KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester – II

18MMU202 THEORY OF EQUATIONS 7H – 6C

Instruction Hours / week: L: 6 T: 1 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- The solution of Reciprocal and Binomial Equations and properties of the derived functions.
- About the relations between the roots and coefficients**,**

Course Outcomes (COs)

On successful completion of this course, the student will be able to

1. Learn about the properties of polynomials.
2. Find positive, negative and imaginary roots using Descartes rule.
3. Identify the relation between coefficients of the equation and its roots.
4. Familiarize about the transformations of equations.
5. Know about the algebraic solutions of cubic and biquadratic equations.

UNIT I

GENERAL PROPERTIES OF POLYNOMIALS

Theorem relating to polynomials when the variable receives large values, similar theorem when the variable receives small values. Continuity of a rational integral function - Form of the quotient and remainder when a polynomial is divided by a Binomial - Tabulation of functions - Graphic representation of a polynomial - Maximum and minimum values of polynomials

UNIT II

GENERAL PROPERTIES OF EQUATIONS

Theorems relating to the real roots of equations - Existence of a root in the general equation. Imaginary roots - Theorem determining the number of roots of an equation.

Descartes' rule of signs for positive roots - Descartes' rule of signs for negative roots - Use of Descartes' rule in proving the existence of imaginary roots - Theorem relating to the substitution of two given numbers for the variable.

UNIT III

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS

Theorem - Applications of the theorem - Depression of an equation when a relation exists between two of its roots - The cube roots of unity - Symmetric functions of the roots – Examples - Theorems relating to symmetric functions - Examples.

UNIT IV

TRANSFORMATION OF EQUATIONS

Roots with signs changed - Roots multiplied by a given quantity - Reciprocal roots and reciprocal equations - To increase or diminish the roots by a given quantity - Removal of terms - Binomial coefficients.

Solution of reciprocal and binomial equations: Reciprocal equations - Binomial equations. Propositions embracing their leading general Properties - The special roots of the equation - Solution of binomial equations by circular functions - Examples.

UNIT V

ALGEBRAIC SOLUTION OF THE CUBIC AND BIQUADRATIC

The algebraic solution of the cubic equation - Application to numerical equations - Expression of the cubic as the difference of two cubes - Solution of the cubic by symmetric functions of the roots – Examples .

Properties of the Derived Functions: Graphic representation of the derived function - Theorem relating to the maxima and minima of a polynomial - Rolle's Theorem. Corollary - Constitution of the derived functions

SUGGESTED READINGS

1. Burnside W.S., and Panton A.W.,(1954). The Theory of Equations, Eighth Edition, Dublin University Press.
2. Leonard Eugene Dickson (2012). First Course in the theory of Equations., J. Wiley & sons, London: Chapman & Hall, Limited, New York.
3. Turnbull,H.W (2013)., Theory Of Equations, Fourth Edition, Published In Great Britain Bt, Oliver And Boyd Ltd., Edinburgh.
4. James Víctor Uspensky., (2005). Theory of Equations, McGraw-Hill Book Co, New York.
5. Mac Duffee C.C., (1962). Theory of Equations, John Wiley & Sons Inc., New York.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: U.R.RAMAKRISHNAN
SUBJECT NAME: THEORY OF EQUATIONS

SUB.CODE:18MMU202

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
		UNIT-I	
1	1	Introduction to polynomials	T: Ch 2, P.No:33
2	1	Theorem relating to polynomials when the variable receives large value	T: Ch 1, P.No:5
3	1	Theorem when the variable receives small values	T: Ch 1, P.No:6
4	1	Continuity of a rational integral function	T: Ch 1, P.No:11
5	1	Form of the quotient and remainder when a polynomial is divided by a Binomial	T: Ch 1, P.No:11
6	1	Tabulation of functions.	T: Ch 1, P.No:13
7	1	Tutorial-1	
8	1	Tabulation of functions.	T: Ch 1, P.No:13
9	1	Graphic representation of a polynomial	T: Ch 1, P.No:14
10	1	Graphic representation of a polynomial	T: Ch 1, P.No:15
11	1	Graphic representation of a polynomial	R1: Ch 3, P.No:38
12	1	Maximum and minimum values of polynomials	T: Ch 1, P.No:18
13	1	Maximum and minimum values of polynomials	T: Ch 1, P.No:18
14	1	Tutorial-2	
15	1	Recapitulation and Discussion of possible questions	
		Total No of Hours Planned For Unit 114 hours	
		UNIT-II	
1	1	Theorems relating to the real roots of equations	T: Ch 2, P.No:19-20
2	1	Existence of a root in the general equation	T: Ch 2, P.No:21
3	1	Imaginary roots	T: Ch 2, P.No:22
4	1	Theorem determining the number of roots of an equation	T: Ch 2, P.No:22
5	1	Theorem determining the number of roots of an equation	T: Ch 2, P.No:23

6	1	Tutorial	
7	1	Descartes' rule of signs for positive roots	T: Ch 2, P.No:28
8	1	Descartes' rule of signs for negative roots	T: Ch 2, P.No:28
9	1	Use of Descartes' rule in proving the existence of imaginary roots	R2: Ch 4: P.No:71
10	1	Use of Descartes' rule in proving the existence of imaginary roots	T: Ch 2, P.No:30
11	1	Theorem relating to the substitution of two given numbers for the variable	T: Ch 2, P.No:31
12	1	Theorem relating to the substitution of two given numbers for the variable	T: Ch 2, P.No:32
13	1	Tutorial	
14	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit II 14 hours			
Unit III			
1	1	Relations between the roots and coefficients	T: Ch 3,P.No:35
2	1	Applications of the theorem	T: Ch 3,P.No:37-38
3	1	Applications of the theorem	T: Ch 3,P.No:39
4	1	relation exists between two of its roots	T: Ch 3,P.No:40
5	1	relation exists between two of its roots	T: Ch 3,P.No:41-42
6	1	Tutorial	
7	1	The cube roots of unity	T: Ch 3,P.No:44
8	1	The cube roots of unity	T: Ch 3,P.No:45
9	1	Symmetric functions of the roots	T: Ch 3,P.No:46-47
10	1	Symmetric functions of the roots	T: Ch 3,P.No:48-49
11	1	Theorems relating to symmetric functions	R3: Ch 8, P.No:173
12	1	Theorems relating to symmetric functions	T: Ch 3, P.No:51-53
13	1	Tutorial	
14	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit 1II14 **hours**			
UNIT-IV			
1	1	Transformation of equations	T: Ch 4, P.No:60
2	1	Roots with signs changed	T: Ch 4, P.No:62-63
3	1	Roots multiplied by a given quantity	T: Ch 4, P.No:64-65
4	1	Reciprocal roots and reciprocal equations	T: Ch 4, P.No:65-66
5	1	Reciprocal roots and reciprocal equations	T: Ch 4, P.No:67
6	1	Tutorial	
7	1	increase or diminish the roots by a given quantity	T: Ch 4, P.No:68-69
8	1	Removal of terms, Binomial coefficients	T: Ch 4, P.No:70
9	1	Reciprocal equations	T: Ch 4, P.No:71

10	1	Binomial equations	T: Ch 4, P.No:72-73
11	1	Propositions embracing their leading general Properties	T: Ch 4, P.No:73
12	1	The special roots of the equation - Solution of binomial equations by circular functions	T: Ch 4, P.No:75
13	1	Tutorial	
14	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit IV 14 Hours			
Unit V			
1	1	The algebraic solution of equations	T: Ch 6,P.No:101-103
2	1	The algebraic solution of the cubic equation	T: Ch 6,P.No:104-105
3	1	Application to numerical equations	T: Ch 6,P.No:105-106
4	1	Expression of the cubic as the difference of two cubes	T: Ch 6,P.No:107-109
5	1	Solution of the cubic by symmetric functions of the roots	T: Ch 6,P.No:109-110
6	1	Tutorial	
7	1	Graphic representation of the derived function	T: Ch 6,P.No:111
8	1	Rolle's Theorem and Corollary	T: Ch 7,P.No:148
9	1	Constitution of the derived functions	T: Ch 7,P.No:149
10	1	Recapitulation and Discussion of possible questions	
11	1	Discuss on Previous ESE Question Papers	
12	1	Discuss on Previous ESE Question Papers	
13	1	Discuss on Previous ESE Question Papers	
Total	**13 Hours**		

SEMESTER: II CLASS: I B.Sc., MATHEMATICS

Total no. of Hours for the Course: 60 hours

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TEXT BOOK

T: Burnside W.S., and Panton A.W.,(1954). The Theory of Equations, Eighth Edition , Dublin University Press.

REFERENCES

R1: Leonard Eugene Dickson (2012). First Course in the theory of Equations. J. Wiley & sons, London: Chapman & Hall, Limited, New York.
R2: Turnbull,H.W (2013).,Theory Of Equations, Fourth Edition, Published In Great Britain Bt, Oliver And Boyd Ltd., Edinburgh.

R3. James Víctor Uspensky., (2005).Theory of Equations,McGraw-Hill Book Co, New York.

R4. MacDuffee C.C., (1962). Theory of Equations, John Wiley & Sons Inc., New York.

UNIT-I

SYLLABUS

General properties of polynomials:Theorem relating to polynomials when the variable receives large values, similar theorem when the variable receives small values.

Continuity of a rational integral function - Form of the quotient and remainder when a polynomial is divided by a Binomial - Tabulation of functions - Graphic representation of a polynomial - Maximum and minimum values of polynomials

INTRODUCTION.

1. **Definitions.**—Any mathematical expression involving a quantity is called a *function* of that quantity.

We shall be employed mainly with such algebraical functions as are *rational* and *integral*. By a *rational* function of a quantity is meant one which contains that quantity in a rational form only; that is, a form free from fractional indices or radical signs. By an *integral* function of a quantity is meant one in which the quantity enters in an integral form only; that is, never in the denominator of a fraction. The following expression, for example, in which n is a positive integer, is a *rational* and *integral algebraical function* of x:—

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots + kx + l.$$

Here it is to be observed that our definition has reference to the quantity x only, of which the expression is a function. The several coefficients a, b, c, &c., may be irrational or fractional, and the function still remain rational and integral in x.

A function of x is represented for brevity by $F(x)$, $f(x)$, $\phi(x)$, or some similar symbol.

The name *polynomial* is given to the algebraical function to express the fact that it is constituted of a number of terms containing different powers of x connected by the signs plus or minus. For certain values of the variable quantity x, one given polynomial may become equal to another differently constituted. The algebraical expression of such a relation is called an *equation;* and any value of the quantity x which satisfies this equation is called a *root* of the equation. The determination of all possible roots constitutes the complete *solution of the equation.*

It is obvious that, by bringing all the terms to one side, we may arrange any equation according to descending powers of x in the following manner :—

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0.$$

The highest power of x in this equation being n, it is said to be an equation of the n^{th} *degree* in x. For an equation of the n^{th} degree we shall, in general, employ the form here written. The suffix attached to the letter a indicates the power of x which each coefficient accompanies, the sum of the exponent of x and the suffix of a being equal to n for each term. An equation is not altered if all its terms be divided by any quantity. We may thus, if we please, dividing by a_0, make the coefficient of x^n in the above equation equal to unity. We shall find it often convenient to make this supposition ; and in such cases we shall write the equation in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0.$$

An equation is said to be *complete* when it contains terms involving x in all its powers from n to 0, and *incomplete* when some of the terms are absent; or, in other words, when some of the coefficients p_1, p_2, &c., are equal to zero. The term p_n, which does not contain x, is called the *absolute term.* An equation is *numerical*, or *algebraical*, according as its coefficients are numbers, or algebraical symbols.

2. **Numerical and Algebraical Equations.**— In many

therefore, that the attention of mathematicians should have been at an early stage in the history of the science directed towards inquiries of this nature. The science of the Theory of Equations, as it now stands, has grown out of the successive attempts of mathematicians to discover general methods for the solution of equations of any degree. When the coefficients of an equation are given numbers, the problem is to determine a numerical value, or perhaps several different numerical values, which will satisfy the equation. In this branch of the science very great progress has been made; and the best methods hitherto advanced for the discovery, either exactly or approximately, of the numerical values of the roots will be explained in their proper places in this work.

Equal progress has not been made in the general solution of equations whose coefficients are algebraical symbols. The student is aware that the root of an equation of the second degree, whose coefficients are such symbols, may be expressed in terms of these coefficients in a general formula; and that the numerical roots of any particular numerical equation may be obtained by substituting in this formula the particular numbers for the symbols. It was natural to inquire whether it was possible to discover any such formula in the case of equations of higher degrees. Such results have been attained in the case of equations of the third and fourth degrees. It will be shown that in certain cases these formulas fail to give us the solution of a numerical equation by substitution of the numerical coefficients for the general symbols, and are, therefore, in this respect, inferior to the corresponding algebraical solution of the quadratic.

Many attempts have been made to arrive at similar general formulas for equations of the fifth and higher degrees; but it may now be regarded as established by the researches of modern analysts that it is not possible by means of radical signs, and other signs of operation employed in common algebra, to ex-

3. **Polynomials.**—One important object of the science of the Theory of Equations is thus the discovery of those values of the quantity x which give to the polynomial $f(x)$ the particular value zero. In attempting to discover such values of the variable we shall be led into many inquiries concerning the values assumed by the polynomial for other different values of x. We shall, in fact, see in the next Chapter that, corresponding to a continuous series of values of x varying from an infinitely great negative quantity $(-\infty)$ to an infinitely great positive quantity $(+\infty)$, $f(x)$ will assume also values continuously varying. The study of such variations is a very important part of the subject on which we are engaged. The general solution of numerical equations is, in fact, a tentative process; and by examining the values assumed by the polynomial for certain arbitrarily assumed values of the variable, we shall be led, if not to the root itself, at least to an indication of the neighbourhood in which it exists, and within which our further approximation must be carried on.

A polynomial is sometimes called a *quantic*. It is convenient to have distinct names for quantics of the 2nd, 3rd, 4th, 5th, &c., degrees. That of the 2nd degree is called a *quadratic* or *quadric;* that of the 3rd is called a *cubic;* that of the 4th a *quartic* or *biquadratic;* that of the 5th a *quintic*; and so on. The equations obtained by equating these quantics to zero are called *quadratic*, *cubic*, *biquadratic*, &c., *equations*, respectively.

4. In tracing the changes of value of a polynomial corresponding to changes in the variable, we shall first inquire what terms in the polynomial are most important when values very great or very small are assigned to x. This inquiry will form the subject of the present and succeeding Articles.

Writing the polynomial in the form

$$a_0x^n\left\{1+\frac{a_1}{a_0}\frac{1}{x}+\frac{a_2}{a_0}\frac{1}{x^2}+\ldots+\frac{a_{n-1}}{a_0}\frac{1}{x^{n-1}}+\frac{a_n}{a_0}\frac{1}{x^n}\right\},$$

it is plain that its value tends to become equal to a_0x^n, as x tends towards ∞. We proceed, then, to inquire what is the value of x nearest to zero which will have the effect of making the term a_0x^n exceed the sum of all the others.

Theorem.—*If in the polynomial*

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$$

the value $\frac{a_k}{a_0}+1$, *or any greater value, be substituted for* x, *where* a_k *is that one of the coefficients* $a_1, a_2, \ldots a_n$ *whose numerical value is greatest, irrespective of sign, the term containing the highest power of* x *will exceed the sum of all the terms which follow.*

The inequality

$$a_0x^n > a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$$

is satisfied by the following:—

$$a_0x^n > a_k(x^{n-1}+x^{n-2}+\ldots+x+1),$$

where a_k is the greatest among the coefficients

$$a_1, a_2, \ldots a_{n-1}, a_n,$$

without regard to sign.

This leads, summing the geometric series, to the condition

$$a_0 x^n > a_k \frac{x^n - 1}{x - 1}, \quad \text{or} \quad x^n > \frac{a_k}{a_0(x-1)}(x^n - 1),$$

which is satisfied if $a_0(x-1)$ be $>$ or $= a_k$,

or $$x > \text{or} = \frac{a_k}{a_0} + 1;$$

which proves the theorem.

Theorem.—*If in the polynomial*

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

the value $\frac{a_n}{a_n + a_k}$, *or any smaller value, be substituted for* x, *where* a_k *is the greatest coefficient exclusive of* a_n, *the term* a_n *will be numerically greater than the sum of all the others.*

To prove this, let $x = \frac{1}{y}$; then by the theorem of Art. 4, a_k being now the greatest among the coefficients $a_0, a_1, \ldots a_{n-1}$, without regard to sign, the value $\frac{a_k}{a_n} + 1$, or any greater value of y, will make

$$a_n y^n > a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + \ldots + a_1 y + a_0,$$

that is, $$a_n > a_{n-1}\frac{1}{y} + a_{n-2}\frac{1}{y^2} + \ldots a_0 \frac{1}{y^n};$$

hence the value $\frac{a_n}{a_n + a_k}$, or any less value of x, will make

$$a_n > a_{n-1} x + a_{n-2} x^2 + \ldots + a_0 x^n.$$

6. **Change of form of a Polynomial corresponding to an increase or diminution of the Variable. Derived Functions.**—We shall now examine the form assumed by the polynomial when $x + h$ is substituted for x.

If we suppose h essentially positive, the resulting form will correspond to an increase of the variable; and by changing the sign of h in the result we obtain the form corresponding to a diminution of x.

When x is changed to $x + h$, $f(x)$ becomes $f(x + h)$, or

$$a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \ldots + a_{n-2}(x+h)^2 + a_{n-1}(x+h) + a_n.$$

Let each term of this expression be expanded by the binomial theorem, and the result arranged according to ascending powers of h. We then have

$$\begin{aligned} & a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n \\ & + h\left\{na_0x^{n-1} + (n-1)\,a_1x^{n-2} + (n-2)\,a_2x^{n-3} + \ldots + 2a_{n-2}x + a_{n-1}\right\} \\ & + \frac{h^2}{1\,.\,2}\left\{n\,(n-1)a_0x^{n-2} + (n-1)(n-2)a_1x^{n-3} + \ldots + 2a_{n-2}\right\} \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & + \frac{h^n}{1\,.\,2\,.\,3\ldots n}\left\{n\,.\,n-1\ldots 2\,.\,1\right\}a_0. \end{aligned}$$

Example.

Find what the polynomial $4x^3 + 6x^2 - 7x + 4$ becomes when x is changed into $x + h$.

Here

$$\begin{aligned} f(x) &= 4x^3 + 6x^2 - 7x + 4, \\ f'(x) &= 12x^2 + 12x - 7, \\ f''(x) &= 24x + 12, \\ f'''(x) &= 24; \end{aligned}$$

and the result is

$$4x^3 + 6x^2 - 7x + 4 + (12x^2 + 12x - 7)h + (24x + 12)\frac{h^2}{1 \cdot 2} + 24\frac{h^3}{1 \cdot 2 \cdot 3}.$$

This example shows how the absolute term of each function disappears when its derived is formed, the degree of the function diminishing, till finally $f_n(x)$ is reached, which is equal in general to $\{n \cdot n-1 \cdot n-2 \ldots 2 \cdot 1\}a_0$; in this case $f_3(x) = \{3 \cdot 2 \cdot 1\}4$.

7. **Continuity of a Rational Integral Function of** x. —If the value of x be made to vary, by indefinitely small increments, from one quantity a to a greater quantity b, it becomes an inquiry how the polynomial $f(x)$ varies at the same time. The object of the present Article is to show that $f(x)$ passes at the same time through all values between $f(a)$ and $f(b)$; in other words, that it varies continuously along with x. Let x be increased from a to $a+h$. The corresponding increment of $f(x)$ is

$$f(a+h)-f(a),$$

which is equal, by Art. 6, to

$$f'(a)h+f''(a)\frac{h^2}{1\,.\,2}+\ldots+a_0h^n,$$

in which all the coefficients $f'(a)$, $f''(a)$, &c., are finite quantities. Now, by the theorem of Art. 5, this latter expression may, by taking h small enough, be made to assume a value less than any assigned quantity; so that the difference between $f(a+h)$ and $f(a)$ may be made as small as we please, and will ultimately vanish with h. The same is true during all stages of the variation of x from a to b; thus the continuity of the function $f(x)$ is established.

EXAMPLES.

1. Find the quotient and remainder when $3x^4 - 5x^3 + 10x^2 + 11x - 61$ is divided by $x - 3$.

The calculation is arranged as follows :—

$$\begin{array}{rrrrr} 3 & -5 & 10 & 11 & -61. \\ & 9 & 12 & 66 & 231. \\ \hline & 4 & 22 & 77 & 170. \end{array}$$

Thus the quotient is $3x^3 + 4x^2 + 22x + 77$, and the remainder 170.

2. Find the quotient and remainder when $x^3 + 5x^2 + 3x + 2$ is divided by $x - 1$.

Ans. $Q = x^2 + 6x + 9$, $R = 11$.

3. Find Q and R when $x^5 - 4x^4 + 7x^3 - 11x - 13$ is divided by $x - 5$.

[N. B.—When any term in a polynomial is absent, care must be taken to supply the place of its coefficient by zero in writing down the coefficients of $f(x)$. In this example, therefore, the series in the first line will be

$$1 \quad -4 \quad 7 \quad 0 \quad -11 \quad -13.]$$

Ans. $Q = x^4 + x^3 + 12x^2 + 60x + 289$, $R = 1432$.

4. Find the quotient and remainder when $x^9 + 3x^7 - 15x^2 + 2$ is divided by $x - 2$.

Ans. $Q = x^8 + 2x^7 + 7x^6 + 14x^5 + 28x^4 + 56x^3 + 112x^2 + 209x + 418$, $R = 838$.

5. Find the quotient and remainder when $x^5 + x^2 - 10x + 113$ is divided by $x + 4$.

Ans. $Q = x^4 - 4x^3 + 16x^2 - 63x + 242$; $R = -855$.

EXAMPLES.

1. Tabulate the trinomial $2x^2 + x - 6$, for the values of x

$$-4, -3, -2, -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4.$$

Values of x,	-4	-3	-2	-1	0	1	2	3	4
Values of $f(x)$,	22	9	0	-5	-6	-3	4	15	30

2. Tabulate the polynomial $10x^3 - 17x^2 + x + 6$ for the values of x

$$-4, -3, -2, -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4.$$

Values of x,	-4	-3	-2	-1	0	1	2	3	4
Values of $f(x)$	-910	-420	-144	-22	6	0	20	126	378

10. **Graphic Representation of a Polynomial.**— Whenever we have to deal with a great number of values of any varying quantity, it is important to be able to represent them in some simple and expressive manner. This in the present instance can be effected, and the general character of the function made apparent to the eye, by means of a graphic representation.

We proceed to explain such a representation of the function $f(x)$.

Let two right lines OX, OY (fig. 1), cut one another at right angles, and be produced indefinitely in both directions. These lines are called the *axis of x* and *axis of y*, respectively. Lines, such as OA, measured on the axis of x at the right-hand side of O, are regarded as positive, and those, such as OA', measured at the left-hand side, as negative. Lines parallel to OY which are above XX', such as AP or $B'Q'$, are positive; and those below it, such as AT or $A'P'$, are negative. These conventions are already familiar to the student acquainted with Trigonometry.

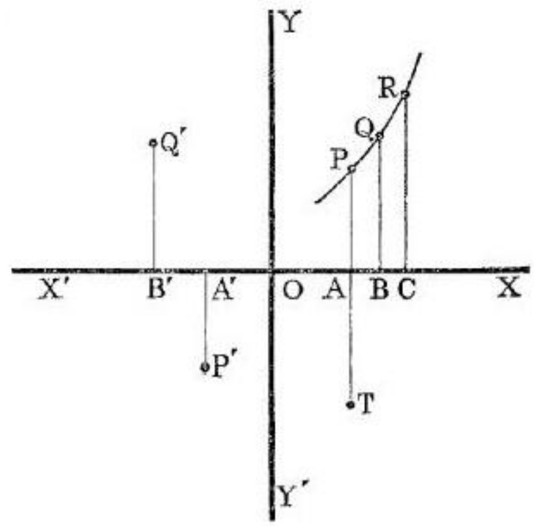

Fig. 1.

EXAMPLES.

1. Let it be required to trace the trinomial $f(x) = 2x^2 + x - 6$.

From Ex. 1, Art. 9, we have, for the values of x

$$\ldots\ldots -4, -3, -2, -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \ldots\ldots$$

the corresponding values of $f(x)$

$$\ldots\ldots \quad 22, \quad 9, \quad 0, -5, -6, -3, \quad 4, \ 15, \ 30, \ldots\ldots$$

The unit of length taken is one-sixth of the line OD in fig. 2.

By means of these values we obtain the positions of nine points on the curve; seven of which, A, B, C, D, E, F, G, are here represented, the other two corresponding to values of $f(x)$ which lie out of the limits of our figure.

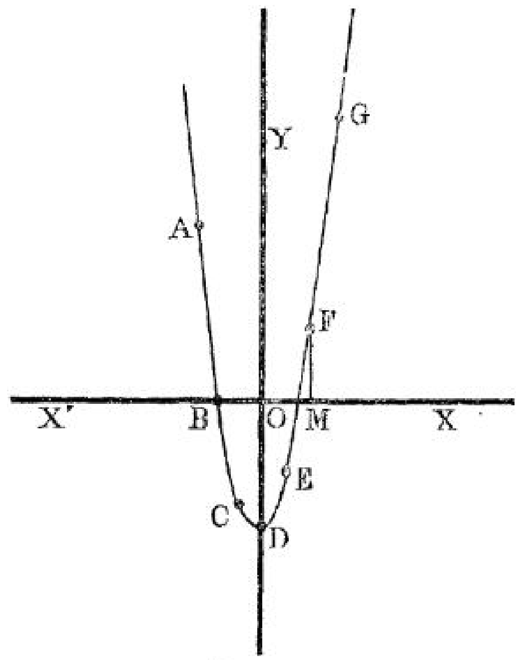

Fig. 2.

It may occur to the student that we have here exercised considerable imagination in drawing that part of the curve which lies between the points determined by calculation; and that much closer numerical values must be substituted for x in order to obtain the shape of the curve with any accuracy. He will learn, however, as he proceeds, that we are assisted in our approximation to the form of the curve by many other considerations besides the ascertained values of $f(x)$. Cases undoubtedly occur in which the portion of the curve between two values of x must be more closely examined, and then the substitution of nearer values of x will become necessary. The next example will furnish an illustration of such cases.

11. Maxima and Minima Values of Polynomials.—It is apparent from the considerations established in the preceding Articles, that as the variable x changes from $-\infty$ to $+\infty$, the function $f(x)$ may undergo many variations. It may go on for a certain period increasing, and then, ceasing to increase, may commence to diminish; it may then cease to diminish and commence again to increase; after which another period of diminution may arrive, or the function may (as in the last example of the preceding Art.) go on then continually increasing. At a stage where the function ceases to increase and commences to diminish, it is said to have attained a *maximum* value; and when it ceases to diminish and commences to increase, it is said to have attained a *minimum* value. A polynomial may have several maxima or several minima values, or both, the number depending on its degree. Nothing exhibits so well as a graphic representation the occurrence of such a maximum or minimum value; as well as the various fluctuations of which the values of a polynomial are susceptible. We shall give in a subsequent Chapter the method of finding the values of x corresponding to the maxima or minima values of $f(x)$, together with criteria to decide between maxima and minima. These are among the considerations alluded to in Art. 10, as aiding us in the graphic construction of the polynomial. Another very material aid to such a construction would be a knowledge of the values of x corresponding to the points (if any) in which the line XX' is cut by the curve; that is to say, of the values of x which render $f(x) = 0$. Such a value of x is a root of the equation $f(x) = 0$.

UNIT-II

SYLLABUS

General properties of equations: Theorems relating to the real roots of equations - Existence of a root in the general equation. Imaginary roots - Theorem determining the number of roots of an equation.

Descartes' rule of signs for positive roots - Descartes' rule of signs for negative roots - Use of Descartes' rule in proving the existence of imaginary roots - Theorem relating to the substitution of two given numbers for the variable

Theorem.—*If two real quantities a and b be substituted for the unknown quantity x in any polynomial $f(x)$, and if they furnish results having different signs, one plus and the other minus; then the equation $f(x) = 0$ must have at least one real root intermediate in value between a and b.*

This theorem is an immediate consequence of the property of the continuity of the function $f(x)$ established in Art. 7; for since $f(x)$ changes continuously from $f(a)$ to $f(b)$, i. e. passes through all the intermediate values, while x changes from a to b; and since one of these quantities, $f(a)$ or $f(b)$, is positive, and the other negative, it follows that for some value of x intermediate between a and b, $f(x)$ must attain the value zero which is intermediate between $f(a)$ and $f(b)$.

Corollary.—*If there exist no real quantity which, substituted for x, makes $f(x) = 0$, then $f(x)$ must be positive for every real value of x.*

13. **Theorem.**—*Every equation of an odd degree has at least one real root of a sign opposite to that of its last term.*

This is an immediate consequence of the theorem in the last Article. Substitute in succession $-\infty$, 0, ∞ for x in the polynomial $f(x)$. The results are, n being odd (see Art. 4),

$x = -\infty$, $f(x)$ is negative;

$x = 0$, sign of $f(x)$ is the same as that of a_n;

$x = +\infty$, $f(x)$ is positive.

If a_n is positive, the equation must have a real root between $-\infty$ and 0, *i.e.* a real negative root; and if a_n is negative, the equation must have a real root between 0 and ∞, *i.e.* a real positive root. The theorem is thus proved.

14. **Theorem.**—*Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other negative.*

The results of substituting $-\infty$, 0, ∞ are in this case

$$\begin{array}{rc} -\infty, & +, \\ 0, & -, \\ +\infty, & +; \end{array}$$

hence there is a real root between $-\infty$ and 0, and another between 0 and $+\infty$; *i.e.*, there exist at least one real negative, and one real positive root.

In this simple instance we observe that, in the absence of any real values, there are two imaginary expressions which reduce the polynomial to zero. The general proposition of which this is a very particular illustration is, that *every rational integra equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

must have a root of the form

$$\alpha + \beta\sqrt{-1},$$

a and β being real finite quantities. This proposition includes both real and imaginary roots, the former corresponding to the value $\beta = 0$.

16. **Theorem.**—*Every equation of n dimensions has n roots, and no more.*

We first observe that if any quantity h is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $x - h$ without a remainder. This is evident from Art. 9; for if $f(h) = 0$, *i.e.* if h is a root of $f(x) = 0$, R must be $= 0$.

The converse of this is also obviously true.

Let, now, the given equation be

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

This equation must have a root, real or imaginary (see Art. 15), which we shall denote by the symbol a_1. Let the quotient, when $f(x)$ is divided by $x - a_1$, be $\phi_1(x)$; we have then the identical equation

$$f(x) = (x - a_1)\,\phi_1(x).$$

Again, the equation $\phi_1(x) = 0$, which is of $n-1$ dimensions, must have a root, which we represent by a_2. Let the quotient obtained by dividing $\phi_1(x)$ by $x - a_2$ be $\phi_2(x)$. Hence

$$\phi_1(x) = (x - a_2)\,\phi_2(x),$$

$$\text{and} \quad \therefore f(x) = (x - a_1)(x - a_2)\,\phi_2(x),$$

where $\phi_2(x)$ is of $n-2$ dimensions.

Proceeding in this manner, we prove that $f(x)$ consists of the product of n factors, each containing x in the first degree, and a numerical factor $\phi_n(x)$. Comparing the coefficients of x^n, it is plain that $\phi_n(x) = 1$. Thus we prove the identical equation

$$f(x) = (x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_{n-1})(x - a_n).$$

It is evident that the substitution of any one of the quantities $a_1, a_2, \ldots a_n$ for x in the right-hand member of this equation will reduce that member to zero, and will therefore reduce $f(x)$ to zero; that is to say, the equation $f(x) = 0$ has for roots the n quantities $a_1, a_2, a_3 \ldots a_{n-1}, a_n$. And it can have no other roots; for if any quantity other than one of the quantities $a_1, a_2, \ldots a_n$ be substituted in the right-hand member of the above equation, the factors will be all different from zero, and therefore the product cannot vanish.

Corollary.—*Two polynomials of the n^{th} degree cannot be equal to one another for more than n values of the variable without being completely identical.*

1. Find the equation whose roots are

$$-3, \quad -1, \quad 4, \quad 5.$$

Ans. $x^4 - 5x^3 - 13x^2 + 53x + 60 = 0.$

2. The equation

$$x^4 - 6x^3 + 8x^2 - 17x + 10 = 0$$

has a root 5; find the equation containing the remaining roots.

[N. B.—Use the method of division of Art. 8.]

Ans. $x^3 - x^2 + 3x - 2 = 0.$

3. Solve the equation

$$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0,$$

two roots being 1 and 7.

Ans. The other two roots are 3, 5.

4. Form the equation whose roots are

$$-\frac{3}{2}, \quad 3, \quad \frac{1}{7}.$$

Ans. $14x^3 - 23x^2 - 60x + 9 = 0.$

5. Solve the cubic equation

$$x^3 - 1 = 0.$$

Here it is evident that $x = 1$ satisfies the equation. Divide by $x - 1$, and solve the resulting quadratic. The two roots are found to be

$$-\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \quad -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

It can be easily shown that if either of these imaginary roots is squared, the other results. It is usual to represent these roots by ω and ω^2. They are called the two *imaginary cube roots of unity*. We have the identical equation

$$x^3 - 1 \equiv (x - 1)(x - \omega)(x - \omega^2).$$

18. **Imaginary Roots enter Equations in Pairs.**—The proposition we have to prove may be stated as follows:—*If an equation $f(x) = 0$, whose coefficients are all real quantities, have for a root the imaginary expression* $\alpha + \beta\sqrt{-1}$, *it must also have for a root the conjugate imaginary expression* $\alpha - \beta\sqrt{-1}$.

The product

$$(x - \alpha - \beta\sqrt{-1})(x - \alpha + \beta\sqrt{-1}) = (x - \alpha)^2 + \beta^2.$$

Let the polynomial $f(x)$ be divided by the second member of this identity, and if possible let there be a remainder $Rx + R'$. We have then the identical equation

$$f(x) = \{(x - \alpha)^2 + \beta^2\} Q + Rx + R',$$

where Q is the quotient, of $n - 2$ dimensions in x. Substitute in this identity $\alpha + \beta\sqrt{-1}$ for x. This, by hypothesis, causes $f(x)$ to vanish. It also causes $(x-\alpha)^2 + \beta^2$ to vanish. Hence

$$R(\alpha + \beta\sqrt{-1}) + R' = 0.$$

From this we obtain the two equations

$$R\alpha + R' = 0, \qquad R\beta = 0,$$

since the real and imaginary parts cannot destroy one another; hence

$$R = 0, \qquad R' = 0.$$

Thus the remainder $Rx + R'$ vanishes; and, therefore, $f(x)$ is divisible without remainder by the product of the two factors

$$x - \alpha - \beta\sqrt{-1}, \qquad x - \alpha + \beta\sqrt{-1}.$$

The equation has, consequently, the root $\alpha - \beta\sqrt{-1}$ as well as the root $\alpha + \beta\sqrt{-1}$.

Thus the total number of imaginary roots in an equation with real coefficients will always be even; and every polynomial may be regarded as composed of real factors, each pair of imaginary roots producing a real quadratic factor, and each real root producing a real simple factor. The actual resolution of the polynomial into these factors constitutes the complete solution of the equation.

19. **Descartes' Rule of Signs—Positive Roots.**—This rule, which enables us, by the mere inspection of a given equation, to assign a superior limit to the number of its positive roots, may be enunciated as follows:—*No equation can have more positive roots than it has changes of sign from + to −, and from − to +, in the terms of its first member.*

We shall content ourselves for the present with the proof which is usually given, and which is more a verification than a general demonstration of this celebrated theorem of Descartes. It will be subsequently shown that this rule of Descartes, and other similar rules which were discovered by early investigators relative to the number of the positive, negative, and imaginary roots of equations, are immediate deductions from the more general theorems of Budan and Fourier.

Let the signs of a polynomial taken at random succeed each other in the following order:—

$$+ + - + - - - + + - + -.$$

20. Descartes' Rule of Signs—Negative Roots.—In order to give the most advantageous statement to Descartes' rule in the case of negative roots, we first prove that if $-x$ be substituted for x in the equation $f(x)=0$, the resulting equation will have the same roots as the original except that their signs will be changed. This follows from the identical equation of Art. 16

$$f(x) \equiv (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n),$$

from which we derive

$$f(-x) \equiv (-1)^n (x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n).$$

From this it is evident that the roots of $f(-x)=0$ are

$$-a_1, \quad -a_2, \quad -a_3, \ldots -a_n.$$

Hence the negative roots of $f(x)$ are positive roots of $f(-x)$, and we may enunciate Descartes' rule for negative roots as follows:—*No equation can have a greater number of negative roots than there are changes of sign in the terms of the polynomial* $f(-x)$.

22. **Theorem.**—We shall close this chapter with the following theorem, which defines fully the conclusions which can be drawn as to the roots of an equation from the signs furnished by its first member when two given numbers are substituted for x:—*If two numbers a and b, substituted for x in the polynomial $f(x)$, give results with contrary signs, an odd number of real roots of the equation $f(x) = 0$ lies between them; and if they give results with the same sign, either no real root or an even number of real roots lies between them.*

We proceed to prove the first part of this proposition: the second is proved in an exactly similar manner.

Let the following m roots $\alpha_1, \alpha_2, \ldots \alpha_m$, and no others, of the equation $f(x) = 0$ lie between the quantities a and b, of which, as usual, we take a to be the lesser.

Let $\phi(x)$ be the quotient when $f(x)$ is divided by the product of the m factors $(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m)$. We have, then, the identical equation

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m)\,\phi(x).$$

Putting in this successively $x = a$, $x = b$, we obtain

$$f(a) = (a - \alpha_1)(a - \alpha_2) \ldots (a - \alpha_m)\,\phi(a),$$
$$f(b) = (b - \alpha_1)(b - \alpha_2) \ldots (b - \alpha_m)\,\phi(b).$$

Now $\phi(a)$ and $\phi(b)$ have the same sign; for if they had different signs there would be, by Art. 12, one root at least of the equation $\phi(x) = 0$ between them. By hypothesis, $f(a)$ and $f(b)$ have different signs; hence the signs of the products

$$(a - \alpha_1)(a - \alpha_2) \ldots (a - \alpha_m),$$
$$(b - \alpha_1)(b - \alpha_2) \ldots (b - \alpha_m),$$

are different; but the sign of the second is positive, since all its factors are positive; hence the sign of the first is negative; but all the factors of the first are negative; therefore their number must be odd; which proves the proposition.

EXAMPLES.

1. If the signs of the terms of an equation be all positive, it cannot have a positive root.

2. If the signs of the terms of any complete equation be alternately positive and negative, it cannot have a negative root.

3. If an equation consist of a number of terms connected by + signs followed by a number of terms connected by − signs, it has one positive root and no more.

[Apply Art. 12, substituting 0 and ∞; and Art. 19.]

4. If an equation involve only even powers of x, and if all the coefficients have positive signs, it cannot have a real root.

[Apply Arts. 19 and 20.]

5. If an equation involve only odd powers of x, and if the coefficients have all positive signs, it has the root zero and no other real root.

6. If an equation be complete, the number of continuations of sign in $f(x)$ is the same as the number of variations of sign in $f(-x)$.

7. When an equation is complete; if all its roots are real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of continuations of sign.

8. An equation having an even number of variations of sign must have its last sign positive, and one having an odd number of variations must have its last sign negative.

[N. B.—The sign + is always given to the highest power of x.]

9. Hence prove that if an equation has an even number of variations it must have an equal or less even number of positive roots; and if it has an odd number of variations it must have an equal or less odd number of positive roots; in other

20. Form an equation with rational coefficients which shall have for roots all the values of the expression

$$\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r},$$

where $$\theta_1^2=1, \quad \theta_2^2=1, \quad \theta_3^2=1.$$

There are eight different values of this expression, viz.,

$$\sqrt{p}+\sqrt{q}+\sqrt{r}, \qquad -\sqrt{p}-\sqrt{q}-\sqrt{r},$$

$$\sqrt{p}-\sqrt{q}-\sqrt{r}, \qquad -\sqrt{p}+\sqrt{q}+\sqrt{r},$$

$$-\sqrt{p}+\sqrt{q}-\sqrt{r}, \qquad \sqrt{p}-\sqrt{q}+\sqrt{r},$$

$$-\sqrt{p}-\sqrt{q}+\sqrt{r}, \qquad \sqrt{p}+\sqrt{q}-\sqrt{r}.$$

Assume

$$x=\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}.$$

Squaring this, we have

$$x^2=p+q+r+2\left(\theta_2\theta_3\sqrt{qr}+\theta_3\theta_1\sqrt{rp}+\theta_1\theta_2\sqrt{pq}\right).$$

Transposing, and squaring again,

$$(x^2-p-q-r)^2=4(qr+rp+pq)+8\theta_1\theta_2\theta_3\sqrt{pqr}\left(\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}\right). \quad (1)$$

Transposing, substituting x for $\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}$, and squaring, we obtain the final equation free from radicals

$$\{x^4-2x^2(p+q+r)+p^2+q^2+r^2-2qr-2rp-2pq\}^2=64pqrx^2.$$

This is an equation of the eighth degree, whose roots are the values above written. Since θ_1, θ_2, θ_3 have disappeared, it is indifferent which of the eight roots $\pm\sqrt{p}\pm\sqrt{q}\pm\sqrt{r}$ is assumed equal to x in the first instance. The final equation is that which would have been obtained if each of the 8 roots had been subtracted from x, and the continued product formed, as in Ex. 6, Art. 16.

UNIT-III

SYLLABUS

Relations between the roots and coefficients-Theorem - Applications of the theorem - Depression of an equation when a relation exists between two of its roots - The cube roots of unity - Symmetric functions of the roots – Examples - Theorems relating to symmetric functions - Examples

23. **Relations between the Roots and Coefficients.**—Taking for simplicity the coefficient of the highest power of x as unity, and representing, as in Art. 16, the n roots of an equation by $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$, we have the following identity :—

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1}x + p_n$$
$$= (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n). \quad (1)$$

When the factors of the second member of this identity are multiplied together, the product will consist, as is proved in elementary works on Algebra, of a highest term x^n; plus a term x^{n-1} multiplied by the factor

$$-(\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n),$$

i. e. the sum of the roots with their signs changed; plus a term x^{n-2} multiplied by the factor

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \ldots + \alpha_{n-1}\alpha_n,$$

i. e. the sum of the products of the roots taken in pairs; plus a term x^{n-3} multiplied by the factor

$$-(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \ldots + \alpha_{n-2}\alpha_{n-1}\alpha_n),$$

i. e. the sum of the products of the roots with their signs changed taken three by three; and so on. It is plain that the sign of each coefficient will be negative or positive according as the number of roots in each product is odd or even. The last term is

$$\pm \alpha_1\alpha_2\alpha_3 \ldots \alpha_{n-1}\alpha_n,$$

the sign being $-$ if n is odd, and $+$ if n is even. Equating the coefficients of x on each side of the identity (1), we have the following series of equations:—

$$\left.\begin{aligned} p_1 &= -(a_1 + a_2 + a_3 + \ldots + a_n), \\ p_2 &= (a_1 a_2 + a_1 a_3 + a_2 a_3 + \ldots + a_{n-1} a_n), \\ p_3 &= -(a_1 a_2 a_3 + a_1 a_3 a_4 + \ldots + a_{n-2} a_{n-1} a_n), \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_n &= (-1)^n a_1 a_2 a_3 \ldots a_{n-1} a_n, \end{aligned}\right\} \quad (2)$$

which furnish us with the following

Theorem.—*In every algebraic equation, the coefficient of whose highest term is unity, the coefficient p_1 of the second term with its sign changed is equal to the sum of the roots.*

The coefficient p_2 of the third term is equal to the sum of the products of the roots taken two by two.

The coefficient p_3 of the fourth term with its sign changed is equal to the sum of the products of the roots taken three by three; and so on, the signs of the coefficients being taken alternately negative and positive, and the number of roots multiplied together in each term of the corresponding function of the roots increasing by unity, till finally that function is reached which consists of the product of the n roots.

Cor. 1.—Every root of an equation is a divisor of the last term.

Cor. 2.—If the roots of an equation be all positive, the coefficients will be alternately positive and negative; and if the roots be all negative, the coefficients will be all positive. This is obvious from the equations (2) [cf. Arts. 19 and 20].

1. Solve the equation

$$x^3 - 5x^2 - 16x + 80 = 0,$$

the sum of two of its roots being equal to nothing.

Let the roots be α, β, γ. We have, then,

$$\alpha + \beta + \gamma = 5,$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -16,$$

$$\alpha\beta\gamma = -80.$$

Taking $\beta + \gamma = 0$, we have, from the first of these, $\alpha = 5$, and from either second or third we obtain $\beta\gamma = -16$. We find for β and γ the values 4 and -4. the three roots are 5, 4, -4.

2. Solve the equation

$$x^3 - 3x^2 + 4 = 0,$$

two of its roots being equal.

Let the roots be α, α, β. We have

$$2\alpha + \beta = 3,$$

$$\alpha^2 + 2\alpha\beta = 0,$$

from which we find $\alpha = 2$, and $\beta = -1$. The roots are 2, 2, -1.

3. The equation

$$x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$$

has two pairs of equal roots; find them.

Let the roots be α, α, β, β; we have

$$2\alpha + 2\beta = -4,$$

$$\alpha^2 + \beta^2 + 4\alpha\beta = -2,$$

from which we obtain for α and β the values 1 and -3.

4. Solve the equation

$$x^3 - 9x^2 + 14x + 24 = 0,$$

two of whose roots are in the ratio of 3 to 2.

25. **Depression of an Equation when a relation exists between two of its Roots.**—The examples given under the preceding Article illustrate the use of the equations connecting the roots and coefficients in determining the roots in particular cases when known relations exist among them. The object of the present Article is to show that, in general, *if a relation of the form $\beta = \phi(\alpha)$ exist between two of the roots of an equation $f(x) = 0$, the equation may be depressed two dimensions.*

Let $\phi(x)$ be substituted for x in the identity

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_n,$$

then $f(\phi(x)) \equiv a_0(\phi(x))^n + a_1(\phi(x))^{n-1} + \ldots + a_{n-1}\phi(x) + a_n.$

We represent, for convenience, the second member of this identity by $F(x)$. Substitute α for x, then

$$F(\alpha) = f(\phi(\alpha)) = f(\beta) = 0;$$

hence α satisfies the equation $F(x) = 0$, and it also satisfies the equation $f(x) = 0$; hence the polynomials $f(x)$ and $F(x)$ have a common measure $x - \alpha$; thus α can be determined, and from it $\phi(\alpha)$ or β, and the given equation can be depressed two dimensions.

EXAMPLES.

1. The equation

$$x^3 - 5x^2 - 4x + 20 = 0$$

has two roots whose difference $= 3$: find them.

Here $\beta - \alpha = 3$, $\beta = 3 + \alpha$; substitute $x + 3$ for x in the given polynomial $f(x)$; it becomes $x^3 + 4x^2 - 7x - 10$; the common measure of this and $f(x)$ is $x - 2$; from which $\alpha = 2$, $\beta = 5$; the third root is -2.

2. The equation

$$x^4 - 5x^3 + 11x^2 - 13x + 6 = 0$$

has two roots connected by the relation $2\beta + 3\alpha = 7$: find all the roots.

Ans. 1, 2, $1 \pm \sqrt{-2}$.

Examples.

1. The equations

$$2x^3 + 5x^2 - 6x - 9 = 0,$$

$$3x^3 + 7x^2 - 11x - 15 = 0,$$

have two common roots, find them. *Ans.* $-1, \quad -3.$

2. The equations

$$x^3 + px^2 + qx + r = 0,$$

$$x^3 + p'x^2 + q'x + r' = 0,$$

have two common roots; find the quadratic which furnishes them, and also the 3rd root of each.

Ans. $x^2 + \frac{q-q'}{p-p'}x + \frac{r-r'}{p-p'} = 0, \quad \frac{-r(p-p')}{r-r'}, \quad \frac{-r'(p-p')}{r-r'}.$

26. **The Cube Roots of Unity.**—Equations of the forms

$$x^n - 1 = 0, \quad x^n + 1 = 0$$

are called *binomial*. The roots of the former are called the n n^{th} *roots of unity*. A general discussion of these forms will be given in a subsequent Chapter. We confine ourselves at present to the simple case of the binomial cubic, for which certain useful properties of the roots can be easily established. It has been already shown (see Ex. 5, Art. 16), that the roots of the cubic

$$x^3 - 1 = 0$$

are

$$1, \quad -\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \quad -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

If ω be a root of the cubic, ω^2 must also be a root; for, since $\omega^3 = 1$, we get, by squaring, $\omega^6 = 1$, which is $(\omega^2)^3 = 1$, thus showing that ω^2 satisfies the cubic $x^3 - 1 = 0$. We have then the identity

$$x^3 - 1 \equiv (x-1)(x-\omega)(x-\omega^2).$$

Changing x into $-x$, we get the following identity also :—

$$x^3 + 1 \equiv (x+1)(x+\omega)(x+\omega^2),$$

which furnishes the roots of

$$x^3 + 1 = 0.$$

Whenever in any product of quantities involving the imaginary cube roots of unity any power higher than the second presents itself, it can be replaced by ω, or ω^2, or by unity; for example,

$$\omega^4 = \omega^3 . \omega = \omega, \quad \omega^5 = \omega^3 . \omega^2 = \omega^2, \quad \omega^6 = \omega^3 . \omega^3 = 1, \text{ \&c.}$$

The first or second of equations (2), Art. 23, gives the following property of the imaginary cube roots :—

$$1 + \omega + \omega^2 = 0.$$

By the aid of this equation any expression involving real quantities and the imaginary cube roots can be written in either of the forms $P + \omega Q$, $P + \omega^2 Q$.

Examples.

1. Show that the product

$$(\omega m + \omega^2 n)(\omega^2 m + \omega n)$$

is rational. *Ans.* $m^2 - mn + n^2$.

2. Prove the following identities :—

$$m^3 + n^3 \equiv (m+n)(\omega m + \omega^2 n)(\omega^2 m + \omega n),$$

$$m^3 - n^3 \equiv (m-n)(\omega m - \omega^2 m)(\omega^2 m - \omega n).$$

3. Show that the product

$$(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma)$$

is rational. *Ans.* $\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta$.

4. Prove the identity

$$(\alpha + \beta + \gamma)(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma) \equiv \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma.$$

5. Prove the identity

$$(\alpha + \omega\beta + \omega^2\gamma)^3 + (\alpha + \omega^2\beta + \omega\gamma)^3 \equiv (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta).$$

[Apply Ex. 2.]

6. Prove the identity

$$(\alpha + \omega\beta + \omega^2\gamma)^3 - (\alpha + \omega^2\beta + \omega\gamma)^3 \equiv -3\sqrt{-3}\,(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

[Apply Ex. 2, and substitute for $\omega - \omega^2$ its value $\sqrt{-3}$.]

7. Prove the identity

$$\alpha'^3 + \beta'^3 + \gamma'^3 - 3\alpha'\beta'\gamma' \equiv (\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma)^2,$$

where

$$\alpha' \equiv \alpha^2 + 2\beta\gamma, \quad \beta' \equiv \beta^2 + 2\gamma\alpha, \quad \gamma' \equiv \gamma^2 + 2\alpha\beta.$$

8. Find the equation whose roots are

$$m+n, \quad \omega m + \omega^2 n, \quad \omega^2 m + \omega n.$$

Ans. $x^3 - 3mnx - (m^3 + n^3) = 0$.

9. Find the equation whose roots are

$$l+m+n, \quad l + \omega m + \omega^2 n, \quad l + \omega^2 m + \omega n.$$

Ans. $x^3 - 3lx^2 + 3(l^2 - mn)x - (l^3 + m^3 + n^3 - 3lmn) = 0$.

11 Form an equation with rational coefficients which shall have

$$\theta_1\sqrt[3]{P} + \theta_2\sqrt[3]{Q}$$

for a root, where $\theta_1{}^3 = 1$, and $\theta_2{}^3 = 1$.

Cubing the equation

$$x = \theta_1\sqrt[3]{P} + \theta_2\sqrt[3]{Q},$$

and substituting x for its value on the right-hand side, we get

$$x^3 - P - Q = 3\theta_1\theta_2\sqrt[3]{PQ}\,.\,x.$$

Cubing again, we have

$$(x^3 - P - Q)^3 = 27\,PQx^3.$$

Since θ_1 and θ_2 may each have any one of the values 1, ω, ω^2, the nine roots of this equation are

$$\begin{array}{lll} \sqrt[3]{P} + \sqrt[3]{Q}, & \omega\sqrt[3]{P} + \omega\sqrt[3]{Q}, & \omega^2\sqrt[3]{P} + \omega^2\sqrt[3]{Q}, \\ \omega\sqrt[3]{P} + \omega^2\sqrt[3]{Q}, & \omega^2\sqrt[3]{P} + \sqrt[3]{Q}, & \omega\sqrt[3]{P} + \sqrt[3]{Q}, \\ \omega^2\sqrt[3]{P} + \omega\sqrt[3]{Q}, & \sqrt[3]{P} + \omega^2\sqrt[3]{Q}, & \sqrt[3]{P} + \omega\sqrt[3]{Q}. \end{array}$$

We see also that, since θ_1 and θ_2 have disappeared from the final equation, it is indifferent which of these nine roots is assumed equal to x in the first instance. The resulting equation is that which would have been obtained by multiplying together the nine factors of the form $x - \sqrt[3]{P} - \sqrt[3]{Q}$ obtained from the nine roots above written.

Examples.

1. Find the value of $\Sigma\alpha^2\beta$ of the roots of the cubic

$$x^3+px^2+qx+r=0.$$

Multiplying together the equations

$$\alpha+\beta+\gamma=-p,$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=\quad q,$$

we obtain

$$\Sigma\alpha^2\beta+3\alpha\beta\gamma=-pq;$$

hence

$$\Sigma\alpha^2\beta=3r-pq.$$

2. Find for the same cubic the value of

$$\alpha^2+\beta^2+\gamma^2. \qquad \textit{Ans. } \Sigma\alpha^2=p^2-2q.$$

3. Find for the same cubic the value of

$$\alpha^3+\beta^3+\gamma^3.$$

Multiplying the values of $\Sigma\alpha$ and $\Sigma\alpha^2$, we obtain

$$\alpha^3+\beta^3+\gamma^3+\Sigma\alpha^2\beta=-p^3+2pq;$$

hence, by Ex. 1,

$$\Sigma\alpha^3=-p^3+3pq-3r.$$

4. Find for the same cubic the value of

$$\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2.$$

We easily obtain

$$\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)=q^2,$$

from which

$$\Sigma\alpha^2\beta^2=q^2-2pr.$$

5. Find for the same cubic the value of

$$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta).$$

This is equal to

$$2\alpha\beta\gamma+\Sigma\alpha^2\beta. \qquad \textit{Ans. } r-pq.$$

6. Find the value of the symmetric function

6. Find the value of the symmetric function

$$\alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \beta^2\gamma\delta$$
$$+ \gamma^2\alpha\beta + \gamma^2\alpha\delta + \gamma^2\beta\delta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma$$

of the roots of the biquadratic

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

Multiplying together

$$\alpha + \beta + \gamma + \delta = -p,$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r,$$

we obtain

$$\Sigma\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta = pr;$$

hence

$$\Sigma\alpha^2\beta\gamma = pr - 4s.$$

12. Find the sum of the reciprocals of the roots of the equation in the preceding example.

From the second last, and last of the equations of Art. 23, we have

$$\alpha_2\alpha_3 \ldots \alpha_n + \alpha_1\alpha_3 \ldots \alpha_n + \ldots + \alpha_1\alpha_2 \ldots \alpha_{n-1} = (-1)^{n-1}p_{n-1},$$
$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n p_n;$$

dividing the former by the latter, we have

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \ldots + \frac{1}{\alpha_n} = \frac{-p_{n-1}}{p_n},$$

or

$$\Sigma\frac{1}{\alpha_1} = \frac{-p_{n-1}}{p_n}.$$

In a similar manner the sum of the products in pairs, in threes, &c. of the reciprocals of the roots can be found by dividing the 3rd last, or 4th last, &c. coefficient by the last.

13. Find for the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

the values, in terms of the coefficients, of the following three functions of the roo
α, β, γ:—

$$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2,$$
$$\alpha(\beta-\gamma)^2 + \beta\ (\gamma-\alpha)^2 + \gamma\ (\alpha-\beta)^2,$$
$$\alpha^2(\beta-\gamma)^2 + \beta^2(\gamma-\alpha)^2 + \gamma^2(\alpha-\beta)^2.$$

It will be often found convenient to write, as in the present example, an equa tion with *binomial coefficients*, that is, numerical coefficients corresponding to those i the expansion by the binomial theorem, in addition to the literal coefficients a_0, a &c.

We easily obtain

$$a_0^2\{(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2\} = 18\,(a_1^2 - a_0a_2),$$
$$a_0^2\{\alpha(\beta-\gamma)^2 + \beta(\gamma-\alpha)^2 + \gamma(\alpha-\beta)^2\} = 9\,(a_0a_3 - a_1a_2),$$
$$a_0^2\{\alpha^2(\beta-\gamma)^2 + \beta^2(\gamma-\alpha)^2 + \gamma^2(\alpha-\beta)^2\} = 18\,(a_2^2 - a_1a_3).$$

14. Find in terms of the coefficients of the cubic in the preceding example th quadratic

$$(x-\alpha)^2(\beta-\gamma)^2 + (x-\beta)^2(\gamma-\alpha)^2 + (x-\gamma)^2(\alpha-\beta)^2 = 0,$$

where α, β, γ are the roots of the cubic.

Ans. $(a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)x + (a_1a_3 - a_2^2) = 0.$

15. Find for the cubic of Example 13 the value of

$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta).$$

Since $$2\alpha-\beta-\gamma = 3\alpha - (\alpha+\beta+\gamma) = 3\alpha + \frac{3a_1}{a_0},$$

the required value is easily obtained by substituting $-\frac{a_1}{a_0}$ for x in the identity

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 \equiv a_0(x-\alpha)(x-\beta)(x-\gamma).$$

Ans. $a_0^3(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta) = -27\,(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3).$

28. *Remark.*—We close this chapter with certain observations which will be found useful in verifying the results of the calculation of symmetric functions. The first is, that *the degree of any symmetric function in the roots is always equal to the sum of the suffixes in each term of its value in terms of the coefficients.* The student will observe that this is true in the case of the results of Examples 13, 15, 16, 17, 18, 19, 21, 22; and that it must be so in general appears from the equations (2) of Art. 23, for the suffix of each coefficient in those equations is equal to the degree in the roots of the corresponding function of the roots; hence in any product of any powers of the coefficients the sum of the suffixes must be equal to the degree of the corresponding function of the roots.

1. Find the value of the symmetric function

$$\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$$

of the roots of the cubic equation

$$x^3+px^2+qx+r=0.$$

Ans. $\frac{pq}{r}-3.$

2. Find for the same cubic the value of

$$(\beta+\gamma-\alpha)^3+(\gamma+\alpha-\beta)^3+(\alpha+\beta-\gamma)^3.$$

Ans. $24r-p^3.$

3. Find the value of $\Sigma\alpha^3\beta^3$ of the roots of the same equation.

Here $\Sigma\alpha\beta\,\Sigma\alpha^2\beta^2=\Sigma\alpha^3\beta^3+\alpha\beta\gamma\,\Sigma\alpha^2\beta$; hence &c.

Ans. $q^3-3pqr+3r^2.$

4. Find for the same cubic the symmetric function

$$(\beta^3-\gamma^3)^2+(\gamma^3-\alpha^3)^2+(\alpha^3-\beta^3)^2.$$

$\Sigma\alpha^6$ is easily obtained by squaring $\Sigma\alpha^3$ (see Ex. 3, Art. 27).

Ans. $2p^6-12p^4q+12p^3r+18p^2q^2-18pqr-6q^3.$

5. Find for the same cubic the value of

$$\frac{\beta^2+\gamma^2}{\beta+\gamma}+\frac{\gamma^2+\alpha^2}{\gamma+\alpha}+\frac{\alpha^2+\beta^2}{\alpha+\beta}.$$

Ans. $\frac{2p^2q-4pr-2q^2}{r-pq}.$

6. Find for the same cubic the value of

$$\frac{\alpha^2+\beta\gamma}{\beta+\gamma}+\frac{\beta^2+\gamma\alpha}{\gamma+\alpha}+\frac{\gamma^2+\alpha\beta}{\alpha+\beta}.$$

Ans. $\dfrac{p^4-3p^2q+5pr+q^2}{r-pq}$.

7. Find for the same cubic the value of

$$\frac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}+\frac{2\gamma\alpha-\beta^2}{\gamma+\alpha-\beta}+\frac{2\alpha\beta-\gamma^2}{\alpha+\beta-\gamma}.$$

Ans. $\dfrac{p^4-2p^2q+14pr-8q^2}{4pq-p^3-8r}$.

8. Find the symmetric function $\Sigma\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)^2$ for the same cubic.

Ans. $\dfrac{-p^2q^2-4p^3r+8q^3-2pqr-9r^2}{(r-pq)^2}$.

UNIT-IV

SYLLABUS

Transformation of Equations: Transformation of equations - Roots with signs changed - Roots multiplied by a given quantity - Reciprocal roots and reciprocal equations - To increase or diminish the roots by a given quantity - Removal of terms - Binomial coefficients.

Solution of reciprocal and binomial equations: Reciprocal equations - Binomial equations. Propositions embracing their leading general Properties - The special roots of the equation - Solution of binomial equations by circular functions – Examples.

29. **Transformation of Equations.**—We can, without knowing the roots of an equation, transform it into another whose roots shall have certain assigned relations to those of the proposed. The utility of this process consists in the fact that the discussion of the transformed equation will often be more simple than that of the original. We proceed to explain the most important transformations of equations.

30. **Roots with Signs changed.**—To transform an equation into another whose roots are those of the given equation with contrary signs, let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the roots of

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots p_{n-1}x + p_n = 0;$$

then

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n \equiv (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n);$$

change x into $-y$; we have, then, whether n be even or odd,

$$y^n - p_1y^{n-1} + p_2y^{n-2} - \ldots \pm p_{n-1}y \mp p_n = (y+\alpha_1)(y+\alpha_2)\ldots(y+\alpha_n).$$

The polynomial in y equated to zero is an equation whose roots are $-\alpha_1, -\alpha_2, \ldots -\alpha_n$; and to effect the required transformation we have only to *change the signs of every alternate term of the given equation beginning with the second.*

31. **Roots Multiplied by a Given Quantity.**—To transform an equation whose roots are $a_1, a_2, \ldots a_n$ into another whose roots are $ma_1, ma_2, \ldots ma_n$, we change x into $\frac{y}{m}$ in the identity of the preceding article. We have then, after multiplication by m^n,

$$y^n + mp_1 y^{n-1} + m^2 p_2 y^{n-2} + \ldots + m^{n-1} p_{n-1} y + m^n p_n$$
$$= (y - ma_1)(y - ma_2) \ldots (y - ma_n).$$

Hence, to multiply the roots of an equation by a given quantity m, *we have only to multiply the successive coefficients, beginning with the second, by* $m, m^2, m^3, \ldots m^n$.

The present transformation is useful for getting rid of the coefficient of the first term of an equation when it is not unity; and generally for removing fractional coefficients from an equation. If there is a coefficient a_0 of the first term, we form the equation whose roots are $a_0 a_1, a_0 a_2, \ldots a_0 a_n$; the transformed equation will be divisible by a_0, and after such division the coefficient of x^n will be unity.

When there are fractional coefficients, we can get rid of them by multiplying the roots by a quantity m, which is the least common multiple of all the denominators of the fractions. In many cases, multiplication by a quantity less than the least common multiple will be sufficient for this purpose, as will appear in the following examples:—

1. Change the equation

$$3x^4 - 4x^3 + 4x^2 - 2x + 1 = 0$$

into another the coefficient of whose highest term will be unity. We multiply the roots by 3.

Ans. $x^4 - 4x^3 + 12x^2 - 18x + 27 = 0.$

32. **Reciprocal Roots.**—To transform an equation into one whose roots are the reciprocals of the roots of the proposed equation, we change x into $\frac{1}{y}$ in the identity of Art. 30. This gives, after certain easy reductions,

$$\frac{1}{y^n}+\frac{p_1}{y^{n-1}}+\frac{p_2}{y^{n-2}}+\ldots+\frac{p_{n-1}}{y}+p_n=\frac{p_n}{y^n}\left(y-\frac{1}{a_1}\right)\left(y-\frac{1}{a_2}\right)\ldots\left(y-\frac{1}{a_n}\right),$$

or

$$y^n+\frac{p_{n-1}}{p_n}y^{n-1}+\frac{p_{n-2}}{p_n}y^{n-2}+\ldots+\frac{p_1}{p_n}y+\frac{1}{p_n}=\left(y-\frac{1}{a_1}\right)\left(y-\frac{1}{a_2}\right)\ldots\left(y-\frac{1}{a_n}\right);$$

hence, if in the given equation we replace x by $\frac{1}{y}$, and multiply by y^n, the resulting polynomial in y equated to zero will have for roots $\frac{1}{a_1}, \frac{1}{a_2}, \ldots \frac{1}{a_n}$.

33. **Reciprocal Equations.**—There is a certain class of equations which remain unaltered when x is changed into $\frac{1}{x}$. These are called *reciprocal equations*. The conditions which must obtain among the coefficients of an equation in order that it should be one of this class are, from the preceding Article, plainly the following:—

$$\frac{p_{n-1}}{p_n} = p_1, \ \frac{p_{n-2}}{p_n} = p_2, \ \&c. \ldots \frac{p_1}{p_n} = p_{n-1}, \ \ \frac{1}{p_n} = p_n.$$

The last of these conditions gives $p_n^2 = 1$, or $p_n = \pm 1$. Reciprocal equations are divided into two classes, according as p_n is equal to $+1$, or to -1.

(1). In the first case

$$p_{n-1} = p_1, \ p_{n-2} = p_2, \ \ldots \ p_1 = p_{n-1};$$

and we have the *first class of reciprocal equations, in which the coefficients of the corresponding terms taken from the beginning and end are equal in magnitude and have the same signs.*

(2). In the second case, when $p_n = -1$,

$$p_{n-1} = -p_1, \quad p_{n-2} = -p_2, \ \&c. \ldots p_1 = -p_{n-1};$$

and we have the *second class of reciprocal equations, in which corresponding terms counting from the beginning and end are equal in magnitude but different in sign.* It is to be observed that in this case when the degree of the equation is even, say $n = 2m$, one of the conditions becomes $p_m = -p_m$, or $p_m = 0$; so that in reciprocal equations of the second class, whose degree is even, the middle term is absent.

1. Find the equation whose roots are those of

$$x^4 - 5x^3 + 7x^2 - 17x + 11 = 0,$$

each diminished by 4.

The operation is best exhibited as follows :—

1	-5	7	-17	11
	4	-4	12	-20
	-1	3	-5	**-9**
	4	12	60	
	3	15	**55**	
	4	28		
	7	**43**		
	4			
	11			

Here the first division of the given polynomial by $x - 4$ gives the remainder -9 ($= A_4$), and the quotient $x^3 - x^2 + 3x - 5$ (cf. Art. 8). Dividing this again by $x - 4$, we get the remainder 55 ($= A_3$), and the quotient $x^2 + 3x + 15$. Dividing this again, we get the remainder 43 ($= A_2$), and quotient $x + 7$; and dividing this we get $A_1 = 11$, and $A_0 = 1$; hence the required transformed equation is

$$y^4 + 11y^3 + 43y^2 + 55y - 9 = 0.$$

$x - 4$, we get the remainder 55 ($= A_3$), and the quotient $x^2 + 3x + 15$. Dividing this again, we get the remainder 43 ($= A_2$), and quotient $x + 7$; and dividing this we get $A_1 = 11$, and $A_0 = 1$; hence the required transformed equation is

$$y^4 + 11y^3 + 43y^2 + 55y - 9 = 0.$$

2. Find the equation whose roots are those of

$$x^5 + 4x^3 - x^2 + 11 = 0,$$

each diminished by 3.

1	0	4	− 1	0	11
	3	9	39	114	342
	3	13	38	114	**353**
	3	18	93	393	
	6	31	131	**507**	
	3	27	174		
	9	58	**305**		
	3	36			
	12	**94**			
	3				
	15				

The transformed equation is, therefore,

$$y^5 + 15y^4 + 94y^3 + 305y^2 + 507y + 353 = 0.$$

3. Find the equation whose roots are those of

$$4x^5 - 2x^3 + 7x - 3 = 0,$$

each increased by 2.

[The multiplier in this operation is, of course, −2.]

Ans. $4y^5 - 40y^4 + 158y^3 - 308y^2 + 303y - 129 = 0.$

4. Increase by 7 the roots of the equation

$$3x^4 + 7x^3 - 15x^2 + x - 2 = 0.$$

Ans. $3y^4 - 77y^3 + 720y^2 - 2876y + 4058 = 0.$

5. Diminish by 23 the roots of the equation

$$5x^3 - 13x^2 - 12x + 7 = 0.$$

35. **Removal of Terms.**—One of the chief uses of the transformation of the preceding Article is to remove a certain specified term from an equation. Such a step often facilitates its solution. Writing the transformed equation in descending powers of y, we have

$$a_0 y^n + (na_0 h + a_1) y^{n-1} + \left\{\frac{n(n-1)}{1.2} a_0 h^2 + (n-1) a_1 h + a_2\right\} y^{n-2} + \ldots = 0.$$

If h be such as to satisfy the equation $na_0 h + a_1 = 0$, the transformed equation will want the second term. If h be either of the values which satisfy the equation

$$\frac{n(n-1)}{1.2} a_0 h^2 + (n-1) a_1 h + a_2 = 0,$$

the transformed equation will want the third term; the removal of the fourth term will require the solution of a cubic for h; and so on. To remove the last term we must solve the equation $f(h) = 0$, which is the original equation itself.

EXAMPLES.

1. Transform the equation

$$x^3 - 6x^2 + 4x - 7 = 0$$

into one which shall want the second term.

$$na_0 h + a_1 = 0 \quad \text{gives } h = 2.$$

Diminish the roots by 2. *Ans.* $y^3 - 8y - 15 = 0.$

2. Transform the equation

$$x^4 + 8x^3 + x - 5 = 0$$

into one which shall want the second term.

Increase the roots by 2. *Ans.* $y^4 - 24y^2 + 65y - 55 = 0.$

3. Transform the equation

$$x^4 - 4x^3 - 18x^2 - 3x + 2 = 0$$

into one which shall want the third term.

The quadratic for h is

$$6h^2 - 12h - 18 = 0, \quad \text{giving } h = 3,\ h = -1.$$

Thus there are two ways of effecting the transformation.

Diminishing the roots by 3, we obtain

$$(1) \quad y^4 + 8y^3 - 111y - 196 = 0.$$

Increasing the roots by 1, we obtain

$$(2) \quad y^4 - 8y^3 - 17y - 8 = 0.$$

EXAMPLES.

1. Find the result of substituting $y+h$ for x in the polynomial

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3.$$

Ans. $a_0y^3 + 3(a_0h + a_1)y^2 + 3(a_0h^2 + 2a_1h + a_2)y + a_0h^3 + 3a_1h^2 + 3a_2h + a_3.$

The student will find it a useful exercise to verify this result by the process of operation explained in Art. 34, which may often be employed with advantage in the case of algebraical as well as numerical examples.

2. Remove the second term from the equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

We must diminish the roots by a quantity h obtained from the equation

$$a_0h + a_1 = 0, \quad \text{i. e. } h = \frac{-a_1}{a_0}.$$

Substituting this value of h in A_2, and A_3, the resulting equation in y is

$$y^3 + \frac{3(a_0a_2 - a_1^2)}{a_0^2}y + \frac{a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3}{a_0^3} = 0.$$

3. Find the condition that the second and third terms of the equation $U_n = 0$ should be capable of being removed by the same substitution.

Here A_1 and A_2 must vanish for the same value of h; and eliminating h between them we find the required condition.

Ans. $a_0a_2 - a_1^2 = 0$

4. Solve the equation

$$x^3 + 6x^2 + 12x - 19 = 0$$

by removing its second term.

The third term is removed by the same substitution, which gives

$$y^3 - 27 = 0.$$

37. **The Cubic.**—On account of their peculiar interest, we shall consider in this and the next following Articles the equations of the third and fourth degrees, in connexion with the transformation of the preceding Article. When $y + h$ is substituted for x in the equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0, \tag{1}$$

we obtain

$$a_0y^3 + 3A_1y^2 + 3A_2y + A_3 = 0,$$

where A_1, A_2, A_3 have the values of Art. 36.

If the transformed equation wants the second term,

$$A_1 = 0, \quad \text{or } h = -\frac{a_1}{a_0};$$

substituting this value for h in A_2 and A_3, we find, as in Ex. 2, Art. 36,

$$a_0 A_2 = a_0 a_2 - a_1^2, \quad a_0^2 A_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3;$$

hence the transformed cubic, wanting the second term, is

$$y^3 + \frac{3}{a_0^2}(a_0 a_2 - a_1^2)\,y + \frac{1}{a_0^3}\,(a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3) = 0.$$

The functions of the coefficients here involved are of such importance in the theory of algebraic equations, that it is customary to represent them by single letters. We accordingly adopt the notation

$$a_0 a_2 - a_1^2 = H, \quad a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = G;$$

and then write the transformed equation in the form

$$y^3 + \frac{3H}{a_0^2}\,y + \frac{G}{a_0^3} = 0. \qquad (2)$$

We here observe that if the roots of this equation be multiplied by a_0 it becomes

$$z^3 + 3Hz + G = 0. \qquad (3)$$

This is the form of the cubic we shall employ when we come to treat of its algebraical solution. The variable

$$z = a_0 y = a_0 x + a_1.$$

The original cubic is in fact identical with

$$(a_0 x + a_1)^3 + 3H(a_0 x + a_1) + G = 0,$$

after the factor a_0^2 is removed from this, as the student can easily verify.

38. **The Biquadratic.**—The transformed equation, wanting the second term, is

$$a_0y^4 + 6A_2y^2 + 4A_3y + A_4 = 0,$$

where A_2 and A_3 have the same values as in the preceding Article; and for A_4 we have

$$a_0^3A_4 = a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4.$$

The transformed equation is then

$$y^4 + \frac{6}{a_0^2}Hy^2 + \frac{4}{a_0^3}Gy + \frac{1}{a_0^4}(a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4) = 0.$$

39. **Homographic Transformation.**—The transformation considered in Art. 34 is a particular case of the following, in which x is connected with the new variable y by the equation

$$y = \frac{\lambda x + \mu}{\lambda' x + \mu'}.$$

If $\lambda = 1$, $\mu = -h$, $\lambda' = 0$, $\mu' = 1$, we have $y = x - h$, as in Art. 34. Solving for x in terms of y, we have

$$x = \frac{\mu - \mu' y}{\lambda' y - \lambda}.$$

UNIT-V

SYLLABUS

Algebraic Solution Of the Cubic and Biquadratic: On the algebraic solution of equations - The algebraic solution of the cubic equation - Application to numerical equations - Expression of the cubic as the difference of two cubes - Solution of the cubic by symmetric functions of the roots – Examples .

Properties of the Derived Functions: Graphic representation of the derived function - Theorem relating to the maxima and minima of a polynomial - Rolle's Theorem. Corollary- Constitution of the derived functions

(1). *First method of solution : by resolving into factors.* Let it be required to resolve the quadratic $x^2 + Px + Q$ into its simple factors. For this purpose we put it under the form

$$x^2 + Px + Q + \theta - \theta,$$

and determine θ so that

$$x^2 + Px + Q + \theta$$

may be a perfect square, *i.e.* we make

$$\theta + Q = \frac{P^2}{4}, \quad \text{or} \quad \theta = \frac{P^2 - 4Q}{4};$$

whence, putting for θ its value, we have

$$x^2 + Px + Q = \left(x + \frac{P}{2}\right)^2 - \left(0x + \frac{\sqrt{P^2 - 4Q}}{2}\right)^2.$$

Thus we have reduced the quadratic to the form $u^2 - v^2$; and its simple factors are $u + v$, and $u - v$.

Subsequently we shall reduce the cubic to the form

$$(lx + m)^3 - (l'x + m')^3, \quad \text{or} \quad u^3 - v^3,$$

and obtain its solution from the simple equations

$$u - v = 0, \quad u - \omega v = 0, \quad u - \omega^2 v = 0.$$

It will be shown also that the biquadratic may be reduced to either of the forms

$$(lx^2 + mx + n)^2 - (l'x^2 + m'x + n')^2,$$
$$(x^2 + px + q)(x^2 + p'x + q'),$$

by solving a cubic equation; and, consequently, the solution of the biquadratic completed by solving two quadratics, viz., in the first case, $lx^2 + mx + n = \pm(l'x^2 + m'x + n')$; and in the second case, $x^2 + px + q = 0$, and $x^2 + p'x + q' = 0$.

(2). *Second method of solution : by assuming for a root a general form involving radicals.*

Assuming $x = p + \sqrt{q}$ to be a root of the equation $x^2 + Px + Q = 0$, and rationalizing the equation $x = p + \sqrt{q}$, we have

$$x^2 - 2px + p^2 - q = 0.$$

Now, if this equation be identical with $x^2 + Px + Q = 0$, we have

$$2p = -P, \quad p^2 - q = Q,$$

giving
$$x = p + \sqrt{q} = \frac{-P \pm \sqrt{P^2 - 4Q}}{2},$$

which is the solution of the quadratic equation.

In the case of the cubic equation we shall find that

$$x = \sqrt[3]{p} + \frac{A}{\sqrt[3]{p}}$$

is the proper form to represent a root; this formula giving precisely three values for x, in consequence of the manner in which the cube root enters into it.

In the case of the biquadratic equation we shall find that

$$\sqrt{p} + \sqrt{q} + \frac{A}{\sqrt{p}\sqrt{q}}, \quad \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q}$$

are forms which represent a root; these formulas each giving four, and only four, values of x when the square roots receive their double signs.

(3). *Third method of solution : by symmetric functions of the roots.*

Consider the quadratic equation $x^2 + Px + Q = 0$, of which the roots are α, β.

Then
$$\alpha + \beta = - P,$$
$$\alpha\beta = Q.$$

If we attempt to determine α and β by these equations, we fall back on the original equation (see Art. 24); but if we could obtain a second equation between the roots and coefficients, of the form $l\alpha + m\beta = f(P, Q)$, we could easily find α and β by means of this equation and the equation $\alpha + \beta = -P$.

Now in the case of the quadratic there is no difficulty in finding the required equation; for, obviously,

$$(\alpha-\beta)^2 = P^2 - 4Q\text{; and, therefore, } \alpha - \beta = \sqrt{P^2 - 4Q}.$$

In the case of the cubic equation $x^3 + Px^2 + Qx + R = 0$, we require *two* simple equations of the form

$$l\alpha + m\beta + n\gamma = f(P, Q, R),$$

in addition to the equation $\alpha + \beta + \gamma = - P$, to determine the roots α, β, γ. It will subsequently be proved that the functions

$$(\alpha + \omega\beta + \omega^2\gamma)^3, \quad (\alpha + \omega^2\beta + \omega\gamma)^3$$

may be expressed in terms of the coefficients by solving a *quadratic* equation; and when their values are known the roots of the cubic may be easily found.

In the case of the biquadratic equation

$$x^4 + Px^3 + Qx^2 + Rx + S = 0$$

we require *three* simple equations of the form

$$l\alpha + m\beta + n\gamma + r\delta = f(P, Q, R, S),$$

in addition to the equation

$$\alpha + \beta + \gamma + \delta = - P,$$

56. The Algebraic Solution of the Cubic Equation.—Let the general cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

be put under the form

$$z^3 + 3Hz + G = 0,$$

where $z = ax + b$, $H = ac - b^2$, $G = a^2d - 3abc + 2b^3$ (see Art. 37).

To solve this equation, assume*

$$z = \sqrt[3]{p} + \sqrt[3]{q};$$

hence, cubing,

$$z^3 = p + q + 3\sqrt[3]{p}\,\sqrt[3]{q}\,(\sqrt[3]{p} + \sqrt[3]{q}),$$

therefore

$$z^3 - 3\sqrt[3]{p}\,\sqrt[3]{q}\,.\,z - (p + q) = 0.$$

Now, comparing coefficients, we have

$$\sqrt[3]{p}\,.\,\sqrt[3]{q} = -H, \quad p + q = -G;$$

from which equations we obtain

$$p = \tfrac{1}{2}(-G + \sqrt{G^2 + 4H^3}), \quad q = \tfrac{1}{2}(-G - \sqrt{G^2 + 4H^3});$$

58. **Expression of the Cubic as the Difference of two Cubes.**—Let the given cubic

$$ax^3 + 3bx^2 + 3cx + d = \phi(x)$$

be put under the form

$$z^3 + 3Hz + G,$$

where $z = ax + b$.

Now assume

$$z^3 + 3Hz + G = \frac{1}{\mu - \nu}\{\mu(z+\nu)^3 - \nu(z+\mu)^3\}, \qquad (1)$$

where μ and ν are quantities to be determined; the second side of this identity becomes, when reduced,

$$z^3 - 3\mu\nu z - \mu\nu(\mu+\nu).$$

Comparing coefficients,

$$\mu\nu = -H, \quad \mu\nu(\mu+\nu) = -G;$$

therefore

$$\mu + \nu = \frac{G}{H}, \quad \mu - \nu = \frac{a\sqrt{\Delta}}{H};$$

where $a^2\Delta = G^2 + 4H^3$, as in Art. 41;

also
$$(z+\mu)(z+\nu) = z^2 + \frac{G}{H}z - H. \qquad (2)$$

Whence, putting for z its value, $ax + b$, we have from (1)

$$a^3\phi(x) = \left(\frac{G + a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax + b + \frac{G - a\Delta^{\frac{1}{2}}}{2H}\right)^3 - \left(\frac{G - a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax + b + \frac{G + a\Delta^{\frac{1}{2}}}{2H}\right)^3,$$

which is the required expression of $\phi(x)$ as the difference of two cubes.

The function (2), when transformed and reduced, becomes

$$\frac{a^2}{H}\{(ac - b^2)x^2 + (ad - bc)x + (bd - c^2)\},$$

which contains the two factors $ax + b + \mu$, $ax + b + \nu$.

The expression of the roots of this quadratic in terms of the roots of the given cubic may be seen on referring to Ex. 23, p. 57.

60. **Homographic Relation between two Roots of a Cubic.**—Before proceeding to the discussion of the biquadratic we prove the following important proposition relative to the cubic :—

The roots of the cubic are connected in pairs by a homographic relation in terms of the coefficients.

Referring to Example 13, Art. 27, we have the relations

$$a_0^2\{\ (\beta-\gamma)^2+\ \ (\gamma-\alpha)^2+\ \ (\alpha-\beta)^2\}=18\,(a_1^2-a_0a_2),$$
$$a_0^2\{\alpha\,(\beta-\gamma)^2+\beta\,(\gamma-\alpha)^2+\gamma\,(\alpha-\beta)^2\}=\ \ 9\,(a_0a_3-a_1a_2),$$
$$a_0^2\{\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2\}=18\,(a_2^2-a_1a_3).$$

We adopt the notation

$$a_0a_2-a_1^2=H,\quad a_0a_3-a_1a_2=2H_1,\quad a_1a_3-a_2^2=H_2.$$

Now, multiplying the above equations by $\alpha\beta$, $-(\alpha+\beta)$, 1, respectively, and adding, since

$$\alpha^2-\alpha(\alpha+\beta)+\alpha\beta=0,\quad \beta^2-\beta(\alpha+\beta)+\alpha\beta=0,$$

we have

$$a_0^2(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)^2=18\{H\alpha\beta+H_1(\alpha+\beta)+H_2\};$$

but

$$a_0^4(\beta-\gamma)^2(\gamma-\alpha)^2\,(\alpha-\beta)^2=-\,27\Delta=108\,(HH_2-H_1^2)$$

(see Art. 41) ; whence

$$\pm\sqrt{-\frac{\Delta}{3}}\left(\frac{\alpha-\beta}{2}\right)=H\alpha\beta+H_1(\alpha+\beta)+H_2,$$

and, therefore,

$$H\alpha\beta+\left(H_1+\frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\alpha+\left(H_1-\frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\beta+H_2=0,$$

which is the required homographic relation (see Art. 39).

I 2

EXAMPLES.

1. Resolve into simple factors the expression

$$(\beta-\gamma)^2(x-\alpha)^2+(\gamma-\alpha)^2(x-\beta)^2+(\alpha-\beta)^2(x-\gamma)^2.$$

Let $U=(\beta-\gamma)(x-\alpha),\quad V=(\gamma-\alpha)(x-\beta),\quad W=(\alpha-\beta)(x-\gamma).$

Ans. $\frac{2}{3}(U+\omega V+\omega^2 W)(U+\omega^2 V+\omega W).$

2. Prove that the several equations of the system

$$(\beta-\gamma)^3(x-\alpha)^3=(\gamma-\alpha)^3(x-\beta)^3=(\alpha-\beta)^3(x-\gamma)^3$$

have two factors common.

Making use of the notation in the last Example, we have

$$U^3=V^3=W^3;$$

whence

$$U^3-V^3=(U-V)(U^2+UV+V^2)\equiv\tfrac{1}{2}(U-V)(U^2+V^2+W^2),$$

since

$$U+V+W\equiv 0;$$

therefore

$$(\beta-\gamma)^2(x-\alpha)^2+(\gamma-\alpha)^2(x-\beta)^2+(\alpha-\beta)^2(x-\gamma)^2$$

is the common quadratic factor required.

3. Resolve into simple factors the expression

$$(\beta-\gamma)^3(x-\alpha)^3+(\gamma-\alpha)^3(x-\beta)^3+(\alpha-\beta)^3(x-\gamma)^3.$$

Ans. $3(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma).$

4. Resolve

$$(x-\alpha)(x-\beta)(x-\gamma)$$

into the difference of two cubes.

Assume

$$(x-\alpha)(x-\beta)(x-\gamma) = U_1^3 - V_1^3;$$

whence

$$U_1 - V_1 = \lambda(x-\alpha),$$

$$\omega U_1 - \omega^2 V_1 = \mu(x-\beta),$$

$$\omega^2 U_1 - \omega V_1 = \nu(x-\gamma):$$

adding these we have

$$\lambda + \mu + \nu = 0, \quad \lambda\alpha + \mu\beta + \nu\gamma = 0;$$

and, therefore,

$$\lambda = \rho(\beta-\gamma), \quad \mu = \rho(\gamma-\alpha), \quad \nu = \rho(\alpha-\beta);$$

but $\lambda\mu\nu = 1$; whence

$$\frac{1}{\rho^3} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Substituting these values of λ, μ, ν; and using the notation of Ex. 1,

$$U_1 - V_1 = \rho U, \quad \omega U_1 - \omega^2 V_1 = \rho V, \quad \omega^2 U_1 - \omega V_1 = \rho W;$$

whence

$$3U_1 = \rho(U + \omega^2 V + \omega W),$$

$$-3V_1 = \rho(U + \omega V + \omega^2 W);$$

and U_1 and V_1 are completely determined.

EXAMPLES.

1. The functions L and M are functions of the differences of the roots.

For, $$L = \alpha + \omega\beta + \omega^2\gamma = \alpha - h + \omega(\beta - h) + \omega^2(\gamma - h)$$

for all values of h, since $1 + \omega + \omega^2 = 0$; and giving to h the values α, β, γ, in succession, we obtain three forms for L in terms of the differences $\beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$. Similarly for M.

2. To express the product of the squares of the differences of the roots in terms of the coefficients.

We have

$$L + M = 2\alpha - \beta - \gamma, \quad L + \omega^2 M = (2\beta - \gamma - \alpha)\,\omega, \quad L + \omega M = (2\gamma - \alpha - \beta)\,\omega^2;$$

and, again,

$$L - M = (\beta - \gamma)(\omega - \omega^2), \quad \omega^2 L - \omega M = (\gamma - \alpha)(\omega - \omega^2), \quad \omega L - \omega^2 M = (\alpha - \beta)(\omega - \omega^2);$$

from which we obtain, as in Art. 26,

$$L^3 + M^3 = (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta),$$

$$L^3 - M^3 = -3\sqrt{-3}\,(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta);$$

and since

$$(L^3 - M^3)^2 = (L^3 + M^3)^2 - 4L^3M^3,$$

we have, substituting the previous results,

$$a_0^6(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = -27\,(G^2 + 4H^3).$$

(See Art. 41.)

3. Prove the following identities :—

$$L^3 + M^3 = \tfrac{1}{3}\{(2\alpha - \beta - \gamma)^3 + (2\beta - \gamma - \alpha)^3 + (2\gamma - \alpha - \beta)^3\},$$

$$L^3 - M^3 = \sqrt{-3}\,\{(\beta - \gamma)^3 + (\gamma - \alpha)^3 + (\alpha - \beta)^3\}.$$

These are easily obtained by cubing and adding the values of

$$L + M, \text{ \&c. }; \quad L - M, \text{ \&c.}$$

in the preceding example.

hence they are the roots of the equation

$$(\phi - L)(\phi - \omega L)(\phi - \omega^2 L)(\phi - M)(\phi - \omega M)(\phi - \omega^2 M) = 0,$$

or

$$\phi^6 - (L^3 + M^3)\phi^3 + L^3 M^3 = 0.$$

Substituting for L and M from the equations

$$LM = -\frac{9H}{a^2}, \quad L^3 + M^3 = -27\frac{G}{a^3},$$

we have this equation expressed in terms of the coefficients as follows :—

$$\phi^6 + 3^3\frac{G}{a^3}\phi^3 - 3^6\frac{H^3}{a^6} = 0.$$

5. To obtain expressions for L^2, M^2, &c., in terms of α, β, γ.

The following forms for L^2 and M^2 are obtained by subtracting

$$(\alpha^2 + \beta^2 + \gamma^2)(1 + \omega + \omega^2) \equiv 0 \text{ from } (\alpha + \omega\beta + \omega^2\gamma)^2, \text{ and } (\alpha + \omega^2\beta + \omega\gamma)^2:$$

$$-L^2 = (\beta - \gamma)^2 + \omega^2(\gamma - \alpha)^2 + \omega(\alpha - \beta)^2,$$

$$-M^2 = (\beta - \gamma)^2 + \omega(\gamma - \alpha)^2 + \omega^2(\alpha - \beta)^2.$$

In a similar manner, we find from these formulas

$$-L^4 = (\beta - \gamma)^2(2\alpha - \beta - \gamma)^2 + \omega(\gamma - \alpha)^2(2\beta - \gamma - \alpha)^2 + \omega^2(\alpha - \beta)^2(2\gamma - \alpha - \beta)^2,$$

$$-M^4 = (\beta - \gamma)^2(2\alpha - \beta - \gamma)^2 + \omega^2(\gamma - \alpha)^2(2\beta - \gamma - \alpha)^2 + \omega(\alpha - \beta)^2(2\gamma - \alpha - \beta)^2.$$

Also, without difficulty, we have the following forms for LM, and L^2M^2:—

$$2LM = (\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2,$$

$$L^2M^2 = (\alpha - \beta)^2(\alpha - \gamma)^2 + (\beta - \gamma)^2(\beta - \alpha)^2 + (\gamma - \alpha)^2(\gamma - \beta)^2.$$

8. Form the equation whose roots are the several values of ρ, where

$$\rho = \frac{\alpha - \beta}{\beta - \gamma}.$$

Since

$$\alpha - (1 + \rho)\beta + \rho\gamma = 0,$$

substituting for α, β, γ, their values in terms of p, q; and putting

$$\lambda = 1 - (1 + \rho)\omega + \rho\omega^2, \quad \mu = 1 - (1 + \rho)\omega^2 + \rho\omega,$$

we have

$$\lambda\sqrt[3]{p} + \mu\sqrt[3]{q} = 0.$$

Cubing, and substituting for p, q their values,

$$G(\lambda^3 + \mu^3) + a\sqrt{\Delta}(\lambda^3 - \mu^3) = 0.$$

Squaring,

$$a^2\Delta\lambda^3\mu^3 = H^3(\lambda^3 + \mu^3)^2,$$

and by previous results

$$\lambda\mu = 3(1 + \rho + \rho^2), \quad \lambda^3 + \mu^3 = -27\rho(1 + \rho);$$

substituting these values, we have the required equation

$$a^2\Delta(1 + \rho + \rho^2)^3 - 27H^3(\rho + \rho^2)^2 = 0.$$

9. Find the relation between the coefficients of the cubics

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

$$a'x'^3 + 3b'x'^2 + 3c'x' + d' = 0,$$

when the roots are connected by the equation

$$\alpha(\beta' - \gamma') + \beta(\gamma' - \alpha') + \gamma(\alpha' - \beta') = 0.$$

Multiplying by $\omega - \omega^2$, this equation becomes

$$LM' = L'M.$$

Cubing and introducing the coefficients, we find

$$G^2H'^3 = G'^2H^3,$$

the required relation.

10. Determine the condition in terms of the roots and coefficients that the cubics of Ex. 9 should become identical by the linear transformation

$$x' = px + q.$$

In this case

$$\alpha' = p\alpha + q, \quad \beta' = p\beta + q, \quad \gamma' = p\gamma + q.$$

Eliminating p and q, we have

$$\beta\gamma' - \beta'\gamma + \gamma\alpha' - \gamma'\alpha + \alpha\beta' - \alpha'\beta = 0,$$

which is the function of the roots considered in the last example. This relation, moreover, is unchanged if for α, β, γ; α', β', γ', we substitute

$$l\alpha + m, \quad l\beta + m, \quad l\gamma + m,$$
$$l'\alpha' + m', \quad l'\beta' + m', \quad l'\gamma' + m',$$

whence we may consider the cubics in the last example under the simple forms

$$z^3 + 3Hz + G = 0, \quad z'^3 + 3H'z' + G' = 0,$$

obtained by the linear transformations $z = ax + b$, $z' = a'x' + b'$; for if the condition

holds for the roots of the former equations, it must hold for the roots of the latter. Now putting $z' = kz$, these equations become identical if

$$H' \equiv k^2 H, \quad G' \equiv k^3 G;$$

whence, eliminating k,

$$G^2 H'^3 = G'^2 H^3$$

is the required condition, the same as that obtained in Ex. 9. It may be observed that the reducing quadratics of the cubics necessarily become identical by the same transformation, viz.,

$$\frac{H'}{G'}(a'x' + b') = \frac{H}{G}(ax + b).$$

61. **First Solution by Radicals of the Biquadratic. Euler's Assumption :**—Let the biquadratic equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

be put under the form

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

where $z \equiv ax + b$,

$$H \equiv ac - b^2, \quad I \equiv ae - 4bd + 3c^2, \quad G \equiv a^2d - 3abc + 2b^3.$$

(See Art. 38.)

To solve this equation (a biquadratic wanting the second term) Euler assumes as the general expression for a root

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

Squaring,

$$z^2 - p - q - r = 2(\sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q}).$$

Squaring again, and reducing, we obtain the equation

$$z^4 - 2(p+q+r)z^2 - 8\sqrt{p}.\sqrt{q}.\sqrt{r}.z + (p+q+r)^2 - 4(qr+rp+pq) = 0.$$

Comparing this equation with the former equation in z, we have

$$p+q+r = -3H, \quad qr+rp+pq = 3H^2 - \frac{a^2I}{4}, \quad \sqrt{p}.\sqrt{q}.\sqrt{r} = -\frac{G}{2};$$

and consequently p, q, r are the roots of the equation

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0;$$

or, since

$$-G^2 \equiv 4H^3 - a^2HI + a^3J \qquad \text{(see Art. 38)},$$

where

$$J \equiv ace + 2bcd - ad^2 - eb^2 - c^3,$$

we may write this equation under the form

$$4(t+H)^3 - a^2 I(t+H) + a^3 J = 0;$$

and finally, putting $t + H \equiv a^2\theta$, we obtain the equation

$$4a^3\theta^3 - Ia\theta + J = 0,$$

which we call *the reducing cubic* of the biquadratic equation.

Also, since $t \equiv b^2 - ac + a^2\theta$; if $\theta_1, \theta_2, \theta_3$ be the roots of the reducing cubic, we have

$$p \equiv b^2 - ac + a^2\theta_1, \quad q \equiv b^2 - ac + a^2\theta_2, \quad r \equiv b^2 - ac + a^2\theta_3;$$

and, therefore,

$$z = \sqrt{b^2 - ac + a^2\theta_1} + \sqrt{b^2 - ac + a^2\theta_2} + \sqrt{b^2 - ac + a^2\theta_3}.$$

The radicals in this formula have not complete generality; for if they had, eight values of z in place of four would be given by the formula. This limitation is imposed by the relation

$$\sqrt{p} \cdot \sqrt{q} \cdot \sqrt{r} = -\frac{G}{2},$$

which (lost sight of in squaring to obtain the value of pqr) requires such signs to be attached to each of the quantities $\sqrt{p}, \sqrt{q}, \sqrt{r}$, that their product may maintain the sign determined by the above equation; thus,

$$\sqrt{p}\,\sqrt{q}\,\sqrt{r} = \sqrt{p}(-\sqrt{q})(-\sqrt{r}) = (-\sqrt{p})\sqrt{q}\,(-\sqrt{r})$$
$$= (-\sqrt{p})(-\sqrt{q})\sqrt{r}$$

are all the possible combinations of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ fulfilling this condition, provided $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ retain the same signs throughout, whatever those signs may be. But we may avoid all ambiguity as regards sign, and express in a single algebraic formula the four values of z, by eliminating one of the quantities $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ from the formula

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r}$$

by means of the relation given above, and leaving the other two quantities unrestricted in sign. We have then

$$z = \sqrt{p} + \sqrt{q} - \frac{G}{2\sqrt{p}\sqrt{q}},$$

a formula free from all ambiguity, since it gives four, and only four, values of z when $\sqrt{p}$ and $\sqrt{q}$ receive their double signs:

the sign given to each of these in the first two terms determining that which must be attached to it in the denominator of the third term. And finally, restoring p, q, and z their values given before, we have

$$ax + b = \sqrt{b^2 - ac + a^2\theta_1} + \sqrt{b^2 - ac + a^2\theta_2} - \frac{G}{2\sqrt{(b^2 - ac + a^2\theta_1)}\,.\,\sqrt{b^2 - ac + a^2\theta_2}}$$

as *the complete algebraic solution of the biquadratic equation;* θ_1 and θ_2 being roots of the equation

$$4a^3\theta^3 - Ia\theta + J = 0.$$

To assist the student in justifying Euler's apparently arbitrary assumption as to the form of solution of the biquadratic, we remark, that since the second term of the equation in z is absent, the sum of the four roots is zero, or $z_1 + z_2 + z_3 + z_4 = 0$; and consequently the functions $(z_1 + z_2)^2$, &c., of which there are in general *six* (the combinations of four quantities two and two), are in this case reduced to *three* only; so that we may assume

$$(z_2 + z_3)^2 = (z_1 + z_4)^2 = 4p,$$
$$(z_3 + z_1)^2 = (z_2 + z_4)^2 = 4q,$$
$$(z_1 + z_2)^2 = (z_3 + z_4)^2 = 4r;$$

from which we have z_1, z_2, z_3, z_4, included in the formula

$$\sqrt{p} + \sqrt{q} + \sqrt{r}.$$

EXAMPLES.

1. Show that the two biquadratic equations

$$A_0x^4 + 6A_2x^2 \pm 4A_3x + A_4 = 0$$

have the same reducing cubic.

2. Find the reducing cubic of the two biquadratic equations

$$x^4 - 6lx^2 \pm 8\sqrt{(l^3 + m^3 + n^3 - 3lmn)} \,.\, x + 3(4mn - l^2) = 0.$$

Ans. $\theta^3 - 3mn\theta - (m^3 + n^3) = 0.$

3. Prove that the eight roots of the equation

$$\{x^4 - 6lx^2 + 3(4mn - l^2)\}^2 = 64(l^3 + m^3 + n^3 - 3lmn)x^2$$

are given by the formula

$$\sqrt{l + m + n} + \sqrt{l + \omega m + \omega^2 n} + \sqrt{l + \omega^2 m + \omega n}.$$

(Compare Ex. 20, p. 34.)

4. If the expression

$$\sqrt{l + m + n} + \sqrt{l + \omega m + \omega^2 n} + \sqrt{l + \omega^2 m + \omega n}$$

be a root of the equation

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

determine H, I, J in terms of l, m, n.

Ans. $H = -l, \quad I = 12mn, \quad J = -4(m^3 + n^3).$

5. Prove that J vanishes for the biquadratic

$$m(x - n)^4 - n(x - m)^4.$$

6. Write down the formulas expressing the root of a biquadratic in the particular cases when $I = 0$, and $J = 0$.

7. What is the quantity under the *final* square root in the formula expressing a root?

Ans. $27J^2 - I^3.$

8. Prove that the coefficients of the equation of the squares of the differences of the roots of the biquadratic equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

may be expressed in terms a_0, H, I, and J.

Removing the second term from the equation, we obtain

$$y^4 + \frac{6H}{a_0^2}y^2 + \frac{4G}{a_0^3}y + \frac{a_0^2I - 3H^2}{a_0^4} = 0\ ;$$

and changing the signs of the roots, we have

$$y^4 + \frac{6H}{a_0^2}y^2 - \frac{4G}{a_0^3}y + \frac{a_0^2I - 3H^2}{a_0^4} = 0.$$

These transformations leave the functions $(\alpha - \beta)^2$, &c., unaltered; but G becomes $-G$, the other coefficients of the latter equation remaining unchanged; therefore G can enter the coefficients of the equation of the squares of the differences in *even* powers only. And since

$$-G^2 \equiv 4H^3 - a_0^2HI + a_0^3J,$$

G^2 may be eliminated, introducing a_0, H, I, J. In a similar manner we may prove that every even function of the differences of the roots α, β, γ, δ may be expressed in terms of a_0, H, I, J, the function G of odd degree not entering.