KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester - II

		L	T	P	C
18MMU203	**REAL ANALYSIS**	**6**	**2**	**0**	**6**

Course Objective

To enable the students to learn and gain knowledge about extreme points, Root test, Ratio test, alternating series, and series of functions.

Course Outcomes

On successful completion of this course, the student will be able to

- Understand about the categories of sets.
- Acquire the knowledge on limits and convergence of sequences.
- Know the types of test of convergence for series.
- Familiarize about the basic theorems on monotone sequences.
- Know about the radius of convergence.

UNIT I

Finite and infinite sets, examples of countable and uncountable sets. Real line, bounded sets, suprema and infima, completeness property of R, Archimedean property of R, intervals.

UNIT II

Real Sequence, Bounded sequence, Cauchy convergence criterion for sequences. Limit of a sequence. Limit Theorems. Cauchy's theorem on limits, order preservation and squeeze theorem, monotone sequences and their convergence (monotone convergence theorem without proof).

UNIT III

Infinite series. Cauchy convergence criterion for series, positive term series, geometric series, comparison test, convergence of p-series, Root test, Ratio test, alternating series, Leibnitz's test(Tests of Convergence without proof). Definition and examples of absolute and conditional convergence.

UNIT IV

Monotone Sequences, Monotone Convergence Theorem. Subsequences, Divergence Criteria, Monotone Subsequence Theorem (statement only), Bolzano Weierstrass Theorem for Sequences. Cauchy sequence, Cauchy's Convergence Criterion. Concept of cluster points and statement of Bolzano -Weierstrass theorem.

UNIT V

Sequence of functions, Series of functions, Point wise and uniform convergence. M-test, Statements of the results about uniform convergence and integrability and differentiability of functions, Power series and radius of convergence.

SUGGESTED READINGS

TEXT BOOK

1. Bartle R.G. and Sherbert D. R., 2013. Introduction to Real Analysis, John Wiley and Sons (Asia) Pvt. Ltd.

REFERENCES

1. Fischer E., (2012). Intermediate Real Analysis, Springer Verlag.

2. Ross K.A., (2003).Elementary Analysis- The Theory of Calculus Series - Undergraduate Texts in Mathematics, Springer Verlag.

3. Apostol T. M., (2010). Calculus (Vol.II), John Wiley and Sons (Asia) P. Ltd.

4. Goldberg R.,(2012). Methods of Real Analysis, Oxford and IBH publishing Co. Pvt. Ltd. New Delhi.

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE-21

DEPARTMENT OF MATHEMATICS

Real analysis-18MMU203

Unit I

S.No	Lecture Duration	Topics to be covered	Support Materials
1	1	Introduction to set	R1: Ch 1, 1-5
2	1	Set operations	R1: Ch 1, 5-10
3	1	Finite and infinite sets	R1: Ch 1, 16-17
4	1	Tutorial	
5	1	Examples of countable and uncountable sets.	R1: Ch 1,18-19
6	1	Real line	R1: Ch 2, 33-34
7	1	Bounded sets	R1: Ch 2, 34
8	1	Tutorial	
9	1	Supremum of a set	R1: Ch 2, 34-35
10	1	infimum of a set	R1: Ch 2, 34-35
11	1	Completeness property of R	R1: Ch 2, 37
12	1	Tutorial	
13	1	Archimedean property of R	R1: Ch 2, 40
14	1	Intervals	R1: Ch 2, 44-47
15	1	Tutorial	
16	1	Recapitulation and Disscussion of possible questions	
Total No.of hrs 16			

Unit II

S.No	Lecture Duration	Topics to be covered	Support Materials
1	1	Introduction to sequence	R1: Ch 3, 52
2	1	Real Sequence	R1: Ch 3, 53
3	1	Bounded sequence	R1: Ch 3, 54
4	1	Tutorial	
5	1	Cauchy convergence criterion for sequences	R1: Ch 3, 54-56
6	**1**	**Limit of a sequence**	**R4:Ch 2, 27-29**
7		**Limits theorem**	**R3:Ch 3, 63-70**
8		Tutorial	
9	1	Cauchy’s theorem on limits	R1: Ch 3, 56-59
10	1	Order preservation	R1: Ch 3, 62-64
11	1	Squeeze theorem and Monotone sequences	R1: Ch 3, 64-69
12	1	Tutorial	
13	1	Convergence of monotone sequences	R1: Ch 3, 69-70
14	1	Monotone convergence theorem	R1: Ch 3, 72-73
15	1	Tutorial	
16	1	Recapitulation and Disscussion of possible questions	
Total No.of hrs 16			

Unit III

S.No	Lecture Duration	Topics to be covered	Support Materials
1.	1	Introduction to Infinite series	R1: Ch 3, 89
2.	1	Cauchy convergence criterion for series	R1: Ch 3, 90
3.	1	Positive term series	R1: Ch 3, 91
4.	1	Tutorial	
5.	1	Geometric series	R1: Ch 3, 92
6.	1	Comparison test	R1: Ch 3, 92-93
7.	1	Convergence of p-series	R1: Ch 3, 93-94
8.	1	Tutorial	
9.	1	Root test and Ratio test	R1: Ch 3, 95, T: Ch 10, 399
10.	1	Alternating series	R1: Ch 3, 95-96
11.	1	Tutorial	
12.	1	Leibnitz’s test	R1: Ch 3, 96
13.	1	Definition and examples of absolute convergence	R3: Ch 10, 406
14.	1	Definition and examples of conditional convergence	R3: Ch 10,407
15.	1	Tutorial	
16.	1	Recapitulation and Disscussion of possible questions	
Total No.of hrs 16			

Unit IV

S.No	Lecture Duration	Topics to be covered	Support Materials
1.	1	Introduction to Sequences	R3: Ch 4, 257-260
2.	1	Monotone Sequences	T1: Ch 3, 66-67
3.	1	Monotone Convergence Theorem	T1: Ch 3, 78
4.	1	Tutorial	
5.	1	Subsequences	T1: Ch 3,79
6.	1	Divergence Criteria	T1: Ch 4, 125
7.	1	Monotone Subsequence Theorem	T1: Ch 3,79-80
8.	1	Tutorial	
9.	1	Bolzano Weierstrass Theorem for Sequences	T1: Ch 3,80-81
10.	1	Bolzano Weierstrass Theorem for Sequences	T1: Ch 3,80
11.	1	Cauchy sequence	T1: Ch 3, 81
12.	1	Tutorial	
13.	1	Cauchy's Convergence Criterion	T1: Ch 3, 81-83
14.	1	Concept of cluster points	R1: Ch 2, 50
15.	1	Tutorial	
16.	1	Recapitulation and Discussion of possible questions	
Total No.of hrs 16			

Unit V

S.No	Lecture Duration	Topics to be covered	Support Materials
1.	1	Introduction to Sequence of functions	R4: Ch 9,252
2.	1	Introduction to Series of functions	T1: Ch 9, 266
3.	1	Pointwise and Uniform convergence	T1: Ch 9, 266-267
4.	1	Tutorial	
5.	1	M-test	T1: Ch 9, 267
6.	1	Results about uniform convergence	T1: Ch 9, 267-268
7.	1	Titorial	
8.	1	Integrability and differentiability of functions	T1: Ch 9, 2669-270
9.	1	Tutorial	
10.	1	Power series	T1: Ch 9, 270-271
11.	1	Radius of convergence	T1: Ch 9, 271-272
12.	1	Tutorial	
13.	1	Discussion of Previous year ESE question paper	
14.	1	Discussion of Previous year ESE question paper	
15.	1	Discussion of Previous year ESE question paper	
16.	1	Recapitulation and Discussion of possible questions	
Total No.of hrs 16			

SUGGESTED READINGS

TEXT BOOK

1. Bartle R.G. and Sherbert D. R., 2013. Introduction to Real Analysis, John Wiley and Sons (Asia) Pvt. Ltd.

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CHAPTER 1

SETS AND FUNCTIONS

1.1 Introduction

[1] Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used in the definitions of nearly all mathematical objects.

The modern study of set theory was initiated by **Georg Cantor** and **Richard Dedekind** in the 1870s. After the discovery of paradoxes in naive set theory, such as the Russell's paradox, numerous axiom systems were proposed in the early twentieth century, of which the ZermeloFraenkel axioms, with the axiom of choice, are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of ZermeloFraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

[1]source from wikipedia

1.2 Basics of sets

Definition 1.2.1 *A collection of well defined objects is called a set.*

Definition 1.2.2 *Objects of a set are called elements or members.*

Remark 1.2.1
- *If x is an element of A, we say that $x \in A$.*
- *If x is not an element of A, we say that $x \notin A$.*

Example 1.2.1
- $A = \{x \colon x \text{ is an integer}\}$
- $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, *set of all natural numbers.*
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, *set of all integers.*[2]
- $\mathbb{Q} = \left\{\frac{p}{q} \colon p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$, *set of rational numbers*[3]

Definition 1.2.3 *A set that contains no elements is called the null set. It is denoted by $\emptyset$.*

Definition 1.2.4 *A set consisting of only one element is called a singleton set.*

Definition 1.2.5 *If every element of a set A also belongs to a set B, we say that $A \subseteq B$ (or) $B \supseteq A$.*

Definition 1.2.6 *A set A is a proper subset of B if $A \subseteq B$ and there is atleast one element of B which is not in A.*

Definition 1.2.7 *Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$.*

Definition 1.2.8 *The union of sets A and B is the set $A \cup B = \{x \colon x \in A \text{ or } x \in B\}$.*

Example 1.2.2 *Since $\mathbb{N}$ is the set of all natural numbers and $\mathbb{Z}$ is the set of all integers, we have $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.*
Then $\mathbb{N} \cup \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
and $\mathbb{N} \cup \mathbb{Z} = Z$

Remark 1.2.2 *(i) If $A \subset B$, then $A \cup B = B$*
(ii) Since $\emptyset \subset A$, then $\emptyset \cup A = A$.
(iii) Union of two sets is commutative.

Definition 1.2.9 *The intersection of the sets A and B is the set $A \cap B = \{x \colon x \in A \text{ and } x \in B\}$.*

Example 1.2.3 *Suppose $A = \{1, 2, 3\}$ and $B = \{-2, -1, 0, 1\}$. Then $A \cap B = \{1\}$.*

Definition 1.2.10 *The complement of B relative to A is the set $A - B = \{x \colon x \in A \text{ and } x \notin B\}$.*

[2] Z is for Zahlen - the German word for integers.
[3] Q is for quotient - which is how rational numbers are identified.

Example 1.2.4 *Suppose* $A = \{1, 2, 3, 4\}$. *and* $B = \{-2, -1, 0, 1\}$. *Then* $A - B = \{2, 3, 4\}$.

Theorem 1.2.1 *For any three sets* A, B *and* C, *we have*
(i) $A \cup A = A$
(ii) $A \cup \emptyset = \emptyset \cup A = A$
(iii) $A \cup B = B \cup A$
(iv) $A \cup (B \cup C) = (A \cup B) \cup C$
(v) $A \cup B = B$ *if and only if* $A \subseteq B$

Proof
(iv) Let $x \in A \cup (B \cup C)$ be arbitrary

$$\begin{aligned}
&\Rightarrow x \in A \text{ (or) } x \in (B \cup C)\\
&\Rightarrow x \in A \text{ (or) } x \in B \text{ (or) } x \in C\\
&\Rightarrow (x \in A \text{ (or) } x \in B) \text{ (or) } x \in C\\
&\Rightarrow x \in (A \cup B) \text{ (or) } x \in C\\
&\Rightarrow x \in (A \cup B) \cup C\\
&\Rightarrow A \in (B \cup C) \subseteq (A \cup B) \cup C \qquad (1.1)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow (x \in A \cup x \in B) \text{ (or) } x \in C\\
&\Rightarrow x \in A \text{ (or) } x \in B \text{ (or) } x \in C\\
&\Rightarrow x \in A \text{ (or) } (x \in B \text{ (or) } x \in C)\\
&\Rightarrow x \in A \text{ (or) } x \in (B \cup C)\\
&\Rightarrow x \in A \cup (B \cup C)\\
&\Rightarrow (A \cup B) \cup C \subseteq A \cup (B \cup C) \qquad (1.2)
\end{aligned}$$

From (1.1) and (1.2), we have
$A \in (B \cup C) = (A \cup B) \cup C$

Theorem 1.2.2 *For any three sets* A, B *and* C, *we have*
(i) $A \cap A = A$.
(ii) $A \cap \emptyset = \emptyset \cap A = A$.
(iii) $A \cap B = B \cap A$.
(iv) $A \cap (B \cap C) = (A \cap B) \cap C$
(v) $A \cap B = B$ *if and only if* $A \subseteq B$

Definition 1.2.11 *Two sets* A *and* B *are said to be disjoint if* $A \cap B = \phi$

Example 1.2.5 *Let* $A = \{1, 3, 4\}$ *and* $B = \{5, 8, 9\}$ *then* $A \cap B = \phi$

Remark 1.2.3 *1.* $x \notin A \cup B \Leftrightarrow x \notin A$ *and* $x \notin B$
2. $x \notin A \cap B \Leftrightarrow x \notin A$ *(or)* $x \notin B$

Theorem 1.2.3 *If A,B and C are sets then*
(i) $A - (B \cup C) = (A - B) \cap (A - C)$
(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

Proof
(i) Let $x \in A - (B \cup C)$ be arbitrary

$$\begin{aligned}
&\Rightarrow \quad x \in A \text{ and } x \notin (B \cup C) \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ and } x \notin C \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ and } x \in A \text{ and } x \notin C \\
&\Rightarrow \quad x \in (A - B) \text{ and } x \in (A - C) \\
&\Rightarrow \quad x \in (A - B) \cap (A - C)
\end{aligned}$$

Therefore, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$
similarly, we can prove
$(A - B) \cap (A - C) \subseteq A - (B \cup C)$
From the above, we have
$A - (B \cup C) = (A - B) \cap (A - C)$
(ii) Let $x \in A - (B \cap C)$ be arbitrary

$$\begin{aligned}
&\Rightarrow \quad x \in A \text{ and } x \notin (B \cap C) \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ or } x \notin C \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ or } x \in Aandx \notin C \\
&\Rightarrow \quad x \in (A - B) \text{ or } x \in (A - C) \\
&\Rightarrow \quad x \in (A - B) \cup (A - C)
\end{aligned}$$

Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$
similarly, we can prove
$(A - B) \cup (A - C) \subseteq A - (B \cap C)$
From the above, we have
$A - (B \cap C) = (A - B) \cup (A - C)$
Hence proved.

Definition 1.2.12 *If A and B are nonempty sets, then the cartesian product of A and B is denoted by AXB and is defined by* $AXB = \{(a, b) : a \in Aandb \in B\}$

Definition 1.2.13 *A set S is said to be finite if it is either empty set (or) it has n elements for some* $n \in N$.

1.3 Functions

Definition 1.3.1 *Let A and B be nonempty sets. A function* $f : A \to B$ *which assigns to each element* $a \in A$*, a unique element* $b \in B$.

Remark 1.3.1 *The element b is called the image of a under f.*

Remark 1.3.2 *The element a is called preimage of b under f.*

Remark 1.3.3 *The set A is called domain of f and the set B is called co domain of f.*

Remark 1.3.4 *The set $\{f(a) : a \in A\}$ is called range of f, and is denoted by $R(f)$.*

Definition 1.3.2 *A function $f : A \to A$ is given by $f(x) = x\ \forall x$, is called identity function.*

Definition 1.3.3 *A function $f : A \to B$ is given by $f(x) = c$, a constant is called constant function.*

Remark 1.3.5

- *The range of constant function is always singleton set.*
- *Suppose $f : A \to B$ is an identity function, then $A = B$ or $A \subseteq B$.*

Definition 1.3.4 *A function $f : A \to B$ is one-one (injective) if distinct elements of A have distinct image in B.*

Remark 1.3.6 *f is one-one if $f(x) = f(y) \Rightarrow x = y$.*

Remark 1.3.7 *f is one-one if $x \neq y \Rightarrow f(x) \neq f(y)$.*

Definition 1.3.5 *A function $f : A \to B$ is onto(surjective) if range of f is equal to B.*

Definition 1.3.6 *A function $f : A \to B$ is called bijection if f is both one-one and onto function.*

Example 1.3.1 *Let $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = |x|\ \forall x \in \mathbb{Z}$.*
Here $f(-2) = f(2)$ but $-2 \neq 2$
Therefore, f is not one-one.

Example 1.3.2 *Consider $f : Z \to Z$ given by $f(x) = x + 3\ \forall\ x \in \mathbb{Z}$.*
Suppose

$$\begin{aligned} f(x) &= f(y) \\ x + 3 &= y + 3 \\ x &= y \end{aligned}$$

Therefore, f is one-one. Also $R_f = \mathbb{Z}$
Therefore, f is onto. Hence, f is bijection.

Definition 1.3.7 *Let $f : A \to B$ be a bijection. Then for each $b \in B$, there exists a unique element $a \in A$ such that $f(a) = b$.*
Define $f^{-1} : B \to A$ by $f^{-1}(b) = a$ Therefore, f^{-1} is called the inverse function of f.

Remark 1.3.8 *Suppose $f : A \to B$ is a bijection. Then A and B are said to be equivalent.*

1.4 Countable sets

Definition 1.4.1 *A set S is said to be countably infinite if there is a bijection between $\mathbb{N}$ and S,*

Example 1.4.1 *Let $E = \{2n : n \in \mathbb{N}\}$ is a even function.*
Let $f : \mathbb{N} \to E$ such that $f(x) = 2x$.
suppose

$$\begin{aligned} f(x) &= f(y) \\ 2x &= 2y \\ x &= y \end{aligned}$$

Therefore, f is one-one.
Also, $R_f = \{2, 4, 6, \cdots\} = E$
Therefore, f is onto.
$\therefore$ f is bijection.
$\therefore$ E is countably finite.

Example 1.4.2 *Let $A = \{\frac{1}{2}, \frac{2}{3}, \cdots\}$*
Solution
Let f be a function from $\mathbb{N} \to A$, such that
$f(n) = \frac{n}{n+1}$.
Suppose

$$\begin{aligned} f(n) &= f(m) \\ \frac{n}{n+1} &= \frac{m}{m+1} \\ n(m+1) &= m(n+1) \\ nm + n &= mn + m \end{aligned}$$

clearly f is one-one and onto function.
Therefore f is bijection.
Hence A is countably infinite.

Remark 1.4.1 *A subset of a countable set is countable.*

Theorem 1.4.1 *$N \times N$ is countable*

Proof
$N \times N = \{(a, b) : a, b \in N\}$
Take all orederd pairs (a, b) such that $a + b = 2$
There is only one element namely $(1, 1)$
Take all ordered pairs (a, b) such that $a + b = 3$
we have $(1, 2)$ and $(2, 1)$.
Next take all the ordered pairs (a, b) such that $a + b = 4$
we have $(1, 3), (2, 2)$ and $(3, 1)$
Proceeding like this and listing all the ordered pairs together from the begining, we get
$\{(1, 1), (1, 2), (2, 1), (1, 3), \cdots\}$
The set contains every ordered pair belonging to $\mathbb{N} \times \mathbb{N}$ exactly once
$\therefore$ $\mathbb{N} \times \mathbb{N}$ is countable (or) countably infinite.

Remark 1.4.2 *If A and B are countable sets then $A \times B$ is also countable.*

Remark 1.4.3 *The set of all natural numbers is countable.*

Definition 1.4.2 *A set which is not countable is called uncountable.*

Theorem 1.4.2 $(0, 1]$ *is uncountable.*

Proof
Suppose $(0, 1]$ is countable.
The elements of $(0, 1]$ can be listed.
i.e., $(0, 1] = \{x_1, x_2, \dots\}$, where

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\dots \\ x_2 &= 0.a_{21}a_{22}a_{23}\dots \\ &\vdots \end{aligned}$$

with $0 \le a_{ij} \le 9$
Let $y = 0.b_1b_2b_3\dots$, clearly $y \in (0, 1]$
Now for each positive integer n select b_n such that $0 \le b_n \le 9$ and $b_n \neq a_{nn}$
Here y is different from each x_i atleast in the i^{th} place.
Which is contradiction to every elements of $(0, 1]listed$.
Hence, $(0, 1]$ is uncountable.

Remark 1.4.4 *The set of all real numbers* $\mathbb{R}$ *is uncountable.*

Remark 1.4.5 *The set of all irrational numbers is uncountable.*

1.5 The absolute value of a real number

Definition 1.5.1 *The absolute value of a real number a is denoted by* $|a|$ *is defined by*

$$|a| = \begin{cases} a & if \quad a > 0 \\ -a & if \quad a < 0 \end{cases}$$

Remark 1.5.1 *Suppose a is a real number* $|a| \ge 0$

Remark 1.5.2 $|a| = |-a|$

Theorem 1.5.1 *(a)* $|ab| = |a||b| foralla, b \in R$
(b) $|a|^2 = a^2 foralla \in R$
(c) If $c \ge 0$ *, then* $|a| \le c \Leftrightarrow -c \le a \le c$
(d) $-|a| \le a \le |a|$ *for all* $a \in R$.

Proof
(a) Case (i): Suppose

$$\begin{aligned} a &= 0 \\ |a| &= 0 \\ |a| \cdot |b| &= 0 \cdot |b| \\ &= 0 \end{aligned}$$

$$\begin{aligned} |a \cdot b| &= |0 \cdot b| \\ &= |0| \\ &= 0 \\ \text{Hence}|ab| &= |a||b| \end{aligned}$$

Case (ii): Suppose $b = 0$
$|b| = 0$
$|a|.|b| = |a|.0 = 0$
$|a.b| = |a.0| = |0| = 0$
$|ab| = 0 = |a|.|b|$
$|ab| = |a||b|$
Case (i): Suppose $a > 0$ and $b > 0$
$|a| = a$ and $|b| = b$
$|ab| = ab$, $(ab > 0)$
$= |a||b|$
$|ab| = |a||b|$
Case(iv): Suppose $a > 0$ and $b < 0$
Therefore, $|a| = a$ and $|b| = -b$
we have $ab < 0$
$|ab| = -(ab)$
$= a.(-b)$
$= |a||b|$
$|ab| = |a||b|$ case(v): Suppose $a < 0$ and $b < 0$
Therefore, $|a| = -a$ and $|b| = -b$
we have $ab > 0$
$|ab| = (ab)$
$= (-a).(-b)$
$= |a||b|$
$|ab| = |a||b|$ Hence $|ab| = |a.b|$ for all $a, b \in R$ (b) Let $a \in R$ be arbitrary
Then $a^2 \geq 0$
Now $|a^2| = a^2$
$= a.a$
$= |a||a|$
$= |a|^2|$
Hence, $|a|^2 = a^2$ for all $a \in R$
(c) Let us assume $c \geq 0$
Suppose $a \leq 0$
Then we have both $a \leq c$ and $-a \leq c$
since, $a \leq c$ and $-a \leq c$
$-c \leq a \leq -a \leq c$
$-c \leq a \leq c$
conversely, suppose $-c \leq a \leq c$
since $-c \leq a, c \geq -a$
$\therefore$ we have $a \leq c$ and $-a \leq c$, Then $|a| \leq c$
(d) Let $a \in R$ be arbitrary , Then $|a| \geq 0$
Let $c = |a|$ we know that, $|a| \leq |a|$
$\therefore -|a| \leq a \leq |a|$

1.6 Triangle inequality

Theorem 1.6.1 *If $a, b \in R$, then $|a+b| \leq |a|+|b|$*

Proof By (d) $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$
By adding above inequalities
$-|a|-|b| \leq a+b \leq |a|+|b|$
$-(|a|+|b|) \leq a+b \leq |a|+|b|$
$|a+b| \leq |a|+|b|$(by (c))

Remark 1.6.1 $|a+b| = |a|+|b|$ *iff* $ab > 0$

Theorem 1.6.2 *If $a, b \in R$ be arbitrary (a) $||a|-|b|| \leq |a-b|$ (b) $|a-b| \leq |a|+|b|$.*

Proof (a) Let $a, b \in R$ be arbitrary
Now

$$\begin{aligned} a &= a-b+b \\ |a| &= |a-b+b| \\ |a| &= |(a-b)+b| \\ |a| &\leq |a-b|+|b| (Bytriangleinequality) \\ |a|-|b| &\leq |a-b| \end{aligned} \tag{1.3}$$

Now

$$\begin{aligned} b &= b-a+a \\ |b| &= |b-a+a| \\ |b| &= |(b-a)+a| \\ |b| &\leq |b-a|+|a| (Bytriangleinequality) \\ |b|-|a| &\leq |b-a| \\ -|b|+|a| &\geq -|b-a| \end{aligned} \tag{1.4}$$

From (1.3) and (1.4)
$-|a-b| \leq |a|-|b| \leq |a-b|$
$\therefore ||a|-|b|| \leq |a-b|$
Hence proved.
(b) Let a and b be any real numbers
since $b \in R$, $-b \in R$ (by triangle inequality)
$\therefore |a+(-b)| \leq |a|+|-b|$
$|a-b| \leq |a|+|b|$
Hence proved.
Let S be a non-empty subset of R.

1.7 Bounded sets

Definition 1.7.1 *Let S is said to be bounded above if there exists a number $u \in R$ such that $s \leq u \forall s \leq S$. Each such number u is called an upper bound of S.*

Definition 1.7.2 *The set S is said to be bounded below if there exists a number $u \in R$ such that $u \leq s \forall s \in S$. Each such number u is called as lower bound of S.*

Definition 1.7.3 *A set S is said to be bounded if it is both bounded above and bounded below.*

Definition 1.7.4 *A set S is said to be unbounded if it is not bounded.*

Example
Let $A = \{x \in R : 0 < x < 1\} = (0, 1)$
since all the elements of $A \geq 0$.
Therefore, A is bounded below.
since all the elements of $A \leq 1$
Therefore A is bounded above
Hence A is bounded.
Note
1. Every interval of the form (a, b),$[a, b)$,$(a, b]$ and $[a, b]$ are bounded subsets of R.
2. Any finite subset of R is a bounded set.

Definition 1.7.5 *Let S be a nonempty subset of R. If S is bounded above, then a number u is said to be supremum (or) a least upper bound of S if (i) u is an upperbound of S. (ii) if v is an upperbound of S, then $u \leq v$*

Definition 1.7.6 *Let S be a nonempty subset of R. If S is bounded below, then a number w is said to be infimum (or) a greatest lower bound of S if (i) w is an lowerbound of S. (ii) if v is an lowerbound of S, then $v \leq w$*

Note
1. There can be only one supremum (infimum) of a given subset of R.
2. If the supremum (or) the infimum of a set S exists, we will denote them by $supS$ or $infS$.
Lemma
A number u is the supremum of a nonempty set S of R iff u satisfies the condition
(i) $s \leq u$ for all $s \in S$
The completeness property of R
(i) Every nonempty set of real numbers that has an upper bound and also has an supremum in R.
(ii) Every nonempty subset or real numbers that has a lower bound also has an infimum in R.
Example
1. Let $S = \{dfrac1n : n \in N\}$
$S = \{1, dfrac12, dfrac13, \dots\}$
$infS = 0$ and $SupS = 1$.

Definition 1.7.7 *Let S be a nonempty subset of R that is bounded above and let a be any number in R. Define $s = \{a + s : s \in S\}$.*

Theorem 1.7.1 *S be a nonempty subset of R. Suppose S is bounded above and $a \in R$. Then prove that $sup(a + S) = a + supS$.*

Proof
Let S be a nonempty bounded above subset of R. Therefore S has an upper bound.
By completeness property of R, we have supremum of S exists.
Let $u \in supS$, Then $x \in u$ for all $x \in S$
Therefore, $a + x \leq u + a \forall x \in S$
$\therefore$ $u + a$ is an upperbound of $a + S$.
Let

$$m = sup(a + S)$$
$$\therefore m \leq u + a \tag{1.5}$$

suppose v is an upperbound of $a + S$
$\therefore a + x \leq v$ for all $x \in S$
$\therefore x \leq v - a$ for all $x \in S$
Therefore $v - a$ is an upperbound of S
$u \leq v - a$
$a + u \leq v$
since v is an upperbound of $a + S$

$$a + u \leq m \tag{1.6}$$

From (1.4) and (1.5), we get
$a + u = m$
$a + supS = sup(a + S)$.

Theorem 1.7.2 *Suppose that A and B are nonempty subset of R, such that $a \leq b \forall a \in A$ and $b \in B$*
Then $supA \leq infB$.

Proof
Let B be arbitrary .
Then $a \leq b$ for all $a \in A$
b is an upper bound of A
$supA \leq b$
Therefore sup A is a lower bound of B
$\therefore supA \leq infB$
Archimedian property
If $x \in R$ then there exist $n_x \in N$ such that $x < n_x$
Proof
Let $x \in R$ be an arbitrary
To prove : There is atleast one $n_x \in N$ such that $x < n_x$
Suppose $n \leq x$ for all $n \in N$
$\therefore$ x is an upper bound of N.
By completeness property of R
$supN$ exists.
Let $u = supN$
Then $u - 1$ is not an upper bound of N
$\therefore m \in N$ such that

$u - 1 < m$, $u < m + 1$
since $m + 1 \in N$, we must have $m + 1 \leq u$
$\therefore$ there exist $n_x \in N$ such that $x < n_x$
Example 1
f(x) = 0, if x is even ,
f(x) =1, if x is odd . $\therefore$ Range of $f = R_f$ ={0, 1} $\subseteq R$. Example 2
$f(x) = |x|$
Range of $f = R_f = \{0, 1, 2, \dots\} \subseteq R$

Definition 1.7.8 *Given a function $f : D \to R$, we say that f is bounded if the set $f(D)$ = range of $f = \{f(x) \colon x \in D\}$ is bounded above in R. similarly, the function f is bounded below if $f(D)$ is bounded below in R. we say that, f is bounded if $f(D)$ is bounded below and bounded above (or) $|f(x)| \leq B, B \in RR$*

Example 1.7.1 *Let $f : N \to Q$ be a function defined by $f(n) = \frac{n}{n+1}$*
The range of $f = R_f = \{dfrac12, dfrac23, dfrac34, \dots\} \subseteq Q$
$SupR_f = supf(N) = 1$
$infR_f = inff(N) = \frac{1}{2}$
$\therefore$ *The given function is bounded.*

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The set of all points between a and b is called ------------	integer	interval	elements	set	interval
The set {x: a < x < b} is --------------	(a, b)	[a, b]	(a, b]	[a, b)	(a, b)
A real number is called a positive integer if it belongs to -----------	interval	open interval	closed interval	inductive set	inductive set
Rational numbers is of the form -----------------	pq	p + q	p/q	p - q	p/q
e is ---------------	rational	irrational	prime	composite	irrational
An integer n is called ----------- if the only possible divisors of n are 1 and n	rational	irrational	prime	composite	prime
A set with no upper bound is called ------------	bounded above	bounded below	prime	function	bounded above
A set with no lower bound is called ------------	bounded above	bounded below	prime	function	bounded below
The least upper bound is called -----------	bounded above	bounded below	supremum	infimum	supremum
The greatest lower bound is called -----------	bounded above	bounded below	supremum	infimum	infimum
The supremum of {3, 4} is ----------	3	4	(3, 4)	[3, 4]	4
Every finite set of numbers is ----------	bounded	unbounded	prime	bounded above	bounded
A set S of real numbers which is bounded above and bounded below is called --------	bounded set	inductive set	super set	subset	bounded set
The set N of natural numbers is ----------	bounded	not bounded	irrational	rational	not bounded
The infimum of {3, 4} is ------------	3	4	(3, 4)	[3, 4]	3
Sup C = Sup A + Sup B is called ------------- property	approximation	additive	archimedean	comparison	additive
For any real x, there is a positive integer n such that -----------	n > x	n < x	n = x	n = 0	n > x
If x > 0 and if y is an arbitrary real number, there is a positive number n such that nx > y is -------------- property	approximation	additive	archimedean	comparison	archimedean
The set of positive integers is -------------	bounded above	bounded below	unbounded above	unbounded below	unbounded above
The absolute value of x is denoted by --------------	|x|	||x||	x < 0	x > 0	|x|
If x < 0 then ---------------	|x| = x	||x|| = |x|	||x|| = -x	|x| = -x	|x| = -x
If S = [0, 1) then sup S = ----------------	0	1	(0, 1)	[0,1]	1
Triangle inequality is -------------------	|a| + |b| greater than equal to |a + b|	|a| > |a + b|	|b| > |a + b|	|a + b| less than equal to |a | + |b|	|a + b| less than equal to |a | + |b|
|x + y| greater than equal to -----------------	|x| + |y|	|x| |y|	|x| - |y|	| |x| - |y| |	| |x| - |y| |
If (x, y) belongs to F and (x, z) belongs to F, then -------------	x = z	x = y	xy = z	y = z	y = z
A mapping S into itself is called ------------	function	relation	domain	transformation	transformation
If F(x) = F(y) implies x =y is a --------------- function	one-one	onto	into	inverse	one-one
One-one function is also called -----------	injective	bijective	transformation	codomain	injective
***S* =** {(a,b) : (b,a) is in S} is called ---------------	inverse	domain	codomain	converse	converse
If A and B are two sets andif there exists a one-one correspondence between them,then it is called ------------- set	denumerable	uncountable	finite	equinumerous	equinumerous
A set which is equinumerous with the set of all positive integers is called ---------------- set	finite	infinite	countably infinite	countably finite	countably infinite
A set which is either finite or countably infinite is called ------------ set	countable	uncountable	similar	equal	countable
Uncountable sets are also called ------------- set	denumerable	non-denumerable	similar	equal	non-denumerable
Countable sets are also called --------------- set	denumerable	non-denumerable	similar	equal	denumerable

Every subset of a countable set is ------------	countable	uncountable	rational	irrational	countable
The set of all real numbers is ----------------	countable	uncountable	rational	irrational	uncountable
The cartesian product of the set of all positive integers is ----------	countable	uncountable	rational	irrational	countable
The set of those elements which belong either to A or to B or to both is called ---------	complement	intersection	union	disjoint	union
The set of those elements which belong to both A and B is called ------------	complement	intersection	union	disjoint	intersection

CHAPTER 2

REAL SEQUENCES

2.1 sequences and their limits

Definition 2.1.1 *A sequence in R is a function from N into R.*

Remark 2.1.1 *(i) The sequence is denoted by the symbol $\{S_n\}$.*
(ii) The image of of n, S_n is called the n^th term of the sequence.

Example 2.1.1 *Let f be function from $N \to R$ such that $f(n) = 0$*
Range of f ={0} = {0,0,0,...}

Definition 2.1.2 *If $b \in R$, the sequence $B = \{b, b, b, \dots\}$ is called constant sequence.*

Definition 2.1.3 *The Fibnacci sequence $F = (f_n)$ is given by*
$f_1 = 1, f_2 = 2$
$f_{n+1} = f_n + f_{n-1}, n \geq 2$

Definition 2.1.4 *A sequence (x_n) in R is said to coverage to $x \in R$ or x is said to be a limit of (x_n) if for every $\in > 0$ there exists a positive integers N such that $|x_n - x| < \in$ for all $n \geq N$.*
If a sequence has a limit , we say that the sequence is convergent, if it has no limit, we say that the sequence is divergent.

Remark 2.1.2 *Suppose a sequence (x_n) has limit x, Then we can write*

$limx_n = x$ *or* $x_n \to x$ *as* $n \to \infty$

Theorem 2.1.1 *Let (x_n) be a sequence of real numbers and let $x \in R$. If (a_n) be a sequence of positive real numbers with*
$lim a_n = 0$ and if for some constant

$c > 0$ and some $m \in N$ we have
$|x_n - x| \leq ca_n \forall n \geq m$, then $\lim x_n = x$
suppose let $\in > 0$ be given, then $\frac{\in}{c} > 0$
Given that $\lim a_n = 0$
Therefore for $\frac{\in}{c} > 0$, There exist a positive integer N.
so that $|a_n - 0| < \frac{\in}{c} \forall n \geq N$, that is $|a_n| \frac{\in}{c}\ \forall n \geq N$
$a_n < \frac{\in}{c} \forall n \geq N$
Suppose for some $m \in N$ such that
$|x_n - x| \leq c.a_n \forall n \geq N$
$\leq c.\frac{\in}{c} = \in$
$\therefore x_n \to \infty$

Example 2.1.2 *If $a > 0$, then*
$lim(\frac{1}{1+na}) = 0$
Solution
since $a > 0$, $na > 0$, Then $0 < na < 1 + na$
Hence $\frac{1}{na} > \frac{1}{1+na}$
Now $|\frac{1}{1+na} - 0| = |\frac{1}{1+na}|$
$= \frac{1}{1+na} < \frac{1}{na}$
$\therefore |\frac{1}{1+na} - 0| < \frac{1}{a}(\frac{1}{n})$
since
$lim(\frac{1}{n}) = 0$

$lim(\frac{1}{1+na}) = 0$

Remark 2.1.3 *Convergence of $(|x_n|)$ need not imply the convergence of (x_n).*
consider a sequence $((-1)^n)$
Then $(|(-1)^n|) = (1, 1, \dots)$
clearly, $\lim |x_n| = 1$
Now $((-1)^n) = (-1, 1, -1, 1, \dots)$
This is not a convergent sequence.

2.2 limit theorems

Theorem 2.2.1 *If $0 < b < 1$, then* $\lim(b^n) = 0$

Proof
suppose $0 < b < 1$
Then $b = \frac{1}{1+a}$ if $a > 0$
Now $|b^n - 0| = |b^n| = b^n = [\frac{1}{1+a}]^n]$
$= \frac{1^n}{(1+a)^n}$
$= \frac{1}{(1+a)^n}$
$\leq 11 + na$

$\leq 1na$
$= c\frac{1}{n} = 0$
By previous theorem, $\lim x_n = \infty$
i.e., $\lim(b^n) = 0$
Example
$\lim \frac{1}{3^n} = \lim \frac{1^n}{3^n}$
$=$
$lim(\frac{1}{3})^n = 0$

Theorem 2.2.2 *If $c > 0$, then*
$lim(c^{\frac{1}{n}})^{=1}$

Proof
case(i)
suppose $c = 1$ then $(c^{\frac{1}{n}})$ is a constant sequene and $\lim(c^{\frac{1}{n}} = 1$
case(ii)
suppose $0 < c < 1$
Then $c^{\frac{1}{n}} = \frac{1}{1+h_n}$ where $h_n > 0$
$(c^{\frac{1}{n}})^n = (c^{\frac{1}{1+h_n}})^n$
$c = \frac{1}{(1+h_n)^n}$
$< \frac{1}{n.h_n}$
Now $|c^{\frac{1}{n}-1}| = |1 - c^{\frac{1}{n}}|$
$= |1 - \frac{1}{1+h_n}|$
$= |\frac{1+h_n-1}{1+h_n}$
$=|\frac{h_n}{1+h_n}|$
$< h_n$
since $c < \frac{1}{nh_n}$, $h_n < \frac{1}{nc}$
$\therefore |c^{\frac{1}{n}} - 1| < \frac{1}{nc}$
since $\frac{1}{c} > 0$ and
$lima_n = 0$ if $a_n = \frac{1}{n}$
Then
$lim(c^{\frac{1}{n}}) = 1$
case(iii)
suppose $c > 1$
Then $c^{\frac{1}{n}} = 1 + d_n$ where $d_n > 0$
Now $c = (1 + d_n)^n$
$= 1 + n.d_n + \cdots + {d_n}^n$
$\geq 1 + nd_n$
$\therefore c - 1 \geq nd_n$
$\frac{c-1}{n} \geq d_n$
Now $|c^{\frac{1}{n}-1}| = |d_n|$
$= d_n$
$\frac{c-1}{n}$
$= (c - 1).\frac{1}{n}$
Hence

$lim(c^{\frac{1}{n}}) = 1$

2.3 Bounded sequences

Definition 2.3.1 *A (x_n) of real numbers is said to bounded if there exists a real number $M > 0$ such that*
$|x_n| \leq M$ *for all* $n \in N$, $-M \leq x_n \leq M$

Theorem 2.3.1 *A convergent sequence of real numbers is bounded.*

Proof
suppose that
$lim(x_n) = x$
Let $\in = 1 > 0$
Then there exists a positive integer N such that $|x_n - x| < 1$ if $n \geq N$
Now $|x_n| = |x_n - x + x|$
$\leq |x_n - x| + |x|$
$\leq 1 + |x|$ if $n \geq N$
Then $|x_n| \leq M$ for all $n \geq 1$
Therefore (x_n) is bounded.

Definition 2.3.2 *If $x = (x_n)$ and $y = (y_n)$ are sequences of real number, we define their sum to be the sequene $x + y = (x_n + y_n)$, their difference to be the sequence $x - y = (x_n - y_n)$ and their product to be the sequence $xy = (x_n y_n)$.*
If $c \in R$, we define the sequence $cx = (cx_n)$
If $z = (z_n)$ is a sequence of non-zero real numbers, then we define the quotient of x and Z to be the sequence
$\frac{x}{Z} = \frac{(}{x_n} z_n)$.

Theorem 2.3.2 *Let $X = (x_n)$ and $Y = (y_n)$ converge to x and y respectively and $c \in R$. Then the sequence $x + y$ $x - y$, xy and cx converge to $x + y$, $x - y$, xy and cx respectively.*

Proof
Let $\in > 0$ be given. suppose $x_n \to x$ and $y_n \to y$
$\frac{\in}{2} > 0$ and $x_n \to x$
There exist a positive integer N_1 such that
$|x_n - x| < \frac{\in}{2}$
since $\frac{\in}{2} > 0$ and $y_n \to y$
Thereexist a positive integer N_2 such that
$|y_n - y| < \frac{\in}{2}$ $\forall n \geq N_2$
Now $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$
$\leq |x_n - x| + |y_n - y|$
let $N = max\{N_1, N_2\}$
$|(x_n + y_n) - (x + y)| < \frac{\in}{2} + \frac{\in}{2}$
$= \in$
Therefore, $(x_n + y_n) \to x + y$

By using similar arguments, we have
The sequence $(x_n - y_n)$ converges to $x - y$
consider $|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$
$= |x_n(y_n - Y) + y(x_n - x)|$
$\leq |x_n(y_n - y)| + |y(x_n - x)|$
$= |x_n||y_n - y| + |y||x_n - x|$
since $(x_n) \to x$, Thereexist a positive real number M_1 such that
$|x_n| \leq M, \forall n \geq 1$
Hence $|x_n y_n - xy| \leq M$, $|y_n - y| + |y||x_n - x|$
let $M = \sup\{M_1, |y|\}$
$|x_n y_n - xy| \leq yM|y_n - y| + M|x_n - x|$
let $\in > 0$ be given
since $(x_n) \to x$, there exist a positive integer N_1 such that
$|x_n - x| < \frac{\in}{2M} \; \forall n \geq N_1$
since $(y_n) \to y$, there exist a positive integer N_2 such that
$|y_n - y| < \frac{\in}{2M} \; \forall n \geq N_2$
$N = sup\{N_1, N_2\}$
Therefore $|x_n y_n - xy| < M\frac{(}{\in}2M) + M\frac{(}{\in}2M)$ if $n \geq N$
Therefore $|x_n y_n - xy| < \in$ if $n \geq N$
i.e., $(x_n y_n) \to xy$
Let (y_n) be a constant sequence(c)
Then $(y_n) \to c$
By the above argument, $(x_n y_n) \to xc$
i.e.,$(x_n c) \to xc$
i.e.,$(cx_n) \to cx$

Theorem 2.3.3 *If $X = (x_n)$ converges to x and $z = (z_n)$ is a sequence of non-zero real numbers that converge to z and if $z \neq 0$, then the quotient sequence $(\frac{x_n}{z_n}) \to \frac{x}{z}$*

Proof
Let $\alpha = \frac{1}{z} > 0$
since $(z_n) \to z$, there exist a positive integer N_1 such that $|z_n - z| < \alpha$ if $n \geq N_1$
$-|z_n - z| > -\alpha$ if $n \geq N_1$ Therefore, $-\alpha < -|z_n - z| \leq |z_n| - |z|$ if $n \geq N_1$
$-\alpha < |z_n| - |z|$ if $n \geq N_1$
$\frac{1}{2}|z| = |z| - \frac{1}{2}|z|$
$= |z| - \alpha$
$< |z_n|$ if $n \geq N_1$
$\frac{1}{2}|z| \leq z_n$ if $n \geq N_1$
$\frac{2}{|z|} \geq \frac{1}{|z_n|}$ if $n \geq N_1$
Now $|\frac{1}{z_n} - \frac{1}{z}|$
$= \frac{|z - z_n|}{|z_n z|}$
$= \frac{|z - z_n|}{|z_n||z|} \leq \frac{|z_n - z}{|z|} . \frac{2}{|z|}$
$= \frac{2|z_n - 2|}{|z|^2}$
let $\in > 0$ be given
since $(z_n) \to z_1$ there exist a positive integer N_2 such that
$|z_n - z| < \frac{\in}{2}|z|^2$ if $n \geq N_2$
Hence $|\frac{1}{z_n} - \frac{1}{z}| \leq \frac{2}{|z|^2}\in|z|^2 2$ if $n \geq N = sup\{N_1, N_2\}$

Therefore, $(\frac{1}{z_n}) \to (\frac{1}{z})$

Theorem 2.3.4 *If (x_n) is a convergent sequence of real number and if $x_n \geq 0$ for all $n \in N$, then $x = \lim(x_n) \geq 0$.*

Proof
suppose $(x_n) \to x$
To prove
$x \geq 0$
suppose $x < 0$
Then $-x > 0$
Let $\in= -x > 0$
since $(x_n) \to x$, There esixt a positive integer N such that
$|x_n - x| < -x$ if $n \geq N$
Then $x < x_n - x < -x$ if $n \geq N$
Therefore, $x_n - x < -x$ if $n \geq N$
$x_n < -x + x$ if $n \geq N$
$x_n < 0$ if $n \geq N$
i.e., $x_N < 0, x_{N+1} < 0, \ldots$
$\Rightarrow x_n \geq 0 \ \forall n$
Hence $x_n \geq 0$
Note
(i) suppose sequence (x_n) is convergent to x and $x_n > 0$. Then $\lim(x_n) = x$ need not be greater than zero.

Theorem 2.3.5 *If (x_n) and (y_n) are convergent sequence of real numbers and if $x_n \leq y_n$ forall $n \in N$, then $\lim(x_n) \leq \lim(y_n)$.*

Proof
Let $z_n = y_n - x_n$
Then (z_n) is a sequence of real numbers and $z_n \geq 0$.
By previous theorem,
$\lim(z_n) \geq 0$
$\lim(y_n - x_n) \geq 0$
$\lim(y_n) - \lim(x_n) \geq 0$
$\lim(y_n) \geq \lim(x_n)$

Theorem 2.3.6 *If (x_n) is a convergent sequence and if $a \leq x_n \leq b$ for all $n \in N$, then $a \leq \lim(x_n) \leq b$.*

Proof
Let (y_n) be q sequence such that $y_n = b \forall n \in N$
since $a \leq x_n \leq b$, we have $n \leq y_n \ \forall n \in N$
By previous theorem, $\lim(a) \leq \lim(y_n) \leq \lim(b)$
$a \leq \lim(y_n) \leq b$

2.4 Squeeze theorem

Theorem 2.4.1 *suppose that* (x_n), (y_n) *and* (z_n) *are sequences of real numbers such that* $x_n \leq y_n \leq z_n \ \forall n \in N$
and $\lim(x_n) = \lim(z_n)$
$\lim(x_n) = \lim(y_n) = \lim(z_n)$

Proof
Given that $\lim(x_n) = \lim(z_n)$
Then $\lim(x_n) = \lim(z_n) = w$
Let $\in > 0$ be given.
Then there exist positive integer N such that
$|x_n - w| < \in$ if $n \geq N$
and $|z_n - w| < \in$ if $n \geq N$
Also given that $x_n \leq y_n \leq z_n$, Then $x_n - w \leq y_n - w \leq z_n - w$
$\in < x_n - w < y_n - w < z_n - w < \in$
$- \in < y_n - w < \in$
$|y_n - w| < \in$ if $n \geq N$
Therefore, $\lim(y_n) = w$

Theorem 2.4.2 *Let the sequence* (x_n) *converges to* x. *Then the sequence* $(|x_n|)$ *of absolute values converges to* $|x|$.

Proof
Let $\in > 0$ be given
There exist a positive integer N such that
$|x_n - x| < \in$ for all $n \geq N$
Now, $||x_n| - |x|| \leq |x_n - x| < \in$
$\therefore \lim(x_n) = |x|$

2.5 Monotone sequence

Definition 2.5.1 *Let* (x_n) *be a sequence of real numbers. we say that sequence* (x_n) *is increasing if* $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots$
we say that sequence (x_n) *is decreasing if* $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots$, *we say that* (x_n) *is monotone if it is either increasing or decreasing.*

Problem
Give an example of two divergent sequences X and Y such that (i) sum $x + y$ converges (ii) Product $X.Y$ converges.
Solution
Let $X = (-1)^n = (-1, 1, -1, 1, \ldots)$ $Y = (-1)^{n+1} = (-1, 1, -1, 1, \ldots)$
clearly X and Y are divergent
Now $X + Y = (0, 0, 0, \ldots)$ converges
$X.Y = (-1, -1, -1, \ldots)$ converges
Problem
Show that if X and Y are sequences such that X and Y $X + Y$ are convergent then

Y is convergent.
Solution
Given X and $X + Y$ are convergent. Then $X + Y - X$ is also convergent. i.e., Y is convergent.
Monotone convergence theorem
A monotone sequene of real numbers is convergent if and only if it is bounded. Moreover (i) If $X = (x_n)$ is a bounded increasing sequene, then $\lim(x_n) = sup\{x_n : n \in N\}$ (ii) If $Y = (y_n)$ is a bounded decreasing sequence, then $\lim(y_n) = inf\{y_n : n \in N\}$
Proof
Suppose a monotone sequence is convergent then the sequence is bounded. conversely, suppose a monotone sequence is bounded. since given sequence is monotone, we have either increasing or decreasing.
(i) Let X be a increasing sequence and bounded.
since, X is bounded, there is a real number M such that $x_n \leq M\ \forall n \in N$
Therefore, $\{x_n : n \in N\}$ is bounded above.
By completeness property of R, there exist the $sup\{x_n : n \in N\}$
$\in > 0$ be given
Then $x^* - \in$ is not an upper bound.
Therefore there exist a member of set x_n such that $x^* - \in$ ¡ x_k
Then $x^* - \in < x_n\ \forall n \geq k$
Hence $x^* - \in < x_k \leq x_n \leq x^* < x^* + \in$
$- \in < x_n - x^* < \in$ if $n \geq k$
$|x_n - x^*| < \in$ if $n \geq k$
$\lim(x_n) = x^*$
(ii) Let $Y = (y_n)$ be a bounded decreasing sequence
Then $X = -Y = (-y_n)$ is an increasing sequence
By (i) $\lim(-y_n) = sup\{-y_n : n \in N\}$
$= -inf\{y_n : n \in N\}$
$\lim X = -inf\{y_n : n \in N\}$
$\lim(-y) = -inf\{y_n : n \in N\}$
$-\lim(y) = -inf\{y_n : n \in N\}$
$\lim(y) = inf\{y_n : n \in N\}$
problem
show that $\lim(\frac{1}{\sqrt{n}}) = 0$
Solution
$\lim(\frac{1}{\sqrt{n}}) = x$ and $x = (\frac{1}{\sqrt{n}})$
Now $X.X = (\frac{1}{\sqrt{n}})(\frac{1}{\sqrt{n}})$
$= (\frac{1}{\sqrt{n}} \to 0)$
Therefore $x^2 = 0$ and $x = 0$
Problem
consider a (x_n) with $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}, n \in N$. Find the limit of the sequence (x_n).
Solution
Let $\lim(x_n) = x$
since $x_n \geq 0\ \forall n$, we have $x \geq 0$
Moreover $x_n \geq 2$ and $x \neq 0$
Now $x = \lim(x_n)$
$= \lim(x_{n+1})$

$= \lim(2 + \frac{1}{x_n})$
Let $y_n = 2$ and $z_n = 1$
Then $\lim(y_n) = 2$ and $\lim(z_n) = 1$
$x = \lim(y_n + \frac{z_n}{x_n})$
$= \lim(y_n) + \lim(\frac{z_n}{x_n})$
$= \lim(y_n) + \frac{\lim(z_n)}{\lim(x_n)}$
$x = 2 + \frac{1}{x}$
$x^2 = 2x + 1$
$x^2 - 2x - 1 = 0$
Therefore, $x = 1 + \sqrt{2}$ (or) $x = 1 - \sqrt{2} < 0$
Show that $(-1)^n$ is divergent
Solution
Suppose sequence $(-1)^n$ is convergent and $\lim(-1)^n = a$
Let $\in = 1 > 0$
There exists a positive integer N such that
$|(-1)^n - a| < 1$ if $n \geq N$
suppose n is even
$|1 - a|$ ¡ 1 if $n \geq N$
$-1 < 1 - a < 1$ if $n \geq N$
$-2 < -a < 0$ if $n \geq N$
$2 > a > 0$ if $n \geq N$ suppose n is odd
$|-1 - a|$ ¡ 1 if $n \geq N$
$-1 < -1 - a < 1$ if $n \geq N$
$-1 + 1 < -a < 1 + 1$ if $n \geq N$
$0 > a > -2$ if $n \geq N$
Therefore we have $a > 0$ and $a < 0$
Hence $(-1)^n$ is diverges.

Theorem 2.5.1 *Let (x_n) be a sequence of positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) = L$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$*

Proof
since (x_n) is a sequence of positive real numbers. we have $(\frac{x_{n+1}}{x_n})$ is also a sequence of positive real numbers.
By previous theorem, $L \geq 0$
suppose $L < 1$, then $0 \leq L < 1$
let $r \in R$ such that $L < r < 1$
let $\in = r - L > 0$
since $(\frac{x_{n+1}}{x_n})$ converges, there exist a positive integer N, such that
$|\frac{x_{n+1}}{x_n} - L| < \in$ if $n \geq N$
Then $\frac{x_{n+1}}{x_n} < \in + L$ if $n \geq N$
$\frac{x_{n+1}}{x_n} < (r - L) + L$ if $n \geq N$
$\frac{x_{n+1}}{x_n} < r$ if $n \geq N$
Therefore $x_{n+1} < rx_n$ if $n \geq N$
$\therefore 0 \leq x_{n+1} < r.x_n < r^2 x_{n-1} < \cdots < r^{n-N+1} x_N$
Let $C = \frac{x_N}{r^N}$
$\therefore 0 \leq x_{n+1} < c.r^{n+1}$
since $0 < r < 1$

$\therefore \lim(x_n) = c$
consider $a(x_n)$ with $x_n = \frac{n}{2^n}$.
Discuss about the convergent of (x_n) and find the limit $x_n = \frac{n}{2^n}$, $x_{n+1} = \frac{n+1}{2^{n+1}}$
By previous theorem $\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}}.\frac{2^n}{n}$
$= \frac{2^n.n+1}{2^n.2n}$
$= \frac{n+1}{2n}$
$\lim(\frac{x_{n+1}}{x_n}) = \frac{1}{2} < 1$
By previous theorem, we have
x_n converges and $\lim(x_n) = 0$
2. Let $a > 0$ construct a sequence (s_n) of real numbers such that $\lim(s_n) = \sqrt{a}$
Solution
Let $s_1 > 0$ be arbitrary and define $s_{n+1} = \frac{1}{2}(s_n + \frac{a}{s_n})$ for $n \in N$
Now $s_{n+1} = \frac{1}{2}(\frac{s_n^2+a}{s_n})$
$2s_{n+1} = \frac{s_n^2+a}{s_n}$
$2s_{n+1}s_n = s_n^2 + a$
$s_n^2 - 2s_{n+1}.s_n + a = 0$
since the quadratic has real roots, we must have
$4.s_{n+1}^2 - 4a \geq 0$
$4s_{n+1}^2 \geq 4a$
$s_{n+1}^2 \geq a, n \in N$
Now $s_n - s_{n+1}$
$= s_n - \frac{1}{2}(s_n + \frac{a}{2n})$
$= \frac{1}{2}(\frac{s_n^2-a}{s_n})$
Therefore $s_n - s_{n+1} \geq 0$
$s_n \geq s_{n+1}, n \in N$
clearly (s_n) is a monotone decreasing sequence.
$\therefore (s_n)$ is convergent.
$\lim(s_n) = s$
$\lim(s_n) = \lim(s_{n+1})$
$= \lim[\frac{1}{2}(s_n + \frac{a}{s_n})]$
$= \lim[\frac{1}{2}(s_n + \frac{a}{2}\frac{1}{s_n})]$
$=[\frac{1}{2}\lim(s_n) + \frac{a}{2}.\frac{1}{\lim(s_n)}]$
$= \frac{1}{2}s + \frac{a}{2}.\frac{1}{s}$
$= \frac{1}{2}(s + \frac{a}{s})$
$2s^2 = s^2 + a$
$s^2 = a$
$s = \sqrt{a}$ and $s = -\sqrt{a}$
$\therefore s > 0$
$\therefore \lim(s_n) = \sqrt{a}$

Theorem 2.5.2 *Let* $e_n = (1 + \frac{1}{n})^n$, $n \in N$ *then,* $\lim(e_n) = e$

Proof
Given $e_n = (1 + \frac{1}{n})^n$
since, the expression for e_n contains $n + 1$ terms, and the expression for e_{n+1} contains $n + 2$ terms and each term appearing in $e_n \leq e_{n+1}$. Therefore (e_n) is monotone

increasing sequence
since $2^{p-1} \leq p!$, $(p = 1, 2, \ldots, n)$
$\frac{1}{2^{p-1}} \geq \frac{1}{p!}$
Hence $2 \leq e_n = 3$
$\therefore$ (e_n) is bounded.
Hence (e_n) is convergent and $\lim(e_n)$ lies between 2 and 3. We define the number e to be the limit of this sequence.
$\therefore$ $\lim(e_n) = e$

Questions	Opt 1	Opt 2	Opt 3	Opt 4	Answers
If A is the set of even prime numbers and B is the set of odd prime numbers. Then	A is a subset of B	B is a subset of A	A and B are disjoint	A and B are not disjoint	A and B are disjoint
which relation is not a function?	{(2,5),(3,6).(4,7)}	{(2,1),(3,2).(4,7)}	{(2,1),(2,3).(3,4),(4,1))}	{(2,1),(3,3),(4,1)}	{(2,1),(2,3).(3,4),(4,1))}
Given the relation A={(5,2),(7,4),(9,10),(x,5)}. Which of the following value for x will make relation on A as a function?	7	9	4	5	4
Let A be the set of letters in the word " trivial" and let B be the set of letters in the word difficult. Then A-B=	{a,r,v}	{d,f,c,u}	{I,l.t}	{a,I,l,r,t,v}	{a,r,v}
Let S be the set of of all 26 letters in the alphabet and let A be the set of letters in the word "trivial". Then the number of elements in is	19	20	21	22	21
Let A={1,2}. Then A X A =	{(1,1),(2,2)}	{(1,2),(2,1)}	{(1,1)(1,2),(2,1),(2,2)}	{(1,1),(2,2),(2,1)}	{(1,1)(1,2),(2,1),(2,2)}
Let A={1,2} and B={a,b,c}. Then number of elements in A X B =	2	3	2*2*2	2*3	2*3
Suppose n(A)=a and n(B)=b. Then number of elements in A X B is	a	b	ab	a+b	ab
Let A={1,2} and B={a,b,c}. Then which of the following element does not belongs to A X B =	(1,a)	(3,c)	(c,2)	(1,c)	(c,2)
Let F be a function and (x,y) in F and (x,z) in F. Then we must have	x=y	y=z	z=x	x=x	y=z
If the number of elements in a set S are %. Then the number of elements of the power set P(S)=	5	6	16	32	32
If range of f is equal to codain set, then f is	into	onto	one-one	many to one	onto
Converse of function is a function only if f is	into	onto	one-one	bijection	bijection
Inverse function is always	into	onto	one-one	bijection	bijection
If A and B contains n elements then number bijection between A and B is	n!	n	n+1	n-1	n!
Let f be a function from A to B. Then we call f as a sequence only if A is a	set of positive integers	set of all real numbers	set of all rationals	set of irrationals	set of positive integers
Two sets A and B are said to be similar iff there is a function f exists such that f is	into	one-one	onto	bijection	bijection
If two sets A={1,2,…,m} and B={1,2,..,n} are smilar then	m<n	n<m	n=m	n>0	n=m
Which of the following is an example for countable?	set of real numbers	set of all irrationals	set of all rationals	(0,1)	set of all rationals
Number of elements in the set of all real numbers is	finite	countably infinite	10000000000	uncountable	uncountable
The union of elements A and B is the set of elements belongs to	either A or B	neither A not B	both A and B	A and not in B	either A or B
The set of elements belongs A and not in B is	B	A	B-A	A-B	A-B
The set of elements belongs B and not in A is	B	A	B-A	A-B	B-A
Countable union of countable set is	uncountable	countable	finite	countably infinite	countable
N X N is	uncountable	countable	finite	countably infinite	countable
Z X R is	uncountable	countable	finite	countably infinite	uncountable
R x R is	uncountable	countable	finite	countably infinite	uncountable
The set of sequences consists of only 1 and 0 is	uncountable	countbale	finite	countably infinite	uncountable

Every subset of a countable set is	uncountable	countable	finite	countably infinite	countable
Every subset of a finite set is	uncountable	countable	finite	countably infinite	finite
Fibonnaci numbers is an example for	uncountable set	countable set	finite set	infinte set	countable
Suppose A and B is countable then A X B is	uncountable	countable	finite	infinite	countable
A X B is similar to	A	B	A XA	A X B	A X B
The set of all even integers is	uncountable	countable	finite	infinite	countable
(0,1] is	uncountable	countable	finite	countably infinite	uncountable
{1,2,…..,100000}	uncountable	countable	infinite	countably infinite	countable
Suppose f is a one to one function. Then x not eqaul y implies	f(x) is not equal to f(y)	f(x)=f(y)	f(x)<f(y)	f(x)>f(y)	f(x) is not equal to f(y)
Suppose f is a one to one function. Then f(x)=f(y) implies	x=-y	y=x+10	x=y	x is not eqaul y	x=y
Let f be a bijection between A and B and A is counatble then B is	uncountable	countable	finite	similar to R	countable
Let f be a function defined on A and itself such that f(x)=x. Then f is	onto	one to one	bijection	neither one to one nor onto	bijection
Constant function is an example for	onto	one to one	many to one	bijection	many to one
Stricly increasing function is	an onto function	one to one	many to one	bijection	one to one
Strictly decreasing function is	an onto function	one to one	many to one	bijection	one to one
If *g(x)* = 3x + x + 5, evaluate *g* (2)	8	9	13	17	13
A = {x: x ≠ x }represents	{1}	{}	{0}	{2}	{}
If a set A has n elements, then the total number of subsets of A is	n!	2n	2^n	n	2^n

CHAPTER 3

INFINITE SERIES

3.1 Introduction

If $x = (x_n)$ is a sequence in R then the infinite series or series generated by x is the squence $s = s_n$ defined by
$s_1 = x_1$
$s_2 = x_1 + x_2$
$s_3 = x_1 + x_2 + x_3$

Remark 3.1.1 *1. clearly* $s_n = x_1 + x_2 + \cdots + x_n$
$s_n = x_1 + x_2 + \cdots + x_{n-1} + x_n$
$= s_{n-1} + x_n$
2. The numbers x_n *are called the terms of the series and the numbers* s_n *is called the partial sum of this series.*
3. If $\lim S$ *exists, we say that the series is convergent and this limit is the sum or the value of this series.*
4. If this limit does not exists, we say that the series is divergent.
5. It is convergent to use symbols such as $\sum(x_n)$ *to denote the infinite series.*

Example 3.1.1 *consider the series* $\sum \frac{1}{n(n+1)}$
Solution $\sum \frac{1}{n(n+1)}$
Now $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Then $s_n = \frac{1}{1.2} + \frac{1}{2.3} + \dots$
$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots$
$= 1 - \frac{1}{n+1}$
$\lim s_n = \lim(1 - \frac{1}{n+1})$
$= \lim(1) - \lim(\frac{1}{n+1})$
$= 1 - 0$
$= 1$
$\therefore \sum \frac{1}{n(n+1)}$ *is converges.*

3.2 Geometric series

Example 3.2.1 *Consider the series*
$limr^n = 1 + r + r^2 + \dots$
Solution

$limr^n = 1 + r + r^2 + \dots$
Now $S_n = 1 + r + r^2 + \dots + r^{n-1}$
$s_n(1 - r) = s_n - s_n r$
$= 1 + r + r^2 + \dots + r^{n-1} - (1 + r + r^2 + \dots + r^{n-1}).r$
$= 1 - r^n$
$S_n(1 - r) = 1 - r^n$
$s_n = \frac{1}{1-r} - \frac{r^n}{1-r}$
$s_n - \frac{1}{1-r} = -\frac{r^n}{1-r}$
$s_n - \frac{1}{1-r} = -\frac{r^n}{1-r}$
$\lim(s_n - \frac{1}{1-r}) = \lim(-\frac{r^n}{1-r})$
$\lim(s_n(1 - r)) = \lim(1 - r^n)$
$= \lim(1) - \lim(r^n) = 1$
$\therefore \sum r^n$ *coverges if* $|r| < 1$

3.3 The nth term test

Theorem 3.3.1 *If the series* $\sum x_n$ *converges then*
$lim(x_n) = 0$

Proof
Suppose $\sum x_n$ converges
Let S_n be the partial sum of α_n
By definition of convergence of $\sum x_n$, we have

$lim(s_n) = x$
Now $s_n = s_{n-1}$
$(x_1 + x_2 + \dots + x_n) - (x_1 + x_2 + \dots x_{n-1})$
i.e., $x_n = s_n - s_{n-1}$

$lim(x_n) =$
$lim(s_n - s_{n-1})$

=
$lim(s_n) -$
$lim(s_{n-1)}$
$x - x = 0$
Therefore
$lim(x_n) = 0$

Example 3.3.1 *Consider* $\sum \frac{1}{r(r+1)}$
$\sum \frac{1}{r(r+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{r(r+1)}$
$\lim(\frac{1}{r(r+1)}) = \lim(\frac{1}{r} - \frac{1}{r+1})$
$= \lim(\frac{1}{r}) - \lim(\frac{1}{r+1})$
$= 0$

Example 3.3.2 *consider the series* $\sum r^n$, $|r| < 1$
$\sum r^n = r^0 + r^1 + \cdots + r^n + \ldots$

$lim(r^n) = 0$

Example 3.3.3 *Consider* $\sum(-1)^n$
$\sum(-1)^n = (1)^0 + (-1)^1 + \ldots$
$= 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

$lim(s_n)$ *does not exist.*

Theorem 3.3.2 *Let* (x_n) *be a sequence of nonnegative real numbers. Then the series* $\sum x_n$ *converges if and only if the sequence* $S = (S_k)$ *of partial sums is bounded. In this case* $\sum x_n = \lim(S_k) = sup\{S_k : k \in N\}$

Proof
since $x_n > 0$, we have
$S_1 = x_1$
$S_2 = x_1 + x_2$
$= S_1 + x_2$
$S_2 > s_!$
$S_3 = x_1 + x_2 + x_3$
$= S_2 + x_3$
$S_3 > s_2$
$\therefore$, the sequence of partial sums satisfies
$S_1 < s_2 < S_3 < \ldots$
$\therefore (S_k)$ is monotone sequence.
Suppose $\sum x_n$ converges
By convergence definition
(S_k) converges , $\therefore (S_k)$ is bounded.
Conversely (s_k) is bounded
i.e., (S_k) is monotone and bounded.
By monotone convergence theorem, (S_k) converges
$\sum x_k$ converges.
Moreover, $\lim(s_k) = sup\{S_k : k \in N\}$
$\therefore \sum x_k = sup\{S_k : k \in N\}$
Example

Consider the series $\sum(\frac{1}{n})$
Solution
$S_1 = 1$
$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$
$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$
$\ldots$
$S_1 < S_2 < \ldots$
clearly (S_k) is not bounded. $\therefore \sum(\frac{1}{n})$ is divergent.
2. Show that $\sum \frac{1}{(n+1)(n+2)} = 1$
Solution
$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n+1)(n+2)}$
we know that
$\frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$
$S_n = 1 - \frac{1}{n+2}$
$\lim(S_n) = \lim(1 - \frac{1}{n+2})$
$\lim(1) = \lim(\frac{1}{n+2})$
$= 1 - \lim(\frac{1}{n})$
$= 1 - 0$
$= 1$
$\sum \frac{1}{(n+1)(n+2)}$ converges and $\sum \frac{1}{(n+1)(n+2)} = 1$

Theorem 3.3.3 *The p-series $\sum \frac{1}{n^p}$ diverges when $0 < p \leq 1$*

Proof
We know that $n^p \leq n$ if $0 < p \leq 1$
Then $\frac{1}{n^p} \geq \frac{1}{n}$
$\frac{1}{n} \leq \frac{1}{n^p}$
since the harmonic series, $\sum \frac{1}{n}$ diverges, we have
$\sum \frac{1}{n^p}$ diverges.
Cauchy criterion series
The series $\sum x_n$ converges if and only if for every $\in > 0$ there exist $M(\in) \in N$ such that if $m > \geq M(\in)$ then $|S_m - S_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \in$

3.4 Comparison test

Let $X = (x_n)$ and $Y = (y_n)$ be real sequences and suppose that for some $k \in N$ we have $0 \leq x_n \leq y_n$ for $n \geq k$
(a) Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$. (b) The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.
Proof
(a) suppose that $\sum y_n$ converges. By cauchy criterion, given $\in > 0$ There exist $M(\in) \in N$ such that $|y_{n+1} + y_{n+2} + \cdots + y_m| < \in$ if $m > n \geq M(\in)$. Therefore
$y_{n+1} + y_{n+2} + \cdots + y_m < \in$
$x_{n+1} + x_{n+2} + \cdots + x_m < y_{n+1} + y_{n+2} + \cdots + y_m < \in$
$x_{n+1} + x_{n+2} + \cdots + x_m < \in$
$|x_{n+1} + x_{n+2} + \cdots + x_m| < \in$ if $m > n \geq M(\in)$

By cauchy criterion, $\sum x_n$ converges.
(b) Suppose $\sum x_n$ diverges
To prove $\sum y_n$ diverges
suppose $\sum y_n$ converges
by(a) $\sum x_n$ converges
$\Leftrightarrow \sum x_n$ diverges.
$\therefore \sum y_n$ diverges.

3.5 limit comparison test

Theorem 3.5.1 *suppose that $X = (x_n)$ and $Y = (y_n)$ are strictly positive sequences and suppose that the following limit exists in R. $r =$*
$lim(\frac{x_n}{y_n})$
(a) If $r \neq 0$ then $\sum x_n$ is convergent if and only if $\sum y_n$ converges.
(b) If $r = 0$ and if $\sum y_n$ is convergent the $\sum x_n$ converges.

Proof
(a) Suppose $r = \lim(\frac{x_n}{y_n})$ and $r \neq 0$ then, clearly $r > 0$.
By convergence of sequence $(\frac{x_n}{y_n})$, we have $\frac{r}{2} > 0$ there exist a N such that
$|\frac{x_n}{y_n} - r| < \frac{r}{2}$ if $n \geq N$
Therefore, $\frac{-r}{2} < \frac{x_n}{y_n} - r < \frac{r}{2}$ if $n \geq N$
$\frac{-r}{2} + r < \frac{x_n}{y_n} - r + r < \frac{r}{2} + r$
$\frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2}$
Therefore $\frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2} < 2r$
$\frac{r}{2} < \frac{x_n}{y_n} < 2r$ if $n \geq N$
$\frac{r}{2}y_n \leq x_n < 2r.y_n$ if $n \geq N$
suppose $\sum y_n$ convergent.
$\sum(2r)y_n$ converges.
By comparison test, $\sum x_n$ converges.
By comparison test, $\sum(\frac{r}{2})y_n$ converges.
Therefore $\sum y_n$ converges.
(b) suppose $r = \lim(\frac{x_n}{y_n})$ and $r = 0$
for given $\in > 0$ $\in = 1 > 0$, there exist N such that
$|\frac{x_n}{y_n} - r| < 1$ if $n \geq N$
$|\frac{x_n}{y_n}| < 1$ if $n \geq N$
$x_n < y_n$ if $n \geq N$
Therefore, $0 < x_n < y_n$ if $n \geq N$
suppose $\sum yn$ converges, by comparison test, $\sum x_n$ converges.

Theorem 3.5.2 $\sum \frac{1}{n^2}$ *is convergent.*

Proof
Let $k_1 = 2^1 - 1 = 2 - 1 = 1$

$$S_{k_1} = S_1 = 1(\text{sum of first term})$$

Let $k_2 = 2^2 - 1 = 4 - 1 = 3$

$$\begin{aligned} Sk_2 &= S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2} \\ &< Sk_2 = S_3 = 1 + \frac{1}{2^2} + \frac{1}{2^2} \\ &= 1 + \frac{2}{2^x} \\ &= 1 + \frac{1}{2} \end{aligned}$$

Therefore $Sk_2 < 1 + (\frac{1}{2})^1$

$$\begin{aligned} Sk_3 &= 7 \text{ sum of first 7 terms} \\ &= Sk_2 + (\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}) \\ &< 1 + \frac{1}{2} + (\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}) \\ &< 1 + \frac{1}{2} + \frac{1}{4} \\ &= 1 + \frac{1}{2} + \frac{1}{2^2} \end{aligned}$$

Therefore $Sk_3 < 1 + (\frac{1}{2})^1 + (\frac{1}{2})^2$
By mathematical induction , $Sk_j < 1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^{j-1}$
since the terms in the (R.H.S) is a partial sum of a geometric series $\sum r^n$ with $r = \frac{1}{2} < 1$
Also

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n &= \frac{1}{1 - (\frac{1}{2})} \\ &= 2 \end{aligned}$$

$\therefore$ The partial sum of $\sum \frac{1}{n^2}$ is bounded also $s_1 \leq s_2 \leq \ldots$
$\therefore$ The sequence of partial sum is monotone.
By previous theorem, $\sum 1n^2$ converges.

Problem 3.5.1 *Prove that* $\sum \frac{1}{n^2+n}$ *converges.*

Solution clearly $0 < \frac{1}{n^2+n} < \frac{1}{n^2}, n \in N$
since the series $\sum \frac{1}{n^2}$ converges, by comparison test, $\sum \frac{1}{n^2+n}$ converges.

Problem 3.5.2 *Prove that the series* $\sum 1n^2 - n + 1$ *is convergent.*

Solution Let $x_n = \frac{1}{n^2-n+1}$ and $y_n = \frac{1}{n^2}$
Then $\frac{x_n}{y_n} = \frac{\frac{1}{n^2-n+1}}{\frac{1}{n^2}}$
$= \frac{n^2}{n^2-n+1}$
By limit comparison test, since $\sum \frac{1}{n^2}$ converges, we have
$\sum \frac{1}{n^2-n+1}$ converges.

Problem 3.5.3 *Prove that the series* $\sum \frac{1}{\sqrt{n+1}}$ *is divergent.*

Solution Let $x_n = \frac{1}{\sqrt{n+1}}$ and $y_n = \frac{1}{\sqrt{n}}$
Then $\frac{x_n}{y_n} = \sqrt{\frac{1}{\sqrt{n+1}} \frac{1}{\sqrt{n}}}$
$= \frac{\sqrt{n}}{\sqrt{n+1}}$
$= \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 \neq 0$
By limit comparison test, since $\sum \frac{1}{\sqrt{n}}$ diverges then $\sum \frac{1}{\sqrt{n+1}}$ is also divergent.

3.6 Root Test

Theorem 3.6.1 *Given a series* $\sum a_n$ *of non-negative terms, Let* $\rho = \lim \sqrt{a_n}$
(a) The series $\sum a_n$ *converges if* $\rho < 1$
(b) The series $\sum a_n$ *diverges if* $\rho > 1$
(c) The test is inconclusive if $\rho = 1$

Proof
(a) suppose $\rho < 1$
Let x be a real number such that $e < x < 1$ given that $\rho = \lim \sqrt{a_n}$
Therefore there exist a positive integer N such that
$\sqrt{a_n} < \rho$ for all $n \geq N$
$\sqrt{a_n} < x < 1$ for all $n \geq N$
$a_n < x^n < 1$ for all $n \geq N$
since $\sum x^n$ converges, we have $\sum a_n$ converges.
(b) suppose $\rho > 1$
Then $(a_n)^{\frac{1}{n}} > 1$ for infintely many.
$\therefore (a_n) > 1$ for infintely many.
$\lim(a_n) > 1 \neq 0$
$\therefore \sum a_n$ diverges.
(c) consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ for both series $\rho = 1$
clearly, $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges
Therefore, the test is inconclusive.

Problem 3.6.1 *Discuss about the convergence of* $\sum[\frac{n}{n+1}]^{n^2}$

Solution Let $a_n = \sum[\frac{n}{n+1}]^{n^2}$
Therefore, $\sqrt{a_n} = (a_n)^{\frac{1}{n}}$
$= [[\frac{n}{n+1}]^{n^2}]^{\frac{1}{n}}$
$= (\frac{n}{n+1})^n$
$\therefore \lim \sqrt{a_n} = \lim[\frac{1}{(1+\frac{1}{n})^n}]$
$= \frac{\lim(1)}{\lim(1+\frac{1}{n})^n}$
$= \frac{1}{\rho}$ ¡ 1
Therefore, $\rho < 1$
By root test, $\sum[\frac{n}{n+1}]^{n^2}$ converges.

Problem 3.6.2 *Discuss about the convergence of* $\sum(logn)^{-n}$

Solution Let $a_n = (logn)^{-n}$
$\sqrt{a_n} = (a_n)^{\frac{1}{n}}$
since $a_n = (logn)^{-n}$
$= (logn)^{-1} = \frac{1}{logn}$
$\lim \sqrt{(a_n)} = \lim(\frac{1}{logn}) < 1$
$\rho < 1$, by root test, $\sum(logn)^{-n}$ converges.

3.7 Ratio test

Theorem 3.7.1 *Let $\sum a_n$ be a series of positive terms such that* $\lim \frac{a_{n+1}}{a_n} = L$
(a) The series $\sum a_n$ converges if $L < 1$. (b) The series $\sum a_n$ diverges if $L > 1$. (c) The test is inconclusive if $L = 1$

Proof
(a) suppose $L < 1$ Let x be a real number such that $L < x < 1$
Then there exist a positive integer N such that

$\frac{a_{n+1}a_n}{<}x$ for all $n \geq N$ $\frac{a_{n+1}x^{n+1}}{<}\frac{a_n}{x^n}$ for all $n \geq N$
$\frac{a_{n+1}x^{n+1}}{<}\frac{a_n}{x^n} \leq a_N x^N$ for all $n \geq N$
$\frac{a_{n+1}x^{n+1}}{<}\frac{a_N}{x_N}$ if $n \geq N$
$\frac{a_{n+1}x^{n+1}}{<}c$
$a_{n+1} < c.x^{n+1}$ if $n \geq N$
since $x < 1$ and $\sum x^n$ converges for $|x| < 1$, we have $\sum a_n$ converges.
(b) suppose $L > 1$
$\frac{a_{n+1}}{a_n} > 1$ for infinitely many
Therefore $a_{n+1} > a_n$ for infintely many
$\therefore \sum a_n$ diverges.
(c) consider the series $\sum \frac{1}{n}$ and $\frac{1}{n^2}$
for both series $L = 1$
clearly, $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges.
$\therefore$ The test is inconclusive.

Remark 3.7.1 *Let $\sum a_n$ be a series of positive terms such that*
$lim\frac{a_n}{a_{n+1}} = L$
(a) The series $\sum a_n$ converges if $L > 1$ (b) The series $\sum an$ diverges if $L < 1$
(c) The test is inconclusive if $L = 1$

Problem 3.7.1 *Test the convergence of the series* $\sum \frac{5^{n-1}}{n!}$

Solution Here $a_n = n^{th}term = \frac{5^{n-1}}{n!}$
$a_{n+1} = n^{th}term = \frac{5^n}{(n+1)!}$
$= \frac{5^n}{n!(n+1)}$
$\frac{a_n}{a_{n+1}} = \frac{5^{n-1}}{n!}\frac{n!(n+1)}{5^n}$
$= \frac{n+1}{5}$

$lim(\frac{a_n}{a_{n+1}}) =$
$lim(\frac{n+1}{5})$
Therefore, by ratio test, $\sum \frac{5^{n-1}}{n!}$ converges

Problem 3.7.2 *Test the convergence of the series* $\sum \frac{2^n}{n^3+1}$

Solution Here $a_n = n^{th} term = \frac{2^n}{n^3+1}$
$a_{n+1} = n^{th} term = \frac{2^{n+1}}{(n+1)^3+1}$
$\frac{a_n}{a_{n+1}} = (\frac{a^n}{n^3+1}).\frac{(n+1)^3+1}{2^n.2}$
$= \frac{1}{2} < 1$

$lim \frac{a_n}{a_{n+1}} = \frac{1}{2}$
By ratio test $\sum \frac{2^n}{n^3+1}$ is divergent.

Problem 3.7.3 *Test the convergence of the series* $\sum \frac{(n+1)^n}{n!}$

Solution $a_n = \sum \frac{(n+1)^n}{n!}$
$a_{n+1} = \sum \frac{(n+2)^{n+1}}{(n+1)!}$
$\frac{a_n}{a_{n+1}} = \frac{(n+1)^n}{n!} \frac{n!(n+1)}{(n+2)^{n+1}}$
$= \frac{(n+1)^{n+1}}{(n+2)^{n+1}}$
$= \frac{(n+1)^{n+1}}{[(n+1)+1]^{n+1}}$
$= \frac{1}{[1+\frac{1}{n+1}]^{n+1}}$
$\frac{a_n}{a_{n+1}} = \frac{1}{e}$ ¡ 1
$\therefore$ By ratio test $\sum \frac{(n+1)^n}{n!}$ is diverges.

Problem 3.7.4 *Test the convergence of the series* $\frac{2!}{3} + \frac{3!}{3^2} + \dots$

Solution Here $a_n = \frac{(n+1)!}{3^n}$
$a_{n+1} = \frac{(n+2)!}{3^{n+1}}$
$\frac{a_n}{a_{n+1}} = \frac{3}{n+2}$
$\lim(\frac{a_n}{a_{n+1}}) = \lim(\frac{3}{n+2}) = 0 < 1$
$\frac{(n+1)!}{3^n}$ is diverges.

Problem 3.7.5 *Test the convergene of the series* $\frac{1}{1+2} + \frac{2}{1+2^2} + \dots$

Solution Here $a_n = \frac{n}{1+2^n}$
$a_{n+1} = \frac{n+1}{1+2^{n+1}}$
$\frac{a_n}{a_{n+1}} = \frac{n(1+2^{n+1})}{1+2^n(n+1)}$

$lim \frac{a_n}{a_{n+1}} =$
$lim \frac{n(1+2^{n+1})}{1+2^n(n+1)}$
$= 2 > 1$
$\therefore$ The above series is convergent.

3.8 Alternating series

The series $\sum(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$ is alternating series where each $a_0 > 0$.

3.9 Leibniz's rule

Theorem 3.9.1 *If $\{a_n\}$ is an monotone decreasing sequence with limit 0, the alternating series $\sum(-1)^{n-1}a_n$ converges. If S denotes its sum and S_n its n^{th} partial sum, we also have $0 < (-1)^n(S - S_n) < a_{n+1}$ for all $n \geq 1$*

Proof
The partial sums S_{2n} form an increasing sequence.
$S_{2n+2} - S_{2n} = (a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} + a_{2n+1} - a_{2n+2}) - (a_1 - a_2 + a_3 - \cdots + a_{2n-1} - a_{2n}$
$= a_{2n+1} - a_{2n+2} > 0$
$= S_{2n+2} - S_{2n} > 0$
$\therefore S_{2n+2} > S_{2n}$
Also the partial sums S_{2n-1} form a decreasing sequence.
Both sequenes are bounded below by S_2 and bounded above by S_1.
$\therefore$ Each sequence (S_{2n}) and (S_{2n-1}) are monotone and bounded.
$\therefore$ By monotone convergence theorem (S_{2n}) and (S_{2n-1}) converges
$\therefore \lim S_{2n} = S'$ and $\lim S_{2n-1} = S''$
Now, $S' - S'' = \lim S_{2n} - \lim S_{2n-1}$
$= \lim(S_{2n} - S_{2n-1})$
$= \lim(-a_{2n}) = -\lim a_{2n} = 0$
Therefore $S' = S'' = S$ Therefore sequence of partial sums converges.
$\therefore \sum(-1)^{n-1}a_n$ converges.
since (S_{2n}) is a monotonicallly increasing sequence, we have
$S_{2n} < S_{2n+2} \leq S$
since (S_{2n-1}) is a monotonically decreasing sequence, we have
$S_{2n} < S_{2n+2} < S_{2n-1}$
$\therefore$ we have
$0 < S_{2n-1} - S \leq S_{2n-1} - S_{2n} = a_{2n+1}$
and $0 < S_{2n-1} - S \leq S_{2n-1} - S_{2n} = a_{2n}$
Hence we have. $0 < (-1)^n(S - S_n) < a_{n+1}$

3.10 Absolute convergence

Let $X = (x_n)$ be a sequence in R. we say that the series $\sum x_n$ is absolutely convergent if $|x_n|$ is convergent in R.
Conditional convergent
A series is said to be conditionally convergent but not absolutely convergent.

Example 3.10.1 *Consider a series $\sum \frac{(-1)^n}{n}$*
By Leibnitz's test, $\sum \frac{(-1)^n}{n}$ converges.
Now $\sum |\frac{(-1)^n}{n}| = \sum \frac{|(-1)^n|}{|n|} = \sum \frac{1}{n}$ diverges is conditionally convergent.

Remark 3.10.1 *A series of positive terms is absolutely convergent if and only if it is convergent.*

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If an increasing sequence is bounded above then	seqeunce converges to inf of	sequence converges to sup	sequence converges to 1	sequence converges to 0	sequence converges to sup
If an decreasing sequence is bounded below then	seqeunce converges to inf of	sequence converges to sup	sequence converges to 2	sequence converges to 1	sequence converges to inf of
Fibonacci sequence is	an increasing sequence	a decresing sequence	constant sequence	bounded sequence	an incresing sequence
If an increasing sequence is bounded above then	seqeunce converges to inf of	sequence converges to sup	sequence converges to 3	sequence converges to 2	sequence converges to sup
Suppose a sequence in a metric space (S,d) converges to both a and b. Then we must have	a<b	a>b	a-b=1	a=b	a=b
In a metric space (S,d), a sequence converges to p. Then range of the sequence is	bounded	unbounded	finite	infinite	bounded
The range of a constant sequence is	infinite	countably infinite	uncountable	singlton set	singleton set
Suppose in a metric space (S,d), a sequence converges to p. Then the point p is	an adherent point of S	an accumulation point of S	an isolated point of S	not an adherent point of S	an adherent point of S
Suppose in a metric space (S,d) , a sequence converges to p and the rnage of the sequence is infinite. Then p is	an adherent point of S	an accumulation point of S	an isolated point of S	not an accumulation	an accumulation point of S
Suppose in a metric space, a sequence converges. Then	every sequence in a metric space converges	every subsequence of convergent sequence converges	some subsequence of convergent sequence converges	some sequence in a metric space converges	every subsequence of convergent sequence converges
A sequence is said to be bounded if if its range is	unbounded	bounded	countable	uncountable	bounded
The range of the sequence {1/n} is	finite	{1}	{}	infinite	infinite
The range of the sequence {1/n} is	unbounded	bounded	{}	{1,0}	bounded
The esequence {1/n}	converges	diverges	oscilates	converges to 1	converges
In Euclidean metric space every cauchy sequence is	convergent	divergent	oscilates	convergent to 0	converges
Every convergent sequence is a	constant seqeunce	cauchy sequence	increasing	decreasing	cauchy sequence
The sequence {n^2}	converges	diverges	oscilates	converges to 2	diverges
The range of the sequence {n^2} is	unbounded	bounded	{}	{0.1}	unbounded
The range of the sequence {n^2} is	finite	{1}	{}	infinite	infinite
The sequence {i^n}	converges	diverges	oscilates	converges to 0	diverges
The range of the sequence {i^n} is	unbounded	bounded	{}	{0,1}	bounded
The range of the sequence {i^n} is	finite	infinite	{}	{0,1}	finite
The sequence {1}	converges	diverges	oscilates	converges to 0	converges
The range of the sequence {1} is	{}	{1}	{1,0}	{1,2,3}	{1}
The range of the sequence {1} is	bounded	unbounded	{1,0}	{0}	bounded

CHAPTER 4

SUBSEQUENCES

4.1 Subsequences

Definition 4.1.1 *Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_n k)$ given by $(x_{n_1}, x_{n_2}, \dots)$ is called a subsequence of X*

Example 4.1.1 *Consider a sequence $X = (1, \frac{1}{2}, \frac{1}{3}, \dots)$*
Let $X' = (\frac{1}{2}, \frac{1}{4}, \dots)$
clearly, x' is a subsequence of X. note that $n_1 = 2, n_2 = 4, \dots$

Definition 4.1.2 *If $X(x_1, x_2, \dots)$ is a sequence of real numbers and if m is a given natural numbers, then the m-tail of X is the sequence. $X_m = (x_{m+1}, x_{m+2}, \dots)$*

Remark 4.1.1 *A tail of a sequence is a special type of subsequence. (ii) Not every subsequence of a given sequence need be a tail of the sequence.*

Theorem 4.1.1 *If a sequence $X = (x_n)$, of real numbers converges to a real number x, then any subsequencece $x' = (x_{n_k})$ of x, also converges to x.*

Proof
Given that,
$lim x_n = x$
$\therefore$ for given $\in > 0$, there exist a positive integer N such that $|x_n - x| < \in$ if $n \geq N$

Let $X^{'} = (x_{n_k})$ be a subsequence of X. The $n_1 < n_2 < n_3 < \ldots$ clealy $n_k \geq k$
suppose $k \geq N$, then $n_k \geq N$
$|x_{n_k} - x| <\in$
Therefore (x_{n_k}) converges to x

Definition 4.1.3 *For a sequence (x_n), we say that the m^{th} term x_m of (x_n) if $x_m \geq x_n$ for all $n \geq M$.*

Remark 4.1.2 *In a decreasing sequence, every term is peak and in an increasing sequence no term is peak.*

4.2 The cauchy sequences

Definition 4.2.1 *A sequence $X = (x_n)$ of real number is said to be a cauchy sequence if for every $\in> 0$, there exist a natural number N such that $|x_n - x_m| \in$ if $n, m \geq N$*

Theorem 4.2.1 *If $X = (x_n)$ is a convergent sequence of real numbers then X is a cauchy sequence.*

Proof
Let $X = (x_n)$ be a convergent sequence. Let $\lim x_n = x$
Let $\in> 0$ be arbitrary, then for $\frac{\in}{2} > 0$, there exist a positive integer N such that
$|x_n - x| < \frac{\in}{2}$ if $n \geq N$
Let $n, m \geq N$
Now $|x_n - x_m| = |x_n - x + x - x_m|$
$\leq |x_n - x| + |x - x_m|$ ¡ $\frac{\in}{2} + \frac{\in}{2} = \in$
$|x_n - x_m| <\in$ if $n, m \geq N$
Therefore (x_n) is a cauchy sequence.

Theorem 4.2.2 *A caushy sequence of real number is bounded*

Proof
Let $X = (x_n)$ be a cauchy sequence
Let $\in= 1$, then there exist a positive integer N such that
$|x_n - x_m| < 1$ if $n, m \geq N$
In particular, $|x_n - x_m| < 1$ if $n, m \geq N$
Now $|x_n| - |x_N| \leq |x_n - x_N| < 1$ if $n \geq N$
$\therefore |x_n| - |x_N| < 1$ if $n \geq N$
$|x_n| < 1 + |x_N|$ if $n \geq N$
Let $M = sup\{|x_1|, |x_2|, \ldots, |x_{N+1}|, 1 + |x_N|\}$
Then $|x_n| < M$ for all n
Therefore $-M < x_n < m$ for all n
Therefore (x_n) is bounded.

4.3 Cauchy convergence criterion

Theorem 4.3.1 *A sequence of real number is convergent if and only if it is cauchy sequence.*

Proof
Suppose $X = (x_n)$ is a convergent sequence
by previous theorem, X is a cauchy sequence.
Conversely suppose $X = (x_n)$ is a cauchy sequence. by previous theorem, X is bounded
By Bolzono theorem, X has a convergent subsequence.
Let $x_{n_k} \to x$
claim $x_n \to x$
since X is cauchy sequence, for given $\frac{\epsilon}{2} > 0$, there exist a positive integer N such that
$|x_n - x_m| < \frac{\epsilon}{2}$ if $n, m \geq N$
since (x_{n_k}) converges to x, for $\frac{\epsilon}{2} > 0$, there exist a positive integer $k \geq N$ such that
$|x_k - x| < \frac{\epsilon}{2}$ if $n \geq N$
Now $|x_n - x| = |x_n - x_k + x_k - x|$
$\leq |x_n - x_k| + |x_k - x| = \epsilon$
i,e., $|x_n - x| < \epsilon$ if $n \geq N$, therefore $x_n \to x$
i.e., X is a convergent sequence.

Problem 4.3.1 *Discuss the convergence of the series* $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \dots$

Solution
Given series is an alternating series.
Let $a_n = \frac{1}{\sqrt{n}}$
$a_{n+1} = \frac{1}{\sqrt{n+1}}$
$a_{n+1} - a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}$
$= \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n}\sqrt{n+1}} < 0$
$a_{n+1} - a_n < 0$
$A_{n+1} < a_n$
$\therefore \{a_n\}$ is monotonically decreasing also $\lim a_n = \frac{1}{\sqrt{n}} = 0$
$\therefore$ The given **Solution** satisfies all the conditions of Leibnitz rule. The given series converges.

Problem 4.3.2 *Discuss the convergence of* $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \dots$

Solution
Given series is an alternating series
LEt $a_n = \frac{2n+3}{2n}$
$a_{n+1} = \frac{2(n+1)+3}{2(n+1)}$
$= \frac{2n+5}{2n+2}$
$a_{n+1} - a_n = \frac{-6}{2n(2n+2)} < 0$
$a_{n+1} < a_n$
$\therefore \{a_n\}$ is monotonically decreasing. Also
$lim a_n = \frac{2n+3}{2n}$
$\frac{2+0}{2} = 1 \neq 0$
$\therefore$ the given series does not satisfies one of the condition of Leibnitz test.
$\therefore$ the given series diverges.

Problem 4.3.3 *Discuss the convergence of the series* $\frac{1}{log 2} - \frac{1}{log 3} + \frac{1}{log 4} - \dots$

Solution
Given series is alternating series
Let $a_n = \frac{1}{log(n+1)}$
$a_{n+1} = \frac{1}{log(n+2)}$
$a_{n+1} - a_n = \frac{1}{log(n+2)} - \frac{1}{log(n+1)}$ ¡ 0
$a_{n+1} - a_n < 0$
$a_{n+1} < a_n$
$\therefore \{a_n\}$ is a monotonically decreasing.

$lim a_n = \frac{1}{log(n+1)}$
$= \frac{1}{\infty} = 0$
Therefore the given series satisfies all the condition of leibnitz test.
The given series is convergent.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Constant sequence	converges	oscillates	diverges	converges to 1	Converges
The sequence {1,1,1,1,1,…..}	converges	oscillates	diverges	converges to 1	converges to 1
The sequence {1,0,1,0,1,0,…}	converges	oscillates	diverges	converges to 1	Oscillates
The harmonic series converges if	P=1	p>1	P<1	P=0	p>1
In limit comparison test both the series converges absolutely if	r=1	r=0	r is not equal to zero	R=2	r is not equal to zero
For the absolute convergence of the series, the ratio between n+1th term and nth term must be	Less than r	Greater than r	Less than or equal to r	Greater than equal to r	Less than or equal to r
For the absolute convergence of the series, the nth root of nth term must be	Less than r	Greater than r	Less than or equal to r	Greater than equal to r	Less than or equal to r
The alternating harmonic series	converges	oscillates	diverges	converges to 1	Converges
If a series converges absolutely, the series	converges	oscillates	diverges	converges to 1	Converges
A series converges iff converges absolutely if the series consists of ----terms	positive	negative	Non zero	Either a or b	Positive
The series 1-1+1-1+1-1+…	converges	oscillates	diverges	converges to 1	Diverges
{1,2,…..,100000}	uncountable	countable	infinite	countably infinite	countable
Suppose f is a one to one function. Then x not eqaul y implies	f(x) is not equal to f(y)	f(x)=f(y)	f(x)<f(y)	f(x)>f(y)	f(x) is not equal to f(y)
Suppose f is a one to one function. Then f(x)=f(y) implies	x=-y	y=x+10	x=y	x is not eqaul y	x=y
Let f be a bijection between A and B and A is counatble then B is	uncountable	countable	finite	similar to R	countable
Let f be a function defined on A and itself such that f(x)=x. Then f is	onto	one to one	bijection	neither one to one nor onto	bijection
Constant function is an example for	onto	one to one	many to one	bijection	many to one
Stricly increasing function is	an onto function	one to one	many to one	bijection	one to one
Strictly decreasing function is	an onto function	one to one	many to one	bijection	one to one
If *g(x)* = 3x + x + 5, evaluate *g* (2)	8	9	13	17	13
A = {x: x ≠ x }represents	{1}	{}	{0}	{2}	{}
If a set A has n elements, then the total number of subsets of A is	n!	2n	2^n	n	2^n

M. Indhumathi

CHAPTER 5

SEQUENCES AND SERIES OF FUNCTIONS

5.1 Sequences of functions

Definition 5.1.1 *Let $A \subseteq R$ be given and suppose that for each $n \in N$ there is a function $f_n : A \to R$, we say that (f_n) is a sequence of functions A to $B \to R$.*

Definition 5.1.2 *A sequence (f_n) of functions on $A \subseteq R$ to R, converges to a function $f : A \to B$ if for every $\in > 0$ there exist a positive integer $N(\in, x)$ such that $|f_n(x) - f(x)| < \in$ if $x \in A$ and $n \geq N$*

Remark 5.1.1 *(i) The positive integer N will depend on both $\in$ and $x \in A$.*
(ii) The sequence (f_n) converges on A to f, we have $f_n \to f$ (or) $f(x) = \lim f_n(x)$

Example 5.1.1 *Let $f(x) = \frac{x}{n}, x \in R$*
Now $f(x) = \lim f_n(x) =$
$lim(\frac{x}{n})$
$= \frac{\lim x}{\lim n} = \frac{x}{\infty} = 0$
Therefore, $f_n \to f$ for all $x \in R$.

Example 5.1.2 *Let $f_n(x) = x^n, x \in R$*
$f(x) = \lim f_n(x) = \lim x^n$
$f_n \to f(x) = 0, -1 < x < 1$ *(or)* $f_n \to f(x) = 1, x = 1$

Example 5.1.3 $f_n(x) = \frac{sin(nx+n)}{n}, x \in R$
$f(x) = \lim f_n(x) = \lim \frac{sin(nx+n)}{n} = 0$

5.2 Uniform convergence

A sequence (f_n) of functions on $A \subseteq R$ to R converges uniformly on A to a function $f : A \to R$ if for every $\in > 0$ there exist a positive integer N such that $|f_n(x) - f(x)| < \in$ if $n \geq N$
Uniform norm
If $A \subseteq R$ and $f : A \to B$ is a function an f is bounded we define the uniform norm of f on A by $||f||_A = sup\{|f(x)| : x \in A\}$
Example
Let $f(x) = frac1x$
Then $||f|| = 1$
Note
Suppose $\in > 0$, and $||f||_A \leq \in$
By definiton of norm of f,
$||f|| = sup\{|f(x)| : x \in A\} \leq \in$
$|f(x)| \leq \in$
suppose $|f(x)| \leq \in$
for all $x \in A$
$||f|| \leq \in$
Hence, $||f||_A \leq \in \Leftrightarrow |f(x)| \leq \in$ for all $x \in A$

Theorem 5.2.1 *A sequence (f_n) of bounded function on $A \subseteq R$ converges uniformly on A to $f \Leftrightarrow ||f_n - f|| \to 0$.*

Proof
Suppose $f_n \to f$ uniformly on A. Then for $\in > 0$, there exist a positive integer N such that $|f_n(x) - f(x)| < \in$ if $n \geq N$
by previous theorem, $||f_n - f|| < \in$ if $n \geq N$
$||f_n - f|| \to 0$
conversely suppose $||f_n - f|| \to 0$
on A
Then for given $\in > 0, thereexistapositiveintegerN$ such that
$|(||f_n - f||) - 0| < \in$ if $n \geq N$
$||f_n - f|| < \in$ if $n \geq N$
i.e., $||f_n - f|| < \in$ if $n \geq N$
i.e., $|f_n(x) - f(x)| < \in$ if $n \geq N$
$\therefore f_n \to f$ uniformly on A.

5.3 Series of functions

Definition 5.3.1 *If (f_n) is a sequence of functions defined on a subset D of R with values in R, the sequence of partial sums (S_n) of the infinite series $\sum f_n$ is efined for x in D by*

$$\begin{aligned} S_1(x) &= f_1(x) \\ S_2(x) &= f_1(x) + f_2(x) \\ S_3(x) &= f_1(x) + f_2(x) + f_3(x) \\ &\vdots \end{aligned}$$

In case, the sequence (S_n) of functions converges on D to a function f, we say that the infinte series of functions $\sum f_n$ converges to f on D.

Definition 5.3.2 *If the series $\sum |f_n(x)|$ converges for each α in D, we say that $\sum f_n$ is absolutely convergent on D.*

Definition 5.3.3 *If the sequence (S_n) of partial sums is uniformly convergent on D to a function f, we say that $\sum f_n$ is uniformly convergent on D to f.*

5.4 Weierstross M - test

Theorem 5.4.1 *Let (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in D, n \in N$*
If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.

Proof
Suppose $m > n$
$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)|$
$\leq |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_m(x)|$
$\leq M_{n+1} + M_{n+2} + \cdots + M_m$
By cauchy criterion for series, The series $\sum x_n$ converges if and only if for every $\in > 0$ there exist a positive integer M that if $m > n \geq M(\in)$ then
$|S_m - S_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \in$
since $\in M_n$ converges, $|M_{n+1} + M_{n+2} + \cdots + M_m| < \in$
$M_{n+1} + M_{n+2} + \cdots + M_m < \in$
Therefore $|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| < \in$
By cauchy criterion for sequence of functions $|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| < \in$
$\therefore \sum f_n$ uniformly convergent on D.

5.5 Power series

Definition 5.5.1 *A series of real functions $\sum f_n$ is said to be a power series around $x = c$ if the function f_n is of the form $f_n(x) = a_n(x - c)^n$ where a_n and c belong to R and where $n = 0, 1, 2, 3, \ldots$*

Definition 5.5.2 *Let $\sum a_n X^n$ be a power series. If the sequence $(|a_n|^{\frac{1}{n}})$ is bounded, we get* $\rho = \lim sup(|a_n|^{\frac{1}{n}}$

If this sequence is not bounded, we get $\rho = +\infty$. we define the radius of convergence of $\sum a_n x^n$ to be given by
$R = 0$ *if* $\rho = +\infty$
$= \frac{1}{\rho} if 0 < \rho < \infty$
$= \infty$ *if* $\rho = 0$

Remark 5.5.1 *The radius of convergence of the series $\sum a_n x^n$ is also given by* $\lim(|\frac{a_n}{a_{n+1}})$ *provided the limits exists.*

Problem 5.5.1 *Find the radius of convergence of the series $\sum a_n x^n$ there $a_n = \frac{1}{n!}$*

Solution
$a_n = \frac{1}{n!}$
$a_{n+1} = \frac{1}{(n+1!)}$
$|\frac{a_n}{a_{n+1}}| = |\frac{1}{n!} x \frac{(n+1)!}{1}|$
$= |n + 1| = n + 1$
$\lim \frac{1}{a_n} a_{n+1}| = lim(n + 1) = \infty$
Therefore, The radius of convergence is $+\infty$

5.6 Cauchy-Hadmard Theorem

Theorem 5.6.1 *If R is the radius of convergence of the power series $\sum a_n x^n$, then the series $\sum a_n x^n$ is absolutely convergent if $|x| < R$ an is divergent if $|x| > R$*

Proof
Suppose $0 < R < +\infty$ suppose $|x| < R$
i.e., $0 < |x| < R$, then there is a positive real number $c < 1$ such that $|x| < c.R$
Therefore $|x| < c.\frac{1}{\rho}$
$\Rightarrow \rho < \frac{c}{|x|}$
$\Rightarrow \lim sup \sqrt{|a_n|} < \frac{c}{|x|}$
Therefore $|a_n| < \frac{c^n}{|x|^n}$
$\Rightarrow |a_n||x|^n < c^n$
$\Rightarrow |a_n x^n| < c^n$
since $c < 1$, the geometric series $\sum c^n$ converges.
By comparison test, $\sum |a_n x^n|$ converges.
Therefore $\sum a_n x^n$ converges absolutely.
Suppose $|x| > R$
$|x| > \frac{1}{\rho}$
$\therefore \lim sup \sqrt{a_n} > \frac{1}{|x|}$
$\Rightarrow |a_n| \geq \frac{1}{|x|^n}$
$\Rightarrow |a_n x^n| \geq 1$ for infintely many n
By comparison test, $\sum a_n x^n$ diverges.

Problem 5.6.1 *Discuss the uniform convergence of $\sum \frac{sinnx}{n^2}$*

Solution
Given $f_n(x) = \frac{sinnx}{n^2}$

$|f_n(x)| = |\frac{sinnx}{n^2}|$
$= \frac{|sinnx|}{n^2}$
$\leq \frac{1}{n^2}$
since $\sum \frac{1}{n^2}$ converges, we have $\sum sinnxn^2$ converges uniformly.

5.7 Cluster Point

Definition 5.7.1 *Let $A \subseteq R$. A point $C \in R$ is a cluster point of A if for every $\in> 0$ there exist atleast one point $x \in A$, $x \neq C$, such that*
$|x - c| <\in$

Example 5.7.1 *Let $A = \{1, 2\}$ 1 and 2 are not cluster point of A. Moreover A has no cluster points of A.*

Remark 5.7.1 *Finite set has no cluster points. Cluster point is also called limit point.*

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If R is the radius of convergence of the series, the series converges absolutely if |x|	>R	=R	<R	Less than or equal to R	<R
If rho=infinity, the radius of convergence R is	0	1	2	3	0
If rho=0, the radius of convergence R is	0	1	2	infinity	Infinity
If rho is finite, the radius of convergence R is	0	Rho	Reciprocal of rho	infinity	Reciprocal of rho
If R is the radius of convergence of the series, the series diverges if |x|	>R	=R	<R	Less than or equal to R	>R
If R is the radius of convergence then the interval of convergence is	(-R,R]	[-R,R]	(-R,R)	[-R,R)	(-R,R)
The sequence of functions (x/n) converges to a function x=	0	1	2	3	0
The sequence of functions x power n converges to a function x=0 if x lies between	1 and 2	-1 and 1	0 and 1	-1 and 0	-1 and 1
A series of positive terms converges then the series	converges only	converges absolutely	both A and B	neither A nor B	both A and B
A convergent series contains only finite number of negative terms then it is	converges only	converges absolutely	both A and B	neither A nor B	converges absolutely
A convergent series contains only -------- number of negative terms then it is converges absolutely	infinite	10	finite	countable	finite
A convergent series contains only finite number of --------- terms then it is converges absolutely	negative	positive	zero	1	negative
The power series converges to a continuous function on --------------	(-R,R)	[-R,R]	(1,2)	[1,2]	(-R,R)
If the series converges at x = R, then f is continuous at ------------------	x=R	x<R	x>R	x≠R	x=R
If the series converges at x = −R, then f is continuous at ---------------	x= -R	x=R	x<R	x≠R	x= -R
Union of sets is --------------	commutative	not commutative	not associative	disjoint	commutative
The complement of A relative to B is denoted by --------------	B - A	B	A	A - B	B - A
If A intersection B is the empty set, then A and B are called --------	commutative	not commutative	not associative	disjoint	disjoint
B - (union A) = -----------------	union (B -A)	B - (intersection A)	intersection (B - A)	{}	intersection (B - A)
B - (intersection A) = -----------------------	union (B -A)	B - (union A)	intersection (B - A)	{}	union (B -A)
Union of countable sets is -----------------	uncountable	infinite	countable	disjoint	countable
The set of all rational numbers is ---------------	uncountable	infinite	countable	disjoint	countable
The set S of intervals with rational end points is ---------- set	uncountable	infinite	countable	disjoint	countable
The product of two prime numbers will always be	even number	odd number	neither prime nor composite	composite	composite
Let A be the set of all prime numbers. Then number of elements in A is	countable	uncountable	finite	empty	countable

Reg. No..................

18MMU203

Karpagam Academy of Higher Education
Coimbatore-21
Department of Mathematics
Second Semester- I Internal test
Real Analysis

Date:18.12.18(FN) Time: 2 hours
Class: I B.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x$. Then f is ——
   a.one-one b. onto
   c. bijection d. neither onto nor one-one

2. The set of all positive integers $\{1, 2, \cdots\}$ is ——
   a. finite b. infinite
   c. countable d. uncountable

3. Greatest lower bound of set of all positive even integers is ——
   a. 2 b. 0
   c. 1 d. 4

4. Let S be a bounded above set of real numbers and sup $S = u$. Then for $x \in S$, we have ——
   a. $x > u$ b. $x < u$
   c. $x \leq u$ d. $x \geq u$

5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(x) = x^2$ where $\mathbb{Z}$ is a set of all real numbers. Then the range of f is ——
   a. $\mathbb{Z}$ b. $\mathbb{N}$
   c. $\mathbb{W}$ d. $\{0, 1, 4, 9, \cdots\}$

6. Which equation does not represent a function?
   a. $y = 2x$ b. $y = x^2 + 10$
   c. $y = \frac{10}{x}$ d. $x^2 + y^2 = 95$

7. $B - (B - A) = A$ if ——
   a. $B \subset A$ b. $A \subset B$
   c. $A \cup B = A$ d. $A \cup B = A$

8. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then the number of distinct functions from A into B is ——
   a. 8 b. 9
   c. 6 d. 5

9. Which of the following sets is countable?
   a. $(0, \infty)$
   b. $\mathbb{R}$
   c. set of all irrational numbers
   d. set of all Fibonacci numbers

10. sup $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$= ——
    a. -1 b. 1
    c. 0 d. $\frac{1}{2}$

11. Suppose $\lim(x_n) = x$ and $\lim(-x_n) = x$. Then $x =$ ——
    a. 1 b. $\frac{1}{2}$
    c. 0 d. -1

12. Suppose $\lim(x_n) = x$. For every $\epsilon > 0$, there is a +ve integer N such that we have ——
    a. $x - \epsilon < x_n$ b. $x + \epsilon > x_n$
    c. both A and B d. neither A nor B

13. The sequence $((-1)^n)$ is ——
    a. convergent b. bounded
    c. both A and B d. neither A nor B

14. Constant sequence is ——
a. increasing b. decreasing
c. both A and B d. neither A nor B

15. If $X = ((-1)^n))$ and $Y = \left((-1)^{n+1}\right)$ then $X+Y$ is ——
a. coverges b. diverges
c. both A and B d. neither A nor B

16. If X and $X + Y$ are convergent, then Y ——
a. coverges b. diverges
c. both A and B d. neither A nor B

17. If $x_1 = 8$ and $x_{n+1} = \frac{x_n}{2} + 2$, (x_n) is ——
a.monotone b.bounded
c. both A and B d. neither A nor B

18. The sequence $\left(\frac{1}{n}\right)$ is ——
a. convergent b. bounded
c. both A and B d. neither A nor B

19. If $z_n = (a^n + b^n)^n$ and $0 < a < b$, then $\lim(z_n) =$ ——
a. 0 b. 1
c. a d. b

20. If X converges to x and XY converges then Y converges if —— a. $x \neq 0$ b. $x_n \neq 0$
c. both A and B d. neither A nor B

Part B-($3 \times 2 = 6$ marks)

21. If $a, b \in \mathbb{R}$, prove that $|a + b| = |a| + |b|$ iff $ab \geq 0$

22. Define Upper bound.

23. State the completness property of $\mathbb{R}$.

Part C-($3 \times 8 = 24$ marks)

24. a) (i) State and prove triangle inequality.
(ii) State and prove Archimedean property.

OR

b) Prove that $\mathbb{R}$ is uncountable

25. a) (i) State an prove uniqueness of limits.
(ii) Prove that if $c > 0$ then $lim(c^{\frac{1}{n}}) = 1$

OR

b) Prove that a convergent sequence of real numbers is bounded.

26. a) State and prove monotone convergence theorem

OR

b) State and prove squeeze theorem.

Reg no-------------

(18MMU203)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore-21

DEPARTMENT OF MATHEMATICS

Second Semester

III Internal Test - Mar'2019

Real Analysis

Date: 12 -03-2019 (AN) **Time: 2 Hours**

Class: I-B.Sc Mathematics **Maximum Marks:50**

PART-A(20×1=20 Marks)

1. A convergent sequence is always------sequence
   a) constant b) unbounded
   c) Cauchy d) non constant
2. Which of the following is not a Cauchy sequence?
   a) $(\frac{1}{n})$ b) (n) c) $(\frac{1}{2\sqrt{n}})$ d) $(\frac{1}{\sqrt{n}})$
3. The series $\sum_n^\infty n^2 \cdot e^{-n}$
   a) converges b) diverges
   c) oscillates d) converges to 0
4. The series $\sum_n^\infty \frac{1}{n(\log n)^p}$, p>1
   a) converges b) diverges
   c) oscillates d) converges to 0
5. The series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$
   a) converges b) diverges
   c) oscillates d) converges to 1
6. The series $\sum_n^\infty \frac{1}{n\cdot(n+1)\cdot(n+2)}$
   a) converges b) diverges
   c) oscillates d) converges to 1
7. If $\sum(\sqrt{n^2+1} - n)$is __________
   a) diverges b) uniformly converges
   c) absolutely converges d) all the above.
8. A convergent sequence is ________ sequence
   a) unbounded b) constant
   c) bounded d) non constant
9. The series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is
   a) diverges b) converges
   c) converges to 1 d) converges to 2
10. The series $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \cdots$
    a) diverges b) converges
    c) oscillates d) converges to 0
11. The series $\frac{1}{2} - \frac{1}{\log 2} - \frac{1}{2} + \frac{1}{\log 3} + \frac{1}{2} - \frac{1}{\log 4} + \cdots$
    a) converges b) diverges
    c) oscillates d) converges to 0
12. The series $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \cdots$
    a) converges b) diverges
    c) oscillates d) converges to 1
13. $\sum x_n$ is absolutely convergent if ∃ ------- and $n \geq N$ such that $\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}$
    a) $a > 1$ b) $a \leq 1$ c) $a > 1$ d) $a \geq 1$
14. If $a = \lim(n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right))$ exists the $\sum x_n$ converges absolutely when ______
    a) $a > 1$ b) $a \leq 1$ c) $a > 1$ d) $a \geq 1$

15. If $\sum c_n \sin nx$ converges uniformly and (c_n) is a decreasing sequence then $\lim nc_n =$ ____

a) 1 b) 2 c) 3 d) 0

16. Which of the following is a subsequence of $\left(\frac{1}{n}\right)$?

a) $\left(\frac{1}{\sqrt{n}}\right)$ b) $\left(\frac{n}{n+1}\right)$ c) $\left(\frac{1}{n^2}\right)$ d) $\left(\frac{\sqrt{n}}{n}\right)$

17. The series $\sum_n^\infty \frac{1}{n(\log n)^p}$, $p \leq 1$ is ________

a) converges b) diverges

c) oscillates d) converges to 0

18. The series $\sum_n^\infty \frac{1}{n \log n}$ is __________

a) converges b) diverges

c) oscillates d) converges to 0

19. The series $\sum_n^\infty (\sqrt{n^4+3} - n^2)$ is _____________

a) converges b) diverges

c) oscillates d) converges to 0

20. If n is odd, $\sum_n^\infty (-1)^n =$ ___________

a) n b) $-\frac{n}{2}$ c) $\frac{1}{2}(n+1)$ d) 0

PART-B(3×2=6 Marks)

21. Give an example for Cauchy sequence.
22. Define power series.
23. Define uniformly convergent of a series.

PART-C(3×8=24 Marks)

24. a) State and prove monotone subsequence theorem.

(OR)

b) Prove that a bounded sequence converges to x if every subsequence converges to x.

25. a) State and prove Bolzano- Weirstrass theorem.

(OR)

b) Prove that a Cauchy sequence of real numbers is bounded.

26. a) State and prove M test

(OR)

b) State and prove Cauchy criterion for series of functions.

Reg. No................................

[18MMU203]

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University)
(Established Under Section 3 of UGC Act, 1956)
Pollachi Main Road, Eachanari Post, Coimbatore – 641 021
(For the candidates admitted from 2018 onwards)

B.Sc., DEGREE EXAMINATION, APRIL 2019

Second Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 x 2 = 10 Marks) (2 ½ Hours)
Answer ALL the Questions

21. Determine the set A of all real numbers x such that $2x + 3 \leq 6$.
22. Define Bounded sequence.
23. If a series in $\mathbb{R}$ is absolutely convergent then prove that it is convergent.
24. Define Divergent.
25. Define Uniform convergent of a function.

PART C (5 x 6 = 30 Marks)
Answer ALL the Questions

26. a. Prove that the following statements are equivalent:
   i. S is a countable set. ii. There exist a surjection of N onto S.
   iii. There exist an injection of S onto N.

Or

b. Prove that an upper bound u of a non empty set S in R is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

27. a. If $X = (x_n)$ is a convergent sequence of real numbers and if $x_n \geq 0$ for all $n \in N$ then prove that $x = \lim (x_n) \geq 0$.

Or

b. Prove that $\{(1 + \frac{1}{n})^n\}_{n=1}^{\infty}$ is convergent.

28. a. If $0 < x < 1$ then prove that $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.

Or

b. If $\sum_{n=1}^{\infty} a_n$ converges absolutely to A, then prove that any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ also converges absolutely to A.

29. a. State and prove monotone convergence theorem.

Or

b. prove that a Cauchy sequence of real numbers is bounded.

30. a. State and prove Cauchy criterion for uniform convergence.

Or

b. If the power series $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = x_0$ then prove that $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-x_1, x_1]$ where x_1 is any number such that $0 < x_1 < |x_0|$.

KARPAGAM ACADEMY OF HIGHER EDUCATION

Department of Mathematics

I Internal Answerkey

Subject : Real Analysis

Class : I B.Sc Mathematics

Subject code : 18MMU203

PART-A

1. bijection
2. Countable
3. 2
4. $x \geq u$
5. $\{0, 1, 4, 9, \ldots\}$
6. $x^2 + y^2 = 95$
7. $(20 - x)e^{-x}$
8. 0
9. $\{0, 1, -1\}$
10. $f(c) \geq f(x)$
11. 0
12. $f''(c) = 0$
13. $1/2$
14. 1
15. 1
16. 3
17. 2
18. $\sqrt{2}$
19. $-1/3$
20. $(-3, 0) \cup (0, \infty)$

PART-B

21. Necessary Part

If $ab \geq 0$ $|a+b| = |a| + |b|$ $\longrightarrow$ 1 mark

Sufficient Part $\longrightarrow$ 1 mark.

If $(|a+b|) = |a| + |b|$ then $ab \geq 0$.

22. upperbound:

S - not empty set

$S \subseteq \mathbb{R}$

If S is bounded above, then

u is ~~supremum~~, upperbound if $\exists$ $u \in \mathbb{R}$

$\ni$ $s \leq u$ $\forall s \in S$.

$\longrightarrow$ 2 marks

23. Completeness Property of $\mathbb{R}$:

* Every nonempty set of $\mathbb{R}$ has upperbound also has supremum. $\longrightarrow$ 1 mark

* Every non empty set of $\mathbb{R}$ has lowerbound also has infimum $\longrightarrow$ 1 mark.

PART- C

24 a) (i) Triangle Inequality :

Proof of $|a+b| \leq |a| + |b|$

$\longrightarrow$ (3 marks)

(ii) Archimedean Property :

Proof of , If $x \in \mathbb{R}$ then $\exists\ n_x \in \mathbb{N}$

$\ni\ x \in n_x$

$\longrightarrow$ (3 marks)

b) To Prove :

$\mathbb{R}$ is uncountable

Proof :

It is enough to Prove that $(0,1)$ is uncountable

listing the elements in $(0,1)$ as

$(0,1) = \{x_1, x_2, \cdots\}$

defining y, $\longrightarrow$ 3 marks

defining a function

Proving by contradiction $\longrightarrow$ 3 marks .

Q5

a) i) Proof of uniqueness of limits

→ (3 marks)

ii) Proof of $c > 0$ then $\lim (c^{1/n}) = 1$

→ (3 marks)

b) Proof :

Suppose $\lim (x_n) = x$

Let $\varepsilon = 1$

$\exists$ a +ve integer $N \ni$

$|x_n - x| < 1$ if $n \geq N$

$|x_n| < 1 + |x|$ if $n \geq N$

→ 3 marks

Let $M = \sup \{ |x_1|, \ldots, 1 + |x| \}$

Then $|x_n| \leq M$

Hence Proved

→ 3 marks

2b

a) Monotone Convergence theorem

Statement: A monotone seq is convergent iff it is bdd

$X = (x_n)$ is bdd $\uparrow$ seq. then

$\lim (x_n) = \sup \{x_n : n \in N\}$

$Y = (y_n)$ is bdd $\downarrow$ seq. then

$\lim (y_n) = \inf \{y_n : n \in N\}$.

$\longrightarrow$ 2 marks.

Proof:

Proof of Necessary Part

$\longrightarrow$ 2 marks

Proof of Sufficient Part

$\longrightarrow$ 2 marks.

b) Statement: Squeeze theorem

Sequences $(x_n)(y_n), (z_n)$ $\ni$

$x_n \leq y_n \leq z_n$ & $\lim (x_n) = \lim (z_n)$

then $\lim(x_n) = \lim (y_n) = \lim (z_n)$

$\longrightarrow$ (2 mark)

Proof:

$|x_n - \omega| < \varepsilon$ &

$|z_n - \omega| < \varepsilon$

$\longrightarrow$ (2 marks)

$\Rightarrow \lim(y_n) = \omega$ $\longrightarrow$ 2 marks.

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KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore

Department of Mathematics

II Internal Answer key

Subject : Real Analysis class: I B-Sc Mathematics

Subject Code : 18MMU203

PART-A

1. a) $r \geq 1$
2. 0
3. $\frac{1}{1-r}$
4. diverges
5. $f_n = f_{n-2} + f_{n-1}$
6. converges
7. at most
8. 0
9. converges
10. 0
11. 0
12. $x < n$
13. $q \neq 0$
14. $|a+b| \leq |a|+|b|$
15. $[0,\infty)$
16. some $n \in N$
17. 0
18. even
19. 0
20. $P > 1$

PART – B

21. Convergent Sequence:

If a sequence has limit then it is called convergent → 1 mark.

Eg: → 1 mark.

22. n^{th} term test:

If the series $\sum x_n$ converges then $\lim(x_n) = 0$ → 2 marks.

23: Eg:

$\mathbb{R}, \mathbb{Q}^c$ → 2 marks

PART – C

24 a) Comparison test

Statement:

$0 \leq x_n \leq y_n$

i) Convergence of $\sum y_n$ implies convergence of $\sum x_n$

ii) divergence of $\sum x_n \Rightarrow$ divergence of $\sum y_n$ → (2 marks)

Proof: Part (i) → 2 marks

Part (ii) → 2 marks.

b) Root test :

Statement : $\sum a_n$ is nonegative

Let

$$l = \lim \sqrt[n]{a_n}$$

i) $\sum a_n$ converges if $l < 1$

ii) diverges if $l > 1$

iii) Inconclusive $l = 1$

$\longrightarrow$ 2 marks

Proof :

Part (i) $\longrightarrow$ 2 marks

Part (ii) $\longrightarrow$ 1 mark

Part (iii) $\longrightarrow$ 1 mark.

25.

a) Solution :

Defining a_n and a_{n+1}

using Ratio test

$\longrightarrow$ 4 marks

Testing convergence of $\sum_{1}^{\infty} \frac{n!(2n)}{n^n}$.

$\longrightarrow$ 2 marks

b). P - Series $\sum_{n=1}^{\infty} \frac{1}{n^P}$ diverges when $0 \leq P \leq 1$

WKT $n^P \leq n$

then $\frac{1}{n^P} \geq \frac{1}{n}$

$\frac{1}{n} \leq \frac{1}{n^P}$ $\longrightarrow$ 3 marks

$\Sigma 1/n$ diverges

$\Rightarrow \frac{1}{n^P}$ diverges when $0 < P \leq 1$

$\therefore$ If $P > 1$ then $\frac{1}{n^P}$ converges $\longrightarrow$ 3 marks.

2b

b) Cauchy criterion for Series :

Σx_n converges iff $\forall \varepsilon > 0 \exists$

$M(\varepsilon) \in N \ni$ if $m > n \geq N(\varepsilon)$ then

$|S_m - S_n| = |x_{n+1} + x_{n+2} + \dots + x_m| < \varepsilon$.

$\longrightarrow$ (2 marks)

Proof :

Necessary Part

If $|S_m - S_n| = |x_{n+1} + x_{n+2} + \dots + x_m| < \varepsilon$

then Σx_n Converges $\longrightarrow$ 2 marks

Sufficient Part $\longrightarrow$ 2 marks.

KARPAGAM ACADEMY OF HIGHER EDUCATION

Department of Mathematics

II Internal Answer key

Subject : Real Analysis   Class : I B.Sc Mathematics

Subject code : 18MMU203

PART- A

1. a) Cauchy
2. b) (n)
3. a) Converges
4. a) converges
5. b) diverges
6. b) diverges
7. a) diverges
8. c) bounded
9. b) converges
10. a) diverges
11. c) Oscillates
12. a) converges
13. c) $a>1$
14. c) $a>1$
15. d) 0
16. c) $\left(\frac{1}{n^2}\right)$
17. d) diverges.
18. d) diverges
19. a) converges
20. d) 0.

PART- B

21. Eg :

$1/n$, $1/n^2$

$\rightarrow$ (2 marks)

22) Power Series:

A Series of real function Σf_n is said to be Power Series, around $x=c$ if

$f_n(x) = a_n (x-c)^n$.

$c \in \mathbb{R}$. where $n = 0, 1, 2, \ldots$

$\longrightarrow$ 2 marks.

23). Uniform convergent

$f: A \to R$ if $\forall\ \varepsilon > 0\ \exists$ a +ve integer $n \ni$ $|f_n(x) - f(x)| < \varepsilon$. if $n \geq N$

$\rightarrow$ 2 marks.

PART- C.

24

a) Monotone Subsequence theorem.

If $X = (x_n)$ then $\exists$ a subsequence which is monotone.

$\longrightarrow$ 2 marks

Proof:

Case(i) X has infinitely many Peaks $\rightarrow$ 2 marks

Case(ii) . X has finite Peaks $\longrightarrow$ 2 marks.

35
a) Bolzano - Weirstrass theorem :-

Statement :

A bdd Sequence of real number has a convergent sequence → 2 marks

Proof:

Using monotone subsequence theorem X has monotone Subsequence.

→ 2 marks

By monotone convergence theorem (x_{n_k}) is converges → 2 marks.

b) Proof :

$X = (x_n)$ be Cauchy sequence.

Let $\varepsilon = 1$

$|x_n - x_m| < 1$

$|x_n - x_N| < 1$

$|x_n| - |x_N| < 1$

→ 3 marks

$|x_n| < M$

(x_n) is bdd

→ 3 marks.

2b

a) M test :

Statement :

(M_n) be a sequence of +ve real numbers

$|f_n(x)| \leq M_n$ for $x \in D, N \in N$.

If ΣM_n converges then Σf_n is uniformly convergent on D

Proof :

Using Cauchy criterion for series and

$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \varepsilon$

→ (2 marks)

By Cauchy criterion sequence of functions

→ 2 (marks)

b)

Statement

$(f_n) \to$ sequence

$D \subseteq R$ to R

Σf_n uniformly converges on D $\Leftrightarrow$ $\forall \varepsilon > 0 \; \exists$

$M(\varepsilon) \ni |f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \varepsilon$.

→ 2 marks

Proof : Part (i) → 2 marks

Converse part — 2 marks

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KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE - 21

Answer key for ESE

Subject : Real Analysis

Subject code : 18MMU203

class : I B.Sc Mathematics.

PART – A

(Question Nos. 1 to 20 Online Examinations)

PART - B (5×2 = 10 Marks)

21.

$A = (-\infty, 3/2]$ $\quad [\because x \leq 3/2]$

22. A sequence is said to be bounded if it is both bounded above and bounded below.

23. By defn. of absolutely convergence,

We've that $\sum_{n=1}^{\infty} |a_n|$ is convergent

From the comparison Test We've that $\sum_{n=1}^{\infty} a_n$ converges iff $\forall\, n \in N > 0 : |a_n| \leq b_n$.

where $\sum_{n=1}^{\infty} b_n$ is convergent.

So substituting $|a_n|$ for b_n in the above,

the result follows.

24. If a sequence has no limit we say that the sequence is divergent'

25. <u>Uniform convergence</u>:

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges uniformly on A to a functio $f : A \to \mathbb{R}$ if $\forall \; \varepsilon > 0 \; \exists$ a +ve integer $N \ni$

$$|f_n(x) - f(x)| < \varepsilon \quad \text{if } n \geq N.$$

PART- C

(5x6 = 30 Marks)

26) a)

(i)⇒(ii) If S is finite $\exists$ a bijection h of some set N_n onto S and we define, H on $\mathbb{N}$ by

$$H(k) = \begin{cases} h(k) & \text{for } k = 1, \ldots, n \\ h(n) & \text{for } k > n \end{cases}$$

Then H is a surjection on $\mathbb{N}$ onto S

If S is denumerable, $\exists$ a bijection H of $\mathbb{N}$ onto S, which is also a surjection of $\mathbb{N}$ onto S

(ii) $\Rightarrow$ iii)

If H is surjection of $\mathbb{N}$ onto S, we define $H_1 : S \to \mathbb{N}$ by letting $H_1(s)$ be the least element in $H^{-1}(s) = \{n \in \mathbb{N} : H(n) = s\}$.

To show H_1 is an injection of S into $\mathbb{N}$, note if $s, t \in S$ and $n_{st} = H_1(s) = H_1(t)$ then $s = H(n_{st}) = t$.

iii) $\Rightarrow$ i) If H_1 is an injection of S into $\mathbb{N}$ Then it is a bijection of S onto $H_1(S) \subseteq \mathbb{N}$

$H_1(S)$ is countable, whence the set S is countable.

26.

b) Pro

Given:

$S \to$ not empty set

Proof:

i) if u is an upperbound of S satisfies the stated condition

if $v < u$, then $\varepsilon := u - v$. Then $\varepsilon > 0$,

so $\exists\ s_\varepsilon \in S \ni v = u - \varepsilon < s_\varepsilon$

$\therefore$ v is not upperbound of S

$\therefore$ u = sup S.

Conversely,

Suppose u = Sup S & let $\epsilon > 0$.

$\because$ $u - \epsilon < u$, then $u - \epsilon$ is not an upper bound of S.

$\therefore$ Some element s_ϵ of S must be greater than

$u - \epsilon$ ie, $u - \epsilon < s_\epsilon$.

27

a) Given :

$X = (x_n)$ is convergent sequence

$x_n \geq 0 \ \forall \ n \in \mathbb{N}$

To Prove :

$x = \lim (x_n) \geq 0$.

Proof :

Suppose $x < 0$, then $\varepsilon = -x$ is positive.

$\because$ $X \to x$, $\exists$ a Natural number K $\ni$

$x - \varepsilon < x_n < x + \varepsilon \ \forall \ n \geq K$.

We've $x_K < x + \varepsilon = x + (-x) = 0$, $\Rightarrow\Leftarrow$ to $x_n \geq 0$

$\therefore$ that $x \geq 0$

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$$a_n = \left(1+\frac{1}{n}\right)^n$$

$$a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} \cdot \left(1+\frac{1}{n}\right)$$

$$= \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} (1+1/n) \rightarrow (i)$$

$$\geq \left(1 - \frac{1}{n+1}\right)(1+1/n)$$

$$= 3$$

$\therefore$ the series converges.

28

b). Suppose Σa_n converges to $A \in \mathbb{R}$: Thus if $\varepsilon > 0$, let N be $\ni$ if $n, q > N$ and $S_n = a_1 + \dots + a_n$ then

$$|A - S_n| < \varepsilon \quad \text{and} \quad \sum_{k=N+1}^{q} |a_k| < \varepsilon$$

Let $M \in \mathbb{N}$

$$t_M := b_1 + b_2 + \dots + b_M$$

It follows if $m \geq M$

then $t_m - S_n$

We've

$$|t_m - S_n| \leq \sum_{k=N+1}^{q} |a_k| < \varepsilon$$

$\therefore$ if $m \geq M$ then we've

$$|t_m - A| \leq |t_m - S_n| + |S_n - x| < \varepsilon + \varepsilon = 2\varepsilon$$

$\because$ $\varepsilon > 0$ is arbitrary we conclude that $\sum b_n$ converges to A.

29
a) Statement:

Monotone sequence of real numbers is convergent iff it is bounded. Moreover

i) If $X = (x_n)$ is a bdd increasing sequence then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

ii) If $Y = (y_n)$ is a bdd decreasing sequence then

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

Proof:

Suppose monotone sequence is convergent. Then, ie, bdd

Conversely it is bdd.

$\because$ y_n sequence is monotone, we've either increasing or decreasing.

$X \to$ be increasing and bdd

$\because$ X is bdd $\exists$ a real number M

$\ni$ $x_n \leq M$ $\forall$ $n \in N$

$\therefore \{x_n : n \in N\}$ is bounded above

By completeness Property of $\mathbb{R}$. $\exists$ a supremum for

$\{x_n : n \in N\}$

$x^* = \sup\{x_n : n \in N\}$

Let $\varepsilon > 0$ be given,

Then $x^* - \varepsilon$ is not an upperbound

$\therefore$ $\exists$ a set x_n $\ni$ $x^* - \varepsilon < x_k$.

$\lim(x_n) = x^*$

ii) Let $Y = (y_n)$ be a bdd decreasing sequence.

Then $X = -Y = -(y_n)$ is an increasing

Sequence

$\lim Y = +\inf\{y_n : n \in N\}$

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b) To Prove:

A convergent sequence of real number is bounded

Proof:

Suppose $\lim (x_n) = x$

Let $\varepsilon = 1 > 0$

$\exists$ a +ve integer $N \ni$

$|x_n - x| < 1$

$\therefore |x_n| < 1 + |x|$

Let $M = \sup \{ |x_1|, |x_2|, \ldots |x_{N+1}| ; 1 + |x| \}$

Then $|x_n| \leq M \quad \forall n \geq 1$

Hence Proved

10

a). Statement:

(f_n) be sequence of bdd functions on $A \subseteq \mathbb{R}$. This sequence converges uniformly on A to a bdd function f. iff for each $\varepsilon > 0$ $\exists$ $H(\varepsilon)$ in $\mathbb{N}$ $\ni$ all $m, n \geq H(\varepsilon)$ then $\|f_m - f_n\|_A \leq \varepsilon$.

If $f_n \rightrightarrows f$ on A, Given $\varepsilon > 0$ $\exists$ $K(\frac{1}{2}\varepsilon)$ $\ni$

if $n \geq K(\frac{1}{2}\varepsilon)$ then $\|f_n - f\|_A \leq \frac{1}{2}\varepsilon$.

Hence if both $m, n \geq K(\frac{1}{2}\varepsilon)$ we conclu

$|f_m - f_n| \leq \varepsilon \quad \forall\ x \in A$.

$\Rightarrow \|f_m - f_n\| \leq \varepsilon$

($\Leftarrow$) Suppose $\|f_m - f_n\| \leq \varepsilon$,

We've. $f : A \to \mathbb{R}$ by

$f(x) = \lim (f_n(x))$ for $x \in A$

$|f_m(x) - f(x)| \leq \varepsilon$ for $m \geq H(\varepsilon)$

Hence Proved.

CHAPTER 4

SUBSEQUENCES

4.1 Subsequences

Definition 4.1.1 *Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_n k)$ given by $(x_{n_1}, x_{n_2}, \dots)$ is called a subsequence of X*

Example 4.1.1 *Consider a sequence $X = (1, \frac{1}{2}, \frac{1}{3}, \dots)$*
Let $X' = (\frac{1}{2}, \frac{1}{4}, \dots)$
clearly, x' is a subsequence of X. note that $n_1 = 2, n_2 = 4, \dots$

Definition 4.1.2 *If $X(x_1, x_2, \dots)$ is a sequence of real numbers and if m is a given natural numbers, then the m-tail of X is the sequence. $X_m = (x_{m+1}, x_{m+2}, \dots)$*

Remark 4.1.1 *A tail of a sequence is a special type of subsequence. (ii) Not every subsequence of a given sequence need be a tail of the sequence.*

Theorem 4.1.1 *If a sequence $X = (x_n)$, of real numbers converges to a real number x, then any subsequencece $x' = (x_{n_k})$ of x, also converges to x.*

Proof
Given that,
$lim x_n = x$
$\therefore$ for given $\in > 0$, there exist a positive integer N such that $|x_n - x| < \in$ if $n \geq N$

Let $X^{'} = (x_{n_k})$ be a subsequence of X. The $n_1 < n_2 < n_3 < \ldots$ clealy $n_k \geq k$
suppose $k \geq N$, then $n_k \geq N$
$|x_{n_k} - x| <\in$
Therefore (x_{n_k}) converges to x

Definition 4.1.3 *For a sequence (x_n), we say that the m^{th} term x_m of (x_n) if $x_m \geq x_n$ for all $n \geq M$.*

Remark 4.1.2 *In a decreasing sequence, every term is peak and in an increasing sequence no term is peak.*

4.2 The cauchy sequences

Definition 4.2.1 *A sequence $X = (x_n)$ of real number is said to be a cauchy sequence if for every $\in> 0$, there exist a natural number N such that $|x_n - x_m| \in$ if $n, m \geq N$*

Theorem 4.2.1 *If $X = (x_n)$ is a convergent sequence of real numbers then X is a cauchy sequence.*

Proof
Let $X = (x_n)$ be a convergent sequence. Let $\lim x_n = x$
Let $\in> 0$ be arbitrary, then for $\frac{\in}{2} > 0$, there exist a positive integer N such that
$|x_n - x| < \frac{\in}{2}$ if $n \geq N$
Let $n, m \geq N$
Now $|x_n - x_m| = |x_n - x + x - x_m|$
$\leq |x_n - x| + |x - x_m|$ ¡ $\frac{\in}{2} + \frac{\in}{2} = \in$
$|x_n - x_m| <\in$ if $n, m \geq N$
Therefore (x_n) is a cauchy sequence.

Theorem 4.2.2 *A caushy sequence of real number is bounded*

Proof
Let $X = (x_n)$ be a cauchy sequence
Let $\in= 1$, then there exist a positive integer N such that
$|x_n - x_m| < 1$ if $n, m \geq N$
In particular, $|x_n - x_m| < 1$ if $n, m \geq N$
Now $|x_n| - |x_N| \leq |x_n - x_N| < 1$ if $n \geq N$
$\therefore |x_n| - |x_N| < 1$ if $n \geq N$
$|x_n| < 1 + |x_N|$ if $n \geq N$
Let $M = sup\{|x_1|, |x_2|, \ldots, |x_{N+1}|, 1 + |x_N|\}$
Then $|x_n| < M$ for all n
Therefore $-M < x_n < m$ for all n
Therefore (x_n) is bounded.

4.3 Cauchy convergence criterion

Theorem 4.3.1 *A sequence of real number is convergent if and only if it is cauchy sequence.*

Proof
Suppose $X = (x_n)$ is a convergent sequence
by previous theorem, X is a cauchy sequence.
Conversely suppose $X = (x_n)$ is a cauchy sequence. by previous theorem, X is bounded
By Bolzono theorem, X has a convergent subsequence.
Let $x_{n_k} \to x$
claim $x_n \to x$
since X is cauchy sequence, for given $\frac{\epsilon}{2} > 0$, there exist a positive integer N such that
$|x_n - x_m| < \frac{\epsilon}{2}$ if $n, m \geq N$
since (x_{n_k}) converges to x, for $\frac{\epsilon}{2} > 0$, there exist a positive integer $k \geq N$ such that
$|x_k - x| < \frac{\epsilon}{2}$ if $n \geq N$
Now $|x_n - x| = |x_n - x_k + x_k - x|$
$\leq |x_n - x_k| + |x_k - x| = \epsilon$
i,e., $|x_n - x| < \epsilon$ if $n \geq N$, therefore $x_n \to x$
i.e., X is a convergent sequence.

Problem 4.3.1 *Discuss the convergence of the series* $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \ldots$

Solution
Given series is an alternating series.
Let $a_n = \frac{1}{\sqrt{n}}$
$a_{n+1} = \frac{1}{\sqrt{n+1}}$
$a_{n+1} - a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}$
$= \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n}\sqrt{n+1}} < 0$
$a_{n+1} - a_n < 0$
$A_{n+1} < a_n$
$\therefore \{a_n\}$ is monotonically decreasing also $\lim a_n = \frac{1}{\sqrt{n}} = 0$
$\therefore$ The given **Solution** satisfies all the conditions of Leibnitz rule. The given series converges.

Problem 4.3.2 *Discuss the convergence of* $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \ldots$

Solution
Given series is an alternating series
LEt $a_n = \frac{2n+3}{2n}$
$a_{n+1} = \frac{2(n+1)+3}{2(n+1)}$
$= \frac{2n+5}{2n+2}$
$a_{n+1} - a_n = \frac{-6}{2n(2n+2)} < 0$
$a_{n+1} < a_n$
$\therefore \{a_n\}$ is monotonically decreasing. Also
$lim a_n = \frac{2n+3}{2n}$
$\frac{2+0}{2} = 1 \neq 0$
$\therefore$ the given series does not satisfies one of the condition of Leibnitz test.
$\therefore$ the given series diverges.

Problem 4.3.3 *Discuss the convergence of the series* $\frac{1}{log2} - \frac{1}{log3} + \frac{1}{log4} - \ldots$

Solution
Given series is alternating series
Let $a_n = \frac{1}{log(n+1)}$
$a_{n+1} = \frac{1}{log(n+2)}$
$a_{n+1} - a_n = \frac{1}{log(n+2)} - \frac{1}{log(n+1)}$ ¡ 0
$a_{n+1} - a_n < 0$
$a_{n+1} < a_n$
$\therefore \{a_n\}$ is a monotonically decreasing.

$lim a_n = \frac{1}{log(n+1)}$
$= \frac{1}{\infty} = 0$
Therefore the given series satisfies all the condition of leibnitz test.
The given series is convergent.

Reg no-------------

(18MMU203)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore-21

DEPARTMENT OF MATHEMATICS

Second Semester

II Internal Test - Feb'2019

Real Analysis

Date:05 -02-2019 (AN) **Time: 2 Hours**

Class: I-B.Sc Mathematics **Maximum Marks:50**

PART-A(20×1=20 Marks)

ANSWER ALL THE QUESTIONS

1. Geometric series $\sum_{n=1}^{\infty} r^n$diverges if ______
   a) $r \geq 1$ b) $r < 1$ c) $r = 1$ d) $r \leq 1$
2. If $\lim |x_n| = 0$ then $\lim x_n =$________
   a) -1 b) 0 c) 1 d) 2
3. For the series $\sum_{n}^{\infty} r^n$, $s_n =$ ___________
   a) $\frac{1}{1-r}$ b) $\frac{1-r^{n+1}}{1-r}$ c) $\frac{1-r^n}{1+r}$ d) $\frac{1-r^{n+1}}{1+r}$
4. The series $\sum_{n}^{\infty} \frac{1}{\sqrt{(n+1)}}$ is ____________
   a) diverges b) oscillates
   c) converges d) converges to 0
5. The nth $(n \geq 3)$ term of the Fibonacci sequence is ___________
   a) $f_n = f_{n-2} + f_{n-1}$ b) $f_n = f_{n-2} - f_{n-1}$
   c) $f_n = f_{n-2} \times f_{n-1}$ d) n
6. If $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots =$______
   a) diverges b) converges
   c) oscillates d) converges to 1
7. A sequence in R has ------- one limit
   a) at most b) at least
   c) no d) all the above
8. For any $b \in R$, $\lim \left(\frac{b}{n}\right) =$
   a) -1 b) 1 c) 2 d) 0
9. If $\{x_n\}$ is a constant sequence with $x_n = c$, a constant, $\{x_n\}$ _____
   a) diverges b) converges
   c) oscillates d) converges to 1
10. $\lim \frac{3n+2}{n+1} =$ ________
    a) 1 b) 2 c) 3 d) 0
11. $\lim \left(\frac{2n}{n+2}\right) =$ __________
    a) -1 b) 1 c) 2 d) 0
12. For every real x there is an integer n such that ____________
    a) $n < x$ b) $x < n$ c) $n = x$ d) $x \leq n$
13. A real number $\frac{p}{q}$, $(p, q \in Z)$ is a rational number if ______
    a) $q > 0$ b) $q \neq 0$ c) $q < 0$ d) $q = 0$
14. Which of the following is not true?
    a) $|a| + |b| \leq |a + b|$ b) $|a| - |b| \leq |a + b|$
    c) $|a + b| \leq |a| + |b|$ d) $|ab| = |a| \cdot |b|$
15. Let $f: R \to R$ be a function defined by $f(x) = x^2$. Then range of f is ________
    a) $[0, \infty)$ b) $(-\infty, 0)$ c) $(0, \infty)$ d) R
16. $\{x_n\}$ is a constant sequence if $x_n = c$, a constant for ____
    a) some $n \in N$ b) all $n \in N$
    c) no $n \in N$ d) only one $n \in N$
17. For any $b \in R$, $\lim \left(\frac{b}{n}\right) =$________ as n tends to ∞.
    a) -1 b) 1 c) 2 d) 0
18. For the series $\sum_{n}^{\infty} (-1)^n$, $s_n = 1$ if n is ________
    a) odd b) even c) prime d) composite
19. If the series $\sum_{n}^{\infty} x_n$ converges, $\lim x_n =$
    a) 0 b) -1 c) 1 d) 2
20. The series $\sum_{n}^{\infty} \frac{1}{n^p}$ converges if _____
    a) p <1 b) p=1 c) p > 1 d) $p \geq 1$

PART-B($3\times2=6$ Marks)
ANSWER ALL THE QUESTIONS

21. Define a convergent sequence.
22. State the n^{th} term test.
23. Give two examples for uncountable sets.

PART-C($3\times8=24$ Marks)
ANSWER ALL THE QUESTIONS

24. a) State and prove the comparison test for the series.

(OR)

b) State and prove Root test for series.

25. a) Test the convergence of series $\sum_1^\infty \frac{n!(2n)}{n^n}$

(OR)

b) Prove the $p - series$ converges if $p > 1$.

26. a) Let (x_n) be a sequence of non-negative real numbers. Then show that the series $\sum x_n$ converges iff the sequence $S= (s_k)$ of partial sums is bounded. In this case, $\sum(x_n) = \lim(s_k) = \sup \{s_k: k\epsilon N\}$

(OR)

b) State and prove Cauchy criterion for series.