		Semester II			
		L	**T**	**P**	**C**
18MMU201	**DIFFERENTIAL EQUATIONS**	**4**	**0**	**0**	**4**

Course Objective

To enable the students to learn and gain knowledge about first order exact differential equations, linear homogeneous and non homogeneous equations of higher order with constant coefficients.

Course Outcomes

On successful completion of this course, the student will be able to

- Understand the concepts of explicit, implicit and singular solutions of a differential equation.
- Acquire knowledge on linear and bernoulli's equaitons.
- Know the concepts of population model.
- Understand the method of solving differential equation using variation of parameters.
- Identify the applications of differential equations.

UNIT I

Differential equations and mathematical models. General, particular, explicit, implicit and singular solutions of a differential equation.

UNIT II

Exact differential equations and integrating factors, separable equations and equations reducible to this form, linear equation and Bernoulli equations, special integrating factors and transformations.

UNIT III

General solution of homogeneous equation of second order, principle of super position for homogeneous equation, Wronskian: its properties and applications, Linear homogeneous and non-homogeneous equations of higher order with constant coefficients, Euler's equation, method of undetermined coefficients, method of variation of parameters.

UNIT IV

Laplace transforms: Definition-Sufficient conditions for the existence of the Laplace Transform, Laplace Transform of periodic functions- Some general theorems-Evaluation of integrals using Laplace Transform.

UNIT V

Inverse Laplace Transforms: Solving ordinary differential equations with constant coefficients using Laplace Transforms-Solving a system of differential equations using Laplace Transforms.

SUGGESTED READINGS

TEXT BOOK

1. Ross S.L., (2016). Differential Equations, Third Edition, John Wiley and Sons, India.

REFERENCES

1. Martha L Abell., and James P Braselton., (2004). Differential Equations with MATHEMATICA, Third Edition, Elsevier Academic Press.
2. Sneddon I.,(2006). Elements of Partial Differential Equations, McGraw-Hill, International Edition, New Delhi.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University)

(Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LESSON PLAN

DEPARTMENT OF MATHEMATICS

NAME OF THE FACULTY : Y.Sangeetha

SUBJECT : DIFFERENTIAL EQUATIONS

SUBJECT CODE : 18MMU201

CLASS : I B.Sc MATHEMATICS

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
UNIT I			
1	1	Introduction of Differential equations	R1:Chap1:Pg.No:3-6
2	1	Mathematical models related examples	
3	1	General solutions of a differential equation Problems	R1:Chap1:Pg.No:7-8
4	1	Particular solutions of a differential equation Problems	R1:Chap1:Pg.No:9-10
5	1	Explicit solutions of a differential equation Problems	R1:Chap1:Pg.No:11-12
6	1	Implicit solutions of a differential equation Problems	R2:Chap1:Pg.No:6-9
7	1	Singular solutions of a differential equation Problems	R2:Chap1:Pg.No:10-13
8	1	Recapitulation and discussion of important questions.	
Total	8 hrs		

UNIT II			
1	1	Introduction on concept of Exact differential equations	R1:Chap2:Pg.No:36-40
2	1	Integrating factors Problems	R1:Chap2:Pg.No:42-44
3	1	Separable equations Problems	R2:Chap2.1:Pg.No:46-49
4	1	Equations reducible to this form linear equation Problems	R1:Chap2:Pg.No:50-53
5	1	Bernoulli equations related Problems	R1:Chap2:Pg.No:56-59
6	1	Continuation on Bernoulli equations related Problems	R1:Chap2:Pg.No:60-62
7	1	Special integrating factors and transformations related Problems	R1:Chap2:Pg.No:68-74
8	1	Recapitulation and discussion of important questions	
Total 8 hrs			
UNIT III			
1	1	Introduction on general solution of homogeneous equation of second order related Problems	R2:Chap:4:Pg.No:196-199
2	1	Principle of super position for homogeneous equation	R2:Chap:4:Pg.No:200-202
3	1	Wronskian: its properties and applications	R2:Chap:4:Pg.No:239-242
4	1	Linear homogeneous equations and non-homogeneous of higher order with constant coefficients related Problems	R2:Chap:4:Pg.No:200-205
5	1	Euler's equation related Problems	R2:Chap:4:Pg.No:255-258
6	1	Method of undetermined coefficients related Problems	R2:Chap:4:Pg.No:222-223

7	1	Method of variation of parameters related Problems	R2:Chap:4:Pg.No:248-251
8	1	Recapitulation and discussion of important questions	
Total 8 hrs			
Unit- IV			
1	1	Definition of Laplace Transform	R3:Chap:4:Pg.No:141-142
2	1	Sufficient conditions for the existence of the Laplace Transform	R3:Chap:4:Pg.No:143-145
3	1	Laplace Transform of periodic functions	R1:Chap:9:Pg.No:428-429
4	1	Continuation on Laplace Transform of periodic functions	R1:Chap 9:Pg.No:429-430
5	1	Some general theorems	R3:Chap 4:Pg.No:164-167
6	1	Continuation on some general theorems	R1:Chap 9:Pg.No:437-438
7	1	Evaluation of integrals using Laplace Transform	R1:Chap 9:Pg.No:439-440
8	1	Recapitulation and discussion of important questions	
Unit- V			
1	1	Solving ordinary differential equations with constant coefficients using Laplace Transforms	R1:Chap:9:Pg.No:441-447
2	1	Continuation on Solving ordinary differential equations with constant coefficients using Laplace Transforms	R1:Chap:9:Pg.No:447-452
3	1	Solving a system of differential equations using Laplace Transforms.	R1:Chap:9:Pg.No:453-455
4	1	Continuation on Solving a system of differential equations using Laplace	R1:Chap:9:Pg.No:456-458

		Transforms	
5	1	Continuation on Solving a system of differential equations using Laplace Transforms	R1:Chap:9:Pg.No:459-460
6	1	Discuss on Previous ESE question papers	
7	1	Discuss on Previous ESE question papers	
8	1	Discuss on Previous ESE question papers	
Total	8 hrs		

SUGGESTED READINGS

R1. Ross S.L., (2004). Differential Equations, Third Edition, John Wiley and Sons, India.

R2. Martha L Abell., and James P Braselton., (2004). Differential Equations with MATHEMATICA, Third Edition, Elsevier Academic Press.

R3. Sneddon I.,(2006). Elements of Partial Differential Equations, McGraw-Hill, International Edition, New Delhi.

UNIT-I

SYLLABUS

Differential equations and mathematical models. General, particular, explicit, implicit and singular solutions of a differential equation.

Introduction

Differential equations finds its application in a variety of real world problems such as growth and decay problems. Newton's law of cooling can be used to determine the time of death of a person. Torricelli's law can be used to determine the time when the tank gets drained off completely and many other problems in science and engineering can be solved by using differential equations. In this chapter, we will first discuss the concept of differential equations and the method of solving a first order differential equation. In the next section, we will discuss various applications of differential equations.

Basic Terminology

Variable: Variable is that quantity which takes on different quantitative values. Example: memory test scores, height of individuals, yield of rice etc.

Dependent Variable: A variable that depends on the other variable is called a dependent variable. For instance, if the demand of gold depends on its price, then demand of gold is a dependent variable.

Independent Variable: Variables which takes on values independently are called independent variables. In the above example, price is an independent variable.

Derivative: Let y = f(x) be a function. Then the derivative $\frac{dy}{dx} = f'(x)$ of the function f is the rate at which the function y = f(x) is changing with respect to the independent variable.

Differential Equation: An equation which relates an independent variable, dependent variable and one or more of its derivatives with respect to independent variable is called a differential equation.

Ordinary differential equation: A differential equation in which the dependent variable (unknown function) depends only on a single independent variable is called an ordinary differential equation.

Partial Differential equation: A differential equation in which the dependent variable is a function of two or more independent variables is called a partial differential equation.

Order of a differential equation: The order of a differential equation is defined as the order of the highest order derivative appearing in the differential equation. The order of a differential equation is a positive integer.

First order differential equation

A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is called a differential equation of first order. If initial condition $y(x_0) = y_0$ is also specified , then it is called an initial value problem.

Degree of a differential equation: The exponent of the highest order derivative appearing in the differential equation, when all derivatives are made free from radicals and fractions, is called degree of the differential equation. In other words, it is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in derivatives.

Differential Equations and Mathematical Models

In this section, we illustrate the use of differential equations in science and engineering and in coordinate geometry through the following examples.

Application in coordinate geometry

Example: In the following problems, a function y = h(x) is described by some geometric property of its graph. Write a differential equation of the form $\frac{dy}{dx} = f(x, y)$ having the function h as its solution.

(a) Every straight line normal to the graph of h passes through the point (0,1).

(b) The line tangent to the graph of h at (x,y) passes through the point (-y , x)

(c) The graph of h is normal to every curve of the form $y = x^2 + k$, k is a constant ,where they meet.

Solution: (a) Slope of tangent at the point (x,y) = $\frac{dy}{dx}$. Then slope of the normal = $\frac{-1}{dy/dx}$

Equation of straight line passing through the point (0,1) and slope $\frac{-1}{dy/dx}$ is

$$(y-1)=\frac{-1}{dy/dx}(x-0)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{y-1}.$$

Thus, the equation of the normal passes through the point (0,1) is $\frac{dy}{dx}=\frac{-x}{y-1}$.

(b) Slope of tangent to the graph at (x, y) = dy/dx. Equation of tangent line with slope $\frac{dy}{dx}$ and passing through the point (-y , x) is

$$y-x=\frac{dy}{dx}(x+y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-x}{y+x}$$

(c) Slope of the tangent = $m=\frac{dy}{dx}$

Slope of the normal to the curve $y=x^2+k$ is $m'=\frac{d(x^2+k)}{dx}=2x$

By condition of orthogonality, $mm'=-1 \quad \Rightarrow \frac{dy}{dx}.2x=-1 \quad \Rightarrow \frac{dy}{dx}=\frac{-1}{2x}$

Therefore, the required differential equation is $\frac{dy}{dx}=\frac{-1}{2x}$

Applications of Differential Equation in science and Engineering

Velocity: The rate of change of displacement with time is called velocity. It is given by dx/dt where x =x(t) gives the position of a moving particle at any time t .

Acceleration: The rate of change of velocity with time is called acceleration. It is given by dv/dt where v =v(t) gives the velocity of a moving particle at any time t .

Let the motion of a particle is given by the position function x = f(t)

Then velocity = $v(t)=\frac{dx}{dt}=f'(t)$ and acceleration = $a(t)=\frac{d^2x}{dt^2}=\frac{dv}{dt}$

By Newton's second law of motion,

F = ma

where F is the force, m is the mass of the particle, a is the acceleration.

Then, $F=m\frac{dv}{dt}$ or $\frac{dv}{dt}=\frac{\mathrm{F}}{\mathrm{m}}$ ....(1)

For instance, suppose that the force F, and therefore acceleration a = F/m are constant.

Then (1) gives $\frac{dv}{dt}=\mathrm{a}$.

Integrating both sides we get

$v=at+c$, where c is constant of integration. ....(2)

Let $v=v_0$ at $\mathrm{t}=0$. Then (2) gives $c=v_0$.

Put this value of c in (2) we get

$$v=at+v_0 .$$

This is the velocity function.

Now, put $v=\frac{dx}{dt}$ in it we get

$\frac{dx}{dt}=at+v_0 \quad \Rightarrow \mathrm{dx}=(at+v_0)dt$ ......(3)

Integrating (3) on both sides we get

$x(t)=\frac{1}{2}at^2+v_0t+k,$ *where* k is a constant of integration.

Put $x=x_0$ at $\mathrm{t}=0$ in the above equation we get k = x_0

Then, $x(t)=\frac{1}{2}at^2+v_0t+x_0$ is the position of the particle at any time t.

Example A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

Solution: We are given $x_0 = 400, v_0 = 0, a = -32\,ft/s^2$ (acceleration is negative because height is decreasing).When the ball strikes the ground, x = 0

We know that $x = \frac{1}{2}at^2 + v_0 t + x_0$

$$0 = \frac{1}{2}(-32)t^2 + 0 \times t + 400$$

$\Rightarrow t = \frac{400}{16} = 5$ sec. Therefore, it will take 5 seconds to reach the ground.

We have $v = v_0 + at$

$\Rightarrow v = 0 - 32 \times 5 = -160$ ft/s. Therefore, the ball will strike the ground with a velocity of 160 ft/s.

Example Find the velocity function v(t) and position function x(t) of a moving particle with the given acceleration a(t), initial position x_0 =x(0), and initial velocity v_0 = v(0) where $a(t) = 50, v_0 = 10, x_0 = 20$

Solution: We know that $a(t) = \frac{dv}{dt}$ ....(1)

Put a(t)=50 in (1) we get $\frac{dv}{dt} = 50$ ....(2)

We rewrite (2) as dv=50dt ....(3)

Integrating both sides of (3) we get

$\int dv = 50\int dt$

$v = 50t + c$ where c is a constant

Put $v_0 = 10$ i.e., v =10 at t = 0 we get

10 = 50(0)+ c or c = 10

Then v = 50 t +10 is the velocity function.

Also, $v = \frac{dx}{dt}$.

Put v = 50 t +10 in it we get

$$\frac{dx}{dt} = 50t + 10$$

$$\Rightarrow dx = (50t + 10)dt \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

Integrating (4) on both sides, we get

$$\int dx = \int (50t + 10)dt$$

$$\Rightarrow x = 25t^2 + 10t + c \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Put $x_0 = 20$ i.e., x = 20 at t =0 in (5) we get c = 20.
Then (5) gives $x = 25t^2 + 10t + 20$ as the required position function.

Example Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $\frac{dv}{dt} = kv^2$. The initial speed of the motorboat is v(0)=10 m/s and v is decreasing at the rate of 1 m/s^2 when v = 5 m/s. How long does it take for the velocity of the boat to decrease to 1 m/s ? To 1/10 m/s? When does the boat come to a stop?

Solution: We are given that $\frac{dv}{dt} = kv^2$ ......(1)

$$\Rightarrow \frac{dv}{v^2} = kdt \qquad \ldots\ldots.(2)$$

Integrating both sides of (2) we get

$$\int \frac{dv}{v^2} = k\int dt$$

$\Rightarrow -\frac{1}{v} = kt + c$, where c is the constant of integration. ....(3)

Put v(0) = 10 i.e., v = 10 at t = 0 in (3), we get

$$\Rightarrow -\frac{1}{10} = k(0) + c \qquad \Rightarrow c = -\frac{1}{10}$$

Put this value of c in (3) , we get

$$-\frac{1}{v} = kt - \frac{1}{10} \qquad \ldots\ldots..(4)$$

Since v is decreasing at the rate of 1 m/s^2 when v = 5, it means

$$\frac{dv}{dt} = -1 \text{ when } v = 5.$$

Put these values in (1) we get $1 = k(5)^2 \quad \Rightarrow k = \frac{-1}{25}$

Put this value of k in (4) we get

$$-\frac{1}{v} = \frac{-1}{25}t - \frac{1}{10} \qquad \text{........(5)}$$

Now we find t when v = 1. For this put v=1 in (5) we get

$$-\frac{1}{1} = \frac{-1}{25}t - \frac{1}{10} \qquad \Rightarrow t = 22.5$$

Therefore, the motorboat will take 22.5 seconds for the velocity of the boat to decrease to 1 m/s.

Now put v = 1/10 in (5), we get

$$-\frac{1}{1/10} = \frac{-1}{25}t - \frac{1}{10} \qquad \Rightarrow t = 247.5$$

Therefore, the motorboat will take 247.5 seconds for the velocity of the boat to decrease to 1/10 m/s.

The boat comes to stop when v $\rightarrow 0$.
It is clear from (5) that when v $\rightarrow 0$ then $t \to \infty$. It means that v(t) approaches zero as t increases without bound.

Example Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

Solution: When the car skids 15m while moving at 50 km/h and the brakes are applied ,then $x(t) = \frac{15}{1000} km, x_0 = 0, v_0 = 50, v = 0, a = ?$

Now, $v = v_0 + at \quad \Rightarrow 0 = 50 + at \quad \Rightarrow at = -50 \quad$ ....(1)

Also, $x = \frac{1}{2}at^2 + v_0 t + x_0$

$$\Rightarrow \frac{15}{1000} = \frac{1}{2}(-50)t + 50t + 0$$

$$\Rightarrow t = 6 \times 10^{-4}$$

Put the value of t in (1) we get $a = \frac{-50}{6 \times 10^{-4}} = -83333.3$

Now if the car is moving with a speed of 100 km/h when the brakes are applied then,

$$v = 0, v_0 = 100, a = -83333.3, x_0 = 0$$

Then, $v = v_0 + at \Rightarrow 0 = 100 - 83333.3t \Rightarrow t = 1.2 \times 10^{-3}$

Now, $x = \frac{1}{2}at^2 + v_0 t + x_0$

$$\Rightarrow x = \frac{1}{2} \times (-83333.3) \times \left(1.2 \times 10^{-3}\right)^2 + 100 \times \left(1.2 \times 10^{-3}\right) + 0$$

$$\Rightarrow x = 0.061km = 61m .$$

Example A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

Solution: When a stone is dropped from rest at an initial height h above the surface of the earth, then $v_0 = 0, x_0 = 0, a = g, x = h, v = ?$

Now, $v = v_0 + at \Rightarrow v = 0 + gt$ ....(1)

Also, $x = \frac{1}{2}at^2 + v_0 t + x_0$

$$\Rightarrow h = \frac{1}{2} \times g \times t^2 + 0 \times t + 0$$

$$\Rightarrow t^2 = \frac{2h}{g} \qquad \Rightarrow t = \sqrt{\frac{2h}{g}}$$

Put this value in (1) we get $v = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}$

Hence proved.

Growth and Decay

Natural Growth Equation: The differential equation $\frac{dx}{dt} = kx,\ x(t) > 0,\ k > 0$ is called a natural growth equation or exponential equation.

Natural Decay equation: The differential equation $\frac{dx}{dt} = kx,\ x(t) > 0,\ k < 0$ is called a natural decay equation.

Population growth: Let P(t) be the population having constant birth and death rates. Then the time rate of change of population P(t) is proportional to the size of the population. Then, we have

$$\frac{dP}{dt} = kP, \text{ where } k \text{ is a constant of proportionality.}$$

$$\Rightarrow \frac{dP}{P} = kdt \qquad \text{..........(1)}$$

Integrating (1) on both sides , we get

$$\int \frac{dP}{P} = k\int dt$$

$$\Rightarrow \log P = kt + c, \text{ where c is the constant of integration.} \qquad \text{.........(2)}$$

Let the population be P_0 initially. It means $P(0)=P_0$ i.e., $P=P_0$ at $t = 0$.

Put this value in (2) we get $\log P_0 = c$.

Then (2) gives $\log P = kt + \log P_0 \Rightarrow P = P_0 e^{kt}$. This is the population at any time t if the initial population is P_0.

Solution of a differential equation:

It is a relation between the variables involved in the differential equation which satisfies the differential equation. Such a relation when substituted in the differential equation with its derivatives, makes left hand side and right hand side identically equal.

Example1: $\frac{dy}{dx} = 2y$ is a differential equation which involves an independent variable x, dependent variable y, first derivative of y with respect to x. This equation involves the unknown function y of the independent variable x and first derivative $\frac{dy}{dx}$ of y w.r.t. x

Example2: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ is a differential equation which consists of an unknown function y of the independent variable x and the first two derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of y w.r.t. x.

Example3: In the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx} + 3\right)^4 = 0$, the order of the highest order derivative is 3, so it is a differential equation of order 3.

Example 4: In the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 - 6\left(\frac{d^2y}{dx^2}\right)^4 - 4y = 0$, the highest order derivative is $\frac{d^3y}{dx^3}$ and its exponent or power is 2. So, it is a differential equation of order 3 and degree 2.

Example 5: Consider the differential equation $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\left(c\frac{d^2y}{dx^2}\right)^{1/3}$. To express the differential equation as a polynomial in derivatives, we proceed as follows:

Squaring both sides, we get

$$1+\left(\frac{dy}{dx}\right)^2=\left(c\frac{d^2y}{dx^2}\right)^{2/3}$$

Cubing both sides , we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(c\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^6+3\left(\frac{dy}{dx}\right)^2+3\left(\frac{dy}{dx}\right)^4=c^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow c^2\left(\frac{d^2y}{dx^2}\right)^2-\left(\frac{dy}{dx}\right)^6-3\left(\frac{dy}{dx}\right)^4-3\left(\frac{dy}{dx}\right)^2-1=0$$

Now, the highest order derivative appearing in the polynomial form of the given differential equation is $\frac{d^2y}{dx^2}$. Its exponent is 2. Therefore, degree of the given differential equation is 2. Infact, its order is also 2.

Example6: Consider a differential equation $y' + 2y = 0$ where $y' = \frac{dy}{dx}$. Then it can be easily verified that $y = 3e^{-2x}$ is the solution of the given differential equation by proceeding as follows.

Differentiating $y = 3e^{-2x}$ w.r.t. x, we get

$$y' = -6e^{-2x}$$

Substituting the values of y *and* y' in the L.H.S of the given differential equation, we get

$$L.H.S = y' + 2y = -6e^{-2x} + 2\left(3e^{-2x}\right) = -6e^{-2x} + 6e^{-2x} = 0 = R.H.S$$

$\therefore y = 3e^{-2x}$ satisfy the given differential equation and thus is a solution of it.

Example7: Consider a differential equation $y'' + y = 3\cos 2x$. Then $y = \cos x - \cos 2x$ is the solution of this differential equation. It can be seen as follows.

We have

$$y = \cos x - \cos 2x \qquad \text{........(1)}$$

Differentiating (1) w.r.t. x on both sides , we get

$$y' = -\sin x + 2\sin 2x \qquad \text{.........(2)}$$

Differentiating (2) w.r.t. x we get

$$y'' = -\cos x + 4\cos 2x$$

Substituting the values of y and y'' in the L.H.S of the given differential equation $y'' + y = 3\cos 2x$, we get

L.H.S = -cosx +4 cos2x+cosx –cos2x

= 3 cos2x = R.H.S

Therefore, $y = \cos x - \cos 2x$ is the solution of the given differential equation.

One more thing to be noted here is that y = sinx - cos2x is also a solution of the given differential equation.

It can be seen as follows. We have

$$y = \sin x - \cos 2x \quad ....(3)$$

Differentiating (3) w.r.t. x , we get

$$y' = \cos x + 2\sin 2x \quad ....(4)$$

Differentiating (4) w.r.t x , we get

$$y'' = -\sin x + 4\cos 2x$$

Substituting the values of y and y'' in the L.H.S of the given differential equation $y'' + y = 3\cos 2x$, we get

L.H.S = -sinx + 4 cos2x +sinx – cos2x

= 3cos2x = R.H.S

Therefore, y = sinx - cos2x is also a solution of the given differential equation.

Example8: Substitute $y = e^{rx}$ in to the following differential equation to determine all values of the constant *r* for which $y = e^{rx}$ is the solution of the equation $3y'' + 3y' - 4y = 0$.

Solution: Consider $3y'' + 3y' - 4y = 0$ ....(1)

We have

$$y = e^{rx} \quad ....(2)$$

Differentiating (2) w.r.t. x, we get

$$y' = re^{rx}$$

Again differentiating w.r.t. x, we get

$$y'' = r^2 e^{rx}$$

Substituting the values of y, y' and y'' in the given differential equation, we get

$$3r^2 e^{rx} + 3re^{rx} - 4e^{rx} = 0$$

$$\Rightarrow e^{rx}\left(3r^2 + 3r - 4\right) = 0$$

$$\Rightarrow 3r^2 + 3r - 4 = 0 \quad \because e^{rx} \neq 0 \text{ forany real value of } r$$

$$\Rightarrow r = \frac{-3 \pm \sqrt{57}}{6}$$

Therefore, e^{rx} is the solution of (1) for $r = \frac{-3 \pm \sqrt{57}}{6}$.

Example 9: If k is a constant , show that a general (1–parameter)solution of the differential equation $\frac{dx}{dt} = kx^2$ is given by $x(t) = \frac{1}{C - kt}$ where C is an arbitrary constant.

Solution: We have $x(t) = \frac{1}{C - kt}$ ....(1)

Differentiating both sides of (1) we get

$$\frac{dx}{dt} = \frac{k}{(C - kt)^2}$$

Put this value in the L.H.S of $\frac{dx}{dt} = kx^2$, we get

$$L.H.S = \frac{dx}{dt} = \frac{k}{(C - kt)^2} = kx^2 = R.H.S.$$

Therefore, $x(t) = \frac{1}{C - kt}$ is the solution of the differential equation.

Actually, $x(t) = \frac{1}{C - kt}$ defines a one parameter family of solution of $\frac{dx}{dt} = kx^2$, one for each value of the arbitrary constant or parameter C.

Integrals as General, Particular and Singular Solutions

General solution: A solution which contains as many arbitrary constants as the order of the differential equation is called a general solution of the differential equation.

Particular solution: A solution obtained by giving particular values to the arbitrary constants in the general solution of the differential equation is called a particular solution.

Singular Solution: A solution which cannot be obtained from the general solution by any choice of the arbitrary constants is called a singular solution.

Example 10: Consider a differential equation $\frac{dy}{dx} = 2\sqrt{y}$ .....(1).

We can rewrite (1) as $\frac{dy}{\sqrt{y}} = 2dx$ .....(2)

Integrating both sides of (2), we get

$y = (x+c)^2$, c is a constant of integration. ....(3)

This solution contains one arbitrary constant c .This is the general solution as it contains only one arbitrary constant which is same as the order of the given differential equation.

If we put the initial condition y(0)=0 , i.e, y = 0 at x=0 in (3) then we get c = 0. In such a case $y = x^2$ is a particular solution.

Evidently, y = 0 is also a solution of (1) but it cannot be obtained from (3) by any choice of c. Thus the function y = 0 is a singular solution of (1).

Example 11: Solve the initial value problem $\frac{dy}{dx} = x\sqrt{x^2+9}, y(-4) = 0$.

Solution: We have $\frac{dy}{dx} = x\sqrt{x^2+9}$ ....(1)

$\Rightarrow dy = x\sqrt{x^2+9}dx$ ....(2)

Integrating both sides of (2) we get

$$\int dy = \int x\sqrt{x^2+9}dx \qquad ....(3)$$

Let $I = \int x\sqrt{x^2+9}dx$

Put $x^2+9=t \quad \Rightarrow \quad 2xdx=dt$

Then $I=\frac{1}{2}\int\sqrt{t}dt=\frac{1}{2}t^{3/2}.\frac{2}{3}+c=\frac{1}{3}t^{3/2}+c=\frac{1}{3}\left(x^2+9\right)^{3/2}+c$, where c is a constant.

Now, from (3) we get $y=\frac{1}{3}\left(x^2+9\right)^{3/2}+c$ ....(4)

Put y(-4)=0 i.e. x= -4 and y = 0 in (4) we get

$$0=\frac{1}{3}(16+9)^{3/2}+c \quad \Rightarrow c=\frac{-125}{3}.$$

Put this value of c in (4) we get

$$y=\frac{1}{3}\left(x^2+9\right)^{3/2}-\frac{125}{3}.$$

It is the required solution.

Example 12: Find the general solutions of the following differential equations.

(a) $(1-x^2)\frac{dy}{dx}=2y$

(b) $\frac{dy}{dx}=\frac{(x-1)y^5}{x^2\left(2y^3-y\right)}$

(c) $x^2\frac{dy}{dx}=1-x^2+y^2-x^2y^2$

Solution: (a) We have $(1-x^2)\frac{dy}{dx}=2y$

$$\Rightarrow \frac{dy}{2y}=\frac{dx}{(1-x^2)} \qquad ....(1)$$

Integrating (1) on both sides, we get

$$\int\frac{dy}{2y}=\int\frac{dx}{(1-x^2)}$$

$$\Rightarrow \frac{1}{2}\log y=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)+\log c$$

$$\Rightarrow y=c\left(\frac{1+x}{1-x}\right)$$ is the general solution.

(b)We have $\frac{dy}{dx}=\frac{(x-1)y^5}{x^2(2y^3-y)}$

$$\Rightarrow \frac{(2y^3-y)dy}{y^5}=\frac{(x-1)dx}{x^2}.$$

$$\Rightarrow \left(\frac{2}{y^2}-\frac{1}{y^4}\right)dy=\left(\frac{1}{x}-\frac{1}{x^2}\right)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Integrating (1) on both sides we get

$$\int\left(\frac{2}{y^2}-\frac{1}{y^4}\right)dy=\int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$\Rightarrow \frac{-2}{y}+\frac{1}{3y^3}=\log|x|+\frac{1}{x}+c$, where c is constant of integration.

(c) We have $x^2\frac{dy}{dx}=1-x^2+y^2-x^2y^2$

$$\Rightarrow x^2\frac{dy}{dx}=(1-x^2)+y^2(1-x^2)$$

$$\Rightarrow x^2\frac{dy}{dx}=(1-x^2)(1+y^2)$$

$$\Rightarrow \frac{dy}{1+y^2}=\frac{(1-x^2)}{x^2}dx$$

$$\Rightarrow \frac{dy}{1+y^2}=\left(\frac{1}{x^2}-1\right)dx$$

Integrating both sides, we get

$$\tan^{-1}y=-\frac{1}{x}-x+c$$

$\Rightarrow y=\tan\left(-\frac{1}{x}-x+c\right)$, where c is a constant of integration.

Example 13: Find the explicit particular solution of the initial value problem $2y\frac{dy}{dx}=\frac{x}{\sqrt{x^2-16}}, y(5)=2$.

Solution: We have $2y\frac{dy}{dx}=\frac{x}{\sqrt{x^2-16}}$

$$\Rightarrow 2ydy=\frac{x}{\sqrt{x^2-16}}dx$$

Integrating on both sides we get

$$\int 2ydy=\int\frac{x}{\sqrt{x^2-16}}dx \qquad \text{...........(1)}$$

Let I = $\int\frac{x}{\sqrt{x^2-16}}dx$

Put $x^2-16=t \qquad \Rightarrow 2xdx=dt$

$$I=\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\sqrt{t}+c=\sqrt{x^2-16}+c$$

From (1) we get,

$$y^2=\sqrt{x^2-16}+c \qquad \text{.............(2)}$$

Put y(5)=2 in (2) i.e., y = 2 at x =5 .

$$4=\sqrt{(5)^2-16}+c \qquad \Rightarrow c=1$$

$$y^2=\sqrt{x^2-16}+1$$

POSSIBLE QUESTIONS

PART - B (5 x 2 = 10 Marks)

1. Define Differential equation with example.
2. Define Partial Differential equation with example.
3. Expalin linear differential equation.
4. Explain singular solutions of the differential equation.
5. Explain the order of the differential equation with example

PART – C (5 x 6 = 30 Marks)

1. Show that $5x^2y^2 - 2x^3y^2 = 1$ is an implicit solution of the differential equation $x\frac{dy}{dx} + y = x^3y^3$ on the interval
   0< x < 5/2.
2. Write the definition of general, particular, explicit, implicit and singular solutions of Differential equations.
3. Show that every function f defined by $f(x) = (x^3 + c)e^{-3x}$ where c is arbitrary equation is a solution of the
   Differential equation $\frac{dy}{dx} + 3y = 3\,x^2e^{-3x}$.
4. Show that the function f defined by f(x)=$3e^{2x} - 2xe^{2x} - cos2x$ satisfies the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + \quad 4y = -8\sin 2x$ and also the condition that f(0)=2 and f'(0)=4
5. Write a note on solution of differential equations.
6. Show that the function for all x by f(x)= 2 sin x +3cos x is an explicit solution of the
7. Differential equation $\frac{d^2y}{dx^2} + y = 0$ for all real x.
8. Show that the function defined by f(x) = x + 3e^{-x} is a solution of differential equation $\frac{dy}{dx}$ + y = x + 1 on every interval a < x < b of the x-axis.
9. Briefly explain linear and nonlinear differential equations with examples.
10. Find the general solutions of the differential equations (1- x^2)$\frac{dy}{dx} = 2y$.
11. Show that $x^3 + 3xy^2$ is an implicit solution of the differential equation 2xy ($\frac{dy}{dx}$) + x^2 +y^2=0 on the interval 0 < x < 1.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
An equation involving one or more dependent variables with respect to one or more independent variables is called……………………..	differential equations	intergral equation	constant equation	Eulers equation	differential equations
An equation involving one or more ……….. variables with respect to one or more independent variables is called differential equations	single	dependent	independent	constant	dependent
An equation involving one or more dependent variables with respect to one or more……….variables is called differential equations	dependent	independent	single	different	independent
A differential equation involving ordinary derivatives of one or moredependentvariables with respect to single independent variables is called ……………	differential equations	partial differential equations	ordinary differential equations	total differential equations	ordinary differential equations
A differential equation involving ordinary derivatives of one or more dependentvariables with respect to………. independent variables is called ordinary differential equations	zero	single	different	one or more	single
A differential equation involving ………… derivatives of one or more dependentvariables with respect to single independent variables is called ordinary differential equations	partial	different	total	ordinary	ordinary
A differential equation involving partial derivatives of one or more dependent variables with respect to oneor more independent variables is called ……………	differential equations	partial differential equations	ordinary differential equations	total differential equations	partial differential equations
A differential equation involving partial derivatives of one or more dependentvariables with respect to………. independent variables is called partial differential equations	zero	single	different	one or more	oneormore
A differential equation involving ………… derivatives of one or more dependentvariables with respect to one or moreindependent variables is called partial differential equations	partial	different	total	ordinary	partial
The order of ……...derivatives involvedin the differential equations is called order of the differential equation	zero	lowest	highest	infinite	highest
The order of highest derivatives involvedin the differential equations is called ……………... of the differential equation	order	power	value	root	order
The order of highest ………………... involvedin the differential equations is called order of the differential equation	derivatives	intergral	power	value	derivatives
The order of the differential equations is (d^2 y)/ dx ^2 +xy(dy/dx)^2=1	0	1	2	4	2
A non linear ordinary differential equation is an ordinary differential equation that is not …………	linear	non linear	differential	intergral	linear
A ……………..ordinary differential equation is an ordinary differential equation that is not linear	linear	non linear	differential	intergral	non linear
A non linear ordinary differential equation is an ……………. differential equation that is not linear	ordinary	partial	single	constant	ordinary
…………... ordinary differential equations are further classified according to the nature of the coefficients of the dependent variables and its derivatives	linear	non linear	differential	intergral	linear
Linear ………………. differential equations are further classified according to the nature of the coefficients of the dependent variables and its derivatives	ordinary	partial	single	constant	ordinary

Linear ordinary differential equations are further classified according to the nature of the coefficients of the ……………….variables and its derivatives	single	dependent	independent	constant	dependent
Linear ordinary differential equations are further classified according to the nature of the coefficients of the dependent variables and its ……………………..	integrals	constant	derivatives	roots	derivatives
Both explicit and implicit solutions will usually be called simply ……………	solutions	constant	equations	values	solutions
Both ……………….. solutions will usually be called simply solutions.	general and particular	singular and non singular	ordinary and partial	explicit and implicit	explicit and implicit
Let f be a real function defined for all x in a real interval I and having nth order derivatives then the function f is called ………….solution of the differential equations	constant	implicit	explicit	general	explicit
Let f be a real function defined for all x in a real interval I and having …………..order derivatives then the function f is called explicit solution of the differential equations	1st	2nd	nth	(n+1)th	nth
The relation g(x,y)=0 is called the ………solution of F[x,y,(dy/dx)………..(dy/dx)^n]=0	constant	implicit	explicit	general	implicit

UNIT – II

SYLLABUS

Exact differential equations and integrating factors, separable equations and equations reducible to this form, linear equation and Bernoulli equations, special integrating factors and transformations.

2.1 Separable Variables

Definition 2.1: A first order differential equation of the form

$$\frac{dx}{dy} = g(x)h(y), \text{ where } g(x) \text{ and } h(y) \text{ are functions of } x \,\&\, y \text{ only, respectively.}$$

is called **separable** or to have **separable variables**.

Method or Procedure for solving separable differential equations

(i) If h(y) = 1, then

$$\frac{dy}{dx} = g(x)$$

or $dy = g(x)\,dx$

Integrating both sides we get

$$\int dy = \int g(x)d(x) + c$$

or

$$y = \int g(x)d(x) + c$$

where c is the constant of integral

We can write

$$y = G(x) + c$$

where *G(x)* is an anti-derivative (indefinite integral) of *g(x)*

(ii) Let $\frac{dy}{dx} = f(x, y)$

where $f(x, y) = g(x)h(y)$,

that is f(x,y) can be written as the product of two functions, one function of variable x and other of y. Equation

$$\frac{dy}{dx} = g(x)h(y)$$

can be written as

$$\frac{1}{h(y)}dy = g(x)dx$$

By integrating both sides we get

$$\int p(y)dy = \int g(x)dx + C$$

where $p(y) = \frac{1}{h(y)}$

or $H(y) = G(x) + C$

where *H(y)* and *G(x)* are anti-derivatives of $p(y) = \frac{1}{h(y)}$ and $g(x)$, respectively.

Example 2.1: Solve the differential equation

$$y' = y/x$$

Solution: Here $g(x) = \frac{1}{x}$, $h(y) = y$ and $p(y) = \frac{1}{y}$

$H(y) = \ln y$, $G(x) = lnx$

Hence

H(y) = G(x) + C

or lny = lnx + lnc **(See Appendix)**

lny – lnx = lnc

$$\ln\frac{y}{x} = \ln c$$

$$\frac{y}{x} = c$$

y = cx

Example 2.2: Solve the initial-value problem

$$\frac{dy}{dx} = -\frac{x}{y}, y(4) = 3$$

Solution: g(x) = x, h(y) = -1/y, p(y) = -y

H(y) = G(x) + c

$$-\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

or $y^2 = -x^2 - 2c$

or $x^2 + y^2 = c_1^2$

where $c_1^2 = -2c$

By given initial-value condition

$16+9 = c_1^2$

or $c_1 = \pm 5$

or $x^2 + y^2 = 25$

Thus the initial value problem determines

$$x^2 + y^2 = 25$$

Example 2.3: Solve the following differential equation

$$\frac{dy}{dx} = \cos 5x$$

Solution: dy = cos5xdx

Integrating both sides we get

$$\int dy = \int \cos 5x dx + c$$

$$y = \frac{\sin 5x}{5} + c$$

2.2 Exact Differential Equations

We consider here a special kind of non-separable differential equation called an **exact differential equation.** We recall that the **total differential** of a function of two variables U(x,y) is given by

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \qquad (2.1)$$

Definition 2.2.1 : The first order differential equation

M(x,y)dx + N(x,y)dy=0 *(2.2)*

is called an **exact differential equation** if left hand side of (2.2) is the total differential of some function U(x,y).

Remark 2.2.1: (a) It is clear that a differential equation of the form (2.2) is exact if there is a function of two variables U(x,y) such that

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = M(x,y)dx + N(x,y)dy$$

or $$\frac{\partial U}{\partial x} = M(x,y), \qquad \frac{\partial U}{\partial y} = N(x,y)$$

(b) Let M(x,y) and N(x,y) be continuous and have continuous first derivatives in a rectangular region R defined by a<x<b, c<y<d. Then a necessary and sufficient condition that M(x,y)dx + N(x,y)dy be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (2.3)$$

For proof of Remark 2.2.1(a) see solution of Exercise 22 of this chapter.

Procedure of Solution 2.2:

Step 1: Check whether differential equation written in the form (2.2) satisfies (2.3) or not.

Step 2: If for given equation (2.3) is satisfied then there exists a function *f* for

which

$$\frac{\partial f}{\partial x} = M(x, y) \qquad (2.4)$$

Integrating (2.4) with respect to x, while holding y constant, we get

$$f(x, y) = \int M(x, y)dx + g(y) \qquad (2.5)$$

where the arbitrary function g(y) is constant of integration.

Step 3: Differentiate (2.5) with respect to y and assume $\frac{\partial f}{\partial y}$ =N(x,y), we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)dx + g'(y) = N(x, y)$$

or

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx \qquad (2.6)$$

Step 4: Integrate (2.6) with respect to y and substitute this value in (2.5) to obtain f(x,y)=c, the solution of the given equation.

Remark 2.2.2: (a) Right hand side of (2.6) is independent of variable x, because

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x, y)dx\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

(b) We could just start the above mentioned procedure with the assumption that

$$\frac{\partial f}{\partial y} = N(x, y)$$

By integrating N(x,y) with respect to y and differentiating the resultant expression, we would find the analogues of (2.5) and (2.6) to be, respectively,

$$f(x,y) = \int N(x,y)dy + h(x) \quad \text{and}$$

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy$$

Example 2.4: Check whether $x^2y^3dx + x^3y^2dy = 0$ is exact or not?

Solution: In view of Remark 2.2.1(b) we must check whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, where $M(x,y)= x^2y^3$, $N(x,y)=x^3y^2$

$$\frac{\partial M}{\partial y} = 3x^2y^2, \quad \frac{\partial N}{\partial x} = 3x^2y^2$$

This shows that $\frac{\partial M}{\partial y} = 3x^2y^2 = \frac{\partial N}{\partial x}$

Hence the given equation is exact.

Example 2.5: Determine whether the following differential equations are exact. If they are exact solve them by the procedure given in this section.

(a) $(2x-1)dx + (3y+7)dy=0$

(b) $(2x+y)dx - (x+6y)dy=0$

(c) $(3x^2y+e^y)dx + (x^3+xe^y-2y)dy=0$

Solution of (a) $M(x,y) = 2x-1$, $N(x,y)=3y+7$

$$\frac{\partial M}{\partial y} = 0, \qquad \frac{\partial N}{\partial x} = 0. \quad \text{Thus}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the given equation is exact.

Apply procedure of solution 2.2 for finding the solution.

Put $\frac{\partial f}{\partial x} = 2x - 1.$ Integrating and choosing h(y) as the constant of integration we get

$$\int \frac{\partial f}{\partial x} = f(x,y) = x^2 - x + h(y)$$

$h'(y) = N(x, y) = 3y + 7,$ and by integrating with respect to y we obtain

$$h(y) = \frac{3}{2}y^2 + 7y$$

The solution is

$$f(x, y) = x^2 - x + \frac{3}{2}y^2 + 7y = c$$

Solution of (b): It is not exact as

$$M(x, y) = 2x + y, N(x, y) = -x - 6y$$

and $\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1$

Solution of (c): $M(x,y) = 3x^2y + e^y$

$$N(x,y) = x^3 + xe^y - 2y$$

$$\frac{\partial M}{\partial y} = 3x^2 + e^y$$

$$\frac{\partial N}{\partial x} = 3x^2 + e^y$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$ that is

the equation is exact.

Apply procedure of solution 2.2

Let $\frac{\partial f}{\partial x} = 3x^2y + e^y$

Integrating with respect to x, we obtain

$$f(x, y) = x^3y + xe^y + g(y)$$

where g(y) is a constant of integration

Differentiating with respect to y we obtain

$$\frac{\partial f}{\partial y} = x^3 + xe^y + g'(y)$$

This gives $$N(x,y) = \frac{\partial f}{\partial y} = x^3 + xe^y + g'(y)$$

or g'(y) = $-2y$

or g(y) = $-y^2$

Substituting this value of g(y) we get

$f(x,y) = x^3y + xe^y - y^2 = c$. Thus

$x^3y + xe^y - y^2 = c$ is the solution of the given differential equation.

2.2.1 Equations Reducible to Exact Form

There are non-exact differential equations of first-order which can be made into exact differential equations by multiplication with an expression called an integrating factor. Finding an integrating factor for a non-exact equation is equivalent to solving it since we can find the solution by the method described in Section 2.2. There is no general rule for finding integrating factors for non-exact equations. We mention here two important cases for finding integrating factors. It may be seen from examples given below that integrating factors are not unique in general.

Computation of Integrating Factor

Let M(x.y)dx+N(x,y)dy=0

be a non-exact equation.

Then

(i) $$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

is an integrating factor, where M_y, N_x are partial derivatives of M and N with respect to y and x and $\frac{M_y - N_x}{N}$ is a function of x alone.

(ii) $$\mu(x) = e^{\int \frac{N_x - M_y}{M} dy}$$

is an integrating factor, where M_y and N_x are as in the case (i) and $\frac{N_x - M_y}{M}$ is a function of y alone.

Example 2.6: (a) Let us consider non-exact differential equation.

$(x^2/y)\, dy+2xdx=0$

$\frac{1}{x^2}$ and y are integrating factors of this equation.

(b) e^x is an integrating factor of the equation

$$\frac{dy}{dx} + y = x$$

Example 2.7: Solve the differential equation of the first-order:

$xydx+(2x^2+3y^2-20)dy=0$

Solution: $M(x,y)=xy$, $N(x,y)=2x^2+3y^2-20$

$M_y=x$ and $N_x=4x$. This shows that the differential equation is not exact.

$$\frac{M_y - N_x}{N} = \frac{-3x}{2x^2+3y^2-20}$$

leads us nowhere, as $\frac{M_y - N_x}{N}$ is a function of both x and y. However,

$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y}$ is a function of y only. Hence

$e^{\int 3\frac{dy}{y}} = e^{3\ln y} = e^{\ln y^3} = y^3$ is an integrating factor.

After multiplying the given differential equation by y^3 we obtain

$xy^4dx+(2x^2y^3+3y^5-20y^3)dy=0$

This is an exact differentiation equation. Applying the method of the previous section we get

$$\frac{1}{2}x^2y^4 + \frac{3}{6}y^6 - 5y^4 = C$$

Example 2.8: Solve the following differential equation:

$(2y^2+3x)dx+2xydy=0$

Solution: The given differential equation is not exact, that is

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, where

$M(x,y)=2y^2+3x$

$N(x,y)=2xy$

$(M_y-N_x)/N = 1/x$ is a function of x only.

Hence $e^{\int dx/x} = x$ is an integrating factor.

By multiplying the given equation by x we get $(2y^2x+3x^2)dx+2x^2ydy=0$

This is an exact equation as

$$\frac{\partial}{\partial y}(2y^2x + 3x^2) = \frac{\partial}{\partial x}(2x^2y)$$

Applying the method for solving exact differential equation, we get $f=x^2y^2+x^3+h(y)$, $h'(y)=0$, and $h(y)=c$ if we put $f_x=2xy^2+3x^2$. The solution of the differential equation is $x^2y^2+x^3=c$.

2.3 Linear Equations

Definition 2.3.1: A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is called a **linear equation.**

if $a_1(x) \neq 0$, we can write this differential equation in the form

$$\frac{dy}{dx} + P(x)y = f(x) \qquad (2.7),$$

where $P(x) = \dfrac{a_0(x)}{a_1(x)}$, $f(x) = \dfrac{g(x)}{a_1(x)}$

(2.7) is called the standard form of a linear differential equation of the first order

Definition 2.3.2: $e^{\int P(x)dx}$ is called the integrating factor of the standard form of a linear differential equation (2.7).

Remark 2.3.1: (a) A linear differential equation of first order can be made exact by multiplying with the integrating factor. Finding the integrating factor is equivalent to solving the equation.

(b) Variation of parameters method is a procedure for finding a particular solution of 2.7. For details of **variation of parameters method** see the solution of Exercise 39 of this chapter.

Procedure of Solution 2.3:

Step 1: Put the equation in the standard form (2.7) if it is not given in this form.

Step 2: Identify P(x) and compute the integrating factor I(x) = $e^{\int P(x)dx}$

Step 3: Multiply the standard form by I(x).

Step 4: The solution is

$$y.I(x) = \int f(x).I(x)dx + c$$

Example 2.9: Find the general solution of the following differential equations:

(a) $\dfrac{dy}{dx} = 8y$ (b) $x\dfrac{dy}{dx} + 2y = 3$

(c) $x\dfrac{dy}{dx} + (3x+1)y = e^{-3x}$

Solution: (a) $\dfrac{dy}{dx} - 8y = 0$

P(x)= – 8

Integrating function = $I(x) = e^{\int -8dx} = e^{-8x}$

$$y.e^{-8x} = \int 0.e^{-8x}dx + c$$

or $y = ce^{8x}$, $-\infty<x<\infty$

(b) $\frac{dy}{dx} + \frac{2}{x}y = \frac{3}{x}$

Integrating factor = $I(x) = e^{\int P(x)dx}$, where

$$P(x) = \frac{2}{x}$$

$$I(x) = e^{\int \frac{2}{x}dx} = x^2$$

Solution is given by

$$y.I(x) = \int f(x).I(x)dx + c$$

where $I(x) = x^2$, $f(x) = \frac{3}{x}$

Thus

$$yx^2 = \int \frac{3}{x}.x^2dx + c$$

$$= \int 3xdx + c = \frac{3}{2}x^2 + c$$

or $y = \frac{3}{2} + \frac{c}{x^2}$, $0<x<\infty$

(c) Standard form is

$$\frac{dy}{dx} = \left(3 + \frac{1}{x}\right)y = \frac{e^{-3x}}{x}$$

$$P(x) = 3 + \frac{1}{x}, \quad f(x) = \frac{e^{-3x}}{x}$$

Integrating factor = $I(x) = e^{\int P(x)dx}$

$= e^{\int\left(3+\frac{1}{x}\right)dx} = xe^{3x}$

$$y.xe^{3x} = \int \frac{e^{-3x}}{x}.xe^{3x}dx + c$$

$$= \int e^{0}dx + c = x + c$$

or $y = e^{-3x} + \frac{c}{x}e^{-3x}$ for $0<x<\infty$.

2.4 Solutions by Substitutions

A first-order differential equation can be changed into a separable differential equation (Definition 2.1) or into a linear differential equation of standard form (Equation (2.7)) by appropriate substitution. We discuss here two classes of differential equations, one class comprises homogeneous equations and the other class consists of Bernoulli's equation.

2.4.1 Homogenous Equations

A function f(.,.) of two variables is called homogeneous function of degree α if

$f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α.

A first order differential equation, M(x,y)dx + N(x,y)dy = 0 is called **homogenous** if both coefficients M(x,y) and N(x,y) are homogenous functions of the same degree.

Method of Solution for Homogenous Equations: A homogeneous differential equation can be solved by either substituting y=ux or x=vy, where u and v are new dependent variables. This substitution will reduce the equation to a separable first-order differential equation.

Example 2.10: Solve the following homogenous equations:

(a) (x-y)dx + xdy = 0

(b) $(y^2+yx)dx + x^2dy = 0$

Solution: (a) Let y=ux, then the given equation takes the form

(x-ux)dx + x(udx + xdu) = 0

or $dx + xdu = 0$

or $$\frac{dx}{x} + du = 0$$

or $\ln|x| + u = c$

or $x\ln|x| + y = cx$

(b) Let $y=ux$, then the given equation takes the form

$(u^2x^2 + ux^2)dx + x^2(udx + xdu) = 0$

or $(u^2 + 2u)dx + xdu = 0$

or $$\frac{dx}{x} + \frac{du}{u(u+2)} = 0$$

Solving this separable differential equation, we get

$$\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| = c$$

or $$\frac{x^2u}{u+2} = c_1$$ where $c_1=2c$

or $$x^2\frac{y}{x} = c_1\left(\frac{y}{x} + 2\right)$$

or $$x^2y = c_1(y + 2x)$$

2.4.2 Bernoulli's Equation

An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n \qquad (2.8)$$

is called a **Bernoulli's differential equation**. If $n\neq 0$ or 1, then the Bernoulli's equation (2.8) can be reduced to a linear equation of first-order by the substitution.

$$v = y^{1-n}$$

The linear equation can be solved by the method described in the previous section.

Example 2.11: Solve the following differential equations:

(a) $\frac{dy}{dx} + \frac{1}{x} y = 3y^3$

(b) $\frac{dy}{dx} - y = e^x y^2$

Solution: (a) Let $v = y^{1-n} = y^{-2}$ (n=3)

$$\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$$

or $$\frac{dy}{dx} . \frac{1}{y^3} = -\frac{1}{2} \frac{dv}{dx}$$

Substituting these values into the given differential equation, we get

$$-\frac{1}{2} \frac{dv}{dx} + \frac{1}{x} v = 3$$

or $$\frac{dv}{dx} - \frac{2}{x} v = -6$$

This equation is of the standard form, (2.7) and so the method of Section 2.3 is applicable.

Integrating factor $I(x) = e^{\int P(x)dx}$

where $P(x) = -\frac{2}{x}$. Therefore $I(x) = x^{-2}$

Solution is given by

$$v.x^{-2} = \int -6x^{-2} \, dx + c$$

or

$v.x^{-2} = 6x^{-1} + c$

or

$v = 6x + cx^2$

Since

$v = y^{-2}$ we get

$y^{-2} = 6x + cx^2$

or $y = \pm \frac{1}{\sqrt{6x + cx^2}}$

(b) Let $w = y^{-1}$, then the equation

$$\frac{dy}{dx} - y = e^x y^2$$

takes the form

$$\frac{dw}{dx} + w = -e^x$$

integrating factor $I(x) = = e^{\int P(x)dx}$, where $P(x) = 1$

or $I(x) = e^{\int P(x)dx} = e^x$

Solution is given by

$e^x.w = - \int e^{2x} dx + c$

$$= -\frac{1}{2}e^{2x} + c$$

$$or \qquad e^x \frac{1}{y} = -\frac{1}{2}e^{2x} + c$$

$$or \qquad y^{-1} = -\frac{1}{2}e^x + ce^{-x}$$

Special Integrating Factors

Given the O.D.E. M(x,y) dx + N(x,y) dy = 0, assume it is non-exact. Suppose that n(x,y) is an integrating factor of the equation, then

$$n(x,y)\,M(x,y)\,dx + n(x,y)\,N(x,y)\,dy = 0$$

is an exact equation.

Therefore,

$$\frac{\partial}{\partial y}[n(\mathbf{x},y)M(\mathbf{x},y)] = \frac{\partial}{\partial \mathbf{x}}[n(\mathbf{x},y)N(\mathbf{x},y)]$$

or

$$n(\mathbf{x},y)\frac{\partial M(\mathbf{x},y)}{\partial y} + \frac{\partial n(\mathbf{x},y)}{\partial y}M(\mathbf{x},y) = n(\mathbf{x},y)\frac{\partial N(\mathbf{x},y)}{\partial \mathbf{x}} + \frac{\partial n(\mathbf{x},y)}{\partial \mathbf{x}}N(\mathbf{x},y)$$

or

$$n(\mathbf{x},y)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial \mathbf{x}}\right] = N\frac{\partial n}{\partial \mathbf{x}} - M\frac{\partial n}{\partial y} \quad (1)$$

n(x,y) is an unknown function that satisfies equation (1), but equation (1) is a partial differential equation. So, in order to find n(x,y) we have to solve a P.D.E. and we do not know how to do it.

Therefore, we have to impose some restriction on n(x,y).

Assume that n is function of only one variable, let's say of the variable x,

then n(x) and $\frac{\partial n}{\partial y} = 0,\ \frac{\partial n}{\partial \mathbf{x}} = \frac{dn}{d\mathbf{x}}$

So, equation (1) reduces to

$$n(\mathbf{x})\left[\frac{\partial M(\mathbf{x},y)}{\partial y} - \frac{\partial N(\mathbf{x},y)}{\partial \mathbf{x}}\right] = N(\mathbf{x},y)\frac{dn}{d\mathbf{x}}$$

or

$$\frac{1}{N(\mathbf{x},y)}\left[\frac{\partial M(\mathbf{x},y)}{\partial y} - \frac{\partial N(\mathbf{x},y)}{\partial \mathbf{x}}\right]d\mathbf{x} = \frac{dn}{n}$$

If the left hand side of the above equation is only function of x, then the equation is separable and $\mathbf{n(x) = exp}\left(\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial \mathbf{x}}\right)d\mathbf{x}\right)$.

<u>Conclusion</u>: The equation M(x,y) dx + N(x,y) dy = 0 has an integrating factor n(x) that depends only on x if the expression $\frac{1}{N(\mathbf{x},y)}\left[\frac{\partial M(\mathbf{x},y)}{\partial y} - \frac{\partial N(\mathbf{x},y)}{\partial \mathbf{x}}\right]$ depends only on x.

Now, let's assume the n depends only on the variable y,

then n(y) and $\frac{\partial n}{\partial x} = 0$, $\frac{\partial n}{\partial y} = \frac{dn}{dy}$

So, equation (1) reduces to

$$n(y)\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = -M(x,y)\frac{dn}{dy}$$

or

$$-\frac{1}{M(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]dy = \frac{dn}{n}$$

or

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]dy = \frac{dn}{n}$$

If the left hand side of the above equation is only function of y, then the equation is separable and $n(y) = \exp\left(\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\right)$.

Conclusion: The equation M(x,y) dx + N(x,y) dy = 0 has an integrating factor n(y) that depends only on y if the expression $\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$ depends only on y.

Examples: Find the integrating factor

1) $(4xy + 3y^2 - x)\,dx + x(x + 2y)\,dy = 0$

$M(x,y) = 4xy + 3y^2 - x$ and $N(x,y) = x(x + 2y)$

$$\frac{\partial M(x,y)}{\partial y} = 4x + 6y \text{ and } \frac{\partial N(x,y)}{\partial x} = 2x + 2y$$

$$\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = 4x + 6y - 2x - 2y = 2x + 4y$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{1}{x(x+2y)}(2x+4y) = \frac{2(x+2y)}{x(x+2y))} = \frac{2}{x}$$

Since it depends on x, only, there is an integrating factor n(x), given by

$$n(x) = \exp\left(\int 2\frac{dx}{x}\right) = \exp(2\ln|x|) = x^2$$

Multiply the original equation by n(x), we get the exact equation

$(4x^3y + 3x^2y^2 - x^3)\,dx + (x^4 + 2x^3y)\,dy = 0$

by grouping we get

$(4x^3y\,dx + x^4\,dy) + (3x^2y^2\,dx + 2x^3y\,dy) - x^3\,dx = 0$

$d(x^4y) + d(x^3y^2) - d(\frac{1}{4}x^4) = d(c)$

$x^4y + x^3y^2 - \frac{1}{4}x^4 = c$

2) $y(x + y)\,dx + (x + 2y - 1)\,dy = 0$

$M(x,y) = y(x + y)$ and $N(x,y) = x + 2y - 1$

$$\frac{\partial M(x,y)}{\partial y} = x + 2y \text{ and } \frac{\partial N(x,y)}{\partial x} = 1$$

$$\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = x + 2y - 1$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{1}{x+2y-1}(x + 2y - 1) = 1$$

Since, the expression is constant, there is an integrating factor n(x)

$$n(x) = e^{\int dx} = e^x$$

Multiplying the original equation by n(x), we obtain the exact equation

$$ye^x(x + y)\,dx + e^x(x + 2y - 1)\,dy = 0$$

$$F(x,y) = \int M(x,y)dx = \int\left(xye^x + y^2e^x\right)dx = y\left(xe^x - e^x\right) + y^2e^x + B(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\left[y\left(xe^x - e^x\right) + y^2e^x + B(y)\right] = xe^x - e^x + 2ye^x + B'(y)$$

then

$$B'(y) = 0 \text{ and } B(y) = c$$

The solution is: $xe^x - e^x + 2ye^x = k$

3) $y(x + y + 1)\,dx + x(x + 3y + 2)\,dy = 0$

$M(x,y) = y(x + y + 1)$ and $N(x,y) = x(x + 3y + 2)$

$$\frac{\partial M(x,y)}{\partial y} = x + 2y + 1 \text{ and } \frac{\partial N(x,y)}{\partial x} = 2x + 3y + 2$$

$$\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = x + 2y + 1 - 2x - 3y - 2 = -(x + y + 1)$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{-(x+y+1)}{2x+3y+2} \text{ depends on } x \text{ and } y$$

consider

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x}-\frac{\partial M(x,y)}{\partial y}\right]=\frac{1}{y(x+y+1)}(x+y+1)=\frac{1}{y}$$

Since, it depends only on y, there is an integrating factor n(y)

$$n(y) = e^{\int \frac{dy}{y}} = e^{\ln y} = y$$

Multiplying the original equation by n(y), we obtain the exact equation

$$y^2(x + y + 1)\, dx + yx(x + 3y + 2)\, dy = 0$$

$$F(x,y)=\int M(x,y)dx=\int\left(xy^2+y^3+y^2\right)dx=\frac{x^2}{2}y^2+xy^3+xy^2+B(y)$$

$$\frac{\partial F(x,y)}{\partial y}=N(x,y)=\frac{\partial}{\partial y}\left[\frac{x^2}{2}y^2+xy^3+xy^2+B(y)\right]=x^2y+3xy^2+2xy+B'(y)$$

then B'(y) = 0 and therefore B(y) = c

The solution is: $\frac{1}{2} x^2y + xy^3 + xy^2 = k$.

Special Transformation

There are certain equations that can be transformed into a more basic type using a suitable transformation.

The equations have the form:

$$(a_1x + b_1y + c_1)\, dx + (a_2x + b_2y + c_2)\, dy = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants.

There are two different kind of transformations according to relationships among the constants.

Case 1: $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$

Solve the system

$$a_1h + b_1k + c_1 = 0$$
$$a_2h + b_2k + c_2 = 0$$

because of the imposed condition the system has a unique solution (h,k).

Then, the transformation:

$$x = X + h$$
$$y = Y + k$$

will change the original equation into a homogeneous equation in the variable X and Y,

$$(a_1X + b_1Y)\, dX + (a_2X + b_2Y)\, dY = 0$$

Case 2: $\frac{a_2}{a_1} = \frac{b_1}{b_2} = k$

The transformation $z = a_1x + b_1y$ changes the original equation into a separable equation in the variables z and x.

Examples: Solve the equations

1) $(2x - 5x + 3)\, dx - (2x + 4y - 6)\, dy = 0$

Since $2/2 \neq 4/-5$, let's solve the system

$$2h - 5k + 3 = 0$$
$$2h + 4k - 6 = 0$$

Subtract the second equation from the first one, to get

$$2h - 5h = -3$$
$$\underline{2h + 4k = 6}$$
$$0 - 9k = -9$$

then $k = 1$ and $2h = -3 + 5$ or $h = 1$.

So, the transformation:

$$x = X + 1, \qquad dx = dX$$
$$y = Y + 1, \qquad dy = dY$$

reduces the given equation to

$$(2X + 2 - 5Y - 5 + 3)\, dX - (2X + 2 + 4Y + 4 - 6)\, dY = 0$$
$$(2X - 5Y)\, dX - (2X + 4Y)\, dY = 0$$

which is homogeneous.

Using the transformation $Y = VX$, and $dY = VdX + XdV$,

We get the equation

$$(2 - 5V)\, dX - (2 + 4V)(VdX + XdV) = 0$$
$$(2 - 7V - 4V^2)\, dX - X(2 + 4V)\, dV = 0$$

$$\frac{dX}{X} - \frac{2+4V}{2-7V-4V^2} dV = 0$$

$$\frac{dX}{X} + \frac{4V+2}{4V^2+7V-2} dV = 0$$

$$\frac{4V+2}{4V^2+7V-2} = \frac{A}{4V-1} + \frac{B}{V+2}$$

$$A = \frac{4}{3}, \quad B = \frac{2}{3}$$

$$\frac{dX}{X} + \frac{4}{3}\frac{dV}{4V-1} + \frac{2}{3}\frac{dV}{V+2} = 0$$

Integrating

$$\ln|X| + \frac{1}{3}\ln|4V-1| + \frac{2}{3}\ln|V+2| = \ln|c|$$

$$X^3(4V-1)(V+2)^2 = k$$

replacing V by $\frac{Y}{X}$

$$X^3\left(4\frac{Y}{X}-1\right)\left(\frac{Y}{X}+2\right)^2 = (4Y-X)(Y+2X)^2 = K$$

replacing X by $\mathbf{x}-1$ and Y by $y-1$,

$$(4y-\mathbf{x}-3)(y+2\mathbf{x}-3)^2 = K$$

2) $(x + y)\, dx + (3x + 3y - 4)\, dy = 0$

Since $3/1 = 3/1$, we must take the transformation $z = x + y$.

We use $y = z - x$, $dy = dz - dx$ to obtain

$$z\, dx + (3z - 4)(dz - dx) = 0$$

or $$z\, dx - 3z\, dx + 4\, dx + (3z - 4)\, dz = 0$$

or $$(4 - 2z)\, dx + (3z - 4)\, dz = 0$$

and this is a separable equation,

POSSIBLE QUESTIONS

PART - B (5 x 2 = 10 Marks)

1. Write the standard forms of the Second order differential equations.
2. Explain integrating factor of the differential equation.
3. Define separable equations with examples.
4. Write the general form of Bernoulli's equation.
5. Define integrating factor of the differential equation.

PART – C (5 x 6 = 30 Marks)

1. Explain about exact differential equations with examples.
2. Solve the equation $(3x^2 + 4xy)\, dx + (2x^2 + 2y)\, dy = 0$.
3. Write a note on integration factor of differential equations.
4. Determine whether the given equations are exact or not and also solve that there is exact.

   $(2xy+1)\, dx + (x^2 + 4y)dy = 0$.
5. Determine the most general function N(x,y) such that the equation is exact

   $(x^3 + xy^2)\, dx + N(x,y)\, dy = 0$.
6. Explain Separable equations with examples.
7. Solve the equation $(x-4)\, y^4\, dx - x^3(y^2 - 3)\, dy = 0$.
8. Determine whether the differential equation is homogeneous or not

   $(x^2 - 3y^2)dx + 2xy\, dy = 0$.
9. Define Bernoulli's equation with example.
10. Solve the differential equation $\frac{dy}{dx} + y = xy^3$.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The standard form of first order differential equations derivative form is…………	(dy/dx)=f(x,y)	(dx/dy)=f(x,y)	(dy/dx)=-f(x,y)	(dy/dx)=0	(dy/dx)=f(x,y)
The standard form of first order differential equations differential form is…………	M(x,y)dx-N(x,y)dy=0	M(x,y)dx*N(x,y)dy=0	M(x,y)dx/N(x,y)dy=0	M(x,y)dx+N(x,y)dy=0	M(x,y)dx+N(x,y)dy=0
The expression M(x,y)dx+N(x,y)dy=0 is called an ………………... differential equations in a domain D.	ordinary	partial	exact	different	exact
The expression ………………….. is called an exact differential equations in a domain D.	M(x,y)dx+N(x,y)dy=0	M(x,y)dx*N(x,y)dy=0	M(x,y)dx/N(x,y)dy=0	M(x,y)dx-N(x,y)dy=0	M(x,y)dx+N(x,y)dy=0
The expression M(x,y)dx+N(x,y)dy=0 is called an exact differential equations in a domain D if there exists a function of………. variable such that the expression equals the total differential for all (x,y)in D	zero	one	two	three	two
The expression M(x,y)dx+N(x,y)dy=0 is called an exact differential equations in a domain D if there exists a function of two variable such that the expression equals the …………for all (x,y)in D	differential	ordinary differential	partial differential	total differential	total differential
If M(x,y)dx+N(x,y)dy is an exact differential then the differential equation M(x,y)dx+N(x,y)dy=…………is called exact differential equation	0	1	2	3	0
If M(x,y)dx+N(x,y)dy is an …………..differential then the differential equation M(x,y)dx+N(x,y)dy=0 is called exact differential equation	ordinary	partial	exact	different	exact
If M(x,y)dx+N(x,y)dy is not an exact differential in D then the differential equation …………………. in D the μ(x,y) is called integrating factor of the differentialequation	μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx-μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx*μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx/μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0
If M(x,y)dx+N(x,y)dy is not an ……………. differential in D then the differential equation μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0 in D the μ(x,y) is called integrating factor of the differentialequation	ordinary	partial	exact	different	exact
If M(x,y)dx+N(x,y)dy is not an exact differential in D then the differential equation μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0 in D the μ(x,y) is called …………factor of the differentialequation	differential	integrating	common	exact	integrating
An equation of the form ……………...is called an equation with variables separable orsimply a separable equations.	F(x)G(y) dx+f(x)g(y) dy=0	F(x)G(y) dx/f(x)g(y) dy=0	F(x) dx+g(y) dy=0	G(y) dx+f(x) dy=0	F(x)G(y) dx+f(x)g(y) dy=0
An equation of the form F(x)G(y) dx+f(x)g(y) dy=0 is called an ……………...or simply a separable equations.	equation with function separable	equation with constant separable	equation with roots separable	equation with variables separable	equation with variables separable
An equation of the form F(x)G(y) dx+f(x)g(y) dy=0 is called an equation with variables separable or simply a…………….equations.	differential	integral	separable	variable	separable
The first order differential equation M(x,y)dx+N(x,y)dy=0 is said to be………………... if the derivativeof the form (dy/dx)=f(x,y) there exists a function g suchthat f(x,y) can be expressed in the form g(y/x)	homogeneous	non homogeneous	singular	non singular	homogeneous

The first order differential equation M(x,y)dx+N(x,y)dy=0 is said to be homogeneous if the derivativeof the form ………….. there exists a function g suchthat f(x,y) can be expressed in the form g(y/x)	(dy/dx)=0	(dy/dx)=f(x,y)	(dy/dx)=1/f(x, y)	(dy/dx)= -f(x,y)	(dy/dx)=f(x,y)
The first order differential equation M(x,y)dx+N(x,y)dy=0 is said to be……………….... if the derivativeof the form (dy/dx)=f(x,y) there exists a function g suchthat f(x,y) can be expressed in the form…………	g(x/y)	g(1/x)	g(1/y)	g(y/x)	g(y/x)
A first order differential equation is linear in the dependent variable y and the independent variable x if it is can be written in the form ……………………	(dy/dx)=P(x)y+Q(x).	(dy/dx)+P(x)y/Q(x)=0.	(dy/dx)+P(x)y=Q(x).	(dy/dx)+P(x)y=0	(dy/dx)+P(x)y=Q(x).
A first order differential equation is ………………. in the dependent variable y and the independent variable x if it is can be written in the form (dy/dx)+P(x)y=Q(x).	linear	nonlinear	zero	separable	linear
A …………..order differential equation is linear in the dependent variable y and the independent variable x if it is can be written in the form (dy/dx)+P(x)y=Q(x).	first	second	third	n th	first
An equation of the form ……………………….is called a Bernoulli differential equation .	(dy/dx=P(x) y^n	(dy/dx)+P(x)y/Q(x)=0.	(dy/dx)+P(x)y=Q(x) y^n	(dy/dx)+P(x)y=0	(dy/dx)+P(x)y=Q(x) y^n
In bernoulli equation when n=……….. then the equation is called linear equation.	0 or 1	1 or 2	0 or 2	0 or -1	0 or 1
In bernoulli equation when n=0 or 1 then the equation is called …………….. equation.	ordinary	partial	nonlinear	linear	linear
In ……………….. equation when n=0 or 1 then the equation is called linear equation.	ordinary	Bernoulli	Euler	partial	Bernoulli

UNIT – III

SYLLABUS

General solution of homogeneous equation of second order, principle of super position for homogeneous equation, Wronskian: its properties and applications, Linear homogeneous and non-homogeneous equations of higher order with constant coefficients, Euler's equation, method of undetermined coefficients, method of variation of parameters.

Linear Differential Equations:

A differential equation of the form

$$F(x, y, y', y'', \ldots, y^n) = R(x)$$

is called the linear differential equation provided that F is linear differential equation of order n in the dependent variable y and its derivatives $y', y'', \ldots, y^n$.

Second Order Linear Differential Equation:

A differential equation of the form

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = R(x)$$

Where $a_0(x), a_1(x), a_2(x)$ and R(x) are continuous functions of x only on some open interval I is called second order linear differential equation.

Homogeneous and Non-homogeneous Linear Differential Equation:

If R(x) = 0, then the differential equation of the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

Where $a_0(x), a_1(x), a_2(x), \ldots, a_{n-1}(x)$ and $a_n(x)$ are continuous functions of x only on some open interval I is called homogeneous linear differential equation of order n.

If $R(x) \neq 0$, then the differential equation of the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = R(x)$$

Where $a_0(x), a_1(x), a_2(x), \ldots, a_{n-1}(x), a_n(x)$ and $R(x)$ are continuous functions of x only on some open interval I is called the non-homogeneous linear differential equation of order n.

Example 1: Consider the differential equation $\cos x \frac{d^2y}{dx^2} + (1+x^{3/2})\frac{dy}{dx} + e^x y = \sin^{-1} x$, in this differential equation dependent variable y and its derivatives y' *and* y'' appears linearly and the highest order derivative term in the equation is 2. Therefore this equation is called linear differential equation of order 2.

Example 2: Consider the differential equation $x^2 y'' + \cos x\, y' + \sin x\, y = 0$, in this differential equation dependent variable y and its derivatives y' *and* y'' appear linearly also the right hand side of the equation is zero. Therefore this equation is called homogeneous linear differential equation of order 2.

Principle of Superposition for Homogeneous Equations:

Principle of superposition states that linear combination of any solutions of a homogeneous linear differential equation of order two is also a solution of the given differential equation.

Theorem 1: Let y_1 and y_2 be two solutions of the homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution of the equation $y'' + p(x)y' + q(x)y = 0$ on the interval I.

Proof: Let y_1 and y_2 are the solutions of the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

on the interval I.

Then they must satisfy the equation (1), then

$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \qquad (2)$$

and $$y_2'' + p(x)y_2' + q(x)y_2 = 0 \qquad (3)$$

Let c_1 and c_2 are the constants, let

$$y = c_1 y_1 + c_2 y_2$$

on differentiating, we have

$$y' = c_1 y_1' + c_2 y_2'$$

on again differentiating, we have

$$y'' = c_1 y_1'' + c_2 y_2''$$

Now, putting these values in equation (1), we have

$$y'' + p(x)y' + q(x)\,\mathrm{y} = (c_1 y_1'' + c_2 y_2'') + \mathrm{p(x)}(c_1 y_1' + c_2 y_2') + \mathrm{q(x)}(c_1 y_1 + c_2 y_2)$$

$$\Rightarrow \quad y'' + p(x)y' + q(x)\,\mathrm{y} = c_1(y_1'' + \mathrm{p(x)}y_1' + \mathrm{q(x)}y_1) + c_2(y_2'' + \mathrm{p(x)}y_2' + \mathrm{q(x)}y_2)$$

$$\Rightarrow \quad y'' + p(x)y' + q(x)\,\mathrm{y} = c_1.0 + c_2.0$$ [using equation (2) and (3)]

$$\Rightarrow \quad y'' + p(x)y' + q(x)\,\mathrm{y} = 0$$

Thus, $y = c_1 y_1 + c_2 y_2$ also satisfy the equation therefore is a solution of the equation

$$y'' + p(x)y' + q(x)\,\mathrm{y} = 0$$

on interval I.

Example 3: Show that $y_1(x) = e^x$ *and* $y_2(x) = e^{-x}$ are two solutions of the equation

$$y'' - y = 0.$$

Solution: Given differential equation is

$$y'' - y = 0 \qquad (1)$$

Now let $y(x) = e^x$

on differentiating w.r.t. x

$$y'(x) = e^x$$

on again differentiating w.r.t. x

$$y''(x) = e^x$$

Now, putting these values in equation (1), we have

$$y''-y=e^{x}-e^{x}=0$$

Hence, e^{x} is the solution of the differential equation

$$y''-y=0$$

Now let $y(x)=e^{-x}$

on differentiating w.r.t. x

$$y'(x)=-e^{-x}$$

on again differentiating w.r.t. x

$$y''(x)=e^{-x}$$

Now, putting these values in equation (1), we have

$$y''-y=e^{-x}-e^{-x}=0$$

Hence, e^{-x} is the solution of the differential equation

$$y''-y=0$$

Thus, $y_1(x)=e^{x}$ *and* $y_2(x)=e^{-x}$ are two solutions of the differential equations

$$y''-y=0.$$

Example 4:

Verify that $y_1=e^{x}$ *and* $y_2=e^{2x}$ are solutions of the differential equation $y''-3y'+2y=0$.

Find a solution satisfying the initial conditions $y(0)=1$ *and* $y'(0)=0$.

Solution: Given differential equation is

$$y''-3y'+2y=0 \qquad (1)$$

Now let $y(x)=e^{x}$

on differentiating w.r.t. x

$$y'(x) = e^x$$

on again differentiating w.r.t. x

$$y''(x) = e^x$$

Now, putting these values in equation (1), we have

$$y'' - 3y' + 2y = e^x - 3e^x + 2e^x = 0$$

Hence, e^x is the solution of the differential equation $y'' - 3y' + 2y = 0$.

Now let $y(x) = e^{2x}$

on differentiating w.r.t. x

$$y'(x) = 2e^{2x}$$

on again differentiating w.r.t. x

$$y''(x) = 4e^{2x}$$

Now, putting these values in equation (1), we have

$$y'' - 3y' + 2y = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$$

Hence, e^{2x} is the solution of the differential equation

$$y'' - 3y' + 2y = 0$$

Thus, $y_1(x) = e^x$ *and* $y_2(x) = e^{2x}$ are two solutions of the differential equations

$$y'' - 3y' + 2y = 0.$$

By the principle of superposition we know that

$$y = c_1 e^x + c_2 e^{2x}$$

Is also a solution of equation (1).

On differentiating w.r.t. x, we have

$$y' = c_1 e^x + 2c_2 e^{2x}$$

Now using the initial conditions, we have

$$y(0) = 1$$

$$\Rightarrow \quad c_1 e^{(0)} + c_2 e^{(0)} = 1$$

$$\Rightarrow \quad c_1 + c_2 = 1 \qquad (2)$$

and $\quad y'(0) = 0$

$$\Rightarrow \quad c_1 e^{(0)} + 2c_2 e^{(0)} = 0$$

$$\Rightarrow \quad c_1 + 2c_2 = 0 \qquad (3)$$

On solving equation (2) and (3), we have

$$c_1 = 2 \ \ and \ \ c_2 = -1$$

Thus, $y(x) = 2e^x - e^{2x}$

is the required solution.

Linearly Independent or Linearly Dependent Functions:

Two functions defined on an open interval I are said to be linearly independent on interval I provided that neither is a constant multiple of the other. If one can be written as a constant multiple of other then they are called linearly dependent functions.

Let f and g are two functions defined on an open interval I. Then f and g are called linearly dependent on I, if one can be written as a constant multiple of other i.e. there exists a constant $\lambda \in R$ such that

$$f(x) = \lambda g(x) \quad for\ each\ x \in I$$

If they cannot be written as constant multiple of each other then they are called linearly independent functions.

In general, the functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ defined on an open interval I are said to be linearly dependent on the interval I provided that there exists constants $c_1, c_2, c_3, \ldots, c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \ldots + c_n f_n(x) = 0$$

The functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ defined on an open interval I are said to be linearly independent on the interval I, if

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \ldots + c_n f_n(x) = 0$$

Then $c_1 = c_2 = c_3 = \ldots = c_n = 0$ are all zero.

Wronskian:

Let f(x) and g(x) are two functions defined on an interval I. Then the Wronskian of f(x) and g(x) is denoted by W(f, g) and determined by the determinant.

$$W(f,g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

$$\Rightarrow \quad W(f,g) = f(x)g'(x) - f'(x)g(x)$$

If the Wronskian of the functions f(x) and g(x) is zero then the function f(x) and g(x) are called linearly dependent functions.

If the Wronskian of the functions f(x) and g(x) is non-zero then the function f(x) and g(x) are called linearly independent functions.

In general, let $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ are n functions defined on an open interval I. Then the Wronskian of $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ is denoted by $W(f_1(x), f_2(x), f_3(x), \ldots, f_n(x))$ and defined as

$$W(f_1(x), f_2(x), f_3(x), \ldots, f_n(x)) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & f_3'(x) & \ldots & f_n'(x) \\ . & & & & \\ . & & & & \\ . & & & & \\ f_1^{n-1}(x) & f_2^{n-1}(x) & f_3^{n-1}(x) & \ldots & f_n^{n-1}(x) \end{vmatrix}$$

If wronskian $W(f_1(x), f_2(x), f_3(x), \ldots, f_n(x))$ is zero then the functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ are called linearly dependent functions. If the wronskian is non-zero then the functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ are called linearly independent functions.

Theorem : Let $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$ are n solutions of the homogeneous nth order linear differential equation

$$y^{(n)} + a_1(x)y(x)^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a_{n-1}(x)y' + a_n(x)y = 0$$

on an interval I where each $a_i(x)$ is continuous function on I. Let wronskian is defined as

$$W = W(y_1(x), y_2(x), y_3(x), \ldots, y_n(x))$$

(i) If $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$ are linearly dependent then W = 0 on I.

(ii) If $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$ are linearly independent then $W \neq 0$ at each point of I.

Proof: Given that $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$ are n solutions of the homogeneous nth order linear differential equation

$$y^{(n)} + a_1(x)y(x)^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a_{n-1}(x)y' + a_n(x)y = 0 \qquad (1)$$

on an interval I.

(I) Let $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$ are linearly dependent on I, then for some choice of the constants $c_1, c_2, c_3, \ldots, c_n$ not all zero, we have

$$c_1y_1 + c_2y_2 + c_3y_3 + \ldots + c_ny_n = 0 \qquad (2)$$

On differentiating this equation (n-1) times, we have

$$\left.\begin{aligned} &c_1y_1' + c_2y_2' + c_3y_3' + \ldots + c_ny_n' = 0 \\ &c_1y_1'' + c_2y_2'' + c_3y_3'' + \ldots + c_ny_n'' = 0 \\ &. \\ &. \\ &. \\ &c_1y_1^{(n-1)} + c_2y_2^{(n-1)} + c_3y_3^{(n-1)} + \ldots + c_ny_n^{(n-1)} = 0 \end{aligned}\right\} \qquad (3)$$

Which holds for all x in I.

We know that the system of equations in equation (2) and (3) represents the n linear homogeneous equations in n unknowns has a non-trivial solution if and only if the determinant of coefficients is zero. Since the unknown in the equation (2) and (3) are the constants $c_1, c_2, c_3, \ldots, c_n$.

Thus for the non-trivial solution we have

$$\begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ . & & & & \\ . & & & & \\ . & & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0$$

$$\Rightarrow \quad W\left(y_1(\mathrm{x}), y_2(\mathrm{x}), y_3(\mathrm{x}), \ldots, y_n(x)\right) = 0$$

$$\Rightarrow \quad W = 0$$

Thus, if c_i's are not all zero then $W = 0$.

Hence, if $y_1(\mathrm{x}), y_2(\mathrm{x}), y_3(\mathrm{x}), \ldots, y_n(x)$ are linearly dependent then $W = 0$ on I.

(II) To prove that if $y_1(\mathrm{x}), y_2(\mathrm{x}), y_3(\mathrm{x}), \ldots, y_n(x)$ are linearly independent, then $W \neq 0$ at each point of I.

Suppose if possible there exists an element $a \in I$ such that

$$W(a) = 0$$

Since W(a) represents the determinant of coefficients of the system of n homogeneous linear equations, then

$$\left.\begin{aligned} &c_1 y_1(a) + c_2 y_2(a) + c_3 y_3(a) + \ldots + c_n y_n(a) = 0 \\ &c_1 y_1'(\mathrm{a}) + c_2 y_2'(\mathrm{a}) + c_3 y_3'(\mathrm{a}) + \ldots + c_n y_n'(\mathrm{a}) = 0 \\ &c_1 y_1''(\mathrm{a}) + c_2 y_2''(\mathrm{a}) + c_3 y_3''(\mathrm{a}) + \ldots + c_n y_n''(\mathrm{a}) = 0 \\ &. \\ &. \\ &. \\ &c_1 y_1^{(n-1)}(a) + c_2 y_2^{(n-1)}(a) + c_3 y_3^{(n-1)}(a) + \ldots + c_n y_n^{(n-1)}(a) = 0 \end{aligned}\right\} \quad (4)$$

In the n unknowns $c_1, c_2, c_3, \ldots, c_n$.

Since W(a) determinant of coefficients in equation (4) is 0, thus the system of equations in (4) have a nontrivial solution i.e., the numbers $c_1, c_2, c_3, \ldots, c_n$ are not all zero.

Now, let

$$y(x) = c_1 y_1(\mathrm{x}) + c_2\, y_2(\mathrm{x}) + c_3 y_3(\mathrm{x}) + \ldots + c_n\, y_n(\mathrm{x}) \tag{5}$$

Is a particular solution of equation (1).

Then equation (4) implies that Y(x) satisfy the trivial initial conditions

$$y(\mathrm{a}) = y'(\mathrm{a}) = y''(\mathrm{a}) = y'''(\mathrm{a}) = \ldots = y^{(n-1)}(\mathrm{a})$$

Thus by the uniqueness theorem, we have y(x) = 0 on I. Thus from equation (5) and the fact that $c_1, c_2, c_3, \ldots, c_n$ are not all zero. It implies that $y_1(\mathrm{x}), y_2(\mathrm{x}), y_3(\mathrm{x}), \ldots, y_n(x)$ are linearly dependent. This contradicts the fact that functions $y_1(\mathrm{x}), y_2(\mathrm{x}), y_3(\mathrm{x}), \ldots, y_n(x)$ are linearly independent. Hence our assumption that W(a)=0 for some a in I is wrong.

Therefore, if $y_1(\mathrm{x}), y_2(\mathrm{x}), y_3(\mathrm{x}), \ldots, y_n(x)$ are linearly independent, then $W \neq 0$ for each point on I.

Example Show that $\sin ax \;\; and \;\; \cos ax$ are linearly independent functions.

Solution: Let $\mathrm{y}_1 = \sin ax \;\; and \;\; y_2 = \cos ax$

on differentiating w.r.t. x we have

$$\mathrm{y}_1' = a\cos ax \;\; and \;\; y_2' = -a\sin ax$$

Wronskian of $\mathrm{y}_1 = \sin ax \;\; and \;\; y_2 = \cos ax$ is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2) = \begin{vmatrix} \sin ax & \cos ax \\ a\cos ax & -a\sin ax \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2) = -a\sin^2 ax - a\cos^2 ax = -a$$

Thus, if $a \neq 0$ then $W(y_1, y_2) \neq 0$.

Hence, if $a \neq 0$, then $\sin ax \;\; and \;\; \cos ax$ are linearly independent functions.

Example Show that the functions e^{ax}, e^{bx}, e^{cx} $(a \neq b \neq c)$ are linearly independent.

Solution: Let $y_1 = e^{ax}, y_2 = e^{bx}$ *and* $y_3 = e^{cx}$

on differentiating w.r.t. x we have

$$y_1' = ae^{ax}, y_2' = be^{bx} \text{ and } y_3' = ce^{cx}$$

Again differentiating w.r.t. x we have

$$y_1'' = a^2e^{ax}, y_2'' = b^2e^{bx} \text{ and } y_3'' = c^2e^{cx}$$

Wronskian of $y_1 = e^{ax}, y_2 = e^{bx}$ *and* $y_3 = e^{cx}$ is

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2, y_3) = \begin{vmatrix} e^{ax} & e^{bx} & e^{cx} \\ ae^{ax} & be^{bx} & ce^{cx} \\ a^2e^{ax} & b^2e^{bx} & c^2e^{cx} \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2, y_3) = e^{ax}e^{ax}e^{ax} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2, y_3) = e^{(a+b+c)x} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2, y_3) = e^{(a+b+c)x}(b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2, y_3) = e^{(a+b+c)x}(b-a)(c-a)[(c+a)-(b+a)]$$

$$\Rightarrow \quad W(y_1, y_2, y_3) = (b-a)(c-a)(c-b)e^{(a+b+c)x}$$

Since $a \neq b \neq c$ thus $W(y_1, y_2, y_3) \neq 0$.

Hence, e^{ax}, e^{bx}, e^{cx} $(a \neq b \neq c)$ are linearly independent functions.

Solutions of Homogeneous Linear Differential Equations with Constant Coefficients:

A differential equation of the form

$$a_0y^n + a_1y^{n-1} + a_2y^{n-2} + \ldots + a_{n-1}y' + a_ny = 0 \qquad a_0 \neq 0 \tag{A}$$

Where $a_0, a_1, a_2, \ldots, a_{n-1}$ *and* a_n are constants is called a homogeneous linear differential equation with constant coefficients.

In order to solve the homogeneous linear differential equation, put $y = 1, \frac{dy}{dx} = y' = \mathrm{m},\ \frac{d^2y}{dx^2} = y'' = \mathrm{m}^2, \ldots, \frac{d^ny}{dx^n} = y^n = \mathrm{m}^n$ and so on in equation (A), we have

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots + a_{n-1}\mathrm{m} + a_n = 0 \qquad a_0 \neq 0 \tag{B}$$

An algebraic equation of m with degree n. This equation is called auxiliary equation or characteristic equation corresponding to the homogeneous equation (A).

Finding the roots of the equation (B) there may arise three different cases

Case (I): Roots of the auxiliary equation are real and distinct:

Let the roots of the auxiliary equation are $m_1, m_2, m_3, \ldots, m_n$ all are real and distinct then the solution of the equation (A) is

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}$$

This solution is called the general solution of equation (A).

Example : Solve the differential equation $y'' + y' - 6y = 0$.

Solution: Given differential equation is

$$y'' + y' - 6y = 0 \tag{1}$$

Corresponding auxiliary equation is

$$m^2 + m - 6 = 0$$

$$\Rightarrow \quad (m-3)(m+2) = 0$$

$$\Rightarrow \quad \mathrm{m} = 3, -2$$

Thus, the general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-2x} \text{ .}$$

Example : Solve the differential equation $2y''' - y'' - 5y' - 2y = 0$.

Solution: Given differential equation is

$$2y''' - y'' - 5y' - 2y = 0 \qquad (1)$$

Corresponding auxiliary equation is

$$2m^3 - m^2 - 5m - 2 = 0 \text{ [putting } y = 1,\ y' = m,\ y'' = m^2 \text{ and } y''' = m^3 \text{ in equation (1)]}$$

$$\Rightarrow \quad (m-2)(m+1)(2m+1) = 0$$

$$\Rightarrow \quad m = 2,\ -1,\ -\frac{1}{2} \qquad \text{[roots are real and distinct]}$$

Thus, the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-\frac{1}{2}x} \text{ .}$$

Case (II): Roots of the auxiliary equation are real but some roots are equal:

Let the roots of the auxiliary equation are $m_1, m_2, m_3, \ldots, m_n$ all are real and let two roots are equal i.e., $m_1 = m_2$ and all other roots are distinct then the solution of the equation (A) is

$$y(x) = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

This solution is called the general solution of equation (A).

Example : Solve the differential equation $y'' - 2y' + y = 0$.

Solution: Given differential equation is

$$y'' - 2y' + y = 0 \qquad (1)$$

Corresponding auxiliary equation is

$$m^2 - 2m + 1 = 0 \qquad \text{[putting } y = 1,\ y' = m \text{ and } y'' = m^2 \text{ in equation (1)]}$$

$\Rightarrow \quad (m-1)^2 = 0$

$\Rightarrow \quad m = 1, 1$

Thus, the general solution is

$$y(x) = (c_1 + c_2 x)e^x .$$

Example : Solve the differential equation $y'''' - 8y''' + 16y'' = 0$.

Solution: Given differential equation is

$$y'''' - 8y''' + 16y'' = 0 \qquad (1)$$

Corresponding auxiliary equation is

$m^4 - 8m^3 + 16m^2 = 0$ [putting $y'' = m^2,\ y''' = m^3$ *and* $y'''' = m^4$ in equation (1)]

$\Rightarrow \quad m^2(m^2 - 8m + 16) = 0$

$\Rightarrow \quad m^2(m-4)^2 = 0$

$\Rightarrow \quad m = 0,\ 0,\ 4,$ *and* 4

Thus, the general solution is

$$y(x) = (c_1 + c_2 x)e^{0.x} + (c_3 + c_4 x)e^{4x}$$

$\Rightarrow \quad y(x) = c_1 + c_2 x + (c_3 + c_4 x)e^{4x}$.

Case (III): Roots of the auxiliary equation are complex:

Let the roots of the auxiliary equation are $m_1, m_2, m_3, \ldots, m_n$ such that two roots are complex i.e., $m_1 \pm im_2$ and all other roots are real and distinct then the solution of the equation (A) is

$$y(x) = e^{m_1 x}(c_1 \cos m_2 x + c_2 \sin m_2 x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

This solution is called the general solution of equation (A).

Example : Solve the differential equation $y''''+3y''-4y=0$.

Solution: Given differential equation is

$$y''''+3y''-4y=0 \qquad (1)$$

Corresponding auxiliary equation is

$$m^4+3m^2-4=0$$

$$\Rightarrow \quad (m^2+4)(m^2-1)=0$$

$$\Rightarrow \quad m=\pm 2i,\ \pm 1 \qquad \text{[roots are real but equal]}$$

Thus, the general solution is

$$y(x)=e^{0.x}(c_1\cos 2x+c_2\sin 2x)+c_3e^{-x}+c_4e^{x}$$

$$\Rightarrow \quad y(x)=c_1\cos 2x+c_2\sin 2x+c_3e^{-x}+c_4e^{x}.$$

Example : Solve the initial value problem

$$9y''+6y'+4y=0; \quad y(0)=3,\ y'(0)=4.$$

Solution: Given differential equation is

$$9y''+6y'+4y=0$$

Corresponding characteristic equation is

$$9m^2+6m+4=0$$

$$\Rightarrow \quad m=-\frac{1}{3}\pm\frac{1}{\sqrt{3}}i \qquad \text{[roots are real but equal]}$$

Thus, the general solution is

$$y(x)=e^{-\frac{1}{3}x}(c_1\cos\frac{1}{\sqrt{3}}x+c_2\sin\frac{1}{\sqrt{3}}x) \qquad (1)$$

Now differentiating y(x) w.r.t. x we have

$$y'(x)=-\frac{1}{3}e^{-\frac{1}{3}x}(c_1\cos\frac{1}{\sqrt{3}}x+c_2\sin\frac{1}{\sqrt{3}}x)+e^{-\frac{1}{3}x}(-\frac{1}{\sqrt{3}}c_1\sin\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}c_2\cos\frac{1}{\sqrt{3}}x)$$

$$\Rightarrow \quad y'(x) = -\frac{1}{3}e^{-\frac{1}{3}x}(c_1 \cos\frac{1}{\sqrt{3}}x + c_2 \sin\frac{1}{\sqrt{3}}x) + -\frac{1}{\sqrt{3}}e^{-\frac{1}{3}x}(c_1 \sin\frac{1}{\sqrt{3}}x + c_2 \cos\frac{1}{\sqrt{3}}x) \qquad (2)$$

Now using the initial values we have

$$y(0) = 3$$

$$\Rightarrow \quad e^{-\frac{1}{3}.0}(c_1 \cos\frac{1}{\sqrt{3}}.0 + c_2 \sin\frac{1}{\sqrt{3}}.0) = 3$$

$$\Rightarrow \quad c_1.1 + c_2.0 = 3$$

$$\Rightarrow \quad c_1 = 3$$

And $y'(0) = 4$

$$\Rightarrow \quad -\frac{1}{3}e^{-\frac{1}{3}.0}(c_1 \cos\frac{1}{\sqrt{3}}.0 + c_2 \sin\frac{1}{\sqrt{3}}.0) + \frac{1}{\sqrt{3}}e^{-\frac{1}{3}.0}(-c_1 \sin\frac{1}{\sqrt{3}}.0 + c_2 \cos\frac{1}{\sqrt{3}}.0) = 4$$

$$\Rightarrow \quad -\frac{1}{3}(c_1.1 + c_2.0) + \frac{1}{\sqrt{3}}(-c_1.0 + c_2.1) = 4$$

$$\Rightarrow \quad -\frac{1}{3}c_1 + \frac{1}{\sqrt{3}}.c_2 = 4$$

$$\Rightarrow \quad -\frac{1}{3}.3 + \frac{1}{\sqrt{3}}.c_2 = 4$$

$$\Rightarrow \quad c_2 = 5\sqrt{3}$$

Putting the values of c_1 *and* c_2 in equation (1), we have

$$y(x) = e^{-\frac{1}{3}x}(3\cos\frac{1}{\sqrt{3}}x + 5\sqrt{3}\sin\frac{1}{\sqrt{3}}x).$$

Euler Equation:

A differential equation of the form

$$a_0(x-a)^n\frac{d^ny}{dx^n}+a_1(x-a)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-1}(x-a)\frac{dy}{dx}+a_ny=0 \quad a_0\neq 0 \qquad (1)$$

Or $$a_0(x-a)^ny^n+a_1(x-a)^{n-1}y^{n-1}+a_2(x-a)^{n-2}y^{n-2}+\ldots+a_{n-1}(x-a)y'+a_ny=0, \quad a_0\neq 0$$

is called the Euler's equation of the order n.

In order to solve the Euler's equation put

$$(x-a)=e^z \quad or \quad z=\ln(x-a)$$

$$\Rightarrow \quad \frac{dz}{dx}=\frac{1}{(x-a)}$$

Thus, $\frac{dy}{dx}=\frac{dy}{dz}.\frac{dz}{dx}$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{dy}{dz}.\frac{1}{(x-a)}$$

$$\Rightarrow \quad (x-a)\frac{dy}{dx}=\frac{dy}{dz}=Dy \quad where\ D=\frac{d}{dz} \qquad (2)$$

or $$(x-a)y'=Dy \quad where\ D=\frac{d}{dz}$$

again differentiating w.r.t. x we have

$$(x-a)\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\Rightarrow \quad (x-a)\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d}{dz}\left(\frac{dy}{dz}\right).\frac{dz}{dx}$$

$$\Rightarrow \quad (x-a)\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d^2y}{dz^2}.\frac{1}{(x-a)}$$

$$\Rightarrow \quad (x-a)^2\frac{d^2y}{dx^2}+(x-a)\frac{dy}{dx}=\frac{d^2y}{dz^2}$$

$$\Rightarrow \quad (x-a)^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$ [using equation (2)]

$$\Rightarrow \quad (x-a)^2\frac{d^2y}{dx^2}=D^2y-Dy=D(D-1)y \qquad where\,\frac{d^2}{dz^2}=D^2 \;\; and \;\; \frac{d}{dz}=D$$

Or $$(x-a)^2y''=D^2y-Dy=D(D-1)y \qquad where\,\frac{d^2}{dz^2}=D^2 \;\; and \;\; \frac{d}{dz}=D$$

Continuing in this way we have

$$(x-a)^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$$

Or $$(x-a)^3y'''=D(D-1)(D-2)y$$

Or in general

$$(x-a)^n\frac{d^ny}{dx^n}=D(D-1)(D-2)\,.\,.\,.\,(D-n+1)y$$

Or $$(x-a)^ny^n=D(D-1)(D-2)\,.\,.\,.\,(D-n+1)y$$

Now, replacing these values in equation (1) and then solve the equation by finding the auxiliary for the variable z, then replace the value of z in the general equation, we obtain the general solution for x and y.

Example : Solve the Euler's equation $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$

Solution: Given equation is

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0 \qquad (1)$$

Putting $x = e^z$ and putting

$$xy' = Dy \quad \text{where } D = \frac{d}{dz}$$

$$x^2y'' = D(D-1)\,y$$

And $x^3y''' = D(D-1)(D-2)\,y$

Putting these values in equation (1) We have

$$D(D-1)(D-2)\,y - 3D(D-1)\,y + 6\,Dy - 6\,y = 0$$

$$\Rightarrow \quad D(D^2 - 3D + 2)\,y - 3D(D-1)\,y + 6\,Dy - 6\,y = 0$$

$$\Rightarrow \quad (D^3 - 3D^2 + 2 - 3D^2 + 3D + 6D - 6)\,y = 0$$

$$\Rightarrow \quad (D^3 - 6D^2 + 11D - 6)\,y = 0$$

Corresponding characteristic equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\Rightarrow \quad (m-1)(m-2)(m-3) = 0$$

$$\Rightarrow \quad m = 1,\ 2,\ 3$$

Solution of the equation is

$$y(z) = c_1e^z + c_2e^{2z} + c_3e^{3z}$$

Putting $e^z = x$

$$\Rightarrow \quad y(x) = c_1x + c_2x^2 + c_3x^3$$

is the required solution.

Example : Solve the Euler's equation $(x+1)^2 y''+(x+1)y'-y=0$

Solution: Given equation is

$$(x+1)^2 y''+(x+1)y'-y=0 \qquad (1)$$

Putting $(x+1)=e^z$ and putting

$$(x+1)y'=Dy \quad where\ D=\frac{d}{dz}$$

and $(x+1)^2 y''=D(D-1)\,y$

Putting these values in equation (1) We have

$$D(D-1)\,y+Dy-y=0$$

$$\Rightarrow \quad (D^2-1)\,y=0$$

Corresponding auxiliary equation is

$$m^2-1=0$$

$$\Rightarrow \quad m=\pm 1$$

Solution of the equation is

$$y(z)=c_1e^z+c_2e^{-z}$$

Putting $e^z=(x+1)$

$$\Rightarrow \quad y(x)=c_1(x+1)+c_2(x+1)^{-1}$$

$$\Rightarrow \quad y(x)=c_1(x+1)+\frac{c_2}{(x+1)}$$

is the required solution.

Method of Undetermined Coefficients:

The method of undetermined coefficients is applied to find the particular solution of the non-homogeneous differential equation if the function R(x) in the non-homogeneous differential equation is a linear combination of finite products of functions of the following three types:

(i) A polynomial in x

(ii) An exponential function of the form e^{kx}

(iii) A trigonometric function of the form $\cos nx \ or \ \sin nx$

Rule to find the Particular Solution by Method of Undetermined Coefficients:

If no term appearing either in R(x) or in any of its derivatives satisfies the homogeneous differential equation associated with the non-homogeneous differential equation (A). Then the particular solution y_p is considered as a linear combination of all linearly independent such terms and their derivatives. Since y_p is a particlar solution of the non-homogeneous differential equation (A). Hence, coefficients of y_p are determined by substituting it into the non-homogeneous equation (A) by comparing the coefficients of like terms of both sides.

Case (I): If R(x) is in the form of a Polynomial:

If R(x) is in the form of a polynomial i.e.

$$R(x) = b_0 + b_1 x + b_2 x^2 + \ . \ . \ . + b_n x^n$$

Then y_p is considered as follows

$$y_p = (A_0 + A_1 x + A_2 x^2 + \ . \ . \ . + A_n x^n) x^s$$

Where the coefficients $A_0, A_1, A_2, \ . \ . \ ., A_n \ and \ s$ are to be determined.

Example

Solve the differential equation by finding the particular solution of the differential equation $y'' - y' - 2y = 3x + 4$.

Solution: Given differential equation is

$y'' - y' - 2y = 3x + 4$ (1)

Associated homogeneous differential equation is

$$y'' - y' - 2y = 0$$

Corresponding auxiliary equation is

$$m^2 - m - 2 = 0$$

$$\Rightarrow \quad (m-2)(m+1) = 0$$

$$\Rightarrow \quad m = -1, \; 2$$

Thus, complementary solution is

$$y_c(x) = c_1 e^{-x} + c_2 e^{2x} \qquad (2)$$

Since $R(x) = 3x + 4$, thus particular solution must be of the form $A_0 + A_1 x$ then there is no duplication of any term $y_c(x)$ with the particular solution. Then consider

$$y_p(x) = A_0 + A_1 x \qquad (3)$$

on differentiating w.r.t. x we have

$$y_p' = A_1$$

again differentiating w.r.t. x we have

$$y_p'' = 0$$

Now putting these values in equation (1) we have

$$0 - A_1 - (A_0 + A_1 x) = 3x + 4$$

$$\Rightarrow \quad -A_1 - A_0 - A_1 x = 3x + 4$$

comparing the coefficients of like terms we have

$$A_0 = -1 \; and \; A_1 = -3$$

putting the values of $A_0 = -1 \; and \; A_1 = -3$ in equation (3) particular solution is

$$y_p(x) = -1 - 3x$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \quad y(x) = c_1 e^{-x} + c_2 e^{2x} - 3x - 1$$

is the required solution.

Case (II): If R(x) is in the form of sin mx or cosmx:

If R(x) is in the form of

$$R(x) = a\cos mx \ or \ b\sin mx \ or \ a\cos mx + b\sin mx$$

Then y_p is considered as follows

$$y_p = (A\cos mx + B\sin mx)x^s$$

Where the coefficients $A, \ B \ and \ s$ are to be determined.

Example

Solve the differential equation by finding the particular solution of the differential equation $y'' - 3y' + 2y = 10\cos 3x$.

Solution: Given differential equation is

$$y'' - 3y' + 2y = 10\cos 3x \tag{1}$$

Associated homogeneous differential equation is

$$y'' - 3y' + 2y = 0$$

Corresponding auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow \quad (m-1)(m-2) = 0$$

$$\Rightarrow \quad m = 1, \ 2$$

Thus, complementary solution is

$$y_c(x) = c_1 e^{x} + c_2 e^{2x} \tag{2}$$

Since $R(x)=10\cos 3x$, thus particular solution must be of the form $A\cos 3x+B\sin 3x$ then there is no duplication of the term $y_c(x)$ with the term $A\cos 3x+B\sin 3x$ in particular solution. Then consider

$$y_p(x)=A\cos 3x+B\sin 3x \qquad (3)$$

on differentiating w.r.t. x we have

$$y_p'=-3A\sin 3x+3B\cos 3x$$

again differentiating w.r.t. x we have

$$y_p''=-9A\cos 3x-9B\sin 3x$$

Now putting these values in equation (1) we have

$$(-9A\cos 3x-9B\sin 3x)-3(-3A\sin 3x+3B\cos 3x)+2(A\cos 3x+B\sin 3x)=10\cos 3x$$

$$\Rightarrow \quad (-7A-9B)\cos 3x+(9A-7B)\sin 3x=10\cos 3x$$

comparing the coefficients of like terms we have

$$A=-\frac{7}{13} \text{ and } B=-\frac{9}{13}$$

putting the values of $A=-\frac{7}{13}$ and $B=-\frac{9}{13}$ in equation (3) particular solution is

$$y_p(x)=-\frac{7}{13}\cos 3x-\frac{9}{13}\sin 3x=-\frac{1}{13}(7\cos 3x+9\sin 3x)$$

Thus the complete solution is

$$y(x)=y_c(x)+y_p(x)$$

$$\Rightarrow \quad y(x)=c_1e^x+c_2e^{2x}-\frac{1}{13}(7\cos 3x+9\sin 3x)$$

is the required solution.

Case (III): If R(x) is in the form of $e^{kx}\cos mx \; or \; e^{kx}\sin mx$:

If R(x) is in the form of

$$R(x) = e^{kx}\cos mx \;\; or \;\; e^{kx}\sin mx \; or \;\; e^{kx}(a\cos mx + b\sin mx)$$

Then y_p is considered as follows

$$y_p = e^{kx}(A\cos mx + B\sin mx)x^s$$

Where the coefficients $A, B \; and \; s$ are to be determined.

Case (IV): If R(x) is in the form of $e^{kx}(b_0 + b_1x + b_2x^2 + \; . \; . \; . \; + b_nx^n)$:

If R(x) is in the form of

$$R(x) = e^{kx}(b_0 + b_1x + b_2x^2 + \; . \; . \; . \; + b_nx^n)$$

Then y_p is considered as follows

$$y_p = e^{kx}(A_0 + A_1x + A_2x^2 + \; . \; . \; . \; + A_nx^n)x^s$$

Where the coefficients $A_0, A_1, A_2, \; . \; . \; ., A_n \; and \; s$ are to be determined.

Method of Variation of Parameters:

Consider the second order non-homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

Where p(x) and q(x) are continuous functions on an open interval I. Then the complementary solution is of the form

$$y_c(x) = c_1y_1(x) + c_2y_2(x)$$

Where $y_1(x)$ and $y_2(x)$ are linearly independent functions.

Then the particular solution of the equation (1) is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x).r(x)}{W(y_1,\, y_2)}dx + y_2(x)\int \frac{y_1(x).r(x)}{W(y_1,\, y_2)}dx$$

Where $W(y_1, y_2)$ is the Wronskian of two independent solutions $y_1(x)$ and $y_2(x)$ of the associated homogeneous equation of the non-homogeneous equation given by (1).

Example: Using the method of variation of parameters solve the differential equation $y''+9y=2\sec 3x$

Solution: Given differential equation is

$$y''+9y=2\sec 3x \qquad (1)$$

Associated homogeneous differential equation is

$$y''+9y=0$$

Corresponding auxiliary equation is

$$m^2+9=0$$

$$\Rightarrow \quad m=\pm 3i$$

Thus, the complementary solution is

$$y_c(x)=c_1\cos 3x+c_2\sin 3x \qquad (2)$$

On comparing equation (2) with

$$y_c(x)=c_1y_1+c_2y_2$$

$$\Rightarrow \quad y_1(x)=\cos 3x \quad and \quad y_2(x)=\sin 3x$$

Then wronskian of y_1 *and* y_2 is

$$W(y_1, y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1, y_2)=\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$=3\cos^2 3x+3\sin^2 3x$$

$$=3(\cos^2 3x+\sin^2 3x)$$

$W(y_1, y_2) = 3$

Given $r(x) = 2\sec 3x$

Using the method of variation of parameter we have

$$y_p(x) = -y_1(x)\int \frac{y_2(x).r(x)}{W(y_1, y_2)}dx + y_2(x)\int \frac{y_1(x).r(x)}{W(y_1, y_2)}dx$$

$$y_p(x) = -\cos 3x\int \frac{\sin 3\text{x}\,.2\sec 3\text{x}}{3}dx + \sin 3x\int \frac{\cos 3\text{x}\,.2\sec 3\text{x}}{3}dx$$

$$y_p(x) = -\frac{2}{3}\cos 3x\int \tan 3x dx + \frac{2}{3}\sin 3x\int dx$$

$$y_p(x) = -\frac{2}{3}\cos 3x.\frac{1}{3}\ln\sec 3x + \frac{2}{3}\sin 3x.x$$

$$y_p(x) = -\frac{2}{9}\cos 3x.\ln\sec 3x + \frac{x}{2}\sin 3x$$

Hence the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1\cos 3x + c_2\sin 3x - \frac{2}{9}\cos 3x.\ln\sec 3x + \frac{x}{2}\sin 3x$$

POSSIBLE QUESTIONS

PART – B (5 x 2 = 10)

1. Define linear combination of functions.
2. Explain a fundamental solution of function
3. Briefly explain Wronskian of functions.
4. Write any two properities of Wronskian of functions.
5. Write the general form of Euler's equation.

PART – C (5x 6 = 30 Marks)

1. Prove that the Wronskin of n solutions $f_1, f_2, \ldots \ldots \ldots \ldots . f_n$ of homogeneous equation is either identically zero on $a \leq x \leq b$ or else never zero on a $a \leq x \leq b$.
2. Given that y=x is the solution of $(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ find a linearly independent solution by reducing order.

3.Find the general solution of i) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$ ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$.

4. Solve the initial value problems $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$, y(0) = 3 , y'(0) = 5.

5.Find the general solutions of the differential equations $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 4e^{2x} - 21e^{-3x}$

6. Determine the linear combinations of functions with undetermined literal coefficients to use in finding a particular integral by the method of undetermined coefficients

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = x^3 + x + e^{-2x} \quad .$$

7.Explain briefly variation of parameters of differential equation.

8.Find the general solution of the differential equation

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y = 6(x^2+1)^2$$

9.Find the general solution of $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$

10. Consider the second order homogeneous linear differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2 = 0$

i) Find the two linearly independent solutions f_1 and f_2 of this equation which are such that $f_1(0) = 1$, $f_1'(0) = 0$ and $f_2(0) = 0$, $f_2'(0) = 1$.

ii) Express the solution $3e^x + 2e^{2x}$ as a linear combination of the two linearly independent Solutions of f_1 and f_2 defined in part (i).

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If f1,f2… … ….fm are m given functions and c1,c2… … … …cm are m constants then the expression ………………...is called a linear combination of f1,f2… … …fm.	c1 f1+c2 f2+…………+ cm fm	c1 f1*c2 f2*…………* cm fm	c1 f1/c2 f2/…………/c m fm	c1 f1-c2 f2- …………-cm fm	c1 f1+c2 f2+…………+ cm fm
If f1,f2… … ….fm are m given functions and c1,c2… … … …cm are m constants then the expression c1 f1+c2 f2+ … … … ….+cm fm is called a …………….. of f1,f2… … …fm.	non linear combination	homogeneous equation	non homogeneous equation	linear combination	linear combination
Any ………….. combination of solutions of the homogeneous linear differential equation is also a solution of homogeneous equation.	linear	nonlinear	zero	separable	linear
Any lienar combination of solutions of the ………………. linear differential equation is also a solution of homogeneous equation.	homogeneous	non homogeneous	singular	non singular	homogeneous
Any lienar combination of solutions of the homogeneous linear differential equation is also a …………………...of homogeneous equation.	value	separable	solution	exact	solution
The n functions f1,f2… … ….fn are called …………….. on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not all zero,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x.	linearly dependent	linearly independent	finite	infinite	linearly dependent
The n functions f1,f2… … ….fn are called linearly dependent on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not ………………..,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x.	all zero	one zero	two zero	n zero	all zero
The n functions f1,f2… … ….fn are called linearly dependent on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not all zero,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=…….... for all x.	1	2	3	0	0
The functions f1,f2… … ….fn are called …………………... on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x implies that c1=c2=… … … …=cn=0.	linearly dependent	linearly independent	finite	infinite	linearly independent
The functions f1,f2… … ….fn are called linearly independent on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x implies that c1=c2=… … … …=cn=…………...	0	1	2	3	0
The functions f1,f2… … ….fn are called linearly independent on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)…………………….. for all x implies that c1=c2=… … … …=cn=0	equal to 0	< 0	> 0	not equal to 0	equal to 0
The nth order ………………….linear differential equations always possess n solutions that are linealy independent.	homogeneous	non homogeneous	singular	non singular	homogeneous
The nth order homogeneous linear …………………. equations always possess n solutions that are linealy independent.	differential	integral	bernoulli	euler	differential
The nth order homogeneous linear differential equations always possess …………………….solutions that are linealy independent.	zero	finite	inifinite	n	n
The nth order homogeneous linear differential equations always possess n solutions that are …………………	linearly dependent	linearly independent	finite	infinite	linearly independent

Let f1, f2,.. .. ..fn be n……………..functions each of which has an (n-1)st derivative on real interval $a \le x \le b$	real	complex	finite	infinite	real
Let f1, f2,.. .. ..fn be n real functions each of which has an ------------derivative on real interval $a \le x \le b$	n	n-1	n+1	n+2	n-1
Let f1, f2,.. .. ..fn be n real functions each of which has an (n-1)st derivative on ------------ interval $a \le x \le b$	real	complex	finite	infinite	real
The ……………...solution of homogeneous equation is called the complementary function of equation.	explicit	implicit	general	particular	general
The general solution of -------------- equation is called the complementary function of equation.	homogeneous	non homogeneous	singular	non singular	homogeneous
The general solution of homogeneous equation is called the ------------------ function of equation.	real	complex	complementary	particular	complementary
Any ………………….solution of linear differential equation involving no arbitrary constants is called particular integralof this equation.	explicit	implicit	general	particular	particular
Any particular solution of linear differential equation involving -------------- arbitrary constants is called particular integralof this equation.	finite	infinite	no	one	no
Any particular solution of linear differential equation involving no arbitrary constants is called ……………. integralof this equation.	general	particular	finite	infinite	particular
The soluation--------------- is called the general solutionsof linear differential equations.	yc-yp	yc+yp	yc*yp	yc/yp	yc+yp
The soluation yc+yp is called the --------------- solutionsof linear differential equations.	explicit	implicit	general	particular	general
In general solution yc+yp where yc is ……………..function	real	complex	complementary	particular	complementary
In general solution yc+yp where yp is ……………..function	explicit	implicit	general	particular	particular

Unit-IV

Syllabus

LAPLACE TRANSFORMS
Definition-Sufficient conditions for the existence of the Laplace Transform, Laplace Transform of periodic functions- Some general theorems-Evaluation of integrals using Laplace Transform.

DEFINITION, EXISTENCE, AND BASIC PROPERTIES OF THE LAPLACE TRANSFORM

DEFINITION

Let f be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

for all values of s for which this integral exists. The function F defined by the integral (9.1) is called the Laplace transform *of the function f. We shall denote the Laplace transform F of f by $\mathscr{L}\{f\}$ and shall denote $F(s)$ by $\mathscr{L}\{f(t)\}$.*

Example

Consider the function f defined by

$$f(t) = 1, \qquad \text{for } t > 0.$$

Then

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}\cdot 1\,dt = \lim_{R\to\infty}\int_0^R e^{-st}\cdot 1\,dt = \lim_{R\to\infty}\left[\frac{-e^{-st}}{s}\right]_0^R$$

$$= \lim_{R\to\infty}\left[\frac{1}{s} - \frac{e^{-sR}}{s}\right] = \frac{1}{s}$$

for all $s > 0$. Thus we have

$$\mathscr{L}\{1\} = \frac{1}{s} \qquad (s > 0).$$

Example

Consider the function f defined by

$$f(t) = t, \qquad \text{for } t > 0.$$

Then

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\,dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot t\,dt = \lim_{R\to\infty} \left[-\frac{e^{-st}}{s^2}(st+1) \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{1}{s^2} - \frac{e^{-sR}}{s^2}(sR+1) \right] = \frac{1}{s^2}$$

for all $s > 0$. Thus

$$\mathscr{L}\{t\} = \frac{1}{s^2} \qquad (s > 0).$$

Example

Consider the function f defined by

$$f(t) = e^{at}, \qquad \text{for } t > 0.$$

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \lim_{R\to\infty} \int_0^R e^{(a-s)t}\,dt = \lim_{R\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s} \right] = -\frac{1}{a-s} = \frac{1}{s-a} \qquad \text{for all } s > a.$$

Thus

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \qquad (s > a).$$

Example

Consider the function f defined by

$$f(t) = \sin bt \qquad \text{for } t > 0.$$

$$\mathscr{L}\{\sin bt\} = \int_0^\infty e^{-st} \cdot \sin bt\, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot \sin bt\, dt$$

$$= \lim_{R\to\infty} \left[-\frac{e^{-st}}{s^2+b^2} (s \sin bt + b \cos bt) \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{b}{s^2+b^2} - \frac{e^{-sR}}{s^2+b^2} (s \sin bR + b \cos bR) \right]$$

$$= \frac{b}{s^2+b^2} \qquad \text{for all } s > 0.$$

Thus

$$\mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \qquad (s > 0).$$

Example

Consider the function f defined by

$$f(t) = \cos bt \qquad \text{for } t > 0.$$

$$\mathscr{L}\{\cos bt\} = \int_0^\infty e^{-st} \cdot \cos bt\, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cos bt\, dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{-st}}{s^2+b^2} (-s \cos bt + b \sin bt) \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{e^{-sR}}{s^2+b^2} (-s \cos bR + b \sin bR) + \frac{s}{s^2+b^2} \right]$$

$$= \frac{s}{s^2+b^2} \qquad \text{for all } s > 0.$$

Thus

$$\mathscr{L}\{\cos bt\} = \frac{s}{s^2+b^2} \qquad (s > 0).$$

THEOREM **Comparison Test for Improper Integrals**

Hypothesis

1. Let g and G be real functions such that

$$0 \le g(t) \le G(t) \qquad on \quad a \le t < \infty.$$

2. Suppose $\int_a^\infty G(t)\,dt$ exists.
3. Suppose g is integrable on every finite closed subinterval of $a \le t < \infty$.

Conclusion. *Then $\int_a^\infty g(t)\,dt$ exists.*

THEOREM

Hypothesis

1. Suppose the real function g is integrable on every finite closed subinterval of $a \le t \le \infty$.
2. Suppose $\int_a^\infty |g(t)|\,dt$ exists.

Conclusion. *Then $\int_a^\infty g(t)\,dt$ exists.*

We now state and prove an existence theorem for Laplace transforms.

THEOREM

Hypothesis. *Let f be a real function that has the following properties:*

1. f is piecewise continuous in every finite closed interval $0 \le t \le b$ $(b > 0)$.

2. f is of exponential order; that is, there exists α, $M > 0$, and $t_0 > 0$ such that

$$e^{-\alpha t}|f(t)| < M \qquad \textit{for} \quad t > t_0.$$

Conclusion. *The Laplace transform*

$$\int_0^\infty e^{-st} f(t)\, dt$$

of f exists for $s > \alpha$.

Proof. We have

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^{t_0} e^{-st} f(t)\, dt + \int_{t_0}^\infty e^{-st} f(t)\, dt.$$

By Hypothesis 1, the first integral of the right member exists. By Hypothesis 2,

$$e^{-st}|f(t)| < e^{-st} M e^{\alpha t} = M e^{-(s-\alpha)t}$$

for $t > t_0$. Also

$$\int_{t_0}^\infty M e^{-(s-\alpha)t}\, dt = \lim_{R\to\infty} \int_{t_0}^R M e^{-(s-\alpha)t}\, dt = \lim_{R\to\infty} \left[-\frac{M e^{-(s-\alpha)t}}{s-\alpha} \right]_{t_0}^R$$

$$= \lim_{R\to\infty} \left[\frac{M}{s-\alpha}\right] [e^{-(s-\alpha)t_0} - e^{-(s-\alpha)R}]$$

$$= \left[\frac{M}{s-\alpha}\right] e^{-(s-\alpha)t_0} \qquad \text{if} \quad s > \alpha.$$

Thus

$$\int_{t_0}^\infty M e^{-(s-\alpha)t}\, dt \qquad \text{exists for} \quad s > \alpha.$$

Finally, by Hypothesis 1, $e^{-st}|f(t)|$ is integrable on every finite closed subinterval of $t_0 \le t < \infty$. Thus, applying Theorem A with $g(t) = e^{-st}|f(t)|$ and $G(t) = Me^{-(s-\alpha)t}$, we see that

$$\int_{t_0}^{\infty} e^{-st}|f(t)|\,dt \qquad \text{exists if} \quad s > \alpha.$$

In other words,

$$\int_{t_0}^{\infty} |e^{-st} f(t)|\,dt \qquad \text{exists if} \quad s > \alpha,$$

and so by Theorem B

$$\int_{t_0}^{\infty} e^{-st} f(t)\,dt$$

also exists if $s > \alpha$. Thus the Laplace transform of f exists for $s > \alpha$.

Let us look back at this proof for a moment. Actually we showed that if f satisfies the hypotheses stated, then

$$\int_{t_0}^{\infty} e^{-st}|f(t)|\,dt \qquad \text{exists if} \quad s > \alpha.$$

Further, Hypothesis 1 shows that

$$\int_{0}^{t_0} e^{-st}|f(t)|\,dt \qquad \text{exists.}$$

Thus

$$\int_{0}^{\infty} e^{-st}|f(t)|\,dt \qquad \text{exists if} \quad s > \alpha.$$

In other words, if f satisfies the hypotheses of Theorem 9.1, then not only does $\mathscr{L}\{f\}$ exist for $s > \alpha$, but also $\mathscr{L}\{|f|\}$ exists for $s > \alpha$. That is,

$$\int_{0}^{\infty} e^{-st} f(t)\,dt \qquad \text{is absolutely convergent for} \quad s > \alpha.$$

Basic Properties of the Laplace Transform

Let f_1 and f_2 be functions whose Laplace transforms exist, and let c_1 and c_2 be constants. Then

$$\mathscr{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathscr{L}\{f_1(t)\} + c_2 \mathscr{L}\{f_2(t)\}.$$

Translation Property

Hypothesis. *Suppose f is such that $\mathscr{L}\{f\}$ exists for $s > \alpha$.*

Conclusion. *For any constant a,*

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

for $s > \alpha + a$, where $F(s)$ denotes $\mathscr{L}\{f(t)\}$.

Example

Find $\mathscr{L}\{e^{at}\sin bt\}$. We let $f(t) = \sin bt$. Then $\mathscr{L}\{e^{at}\sin bt\} = F(s-a)$, where

$$F(s) = \mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \qquad (s > 0).$$

Thus

$$F(s-a) = \frac{b}{(s-a)^2+b^2}$$

and so

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \qquad (s > a).$$

Example

Find the Laplace transform of

$$g(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \sin t, & t > \dfrac{\pi}{2}. \end{cases}$$

$$g(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \cos\left(t - \dfrac{\pi}{2}\right), & t > \dfrac{\pi}{2}. \end{cases}$$

$$u_{\pi/2}(t) f(t - \pi/2) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \cos\left(t - \dfrac{\pi}{2}\right), & t > \dfrac{\pi}{2}, \end{cases}$$

$$F(s) = \mathscr{L}\{\cos t\} = \frac{s}{s^2 + 1}.$$

$a = \pi/2$, we obtain

$$\mathscr{L}\{g(t)\} = \mathscr{L}\{u_{\pi/2}(t) f(t - \pi/2)\} = \frac{s e^{-(\pi/2)s}}{s^2 + 1}.$$

THEOREM

Hypothesis. *Suppose f is a periodic function of period P which satisfies the hypotheses of Theorem :*

Then

$$\mathscr{L}\{f(t)\} = \frac{\int_0^P e^{-st} f(t)\, dt}{1 - e^{-Ps}}.$$

By definition of the Laplace transform,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

The integral on the right can be broken up into the infinite series of integrals

$$\int_0^P e^{-st} f(t)\, dt + \int_P^{2P} e^{-st} f(t)\, dt + \int_{2P}^{3P} e^{-st} f(t)\, dt + \cdots + \int_{nP}^{(n+1)P} e^{-st} f(t)\, dt \cdots. \quad (9.28)$$

We now transform each integral in this series. For each $n = 0, 1, 2, \ldots$, let $t = u + nP$ in the corresponding integral

$$\int_{nP}^{(n+1)P} e^{-st} f(t)\, dt.$$

Then for each $n = 0, 1, 2, \ldots$, this becomes

$$\int_0^P e^{-s(u+nP)} f(u + nP)\, du.$$

$$e^{-nPs} \int_0^P e^{-su} f(u)\, du.$$

Hence the infinite series takes the form

$$\int_0^P e^{-su} f(u)\, du + e^{-Ps} \int_0^P e^{-su} f(u)\, du + e^{-2Ps} \int_0^P e^{-su} f(u)\, du + \cdots + e^{-nPs} \int_0^P e^{-su} f(u)\, du + \cdots$$

$$= [1 + e^{-Ps} + e^{-2Ps} + \cdots + e^{-nPs} + \cdots] \int_0^P e^{-su} f(u)\, du.$$

Now observe that the infinite series in brackets is a geometric series of first term 1 and common ratio $r = e^{-Ps} < 1$. Such a series converges to $1/(1 - r)$, and hence the series in brackets converges to $1/(1 - e^{-Ps})$. Therefore the right member of (9.30), and hence that of reduces to

$$\frac{\int_0^P e^{-su} f(u)\, du}{1 - e^{-Ps}}.$$

we have

$$\mathscr{L}\{f(t)\} = \frac{\int_0^P e^{-st} f(t)\, dt}{1 - e^{-Ps}}$$

Find the Laplace transform of f defined on $0 \leq t < 4$ by

$$f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ -1, & 2 \leq t < 4, \end{cases}$$

and for all other positive t by the periodicity condition

$$f(t + 4) = f(t).$$

The graph of f is shown in Figure 9.5. Clearly this function f is periodic of period $P = 4$. Applying formula (9.26) of Theorem 9.8, we find

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{\int_0^4 e^{-st} f(t)\,dt}{1 - e^{-4s}} \\
&= \frac{1}{1 - e^{-4s}} \left[\int_0^2 e^{-st}(1)\,dt + \int_2^4 e^{-st}(-1)\,dt \right] \\
&= \frac{1}{1 - e^{-4s}} \left[\frac{-e^{-st}}{s}\Bigg|_0^2 + \frac{e^{-st}}{s}\Bigg|_2^4 \right] \\
&= \frac{1}{1 - e^{-4s}} \left(\frac{1}{s}\right)[-e^{-2s} + 1 + e^{-4s} - e^{-2s}] \\
&= \frac{1 - 2e^{-2s} + e^{-4s}}{s(1 - e^{-4s})} = \frac{(1 - e^{-2s})^2}{s(1 - e^{-2s})(1 + e^{-2s})} \\
&= \frac{1 - e^{-2s}}{s(1 + e^{-2s})}.
\end{aligned}$$

POSSIBLE QUESTIONS

PART - B (5 x 2 = 10 Marks)

1. Define Laplace Transform
2. Give sufficient condition for the existence of Laplace Transform
3. Find L(1),L(t) and L(t^2) .
4. Find L(t^2 +2t+3)
5. Define piecewise continuity.

PART – C (5 x 6 = 30 Marks)

1. Find $L\{f(t)\}$ where $f(t) = 0$ when $0 < t \leq 2$

$$= 3 \; when \; t > 2.$$

2. Find $L\{f(t)\}$ where $f(t) = 1$ when $0 < t < b$

$$= -1 \; when \; b < t < 2b.$$

3. Find $L\{f(t)\}$ where $f(t) = t$ when $0 < t < b$

$$= 2b - t \; when \; b < t < 2b.$$

4. Find $L(te^{-t} sin\; t)$.

5. Find $L(\sin^3 2t)$.

6. Prove that $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\, dt = \log 2.$

7. Evaluate $\int_0^\infty t\, e^{-3t} \cos t\; dt.$

8. If $L\{f(t)\} = F(s)$ then prove that $L\{tf(t)\} = -\frac{d}{ds} F(S).$

9. Find the Laplace transform of $\frac{\sin at}{t}$.

10. Find the Laplace transform of $\frac{1-e^{t}}{t}$.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A Laplace Transform exists when ______	The function is piece-wise continuous	The function is non-linear	The function is piecewise discrete	The function is of differential order	The function is piece-wise continuous
Where is the ROC defined or specified for the signals containing causal as well as anti-causal terms?	Greater than the largest pole	Between two poles	Less than the smallest pole	Cannot be defined	Between two poles
Which result is generated/ obtained by the addition of a step to a ramp function ?	Step Function shifted by an amount equal to ramp	Ramp Function shifted by an amount equal to step	Step function of zero slope	Step function of zero slope	Ramp Function shifted by an amount equal to step
Unilateral Laplace Transform is applicable for the determination of linear constant coefficient differential equations with ________	Zero initial condition	Non-zero initial condition	Zero final condition	Non-zero final condition	Non-zero initial condition
What should be location of poles corresponding to ROC for bilateral Inverse Laplace Transform especially for determining the nature of time domain signal?	On L.H.S of ROC	On R.H.S of ROC	On both sides of ROC	None of the above	On both sides of ROC
Generally, the convolution process associated with the Laplace Transform in time domain results into________	Simple multiplication in complex frequency domain	Simple division in complex frequency domain	Simple multiplication in complex time domain	Simple division in complex time domain	Simple multiplication in complex frequency domain
When is the system said to be causal as well as stable in accordance to pole/zero of ROC specified by system transfer function?	Only if all the poles of system transfer function lie in left-half of S-plane	Only if all the poles of system transfer function lie in right-half of S-plane	Only if all the poles of system transfer function lie at the centre of S-	None of the above	Only if all the poles of system transfer function lie in left-half of S-plane
Transformation in which function in one space is transformed to another space by process of integration that involves kernel is termed as	differential transform	integral transform	algebraic transform	rational transform	integral transform
(a,b] or [a,b) represents	infinite interval	closed interval	half open interval	open interval	half open interval
A function has a Laplace transform if	$t<0$	$t>0$	$t\leq 0$	$t \geq 0$	$t \geq 0$
Laplace transform of function $f(t)=\cos(\pi t)$ is	$s/(s+\pi)$	$s/(s-\pi)$	$s/(s^2+\pi^2)$	$s/((s-a)^2+\pi^2)$	$s/(s^2+\pi^2)$
If Laplace transform of function exists, it is determined	similarly	constantly	uniquely	identically	uniquely
If two continuous functions have same transform, they are completely	unique	constant	identical	zero	identical
When taking Laplace transform of function $f(t)=1$ where $t \geq 0$, integral limit will be from 0 to ∞ results in	proper integral	improper integral	singular integral	finite integral	improper integral
Laplace transform of function $f(t)=\sin(wt)$ is	$s/(s^2+w^2)$	$s/(s-w)$	$s/(s+w)$	$w/(s^2+w^2)$	$w/(s^2+w^2)$
Laplace transform of function $f(t)=e^{at}$ where $t \geq 0$ is	$1/s$	$1/(s+a)$	$1/(s-a)$	$s-a$	$1/(s-a)$
Laplace transform of function $f(t)=\cos(wt)$ is	$s/(s+w)$	$s/(s-w)$	$s/(s^2+w^2)$	$w/(s^2+w^2)$	$s/(s^2+w^2)$
Laplace transform when applied to function, changes that function into new function by using a process that involves	integration	differentiation	binary manipulation	logical manipulation	integration
In Laplace transform, kernel is	integral x e	integral x e^{st}	e^{-st}	integral x e^{-st}	e^{-st}
If f(t) is a function defined for all $t \geq 0$; its Laplace transform limit will be	0	∞	0 to ∞	−∞ to 0	0 to ∞
Kernel e^{-st} in Laplace transform is represented as	k(s,t)	k(s)	k(t)	k(t;s)	k(s,t)

Unit-V

Syllabus

INVERSE LAPLACE TRANSFORMS
Solving ordinary differential equations with constant coefficients using Laplace Transforms-Solving a system of differential equations using Laplace Transforms.

The Inverse Transform

Thus far in this chapter we have been concerned with the following problem: Given a function f, defined for $t > 0$, to find its Laplace transform, which we denoted by $\mathscr{L}\{f\}$ or F. Now consider the inverse problem: Given a function F, to find a function f whose Laplace transform is the given F. We introduce the notation $\mathscr{L}^{-1}\{F\}$ to denote such a function f, denote $\mathscr{L}^{-1}\{F(s)\}$ by $f(t)$, and call such a function an *inverse transform* of F. That is,

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

means that $f(t)$ is such that

$$\mathscr{L}\{f(t)\} = F(s).$$

THEOREM

Hypothesis. *Let f and g be two functions that are continuous for $t \geq 0$ and that have the same Laplace transform F.*

Conclusion. $f(t) = g(t)$ *for all* $t \geq 0$.

Thus if it is known that a given function F has a *continuous* inverse transform f, then f is the *only* continuous inverse transform of F. Let us consider the following example.

$$g(t) = \begin{cases} 1, & 0 < t < 3, \\ 2, & t = 3, \\ 1, & t > 3. \end{cases}$$

Then

$$\mathscr{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\, dt = \int_0^3 e^{-st}\, dt + \int_3^\infty e^{-st}\, dt$$
$$= \left[-\frac{e^{-st}}{s}\right]_0^3 + \lim_{R\to\infty}\left[-\frac{e^{-st}}{s}\right]_3^R = \frac{1}{s} \qquad \text{if } s > 0.$$

Thus this discontinuous function g is also an inverse transform of F defined by $F(s) = 1/s$. However, we again emphasize that the only *continuous* inverse transform of F defined by $F(s) = 1/s$ is f defined for all t by $f(t) = 1$. Indeed we write

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

with the understanding that f defined for all t by $f(t) = 1$ is the *unique continuous* inverse transform of F defined by $F(s) = 1/s$.

find $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + 6s + 13}\right\}$.

Solution. ' we would first look for $F(s) = \dfrac{1}{as^2 + bs + c}$. However, we find no such $F(s)$; but we do find $F(s) = \dfrac{b}{(s+a)^2 + b^2}$ (number 11). We can put the given expression $\dfrac{1}{s^2 + 6s + 13}$ in this form as follows:

$$\frac{1}{s^2 + 6s + 13} = \frac{1}{(s+3)^2 + 4} = \frac{1}{2} \cdot \frac{2}{(s+3)^2 + 2^2}.$$

Thus, using number 11 of Table 9.1, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2 + 6s + 13}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+3)^2 + 2^2}\right\} = \frac{1}{2} e^{-3t} \sin 2t.$$

find $\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$.

Solution. No entry of this form appears in the $F(s)$ column of Table 9.1. We employ the method of partial fractions. We have

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

and hence

$$1 = (A+B)s^2 + Cs + A.$$

Thus

$$A + B = 0, \qquad C = 0, \quad \text{and} \quad A = 1.$$

LAPLACE TRANSFORMS

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1	1	$\frac{1}{s}$
2	e^{at}	$\frac{1}{s-a}$
3	$\sin bt$	$\frac{b}{s^2+b^2}$
4	$\cos bt$	$\frac{s}{s^2+b^2}$
5	$\sinh bt$	$\frac{b}{s^2-b^2}$
6	$\cosh bt$	$\frac{s}{s^2-b^2}$
7	$t^n (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
8	$t^n e^{at} (n = 1, 2, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$

9	$t \sin bt$	$\dfrac{2bs}{(s^2+b^2)^2}$
10	$t \cos bt$	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$
11	$e^{-at} \sin bt$	$\dfrac{b}{(s+a)^2+b^2}$
12	$e^{-at} \cos bt$	$\dfrac{s+a}{(s+a)^2+b^2}$
13	$\dfrac{\sin bt - bt \cos bt}{2b^3}$	$\dfrac{1}{(s^2+b^2)^2}$
14	$\dfrac{t \sin bt}{2b}$	$\dfrac{s}{(s^2+b^2)^2}$
15	$u_a(t)$ [see equations (9.19) and (9.21)]	$\dfrac{e^{-as}}{s}$
16	$u_a(t)f(t-a)$ [see Theorem 9.7]	$e^{-as}F(s)$

From these equations, we have the partial fractions decomposition

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

By number 1 of Table 9.1, $\mathscr{L}^{-1}\{1/s\} = 1$ and by number 4, $\mathscr{L}^{-1}\{s/(s^2+1)\} = \cos t$.
Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

Find

$$\mathscr{L}^{-1}\left\{\frac{5}{s}-\frac{3e^{-3s}}{s}-\frac{2e^{-7s}}{s}\right\}.$$

Solution. By number 1 of Table 9.1, we at once have

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}=1.$$

By number 15, we see that

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}=u_a(t).$$

Here u_a is the unit step function
by

$$u_a(t)=\begin{cases}0, & 0<t<a,\\ 1, & t>a,\end{cases}$$

and for $a=0$ by

$$u_0(t)=1 \quad \text{for } t>0.$$

Applying with $a=3$ and $a=7$, respectively, we have

$$\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}=u_3(t)=\begin{cases}0, & 0<t<3,\\ 1, & t>3,\end{cases}$$

and

$$\mathscr{L}^{-1}\left\{\frac{e^{-7s}}{s}\right\}=u_7(t)=\begin{cases}0, & 0<t<7,\\ 1, & t>7.\end{cases}$$

Thus we obtain

$$\mathscr{L}^{-1}\left\{\frac{5}{s}-\frac{3e^{-3s}}{s}-\frac{2e^{-7s}}{s}\right\}=5-3u_3(t)-2u_7(t).$$

we see that this equals

$$\begin{cases}5-0-0, & 0<t<3,\\ 5-3-0, & 3<t<7,\\ 5-3-2, & t>7;\end{cases}$$

and hence

$$\mathscr{L}^{-1}\left\{\frac{5}{s}-\frac{3e^{-3s}}{s}-\frac{2e^{-7s}}{s}\right\}=\begin{cases}5, & 0<t<3,\\ 2, & 3<t<7,\\ 0, & t>7.\end{cases}$$

Find

$$\mathscr{L}^{-1}\left\{e^{-4s}\left(\frac{2}{s^2}+\frac{5}{s}\right)\right\}.$$

Solution. This is of the form $\mathscr{L}^{-1}\{e^{-as}F(s)\}$, where $a=4$ and $F(s)=2/s^2+5/s$. By number 16 of Table 9.1, we see that

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}=u_a(t)f(t-a).$$

Here u_a is the unit step function defined for $a>0$ by (9.32) and $f(t)=\mathscr{L}^{-1}\{F(s)\}$ [see Theorem 9.7]. By number 1 of Table 9.1, we again find $\mathscr{L}^{-1}\{1/s\}=1$; and by number 7 with $n=1$, we obtain $\mathscr{L}^{-1}\{1/s^2\}=t$. Thus

$$f(t)=\mathscr{L}^{-1}\{F(s)\}=\mathscr{L}^{-1}\left\{\frac{2}{s^2}+\frac{5}{s}\right\}=2t+5,$$

and so $f(t-4)=2(t-4)+5=2t-3$. Then by (9.35), with $a=4$,

$$\mathscr{L}^{-1}\{e^{-4s}F(s)\}=u_4(t)f(t-4);$$

that is,

$$\mathscr{L}^{-1}\left\{e^{-4s}\left(\frac{2}{s^2}+\frac{5}{s}\right)\right\}=u_4(t)[2t-3]=\begin{cases}0, & 0<t<4,\\ 2t-3, & t>4.\end{cases}$$

The Convolution

DEFINITION

Let f and g be two functions that are piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order. The function denoted by $f * g$ and defined by

$$f(t) * g(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau$$

is called the convolution *of the functions f and g.*

Let us change the variable of integration in by means of the substitution $u = t - \tau$. We have

$$f(t) * g(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau = -\int_t^0 f(t - u) g(u)\, du$$

$$= \int_0^t g(u) f(t - u)\, du = g(t) * f(t).$$

Thus we have shown that

$$f * g = g * f$$

Hypothesis. *Let the functions f and g be piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order e^{at}.*

Conclusion

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$$

for $s > a$.

Proof. By definition of the Laplace transform, $\mathscr{L}\{f * g\}$ is the function defined by

$$\int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right]dt.$$

The integral (9.39) may be expressed as the iterated integral

$$\int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)\,d\tau\,dt.$$

Further, the iterated integral (9.40) is equal to the double integral

$$\iint_{R_1} e^{-st} f(\tau)g(t-\tau)\,d\tau\,dt,$$

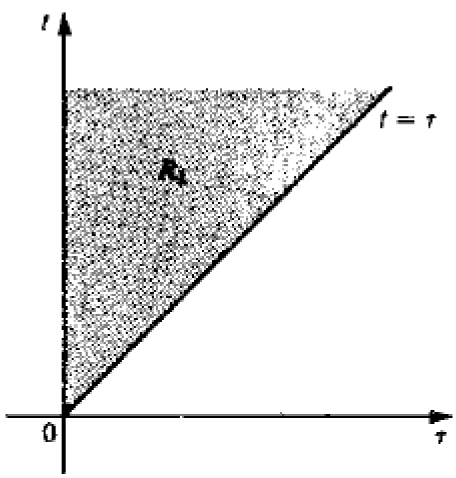
where R_1 is the 45° wedge bounded by the lines $\tau = 0$ and $t = \tau$ (see Figure 9.6).

We now make the change of variable

$$u = t - \tau,$$

$$v = \tau,$$

to transform the double integral (9.41). The change of variables (9.42) has Jacobian 1 and transforms the region R_1 in the τ, t plane into the first quadrant of the u, v plane.



Thus the double integral (9.41) transforms into the double integral

$$\iint_{R_2} e^{-s(u+v)} f(v) g(u)\, du\, dv,$$

where R_2 is the quarter plane defined by $u > 0$, $v > 0$ (see Figure 9.7).
The double integral (9.43) is equal to the iterated integral

$$\int_0^\infty \int_0^\infty e^{-s(u+v)} f(v) g(u)\, du\, dv.$$

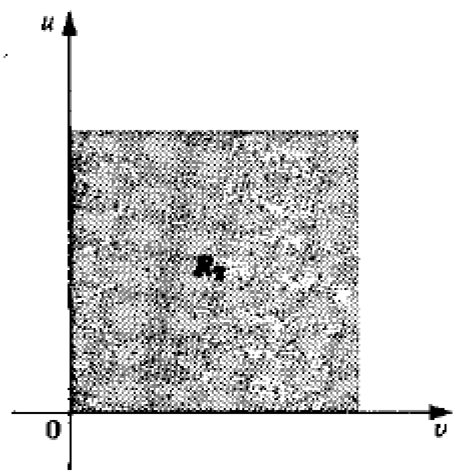
But the iterated integral (9.44) can be expressed in the form

$$\int_0^\infty e^{-sv} f(v)\, dv \int_0^\infty e^{-su} g(u)\, du.$$

But the left-hand integral in (9.45) defines $\mathscr{L}\{f\}$ and the right-hand integral defines $\mathscr{L}\{g\}$. Therefore the expression (9.45) is precisely $\mathscr{L}\{f\}\mathscr{L}\{g\}$.

We note that since the integrals involved are absolutely convergent for $s > a$, the operations performed are indeed legitimate for $s > a$. Therefore we have shown that

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f\}\mathscr{L}\{g\} \qquad \text{for } s > a.$$



Denoting $\mathscr{L}\{f\}$ by F and $\mathscr{L}\{g\}$ by G, we may write the conclusion (9.38) in the form

$$\mathscr{L}\{f(t) * g(t)\} = F(s)G(s).$$

Hence, we have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau,$$

and using (9.37), we also have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = g(t) * f(t) = \int_0^t g(\tau)f(t-\tau)\, d\tau.$$

Suppose we are given a function H and are required to determine $\mathscr{L}^{-1}\{H(s)\}$. If we can express $H(s)$ as a product $F(s)G(s)$, where $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$ are known, then we can apply either (9.46) or (9.47) to determine $\mathscr{L}^{-1}\{H(s)\}$.

Find $\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ using the convolution

Solution. We write $1/s(s^2+1)$ as the product $F(s)G(s)$, where $F(s) = 1/s$ and $G(s) = 1/(s^2+1)$. By Table 9.1, number 1, $f(t) = \mathscr{L}^{-1}\{1/s\} = 1$, and by number 3, $g(t) = \mathscr{L}^{-1}\{1/(s^2+1)\} = \sin t$. Thus by (9.46),

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = f(t) * g(t) = \int_0^t 1 \cdot \sin(t-\tau)\, d\tau,$$

and by (9.47),

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = g(t) * f(t) = \int_0^t \sin\tau \cdot 1\, d\tau.$$

The second of these two integrals is slightly more simple. Evaluating it, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

LAPLACE TRANSFORM SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Solve the initial-value problem

$$\frac{dy}{dt} - 2y = e^{5t},$$

$$y(0) = 3$$

Taking the Laplace transform of both sides of the differential equation we have

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} - 2\mathscr{L}\{y(t)\} = \mathscr{L}\{e^{5t}\}.$$

Using Theorem 9.4 with $n = 1$ (or Theorem 9.3) and denoting $\mathscr{L}\{y(t)\}$ by $Y(s)$, we may express $\mathscr{L}\{dy/dt\}$ in terms of $Y(s)$ and $y(0)$ as follows:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0).$$

Applying the initial condition (9.53), this becomes

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - 3.$$

Using this, the left member of Equation (9.54) becomes $sY(s) - 3 - 2Y(s)$. From Table 9.1, number 2, $\mathscr{L}\{e^{5t}\} = 1/(s - 5)$. Thus Equation (9.54) reduces to the algebraic equation

$$[s - 2]Y(s) - 3 = \frac{1}{s - 5}$$

in the unknown $Y(s)$.

$$[s-2]\,Y(s) = \frac{3s-14}{s-5}$$

and so

$$Y(s) = \frac{3s-14}{(s-2)(s-5)}.$$

We must now determine

$$\mathscr{L}^{-1}\left\{\frac{3s-14}{(s-2)(s-5)}\right\}.$$

We employ partial fractions. We have

$$\frac{3s-14}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5},$$

and so $3s - 14 = A(s-5) + B(s-2)$. From this we find that

$$A = \tfrac{8}{3} \quad \text{and} \quad B = \tfrac{1}{3},$$

and so

$$\mathscr{L}^{-1}\left\{\frac{3s-14}{(s-2)(s-5)}\right\} = \frac{8}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\}.$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} \quad \text{and} \quad \mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\} = e^{5t}.$$

Thus the solution of the given initial-value problem is

$$y = \tfrac{8}{3}e^{2t} + \tfrac{1}{3}e^{5t}.$$

LAPLACE TRANSFORM SOLUTION OF LINEAR SYSTEMS

Use Laplace transforms to find the solution of the system

$$\frac{dx}{dt} - 6x + 3y = 8e^t,$$

$$\frac{dy}{dt} - 2x - y = 4e^t,$$

that satisfies the initial conditions

$$x(0) = -1,$$

$$y(0) = 0.$$

Step 1. Taking the Laplace transform of both sides of each differential equation of system we have

$$\mathscr{L}\left\{\frac{dx}{dt}\right\} - 6\mathscr{L}\{x(t)\} + 3\mathscr{L}\{y(t)\} = \mathscr{L}\{8e^t\},$$

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} - 2\mathscr{L}\{x(t)\} - \mathscr{L}\{y(t)\} = \mathscr{L}\{4e^t\}.$$

Denote $\mathscr{L}\{x(t)\}$ by $X(s)$ and $\mathscr{L}\{y(t)\}$ by $Y(s)$. Then applying and the initial conditions (9.87), we have

$$\mathscr{L}\left\{\frac{dx}{dt}\right\} = sX(s) - x(0) = sX(s) + 1,$$

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s).$$

$$\mathscr{L}\{8e^t\} = \frac{8}{s-1} \quad \text{and} \quad \mathscr{L}\{4e^t\} = \frac{4}{s-1}.$$

$$sX(s) + 1 - 6X(s) + 3Y(s) = \frac{8}{s-1},$$
$$sY(s) - 2X(s) - Y(s) = \frac{4}{s-1},$$

which simplify to the form

$$(s-6)X(s) + 3Y(s) = \frac{8}{s-1} - 1,$$
$$-2X(s) + (s-1)Y(s) = \frac{4}{s-1},$$

or

$$(s-6)X(s) + 3Y(s) = \frac{-s+9}{s-1},$$
$$-2X(s) + (s-1)Y(s) = \frac{4}{s-1}.$$

Step 2. We solve the linear algebraic system of the two equations in the two "unknowns" $X(s)$ and $Y(s)$. We have

$$(s-1)(s-6)X(s) + 3(s-1)Y(s) = -s+9,$$

$$-6X(s) + 3(s-1)Y(s) = \frac{12}{s-1}.$$

Subtracting we obtain

$$(s^2 - 7s + 12)X(s) = -s + 9 - \frac{12}{s-1},$$

from which we find

$$X(s) = \frac{-s^2 + 10s - 21}{(s-1)(s-3)(s-4)} = \frac{-s+7}{(s-1)(s-4)}.$$

In like manner, we find

$$Y(s) = \frac{2s-6}{(s-1)(s-3)(s-4)} = \frac{2}{(s-1)(s-4)}.$$

We must now determine

$$x(t) = \mathscr{L}^{-1}\{X(s)\} = \mathscr{L}^{-1}\left\{\frac{-s+7}{(s-1)(s-4)}\right\}$$

and

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{2}{(s-1)(s-4)}\right\}.$$

We first find $x(t)$. We use partial fractions and write

$$\frac{-s+7}{(s-1)(s-4)} = \frac{A}{s-1} + \frac{B}{s-4}.$$

From this we find

$$A = -2 \quad \text{and} \quad B = 1.$$

Thus

$$x(t) = -2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\},$$

we obtain

$$x(t) = -2e^{t} + e^{4t}.$$

In like manner, we find $y(t)$. Doing so, we obtain

$$y(t) = -\tfrac{2}{3}e^{t} + \tfrac{2}{3}e^{4t}.$$

The pair defined by (9.92) and (9.93) constitute the solution of the given system (9.86) that satisfies the given initial conditions (9.87).

UNIT V

POSSIBLE QUESTIONS

PART - B (5 x 2 = 10 Marks)

1. Find $L^{-1}[\frac{1}{(s+a)^2}]$
2. Find $L^{-1}[\frac{s-3}{(s-3)^2+4}]$
3. Find $L^{-1}[\frac{s}{(s+2)^2}]$.
4. Find the inverse Laplace transform of $\frac{1}{s(s+a)}$.
5. Find the inverse Laplace transform of $\frac{1}{(s-3)^5}$

PART – C (5 x 6 = 30 Marks)

1. Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$.
2. Find $L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$.
3. Find $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^5}\right]$.
4. Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$.
5. Find the inverse Laplace transform of $\frac{1}{(s^2+a^2)^2}$.
6. Find the inverse Laplace transform of $\frac{1}{s(s+1)(s+2)}$.
7. Solve the equation $\frac{d^2y}{dt^2}+2\frac{dy}{dx}-3y=\sin t$ given that $y=\frac{dy}{dx}=0$ when $t=0$.
8. Show the solution of the differential equation $\frac{d^2y}{dt^2}+4y=A\sin kt$ which is such that $y=0$ and $\frac{dy}{dx}=0$ when $t=0$ is $y=A\,\frac{\sin kt-\frac{k}{2}\sin 2t}{4-k^2}$ if k ≠ 2. If $k=2$, $y=\frac{A(\sin 2t-2t\cos 2t)}{8}$.
9. Solve the simultaneous equations
$3\frac{dx}{dt}+\frac{dy}{dt}+2x=1.$
$\frac{dx}{dt}+4\frac{dy}{dt}+3y=0.$
Given $x=0=y$ at $t=0$.
10. Solve the simultaneous equations
$\frac{dx}{dt}-\frac{dy}{dt}-2x+2y=1-2t.$
$\frac{d^2x}{dt^2}+2\frac{dy}{dt}+x=0.$
with the conditions $x=0,\ y=0, \frac{dx}{dt}=0$ when $t=0$.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
A function in which interval can be broken into a finite number of sub-intervals on which function is continuous on each open sub-interval and has a finite limit at endpoints of each sub-interval is called ………	piecewise continuous	piecewise discontinuous	single discontinuous	single continuous	piecewise continuous
Laplace transform is basically an …….	differential	algebraic	integral	rational	integral
Inverse Laplace transform of $F(s)=(5s+1)/(s^2-25)$ is …….	5cosh5t+1/5sin	5cos5t+1/5sin5	cos5t+1/5sin5t	5cosh5t+sinh5t	5cosh5t+1/5sin
Initial Value Problems 'IVP' are solved without first determing a general solution in ………	differentiation	integration	Laplace transform	None of these	Laplace transform
In Laplace transform, subsidiary equation can only be solved by …..	integration	differentiation	algebraic manipulation	logical manipulation	algebraic manipulation
When ODE is transformed into algebraic equation, resultant equation is called	real equation	primary equation	subsidiary equation	diary equation	subsidiary equation
A definite integral that has either or both limits infinite or an integrand that approaches infinity at one or more points in range of integration is called	improper integral	proper integral	singular integral	finite integral	improper integral
A piecewise continuous function is a function that have breaks of	infinite number	finite number	complex number	real number	finite number
The inverse Laplace transform of 1/s is	sin t	cos t	1	t sin t.	1
The inverse Laplace transform of F(s-a) where F(s) is the Laplace transform of f(t) is	e^t	e^s	e^(at) f(t)	1	e^(at) f(t)
. The inverse Laplace transform of s/(s^2- 4) is …….	sinh 2t	sinh 4t	cosh 2t	cosh 4t	cosh 2t
The inverse Laplace transform of 1/(s^2 - 9) is ……..	sin at	sinh at	1/3 sinh 3t	1/9 sinh 9t	1/3 sinh 3t
The inverse Laplace transform of 1/ (s-4) is ……….	1	e^ (2t)	e^ (4t)	e^ (at)	e^ (4t)
The inverse Laplace transform of 1/ ((s-5)^2 + 1) is ………	e^(5t) sin t.	e^t sin 5t	sin 5t	sin t	e^(5t) sin t.
The inverse Laplace transform of 1/ s^2 is ……….	e^ (2t)	t	t^2	e^t	t
The inverse Laplace transform of s/ (s^2 + 4) is ……..	cos 2t	sin 2t	cosh 2t	sinh 2t	cos 2t
The inverse Laplace transform of 1/ s^4 is ……..	e^(4t)	t^3 / 6	t^4	sin 4t	t^3 / 6
The inverse Laplace transform of F’(s) where F(s) is the Laplace transform of f(t) is ………..	f(t)	–t f(t)	t f(t)	t.	–t f(t)
What is the value of L[t f(t)] ?	F(s)	– F’(s)	1	f(t)	– F’(s)
What is the value of L[t^2 f(t)] ?	F(s)	F’(s)	F’’(s)	–F’(s).	**F’’(s)**

Reg no-------------

(I8MMU201)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore-21

DEPARTMENT OF MATHEMATICS

I Internal Test - Dec'2018

Differential Equations

Date: 17.12.18(AN) **Time: 2 Hours**

Class: I-B.ScMathematics **Maximum Marks:50**

PART-A(20×1=20 Marks)

Answer all the Questions:

1. An equation involving one or more dependent variables with respect to one or more independent variables is called……………………...
   a) differential equations b) intergral equation
   c) Eulers equation d) Laplace equation
2. A partial differential equation requires ……..
   a) exactly one independent variable
   b) two or more independent variables
   c) more than one dependent variable
   d) equal number of dependent and independent variables
3. The order of the differential equation $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^5 - 5y = 0$ is ……………
   a)1 b)3 c)5 d)7
4. Linear ordinary differential equations are further classified according to the nature of the coefficients of the ……………….variables and its derivatives.
   a) single b) dependent
   c) independent d) constant
5. The expression $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equations in a domain D if there exists a function of two variable such that the expression equals the ……….. for all (x,y) in D.
   a) differential b) ordinary differential
   c) partial differential d) total differential
6. A first order differential equation is ……………… in the dependent variable y and the independent variable x if it is can be written in the form $(dy/dx) + P(x)y = Q(x)$.
   a)differential b)integral c)linear d) non linear
7. Polynomial $ar^2 + br + c = 0$ is called………
   a) characteristic polynomial b) trivial polynomial
   c)determinant polynomial d) singular polynomial
8. Let f be a real function defined for all x in a real interval I and having ………….order derivatives then the function f is called explicit solution of the differential equations.
   a)1^{st} b)2^{nd} c)n^{th} d)$n + 1^{th}$
9. The standard form of first order differential equations differential form is…………
   a) $M(x,y)dx + N(x,y)dy = 0$ b) $M(x,y)dx - N(x,y)dy = 0$
   c) $M(x,y)dx * N(x,y)dy = 0$ d) $M(x,y)dx\ /\ N(x,y)dy = 0$
10. A ordinary differential equation requires ……..
    a) exactly one independent variable
    b) two or more independent variables
    c) more than one dependent variable
    d) equal number of variables
11. The order of highest derivatives involved in the differential equations is called …………….. of the differential equation.
    a)power b)value c)order d)root
12. General solution of higher order linear differential equation depends on ....
    a) arbitrary constant b) coefficient
    c) type of roots d) method to which solved
13. Both ……………….. solutions will usually be called simply solutions.
    a)general and particular b)singular and non singular
    c)ordinary and partial d)explicit and implicit
14. A solution which cannot be obtained from the general solution by any choice of the arbitrary constants is called ……… solution.
    a) general b) singular c) particular d) zero

15. Let f be a real function defined for all x in a real interval I and having nth order derivatives then the function f is called …………solution of the differential equations
a)implicit b)explicit c)finite d) infinite

16. The first order differential equation M(x,y)dx+N(x,y)dy=0 is said to be……………….... if the derivativeof the form (dy/dx)=f(x,y) there exists a function g suchthat f(x,y) can be expressed in the form g(y/x).
a) homogeneous b) non homogeneous
c) singular d) non singular

17. The standard form of first order differential equations derivative form is…………
a)$(dy/dx) = f(x)$ b)$(dx/dy) = f(x, y)$
c) $(dy/dy) = f(x, y)$ d)$(dx/dy) = f(y)$

18. A non linear ordinary differential equation is an ordinary differential equation that is not….. …………
a)differential b)integral c)linear d)non linear

19. A solution which contains as many arbitrary constants as the order of the differential equation is called a ……..solution of the differential equation.
a) general b) singular c) particular d) zero

20. Variable is that ……which takes on different quantitative values
a) quantity b) order c) quality d) values

PART-B (3×2=6 Marks)
Answer all the Questions

21. Define Partial Differential equation with example.
22. Explain singular solutions of the differential equation.
23. Explain the order of the differential equation with example.

PART- C (3×8=24 Marks)
Answer all the Questions

24. a) Write the definition of general, particular, explicit, implicit and singular solutions of differential equations.
(OR)
b) Show that every function f defined by $f(x) = (x^3 + c)e^{-3x}$ where c is arbitrary equation is a solution of the differential equation $\frac{dy}{dx} + 3y = 3\,x^2 e^{-3x}$.

25. a) Show that the function for all x by f(x)= 2 sin x +3cos x is an explicit solution of the Differential equation $\frac{d^2y}{dx^2} + y = 0$ for all real x.
(OR)
b) Determine whether the given equation is exact or not and solve $(2\,xy + 1)\,dx + (\,x^2 + 4y)dy = 0.$

26. a) Determine the most general function $N(x, y)$ such that the equation is exact $(x^3 + xy^2)\,dx + N(x, y)\,dy = 0.$
(OR)
b) Find the explicit particular solution of the initial value problem $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}$,$y(5) = 2.$

Reg No ……………….

(18MMU201)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore - 21

DEPARTMENT OF MATHEMATICS

Second Semester

II Internal Test –Feb'2019

Differential Equations

Date : 4.2.19(AN) **Time: 2 Hours**

Class: I B.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 × 1 = 20 Marks)

Answer all the questions

1. Any particular solution of linear differential equation involving ……… arbitrary constants is called particular integral of this equation.
   a) finite b) infinite c) no d) one

2. The solution………… is called the general solutions of linear differential equations.
   a) $yc - yp$ b) $yc + yp$ c) $yc * yp$ d) yc/yp

3. The general solution of ………..equation is called the complementary function of equation.
   a) non homogeneous b) singular
   c) homogeneous d) non singular

4. Rate of change of population=……………
   a) Rate of births+Rate of deaths
   b) Rate of births-Rate of deaths
   c) Rate of births*Rate of deaths
   d) Rate of births/Rate of deaths

5. The n functions $f_1, f_2, \dots, f_n$ are called linearly dependent on $a \le x \le b$ if there exists a constants $c_1, c_2 \dots c_n$ not …… ,such that $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for all x.
   a) all zero b) one zero c) two zero d) n zero

6. The functions $f_1, f_2, \dots, f_n$ are called linearly independent on $a \le x \le b$ if the relation $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for all x implies that $c_1 = c_2 = \dots = c_n =$………
   a) 1 b) 0 c) 2 d) 3

7. Let $f_1, f_2, \dots, f_n$ be n real functions each of which has an $(n-1)^{st}$ derivative on ----------- interval $a \le x \le b$
   a) real b) complex c) finite d) infinite

8. Any linear combination of solutions of the homogeneous linear differential equation is also a ………...of homogeneous equation.
   a) value b) separable c) solution d) exact

9. The n^{th} order ………………….linear differential equations always possess n solutions that are linearly independent.
   a)homogeneous b) nonhomogeneous
   c)singular d)non singular

10. In bernoulli equation when n=0 or 1 then the equation is called …………….. equation.
    a) ordinary b)partial c)linear d)separable

11. If $M(x,y)dx + N(x,y)dy$ is not an exact differential in D then the differential equation $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$ in D then $\mu(x,y)$ is called ……..of the differential equation.
a) integrating factor b)singular
c) general d) exact

12. If $f_1, f_2, \ldots f_m$ are m given functions and $c_1, c_2 \ldots c_m$ are m constants then the expression $c_1f_1 + c_2f_2 + \ldots + c_mf_m$ is called a ……………... of $f_1, f_2, \ldots f_m$
a) homogeneous equation b) non homogeneous equation
c) linear combination d) separable equation

13. A first order differential equation is ………………. in the dependent variable y and the independent variable x if it is can be written in the form $\left(\frac{dy}{dx}\right) + P(x)y = Q(x)$.
a)differential b)integral
c)linear d)non linear

14. The first order differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be……………….if the derivative of the form $\left(\frac{dy}{dx}\right) = f(x,y)$ there exists a function g such that $f(x,y)$ can be expressed in the form $g(y/x)$.
a)homogeneous b) non homogeneous
c)singular d) non singular

15. An equation of the form ………………………is called a Bernoulli differential equation
a)$\left(\frac{dy}{dx}\right) = P(x)y^n$ b)$\left(\frac{dy}{dx}\right) + P(x)y/Q(x) = 0$
c)$\left(\frac{dy}{dx}\right) + P(x)y = Q(x)y^n$ d)$\left(\frac{dy}{dx}\right) + P(x)y = 0$

16. The standard form of first order differential equations derivative form is…………
a)$(dy/dx) = f(x)$ b)$(dx/dy) = f(x,y)$
c)$(dy/dx) = f(x,y)$ d)$(dx/dy) = f(y)$

17. e^{-i3x}, e^{i3x} be solution of
a) $y'' + 6y' + 9y = 0$ b) $y'' - 6y' + y = 0$
c) $y'' + 9y = 0$ d) $y'' - 9y = 0$

18. If e^{kx} is solution of $4y'' + 7y' + 3y = 0$ then k is _____
a) 1 b) 2 c) ¾ d) $-3/4$

19. The expression $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equations in a domain D if there exists a function of two variable such that the expression equals the ………...for all (x,y)in D.
a) differential b) ordinary differential
c) partial differential d) total differential

20. The general solution of the differential equation $y'' + 4y = 0$ is _______
a) $a\cos 2x + b\sin 2x$ b) $ae^{-2x} + bxe^{2x}$
c) $ae^{-2x} + bx^2e^{2x}$ d) $ae^{-2x} + be^{2x}$

PART –B (3×2=6 Marks)

Answer all the questions

21. Explain integrating factor of the differential equation.
22. Define linear combination of functions.
23. Define separable equations with examples

PART-C (3× 8=24 Marks)

Answer all the questions

24. a)Solve the differential equation $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$.

(OR)

b) Find the general solution of

i) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$.

25. a) Solve $(5xy + 4y^2 + 1)dx + (x^2 + 2xy)dy$=0

(OR)

b) Solve $\frac{dy}{dx} + 3y = 3x^2e^{-3x}$

26. a) Solve $y'' + 9y = 2\, sec\, 3x$ by using the method of variations of parameter.

(OR)

b) Solve the Euler's equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0$$