} + F_{pz}\frac{\partial z}{\partial x} + F_{pp}\frac{\partial p}{\partial x} + F_{pq}\frac{\partial q}{\partial x}$$

and similarly

$$\frac{\partial}{\partial y}\{F_q\} = F_{qy} + F_{qz}\frac{\partial z}{\partial x} + F_{qp}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y}.$$

Using the familiar Green's function

$$\iint_D \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dx\,dy = \int_C (N\,dy - M\,dx)$$

we get

$$\iint_D \left[\frac{\partial}{\partial x}\{F_p\,\delta z\} + \frac{\partial}{\partial y}\{F_q\,\delta z\}\right] dx\,dy = \int_C (F_p\,dy - F_q\,dx)\,\delta z = 0.$$

The last integral is equal to zero, since on the contour C the variation $\delta z = 0$ because all permissible surfaces pass through one and the same spatial contour $\tilde{C}$. Consequently,

$$\iint_D [F_p\,\delta p + F_q\,\delta q]\,dx\,dy = -\iint_D \left[\frac{\partial}{\partial x}\{F_p\} + \frac{\partial}{\partial y}\{F_q\}\right]\delta z\,dx\,dy,$$

and the necessary condition for an extremum,

$$\iint_D (F_z\,\delta z + F_p\,\delta p + F_q\,\delta q)\,dx\,dy = 0,$$

takes the form

$$\iint_D \left(F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\}\right)\delta z\,dx\,dy = 0.$$

Since the variation δz is arbitrary (only restrictions of a general nature are imposed on δz that have to do with continuity and differentiability, vanishing on the contour C, etc.) and the first factor is continuous, it follows from the fundamental lemma (page 308) that on the extremizing surface $z = z(x, y)$

$$F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\} \equiv 0.$$

Consequently, $z(x, y)$ is a solution of the equation

$$F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\} = 0.$$

This second-order partial differential equation that must be satisfied by the extremizing function $z(x, y)$ is called the *Ostrogradsky equation* after the celebrated Russian mathematician M. Ostrogradsky who in 1834 first obtained the equation (for rectangular domains D it had already appeared in the works of Euler).

Example 1.

$$v[z(x, y)] = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] dx\,dy,$$

the values of the function z are given on the boundary C of the domain D: $z = f(x, y)$. Here the Ostrogradsky equation is of the form

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

or, in abbreviated notation,

$$\Delta z = 0,$$

which is the familiar *Laplace equation*; we have to find a solution, continuous in D, of this equation that takes on specified values on the boundary of the domain D. This is one of the basic problems of mathematical physics, called the *Dirichlet problem.*

6. Variational Problems in Parametric Form

In many variational problems the solution is more conveniently sought in parametric form. For example, in the isoperimetric problem (see page 295) of finding a closed curve of given length l bounding a maximum area S, it is inconvenient to seek the solution in the form $y = y(x)$, since by the very meaning of the problem the function $y(x)$ is ambiguous (Fig. 6.14). Therefore, in this problem it is advisable to seek the solution in parametric form: $x = x(t)$, $y = y(t)$. Hence, in the given case we have to seek the extremum of the functional

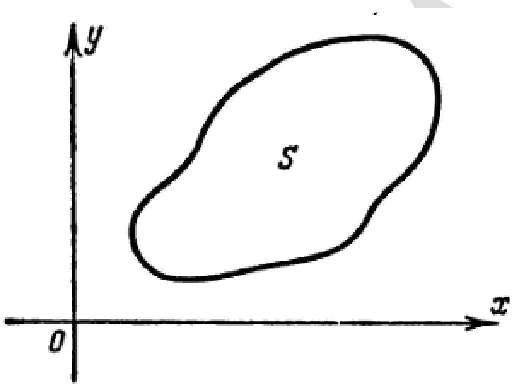


Fig. 6-14

$$S[x(t), y(t)] = \frac{1}{2}\int_0^T (x\dot{y} - y\dot{x})\, dt$$

provided that $l = \int_0^T \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$, where l is a constant.

In the investigation of a certain functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

for an extremum let it be more advisable to seek the solution in the parametric form $x = x(t)$, $y = y(t)$; then the functional will be reduced to the following form:

$$v[x(t), y(t)] = \int_{t_0}^{t_1} F\left(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)}\right)\dot{x}(t)\, dt.$$

Note that after transformation of the variables, the integrand

$$F\left(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)}\right)\dot{x}(t)$$

does not contain t explicitly and, with respect to the variables $\dot{x}$ and $\dot{y}$, is a homogeneous function of the first degree.

Thus, the functional $v[x(t), y(t)]$ is not an arbitrary functional of the form

$$\int_{t_0}^{t_1} \Phi(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\, dt$$

that depends on two functions $x(t)$ and $y(t)$, but only an extremely particular case of such a functional, since its integrand does not contain t explicitly and is a homogeneous function of the first degree in the variables $\dot{x}$ and $\dot{y}$.

If we were to go over to any other parametric representation of the desired curve $x=x(\tau)$, $y=y(\tau)$, then the functional $v[x, y]$ would be reduced to the form $\int_{\tau_0}^{\tau_1} F\left(x, y, \frac{\dot{y}_\tau}{\dot{x}_\tau}\right)\dot{x}_\tau\, d\tau$. Hence, the integrand of the functional v does not change its form when the parametric representation of the curve is changed. Thus, the functional v depends on the type of curve and not on its parametric representation.

It is easy to see the truth of the following assertion: if the integrand of the functional

$$v[x(t), y(t)]=\int_{t_0}^{t_1} \Phi(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\,dt$$

does not contain t explicitly and is a homogeneous function of the first degree in $\dot{x}$ and $\dot{y}$, then the functional $v[x(t), y(t)]$ depends solely on the kind of curve $x=x(t)$, $y=y(t)$, and not on its parametric representation. Indeed, let

$$v[x(t), y(t)]=\int_{t_0}^{t_1} \Phi(x(t), y(t), \dot{x}(t), \dot{y}(t))\, dt,$$

where

$$\Phi(x, y, k\dot{x}, k\dot{y})=k\Phi(x, y, \dot{x}, \dot{y}).$$

Let us pass to a new parametric representation putting

$$\tau=\varphi(t) \quad (\varphi(t)\neq 0), \quad x=x(\tau), \quad y=y(\tau).$$

Then

$$\int_{t_0}^{t_1} \Phi(x(t), y(t), \dot{x}(t), \dot{y}(t))\, dt = \int_{\tau_0}^{\tau_1} \Phi(x(\tau), y(\tau), \dot{x}_\tau(\tau)\dot{\varphi}(t), \dot{y}_\tau(\tau)\dot{\varphi}(t)) \frac{d\tau}{\dot{\varphi}(t)}.$$

By virtue of the fact that Φ is a homogeneous function of the first degree in $\dot{x}$ and $\dot{y}$, we have

$$\Phi(x, y, \dot{x}_\tau\dot{\varphi}, \dot{y}_\tau\dot{\varphi}) = \dot{\varphi}\Phi(x, y, \dot{x}_\tau, \dot{y}_\tau),$$

whence

$$\int_{t_0}^{t_1} \Phi(x, y, \dot{x}_t, \dot{y}_t)\, dt = \int_{\tau_0}^{\tau_1} \Phi(x, y, \dot{x}_\tau, \dot{y}_\tau)\, d\tau,$$

that is, the integrand has not changed with a change in the parametric representation.

The arc length $\int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$ * and an area bounded by a certain curve $\frac{1}{2}\int_{t_0}^{t_1} (x\dot{y} - y\dot{x})\, dt$ are examples of such functionals.

In order to find the extremals of functionals of this kind,

$$v[x(t), y(t)] = \int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\, dt,$$

where Φ is a homogeneous function of the first degree in $\dot{x}$ and $\dot{y}$, and also for functionals with an arbitrary integrand function $\Phi(t, x, y, \dot{x}, \dot{y})$, one has to solve a system of Euler's equations:

$$\Phi_x - \frac{d}{dt}\Phi_{\dot{x}} = 0; \quad \Phi_y - \frac{d}{dt}\Phi_{\dot{y}} = 0.$$

However, in the case under consideration, these equations are not independent, since they must be satisfied by a certain solution $x = x(t)$, $y = y(t)$ and also by any other pairs of functions that yield a different parametric representation of the same curve, which, in the case of Euler's equations being independent, would lead to a contradiction with the theorem of the existence and uniqueness of a solution of a system of differential equations. This is an indication that for functionals of the form

$$v[x(t),\ y(t)] = \int_{t_0}^{t_1} \Phi(x,\ y,\ \dot{x},\ \dot{y})\,dt,$$

where Φ is a homogeneous function of the first degree in $\dot{x}$ and $\dot{y}$, one of the Euler equations is a consequence of the other. To find the extremals, we have to take one of the Euler equations and integrate it together with the equation defining the choice of parameter. For example, to the equation $\Phi_x - \frac{d}{dt}\Phi_{\dot{x}} = 0$ we can adjoin the equation $\dot{x}^2 + \dot{y}^2 = 1$, which indicates that the arc length of the curve is taken as the parameter.

Possible Questions

PART-B (Six Mark)

1) Explain about Euler equation.
2) Find the extremals of the functional $V[y(x)] = \int_0^1 1 + (y")^2\, dx$, $y(0) = 0\,, y(1) = 1\,, y'(0) = 1\,, y'(1) = 1.$
3) Solve $V[y(x)] = \int_{x_0}^{x_1}(y^2 + 2xyy')dx$ with $y(x_0) = y_0\ and\ y(x_1) = y_1$.
4) Solve $V[y(x)] = \int_{x_0}^{x_1}(y + xy')dx$ with $y(x_0) = y_0\ and\ y(x_1) = y_1$.
5) Obtain variational problem in parametric form.
6) Find the curve joining two body points rotated about absicca's axis generated.
7) Find the curve joining the points (0,0) and (1,0) for which the integral $\int_0^1 y"^2 dx$ is minimum if y'(0) = a and y'(1) = b.
8) Obtain the differential equation of the vibrating string.
9) Find the extremals of the functional $V[y(x), Z(x)] = \int_0^{\frac{\pi}{2}}(y'^2 + Z'^2 + 2yZ)dx$, $y(0) = 0\,, y\left(\frac{\pi}{2}\right) = 1\ and\ Z(0) = 0\,, Z\left(\frac{\pi}{2}\right) = -1.$
10) Obtain the equation of vibrating of a rectilinear bar.
11) Explain the functional dependent on the functions of several independent variables.

PART-C (Ten Mark)

1.On what curve can the functional $V[y(x)] = \int_0^1(y'^2 + 12xy)dx$, $y(0) = 0\ and$ y(1) = 1 be extremized.