KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN

DEPARTMENT OF MATHEMATICS

STAFF NAME: Dr. K.KALIDASS SUBJECT NAME: TOPOLOGY
SUB.CODE:18MMP202 SEMESTER: II
CLASS: I M. Sc. MATHEMATICS

S. No	Lecture Duration Hour	Topics to be covered	Support Materials
UNIT-I			
1	1	Introduction to topological spaces	R1: Ch 2, 75
2	1	Definitions and Examples on topology	R1: Ch 2, 76-77
3	1	Theorems on basis	R1: Ch 2, 78
4	1	Continuation of theorems on basis	R1: Ch 2, 79-80
5	1	Theorems on the order topology	R1: Ch 2, 84-86
6	1	Theorems on product topology	R1: Ch 2,86
7	1	Continuation of theorems on product topology	R1: Ch 2,87-88
8	1	Theorems on the subspace topology	R2: Ch 3,101
9	1	Recapitulation and Discussion of possible questions	
			Total 9 Hours
UNIT-II			
1	1	Introduction to closed set	R1: Ch 2, 92
2	1	Theorems on closed set	R1: Ch 2, 93
3	1	Continuation of theorems on closed set	R1: Ch 2, 94-95
4	1	Theorems on limit points	R3: Ch 3,110
5	1	Theorems on continuous functions	R1: Ch 2, 101-102
6	1	Continuation of theorems on continuous functions	R1: Ch 2, 103-104
7	1	Theorems on product topologies	R1: Ch 2, 114-116
8	1	Theorems on metric topologies	R1: Ch 2, 117-118
9	1	Recapitulation and Discussion of possible questions	
			Total 9 Hours
UNIT-III			
1	1	Introduction to connected spaces	R1: Ch 3,147
2	1	Theorems on connected spaces	R4: Ch 5,107
3	1	Continuation of theorems on connected spaces	R1: Ch 3,150-151
4	1	Theorems on connected subspaces of R	R1: Ch 3,152-158
5	1	Theorems on components	R1: Ch 3, 160-162
6	1	Theorems on local connectedness	R1: Ch 3, 163-164
7	1	Continuation of theorems on local connectedness	R1: Ch 3, 164-165
8	1	Recapitulation and Discussion of possible questions	

			Total 8 Hours
		UNIT-IV	
1	1	Introduction to compact spaces	R1: Ch 3,164-166
2	1	Theorems on compact spaces	R1: Ch 3,166-168
3	1	Theorems on compact subspaces of *R*	R1: Ch 3,169-172
4	1	Theorems on limit point compactness	R1: Ch 3,173-174
5	1	Continuation of theorems on limit point compactness	R1: Ch 3,175-181
6	1	Theorems on local compactness	R5: Ch 3, 183
7	1	Continuation of theorems on local compactness	R1: Ch 3,184-185
8	1	Recapitulation and discussion of possible questions	
			Total 8 Hours
		UNIT-V	
1	1	Theorems on countability axioms	R1: Ch 4, 190-191
2	1	Some examples of the separation axioms	R1: Ch 4, 192-194
3	1	Theorems on normal spaces	R1: Ch 4, 198-202
4	1	The Urysohn lemma	R1: Ch 4, 203-206
5	1	The Urysohn metrization theorem	R1: Ch 4, 208-210
6	1	The Tietze Extension theorem	R1: Ch 4, 210-212
7	1	Recapitulation and discussion of possible questions	
8	1	Discussion on Previous ESE Question Papers	
9	1	Discussion on Previous ESE Question Papers	
10	1	Discussion on Previous ESE Question Papers	
			Total 9 Hours

Total no. of Hours for the Course: 44 hours

Suggested readings

R1 James R. Munkres., (2008). Topology, Second edition, Pearson Prentice Hall, New Delhi.
R2 Simmons, G. F., (2004). Introduction to Topology and Modern Analysis, Tata Mc Graw Hill, New Delhi.
R3 Deshpande, J. V., (1990). Introduction to topology, Tata Mc Graw Hill, New Delhi.
R4 James Dugundji., (2002). Topology, Universal Book Stall, New Delhi.
R5 Joshi, K. D.(2004). Introduction to General Topology, New Age International Pvt Ltd, New Delhi

UNIT-I

SYLLABUS

Topological spaces, Basis for a topologies, the order topology, the product topology X x Y, the subspace topology.

§1 Definition and Examples:

Definition 1.1: Let X be any non-empty set. A family $\mathfrak{I}$ of subsets of X is called a topology on X if it satisfies the following conditions:

(i) $\phi \in \mathfrak{I}$ *and* $X \in \mathfrak{I}$

(ii) $A, B \in \mathfrak{I} \implies A \cap B \in \mathfrak{I}$

(iii) $A_\lambda \in \mathfrak{I}$, $\forall \lambda \in \Lambda$ (where Λ is any indexing set) $\implies \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{I}$

If $\mathfrak{I}$ is a topology on X, then the ordered pair $\langle X, \mathfrak{I} \rangle$ is called a topological space (or T-space)

Examples 1.2: Throughout X denotes a non-empty set.

1) $\mathfrak{I} = \{\emptyset, X\}$ is a topology on X. This topology is called **indiscrete topology** on X and the T-space $\langle X, \mathfrak{I} \rangle$ is called indiscrete topological space.

2) $\mathfrak{I} = \wp(X)$, ($\wp(X)$ = power set of X is a topology on X and is called **discrete topology** on X and the T-space $\langle X, \mathfrak{I} \rangle$ is called discrete topological space.

Remark: If $|X| = 1$, then discrete and indiscrete topologies on X coincide, otherwise they are different.

3) Let $X = \{a, b, c\}$ then $\mathfrak{I}_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathfrak{I}_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ are topologies on X whereas $\mathfrak{I}_3 = \{\emptyset, \{a\}, \{b\}, X\}$ is a not a topology on X.

4) Let X be an infinite set. Define $\mathfrak{I} = \{\emptyset\} \cup \{A \subseteq X \mid X - A \text{ is finite}\}$ then $\mathfrak{I}$ is topology on X.

(i) $\emptyset \in \mathfrak{I}$ …… (by definition of $\mathfrak{I}$)

As X – X = $\emptyset$, a finite set, $X \in \mathfrak{I}$

(ii) Let $A, B \in \mathfrak{I}$. If either $A = \emptyset$ or $B = \emptyset$, then $A \cap B \in \mathfrak{I}$. Assume that $A \neq \emptyset$ and $B \neq \emptyset$.

Then $X - A$ is finite and $X - B$ is finite. Hence $X - (A \cap B) = (X - A) \cup (X - B)$ is finite set. Therefore $A \cap B \in \mathfrak{J}$. Thus $A, B \in \mathfrak{J} \Rightarrow A \cap B \in \mathfrak{J}$.

(iii) Let $A_\lambda \in \mathfrak{J}$, for each $\lambda \in \Lambda$ (where Λ is any indexing set). If each $A_\lambda = \emptyset$, then

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \emptyset \in \mathfrak{J}.$$

If $\exists\, \lambda_0 \in \Lambda$ such that $A_{\lambda_0} \neq \emptyset$, then $A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda \Rightarrow X - A_{\lambda_0} \supseteq X - \bigcup_{\lambda \in \Lambda} A_\lambda$.

As $X - A_{\lambda_0}$ is a finite set and subset of finite set being finite we get $X - \bigcup_{\lambda \in \Lambda} A_\lambda$ is finite

and hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$. Thus in either case,

$$A_\lambda \in \mathfrak{J}, \quad \forall\, \lambda \in \Lambda \quad \Rightarrow \quad \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}.$$

From (i), (ii) and (iii) is a topology on X. This topology is called **co-finite topology** on X and t topological space is called co-finite topological space.

Remark: If X is finite set, then co-finite topology on X coincides with the discrete topology X.

5) Let X be any uncountable set. Define $\mathfrak{J} = \{\emptyset\} \cup \{A \subseteq X \mid X - A \text{ is countable}\}$ Then $\mathfrak{J}$ is topology on X.

i. $\emptyset \in \mathfrak{J}$ (by definition).

As X – X = $\emptyset$ and $\emptyset$ is countable (Since $\emptyset$ is finite) we get X $\in \mathfrak{J}$.

ii. Let $A, B \in \mathfrak{J}$. If either $A = \emptyset$ or $B = \emptyset$ we get $A \cap B \in \mathfrak{J}$.

Let $A \neq \emptyset$ and $B \neq \emptyset$.

Then by definition of $\mathfrak{J}$, X – A and X – B both are countable sets and hence

$X - (A \cap B) = (X - A) \cup (X - B)$ is countable. This shows that $A \cap B \in \mathfrak{J}$. Thus $A, B \in \mathfrak{J}$ implies $A \cap B \in \mathfrak{J}$.

iii. Let $A_\lambda \in \mathfrak{J}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set. If for each $\lambda \in \Lambda$, $A_\lambda = \emptyset$

then $\bigcup_{\lambda\in\Lambda} A_\lambda = \emptyset$ will imply $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$. Let $A_{\lambda_0} \neq \emptyset$ for some $\lambda_0 \in \Lambda$.

Then $A_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda \Rightarrow X - A_{\lambda_0} \supseteq X - \bigcup_{\lambda\in\Lambda} A_\lambda$

$\Rightarrow X - \bigcup_{\lambda\in\Lambda} A_\lambda$ is a subset of a countable set $X - A_{\lambda_0}$ (Since $A_{\lambda_0} \in \mathfrak{I}$ and $A_{\lambda_0} \neq \emptyset$)

$\Rightarrow X - \bigcup_{\lambda\in\Lambda} A_\lambda$ is a countable set. (since subset of countable set is countable)

$\Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$

Thus in either case, $A_\lambda \in \mathfrak{I}, \forall \lambda \in \Lambda \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{I}$

From (i) , (ii) and (iii) we get $\mathfrak{I}$ is a topology on X. This topology on X is called **co-countable topology** on X and the T-space $\langle X, \mathfrak{I}\rangle$ is called co-countable topological space.

Remark: If X is a countable set, the co-countable topology on X coincides with the discrete topology on X.

6) Let $A \subseteq X$. Define $\mathfrak{I} = \{\emptyset\} \cup \{B \subseteq X \mid A \subseteq B\}$. Then $\mathfrak{I}$ is a topology on X.

(i) $\emptyset \in \mathfrak{I}$ by definition. As $A \subseteq X$, $X \in \mathfrak{I}$.

(ii) Let $B, C \in \mathfrak{I}$. If $B = \emptyset$ or $C = \emptyset$, then $B \cap C = \emptyset$ will give $B \cap C \in \mathfrak{I}$. Let $B \neq \emptyset$ or $C \neq \emptyset$. Then $A \subseteq B \cap C$ will imply $B \cap C \in \mathfrak{I}$.

(iii) Let $B_\lambda \in \mathfrak{I}\ \forall \lambda \in \Lambda$, where Λ is any indexing set. If for each $\lambda \in \Lambda$, $B_\lambda = \phi$ then $\bigcup_{\lambda\in\Lambda} B_\lambda = \emptyset$ and in this case $\bigcup_{\lambda\in\Lambda} B_\lambda \in \mathfrak{I}$.

Assume that $B_{\lambda_0} \neq \emptyset$ for some $\lambda_0 \in \Lambda$. Then $A \subseteq B_{\lambda_0}$ and $B_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} B_\lambda$ imply $A \subseteq \bigcup_{\lambda\in\Lambda} B_\lambda$.

Therefore $\bigcup_{\lambda\in\Lambda} B_\lambda \in \mathfrak{I}$.

From (i), (ii) and (iii) $\mathfrak{I}$ is a topology on X.

Remarks: (1) If $A = \emptyset$ then $\mathfrak{J}$ is discrete topology on X.

(2) If $A = X$ then $\mathfrak{J}$ is indiscrete topology on X.

(3) If $A = \{p\}$, then $\mathfrak{J} = \{\emptyset\} \cup \{B \subseteq X \mid p \in B\}$ is called ***p*-inclusive topology on X**.

7) Let $p \in X$. Define $\mathfrak{J} = \{X\} \cup \{A \subseteq X \mid p \notin A\}$. Then $\mathfrak{J}$ is topology on X.

(i) $p \notin \emptyset$ implies $\emptyset \in \mathfrak{J}$. By definition $X \in \mathfrak{J}$.

(ii) Let $A, B \in \mathfrak{J}$. If $A = X$ or $B = X$, then $A \cap B = X$. In this case $A \cap B \in \mathfrak{J}$. Assume that either $A \neq X$ or $B \neq X$. Then $p \notin A$ or $p \notin B$ and hence $p \notin A \cap B$ which proves $A \cap B \in \mathfrak{J}$.

Thus $A, B \in \mathfrak{J}$ implies $A \cap B \in \mathfrak{J}$.

(iii) Let $A_\lambda \in \mathfrak{J}\ \forall \lambda \in \Lambda$, where Λ is any indexing set. If for some $\lambda \in \Lambda$, $A_\lambda = X$ then $\bigcup_{\lambda\in\Lambda} A_\lambda = X$ will give $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

Assume that $A_\lambda \neq X$ for each $\in \Lambda$. Then $p \notin A_\lambda$ for each $\lambda \in \Lambda$ will imply,

$p \notin \bigcup_{\lambda\in\Lambda} A_\lambda$ and hence $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

Thus in either case, $A_\lambda \in \mathfrak{J}\ \forall \lambda \in \Lambda \implies \bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}$.

From (i), (ii) and (iii) $\mathfrak{J}$ is a topology on X.

This topology on X is called ***p*-exclusive topology on X.**

8) Let $\langle X, \mathfrak{J}\rangle$ be topological space and $A \subseteq X$. Define $\mathfrak{J}^* = \{G \cup (A \cap H) \mid G, H \in \mathfrak{J}\}$. Then $\mathfrak{J}^*$ is a topology on X.

(i) Take $G = \emptyset$ and $H = \emptyset$. Then $G \cup (A \cap H) = \emptyset \cup (A \cap \emptyset) = \emptyset \Rightarrow \emptyset \in \mathfrak{J}^*$. Take $G = X$. Then for any $H \in \mathfrak{J}$ we get $X \cup (A \cap H) = X$. Hence $X \in \mathfrak{J}^*$.

(ii) Let $G_1 \cup (A \cap H_1) \in \mathfrak{J}^*$ and $G_2 \cup (A \cap H_2) \in \mathfrak{J}^*$ for $G_1, H_1, G_2, H_2 \in \mathfrak{J}$.

Then $[G_1 \cup (A \cap H_1)] \cap [G_2 \cup (A \cap H_2)]$

$= (G_1 \cap G_2) \cup (G_1 \cap A \cap H_2) \cup (A \cap H_1 \cap G_2) \cup (A \cap H_1 \cap H_2)$

$= (G_1 \cap G_2) \cup [A \cap [(G_1 \cap H_2) \cup (H_1 \cap G_2) \cup (H_1 \cap H_2)]]$

As $(G_1 \cap G_2) \in \mathfrak{J}$ and $[(G_1 \cap H_2) \cup (H_1 \cap G_2) \cup (H_1 \cap H_2)] \in \mathfrak{J}$ we get,

$[G_1 \cup (A \cap H_1)] \cap [G_2 \cup (A \cap H_2)] \in \mathfrak{J}$.

(iii) Let $G_\lambda \cup (A \cap H_\lambda) \in \mathfrak{J}^*$ for $\lambda \in \Lambda$, where Λ is any indexing set. Then $G_\lambda \in \mathfrak{J}$ and $H_\lambda \in \mathfrak{J}, \forall\, \lambda \in \Lambda$.

$$\bigcup_{\lambda\in\Lambda}[G_\lambda \cup (A \cap H_\lambda)] = \left[\bigcup_{\lambda\in\Lambda} G_\lambda\right] \cup \left[A \cap \left[\bigcup_{\lambda\in\Lambda} H_\lambda\right]\right]$$

As $\bigcup_{\lambda\in\Lambda} G_\lambda \in \mathfrak{J}$ and $\bigcup_{\lambda\in\Lambda} H_\lambda \in \mathfrak{J}$, we get $\bigcup_{\lambda\in\Lambda}[G_\lambda \cup (A \cap H_\lambda)] \in \mathfrak{J}^*$.

From (i), (ii) and (iii) we get $\mathfrak{J}^*$ is a topology on X.

Remark: This example shows that every topology on X induces another topology on X.

9) Let X and Y be any two non-empty sets and let $f : X \longrightarrow Y$ be any function. Let $\mathfrak{J}$ be topology on Y. Define $\mathfrak{J}^* = \{f^{-1}(G) \mid G \in \mathfrak{J}\}$, where $f^{-1}(G) = \{x \in X \mid f(x) \in G\}$. Then $\mathfrak{J}^*$ is topology on X.

(i) $f^{-1}(\emptyset) = \emptyset \quad \Rightarrow \quad \emptyset \in \mathfrak{J}^*$ and $f^{-1}(Y) = X \quad \Rightarrow \quad X \in \mathfrak{J}^*$

(ii) Let $f^{-1}(G) \in \mathfrak{J}^*$ and $f^{-1}(H) \in \mathfrak{J}^*$ for $, H \in \mathfrak{J}$. Then $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$ and $G, H \in \mathfrak{J}$ will imply $f^{-1}(G) \cap f^{-1}(H) \in \mathfrak{J}^*$.

(iii) Let $f^{-1}(G_\lambda) \in \mathfrak{J}^*\ \forall\, \lambda \in \Lambda$, where Λ any indexing set is. Then

$$f^{-1}\left(\bigcup_{\lambda\in\Lambda} G_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{-1}(G_\lambda)\,. \quad \text{As } \bigcup_{\lambda\in\Lambda} G_\lambda \in \mathfrak{J}, \text{we get } \bigcup_{\lambda\in\Lambda} f^{-1}(G_\lambda) \in \mathfrak{J}^*.$$

Thus from (i), (ii) and (iii) we get $\mathfrak{J}^*$ is a topology on X.

10) Let X be any uncountable set and let ∞ be a fixed point of X. Let $\mathfrak{J} = \{G \subseteq X \mid \infty \notin G\} \cup \{G \subseteq X \mid \infty \in G$ and $X - G$ is finite$\}$. Then $\mathfrak{J}$ is a topology on X.

Define $\mathfrak{J}_1 = \{G \subseteq X \mid \infty \notin G\}$ and $\mathfrak{J}_2 = \{G \subseteq X \mid \infty \in G$ and $X - G$ is finite$\}$ then $\mathfrak{J} = \mathfrak{J}_1 \cup \mathfrak{J}_2$.

(i) $\infty \notin \emptyset \Rightarrow \emptyset \in \mathfrak{J}$. $\infty \in X$ and $X - X = \emptyset$ is a finite set $\Rightarrow X \in \mathfrak{J}_2 \Rightarrow X \in \mathfrak{J}$.

(ii) Let $A, B \in \mathfrak{J}$.

Case 1: $A, B \in \mathfrak{J}_1$. Then $\infty \notin A$ and $\infty \notin B$. Hence $\infty \notin A \cap B$.

Therefore $A \cap B \in \mathfrak{J}_1 \Rightarrow A \cap B \in \mathfrak{J}$.

Case 2 : $A, B \in \mathfrak{J}_2$. Then $A \in \mathfrak{J}_2 \implies \infty \in A$ and $X - A$ is finite. $B \in \mathfrak{J}_2 \implies \infty \in B$ and $X - B$ is finite. Then $\infty \in A \cap B$ and $X - (A \cap B) = (X - A) \cup (X - B)$ is finite. Thus $A \cap B \in \mathfrak{J}_2$ which gives $A \cap B \in \mathfrak{J}$.

Case 3 : $A \in \mathfrak{J}_1$ and $B \in \mathfrak{J}_2$. Then $\infty \notin A$ will imply $\infty \notin A \cap B$.

Hence $A \cap B \in \mathfrak{J}_1 \Rightarrow A \cap B \in \mathfrak{J}$.

Case 4 : $A \in \mathfrak{J}_2$ and $B \in \mathfrak{J}_1$. Then $\infty \notin B$ will imply $\infty \notin A \cap B$.

Hence $A \cap B \in \mathfrak{J}_1 \Rightarrow A \cap B \in \mathfrak{J}$.

Thus in all the cases $A, B \in \mathfrak{J} \implies A \cap B \in \mathfrak{J}$.

(iii) $A_\lambda \in \mathfrak{J}\ \forall\, \lambda \in \Lambda$, where Λ is any indexing set. If $A_\lambda \in \mathfrak{J}_1\ \forall\, \lambda \in \Lambda$ then $\infty \notin A_\lambda\ \forall\, \lambda \in \Lambda$ will imply $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}_1$. Hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

If $\exists\ \lambda_0 \in \Lambda$ such that $A_{\lambda_0} \notin \mathfrak{J}_1$ then $A_{\lambda_0} \in \mathfrak{J}_2$. In this case $\infty \in A_{\lambda_0}$ and $X - A_{\lambda_0}$ is a finite set.

$A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$ implies $\infty \in \bigcup_{\lambda \in \Lambda} A_\lambda$ and $X - \bigcup_{\lambda \in \Lambda} A_\lambda \subseteq X - A_{\lambda_0}$.

As $X - A_{\lambda_0}$ is finite we get $X - \bigcup_{\lambda \in \Lambda} A_\lambda$ a is finite set. Thus in this case $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}_2$ and hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

Thus in either case, $A_\lambda \in \mathfrak{J}, \forall\, \lambda \in \Lambda \implies \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{J}$.

From (i), (ii) and (iii) $\mathfrak{J}$ is a topology on X .

This topology $\mathfrak{J}$ is called **Fort's topology** on X and the T-space $\langle X, \mathfrak{J} \rangle$ is called **Fort's space**.

Some Special Topologies on Special sets .

Apart from the topologies given in the above examples there exist some special topologies on $\mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{N}$. ($\mathbb{R}$ = the set of all real numbers , $\mathbb{Z}$ = the set of all integers and $\mathbb{N}$ = the set of all natural numbers). We list some of them in the following examples.

(11) Let $\mathfrak{J}_u = \{\emptyset\} \cup \{A \subseteq \mathbb{R} \mid \forall\, a \in A\ \exists\ \ r > 0$ such that $(a - r\,, a + r) \subseteq A\}$. Then $\mathfrak{J}_u$ is a topology on $\mathbb{R}$.

(i) $\emptyset \in \mathfrak{J}_u$ (by definition) and $\mathbb{R} \in \mathfrak{J}_u$ as for any $a \in \mathbb{R}$, $(a - 1\,, a + 1) \subseteq \mathbb{R}$.

(ii) Let $A, B \in \mathfrak{J}_u$. If $A = \emptyset$ or $B = \emptyset$, then $A \cap B \in \mathfrak{J}_u$. Let $A \neq \emptyset$ and $B \neq \emptyset$.

Then $x \in A \cap B \ \Rightarrow\ x \in A$ and $x \in B \ \Rightarrow\ \exists\ \ r_1 > 0$ such that $(x - r_1\,,\ x + r_1) \subseteq A$ and $\exists\ \ r_2 > 0$ such that $(x - r_2\,,\ x + r_2) \subseteq B$.

Define $r = min(r_1, r_2)$. Then $r > 0$ and $(x - r\,, x + r) \subseteq A \cap B$. But this shows that $A \cap B \in \mathfrak{J}_u$. Thus in either case $A, B \in \mathfrak{J}_u \ \Rightarrow\ A \cap B \in \mathfrak{J}_u$.

(iii) $A_\lambda \in \mathfrak{J}_u\ \ \forall\, \lambda \in \Lambda$, where Λ is any indexing set .

If $\displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda = \emptyset$, then obviously, $\displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}_u$.

Hence, assume that $\displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda \neq \emptyset$. Let $x \in \displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda$. Then $x \in A_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

As $A_{\lambda_0} \in \mathfrak{J}_u\ \ \exists\ \ r > 0$ such that $(x - r\,,\ x + r) \subseteq A_{\lambda_0}$.

But then $(x - r\,, x + r) \subseteq \displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda$. But this shows that $\displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}_u$.

Thus in either case $A_\lambda \in \mathfrak{J}_u$, $\forall\, \lambda \in \Lambda \ \Rightarrow\ \displaystyle\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathfrak{J}_u$.

From (i), (ii) and (iii) $\mathfrak{J}_u$ is a topology on $\mathbb{R}$.

This topology is called **usual topology** on $\mathbb{R}$.

Remarks: (1) The usual topology on $\mathbb{R}$ is also called standard topology or Euclidean topology .

(2) Any open interval in $\mathbb{R}$ is a member of $\mathfrak{J}_u$. Consider the open interval (a, b) and $x \in (a, b)$. Take $r = min(x - a\,, b - x)$. Then $(x - r\,, x + r) \subseteq (a\,, b)$. This shows that $(a\,, b) \in \mathfrak{J}_u$.

(12) Let $\mathfrak{J}_r = \{\emptyset\} \cup \{A \subseteq \mathbb{R} \mid \forall\, p \in A\ \exists\, a, b \in \mathbb{R}$ such that $p \in [a\,,\ b) \subseteq A\}$. Then $\mathfrak{J}_r$ is a topology on $\mathbb{R}$.

(i) $\emptyset \in \mathfrak{J}_r$ (by definition). $\mathbb{R} \in \mathfrak{J}_r$ as for any $p \in \mathbb{R}\ \exists\, a, b \in \mathbb{R}$ such that $p \in [p\,,\ p + 1) \subseteq \mathbb{R}$.

(ii) Let $A, B \in \mathfrak{J}_r$. If $A \cap B = \emptyset$, then $A \cap B \in \mathfrak{J}_r$. If $A \cap B \neq \emptyset$ then for $x \in A \cap B$ there exist half open intervals H_1 and H_2 in $\mathbb{R}$ such that $x \in H_1 \subseteq A$ and

UNIT-II

SYLLABUS

Closed set and limit points, continuous functions, the product topologies, the metric topologies.

(1.1) Definition: Let $(X, \mathfrak{J})$ be a topological space. Then a subset A of X is said to be **closed** in X if its complement $X - A$ is open in X.

The definition is fairly straightforward and one can cite as many examples of closed sets as of open sets. It is fortunate that all closed intervals (bounded or not) of real numbers are indeed closed in the usual topology on the real line. If (X, d) is a metric space, $x \in X$ and $r > 0$, then the **closed ball** with centre x and radius r is defined as the set $\{y \in X : d(x, y) \leq r\}$. We leave it to the reader to verify that each such closed ball is a closed subset in the topology induced by the metric.

A word of warning is perhaps in order. In analogy with everyday usage, a biginner is likely to think that 'closed' is the negation of 'open', that is to say, a set is closed if and only if it is not open. But this is not so. The reason for the misleading terminology is probably that complements of sets are defined in terms of negation. The fact is that the possibilities of a set being open and its being closed are neither mutually exclusive nor exhaustive. Note for example that the empty set and the whole set are always open as well as closed in every space. On the other hand, the set of rationals is neither open nor closed in the usual topology on the real line. A set which is both open and closed is sometimes called a **clopen** set.

It is immediate that a set is open iff its complement is closed. As a result, any statement about open sets can be immediately translated into a corresponding statement about closed sets and vice-versa, as we do in the following theorem.

(1.2) Theorem: Let $\mathcal{C}$ be the family of all closed sets in a topological space $(X, \mathfrak{J})$. Then $\mathcal{C}$ has the following properties:

(i) $\phi \in \mathcal{C}$, $X \in \mathcal{C}$.

(ii) $\mathcal{C}$ is closed under arbitrary intersections.

(iii) $\mathcal{C}$ is closed under finite unions.

Conversely, given any set X and a family $\mathcal{C}$ of its subsets which satisfies these three properties, there exists a unique topology $\mathfrak{J}$ on X such that $\mathcal{C}$ coincides with the family of closed subsets of $(X, \mathfrak{J})$.

Proof: The first part follows trivially from the definition of a topology and De Morgan's laws. The converse part is equally trivial once it is clearly understood what it says. Here we are given a set X (just a bare set with no topology on it) and some collection $\mathcal{C}$ of its subsets. We are given that properties (i) to (iii) hold for $\mathcal{C}$. We do not know how $\mathcal{C}$ originated, nor do we know whether its members are closed subsets of X. Actually it is meaningless to talk about closed subsets of X unless a topology on X is specified. The theorem says that given such a family $\mathcal{C} \subset P(X)$ we can define a suitable topology $\mathfrak{J}$ on X such that members of $\mathcal{C}$ are precisely the closed subsets of X (w.r.t. the topology $\mathfrak{J}$), and that such a topology is unique.

Having understood what the theorem says, the proof itself is trivial as we have no choice but to let $\mathfrak{J}$ consist of complements (in X) of members of $\mathcal{C}$, i.e. $\mathfrak{J} = \{B \subset X : X - B \in \mathcal{C}\}$. That $\mathfrak{J}$ is a topology on X follows by applying De Morgan's laws. The open subsets of X are precisely the complements of members of $\mathcal{C}$, and hence the closed subsets of X are precisely the members of $\mathcal{C}$ as asserted. Also this condition determines $\mathfrak{J}$ uniquely. ∎

Trivial as the theorem is, its significance is noteworthy. In the definition of a topological space we took 'open set' as a primitive term, that is to say, open sets are not defined (except as members of the topology on the set in question) and nothing is known about their nature save what is implied by the definition of a topology. Everything we do with topological spaces is in terms of open sets. For example, we defined convergence of sequences in a topological space in terms of open sets, and we defined closed sets as complements of open sets. The preceding theorem asserts that this procedure could be reversed. That is, we could as well take 'closed sets' as a primitive concept and then define open sets as complements of closed sets. With this approach our definition of a topological space would be that it is a pair $(X, \mathcal{C})$ where X is a set, $\mathcal{C} \subset P(X)$ and conditions (i), (ii), (iii) above are satisfied. Although nothing is to be gained and nothing is to be lost by adopting this new approach over the usual one, in particular examples of topological spaces it may be more natural to specify the closed sets rather than the open sets. For instance, in the cofinite topology on a set X, it is so easy to tell what the closed subsets are, they are precisely all finite subsets of X and the set X itself.

Any subset of a topological space generates a closed subset called its closure. The definition is as follows:

$\bigcap \{C \subset X : C \text{ closed in } X, C \supset A\}$. It is denoted by $\bar{A}$. Obviously it depends on the topology $\mathfrak{J}$ and when it is important to stress this, it is customary to write $\bar{A}^{\mathfrak{J}}$ or $(\bar{A})_{\mathfrak{J}}$ instead of mere $\bar{A}$. Note further that if $Y \subset X$ and $A \subset Y$ then the closure of A in the space $(X, \mathfrak{J})$ is in general different from its closure in the subspace $(Y, \mathfrak{J}/Y)$. We leave it to the reader to verify that the latter is the intersection of the former with Y. When confusion is likely to arise otherwise, it is usual to write $\bar{A}^Y$ or $(\bar{A})_Y$ to indicate the subspace w.r.t. which the closure is intended. The notations $Cl(A)$ or $C(A)$ or $c(A)$ are also used sometimes to denote the closure. In the next proposition we list down a few properties of closures.

(1.4) Proposition: Let A, B be subsets of a topological space $(X, \mathfrak{J})$.

(i) $\bar{A}$ is a closed subset of X. Moreover it is the smallest closed subset of X containing A i.e. if C is closed in X and $A \subset C$ then $\bar{A} \subset C$.

(ii) $\bar{\phi} = \phi$

(iii) A is closed in X iff $\bar{A} = A$

(iv) $\bar{\bar{A}} = \bar{A}$ or in other words, $c(c(A)) = c(A)$

(v) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof: (i) and (ii) are immediate consequences of the definition and properties of closed sets. For (iii) we note that if A is closed then it is clearly the smallest closed set containing A and consequently $\bar{A} = A$. Conversely if $\bar{A} = A$ then A is closed since $\bar{A}$ is always a closed set, being the intersection of closed sets. Property (iv) follows by applying (iii) to $\bar{A}$ which is known to be closed. Finally, for (v), note that $\bar{A} \cup \bar{B}$ is first of all a closed set containing $A \cup B$; as $A \subset \bar{A}$ and $B \subset \bar{B}$, and hence $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. For the other way inclusion, we first observe that whenever $A_1 \subset A_2$, $\bar{A}_1 \subset \bar{A}_2$ (prove !). Now $A \cup B$ contains A as well as B and so $\bar{A}$, $\bar{B}$ are both subsets of $\overline{A \cup B}$. Hence $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. This completes the proof. ∎

(1.5) Theorem: Let X be a set, $\theta : P(X) \to P(X)$ a function such that

(1) for every $A \in P(X)$, $A \subset \theta(A)$ (this condition is sometimes expressed by saying that θ is an **expansive** operator),

(2) ϕ is a fixed point of θ,

(3) θ is idempotent, and

(4) θ commutes with finite unions.

Then there exists a unique topology $\mathfrak{J}$ on X such that θ coincides with the closure operator associated with $\mathfrak{J}$. Conversely, any closure operator satisfies these properties.

Proof: The converse part is already established. For the direct implication, suppose $\theta : P(X) \to P(X)$ satisfies (1) to (4). We want to find a topology $\mathfrak{J}$ on X such that for every $A \subset X$, $\theta(A) = \bar{A}^{\mathfrak{J}}$. If at all such a topology exists then its closed subsets must be precisely the fixed points of θ as we saw above. This gives us a clue to the construction of $\mathfrak{J}$. We let $\mathcal{C} = \{A \subset X : \theta(A) = A\}$ and contend that $\mathcal{C}$ has properties (i) to (iii) of Theorem (1.2). Condition (2) shows that $\phi \in \mathcal{C}$ while condition (4) implies that $\mathcal{C}$ is closed under finite unions. To prove that $X \in \mathcal{C}$, we merely note that by (1), $X \subset \theta(X)$ and hence $X = \theta(X)$ since $\theta(X) \subset X$ anyway. It only remains to verify that $\mathcal{C}$ is closed under arbitrary intersections. For this we first note that θ is monotonic, i.e., whenever $A \subset B$, $\theta(A) \subset \theta(B)$, which follows by writing B as $A \cup (B - A)$ and applying (4). Now let $A = \bigcap\limits_{i \in I} A_i$ where I is an index set and $A_i \in \mathcal{C}$ for each $i \in I$. We want to show that $A \in \mathcal{C}$, i.e. $\theta(A) = A$. By (1) we already know $A \subset \theta(A)$. Also $\theta(A) \subset \theta(A_i)$ for each $i \in I$ since θ is monotonic, and so $\theta(A) \subset \bigcap\limits_{i \in I} \theta(A_i)$. But $\theta(A_i) = A_i$ since $A_i \in \mathcal{C}$ for all $i \in I$. Consequently, $\theta(A) \subset A$ and hence $\theta(A) = A$ as desired. So by theorem (1.2), the family $\mathfrak{J}$ of complements of members of $\mathcal{C}$ is a topology on X.

It remains to be verified that the closure operator associated with $\mathfrak{J}$ coincides with θ. Let $A \subset X$. Then $\bar{A}^{\mathfrak{J}}$ (i.e. $\bar{A}$ w.r.t. $\mathfrak{J}$) is the intersection of all closed subsets of X containing A. But by very construction, closed subsets of X are precisely the fixed points of θ. Hence $\bar{A} = \bigcap \{B \subset X : A \subset B;$

$\theta(B) = B\}$. Now, whenever $B \supset A$, $\theta(B) \supset \theta(A)$ by monotonicity of θ. So if $B \supset A$ and $\theta(B) = B$ then $B \supset \theta(A)$. But $\bar{A}$ is the intersection of such B's and so $\bar{A} \supset \theta(A)$. For the other way inclusion we note that by condition (3), $\theta(A) \in \mathcal{C}$ while by (1) $A \subset \theta(A)$ whence $\bar{A} \subset \theta(A)$, $\bar{A}$ being the smallest member of $\mathcal{C}$ containing A. Hence for all $A \subset X$, $\theta(A) = \bar{A}$ completing the proof. ∎

(3.1) Definition: Let $f : X \to Y$ be a function; $x_0 \in X$ and $\mathfrak{J}$, $\mathcal{U}$ be topologies on X, Y respectively. Then f is said to be **continuous** (or more precisely **$\mathfrak{J}$-$\mathcal{U}$ continuous**) at x_0 if for every $V \in \mathcal{U}$ such that $f(x_0) \in V$, there exists $U \in \mathfrak{J}$ such that $x_0 \in U$ and $f(U) \subset V$.

It is convenient to have some other formulations of continuity at a point

(3.2) Proposition: With the notation above, the following statements are equivalent.

1. f is continuous at x_0.
2. The inverse image (under f) of every neighbourhood of $f(x_0)$ in Y is a neighbourhood of x_0 in X.
3. For every subset $A \subset X$, $x_0 \in \bar{A}$ implies $f(x_0) \in \overline{f(A)}$.
4. For every subset $A \subset X$, $x_0 \, \delta \, A$ implies $f(x_0) \, \delta \, f(A)$.

Proof (1) $\Rightarrow$ (2). Let N be a neighbourhood of $f(x_0)$ in Y. Then there is an open set V in Y such that $f(x_0) \in V$ and $V \subset N$. Since f is continuous at x_0, there is an open set U in X such that $x_0 \in U$ and $f(U) \subset V$. This means $x_0 \in U \subset f^{-1}(V) \subset f^{-1}(N)$ thus showing that $f^{-1}(N)$ is a neighbourhood of x_0.

(2) $\Rightarrow$ (3). Suppose $x_0 \in \bar{A}$ where $A \subset X$. If $f(x_0) \notin \overline{f(A)}$ then by Theorem (2.10) in the last section, there is a neighbourhood N of $f(x_0)$ such that $f(A) \cap N = \emptyset$. This means $f^{-1}(\overline{f(A)}) \cap f^{-1}(N) = \emptyset$ and hence that $A \cap f^{-1}(N) = \emptyset$ since $A \subset f^{-1}(f(A))$. But by (2), $f^{-1}(N)$ is a neighbourhood of x_0 and so $A \cap f^{-1}(N) \neq \emptyset$, since $x_0 \in \bar{A}$. This is a contradiction.

(3) $\Leftrightarrow$ (4). This is immediate since the nearness relation corresponding to a topology is defined by saying that a point is near a set iff it is in the closure of that set.

(3) $\Rightarrow$ (1). Let V be an open set containing $f(x_0)$. Let $A = X - f^{-1}(V) = f^{-1}(Y - V)$. Then $f(A) \subset Y - V$ and so $\overline{f(A)} \subset Y - V$ as $Y - V$ is closed. So $f(x_0) \notin \overline{f(A)}$ whence $x_0 \notin \bar{A}$ by (3). Hence there is a neighbourhood N of x_0 such that $N \cap A = \emptyset$. Clearly then $f(N) \subset V$ and the proof is completed if we let $U = \text{int}\,(N)$. ∎

UNIT-III

SYLLABUS

Connected spaces, connected subspaces of the real line, components and local connectedness.

(2.1) Definition: A space X is said to be **connected** if it is impossible to find non-empty subsets A and B of it such that $X = A \cup B$ and $\bar{A} \cap \bar{B} = \emptyset$. A space which is not connected is called **disconnected**.

(2.2) Proposition: Let X be a space and A, B subsets of X. Then the following statements are equivalent:

1. $A \cup B = X$ and $\bar{A} \cap \bar{B} = \emptyset$.
2. $A \cup B = X$, $A \cap B = \emptyset$ and A, B are both closed in X.
3. $B = X - A$ and A is clopen (i.e. closed as well as open) in X.
4. $B = X - A$ and ∂A (that is, the boundary of A) is empty.
5. $A \cup B = X$, $A \cap B = \emptyset$ and A, B are both open in X.

Proof: (1) ⇒ (2). Clearly $\bar{A} \cap \bar{B} = \emptyset$ implies that $A \cap B = \emptyset$ since $A \subset \bar{A}$ and $B \subset \bar{B}$. Also $\bar{A} \subset X - \bar{B} \subset X - B = A$ and so $\bar{A} = A$ showing that A is closed. Similarly B is closed.

(2) ⇒ (3) is immediate since the complement of a closed set is open.

(3) ⇒ (4). This follows from the fact that the boundary of a clopen set is empty (see Exercise (5.2.7).)

(4) ⇒ (5). This requires the converse, viz., that a set with empty boundary is clopen. Also if A is closed, then its complement B is open.

(5) ⇒ (1). Assume $X = A \cup B$ where $A \cap B = \emptyset$ and A, B are open. Then $A = X - B$ and $B = X - A$ whence A, B are closed as well and so $\bar{A} = A$, $\bar{B} = B$, showing $\bar{A} \cap \bar{B} = \emptyset$. ∎

(2.3) Proposition: Let X be a space. Then the following are equivalent:

1. X is connected.
2. X cannot be written as the disjoint union of two nonempty closed subsets.
3. The only clopen subsets of X are $\emptyset$ and X.
4. Every nonempty proper subset of X has a nonempty boundary.
5. X cannot be written as the disjoint union of two nonempty open subsets.

Proof: The result is immediate from the definition and the last proposition. ∎

From the definitions we see immediately that every indiscrete space is connected and that the only connected discrete spaces are those which consist of at most one point. The space of rational numbers is disconnected; given any irrational number α the sets $\{x \in \mathbb{Q} : x < \alpha\}$ and $\{x \in \mathbb{Q} : x > \alpha\}$ are both open in the relative topology on $\mathbb{Q}$ and $\mathbb{Q}$ is clearly their disjoint union. Similarly the set of irrational numbers is disconnected. The Sierpinsky space defined in Chapter 4, Section 2 is connected, although it is not indiscrete. It is clear that if a set is connected w.r.t. a topology $\mathfrak{J}$ on it, then it is connected w.r.t. every topology weaker than $\mathfrak{J}$. The following proposition shows that connectedness is preserved under continuous functions.

(2.5) Theorem: A subset of $\mathbf{R}$ is connected iff it is an interval.

Proof: First note that a subset $X \subset \mathbf{R}$ is an interval iff it has the property that for any $a, b \in X$, $(a, b) \subset X$. (Prove.) Now if X is not an interval then there exist real numbers a, b, c such that $a < c < b$; $a, b \in X$ and $c \notin X$. Let $A = \{x \in X : x < c\}$ and $B = \{x \in X : x > c\}$. Clearly A, B are disjoint, open subsets of X (in the relative topology) since $A = X \cap (-\infty, c)$ and $B = (c, \infty) \cap X$ and $A \cup B = X$. Further $a \in A$, $b \in B$ and hence A, B are nonempty. This shows that X is not connected.

Conversely suppose X is an interval and that $X = A \cup B$ where $\bar{A} \cap \bar{B} = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ where the closure is relative to X. Let $a_0 \in A$, $b_0 \in B$. Without loss of generality we may suppose that $a_0 < b_0$. Now let x be the mid-point of the interval from a_0 to b_0, i.e. $x = \dfrac{a_0 + b_0}{2}$. Then $x \in X$ and so x is exactly in one of the sets A and B. If $x \in A$ we rename it as a_1 and rename b_0 as b_1. If $x \in B$, we rename a_0 as a_1 and x as b_1. In any case $[a_1, b_1]$ is an interval with its left end-point in A and the right end-point in B. We can now take the mid-point of $[a_1, b_1]$ and get an interval $[a_2, b_2]$ of half the length with $a_2 \in A$, $b_2 \in B$. Repeating this process ad infinitum, we get a nested sequence of intervals $\{[a_n, b_n] : n = 0, 1, 2, 3, \ldots\}$ such that $a_n \in A$ and $b_n \in B$ for all n. Note that $\{a_n\}$ is a bounded monotonically increasing sequence while $\{b_n\}$ is a bounded monotonically decreasing sequence and that $(b_n - a_n) \to 0$ as $n \to \infty$. By the order completeness of $\mathbf{R}$, both sequences converge to a common limit, say c. Note that $c \in X$ since $a_0 \leqslant c \leqslant b_0$. Also every neighbourhood of c intersects A as well as B. So $c \in \bar{A} \cap \bar{B}$, a contradiction. Hence X is connected. ∎

(2.7) Definition: Two subsets A and B of a space X are said to be **(mutually) separated** if $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$.

(2.8) Proposition: Let X be a space and C be a connected subset of X (that is, C with the relative topology is a connected space). Suppose $C \subset A \cup B$ where A, B are mutually separated subsets of X. Then either $C \subset A$ or $C \subset B$.

Proof: Let $G = C \cap A$ and $H = C \cap B$. Then G, H are closed subsets of C since, A, B are closed in $A \cup B$. Also $G \cap H = \emptyset$. But C is connected. So either $G = \emptyset$ or $H = \emptyset$. In the first case $C \subset B$ while in the second, $C \subset A$. ■

(2.9) Theorem: Let $\mathcal{C}$ be a collection of connected subsets of a space X such that no two members of $\mathcal{C}$ are mutually separated. Then $\bigcup_{C \in \mathcal{C}} C$ is also connected.

Proof: Let $M = \bigcup_{C \in \mathcal{C}} C$. If M is not connected then we could write M as a $A \cup B$ where A, B are nonempty and mutually separated subsets of X. By the proposition above, for each $C \in \mathcal{C}$ either $C \subset A$ or $C \subset B$. We contend that the same possibility holds for all $C \in \mathcal{C}$, i.e. either $C \subset A$ for all $C \in \mathcal{C}$ or $C \subset B$ for all $C \in \mathcal{C}$. If this is not the case, then there exist $C, D \in \mathcal{C}$ such that $C \subset A$ and $D \subset B$. But, A, B are mutually separated and hence their subsets C, D are also mutually separated contradicting the hypothesis. Thus all members of $\mathcal{C}$ are contained in A or all are contained in B. Accordingly $M = A$ or $M = B$, contradicting that A, B are both non-empty. ■

(2.10) Corollary: Let $\mathcal{C}$ be a collection of connected subsets of a space X and suppose K is a connected subset of X (not necessarily a member of $\mathcal{C}$) such that $C \cap K \neq \emptyset$ for all $C \in \mathcal{C}$. Then $(\bigcup_{C \in \mathcal{C}} C) \cup K$ is connected.

Proof: Let $M = (\bigcup_{C \in \mathcal{C}} C) \cup K$. Let $\mathcal{D} = \{K \cup C : C \in \mathcal{C}\}$. Clearly $M = \bigcup_{D \in \mathcal{D}} D$. By the theorem above, each member of $\mathcal{D}$ is connected since it is a union of two connected sets which intersect (and which are therefore not separated). Now any two members of $\mathcal{D}$ have at least points of K in common and so are not mutually separated. So again by the theorem above, M is connected. ∎

(2.12) Corollary: The topological product of any finite number of connected spaces is connected.

Proof: If $X_1, X_2, \ldots, X_{n-1}, X_n$ are spaces (with $n \geq 2$) then $X_1 \times X_2 \times \ldots \times X_n$ is homeomorphic to $(X_1 \times \ldots \times X_{n-1}) \times X_n$ (see Exercise (5.3.6)). The result follows by induction on n and the last proposition. ∎

(2.13) Proposition: The closure of a connected subset is connected. More generally if C is a connected subset of a space X then any set D such that $C \subset D \subset \bar{C}$ is connected.

Proof: Suppose C is connected and $C \subset D \subset \bar{C}$. If D is not connected then we can write $D = A \cup B$ where A, B are nonempty, disjoint and closed relative to D. Then $C \cap A$, $C \cap B$ are disjoint closed subsets of C whose union is C. But C is connected. So one of them, say, $C \cap B$ is empty. This means $C \subset A$, and hence $\bar{C}^D \subset A$ where the closure is w.r.t. D. But $\bar{C}^D = \bar{C}^X \cap D = D$ since $D \subset \bar{C}^X$. Hence $A = D$ contradicting that B is nonempty. So D is connected. ∎

UNIT-IV

SYLLABUS

Compact spaces, compact subspaces of the Real line, limit point compactness, local compactness.

Definition. A collection $\mathcal{A}$ of subsets of a space X is said to ***cover*** X, or to be a ***covering*** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an ***open covering*** of X if its elements are open subsets of X.

Definition. A space X is said to be ***compact*** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

EXAMPLE 1. The real line $\mathbb{R}$ is not compact, for the covering of $\mathbb{R}$ by open intervals

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

contains no finite subcollection that covers $\mathbb{R}$.

Lemma 26.1. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite subcollection

$$\{A_{\alpha_1} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a subcollection of $\mathcal{A}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By hypothesis, some finite subcollection $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, \ldots, A'_{\alpha_n}\}$ is a subcollection of $\mathcal{A}'$ that covers Y. ∎

Theorem 26.2. *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let us form an open covering $\mathcal{B}$ of X by adjoining to $\mathcal{A}$ the single open set $X - Y$, that is,

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\}.$$

Some finite subcollection of $\mathcal{B}$ covers X. If this subcollection contains the set $X - Y$, discard $X - Y$; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of $\mathcal{A}$ that covers Y. ∎

Theorem 26.3. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the Hausdorff space X. We shall prove that $X - Y$ is open, so that Y is closed.

Let x_0 be a point of $X - Y$. We show there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (using the Hausdorff condition). The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X; therefore, finitely many of them $V_{y_1}, \dots, V_{y_n}$ cover Y. The open set

$$V = V_{y_1} \cup \dots \cup V_{y_n}$$

contains Y, and it is disjoint from the open set

$$U = U_{y_1} \cap \dots \cap U_{y_n}$$

formed by taking the intersection of the corresponding neighborhoods of x_0. For if z is a point of V, then $z \in V_{y_i}$ for some i, hence $z \notin U_{y_i}$ and so $z \notin U$. See Figure 26.1.

Then U is a neighborhood of x_0 disjoint from Y, as desired. ∎

Lemma 26.4. *If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.*

Theorem 26.5. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \to Y$ be continuous; let X be compact. Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them, say

$$f^{-1}(A_1), \ldots, f^{-1}(A_n),$$

cover X. Then the sets $A_1, \ldots, A_n$ cover $f(X)$. ■

Theorem 26.6. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We shall prove that images of closed sets of X under f are closed in Y; this will prove continuity of the map f^{-1}. If A is closed in X, then A is compact, by Theorem 26.2. Therefore, by the theorem just proved, $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed in Y, by Theorem 26.3. ■

Lemma 26.8 (The tube lemma). *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.*

Definition. A collection $\mathcal{C}$ of subsets of X is said to have the ***finite intersection property*** if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 26.9. *Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.*

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$$

be the collection of their complements. Then the following statements hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X if and only if the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is empty.

(3) The finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ covers X if and only if the intersection of the corresponding elements $C_i = X - A_i$ of $\mathcal{C}$ is empty.

The first statement is trivial, while the second and third follow from DeMorgan's law:

$$X - (\bigcup_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in J} (X - A_\alpha).$$

The proof of the theorem now proceeds in two easy steps: taking the *contrapositive* (of the theorem), and then the *complement* (of the sets)!

The statement that X is compact is equivalent to saying: "Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then some finite subcollection of $\mathcal{A}$ covers X." This statement is equivalent to its contrapositive, which is the following: "Given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X." Letting $\mathcal{C}$ be, as earlier, the collection $\{X - A \mid A \in \mathcal{A}\}$ and applying (1)–(3), we see that this statement is in turn equivalent to the following: "Given any collection $\mathcal{C}$ of closed sets, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection of all the elements of $\mathcal{C}$ is nonempty." This is just the condition of our theorem. ■

Theorem 27.1. *Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.*

Proof. *Step 1.* Given $a < b$, let $\mathcal{A}$ be a covering of $[a, b]$ by sets open in $[a, b]$ in the subspace topology (which is the same as the order topology). We wish to prove the existence of a finite subcollection of $\mathcal{A}$ covering $[a, b]$. First we prove the following: If x is a point of $[a, b]$ different from b, then there is a point $y > x$ of $[a, b]$ such that the interval $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

If x has an immediate successor in X, let y be this immediate successor. Then $[x, y]$ consists of the two points x and y, so that it can be covered by at most two elements of $\mathcal{A}$. If x has no immediate successor in X, choose an element A of $\mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x, c)$, for some c in $[a, b]$. Choose a point y in (x, c); then the interval $[x, y]$ is covered by the single element A of $\mathcal{A}$.

Step 2. Let C be the set of all points $y > a$ of $[a, b]$ such that the interval $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$. Applying Step 1 to the case $x = a$, we see that there exists at least one such y, so C is not empty. Let c be the least upper bound of the set C; then $a < c \leq b$.

Step 3. We show that c belongs to C; that is, we show that the interval $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Choose an element A of $\mathcal{A}$ containing c; since A is open, it contains an interval of the form $(d, c]$ for some d in $[a, b]$. If c is not in C, there must be a point z of C lying in the interval (d, c), because otherwise d would be a smaller upper bound on C than c. See Figure 27.1. Since z is in C, the interval $[a, z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$. Now $[z, c]$ lies in the single element A of $\mathcal{A}$, hence $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$. Thus c is in C, contrary to assumption.

Step 4. Finally, we show that $c = b$, and our theorem is proved. Suppose that $c < b$. Applying Step 1 to the case $x = c$, we conclude that there exists a point $y > c$ of $[a, b]$ such that the interval $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$. See Figure 27.2. We proved in Step 3 that c is in C, so $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Therefore, the interval

$$[a, y] = [a, c] \cup [c, y]$$

can also be covered by finitely many elements of $\mathcal{A}$. This means that y is in C, contradicting the fact that c is an upper bound on C. ■

Corollary 27.2. *Every closed interval in $\mathbb{R}$ is compact.*

Now we characterize the compact subspaces of $\mathbb{R}^n$:

UNIT-V

SYLLABUS

The countability axioms, the separation axioms, normal spaces, The Urysohn lemma, The Urysohn metrization theorem, the Tietze Extension theorem

Definition. A space X is said to have a ***countable basis at x*** if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the ***first countability axiom***, or to be ***first-countable***.

Definition. If a space X has a countable basis for its topology, then X is said to satisfy the ***second countability axiom***, or to be ***second-countable***.

Theorem 30.2. *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.*

Proof. Consider the second countability axiom. If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A of X. If $\mathcal{B}_i$ is a countable basis for the space X_i, then the collection of all products $\prod U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\prod X_i$.

The proof for the first countability axiom is similar. ■

Definition. A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$.

Theorem 30.3. *Suppose that X has a countable basis. Then:*

(a) Every open covering of X contains a countable subcollection covering X.

(b) There exists a countable subset of X that is dense in X.

Proof. Let $\{B_n\}$ be a countable basis for X.

(a) Let $\mathcal{A}$ be an open covering of X. For each positive integer n for which it is possible, choose an element A_n of $\mathcal{A}$ containing the basis element B_n. The collection $\mathcal{A}'$ of the sets A_n is countable, since it is indexed with a subset J of the positive integers. Furthermore, it covers X: Given a point $x \in X$, we can choose an element A of $\mathcal{A}$ containing x. Since A is open, there is a basis element B_n such that $x \in B_n \subset A$. Because B_n lies in an element of $\mathcal{A}$, the index n belongs to the set J, so A_n is defined; since A_n contains B_n, it contains x. Thus $\mathcal{A}'$ is a countable subcollection of $\mathcal{A}$ that covers X.

(b) From each nonempty basis element B_n, choose a point x_n. Let D be the set consisting of the points x_n. Then D is dense in X: Given any point x of X, every basis element containing x intersects D, so x belongs to $\bar{D}$. ■

Definition. Suppose that one-point sets are closed in X. Then X is said to be ***regular*** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be ***normal*** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Lemma 31.1. *Let X be a topological space. Let one-point sets in X be closed.*

(a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proof. (a) Suppose that X is regular, and suppose that the point x and the neighborhood U of x are given. Let $B = X - U$; then B is a closed set. By hypothesis, there exist disjoint open sets V and W containing x and B, respectively. The set $\bar{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V. Therefore, $\bar{V} \subset U$, as desired.

To prove the converse, suppose the point x and the closed set B not containing x are given. Let $U = X - B$. By hypothesis, there is a neighborhood V of x such that $\bar{V} \subset U$. The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively. Thus X is regular.

(b) This proof uses exactly the same argument; one just replaces the point x by the set A throughout. ■

Theorem 32.1. *Every regular space with a countable basis is normal.*

Proof. Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. Each point x of A has a neighborhood U not intersecting B. Using regularity, choose a neighborhood V of x whose closure lies in U; finally, choose an element of $\mathcal{B}$ containing x and contained in V. By choosing such a basis element for each x in A, we construct a countable covering of A by open sets whose closures do not intersect B. Since this covering of A is countable, we can index it with the positive integers; let us denote it by $\{U_n\}$.

Similarly, choose a countable collection $\{V_n\}$ of open sets covering B, such that each set $\bar{V}_n$ is disjoint from A. The sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B, respectively, but they need not be disjoint. We perform the following simple trick to construct two open sets that *are* disjoint. Given n, define

$$U'_n = U_n - \bigcup_{i=1}^{n} \bar{V}_i \quad \text{and} \quad V'_n = V_n - \bigcup_{i=1}^{n} \bar{U}_i .$$

Theorem 32.2. *Every metrizable space is normal.*

Proof. Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B. Similarly, for each b in B, choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A. Define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

Then U and V are open sets containing A and B, respectively; we assert they are disjoint. For if $z \in U \cap V$, then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$ and some $b \in B$. The triangle inequality applies to show that $d(a, b) < (\epsilon_a + \epsilon_b)/2$. If $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$, so that the ball $B(b, \epsilon_b)$ contains the point a. If $\epsilon_b \leq \epsilon_a$, then $d(a, b) < \epsilon_a$, so that the ball $B(a, \epsilon_a)$ contains the point b. Neither situation is possible. ∎

Theorem 32.3. *Every compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space. We have already essentially proved that X is regular. For if x is a point of X and B is a closed set in X not containing x, then B is compact, so that Lemma 26.4 applies to show there exist disjoint open sets about x and B, respectively.

Essentially the same argument as given in that lemma can be used to show that X is normal: Given disjoint closed sets A and B in X, choose, for each point a of A, disjoint open sets U_a and V_a containing a and B, respectively. (Here we use regularity of X.) The collection $\{U_a\}$ covers A; because A is compact, A may be covered by finitely many sets $U_{a_1}, \dots, U_{a_m}$. Then

$$U = U_{a_1} \cup \cdots \cup U_{a_m} \quad \text{and} \quad V = V_{a_1} \cap \cdots \cap V_{a_m}$$

are disjoint open sets containing A and B, respectively. ∎

Reg. No..................

18MMP202

Karpagam Academy of Higher Education
Coimbatore-21
Department of Mathematics
Second Semester- I Internal test
Topology

Date: 04.02.2019(AN) Time: 2 hours
Class: I M.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. Which of the following is a topology on $X = \{a, b, c\}$
A. $\{X, \{a\}\emptyset\}$ B. $\{X, \{a, b\}, \{b, c\}, \emptyset\}$
C. $\{X, \{a\}, \{b\}, \emptyset\}$ D. $\{X, \{a\}, \{b\}, \{c\}, \emptyset\}$

2. The maximum number of topology exists on $X = \{a, b\}$ is
A. 2 B. 1
C. 16 D. 13

3. Total number of topology exists on $X = \{a, b, c\}$ is
A. 20 B. 30
C. 39 D. 29

4. If $X = \{a, b, c\}$ and $\mathcal{B} = \{\{a, b\}, \{b.c\}, X\}$ then $\mathcal{B}$ satisfies basis condition
A. (i) B. (ii)
C. neither (i) nor (ii) D. both (i) and (ii)

5. If X is any set, the collection of all one point subsets of X is a basis for the —— topology
A. cofinite B. discrete
C. indiscrete D. cocountable

6. Which of the following is true?
A. $\mathcal{T} \subset \mathcal{B}$ B. $\mathcal{B} \subset \mathcal{T}$
C. $\mathcal{B} = \mathcal{T}$ D. $\mathcal{B} \not\subseteq \mathcal{T}$

7. Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all —— of elements of $\mathcal{B}$
A. union B. intersection
C. both A and B D. neither A nor B

8. If $\mathcal{T}_\infty$ and $\mathcal{T}_\in$ are two topologies on non-empty set X, then —— is topology
A. $\mathcal{T}_\infty \cap \mathcal{T}_\in$ B. $\mathcal{T}_\infty \cup \mathcal{T}_\in$
C. $\mathcal{T}_\infty - \mathcal{T}_\in$ D. $\mathcal{T}_\infty \times \mathcal{T}_\in$

9. If $\mathcal{T}$ is topology on non-empty set X, then arbitrary —— of member of $\mathcal{T}$ belong to $\mathcal{T}$.
A. union B. intersection
C. both A and B D. neither A nor B

10. If $\mathcal{T}$ is topology on non-empty set X, then finite . of member of $\mathcal{T}$ belong to $\mathcal{T}$.
A. union B. intersection
C. both A and B D. neither A nor B

11. Let $\mathcal{T}$ be a topology on non-empty set X. Which of the following is true?
A. $\emptyset \notin \mathcal{T}$ B. $X \in \mathcal{T}$
C. X $\notin \mathcal{T}$ D. $P(X) \in \mathcal{T}$

12. If $X = \{a, b, c\}$ and $\mathcal{T}$ be the discrete topology. Then number of elements in basis for $\mathcal{T}$ is
A. 1 B. 2
C. 3 D. 4

13. If $X = \{a, b, c\}$ and $\mathcal{T}$ be the indiscrete topology. Then number of open sets related to $\mathcal{T}$ is
A. 1 B. 2
C. 3 D. 4

14. Let X be a set, and let $\mathcal{B}$ is a basis for a topology on X. For each $x \in X$, there is atleast——— $B \in \mathcal{B}$ such that $x \in B$
A. 1 B. 2
C. 3 D. 4

15. Let X be a set, and let $\mathcal{B}$ is a basis for a topology on X. If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there is atlaest —— $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.
A. 1 B. 2
C. 3 D. 4

16. If $\mathcal{B}$ is the collection of all open intervals in the real line, then $\mathcal{B}$ satisfies basis condition
A. (i) B. (ii)
C. neither (i) nor (ii) D. both (i) and (ii)

17. If $\mathcal{B}$ is the collection of all half open intervals in the real line, then $\mathcal{B}$ satisfies basis condition
A. (i) B. (ii)
C. neither (i) nor (ii) D. both (i) and (ii)

18. Let X be a set. $\mathcal{T}$ be the collection of all subsets U of X such that $X - U$ is either —— or X. Then $\mathcal{T}$ is a topology.
A. finite B. countable
C. both A and B D. neither A nor B

19. Arbitrary union of open sets is—— set
A. open B. closed
C. both A and B D. neither A nor B

20. Suppsoe $\mathcal{T}_\infty$ and $\mathcal{T}_\in$ are discrete and indiscrete topologies on non-empty set X. Which of the following is true?
A. $\mathcal{T}_\infty \subset \mathcal{T}_\in$ B. $\mathcal{T}_\infty \supset \mathcal{T}_\in$
C. $\mathcal{T}_\infty = \mathcal{T}_\in$ D. $\mathcal{T}_\infty \not\supseteq \mathcal{T}_\in$

Part B-($3 \times 2 = 6$ **marks)**

21. Define K topology

22. Define continuous function.

23. Define subbasis

Part C-($3 \times 8 = 24$ **marks)**

24. a) Prove that intersection of topologies is a topology on X.

OR

b) Let X be a set; let

$$\mathcal{T}_\infty = \{U \,|\, X - U \text{ is infinite or } \phi \text{ or } X\}.$$

Is this a topology on X?

25. a) Find the all the topologies for (i) $X = \{a, b\}$ (ii) $X = \{a, b, c\}$

OR

b) Let $\mathcal{T}$ be the collection of subsets U of X if for each $x \in U$ there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$. Then prove that $\mathcal{T}$ is the topology

26. a) Show that the set $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$ is not a Hausdorff space

OR

b) Prove that every finite set in a Hausdorff space is closed.

Reg. No..................

18MMP202

Karpagam Academy of Higher Education
Coimbatore-21
Department of Mathematics
Second Semester
II Internal test - March 2019
Topology

Date:11.03.2019(AN) Time: 2 hours
Class: I M.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. Let $X = \{a, b, c\}$ be a topological space with $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then limit points of $A = \{b, c\}$ is $---$

   a. $\emptyset$ b. A
   c. X d. $\{a\}$

2. Which of the following is true?

   a. $\overline{A} \subset A$ b. $A \subset \overline{A}$
   c. $\overline{A} = A$ d. $\overline{A} \neq A$

3. Every finite set point in Hausdorff space is $---$

   a. *open* b. *closed*
   c. both a and b d. neither a nor b

4. In a disconnected space X, every nonempty proper subset of X is $---$

   a. *open* b. *closed*
   c. both a and b d. neither a nor b

5. Let $X = \{a, b, c\}$ be a topological space with discrete topology. Then X is $---$

   a. connected b. disconnected
   c. Hausdroff d. both b and c.

6. If X is a finite Hausdorff space, then $\mathcal{T}$ is——- topology

   a. indiscrete b. discrete
   c. finite complement d. co countable

7. A space is totally connected space if every —— sets are connected

   a. one point b. two points
   c. three points d. four points.

8. A space X is said to be compact if every open cover has a —— subscover
   a. infinite b. finite
   c. countable d. uncountable.

9. If every infinite subset of X has a limit point then X is $---$

   a. connected b. limit point compact
   c. Hausdroff d. compact.

10. Every ——- space is totally connected

    a. indiscrete b. discrete
    c. finite complement d. co countable

11. Every compact subset of a Hausdorff space is ——-

    a. *open* b. *closed*
    c. both a and b d. neither a nor b

12. Let $X = \{a, b, c\}$ be a topological space with $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then X is $---$

a. compact b. disconnected
c. Hausdroff d. both a and c.

13. Let X be a infinite set with discrete topology. Then X is $---$

a. compact b. non compact
c. Hausdroff d. both a and c.

14. Let X be a Hausdorff space. Then X is locally compact at x if —— nbd U of x there is a nbd V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$

a. no b. all
c. some d. finite

15. A subset A of a space X is said to be dense in X if $\overline{A} = ---$

a. A b. $\emptyset$
c. X d. both a and c

16. Every compact Hausdorff space is ——

a. compact b. normal
c. Hausdroff d. both a and c.

17. Every regular space with ——— basis is normal

a. infinite b. finite
c. countable d. uncountable.

18. A topological space X is said to satisfy —— countability axioms if X has a countable basis for its topology

a. first b. second
c. both a and b d. neither a nor b

19. Let X be a compact metrizable space. Then X is $---$

a. limit point compact b. sequentially compact
c. both a and b d. neither a nor b

20. If —— sequence has a convergent subsequence in X, then X sequentially compact

a. no b. all
c. some d. finite

Answer ALL questions
Part B-($3 \times 2 = 6$ marks)

21. Define path connected space

22. Define components of a topological space X

23. Define compact space

Answer ALL questions
Part C-($3 \times 8 = 24$ marks)

24. a) Let Y be a subspace of X. Prove that two disjoint nonempty sets A and B whose union is Y form a separation of Y if and only if A contains no limit points of B and B contains not limit points of A.

OR

b) Prove that the closure of a connected set is connected.

25. a) Prove that every compact subset of a Hausdorff space is closed

OR

b) Prove that the continuous image of a connected space is connected

26. a) Prove that the union of a collection of connected subspaces of X that have a point in common is connected.

OR

b) Prove that a space is connected if and only if the only subsets of X that are both open and closed in X are X and $\emptyset$

Reg. No.................................

[17MMP202]

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari Post, Coimbatore – 641 021.
(For the candidates admitted from 2017 onwards)

M.Sc., DEGREE EXAMINATION, APRIL 2018

Second Semester

MATHEMATICS

TOPOLOGY

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21.a. Is the collection $T_{\infty} = \{U/X$ – is infinite or empty or all of X$\}$ a topology on X?

Or

b. Prove that the collection s = $\{\pi_1^{-1}(U)/U$ is open in X$\} \cup \{\pi_2^{-1}(V)/V$ is open in Y$\}$ is a subbasis for the product topology X x Y

22. a. Let A' be the set of all limit points of A. Then prove that $\overline{A} = A \cup A^{l}$

Or

b. State and prove pasting lemma

23.a. Prove that the Cartesian product of connected spaces is connected.

Or

b. State and prove intermediate value theorem

24.a. Prove that every compact subspace of a Hausdorff space is closed

Or

b. State and prove tube lemma.

25. a. State and prove Tietze extension theorem

Or

b. Prove that every well-ordered set X is normal in the order topology.

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Let A, B and A_{α} denote subsets of a space X. Prove the following
i. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ ii. $\overline{\cup A_{\alpha}} \supset \cup \overline{A_{\alpha}}$; give an example where equality fai

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