Semester-II

18MMP201 **COMPLEX ANALYSIS** **L T P C**
4 0 0 4

Course Objectives

To study about the development of the functions of one complex variable and some important concepts such as Complex integration, Harmonic functions and Riemann Mapping Theorem etc.

Course Outcomes

On successful completion of this course, students will be able to:

- Give an account of the concepts of analytic function and harmonic function and to explain the role of the Cauchy-Riemann equations.
- Evaluate complex contour integrals, and be familiar with the Cauchy integral theorem, the Cauchy integral formula and some of their consequences.
- Describe the convergence properties of a power series and to determine the Taylor series or the Laurent series of an analytic function in a given region.
- Know the basic properties of singularities of analytic functions, how to determine the order of zeros and poles, how to compute residues and how to evaluate integrals using residue techniques.

UNIT I

Conformal mapping-Linear transformations- cross ratio- symmetry- Oriented circles-families of circles-level curves.

UNIT II

Complex integration-rectifiable Arcs- Cauchy's theorem for Rectangle and disc-Cauchy's integral formula-higher derivatives.

UNIT III

Harmonic functions-mean value property-Poisson's formula-Schwarz theorem, Reflection principle-Weierstrass theorem- Taylor series and Laurent series.

UNIT IV

Partial Fractions- Infinite products – Canonical products-The gamma function – Stirling's Formula – Entire functions – Jensen's formula.

UNIT V

Riemann Mapping Theorem – Boundary behaviour – Use of Reflection Principle – Analytical arcs
– Conformal mapping of polygons- The Schwartz - Christoffel formula.

SUGGESTED READINGS

TEXT BOOK

1. Lars V .Ahlfors., (1979). Complex Analysis, Third edition, Mc-Graw Hill Book Company, New Delhi.

REFERENCES

1. Ponnusamy, S., (2005). Foundation of Complex Analysis, Second edition, Narosa publishing house, New Delhi.
2. Choudhary, B.,(2005). The Elements of Complex Analysis ,New Age International Pvt. Ltd , New Delhi.
3. Vasishtha, A. R.,(2005). Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.
4. Walter Rudin., (2017) .Real and Complex Analysis,3rd edition, Mc Graw Hill Book Company, New york.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Coimbatore – 641 021.

LESSON PLAN
DEPARTMENT OF MATHEMATICS

Class : **I-M.Sc Mathematics**
Subject Code : **18MMP201**
Subject Name : **Complex Analysis**
Name of the Faculty : **Y.Sangeetha**

S.No.	Lecture Duration (Hr)	Topics to be covered	Support Materials
UNIT - I			
1	1	Conformal Mapping- Definition and theorems	R1:Chapter-3 Pg.No:-73-76
2	1	Linear Transformation - Definition and theorems	R1:Chapter-3 Pg.No:-76-78
3	1	Cross ratio- theorems	R1:Chapter-3 Pg.No:-78-80
4	1	Symmetry	R1:Chapter-3 Pg.No:-80-83
5	1	Oriented Circles- theorems, Families of Circles- theorems	R1:Chapter-3 Pg.No:-83-84, 84-85
6	1	Continuation on Families of circles- theorems	R1:Chapter-3 Pg.No:-86-87
7	1	Level Curves, Use of Level Curves	R1:Chapter-3 Pg.No:-89-91, 91 -93
8	1	Use of Level Curves	R1:Chapter-3 Pg.No:91 -93
9	1	Recapitulation and Discussion of possible questions	
Total	**9 Hours**		
UNIT-II			
1	1	Introduction to Complex Integration and Definite integrals	R1:Chapter-4 Pg.No:-101-103

2	1	Rectifiable Arcs-Problems	R1:Chapter-4 Pg.No:-104-109
3	1	Cauchy theorem for Rectangle and Disc	R1:Chapter-4 Pg.No:-109-110
4	1	Cauchy theorem for Disc	R1:Chapter-4 Pg.No:-112-114
5	1	Cauchy's integral formula	R1:Chapter-4 Pg.No:-114-117
6	1	Theorems for higher Derivatives	R1:Chapter-4 Pg.No:-120-122
7	1	Continuation of theorems on higher Derivatives	R1:Chapter-4 Pg.No:-122-125
8	1	Recapitulation and Discussion of possible questions	
Total	**8 Hours**		
UNIT-III			
1	1	Introduction to Harmonic function	R1:Chapter-4 Pg.No:-162-165
2	1	Geometric Interpretation and and Mean value property	R1:Chapter-4 Pg.No:-165-166
3	1	Poisson Formula	R1:Chapter-4 Pg.No:-166-168
4	1	Schwarz theorem and Problems	R1:Chapter-4 Pg.No:-168-171
5	1	Geometric Interpretation of Poisson Formula and Reflection Principle	R1:Chapter-4 Pg.No:-172-174
6	1	Problems based on Weierstrass theorem	R1:Chapter-4 Pg.No:-174-179
7	1	Problems based on Tayler Series and Laurent's Series- introduction	R1:Chapter-4 Pg.No:-179-186
8	1	Problems based on Laurent's Series	R1:Chapter-4 Pg.No:-184-186
8	1	Recapitulation and Discussion of possible questions	
Total	**9 Hours**		
UNIT-IV			
1	1	Theorems on Partial Fraction	R1:Chapter-5 Pg.No:-187-190
2	1	Infinite Product	R1:Chapter-5 Pg.No:-191-193
3	1	Problems on Canonical Product	R1:Chapter-5 Pg.No:-193-197
4	1	Gamma Functions and their products	R4: chapter -9 Pg.No:603-

			606
5	1	Properties of Gamma Functions	R4: chapter -9 Pg.No:607-610
6	1	Stirling's Formula	R1:Chapter-5 Pg.No:-201-206
7	1	Entire Function and Jensen's Formula	R4: chapter -9 Pg.No:582-589
8	1	Recapitulation and Discussion of possible questions	
Total	**8 Hours**		
		UNIT-V	
1	1	Riemann Mapping theorem	R5: chapter -14 Pg.No:282-288
2	1	Boundary behaviour, Use of Reflection Principle	R1:Chapter-6 Pg.No:232-234
3	1	Analytical Arcs and Conformal Mapping of Polygon	R1:Chapter-6 Pg.No:234-235
4	1	The Schwarz-Christoffel Formula	R3: chapter -9 Pg.No:143-146
5	1	Mapping on rectangle	R2: chapter -12 Pg.No:429-432
6	1	The triangle function of Schwarz	R1:Chapter-6 Pg.No:240-241
7	1	Recapitulation and Discussion of possible questions	
8	1	Discuss on Previous ESE Question Papers	
9	1	Discuss on Previous ESE Question Papers	
19	1	Discuss on Previous ESE Question Papers	
Total	**10 Hours**		

REFERENCES

1. **Lars V .Ahlfors., (1979). Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi.**
2. **Ponnusamy, S., (2005). Foundation of Complex Analysis, second edition, Narosa publishing House, New Delhi**
3. **Choudhary, B., (2003). The Elements of Complex Analysis, New age International Pvt. Ltd, New Delhi.**
4. **Vasistha, A.R., (2005). Complex Analysis, Krishna Prakashan media, Pvt, Ltd., Meerut.**
5. **Walter Rudin., (2012). Ral and Complex Analysis, 3rd edition, Mc-Graw Hill Book Company, Newyark.**

UNIT I

SYLLABUS

Conformal mapping-Linear transformations- cross ratio- symmetry- Oriented circles- families of circles-level curves.

Introduction:

A **complex number** is a number that can be expressed in the form *a* + *bi*, where *a*and *b* are real numbers, and *i* is a solution of the equation $x^2 = -1$, which is called an imaginary number because there is no real number that satisfies this equation. For the complex number *a* + *bi*, *a* is called the *real part*, and *b* is called the *imaginary part*. Despite the historical nomenclature "imaginary", complex numbers are regarded in the mathematical sciences as just as "real" as the real numbers, and are fundamental in many aspects of the scientific description of the natural world

PROPERTIES:

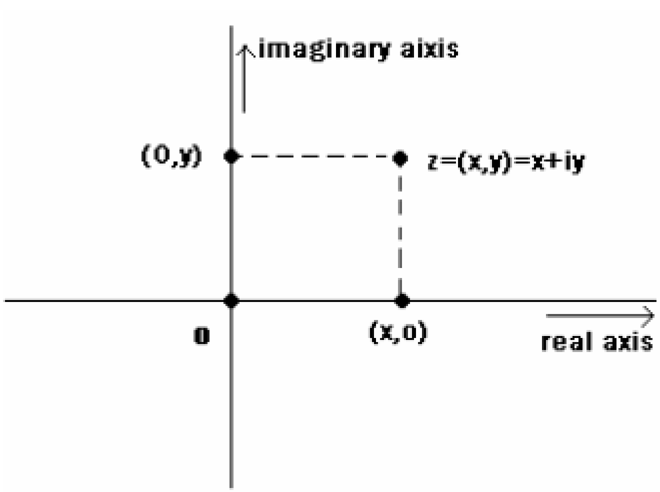
1. **Commutative law for addition :** $z_1 + z_2 = z_2 + z_1$.
2. **Associative law for addition :** $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
3. **Additive identity :** There is a complex number z' such that $z + z_0 = z$ for all complex number z. The number z_0 is an ordered pair $(0,0)$.
4. **Additive inverse :** For any complex number z there is a complex number $-z$ such that $z + (-z) = (0,0)$. The number $-z$ is $(-x, -y)$.
5. **Commutative law for multiplication :** $z_1 z_2 = z_2 z_1$.
6. **Associative law for multiplication :** $z_1(z_2 z_3) = (z_1 z_2) z_3$.
7. **Multiplicative identity :** There is a complex number z' such that $zz' = z$ for all complex number z. The number z' is an ordered pair $(1,0)$.
8. **Multiplicative inverse :** For any non-zero complex number z there is a complex number z^{-1} such that $zz^{-1} = (1,0)$. The number z^{-1} is $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$.

9. **The distributive law :** $z_1(z_2+z_3)=z_1z_2+z_1z_3$.

If we write x for the complex number $(x,0)$. This mapping $x \to (x,0)$ defines a field isomorphism of □ into □ so we may consider □ as a subset of □ .

If we put $i=(0,1)$, then $z=(x,y)=(x,0)+(0,y)=(x,0)+(0,1)(y,0)=x+iy$.

Let $z=x+iy$, $x,y \in$ □ , then x and y are called the real and imaginary parts of z and denote this by $x=\operatorname{Re} z$, $y=\operatorname{Im} z$. If $x=0$, the complex number z is called purely imaginary and if $y=0$, then z is real. Note that zero is the only number which is at once real and purely imaginary. Two complex numbers are equal iff they have the same real part and the same imaginary part.



Definition 2 Let $z=x+iy$, $x,y \in$ □ then the complex number $x-iy$ is called the conjugate of z and is denoted by $\bar{z}$.

Following are the basic properties of conjugates.

1. $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$.
2. z is real iff $z = \bar{z}$.
3. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
4. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
5. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$.
6. $\bar{\bar{z}} = z$.

Definition 3 Let $z = x + iy$, $x, y \in \mathbb{R}$ then modulus or absolute value of z is a non-negative real number denoted by $|z|$ and is given by $|z| = (x^2 + y^2)^{\frac{1}{2}}$. The number $|z|$ is the distance between the origin and the point (x, y).

Following are the basic properties of Modulus.

1. $|z|^2 = z\bar{z}$
2. $|z_1 z_2| = |z_1||z_2|$
3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.
4. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
5. $|\bar{z}| = |z|$.
6. $|x| = |\operatorname{Re}(z)| \leq |z|$ and $|y| = |\operatorname{Im}(z)| \leq |z|$.

7. $|z_1 + z_2| \le |z_1| + |z_2|$.

8. $|z_1 - z_2| \ge |z_1| - |z_2|$.

9. Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ then

$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{\frac{1}{2}}$ which is the distance between the points $(x_1, y_1), (x_2, y_2)$. Hence distance between the points z_1 and z_2 is given by $|z_1 - z_2|$.

Polar representation of complex numbers

Consider the point $z = x + iy$ in the complex plane □. This point has polar co-ordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$. Thus $z = x + iy = r(\cos\theta + i\sin\theta)$.

Clearly $r = |z| = (x^2 + y^2)^{\frac{1}{2}}$ which is **magnitude** of the complex number and θ(undefined if $z = 0$) is the angle between the positive real axis and the line segment from 0 to z and is called the **argument** of z, denoted by $\theta = \arg z$.

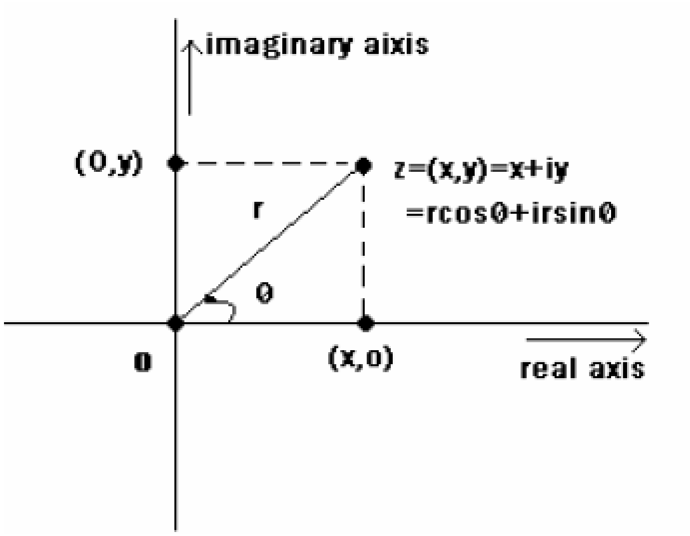
We note that the value of argument of z is not unique. If $\theta = \arg z$, then $\theta + 2\pi n$, where n is an integer is also $\arg z$. The value of $\arg z$ that lies in the range $-\pi < \theta \le \pi$ is called the **principal value** of $\arg z$.

If z_1, z_2 are any two non-zero complex numbers then

1. $\arg z_1 = -\arg \overline{z_1}$

2. $\arg z_1 z_2 = \arg z_1 + \arg z_2$.

3. $\arg\left[\dfrac{z_1}{z_2}\right] = \arg z_1 - \arg z_2$.



Conformal mapping

Let S be a domain in a plane in which x and y are taken as rectangular Cartesian co-ordinates. Let us suppose that the functions u(x, y) and v(x, y) are continuous and possess continuous partial derivatives of the first order at each point of the domain S. The equations

$$u = u(x, y), \quad v = v(x, y)$$

set up a correspondence between the points of S and the points of a set T in the (u, v) plane. The set T is evidently a domain and is called a map of S. Moreover, since the first order partial derivatives of u and v are continuous, a curve in S which has a continuously turning tangent is mapped on a curve with the same property in T. The correspondence between the two domains is not, however, necessarily a one-one correspondence.

For example, if we take $u = x^2$, $v = y^2$, then the domain $x^2 + y^2 < 1$ is mapped on the triangle bounded by u = 0, v = 0, u + v = 1, but there are four points of the circle corresponding to each point of the triangle.

2.1 Definition : A mapping from S to T is said to be isogonal if it has a one-one transformation which maps any two intersecting curves of S into two curves of T which cut at the same angle. Thus in an isogonal mapping, only the magnitude of angle is preserved.

An isogonal transformation which also conserves the sense of rotation is called conformal mapping. Thus in a conformal transformation, the sense of rotation as well as the magnitude of the angle is preserved.

The following theorem provides the necessary condition of conformality which briefly states that if f(z) is analytic, mapping is conformal.

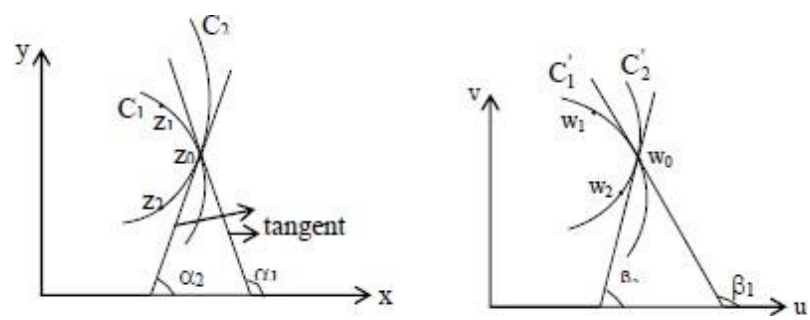
2.2. Theorem : Prove that at each point z of a domain D where f(z) is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

Proof. Let $w = f(z)$ be an analytic function of z, regular and one valued in a region D of the z-plane. Let z_0 be an interior point of D and let C_1 and C_2 be two continuous curves passing through z_0 and having definite tangents at this point, making angles α_1, α_2, say, with the real axis.

We have to discover what is the representation of this figure in the w-plane. Let z_1 and z_2 be points on the curves C_1 and C_2 near to z_0. We shall suppose that they are at the same distance r from z_0, so we can write

$$z_1 - z_0 = re^{i\theta 1},\ z_2 - z_0 = re^{i\theta 2}.$$

Then as $r \to 0$, $\theta_1 \to \alpha_1$, $\theta_2 \to \alpha_2$. The point z_0 corresponds to a point w_0 in the w-plane and z_1 and z_2 correspond to point w_1 and w_2 which describe curves C'_1 and C_2', making angles β_1 and β_2 with the real axis.



Let $w_1 - w_0 = \rho_1\ e^{i\phi_1}$, $w_2 - w_0 = \rho_2\ e^{i\phi_2}$,
where $\rho_1, \rho_2 \to 0 \Rightarrow \phi_1,\ \phi_2 \to \beta_1, \beta_2$
respectively.
Now, by the definition of an analytic function,

$$\lim_{z\to z_0} \frac{w_1 - w_0}{z_1 - z_0} = f'(z_0)$$

Since $f'(z_0) \neq 0$, we may write it in the form $Re^{i\lambda}$ and thus

$$\lim \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}} = Re^{i\lambda} \text{ i.e. } \lim \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)} = Re^{i\lambda}$$

$$\Rightarrow \qquad \lim \frac{\rho_1}{r} = R = |f'(z_0)|$$

and $\lim(\phi_1 - \theta_1) = \lambda$
i.e. $\lim \phi_1 - \lim \theta_1 = \lambda$
i.e. $\beta_1 - \alpha_1 = \lambda \Rightarrow \beta_1 = \alpha_1 + \lambda$
Similarly, $\beta_2 = \alpha_2 + \lambda$.

Hence the curves C'_1 and C'_2 have definite tangents at w_0 making angles $\alpha_1 + \lambda$ and $\alpha_2 + \lambda$ respectively with the real axis. The angle between C'_1 and C'_2 is

$$\beta_1 - \beta_2 = (\alpha_1 + \lambda) - (\alpha_2 - \lambda) = \alpha_1 - \alpha_2$$

which is the same as the angle between C_1 and C_2. Hence the curve C_1' and C_2' intersect at the same angle as the curves C_1 and C_2. Also the angle between the curves has the same sense in the two figures. So the mapping is conformal.

Special Case : When $f'(z_0) = 0$, we suppose that $f'(z)$ has a zero of order n at the point z_0. Then in the neighbourhood of this point (by Taylor's theorem)

$$f(z) = f(z_0) + a(z - z_0)^{n+1} + \ldots, \text{ where } a \neq 0$$

Hence $\quad w_1 - w_0 = a(z - z_0)^{n+1} + \ldots$

i.e. $\quad \rho_1 \, e^{i\phi_1} = |a| \, r^{n+1} \, e^{i[\delta + (n+1)\theta_1]} + \ldots$

where $\delta = \arg a$

Hence $\quad \lim \phi_1 = \lim \, [\delta + (n + 1)\, \theta_1] = \delta + (n + 1)\, \alpha_1 \qquad | \, \delta$ is constant

Similarly $\quad \lim \phi_2 = \delta + (n + 1)\, \alpha_2$

Thus the curves C'_1 and C'_2 still have definite tangent at w_0, but the angel between the tangents is

$$\lim(\phi_2 - \phi_1) = (n + 1)\,(\alpha_2 - \alpha_1)$$

Thus, the angle is magnified by $(n + 1)$.

Also the linear magnification, $R = \lim \dfrac{\rho_1}{r} = 0 \qquad |\because \lim \dfrac{\rho_1}{r} = R = |f'(z_0)| = 0$

Therefore, the conformal property does not hold at such points where $f'(z) = 0$

A point z_0 at which $f'(z_0) = 0$ is called a critical point of the mapping. The following theorem is the converse of the above theorem and is sufficient condition for the mapping to be conformal.

2.3. Theorem : If the mapping $w = f(z)$ is conformal then show that $f(z)$ is an analytic function of z.

Proof. Let $w = f(z) = u(x, y) + iv(x, y)$

Here, $u = u(x, y)$ and $v = v(x, y)$ are continuously differentiable equations defining conformal transformation from z-plane to w-plane. Let ds and $d\sigma$ be the length elements in z-plane and w-plane respectively so that

$$ds^2 = dx^2 + dy^2, \quad d\sigma^2 = du^2 + dv^2$$

Since u, v are functions of x and y, therefore

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \qquad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$\therefore$
$$du^2 + dv^2 = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)^2$$

i.e.
$$d\sigma^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]dx^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dy^2$$

$$+2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)dxdy \qquad \ldots(2)$$

Since the mapping is given to be conformal, therefore the ratio $d\sigma^2 : ds^2$ is independent of direction, so that from (1) and (2), comparing the coefficients, we get

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}{1}=\frac{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}{1}=\frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0}$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2 \qquad \ldots(3)$$

and
$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}=0 \qquad \ldots(4)$$

Equations (3) and (4) are satisfied if

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \qquad \ldots(5)$$

or
$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y} \qquad \ldots(6)$$

Equation (6) reduces to (5) if we replace v by –v i.e. by taking as image figure obtained by the reflection in the real axis of the w-plane.

Thus the four partial derivatives u_x, u_y, v_x, v_y exist, are continuous and they satisfy C-R equations (5). Hence f(z) is analytic.

2.4. Remarks

(i) The mapping w = f(z) is conformal in a domain D if it is conformal at each point of the domain.

(ii) The conformal mappings play an important role in the study of various physical phenomena defined on domains and curves of arbitrary shapes. Smaller portions of these domains and curves are conformally mapped by analytic function to well-known domains and curves.

2.5. Example : Discuss the mapping $w = \bar{z}$.

Solution. We observe that the given mapping replaces every point by its reflection in the real axis. Hence angles are conserved but their signs are changed and thus the mapping is isogonal but not conformal. If the mapping $w = \bar{z}$ is followed by a conformal transformation, then resulting transformation of the form $w = f(\bar{z})$ is also isogonal but not conformal, where f(z) is analytic function of z.

2.6. Example : Discuss the nature of the mapping $w = z^2$ at the point $z = 1 + i$ and examine its effect on the lines Im z = Re z and Re z = 1 passing through that point.

Solution. We note that the argument of the derivative of $f(z) = z^2$ at $z = 1 + i$ is

$$[\arg 2z]_{z=1+i} = \arg(2 + 2i) = \pi/4$$

Hence the tangent to each curve through $z = 1 + i$ will be turned by the angle $\pi/4$. The co-efficient of linear magnification is $|f'(z)|$ at $z = 1 + i$, i..e $|2 + 2i| = 2\sqrt{2}$. The mapping is

$$w = z^2 = x^2 - y^2 + 2ixy = u(x, y) + iv(x, y)$$

We observe that mapping is conformal at the point $z = 1 + i$, where the half lines $y = x (y \geq 0)$ and $x = 1 (y \geq 0)$ intersect. We denote these half lines by C_1 and C_2, with positive sense upwards and observe that the angle from C_1 to C_2 is $\pi/4$ at their point of intersection. We have

$$u = x^2 - y^2, \qquad v = 2xy$$

The half line C_1 is transformed into the curve C'_1 given by

$$u = 0, \quad v = 2y^2 (y \geq 0)$$

Thus C'_1 is the upper half $v \geq 0$ of the v-axis.

The half line C_2 is transformed into the curve C'_2 represented by

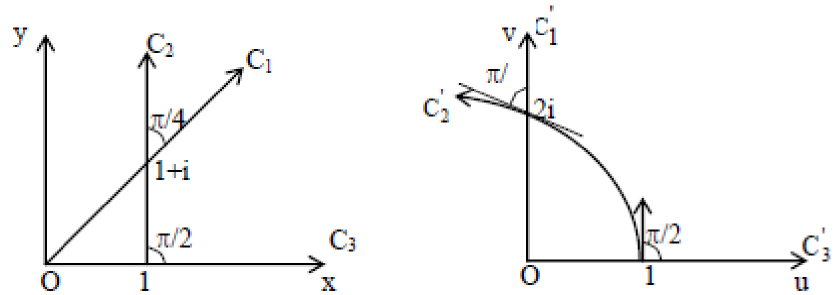
$$u = 1 - y^2, \qquad v = 2y \ (y \geq 0)$$

Hence C'_2 is the upper half of the parabola $v^2 = -4(u - 1)$. We note that, in each case, the positive sense of the image curve is upward.

For the image curve C'_2,

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = \frac{2}{-2y} = -\frac{2}{v}$$

In particular $\frac{dv}{du} = -1$ when $v = 2$. Consequently, the angle from the image curve C'_1 to the image curve C'_2 at the point $w = f(1 + i) = 2i$ is $\frac{\pi}{4}$, as required by the conformality of the mapping there.



Note. The angle of rotation and the scalar factor (linear magnification) can change from point to point. We note that they are 0 and 2 respectively, at the point $z = 1$, since $f'(1) = 2$, where the curves C_2 and C'_2 are the same as above and the non-negative x-axis (C_3) is transformed into the non-negative u-axis (C'_3).

2.7. Example. Discuss the mapping $w = z^a$, where a is a positive real number.

Solution. Denoting z and w in polar as

$$z = re^{i\theta},\ w = \rho e^{i\phi}, \text{ the mapping gives } \rho = r^a,\ \phi = a\theta.$$

Thus the radii vectors are raised to the power a and the angles with vertices at the origin are multiplied by the factor a. If a > 1, distinct lines through the origin in the z-plane are not mapped onto distinct lines through the origin in the w-plane, since, e.g. the straight line through the origin at an angle $\frac{2\pi}{a}$ to the real axis of the z-plane is mapped onto a line through the origin in the w-plane at an angle 2π to the real axis i.e. the positive real axis itself. Further $\frac{dw}{dz} = az^{a-1}$, which vanishes at the origin if a > 1 and has a singularity at the origin if a < 1. Hence the mapping is conformal and the angles are therefore preserved, excepting at the origin. Similarly the mapping $w = e^z$ is conformal.

2.8. Example. Prove that the quadrant $|z| < 1,\ 0 < \arg z < \frac{\pi}{2}$ is mapped conformally onto a domain in the w-plane by the transformation $w = \frac{4}{(z+1)^2}$.

Solution. If $w = f(z) = \frac{4}{(z+1)^2}$, then $f'(z)$ is finite and does not vanish in the given quadrant. Hence the mapping $w = f(z)$ is conformal and the quadrant is mapped onto a domain in the w-plane provided w does not assume any value twice i.e. distinct points of the quadrant are mapped to distinct points of the w-plane. We show that this indeed is true. If possible, let $\frac{4}{(z_1+1)^2} = \frac{4}{(z_2+1)^2}$, where $z_1 \neq z_2$ and both z_1 and z_2 belong to the quadrant in the z-plane.

Then, since $z_1 \neq z_2$, we have $(z_1 - z_2)(z_1 + z_2 + 2) = 0$

$\Rightarrow z_1 + z_2 + 2 = 0$ i.e. $z_1 = -z_2 - 2$. But since z_2 belongs to the quadrant, $-z_2 - 2$ does not, which contradicts the assumption that z_1 belongs to the quadrant. Hence w does not assume any value twice.

LINEAR TRANSFORMATION:

Bilinear Transformation. The transformation

$$w = \frac{az+b}{cz+d},\ ad - bc \neq 0$$

where a, b, c, d are complex constants, is called bilinear transformation or a linear fractional transformation or Mōbius transformation. We observe that the condition $ad - bc \neq 0$ is necessary for (1) to be a bilinear transformation, since if

$$ad - bc = 0, \text{ then } \frac{b}{a} = \frac{d}{c} \text{ and we get}$$

$$w = \frac{a(z + b/a)}{c(z + d/c)} = \frac{a}{c}$$ i.e. we get a constant function which is not linear.

Equation (1) can be written in the form

$$cwz + dw - az - b = 0 \qquad \ldots(2)$$

Since (2) is linear in z and linear in w or bilinear in z and w, therefore (1) is termed as bilinear transformation.

When $c = 0$, the condition $ad - bc \neq 0$ becomes $ad \neq 0$ and we see that the transformation reduces to general linear transformation. When $c \neq 0$, equation (1) can be written as

$$w = \frac{a\,(z + b/a)}{c\,(z + d/c)} = \frac{a}{c}\left[1 + \frac{b/a - d/c}{z + d/c}\right]$$

$$= \frac{a}{c} + \frac{bc - ad}{c^2} = \frac{1}{z + d/c} \qquad \ldots(3)$$

We note that (3) is a composition of the mappings

$$z_1 = z + \frac{d}{c}, \qquad z_2 = \frac{1}{z_1}, \qquad z_3 = \frac{bc - ad}{c^2} z_2$$

and thus we get $w = \frac{a}{c} + z_3.$

The above three auxiliary transformations are of the form

$$w = z + \alpha, \qquad w = \frac{1}{z}, \qquad w = \beta z \qquad \ldots(4)$$

Hence every bilinear transformation is the resultant of the transformations in (4).

But we have already discussed these transformations and thus we conclude that a bilinear transformation always transforms circles and lines into circles and lines because the transformations in (4) do so.

From (1), we observe that if $c = 0$, $a, d \neq 0$, each point in the w plane is the image of one and only one point in the z-plane. The same is true if $c \neq 0$, except when $z = -\frac{d}{c}$ which makes the denominator zero. Since we work in extended complex plane, so in case $z = -\frac{d}{c}$, $w = \infty$ and thus we may regard the point at infinity in the w-plane as corresponding to the point $z = -\frac{d}{c}$ in the z-plane.

Thus if we write

$$T(z) = w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \qquad \ldots(5)$$

Then $T(\infty) = \infty, \quad \text{if } c = 0$

and $T(\infty) = \frac{a}{c}, \quad T\left(-\frac{d}{c}\right) = \infty, \quad \text{if } c \neq 0$

Thus T is continuos on the extended z-plane. When the domain of definition is enlarged in this way, the bilinear transformation (5) is one-one mapping of the extended z-plane onto the extended w-plane.

Hence, associated with the transformation T, there is an inverse transformation T^{-1} which is defined on the extended w-plane as

$$T^{-1}(w) = z \text{ if and only if } T(z) = w.$$

Thus, when we solve equation (1) for z, then

$$z = \frac{-dw+b}{cw-a}, \; ad - bc \neq 0 \qquad \ldots(6)$$

and thus

$$T^{-1}(w) = z = \frac{-dw+b}{cw-a}, \; ad - bc \neq 0$$

Evidently T^{-1} is itself a bilinear transformation, where

$$T^{-1}(\infty) = \infty \quad \text{if } c = 0$$

and $T^{-1}\left(\frac{a}{c}\right) = \infty, \quad T^{-1}(\infty) = -\frac{d}{c}, \text{ if } c \neq 0$

From the above discussion, we conclude that inverse of a bilinear transformation is bilinear. The points $z = -\frac{d}{c}\,(w = \infty)$ and $z = \infty(w = \frac{a}{c})$ are called critical points.

$$w_1 = \frac{a_1\left(\frac{az+b}{cz+d}\right)+b_1}{c_1\left(\frac{az+b}{cz+d}\right)+d_1} = \frac{(a_1a+b_1c)z+(b_1d+a_1b)}{(c_1a+d_1c)z+(d_1d+c_1b)}$$

Taking $A = a_1a + b_1c, \quad B = b_1d + a_1b,$
$C = c_1a + d_1c, \quad D = d_1d + c_1b,$ we get

$$w_1 = \frac{Az+B}{Cz+D}$$

Also

$$\begin{aligned} AD - BC &= (a_1a + b_1c)(d_1d + c_1b) - (b_1d + a_1b)(c_1a + d_1c) \\ &= (a_1ad_1d + a_1ac_1b + b_1cd_1d + b_1cc_1b) \\ &\quad - (b_1dc_1a + b_1d\,d_1c + a_1bc_1a + a_1bd_1c) \\ &= a_1ad_1d + b_1bc_1c - b_1dc_1a - a_1bd_1c \\ &= ad(a_1d_1 - b_1c_1) - bc(a_1d_1 - b_1c_1) \\ &= (ad - bc)(a_1d_1 - b_1c_1) \neq 0 \end{aligned}$$

Thus $w_1 = \frac{Az+B}{Cz+D}, \quad AD - BC \neq 0$

Is a bilinear transformation.

This bilinear transformation is called the resultant (or product or composition) of the bilinear transformations (1) and (2).

The above property is also expressed by saying that bilinear transformations form a group.

1.2. Theorem. Composition (or resultant or product) of two bilinear transformations is a bilinear transformation.

Proof. We consider the bilinear transformations

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \qquad \ldots(1)$$

and

$$w_1 = \frac{a_1w+b_1}{c_1w+d_1}, \quad a_1d_1 - b_1c_1 \neq 0 \qquad \ldots(2)$$

Putting the value of w from (1) in (2), we get

1.3. Definitions. (i) The points which coincide with their transforms under bilinear transformation are called its fixed points. For the bilinear transformation $w = \frac{az+b}{cz+d}$, fixed points are given by $w = z$ i.e. $z = \frac{az+b}{cz+d}$ ...(1)

$\Rightarrow \quad (1 - K^2)\, z\bar{z} - (p - qK^2)\,\bar{z} - (\bar{p} - \bar{q}K^2)z = K^2 q\bar{q} - p\bar{p}$

$$\Rightarrow \quad z\bar{z} - \left(\frac{p - qK^2}{1 - K^2}\right)\bar{z} - \left(\frac{\bar{p} - \bar{q}K^2}{1 - K^2}\right)z + \frac{|p|^2 - K^2|q|^2}{1 - K^2} = 0 \qquad \ldots(2)$$

Equation (2) is of the form

$z\bar{z} + b\bar{z} + \bar{b}z + c = 0$ (c is being a real constant)

which always represents a circle.

Thus equation (2) represents a circle if $K \neq 1$.

If $K = 1$, then it represents a straight line

$$|z - p| = |z - q|$$

Further, we observe that in the form (1), p and q are inverse points w.r.t. the circle. For this, if the circle is $|z - z_0| = \rho$ and p and q are inverse points w.r.t. it, then

$$z - z_0 = \rho e^{i\theta},\ p - z_0 = qe^{i\lambda},$$

$$q - z_0 = \frac{\rho^2}{a} e^{i\lambda}$$

Therefore,

$$\left|\frac{z-p}{z-q}\right| = \left|\frac{\rho e^{i\theta} - ae^{i\lambda}}{\rho e^{i\theta} - \frac{\rho^2}{a} e^{i\lambda}}\right| = \frac{a}{\rho}\left|\frac{\rho e^{i\theta} - ae^{i\lambda}}{ae^{i\theta} - \rho e^{i\lambda}}\right|$$

Since (1) is a quadratic in z and has in general two different roots, therefore there are generally two invariant points for a bilinear transformation.

(ii) If z_1, z_2, z_3, z_4 are any distinct points in the z-plane, then the ratio

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

is called cross ratio of the four points z_1, z_2, z_3, z_4. This ratio is invariant under a bilinear transformation i.e.

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

1.4. Transformation of a Circle. First we show that if p and q are two given points and K is a constant, then the equation

$$\left|\frac{z-p}{z-q}\right| = K, \qquad (1)$$

represents a circle. For this, we have

$$|z - p|^2 = K^2|z - q|^2$$

$\Rightarrow \quad (z - p)\,\overline{(z - p)} = K^2(z - q)\,\overline{(z - q)}$

$\Rightarrow \quad (z - p)\,(\bar{z} - \bar{p}) = K^2(z - q)\,(\bar{z} - \bar{q})$

$\Rightarrow \quad z\bar{z} - \bar{p}z - p\bar{z} + p\bar{p} = K^2(z\bar{z} - \bar{q}z - q\bar{z} + q\bar{q})$

$$= K\left|\frac{\rho(\cos\theta + i\sin\theta) - a(\cos\lambda + i\sin\lambda)}{a(\cos\theta + i\sin\theta) - \rho(\cos\lambda + i\sin\lambda)}\right|, \; K = \frac{a}{\rho}$$

$$= K\left|\frac{(\rho\cos\theta - a\cos\lambda) + i(\rho\sin\theta - a\sin\lambda)}{(a\cos\theta - \rho\cos\lambda) + i(a\sin\theta - \rho\sin\lambda)}\right|$$

$$= K\left[\frac{(\rho\cos\theta - a\cos\lambda)^2 + (\rho\sin\theta - a\sin\lambda)^2}{(a\cos\theta - \rho\cos\lambda)^2 + (a\sin\theta - \rho\sin\lambda)^2}\right]^{1/2}$$

$= K$, where $K \neq 1$, since $a \neq \rho$

Thus, if p and q are inverse points w.r.t. a circle, then its equation can be written as

$$\left|\frac{z-p}{z-q}\right| = K, \quad K \neq 1, \text{ K being a real constant.}$$

1.5 Theorem. In a bilinear transformation, a circle transforms into a circle and inverse points transform into inverse points. In the particular case in which the circle becomes a straight line, inverse points become points symmetric about the line.

Proof : We know that $\left|\frac{z-p}{z-q}\right| = K$ represents a circle in the z-plane with p and q as inverse points, where $K \neq 1$. Let the bilinear transformation be

$$w = \frac{az+b}{cz+d} \qquad \text{so that} \qquad z = \frac{dw-b}{-cw+a}$$

Then under this bilinear transformation, the circle transforms into

$$\left|\frac{\frac{dw-b}{-cw+a} - p}{\frac{dw-b}{-cw+a} - q}\right| = K \quad \Rightarrow \quad \left|\frac{dw-b-p(q-cw)}{dw-b-q(a-cw)}\right| = K$$

$$\Rightarrow \quad \left|\frac{w(d+cp)-(ap+b)}{w(d+cq)-(aq+b)}\right| = K \quad \Rightarrow \quad \left|\frac{w - \frac{ap+b}{cp+d}}{w - \frac{aq+b}{cq+d}}\right| = K\left|\frac{cq+d}{cp+d}\right| \qquad \ldots(1)$$

The form of equation (1) shows that it represents a circle in the w-plane whose inverse points are $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$. Thus, a circle in the z-plane transforms into a circle in the w-plane and the inverse points transform into the inverse points.

Also if $K\left|\frac{cq+d}{cp+d}\right| = 1$, then equation (1) represents a straight line bisecting at right angle the join of the points $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$ so that these points are symmetric about this line. Thus in a particular case, a circle in the z-plane transforms into a straight line in the w-plane and the inverse points transform into points symmetrical about the line.

1.6. Example. Find all bilinear transformations of the half plane Im $z \ge 0$ into the unit circle $|w| \le 1$.

Solution. We know that two points $z, \bar{z}$, symmetrical about the real z-axis(Im $z = 0$) correspond to points $w, \frac{1}{\bar{w}}$, inverse w.r.t. the unit w-circle. ($|w|\frac{1}{|\bar{w}|} = 1$). In particular, the origin and the point at infinity in the w-plane correspond to conjugate values of z.

Comment [a1]:

Let

$$w = \frac{az+b}{cz+d} = \frac{a(z+b/a)}{c(z+d/c)} \qquad \ldots(1)$$

be the required transformation.

Clearly $c \neq 0$, otherwise points at ∞ in the two planes would correspond.

Also, $w = 0$ and $w = \infty$ are the inverse points w.r.t. $|w| = 1$. Since in (1), $w = 0$, $w = \infty$ correspond respectively to $z = -\frac{b}{a}$, $z = -\frac{d}{c}$, therefore these two values of z-plane must be conjugate to each other. Hence we may write

$$-\frac{b}{a} = \alpha, -\frac{d}{c} = \bar{\alpha} \text{ so that}$$

$$w = \frac{a}{c}\frac{z-\alpha}{z-\bar{\alpha}} \qquad (2)$$

The point $z = 0$ on the boundary of the half plane Im $z \ge 0$ must correspond to a point on the boundary of the circle $|w| = 1$, so that

$$1 = |w| = \left|\frac{a}{c}\right|\left|\frac{0-\alpha}{0-\bar{\alpha}}\right| = \left|\frac{a}{c}\right|$$

$$\Rightarrow \qquad \frac{a}{c} = e^{i\lambda} \quad \Rightarrow \quad a = ce^{i\lambda}, \text{ where } \lambda \text{ is real.}$$

Thus, we get

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \qquad \ldots(3)$$

Since $z = \alpha$ gives $w = 0$, α must be a point of the upper half plane i.e. Im $\alpha > 0$. With this condition, (3) gives the required transformation. In (3), if z is real, obviously $|w| = 1$ and if Im $z > 0$, then z is nearer to α than to $\bar{\alpha}$ and so $|w| < 1$. Hence the general linear transformation of the half plane Im $z \ge 0$ on the circle $|w| \le 1$ is

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right), \text{ Im } \alpha > 0.$$

1.7. Example. Find all bilinear transformations of the unit $|z| \le 1$ into the unit circle $|w| \le 1$.

OR

Find the general homographic transformations which leaves the unit circle invariant.

Solution. Let the required transformation be

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\frac{(z+b/a)}{(z+d/c)} \qquad \ldots(1)$$

Here, w = 0 and w = ∞, correspond to inverse points

$$z = -\frac{b}{a},\ z = -\frac{d}{c}, \qquad \text{so we may write}$$

$$-\frac{b}{a} = \alpha, \qquad -\frac{d}{c} = \frac{1}{\overline{\alpha}} \qquad \text{such that } |\alpha| < 1.$$

So,

$$w = \frac{a}{c}\left(\frac{z-\alpha}{z-1/\overline{\alpha}}\right) = \frac{a\overline{\alpha}}{c}\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right) \qquad \ldots(2)$$

The point z = 1 on the boundary of the unit circle in z-plane must correspond to a point on the boundary of the unit circle in w-plane so that

$$1 = |w| = \left|\frac{a\overline{\alpha}}{c}\,\frac{1-\alpha}{\overline{\alpha}-1}\right| = \left|\frac{a\overline{\alpha}}{c}\right|$$

or $a\,\overline{\alpha} = c\,e^{i\lambda}$, where λ is real.

Hence (2) becomes,

$$w = e^{i\lambda}\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right), \qquad |\alpha| < 1 \qquad \ldots(3)$$

This is the required transformation, for if $z = e^{i\theta}$, $\alpha = be^{i\beta}$, then

$$|w| = \left|\frac{e^{i\theta} - be^{i\beta}}{be^{i(\theta-\beta)} - 1}\right| = 1.$$

If $z = re^{i\theta}$, where r < 1, then

$$|z-\alpha|^2 - |\overline{\alpha}\,z - 1|^2$$
$$= r^2 - 2rb\cos(\theta-\beta) + b^2 - \{b^2r^2 - 2br\cos(\theta-\beta) + 1\}$$
$$= (r^2-1)(1-b^2) < 0$$

and so

$$|z-\alpha|^2 < |\overline{\alpha}\,z - 1|^2 \quad \Rightarrow \quad \frac{|z-\alpha|^2}{|\overline{\alpha}z-1|^2} < 1$$

i.e. $|w| < 1$

Hence the result.

1.8. Example. Show that the general transformation of the circle $|z| \le \rho$ into the circle $|w| \le \rho'$ is

$$w = \rho\rho' e^{i\lambda}\left(\frac{z-\alpha}{\overline{\alpha}z-\rho^2}\right), \qquad |\alpha| < \rho.$$

Solution. Let the transformation be

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\left(\frac{z+b/a}{z+d/c}\right) \qquad \ldots(1)$$

The points w = 0 and w = ∞, inverse points of $|w| = \rho'$ correspond to inverse point z = –b/a, z = –d/c respectively of $|z| = \rho$, so we may write

$$-\frac{b}{a} = \alpha, \quad -\frac{d}{c} = \frac{\rho^2}{\overline{\alpha}}, \qquad |\alpha| < \rho$$

Thus, from (1), we get

$$w = \frac{a}{c}\left(\frac{z-\alpha}{z-\dfrac{\rho^2}{\overline{\alpha}}}\right) = \frac{a\overline{\alpha}}{c}\left(\frac{z-\alpha}{\overline{\alpha}z-\rho^2}\right) \qquad \ldots(2)$$

Equation (2) satisfied the condition $|z| \le \rho$ and $|w| \le \rho'$. Hence for $|z| = \rho$, we must have $|w| = \rho'$ so that (2) becomes

$$\rho' = |w| = \left|\frac{a\overline{\alpha}}{c}\right|\left|\frac{z-\alpha}{\overline{\alpha}z - z\overline{z}}\right|, \qquad z\overline{z} = \rho^2$$

$$= \left|\frac{a\overline{\alpha}}{c}\right|\left|\frac{1}{z}\right|\left|\frac{z-\alpha}{\overline{z}-\overline{\alpha}}\right| = \left|\frac{a\overline{\alpha}}{c}\right|\left|\frac{1}{z}\right|\left|\frac{z-\alpha}{\overline{z-\alpha}}\right|$$

$$= \left|\frac{a\overline{\alpha}}{c}\right|\frac{1}{\rho}, \qquad |z-\alpha| = |\overline{z-\alpha}|$$

$$\Rightarrow \qquad \rho\rho' = \left|\frac{a\overline{\alpha}}{c}\right| \qquad \Rightarrow \qquad \frac{a\overline{\alpha}}{c} = \rho\rho' e^{i\lambda}, \ \lambda \text{ being real.}$$

Thus, the required transformation becomes

$$w = \rho\rho' e^{i\lambda}\left(\frac{z-\alpha}{\overline{\alpha}z-\rho^2}\right), \quad |\alpha| < \rho.$$

1.9. Example. Find the bilinear transformation which maps the point 2, i, –2 onto the points 1, i, –1.

Solution. Under the concept of cross-ratio, the required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Using the values of z_i and w_i, we get

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

or
$$\frac{w-1}{w+1} = \left(\frac{z-2}{z+2}\right)\left(\frac{2+i}{2-i}\right)\left(\frac{1-i}{1+i}\right)$$

or
$$\frac{w-1}{w+1} = \frac{4-3i}{5}\frac{z-2}{z+2}$$

or
$$\frac{w-1+w+1}{w-1-(w+1)} = \frac{(4-3i)(z-2)+5(z+2)}{(4-3i)(z-2)-5(z+2)}$$

or
$$-w = \frac{3z(3-i)+2i(3-i)}{-iz(z-i)-6(3-i)} = \frac{3z+2i}{-(iz+6)}$$

or
$$w = \frac{3z+2i}{iz+6}$$

which is the required transformation.

Cross-ratio

In geometry, the **cross-ratio**, also called double **ratio** and anharmonic **ratio**, is a number associated with a list of four collinear points, particularly points on a projective line. Given four points A, B, C and D on a line, their **cross ratio** is defined as.

$$(A,B;C,D)= \frac{AC.BD}{BC.AD}$$

where an orientation of the line determines the sign of each distance and the distance is measured as projected into Euclidean space. (If one of the four points is the line's point at infinity, then the two distances involving that point are dropped from the formula.)

DEFINITION:

The cross-ratio of a 4-tuple of distinct points on the real line with coordinates z_1, z_2, z_3, z_4 is given by

$$(Z_1,Z_2;Z_3,Z_4)= \frac{(Z_3-Z_1)(Z_4-Z_2)}{(Z_3-Z_2)(Z_4-Z_1)}$$

A mapping of the form $S(z)=\dfrac{az+b}{cz+d}$ is called bilinear or linear fractional

Transformation a,b,c,d∈ z

A bilinear transformation $S(z)=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$ is called Möbius

map or Möbius transformation.

1) Möbius transformation is one-one and onto.

2) If $S(z)=\dfrac{az+b}{cz+d}$, then $S^{-1}(w)=\dfrac{-dw+b}{cw-a}$.

3) If S and T are Möbius transformations then $S\circ T$ is also Möbius transformation.

4) $S(z)=z+a$ (Translation)

$S(z)=az$ (Dilation/Magnification)

$S(z)=e^{i\theta}z$ (Rotation)

$S(z)=\dfrac{1}{z}$ (Inversion).

Theorem 22 If S is a Möbius transformation then S is composition of translation , dilation and inversion.

Proof. Let $S(z)=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$ be Möbius transformation.

Case 1. When $c=0$ then $S(z)=\left(\dfrac{a}{d}\right)z+\left(\dfrac{b}{d}\right)$

Let $S_1(z)=\left(\dfrac{a}{d}\right)z$, $S_2(z)=z+\left(\dfrac{b}{d}\right)$

Then $S_2\circ S_1(z)=S_2\left(S_1(z)\right)=S_2\left(\left(\dfrac{a}{d}\right)z\right)=\left(\dfrac{a}{d}\right)z+\left(\dfrac{b}{d}\right)=S(z)$

Thus $S=S_2\circ S_1$.

Case 2. When $c \neq 0$

Let $S_1(z) = z + \frac{d}{c}$, $S_2(z) = \frac{1}{z}$, $S_3(z) = \frac{bc - ad}{c^2} z$, $S_4(z) = z + \frac{a}{c}$.

Then $S_4 \circ S_3 \circ S_2 \circ S_1(z) = S_4 \circ S_3 \circ S_2 \left(S_1(z) \right)$

$$= S_4 \circ S_3 \circ S_2 \left(z + \frac{d}{c} \right)$$

$$= S_4 \circ S_3 \left(S_2 \left(z + \frac{d}{c} \right) \right)$$

$$= S_4 \circ S_3 \left(\frac{1}{z + \frac{d}{c}} \right)$$

$$= S_4 \left(S_3 \left(\frac{1}{z + \frac{d}{c}} \right) \right)$$

$$= S_4 \left(\frac{bc - ad}{c^2} \left(\frac{1}{z + \frac{d}{c}} \right) \right)$$

$$= \left(\frac{bc - ad}{c(cz + d)} \right) + \frac{a}{c}$$

$$= \frac{az + b}{cz + d} = S(z).$$

Thus $S = S_4 \circ S_3 \circ S_2 \circ S_1$.

Theorem 23 Every Möbius transformation can have at most two fixed points.

Proof. Let $S(z) = \dfrac{az+b}{cz+d}$ with $ad - bc \neq 0$ be Möbius transformation.

Let z be fixed point of $S(z)$ then $S(z) = z$

$$\frac{az+b}{cz+d} = z$$

$$cz^2 + (d-a)z - b = 0$$

which is quadratic in z. Hence it can have at most two roots. Therefore every Möbius transformation can have at most two fixed points otherwise $S(z) = z$ for all z (Identity map).

THEOREM:

The cross ratio (Z_1,Z_2,Z_3,Z_4) is real ,iff four points lies ona circle or a straight line.

Suppose $S(w) = \text{real}$, then $S(w) = \overline{S(w)}$.

Let $S(w) = \dfrac{aw+b}{cw+d}$ with $ad - bc \neq 0$.

Thus, $\dfrac{aw+b}{cw+d} = \dfrac{\overline{a}\overline{w}+\overline{b}}{\overline{c}\overline{w}+\overline{d}}$

Therefore, $(a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - \overline{a}d)\overline{w} + (b\overline{d} - \overline{b}d) = 0$ …(1)

Case 1. When $a\overline{c}$ is real.

Therefore, $a\overline{c} = \overline{a}c$, then from (1) we have,

$(a\overline{d} - \overline{b}c)w + (b\overline{c} - \overline{a}d)\overline{w} + (b\overline{d} - \overline{b}d) = 0$ …(2)

Let $\alpha = 2(a\overline{d} - \overline{b}c)$, $\beta = i(b\overline{d} - \overline{b}d)$ then (2) becomes,

$$\frac{\alpha}{2}w+\frac{\overline{\alpha}}{2}\overline{w}+\frac{\beta}{i}=0$$

$$i\left(\alpha w+\overline{\alpha w}\right)+2\beta=0$$

$$i.2i.\operatorname{Im}\left(\alpha w\right)+2\beta=0$$

$$\operatorname{Im}\left(\alpha w\right)-\beta=0 \qquad \ldots(3)$$

Let $\alpha=p+iq$, $w=x+iy$ then $\alpha w=px-qy+i(qx+py)$.

Therefore, $\operatorname{Im}\left(\alpha w\right)-\beta=(qx+py)-\beta=0$. Thus (3) represents a line $y=\left(\frac{-q}{p}\right)x+\beta$.

That is, w lies on the line determined by (3) for fixed α and β. We know that straight line may be regarded as circle with infinite radius. Therefore, w lies on the circle.

Case 2. When $a\overline{c}$ is not real.

Therefore, $a\overline{c}\neq\overline{a}c$, then from (1) we have,

$$|w|^2+\frac{(a\overline{d}-\overline{b}c)}{(a\overline{c}-\overline{a}c)}w+\frac{(b\overline{c}-\overline{a}d)}{(a\overline{c}-\overline{a}c)}\overline{w}+\frac{(b\overline{d}-\overline{b}d)}{(a\overline{c}-\overline{a}c)}=0$$

Let $\overline{\gamma}=\left(\frac{a\overline{d}-\overline{b}c}{a\overline{c}-\overline{a}c}\right)$, $\delta=-\left(\frac{b\overline{d}-\overline{b}d}{a\overline{c}-\overline{a}c}\right)$.

Therefore, $|w|^2+\overline{\gamma}w+\gamma\overline{w}-\delta=0$

$$w\overline{w}+\overline{\gamma}w+\gamma\overline{w}+\gamma\overline{\gamma}=\delta+\gamma\overline{\gamma}$$

$$(w+\gamma)\left(\overline{w}+\overline{\gamma}\right)=\delta+\gamma\overline{\gamma}$$

$$|w+\gamma|^2=\delta+\gamma\overline{\gamma}$$

$$|w+\gamma|^2=\delta+\gamma\overline{\gamma}$$

Therefore, $|w+\gamma|=\lambda \qquad \ldots(4)$

where $\lambda = \left(\delta + \gamma\bar{\gamma}\right)^{1/2} = \left|\frac{ad - bc}{a\bar{c} - \bar{a}c}\right| > 0$

Since γ and λ are independent of w, (4) represents a circle on which w lies.

Symmetry:

If a linear transformation T carries a real axis into a circle c we shall say that the points $w=Tz$,$w^*=T\bar{z}$ are symmetric between w and w^* and c which does not depends for if s is another transformation which carries the real axis in c then $s^{-1}(T)$ is a linear transformation and hence $s^{-1}(w) = s^{-1}(Tz)$ and $s^{-1}(w^*)= s^{-1}(T\bar{z})$ are also conjugate,symmetric with respect with respect to circle centre $o(z,z^*)$ lie on same line and multiple of $oz \rightarrow oz^*$ in R.(where R is radius).

Theorem: Symmetric principle

If a liner transformation carries a circle c into a circle c' then it transforms any pair of symmetric points with respect to c in to a pair of symmetric points with respect to c'.

Proof:

We can determine the transformation by requiring that 3 points Z_1,Z_2,Z_3 and c,go over into 3 points w_1,w_2,w_3 on c'.The transformation is $(w,w_1,w_2,w_3) = (Z,Z_1,Z_2,Z_3)$.

But the transformation is also determined that a point z on C shall correspond to a point w on c' and that a point Z_2 not c shall be carried into a point w_2 not on c' we know that Z_2^* the symmetric point of z with respect to c must correspond to w_2^* the symmetric of w_2 with respect to c'.Hence the transformation will be obtained from the relation $(w,w_1,w_2,w_2^*)= (Z,Z_1,Z_2,Z_2^*)$.

Oriented circle :

An orientation of circle c is determined by an ordered tripule of points (Z_1,Z_2,Z_3) on c with respect to this orientation a point z not on c is said to be lie to the right side of c.If $Im(Z,Z_1,Z_2,Z_3)=0$ and to the left of c if $Im(Z,Z_1,Z_2,Z_3)<0$

Note:

It is essential to show that there are only two different orientation.

Level curves:

when a conformal mapping is defined by an analytic w=f(z).In more general cases the image of curves in the line x=x0 and y=y0 .we write the transformation f(z)=u(x,y)+iV(x,y) , the image of x=x0 is given the parametric equation u=u(x0,y) and v=v(x0,y).Also the image of y=y0 is determined in the image of y=y0 is determinant in the same way the above curves form a orthogonal net in w plane.similarly we may consider the curves u(x,y) =u0 and v(x,y)=v0 is the z plane,They are also orthogonal and are called the level curves of u and v.

FAMILY OF CIRCLE:

Consider the linear transformation of the form w =k$\frac{z-}{z-b}$. Here z=a corresponds to w=0 and z=b to w=∞ the straight line through the origin of w plane are image of circles through a and b.

The concentric circles when arg(k) varies the point to move along the circle c.The corresponding flow circle depends a and b in different direction.

On the otherhand the concentric circles about the origin |w|=ρ corresponding to the circles with equation $|\frac{z-a}{z-b}|$ =-ρ |k| these all the circles with anypoint A and B by there equation loci of points whose distances from A and B have a constant ratio.

Denote by c1 the circles through A ,B and by c2 the circles in these the limit points A,B.These circles c1 & c2 wii be refer to as the circles net the steiner circles alternate by A and B.There are many interesting properties given below.

- There is exactly one c1 and c2 through each points in the plane with the exception of on limit point.
- Every c1 meets every c2 under right angles.
- Reflection in c1 transforms every c2 into itself and every c1 into another c1.Reflextion in a c2 transforms every c1 into itself and every c2 into another c2.
- The limit points of symmetric with respect to each c2 but not with respect to any other circles.
- These properties are all trivial with limit points are 0 and ∞ .That is when the c1 are lines through the origin and the c2 concentric circles. since with properties are invariant under linear transformation in given general case. It can be written in the form $\frac{w-a}{w-b}$=k$\frac{z-a}{z-b}$ It is clear that T transformation the circles c1 and c2 into circles $c1^1$ and $c2^1$with limit points A' ,B'
  Case (i):
  We have $c1^1$ =c1 for all c,if k>0(if k<0 these circles are orientation in this transforms is said to be hyperbolic).
  Case (ii):

In this case c2'=c2 when |k|=1.This transformation with property are called elliptic.

SIX MARKS QUESTIONS:

1. Show that any linear transformation which transforms a real axis into itself can be written with real coefficient.

2. Show that the cross ratio (z_1, z_2, z_{3z} z_4) is real if and only if the four points lie on a circle or on a straight line.

3. Show that a function f(z) be an analytic in the region Ω of the z-plane. If f '(z) ≠ 0 in Ω then the mapping w = f(z) conformal at all points of Ω.

4. Show that the set of all linear transformation forms a group under the product of transformation

5. Show that an analytic function in a region **Ω** whose derivative vanishes identically must reduce to a constant . The same is true if either the part, the imaginary part , the modulus the argument is constant.

6. Find the linear transformation which carries 0, i, -i into 1 , -1, 0.

7. Show that If c=0 then inverse doesn't exist the reflextion Z→ $\bar{Z}$ is not a linear transformation.

8. Show that the set of all linear transformation forms a group under the product of transformation

9. State and prove the symmetry principle

TEN MARKS:

1. Discuss about the Family of circles.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
UNIT I					
The additive identity of complex number is………	(1,1)	(1,0)	(0,0)	(0,1)	(0,0)
The multiplicative identity of complex number is…….	(0,1)	(1,0)	(0,0)	(0,1)	(1,0)
The inverse of (α,β) under addition is ……	(-α,β)	(-α,-β)	(α,β)	(α,-β)	(-α,-β)
$\lvert Z_1 . Z_2 \rvert$=……….	$\lVert z_1 \rVert \lVert z_2 \rVert$	$\lvert z_1 \rvert \lVert z_2 \rVert$	$\lvert z_1 \rvert \lvert z_2 \rvert$	$\lvert z_1 \rvert + \lvert z_2 \rvert$	$\lvert z_1 \rvert \lvert z_2 \rvert$
The value of i^2 is………..	1	-1	0	i	-1
If Z_1 and Z_2 are any two complex numbers ,then………..	$\arg(Z_1Z_2) = \arg(Z_1)+\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)-\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)/\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)*\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)/\arg(Z$
The Equation of the unit sphere is…………..	$x^2+y^2+z^2=1$	$x^2+y^2+z^2=2$	$x^2-y^2+z^2=1$	$x^2-y^2-z^2=1$	$x^2+y^2+z^2=1$
The element (1,0) is the -------	Additive	Multiplicative	identity	unique	Multiplicativ
The element (0,0) is the ----------	Additive	Multiplicative	identity	unique	Additive
If $\lvert Z_1 \rvert = \lvert Z_2 \rvert$ and $\arg(Z_1)= \arg(Z_2)$ then -----	$Z_1 \neq Z_2$	$Z_1 < Z_2$	$Z_1 > Z_2$	$Z_1 = Z_2$	$Z_1 = Z_2$
The Equation of the unit circle whose centre is the origin is…………	$\lvert Z \rvert =1$	$\lvert Z-a \rvert =1$	$\lvert Z \rvert =0$	$\lvert Z \rvert \neq 1$	$\lvert Z \rvert =1$
The complex plane containing all the finite complex numbers and infinity is called the	infinite complex plane	extended complex plane	complex plane	finite complex plane	extended complex plane
The inversion w = 1/z maps the region $\lvert z \rvert < 1$ into the region………	$\lvert w \rvert <1$	$\lvert w \rvert >1$	$\lvert w \rvert =1$	$\lvert w \rvert \leq 1$	$\lvert w \rvert >1$
The square of real number is ---------	Non negative	Non positive	Negative	absolute value	absolute
The absolute value of z = x+iy is………..	√x	√y	$\sqrt{x^2-y^2}$	$\sqrt{x^2+y^2}$	$\sqrt{x^2+y^2}$
If Z_1 and Z_2 are any two complex numbers ,then………..	$\lvert Z_1 +Z_2 \rvert \leq \lvert Z_1 \rvert + \lvert Z_2 \rvert$	$\lvert Z_1 +Z_2 \rvert = \lvert Z_1 \rvert + \lvert Z_2 \rvert$	$\lvert Z_1 +Z_2 \rvert \geq \lvert Z_1 \rvert + \lvert Z_2 \rvert$	$\lvert Z_1 +Z_2 \rvert \neq \lvert Z_1 \rvert + \lvert Z_2 \rvert$	$\lvert Z_1 +Z_2 \rvert \leq \lvert Z_1 \rvert + \lvert Z_2 \rvert$
The mapping W=1/Z is called an ………	Linear transformation	Translation	Inversion	Rotation	Inversion
The polar form of x+iy is …………	r(cos θ +isinθ)	r(cos θ -isinθ)	cos θ +isinθ	r(cos θ - sinθ)	r(cos θ
If Z_1 and Z_2 are any two complex numbers ,then ……….	$\lvert Z_1 -Z_2 \rvert \leq \lvert Z_1 \rvert + \lvert Z_2 \rvert$	$\lvert Z_1 -Z_2 \rvert = \lvert Z_1 \rvert + \lvert Z_2 \rvert$	$\lvert Z_1 -Z_2 \rvert \geq \lvert \lvert Z_1 \rvert - \lvert Z_2 \rvert \rvert$	$\lvert Z_1 -Z_2 \rvert \neq \lvert Z_1 \rvert + \lvert Z_2 \rvert$	$\lvert Z_1 -Z_2 \rvert \geq \lvert \lvert Z_1 \rvert - \lvert Z_2 \rvert \rvert$
The complex plane containing all the finite complex numbers is called the…….	infinite complex plane	extended complex plane	complex plane	finite complex plane	finite complex plane
The conjugation of 5+i3 is……..	5	3	5+i3	5-i3	5-i3
If Z_1 and Z_2 are any two complex numbers ,then ……..	$\arg(Z1/Z2) = \arg(Z_1)+\arg(Z_2)$	$\arg(Z_1 /Z_2) = \arg(Z_1)-\arg(Z_2)$	$\arg(Z_1 /Z_2) = \arg(Z_1)/\arg(Z_2)$	$\arg(Z_1 /Z_2) = \arg(Z_1)*\arg(Z_2)$	$\arg(Z_1 /Z_2) = \arg(Z_1)-$
The mapping W=Z+b ,b is a complex number, is called the……..	Linear transformation	Translation	Inversion	Rotation	Translation
All the complex numbers except infinity are called…...	Complex numbers	Complex plane	finite complex numbers	infinite complex	finite complex
If x= rcosθ , y = rsinθ then for z we get……..	z= rcosθ+ r sinθ	z= rsinθ + ircosθ	z= rcosθ+irsinθ	z= rcosθ-irsinθ	z= rcosθ+irsinθ
The angle made by the vector (x,y)measured in the anticlockwise direction is …….	mod of z	norm of z	argument of z	0	argument of z
The argument θ is ------------- as it can take infinite values	unique	not unique	finite	infinite	not unique
From x= rcosθ and y = rsinθ weget θ =	$\sin^{-1} y/x$	$\cos^{-1} y/x$	$\tan^{-1} y/x$	$\cot^{-1} y/x$	$\tan^{-1} y/x$
$\arg \bar{z}$………	arg z	- arg z	arg(-z)	arg 1/z	- arg z

The argument of the product of two complex numbers is---- of the complex number	The sum of the arguments	the argument of the sum	the argument of the division	the product of the arguments	the argument of the sum
arg (z1 . z2)=………	arg z_1+ arg z_2	arg $z_{1.}$argz_2	argz_1/argz_2	arg(z_1 +z_2)	arg $z_{1.}$argz_2
The cross ratio of the form…..	(z_1-z_2)(z_2-z_4)/(z_1-z_4)(z_2-z_3)	(z_1-z3)(z_2-z_4)/(z_1-z_4)(z_2-	(z_1-z_2)(z_2-z_4)/(z_1-z_4)	(z_1-z_2)/(z_1-z_4)(z_2-z_3)	(z_1-z3)(z_2-z_4)/(z_1-z_4)(
If z = -1 + i , then z-1 = ……..	-1+i	-1-i	(-1)/2 + i 1/2	(-1)/2 - i 1/2	-1-i
The stereographic projection of the complex point z = (√2 ,1) is	(1/√2, 1/√2, 0)	(0, √2, 1)	(1/√2, 1/2 , 1/2)	(0, 0,1)	(1/√2, 1/2 , 1/2)
The inversion w = 1/z maps the region │z│>1 into the region	│w│<1	│w│>1	│w│=1	│w│≤1	│w│<1
Under the transformation w = az there are ------ fixed points	one	two	zero	∞	two
According to De Moivre's theorem (cos θ +isin θ)^n = ………………	$\cos^n θ + i\sin^n θ$	cosn θ+isinn θ	ncos θ+insin θ	1	cosn θ+isinn θ
The transformation w = az=b , where a, b are complex constants ,is a composition of …… tranformations	Rotation and Homothetic	Translation and Rotation	Rotation , Homothetic and Translation	Homothetic and Translation	Rotation , Homothetic and
The equation $z\bar{z} + \bar{a}z + a\bar{z} + c = 0$, where c is real and a is complex , is a equation of a	Line	Ray	Ellipse	circle	circle

UNIT II

Complex integration-rectifiable Arcs- Cauchy's theorem for Rectangle and disc-Cauchy's integral formula-higher derivatives.

INTRODUCTION:

In this section we shall study complex integration of complex functions and established fundamental theorem of calculus for line integral.we show that an analytic function has a power series expansion as a Taylor theorem .Form then we established cauchy's estimate to prove Cauchy theorem.

2. Complex Integration

Let [a, b] be a closed interval, where a, b are real numbers. Divide [a, b] into subintervals

$$[a = t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n = b] \qquad (1)$$

by inserting n−1 points $t_1, t_2, \ldots, t_{n-1}$ satisfying the inequalities

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$$

Then the set $P = \{t_0, t_1, \ldots, t_n\}$ is called the partition of the interval [a, b] and the greatest of the numbers t_1 t_0, $t_2 - t_1, \ldots, t_n - t_{n-1}$ is called the norm of the partition P. Thus the norm of the partition P is the maximum length of the subintervals in (1).

We say that an arc is simple or Jordan arc if $z(t_1) = z(t_2)$ only when $t_1 = t_2$ i.e. the arc does not intersect itself. If the points corresponding to the values a and b coincide, the arc is said to be a closed arc (closed curve). An arc is said to be piecewise continuous in [a,b] if it is continuous in every subinterval of [a, b].

2.1. Arcs and Curves in the Complex Plane. An arc (path) L in a region $G \subset \forall$ is a continuous function $z(t) : [a, b] \rightarrow G$ for t ε [a, b] in R. The arc L, given by $z(t) = x(t) + iy(t)$, t ε [a, b], where x(t) and y(t) are continuous functions of t, is therefore a set of all image points of a closed interval under a continuous mapping. The arc L is said to be differentiable if $z'(t)$ exists for all t in [a, b]. In addition to the existence of $z'(t)$, if $z'(t) : [a, b] \rightarrow \forall$ is continuous, then z(t) is a smooth arc. In such case, we may say that L is regular and smooth. Thus a regular arc is characterized by the property that $\dot{x}(t)$ and $\dot{y}(t)$ exist and are continuous over the whole range of values of t.

RECTIFIABLE ARCS

2.2. Rectifiable Arcs. Let $z = x(t) + iy(t)$ be the equation of the Jordan arc L, the range for the parameter t being $t_0 \le t \le T$.

Let $z_0, z_1, \ldots, z_n$ be the points of this arc corresponding to the values $t_0, t_1, \ldots, t_n$ of t, where $t_0 < t_1 < t_2 < \ldots < t_n = T$. Evidently, the length of the polygonal arc obtained by joining successively z_0 and z_1, z_1 and z_2 etc by st. line segments is given by

$$\Sigma_n = \sum_{r=1}^{n} |z_r - z_{r-1}|$$
$$= \sum_{r=1}^{n} |(x_r + iy_r) - (x_{r-1} + iy_{r-1})|$$
$$= \sum_{r=1}^{n} |(x_r - x_{r-1}) + i(y_r - y_{r-1})|$$
$$= \sum_{r=1}^{n} [(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2]^{1/2}$$

If this sum Σ_n tends to a unique limit $l<\infty$, as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero, we say that the arc L defined by $z = x(t) + iy(t)$ is rectifiable and that its length is l. In this connection, we have the following result.

"A regular arc $z = x(t) + iy(t)$, $t_0 \le t \le T$ is rectifiable and its length is

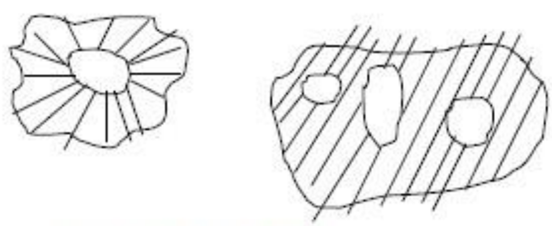
$$\int_{t_0}^{T} [(\dot{x}(t))^2 + (\dot{y}(t))^2]^{1/2}\, dt".$$

2.3. Contours. Let PQ and QR to be two rectifiable arcs with only Q as common point, then the arc PR is evidently rectifiable and its length is the sum of lengths of PQ and QR. Thus it follows that Jordan arc which consists of a finite number of regular arcs is rectifiable, its length being the sum of lengths of regular arcs of which it is composed. Such an arc is called contour. Thus a contour C is continuous chain of finite number of regular arcs. i.e. a contour is a piecewise smooth arc.

By a closed contour we shall mean a simple closed Jordan arc consisting of a finite number of regular arcs. Clearly, every closed contour is rectifiable. Circle rectangle, ellipse etc. are examples of closed contour.

2.4. Simply Connected Region A region D is said to be simply connected if every simple closed contour within it encloses only points of D. In such a region every closed curve can be shrunk (contracted) to a point without passing out of the region(Fig.1). If the region is not simply connected, then it is called multiply connected(Fig. 2).

Simply connected region
Fig. 1



Multiply connected regions
Fig. 2

2.5. Riemann's Definition of Complex Integration

First, we define the integral as the limit of a sum and later on, deduce it as the operation inverse to that of differentiation.

Let us consider a function $f(z)$ of the complex variable z. We assume that $f(z)$ has a definite value at each point of a rectifiable arc L having equation

$$z(t) = x(t) + iy(t),\ t_0 \le t \le T.$$

We divide this arc into n smaller arcs by points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n$ (= Z, say) which correspond to the values

$t_0 < t_1 < t_2, \ldots, < t_{n-1} < t_n$ (= T) of the parameter t and then form the sum

$$\Sigma = \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$$

where ξ_r is a point of L between z_{r-1} and z_r. If this sum Σ tends to a unique limit I as $n \to \infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero, we say that $f(z)$ is integrable from z_0 to Z along the arc L, and we write

$$I = \int_L f(z)\,dz$$

The direction of integration is from z_0 to Z, since the points on $x(t) + iy(t)$ describe the arc L in this sense when t increases.

2.6. Remarks. (i) Some of the most obvious properties of real integrals extend at once to complex integrals, for example,

$$\int_L [f(z) + g(z)]\,dz = \int_L f(z)\,dz + \int_L g(z)\,dz,$$

$$\int_L K f(z)\,dz = K \int_L f(z)\,dz, \text{ K being constant}$$

and

$$\int_{L'} f(z)\,dz = -\int_L f(z)\,dz,$$

where L′ denotes the arc L described in opposite direction.

(ii) In the above definition of the complex integral, although z_0, Z play much the same parts as the lower and upper limits in the definite integral of a function of a real variable, we do not write

$$I = \int_{z_0}^{Z} f(z)\,dz$$

This is dictated essentially by the fact that the value of I depends, in general, not only on the initial and final points of the arc L but also on its actual form.

In special circumstances, the integral may be independent of path from z_0 to Z as shown in the following example.

2.7. Example. Using the definition of an integral as the limit of a sum, evaluate the integrals

(i) $\int_L dz$ (ii) $\int_L |dz|$ (iii) $\int_L z\,dz$

where L is a rectifiable arc joining the points $z = \alpha$ and $z = \beta$.

Solution. We first observe that the integrals exist since the integrand is continuous on L in each case.

(i) By definition we have.

$$\int_L dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1})\, 1$$
$$= \lim_{n\to\infty} [z_1 - z_0 + z_2 - z_1 + \ldots + z_n - z_{n-1}]$$
$$= \lim_{n\to\infty} (z_n - z_0) = \beta - \alpha$$

(ii) $$\int_L |dz| = \lim_{n\to\infty} \sum_{r=1}^{n} |z_r - z_{r-1}|$$
$$= \lim_{n\to\infty} [|z_1 - z_0| + |z_2 - z_1| + \ldots + |z_n - z_{n-1}|]$$

= Arc length of L
= l (say)

(iii) Let $$I = \int_L z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1})\, \xi_r \quad (1)$$

where ξ_r is any point on the sub arc joining z_{r-1} and z_r.
Since ξ_r is arbitrary, we set $\xi_r = z_r$ and $\xi_{r-1} = z_{r-1}$ successively in (1) to find

$$I = \lim_{n\to\infty} \sum_{r=1}^{n} z_r (z_r - z_{r-1})$$
$$I = \lim_{n\to\infty} \sum_{r=1}^{n} z_{r-1} (z_r - z_{r-1})$$

Adding these two results, we get

$$2I = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r + z_{r-1})(z_r - z_{r-1})$$
$$= \lim_{n\to\infty} \sum_{r=1}^{n} (z_r^2 - z_{r-1}^2) = \lim_{n\to\infty} (z_n^2 - z_0^2) = \beta^2 - \alpha^2$$
$$\therefore \quad I = \frac{1}{2}(\beta^2 - \alpha^2)$$

In particular, if L is closed, then $\beta = \alpha$ and thus

$$\int_L dz = 0, \int_L z\,dz = 0.$$

2.8. Theorem (Integration along a regular arc). Let $f(z)$ be continuous on the regular arc L whose equation is $z(t) = x(t) + iy(t)$, $t_0 \le t \le T$. Prove that $f(z)$ is integrable along L and that

$$\int_L f(z)\,dz = \int_{t_0}^{T} F(t)\,[\dot{x}(t) + i\,\dot{y}(t)]\,dt,$$

where F(t) denotes the value of $f(z)$ at the point of L corresponding to the parametric value t.

Proof. Let us consider the sum

$$\Sigma = \sum_{r=1}^{n} f(\xi_r)\,(z_r - z_{r-1})$$

where ξ_r is a point of L between z_{r-1} and z_r. If τ_r is the value of the parameter t corresponding to ξ_r, then τ_r lies between t_{r-1} and t_r. Writing $F(t) = \phi(t) + i\psi(t)$, where ϕ and ψ are real, we find that

$$\Sigma = \sum_{r=1}^{n} [\phi(\tau_r) + i\psi(\tau_r)]\,[x_r - x_{r-1}) + i(y_r - y_{r-1})]$$

$$= \sum_{r=1}^{n} \phi(\tau_r)\,(x_r - x_{r-1}) + i\sum_{r=1}^{n} \phi(\tau_r)\,(y_r - y_{r-1})$$

$$+ i\sum_{r=1}^{n} \psi(\tau_r)\,(x_r - x_{r-1}) - \sum_{r=1}^{n} \psi(\tau_r)\,(y_r - y_{r-1})$$

$$= \Sigma_1 + i\,\Sigma_2 + i\,\Sigma_3 - \Sigma_4 \text{ (say)}$$

$$= \Sigma_1 - \Sigma_4 + i\,(\Sigma_2 + \Sigma_3)$$

We consider these four sums separately.

By the mean value theorem of differential calculus, the first sum is

$$\Sigma_1 = \sum_{r=1}^{n} \phi(\tau_r)\,(x_r - x_{r-1})$$

$$= \sum_{r=1}^{n} \phi(\tau_r)\,\dot{x}(\tau_r')\,(t_r - t_{r-1})$$

$$(f(a + h) - f(a) = hf'(a + \theta h),\ 0 \le \theta \le 1$$

$$x_r - x_{r-1} = x(t_r) - x(t_{r-1})$$

$$= (t_r - t_{r-1})\,\dot{x}(\tau_r'))$$

where τ_r' his between t_{r-1} and t_r.

We first show that Σ_1 can be made to differ by less than an arbitrary positive number, however small, from the sum

$$\Sigma_1' = \sum_{r=1}^{n} \phi(t_r)\,\dot{x}(t_r)\,(t_r - t_{r-1})$$

by making the maximum of the differences $t_r - t_{r-1}$ sufficiently small.

Now, by hypothesis, the functions $\phi(t)$ and $\dot{x}(t)$ are continuous. As continuous functions are necessarily bounded, there exist a positive number K such that the inequalities

$$|\phi(t)| \le K,\ |\dot{x}(t)| \le K$$

hold for $t_0 \le t \le T$.

Moreover, the functions are also uniformly continuous, we can, therefore, preassign an arbitrary positive number $\in$, as small as we please, and then choose a positive number δ, depending on $\in$, such that

$$|\phi(t) - \phi(t')| < \in,\ |\dot{x}(t) - \dot{x}(t')| < \in,$$

whenever $|t - t'| < \delta$

Hence if the maximum of the differences $t_r - t_{r-1}$ is less than δ, we have

$$\begin{aligned} &|\phi(\tau_r)\,\dot{x}(\tau_r') - \phi(t_r)\,\dot{x}(t_r)| \\ &= |\,\phi(\tau_r)\,\{\dot{x}(\tau_r') - \dot{x}(t_r)\} + \dot{x}(t_r)\,\{\phi(\tau_r) - \phi(t_r)\}| \\ &\le |\phi(\tau_r)|.|\,\dot{x}(\tau_r') - \dot{x}(t_r)\,| + |\dot{x}(t_r)|.|\phi(\tau_r) - \phi(t_r)| \\ &< 2K\in \end{aligned}$$

and therefore

$$|\Sigma_1 - \Sigma_1'| < 2K\in (T - t_0)$$

By the definition of the integral of a continuous function of a real variable, Σ_1' tends to the limit

$$\int_{t_0}^{T} \phi(t)\,\dot{x}(t)dt \qquad \Big|\ \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\delta x_i$$

as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero. Since $|\Sigma_1 - \Sigma_1'|$ can be made as small as we please by taking δ small enough, Σ_1 must also tend to the same limit.

Similarly the other terms of Σ tend to limits. Combining these results we find that Σ tends to the limit

$$\begin{aligned} &\int_{t_0}^{T} [\phi(t)\ \dot{x}(t) - \psi(t)\ \dot{y}(t)]\ dt \\ &+ i \int_{t_0}^{T} [\psi(t)\ \dot{x}(t) + \phi(t)\ \dot{y}(t)]\ dt \\ &= \int_{t_0}^{T} F(t)[\dot{x}(t) + i\ \dot{y}(t)]\ dt \end{aligned}$$

and so $f(z)$ is integrable along the regular arc L.

2.12. Cauchy Theorem (Elementary Form). First we consider the elementary form of Cauchy theorem which requires the additional assumption that the derivative of $f(z)$ is continuous. This form of Cauchy theorem is also known as Cauchy fundamental theorem, which has the following statement.

If $f(z)$ is analytic function whose derivative $f'(z)$ exists and is continuous at each point within and on a closed contour C, then

$$\int_C f(z)\,dz = 0$$

Proof. Let D denotes the closed region which consists of all points within and on C. If we write $z = x + iy$, $f(z) = u + iv$, then we have

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \qquad (1)$$

Now, we use the Green's theorem for a plane which states that if $P(x, y)$, $Q(x, y)$, $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous functions within a domain D and if C is any closed contour in D, then

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad (2)$$

By hypothesis $f'(z)$ exists and is continuous in D, so u and v and their partial derivatives u_x, v_x, u_y, v_y are continuous functions of x and y in D. Thus the conditions of Green's theorem are satisfied. Hence applying this theorem in (1), we obtain

$$\int_C f(z)dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy \quad \text{(using C–R equations)}$$

$$= 0 + i\,0 = 0$$

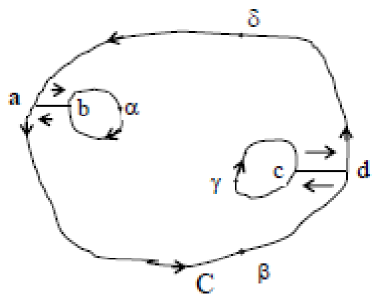
Hence the result.

Extension of Cauchy's Theorem to Contours Defining Multiply Connected Regions.

By adopting a suitable convention as to the sense of integration, Cauchy's theorem can be extended to the case of contours which are made up of several distinct closed contours. Consider, for example, a function $f(z)$ which is analytic in the multiply connected region R bounded by the closed contour C and the two interior contours C_1, C_2 as well as on these contours themselves. The complete contour C^* which is the boundary of the region R is made up of the three contours C, C_1 and C_2 and we adopt the convention that C^* is described in the positive sense if the region R is on the L.H.S. w.r.t. this sense of describing it. Then by Cauchy's theorem

$$\int_{C^*} f(z)\,dz = 0$$

where the integral is taken round the complete contour C^* in the positive sense.



Practically, we deal with this case by drawing transversals like ab, cd and by applying Cauchy's theorem for a simple closed contour abαbaβdcγcdδa. It is found convenient in applications to express the same result in the form

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

where all the three integrals are now taken in the same (positive) sense.

An exactly similar result holds in case there are any finite number of closed contours $C_1, C_2, \ldots, C_m$ inside a closed contour C and $f(z)$ is analytic in the multiply connected region bounded by them as well as on them. We then have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_m} f(z)dz.$$

where all the contours are described in positive sense.

Theorem. (Cauchy's Integral Formula). Let $f(z)$ be analytic inside and on a closed contour C and let z_0 be any point inside C. Then

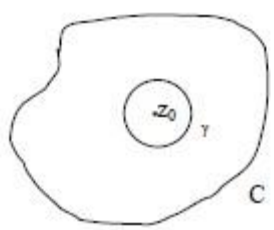
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

Proof. We consider the function $\frac{f(z)}{z-z_0}$ This function is analytic throughout the region bounded by C except at $z = z_0$.

Then, by 2.15, we have

$$\int_C \frac{f(z)}{z-z_0}dz = \int_\gamma \frac{f(z)}{z-z_0}dz$$

where γ is any closed contour inside C including the point z_0 as an interior point.



Let us choose γ to be the circle with centre z_0 and radius ρ. Since $f(z)$ is continuous, we can take ρ so small that on γ,

$$|f(z) - f(z_0)| < \in$$

where $\in$ is any preassigned positive number.

Now,

$$\int_\gamma \frac{f(z)}{z-z_0}dz = \int_\gamma \frac{[f(z)-f(z_0)]+f(z_0)}{z-z_0}dz$$

$$= f(z_0)\int_\gamma \frac{dz}{z-z_0}dz + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz \qquad (1)$$

For any point z on γ,

$$z - z_0 = \rho e^{i\theta} \Rightarrow dz = \rho i e^{i\theta} d\theta$$

$$\therefore \int_\gamma \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{\rho e^{i\theta}d\theta}{\rho e^{i\theta}} = \int_0^{2\pi} i d\theta = 2\pi i$$

and

$$\left|\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz\right| = \left|\int_0^{2\pi} \frac{[f(z)-f(z_0)]}{\rho e^{i\theta}}\rho e^{i\theta} i d\theta\right|$$

$$= \left|\int_0^{2\pi} [f(z)-f(z_0)] i d\theta\right|$$

$$< \in \int_0^{2\pi} d\theta = 2\pi \in$$

Hence from (1), we get

$$\left|\int_C \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0)\right| < 2\pi\in$$

Since $\in$ is arbitrarily small and L.H.S. is independent of $\in$, it follows that

$$\int_C \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = 0$$

i.e. $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$$

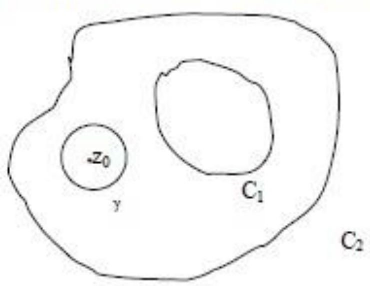
which proves the result.

Cor. (Extension of Cauchy's Integral Formula to Multiply Connected Regions): If $f(z)$ is analytic in a ring shaped region bounded by two closed contours C_1 and C_2 and z_0 is a point in the region between C_1 and C_2, then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}dz.$$

where C_2 is the outer contour.

Proof. Describe a circle γ of radius ρ about the point z_0 such that the circle lies in the ring shaped region. The function $\frac{f(z)}{z-z_0}$ is analytic in the region bounded by three close contours C_1, C_2 and γ.



Thus by 2.15, we have.

$$\int_{C_2} \frac{f(z)}{z-z_0}dz = \int_{C_1} \frac{f(z)}{z-z_0}dz + \int_{\gamma} \frac{f(z)}{z-z_0}dz$$

where the integral along each contour is taken in positive sense. Now, using Cauchy's integral formula, we find.

$$\int_{C_2} \frac{f(z)}{z-z_0}dz = \int_{C_1} \frac{f(z)}{z-z_0}dz + 2\pi i f(z_0)$$

or

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}dz.$$

Theorem (The derivative of an analytic function). Let $f(z)$ be analytic within and on a closed contour C and let z_0 be any point inside C, then

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

Proof. Let $z_0 + h$ be a point in the nighbourhood of z_0 and inside C, ($\Delta z = h$). Then Cauchy's Integral formula at these two points, gives

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$$

and
$$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0-h}dz$$

Subtracting the first result from second, we get

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-h)}dz \qquad (1)$$

We observe in (1) that as $h\to 0$, the required result follows. We have thus only to show that we can proceed to the limit under the integral sign. We consider the difference

$$\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-h)} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

$$= \frac{h}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^2(z-z_0-h)} \qquad (2)$$

Since $f(z)$ is analytic on C so $f(z)$ is bounded on C. Thus $|f(z)| \le M$ on C, M being an absolute positive constant. Let us denote the distance of z_0 from the points nearest to it on C by δ and the length of C by l. Then if $|h|<\delta$,

$$\left| h\int_C \frac{f(z)dz}{(z-z_0)^2(z-z_0-h)} \right| \le \frac{Ml\,|h|}{\delta^2(\delta-|h|)} \qquad (3)$$

which is bounded and tends to zero as $|h|\to 0$. Thus, taking limit as $|h|\to 0$, it follows from (2) that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

Hence $f(z)$ is differentiable at z_0 and

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

which is Cauchy's integral formula for $f'(z)$ at points within C.

Cauchy's Inequality (Cauchy's Estimate). If $f(z)$ is analytic within and on a circle C given by $|z - z_0| = R$ and if $|f(z)| \leq M$ for every z on C, then

$$|f^n(z_0)| \leq \frac{M\lfloor n}{R^n}$$

Proof. Since $f(z)$ is analytic inside C, we have by Cauchy's integral formula for nth derivative of an analytic function

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Since on the circle $|z - z_0| = R$,

$$z - z_0 = Re^{i\theta},\ dz = Re^{i\theta}\, id\theta$$

and the length of the circle is $2\pi R$, therefore

$$|f^n(z_0)| = \frac{\lfloor n}{2\pi}\left|\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}\right|$$

$$\leq \frac{\lfloor n}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}}$$

$$\leq \frac{\lfloor n}{2\pi}\int_0^{2\pi} \frac{M|Re^{i\theta}\, id\theta|}{|Re^{i\theta}|^{n+1}} = \frac{\lfloor n}{2\pi}\int_0^{2\pi} \frac{M}{R^n}d\theta$$

$$= \frac{\lfloor n}{2\pi}\frac{M}{R^n}2\pi = \frac{M\lfloor n}{R^n}$$

Hence $$|f^n(z_0)| \leq \frac{M\lfloor n}{R^n}$$

Liouville's Theorem. A function which is analytic in all finite regions of the complex plane, and is bounded, is identically equal to a constant.

or

If an integral function $f(z)$ is bounded for all values of z, then it is constant

or

The only bounded entire functions are the constant functions.

Proof. Let z_1, z_2 be arbitrary distinct points in z-plane and let C be a large circle with centre at origin and radius R such that C encloses z_1 and z_2 i.e. $|z_1| < R$, $|z_2| < R$.

Since $f(z)$ is bounded, there exists a positive number M such that $|f(z)| \leq M\ \forall$ z.
By Cauchy's integral formula,

$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_1}$$

$$f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_2}$$

$$f(z_2) - f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)(z_2-z_1)}{(z-z_2)(z-z_1)}dz$$

Thus

$$|f(z_2) - f(z_1)| \le \frac{|z_2 - z_1|}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_1||z-z_2|}$$
$$\le \frac{M|z_2 - z_1|}{2\pi}\int_C \frac{|dz|}{|z-z_1||z-z_2|}$$

$$\le \frac{M|z_2 - z_1|}{2\pi}\int_C \frac{|dz|}{(|z|-|z_1|)(|z|-|z_2|)} \quad |\because |z-z_1| \ge |z|-|z_1|$$

Now, on the circle C, $z = R\,e^{i\theta}$, $|z| = R$,
$dz = Re^{i\theta}\, i d\theta$

Therefore,

$$|f(z_2) - f(z_1)| \le \frac{M|z_2 - z_1|}{2\pi}\int_0^{2\pi} \frac{|Re^{i\theta}\, i d\theta|}{(R-|z_1|)(R-|z_2|)}$$
$$= \frac{M|z_2 - z_1|}{2\pi}\frac{R}{(R-|z_1|(R-|z_2|)}2\pi$$
$$= \frac{M|z_2 - z_1|}{\left(1-\frac{|z_1|}{R}\right)\left(1-\frac{|z_2|}{R}\right)}.\frac{1}{R}$$

which tends to zero as $R\to\infty$.
Hence $f(z_2) - f(z_1) = 0$ i.e. $f(z_1) = f(z_2)$
But z_1, z_2 are arbitrary, this holds for all couples of points z_1, z_2 in the z-plane, therefore $f(z) = \text{constant}$.

The Fundamental Theorem of Algebra. Any polynomial

$P(z) = a_0 + a_1 z + \ldots + z_n z^n$, $a_n \ne 0$, $n \ge 1$ has at least one point $z = z_0$ such that $P(z_0) = 0$ i.e. $P(z)$ has at least one zero.

Proof. We establish the proof by contradiction.

If $P(z)$ does not vanish, then the function $f(z) = \dfrac{1}{P(z)}$ is analytic in the finite z-plane. Also when $|z|\to\infty$, $P(z)\to\infty$ and hence $f(z)$ is bounded in entire complex plane, including infinity. Liouville's theorem then implies that $f(z)$ and hence $P(z)$ is a constant which violates $n \ge 1$ and thus contradicts the assumption that $P(z)$ does not vanish. Hence it is concluded that $P(z)$ vanishes at some point $z = z_0$

2.32. Remark. The above form of fundamental theorem of algebra does not tell about the number of zeros of $P(z)$. Another form which tells that $P(z)$ has exactly n zeros, will be discussed later on. Of course, here we can prove this result by using the process of algebra as follows :

By the fundamental theorem of algebra, proved above, P(z) has at least one zero say $z = z_0$ such that $P(z_0) = 0$
Then,

$$\begin{aligned} P(z) - P(z_0) &= a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \\ &\quad - (a_0 + a_1 z_0 + a_2 z_0^2 + \ldots a_n z_0^n) \\ &= a_1 (z - z_0) + a_2(z^2 - z_0^2) + \ldots + a_n (z^n - z_0^n) \\ &= (z - z_0)\, Q(z) \end{aligned}$$

where Q(z) is a polynomial of degree (n–1). Applying the fundamental theorem of algebra again, we note that Q(z) has at least one zero, say z_1 (which may be equal to z_0) and so $P(z) = (z-z_0)(z-z_1) R(z)$, where R(z) is a polynomial of degree (n–2). Continuing in this manner, we see that P(z) has exactly n zeros.

2.1. Cauchy's theorem for a rectangle. We now see the simplest version of Cauchy's Theorem, in the case of a rectangle $R = \{z = x + iy : a \le x \le b,\ c \le y \le d\}$. We denote by ∂R the boundary of R, oriented counter-clockwise.

Theorem 2.5. *Let Ω be a domain containing R. For any f holomorphic on Ω we have*

$$\int_{\partial R} f\, dz = 0\,.$$

Proof. For a rectangle $R' \subseteq \Omega$ we write

$$\eta(R') = \int_{\partial R'} f\, dz\,.$$

We divide the rectangle R into 4 rectangles $R^{(1)}, \ldots, R^{(4)}$ by bisecting each side into two equal segments.

Since the line integrals over the common sides cancel out, we obtain that

$$\eta(R) = \eta(R^{(1)}) + \cdots + \eta(R^{(4)})\,.$$

At least one rectangle $R^{(k)}$, $k = 1, \ldots, 4$ must satisfy

$$|\eta(R^{(k)})| \ge |\tfrac{1}{4}\eta(R)|\,.$$

We call this rectangle R_1. By repeating this construction we obtain a sequence of rectangles $R_1, R_2, \ldots$ such that:

(i) $R \supset R_1 \supset R_2 \supset \cdots$;
(ii) $|\eta(R_n)| \ge \frac{1}{4}|\eta(R_{n-1})|$, so that $|\eta(R_n)| \ge 4^{-n}|\eta(R)|$;
(iii) if p_n and d_n denote the perimeter and the diameter of R_n, respectively, and p, d the ones of R, then $p_n = 2^{-n}p$ and $d_n = 2^{-n}d$.

By the Bolzano-Weierstrass theorem, $\cap_n R_n$ is non-empty, and since $d_n \to 0$, $\cap_n R_n$ cannot contain two distinct points. Therefore, there exists $\zeta \in R$ such that $\cap_n R_n = \{\zeta\}$.

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $D(\zeta, \delta) \subseteq \Omega$ and, since the function f is holomorphic in Ω, such that

$$\big|f(z) - f(\zeta) - (z-\zeta)f'(\zeta)\big| < \varepsilon|z-\zeta|$$

for $z \in D(\zeta, \delta)$.

Recall that, from Cor. 2.4 we know that

$$\int_{\partial R_n} dz = \int_{\partial R_n} (z-\zeta)\, dz = 0\,.$$

Now, there exists n_0 such that for $n \geq n_0$ R_n is contained in $D(\zeta, \delta)$, and then, if $z \in \partial R_n$, $|z-\zeta| \leq d_n$. Therefore, by (ii) and (iii) above,

$$\begin{aligned}|\eta(R_n)| &= \Big|\int_{\partial R_n} \big(f(z) - f(\zeta) - (z-\zeta)f'(\zeta)\big)\, dz\Big| \\ &\leq \varepsilon \int_{\partial R_n} |z-\zeta|\, |dz| \\ &\leq \varepsilon d_n p_n \\ &\leq \varepsilon 4^{-n} dp\,.\end{aligned}$$

It then follows that

$$|\eta(R)| \leq 4^n |\eta(R_n)| \leq \varepsilon dp\,.$$

Since $\varepsilon > 0$ was arbitrary, the theorem is proven.

Theorem 2.6. *Let Ω and R be as in Thm. 2.5. Let f be holomorphic in the domain Ω' obtained removing from Ω a finite number of points ζ_j, $j = 1, \ldots, n$, lying in the interior of R, and assume that*

$$\lim_{z\to\zeta_j} (z-\zeta_j) f(z) = 0$$

for $j = 1, \ldots, n$. Then

$$\int_{\partial R} f\, dz = 0\,.$$

Proof. We first argue that it suffices to consider the case of a single exceptional point ζ. In fact, we can divide the rectangle R as finite union of rectangles R_j, each containing a single exceptional point ζ_j, $j = 1, \ldots, n$, and observe again that

$$\int_{\partial R} f\, dz = \sum_{j=1}^{n} \int_{\partial R_j} f\, dz\,.$$

So, let us assume that we have a single exceptional point ζ inside R. We divide R as union of nine rectangles, in such a way that the central one is a square R_0 centered at ζ and has side lengths to be fixed. Then,

$$\int_{\partial R} f\,dz = \int_{\partial R_0} f\,dz + \sum_{j=1}^{8} \int_{\partial R_j} f\,dz$$
$$= \int_{\partial R_0} f\,dz$$

by applying Thm. 2.5 to the integrals $\int_{\partial R_j} f\,dz$, $j = 1, \ldots, 8$.

Given $\varepsilon > 0$ we fix the side lengths of R_0 to be small enough so that

$$|z - \zeta|\,|f(z)| \le \varepsilon$$

for $z \in \partial R_0$. We then have

$$\Big|\int_{\partial R} f\,dz\Big| = \Big|\int_{\partial R_0} f\,dz\Big| \le \int_{\partial R_0} |f(z)|\,|dz|$$
$$\le \varepsilon \int_{\partial R_0} \frac{1}{|z-\zeta|}\,|dz|$$
$$\le 8\varepsilon\,,$$

since R_0 is a square, as an elementary argument shows. This proves the theorem.

2.2. **Cauchy's theorem in a disk.** We denote by $D = D(z_0, r)$ the open disk having center z_0 and radius $r > 0$; that is,

$$D(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}.$$

Theorem 2.7. *Let f be holomorphic in an open disk D. Then*

$$\int_\gamma f(z)\,dz = 0$$

for all closed curves γ contained in D.

Proof. We are going to use Thm. 2.5. For any $z = x + iy \in D$, let $\sigma = \sigma_z$ be the curve in D consisting of the horizontal segment from (x_0, y_0) to (x, y_0) followed by the vertical segment from (x, y_0) to (x, y). Define

$$F(z) = \int_{\sigma_z} f\,dz\,.$$

Then F is well defined and we can easily compute that

$$\partial_y F(z) = if(z)\,.$$

By Thm. 2.5, since f is holomorphic on D, we have that

$$F(z) = \int_{\sigma_z} f\,dz = \int_{\tau_z} f\,dz$$

where τ_z is the curve consisting of the vertical segment from (x_0, y_0) to (x_0, y) followed by the horizontal segment from (x_0, y) to (x, y). Computing the partial derivatives in x of F we obtain that $\partial_x F(z) = f(z)$. Since the partial derivatives of F are continuous and satisfy the CR-equation, F is holomorphic in D, and its derivative is f.

Therefore, $f(z)dz$ is an exact differential and

$$\int_\gamma f(z)\,dz = 0$$

Theorem 2.9. *Let f be holomorphic in D' obtained removing from an open disk D a finite number of points ζ_j, $j = 1, \ldots, n$, and assume that*

$$\lim_{z\to\zeta_j} (z - \zeta_j) f(z) = 0$$

for $j = 1, \ldots, n$. Then

$$\int_\gamma f\,dz = 0$$

for every closed curve γ contained in D'.

Proof. This proof now follows from the previous arguments. First we can reduce to the case of a single ecceptional point ζ. Then we only need to make sure that the curve γ does not pass through ζ. Having fixed $z_0 \in D'$, given $z \in D'$, if the the rectangle with oppositive vertices in z_0 and z passes through ζ, we can still easily define the indefinite integral F of f on D'. We leave

the simple detail to the reader

Cauchy's formula. We begin with the notion of *index of a point* with respect to a curve.

Let γ be a closed curve and let z_0 be a point not lying on γ. Then the integral

$$\int_\gamma \frac{dz}{z - z_0}$$

is an integral multiple of $2\pi i$.

Proof. Let $\gamma : [a, b] \to \mathbf{C}$ and define

$$h(t) = \int_a^t \frac{\gamma'(\tau)}{\gamma(\tau) - z_0}\, d\tau\,.$$

We wish to show that there exists an integer k such that

$$\int_\gamma \frac{dz}{z - z_0} = h(b) = 2\pi i k\,.$$

The function h is defined and continuous on $[a, b]$, $h(a) = 0$ and

$$h'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}$$

on the interval $[a, b]$ taken away a finite number of points where $\gamma(t)$ is not differentiable. It follows that

$$\begin{aligned}\frac{d}{dt}(e^{-h(t)}(\gamma(t) - z_0)) &= e^{-h(t)}(-h'(t)(\gamma(t) - z_0) + \gamma'(t)) \\ &= 0\end{aligned}$$

except at those points $t_1, \ldots, t_n$ where $\gamma(t)$ is not differentiable. Therefore, $e^{-h(t)}(\gamma(t) - z_0)$ is constant on each connected component of $[a, b] \setminus \{t_1, \ldots, t_n\}$. Since $e^{-h(t)}(\gamma(t) - z_0)$ is also continuous, it follows that it is constant on $[a, b]$; that is,

$$e^{-h(t)}(\gamma(t) - z_0) = c\,.$$

Since $h(a) = 0$, $c = \gamma(a) - z_0$, so that

$$e^{h(t)} = \frac{\gamma(t) - z_0}{\gamma(a) - z_0}\,.$$

Now, using the fact that $\gamma(b) = \gamma(a)$ we have $e^{h(b)} = 1$, so that

$$h(b) = 2\pi i k$$

(Morera) *Let f be continuous on a domain Ω. Suppose that*

$$\int_\gamma f(\zeta)\, d\zeta = 0$$

for all closed curves γ in Ω. Then, f is holomorphic in Ω.

Recall that for a power series expansion $f(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ we have that $f^{(n)}(z_0) = n!a_n$. Then, we just have obtained the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\,,$$

valid when $\gamma = \partial D(z_0, r) \subseteq \Omega$. More generally we have

Definition :

If $f = u + iv$ is a continuous complex-valued function defined on an interval $[a, b]$ on the real line, we set

$$\int_a^b f(t)\, dt = \int_a^b u(t)\, dt + i\int_a^b v(t)\, dt\,.$$

Then the mapping $f \mapsto \int_a^b f(t)dt$ is complex linear.

Theorem:

Let $f \in C\big([a, b]\big)$. *Then*

$$\Big|\int_a^b f(t)\, dt\Big| \leq \int_a^b |f(t)|\, dt\,.$$

Proof. The proof is simple. If $\int_a^b f(t)dt = 0$ we have nothing to prove.
Otherwise, let $\alpha \in \mathbf{C}$, $|\alpha| = 1$. Then

$$\begin{aligned}\mathrm{Re}\left(\alpha\int_a^b f(t)\, dt\right) &= \int_a^b \mathrm{Re}\left(\alpha f(t)\right) dt \leq \int_a^b \big|\mathrm{Re}\left(\alpha f(t)\right)\big|\, dt \\ &\leq \int_a^b \big|\alpha f(t)\big|\, dt = \int_a^b |f(t)|\, dt\,.\end{aligned}$$

SIX MARKS

1. Prove that If f(z) is analytic in an open disk Δ, then $\int_{\gamma} f(z)dz = 0$ for every closed curve γ in Δ.

2.State and prove fundamental theorem of algebra.

3.Prove that f(z) be analytic on the set R' obtained from the rectangle R by omitting a finite number of interior points ζ_j. If it is true that $\lim_{z\to \zeta j}(z - \zeta j)$f(z) $= 0$ for all j, then $\int_{\partial R} f(z)\, dz = 0$.

4. Show that an analytic function f(z) has derivative of all ordered which are analytic can be represented by these formula $\frac{n!}{2\pi i}\int_C \frac{(\zeta)}{(\zeta - z)^{+1}}\, d\zeta$ where c is a circle about a point 'z' and z belongs to an arbitrary region in Ω.

5. State and Prove Cauchy's theorem for rectangle.

6.State and prove Morera's theorem.

7.State and prove Fundamental theorem of algebra.

8. Show that the line $\int_{\gamma} p\, dx + q dy$, defined in Ω depends only on the end points of γ iff there exist a function u(x, y) in Ω with the partial derivative $\frac{\partial U}{\partial x} = p, \frac{\partial U}{\partial y} = q$.

9.State and prove Cauchy's estimate theorem.

10.State and prove Liouville's theorem.

11.State and Prove Cauchy's theorem for disk.

TEN MARKS

1. State and Prove Cauchy theorem for Rectangle.

2.Write about Properties of complex integral.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The functions of the form, $P_n(Z)= a_0+a_1z+a_2z^2+\ldots\ldots+a_nz^n$, $a_n\neq0$ is called a….	polynomial of degree n	polynomial of degree 5	polynomial of degree 2n	polynomial of degree n-1	polynomial of degree n
If f (z) and g(z) are continuous at z_0 then f(z).g(z) is………	Continuous at z_0	differentiable at z_0	Continuous at z	differentiable at z	Continuous at z_0
f(z) = z^2 is a ------------------ valued function.	single	multi	double	many	double
If f(z) of f has only one value it is called ----------- valued function.	single	multi	double	many	single
If │f(z) │ < M for all z in s , then f(z) is said to --------------in S.	multi valued	continuous	bounded	analytic	bounded
The limit of a function is -----------	unique	does not exist	different	multivalued	unique
If $\|f(z) –f(z_0)\| < \epsilon$ for all z in S with $\|z –z_0\| <\delta$ then f(z) is …..	bounded	continuous	unique	does not exist	continuous
If f (z) and g(z) are continuous at z_0 then f(z) ±g(z) is ……….	Continuous at z_0	differentiable at z_0	Continuous at z	differentiable at z	Continuous at z_0
If f (z) and g(z) are continuous at z_0 then f(z) /g(z) is ……..	Continuous at z_0	differentiable at z_0	Continuous at z	differentiable at z	Continuous at z_0
In a compact set every continuous function is………..	bounded in s	uniformly continuous in s	unique	does not exist	bounded in s
If $\|f(z_1) –f(z_2)\| < \epsilon$ for all z_{1}, z_2 S with $\|z_1 –z_2\| <\delta$ then f(z) is…….	bounded in s	uniformly continuous in s	unique	does not exist	uniformly continuous in
If a function is differentiable at all points in some neighbourhood of a point , then the function is said to be ---- at that point.	bounded	analytic	differentiable	compact	analytic
A function which is analytic everywhere in the finite plane is called an ----------------- function.	single	multi	entire	continuous	entire
f(z) is a function differentiable at z0 , then f(z) is	Continuous at z_0	compact at z	Continuous at z	differentiable at z	Continuous at z_0
A ---- point of a function is a point at which the function ceases to be analytic	non singular	Singular	entire	continuous	Singular
f(z) = $\|z\|^2$ is ---------------- everywhere	analytic	not analytic	continuous	exist	not analytic
d/dz{cf(z)}……………	$cf^1(z)$	$f^1(z)$	$f^1(z)+c$	$f^1(z) /c$	$cf^1(z)$
The quotient of two polynomials is called a	Exponential function	logarithmic function	Continuous function	rational function	rational function
If f(z) and g(z) are continuous at z_0 then f(z)/g(z), g(z)≠0 is	Continuous at z_0	differentiable at z_0	Continuous at z	differentiable at z	Continuous at z_0
If f(1/z) is analytic at 0 then f(z) is	Analytic at ∞	Continuous at ∞	Differentiable at	Differentiable at	Analytic at ∞
The cartesian coordinates of C-R equations are	$u_x=v_y$ and $u_y=-v_x$	$u_x=v_y$ and $u_y=-v_x$	$u_x=v_y$ and $u_x=-v_x$	$u_x=1$ and $u_y=-v_x$	$u_x=v_y$ and $u_y=-v_x$
A function of complex variable is sometimes called a	complex variable	variable	complex function	constant	complex function
If the product of the slopes is -1, then the curves cut each other ------	diagonally	orthogonally	at the origin	at the point 1	orthogonally
The function that is multiple valued .	f(z) = z^2	f(z) = e^z	f(z) = 1/z	f(z) = $z^{1/2}$	f(z) = $z^{1/2}$
logz is a ------------ valued function	single	multi	double	three	multi
If f(z) = $1/z^2$ then ………	0	2	1	-1	0
If $f(z_0) = \infty$, the function f(z) is …………… at $z = z_0$ $\lim_{z\to\infty} f(z) =$	continuous	not continuous	differentiable	bounded	not continuous

The function f(z) = Re z/│z │, when z ≠0 ; f(z) = 0 when f(z) = 0 is …………….	continuous	not continuous	differentiable	bounded	not continuous
The function │z │^{2} is ……………. at that point.	continuous	analytic	not analytic	bounded	not analytic
If f(z) = u +iv is analytic , then u(x,y) and v(x,y) are …………….. Functions	harmonic	analytic	continuous	bounded	harmonic
The function f(z) = log z,then u(r,θ) = …… v(r,θ) = ...	log θ , log r	r, log θ	log r , θ	r,θ	log r , θ
If f(z) = 1/z then	∞	-1	0	1	0

UNIT III

SYLLABUS

Harmonic functions-mean value property-Poisson's formula-Schwarz theorem, Reflection principle-Weierstrass theorem- Taylor series and Laurent series.

Harmonic functions

In this section we return to one aspect of the theory that concerns the analysis of harmonic functions, subject often called *potential theory.*

Recall that a C^2 function u on an open set $A \subseteq \mathbf{R}^2$ is said to be harmonic on A if $\Delta u = 0$ on A, where $\Delta = \partial_x^2 + \partial_y^2$ is the *Laplacian.* The next lemma collects the first elementary but fundamental facts about the relation between harmonic and holomorphic functions.

Theorem:

If $f = u + iv$ is holomorphic on an open set $A \subseteq \mathbf{C}$ then its real and imaginary parts u and v are harmonic on A.

If u is a real harmonic function on a simply connected *open set $\mathcal{D}$, then there exists a real harmonic function v on $\mathcal{D}$ such that $u + iv$ is holomorphic on $\mathcal{D}$. In this case, we will say that v is the harmonic* conjugate *of u on $\mathcal{D}$.*

Proof. The first part follows from Subsection 1.2 .

Suppose now u is a real harmonic function on a simply connected open set $\mathcal{D}$. We wish to $v \in C^2(\mathcal{D})$ satisfying the CR-equations on $\mathcal{D}$, that is, such that

$$dv = (-\partial_y u)dx + (\partial_x u)dy \,.$$

The one on the right hand side is a closed differential since u is harmonic. Since $\mathcal{D}$ is simply connected, it is an exact differential, so such a v exists. It immediately follows that $u + iv$ is

holomorphic.

We remark that the hypothesis of $\mathcal{D}$ being simply connected cannot be relaxed. As an example, consider $A = \mathbf{C} \setminus \{0\}$ and $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$. Then u is real and harmonic. On $A \cap \{x + iy : x > 0\}$ is the real part of $\log z$, that cannot be extended to all of A. Hence, there exists no function holomorphic on A whose real part is u.

Maximum principle. We now prove the maximum principle for (real) harmonic functions.

Theorem:

Let $\Omega \subseteq \mathbf{C} \equiv \mathbf{R}^2$ be a domain (connected open set), $u : \Omega \to \mathbf{R}$ be harmonic. If *there exists* $z_0 \in \Omega$ *and* $r_0 > 0$ *such that* $D(z_0, r_0) \subseteq \Omega$ *and* $u(z_0) = \sup\{u(z) : z \in D(z_0, r_0)\}$, *then* u *is constant on* Ω.

Proof. Let

$$\Omega' = \{z \in \Omega : \text{ there exists } r_z > 0 \text{ such that for } w \in D(z, r_z),\, u(w) = u(z_0)\}.$$

We wish to show that Ω' is open, closed in Ω and non-empty, thus showing that $\Omega' = \Omega$.

On $D(z_0, r_0)$ we can find h holomorphic such that $\operatorname{Re} h = u$. Take $f = e^h$. Since $|f| = e^{\operatorname{Re} h} = e^u$, $|f|$ attains its maximum at z_0. Hence f is constant on $D(z_0, r_0)$, so is u. Thus, $\Omega' \neq \emptyset$. Moreover, Ω' is open by construction.

Finally, let $z \in \overline{\Omega'}$. Let $D(z, r_z) \subseteq \Omega$. Since $z \in \overline{\Omega'}$, there exists some open disk on which u is constant. Let h_z be the holomorphic function on $D(z, r_z)$ whose real part is u. Then, h_z is constant on an open disk, hence on all of $D(z, r_z)$, so is u. Thus, $z \in \Omega'$, Ω' is closed, that is,

$\Omega' = \Omega$.

Theorem:

(The Poisson formula for the disk) *Let* $A \subseteq \mathbf{C}$ *be open,* $\overline{D(0, R)} \subseteq A$, u *be harmonic on* A. *Then for every* $z \in D(0, R)$ *we have*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\theta\,.$$

(i) The function

$$P_z(\zeta) = \frac{1}{2\pi} \cdot \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2},$$

defined for $z \in D(0, R)$ and $\zeta \in \partial D(0, R)$ is called the *Poisson kernel* for the disk $D(0, R)$ polar coordinates it has the expression

$$\begin{aligned} P_{re^{i\eta}}(Re^{i\theta}) &= \frac{1}{2\pi} \cdot \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\eta}|^2} \\ &= \frac{1}{2\pi} \cdot \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \eta) + r^2}\,. \end{aligned}$$

(ii) The Poisson kernel $P_{re^{i\eta}}(Re^{i\theta})$ is a positive kernel, that is,

$$P_{re^{i\eta}}(Re^{i\theta}) > 0$$

for all $0 < r < R$, $\eta, \theta \in [0, 2\pi]$.

(iii) If $\zeta = Re^{i\theta}$ and $z = re^{i\eta}$, then

$$P_{re^{i\eta}}(Re^{i\theta}) = \frac{1}{2\pi}\mathrm{Re}\,\frac{\zeta + z}{\zeta - z} = \frac{1}{2\pi}\mathrm{Re}\,\frac{Re^{i\theta} + re^{i\eta}}{Re^{i\theta} - re^{i\eta}}\,.$$

This follows at once from (7.1), since

$$\frac{\zeta + z}{\zeta - z} = \frac{\zeta + z}{\zeta - z} \cdot \frac{\bar{\zeta} - \bar{z}}{\bar{\zeta} - \bar{z}} = \frac{|\zeta|^2 - |z|^2 + (\bar{\zeta}z - \zeta\bar{z})}{|\zeta - z|^2}\,.$$

(iv) Finally, since the constant function $u(z) = 1$ is harmonic, from the reproducing property in Thm. 7.6 we see that

$$\frac{1}{2\pi}\int_0^{2\pi} P_z(Re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,d\theta = 1\,.$$

Proof of Thm. 7.6. Let $s > R$ and h be the holomorphic function on $D(0, s)$ such that $u = \mathrm{Re}\,h$. For $z \in D(0, R)$, by Cauchy's formula, letting $\gamma(\theta) = Re^{i\theta}$, with $\theta \in [0, 2\pi]$, we have

$$h(z) = \frac{1}{2\pi i}\int_\gamma \frac{h(\zeta)}{\zeta - z}\,d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\frac{Re^{i\theta}}{Re^{i\theta} - z}\,d\theta\,.$$

Moreover, if we set $w = R^2/\bar{z}$, we observe that $w = \frac{R^2}{r^2}z$, where $z = re^{i\eta}$, and that the function

$$\zeta \mapsto \frac{h(\zeta)}{\zeta - w}$$

is holomorphic on $\overline{D(0, R)}$, since $|w| = \frac{R^2}{r} > R$. Therefore,

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{h(\zeta)}{\zeta - w}\,d\zeta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\frac{Re^{i\theta}}{Re^{i\theta} - \frac{R^2}{\bar{z}}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\frac{\bar{z}}{\bar{z} - Re^{-i\theta}}\,d\theta\,.$$

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\left(\frac{Re^{i\theta}}{Re^{i\theta}-z} - \frac{\bar{z}}{\bar{z}-Re^{-i\theta}}\right)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\,d\theta.$$

By passing to real and imaginary parts we obtain

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\cdot\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\,d\theta,$$

The Weierstrass factorization theorem.

Definition

We define the *Weierstrass elementary factors* as $E(z,0) = 1-z$ and for

$n = 1,2,\ldots, \quad E(z,n) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^n}{n}}$.

Theorem

Let $\{z_j\} \subseteq \mathbf{C}$, $\{p_j\} \subseteq \mathbf{N}$ be chosen as above. Then the Weierstrass product

$$\prod_{j=1}^{+\infty} E(z/z_j, p_j)$$

converges uniformly on every set $\{|z| \le r\}$, $r > 0$, to a holomorphic entire function F. The zeros of F are precisely the points $\{z_j\}$ counted with the corresponding multiplicity.

Proof. Let $r > 0$ be fixed. Let j_0 be such that $|z_j| > r$ for $j \ge j_0$. Thus,

$$|E(z/z_j, p_j) - 1| \le \left|\frac{z}{z_j}\right|^{p_j+1} \le \left(\frac{r}{|z_j|}\right)^{p_j+1}.$$

By the hypothesis on the p_j's,

$$\sum_{j=j_0}^{+\infty} |E(z/z_j,p_j) - 1| \le \sum_{j=j_0}^{+\infty}\left(\frac{r}{|z_j|}\right)^{p_j+1} < +\infty.$$

Weierstrass's M-test implies that $\sum_{j=j_0}^{+\infty} |E(z/z_j,p_j)-1|$ converges uniformly on $\{|z| \le r\}$, for any $r > 0$. Thm. 8.6 now implies that

$$\prod_{j=1}^{+\infty} E(z/z_j,p_j) = \prod_{j=1}^{j_0-1} E(z/z_j,p_j)\prod_{j=j_0}^{+\infty} E(z/z_j,p_j)$$

converges uniformly on compact subsets of $\mathbf{C}$ to an entire function F whose zeros are precisely the zeros of the $E(z/z_j, p_j)$'s. □

Corollary

Let $\{z_j\}$ be a sequence such that $|z_j| \to +\infty$. Then there exists an entire function F whose zeros are precisely the $\{z_j\}$, counting multiplicity.

Proof. We may assume that $z_1 = \cdots = z_k = 0$, and $z_j \neq 0$ for $j > k$. Let $p_j = j - 1$.
Let $r > 0$ be fixed. Let $N = N(r)$ be such that $|z_j| > 2r$ for $j \geq N$. Then

$$\sum_{j=N}^{+\infty} \left(\frac{r}{|z_j|}\right)^j \leq \sum_{j=N}^{+\infty} \frac{1}{2^j} < +\infty .$$

Thus, by Thm. 8.9, the function

$$F(z) = z^k \prod_{j=k+1}^{+\infty} E(z/z_j, j-1)$$

(Weierstrass' Factorization Theorem) *Let f be an entire function. Suppose that f vanishes of order k at the origin. Let $\{z_j\}$ be the other zeros of f, counting multiplicity. Then there exists an entire function g such that*

$$f(z) = z^k e^{g(z)} \prod_{j=1}^{+\infty} E(z/z_j, j-1) .$$

Proof. By the Cor. 8.10, the function $h(z) = z^k \prod_{j=1}^{+\infty} E(z/z_j, j-1)$ is entire and has the same zeros as f. Hence, the function f/h can be extended to an entire function, with no zero. Since $\mathbf{C}$ is simply connected, $\log(f/h) = g$ is well defined and entire.
Hence, $e^g = f/h$, that is,

$$f(z) = h(z)e^{g(z)} = z^k e^{g(z)} \prod_{j=1}^{+\infty} E(z/z_j, j-1) . \quad \square$$

We now apply this result to describe an identity that describes the factorization of $\sin z$.

We have

$$\sin \pi z = \pi z \prod_{n\neq 0} E(z/n, 1)$$
$$= \pi z \prod_{n\neq 0}\Big(1 - \frac{z}{n}\Big)e^{z/n} = \pi z \prod_{n=1}^{+\infty}\Big(1 - \frac{z^2}{n^2}\Big).$$

In order to prove the above identity we need a preliminary result.

Lemma

(i) $\pi \cot \pi z = \pi \dfrac{\cos \pi z}{\sin \pi z} = \dfrac{1}{z} + \sum_{n\neq 0}\Big(\dfrac{1}{z-n} + \dfrac{1}{n}\Big);$

(ii) $\dfrac{\pi^2}{\sin^2 \pi z} = \sum_{n\in \mathbf{Z}} \dfrac{1}{(z-n)^2}$.

Proof. (i) The function

$$f_1(z) = \frac{1}{z} + \sum_{n\neq 0}\Big(\frac{1}{z-n} + \frac{1}{n}\Big)$$

is meromorphic in $\mathbf{C}$ and having simple poles at the integers, with residues all equal to 1. The function

$$f_2(z) = \pi \frac{\cos \pi z}{\sin \pi z}$$

is also meromorphic in $\mathbf{C}$ and having simple poles at the integers, with residues all equal to 1.

Hence, $h(z) = f_1(z) - f_2(z)$ is entire. It is immediate to check that

$$h'(z) = -\sum_{n\in \mathbf{Z}} \frac{1}{(z-n)^2} + \frac{\pi^2}{\sin^2 \pi z}$$

is periodic of period 1 (and it would not be so obvious that h is periodic of period 1).

We wish to prove that $h(z) \equiv 0$. We begin by showing that h' is constant, and equal to 0. In order to show that h' is constant, we show that h' is bounded and then invoke Liouville's theorem. Being periodic of period 1, h' is bounded if and only if it is bounded in the strip $\{z = x + iy : 0 \le x \le 1\}$. But, on the compact set $\{z = x + iy : 0 \le x \le 1, |y| \le 1\}$ h' is certainly bounded. For $|y| > 1$ and $0 \le x \le 1$, the sum

$$\sum_{n\in\mathbf{Z}} \frac{1}{|x+iy-n|^2}$$

is finite. Moreover, since $\frac{1}{|x+iy-n|^2} \le \frac{1}{y^2+n^2} \le \frac{1}{1+n^2}$, for $n < 0$, while $\frac{1}{|x+iy-n|^2} \le \frac{1}{1+(n-1)^2}$ for $n \ge 2$, we can apply Lebegue's dominated convergence theorem to obtain that

$$\sum_{n\in\mathbf{Z}} \frac{1}{|x+iy-n|^2} \to 0 \qquad \text{as } |y| \to +\infty\,.$$

The same is true for the function $\frac{\pi^2}{\sin^2 \pi z}$. Recall that

$$|\sin \pi z| = \frac{|e^{i\pi z} - e^{-i\pi z}|}{2} \ge \frac{e^{\pi|y|} - e^{-\pi|y|}}{2} = \sinh \pi|y| \to +\infty$$

as $|y| \to +\infty$. Then, $\left|\frac{\pi^2}{\sin^2 \pi z}\right| \to 0$ as $|y| \to +\infty$ in the set $\{z = x + iy : 0 \le x \le 1, |y| > 1\}$.

This proves that h' is bounded, hence constant. But, since h' tends to 0 as $|y| \to +\infty$, the constant must be 0. This proves (ii).

Thus, h is constant, and it is 0, since h vanishes at the integers. This proves (i), and we are done. □

Proof of Prop. 8.12. Notice that the last equality in (8.4) follows at once.

For $n \in \mathbf{Z}$, let $z_n = n$. By Thm. 8.9 the function

$$f(z) = \pi z \prod_{n\neq 0} \Big(1 - \frac{z}{n}\Big) e^{z/n}$$

is entire, having simple zeros at the integers. Let $z_0 \in \mathbf{C} \setminus \mathbf{Z} = \Omega$. Since $f(z_0) \neq 0$, there exists a disk $\overline{D(z_0, r_0)} \subseteq \Omega$ on which $\log f(z)$ is well defined and holomorphic. On such disk,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{d}{dz} \log f(z) = \frac{d}{dz}\Big[\log \pi z + \sum_{n\neq 0}\Big(\log(1 - z/n) + \frac{z}{n}\Big)\Big] \\ &= \frac{1}{z} + \sum_{n\neq 0}\Big(\frac{1}{z-n} + \frac{1}{n}\Big) \\ &= \pi \cot \pi z\,, \end{aligned}$$

by the previous lemma. On the same disk $\overline{D(z_0, r_0)}$ $\log \sin \pi z$ is well defined and its derivative equals $\pi \cot \pi z$. Then, there exists a constant C such that $\log f(z) = \log \sin \pi z + C$, that is, $f(z) = C_1 \sin \pi z$ on $\overline{D(z_0, r_0)}$; hence on Ω and therefore on all of $\mathbf{C}$. Since $\lim_{z\to 0} \frac{f(z)}{\sin \pi z} = 1$,

$C_1 = 1$ and we are done.

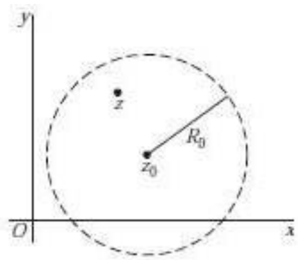
3.2 Taylor Series

We begin with the ***Taylor's theorem***.

Theorem 3.2.1.

Suppose that a function f is analytic throughout a disk $|z-z_0| < R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$f(z) = \Sigma_{n=1}^{\infty} a_n(z-z_0)^n \qquad (|z-z_0| < R_0), \text{ where } a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, ...).$$



i.e., $\Sigma_{n=1}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ when z lies in the open disk $|z-z_0| < R_0$. (This expansion of $f(z)$ is called the **Taylor series** of $f(z)$ about the point z_0.) *Since $f^{(0)}(z_0) = f(z_0)$ and $0! = 1$, the Taylor series of $f(z)$ about the point z_0 can be written as* $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + ...,\ (|z-z_0| < R_0)$.

Remark.

A Taylor's series about the point $z_0 = 0$, $f(z) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad (|z| < R_0)$ is called a ***Maclaurin series***.

Any function which is analytic at a point z_0 must have a Taylor series about z_0. For, if f is analytic at z_0, it is analytic throughout some neighborhood $|z-z_0| < \varepsilon$ of z_0. Therefore by Taylor's theorem, $f(z)$ have a Taylor series about z_0 valid in $|z-z_0| < \varepsilon$. Also, if f is entire, R_0 can be chosen arbitrarily large, and the condition of validity becomes $|z-z_0| < \infty$ and the Taylor series then converges to $f(z)$ at each point z in the finite plane. If f is analytic everywhere inside a circle centered at z_0, then the Taylor series of $f(z)$ about z_0 converges to $f(z)$ for each point z within that circle and in fact, according to Taylor's theorem, the series converges to $f(z)$ within the circle about z_0 whose radius is

the distance from z_0 to the nearest point z_1 at which f fails to be analytic.

Example 7.

Consider the function $f(z) = e^z$. Since $f(z) = e^z$ is an entire function, it has a Maclaurin series representation which is valid for all z. Here, $f^{(n)}(z) = e^z$ $(n = 0, 1, 2, ...) \Rightarrow f^{(n)}(0) = 1$, $(n = 0, 1, 2, ...)$
Therefore, $e^z = \Sigma_{n=0}^{\infty} \frac{z^n}{n!}$, $(|z| < \infty)$.

Example 8.

Let $f(z) = \frac{1}{1-z}$. Then, the derivatives of the function $f(z) = \frac{1}{1-z}$, which fails to be analytic at $z = 1$, are $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ $(n = 0, 1, 2, ...)$. $\Rightarrow f^{(n)}(0) = n!$ $(n = 0, 1, 2, ...)$.. Therefore, $f(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + ...$ $(|z| < 1)$.

3.3 Laurent Series

If a function f fails to be analytic at a point z_0, but it is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, then the power series representation for $f(z)$ involves both positive and negative powers of $z - z_0$. Such a series representation for $f(z)$ is called a ***Laurent's series***.

THEOREM ON LAURENT SERIES

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

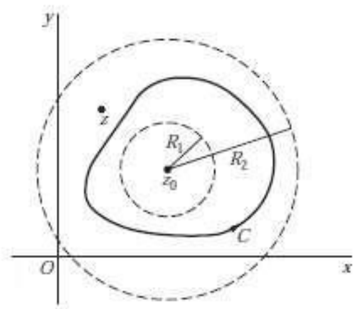
$$f(z) = \Sigma_{n=0}^{\infty} a_n(z - z_0)^n + \Sigma_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2),$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, ...)$$

and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, ...).$$



Remark.

Replacing n by $-n$ in the second series in the above Laurent's series enables us to write that series as

$$\Sigma_{n=-\infty}^{-1} \frac{b_{-n}}{(z - z_0)^{-n}},$$

where

$$b_{-n} = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = -1, -2, ...).$$

Thus, we have

$$f(z) = \Sigma_{n=-\infty}^{-1} b_{-n}(z - z_0)^n + \Sigma_{n=1}^{\infty} a_n(z - z_0)^n \quad (R_1 < |z - z_0| < R_2).$$

Or, we can write

$$f(z) = \Sigma_{n=-\infty}^{\infty} c_n(z - z_0)^n \quad (R_1 < |z - z_0| < R_2),$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, ...).$$

When the annular domain is specified, it can be proved that a Laurent's series for a given function is unique. This fact helps us to found the coefficients in a Laurent's series by means other than appealing directly to their integral representations. We illustrate this through the following examples.

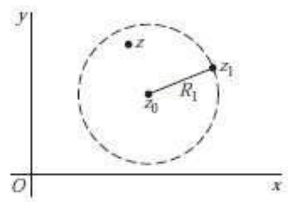
Absolute and Uniform Convergence of Power Series

We will now discuss basic properties of power series.

A natural question is to determine the set of complex numbers z for which a given power series converges. We have the following theorem.

THEOREM:

If a power series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z = z_1$ $(z_1 \neq z_0)$, then it is absolutely convergent at each point z in the open disk $|z-z_0| < R_1$ where $R_1 = |z_1 - z_0|$.



Analogous to the concept of an interval of convergence in real calculus, a complex power series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ has a circle of convergence defined by $|z-z_0| = R$ for some $R \geq 0$.

The above theorem implies that the set of all points inside some circle centered at z_0 is a region of convergence for the above power series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$, provided it converges at some point other than z_0.

The greatest circle centered at z_0 such that series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ converges at each point inside is called the ***circle of convergence*** of the series.

The series cannot converge at any point z_2 outside that circle, according to the theorem ; for if it did, it would converge everywhere inside the circle centered at z_0 and passing through z_2. The first circle could not, then, be the circle of convergence.

The power series converges absolutely for all z satisfying $|z-z_0| < R$ and diverges for $|z-z_0| > R$. Here R is called the radius of convergence of the power series. The radius R of convergence can be (a) zero (in which case $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ converges only at $z = z_0$), (b) a finite number (in which case the given power series converges at all interior points of the circle $|z - z_0| = R$), (c) ∞ (in which case the given power series converges for all z).

A power series may converge at some, all, or none of the points on the actual circle of convergence.

Suppose that the power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ has circle of convergence $|z - z_0| = R$, and let $S(z)$ and $S_N(z)$ represent the sum and partial sums, respectively, of that series:

$$S(z) = \Sigma_{n=0}^{\infty} a_n(z - z_0)^n, \quad S_N(z) = \Sigma_{n=0}^{N-1} a_n(z - z_0)^n \ (|z - z_0| < R).$$

Then, the ***remainder function*** $\rho_N(z)$ is given by $\rho_N(z) = S(z) - S_N(z)$ $(|z - z_0| < R)$. Since the power series converges for any fixed value of z when $|z - z_0| < R$, we know that the remainder $\rho_N(z)$ approaches zero for any such z as N tends to infinity.

This means that corresponding to each positive number ε, there is a positive integer N_ε such that $|\rho_N(z)| < \varepsilon$ whenever $N > N_\varepsilon$.

When the choice of N_ε depends only on the value of ε and is independent of the point z taken in a specified region within the circle of convergence, the convergence is said to be ***uniform*** in that region.

It can be shown that if z_1 is a point inside the circle of convergence $|z-z_0| = R$ of a power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$, then that series must be uniformly convergent in the closed disk $|z - z_0| < R_1$, where $R_1 = |z_1 - z_0|$.

Note that a power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ represents a continuous function

$S(z)$ at each point inside its circle of convergence $|z - z_0| = R$. Furthermore, the sum $S(z)$ of the power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ is actually analytic within the circle of convergence.

Theorem

Let C denote any contour interior to the circle of convergence of the power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$, and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; i.e., $\int_C g(z)S(z)dz = \Sigma_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz$.

If a series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$, then it is the Taylor series expansion for f in powers of $z - z_0$.

If a series $\Sigma_{n=-\infty}^{\infty} c_n(z - z_0)^n = \Sigma_{n=0}^{\infty} a_n(z - z_0)^n + \Sigma_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

An important result in real calculus states that, within a power series's radius of convergence, a power series is differentiable, and its derivative can be obtained by differentiating the individual terms of the power series term–by–term. The same holds true for complex power series:

Schwarz's Theorem:

Suppose f is holomorphic in the open unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ such that

$$f(0) = 0 \quad and \quad f(z) \in D \quad \forall z \in D.$$

Then $|f(z)| \leq |z|$ for all $z \in D$. Strict inequality follows unless f is of the form $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$.

Proof. By the assumptions on f, we have

$$f(z) = zg(z), \qquad z \in D,$$

where g is holomorphic on D and $g(0) = f'(0)$. Since $f(z)| \leq 1$, we have

$$|g(z)| \leq \frac{1}{|z|} = \frac{1}{r} \quad \text{whenever} \quad |z| = r < 1.$$

By maximum modulus principle,

$$|g(z)| \leq \frac{1}{r} \quad \text{whenever} \quad |z| \leq r < 1.$$

Now, let $z \in D$, and $0 < r < 1$ such that $|z| \leq r$. By the above arguments,

$$|g(z)| \leq \frac{1}{r}.$$

Letting $r \to 1$, we obtain $|g(z)| \leq 1$. Thus, $|f(z)| \leq |z|$ for all $z \in D$.

SIX MARKS:

1.Show the function $P_U(z)$ = U(θ_0) provided that U is continuous at θ_0.

2. State and prove Weierstrass theorem.

3.Show that the real part and imaginary part of an analytic function are harmonic.

4.State and prove Schwartz' theorem.

5. Show that u(z) is harmonic for │z│< R and continuous for │z│ $\leq R$, then u(a) = $\frac{1}{2\pi}\int_{|z|=r} \frac{R^2-|a^2|}{|z-a^{\,2}|}$ u(z) dθ,$\forall$ $|a|$<R.

6. State and prove poisson's formula.

7.State and prove Harwitz's theorem.

8.State and prove Laurent's theorem.

9. If u_1 and u_2 are harmonic I a region Ω then $\int u_1{}^*du_2 - u_2{}^*du_1 = 0$, for every cycle ϑ which homologous to zero in Ω.

TEN MARKS:

1.State and prove Weierstrass theorem.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The power series with ………. Coefficients are called geometric series.	two	unit	zero	three	unit
The power series of the form a0 + a1(z –a) + a2(z – a)2 +… converges absolutely in the open disc	$\lvert z-a \rvert = R$	$\lvert z-a \rvert > R$	$\lvert z -a \rvert < R$	$\lvert z -a \rvert = 0$	$\lvert z -a \rvert < R$
The power series of the form $a_0 + a_1(z –a) + a_2(z – a)^2 +$…. is said to be a series about ………	z = 0	z = -a	z = a	z = ∞	z = a
The power series $a_0 + a_1 z + a_2 z^2 +$.. converges absolutely in the open disc ……..	$\lvert z \rvert = R$	$\lvert z \rvert > R$	$\lvert z \rvert < R$	$\lvert z \rvert = 0$	$\lvert z \rvert < R$
The circle of the convergence of the series $a_0 + a_1 z + a_2 z^2 +$……	$\lvert z \rvert > R$	$\lvert z \rvert < R$	$\lvert z \rvert = 0$	$\lvert z \rvert = R$	$\lvert z \rvert = R$
The circle of the convergence of the series $a_0 + a_1(z-a) + a_2(z-a)^2 +$……	$\lvert z-a \rvert > R$	$\lvert z -a \rvert < R$	$\lvert z -a \rvert = 0$	$\lvert z-a \rvert = R$	$\lvert z-a \rvert = R$
A power series … in the exterior of its circle of convergence	absolutely conve	converges	diverges	uniformly convergent	diverges
If R = 0 the series is divergent in the extended plane except at	z = 0	z =1	z = ∞	z = -1	z = 0
The sequence $\{z_n\}$ is bounded if there exists a constant M such that ------ for all n.	$\lvert z_n \rvert = M$	$\lvert z_n \rvert \le M$	$\lvert zn \rvert \ge M$	$\lvert z_n \rvert > M$	$\lvert z_n \rvert \le M$
For all finite z= h + ik, $\lvert e^z \rvert =$ ……	e^{h+k}	e^{h+ik}	e^h	e^k	e^h
Euler's relation $e^{x+iy} =$	e^x(cos y + isin y)	e^x(sin y + icosy)	e^y (cosx + isinx)	e^y(sinx + icosx)	e^x(cos y + isin y)
The polar form r (cos θ + i sinθ) of a complex numbers in exponential form as	re^{θ}	$e^{i\theta}$	$re^{i\theta}$	$1/re^{i\theta}$	$re^{i\theta}$
e^z is not defined at	z = ∞	z =0	z = 1	z= -1	z = ∞
The inverse function of the exponential function is the ……	Trignometric fun	hyberbolic functions	harmonic functions	Logarithmic functions	Logarithmic functions
Logarithamic function log z = ---------------------- n = 0, ±1, ±2	log r + iθ + n(2πi	log 1/r + $ie^{i\theta}$ + n (2πi)	log r + $ie^{i\theta}$ + n(2πi)	logr + iθ + n2π	log r + iθ + n(2πi)
siniz=……	sinz	sinhz	isinz	isinhz	isinhz
cosiz=……..	cosz	icosz	icoshz	coshz	coshz
tanz and secz are analytic in a bounded region in which	tanz ≠ 0	sec z ≠ 0	sinz ≠ 0	cos z ≠ 0	sinz ≠ 0
cot z and cosecz are analytic in a bounded region in which	cot z ≠ 0	cosecz ≠ 0	sinz ≠ 0	cosz ≠0	cosz ≠0
$\cosh^2 z – \sinh^2 z =$	0	1	-1	∞	1
singular points of logz are	z = 0 and z = ∞	z = 1 and z =0	z =0 and z = -1	z =1 and z = ∞	z = 0 and z = ∞
Principle value of logz is obtained when n =	0	-1	1	2	0
The logarithmic function is a ------ valued function	single	multiple	two	zero	multiple
In a complex field z = x + iy then θ = ……….	$\sin^{-1}$ (y/x)	$\cos^{-1}$(y/x)	$\tan^{-1}$(y/x)	$\cot^{-1}$(y/x)	$\tan^{-1}$(y/x)
The sum f(z) of a powerseries is analytic in …………	$\lvert z \rvert > R$	$\lvert z \rvert < R$	$\lvert z \rvert \le R$	$\lvert z \rvert = R$	$\lvert z \rvert < R$
A power series …………… is the interior of the circle of convergence	converges	diverges	uniformly converges	converges absolutely	converges absolutely

The radius of convergence of the series $\sum (2+in)/2^n .z^n$ …………	2	0	∞	1	2
If u+iv is analytic then v+iu is………..	analytic	not analytic	continuous	conjugate	not analytic
a^z is a …………… valued function	single	double	multiple	triple	multiple
The function a^z =	e^{zloga}	e^{loga}	e^{alogz}	e^{-zloga}	e^{zloga}
The radius of convergence of the series $\sum n^2 .z^n$ …………	1	0	2	n	1
Cos $(z_1 + z_2)$ =	$cosz_1\ cosz_2$ - sin	$cosz_1\ sinz_2 - sinz_1cosz_2$	$cosz_1\ cosz_2 + sinz_1sinz_2$	$sinz_1\ cosz_2 - cosz_1sinz_2$	$cosz_1\ cosz_2 - sinz_1sinz_2$
The radius of convergence of the series $\sum n^n .z^n$ …………	1	0	2	n	0

UNIT-IV

SYLLABUS

Partial fraction- Infinite products – Canonical products--The gamma function – Stirling's Formula – Entire functions – Jensen's formula.

PARTIAL FRACTION:

The method for computing partial fraction decompositions applies to all rational functions with one qualification:

The degree of the numerator must be less than the degree of the denominator.

One can always arrange this by using polynomial long division, as we shall see in the examples.

Looking at the example above (in Equation 1), the denominator of the right side is $x^3 - 3x^2 + x - 3 = (x-3)(x^2+1)$. Factoring the denominator of a rational function is the first step in computing its partial fraction decomposition. Note, the factoring must be complete (over the real numbers). In particular this means that each individual factor must either be linear (of the form $ax + b$) or irreducible quadratic (of the form $ax^2 + bx + c$).

When is a quadratic polynomial irreducible? If a quadratic polynomial factors, such as $x^2 - x - 6 = (x-3)(x+2)$, then it has at least one root. Similarly, if it has a root r, then it must have a factor of $x - r$. Thus, a quadratic polynomial is irreducible iff it has no real roots. This is easy to determine using the quadratic formula: the roots of $ax^2 + bx + c$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and these are real numbers iff $b^2 - 4ac \geq 0$. Thus, this quadratic polynomial is irreducible iff its discriminant $b^2 - 4ac < 0$.

Computing the coefficients

Once we have determined the right form for the partial fraction decomposition of a rational function, we need to compute the unknown coefficients $A, B, C, \ldots$. There are basically two methods to choose from for this purpose. We will now look at both methods for the decomposition of

$$\frac{2x-1}{(x+2)^2(x-3)}.$$

By the rules above, its partial fraction decomposition takes the form

$$\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}.$$

Setting these equal and multiplying by the common denominator gives

$$2x - 1 = A(x+2)(x-3) + B(x-3) + C(x+2)^2. \qquad (2)$$

Our first method is to substitute different values for x into Equation 2 and deduce the values of A, B, and C. It helps to start with values of x which are roots of the original denominator since they will make some of the terms on the right side vanish.

- Using $x = 3$ gives $2(3) - 1 = 0 + 0 + C \cdot 5^2$. Thus, $C = 1/5$.
- From $x = -2$, we learn that $-5 = 0 + B(-5) + 0$, and so $B = 1$.
- We have run out of roots of the denominator, and so we pick a simple value of x to finish off. From $x = 0$ we find $-1 = -6A - 3B + 4C$. Using our values for B and C, this becomes $-1 = -6A - 3(1) + 4(1/5)$ and so $A = -1/5$.

Therefore,

$$\frac{2x-1}{(x+2)^2(x-3)} = \frac{-1/5}{x+2} + \frac{1}{(x+2)^2} + \frac{1/5}{x-3}.$$

PROBLEMS: 1.Evaluate

$$\int \frac{x+3}{(x^2-1)(x+5)}\, dx$$

Solution: Factoring the denominator completely yields $(x-1)(x+1)(x+5)$, and so

$$\frac{x+3}{(x^2-1)(x+5)} = \frac{x+3}{(x-1)(x+1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+5}$$

Clearing denominators gives the equation:

$$x + 3 = A(x+1)(x+5) + B(x-1)(x+5) + C(x-1)(x+1)$$

Since the denominator has distinct roots, the quickest way to find A, B, and C will be to plug in the roots of the original denominator:

- $x = 1$ gives $4 = 12A \implies A = 1/3$

- $x = -1$ gives $2 = -8B \implies B = -1/4$
- $x = -5$ gives $-2 = -24C \implies C = 1/12$

Putting it all together, we find

$$\begin{aligned}\int \frac{x+3}{(x^2-1)(x+5)}\,dx &= \int \frac{1/3}{x-1} + \frac{-1/4}{x+1} + \frac{1/12}{x+5}\,dx \\ &= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{12}\int \frac{dx}{x+5} \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{12}\ln|x+5| + C\end{aligned}$$

Note, we use C here for the constant of integration even though C has occured earlier in the problem as a coefficient. However, it is unlikely that confusion will arise by re-using C in this way.

Problems:Evaluate

$$\int \frac{3x^4 + 31x^3 + 122x^2 + 219x + 159}{x^3 + 9x^2 + 27x + 27}\,dx$$

Solution: The first thing we should notice is that the degree of the numerator is **not** less than the degree of the denominator. Applying polynomial long division, we learn that the quotient is $3x + 4$ and that remainder is $5x^2 + 30x + 51$. Thus,

$$\frac{3x^4 + 31x^3 + 122x^2 + 219x + 159}{x^3 + 9x^2 + 27x + 27} = 3x + 4 + \frac{5x^2 + 30x + 51}{x^3 + 9x^2 + 27x + 27}$$

We now find the partial fraction decomposition of the last term. The denominator factors as $(x+3)^3$, and so

$$\frac{5x^2 + 30x + 51}{x^3 + 9x^2 + 27x + 27} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

Clearing denominators leads to

$$5x^2 + 30x + 51 = A(x+3)^2 + B(x+3) + C \qquad (4)$$

We can quickly determine C by evaluating at $x = -3$, which leads to $5(-3)^2 + 30(-3) + 51 = C$, and so $C = 6$. We now pick two simple values of x to obtain relations between A and B. From $x = -2$, we find

$$11 = A(1)^2 + B(1) + 6 \quad \implies \quad 5 = A + B$$

and from $x = -4$, we find

$$11 = A(-1)^2 + B(-1) + 6 \quad \implies \quad 5 = A - B$$

Adding these equations together, we find that $10 = 2A$ and so $A = 5$. Substituting this back into $11 = A + B$ yields $B = 0$. Thus

$$\int \frac{3x^4 + 31x^3 + 122x^2 + 219x + 159}{x^3 + 9x^2 + 27x + 27}\, dx$$

$$= \int 3x + 4 + \frac{5x^2 + 30x + 51}{x^3 + 9x^2 + 27x + 27}\, dx$$
$$= \int 3x + 4\, dx + \int \frac{5}{x+3} + \frac{6}{(x+3)^3}\, dx$$
$$= \frac{3}{2}x^2 + 4x + 5\ln|x+3| + \frac{6}{-2}\frac{1}{(x+3)^2} + C$$
$$= \frac{3}{2}x^2 + 4x + 5\ln|x+3| - \frac{3}{(x+3)^2} + C$$

Infinite products.

Let α_j be complex numbers, $j = 1, 2, \ldots$. We want to give a meaning to the convergence of the infinite product $\prod_{j=1}^{+\infty} \alpha_j$.

Definition

We say that the infinite product $\prod_{j=1}^{+\infty} \alpha_j$ converges if

(i) there exist at most finitely many $\alpha_j = 0$, say $\alpha_j \neq 0$ for $j \geq j_N$;
(ii) for any $N_0 \geq j_N$, the limit

$$\lim_{N \to +\infty} \prod_{j=N_0}^{N} \alpha_j = \beta_{N_0}$$

exists finite and $\neq 0$.

Notice that, if condition (ii) is verified, we may compute the logarithm of β_{N_0}. Let $\beta = \beta_{N_0}$, $\alpha_N = \prod_{j=N_0}^{N} \alpha_j$, and let $D(\beta, \varepsilon)$ not contain the origing and let N_ε be such that $\alpha_N \in D(\beta, \varepsilon)$ for $N \geq N_\varepsilon$.

We may assume that β is not on the negative real axis and let log denote the principal branch of the logarithm (otherwise, chose a different branch cut for the determination of the logarithm.) Then, we have

$$\log \beta_{N_0} = \log \Big(\lim_{N\to+\infty} \prod_{j=N_0}^{N} \alpha_j \Big) = \lim_{N\to+\infty} \log \prod_{j=N_0}^{N} \alpha_j$$
$$= \lim_{N\to+\infty} \sum_{j=N_0}^{N} \log_{(j)} \alpha_j \,,$$

where $\log_{(j)}$ denotes some branch of the logarithm. Since the limit on the right hand side exists finite, $\log_{(j)} \alpha_j \to 0$ as $j \to +\infty$. Hence, in particular the branch of the logarithm must be the principal one, and $\alpha_j \to 1$. This is a necessary condition for the convergence of the infinite product.

Although the next result is not strictly necessary for what that follows, we state it for the sake of clarity.

Lemma

Let α_j be non-zero complex numbers. Then $\prod_{j=1}^{+\infty} \alpha_j$ converges if and only if

$\sum_{j=1}^{+\infty} \log \alpha_j$ converges, where $\log$ denotes the principal branch.

Proof. The previous argument shows that if $\prod_{j=1}^{+\infty} \alpha_j$ converges then also $\sum_{j=1}^{+\infty} \log \alpha_j$ converges.

Conversely, if $\sum_{j=1}^{+\infty} \log \alpha_j$ converges, then, since $e^{\sum_{j=1}^{N} \log \alpha_j} = \prod_{j=1}^{N} \alpha_j$ also $\prod_{j=1}^{+\infty} \alpha_j$ converges. □

For simplicity of notation, we are going to write $\alpha_j = 1 + a_j$.

Lemma

Let $a_j \in \mathbf{C}$ be such that $|a_j| < 1$. Let $Q_N = \prod_{j=1}^{N} (1 + |a_j|)$. Then
$$e^{\frac{1}{2}\sum_{j=1}^{N} |a_j|} \le Q_N \le e^{\sum_{j=1}^{N} |a_j|} \,.$$

Proof. Since $1 + |a_j| \le e^{|a_j|}$,

$$(1 + |a_1|) \cdots (1 + |a_N|) \le e^{\sum_{j=1}^{N} |a_j|} \,.$$

On the other hand, since $e^x \le 1 + 2x$ for $0 \le x \le 1$,

$$e^{\frac{1}{2}\sum_{j=1}^{N} |a_j|} \le \big(1 + 2(|a_1|/2)\big) \cdots \big(1 + 2(|a_N|/2)\big)$$
$$= \prod_{j=1}^{N} (1 + |a_j|) \,. \quad \square$$

Proposition

If the infinite product $\prod_{j=1}^{+\infty}(1+|a_j|)$ *converges, then also* $\prod_{j=1}^{+\infty}(1+a_j)$ *converges. Hence, if the series* $\sum_{j=1}^{+\infty}|a_j|$ *converges, also* $\prod_{j=1}^{+\infty}(1+a_j)$ *converges.*

Proof. Since the product $\prod_{j=1}^{+\infty}(1+|a_j|)$ converges, then $|a_j| \to 0$, so that $1+a_j \neq 0$
We may assume $j_0 = 1$. Let

$$P_N = \prod_{j=1}^{N}(1+a_j), \quad \text{and} \quad Q_N = \prod_{j=1}^{N}(1+|a_j|).$$

Notice that, for a suitable choice of indices j_k,

$$P_N = 1 + \sum_{n=1}^{N}\prod_{k=1}^{n} a_{j_k}.$$

Then,

$$\begin{aligned}|P_N - 1| &= \Big|\sum_{n=1}^{N}\prod_{k=1}^{n} a_{j_k}\Big| \\ &\le \sum_{n=1}^{N}\prod_{k=1}^{n} |a_{j_k}| = Q_N - 1.\end{aligned}$$

Then, for $N, M > 1$, $N > M$,

$$\begin{aligned}|P_N - P_M| &= \Big|\prod_{j=1}^{N}(1+a_j) - \prod_{j=1}^{M}(1+a_j)\Big| \\ &= \Big|\prod_{j=1}^{M}(1+a_j)\Big| \cdot \Big|1 - \prod_{j=M+1}^{N}(1+a_j)\Big| \\ &\le Q_M\Big(\prod_{j=M+1}^{N}(1+|a_j|) - 1\Big) \\ &= Q_N - Q_M.\end{aligned}$$

$$\prod_{j=M}^{N}(1+|a_j|) \le e^{\sum_{j=M}^{N}|a_j|} \le \frac{3}{2}$$

for $M \ge j_0$, and $N > M$. Then, arguing as in (8.1) we see that

$$\left|1-\prod_{j=M}^{N}(1+a_j)\right| \le \prod_{j=M}^{N}(1+|a_j|)-1 \le \frac{1}{2},$$

for $M \ge j_0$, and $N > M$. Hence,

$$\left|\prod_{j=M}^{N}(1+a_j)\right| \ge \frac{1}{2}$$

so that

$$\lim_{N\to+\infty}|P_N| = \lim_{N\to+\infty}\left|\prod_{j=1}^{M}(1+a_j)\right| \cdot \left|\prod_{j=M}^{N}(1+a_j)\right|$$
$$\ge \frac{1}{2}\left|\prod_{j=1}^{j_0}(1+a_j)\right|. \quad \square$$

We apply these results to the infinite product of functions.

7. Canonical Product

We recall the Weierstrass factorization theorem for entire functions. Let $f(z)$ be an entire function with a zero of multiplicity $m \ge 0$ at $z = 0$. Let $\{z_n\}$ be the non-zero zeros of $f(z)$, arranged so that a zero of multiplicity K is repeated in this sequence K times. Also suppose that $|z_1| \le |z_2| \le \ldots$. If $\{p_n\}$ is a sequence of integers such that

$$\sum_{n=1}^{\infty}\left(\frac{R}{|z_n|}\right)^{p_n+1} < \infty, \text{ for every } R > 0, \text{ then}$$

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}(z/z_n) \quad (1)$$

converges uniformly on compact subsets of the plane, where by definition of primary factors, we have

$$E_p(z) = (1-z)\exp\left(z+\frac{z^2}{2}+\ldots+\frac{z^p}{p}\right) \quad (2)$$

for $p \ge 1$ and $E_0(z) = 1-z$

Then the Weierstrass theorem says that

$$f(z) = z^m e^{g(z)} P(z) \quad (3)$$

where $g(z)$ is an entire function.

We are interested in the case in which g(z) and P(z) have certain characteristics which result in properties of $f(z)$ and conversely. A convenient assumption for P(z) is that all the integers p_n are equal. Then we see that this is to assume that there is an integer $p \geq 1$ such that

$$\sum_{n=1}^{\infty} |z_n|^{-(p+1)} < \infty \tag{4}$$

i.e. it is an assumption on the growth rate of the zeros of $f(z)$. Further, if we assume that p is the smallest integer for which the series (4) converges, then the product

$$P(z) = \prod_{n=1}^{\infty} E_p(z/z_n) \tag{5}$$

is called the **canonical product** associated with the sequence $\{z_n\}$ of zeros of $f(z)$ and the integer p is called the **genus** of the canonical product. The restriction on g(z), we impose, is that it is a polynomial. Such an assumption must impose a growth condition on $e^{g(z)}$. When g(z) is a polynomial, then we say that $f(z)$ is of finite genus and we define the genus of $f(z)$ to be the degree of this polynomial or to be the genus of the canonical product whichever is greater.

Now we drive Jensen's formula which says that there is a relation between the growth rate of the zeros of $f(z)$ and the growth of $M(r) = \sup \{|f(re^{i\theta})| : 0 \leq \theta \leq 2\pi\}$ as r increases. For this, we shall use Gauss-Mean Value Theorem which states that if $f(z)$ is analytic in a domain D which contains the disc $|z - z_0| \leq \rho$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

If u is the real part of $f(z)$, the above result gives Gauss-mean value theorem for harmonic function, as

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta$$

7.1. Jensen's Formula. Let $f(z)$ be analytic in the closed disc $|z| \leq R$ and let $f(0) \neq 0$, $f(z) \neq 0$ on $|z| = R$. If $z_1, z_2, \ldots, z_n$ are zeros of $f(z)$ in the open disc $|z| < R$ repeated according to their multiplicity, then

$$\log|f(0)| = -\sum_{i=1}^{n} \log\left(\frac{R}{|z_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\, d\phi.$$

Proof. Consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \tag{1}$$

We observe that F(z) is analytic in any domain in which $f(z)$ is analytic and further $F(z) \neq 0$ for $|z| \leq R$. Hence F(z) is analytic and never vanish on an open disc $|z| < \rho$ for some $\rho > R$.

Also $$|F(z)| = |f(z)| \tag{2}$$

on $|z| = R$, since

$$\left|\prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right| = \prod_{i=1}^{n} \left|\frac{R^2 - \bar{z}_i Re^{i\phi}}{R^2 e^{i\phi} - Rz_i}\right|, z = Re^{i\phi}$$

$$= \prod_{i=1}^{n} \left|\frac{R(R - \bar{z}_i e^{i\phi})}{Re^{i\phi}(R - z_i e^{-i\phi})}\right|$$

$$= \prod_{i=1}^{n} \left| \frac{R - \bar{z}_i e^{i\phi}}{R - z_i e^{-i\phi}} \right|, \ |e^{i\phi}| = 1$$
$$= 1$$

Since F(z) is analytic and non-zero in $|z| < \rho$, log F(z) is analytic in $z| < \rho$ and consequently its real part log |F(z)| is harmonic there. Hence using Gauss-Mean value theorem for log |F(z)|, we get

$$\log |F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log |F(Re^{i\phi})| d\phi \qquad (3)$$

Now, from (1),

$$F(0) = f(0) \prod_{i=1}^{n} \left(\frac{-R}{z_i}\right)$$

so that

$$|F(0)| = |f(0)| \prod_{i=1}^{n} \frac{R}{|z_i|}$$

and thus

$$\log |F(0)| = \log |f(0)| + \sum_{i=1}^{n} \log \frac{R}{|z_i|}$$

Also by (2), $|F(Re^{i\phi})| = |f(Re^{i\phi})|$ on $|z| = R$.
Therefore (3) becomes

$$\log |f(0)| + \sum_{i=1}^{n} \log \frac{R}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\phi})| d\phi$$

or

$$\log |f(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\phi})| d\phi$$

7.2. Poisson-Jensen Formula. Let $f(z)$ be analytic in the closed disc $|z| \le R$ and let $f(z) \neq 0$ on $|z| = R$. If $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$ in the open disc $|z| < R$ repeated according to their multiplicity and $z = re^{i\theta}$, $0 \le r < R$, then

$$\log |f(z)| = -\sum_{i=1}^{n} \log \left| \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log |f(Re^{i\phi})|}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi.$$

Proof. Consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \qquad (1)$$

Clearly F(z) is analytic in any domain in which $f(z)$ is analytic and $F(z) \neq 0$ for $|z| \le R$. Hence F(z) is analytic and never vanish on an open disc $|z| < \rho$ for some $\rho > R$. Also

$$|F(z)| = |f(z)| \text{ on } |z| = R$$

Since F(z) is analytic and non-zero in $|z| < \rho$, log F(z) is analytic in $|z| < \rho$ and consequently its real part log |F(z)| is harmonic there. Hence using Poisson integral formula (unit-I) for log |F(z)|, we get

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|F(Re^{i\phi})|}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\,d\phi \qquad (2)$$

Now, $\log|F(Re^{i\phi})| = \log|f(Re^{i\phi})|$ on $|z| = R$.

Also $$\log|F(z)| = \log|f(z)| \prod_{i=1}^{n} \left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right|$$

$$= \log|f(z)| + \sum_{i=1}^{n} \log\left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right|$$

Therefore (2) becomes

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\,d\phi$$

Entire functions

To begin our study of holomorphic functions in the entire plane, we diskuss the notion of convergence for infinite products.

The gamma function. The subject of this and of the next section is to introduce probably the two most famous and studied non-elementary functions: the Euler *gamma function* $\Gamma(z)$ and the Riemann *zeta function* $\zeta(s)$.

DEFINITION:

For $\operatorname{Re} z > 0$ we set

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\,dt\,.$$

We first state a general result about the holomorphicity of functions defined by integrals. For its proof we refer to [L].

THEOREM:

The function $\Gamma(z)$ is holomorphic for $\operatorname{Re} z > 0$. Moreover, it can be analytically continued in the domain $\Omega = \mathbf{C} \setminus \{0, -1, -2, \dots\}$. At the non-positive integers $z = -n$, with $n = 0, 1, 2, \dots$, the function $\Gamma(z)$ has simple poles with residues $(-1)^n/n!$.

Proof. It follows from the previous proposition that $\Gamma(z)$ is holomorphic for $\operatorname{Re} z > 0$, since for $t > 0$, $|t^z| = t^x$ so that the integral defining $\Gamma(z)$ converges absolutely.

Next we notice that, integrating by parts we have

$$\begin{aligned}\int_0^{+\infty} t^{z-1}e^{-t}\,dt &= \lim_{a\to 0^+,\, b\to+\infty} \int_a^b t^{z-1}e^{-t}\,dt \\ &= \lim_{a\to 0^+,\, b\to+\infty} \frac{1}{z}t^z e^{-t}\Big|_a^b + \frac{1}{z}\int_a^b t^z e^{-t}\,dt \\ &= \frac{1}{z}\int_0^{+\infty} t^z e^{-t}\,dt\,.\end{aligned}$$

Notice that we have obtained the identity

$$z\Gamma(z) = \Gamma(z+1)\,,$$

valid when $\mathrm{Re}\, z > 0$.

The expression $\frac{1}{z}\int_0^{+\infty} t^z e^{-t}\,dt$ on the right hand side above defines a function holomorphic on $\{\mathrm{Re}\, z > -1\} \setminus \{z = 0\}$ that coincides with $\Gamma(z)$ on the set $\{\mathrm{Re}\, z > 0\}$. Hence, the function Γ can be analytically continued on the set $\{\mathrm{Re}\, z > -1\} \setminus \{z = 0\}$.

Assume by induction that, for $n \geq 2$,

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n-1)} \int_0^{+\infty} t^{z+n-1}e^{-t}\,dt\,,$$

for $z \in \{\mathrm{Re}\, z > -n\} \setminus \{0, -1, \ldots, -n+1\}$.

Arguing as before, integrating by parts again we obtain

$$\begin{aligned}\Gamma(z) &= \frac{1}{z(z+1)\cdots(z+n-1)(z+n)} \int_0^{+\infty} t^{z+n}e^{-t}\,dt \\ &= \Big(\prod_{j=0}^{n} \frac{1}{z+j}\Big)\Gamma(z+n+1)\,,\end{aligned}$$

for $\mathrm{Re}\, z > -n-1$ and $z \neq 0, -1, \ldots, -n$.

This shows that, $\Gamma(z)$ is holomorphic for $z \in \mathbf{C}\setminus\{0, -1, -2, \ldots\}$. Moreover, in the non-positive integers Γ has simple poles with residues given by

$$\begin{aligned}\lim_{z\to -n} (z+n)\Gamma(z) &= \lim_{z\to -n} (z+n) \prod_{j=0,\ldots,n} \frac{1}{z+j} \int_0^{+\infty} t^{z+n}e^{-t}\,dt \\ &= \prod_{j=0,\ldots,n} \frac{1}{j-n} \int_0^{+\infty} e^{-t}\,dt \\ &= \frac{(-1)^n}{n!}. \quad \square\end{aligned}$$

In the next proposition we collect a few facts that emerged from the previous proof.

Proposition

Let $\Omega = \mathbf{C}\setminus\{0, -1, -2, \dots\}$. The gamma function $\Gamma(z)$ satisfies the following properties:

(i) $z\Gamma(z) = \Gamma(z+1)$ *for all* $z \in \Omega$;

(ii) $\Gamma(n+1) = n!$;

(iii) $\Gamma(1/2) = \sqrt{\pi}$.

Proof.

Since $\Gamma(1) = 1$, (ii) follows from (i) inductively.

Condition (iii) follows from the well-known identity $\int_{-\infty}^{+\infty} e^{-t^2/2}dt = \sqrt{\pi}$ and the change of variables $x = \sqrt{t}$. □

Corollary

For all $z \in \Omega$ we have

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Proof. It follows from the Thm. that

$$\begin{aligned}\Gamma(z)\Gamma(-z) &= \frac{e^{-\gamma z}}{z}\prod_{n=1}^{+\infty}\Big(1+\frac{z}{n}\Big)^{-1}e^{z/n}\cdot\frac{e^{\gamma z}}{-z}\prod_{n=1}^{+\infty}\Big(1-\frac{z}{n}\Big)^{-1}e^{-z/n}\\ &= -\frac{1}{z^2}\prod_{n=1}^{+\infty}\Big(1-\frac{z^2}{n^2}\Big)^{-1}\\ &= -\frac{\pi}{z\sin \pi z}.\end{aligned}$$

Possible Questions:

Part-B:

1. A necessary and sufficient condition for the absolutely convergence of the product $\prod_1^\infty(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty}|a_n|$. Find the product representation for sin πz.

2. State and prove Legendre's duplication formula

3. Find the power series for the function $\frac{\pi^2}{sin^2\pi z}$

4. Prove that $\pi cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$

5. Show that $\{b_\vartheta\}$ be a sequence of complex numbers with $\lim_{\vartheta\to\infty} b_\vartheta = \infty$ and let p_ϑ(J) be polynomials without constant term then there are functions which are meromorphic in the whole plane with poles at the points b_ϑ, and the corresponding singular parts $p_\vartheta\left(\frac{1}{z-b_\vartheta}\right)$. Moreover, the most general meromorphic function of this kind can be written in the form, f(z) = $[\sum p_\vartheta\left(\frac{1}{z-b_\vartheta}\right) + p_\vartheta(z)]$+g(z) where p_ϑ (z) are suitably chosen polynomials and g(z) is analytic in the whole plane.

6. Prove that $\Gamma z = 1/z\, e^{-\gamma z} \prod (1+z/n)^{-1} e^{z/n}$ using the relations $\Gamma(z) = 1/zH(z)$ and $H(z) = e^{\gamma(z)}$ G(z).where G(z) is the simplest function with negative integers for zero is given by the (a) Find the product representation for sinπz.

7. If f(z) is analytic in |z| < ρ and has zeros at $a_1, a_2, \ldots\ldots\ldots a_n$ in |z| < ρ. Then prove that $\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right)\log(\rho e^{i\theta})d\theta + \sum_{j=1}^{n}\log\left(\frac{\rho^2 - a_j z}{\rho(z - a_j)}\right)$corresponding canonical product.

Part-C:

1. State and prove Poisson-jensen's formula.

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The polar coordinates of C-R equations are……	$u_r=1/r\ v_ϴ$ and $u_ϴ= -r\ v_r$	$u_r= v_ϴ$ and $u_ϴ= v_r$	$u_r=1/r\ v_ϴ$ and $u_ϴ= r\ v_r$	$u_ϴ= -r\ v_r$	$u_r=1/r\ v_ϴ$ and $u_ϴ= -r\ v_r$
Two harmonic functions are said to be ……. Functions if they satisfies the C-R equations.	Conjugate harmonic	harmonic	functions	analytic	analytic
The Laplace equation of the form	$U_{xx}+U_{yy}=0$	$U_{xx}-U_{yy}=0$	$V_{xx}+U_{yy}=0$	$V_{xx}+Vx_y=0$	$U_{xx}+U_{yy}=0$
If $U=x^2-y^2$ then Uyy = ?	3	1	0	2	2
If $u(x,y)=e^x\cos y$ then find u_x= ?	$e^x\cos x$	$e^x\cos y$	cosy	e^x	$e^x\cos y$
The second order partial derivatives exist, continuous and satisfies the laplace equation is called functions	Analytic	Continuous	differentiable	harmonic	harmonic
If $U=x^2-y^2$ then U_{xx} = ?	3	2	0	1	2
The fixed point's transformation is also known as …… points transformation	mobius	invariant	constant	bilinear	constant
The bilinear transformation of the form W=	az+b/cz+d	az+b/c+d	az+b	az+b/c	az+b/cz+d
A function which is ………. in region which is not close may or may not be bounded in it.	Analytic	differentiable	continuous	bounded	Analytic
The function 1/(1+z) is analytic at infinity because the function 1/(1+1/z) is	Analytic at 0	continuous at 0	differentiable at 0	analytic at 1	Analytic at 0
If a function is differentiable at a points then the function is said to be ….	analytic at that point	continuous at that point	differentiabe at that point	not differentiable at that point	differentiabe at that point
The Laplace equation of the format	$U_{xx}+U_{yy}=0$	$U_{xx}-U_{yy}=0$	$V_{xx}+U_{yy}=0$	$V_{xx}+V_{yy}=0$	$U_{xx}+U_{yy}=0$
The bilinear transformation is also known as ……… transformation	non mobius	linear	mobius	non linear	mobius
The equations $u_x=v_y$ and $u_y= -v_x$ are	Polar equation	Euler equation	C - R equation	coordinates	C - R equation
If u or v is not harmonic , then u+iv is ………	analytic	not analytic	conjugate harmonic	diffrentiable	not analytic
If f(z) =u(x,y) + iv(x,y) is analytic in domain d iff u(x,y) and v(x,y) are	harmonic	conjugate harmonic	differentiable	continuous	conjugate harmonic
In a two dimensional flow the stream function is $\tan^{-1}y/x$ then the velocity potential is	$1/2\log(x^2 + y^2)$	Sin^{-1} y/x	$\log(x^2 + y^2)$	$\cos^{-1}$ y/x	$1/2\log(x^2 + y^2)$
By Milne – Thomson method if u (x,y) = $x^2 – y^2$ then f(z) =	Z^2	2x+2y	x+y	z	Z^2
The function f(z) = $z^{1/2}$ is …………. Valued function	single	multi	double	triple	double
The transformation w = z^2 maps the -------------- onto the straight lines	parabola	hyperbola	ellipse	rectangular hyperbola	rectangular hyperbola
If f(z) = u+iv is an analytic function then -if(z) =	u-iv	v+iu	u+v	v+i(-u)	v+i(-u)
The value of m such that $2x – x^2 + my^2$ may be harmonic is ----	1	2	0	3	1
If f(z) = u+iv is an analytic function then(1 -i)f(z) =	(u+v)+i(v-u)	(u+v)-i(v-u)	(u-v)+i(v-u)	(u+v)+i(v+u)	(u+v)+i(v-u)

If f(z) = u+iv is an analytic function then(1+i)f(z) =	(u+v)+i(v-u)	(u+v)-i(v-u)	(u-v)+i(u+v)	(u+v)+i(v+u)	(u-v)+i(u+v)
Harmonic functions in polar coordinates are	$U_{rr} + 1/r\, u_r + 1/r^2\, u_{\theta\theta}$	$U_{rr} + r\, u_r + 1/r^2\, u_{\theta\theta}$	$U_{rr}\ + 1/r^2\, u_{\theta\theta}$	$U_{rr} + 1/r\, u_r + 1/r^2\, u_{\theta\theta}$	$U_{rr} + 1/r\, u_r + 1/r^2\, u_{\theta\theta}$
The function ----------- is called zhukosky's function	1/z	z+1/z	z	sinz	z+1/z
If w = u+iv under w = z+1/z then u = ….	u = (r + 1/r)cosθ	u = (r - 1/r)cosθ	u = (r + 1/r)sinθ	u = r cosθ	u = (r + 1/r)cosθ
If w = u+iv under w = z+1/z then v = ….	v = (r + 1/r)cosθ	v = rsinθ	v= (r - 1/r)sinθ	v = r cosθ	v= (r - 1/r)sinθ
A circle whose centre is origin goes onto an …… whose centre is the origin under the zhukosky's transformation.	parabola	hyperbola	ellipse	rectangular hyperbola	ellipse
A ray emanating from the origin goes onto a ….. Whose centre is the origin under the zhukosky's transformation	parabola	hyperbola	ellipse	rectangular hyperbola	hyperbola
The principle value of log z are ….	logr	logr+iθ	log1/r	logr-iθ	logr+iθ
. The partial derivatives are all ----- in domain D	analytic	not analytic	does not exists	continuous	analytic
w = cos z is a ------ function.	analytic	continuous	not analytic	limit	analytic
. f(z) = xy + iy is --------	analytic	continuous	analytic anywhere	limit	continuous
. The function f(z) = |z| is differentiable -----	on real part	on imaginary part	at the origin	at the point 2	at the origin
If f(z) has the derivative only at the origin, it is ------	analytic everywhere	not analytic nowhere	analytic nowhere	continuous nowhere	not analytic nowhere
f(z) = 1/z is a ------ function.	differentiable	continuous	analytic	not analytic	analytic
An analytic function with constant real part is ------	constant	real	imaginary	not analytic	constant
An analytic function with constant imaginary part is ------	constant	real	imaginary	not analytic	constant
An analytic function with constant modulus part is ------	constant	real	imaginary	not analytic	constant
Both real part and imaginary part of any analytic function satisfies ------	wave equation	polynomial equation	del operator	laplace's equation	laplace's equation

UNIT –V

SYLLABUS

Riemann Mapping Theorem – Boundary behaviour – Use of Reflection Principle – Analytical arcs– Conformal mapping of polygons- The Schwartz - Christoffel formula.

Definition. A metric space is a pair (X, d) where X is a set and d is a function from X × X into R, called the distance function or metric, which satisfy the following conditions for x, y, z ∈ X

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ if $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$

Conditions (iii) and (iv) are called 'symmetry' and 'triangle inequality' respectively. A metric space (X, d) is said to be bounded if there exists a positive number K such that

$$d(x, y) \leq K \text{ for all } x, y \in X.$$

The metric space (X, d), in short, is also denoted by X, the metric being understood. If x and r > 0 are fixed then let us define

$$B(x; r) = \{x \in X : d(x, y) < r\}$$
$$\overline{B}(x; r) = \{y \in X : d(x, y) \leq r\}$$

B(x; r) and $\overline{B}(x; r)$ are called open and closed balls (spheres) respectively, with centre x and radius r. B(x; ∈) is also referred to as the ∈-neighbourhood of x.

Let X = R or ∀ and define $d(z, w) = |z-w|$. This makes both (R,d) and (∀, d) metric spaces. (∀, d) is the case of principal interest for us. In (∀ , d), open and closed balls are termed as open and closed discs respectively.

A metric space (X, d) is said to be complete if every sequence in X converges to a point of X, R and ∀ are examples of complete metric spaces.

If G is an open set in ∀ and (X, d) is complete metric space then the set of all continuous functions from G to X is denoted by C(G, X).

The set C(G, X) is always non empty as it contains the constant functions. However it is possible that C(G, X) contains only the constant functions. For example, suppose that G is connected and X = N = {1, 2, 3, 4,..}. If f ∈ C(G, X) then f(G) must be connected in X and hence, must be singleton as the only connected subsets of N are singleton sets.

3.12. Riemann mapping theorem. Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function $f : G \to \forall$ having the properties :

(a) $f(a) = 0$ and $f'(a) > 0$
(b) f is one one
(c) $f(G) = \{z : |z| < 1\}$

Proof : First we show f is unique.
Let g be another analytic function on $\forall$ such that $g(a) = 0$, $g'(a) > 0$ g is one one and $g(G) = \{z : |z| < 1\} = D$.
Then $f_0g^{-1} : D \to D$ is analytic, one one and onto
Also $f_0g^{-1}(0) = f(a) = 0$. So there is a constant c with $|c| = 1$
and $f_0g^{-1}(z) = cz$ for all z. [Applying theorem (2) with a = 0]
But then $f(z) = c\,g(z)$ gives that $0 < f'(a) = cg'(a)$.

Since $g'(a) > 0$, it follows that $c = 1$. Hence $f = g$ and so f is unique.

Now let $\Phi = \{f \in H(G) : f \text{ is one one}, f(a) = 0, f'(a) > 0, f(G) \subset D\}$

We first show $\Phi \neq \phi$.
Since $G \neq \forall$ so there exists $b \in \forall$ such that $b \in G$

Also G is simply connected so there exists an analytic function g on G such that $[g(z)]^2 = z - b$.
Then g is one-one

For this let $z_1, z_2 \in G$ such that $g(z_1) = g(z_2)$
Then $[g(z_1)]^2 = [g(z_2)]^2$
$\Rightarrow$ $z_1 - b = z_2 - b$
$\Rightarrow$ $z_1 = z_2$
$\Rightarrow$ g is one-one.
So by open mapping theorem, there is a positive number r such that

$$B(g(a)\,;\,r) \subset g(G) \quad \ldots(1)$$

Let z be a point in G such that $g(z) \in B(-g(a)\,;\,r)$
Then $|g(z) + g(a)| < r$
$\Rightarrow$ $|-g(z) - g(a)| < r$
$\Rightarrow$ $-g(z) \in B(g(a)\,;\,r)$
$\Rightarrow$ $-g(z) \in g(G)$ [using (1)]
So $\exists$ some $w \in G$ such that
$-g(z) = g(w)$
$\Rightarrow$ $[g(z)]^2 = [g(w)]^2$
$\Rightarrow$ $z - b = w - b$
$\Rightarrow$ $z = w$
Thus, we get,
$-g(z) = g(z)$
$\Rightarrow$ $g(z) = 0$

But then $z - b = [g(z)]^2 = 0$ implies $b = z \in G$, a contradiction.

Hence $g(G) \cap B(-g(a)\,;\,r) = \phi$

Let $U = B(-g(a)\,;\,r)$. There is a Mobius transformation T such that

$T(C_\infty - \overline{U}) = D$

Let $g_1 = T_0 g$ then g_1 is analytic and $g_1(G) \subset D$.

Consider $g_2(z) = \dfrac{g_1(z) - \alpha}{1 - \overline{\alpha} g_1(z)}$ where $\alpha = g_1(a)$.

Then g_2 is analytic, $g_2(G) \subset D$ and $g_2(a) = 0$

Choose a complex number c, $|c| = 1$, such that

$g_3(z) = c\, g_2(z)$ and $g_3'(a) > 0$

Now $g_3 \in \Phi$ hence $\Phi \neq \phi$.

Next we assume that $\overline{\Phi} = \Phi \cup \{0\}$ ...(2)

Since $f(G) \subset D$, sup $\{| f(z) : z \in G\} \leq 1$ for f in Φ. So by Montel's theorem, Φ is normal.

This gives $\overline{\Phi}$ is compact.

Consider the function $\phi : H(G) \to C$

as $\boldsymbol{\phi(f) = f'(a)}$

Then ϕ is continuous function. Since $\overline{\Phi}$ is compact, there is an f in $\overline{\Phi}$ such that $f'(a) \geq g'(a)$ for all $g \in \Phi$.

As $\Phi \neq \phi$, (2) implies that $f \in \Phi$. We show that $f(G) = D$. Suppose $w \in D$ such that $w \notin f(G)$. Then the function

$$\frac{f(z) - w}{1 - \overline{w} f(z)}$$

is analytic in G and never vanishes. Since G is simply connected, there is an analytic function $h : G \to \forall$ such that

$$[h(z)]^2 = \frac{f(z) - w}{1 - \overline{w} f(z)} \qquad \ldots(3)$$

Since the Mobius transformation $T_\xi = \dfrac{\xi - w}{1 - \overline{w}\xi}$ maps D onto D,

we have $h(G) \subset D$.

Define $g : G \to \forall$ as

$$g(z) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)}$$

Then $g(G) \subset D$, $g(a) = 0$ and g is one-one.

Also
$$g'(a) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h'(a)[1 - |h(a)|^2]}{[1 - |h(a)|^2]^2} = \frac{|h'(a)|}{1 - |h(a)|^2}$$

But
$$|h(a)|^2 = \left|\frac{f(a) - w}{1 - \overline{w} f(a)}\right| = |-w| = |w| \qquad [\because f(a) = 0]$$

Differentiating (3), we get

$$2\, h(a)\, h'(a) = f'(a)\,[1 - |w|^2]$$

$$\Rightarrow \quad h'(a) = \frac{f'(a)(1 - |w|^2)}{2h(a)} = \frac{f'(a)(1 - |w|^2)}{2\sqrt{|w|}}$$

$\therefore \quad g'(a) = \dfrac{f'(a)(1-|w|^2)}{2\sqrt{|w|}} \cdot \dfrac{1}{(1-|w|)} = \dfrac{f'(a)(1+|w|)}{2\sqrt{|w|}} > f'(a)$

Thus $g \in \Phi$. A contradiction to the choice of f.

Hence we must have $f(G) = D$.

Next we prove $\overline{\Phi} = \Phi \cup \{0\}$.
Suppose $\{f_n\}$ is a sequence in Φ and $f_n \to f$ in $H(G)$.
Then $f(a) = \lim_{n\to\infty} f_n(a) = 0$ Also $f_n'(a) \to f'(a)$ so $f'(a) \ge 0$

Let z_1 be an arbitrary element of G and let $w = f(z_1)$. Let $w_n = f_n(z_1)$. Let $z_2 \in G$, $z_2 \ne z_1$ and K be a closed disk centred at z_2 such that $z_1 \equiv K$.
Then $f_n(z) - w_n$ never vanishes on K since f is one one But $f_n(z) - w_n$ converges uniformly to $f(z) - w$ on K as K is compact. So Hurwitz's theorem gives that $f(z) - w$ never vanishes on K or $f(z) = w$.

If $f(z) \equiv w$ on K then f is constant function throughout G and since $f(a) = 0$, we have $f(z) \equiv 0$. Otherwise we have f is one. So f' can never vanish. This gives

$f'(a) > 0 \qquad [\because f' a) \ge 0]$

and so $f \in \Phi$.

3.6. Hurwitz's Theorem. Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f. If $f \ne 0$, $\overline{B}$ $(a ; R) \subset G$ and $f(z) \ne 0$ for $|z - a| = R$ then there is an integer N such that for $n \ge N$, f and f_n have the same number of zeros in $B(a ; R)$.

Proof : Let $\delta = \inf \{|f(z)| : |z - a| = R\}$
Since $f(z) \ne 0$ for $|z - a| = R$, we have $\delta > 0$.

Now $f_n \to f$ uniformly on $\{z : |z - a| = R\}$ so there is an integer N such that if $n \ge N$ and $|z - a| = R$ then

$$|f(z) = f_n(z)| < \frac{\delta}{2} < |f(z)|$$

Hence by Rouche's theorem, f and f_n have the same number of zeros in $B(a; R)$.

Cor : If $\{f_n\} \subset H(G)$ converges to f in $H(G)$ and each f_n never vanishes on G then either $f \equiv 0$ or f never vanishes.

3.2. Theorem : If G is open in $\forall$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions :

(a) $K_n \subset \text{int } K_{n+1}$;
(b) $K \subset G$ and K compact implied $K \subset K_n$ for some n.
Now we define a metric on $C(G, X)$.

Since G is open set in $\forall$, we have $G = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact and $K_n \subset \text{int } K_{n+1}$. For

$n \in N$, we define

$$\rho_n(f, g) = \sup \{d(f(z), g(z)) : z \in K_n\}$$

for all functions f and g in C(G, X).

Also if we define

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \qquad \text{for all } f, g \in C(G, X)$$

then (C(G, X), ρ) is a metric space. In fact (C(G, X), ρ) is a complete metric space.

3.3. Definitions : A set $\Phi \subset C(G, X)$ is normal if each sequence in Φ has a subsequence which converges to a function f in C(G, X).

A set $\Phi \subset C(G, X)$ is normal iff its closure is compact.

A set $\Phi \subset C(G, X)$ is called **equicontinuous** at a point z_0 in G iff for every $\in > 0$ there is a $\delta > 0$ such that for $|z - z_0| < \delta$,

$$d(f(z), f(z_0)) < \in$$

Schwarz's Reflection Principle

We observe that some elementary functions $f(z)$ possess the property that $f(\overline{z}) = \overline{f(z)}$ for all points z in some domain. In other words, if $w = f(z)$, then it may happen that $\overline{w} = f(\overline{z})$ i.e. the reflection of z in the real axis corresponds to one reflection of w in the real axis. For example, the functions

$$z,\ z^2 + 1,\ e^z,\ \sin z \text{ etc}$$

have the above said property, since, when z is replaced by its conjugate, the value of each function changes to the conjugate of its original value. On the other hand, the functions

$$iz,\ z^2 + i,\ e^{iz},\ (1 + i)\sin z \text{ etc}$$

do not have the said property.

Theorem (Schwarz's Reflection Principle). Let G be a region such that G = G* if

$f : G_+ \cup G_0 \to \forall$ is a continuous function which is analytic on G_+ and $f(x)$' is real for x in G_0 then there is an analytic function $g : G \to \forall$ s.t. $g(z) = f(z)$ for all z in $G_+ \cup G_0$.

Proof. For z in G_-, define $g(z) = \overline{f(\overline{z})}$ and for z in $G_+ \cup G_0$, define $g(z) = f(z)$.

Then $g : G \to \forall$ is continuous. We will show that g is analytic. Clearly g is analytic on $G_+ \cup G_-$. To show g is analytic on G_0, let x_0 be a fixed point in G_0 and let $R > 0$ be such that

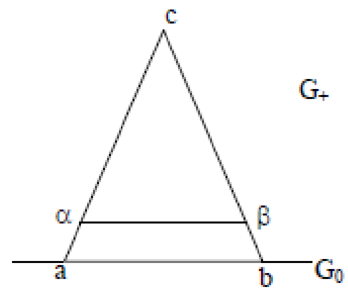
$$B(x_0 ; R) \subset G.$$

It is sufficient to show that g is analytic on $B(x_0 ; R)$ We shall apply Morera's theorem.

Let $T = [a, b, c, a]$ be a triangle in $B(x_0 ; R)$. Assume that $T \subset G_+ \cup G_0$ and $[a, b] \subset G_0$ Let Δ represent T together with its inside. Then $g(z) = f(z)$ for all z in Δ. $[\because T \subset G_+ \cup G_0]$ By hypothesis f is continuous on $G_+ \cup G_0$, so f is uniformly continuous on Δ. So given $\in > 0$, there is a $\delta > 0$ s.t. $z, z' \in \Delta$ implies

$$|f(z) - f(z')| < \in \text{ whenever } |z - z'| < \delta.$$

Choose α and β on the line segments [c, a] and [b, c] respectively so that $|\alpha - a| < \delta$ and $|\beta - b| < \delta$. Let $T_1 = [\alpha, \beta, c, \alpha]$ and $Q = [a, b, \beta, \alpha, a]$. Then $\int_T f = \int_{T_1} f + \int_Q f$



But T_1 and its inside are contained in G_+ and f is analytic there.

So $\int_{T_1} f = 0$

$$\therefore \quad \int_T f = \int_Q f$$

By if $0 \le t \le 1$, then

$$|[t\beta + (1-t)\alpha] - [tb + (1-t)a]| < \delta$$

so that

$$|f(t\beta + (1-t)\alpha) - f(tb + (1-t)a)| < \in.$$

Let $M = \max. \{1 f(z) | : z \in \Delta\}$ and l be the perimeter of T then

$$\left| \int_{[a,b]} f + \int_{[\beta,\alpha]} f \right| = \left|(b-a)\int_0^1 f(tb + (1-t)a)dt - (\beta-\alpha)\int_0^1 f(t\beta + (1-t)\alpha)\,dt\right|$$

$$\le |b-a| \left|\int_0^1 [f(tb + (1-t)a) - f(t\beta + (1-t)\alpha)]\,dt\right|$$

$$+ |b-a) - (\beta-\alpha)| \left|\int_0^1 f(t\beta + (1-t)\alpha)\,dt\right|$$

$$\le \in |b-a| + M\,|(b-\beta) + (\alpha - a)|$$

$$\le \in l + 2M\delta.$$

Also $\quad \left|\int_{[\alpha,a]} f\right| \le M|a-\alpha| \le M\delta$

and $\quad \left|\int_{[b,\beta]} f\right| \le M\delta.$

$$\therefore \quad \left|\int_T f\right| = \left|\int_{[a,b]} f + \int_{[\beta,\alpha]} f + \int_{[\alpha,a]} f + \int_{[b,\beta]} f\right| \le \left|\int_{[a,b]} f + \int_{[\beta,\alpha]} f\right| + \left|\int_{[\alpha,a]} f\right| + \left|\int_{[b,\beta]} f\right|$$

$$\le \in l + 4M\delta$$

Choosing $\delta > 0$ s. t. $\delta < \in$. Then

$\left|\int_T f\right| < \in(l + 4M)$. Since $\in$ is arbitrary it follows that $\int_T f = 0$. Hence f must be analytic.

- **Schwarz-Christoffel Transformation**

We've seen that the transformation $w = f(z)$ with

$$\frac{d f}{d z} = A \prod_{j=1}^{n-1} (z - x_j)^{-p_j} \qquad \text{with} \qquad p_n = 2 - \sum_{j=1}^{n-1} p_j\,, \qquad 0 < p_j < 1$$

maps the real axis onto a convex polygon of the positive sense.

Thus, $\dfrac{d f}{d z} \neq 0$ for finite z.

If all the branch cuts are oriented toward the lower plane, the mapping will be analytic & hence conformal in the finite upper half plane $y \geq 0$ except for the branch points $z = x_j$.

Let us denote the region of analyticity for f by R.

$$\longrightarrow \qquad f(z) = f(z_0) + \int_{z_0}^{z} d s\, \frac{d f(s)}{d s} \qquad \forall\, z,\ z_0 \in R$$

$$= B + A \int_{z_0}^{z} d s \prod_{j=1}^{n-1} (s - x_j)^{-p_j} \qquad B = f(z_0)$$

$w = f(z)$ is known as the **Schwarz-Christoffel Transformation** (SCT).

Some properties of the SCT will be studied in some detail in the following:

- **Existence**

An implicit assumption is that the integral in the SCT exist.

Since $\left| \dfrac{d f}{d z} \right| \underset{z\to\infty}{\longrightarrow} |A|\ |z|^{-\sum p_j}$

a necessary condition is therefore:

$$\sum_{j=1}^{n-1} p_j = 2 - p_n > 1$$

which is automatically satisfied since all p_j, including p_n, obeys criterion $p_j < 1$.

- ***f* is continuous at $z = x_j$**

Near each x_k, we can write

$$\frac{d f}{d z} = (z - x_k)^{-p_k}\, \phi(z)$$

where $\qquad \phi(z) = A \prod_{j \neq k} (z - x_j)^{-p_j}$

is analytic at x_k.

$$\longrightarrow \qquad \phi(z) = \sum_{m=0}^{\infty} \frac{\phi^{(m)}(x_k)}{m\,!}\,(z - x_k)^m$$
$$= \phi(x_k) + (z - x_k)\,\psi(z)$$

where $\psi(z) = \sum_{m=1}^{\infty} \frac{\phi^{(m)}(x_k)}{m\,!}\,(z - x_k)^{m-1}$

is analytic at x_k.

Hence $\qquad \frac{d\,f}{d\,z} = (z - x_k)^{-p_k}\,\phi(x_k) + (z - x_k)^{1-p_k}\,\psi(z)$

$$f(z) = f(z_0) + \int_{z_0}^{z} d\,s \left\{ (s - x_k)^{-p_k}\,\phi(x_k) + (s - x_k)^{1-p_k}\,\psi(z) \right\}$$

where for our purpose here, $z,\ z_0 \approx x_k$.

Now: $\quad |\,p_k\,| < 1 \qquad \longrightarrow \qquad 1 - p_k > 0$

$$\therefore \qquad \int_{z_0}^{z} \mathrm{ds}\,(s - x_k)^{1-p_k}\,\psi(z)$$

is analytic & therefore continuous at x_k as a function of z.

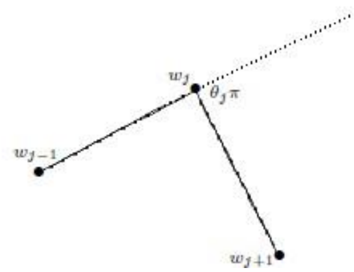
Next $\quad \int_{z_0}^{z} d\,s\ (s - x_k)^{-p_k} = -\frac{1}{p_k}\left\{ (z - x_k)^{1-p_k} - (z_0 - x_k)^{1-p_k} \right\}$

is analytic & therefore continuous at x_k as a function of z.

Thus, $f(z)$ is the sum of 2 continuous functions so that it's continuous at x_k too.

Polygons:

Suppose that the vertices of the polygon P are given by $w_1, \ldots, w_k$ in the anticlockwise direction. Let us follow the edges of the polygon P. At vertex w_j, suppose that we make a right turn of angle $\theta_j\pi$, where $-1 < \theta_j < 1$, with the convention that $\theta_j < 0$ denotes a left turn.



strong Markov property and reflection principle. These are concepts that you can use to compute probabilities for Brownian motion.

Theorem:

Suppose that X_t is Brownian motion. If $t > T$, T a

reflection principle. Suppose X_t is Brownian motion with zero
stopping time, then $X_t - X_T$ is independent of $\mathcal{F}_T$.

An example of a stopping time is the first time that X_t reaches 1.

drift ($\mu = 0$). Then we want to calculate the probability that, starting at $X_0 = 0$, it will reach $X_s = 1$ at some time $0 < s < t$.

$$\mathbb{P}(X_s = 1 \text{ for some } 0 < s < t \,|\, X_0 = 0) = ?$$

Let T = first time that $X_T = 1$. Then X_s reaches 1 for $s < t$ if $T < t$. So, this is the same as

$$\mathbb{P}(T < t)$$

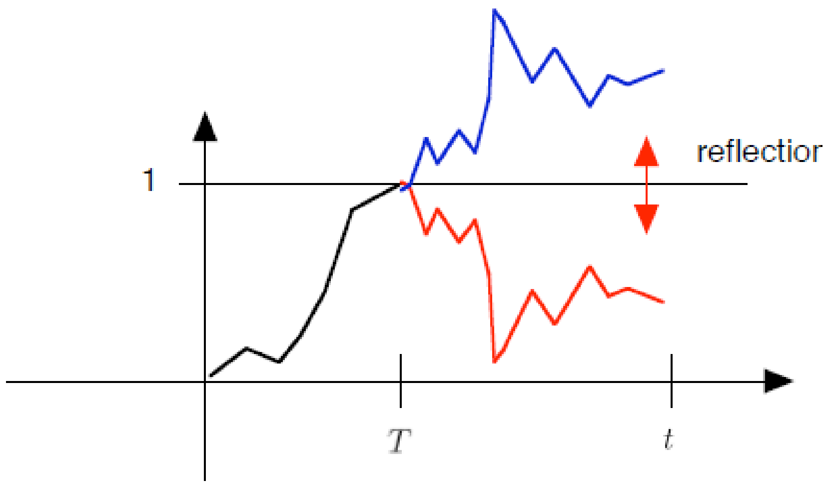
The strong Markov property implies that $X_t - X_T$ is independent of $\mathcal{F}_T$. We also know that $X_t - X_T$ is normal:

$$X_t - X_T \sim N(0, \sigma^2(t - T))$$

(assuming that $t > T$). Since the mean is zero, it is positive half the time and negative half the time (and the probability of being exactly zero is 0):

$$\mathbb{P}(X_t - X_T > 0) = \frac{1}{2}$$

$$\mathbb{P}(X_t - X_T \leq 0) = \frac{1}{2}$$



Half the time X will reach 1 and go up, half the time it will reach 1 and go down. So,

$$\mathbb{P}(T < t) = 2\mathbb{P}(T < t \text{ and } X_t > X_T = 1)$$

But X_t is continuous. So, the intermediate value theorem (IMT) tells us that the second condition implies the first: If $X_t > 1$ and $X_0 = 0$ then $0 < \exists s < 1$ so that $X_s = 1$. So,

$$\mathbb{P}(T < t \,|\, X_0 = 0) = 2\mathbb{P}(X_t > 1 \,|\, X_0 = 0)$$

This is given by an integral

$$= 2\int_1^\infty f_t(x)\,dx$$

where f_t is the density function for $X_t - X_0$.

Schwarz-Christoffel Transformations

Given a polygonal curve Γ, its interior P is a simply connected domain. Thus, by the Riemann Mapping Theorem, there exists a function S that conformally maps the upper half plane onto P. The Schwarz-Christoffel theorem provides a concrete description of such maps.

Here is a typical textbook statement of the theorem:

Theorem: *Let P be the interior of a polygon Γ having vertices $w_1, \ldots w_n$ and interior angles $\alpha_1\pi \ldots \alpha_n\pi$ in counterclockwise order. Let S be any conformal, one-to-one map from the upper half plane $\mathbb{H}$ onto P satisfying $S(\infty) = w_n$. Then S can be written in the form:*

$$S(z) = A + C\int_{z_0}^{z}\prod_{k=1}^{n-1}(\zeta - z_k)^{\alpha_k - 1}d\zeta \tag{1}$$

where A and C are complex constants, and $z_0 < z_1 < \cdots < z_{n-1}$ are real numbers satisfying $S(z_k) = w_k$ for $k = 1, \ldots n-1$.

Functions of the form in Equation (1) are called **Schwarz-Christoffel candidates**[1]. Furthermore, a Schwarz-Christoffel candidate is a **Schwarz-Christoffel Transformation** if it does indeed conformally map the upper half plane $\mathbb{H}$ onto the interior of a polygon.

To make total sense of this theorem, several issues have to be addressed. First, and most fundamentally, the map S from Equation (1) refers to values of S on the extended real axis, but this set is not part of the upper half plane. Therefore, to be able to discuss $S(z_1), \ldots, S(\infty)$, it is important to extend the definition of S to the closure of $\mathbb{H}$.

Secondly, notice that Equation (1) involves improper contour integrals. We need to specify which contours joining z_0 to z, are admissible, show that the resulting integrals converge, and are in fact independent of the particular contour is chosen.

Also, the theorem mandates that $S(\infty) = w_n$. We shall discuss the seriousness of this stipulation, as well as how much freedom we are allowed with the parameters A, C, z_0, … z_{n-1} in the map S.

After these issues have been addressed, we can then formally prove the theorem.

We start the paper with a careful setup of notations and terms in chapter 2. We begin chapter 3 by proving the theorem for prototypical cases when P is a half or quarter plane. This will then motivate us to construct a Schwarz-Christoffel candidate f for the general case. In Chapter 4, we show that f is indeed a Schwarz-Christoffel Transformation if and only if its image curve does not cross itself.

Schwarz's Reflection Principle: Let Ω be a symmetric region, and set $\Omega^+ : \Omega \cap \mathbb{H}$ and $\sigma := \Omega \cap \mathbb{R}$. Suppose that v is continuous on $\Omega^+ \cup \sigma$, harmonic in Ω^+, and zero on σ. Then v has a harmonic extension to Ω which satisfies the symmetry relation $v(\overline{z}) = -v(z)$. In the same situation, if v is the imaginary part of an analytic function f in Ω^+, then f can be extended to an analytic function on all of Ω by the formula $f(\overline{z}) = \overline{f(z)}$.

Boundary Properties of Polygons

Notice that the interior of any polygon P is an open set. Thus, we are guaranteed a conformal S from the upper half plane $\mathbb{H}$ onto P.

Furthermore, as we will discuss later, this function S has a continuous extension that maps the real axis to the boundary of the polygon. Thus, for each of the vertices w_k, there exists a unique prevertex z_k so that $f(z_k) = w_k$.

SIX MARK QUESTIONS:

1. Show that Ω be a bounded simply connected region whose boundary is a closed polygonal line with self-intersection. Let the consecutive vertices be $z_1, z_2, z_3, \ldots\ldots z_n$ in the positive cyclic order. The angle z_k is given by $\alpha_k \pi\, a_n$.

2. If any simply connected region Ω which is not the whole plane and the point $z_0 \in \Omega$ then there exist a unique analytic function f(z) in Ωnormalized by the conditions, $f(z_0) = 0$, $f'(z_0)=0$, such that f(z) defines a one-one mapping of Ω, onto the disk $|\omega| < 1$.

3. Show that any simply connected region Ω which is not the whole plane and the point $z_0 \in \Omega$ thenthere exist a unique analytic function f(z) in Ω normalized by the conditions, $f(z_0) = 0$, $f'(z_0)=0$, such that f(z) defines a one-one mapping of Ω, onto the disk $|\omega|<1$.

4. Show that the boundary of a simply connected region Ω contains a line segment γ as a one sided free boundary arc. Then the function f(z) which maps Ω onto the unit disk can be extended to a function which is analytic and one to one on $\Omega \cup \gamma$. The image of γ is an arc γ' on the unit circleShow that the function z = F(w) which map |w| <1 conformaily onto polygons with angles $\alpha_k\pi$ (k = 1,2,3……n) are of the form $F(w) = c \int_o^W \prod_{k=1}^{n} (w - w_k)^{-\beta k}\, dw + c'$ where $\beta_k = 1 - \alpha_k$, the w_k are points on the unit circle and c ,c' are complex constants.

5. Suppose that the boundary of a simply connected region Ω contains a line segment γ as a one sided free boundary arc. Then the function f(z) which maps Ω onto the unit disk can be extended to a function which is analytic and one to one on $\Omega \cup \gamma$. The image of γ is an arc γ' on the unit circle.

6. Show that an analytic function in a region Ω whose derivative vanishes identically must reduce to a constant . The same is true if either the part, the imaginary part , the modulus the argument is constant.

7. Show that f be a topological mapping of a region Ω onto a region Ω'. If $\{z_n\}$ or z(t) tends to the boundary of Ω then the sequence of $\{f(Z_n)\}$ or f(Z(t)) tends to the boundary of Ω'.

8. Show that f be a topological mapping of a region Ω onto a region Ω'. If $\{z_n\}$ or z(t) tends to the boundary of Ω then the sequence of $\{z_n\}$ or z(t) tends to the boundary of Ω'.

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TEN MARKS

1.State and prove Schwarz's christoffel formula

Questions	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The set of complex points is called ….	arc	simple arc	closed arc	open arc	simple arc
If a curve intersects itself at a point then the point is said to be a……	single	multiple points	double valued	trile	multiple points
The equation z = cost+isint, 0≤t≤π represents a ……………	arc	simple arc	closed arc	curve	simple arc
The unit circle z=cost+isint, 0≤t≤2Π are …..	positively	negatively	circle	unit circle	positively
The unit circle z = cos(-t) +isin(-t), 0≤t≤2π are ……..	positively oriented circle	negatively oriented circle	circle	unit circle	negatively oriented circle
It the region lies to the left of a person when he travels along C, then the curve C is called a ………..	postively oriented simple closed curve	negatively oriented simple closed curve	open curve	simple closed curve	postively oriented simple closed curve
The simple closed rectifible curve is abbreviated as……	curve	scr curve	scro curve	arc	scr curve
In cauchy's fundamental theorem, ∫ f(z) dz=…	1	2	0	4	0
The simple closed rectifiable positively oriented curve is abbreviated as …..	curve	scr curve	scro curve	arc	scro curve
The simple arc is also known as ….	multiple	Jordan	double	multiple	Jordan
The derivative of an analytic function is also …	analytic	continuous	derivative	bounded	continuous
The integral ∫ f(z) dz=F(b)-F(a) is called a…..	integral	indefinite	definite	derivative	derivative
The poles of an analytic function are ……	essential	removable	pole	isolated	isolated
If C is a positively oriented circle then ∫ 1/(z-a) dz=	2Π	2Πi	0	Π	2Πi
When the order of the pole is 2,the pole is said to be ……pole.	double	simple	multiple	triple	multiple
The limit point of zero's of an analytic function is a ……point of the function	singular	nonsingular	poles	zeros	singular
A region which has only one hole is an …..region	origin	set	annular	moment	annular
A region which is not simply connected is called …	connected	compact	multiply-connected	region.	compact
The integrals along scr curves are called….	complex integrals	integrals	contour integrals	partial integrals	contour integrals
If f(z) is a continuous function defined on a simple rectifiable curve then ∫f(z) dz =………..	∫ (u dx- vdy) + i∫(udy-vdx)	∫ (u dx- vdy) - i∫(udy-vdx)	∫ (u dx- vdy) +∫(udy-vdx)	∫ (u dx+ vdy) + i∫(udy+vdx)	∫ (u dx- vdy) + i∫(udy-vdx)
∫ [f_1(z) +f_2(z)]dz on C is………..	∫ f_1(z)dz +	∫ f_1(z)dz -	∫ f_1(z)dz .	∫ f_1(z)dz /	∫ f_1(z)dz +
If f(z) is analytic in a simply connected domain , then the values of the integrals of f(z) along all paths in the region joining ------ fixed points are the same.	one	two	three	multiple	two
The bounded region of C is called ……………	interior	exterior	interior nor	interior and	interior
A region D is said to be …………… for every closed curve in D , Ci is contained in D	connected	simply - connected	disconnected	disjoint	simply - connected
When A is fixed and B(z) moves in D, the integral …..	single - valued	double -valued	multi - valued	zero	single valued
………… of an analytic function are isolated	zeros	poles	residues	points	zeros
If f(z) = (z – a)^m [a_0 + a_1(z-a)………] ,a_0 ≠0 , then z =a is a zero of order …..	m	1	2	0	m
If C is an arc in D , joining a fixed point z_0 and the arbitrary point z then d/dz………	0	1	f(z)	c	f(z)
A function analytic in D has ………….. of all orders in D	derivatives	points	curves	zeros	derivatives
A curve is said to be piece-wise smooth if C is not smooth at a …….. number of points in it.	finite	infinite	zero	one	finite

Reg No ………………

(18MMP201)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore - 21

DEPARTMENT OF MATHEMATICS

Second Semester

I Internal Test

COMPLEX ANALSIS

Date : 04.02.19 **Time: 2 Hours**

Class: I M.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 × 1 = 20 Marks)

Answer all the questions

1. The element $(1,0)$ is the _______________
   (a) additive identity (b) multiplicative identity
   (c) identity (d) unique

2. If $|Z_1| = |Z_2|$ and $arg(Z_1) = arg(Z_2)$ then________
   (a)$Z_1 \neq Z_2$ (b)$Z_1 < Z_2$ (c)$Z_1 > Z_2$ (d)$Z_1 = Z_2$

3. The equation of the unit circle whose centre is the origin is________________
   (a) $|Z| = 1$ (b) $|Z - a| = 1$
   (c) $|Z| = 0$ (d) $|Z| \neq 1$

4. The complex plane containing all the finite complex numbers and infinity is called the_______
   (a) Infinite complex plane (b) extended complex plane
   (c) complex plane (d) finite complex plane

5. If a function is continuous at all points in some neighborhood of a point then the function is said to be __________________
   (a) analytic (b) continuous
   (c) differentiable (d) continuity

6. If $f(z)$ and $g(z)$ are continuous at $z_0$then $f(z) + g(z)$ is___________
   (a) continuous at z_0 (b) differentiable at z_0
   (c) continuous at z (d) differentiable at z

7. In a compact set every continuous function is __________
   (a) uniformly continuous (b) uniformly differentiable
   (c) differentiable (d) continuous

8. The Cartesian coordinates of $C - R$ equations are___________
   (a)$u_x = v_y \; and \; u_y = -v_x$ (b)$u_x = v_y \; and \; u_y = -v_x$
   (c)$u_x = v_y \; and \; u_x = -v_x$ (d) $u_x = 1 \; and \; u_y = -v_x$

9. A function of complex variable is sometimes called a ___________
   (a) complex variable (b) variable
   (c) complex function (d) constant

10. If $u(x,y) = e^x cosy$ then find u_x=________________
    (a) $e^x cosx$ (b) $e^x cosy$ (c)$cos\ x$ (d) e^x

11. The second order partial derivatives exist, continuous and satisfies the laplace equation is called_________functions
    (a) analytic (b) continuous
    (c) differentiable (d)harmonic

12. If $U = x^2 - y^2$ then U_{xx} = ______________
    (a) 3 (b) 2 (c)0 (d)1

13. The mapping $W = Z + b$, b is a complex number, is called the _______________

(a)linear transformation (b)translation
(c)inversion (d)rotation

14. All the complex numbers except infinity are called ________________
(a) complex numbers (b) complex plane
(c) finite complex numbers (d) Infinite complex numbers

15. The cross ratio is of the form____________________
(a) $\frac{(z_1-z_2)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ (b) $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$
(c) $\frac{(z_1-z_2)(z_2-z_4)}{z_1-z_4}$ (d) $\frac{z_1-z_2}{(z_1-z_4)(z_2-z_3)}$

16. The mapping $w = f(z)$ is said to be ______________if it preserves the magnitude of the angle between every two curves.
(a) conformal (b) isogonal
(c) translation (d) not a conformal

17. The fixed point of the transformation $W = Z + b$ is ___________
(a)$Z = 0$ (b)$Z = 1$ (c)$Z = \infty$ (d)$Z \neq \infty$

18. The polar form of $x + iy$ is ________________
(a) $r(cos\ \theta + isin\theta)$ (b) $r(cos\ \theta - i\ sin\theta)$
(c)$(cos\ \theta + isin\theta)$ (d) $r(cos\ \theta -\ sin\theta)$

19. If Z_1 and Z_2 are any two complex numbers then ___________
(a) $|Z_1 - Z_2| \leq |Z_1| + |Z_2|$ (b) $|Z_1 - Z_2| = |Z_1| + |Z_2|$
(c) $|Z_1 - Z_2| \geq |Z_1| - |Z_2||$ (d) $|Z_1 - Z_2| \neq |Z_1| + |Z_2|$

20. The value of i^2 is________________
(a) 1 (b) -1 (c)0 (d)i

PART –B (3×2=6 Marks)
Answer all the questions

21. Define length of arc.
22. Define complex line integral.
23. Define Harmonic function.

PART-C (3x 8=24 Marks)
Answer all the questions

24. (a) Show that an analytic function in a region Ω whose derivative vanishes identically must reduce to a constant . The same is true if either the part, the imaginary part , the modulus the argument is constant.

(OR)

(b) Show that the set of all linear transformation forms a group under the product of transformation.

25. (a) State and prove Liouville'stheorem.

(OR)

(b) i) State and prove Morera's theorem.
ii) State and prove Fundamental theorem of algebra.

26. (a) Show that the real part and imaginary part of an analytic function are harmonic.

(OR)

(b) If u_1 and $u_2$are harmonic in a region Ω then $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$, for every cycle γ which homologous to zero in Ω.

Reg No ……………….

(18MMP201)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore - 21

DEPARTMENT OF MATHEMATICS

Second Semester

II Internal Test

COMPLEX ANALYSIS

Date : 11.03.2019(FN) **Time: 2 Hours**

Class: I M.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 × 1 = 20 Marks)

Answer all the questions

1. The Laplace equation is of the form__________

   (a)$U_{xx} + U_{yy} = 0$ (b)$U_{xx} - U_{yy} = 0$

   (c)$V_{xx} + U_{yy} = 0$ (d)$V_{xx} + V_{yy} = 0$

2. The equation $z = cost + isint, 0 \leq t \leq \pi$ represents a _________

   (a) positively oriented circle (b) negatively oriented circle

   (c)simple arc (d)closed simple curve.

3. The simple closed rectifiable curve is abbreviated as ____________

   (a) curve (b)scr curve

   (b) (c)scro curve (d) normal curve

4. If C is a positively oriented circle then

   $\int_c \frac{1}{(z-a)} dz =$_______________

   (a)2π (b)$2 \pi i$ (c)0 (d)π

5. If C is the unit circle$|Z| = 1$, then $\int_c ze^z\, dz =$ _______

   (a)1 (b) 2 (c)0 (d) -1

6. The Taylor's series is a series of ____________powers.

   (a) positive (b) negative (c) exponential

   (d) logarithmic.

7. The simple arc is also known as __________________

   (a) multiple (b)Jordan (c)double (d)none.

8. The unit circle $z = cos(-t) + i\, sin(-t)$, $0 \leq t \leq 2\pi$ are ____________

   (a) positively oriented circle

   (b) negatively oriented circle

   (c) circle

   (d) unit circle

9. The derivative of an analytic function is also ___________

   (a) analytic (b)continuous

   (c)derivative (d) constant

10. The integral $\int_C$ f(z) dz = $F(b) - F(a)$ is called a __________

    (a)integral (b)indefinite (c)definite (d)derivative

11. The Laurent's series is a series of___________powers.

    (a) positive (b) negative

    (c) exponential (d) both positive and negative

12. The poles of $\frac{2Z+1}{z^2-z-2}$ are ____________

(a) 2,1 (b) 2 (c)1 (d)2,-1

13. If $z = a$ is an isolated singularities of a function $f(z)$ then the singularity is called ___________ singularity.

(a) essential (b) removable

(c) pole (d) isolated

14. If $U = x^2 + y^2$ then U_{yy} = _______________

(a) 3 (b) 2 (c) 0 (d)1

15. When the order of the pole is 1, the pole is said to be _________pole

(a) Simple (b)double (c)multiple (d)triple

16. The poles of an analytic functions are _____________

(a) essential (b) removable (c) pole (d) isolated

17. When the order of the pole is 2,the pole is said to be __________pole

(a) simple (b)double (c)multiple (d)zero

18. Two harmonic functions are said to be ____ Functions if they satisfies the C-R equations.

(a) conjugate harmonic (b) harmonic

(c) functions (d)analytic.

19. The mapping $w = f(z)$ is said to be __________ if it preserves the magnitude of the angle between every two curves and also it preserves the sense of orientation of the angle.

(a) conformal (b) isogonal

(c) translation (d) not a conformal

20. The bilinear transformation is also known as ______ transformation

(a) nonmobius (b) linear

(c) mobius (d) non linear

PART –B (3×2=6 Marks)

Answer all the questions

21. Define Laurent's Series.
22. Write about partial fraction.
23. Give two examples of simple pole and double pole.

PART-C (3 × 8=24 Marks)

Answer all the questions

24. a)State and prove Weierstrass theorem.

(OR)

b)State and prove Poisson's formula.

25. a) Find the power series for the function $\frac{\pi^2}{sin^2 \pi z}$

(OR)

b)Prove that $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$

26. a)State and prove Jensen's formula.

(OR)

b)State and prove Schwartz' theorem.