		Semester – II
18MMU211	**DIFFERENTIAL EQUATIONS (PRACTICAL)**	**3H – 2C**

Instruction Hours / week: L: 0 T: 0 P: 3 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- Problem-solving through programming.
- Hands-on training using lab components.

Course Outcomes (COs)

On successful completion of this course, the student will be able to

1. Demonstrate comprehension in fundamental topics of computing, algorithms, computer organization and software systems.
2. Have applied knowledge of areas of computing to create solutions to challenging problems, including specify, design, implement and validate solutions for new problems.
3. Be aware of current research activity in computing through activities including reading papers, hearing research presentations, and successfully planning and completing an individual research project in computing or its application.

List of Practical (Any 8 programs)

1. Plotting of second order solution family of differential equation.
2. Growth model (exponential case only).
3. Decay model (exponential case only).
4. Lake pollution model (with constant/seasonal flow and pollution concentration).
5. Case of single cold pill and a course of cold pills.
6. Limited growth of population (with and without harvesting).
7. Predatory-prey model (basic volterra model, with density dependence, effect of DDT, two prey one predator).
8. Plotting of recursive sequences.
9. Verify Bolzano-Weierstrass theorem through plotting of sequences and hence identify convergent subsequences from the plot.
10. Study the convergence/divergence of infinite series by plotting their sequences of partial sum.
11. Cauchy's root test by plotting nth roots.
12. Ratio test by plotting the ratio of nth and $(n+1)^{th}$ term.

UNIT I
Partial Differential Equations
SYLLABUS

Partial Differential Equations – Basic concepts and Definitions -Mathematical Problems. First Order Equations: Classification - Construction and Geometrical Interpretation- Method of characteristics for obtaining General Solution of Quasi Linear Equations- Canonical Forms of First-order Linear Equations.

Basic Concepts and Definitions

A differential equation that contains, in addition to the dependent variable and the independent variables, one or more partial derivatives of the dependent variable is called a *partial differential equation*. In general, it may be written in the form

$$f(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0, \qquad (1.2.1)$$

involving several independent variables x, y, $\ldots$, an unknown function u of these variables, and the partial derivatives u_x, u_y, $\ldots$, u_{xx}, u_{xy}, $\ldots$, of the function. Subscripts on dependent variables denote differentiations, e.g.,

$$u_x = \partial u/\partial x, \qquad u_{xy} = \partial^2/\partial y\, \partial x.$$

Here equation (1.2.1) is considered in a suitable domain D of the n-dimensional space R^n in the independent variables x, y, $\ldots$. We seek functions $u = u(x, y, \ldots)$ which satisfy equation (1.2.1) identically in D. Such functions, if they exist, are called *solutions* of equation (1.2.1). From these many possible solutions we attempt to select a particular one by introducing suitable additional conditions.

For instance,

$$\begin{aligned} uu_{xy} + u_x &= y, \\ u_{xx} + 2yu_{xy} + 3xu_{yy} &= 4\sin x, \\ (u_x)^2 + (u_y)^2 &= 1, \\ u_{xx} - u_{yy} &= 0, \end{aligned} \qquad (1.2.2)$$

are partial differential equations. The functions

$$u(x,y) = (x+y)^3,$$
$$u(x,y) = \sin(x-y),$$

are solutions of the last equation of (1.2.2), as can easily be verified.

The *order* of a partial differential equation is the order of the highest-ordered partial derivative appearing in the equation. For example

$$u_{xx} + 2xu_{xy} + u_{yy} = e^y$$

is a second-order partial differential equation, and

$$u_{xxy} + xu_{yy} + 8u = 7y$$

is a third-order partial differential equation.

A partial differential equation is said to be *linear* if it is linear in the unknown function and all its derivatives with coefficients depending only on the independent variables; it is said to be *quasi-linear* if it is linear in the highest-ordered derivative of the unknown function. For example, the equation

$$yu_{xx} + 2xyu_{yy} + u = 1$$

is a second-order linear partial differential equation, whereas

$$u_x u_{xx} + xuu_y = \sin y$$

is a second-order quasi-linear partial differential equation. The equation which is not linear is called a *nonlinear* equation.

We shall be primarily concerned with linear second-order partial differential equations, which frequently arise in problems of mathematical physics. The most general second-order linear partial differential equation in n independent variables has the form

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G, \qquad (1.2.3)$$

where we assume without loss of generality that $A_{ij} = A_{ji}$. We also assume that B_i, F, and G are functions of the n independent variables x_i.

If G is identically zero, the equation is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

The general solution of a linear ordinary differential equation of nth order is a family of functions depending on n independent arbitrary constants. In the case of partial differential equations, the general solution depends on arbitrary functions rather than on arbitrary constants. To illustrate this, consider the equation

$$u_{xy} = 0.$$

If we integrate this equation with respect to y, we obtain

$$u_x(x, y) = f(x).$$

A second integration with respect to x yields

$$u(x, y) = g(x) + h(y),$$

where $g(x)$ and $h(y)$ are arbitrary functions.

Suppose u is a function of three variables, x, y, and z. Then, for the equation

$$u_{yy} = 2,$$

one finds the general solution

$$u(x, y, z) = y^2 + yf(x, z) + g(x, z),$$

where f and g are arbitrary functions of two variables x and z.

Mathematical Problems

A problem consists of finding an unknown function of a partial differential equation satisfying appropriate supplementary conditions. These conditions may be *initial conditions (I.C.)* and/or *boundary conditions (B.C.)*. For example, the partial differential equation (PDE)

$$\begin{array}{llll} & u_t - u_{xx} = 0, & 0 < x < l, & t > 0, \\ \text{with} \quad I.C. & u(x,0) = \sin x, & 0 \le x \le l, & t > 0, \\ B.C. & u(0,t) = 0, & & t \ge 0, \\ B.C. & u(l,t) = 0, & & t \ge 0, \end{array}$$

constitutes a problem which consists of a partial differential equation and three supplementary conditions. The equation describes the heat conduction in a rod of length l. The last two conditions are called the *boundary conditions* which describe the function at two prescribed boundary points. The first condition is known as the *initial condition* which prescribes the unknown function $u(x,t)$ throughout the given region at some initial time t, in this case $t = 0$. This problem is known as the *initial boundary-value problem*. Mathematically speaking, the time and the space coordinates are regarded as independent variables. In this respect, the initial condition is merely a point prescribed on the t-axis and the boundary conditions are prescribed, in this case, as two points on the x-axis. Initial conditions are usually prescribed at a certain time $t = t_0$ or $t = 0$, but it is not customary to consider the other end point of a given time interval.

In considering the problem of unbounded domain, the solution can be determined uniquely by prescribing initial conditions only. The corresponding problem is called the *initial-value problem* or the *Cauchy problem*. The

A mathematical problem is said to be *well-posed* if it satisfies the following requirements:

1. Existence: There is at least one solution.
2. Uniqueness: There is at most one solution.
3. Continuity: The solution depends continuously on the data.

Classification of First-Order Equations

The most general, first-order, partial differential equation in two independent variables x and y is of the form

$$F(x, y, u, u_x, u_y) = 0, \qquad (x, y) \in D \subset R^2, \tag{2.2.1}$$

where F is a given function of its arguments, and $u = u(x, y)$ is an unknown function of the independent variables x and y which lie in some given domain D in R^2, $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. Equation (2.2.1) is often written in terms of standard notation $p = u_x$ and $q = u_y$ so that (2.2.1) takes the form

$$F(x, y, u, p, q) = 0. \tag{2.2.2}$$

Similarly, the most general, first-order, partial differential equation in three independent variables x, y, z can be written as

$$F(x, y, z, u, u_x, u_y, u_z) = 0. \tag{2.2.3}$$

Equation (2.2.1) or (2.2.2) is called a *quasi-linear partial differential equation* if it is linear in first-partial derivatives of the unknown function $u(x, y)$. So, the most general quasi-linear equation must be of the form

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u), \tag{2.2.4}$$

where its coefficients a, b, and c are functions of x, y, and u.

The following are examples of quasi-linear equations:

$$x\left(y^2 + u\right) u_x - y\left(x^2 + u\right) u_y = \left(x^2 - y^2\right) u, \tag{2.2.5}$$

$$u u_x + u_t + n u^2 = 0, \tag{2.2.6}$$

$$\left(y^2 - u^2\right) u_x - xy\, u_y = xu. \tag{2.2.7}$$

Equation (2.2.4) is called a *semilinear partial differential equation* if its coefficients a and b are independent of u, and hence, the semilinear equation can be expressed in the form

$$a(x, y)\, u_x + b(x, y)\, u_y = c(x, y, u). \tag{2.2.8}$$

Examples of semilinear equations are

$$xu_x + yu_y = u^2 + x^2, \tag{2.2.9}$$

$$(x+1)^2 u_x + (y-1)^2 u_y = (x+y) u^2, \tag{2.2.10}$$

$$u_t + au_x + u^2 = 0, \tag{2.2.11}$$

where a is a constant.

Equation (2.2.1) is said to be *linear* if F is linear in each of the variables u, u_x, and u_y, and the coefficients of these variables are functions only of the independent variables x and y. The most general, first-order, *linear* partial differential equation has the form

$$a(x,y) u_x + b(x,y) u_y + c(x,y) u = d(x,y), \tag{2.2.12}$$

where the coefficients a, b, and c, in general, are functions of x and y and $d(x,y)$ is a given function. Unless stated otherwise, these functions are assumed to be continuously differentiable. Equations of the form (2.2.12) are called *homogeneous* if $d(x,y) \equiv 0$ or *nonhomogeneous* if $d(x,y) \neq 0$.

Obviously, linear equations are a special kind of the quasi-linear equation (2.2.4) if a, b are independent of u and c is a linear function in u. Similarly, semilinear equation (2.2.8) reduces to a linear equation if c is linear in u.

Examples of linear equations are

$$xu_x + yu_y - nu = 0, \tag{2.2.13}$$

$$nu_x + (x+y) u_y - u = e^x, \tag{2.2.14}$$

$$yu_x + xu_y = xy, \tag{2.2.15}$$

$$(y-z) u_x + (z-x) u_y + (x-y) u_z = 0. \tag{2.2.16}$$

An equation which is *not* linear is often called a *nonlinear equation*. So, first-order equations are often classified as linear and nonlinear.

Construction of a First-Order Equation

We consider a system of geometrical surfaces described by the equation

$$f(x, y, z, a, b) = 0, \tag{2.3.1}$$

where a and b are arbitrary parameters. We differentiate (2.3.1) with respect to x and y to obtain

$$f_x + p\, f_z = 0, \qquad f_y + q\, f_z = 0, \tag{2.3.2}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

The set of three equations (2.3.1) and (2.3.2) involves two arbitrary parameters a and b. In general, these two parameters can be eliminated from this set to obtain a first-order equation of the form

$$F(x, y, z, p, q) = 0. \tag{2.3.3}$$

Thus the system of surfaces (2.3.1) gives rise to a first-order partial differential equation (2.3.3). In other words, an equation of the form (2.3.1) containing two arbitrary parameters is called a *complete solution* or a *complete integral* of equation (2.3.3). Its role is somewhat similar to that of a general solution for the case of an ordinary differential equation.

On the other hand, any relationship of the form

$$f(\phi, \psi) = 0, \tag{2.3.4}$$

First, we prescribe the second parameter b as an arbitrary function of the first parameter a in the complete solution (2.3.1) of (2.3.3), that is, $b = b(a)$. We then consider the envelope of the one-parameter family of solutions so defined. This envelope is represented by the two simultaneous equations

$$f(x, y, z, a, b(a)) = 0, \qquad (2.3.5)$$

$$f_a(x, y, z, a, b(a)) + f_b(x, y, z, b(a))\, b'(a) = 0, \qquad (2.3.6)$$

where the second equation (2.3.6) is obtained from the first equation (2.3.5) by partial differentiation with respect to a. In principle, equation (2.3.5) can be solved for $a = a(x, y, z)$ as a function of x, y, and z. We substitute this result back in (2.3.5) to obtain

$$f\{x, y, z, a(x, y, z),\ b(a(x, y, z))\} = 0, \qquad (2.3.7)$$

where b is an arbitrary function. Indeed, the two equations (2.3.5) and (2.3.6) together define the general solution of (2.3.3). When a definite $b(a)$ is prescribed, we obtain a *particular solution* from the general solution. Since the general solution depends on an arbitrary function, there are infinitely many solutions. In practice, only one solution satisfying prescribed conditions is required for a physical problem. Such a solution may be called a *particular solution*.

Geometrical Interpretation

To investigate the geometrical content of a first-order, partial differential equation, we begin with a general, quasi-linear equation

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y - c(x, y, u) = 0. \qquad (2.4.1)$$

We assume that the possible solution of (2.4.1) in the form $u = u(x, y)$ or in an implicit form

$$f(x, y, u) \equiv u(x, y) - u = 0 \qquad (2.4.2)$$

represents a possible *solution surface* in (x, y, u) space. This is often called an *integral surface* of the equation (2.4.1). At any point (x, y, u) on the solution surface, the gradient vector $\nabla f = (f_x, f_y, f_u) = (u_x, u_y, -1)$ is normal to the solution surface. Clearly, equation (2.4.1) can be written as the dot product of two vectors

$$a\,u_x + b\,u_y - c = (a, b, c) \cdot (u_x, u_y - 1) = 0. \qquad (2.4.3)$$

This clearly shows that the vector (a, b, c) must be a tangent vector of the integral surface (2.4.2) at the point (x, y, u), and hence, it determines a direction field called the the *characteristic direction* or *Monge axis*. This direction is of fundamental importance in determining a solution of equation (2.4.1). To summarize, we have shown that $f(x, y, u) = u(x, y) - u = 0$, as a surface in the (x, y, u)-space, is a solution of (2.4.1) if and only if the direction vector field (a, b, c) lies in the tangent plane of the integral surface $f(x, y, u) = 0$ at each point (x, y, u), where $\nabla f \neq 0$, as shown in Figure 2.4.1.

A curve in (x, y, u)-space, whose tangent at every point coincides with the characteristic direction field (a, b, c), is called a *characteristic curve*. If the parametric equations of this characteristic curve are

$$x = x(t), \quad y = y(t), \quad u = u(t), \qquad (2.4.4)$$

then the tangent vector to this curve is $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right)$ which must be equal to (a, b, c). Therefore, the system of ordinary differential equations of the characteristic curve is given by

$$\frac{dx}{dt} = a(x, y, u), \quad \frac{dy}{dt} = b(x, y, u), \quad \frac{du}{dt} = c(x, y, u). \qquad (2.4.5)$$

These are called the *characteristic equations* of the quasi-linear equation (2.4.1).

Method of Characteristics and General Solutions

Theorem 2.5.1. The general solution of a first-order, quasi-linear partial differential equation

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \tag{2.5.1}$$

is

$$f(\phi,\psi) = 0, \tag{2.5.2}$$

where f is an arbitrary function of $\phi(x,y,u)$ and $\psi(x,y,u)$, and $\phi =$ constant $= c_1$ and $\psi =$ constant $= c_2$ are solution curves of the characteristic equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{2.5.3}$$

The solution curves defined by $\phi(x,y,u) = c_1$ and $\psi(x,y,u) = c_2$ are called the families of *characteristic curves* of equation (2.5.1).

Proof. Since $\phi(x,y,u) = c_1$ and $\psi(x,y,u) = c_2$ satisfy equations (2.5.3), these equations must be compatible with the equation

$$d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0. \tag{2.5.4}$$

This is equivalent to the equation

$$a\,\phi_x + b\,\phi_y + c\,\phi_u = 0. \tag{2.5.5}$$

Similarly, equation (2.5.3) is also compatible with

$$a\,\psi_x + b\,\psi_y + c\,\psi_u = 0. \tag{2.5.6}$$

We now solve (2.5.5), (2.5.6) for a, b, and c to obtain

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}}. \tag{2.5.7}$$

It has been shown earlier that $f(\phi, \psi) = 0$ satisfies an equation similar to (2.3.14), that is,

$$p \frac{\partial(\phi, \psi)}{\partial(y, u)} + q \frac{\partial(\phi, \psi)}{\partial(u, x)} = \frac{\partial(\phi, \psi)}{\partial(x, y)}. \qquad (2.5.8)$$

Substituting, (2.5.7) in (2.5.8), we find that $f(\phi, \psi) = 0$ is a solution of (2.5.1). This completes the proof.

Theorem 2.5.2. (*The Cauchy Problem for a First-Order Partial Differential Equation*). Suppose that C is a given curve in the (x, y)-plane with its parametric equations

$$x = x_0(t), \quad y = y_0(t), \qquad (2.5.9)$$

where t belongs to an interval $I \subset R$, and the derivatives $x_0'(t)$ and $y_0'(t)$ are piecewise continuous functions, such that $(x_0')^2 + (y_0')^2 \neq 0$. Also, suppose that $u = u_0(t)$ is a given function on the curve C. Then, there exists a solution $u = u(x, y)$ of the equation

$$F(x, y, u, u_x, u_y) = 0 \qquad (2.5.10)$$

in a domain D of R^2 containing the curve C for all $t \in I$, and the solution $u(x, y)$ satisfies the given initial data, that is,

$$u(x_0(t), y_0(t)) = u_0(t) \qquad (2.5.11)$$

for all values of $t \in I$.

Theorem 2.5.3. (*The Cauchy Problem for a Quasi-linear Equation*). Suppose that $x_0(t)$, $y_0(t)$, and $u_0(t)$ are continuously differentiable functions of t in a closed interval, $0 \leq t \leq 1$, and that a, b, and c are functions of x, y, and u with continuous first-order partial derivatives with respect to their arguments in some domain D of (x, y, u)-space containing the initial curve

$$\Gamma : \; x = x_0(t), \quad y = y_0(t), \quad u = u_0(t), \qquad (2.5.12)$$

where $0 \le t \le 1$, and satisfying the condition

$$y_0'(t)\,a(x_0(t), y_0(t), u_0(t)) - x_0'(t)\,b(x_0(t), y_0(t), u_0(t)) \neq 0. \quad (2.5.13)$$

Then there exists a unique solution $u = u(x, y)$ of the quasi-linear equation (2.5.1) in the neighborhood of $C : x = x_0(t)$, $y = y_0(t)$, and the solution satisfies the initial condition

$$u_0(t) = u(x_0(t), y_0(t)), \quad \text{for} \quad 0 \le t \le 1. \qquad (2.5.14)$$

Canonical Forms of First-Order Linear Equations

It is often convenient to transform the more general first-order linear partial differential equation (2.2.12)

$$a(x, y)\; u_x + b(x, y)\; u_y + c(x, y)\; u = d(x, y), \qquad (2.6.1)$$

into a *canonical* (or *standard*) form which can be easily integrated to find the general solution of (2.6.1). We use the characteristics of this equation (2.6.1) to introduce the new transformation by equations

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y), \qquad (2.6.2)$$

where ξ and η are once continuously differentiable and their Jacobian $J(x, y) \equiv \xi_x \eta_y - \xi_y \eta_x$ is nonzero in a domain of interest so that x and y can be determined uniquely from the system of equations (2.6.2). Thus, by chain rule,

$$u_x = u_\xi \xi_x + u_\eta \eta_x, \qquad u_y = u_\xi \xi_y + u_\eta \eta_y, \qquad (2.6.3)$$

we substitute these partial derivatives (2.6.3) into (2.6.1) to obtain the equation

$$A\, u_\xi + B\, u_\eta + cu = d, \qquad (2.6.4)$$

where

$$A = u\xi_x + b\xi_y, \qquad B = a\eta_x + b\eta_y. \qquad (2.6.5)$$

From (2.6.5) we see that $B = 0$ if η is a solution of the first-order equation

This equation has infinitely many solutions.

$$\ln u(\xi, \eta) = -\eta + \ln f(\xi),$$

where $f(\xi)$ is an arbitrary function of ξ only.
Equivalently,

$$u(\xi, \eta) = f(\xi) e^{-\eta}.$$

In terms of the original variables x and y, the general solution of equation (2.6.8) is

$$u(x, y) = f(x + y) e^{-y}, \tag{2.6.10}$$

where f is an arbitrary function.

Example 2.6.1. Reduce each of the following equations

$$u_x - u_y = u, \tag{2.6.8}$$
$$yu_x + u_y = x, \tag{2.6.9}$$

to canonical form, and obtain the general solution.

In (2.6.8), $a = 1$, $b = -1$, $c = -1$ and $d = 0$. The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{u}.$$

The characteristic curves are $\xi = x + y = c_1$, and we choose $\eta = y = c_2$ where c_1 and c_2 are constants. Consequently, $u_x = u_\xi$ and $u_y = u_\xi + u_\eta$, and hence, equation (2.6.8) becomes

$$u_\eta = u.$$

Integrating this equation gives

POSSIBLE QUESTIONS

UNIT-1					
QUESTIONS	**OPTION 1**	**OPTION2**	**OPTION3**	**OPTION4**	**ANSWER**
A differential equation involving ______ of one or more variables with respect to a single independent variable is called ODE	Ordinary derivatives	ordinary variales	partial derivatives	partial variales	Ordinary derivatives
An equaton which is _____ is called non linear	not linear	dependent	linear	independent	not linear
Initial value prolem is also called	one point boundary value prolem	two point boundary value problem	simply boundary value problem	wave equation	one point boundary value prolem
Simply boundary value problem is also called	initial value prolem	one point boundary value prolem	two point boundary value problem	transport equation	two point boundary value problem
of one or more dependent variables with respect to one or more independent variable is called	homogeneous	non-homogeneous	quasi-linear	linear	homogeneous
Non-linear ODE is an ODE that is _____	linear	non-linear	homogeneous	non-homogeneous	non-linear
____ ODE of order n is the dependent variable y and the independent variable x is an equation	linear	non-linear	homogeneous	non-homogeneous	linear
An equation involving ____ of one or more dependent variables with respect to one or more independent variables is called	equation	derivative	euler	linear	derivatives
partial equation of one or more variables with respect to _____ independent variable is called PDE	one or more	two or more	more than one	more than two	more than one
The order of the highest ordered derivatives involves in a differential equation is called	order of differential equation	degree of differential equation	product of differential equation	all the above	order of differential equation
Mathematical problem is said to be well posed if it satisfies the uniqueness then there is ____ one solution	atleast	almost	more than	all the above	almost
Initial value prolem is also called	boundary value problem	cauchy problem	all the above	none of these	cauchy problem

A partial differential equation is the equation involving partial derivatives of one or more dependent variables with respect to ------------------ independent variable.	one most one	atleast one	more than two	more than one	more than one
A differential equation involving ordinary derivatives of one or more dependent variables with respect to a ------------------ independent variable is called an ordinary differential equation.	1	2	3	4	1
The equation $u_x u_{xx} + xuu_y = \sin y$ is ------------------	quasi linear	non linear	Cauchy	boundary	quasi linear
The order of the equation $u_{xx} - u_{yy} = 0$ is ------------------	1	2	3	4	2
The partial differential equation $u_t - u_{xx} = 0$ with $u(x, 0) = \sin x$. The given condition is called ------------------	initial	boundary	Cauchy	linear	Cauchy
$F(x, y, u, u_{xx}, u_{yy}) = 0$ is ------------------ order PDE	first	second	third	fourth	second
The equation $F(x, y, u, u_x, u_y) = 0$ is said to be ------------------	linear	non linear	quasi-linear	non quasi linear	linear
Which one of the following is the homogeneous equation?	xy + x = e	ax + by = 0	nx + my = sinx	yx = y +3x	ax + by = 0
The existence of the mathematical problem have ------------------ solution.	one most one	more than two	atleast one	more than one	atleast one
L + M = M + L & LM = ML are called ------------------ property.	associative	commutative	distributive	closed	commutative
Which of an operator does not satisfies the linear condition ------------------	L(cu) = c L(u)	$L(u_1 + u_2) = L(u_1) + L(u_2)$	$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$	L(c1u1 + c2u2) ≠ c1L(u1) + c2L(u2)	L(c1u1 + c2u2) ≠ c1L(u1) + c2L(u2)
(L + M) + N = L + (M + N) is called ------------------ property.	associative	distriutive	commutative	closed	associative
equation is an ordinary differential equation that is not -----------	linear	non linear	differential	integral	linear
Linear ordinary differential equations are further classified according to the nature of the coefficients of the ------------------ variables and its derivatives	single	dependent	independent	constant	dependent

The order of ------------------ derivatives involved in the differential equations is called order of the differential equation	zero	lowest	highest	infinite	highest
The equation f(x, y, z, a, b) = 0 containing two arbitrary parameters is called --------------- of an equation.	linear	non linear	complete solution	partial solution	complete solution
A solution which is not everywhere differentiable is called a ------------------ solution.	strong	weak	low	high	weak
A curve in (x, y, u)-space, whose tangent at every point coincides with the characteristic direction field (a, b, c), is called a ------------------.	tangent curve	normal curve	characteristic curve	uniform curve	characteristic curve
The characteristic direction is also called as ------------------	monge axis	monge curve	monge line	monge line	monge axis
In the equation $u(x_0(t), y_0(t)) = u_0(t)$, $u_0(t)$ is called the -----------------	final data	initial data	curve value	null value	initial data
The equation f(x, y, z, a, b) = ________ containing two arbitrary parameters is called linear of an equation.	0	1	2	3	**0**
The characteristic direction is also called as monge-----------	axis	line	curve value	curve	axis
Linear ordinary differential equations are further classified according to the nature of the coefficients of the dependent------------ and its derivatives	constant	variable	derivatives	equations	variable
The equation f(x, y, z, a, b) = 0 containing ______ arbitrary parameters is called linear of an equation.	one	two	three	fourth	two
The equation f(x, y, z, a, b) = 0 containing two arbitrary parameters is called linear of an ________.	line	equation	point	graph	equation
_____ value prolem is also called cauchy problem	boundary value problem	initial	all the above	none of these	initial

The order of highest derivatives involved in the differential equations is called ______ of the differential equation	order	variable	constant	linear	order
The order of highest derivatives involved in the _______ equations is called order of the differential equation	differential	linear	non linear	quasi linear	differential

UNIT II
SYLLABUS

Method of Separation of Variables for solving first order partial differential equations.Derivation of Heat equation -Wave equation and Laplace equation. - Classification of second order - linear equations as hyperbolic, parabolic or elliptic.

Method of Separation of Variables

Example 2.7.1. Solve the initial-value problem

$$u_x + 2u_y = 0, \qquad u(0, y) = 4e^{-2y}. \qquad (2.7.1\text{ab})$$

We seek a separable solution $u(x, y) = X(x)Y(y) \neq 0$ and substitute into the equation to obtain

$$X'(x)Y(y) + 2X(x)Y'(y) = 0.$$

This can also be expressed in the form

$$\frac{X'(x)}{2X(x)} = -\frac{Y'(y)}{Y(y)}. \qquad (2.7.2)$$

Since the left-hand side of this equation is a function of x only and the right-hand is a function of y only, it follows that (2.7.2) can be true if both sides are equal to the same constant value λ which is called an arbitrary separation constant. Consequently, (2.7.2) gives two ordinary differential equations

$$X'(x) - 2\lambda X(x) = 0, \qquad Y'(y) + \lambda Y(y) = 0. \qquad (2.7.3)$$

These equations have solutions given, respectively, by

$$X(x) = Ae^{2\lambda x} \quad \text{and} \quad Y(y) = Be^{-\lambda y}, \qquad (2.7.4)$$

where A and B are arbitrary integrating constants.

Consequently, the general solution is given by

$$u(x,y) = AB\exp(2\lambda x - \lambda y) = C\exp(2\lambda x - \lambda y), \qquad (2.7.5)$$

where $C = AB$ is an arbitrary constant.

Using the condition (2.7.1b), we find

$$4e^{-2y} = u(0,y) = Ce^{-\lambda y},$$

and hence, we deduce that $C = 4$ and $\lambda = 2$. Therefore, the final solution is

$$u(x,y) = 4\exp(4x - 2y). \qquad (2.7.6)$$

Classical Equations

The three basic types of second-order partial differential equations are:
(a) The wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0. \qquad (3.1.1)$$

(b) The heat equation

$$u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0. \qquad (3.1.2)$$

(c) The Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0. \qquad (3.1.3)$$

Classification of Second-Order Linear Equations

(A) Hyperbolic Type

If $B^2 - 4AC > 0$, then integration of equations (4.2.5) and (4.2.6) yield two real and distinct families of characteristics. Equation (4.1.11) reduces to

$$u_{\xi\eta} = H_1, \qquad (4.2.7)$$

where $H_1 = H^*/B^*$. It can be easily shown that $B^* \neq 0$. This form is called the *first canonical form of the hyperbolic equation.*

Now if new independent variables

$$\alpha = \xi + \eta, \qquad \beta = \xi - \eta, \tag{4.2.8}$$

are introduced, then equation (4.2.7) is transformed into

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2\left(\alpha, \beta, u, u_\alpha, u_\beta\right). \tag{4.2.9}$$

This form is called the *second canonical form of the hyperbolic equation.*

(B) Parabolic Type

In this case, we have $B^2 - 4AC = 0$, and equations (4.2.5) and (4.2.6) coincide. Thus, there exists one real family of characteristics, and we obtain only a single integral $\xi =$ constant (or $\eta =$ constant).

Since $B^2 = 4AC$ and $A^* = 0$, we find that

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = \left(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y\right)^2 = 0.$$

From this it follows that

$$\begin{aligned} A^* &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\ &= 2\left(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y\right)\left(\sqrt{A}\,\eta_x + \sqrt{C}\,\eta_y\right) = 0, \end{aligned}$$

for arbitrary values of $\eta(x, y)$ which is functionally independent of $\xi(x, y)$; for instance, if $\eta = y$, the Jacobian does not vanish in the domain of parabolicity.

Division of equation (4.1.11) by C^* yields

$$u_{\eta\eta} = H_3\left(\xi, \eta, u, u_\xi, u_\eta\right), \qquad C^* \neq 0. \tag{4.2.10}$$

This is called the *canonical form of the parabolic equation.*

Equation (4.1.11) may also assume the form

$$u_{\xi\xi} = H_3^*\left(\xi, \eta, u, u_\xi, u_\eta\right), \tag{4.2.11}$$

if we choose $\eta =$ constant as the integral of equation (4.2.5).

(C) Elliptic Type

For an equation of elliptic type, we have $B^2 - 4AC < 0$. Consequently, the quadratic equation (4.2.4) has no real solutions, but it has two complex conjugate solutions which are continuous complex-valued functions of the real variables x and y. Thus, in this case, there are no real characteristic curves. However, if the coefficients A, B, and C are analytic functions of x and y, then one can consider equation (4.2.4) for complex x and y. A function of two real variables x and y is said to be analytic in a certain domain if in some neighborhood of every point (x_0, y_0) of this domain, the function can be represented as a Taylor series in the variables $(x - x_0)$ and $(y - y_0)$.

Since ξ and η are complex, we introduce new real variables

$$\alpha = \frac{1}{2}(\xi + \eta), \qquad \beta = \frac{1}{2i}(\xi - \eta), \tag{4.2.12}$$

so that

$$\xi = \alpha + i\beta, \qquad \eta = \alpha - i\beta. \tag{4.2.13}$$

First, we transform equations (4.1.10). We then have

$$A^{**}(\alpha, \beta)\, u_{\alpha\alpha} + B^{**}(\alpha, \beta)\, u_{\alpha\beta} + C^{**}(\alpha, \beta)\, u_{\beta\beta} = H_4(\alpha, \beta, u, u_\alpha, u_\beta), \tag{4.2.14}$$

in which the coefficients assume the same form as the coefficients in equation (4.1.11). With the use of (4.2.13), the equations $A^* = C^* = 0$ become

$$\begin{aligned}\left(A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2\right) - \left(A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2\right)\\ +i\left[2A\alpha_x\beta_x + B\left(\alpha_x\beta_y + \alpha_y\beta_x\right) + 2C\alpha_y\beta_y\right] = 0,\end{aligned}$$

$$\left(A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2\right) - \left(A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2\right)$$
$$-i\left[2A\alpha_x\beta_x + B\left(\alpha_x\beta_y + \alpha_y\beta_x\right) + 2C\alpha_y\beta_y\right] = 0,$$

or,

$$\left(A^{**} - C^{**}\right) + iB^{**} = 0, \qquad \left(A^{**} - C^{**}\right) - iB^{**} = 0.$$

These equations are satisfied if and only if

$$A^{**} = C^{**} \quad \text{and} \quad B^{**} = 0.$$

Hence, equation (4.2.14) transforms into the form

$$A^{**}u_{\alpha\alpha} + A^{**}u_{\beta\beta} = H_4\left(\alpha, \beta, u, u_\alpha, u_\beta\right).$$

Dividing through by A^{**}, we obtain

$$u_{\alpha\alpha} + u_{\beta\beta} = H_5\left(\alpha, \beta, u, u_\alpha, u_\beta\right), \qquad (4.2.15)$$

where $H_5 = (H_4/A^{**})$. This is called the *canonical form of the elliptic equation.*

Example 4.2.1. Consider the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

Here

$$A = y^2, \quad B = 0, \quad C = -x^2.$$

Thus,

$$B^2 - 4AC = 4x^2y^2 > 0.$$

The equation is hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$. From the characteristic equations (4.2.5) and (4.2.6), we have

$$\frac{dy}{dx} = \frac{x}{y}, \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

After integration of these equations, we obtain

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1, \qquad \frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2.$$

The first of these curves is a family of hyperbolas

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1,$$

and the second is a family of circles

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2.$$

To transform the given equation to canonical form, we consider

$$\xi = \frac{1}{2}y^2 - \frac{1}{2}x^2, \qquad \eta = \frac{1}{2}y^2 + \frac{1}{2}x^2.$$

From the relations (4.1.6), we have

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = -x u_\xi + x u_\eta, \\
u_y &= u_\xi \xi_y + u_\eta \eta_y = y u_\xi + y u_\eta, \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
&= x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta.
\end{aligned}$$

$$\begin{aligned} u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy} \\ &= y^2u_{\xi\xi} + 2y^2u_{\xi\eta} + y^2u_{\eta\eta} + u_\xi + u_\eta. \end{aligned}$$

Thus, the given equation assumes the canonical form

$$u_{\xi\eta} = \frac{\eta}{2\left(\xi^2 - \eta^2\right)}\,u_\xi - \frac{\xi}{2\left(\xi^2 - \eta^2\right)}\,u_\eta.$$

Example 4.2.2. Consider the partial differential equation

$$x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = 0.$$

In this case, the discriminant is

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0.$$

The equation is therefore parabolic everywhere. The characteristic equation is

$$\frac{dy}{dx} = \frac{y}{x},$$

and hence, the characteristics are

$$\frac{y}{x} = c,$$

which is the equation of a family of straight lines.

Consider the transformation

$$\xi = \frac{y}{x}, \quad \eta = y,$$

where η is chosen arbitrarily. The given equation is then reduced to the canonical form

$$y^2u_{\eta\eta} = 0.$$

Thus,

$$u_{\eta\eta} = 0 \quad \text{for} \quad y \neq 0.$$

Example 4.2.3. The equation

$$u_{xx} + x^2 u_{yy} = 0,$$

is elliptic everywhere except on the coordinate axis $x = 0$ because

$$B^2 - 4AC = -4x^2 < 0, \quad x \neq 0.$$

The characteristic equations are

$$\frac{dy}{dx} = ix, \qquad \frac{dy}{dx} = -ix.$$

Integration yields

$$2y - ix^2 = c_1, \qquad 2y + ix^2 = c_2.$$

Thus, if we write

$$\xi = 2y - ix^2, \qquad \eta = 2y + ix^2,$$

and hence,

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y, \qquad \beta = \frac{1}{2i}(\xi - \eta) = -x^2,$$

we obtain the canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta} u_\beta.$$

It should be remarked here that a given partial differential equation may be of a different type in a different domain. Thus, for example, *Tricomi's equation*

$$u_{xx} + x u_{yy} = 0, \tag{4.2.16}$$

is elliptic for $x > 0$ and hyperbolic for $x < 0$, since $B^2 - 4AC = -4x$. For a detailed treatment, see Hellwig (1964).

UNIT 2					
QUESTION	**OPTION 1**	**OPTION 2**	**OPTION 3**	**OPTION 4**	**ANSWER**
The method of separation of variables in perhaps the worldest ____ method for soving PDE	systematic	transformation	wave equation	heat equation	systamatic
The required solution of the PDE is then expossed as a product_____ not equal to 0	u(x,t)=X(x)+T(t)	u(x,t)=X(x).T(t)	X(x).Y(y)	u(x,t)=Y(x).Y(y)	u(x,t)=X(x).T(t)
Complementry function is	Ae^m1x+Be^m2x	u(x,t)=Y(x).Y(y)	Ae^m1x	Be^m2x	Ae^m1x+Be^m2x
____ law the tension is constant	translation	Hooke's law	Newton's law	Faraday's law	Hooke's law
The tension in the string is in the direction of the tangent to the ______	existing profile	elongation	tension	center	existing profile
Classification of second order PDE is said to be hyperbolic . Then the equation is ______	negative	equal	equal to zero	positive	positive
If alpha and beta are the angles made by the _____ a and b respectively.	perpendicular	parallel	tangents	center	tangents
The discriminent B^2-4AC=4y^2 x^2 is	parabolic	eliptic	hyperbolic	all the aove	hyperbolic
The string is _____ and elastic	flexible	segment	elongation	none of these	flexible
The complete solution of PDE is u(x,y)=X(x)Y(y) (i.e)u=________	xy	x+y	x/y	y/x	xy
A solution to the partial differential equation (∂^2 u)/(∂x^2)=9(∂^2 u)/(∂y^2) is -------------------	cos(3x – y)	$x^2 + y^2$	sin(3x – y)	$e^{-3\pi x}\sin(\pi y)$	$e^{-3\pi x}\sin(\pi y)$
The partial differential equation 5 (∂^2 z)/(∂x^2)+6(∂^2 z)/(∂y^2)=xy is classified as	elliptic	hyperbolic	parabolic	circle	elliptic
The partial differential equation xy ∂z/∂x=5(∂^2 z)/(∂y^2) is classified as	elliptic	hyperbolic	parabolic	sphere	paraolic
The partial differential equation (∂^2 z)/(∂x^2)-5 (∂^2 z)/(∂y^2)=0 is classified as	elliptic	hyperbolic	parabolic	sphere	hyperbolic
A partial differential equation is one which involves __________ derivatives	single	ordinary	partial	linear	partial
The _______ of PDE satisfies for all values of n	unity	existence	solution	formal	solution
Z=X(x)Y(y) is called ________ of variables	integration	seperation	differention	indution	seperation

The separation principle can readily be extended to ___ number of variables	smaller	unique	larger	contrary	larger
The solution of PDE satisfies for all values of ______	n	1	2	3	n
If f(x,p) = g(y,q) is called _______equation	Clairaut	Charpit	Crout	separable	separable
L(z)+f(x,y,z,p,q)=0 where L is the ________ operator	laplace	differential	lagrange	longdivision	differential
Z=X(x)Y(y)T(z) is the extension of ______________ variables	integration	differentiation	induction	separation	seperation
The use of the theory of integral transforms is the _______ of PDE	unity	existance	solution	formal	solution
The method of separation of variables applied to diffusion equation is similar to_______theory	potential	grad	calculus	electrostatic	potential
The order of PDE to be the order of the derivative of ______ order occurring in it.	lowest	highest	first	second	highest
Z=X(x)Y(y) is separable in the variables_________	x&y	x&z	y&z	x+y	x&y
In the method of integral transforms L denotes ________operator.	nonlinear	linear	constant	variable	linear
The separation principle can readily be extended to larger number of ________.	constant	variable	coefficient	sequence	variable
The variational approach to ____ value problem is useful in the derivation of approximating solution	Euclid	kernal	eigen	node	eigen
The use of the theory of integral transforms is the solution of ______________	ode	pde	C.I	P.I	pde
The use of the _______ of integral transforms is the solution of pde	theory	integral	differentiation	ode	theory
The use of the theory of integral _______is the solution of pde	transforms	integral	differentiation	ode	transforms
The use of the theory of integral transforms is the ______ of pde	solution	integral	differentiation	ode	solution
The use of the theory of ______ transforms is the solution of pde	integral	transforms	differentiation	ode	integral

The ______ principle can readily be extended to larger number of variable	larger	variable	principle	separation	separation
The separation principle can readily be extended to _____ number of variable	larger	variable	principle	separation	larger
The separation principle can readily be _____ to larger number of variable	extended	variable	principle	separation	extended
The separation principle can readily be extended to larger ____ of variable	number	variable	principle	separation	number
The separation _____ can readily be extended to larger number of variable	larger	variable	principle	separation	principle
In the method of _____ transforms L denotes linear operator.	nonlinear	integral	constant	variable	integral

UNIT III
SYLLABUS

Reduction of second order Linear Equations to canonical forms- The Cauchy problem- The Cauchy-Kowaleewskaya theorem -Cauchy problem of an infinite string - Initial Boundary Value Problems -Semi-Infinite String with a fixed end -Semi-Infinite String with a Free end- Equations with non-homogeneous boundary conditions -Non- Homogeneous Wave Equation.

Reduction of second order Linear Equations to canonical forms:

The Cauchy Problem

In the theory of ordinary differential equations, by the initial-value problem we mean the problem of finding the solutions of a given differential equation with the appropriate number of initial conditions prescribed at an initial point. For example, the second-order ordinary differential equation

$$\frac{d^2u}{dt^2} = f\left(t, u, \frac{du}{dt}\right)$$

and the initial conditions

$$u(t_0) = \alpha, \qquad \left(\frac{du}{dt}\right)(t_0) = \beta,$$

constitute an initial-value problem.

An analogous problem can be defined in the case of partial differential equations. Here we shall state the problem involving second-order partial differential equations in two independent variables.

We consider a second-order partial differential equation for the function u in the independent variables x and y, and suppose that this equation can be solved explicitly for u_{yy}, and hence, can be represented in the from

$$u_{yy} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}). \tag{5.1.1}$$

For some value $y = y_0$, we prescribe the initial values of the unknown function and of the derivative with respect to y

$$u(x, y_0) = f(x), \qquad u_y(x, y_0) = g(x). \tag{5.1.2}$$

The problem of determining the solution of equation (5.1.1) satisfying the initial conditions (5.1.2) is known as the *initial-value problem*. For instance, the initial-value problem of a vibrating string is the problem of finding the solution of the wave equation

$$u_{tt} = c^2 u_{xx},$$

satisfying the initial conditions

$$u(x, t_0) = u_0(x), \qquad u_t(x, t_0) = v_0(x),$$

where $u_0(x)$ is the initial displacement and $v_0(x)$ is the initial velocity.

In initial-value problems, the initial values usually refer to the data assigned at $y = y_0$. It is not essential that these values be given along the line $y = y_0$; they may very well be prescribed along some curve L_0 in the xy plane. In such a context, the problem is called the *Cauchy problem* instead of the initial-value problem, although the two names are actually synonymous.

We consider the Euler equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y), \tag{5.1.3}$$

where A, B, C are functions of x and y. Let (x_0, y_0) denote points on a smooth curve L_0 in the xy plane. Also let the parametric equations of this curve L_0 be where A, B, C are functions of x and y. Let (x_0, y_0) denote points on a smooth curve L_0 in the xy plane. Also let the parametric equations of this curve L_0 be

$$x_0 = x_0(\lambda), \qquad y_0 = y_0(\lambda), \tag{5.1.4}$$

where λ is a parameter.

We suppose that two functions $f(\lambda)$ and $g(\lambda)$ are prescribed along the curve L_0. The Cauchy problem is now one of determining the solution $u(x, y)$ of equation (5.1.3) in the neighborhood of the curve L_0 satisfying the Cauchy conditions

$$u = f(\lambda), \tag{5.1.5a}$$

$$\frac{\partial u}{\partial n} = g(\lambda), \tag{5.1.5b}$$

on the curve L_0 where n is the direction of the normal to L_0 which lies to the left of L_0 in the counterclockwise direction of increasing arc length. The function $f(\lambda)$ and $g(\lambda)$ are called the *Cauchy data.*

For every point on L_0, the value of u is specified by equation (5.1.5a). Thus, the curve L_0 represented by equation (5.1.4) with the condition (5.1.5a) yields a twisted curve L in (x, y, u) space whose projection on the xy plane is the curve L_0. Thus, the solution of the Cauchy problem is a surface, called an *integral surface*, in the (x, y, u) space passing through L and satisfying the condition (5.1.5b), which represents a tangent plane to the integral surface along L.

If the function $f(\lambda)$ is differentiable, then along the curve L_0, we have

$$\frac{du}{d\lambda} = \frac{\partial u}{\partial x}\frac{dx}{d\lambda} + \frac{\partial u}{\partial y}\frac{dy}{d\lambda} = \frac{df}{d\lambda}, \tag{5.1.6}$$

and

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\frac{dx}{dn} + \frac{\partial u}{\partial y}\frac{dy}{dn} = g, \tag{5.1.7}$$

but

$$\frac{dx}{dn} = -\frac{dy}{ds} \qquad \text{and} \qquad \frac{dy}{dn} = \frac{dx}{ds}. \tag{5.1.8}$$

Equation (5.1.7) may be written as

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds} = g. \tag{5.1.9}$$

Since

$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ \frac{-dy}{ds} & \frac{dx}{ds} \end{vmatrix} = \frac{(dx)^2 + (dy)^2}{ds\,d\lambda} \neq 0, \tag{5.1.10}$$

it is possible to find u_x and u_y on L_0 from the system of equations (5.1.6) and (5.1.9). Since u_x and u_y are known on L_0, we find the higher derivatives by first differentiating u_x and u_y with respect to λ. Thus, we have

$$\frac{\partial^2 u}{\partial x^2}\frac{dx}{d\lambda} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial x}\right), \tag{5.1.11}$$

$$\frac{\partial^2 u}{\partial x\,\partial y}\frac{dx}{d\lambda} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial y}\right). \tag{5.1.12}$$

From equation (5.1.3), we have

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = F, \tag{5.1.13}$$

where F is known since u_x and u_y have been found. The system of equations can be solved for u_{xx}, u_{xy}, and u_{yy}, if

$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} & 0 \\ 0 & \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ A & B & C \end{vmatrix} = C\left(\frac{dx}{d\lambda}\right)^2 - B\left(\frac{dx}{d\lambda}\right)\left(\frac{dy}{d\lambda}\right) + A\left(\frac{dy}{d\lambda}\right)^2 \neq 0. \tag{5.1.14}$$

The equation

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0, \tag{5.1.15}$$

is called the *characteristic equation*. It is then evident that the necessary condition for obtaining the second derivatives is that the curve L_0 must not be a characteristic curve.

If the coefficients of equation (5.1.3) and the function (5.1.5) are analytic, then all the derivatives of higher orders can be computed by the above process. The solution can then be represented in the form of a Taylor series:

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!\,(n-k)!}\,\frac{\partial^n u_0}{\partial x_0^k\,\partial y_0^{n-k}}\,(x-x_0)^k\,(y-y_0)^{n-k}\,, \quad (5.1.16)$$

which can be shown to converge in the neighborhood of the curve L_0. Thus, we may state the famous Cauchy–Kowalewskaya theorem.

The Cauchy–Kowalewskaya Theorem

Let the partial differential equation be given in the form

$$u_{yy} = F\left(y, x_1, x_2, \ldots, x_n, u, u_y, u_{x_1}, u_{x_2}\ldots, u_{x_n},\; u_{x_1y}, u_{x_2y}, \ldots, u_{x_ny}, u_{x_1x_1}, u_{x_2x_2}, \ldots, u_{x_nx_n}\right), \quad (5.2.1)$$

and let the initial conditions

$$u = f(x_1, x_2, \ldots, x_n)\,, \quad (5.2.2)$$

$$u_y = g(x_1, x_2, \ldots, x_n)\,, \quad (5.2.3)$$

be given on the noncharacteristic manifold $y = y_0$.

If the function F is analytic in some neighborhood of the point $\left(y^0, x_1^0, x_2^0, \ldots, x_n^0, u^0, u_y^0, \ldots\right)$ and if the functions f and g are analytic in some neighborhood of the point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$, then the Cauchy problem has a unique analytic solution in some neighborhood of the point $\left(y^0, x_1^0, x_2^0, \ldots, x_n^0\right)$.

For the proof, see Petrovsky (1954).

The preceding statement seems equally applicable to hyperbolic, parabolic, or elliptic equations. However, we shall see that difficulties arise in formulating the Cauchy problem for nonhyperbolic equations. Consider, for instance, the famous Hadamard (1952) example.

The problem consists of the elliptic (or Laplace) equation

$$u_{xx} + u_{yy} = 0,$$

and the initial conditions on $y = 0$

$$u(x,0) = 0, \qquad u_y(x,0) = n^{-1} \sin nx.$$

The solution of this problem is

$$u(x,y) = n^{-2} \sinh ny \sin nx,$$

which can be easily verified.

It can be seen that, when n tends to infinity, the function $n^{-1} \sin nx$ tends uniformly to zero. But the solution $n^{-2} \sinh ny \sin nx$ does not become small, as n increases for any nonzero y. Physically, the solution represents an oscillation with unbounded amplitude $(n^{-2} \sinh ny)$ as $y \to \infty$ for any fixed x. Even if n is a fixed number, this solution is unstable in the sense that $u \to \infty$ as $y \to \infty$ for any fixed x for which $\sin nx \neq 0$. It is obvious then that the solution does not depend continuously on the data. Thus, it is not a properly posed problem.

Initial Boundary-Value Problems

(A) Semi-infinite String with a Fixed End

Let us first consider a semi-infinite vibrating string with a fixed end, that is,

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \infty, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad 0 \leq x < \infty, \tag{5.4.1}$$

$$u_t(x,0) = g(x), \qquad 0 \le x < \infty,$$
$$u(0,t) = 0, \qquad 0 \le t < \infty.$$

It is evident here that the boundary condition at $x = 0$ produces a wave moving to the right with the velocity c. Thus, for $x > ct$, the solution is the same as that of the infinite string, and the displacement is influenced only by the initial data on the interval $[x - ct,\ x + ct]$, as shown in Figure 5.4.1.

When $x < ct$, the interval $[x - ct,\ x + ct]$ extends onto the negative x-axis where f and g are not prescribed.

But from the d'Alembert formula

$$u(x,t) = \phi(x + ct) + \psi(x - ct), \tag{5.4.2}$$

where

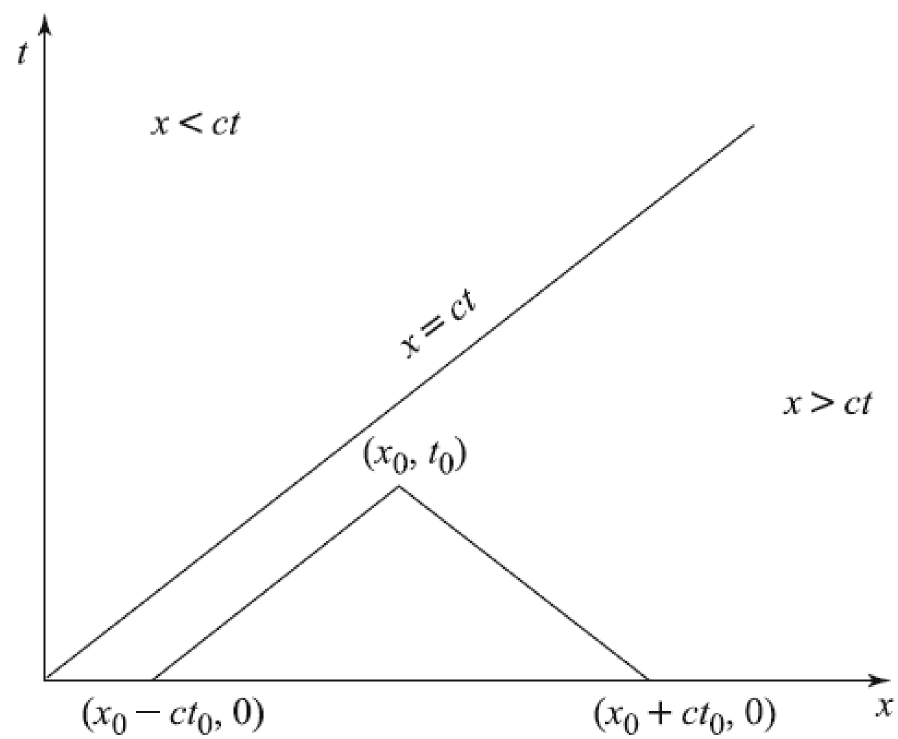

Figure 5.4.1 Displacement influenced by the initial data on $[x - ct,\ x + ct]$.

$$\phi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)\,d\tau + \frac{K}{2}, \tag{5.4.3}$$

$$\psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)\,d\tau - \frac{K}{2}, \tag{5.4.4}$$

we see that

$$u(0,t) = \phi(ct) + \psi(-ct) = 0.$$

Hence,

$$\psi(-ct) = -\phi(ct).$$

If we let $\alpha = -ct$, then

$$\psi(\alpha) = -\phi(-\alpha).$$

Replacing α by $x - ct$, we obtain for $x < ct$,

$$\psi(x-ct) = -\phi(ct-x),$$

and hence,

$$\psi(x-ct) = -\frac{1}{2}f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(\tau)\,d\tau - \frac{K}{2}.$$

The solution of the initial boundary-value problem, therefore, is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau \quad \text{for } x > ct, \tag{5.4.5}$$

$$u(x,t) = \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\tau)\,d\tau \quad \text{for } x < ct. \tag{5.4.6}$$

In order for this solution to exist, f must be twice continuously differentiable and g must be continuously differentiable, and in addition

$$f(0) = f''(0) = g(0) = 0.$$

Solution (5.4.6) has an interesting physical interpretation. If we draw the characteristics through the point (x_0, t_0) in the region $x > ct$, we see, as pointed out earlier, that the displacement at (x_0, t_0) is determined by the initial values on $[x_0 - ct_0, x_0 + ct_0]$.

Example 5.4.1. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= 4\,u_{xx}, \qquad x > 0, \qquad t > 0, \\ u(x,0) &= |\sin x|, \qquad x > 0, \\ u_t(x,0) &= 0, \qquad x \geq 0, \\ u(x,0) &= 0, \qquad t \geq 0. \end{aligned}$$

For $x > 2t$,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+2t) + f(x-2t)\right] \\ &= \frac{1}{2}\left[|\sin(x+2t)| - |\sin(x-2t)|\right], \end{aligned}$$

and for $x < 2t$,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+2t) - f(2t-x)\right] \\ &= \frac{1}{2}\left[|\sin(x+2t)| - |\sin(2t-x)|\right]. \end{aligned}$$

Notice that $u(0,t) = 0$ is satisfied by $u(x,t)$ for $x < 2t$ (that is, $t > 0$).

(B) Semi-infinite String with a Free End

We consider a semi-infinite string with a free end at $x = 0$. We will determine the solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= f(x), \qquad 0 \leq x < \infty, \\ u_t(x,0) &= g(x), \qquad 0 \leq x < \infty, \\ u_x(0,t) &= 0, \qquad 0 \leq t < \infty. \end{aligned} \tag{5.4.7}$$

As in the case of the fixed end, for $x > ct$ the solution is the same as that of the infinite string. For $x < ct$, from the d'Alembert solution (5.4.2)

$$u(x,t) = \phi(x+ct) + \psi(x-ct),$$

we have

$$u_x(x,t) = \phi'(x+ct) + \psi'(x-ct).$$

Thus,

$$u_x(0,t) = \phi'(ct) + \psi'(-ct) = 0.$$

Integration yields

$$\phi(ct) - \psi(-ct) = K,$$

where K is a constant. Now, if we let $\alpha = -ct$, we obtain

$$\psi(\alpha) = \phi(-\alpha) - K.$$

Replacing α by $x-ct$, we have

$$\psi(x-ct) = \phi(ct-x) - K,$$

and hence,

$$\psi(x-ct) = \frac{1}{2} f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(\tau)\,d\tau - \frac{K}{2}.$$

The solution of the initial boundary-value problem, therefore, is given by

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau \quad \text{for } x > ct. \quad (5.4.8)$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(ct-x)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)\,d\tau + \int_0^{ct-x} g(\tau)\,d\tau\right] \quad \text{for } x < ct. \quad (5.4.9)$$

We note that for this solution to exist, f must be twice continuously differentiable and g must be continuously differentiable, and in addition,

$$f'(0) = g'(0) = 0.$$

Example 5.4.2. Find the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= \cos\left(\frac{\pi x}{2}\right), \qquad 0 \le x < \infty, \\ u_t(x,0) &= 0, \qquad 0 \le x < \infty, \\ u_x(x,0) &= 0, \qquad t \ge 0. \end{aligned}$$

For $x > t$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\cos\frac{\pi}{2}(x+t) + \cos\frac{\pi}{2}(x-t)\right] \\ &= \cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}t\right), \end{aligned}$$

and for $x < t$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\cos\frac{\pi}{2}(x+t) + \cos\frac{\pi}{2}(t-x)\right] \\ &= \cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}t\right). \end{aligned}$$

Equations with Nonhomogeneous Boundary Conditions

In the case of the initial boundary-value problems with nonhomogeneous boundary conditions, such as

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x > 0, \qquad t > 0, \\ u(x,0) &= f(x), \qquad x \ge 0, \\ u_t(x,0) &= g(x), \qquad x \ge 0, \\ u(0,t) &= p(t), \qquad t \ge 0, \end{aligned} \tag{5.5.1}$$

we proceed in a manner similar to the case of homogeneous boundary conditions. Using equation (5.4.2), we apply the boundary condition to obtain

$$u(0,t) = \phi(ct) + \psi(-ct) = p(t).$$

If we let $\alpha = -ct$, we have

$$\psi(\alpha) = p\left(-\frac{\alpha}{c}\right) - \phi(-\alpha).$$

Replacing α by $x - ct$, the preceding relation becomes

$$\psi(x-ct) = p\left(t - \frac{x}{c}\right) - \phi(ct - x).$$

Thus, for $0 \le x < ct$,

$$\begin{aligned} u(x,t) &= p\left(t - \frac{x}{c}\right) + \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\tau)\,d\tau \\ &= p\left(t - \frac{x}{c}\right) + \phi(x+ct) - \psi(ct-x), \end{aligned} \tag{5.5.2}$$

where $\phi(x + ct = \xi)$ is given by (5.3.11), and $\psi(\eta)$ is given by

$$\psi(\eta) = \frac{1}{2}f(\eta) + \frac{1}{2c}\int_0^{\eta} g(\tau)\,d\tau. \tag{5.5.3}$$

The solution for $x > ct$ is given by the solution (5.4.5) of the infinite string.

In this case, in addition to the differentiability conditions satisfied by f and g, as in the case of the problem with the homogeneous boundary conditions, p must be twice continuously differentiable in t and

$$p(0) = f(0), \qquad p'(0) = g(0), \qquad p''(0) = c^2 f''(0).$$

We next consider the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && x > 0, \quad t > 0, \\ u(x,0) &= f(x), && x \ge 0, \\ u_t(x,0) &= g(x), && x \ge 0, \\ u_x(0,t) &= q(t), && t \ge 0. \end{aligned}$$

Using (5.4.2), we apply the boundary condition to obtain

$$u_x(0,t) = \phi'(ct) + \psi'(-ct) = q(t).$$

Then, integrating yields

$$\phi(ct) - \psi(-ct) = c\int_0^t q(\tau)\,d\tau + K.$$

If we let $\alpha = -ct$, then

$$\psi(\alpha) = \phi(-\alpha) - c\int_0^{-\alpha/c} q(\tau)\,d\tau - K.$$

Replacing α by $x - ct$, we obtain

$$\psi(x-ct) = \phi(ct-x) - c\int_0^{t-x/c} q(\tau)\,d\tau - K.$$

The solution of the initial boundary-value problem for $x < ct$, therefore, is given by

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(ct-x)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)\,d\tau + \int_0^{ct-x} g(\tau)\,d\tau\right] - c\int_0^{t-x/c} q(\tau)\,d\tau. \quad (5.5.4)$$

Here f and g must satisfy the differentiability conditions, as in the case of the problem with the homogeneous boundary conditions. In addition

$$f'(0) = q(0), \qquad g'(0) = q'(0).$$

The solution for the initial boundary-value problem involving the boundary condition

$$u_x(0,t) + h\,u(0,t) = 0, \qquad h = \text{constant}$$

can also be constructed in a similar manner from the d'Alembert solution.

Nonhomogeneous Wave Equations

We shall consider next the Cauchy problem for the nonhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + h^* (x, t), \tag{5.7.1}$$

with the initial conditions

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g^*(x). \tag{5.7.2}$$

By the coordinate transformation

$$y = ct, \tag{5.7.3}$$

the problem is reduced to

$$u_{xx} - u_{yy} = h(x, y), \tag{5.7.4}$$
$$u(x, 0) = f(x), \tag{5.7.5}$$
$$u_y(x, 0) = g(x), \tag{5.7.6}$$

where $h(x, y) = -h^*/c^2$ and $g(x) = g^*/c$.

Let $P_0(x_0, y_0)$ be a point of the plane, and let Q_0 be the point $(x_0, 0)$ on the initial line $y = 0$. Then the characteristics, $x \pm y = \text{constant}$, of equation (5.7.4) are two straight lines drawn through the point P_0 with slopes ± 1. Obviously, they intersect the x-axis at the points $P_1(x_0 - y_0, 0)$ and $P_2(x_0 + y_0, 0)$, as shown in Figure 5.7.1. Let the sides of the triangle $P_0P_1P_2$ be designated by B_0, B_1, and B_2, and let D be the region representing the interior of the triangle and its boundaries B. Integrating both sides of equation (5.7.4), we obtain

$$\iint_R (u_{xx} - u_{yy})\, dR = \iint_R h(x, y)\, dR. \tag{5.7.7}$$

Now we apply Green's theorem to obtain

UNIT-3					
QUESTIONS	**OPTION 1**	**OPTION2**	**OPTION3**	**OPTION4**	**ANSWER**
The second order partial differential equation is said to be ______ if $B^2-4AC>0$	parabolic	hyperbolic	elliptic	none	hyperbolic
The second order partial differential equation is said to be ______ if $B^2-4AC=0$	parabolic	hyperbolic	elliptic	none	parabplic
The second order partial differential equation is said to be ______ if $B^2-4AC<0$	parabolic	hyperbolic	elliptic	none	elliptic
The second order partial differential equation is said to be elliptic if $B^2-4AC=$____	>0	<0	0	>=0	<0
The second order partial differential equation is said to be paraolic if $B^2-4AC=$____	>0	<0	0	>=0	0
The second order partial differential equation is said to be hyperolic if $B^2-4AC=$____	>0	<0	0	>=0	>0
The second order linear partial differential equation, the coefficients are ______constants	real	imaginary	known	unknown	real
The second order linear partial differential equation, the ____ are real constants	coefficients	constants	numbers	operators	coefficients
The _____ order linear partial differential equation, the coefficients are real constants	first	second	third	fourth	second
The second order linear partial differential equation, the coefficients are real _____	coefficients	constants	numbers	operators	constants
The simplex form by making a change in the independent variable in second order linear partial differential equation is called______	canonical	parabola	hyperbola	elliptic	canonical
The simplex form by making a _______ in the independent variable in second order linear partial differential equation is called canonical	change	parabola	hyperbola	elliptic	change
The _____ form by making a change in the independent variable in second order linear partial differential equation is called canonical	simplex	parabola	hyperbola	elliptic	simplex

The simplex form by making a change in the ______ variable in second order linear partial differential equation is called canonical	independent	parabola	hyperbola	elliptic	independent
The simplex form by making a change in the independent variable in second order linear _______ differential equation is called canonical	partial	parabola	hyperbola	elliptic	partial
The simplex form by making a change in the independent variable in ____ order linear partial differential equation is called canonical	second	first	second	third	second
The set of five functions is called _____	strip	five	functions	set	strip
The set of _____ functions is called strip	five	strip	functions	set	five
The set of five _____ is called strip	functions	five	strip	set	functions
The set of five functions is called strip	set	five	functions	strip	set
The infinite sector --------------- is called the range of influence of the point.	P	Q	R	S	R
The ---------------------- theorem can be applied with continuous data by using polynomial approximations only if a small variation in the initial data leads to a small change in the solution.	Cauchy Kowalewskaya	Cauchy	existence	uniqueness	Cauchy Kowalewskaya
The necessary condition for obtaining the second derivatives is that the curve ------ must not be a characteristic curve.	L1	R0	P0	L0	L0
The equation A (dy/dx)^2-B(dy/dx)+C=0 is called the --------------------	characteristic curve	characteristic equation	finite equation	finite curve	characteristic equation
The solution of the --------------------- is a surface called an integral surface in the (x, y, u) space passing through L.	Cauchy Kowalewskaya	Cauchy problem	existence	uniqueness	Cauchy problem
The function f(λ) and g(λ) in u = f(λ) and ∂u/∂n= g(λ) are called --------------data	initial	final	cauchy	complete	cauchy

The Cauchy problem involves second order partial differential equations in------ independent variables.	0	1	2	3	2
The Cauchy Kowalewskaya theorem can be applied with -------------- data by using polynomial approximations only if a small variation in the initial data leads to a small change in the solution.	discontinuous	continuous	infinite	finite	continuous
The necessary condition for obtaining the ------------ derivatives is that the curve L0 must not be a characteristic curve.	first	second	third	fourth	second
The equation ---------------------- is called the characteristic equation.	A (dy/dx)^2+B(dy/dx)+C=0	A (dy/dx)^2-B(dy/dx)-C=0	A (dy/dx)^2-B(dy/dx)=0	A (dy/dx)^2-B(dy/dx)+C=0	A (dy/dx)^2-B(dy/dx)+C=0
The solution of the Cauchy problem is a surface called an -------------- surface in the (x, y, u) space passing through L.	Complete	integral	bounded	unbounded	integral
The function ------------------- in u = f(λ) and ∂u/∂n= g(λ) are called cauchy data.	f(λ)	g(λ)	both f(λ) and g(λ)	neither f(λ) nor g(λ)	both f(λ) and g(λ)
Linear ordinary differential equations are further classified according to the nature of the coefficients of the -------------------- variables and its derivatives.	single	dependent	independent	constant	dependent
The standard form of first order differential equations derivative form is -------------------	(dy/dx)=f(x)	(dx/dy)=f(x,y)	(dy/dy)=f(x,y)	(dx/dy)=f(y)	(dy/dy)=f(x,y)
The general solution of ----------- equation is called the complementary function of equation.	single	homogeneous	nonsingular	nonhomogeneous	homogeneous
Polynomial ar2+br+c=0 is called --------------------	characteristic polynomial	trivial polynomial	determinant polynomial	singular polynomial	characteristic polynomial
A non linear ordinary differential equation is an ordinary differential equation that is not ------	differential	integral	linear	non linear	linear
General solution of higher order linear differential equation depends on -------------------	arbitrary constant	coefficient	type of roots	method to which solved	type of roots

An equation of the form ------------------- is called an equation with variables separable or simply a separable equations.	F(x)G(y) dx+f(x)g(y) dy=0	G(y) dx+ g(y) dy=0	F(x)G(y) dx- g(y) dy=0	F(x)G(y) dx=0	F(x)G(y) dx+f(x)g(y) dy=0
Any linear combination of solutions of the homogeneous linear differential equation is also a -------------------- of homogeneous equation.	value	separable	solution	exact	solution

UNIT IV
SYLLABUS

Method of separation of variables - Solving the Vibrating String - Problems- Solving the Heat Conduction problem - Systems of linear differential equations - Types of linear systems differential operators - an operator method for linear systems with constant coefficients.

METHOD OF SEPARATION OF VARIABLES

The Vibrating String

One of the most important problems in mathematical physics is the vibration of a stretched string. Simplicity and frequent occurrence in many branches of mathematical physics make it a classic example in the theory of partial differential equations.

Let us consider a stretched string of length l fixed at the end points. The problem here is to determine the equation of motion which characterizes the position $u(x,t)$ of the string at time t after an initial disturbance is given.

In order to obtain a simple equation, we make the following assumptions:

1. The string is flexible and elastic, that is the string cannot resist bending moment and thus the tension in the string is always in the direction of the tangent to the existing profile of the string.
2. There is no elongation of a single segment of the string and hence, by Hooke's law, the tension is constant.
3. The weight of the string is small compared with the tension in the string.
4. The deflection is small compared with the length of the string.
5. The slope of the displaced string at any point is small compared with unity.

6. There is only pure transverse vibration.

We consider a differential element of the string. Let T be the tension at the end points as shown in Figure 3.2.1. The forces acting on the element of the string in the vertical direction are

$$T \sin \beta - T \sin \alpha.$$

By Newton's second law of motion, the resultant force is equal to the mass times the acceleration. Hence,

$$T \sin \beta - T \sin \alpha = \rho \, \delta s \, u_{tt} \tag{3.2.1}$$

where ρ is the line density and δs is the smaller arc length of the string. Since the slope of the displaced string is small, we have

$$\delta s \simeq \delta x.$$

Since the angles α and β are small

$$\sin \alpha \simeq \tan \alpha, \qquad \sin \beta \simeq \tan \beta.$$

Thus, equation (3.2.1) becomes

$$\tan \beta - \tan \alpha = \frac{\rho \, \delta x}{T} u_{tt}. \tag{3.2.2}$$

But, from calculus we know that $\tan \alpha$ and $\tan \beta$ are the slopes of the string at x and $x + \delta x$:

$$\tan \alpha = u_x (x, t)$$

and

$$\tan \beta = u_x (x + \delta x, t)$$

at time t. Equation (3.2.2) may thus be written as

$$\frac{1}{\delta x} \left[(u_x)_{x+\delta x} - (u_x)_x \right] = \frac{\rho}{T} u_{tt}, \quad \frac{1}{\delta x} \left[u_x (x + \delta x, t) - u_x (x, t) \right] = \frac{\rho}{T} u_{tt}.$$

In the limit as δx approaches zero, we find

$$u_{tt} = c^2 u_{xx} \tag{3.2.3}$$

where $c^2 = T/\rho$. This is called the *one-dimensional wave equation*.

If there is an external force f per unit length acting on the string. Equation (3.2.3) assumes the form

$$u_{tt} = c^2 u_{xx} + F, \quad F = f/\rho, \tag{3.2.4}$$

where f may be pressure, gravitation, resistance, and so on.

Conduction of Heat in Solids

We consider a domain D^* bounded by a closed surface B^*. Let $u(x, y, z, t)$ be the temperature at a point (x, y, z) at time t. If the temperature is not constant, heat flows from places of higher temperature to places of lower temperature. Fourier's law states that the rate of flow is proportional to the gradient of the temperature. Thus the velocity of the heat flow in an isotropic body is

$$\mathrm{v} = -K \operatorname{grad} u, \tag{3.5.1}$$

where K is a constant, called the *thermal conductivity* of the body.

Let D be an arbitrary domain bounded by a closed surface B in D^*. Then the amount of heat leaving D per unit time is

$$\iint_B v_n ds,$$

where $v_n = \mathbf{v} \cdot \mathbf{n}$ is the component of $\mathbf{v}$ in the direction of the outer unit normal $\mathbf{n}$ of B. Thus, by Gauss' theorem (Divergence theorem)

$$\iint_B v_n ds = \iiint_D \operatorname{div}(-K \operatorname{grad} u)\, dx\, dy\, dz$$
$$= -K \iiint_D \nabla^2 u\, dx\, dy\, dz. \qquad (3.5.2)$$

But the amount of heat in D is given by

$$\iiint_D \sigma \rho u\, dx\, dy\, dz, \qquad (3.5.3)$$

where ρ is the density of the material of the body and σ is its specific heat. Assuming that integration and differentiation are interchangeable, the rate of decrease of heat in D is

$$-\iiint_D \sigma \rho \frac{\partial u}{\partial t}\, dx\, dy\, dz. \qquad (3.5.4)$$

Since the rate of decrease of heat in D must be equal to the amount of heat leaving D per unit time, we have

$$-\iiint_D \sigma \rho u_t\, dx\, dy\, dz = -K \iiint_D \nabla^2 u\, dx\, dy\, dz,$$

for an arbitrary D in D^*. We assume that the integrand is continuous. If we suppose that the integrand is not zero at a point (x_0, y_0, z_0) in D, then, by continuity, the integrand is not zero in a small region surrounding the point (x_0, y_0, z_0). Continuing in this fashion we extend the region encompassing D. Hence the integral must be nonzero. This contradicts (3.5.5). Thus, the integrand is zero everywhere, that is,

$$u_t = \kappa \nabla^2 u, \qquad (3.5.6)$$

where $\kappa = K/\sigma\rho$. This is known as the *heat equation*.

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

DIFFERENTIAL OPERATORS AND AN OPERATOR METHOD

A. Types of Linear Systems

We start by introducing the various types of linear systems that we shall consider. The general linear system of two first-order differential equations in two unknown functions x and y is of the form

$$\begin{aligned} a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)x + a_4(t)y &= F_1(t), \\ b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)x + b_4(t)y &= F_2(t). \end{aligned} \qquad (7.1)$$

We shall be concerned with systems of this type that have constant coefficients. An example of such a system is

$$2\frac{dx}{dt} + 3\frac{dy}{dt} - 2x + y = t^2, \qquad \frac{dx}{dt} - 2\frac{dy}{dt} + 3x + 4y = e^t.$$

We shall say that a *solution* of system (7.1) is an ordered pair of real functions (f, g) such that $x = f(t)$, $y = g(t)$ simultaneously satisfy both equations of the system (7.1) on some real interval $a \le t \le b$.

The general linear system of three first-order differential equations in three unknown functions x, y, and z is of the form

$$\begin{aligned} a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)\frac{dz}{dt} + a_4(t)x + a_5(t)y + a_6(t)z &= F_1(t), \\ b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)\frac{dz}{dt} + b_4(t)x + b_5(t)y + b_6(t)z &= F_2(t), \\ c_1(t)\frac{dx}{dt} + c_2(t)\frac{dy}{dt} + c_3(t)\frac{dz}{dt} + c_4(t)x + c_5(t)y + c_6(t)z &= F_3(t). \end{aligned} \qquad (7.2)$$

As in the case of systems of the form (7.1), so also in this case we shall be concerned with systems that have constant coefficients. An example of such a system is

$$\frac{dx}{dt} + \frac{dy}{dt} - 2\frac{dz}{dt} + 2x - 3y + z = t,$$

$$\frac{dx}{dt} + \frac{dy}{dt} - 2\frac{dz}{dt} + 2x - 3y + z = t,$$

$$2\frac{dx}{dt} - \frac{dy}{dt} + 3\frac{dz}{dt} + x + 4y - 5z = \sin t,$$

$$\frac{dx}{dt} + 2\frac{dy}{dt} + \frac{dz}{dt} - 3x + 2y - z = \cos t.$$

We shall say that a solution of system (7.2) is an ordered triple of real functions (f, g, h) such that $x = f(t)$, $y = g(t)$, $z = h(t)$ simultaneously satisfy all three equations of the system (7.2) on some real interval $a \leq t \leq b$.

Systems of the form (7.1) and (7.2) contained only first derivatives, and we now consider the basic linear system involving higher derivatives. This is the general linear system of two second-order differential equations in two unknown fucntions x and y, and is a system of the form

$$\begin{aligned} a_1(t)\frac{d^2x}{dt^2} + a_2(t)\frac{d^2y}{dt^2} + a_3(t)\frac{dx}{dt} + a_4(t)\frac{dy}{dt} + a_5(t)x + a_6(t)y &= F_1(t), \\ b_1(t)\frac{d^2x}{dt^2} + b_2(t)\frac{d^2y}{dt^2} + b_3(t)\frac{dx}{dt} + b_4(t)\frac{dy}{dt} + b_5(t)x + b_6(t)y &= F_2(t). \end{aligned} \tag{7.3}$$

We shall be concerned with systems having constant coefficients in this case also, and an example is provided by

$$2\frac{d^2x}{dt^2} + 5\frac{d^2y}{dt^2} + 7\frac{dx}{dt} + 3\frac{dy}{dt} + 2y = 3t + 1,$$

$$3\frac{d^2x}{dt^2} + 2\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 4x + y = 0.$$

For given fixed positive integers m and n, we could proceed, in like manner, to exhibit other general linear systems of n mth-order differential equations in n unknown functions and give examples of each such type of system. Instead we proceed to introduce the standard type of linear system referred to in the introductory paragraph at the start of the chapter, and of which we shall make a more systematic study later. We introduce this standard type as a special case of the system (7.1) of two first-order differential equations in two unknowns functions x and y.

We consider the special type of linear system (7.1), which is of the form

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(t)x + a_{12}(t)y + F_1(t), \\ \frac{dy}{dt} &= a_{21}(t)x + a_{22}(t)y + F_2(t). \end{aligned} \tag{7.4}$$

This is the so-called *normal form* in the case of two linear differential equations in two unknown functions. The characteristic feature of such a system is apparent from the manner in which the derivatives appear in it. An example of such a system with variable coefficients is

$$\frac{dx}{dt} = t^2x + (t+1)y + t^3,$$

$$\frac{dy}{dt} = te^tx + t^3y - e^t,$$

while one with constant coefficients is

$$\frac{dx}{dt} = 5x + 7y + t^2,$$

$$\frac{dy}{dt} = 2x - 3y + 2t.$$

The normal form in the case of a linear system of three differential equations in three unknown functions x, y, and z is

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + a_{13}(t)z + F_1(t),$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + a_{23}(t)z + F_2(t),$$

$$\frac{dz}{dt} = a_{31}(t)x + a_{32}(t)y + a_{33}(t)z + F_3(t).$$

An example of such a system is the constant coefficient system

$$\frac{dx}{dt} = 3x + 2y + z + t,$$

$$\frac{dy}{dt} = 2x - 4y + 5z - t^2,$$

$$\frac{dz}{dt} = 4x + y - 3z + 2t + 1.$$

B. Differential Operators

In this section we shall present a symbolic operator method for solving linear systems with constant coefficients. This method depends upon the use of so-called *differential operators*, which we now introduce.

Let x be an n-times differentiable function of the independent variable t. We denote the operation of differentiation with respect to t by the symbol D and call D a differential operator. In terms of this differential operator the derivative dx/dt is denoted by Dx. That is,

$$Dx \equiv dx/dt.$$

In like manner, we denote the second derivative of x with respect to t by D^2x. Extending this, we denote the nth derivative of x with respect to t by D^nx. That is,

$$D^n x = \frac{d^n x}{dt^n} \qquad (n = 1, 2, \ldots).$$

Further extending this operator notation, we write

$$(D + c)x \quad \text{to denote} \quad \frac{dx}{dt} + cx$$

and

$$(aD^n + bD^m)x \quad \text{to denote} \quad a\frac{d^n x}{dt^n} + b\frac{d^m x}{dt^m},$$

where a, b, and c are constants.

In this notation the general linear differential expression with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$,

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x,$$

is written

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)x.$$

Observe carefully that the operators $D^n, D^{n-1}, \ldots, D$ in this expression do *not* represent quantities that are to be multiplied by the function x, but rather they indicate *operations* (differentiations) that are to be carried out upon this function. The expression

$$a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$$

by itself, where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants, is called a linear differential operator with constant coefficients.

Example 7.1

Consider the linear differential operator

$$3D^2 + 5D - 2.$$

If x is a twice differentiable function of t, then

$$(3D^2 + 5D - 2)x \quad \text{denotes} \quad 3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 2x.$$

For example, if $x = t^3$, we have

$$(3D^2 + 5D - 2)t^3 = 3\frac{d^2}{dt^2}(t^3) + 5\frac{d}{dt}(t^3) - 2(t^3) = 18t + 15t^2 - 2t^3.$$

We shall now discuss certain useful properties of the linear differential operator with constant coefficients. In order to facilitate our discussion, we shall let L denote this operator. That is,

$$L \equiv a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n,$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants. Now suppose that f_1 and f_2 are both n-times differentiable functions of t and c_1 and c_2 are constants. Then it can be shown that

$$L[c_1f_1 + c_2f_2] = c_1L[f_1] + c_2L[f_2].$$

For example, if the operator $L \equiv 3D^2 + 5D - 2$ is applied to $3t^2 + 2\sin t$, then

$$L[3t^2 + 2\sin t] = 3L[t^2] + 2L[\sin t]$$

or

$$(3D^2 + 5D - 2)(3t^2 + 2\sin t) = 3(3D^2 + 5D - 2)t^2 + 2(3D^2 + 5D - 2)\sin t.$$

Now let

$$L_1 \equiv a_0D^m + a_1D^{m-1} + \cdots + a_{m-1}D + a_m$$

and

$$L_2 \equiv b_0D^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_n$$

be two linear differential operators with constant coefficients $a_0, a_1, \ldots, a_{m-1}, a_m$, and $b_0, b_1, \ldots, b_{n-1}, b_n$, respectively. Let

$$L_1(r) \equiv a_0 r^m + a_1 r^{m-1} + \cdots + a_{m-1} r + a_m$$

and

$$L_2(r) \equiv b_0 r^n + b_1 r^{n-1} + \cdots + b_{n-1} r + b_n$$

be the two polynomials in the quantity r obtained from the operators L_1 and L_2, respectively, by formally replacing D by r, D^2 by $r^2, \ldots, D^k$ by r^k. Let us denote the product of the polynomials $L_1(r)$ and $L_2(r)$ by $L(r)$, that is,

$$L(r) = L_1(r)L_2(r).$$

Then, if f is a function possessing $n + m$ derivatives, it can be shown that

$$L_1 L_2 f = L_2 L_1 f = Lf, \tag{7.10}$$

where L is the operator obtained from the "product polynomial" $L(r)$ by formally replacing r by D, r^2 by $D^2, \ldots, r^{m+n}$ by D^{m+n}. Equation (7.10) indicates two important properties of linear differential operators with constant coefficients. First, it states the effect of first operating on f by L_2 and then operating on the resulting function by L_1 is the same as that which results from first operating on f by L_1 and then operating on this resulting function by L_2. Second, Equation (7.10) states that the effect of first operating on f by either L_1 or L_2 and then operating on the resulting function by the other is the same as that which results from operating on f by the "product operator" L.

C. An Operator Method for Linear Systems with Constant Coefficients

We now proceed to explain a symbolic operator method for solving linear systems with constant coefficients. We shall outline the procedure of this method on a strictly formal basis and shall make no attempt to justify it.

We consider a linear system of the form

$$\begin{aligned} L_1 x + L_2 y &= f_1(t), \\ L_3 x + L_4 y &= f_2(t), \end{aligned} \tag{7.11}$$

where L_1, L_2, L_3, and L_4 are linear differential operators with constant coefficients.

That is, L_1, L_2, L_3, and L_4 are operators of the forms

$$L_1 \equiv a_0 D^m + a_1 D^{m-1} + \cdots + a_{m-1} D + a_m,$$
$$L_2 \equiv b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n,$$
$$L_3 \equiv \alpha_0 D^p + \alpha_1 D^{p-1} + \cdots + \alpha_{p-1} D + \alpha_p,$$
$$L_4 \equiv \beta_0 D^q + \beta_1 D^{q-1} + \cdots + \beta_{q-1} D + \beta_q,$$

where the a's, b's, α's, and β's are constants.

A simple example of a system which may be expressed in the form (7.11) is provided by

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t,$$

$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2.$$

Introducing operator notation this system takes the form

$$(2D - 3)x - 2Dy = t,$$
$$(2D + 3)x + (2D + 8)y = 2.$$

This is clearly of the form (7.11), where $L_1 \equiv 2D - 3$, $L_2 \equiv -2D$, $L_3 \equiv 2D + 3$, and $L_4 \equiv 2D + 8$.

Returning now to the general system (7.11), we apply the operator L_4 to the first equation of (7.11) and the operator L_2 to the second equation of (7.11), obtaining

$$L_4 L_1 x + L_4 L_2 y = L_4 f_1,$$
$$L_2 L_3 x + L_2 L_4 y = L_2 f_2.$$

We now subtract the second of these equations from the first. Since $L_4 L_2 y = L_2 L_4 y$, we obtain

$$L_4 L_1 x - L_2 L_3 x = L_4 f_1 - L_2 f_2,$$

or

$$(L_1 L_4 - L_2 L_3)x = L_4 f_1 - L_2 f_2. \tag{7.12}$$

The expression $L_1L_4 - L_2L_3$ in the left member of this equation is itself a linear differential operator with constant coefficients. We assume that it is neither zero nor a nonzero constant and denote it by L_5. If we further assume that the functions f_1 and f_2 are such that the right member $L_4f_1 - L_2f_2$ of (7.12) exists, then this member is some function, say g_1, of t. Then Equation (7.12) may be written

$$L_5x = g_1. \tag{7.13}$$

Equation (7.13) is a linear differential equation with constant coefficients in the single dependent variable x. We thus observe that our procedure has eliminated the other dependent variable y. We now solve the differential equation (7.13) for x using the methods developed in Chapter 4. Suppose Equation (7.13) is of order N. Then the general solution of (7.13) is of the form

$$x = c_1u_1 + c_2u_2 + \cdots + c_Nu_N + U_1, \tag{7.14}$$

where $u_1, u_2, \ldots, u_N$ are N linearly independent solutions of the homogeneous linear equation $L_5x = 0$, $c_1, c_2, \ldots, c_N$ are arbitrary constants, and U_1 is a particular solution of $L_5x = g_1$.

We again return to the system (7.11) and this time apply the operators L_3 and L_1 to the first and second equations, respectively, of the system. We obtain

$$L_3L_1x + L_3L_2y = L_3f_1,$$
$$L_1L_3x + L_1L_4y = L_1f_2.$$

Subtracting the first of these from the second, we obtain

$$(L_1L_4 - L_2L_3)y = L_1f_2 - L_3f_1.$$

Assuming that f_1 and f_2 are such that the right member $L_1f_2 - L_3f_1$ of this equation exists, we may express it as some function, say g_2, of t. Then this equation may be written

$$L_5y = g_2, \tag{7.15}$$

where L_5 denotes the operator $L_1L_4 - L_2L_3$. Equation (7.15) is a linear differential equation with constant coefficients in the single dependent variable y. This time we have eliminated the dependent variable x. Solving the differential equation (7.15) for y, we obtain its general solution in the form

$$y = k_1u_1 + k_2u_2 + \cdots + k_Nu_N + U_2, \tag{7.16}$$

where $u_1, u_2, \ldots, u_N$ are the N linearly independent solutions of $L_5y = 0$ (or $L_5x = 0$) that already appear in (7.14), $k_1, k_2, \ldots, k_N$ are arbitrary constants, and U_2 is a particular solution of $L_5y = g_2$.

UNIT-4					
QUESTIONS	**OPTION 1**	**OPTION2**	**OPTION3**	**OPTION4**	**ANSWER**
A partial differential equation has ____ independent variables	one or more	two or more	more than one	none of these	two or more
The general linear system on the interval of	a<=t<=b	a<t<b	a>t>b	none of these	a<=t<=b
The normal form of linear system of differential equation is the function of	x,y,z	x	y	z	x,y,z
Which method is used for solving linear system with constant coefficient	symbolic operator method	constant method	coefficient method	none of these	symbolic operator method
A linear operator with	constant	constant coefficient	linear	non linear	constant coefficient
The method which depends upon the symbolic operator is called	linear operator	non linear operator	differential operator	none of these	differential operator
A equation involving of______ variables with respect to a one or more independent variable is called differential equation	one or more	one	two	three	one or more
The canonical form is hyperolic if	a=-c	a=0	a=c	a>0	a=-c
The canonical form is paraolic if	a=-c	a=0	a=c	a>0	a=0
The canonical form is elliptic if	a=-c	a=0	a=c	a>0	a=c
The required solution for the product will be in the form of	u(x,y)=X(x)Y(y)	u(x,y)=X(x)/Y(y)	u(x,y)=X(x)-Y(y)	u(x,y)=X(x)+Y(y)	u(x,y)=X(x)Y(y)
The function u(x,y,z,t) is used to represent the displacement at a time of a particle whose position at ____ is (x,y,z)	rest	motion	object	moving	rest
The equation for conduction tells us that the rate of _____in joules per second	melting	heat transfer	heat conduction	cooling	heat transfer
The heat equation is a consequence of ____ law	fourier law	cauchy law	eulers law	conservation law	fourier law
The heat equation is a consequence of fourier law of	constant	melting	heating	conduction	conduction
An ____ is a function where domain is a set of function	operator	method	linear operator	non linear operator	operator
The equation for conduction tells us that the rate of heat transfer in______	joules per second	kilowatts	meter	kilometer	joules per second
The equation of differentiation with respect to t is denoted by the symbol -----------.	A	B	C	D	D
The dependent variable ------------- is expressed in the separable form u(x, y) = X(x) Y(y).	u(y, z)	u(x', y')	u(x, y)	u(y', z')	u(x, y)

The derivative dx/dt is denoted by -------------.	Dy	Dx	Dy'	Dx'	Dx
D is called as ----------------- operator.	differential	logical	relational	boolean	differential
The second derivative of x with respect to t is denoted by _______	D^2y	D^2x	D^2y'	D^2x'	D^2x
The dependent variable u(x, y) is expressed in the separable form u(x, y) = -------------.	X(x) + Y(y)	X(x) - Y(y)	X(x) Y(y)	X(x) / Y(y)	X(x) Y(y)
The method of -------------- of variables is widely used in finding solutions of a large class of initial boundary value problems.	integration	separation	differentiation	induction	separation
In the ------------- functions ω_n represents the discrete spectrum of circular frequencies.	Euclid	Kernal	eigen	node	eigen
The nth derivative of x with respect to t is denoted by -----------------.	$D^n y$	Dnx	Dny'	Dnx'	Dnx
The method of ---------------- of variables is also known as the Fourier method or the method of eigen function expansion.	integration	separation	differentiation	induction	separation
The string displays --------------- loops separated by the nodes.	1	2	n-1	n	n
In the eigen functions γ_n=ω_n/2π=nc/2l represents the ---------------- frequencies.	angular	circular	rectangular	spherical	angular
The harmonic ------------- is called fundamental harmonic	n=0	n=1	n=2	n=3	n=1
The harmonics ------------ are called overtones	n < 1	n=1	n>1	n=0	n>1
The λ_n are called the ----------- values of the problem.	euclid	kernal	eigen	node	eigen
The T* represents ----------------- of a vibrating string.	constant tension	eigen	node	kernel	constant tension
In the method of integral transforms K is the _______ of the function	kennel	k(x,y)	kernel	k(z)	kernel
compatible then it have ________ solution	unique	different	linear	non linear	unique
The ______ derivative of x with respect to t is denoted by D^2x	second	first	third	fourth	second
The second ______ of x with respect to t is denoted by D^2x	derivative	non linear	linear	integral	derivative
The second derivative of ____ with respect to t is denoted by D^2x	x,y,z	x	y	z	x

The second derivative of x with respect to _____ is denoted by D^2x	t	x	y	z	t
The harmonic n=1 is called_______	fundamental harmonic	fundamental	harmonic	none of these	fundamental harmonic

UNIT V
SYLLABUS

Basic Theory of linear systems in normal form : Homogeneous linear systems with constant coefficients -Two Equations in two unknown functions -The method of successive approximations -The Euler method-The modified – Euler method -The Runge-Kutta method.

BASIC THEORY OF LINEAR SYSTEMS IN NORMAL FORM: TWO EQUATIONS IN TWO UNKNOWN FUNCTIONS

A. Introduction

We shall begin by considering a basic type of system of two linear differential equations in two unknown functions. This system is of the form

$$\begin{aligned}\frac{dx}{dt} &= a_{11}(t)x + a_{12}(t)y + F_1(t),\\ \frac{dy}{dt} &= a_{21}(t)x + a_{22}(t)y + F_2(t).\end{aligned} \tag{7.61}$$

We shall assume that the functions a_{11}, a_{12}, F_1, a_{21}, a_{22}, and F_2 are all continuous on a real interval $a \leq t \leq b$. If $F_1(t)$ and $F_2(t)$ are zero for all t, then the system (7.61) is called *homogeneous*; otherwise, the system is said to be *nonhomogeneous*.

► Example 7.8

The system

$$\begin{aligned}\frac{dx}{dt} &= 2x - y,\\ \frac{dy}{dt} &= 3x + 6y,\end{aligned} \tag{7.62}$$

is homogeneous; the system

$$\frac{dx}{dt} = 2x - y - 5t,$$
$$\frac{dy}{dt} = 3x + 6y - 4, \quad (7.63)$$

is nonhomogeneous.

DEFINITION

By a solution *of the system (7.61) we shall mean an ordered pair of real functions*

$$(f, g), \quad (7.64)$$

each having a continuous derivative on the real interval $a \le t \le b$, such that

$$\frac{df(t)}{dt} = a_{11}(t)f(t) + a_{12}(t)g(t) + F_1(t),$$
$$\frac{dg(t)}{dt} = a_{21}(t)f(t) + a_{22}(t)g(t) + F_2(t),$$

for all t such that $a \le t \le b$. In other words,

$$x = f(t),$$
$$y = g(t), \quad (7.65)$$

simultaneously satisfy both equations of the system (7.61) identically for $a \le t \le b$.

Notation. We shall use the notation

$$x = f(t),$$
$$y = g(t), \quad (7.65)$$

to denote a solution of the system (7.61) and shall speak of "the solution

$$x = f(t),$$
$$y = g(t)."$$

Whenever we do this, we must remember that the solution thus referred to is really the ordered pair of functions (f, g) such that (7.65) simultaneously satisfy both equations of the system (7.61) identically on $a \le t \le b$.

Example 7.9

The ordered pair of functions defined for all t by $(e^{5t}, -3e^{5t})$, which we denote by

$$x = e^{5t},\quad y = -3e^{5t}, \tag{7.66}$$

is a solution of the system (7.62). That is,

$$x = e^{5t},\quad y = -3e^{5t}, \tag{7.66}$$

simultaneously satisfy both equations of the system (7.62). Let us verify this by directly substituting (7.66) into (7.62). We have

$$\frac{d}{dt}(e^{5t}) = 2(e^{5t}) - (-3e^{5t}),$$

$$\frac{d}{dt}(-3e^{5t}) = 3(e^{5t}) + 6(-3e^{5t}),$$

or

$$5e^{5t} = 2e^{5t} + 3e^{5t},$$

$$-15e^{5t} = 3e^{5t} - 18e^{5t}.$$

Hence (7.66) is indeed a solution of the system (7.62). The reader should verify that the ordered pair of functions defined for all t by $(e^{3t}, -e^{3t})$, which we denote by

$$x = e^{3t},$$

$$y = -e^{3t},$$

is also a solution of the system (7.62).

We shall now proceed to survey the basic theory of linear systems. We shall observe a close analogy between this theory and that introduced in Section 4.1 for the single linear equation of higher order. Theorem 7.1 is the basic existence theorem dealing with the system (7.61).

THEOREM 7.1

Hypothesis. *Let the functions a_{11}, a_{12}, F_1, a_{21}, a_{22}, and F_2 in the system (7.61) all be continuous on the interval $a \le t \le b$. Let t_0 be any point of the interval $a \le t \le b$; and let c_1 and c_2 be two arbitrary constants.*

Conclusion. *There exists a unique solution*

$$x = f(t),$$
$$y = g(t),$$

of the system (7.61) such that

$$f(t_0) = c_1 \quad \text{and} \quad g(t_0) = c_2,$$

and this solution is defined on the entire interval $a \le t \le b$.

B. Homogeneous Linear Systems

We shall now assume that $F_1(t)$ and $F_2(t)$ in the system (7.61) are both zero for all t and consider the basic theory of the resulting *homogeneous* linear system

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y,$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y.$$

THEOREM 7.2

Hypothesis. *Let*

$$x = f_1(t), \qquad x = f_2(t),$$
$$\text{and} \qquad (7.68)$$
$$y = g_1(t), \qquad y = g_2(t),$$

be two solutions of the homogeneous linear system (7.67). Let c_1 and c_2 be two arbitrary constants.

Conclusion. *Then*

$$x = c_1 f_1(t) + c_2 f_2(t),$$
$$y = c_1 g_1(t) + c_2 g_2(t), \qquad (7.69)$$

is also a solution of the system (7.67).

DEFINITION

The solution (7.69) is called a linear combination *of the solutions (7.68). This definition enables us to express Theorem 7.2 in the following alternative form.*

THEOREM 7.2 RESTATED

Any linear combination of two solutions of the homogeneous linear system (7.67) is itself a solution of the system (7.67).

► Example 7.11

We have already observed that

$$x = e^{5t}, \qquad x = e^{3t},$$
and
$$y = -3e^{5t}, \qquad y = -e^{3t},$$

are solutions of the homogeneous linear system (7.62). Theorem 7.2 tells us that

$$x = c_1 e^{5t} + c_2 e^{3t},$$
$$y = -3c_1 e^{5t} - c_2 e^{3t},$$

where c_1 and c_2 are arbitrary constants, is also a solution of the system (7.62). For example, if $c_1 = 4$ and $c_2 = -2$, we have the solution

$$x = 4e^{5t} - 2e^{3t},$$
$$y = -12e^{5t} + 2e^{3t}.$$

DEFINITION

Let

$$x = f_1(t), \qquad x = f_2(t),$$
and
$$y = g_1(t), \qquad y = g_2(t),$$

be two solutions of the homogeneous linear system (7.67). These two solutions are linearly dependent *on the interval $a \le t \le b$ if there exist constants c_1 and c_2*, not both zero, *such that*

$$c_1 f_1(t) + c_2 f_2(t) = 0,$$
$$c_1 g_1(t) + c_2 g_2(t) = 0, \tag{7.70}$$

for all t such that $a \le t \le b$.

DEFINITION

Let

$$x = f_1(t), \qquad x = f_2(t),$$
$$\text{and}$$
$$y = g_1(t), \qquad y = g_2(t),$$

be two solutions of the homogeneous linear system (7.67). These two solutions are linearly independent *on $a \le t \le b$ if they are not linearly dependent on $a \le t \le b$. That is, the solutions $x = f_1(t)$, $y = g_1(t)$ and $x = f_2(t)$, $y = g_2(t)$ are linearly independent on $a \le t \le b$ if*

$$c_1 f_1(t) + c_2 f_2(t) = 0,$$
$$c_1 g_1(t) + c_2 g_2(t) = 0, \tag{7.71}$$

for all t such that $a \le t \le b$ implies that

$$c_1 = c_2 = 0.$$

► Example 7.12

The solutions

$$x = e^{5t}, \qquad x = 2e^{5t},$$
$$\text{and}$$
$$y = -3e^{5t}, \qquad y = -6e^{5t},$$

of the system (7.62) are linearly dependent on every interval $a \le t \le b$. For in this case the conditions (7.70) become

$$c_1e^{5t} + 2c_2e^{5t} = 0,$$
$$-3c_1e^{5t} - 6c_2e^{5t} = 0, \tag{7.72}$$

and clearly there exist constants c_1 and c_2, not both zero, such that the conditions (7.72) hold on $a \le t \le b$. For example, let $c_1 = 2$ and $c_2 = -1$.

On the other hand, the solutions

$$x = e^{5t}, \quad y = -3e^{5t}, \qquad \text{and} \qquad x = e^{3t}, \quad y = -e^{3t},$$

of system (7.62) are linearly independent on $a \le t \le b$. For in this case the conditions (7.71) are

$$c_1e^{5t} + c_2e^{3t} = 0,$$
$$-3c_1e^{5t} - c_2e^{3t} = 0.$$

If these conditions hold for all t such that $a \le t \le b$, then we must have $c_1 = c_2 = 0$.

We now state the following basic theorem concerning sets of linearly independent solutions of the homogeneous linear system (7.67).

THEOREM 7.3

There exist sets of two linearly independent solutions of the homogeneous linear system (7.67). Every solution of the system (7.67) can be written as a linear combination of any two linearly independent solutions of (7.67).

Example 7.13

We have seen that

$$x = e^{5t}, \quad y = -3e^{5t}, \qquad \text{and} \qquad x = e^{3t}, \quad y = -e^{3t},$$

constitute a pair of linearly independent solutions of the system (7.62). This illustrates the first part of Theorem 7.3. The second part of the theorem tells us that every solution of the system (7.62) can be written in the form

$$x = c_1 e^{5t} + c_2 e^{3t},$$
$$y = -3c_1 e^{5t} - c_2 e^{3t},$$

where c_1 and c_2 are suitably chosen constants.

Recall that in Section 4.1 in connection with the single nth-order homogeneous linear differential equation, we defined the general solution of such an equation to be a linear combination of n linearly independent solutions. As a result of Theorems 7.2 and 7.3 we now give an analogous definition of general solution for the homogeneous linear system (7.67).

DEFINITION

Let

$$x = f_1(t), \qquad x = f_2(t),$$
$$\text{and}$$
$$y = g_1(t), \qquad y = g_2(t),$$

be two linearly independent solutions of the homogeneous linear system (7.67). *Let* c_1 *and* c_2 *be two arbitrary constants. Then the solution*

$$x = c_1 f_1(t) + c_2 f_2(t),$$
$$y = c_1 g_1(t) + c_2 g_2(t),$$

is called a general solution *of the system* (7.67).

▶ Example 7.14

Since

$$x = e^{5t}, \qquad x = e^{3t},$$
$$\text{and}$$
$$y = -3e^{5t}, \qquad y = -e^{3t},$$

are linearly independent solutions of the system (7.62), we may write the general solution of (7.62) in the form

$$x = c_1 e^{5t} + c_2 e^{3t},$$
$$y = -3c_1 e^{5t} - c_2 e^{3t},$$

where c_1 and c_2 are arbitrary constants.

DEFINITION

Let

$$x = f_1(t), \quad y = g_1(t), \quad \text{and} \quad x = f_2(t), \quad y = g_2(t),$$

be two solutions of the homogeneous linear system (7.67). *The determinant*

$$\begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix} \tag{7.73}$$

is called the Wronskian *of these two solutions. We denote it by* $W(t)$.

THEOREM 7.4

Two solutions

$$x = f_1(t), \quad y = g_1(t), \quad \text{and} \quad x = f_2(t), \quad y = g_2(t),$$

of the homogeneous linear system (7.67) *are linearly independent on an interval* $a \le t \le b$ *if and only if their Wronskian determinant*

$$W(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix} \tag{7.73}$$

is different from zero for all t *such that* $a \le t \le b$.

THEOREM 7.5

Let $W(t)$ *be the Wronskian of two solutions of homogeneous linear system* (7.67) *on an interval* $a \le t \le b$. *Then either* $W(t) = 0$ *for all* $t \in [a, b]$ *or* $W(t) = 0$ *for no* $t \in [a, b]$.

UNIT-5					
QUESTIONS	**OPTION 1**	**OPTION2**	**OPTION3**	**OPTION4**	**ANSWER**
The system of the form dx/dt= a11(t)x + a12(t)y + F1(t) if F1(t) is non zero for all t, then the system is called --------------	homogeneous	non homogeneous	linear	non linear	non homogeneous
Any --------------- of two solutions of the homogeneous linear system dx/dt=a11x + a12y and dy/dt=a21x + a22y is itself a solution.	different equations	non linear combination	linear combination	same equations	linear combination
The general solution of the of the form $x = c_1e^{5t} + c_2e^{3t}$ where c_1 and c_2 are -------------	constant	variable	dependent	independent	constant
The homogeneous linear system X' = a1x + b1y and Y' = a2x + b2y where the coefficient a1, b1, a2, & b2 are -------------	variable	real constant	dependent	independent	real constant
The general solution of the system dx/dt=a1x1 + b1x2 is written as x = A1e^(λ_1 t) + A2e^(λ_2 t). Then roots λ_1 and λ_2 are -----------	imaginary and unequal	conjugate complex	real and equal	real and distinct	real and distinct
The general solution of the system dx/dt=a1x1 + b1x2 is written as x = eat (A1cos bt + A2 sin bt). Then roots λ_1 and λ_2 are -----------------------	imaginary and unequal	conjugate complex	real and equal	real and distinct	conjugate complex
The general solution of the system dx/dt=a1x1 + b1x2 is written as x = (A1t + A2) e^λt then roots λ_1 and λ_2 are -----------------------	imaginary and unequal	conjugate complex	real and equal	real and distinct	real and equal
Equations having a common solution are called -------------	linear equations	homogeneous equations	simultaneous equations	non homogeneous equations	simultaneous equations
Complementary function of (D2 + 4)y = tan 200x is --------------------	(Acos2x + Bsin2x)	(Acos2x – Bsin2x)	(Acosh2x + Bsinh2x)	(Acosh2x – Bsinh2x)	(Acos2x + Bsin2x)
If roots of linear second order differential equation is real double root than general solution will contain -----------------	two constants & two exponentials	one constant & two exponentials	two constants & one exponential	d) one constant & one exponential	two constants & one exponential
A particular case of Runge Kutta method of second order is ------------------	Milne’s method	Picard’s method	Modified Euler method	Runge’s method	Modified Euler method

Runge Kutta of first order is nothing but the --------------.	Euler method	modified Euler method	Taylor series	none of these	Euler method
In Runge Kutta second and fourth order methods, the values of k1 and k2 are ----	always positive	always negative	differ	same	same
The ----------------values are calculated in Runge Kutta fourth order method.	k1, k2, k3, k4 and Dy	k_1, k_2 and Dy	k_1, k_2, k_3 and Dy	k_1 and Dy	k1, k2, k3, k4 and Dy
The use of Runge kutta method gives ------------- to the solutions of the differential equation than Taylor's series method.	Slow convergence	quick convergence	oscillation	divergence	quick convergence
In Runge – kutta method the value x is taken as -------	h = x0 – x	x0= x + h	x = x0 + h	h = x0 + x	x = x0 + h
The ------------------------ is nothing but the modified Euler method.	Taylor series method	Runge kutta method of fourth order	Runge kutta method of third order	Runge kutta method of second order	Runge kutta method of second order
If dy/dx is a function x alone, then fourth order Runge – Kutta method reduces to --------------.	Euler method	Taylor series	Simpson method	Trapezoidal rule	Simpson method
In Runge Kutta methods, the derivatives of ----------- order are not require and we require only the given function values at different points.	lower	higher	middle	zero	higher
The formula of Dy in second order Runge Kutta method is given by -------	k1	k2	k3	k4	k2