KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act1956)

Coimbatore – 641 021.

DEPARTMENT OF MATHEMATICS

17MMP401 **MEASURE THEORY**

SEMESTER-IV

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Scope: On successful completion of this course the learners gain a clear knowledge about the Integration, Outer Measure and Product Measure etc which plays an essential role in Operator Theory.

Objectives: To be familiar with the Lebesgue measure, Lebesgue Integral, differentiation of monotone function and be exposed with measure spaces & L^P spaces.

UNIT I

Lebesgue Measure: Introduction – Outer measure – Measurable sets and Lebesgue Measure – A non measurable set – Measurable set – Measurable functions – Littlewoods's three principles.

UNIT II

The Lebesgue Integral: The Riemann integral – The Lebesgue integral of a bounded function over a set finite measure – The integral of a non negative function – The general Lebesgue integral – Convergence in measure.

UNIT III

Differentiation of monotone function, Functions of bounded variation-differentiation of an integral-Absolute continuity.

UNIT IV

Measure spaces-Measurable functions-Integration-General convergence Theorems.

UNIT V

Signed measures-The Radon-Nikodym theorem-the L^P spaces.

SUGGESTED READINGS

TEXT BOOK

1. Royden, H. L, (2004). Real Analysis, Third Edition, Prentice – Hall of India Pvt.Ltd, New Delhi.

REFERENCES

1. Keshwa Prasad Gupta, (2005). Measure Theory, Krishna Prakashan Ltd, Meerut.
2. Donald L. Cohn, (2013). Measure Theory, United States.
3. Paul R. Halmos, (2008). Measure Theory, Princeton University Press Dover Publications, New York .
4. Rudin W, (2006). Real and Complex Analysis, 3 rd Edition, Mcgraw – Hill, New Delhi.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: SANGEETHA.M
SUBJECT NAME: MEASURE THEORY SUB.CODE:17MMP401
SEMESTER: IV CLASS: II M.Sc (MATHEMATICS)

S.No	Lecture Hours (Hr)	Topics to be covered	Support Materials
Unit-I			
1	1	Lebesgue Measure:Introduction on Outer Measure and Problems on outer measure	T1: Chapter 3 , Pg.no:54-58
2	1	Measurable set- Theorems	T1: Chapter 3, Pg.no: 58-59, R4: Chapter 6, Pg.no:46-47
3	1	Theorems on measurable set and lebesgue measure	T1:Chapter 3 , Pg.no:59-61
4	1	Theorems on measurable set and lebesgue measure-continuation	T1:Chapter 3 , Pg.no:62-63 R4:Chapter 8, Pg.no:106-120
5	1	A non measurable set ,Theorems on non measurable set	T1: Chapter 3 ,Pg.no: 64-66
6	1	Measurable function theorem and Problems on Measurable function	T1: Chapter 3,Pg.no: 66-71
7	1	Littlewood's three principle	T1: Chapter 3 : 72-73, R1:Chapter , Pg.no:132-136
8	1	Theorems on Littlewood's three principle	T1: Chapter 3 : 73-74
9	1	Recapitulation and discussion of important questions	
Total	**9 Hrs**		
Unit-II			
1	1	Lebesgue integral:Riemann Integral and Lebesgue integral of a bounded function over a set of finite measure	T1: Chapter 4, , Pg.no 75-78
2	1	Lebesgue integral of a bounded function over a set of finite measure	T1:Chapter 4, Pg.no : 77-78
3	1	Simple function theorem, Bounded	T1: Chapter 4, Pg.no : 78-81

4	1	Theorem on bounded function Bounded convergence theorem	T1: Chapter 4, Pg.no : 82-85
5	1	Integral of a non negative function theorem	T1: Chapter 4 , Pg.no: 85-86
6	1	Fatous lemma, Monotone convergence theorem	T1: Chapter 4, Pg.no : 86-89
7	1	General Lebesgue integral and Lebesgue convergence theorem	T1: Chapter 4 , Pg.no: 89-93
8	1	Convergence in measure and Theorems on Convergence in measure	T1: Chapter 4 , Pg.no: 93-96
9	1	Recapitulation and discussion of important questions	
Total	**9 Hrs**		
Unit-III			
1	1	Differentiation of monotone function	T1: Chapter 5 , Pg.no:97-99
2	1	Continuation on Differentiation of monotone function	T1: Chapter 5 , Pg.no:100-101
3	1	Problems on monotone function	T1: Chapter 5 , Pg.no:101-102
4	1	Functions of bounded variation	T1: Chapter 5 , Pg.no:102-104
5	1	Differentiation of an integral	T1: Chapter 5 , Pg.no:105-108
6	1	Absolute continuity	T1: Chapter 5 , Pg.no:108-110
7	1	Problems on Absolute continuity	T1: Chapter 5 , Pg.no:111 R2: Chapter 5 , Pg.no:131-135
8	1	Recapitulation and discussion of important questions	
Total	**8 Hrs**		
Unit-IV			
1	1	Introduction on Measure spaces and their related conditions	T1: Chapter 11,Pg.no:253-257
2	1	Problems on Measure spaces	T1: Chapter 11,Pg.no:258-259
3	1	Measurable functions	T1: Chapter 11,Pg.no:259-260
4	1	Continuation on Measurable functions	T1: Chapter 11,Pg.no:261-262
5	1	Problems on Measurable functions	T1: Chapter 11,Pg.no:262-264
6	1	Problems on Integration	T1: Chapter 11,Pg.no:264-265
7	1	Theorems on Integration	T1: Chapter 11,Pg.no:266-268
8	1	General convergence Theorems	T1: Chapter 11,Pg.no:268-270
9	1	Recapitulation and discussion of important questions	
Total	**9 Hrs**		
Unit-V			
1	1	Signed measures and theorems on	T1: Chapter 11,Pg.no:270-275

		Signed measures	
2	1	Problems on Signed measures and the Radon-Nikodym theorem	T1: Chapter 11,Pg.no:275-278
3	1	Lebesgue decomposition and Problems on Lebesgue decomposition	T1: Chapter 11,Pg.no:278-282
4	1	The L_P spaces Some theorems on The L_P spaces	T1: Chapter 11,Pg.no:282-287 R4: Chapter 3,Pg.no:61-63
5	1	Problems on The L_P spaces	R4: Chapter 3,Pg.no:64-65
6	1	Recapitulation and discussion of important questions	
7	1	Discussion of previous ESE question papers	
8	1	Discussion of previous ESE question papers	
9	1	Discussion of previous ESE question papers	
Total	**9 Hrs**		

TEXT BOOK
T1. Royden H.L,(2004). Real Analysis, Third Edition, Prentice – Hall of India Pvt.Ltd, New Delhi.
REFERENCES
R1. Keshwa Prasad Gupta,(2005). Measure Theory, Krishna Prakashan Ltd, Meerut. R2. Donald L. Cohn, (2013). Measure Theory, United States.
R3. Paul R. Halmos, (2008). Measure Theory, Princeton University Press Dover Publications.
R4. Rudin W, (2006). Real and Complex Analysis, 3 rd Edition, Mcgraw – Hill, New Delhi.

UNIT – I

Lebesgue Measure: Introduction – Outer measure – Measurable sets and Lebesgue Measure – A non measurable set – Measurable set – Measurable functions – Littlewoods's three principles.

Lebesgue Measure

Note. We "weigh" an interval by its length when setting up the Riemann integral. So to generalize the Riemann integral, we desire a way to weigh sets other than intervals. This weight should be a generalization of the length of an interval.

Note. Since we know an open set is a countable union of disjoint open intervals, we would define its "weight" (or "measure") to be the sum of the lengths of the open intervals which compose it.

Note. We want a function m which maps the collection of all subsets of $\mathbb{R}$, that is the power set of the reals $\mathcal{P}(\mathbb{R})$, into $\mathbb{R}^+ \cup \{0, \infty\} = [0, \infty]$. We would like m to satisfy:

1. For any interval I, $m(I) = \ell(I)$ (where $\ell(I)$ is the length of I).
2. For all E on which m is defined and for all $y \in \mathbb{R}$, $m(E + y) = m(E)$. That is, m is *translation invariant.*
3. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of disjoint sets (on each of which, m is defined), then $m(\cup E_k) = \sum m(E_k)$. That is, m is *countably additive.*
4. m is defined on $\mathcal{P}(\mathbb{R})$.

Here, and throughout, we use the symbol $\dot\cup$ to indicate disjoint union.

Note. We will see in Section 2.6 that there is *not* a function satisfying all four properties. In fact, there is not even a set function satisfying (1), (2), and (4) for which $m\left(\dot{\bigcup_{k=1}^{n}} E_k\right) = \sum_{k=1}^{n} m(E_k)$ for disjoint E_k (this property is called *finite additivity*). See Theorem 2.18 for details.

Note. It is "unknown" whether m exists satisfying properties (1), (3), and (4) (if we assume the Continuum Hypothesis, then there is *not* such a function).

Note. We will weaken Property (4) and try to find a function defined on as large a set as possible. We will require (by (3)) that our collection of sets, $\mathcal{M}$, on which m is defined, be countably additive and therefore $\mathcal{M}$ will be a σ-algebra.

Problem 2.1. Let m' be a set function defined on a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m' is countably additive over countable disjoint collections in $\mathcal{A}$. If A and B are two sets in $\mathcal{A}$ with $A \subset B$, then $m'(A) \leq m'(B)$. This is called *monotonicity*.

Proof. First, $B \setminus A = B \cap A^c$ and since $\mathcal{A}$ is a σ-algebra (and hence closed under countable intersections and complements), then $B \setminus A \in \mathcal{A}$. Next, $B = (B \setminus A) \dot\cup A$, so by the hypothesized Countable Additivity, $m'(B) = m'(B \setminus A) + m'(A)$ since $B \setminus A$ and A are disjoint. Since $m'(B \setminus A) \geq 0$ by hypothesis, then $m'(A) \leq m'(B)$. (NOTICE: We could weaken the hypothesis of "σ-algebra" to "algebra" and weaken the hypothesis of "countable additivity" to "finite additivity," and the result would still hold.) ∎

Note. Another property of measure is the following.

Problem 2.3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in a σ-algebra $\mathcal{A}$ on which a countably additive measure m' is defined. Then $m'\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m'(E_k)$. This is called *countable subadditivity.*

The Riemann integral of a bounded function over a closed, bounded interval is defined using approximations of the function that are associated with partitions of its domain into finite collections of subintervals. The generalization of the Riemann integral to the Lebesgue integral will be achieved by using approximations of the function that are associated with decompositions of its domain into finite collections of sets which we call Lebesgue measurable. Each interval is Lebesgue measurable. The richness of the collection of Lebesgue measurable sets provides better upper and lower approximations of a function, and therefore of its integral, than are possible by just employing intervals. This leads to a larger class of functions that are Lebesgue integrable over very general domains and an integral that has better properties. For instance, under quite general circumstances we will prove that if a sequence of functions converges pointwise to a limiting function, then the integral of the limit function is the limit of the integrals of the approximating functions. In this chapter we establish the basis for the forthcoming study of Lebesgue measurable functions and the Lebesgue integral: the basis is the concept of measurable set and the Lebesgue measure of such a set.

The length $\ell(I)$ of an interval I is defined to be the difference of the endpoints of I if I is bounded, and ∞ if I is unbounded. Length is an example of a *set function*, that is, a function that associates an extended real number to each set in a collection of sets. In the case of length, the domain is the collection of all intervals. In this chapter we extend the set function length to a large collection of sets of real numbers. For instance, the "length" of an open set will be the sum of the lengths of the countable number of open intervals of which it is composed. However, the collection of sets consisting of intervals and open sets is still too limited for our purposes. We construct a collection of sets called **Lebesgue measurable sets**, and a set function of this collection called **Lebesgue measure** which is denoted by m.

The collection of Lebesgue measurable sets is a σ-algebra[1] which contains all open sets and all closed sets. The set function m possesses the following three properties.

The measure of an interval is its length Each nonempty interval I is Lebesgue measurable and

$$m(I) = \ell(I).$$

Measure is translation invariant If E is Lebesgue measurable and y is any number, then the translate of E by y, $E+y=\{x+y \mid x \in E\}$, also is Lebesgue measurable and

$$m(E+y)=m(E).$$

Measure is countably additivity over countable disjoint unions of sets[2] If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right)=\sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers (see page 48). In fact, there is not even a set function defined for all sets of real numbers that possesses the first two properties and is finitely additive (see Theorem 18). We respond to this limitation by constructing a set function on a very rich class of sets that does possess the above three properties. The construction has two stages.

We first construct a set function called **outer–measure**, which we denote by m^*. It is defined for any set, and thus, in particular, for any interval. The outer measure of an interval is its length. Outer measure is translation invariant. However, outer measure is not finitely additive. But it is countably subadditive in the sense that if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k).$$

The second stage in the construction is to determine what it means for a set to be **Lebesgue measurable** and show that the collection of Lebesgue measurable sets is a σ-algebra containing the open and closed sets. We then restrict the set function m^* to the collection of Lebesgue measurable sets, denote it by m, and prove m is countably additive. We call m **Lebesgue measure.**

LEBESGUE OUTER MEASURE

Let I be a nonempty interval of real numbers. We define its length, $\ell(I)$, to be ∞ if I is unbounded and otherwise define its length to be the difference of its endpoints. For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover A, that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of the terms. We define the **outer measure**[3] of A, $m^*(A)$, to be the infimum of all such sums, that is

$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}.$$

It follows immediately from the definition of outer measure that $m^*(\emptyset) = 0$. Moreover, since any cover of a set B is also a cover of any subset of B, outer measure is **monotone** in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Example A countable set has outer measure zero. Indeed, let C be a countable set enumerated as $C = \{c_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$. For each natural number k, define $I_k = (c_k - \epsilon/2^{k+1},\ c_k + \epsilon/2^{k+1})$. The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C. Therefore

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

This inequality holds for each $\epsilon > 0$. Hence $m^*(E) = 0$.

Proposition 1 *The outer measure of an interval is its length.*

Proof We begin with the case of a closed, bounded interval $[a, b]$. Let $\epsilon > 0$. Since the open interval $(a - \epsilon, b + \epsilon)$ contains $[a, b]$ we have $m^*([a, b]) \leq \ell((a - \epsilon, b + \epsilon)) = b - a + 2\epsilon$. This holds for any $\epsilon > 0$. Therefore $m^*([a, b]) \leq b - a$. It remains to show that $m^*([a, b]) \geq b - a$. But this is equivalent to showing that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open, bounded intervals covering $[a, b]$, then

$$\sum_{k=1}^{\infty} \ell(I_k) \geq b - a. \tag{1}$$

By the Heine-Borel Theorem,[4] any collection of open intervals covering $[a, b]$ has a finite subcollection that also covers $[a, b]$. Choose a natural number n for which $\{I_k\}_{k=1}^n$ covers $[a, b]$. We will show that

$$\sum_{k=1}^{n} \ell(I_k) \geq b-a, \tag{2}$$

and therefore (1) holds. Since a belongs to $\bigcup_{k=1}^n I_k$, there must be one of the I_k's that contains a. Select such an interval and denote it by (a_1, b_1). We have $a_1 < a < b_1$. If $b_1 \geq b$, the inequality (2) is established since

$$\sum_{k=1}^{n} \ell(I_k) \geq b_1 - a_1 > b-a.$$

Otherwise, $b_1 \in [a, b)$, and since $b_1 \notin (a_1, b_1)$, there is an interval in the collection $\{I_k\}_{k=1}^n$, which we label (a_2, b_2), distinct from (a_1, b_1), for which $b_1 \in (a_2, b_2)$; that is, $a_2 < b_1 < b_2$. If $b_2 \geq b$, the inequality (2) is established since

$$\sum_{k=1}^{n} \ell(I_k) \geq (b_1 - a_1) + (b_2 - a_2) = b_2 - (a_2 - b_1) - a_1 > b_2 - a_1 > b-a.$$

We continue this selection process until it terminates, as it must since there are only n intervals in the collection $\{I_k\}_{k=1}^n$. Thus we obtain a subcollection $\{(a_k, b_k)\}_{k=1}^N$ of $\{I_k\}_{k=1}^n$ for which

$$a_1 < a,$$

while

$$a_{k+1} < b_k \text{ for } 1 \leq k \leq N-1,$$

and, since the selection process terminated,

$$b_N > b.$$

Thus

$$\begin{aligned}\sum_{k=1}^{n} \ell(I_k) &\geq \sum_{k=1}^{N} \ell((a_i, b_i)) \\ &= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \cdots + (b_1 - a_1) \\ &= b_N - (a_N - b_{N-1}) - \ldots - (a_2 - b_1) - a_1 \\ &> b_N - a_1 > b-a.\end{aligned}$$

Thus the inequality (2) holds.

If I is any bounded interval, then given $\epsilon > 0$, there are two closed, bounded intervals J_1 and J_2 such that

$$J_1 \subseteq I \subseteq J_2$$

while

$$\ell(I) - \epsilon < \ell(J_1) \text{ and } \ell(J_2) < \ell(I) + \epsilon.$$

By the equality of outer measure and length for closed, bounded intervals and the monotonicity of outer measure,

$$\ell(I) - \epsilon < \ell(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = \ell(J_2) < \ell(I) + \epsilon.$$

This holds for each $\epsilon > 0$. Therefore $\ell(I) = m^*(I)$.

If I is an unbounded interval, then for each natural number n, there is an interval $J \subseteq I$ with $\ell(J) = n$. Hence $m^*(I) \geq m^*(J) = \ell(J) = n$. This holds for each natural number n. Therefore $m^*(I) = \infty$. □

Proposition 2 *Outer measure is translation invariant, that is, for any set A and number y,*

$$m^*(A + y) = m^*(A).$$

Proof Observe that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of sets, then $\{I_k\}_{k=1}^{\infty}$ covers A if and only if $\{I_k + y\}_{k=1}^{\infty}$ covers $A + y$. Moreover, if each I_k is an open interval, then each $I_k + y$ is an open interval of the same length and so

$$\sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \ell(I_k + y).$$

The conclusion follows from these two observations. □

Proposition 3 *Outer measure is countably subadditive, that is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then*

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Proof If one of the E_k's has infinite outer measure, the inequality holds trivially. We therefore suppose each of the E_k's has finite outer measure. Let $\epsilon > 0$. For each natural number k, there is a countable collection $\{I_{k,i}\}_{i=1}^{\infty}$ of open, bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,i} \text{ and } \sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \epsilon/2^k.$$

Now $\{I_{k,i}\}_{1\le k,i\le\infty}$ is a countable collection of open, bounded intervals that covers $\bigcup_{k=1}^{\infty} E_k$: the collection is countable since it is a countable collection of countable collections. Thus, by the definition of outer measure,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{1\le k,i<\infty} \ell(I_{k,i}) = \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} \ell(I_{k,i})\right]$$

$$< \sum_{k=1}^{\infty}\left[m^*(E_k)+\epsilon/2^k\right]$$

$$= \left[\sum_{k=1}^{\infty} m^*(E_k)\right]+\epsilon.$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. The proof is complete. □

If $\{E_k\}_{k=1}^{n}$ is any finite collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) \le \sum_{k=1}^{n} m^*(E_k).$$

This **finite subadditivity** property follows from countable subadditivity by taking $E_k = \emptyset$ for $k > n$.

LEBESGUE MEASURABLE SETS

Outer measure has four virtues: (i) it is defined for all sets of real numbers, (ii) the outer measure of an interval is its length, (iii) outer measure is countably subadditive, and (iv) outer measure is translation invariant. But outer measure fails to be countably additive. In fact, it is not even finitely additive (see Theorem 18): there are disjoint sets A and B for which

$$m^*(A\cup B) < m^*(A)+m^*(B). \tag{3}$$

To ameliorate this fundamental defect we identify a σ-algebra of sets, called the Lebesgue measurable sets, which contains all intervals and all open sets and has the property that the restriction of the set function outer measure to the collection of Lebesgue measurable sets is countably additive. There are a number of ways to define what it means for a set to be measurable.[5] We follow an approach due to Constantine Carathéodory.

Definition *A set E is said to be **measurable** provided for any set A,*[6]

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C).$$

We immediately see one advantage possessed by measurable sets, namely, that the strict inequality (3) cannot occur if one of the sets is measurable. Indeed, if, say, A is measurable and B is any set disjoint from A, then

$$m^*(A \cup B) = m^*([A \cup B] \cap A)) + m^*([A \cup B)] \cap A^C) = m^*(A) + m^*(B).$$

Since, by Proposition 3, outer measure is finitely subadditive and $A = [A \cap E] \cup [A \cap E^C]$, we always have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^C).$$

Therefore E is measurable if and only if for each set A we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C). \tag{4}$$

This inequality trivially holds if $m^*(A) = \infty$. Thus it suffices to establish (4) for sets A that have finite outer measure.

Observe that the definition of measurability is symmetric in E and E^C, and therefore a set is measurable if and only if its complement is measurable. Clearly the empty-set $\emptyset$ and the set **R** of all real numbers are measurable.

Proposition 4 *Any set of outer measure zero is measurable. In particular, any countable set is measurable.*

Proof Let the set E have outer measure zero. Let A be any set. Since

$$A \cap E \subseteq E \text{ and } A \cap E^C \subseteq A,$$

by the monotonicity of outer measure,

$$m^*(A \cap E) \leq m^*(E) = 0 \text{ and } m^*(A \cap E^C) \leq m^*(A).$$

Thus,

$$m^*(A) \geq m^*(A \cap E^C) = 0 + m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C),$$

Proposition 5 *The union of a finite collection of measurable sets is measurable.*

Proof As a first step in the proof, we show that the union of two measurable sets E_1 and E_2 is measurable. Let A be any set. First using the measurability of E_1, then the measurability of E_2, we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C)$$

$$= m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C).$$

There are the following set identities:

$$[A \cap E_1^C] \cap E_2^C = A \cap [E_1 \cup E_2]^C$$

and

$$[A \cap E_1] \cup [A \cap E_1^C \cap E_2] = A \cap [E_1 \cup E_2].$$

We infer from these identities and the finite subadditivity of outer measure that

$$m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C)$$

$$= m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^C)$$

$$\geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^C).$$

Thus $E_1 \cup E_2$ is measurable.

Now let $\{E_k\}_{k=1}^n$ be any finite collection of measurable sets. We prove the measurability of the union $\bigcup_{k=1}^n E_k$, for general n, by induction. This is trivial for $n = 1$. Suppose it is true for $n - 1$. Thus, since

$$\bigcup_{k=1}^{n} E_k = \left[\bigcup_{k=1}^{n-1} E_k\right] \cup E_n,$$

and we have established the measurability of the union of two measurable sets, the set $\bigcup_{k=1}^n E_k$ is measurable. □

Proposition 6 *Let A be any set and $\{E_k\}_{k=1}^n$ a finite disjoint collection of measurable sets. Then*

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

In particular,

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k).$$

Proof The proof proceeds by induction on n. It is clearly true for $n = 1$. Assume it is true for $n - 1$. Since the collection $\{E_k\}_{k=1}^n$ is disjoint,

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n = A \cap E_n$$

and

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^C = A \cap \left[\bigcup_{k=1}^{n-1} E_k\right].$$

Hence, by the measurability of E_n and the induction assumption,

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{k=1}^{n-1} E_k\right]\right)$$

$$= m^*(A \cap E_n) + \sum_{k=1}^{n-1} m^*(A \cap E_k)$$

$$= \sum_{k=1}^{n} m^*(A \cap E_k).$$

A collection of subsets of **R** is called an **algebra** provided it contains **R** and is closed with respect to the formation of complements and finite unions; by De Morgan's Identities, such a collection is also closed with respect to the formation of finite intersections. We infer from Proposition 5, together with the measurability of the complement of a measurable set, that the collection of measurable sets is an algebra. It is useful to observe that the union of a countable collection of measurable sets is also the union of a countable disjoint collection of measurable sets. Indeed, let $\{A_k\}_{k=1}^\infty$ be a countable collection of measurable sets. Define $A'_1 = A_1$ and for each $k \geq 2$, define

$$A'_k = A_k \sim \bigcup_{i=1}^{k-1} A_i.$$

Since the collection of measurable sets is an algebra, $\{A'_k\}_{k=1}^\infty$ is a disjoint collection of measurable sets whose union is the same as that of $\{A_k\}_{k=1}^\infty$.

Proposition 7 *The union of a countable collection of measurable sets is measurable.*

Proof Let E be the union of a countable collection of measurable sets. As we observed above, there is a countable disjoint collection of measurable sets $\{E_k\}_{k=1}^{\infty}$ for which $E = \bigcup_{k=1}^{\infty} E_k$. Let A be any set. Let n be a natural number. Define $F_n = \bigcup_{k=1}^{n} E_k$. Since F_n is measurable and $F_n^C \supseteq E^C$,

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^C) \geq m^*(A \cap F_n) + m^*(A \cap E^C).$$

By Proposition 6,

$$m^*(A \cap F_n) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Thus

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^C).$$

The left-hand side of this inequality is independent of n. Therefore

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*\left(A \cap E^C\right).$$

Hence, by the countable subadditivity of outer measure,

$$m^*(A) \geq m^*(A \cap E) + m^*\left(A \cap E^C\right).$$

Thus E is measurable. □

A collection of subsets of **R** is called an σ-**algebra** provided it contains **R** and is closed with respect to the formation of complements and countable unions; by De Morgan's Identities, such a collection is also closed with respect to the formation of countable intersections. The preceding proposition tells us that the collection of measurable sets is a σ-algebra.

Proposition 8 *Every interval is measurable.*

Proof As we observed above, the measurable sets are a σ-algebra. Therefore to show that every interval is measurable it suffices to show that every interval of the form (a, ∞) is measurable (see Problem 11). Consider such an interval. Let A be any set. We assume a does not belong to A. Otherwise, replace A by $A \sim \{a\}$, leaving the outer measure unchanged. We must show that

$$m^*(A_1) + m^*(A_2) \leq m^*(A), \tag{5}$$

where

$$A_1 = A \cap (-\infty, a) \text{ and } A_2 = A \cap (a, \infty).$$

By the definition of $m^*(A)$ as an infimum, to verify (5) it is necessary and sufficient to show that for any countable collection $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals that covers A,

$$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k). \tag{6}$$

Indeed, for such a covering, for each index k, define

$$I_k' = I_k \cap (-\infty, a) \text{ and } I_k'' = I_k \cap (a, \infty)$$

Then I_k' and I_k'' are intervals and

$$\ell(I_k) = \ell(I_k') + \ell(I_k'').$$

Since $\{I_k'\}_{k=1}^{\infty}$ and $\{I_k''\}_{k=1}^{\infty}$ are countable collections of open, bounded intervals that cover A_1 and A_2, respectively, by the definition of outer measure,

$$m^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I_k') \text{ and } m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k'').$$

Therefore

$$m^*(A_1) + m^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I_k') + \sum_{k=1}^{\infty} \ell(I_k'').$$

$$= \sum_{k=1}^{\infty} [\ell(I_k') + \ell(I_k'')]$$

$$= \sum_{k=1}^{\infty} \ell(I_k).$$

Thus (6) holds and the proof is complete.

Every open set is the disjoint union of a countable collection of open intervals.[7] We therefore infer from the two preceding propositions that every open set is measurable. Every closed set is the complement of an open set and therefore every closed set is measurable. Recall that a set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets and said to be an F_σ set provided it is the union of a countable collection of closed sets. We infer from Proposition 7 that every G_δ set and every F_σ set is measurable.

The intersection of all the σ-algebras of subsets of $\mathbf{R}$ that contain the open sets is a σ-algebra called the Borel σ-algebra; members of this collection are called **Borel sets**. The Borel σ-algebra is contained in every σ-algebra that contains all open sets. Therefore, since the measurable sets are a σ-algebra containing all open sets, every Borel set is measurable. We have established the following theorem.

Theorem 9 *The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains the σ-algebra $\mathcal{B}$ of Borel sets. Each interval, each open set, each closed set, each G_δ set, and each F_σ set is measurable.*

Proposition 10 *The translate of a measurable set is measurable.*

Proof Let E be a measurable set. Let A be any set and y be a real number. By the measurability of E and the translation invariance of outer measure,

$$m^*(A) = m^*(A-y) = m^*([A-y] \cap E) + m^*\left([A-y] \cap E^C\right)$$

$$= m^*(A \cap [E+y]) + m^*\left(A \cap [E+y]^C\right).$$

Therefore $E + y$ is measurable. □

NONMEASURABLE SETS

We have defined what it means for a set to be measurable and studied properties of the collection of measurable sets. It is only natural to ask if, in fact, there are any sets that fail to be measurable. The answer is not at all obvious.

We know that if a set E has outer measure zero, then it is measurable, and since any subset of E also has outer measure zero, every subset of E is measurable. This is the best that can be said regarding the inheritance of measurability through the relation of set inclusion: we now show that if E is any set of real numbers with positive outer measure, then there are subsets of E that fail to be measurable.

Lemma 16 *Let E be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers Λ for which the collection of translates of E, $\{\lambda + E\}_{\lambda \in \Lambda}$, is disjoint. Then $m(E) = 0$.*

Proof The translate of a measurable set is measurable. Thus, by the countable additivity of measure over countable disjoint unions of measurable sets,

$$m\left[\bigcup_{\lambda \in \Lambda} (\lambda + E)\right] = \sum_{\lambda \in \Lambda} m(\lambda + E). \tag{15}$$

Since both E and Λ are bounded sets, the set $\bigcup_{\lambda \in \Lambda}(\lambda + E)$ also is bounded and therefore has finite measure. Thus the left-hand side of (15) is finite. However, since measure is translation invariant, $m(\lambda + E) = m(E) > 0$ for each $\lambda \in \Lambda$. Thus, since the set Λ is countably infinite and the right-hand sum in (15) is finite, we must have $m(E) = 0$. □

For any nonempty set E of real numbers, we define two points in E to be **rationally equivalent** provided their difference belongs to $\mathbf{Q}$, the set of rational numbers. It is easy to see that this is an equivalence relation, that is, it is reflexive, symmetric, and transitive. We call it the rational equivalence relation on E. For this relation, there is the disjoint decomposition of E into the collection of equivalence classes. By a **choice set** for the rational equivalence relation on E we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class. We infer from the Axiom of Choice[10] that there are such choice sets. A choice set $\mathcal{C}_E$ is characterized by the following two properties:

(i) the difference of two points in $\mathcal{C}_E$ is not rational;
(ii) for each point x in E, there is a point c in $\mathcal{C}_E$ for which $x = c + q$, with q rational.

This first characteristic property of $\mathcal{C}_E$ may be conveniently reformulated as follows:

$$\text{For any set } \Lambda \subseteq \mathbf{Q},\ \{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda} \text{ is disjoint.} \tag{16}$$

Theorem 17 (Vitali) *Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.*

Proof By the countable subadditivity of outer measure, we may suppose E is bounded. Let $\mathcal{C}_E$ be any choice set for the rational equivalence relation on E. We claim that $\mathcal{C}_E$ is not measurable. To verify this claim, we assume it is measurable and derive a contradiction.

Let Λ_0 be any bounded, countably infinite set of rational numbers. Since $\mathcal{C}_E$ is measurable, and, by (16), the collection of translates of $\mathcal{C}_E$ by members of Λ_0 is disjoint, it follows from Lemma 16 that $m(\mathcal{C}_E) = 0$. Hence, again using the translation invariance and the countable additivity of measure over countable disjoint unions of measurable sets,

$$m\left[\bigcup_{\lambda \in \Lambda_0} (\lambda + \mathcal{C}_E)\right] = \sum_{\lambda \in \Lambda_0} m(\lambda + \mathcal{C}_E) = 0.$$

To obtain a contradiction we make a special choice of Λ_0. Because E is bounded it is contained in some interval $[-b, b]$. We choose

$$\Lambda_0 = [-2b, 2b] \cap \mathbf{Q}.$$

Then Λ_0 is bounded, and is countably infinite since the rationals are countable and dense.[11] We claim that

$$E \subseteq \bigcup_{\lambda \in [-2b, 2b] \cap \mathbf{Q}} (\lambda + \mathcal{C}_E). \tag{17}$$

Indeed, by the second characteristic property of $\mathcal{C}_E$, if x belongs to E, there is a number c in the choice set $\mathcal{C}_E$ for which $x = c + q$ with q rational. But x and c belong to $[-b, b]$, so that q belongs to $[-2b, 2b]$. Thus the inclusion (17) holds. This is a contradiction because E, a set of positive outer measure, is not a subset of a set of measure zero. The assumption that $\mathcal{C}_E$ is measurable has led to a contradiction and thus it must fail to be measurable. □

Theorem 18 *There are disjoint sets of real numbers A and B for which*

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof We prove this by contradiction. Assume $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B. Then, by the very definition of measurable set, every set must be measurable. This contradicts the preceding theorem. □

Lebesgue Measurable Functions

All the functions considered in this chapter take values in the extended real numbers, that is, the set $\mathbf{R} \cup \{\pm\infty\}$. Recall that a property is said to hold **almost everywhere** (abbreviated a.e.) on a measurable set E provided it holds on $E \sim E_0$, where E_0 is a subset of E for which $m(E_0) = 0$.

Given two functions h and g defined on E, for notational brevity we often write "$h \leq g$ on E" to mean that $h(x) \leq g(x)$ for all $x \in E$. We say that a sequence of functions $\{f_n\}$ on E is increasing provided $f_n \leq f_{n+1}$ on E for each index n.

Proposition 1 *Let the function f have a measurable domain E. Then the following statement are equivalent:*

(i) For each real number c, the set $\{x \in E \mid f(x) > c\}$ is measurable.
(ii) For each real number c, the set $\{x \in E \mid f(x) \geq c\}$ is measurable.
(iii) For each real number c, the set $\{x \in E \mid f(x) < c\}$ is measurable.
(iv) For each real number c, the set $\{x \in E \mid f(x) \leq c\}$ is measurable.

Each of these properties implies that for each extended real number c,

$$\text{the set } \{x \in E \mid f(x) = c\} \text{ is measurable.}$$

Proof Since the sets in (i) and (iv) are complementary in E, as are the sets in (ii) and (iii), and the complement in E of a measurable subset of E is measurable, (i) and (iv) are equivalent, as are (ii) and (iii).

Now (i) implies (ii), since

$$\{x \in E \mid f(x) \geq c\} = \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > c - 1/k\},$$

and the intersection of a countable collection of measurable sets is measurable. Similarly, (ii) implies (i), since

$$\{x \in E \mid f(x) > c\} = \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + 1/k\},$$

and the union of a countable collection of measurable sets is measurable.

Thus statements (i)–(iv) are equivalent. Now assume one, and hence all, of them hold. If c is a real number, $\{x \in E \mid f(x) = c\} = \{x \in E \mid f(x)) \geq c\} \cap \{x \in E \mid f(x) \leq c\}$, so $f^{-1}(c)$ is measurable since it is the intersection of two measurable sets. On the other hand, if c is infinite, say $c = \infty$,

$$\{x \in E \mid f(x) = \infty\} = \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > k\}$$

so $f^{-1}(\infty)$ is measurable since it is the intersection of a countable collection of measurable sets. □

Definition *An extended real-valued function f defined on E is said to be* **Lebesgue measurable**, *or simply* **measurable**, *provided its domain E is measurable and it satisfies one of the four statements of Proposition 1.*

Proposition 2 *Let the function f be defined on a measurable set E. Then f is measurable if and only if for each open set $\mathcal{O}$, the inverse image of $\mathcal{O}$ under f, $f^{-1}(\mathcal{O}) = \{x \in E \mid f(x) \in \mathcal{O}\}$, is measurable.*

Proof If the inverse image of each open set is measurable, then since each interval (c, ∞) is open, the function f is measurable. Conversely, suppose f is measurable. Let $\mathcal{O}$ be open. Then[1] we can express $\mathcal{O}$ as the union of a countable collection of open, bounded intervals $\{I_k\}_{k=1}^{\infty}$ where each I_k may be expressed as $B_k \cap A_k$, where $B_k = (-\infty, b_k)$ and $A_k = (a_k, \infty)$. Since f is a measurable function, each $f^{-1}(B_k)$ and $f^{-1}(A_k)$ are measurable sets. On the other hand, the measurable sets are a σ-algebra and therefore $f^{-1}(\mathcal{O})$ is measurable since

$$f^{-1}(\mathcal{O}) = f^{-1}\left[\bigcup_{k=1}^{\infty} B_k \cap A_k\right] = \bigcup_{k=1}^{\infty} f^{-1}(B_k) \cap f^{-1}(A_k).$$ □

The following proposition tells us that the most familiar functions from elementary analysis, the continuous functions, are measurable.

Proposition 3 *A real-valued function that is continuous on its measurable domain is measurable.*

Proof Let the function f be continuous on the measurable set E. Let $\mathcal{O}$ be open. Since f is continuous, $f^{-1}(\mathcal{O}) = E \cap \mathcal{U}$, where $\mathcal{U}$ is open.[2] Thus $f^{-1}(\mathcal{O})$, being the intersection of two measurable sets, is measurable. It follows from the preceding proposition that f is measurable. □

A real-valued function that is either increasing or decreasing is said to be monotone. We leave the proof of the next proposition as an exercise (see Problem 24).

Proposition 4 *A monotone function that is defined on an interval is measurable.*

Proposition 5 *Let f be an extended real-valued function on E.*

(i) If f is measurable on E and $f = g$ a.e. on E, then g is measurable on E.
(ii) For a measurable subset D of E, f is measurable on E if and only if the restrictions of f to D and $E \sim D$ are measurable.

Proof First assume f is measurable. Define $A = \{x \in E \mid f(x) \neq g(x)\}$. Observe that

$$\{x \in E \mid g(x) > c\} = \{x \in A \mid g(x) > c\} \cup \left[\{x \in E \mid f(x) > c\} \cap [E \sim A]\right]$$

Since $f = g$ a.e. on E, $m(A) = 0$. Thus $\{x \in A \mid g(x) > c\}$ is measurable since it is a subset of a set of measure zero. The set $\{x \in E \mid f(x) > c\}$ is measurable since f is measurable on E. Since both E and A are measurable and the measurable sets are an algebra, the set $\{x \in E \mid g(x) > c\}$ is measurable. To verify (ii), just observe that for any c,

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f(x) > c\} \cup \{x \in E \sim D \mid f(x) > c\}$$

and once more use the fact that the measurable sets are an algebra. □

The sum $f + g$ of two measurable extended real-valued functions f and g is not properly defined at points at which f and g take infinite values of opposite sign. Assume f and g are finite a.e. on E. Define E_0 to be the set of points in E at which both f and g are finite. If the restriction of $f + g$ to E_0 is measurable, then, by the preceding proposition, any extension of $f + g$, as an extended real-valued function, to all of E also is measurable. This is the sense in which we consider it unambiguous to state that the sum of two measurable functions that are finite a.e. is measurable. Similar remarks apply to products. The following proposition tells us that standard algebraic operations performed on measurable functions that are finite a.e. again lead to measurable functions

Theorem 6 *Let f and g be measurable functions on E that are finite a.e. on E.*

(Linearity) For any α and β,

$$\alpha f + \beta g \text{ is measurable on } E.$$

(Products)

$$fg \text{ is measurable on } E.$$

Proof By the above remarks, we may assume f and g are finite on all of E. If $\alpha = 0$, then the function αf also is measurable. If $\alpha \neq 0$, observe that for a number c,

$$\{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) > c/\alpha\} \text{ if } \alpha > 0$$

and

$$\{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) < c/\alpha\} \text{ if } \alpha < 0.$$

Thus the measurability of f implies the measurability of αf. Therefore to establish linearity it suffices to consider the case that $\alpha = \beta = 1$.

For $x \in E$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$ and so, by the density of the set of rational numbers $\mathbf{Q}$ in $\mathbf{R}$, there is a rational number q for which

$$f(x) < q < c - g(x).$$

Hence

$$\{x \in E \mid f(x) + g(x) < c\} = \bigcup_{q \in \mathbf{Q}} \{x \in E \mid g(x) < c - q\} \cap \{x \in E \mid f(x) < q\}.$$

The rational numbers are countable. Thus $\{x \in E \mid f(x) + g(x) < c\}$ is measurable, since it is the union of a countable collection of measurable sets. Hence $f + g$ is measurable.

To prove that the product of measurable functions is measurable, first observe that

$$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2].$$

Thus, since we have established linearity, to show that the product of two measurable functions is measurable it suffices to show that the square of a measurable function is measurable. For $c \geq 0$,

$$\{x \in E \mid f^2(x) > c\} = \{x \in E \mid f(x) > \sqrt{c}\} \cup \{x \in E \mid f(x) < -\sqrt{c}\}$$

while for $c < 0$,

$$\{x \in E \mid f^2(x) > c\} = E.$$

Thus f^2 is measurable. □

Many of the properties of functions considered in elementary analysis, including continuity and differentiability, are preserved under the operation of composition of functions. However, the composition of measurable functions may not be measurable.

Example There are two measurable real-valued functions, each defined on all of **R**, whose composition fails to be measurable. By Lemma 21 of Chapter 2, there is a continuous, strictly increasing function ψ defined on $[0, 1]$ and a measurable subset A of $[0, 1]$ for which $\psi(A)$ is nonmeasurable. Extend ψ to a continuous, strictly increasing function that maps **R** onto **R**. The function ψ^{-1} is continuous and therefore is measurable. On the other hand, A is a measurable set and so its characteristic function χ_A is a measurable function. We claim that

the composition $f = \chi_A \circ \psi^{-1}$ is not measurable. Indeed, if I is any open interval containing 1 but not 0, then its inverse image under f is the nonmeasurable set $\psi(A)$.

Despite the setback imposed by this example, there is the following useful proposition regarding the preservation of measurability under composition (also see Problem 11).

Proposition 7 *Let g be a measurable real-valued function defined on E and f a continuous real-valued function defined on all of* **R**. *Then the composition $f \circ g$ is a measurable function on E.*

Proof According to Proposition 2, a function is measurable if and only if the inverse image of each open set is measurable. Let $\mathcal{O}$ be open. Then

$$(f \circ g)^{-1}(\mathcal{O}) = g^{-1}(f^{-1}(\mathcal{O})).$$

Since f is continuous and defined on an open set, the set $\mathcal{U} = f^{-1}(\mathcal{O})$ is open.[3] We infer from the measurability of the function g that $g^{-1}(\mathcal{U})$ is measurable. Thus the inverse image $(f \circ g)^{-1}(\mathcal{O})$ is measurable and so the composite function $f \circ g$ is measurable. □

An immediate important consequence of the above composition result is that if f is measurable with domain E, then $|f|$ is measurable, and indeed

$$|f|^p \text{ is measurable with the same domain } E \text{ for each } p > 0.$$

For a finite family $\{f_k\}_{k=1}^n$ of functions with common domain E, the function

$$\max\{f_1, \ldots, f_n\}$$

is defined on E by

$$\max\{f_1, \ldots, f_n\}(x) = \max\{f_1(x), \ldots, f_n(x)\} \text{ for } x \in E.$$

The function $\min\{f_1, \ldots, f_n\}$ is defined the same way.

Proposition 8 *For a finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E, the functions $\max\{f_1, \ldots, f_n\}$ and $\min\{f_1, \ldots, f_n\}$ also are measurable.*

Proof For any c, we have

$$\{x \in E \mid \max\{f_1, \ldots, f_n\}(x) > c\} = \bigcup_{k=1}^{n} \{x \in E \mid f_k(x) > c\}$$

so this set is measurable since it is the finite union of measurable sets. Thus the function $\max\{f_1, \ldots, f_n\}$ is measurable. A similar argument shows that the function $\min\{f_1, \ldots, f_n\}$ also is measurable. □

For a function f defined on E, we have the associated functions $|f|$, f^+, and f^- defined on E by

$$|f|(x) = \max\{f(x), -f(x)\}, \; f^+(x) = \max\{f(x), 0\}, \; f^-(x) = \max\{-f(x), 0\}.$$

If f is measurable on E, then, by the preceding proposition, so are the functions $|f|$, f^+, and f^-. This will be important when we study integration since the expression of f as the difference of two nonnegative functions,

$$f = f^+ - f^- \text{ on } E,$$

plays an important part in defining the Lebesgue integral.

POSSIBLE QUESTIONS

1. Prove that outer measure of an interval is its length.
2. Prove that the collection M of measurable set is a σ - algebra.
3. (i) Prove that the interval (a,∞) is measurable.
   (ii) Prove that [0,1] is not countable.
4. If f be an extended real valued function whose domain is measurable.Then prove that the following statements are equivalent
   (i) For each real number α the set $\{x/ f(x) > \alpha\}$is measurable.
   (ii) For each real number α the set $\{x/ f(x) \geq\alpha\}$is measurable.
   (iii) For each real number α the set $\{x/ f(x)< \alpha\}$is measurable.
   (iv) For each real number α the set $\{x/ f(x) \leq\alpha\}$is measurable.
   These statements are imply for each extended real number α the set $\{ x/ f(x)= \infty\}$is measurable.
5. If 'c' be a constant and f and g two measurable real valued function defined on the same domain.Then prove that the function f+c, cf ,f+g ,g-f and fg are also measurable.
6. State and prove Little wood's three principles.
7. (i)If f is an measurable function and f = g a.e then g is measurable.
   (ii) Define (a) Almost everywhere (b) Simple function (c) Characterstic function (d) Little wood's principle.
8. If $\{f_n\}$ be a sequence of measurable functions.Then the functions sup $\{ f_1, f_2, \ldots f_n \}$,
9. i) If $m^*(E) = 0$ then E is measurable.
   ii) If E_1 & E_2 are measurable, so is $E_1 \cup E_2$.
10. Prove that every borel set is measurable, inparticular each open set and each closed set is measurable.
11. If $\{E_i\}$be an infinite decreasing sequence of measurable sets. (ie) A sequence with $E_{n+1} \subset En$ for each A and mE be finie. Then $m(\bigcap_{i=1}^{\infty} Ei) = \lim_{n\to\infty} mEn$.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If a and B are two setsin M with AÌB, then MA≤MB thisproperty is called……………….	Additivity	subadditivity	monotonicity	translationinvariant	monotonicity
The outer measure of an interval is its………………………	Length	measure	endpoints	sistance	Length
If A is countable then m*A=………..	1	0	-2	-1	0
The set [0,1] is…………….	Not countable	countable	un countable	measurable	Not countable
If m*E=0 then E is …………………	outer measure	measurable	borel set	σ - algebra	measurable
The complement of a measurable set is ……………..	countable set	σ - algebra	borel set	measurable	measurable
The collection M of measurable sets is…………………..	σ - algebra	measurable	countable set	borel set	σ - algebra
Every………………………. is measurable.	σ - algebra	borel set	countable set	open set	borel set
The union of a countable collection of mesurable sets is…………..	outer measurable set	σ - algebra	borel set	measurable	measurable
Every borel Set is measurable set then the converse?	not true	not false	partially true	partially false	not true
Any element in borel algebra B is called……………..	measurable set	borel set	σ - algebra	borel space	borel set

The intersection of any collection of closed sets is ………………	measure set	open	sub set	closed	closed
The union of any finite collection of …………………………..is closed.	measure set	closed set	sub set	open set	closed set
The set of rational numbers is the union of a countable collection of closed set s each of which contains exactly……………………………………. number.	Zero	finite	one	infinite	one
A set which is a countable union of ……………….. is called an F	sub set	closed set	measure set	open set	closed set
A Set which is a ……………………. of closed sets is called an F	countable union	countble intersection	union	intersection	countable union
The intersection of a countable collection of…………………… is callled an G	closed seet	subset	open set	measue set	open set
Every isolated set of real number is …………………	finite	uncountable	infinite	countable	countable
The collection B of borel set is the smallest σ- algebra which contains all of the…………………	subset	closed set	measure set	open set	open set
If A and B are two sets in whit ACB ,then…………………	A ≥ B	mA ≥ mB	mA ≤mB	mA=mB	mA ≤mB

The sum of the lenghts of the finite subcollection is ………………………the sum of the lengths of the original collection.	no less than	less than	no greates than	greater than	no greates than

UNIT – II

The Lebesgue Integral: The Riemann integral – The Lebesgue integral of a bounded function over a set finite measure – The integral of a non negative function – The general Lebesgue integral – Convergence in measure.

Lebesgue Integration

We now turn to our main object of interest in Part I, the Lebesgue integral. We define this integral in four stages. We first define the integral for simple functions over a set of finite measure. Then for bounded measurable functions f over a set of finite measure, in terms of integrals of upper and lower approximations of f by simple functions. We define the integral of a general nonnegative measurable function f over E to be the supremum of the integrals of lower approximations of f by bounded measurable functions that vanish outside a set of finite measure; the integral of such a function is nonnegative, but may be infinite. Finally, a general measurable function is said to be integrable over E provided $\int_E |f| < \infty$. We prove that linear combinations of integrable functions are integrable and that, on the class of integrable functions, the Lebesgue integral is a monotone, linear functional. A principal virtue of the Lebesgue integral, beyond the extent of the class of integrable functions, is the availability of quite general criteria which guarantee that if a sequence of integrable functions $\{f_n\}$ converge pointwise almost everywhere on E to f, then

$$\lim_{n\to\infty}\int_E f_n = \int_E [\lim_{n\to\infty} f_n] \equiv \int_E f.$$

We refer to that as passage of the limit under the integral sign. Based on Egoroff's Theorem, a consequence of the countable additivity of Lebesgue measure, we prove four theorems that provide criteria for justification of this passage: the Bounded Convergence Theorem, the Monotone Convergence Theorem, the Lebesgue Dominated Convergence Theorem, and the Vitali Convergence Theorem.

THE RIEMANN INTEGRAL

We recall a few definitions pertaining to the Riemann integral. Let f be a bounded real-valued function defined on the closed, bounded interval $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, that is,

$$a = x_0 < x_1 < \ldots < x_n = b.$$

Define the **lower and upper Darboux sums** for f with respect to P, respectively, by

$$L(f, P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}),$$

where,[1] for $1 \leq i \leq n$,

$$m_i = \inf \{f(x) \mid x_{i-1} < x < x_i\} \text{ and } M_i = \sup \{f(x) \mid x_{i-1} < x < x_i\}.$$

We then define the **lower and upper Riemann integrals** of f over $[a, b]$, respectively, by

$$(R)\underline{\int_a^b} f = \sup \{L(f, P) \mid P \text{ a partition of } [a, b]\}$$

and

$$(R)\overline{\int_a^b} f = \inf \{U(f, P) \mid P \text{ a partition of } [a, b]\}.$$

Since f is assumed to be bounded and the interval $[a, b]$ has finite length, the lower and upper Riemann integrals are finite. The upper integral is always at least as large as the lower integral, and if the two are equal we say that f is **Riemann integrable** over $[a, b]$[2] and call this common value the Riemann integral of f over $[a, b]$. We denote it by

$$(R)\int_a^b f$$

to temporarily distinguish it from the Lebesgue integral, which we consider in the next section.

A real-valued function ψ defined on $[a, b]$ is called a **step function** provided there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and numbers $c_1, \ldots, c_n$ such that for $1 \leq i \leq n$,

$$\psi(x) = c_i \text{ if } x_{i-1} < x < x_i.$$

Observe that

$$L(\psi, P) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) = U(\psi, P).$$

From this and the definition of the upper and lower Riemann integrals, we infer that a step function ψ is Riemann integrable and

$$(R)\int_a^b \psi = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

Therefore, we may reformulate the definition of the lower and upper Riemann integrals as follows:

$$(R)\underline{\int_a^b} f = \sup\left\{(R)\int_a^b \varphi \;\middle|\; \varphi \text{ a step function and } \varphi \le f \text{ on } [a,b]\right\},$$

and

$$(R)\overline{\int_a^b} f = \inf\left\{(R)\int_a^b \psi \;\middle|\; \psi \text{ a step function and } \psi \ge f \text{ on } [a,b]\right\}.$$

Example (**Dirichlet's Function**) Define f on [0, 1] by setting $f(x) = 1$ if x is rational and 0 if x is irrational. Let P be any partition of [0, 1]. By the density of the rationals and the irrationals,[3]

$$L(f, P) = 0 \text{ and } U(f, P) = 1.$$

Thus

$$(R)\underline{\int_0^1} f = 0 < 1 = (R)\overline{\int_0^1} f,$$

so f is not Riemann integrable. The set of rational numbers in [0,1] is countable.[4] Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in [0, 1]. For a natural number n, define f_n on [0, 1] by setting $f_n(x) = 1$, if $x = q_k$ for some q_k with $1 \le k \le n$, and $f(x) = 0$ otherwise. Then each f_n is a step function, so it is Riemann integrable. Thus, $\{f_n\}$ is an increasing sequence of Riemann integrable functions on [0, 1],

$$|f_n| \le 1 \text{ on } [0, 1] \text{ for all } n$$

and

$$\{f_n\} \to f \text{ pointwise on } [0, 1].$$

However, the limit function f fails to be Riemann integrable on [0, 1].

THE LEBESGUE INTEGRAL OF A BOUNDED MEASURABLE FUNCTION OVER A SET OF FINITE MEASURE

The Dirichlet function, which was examined in the preceding section, exhibits one of the principal shortcomings of the Riemann integral: a uniformly bounded sequence of Riemann integrable functions on a closed, bounded interval can converge pointwise to a function that is not Riemann integrable. We will see that the Lebesgue integral does not suffer from this shortcoming.

Henceforth we only consider the Lebesgue integral, unless explicitly mentioned otherwise, and so we use the pure integral symbol to denote the Lebesgue integral. The forthcoming Theorem 3 tells us that any bounded function that is Riemann integrable over $[a, b]$ is also Lebesgue integrable over $[a, b]$ and the two integrals are equal.

Recall that a measurable real-valued function ψ defined on a set E is said to be simple provided it takes only a finite number of real values. If ψ takes the distinct values $a_1, \ldots, a_n$ on E, then, by the measurability of ψ, its level sets $\psi^{-1}(a_i)$ are measurable and we have the canonical representation of ψ on E as

$$\psi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i} \text{ on } E, \text{ where each } E_i = \psi^{-1}(a_i) = \{x \in E \mid \psi(x) = a_i\}. \tag{1}$$

The canonical representation is characterized by the E_i's being disjoint and the a_i's being distinct.

Definition *For a simple function ψ defined on a set of finite measure E, we define the integral of ψ over E by*

$$\int_E \psi = \sum_{i=1}^{n} a_i \cdot m(E_i),$$

where ψ has the canonical representation given by (1).

Lemma 1 *Let $\{E_i\}_{i=1}^n$ be a finite disjoint collection of measurable subsets of a set of finite measure E. For $1 \le i \le n$, let a_i be a real number.*

$$\textit{If } \varphi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i} \textit{ on } E, \textit{ then } \int_E \varphi = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

Proof The collection $\{E_i\}_{i=1}^n$ is disjoint but the above may not be the canonical representation since the a_i's may not be distinct. We must account for possible repetitions. Let $\{\lambda_1, \ldots, \lambda_m\}$ be the distinct values taken by φ. For $1 \le j \le m$, set $A_j = \{x \in E \mid \varphi(x) = \lambda_j\}$. By definition of the integral in terms of canonical representations,

$$\int_E \varphi = \sum_{j=1}^{m} \lambda_j \cdot m(A_j).$$

For $1 \leq j \leq m$, let I_j be the set of indices i in $\{1, \ldots, n\}$ for which $a_i = \lambda_j$. Then $\{1, \ldots, n\} = \bigcup_{j=1}^{m} I_j$, and the union is disjoint. Moreover, by finite additivity of measure,

$$m(A_j) = \sum_{i \in I_j} m(E_i) \text{ for all } 1 \leq j \leq m.$$

Therefore

$$\sum_{i=1}^{n} a_i \cdot m(E_i) = \sum_{j=1}^{m} \left[\sum_{i \in I_j} a_i \cdot m(E_i) \right] = \sum_{j=1}^{m} \lambda_j \left[\sum_{i \in I_j} m(E_i) \right]$$
$$= \sum_{j=1}^{m} \lambda_j \cdot m(A_j) = \int_E \varphi.$$

□

One of our goals is to establish linearity and monotonicity properties for the general Lebesgue integral. The following is the first result in this direction.

Proposition 2 (**Linearity and Monotonicity of Integration**) *Let φ and ψ be simple functions defined on a set of finite measure E. Then for any α and β,*

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi.$$

Moreover,

$$\text{if } \varphi \leq \psi \text{ on } E, \text{ then } \int_E \varphi \leq \int_E \psi.$$

Proof Since both φ and ψ take only a finite number of values on E, we may choose a finite disjoint collection $\{E_i\}_{i=1}^{n}$ of measurable subsets of E, the union of which is E, such that φ and ψ are constant on each E_i. For each i, $1 \leq i \leq n$, let a_i and b_i, respectively, be the values taken by φ and ψ on E_i. By the preceding lemma,

$$\int_E \varphi = \sum_{i=1}^{n} a_i \cdot m(E_i) \text{ and } \int_E \psi = \sum_{i=1}^{n} b_i \cdot m(E_i)$$

However, the simple function $\alpha\varphi + \beta\psi$ takes the constant value $\alpha a_i + \beta b_i$ on E_i. Thus, again by the preceding lemma,

$$\int_E (\alpha\varphi + \beta\psi) = \sum_{i=1}^{n} (\alpha a_i + \beta b_i) \cdot m(E_i)$$
$$= \alpha \sum_{i=1}^{n} a_i \cdot m(E_i) + \beta \sum_{i=1}^{n} b_i \cdot m(E_i) = \alpha \int_E \varphi + \beta \int_E \psi.$$

To prove monotonicity, assume $\varphi \leq \psi$ on E. Define $\eta = \psi - \varphi$ on E. By linearity,

$$\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) = \int_E \eta \geq 0,$$

since the nonnegative simple function η has a nonnegative integral. □

The linearity of integration over sets of finite measure of simple functions shows that the restriction in the statement of Lemma 1 that the collection $\{E_i\}_{i=1}^n$ be disjoint is unnecessary.

A step function takes only a finite number of values and each interval is measurable. Thus a step function is simple. Since the measure of a singleton set is zero and the measure of an interval is its length, we infer from the linearity of Lebesgue integration for simple functions defined on sets of finite measure that the Riemann integral over a closed, bounded interval of a step function agrees with the Lebesgue integral.

Let f be a bounded real-valued function defined on a set of finite measure E. By analogy with the Riemann integral, we define the **lower and upper Lebesgue integral**, respectively, of f over E to be

$$\sup \left\{ \int_E \varphi \;\middle|\; \varphi \text{ simple and } \varphi \leq f \text{ on } E, \right\}$$

and

$$\inf \left\{ \int_E \psi \;\middle|\; \psi \text{ simple and } f \leq \psi \text{ on } E. \right\}$$

Since f is assumed to be bounded, by the monotonicity property of the integral for simple functions, the lower and upper integrals are finite and the upper integral is always at least as large as the lower integral.

Definition *A bounded function f on a domain E of finite measure is said to be* **Lebesgue integrable** *over E provided its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the* **Lebesgue integral**, *or simply the integral, of f over E and is denoted by $\int_E f$.*

Theorem 3 *Let f be a bounded function defined on the closed, bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.*

Proof The assertion that f is Riemann integrable means that, setting $I = [a, b]$,

$$\sup\left\{(R)\int_I \varphi \;\middle|\; \varphi \text{ a step function, } \varphi \leq f\right\} = \inf\left\{(R)\int_I \psi \;\middle|\; \psi \text{ a step function, } f \leq \psi\right\}$$

To prove that f is Lebesgue integrable we must show that

$$\sup\left\{\int_I \varphi \;\middle|\; \varphi \text{ simple, } \varphi \leq f\right\} = \inf\left\{\int_I \psi \;\middle|\; \psi \text{ simple, } f \leq \psi\right\}.$$

However, each step function is a simple function and, as we have already observed, for a step function, the Riemann integral and the Lebesgue integral are the same. Therefore the first equality implies the second and also the equality of the Riemann and Lebesgue integrals. □

We are now fully justified in using the symbol $\int_E f$, without any preliminary (R), to denote the integral of a bounded function that is Lebesgue integrable over a set of finite measure. In the case of an interval $E = [a, b]$, we sometimes use the familiar notation $\int_a^b f$ to denote $\int_{[a,b]} f$ and sometimes it is useful to use the classic Leibniz notation $\int_a^b f(x)\,dx$.

Example The set E of rational numbers in $[0, 1]$ is a measurable set of measure zero. The Dirichlet function f is the restriction to $[0, 1]$ of the characteristic function of E, χ_E. Thus f is integrable over $[0, 1]$ and

$$\int_{[0,1]} f = \int_{[0,1]} 1 \cdot \chi_E = 1 \cdot m(E) = 0.$$

We have shown that f is not Riemann integrable over $[0, 1]$.

Theorem 4 *Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.*

Proof Let n be a natural number. By the Simple Approximation Lemma, with $\epsilon = 1/n$, there are two simple functions φ_n and ψ_n defined on E for which

$$\varphi_n \leq f \leq \psi_n \text{ on } E,$$

and

$$0 \leq \psi_n - \varphi_n \leq 1/n \text{ on } E.$$

By the monotonicity and linearity of the integral for simple functions,

$$0 \leq \int_E \psi_n - \int_E \varphi_n = \int_E [\psi_n - \varphi_n] \leq 1/n \cdot m(E).$$

However,

$$0 \leq \inf\left\{\int_E \psi \;\middle|\; \psi \text{ simple}, \psi \geq f\right\} - \sup\left\{\int_E \varphi \;\middle|\; \varphi \text{ simple}, \varphi \leq f\right\}$$
$$\leq \int_E \psi_n - \int_E \varphi_n \leq 1/n \cdot m(E).$$

This inequality holds for every natural number n and $m(E)$ is finite. Therefore the upper and lower Lebesgue integrals are equal and thus the function f is integrable over E. □

It turns out that the converse of the preceding theorem is true; a bounded function on a set of finite measure is Lebesgue integrable if and only if it is measurable: we prove this later (see the forthcoming Theorem 7 of Chapter 5). This shows, in particular, that not every bounded function defined on a set of finite measure is Lebesgue integrable. In fact, for any measurable set E of finite positive measure, the restriction to E of the characteristic function of each nonmeasurable subset of E fails to be Lebesgue integrable over E.

Theorem 5 **(Linearity and Monotonicity of Integration)** *Let f and g be bounded measurable functions on a set of finite measure E. Then for any α and β,*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g. \tag{2}$$

Moreover,

$$\text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g. \tag{3}$$

Proof A linear combination of measurable bounded functions is measurable and bounded. Thus, by Theorem 4, $\alpha f + \beta g$ is integrable over E. We first prove linearity for $\beta = 0$. If ψ is a simple function so is $\alpha\psi$, and conversely (if $\alpha \neq 0$). We established linearity of integration for simple functions. Let $\alpha > 0$. Since the Lebesgue integral is equal to the upper Lebesgue integral,

$$\int_E \alpha f = \inf_{\psi \geq \alpha f} \int_E \psi = \alpha \inf_{[\psi/\alpha] \geq f} \int_E [\psi/\alpha] = \alpha \int_E f.$$

For $\alpha < 0$, since the Lebesgue integral is equal both to the upper Lebesgue integral and the lower Lebesgue integral,

$$\int_E \alpha f = \inf_{\varphi \geq \alpha f} \int_E \varphi = \alpha \sup_{[\varphi/\alpha] \leq f} \int_E [\varphi/\alpha] = \alpha \int_E f.$$

It remains to establish linearity in the case that $\alpha = \beta = 1$. Let ψ_1 and ψ_2 be simple functions for which $f \leq \psi_1$ and $g \leq \psi_2$ on E. Then $\psi_1 + \psi_2$ is a simple function and $f + g \leq \psi_1 + \psi_2$ on E. Hence, since $\int_E (f+g)$ is equal to the upper Lebesgue integral of $f+g$ over E, by the linearity of integration for simple functions,

$$\int_E (f+g) \leq \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \int_E \psi_2.$$

The greatest lower bound for the sums of integrals on the right-hand side, as ψ_1 and ψ_2 vary among simple functions for which $f \le \psi_1$ and $g \le \psi_2$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is a lower bound for these same sums. Therefore,

$$\int_E (f+g) \le \int_E f + \int_E g.$$

It remains to prove this inequality in the opposite direction. Let φ_1 and φ_2 be simple functions for which $\varphi_1 \le f$ and $\varphi_2 \le g$ on E. Then $\varphi_1 + \varphi_2 \le f + g$ on E and $\varphi_1 + \varphi_2$ is simple. Hence, since $\int_E (f+g)$ is equal to the lower Lebesgue integral of $f+g$ over E, by the linearity of integration for simple functions,

$$\int_E (f+g) \ge \int_E (\varphi_1 + \varphi_2) = \int_E \varphi_1 + \int_E \varphi_2.$$

The least upper bound bound for the sums of integrals on the right-hand side, as φ_1 and φ_2 vary among simple functions for which $\varphi_1 \le f$ and $\varphi_2 \le g$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is an upper bound for these same sums. Therefore,

$$\int_E (f+g) \ge \int_E f + \int_E g.$$

This completes the proof of linearity of integration.

To prove monotonicity, assume $f \le g$ on E. Define $h = g - f$ on E. By linearity,

$$\int_E g - \int_E f = \int_E (g-f) = \int_E h.$$

The function h is nonnegative and therefore $\psi \le h$ on E, where $\psi \equiv 0$ on E. Since the integral of h equals its lower integral, $\int_E h \ge \int_E \psi = 0$. Therefore, $\int_E f \le \int_E g$. □

Corollary 6 *Let f be a bounded measurable function on a set of finite measure E. Suppose A and B are disjoint measurable subsets of E. Then*

$$\int_{A \cup B} f = \int_A f + \int_B f. \tag{4}$$

Proof Both $f \cdot \chi_A$ and $f \cdot \chi_B$ are bounded measurable functions on E. Since A and B are disjoint,

$$f \cdot \chi_{A\cup B} = f \cdot \chi_A + f \cdot \chi_B.$$

Furthermore, for any measurable subset E_1 of E (see Problem 10),

$$\int_{E_1} f = \int_E f \cdot \chi_{E_1}.$$

Therefore, by the linearity of integration,

$$\int_{A\cup B} f = \int_E f \cdot \chi_{A\cup B} = \int_E f \cdot \chi_A + \int_E f \cdot \chi_B = \int_A f + \int_B f.$$

Corollary 7 *Let f be a bounded measurable function on a set of finite measure E. Then*

$$\left| \int_E f \right| \leq \int_E |f|. \tag{5}$$

Proof The function $|f|$ is measurable and bounded. Now

$$-|f| \leq f \leq |f| \text{ on } E.$$

By the linearity and monotonicity of integration,

$$-\int_E |f| \leq \int_E f \leq \int_E |f|,$$

that is, (5) holds. □

Proposition 8 *Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E.*

$$\textit{If } \{f_n\} \to f \textit{ uniformly on } E, \textit{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof Since the convergence is uniform and each f_n is bounded, the limit function f is bounded. The function f is measurable since it is the pointwise limit of a sequence of measurable functions. Let $\epsilon > 0$. Choose an index N for which

$$|f - f_n| < \epsilon/m(E) \text{ on } E \text{ for all } n \geq N. \tag{6}$$

By the linearity and monotonicity of integration and the preceding corollary, for each $n \geq N$,

$$\left|\int_E f - \int_E f_n\right| = \left|\int_E [f - f_n]\right| \leq \int_E |f - f_n| \leq [\epsilon/m(E)] \cdot m(E) = \epsilon.$$

Therefore $\lim_{n\to\infty} \int_E f_n = \int_E f$. □

This proposition is rather weak since frequently a sequence will be presented that converges pointwise but not uniformly. It is important to understand when it is possible to infer from

$$\{f_n\} \to f \text{ pointwise a.e. on } E$$

that

$$\lim_{n\to\infty}\left[\int_E f_n\right] = \int_E \left[\lim_{n\to\infty} f\right] = \int_E f.$$

We refer to this equality as **passage of the limit under the integral sign.**[5] Before proving our first important result regarding this passage, we present an instructive example.

Example For each natural number n, define f_n on $[0, 1]$ to have the value 0 if $x \geq 2/n$, have $f(1/n) = n$, $f(0) = 0$ and to be linear on the intervals $[0, 1/n]$ and $[1/n, 2/n]$. Observe that $\int_0^1 f_n = 1$ for each n. Define $f \equiv 0$ on $[0, 1]$. Then

$$\{f_n\} \to f \text{ pointwise on } [0, 1], \text{ but } \lim_{n\to\infty}\int_0^1 f_n \neq \int_0^1 f.$$

Thus, pointwise convergence alone is not sufficient to justify passage of the limit under the integral sign.

The Bounded Convergence Theorem *Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \geq 0$ for which*

$$|f_n| \leq M \text{ on } E \text{ for all } n.$$

$$\text{If } \{f_n\} \to f \text{ pointwise on } E, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof The proof of this theorem furnishes a nice illustration of Littlewood's Third Principle. If the convergence is uniform, we have the easy proof of the preceding proposition. However, Egoroff's Theorem tells us, roughly, that pointwise convergence is "nearly" uniform.

The pointwise limit of a sequence of measurable functions is measurable. Therefore f is measurable. Clearly $|f| \leq M$ on E. Let A be any measurable subset of E and n a natural number. By the linearity and additivity over domains of the integral,

$$\int_E f_n - \int_E f = \int_E [f_n - f] = \int_A [f_n - f] + \int_{E\sim A} f_n + \int_{E\sim A} (-f).$$

Therefore, by Corollary 7 and the monotonicity of integration,

$$\left| \int_E f_n - \int_E f \right| \leq \int_A |f_n - f| + 2M \cdot m(E \sim A). \tag{7}$$

To prove convergence of the integrals, let $\epsilon > 0$. Since $m(E) < \infty$ and f is real-valued, Egoroff's Theorem tells us that there is a measurable subset A of E for which $\{f_n\} \to f$ uniformly on A and $m(E \sim A) < \epsilon/4M$. By uniform convergence, there is an index N for which

$$|f_n - f| < \frac{\epsilon}{2 \cdot m(E)} \text{ on } A \text{ for all } n \geq N.$$

Therefore, for $n \geq N$, we infer from (7) and the monotonicity of integration that

$$\left| \int_E f_n - \int_E f \right| \leq \frac{\epsilon}{2 \cdot m(E)} \cdot m(A) + 2M \cdot m(E \sim A) < \epsilon.$$

Hence the sequence of integrals $\{\int_E f_n\}$ converges to $\int_E f$. □

Remark *Prior to the proof of the Bounded Convergence Theorem, no use was made of the countable additivity of Lebesgue measure on the real line. Only finite additivity was used, and it was used just once, in the proof of Lemma 1. But for the proof of the Bounded Convergence Theorem we used Egoroff's Theorem. The proof of Egoroff's Theorem needed the continuity of Lebesgue measure, a consequence of countable additivity of Lebesgue measure.*

THE LEBESGUE INTEGRAL OF A MEASURABLE NONNEGATIVE FUNCTION

A measurable function f on E is said to vanish outside a set of finite measure provided there is a subset E_0 of E for which $m(E_0) < \infty$ and $f \equiv 0$ on $E \sim E_0$. It is convenient to say that a function that vanishes outside a set of finite measure has finite support and define its support to be $\{x \in E \mid f(x) \neq 0\}$.[6] In the preceding section, we defined the integral of a bounded measurable function f over a set of finite measure E. However, even if $m(E) = \infty$, if f is bounded and measurable on E but has finite support, we can define its integral over E by

$$\int_E f = \int_{E_0} f,$$

where E_0 has finite measure and $f \equiv 0$ on $E \sim E_0$. This integral is properly defined, that is, it is independent of the choice of set of finite measure E_0 outside of which f vanishes. This is a consequence of the additivity over domains property of integration for bounded measurable functions over a set of finite measure.

Definition *For f a nonnegative measurable function on E, we define the integral of f over E by*[7]

$$\int_E f = \sup \left\{ \int_E h \;\middle|\; h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E \right\}. \quad (8)$$

Chebychev's Inequality *Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,*

$$m\{x \in E \mid f(x) \geq \lambda\} \leq \frac{1}{\lambda} \cdot \int_E f. \quad (9)$$

Proof Define $E_\lambda = \{x \in E \mid f(x) \geq \lambda\}$. First suppose $m(E_\lambda) = \infty$. Let n be a natural number. Define $E_{\lambda, n} = E_\lambda \cap [-n, n]$ and $\psi_n = \lambda \cdot \chi_{E_{\lambda, n}}$. Then ψ_n is a bounded measurable function of finite support,

$$\lambda \cdot m(E_{\lambda, n}) = \int_E \psi_n \text{ and } 0 \leq \psi_n \leq f \text{ on } E \text{ for all } n.$$

We infer from the continuity of measure that

$$\infty = \lambda \cdot m(E_\lambda) = \lambda \cdot \lim_{n \to \infty} m(E_{\lambda, n}) = \lim_{n \to \infty} \int_E \psi_n \leq \int_E f.$$

Thus inequality (9) holds since both sides equal ∞. Now consider the case $m(E_\lambda) < \infty$. Define $h = \lambda \cdot \chi_{E_\lambda}$. Then h is a bounded measurable function of finite support and $0 \leq h \leq f$ on E. By the definition of the integral of f over E,

$$\lambda \cdot m(E_\lambda) = \int_E h \leq \int_E f.$$

Divide both sides of this inequality by λ to obtain Chebychev's Inequality. □

Proposition 9 *Let f be a nonnegative measurable function on E. Then*

$$\int_E f = 0 \text{ if and only if } f = 0 \text{ a.e. on } E. \tag{10}$$

Proof First assume $\int_E f = 0$. Then, by Chebychev's Inequality, for each natural number n, $m\{x \in X \mid f(x) \geq 1/n\} = 0$. By the countable additivity of Lebesgue measure, $m\{x \in X \mid f(x) > 0\} = 0$. Conversely, suppose $f = 0$ a.e. on E. Let φ be a simple function and h a bounded measurable function of finite support for which $0 \leq \varphi \leq h \leq f$ on E. Then $\varphi = 0$ a.e. on E and hence $\int_E \varphi = 0$. Since this holds for all such φ, we infer that $\int_E h = 0$. Since this holds for all such h, we infer that $\int_E f = 0$. □

Theorem 10 (Linearity and Monotonicity of Integration) *Let f and g be nonnegative measurable functions on E. Then for any $\alpha > 0$ and $\beta > 0$,*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g. \tag{11}$$

Moreover,

$$\text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g. \tag{12}$$

Proof For $\alpha > 0$, $0 \leq h \leq f$ on E if and only if $0 \leq \alpha h \leq \alpha f$ on E. Therefore, by the linearity of the integral of bounded functions of finite support, $\int_E \alpha f = \alpha \int_E f$. Thus, to prove linearity we need only consider the case $\alpha = \beta = 1$. Let h and g be bounded measurable functions of finite support for which $0 \leq h \leq f$ and $0 \leq k \leq g$ on E. We have $0 \leq h + k \leq f + g$ on E, and $h + k$ also is a bounded measurable function of finite support. Thus, by the linearity of integration for bounded measurable functions of finite support,

$$\int_E h + \int_E k = \int_E (h + k) \leq \int_E (f + g).$$

The least upper bound for the sums of integrals on the left-hand side, as h and k vary among bounded measurable functions of finite support for which $h \leq f$ and $k \leq g$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f + g)$ is an upper bound for these same sums. Therefore,

$$\int_E f + \int_E g \leq \int_E (f + g).$$

It remains to prove this inequality in the opposite direction, that is,

$$\int_E (f+g) \leq \int_E f + \int_E g.$$

By the definition of $\int_E (f+g)$ as the supremum of $\int_E \ell$ as ℓ ranges over all bounded measurable functions of finite support for which $0 \leq \ell \leq f+g$ on E, to verify this inequality it is necessary and sufficient to show that for any such function ℓ,

$$\int_E \ell \leq \int_E f + \int_E g. \tag{13}$$

For such a function ℓ, define the functions h and k on E by

$$h = \min\{f, \ell\} \text{ and } k = \ell - h \text{ on } E.$$

Let x belong to E. If $\ell(x) \leq f(x)$, then $k(x) = 0 \leq g(x)$; if $\ell(x) > f(x)$, then $h(x) = \ell(x) - f(x) \leq g(x)$. Therefore, $h \leq g$ on E. Both h and k are bounded measurable functions of finite support. We have

$$0 \leq h \leq f, 0 \leq k \leq g \text{ and } \ell = h + k \text{ on } E.$$

Hence, again using the linearity of integration for bounded measurable functions of finite support and the definitions of $\int_E f$ and $\int_E g$, we have

$$\int_E l = \int_E h + \int_E k \leq \int_E f + \int_E g.$$

Thus (13) holds and the proof of linearity is complete.

Theorem 11 **(Additivity Over Domains of Integration)** *Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E, then*

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a subset of E of measure zero, then

$$\int_E f = \int_{E \sim E_0} f. \tag{15}$$

Proof Additivity over domains of integration follows from linearity as it did for bounded functions on sets of finite measure. The excision formula (15) follows from additivity over domains and the observation that, by Proposition 9, the integral of a nonnegative function over a set of measure zero is zero. □

The following lemma will enable us to establish several criteria to justify passage of the limit under the integral sign.

Fatou's Lemma *Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E.*

$$\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } \int_E f \le \liminf \int_E f_n. \tag{16}$$

Proof In view of (15), by possibly excising from E a set of measure zero, we assume the pointwise convergence is on all of E. The function f is nonnegative and measurable since it is the pointwise limit of a sequence of such functions. To verify the inequality in (16) it is necessary and sufficient to show that if h is any bounded measurable function of finite support for which $0 \le h \le f$ on E, then

$$\int_E h \le \liminf \int_E f_n. \tag{17}$$

Let h be such a function. Choose $M \ge 0$ for which $|h| \le M$ on E. Define $E_0 = \{x \in E \mid h(x) \neq 0\}$. Then $m(E_0) < \infty$. Let n be a natural number. Define a function h_n on E by

$$h_n = \min\{h, f_n\} \text{ on } E.$$

Observe that the function h_n is measurable, that

$$0 \le h_n \le M \text{ on } E_0 \text{ and } h_n \equiv 0 \text{ on } E \sim E_0.$$

Furthermore, for each x in E, since $h(x) \le f(x)$ and $\{f_n(x)\} \to f(x)$, $\{h_n(x)\} \to h(x)$. We infer from the Bounded Convergence Theorem applied to the uniformly bounded sequence of restrictions of h_n to the set of finite measure E_0, and the vanishing of each h_n on $E \sim E_0$, that

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h.$$

However, for each n, $h_n \le f_n$ on E and therefore, by the definition of the integral of f_n over E, $\int_E h_n \le \int_E f_n$. Thus,

$$\int_E h = \lim_{n\to\infty} \int_E h_n \le \liminf \int_E f_n.$$ □

The inequality in Fatou's Lemma may be strict.

Example Let $E = (0, 1]$ and for a natural number n, define $f_n = n \cdot \chi_{(0, 1/n)}$. Then $\{f_n\}$ converges pointwise on E to $f \equiv 0$ on E. However,

$$\int_E f = 0 < 1 = \lim_{n \to \infty} \int_E f_n.$$

As another example of strict inequality in Fatou's Lemma, let $E = \mathbf{R}$ and for a natural number n, define $g_n = \chi_{(n, n+1)}$. Then $\{g_n\}$ converges pointwise on E to $g \equiv 0$ on E. However,

$$\int_E g = 0 < 1 = \lim_{n \to \infty} \int_E g_n.$$

However, the inequality in Fatou's Lemma is an equality if the sequence $\{f_n\}$ is increasing.

The Monotone Convergence Theorem *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E.*

$$\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } \lim_{n \to \infty} \int_E f_n = \int_E f.$$

Proof According to Fatou's Lemma,

$$\int_E f \leq \liminf \int_E f_n.$$

However, for each index n, $f_n \leq f$ a.e. on E, and so, by the monotonicity of integration for nonnegative measurable functions and (15), $\int_E f_n \leq \int_E f$. Therefore

$$\limsup \int_E f_n \leq \int_E f.$$

Hence

$$\int_E f = \lim_{n \to \infty} \int f_n.$$ □

Corollary 12 *Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E.*

$$\text{If } f = \sum_{n=1}^{\infty} u_n \text{ pointwise a.e. on } E, \text{ then } \int_E f = \sum_{n=1}^{\infty} \int_E u_n.$$

Proof Apply the Monotone Convergence Theorem with $f_n = \sum_{k=1}^{n} u_k$, for each index n, and then use the linearity of integration for nonnegative measurable functions. □

Definition *A nonnegative measurable function f on a measurable set E is said to be* **integrable** *over E provided*

$$\int_E f < \infty.$$

Proposition 13 *Let the nonnegative function f be integrable over E. Then f is finite a.e. on E.*

Proof Let n be a natural number. Chebychev's Inequality and the monotonicity of measure tell us that

$$m\{x \in E \mid f(x) = \infty\} \leq m\{x \in E \mid f(x) \geq n\} \leq \frac{1}{n}\int_E f.$$

But $\int_E f$ is finite and therefore $m\{x \in E \mid f(x) = \infty\} = 0$. □

Beppo Levi's Lemma *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence of integrals $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e. on E and*

$$\lim_{n\to\infty}\int_E f_n = \int_E f < \infty.$$

Proof Every monotone sequence of extended real numbers converges to an extended real number.[8] Since $\{f_n\}$ is an increasing sequence of extended real-valued functions on E, we may define the extended real-valued nonnegative function f pointwise on E by

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ for all } x \in E.$$

According to the Monotone Convergence Theorem, $\{\int_E f_n\} \to \int_E f$. Therefore, since the sequence of real numbers $\{\int_E f_n\}$ is bounded, its limit is finite and so $\int_E f < \infty$. We infer from the preceding proposition that f is finite a.e. on E. □

THE GENERAL LEBESGUE INTEGRAL

For an extended real-valued function f on E, we have defined the positive part f^+ and the negative part f^- of f, respectively, by

$$f^+(x) = \max\{f(x), 0\} \text{ and } f^-(x) = \max\{-f(x), 0\} \text{ for all } x \in E.$$

Then f^+ and f^- are nonnegative functions on E,

$$f = f^+ - f^- \text{ on } E$$

and

$$|f| = f^+ + f^- \text{ on } E.$$

Observe that f is measurable if and only if both f^+ and f^- are measurable.

Proposition 14 *Let f be a measurable function on E. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.*

Proof Assume f^+ and f^- are integrable nonnegative functions. By the linearity of integration for nonnegative functions, $|f| = f^+ + f^-$ is integrable over E. Conversely, suppose $|f|$ is integrable over E. Since $0 \le f^+ \le |f|$ and $0 \le f^- \le |f|$ on E, we infer from the monotonicity of integration for nonnegative functions that both f^+ and f^- are integrable over E. □

Definition *A measurable function f on E is said to be* **integrable** *over E provided $|f|$ is integrable over E. When this is so we define the integral of f over E by*

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Of course, for a nonnegative function f, since $f = f^+$ and $f^- \equiv 0$ on E, this definition of integral coincides with the one just considered. By the linearity of integration for bounded measurable functions of finite support, the above definition of integral also agrees with the definition of integral for this class of functions.

Proposition 15 *Let f be integrable over E. Then f is finite a.e. on E and*

$$\int_E f = \int_{E \sim E_0} f \text{ if } E_0 \subseteq E \text{ and } m(E_0) = 0. \tag{18}$$

Proof Proposition 13, tells us that $|f|$ is finite a.e. on E. Thus f is finite a.e. on E. Moreover, (18) follows by applying (15) to the positive and negative parts of f. □

The following criterion for integrability is the Lebesgue integral correspondent of the comparison test for the convergence of series of real numbers.

Proposition 16 (the Integral Comparison Test) *Let f be a measurable function on E. Suppose there is a nonnegative function g that is integrable over E and dominates f in the sense that*

$$|f| \le g \text{ on } E.$$

Then f is integrable over E and

$$\left|\int_E f\right| \le \int_E |f|.$$

Proof By the monotonicity of integration for nonnegative functions, $|f|$, and hence f, is integrable. By the triangle inequality for real numbers and the linearity of integration for nonnegative functions,

$$\left|\int_E f\right| = \left|\int_E f^+ - \int_E f^-\right| \leq \int_E f^+ + \int_E f^- = \int_E |f|. \qquad \square$$

We have arrived at our final stage of generality for the Lebesgue integral for functions of a single real variable. Before proving the linearity property for integration, we need to address, with respect to integration, a point already addressed with respect to measurability. The point is that for two functions f and g which are integrable over E, the sum $f+g$ is not properly defined at points in E where f and g take infinite values of opposite sign. However, by Proposition 15, if we define A to be the set of points in E at which both f and g are finite, then $m(E \sim A) = 0$. Once we show that $f+g$ is integrable over A, we define

$$\int_E (f+g) = \int_A (f+g).$$

We infer from (18) that $\int_E (f+g)$ is equal to the integral over E of any extension of $(f+g)|_A$ to an extended real-valued function on all of E.

Theorem 17 (Linearity and Monotonicity of Integration) *Let the functions f and g be integrable over E. Then for any α and β, the function $\alpha f + \beta g$ is integrable over E and*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover,

$$\text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g.$$

Proof If $\alpha > 0$, then $[\alpha f]^+ = \alpha f^+$ and $[\alpha f]^- = \alpha f^-$, while if $\alpha < 0$, $[\alpha f]^+ = -\alpha f^-$ and $[\alpha f]^- = -\alpha f^+$. Therefore $\int_E \alpha f = \alpha \int_E f$, since we established this for nonnegative functions f and $\alpha > 0$. So it suffices to establish linearity in the case $\alpha = \beta = 1$. By the linearity of integration for nonnegative functions, $|f| + |g|$ is integrable over E. Since $|f+g| \leq |f| + |g|$ on E, by the integral comparison test, $f+g$ also is integrable over E. Proposition 15 tells us that f and g are finite a.e. on E. According to the same proposition, by possibly excising from E a set of measure zero, we may assume that f and g are finite on E. To verify linearity is to show that

$$\int_E [f+g]^+ - \int_E [f+g]^- = \left[\int_E f^+ - \int_E f^-\right] + \left[\int_E g^+ - \int_E g^-\right].$$

But

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-) \text{ on } E,$$

But

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-) \text{ on } E,$$

and therefore, since each of these six functions takes real values on E,

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+ \text{ on } E.$$

We infer from linearity of integration for nonnegative functions that

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+.$$

Since f, g and $f+g$ are integrable over E, each of these six integrals is finite. Rearrange these integrals to obtain (19). This completes the proof of linearity.

To establish monotonicity we again argue as above that we may assume g and f are finite on E. Define $h = g - f$ on E. Then h is a properly defined nonnegative measurable function on E. By linearity of integration for integrable functions and monotonicity of integration for nonnegative functions,

$$\int_E g - \int_E f = \int_E (g-f) = \int_E h \geq 0. \qquad \square$$

Corollary 18 (Additivity Over Domains of Integration) *Let f be integrable over E. Assume A and B are disjoint measurable subsets of E. Then*

$$\int_{A\cup B} f = \int_A f + \int_B f. \tag{20}$$

Proof Observe that $|f \cdot \chi_A| \leq |f|$ and $|f \cdot \chi_B| \leq |f|$ on E. By the integral comparison test, the measurable functions $f \cdot \chi_A$ and $f \cdot \chi_B$ are integrable over E. Since A and B are disjoint

$$f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B \text{ on } E. \tag{21}$$

But for any measurable subset C of E (see Problem 28),

$$\int_C f = \int_E f \cdot \chi_C.$$

Thus (20) follows from (21) and the linearity of integration. □

The following generalization of the Bounded Convergence Theorem provides another justification for passage of the limit under the integral sign.

The Lebesgue Dominated Convergence Theorem *Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g$ on E for all n.*

If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n \to \infty} \int_E f_n = \int_E f$.

Proof Since $|f_n| \leq g$ on E and $|f| \leq g$ a.e. on E and g is integrable over E, by the integral comparison test, f and each f_n also are integrable over E. We infer from Proposition 15 that, by possibly excising from E a countable collection of sets of measure zero and using the countable additivity of Lebesgue measure, we may assume that f and each f_n is finite on E. The function $g - f$ and for each n, the function $g - f_n$, are properly defined, nonnegative and measurable. Moreover, the sequence $\{g - f_n\}$ converges pointwise a.e. on E to $g - f$. Fatou's Lemma tells us that

$$\int_E (g - f) \leq \liminf \int_E (g - f_n).$$

Thus, by the linearity of integration for integrable functions,

$$\int_E g - \int_E f = \int_E (g - f) \leq \liminf \int_E (g - f_n) = \int_E g - \limsup \int_E f_n,$$

that is,

$$\limsup \int_E f_n \leq \int_E f.$$

Similarly, considering the sequence $\{g + f_n\}$, we obtain

$$\int_E f \leq \liminf \int_E f_n.$$

The proof is complete. □

Theorem 19 (General Lebesgue Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions on* **E** *that converges pointwise a.e. on* **E** *to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that*

$$|f_n| \leq g_n \text{ on } E \text{ for all } n.$$

$$\text{If } \lim_{n \to \infty} \int_E g_n = \int_E g < \infty, \text{ then } \lim_{n \to \infty} \int_E f_n = \int_E f.$$

Remark *In Fatou's Lemma and the Lebesgue Dominated Convergence Theorem, the assumption of pointwise convergence a.e. on E rather than on all of E is not a decoration pinned on to honor generality. It is necessary for future applications of these results. We provide one illustration of this necessity. Suppose f is an increasing function on all of* **R**. *A forthcoming theorem of Lebesgue (Lebesgue's Theorem of Chapter 6) tells us that*

$$\lim_{n \to \infty} \frac{f(x + 1/n) - f(x)}{1/n} = f'(x) \text{ for almost all } x. \quad (22)$$

From this and Fatou's Lemma we will show that for any closed, bounded interval $[a, b]$,

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

In general, given a nondegenerate closed, bounded interval $[a, b]$ and a subset A of $[a, b]$ that has measure zero, there is an increasing function f on $[a, b]$ for which the limit in (22) fails to exist at each point in A (see Problem 10 of Chapter 6).

CONVERGENCE IN MEASURE

We have considered sequences of functions that converge uniformly, that converge pointwise, and that converge pointwise almost everywhere. To this list we add one more mode of convergence that has useful relationships both to pointwise convergence almost everywhere and to forthcoming criteria for justifying the passage of the limit under the integral sign.

Definition *Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. The sequence $\{f_n\}$ is said to* **converge in measure** *on E to f provided for each $\eta > 0$,*

$$\lim_{n\to\infty} m\left\{x\in E \mid |f_n(x)-f(x)|>\eta\right\}=0.$$

When we write $\{f_n\}\to f$ in measure on E we are implicitly assuming that f and each f_n is measurable, and finite a.e. on E. Observe that if $\{f_n\}\to f$ uniformly on E, and f is a real-valued measurable function on E, then $\{f_n\}\to f$ in measure on E since for $\eta>0$, the set $\{x\in E \mid |f_n(x)-f(x)|>\eta\}$ is empty for n sufficiently large. However, we also have the following much stronger result.

Proposition 3 *Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Then $\{f_n\}\to f$ in measure on E.*

Proof First observe that f is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let $\eta>0$. To prove convergence in measure we let $\epsilon>0$ and seek an index N such that

$$m\left\{x\in E \mid |f_n(x)-f(x)|>\eta\right\}<\epsilon \text{ for all } n\geq N. \tag{4}$$

Egoroff's Theorem tells us that there is a measurable subset F of E with $m(E\sim F)<\epsilon$ such that $\{f_n\}\to f$ uniformly on F. Thus there is an index N such that

$$|f_n-f|<\eta \text{ on } F \text{ for all } n\geq N.$$

Thus, for $n\geq N$, $\{x\in E \mid |f_n(x)-f(x)|>\eta\}\subseteq E\sim F$ and so (4) holds for this choice of N. □

The above proposition is false if E has infinite measure. The following example shows that the converse of this proposition also is false.

Example Consider the sequence of subintervals of $[0, 1]$, $\{I_n\}_{n=1}^{\infty}$, which has initial terms listed as

$$[0, 1], [0, 1/2], [1/2, 1], [0, 1/3], [1/3, 2/3], [2/3, 1],$$
$$[0, 1/4], [1/4, 1/2], [1/2, 3/4], [3/4, 1] \ldots$$

For each index n, define f_n to be the restriction to $[0, 1]$ of the characteristic function of I_n. Let f be the function that is identically zero on $[0, 1]$. We claim that $\{f_n\} \to f$ in measure. Indeed, observe that $\lim_{n\to\infty} \ell(I_n) = 0$ since for each natural number m,

$$\text{if } n > 1 + \cdots + m = \frac{m(m+1)}{2}, \text{ then } \ell(I_n) < 1/m.$$

Thus, for $0 < \eta < 1$, since $\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq I_n$,

$$0 \le \lim_{n\to\infty} m\left\{x \in E \;\middle|\; |f_n(x) - f(x)| > \eta\right\} \le \lim_{n\to\infty} \ell(I_n) = 0.$$

However, it is clear that there is no point x in $[0, 1]$ at which $\{f_n(x)\}$ converges to $f(x)$ since for each point x in $[0, 1]$, $f_n(x) = 1$ for infinitely many indices n, while $f(x) = 0$.

Theorem 4 **(Riesz)** *If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.*

Proof By the definition of convergence in measure, there is a strictly increasing sequence of natural numbers $\{n_k\}$ for which

$$m\left\{x \in E \;\middle|\; |f_j(x) - f(x)| > 1/k\right\} < 1/2^k \text{ for all } j \ge n_k.$$

For each index k, define

$$E_k = \left\{x \in E \;\middle|\; |f_{n_k} - f(x)| > 1/k\right\}.$$

Then $m(E_k) < 1/2^k$ and therefore $\sum_{k=1}^{\infty} m(E_k) < \infty$. The Borel-Cantelli Lemma tells us that for almost all $x \in E$, there is an index $K(x)$ such that $x \notin E_k$ if $k \geq K(x)$, that is,

$$|f_{n_k}(x) - f(x)| \leq 1/k \text{ for all } k \geq K(x).$$

Therefore

$$\lim_{k \to \infty} f_{n_k}(x) = f(x).$$

□

Corollary 5 *Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Then*

$$\lim_{n \to \infty} \int_E f_n = 0 \tag{5}$$

if and only if

$$\{f_n\} \to 0 \text{ in measure on } E \text{ and } \{f_n\} \text{ is uniformly integrable and tight over } E. \tag{6}$$

Proof First assume (5). Corollary 2 tells us that $\{f_n\}$ is uniformly integrable and tight over E. To show that $\{f_n\} \to 0$ in measure on E, let $\eta > 0$. By Chebychev's Inequality, for each index n,

$$m\{x \in E \mid f_n > \eta\} \leq \frac{1}{\eta} \cdot \int_E f_n.$$

Thus,

$$0 \leq \lim_{n \to \infty} m\{x \in E \mid f_n > \eta\} \leq \frac{1}{\eta} \cdot \lim_{n \to \infty} \int_E f_n = 0.$$

Hence $\{f_n\} \to 0$ in measure on E.

To prove the converse, we argue by contradiction. Assume (6) holds but (5) fails to hold. Then there is some $\epsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ for which

$$\int_E f_{n_k} \geq \epsilon_0 \text{ for all } k.$$

However, by Theorem 4, a subsequence of $\{f_{n_k}\}$ converges to $f \equiv 0$ pointwise almost everywhere on E and this subsequence is uniformly integrable and tight so that, by the Vitali Convergence Theorem, we arrive at a contradiction to the existence of the above ϵ_0. This completes the proof. □

POSSIBLE QUESTIONS

1. If f be defined and bounded on a measurable set E with mE finite.In order that $\inf_{f\leq\psi} \int_E \psi(x)dx = \sup_{f\geq\varphi} \int_E \varphi(x)dx$ for all simple functions φ and ψ it is necessary & Sufficient that f must be measurable.
2. If f and g are bounded measurable functions defined on a set E of finite measure, then prove that the following
   (i) $\int_E(af + bg) = a\int_E f + b\int_E g$
   (ii) $If\ f = g\ a.e\ then \int_E f = \int_E g$
   (iii) If $f \leq g$ a.e then $\int_E f \ \leq \int_E g$, hence $|\int f| \leq \int|f|$.
   (iv)If A and B are disjoint measurable sets of finite measure then,
   $\int_{AUB} f = \int_A f + \int_B f$
3. If f and g be integrable over E, then prove that
   (i) The function cf is integrable over E and integral $\int_E cf$ = c$\int_E f$.
   (ii) The function f+g is integrable over E and $\int_E f + g = \int_E f + \int_E g$.
4. State and prove lebesgue convergence theorem.
5. If f and g are non-negative measurable functions , then prove that the following
   i) $\int_E cf$ = c$\int_E f$, c > 0
   ii) $\int_E f + g = \int_E f + \int_E g$
   iii) $If\ f = g\ a.e\ then \int_E f = \leq \int_E g$
6. If f be a non negative function,which is integrable over a set 'E' then given $\varepsilon > 0$,there is a δ>0. such that for every set A⊆ E with m(A) < δ,then $\int_A f < \varepsilon$.
7. If f be a bounded function defined on [a,b].If f is Riemann integral on [a,b] then it is measurable and R$\int_a^b f(x)dx = \int_a^b f(x)dx$.
8. If ψ *and* φ be simple function which vanish outside a set of measure then $\int a\,\varphi + b\psi = a\int \varphi + \int \psi$ and if $\varphi \geq \psi$ a.e then $\int \varphi \geq \int \psi$.
9. State and prove bounded convergence theorem.
10. State and prove Fatou's lemma.
11. State and prove monotone convergent theorem.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
A function F defined on a compact interval [a,b] is called a …………….	Jump function	Step function	Characteristic function	Continuity function	Step function
If E is a measurable set then the complement of the set is ……………….	borel set	open set	σ - algebra	measurable set	measurable set
The family of ……………………. is an algebra of sets	measurable set	open set	σ - algebra	borel set	measurable set
Every set with outer measure Zero is …………………………….	open set	measurable	σ - algebra	borel set	measurable
Every convergent sequence of measurable functions is …………………………	convergent	uniformly convergent	uniformly continrous	continuous	uniformly convergent
Every measurable function is nearly ………………	continuous	uniformly convergent	uniformly continrous	discontinuous	continuous
The partition of an interval denoted by P then the sets A_1, A2, A3...A_n are called……..of partition P.	Compact	Closed	Open	Components	Components

A measurable space is said to be _________if б contains all the subset of set of measure zero	complete	closed	borel set	compact	complete
Lebesgue measure is________	complete	closed	borel set	compact	complete
If μ is a complete measure & f is a measurable function then f=g ,a.e	g is measurable	f is measurable	μ is measurable	f & g is measurable	g is measurable
Every measurable sub set of a +ve set is_______	itself +ve	itself –ve	+ve and –ve	null set	itself +ve
The union of countable collection of +ve set is_______	negative	positive	null set	positive and negative	positive
Hahn decomposition is_______	unique	absolute value	not absolute value	not unique	not unique
A measure v is said to be absolutely continuous with respect to the measure μ if _________________	vA=0	vA = 0	vA=1	vA= 1	vA = 1

A collection $\{X_\alpha\}$ of disjoint measurable subsets of X is called a ___________ for μ.	Jordan	measurable	Nikodym	decomposition	decomposition
A measurable μ is called ___________ if it has a decomposition.	measurable	decomposition	decomposable	Jordan	Jordan

UNIT – III

Differentiation of monotone function, Functions of bounded variation-differentiation of an integral-Absolute continuity.

Functions of bounded variation

Let f and F be two functions on $[a, b]$ such that f is continuous and F has a continuous derivative. Then it will be recalled from elementary calculus that the connection between the operations of differentiation and integration is expressed by the familiar formulas

$$\frac{d}{dx}\int_a^x f(t)dt = f(x),$$

$$\int_a^x F'(t)dt = F(x) - F(a).$$

This immediately suggests:

1. Does (7.1) continue to hold almost everywhere for an arbitrary summable function f?

2. What is the largest class of functions for which (7.2) holds?

These questions will be answered in this chapter. We observe that if f is nonnegative, then the indefinite Lebesgue integral

$$\int_a^x f(t)dt, \quad x \in [a, b],$$

as a function of its upper limit, is nondecreasing. Moreover, since every summable function f is the difference of two nonnegative summable functions f^+ and f^-, the integral (7.3) is the difference between two nondecreasing functions. Hence, the study of the indefinite Lebesgue integral is closely related to the study of monotonic functions. Monotonic functions have a number of simple and important properties which we now discuss.

Monotonic functions

Definition

A function $f : [a,b] \to \mathbb{R}$ *is said to be* nondecreasing *if* $a \le x_1 \le x_2 \le b$ *implies* $f(x_1) \le f(x_2)$ *and* nonincreasing *if* $a \le x_1 \le x_2 \le b$ *implies* $f(x_1) \ge f(x_2)$. *By a* monotonic *function is meant a function which is either nondecreasing or nonincreasing.*

Definition

Given a monotonic function $f : [a,b] \to \mathbb{R}$ *and* $x_0 \in [a,b)$, *the limit*

$$f(x_0^+) := \lim_{h \to 0,\, h>0} f(x_0 + h)$$

(which always exists) is said to be the right hand limit *of* f *at the point* x_0. *Similarly, if* $x_0 \in (a,b]$, *the limit*

$$f(x_0^-) = \lim_{h \to 0,\, h>0} f(x_0 - h)$$

is called the left-hand limit *of* f *at* x_0.

Remark

Let f be nondecreasing on $[a,b]$. If $a \le x < y \le b$, then

$$f(x^+) \le f(y^-).$$

Analogously, if f is nonincreasing on $[a,b]$ and $a \le x < y \le b$, then

$$f(x^+) \ge f(y^-).$$

We now establish the basic properties of monotonic functions.

Theorem *Every monotonic function f on $[a,b]$ is Borel and bounded, and hence summable.*

Proof. Assume that f is nondecreasing. Since $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$, f is obviously bounded. For every $c \in \mathbb{R}$ consider the set

$$E_c = \{x \in [a,b] \mid f(x) < c\}.$$

If E_c is empty, then E_c is (trivially) a Borel set. If E_c is nonempty, let y be the least upper bound of all $x \in E_c$. Then E_c is either the closed interval $[a,y]$, if $y \in E_c$, or the half-open interval $[a,y)$, if $y \notin E_c$. In either case, E_c is a Borel set; this proves that f is Borel. Finally we have

$$\int_a^b |f(x)|dx \le \max\{|f(a)|, |f(b)|\}(b-a),$$

by which f is summable. □

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. Then the set of points of $[a, b]$ at which f is discontinuous is at most countable.

Proof. Suppose, for the sake of definiteness, that f is nondecreasing, and let E be the set of points at which f is discontinuous. If $x \in E$ we have $f(x^-) < f(x^+)$; then with every point x of E we associate we associate a rational number $r(x)$ such that

$$f(x^-) < r(x) < f(x^+).$$

Since by Remark 7.3 $x_1 < x_2$ implies $f(x_1^+) \leq f(x_2^-)$, we see that $r(x_1) \neq r(x_2)$. We have thus established a 1-1 correspondence between the set E and a subset of the rational numbers. □

Differentiation of a monotonic function

The key result of this section will be to show that a monotonic function f defined on an interval $[a, b]$ has a finite derivative almost everywhere in $[a, b]$. Before proving this proposition, due to Lebesgue, we must first introduce some further notation. For every $x \in (a, b)$ the following four quantities (which may take infinite values) always exist:

$$D'_L f(x) = \liminf_{h\to 0,\, h<0} \frac{f(x+h) - f(x)}{h}, \quad D''_L f(x) = \limsup_{h\to 0,\, h<0} \frac{f(x+h) - f(x)}{h},$$

$$D'_R f(x) = \liminf_{h\to 0,\, h>0} \frac{f(x+h) - f(x)}{h}, \quad D''_R f(x) = \limsup_{h\to 0,\, h>0} \frac{f(x+h) - f(x)}{h}.$$

These four quantities are called the *derived numbers* of f at x. It is clear that the inequalities

$$D'_L f(x) \leq D''_L f(x), \quad D'_R f(x) \leq D''_R f(x)$$

always hold. If $D'_L f(x)$ and $D''_L f(x)$ are finite and equal, their common value is just the left-hand derivative of f at x. Similarly, if $D'_R f(x)$ and $D''_R f(x)$ are finite and equal, their common value is just the right-hand derivative of f at x. Moreover, f has a derivative at x if and only if all four derived numbers $D'_L f(x)$, $D''_L f(x)$, $D'_R f(x)$ and $D''_R f(x)$ are finite and equal.

Theorem *Let $f : [a, b] \to \mathbb{R}$ be a monotonic function.*

Then f has a derivative almost everywhere on $[a, b]$. Furthermore $f' \in L^1([a, b])$ and

$$\int_a^b |f'(t)|dt \leq |f(b) - f(a)|.$$

Proof. There is no loss of generality in assuming that f is nondecreasing, since if f is nonincreasing, we can apply the result to $-f$ which is obviously nondecreasing. We begin by proving that the derived numbers of f are equal (with possibly infinite value) almost everywhere on $[a, b]$. It will be enough to show that the inequality

$$D'_L f(x) \geq D''_R f(x)$$

holds almost everywhere on $[a, b]$. In fact, setting, $f^*(x) = -f(-x)$, we see that f^* in nondecreasing on $[-b, -a]$; moreover, it is easily verified that

$$D'_L f^*(x) = D'_R f(-x), \quad D''_L f^*(x) = D''_R f(-x).$$

Therefore, applying (7.6) to f^*, we get

$$D'_L f^*(x) \geq D''_R f^*(x)$$

or

$$D'_R f(x) \geq D''_L f(x).$$

Combining this inequality with (7.6), we obtain

$$D''_R f \leq D'_L f \leq D''_L f \leq D'_R f \leq D''_R f,$$

after using (7.4), and the equality of the four derived numbers follows. To prove that (7.6) holds almost everywhere, observe that the set of points where $D_L^- f < D_R^+ f$ can clearly be represented as the union over $u, v \in \mathbb{Q}$ with $v > u > 0$ of the sets

$$E_{u,v} = \{x \in (a,b) \mid D_R'' f(x) > v > u > D_L' f(x)\}.$$

It will then follow that (7.6) holds almost everywhere, if we succeed in showing that $\lambda(E_{u,v}) = 0$. Let $s = \lambda(E_{u,v})$. Then, given $\varepsilon > 0$, according to Proposition 1.53 there is an open set A such that $E_{u,v} \subset A$ and $\lambda(A) < s + \varepsilon$. For every $x \in E_{u,v}$ and $\delta > 0$, since $D_L' f(x) < u$, there exists $h_{x,\delta} \in (0, \delta)$ such that $[x - h_{x,\delta}, x] \subset A$ and

$$f(x) - f(x - h_{x,\delta}) < u h_{x,\delta}.$$

Since the collection of closed intervals $([x - h_{x,\delta}, x])_{x \in (a,b),\, \delta > 0}$ is a fine cover of $E_{u,v}$, by Vitali's covering lemma there exists a finite number of disjoint intervals of such collection, say

$$I_1 := [x_1 - h_1, x_1], \ldots, I_N := [x_N - h_N, x_N],$$

such that, setting $B = E_{u,v} \cap \bigcup_{i=1}^N (x_i - h_i, x_i)$,

$$\lambda(B) = \lambda\Big(E_{u,v} \cap \bigcup_{i=1}^N I_k\Big) > s - \varepsilon.$$

Summing up over these intervals we get

$$\sum_{i=1}^N \big(f(x_i) - f(x_i - h_i)\big) < u \sum_{i=1}^N h_i < u\lambda(A) < u(s + \varepsilon).$$

Now we reason as above and use the inequality $D_R'' f(x) > v$; for every $y \in B$ and $\eta > 0$, since $D_R'' f(x) > v$, there exists $k_{y,\eta} \in (0, \eta)$ such that $[y, y + k_{y,\eta}] \subset I_i$ for some $i \in \{1, \ldots, N\}$ and

$$f(y + k_{y,\eta}) - f(y) > v k_{y,\eta}.$$

Since the collection of closed intervals $([y, y+k_{y,\eta}])_{y\in B,\, \eta>0}$ is a fine cover of B, by Vitali's covering lemma there exists a finite number of disjoint intervals of such collection, say

$$J_1 := [y_1, y_1 + k_1], \ldots, J_M := [y_M, y_M + k_M],$$

such that,

$$\lambda\Big(B \cap \bigcup_{j=1}^{M} J_j\Big) \geq \lambda(B) - \varepsilon > s - 2\varepsilon.$$

Summing up over these intervals we get

$$\sum_{j=1}^{M} \big(f(y_j + k_j) - f(y_j)\big) > v \sum_{j=1}^{M} k_j = v\lambda\Big(\bigcup_{j=1}^{M} J_j\Big) > v(s - 2\varepsilon).$$

For every $i \in \{1, \ldots, N\}$, we sum up over all the intervals J_j such that $J_j \subset I_i$, and, using that f is nondecreasing, we obtain

$$\sum_{j,\, J_j \subset I_i} \big(f(y_j + k_j) - f(y_j)\big) \leq f(x_i) - f(x_i - h_i)$$

by which, summing over i and taking into account that every interval J_j is contained in some interval I_i,

$$\sum_{i=1}^{N} \big(f(x_i) - f(x_i - h_i)\big) \geq \sum_{i=1}^{N} \sum_{j,\, J_j \subset I_i} \big(f(y_j + k_j) - f(y_j)\big) = \sum_{j=1}^{M} \big(f(y_j + k_j) - f(y_j)\big).$$

Combining this with (7.7)-(7.8),

$$u(s + \varepsilon) > v(s - 2\varepsilon).$$

The arbitrariness of ε implies $us \geq vs$; since $u < v$, then $s = 0$. This shows that $\lambda(E_{u,v}) = 0$, as asserted.

We have thus proved that the function

$$\Phi(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is defined almost everywhere on $[a, b]$ and f has a derivative at x if and only if $\Phi(x)$ is finite. Let

$$\Phi_n(x) = n\left(f\left(x + \frac{1}{n}\right) - f(x)\right)$$

where, to make Φ_n meaningful for all $x \in [a, b]$, we get $f(x) = f(b)$ for $x \geq b$, by definition. Since f is summable on $[a, b]$, so is every Φ_n. Integrating Φ_n, we get

$$\begin{aligned}\int_a^b \Phi_n(x)dx &= n\int_a^b \left(f\left(x + \frac{1}{n}\right) - f(x)\right)dx = n\left(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx\right) \\ &= n\left(\int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx\right) = f(b) - n\int_a^{a+\frac{1}{n}} f(x)dx \\ &\leq f(b) - f(a)\end{aligned}$$

where in the last step we use the fact that f is nondecreasing. From Fatou's lemma it follows that

$$\int_a^b \Phi(x)dx \leq f(b) - f(a).$$

In particular Φ is summable, and, consequently, a.e. finite. Then f has a derivative almost everywhere and $f'(x) = \Phi(x)$ a.e. in $[a, b]$. □

Example It is easy to find monotonic functions f for which (7.5) becomes a strict inequality. For example, given points $a = x_0 < x_1 < \ldots < x_n = b$ in the interval $[a, b]$ and $h_1, h_2, \ldots, h_n$ corresponding numbers, consider the function

$$f(x) = \begin{cases} h_1 & \text{if } a \le x < x_1, \\ h_2 & \text{if } x_1 \le x < x_2, \\ \ldots & \\ h_n & \text{if } x_{n-1} \le x \le b. \end{cases}$$

A function of this particularly simple type is called a *step function*. If $h_1 \le h_2 \le \ldots \le h_n$, then f is obviously nondecreasing and

$$0 = \int_a^b f'(x)dx < f(b) - f(a) = h_n - h_1.$$

Example

[Vitali's function] In the preceding example, f is discontinuous. However, it is also possible to find continuous nondecreasing functions satisfying the strict inequality (7.5). To this end let

$$(a_1^1, b_1^1) = \left(\frac{1}{3}, \frac{2}{3}\right)$$

be the middle third of the interval $[0, 1]$, let

$$(a_1^2, b_1^2) = \left(\frac{1}{9}, \frac{2}{9}\right), \quad (a_2^2, b_2^2) = \left(\frac{7}{9}, \frac{8}{9}\right)$$

be the middle thirds of the intervals remaining after deleting (a_1^1, b_1^1) from [0,1], let

$$(a_1^3, b_1^3) = \left(\frac{1}{27}, \frac{2}{27}\right), \quad (a_2^3, b_2^3) = \left(\frac{7}{27}, \frac{8}{27}\right),$$

$$(a_3^3, b_3^3) = \left(\frac{19}{27}, \frac{20}{27}\right), \quad (a_4^3, b_4^3) = \left(\frac{25}{27}, \frac{26}{27}\right)$$

be the middle thirds of the intervals remaining after deleting (a_1^1, b_1^1), (a_1^2, b_1^2), (a_2^2, b_2^2) from $[0, 1]$ and so on. Note that the complement of the union of all the intervals (a_k^n, b_k^n) is the Cantor set constructed in Example 1.49. Now define a function

$$f(0) = 0, \quad f(1) = 1, \quad f(t) = \frac{2k-1}{2^n} \text{ if } t \in (a_k^n, b_k^n),$$

so that

$$f(t) = \frac{1}{2} \quad \text{if } \frac{1}{3} < t < \frac{2}{3},$$

$$f(t) = \begin{cases} \frac{1}{4} & \text{if } \frac{1}{9} < t < \frac{2}{9}, \\ \frac{3}{4} & \text{if } \frac{7}{9} < t < \frac{8}{9}, \end{cases}$$

$$f(t) = \begin{cases} \frac{1}{8} & \text{if } \frac{1}{27} < t < \frac{2}{27}, \\ \frac{3}{8} & \text{if } \frac{7}{27} < t < \frac{8}{27}, \\ \frac{5}{8} & \text{if } \frac{19}{27} < t < \frac{20}{27}, \\ \frac{7}{8} & \text{if } \frac{25}{27} < t < \frac{26}{27}, \end{cases}$$

and so on. Then f is defined everywhere except at points of the Cantor set C; furthermore f is nondecreasing on $[0,1]\setminus C$ and $f([0,1]\setminus C) = \{\frac{2k-1}{2^n} \mid n \in \mathbb{N},\ 1 \le k \le 2^{n-1}\}$ which is dense in $[0,1]$, that is

$$\overline{f([0,1]\setminus C)} = [0,1].$$

Given any point $t^* \in C$, let $(t_n)_n$ be an increasing sequence of points in $[0,1]\setminus C$ converging to t^* and let $(t'_n)_n$ be a decreasing sequence of points in $[0,1]\setminus C$ converging to t^*. Such sequences exist since $[0,1]\setminus C$ is dense in $[0,1]$. Then the limits $\lim_n f(t_n)$ and $\lim_n f(t'_n)$ exist (since f is nondecreasing in $[0,1]\setminus C$); we claim that they are equal. Otherwise, setting $a = \lim_n f(t_n)$ and $b = \lim_n f(t'_n)$, then $(a,b) \subset [0,1]\setminus f([0,1]\setminus C)$, in contradiction with (7.9). Then let

$$f(t^*) = \lim_n f(t_n) = \lim_n f(t'_n).$$

Completing the definition of f in this way, we obtain a continuous nondecreasing function on the whole interval $[0,1]$, known as Vitali's function. The derivatives f' obviously vanishes at every interval (a_k^n, b_k^n), and hence vanishes almost everywhere, since the Cantor set has measure zero. It follows that

$$0 = \int_0^1 f'(x)dx < f(1) - f(0) = 1.$$

Functions of bounded variation

Definition
A function f defined on an interval $[a,b]$ if said to be of bounded variation *if there is a constant $C > 0$ such that*

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \le C$$

for every partition

$$a = x_0 < x_1 < \ldots < x_n = b$$

of $[a,b]$. By the total variation of f on $[a,b]$, denoted by $V_a^b(f)$, is meant the quantity:

$$V_a^b(f) = \sup \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$$

where the least upper bound is taken over all partitions (7.11) *of the interval* $[a, b]$.

Remark It is an immediate consequence of the above definition that if $\alpha \in \mathbb{R}$ and f is a function of bounded variation on $[a, b]$, then so is αf and

$$V_a^b(\alpha f) = |\alpha| V_a^b(f).$$

Example 1. If f is a monotonic function on $[a, b]$, then the left-hand side of (7.10) equals $|f(b) - f(a)|$ regardless of the choice of partition. Then f is of bounded variation and $V_a^b(f) = |f(b) - f(a)|$.

2. If f is a step function of the type considered in Example 7.7 with $h_1, \ldots, h_n \in \mathbb{R}$, then f is of bounded variation, with total variation given by the sum of the jumps, i.e.

$$V_a^b(f) = \sum_{i=1}^{n-1} |h_{i+1} - h_i|.$$

Example Suppose f is a Lipschitz function on $[a, b]$ with Lipschitz constant K; then for any partition (7.11) of $[a, b]$ we have

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \leq K \sum_{k=0}^{n-1} |x_{k+1} - x_k| = K(b-a).$$

Then f is of bounded variation and $V_a^b(f) \leq K(b-a)$.

Example It is easy to find a continuous function which is not of bounded variation. Indeed consider the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$$

and, fixed $n \in \mathbb{N}$, take the following partition

$$0, \frac{2}{(4n-1)\pi}, \frac{2}{(4n-3)\pi}, \ldots, \frac{2}{3\pi}, \frac{2}{\pi}, 1.$$

The sum on the left-hand side of (7.10) associated to such partition is given by

$$\frac{4}{\pi} \sum_{k=1}^{2n-1} \frac{1}{2k+1} + \frac{2}{\pi} + \left| \sin 1 - \frac{2}{\pi} \right|.$$

Taking into account that $\sum_{k=1}^{\infty} \frac{1}{2k+1} = \infty$, we deduce that the least upper bound on the right-hand side of (7.12) over all partitions of $[a, b]$ is infinity.

Proposition *If f and g are functions of bounded variation on $[a, b]$, then so is $f + g$ and* $V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)$.

$$\sum_{k=0}^{n-1} |f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)|$$
$$\leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)| \leq V_a^b(f) + V_a^b(g).$$

Taking the least upper bound on the left-hand side over all partitions of $[a, b]$ we immediately get the thesis. □

It follows from Remark 7.10 and Proposition 7.14 that any linear combination of functions of bounded variation is itself a function of bounded variation. In other words, the set $BV([a,b])$ of all functions of bounded variation on the interval $[a, b]$ is a linear space (unlike the set of all monotonic functions).

Proposition

If f is a function of bounded variation on $[a, b]$ and $a <$

$c < b$, then

$$V_a^b(f) = V_a^c(f) + V_c^b(f).$$

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$$

$$= \sum_{k=0}^{r-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=r}^{n-1} |f(x_{k+1}) - f(x_k)|$$

$$\leq V_a^c(f) + V_c^b(f).$$

Now consider an arbitrary partition of $[a, b]$. It is clear that adding an extra point of subdivision to this partition can never decrease the sum $\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$. Therefore (7.13) holds for any subdivision of $[a, b]$, and hence

$$V_a^b(f) \leq V_a^c(f) + V_c^b(f).$$

On the other hand, given any $\varepsilon > 0$, there are partitions of the intervals $[a, c]$ and $[c, b]$, respectively, such that

$$\sum_i |f(x'_{i+1}) - f(x'_i)| > V_a^c(f) - \frac{\varepsilon}{2},$$

$$\sum_j |f(x''_{j+1}) - f(x''_j)| > V_c^b(f) - \frac{\varepsilon}{2}.$$

Combining all points of subdivision x'_i, x''_j, we get a partition of the interval $[a, b]$, with points of subdivision x_k, such that

$$V_a^b(f) \geq \sum_k |f(x_{k+1}) - f(x_k)| = \sum_i |f(x'_{i+1}) - f(x'_i)| + \sum_j |f(x''_{j+1}) - f(x''_j)|$$
$$> V_a^c(f) + V_c^b(f) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $V_a^b(f) \geq V_a^c(f) + V_c^b(f)$. □

Corollary *If f is a function of bounded variation on $[a, b]$, then the function*

$$x \longmapsto V_a^x(f)$$

is nondecreasing.

Proof. If $a \leq x < y \leq b$, Proposition 7.15 implies

$$V_a^y(f) = V_a^x(f) + V_x^y(f) \geq V_a^x(f).$$

Proposition *A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if f can be represented as the difference between two nondecreasing functions on $[a, b]$.*

Proof. Since, by Example 7.11, any monotonic function is of bounded variation and since the set $BV([a, b])$ is a linear space, we get that the difference of two nondecreasing functions is of bounded variation. To prove the converse, set

$$g_1(x) = V_a^x(f), \quad g_2(x) = V_a^x(f) - f(x).$$

By Corollary 7.16 g_1 is a nondecreasing function. We claim that g_2 is nondecreasing too. Indeed, if $x < y$, then, using Proposition 7.15, we get

$g_2(y) - g_2(x) = V_x^y(f) - (f(y) - f(x))$.

$$|f(y) - f(x)| \le V_x^y(f)$$

and hence the right hand side of (7.14) is nonnegative. Writing $f = g_1 - g_2$, we get the desired representation of f as the difference between two nondecreasing functions. □

Theorem *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation. Then the set of points of $[a,b]$ at which f is discontinuous is at most countable. Furthermore f has a derivative almost everywhere on $[a,b]$, $f' \in L^1([a,b])$ and*

$$\int_a^b |f'(x)|dx \le V_a^b(f).$$

Proof. Combining Theorem 7.5, Theorem 7.6 and Proposition 7.17 we immediately obtain that f has no more than countably many points of discontinuity, has a derivative almost everywhere on $[a,b]$ and $f' \in L^1([a,b])$. Since for $a \le x < y \le b$

$$|f(y) - f(x)| \le V_x^y(f) = V_a^y(f) - V_a^x(f),$$

we get

$$|f'(x)| \le (V_a^x(f))' \quad \text{a.e. in } [a,b].$$

Finally, using (7.5)

$$\int_a^b |f'(x)|dx \le \int_a^b (V_a^x(f))'dx \le V_a^b(f).$$

Proposition

A function $f : [a,b] \to \mathbb{R}$ is of bounded variation if and only if the curve

$$y = f(x) \quad a \le x \le b$$

is rectifiable, i.e. has finite lenght(1).

Proof. For any partition of $[a, b]$ we get

$$\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \leq \sum_{i=0}^{n-1} \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}$$
$$\leq (b-a) + \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|.$$

Taking the least upper bound over all partitions we get the thesis.

Absolutely continuous functions

Definition *A function f defined on an interval $[a, b]$ is said to be* absolutely continuous *if, given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every finite system of pairwise disjoint subintervals

$$(a_k, b_k) \subset [a, b] \quad k = 1, \dots, n$$

of total length $\sum_{k=1}^{n}(b_k - a_k)$ less than δ.

Example Suppose f is a Lipschitz function on $[a, b]$ with Lipschitz constant K; then, choosing $\delta = \frac{\varepsilon}{K}$, we immediately get that f is absolutely continuous.

Remark

Clearly every absolutely continuous function is uniformly continuous, as we see by choosing a single subinterval $(a_1, b_1) \subset [a, b]$. However, a uniformly continuous function need not be absolutely continuous. For example, the Vitali's function f constructed in Example 7.8 is continuous (and hence uniformly continuous) on $[0, 1]$, but not absolutely continuous on $[0, 1]$. In fact, for every n consider the set

$$C_n = \left\{ x \in [0,1] \,\middle|\, x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \text{ with } a_1, \ldots, a_n \neq 1 \right\}$$

which is the union of 2^n pairwise disjoint closed intervals I_i, each of which has measure $\frac{1}{3^n}$ (then the total length is $(\frac{2}{3})^n$). Denoting by C the Cantor set (see Example 1.49), we have $C \subset C_n$; since, by construction, the Vitali's function is constant on the subintervals of $[0,1] \setminus C$, then the sum (7.16) associated

to the system (I_i) is equal to 1. Hence the Cantor set C can be covered by a finite system of subintervals of arbitrarily small length, but the sum (7.16) associated to every such system is equal to 1. The same example shows that a function of bounded variation needs not be absolutely continuous. On the other hand, an absolutely continuous function is necessarily of bounded variation (see Proposition 7.27).

Proposition *If f is absolutely continuous on $[a,b]$, then f is of bounded*

variation on $[a,b]$.

Proof. Given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every finite system of pairwise disjoint subintervals $(a_k, b_k) \subset [a,b]$ such that

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

Hence if $[\alpha, \beta]$ is any subinterval of length less than δ, we have

$$V_\alpha^\beta(f) \leq \varepsilon.$$

Let $a = x_0 < x_1 < \ldots < x_N = b$ be a partition of $[a, b]$ into N subintervals $[x_k, x_{k+1}]$ all of length less than δ. Then, by Proposition 7.15,

$$V_a^b(f) \leq N\varepsilon.$$

□

An immediate consequence of Definition 7.24 and obvious properties of absolute value is the following.

Proposition *If f is absolutely continuous on $[a, b]$, then so is αf, where α is any constant. Moreover, if f and g are absolutely continuous on $[a, b]$, then so is $f + g$.*

It follows from Proposition 7.28 together with Remark 7.26 that the set $AC([a, b])$ of all absolutely continuous functions on $[a, b]$ is a proper subspace of the linear space $BV([a, b])$ of all functions of bounded variation on $[a, b]$.

Lemma *Let $g \in L^1([a, b])$ be such that $\int_I g(t)dt = 0$ for every subinterval $I \subset [a, b]$. Then $g(x) = 0$ a.e. in $[a, b]$.*

Proof. If we denote by $\mathcal{I}$ the family of all finite disjoint union of subintervals of $[a, b]$, it is immediate to see that $\mathcal{I}$ is an algebra and $\int_A g(t)dt = 0$ for every $A \in \mathcal{I}$. Let V be an open set in $[a, b]$; then $V = \cup_{n=1}^{\infty} I_n$ where $I_n \subset [a, b]$ is a subinterval. For every n, since $\cup_{i=1}^{n} I_i \in \mathcal{I}$, we have $\int_{\cup_{i=1}^{n} I_i} g(t)dt = 0$; Lebesgue Theorem implies

$$\int_V g(t)dt = \lim_{n\to\infty} \int_{\cup_{i=1}^{n} I_i} g(t)dt = 0$$

Assume by contradiction the existence of $E \in \mathcal{B}([a, b])$ such that $\lambda(E) > 0$ and $g(x) > 0$ in E. By Theorem 1.55 there exists a compact set $K \subset E$ such that $\lambda(K) > 0$. Setting $V = [a, b] \setminus K$, V is an open set in $[a, b]$; then

$$0 = \int_a^b g(t)dt = \int_V g(t)dt + \int_K g(t)dt = \int_K g(t)dt > 0,$$

and the contradiction follows. □

Returning to the problem of differentiating the indefinite Lebesgue integral, in the following Theorem we evaluate the derivative (7.1), thereby giving an affirmative answer to the first of the two questions posed at the beginning of the chapter.

Theorem *If f is absolutely continuous on $[a,b]$, then f has a derivative almost everywhere on $[a,b]$, $f' \in L^1([a,b])$ and*

$$f(x) = f(a) + \int_a^x f'(t)dt \quad \forall x \in [a,b].$$

Proof. By Proposition 7.27 f is of bounded variation; hence, by Theorem 7.18, f has a derivative almost everywhere and $f' \in L^1([a,b])$. To prove (7.21) consider the function

$$g(x) = \int_a^x f'(t)dt.$$

Then, by Theorem 7.30, g is absolutely continuous on $[a,b]$ and $g'(x) = f'(x)$ a.e. in $[a,b]$. Setting $\Phi = g - f$, Φ is absolutely continuous, being the difference of two absolutely continuous functions, and $\Phi'(x) = 0$ a.e. in $[a,b]$. It follows from the previous lemma that Φ is constant, that is $\Phi(x) = \Phi(a) = f(a) - g(a) = f(a)$, by which

$$f(x) = \Phi(x) + g(x) = f(a) + \int_a^x f'(t)dt \quad \forall x \in [a,b].$$

Proposition

Let $f : [a, b] \to \mathbb{R}$. The following properties are equivalent:

a) f is absolutely continuous on $[a, b]$;

b) f is of bounded variation on $[a, b]$ and

$$\int_a^b |f'(t)|dt = V_a^b(f).$$

Proof. $\boxed{a) \Rightarrow b)}$ For any partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$, by Theorem 7.32 we have

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{n-1} \left| \int_{x_k}^{x_{k+1}} f'(t)dt \right| \le \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |f'(t)|dt = \int_a^b |f'(t)|dt,$$

which implies

$$V_a^b(f) \le \int_a^b |f'(t)|dt.$$

On the other hand, by Theorem 7.18, $\int_a^b |f'(t)|dt \le V_a^b(f)$, and so $V_a^b(f) = \int_a^b |f'(t)|dt$.

$\boxed{b) \Rightarrow a)}$ For every $x \in [a, b]$, using (7.15), we have

$$V_a^x(f) \geq \int_a^x |f'(t)|dt = \int_a^b |f'(t)|dt - \int_x^b |f'(t)|dt = V_a^b(f) - \int_x^b |f'(t)|dt$$
$$\geq V_a^b(f) - V_x^b(f) = V_a^x(f)$$

where last equality follows from Proposition 7.15. Then we get

$$V_a^x(f) = \int_a^x |f'(t)|dt.$$

Since $f' \in L^1([a, b])$, Theorem 7.30 implies that the function $x \mapsto V_a^x(f)$ is absolutely continuous. Given any collection of pairwise disjoint intervals (a_k, b_k), we have

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \leq \sum_{k=1}^n V_{a_k}^{b_k}(f) = \sum_{k=1}^n \left(V_a^{b_k}(f) - V_a^{a_k}(f)\right).$$

By the absolute continuity of $x \mapsto V_a^x(f)$, the last expression on the right approaches zero as the total length of the intervals (a_k, b_k) approaches zero. This proves that f is absolutely continuous on $[a, b]$. □

POSSIBLE QUESTIONS

1. State and prove Vitali theorem.
2. Prove that a function f is bounded variation on [a,b] if and only if f is the difference of two monotone real valued functions on [a,b].
3. If f is integrable on [a,b] , then prove that the function F defined by F(x) = $\int_a^x f(t)dt$ is a continuous function of bounded variation on [a ,b].
4. Prove that if f be an integrable function on [a ,b] , and suppose that F(x) = F(a) + $\int_a^x f(t)dt$ then
   F '(x) =f(x) for almost all x in [a,b].
5. Prove that a function is an indefinite integral if and only if it is absolutely continuous.
6. Prove that every absolutely continuous function is the indefinite integral of its derivative.
7. Prove that if f is absolutely continuous on [a, b], then it is of bounded variation on[a,b].
8. Prove that if f is bounded and measurable on [a,b] and F(x) = $\int_a^x f(t)dt$ + F(a) , then
   F '(x) =f(x) for almost all x in [a,b].
9. If f is integrable on [a,b] and $\int_a^x f(t)dt$ =0 for all x ε [a,b] then prove that f(t)=0 a.e in [a ,b].
10. Prove that if f is absolutely continuous on [a, b] and f '(x) = 0 a.e then f is constant.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The dependence on the interval [a,b] or on the function f, If T<∞,we say that f is of____________ over [a,b]	variance	bounded variation	function	measure	bounded variation
A function f is of bounded variation on [a,b] if and only if f is the difference of ___________function on[a,b]	Tow monotone real valued	Monotone	Measure	Real value	Tow monotone real valued
A monotone function on[0.1] which is discontinuous at each ________	continuous	discontinuous	rational point	point	rational point
The derivative of the indefinite integral of an integral function is equal to the _______almost every where	integrand	positive	non integrand	negative	integrand
A real valued function f defined on [a,b] is said to be _________on[a,b]	continuous	countable	measure	absolutely continuous	absolutely continuous
The ______of tow absolutely continuous function is absolutely continuous	sum	odd	different	sum & difference	sum
If f is absolutely continuous on [a,b] then it is of _______on[a,b]	bounded	continuous	variation	bounded variation	bounded variation

If f is absolutely continuous on [a,b]&f'(x)=0 a.e ,then f is __________	constant	bounded	variance	continuous	constant
A function F is an indefinite integral if and only if it is __________	absolutely continuous	continuous	bounded	measure	absolutely continuous
Every absolutely continuous function is the _________of its derivative	indefinite integral	integral	continuous	bounded	indefinite integral
If X is any uncountable set & б the family of these subset which are either or the complement of a countable set then б is a__________	algebra	countable	uncountable	σ-algebra	σ-algebra
A measure μ is called ________if μ(x)<∞	finite	infinite	uncountable	countable	finite
which one is an ________a finite measure	(0,1)	[1,1]	(1,1)	[0,1]	[0,1]
which one is an σ-finite measure _______	(-∞,∞)	(0,∞)	(∞,0)	(∞,-∞)	(-∞,∞)
the counting measure on an______is a measure which is not σ-finite	uncountable	infinite	finite	countable	uncountable
Any measurable set contained in a set of σ finite measure is it self of _________	infinite measurable	measurable	σ-finite measurable	finite measurable	σ-finite measurable

If μ is _______then every measurable set is of σ-finite measure	σ-algebra	σ-finite	Borel set	countable set	σ-finite
Every σ-finite measure is _______	semi finite	constant	finite	infinite	semi finite

UNIT – IV

Measure spaces-Measurable functions-Integration-General convergence Theorems.

Measure Spaces

The first goal of the present chapter is to abstract the most important properties of Lebesgue measure on the real line in the absence of any topology. We shall do this by giving certain axioms that Lebesgue measure satisfies and base our theory on these axioms. As a consequence our theory will be valid for every system satisfying the given axioms.

To establish that Lebesgue measure on the real line is a countably additive set function on a σ-algebra we employed only the most rudimentary set-theoretic concepts. We defined a primitive set function by assigning length to each bounded interval, extended this set function to the set function outer measure defined for every subset of real numbers, and then distinguished a collection of measurable sets. We proved that the collection of measurable sets is a σ-algebra on which the restriction of outer measure is a measure. We call this the Carathéodory construction of Lebesgue measure. The second goal of this chapter is to show that the Carathéodory construction is feasible for a general abstract set X. Indeed, we show that any nonnegative set function μ defined on a collection $\mathcal{S}$ of subsets of X induces an outer measure μ^* with respect to which we can identify a σ-algebra $\mathcal{M}$ of measurable sets. The restriction of μ^* to $\mathcal{M}$ is a measure that we call the Carathéodory measure induced by μ. We conclude the chapter with a proof of the Carathéodory-Hahn Theorem, which tells us of very general conditions under which the Carathéodory measure induced by a set function μ is an extension of μ.

MEASURES AND MEASURABLE SETS

Recall that a σ-algebra of subsets of a set X is a collection of subsets of X that contains the empty-set and is closed with respect to the formation of complements in X and with respect to the formation of countable unions and therefore, by De Morgan's Identities, with respect to the formation of intersections. By a set function μ we mean a function that assigns an extended real number to certain sets.

Definition *By a* **measurable space** *we mean a couple $(X, \mathcal{M})$ consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X. A subset E of X is called* **measurable** *(or measurable with respect to $\mathcal{M}$) provided E belongs to $\mathcal{M}$.*

Definition *By a* **measure** *μ on a measurable space $(X, \mathcal{M})$ we mean an extended real-valued nonnegative set function $\mu: \mathcal{M} \to [0, \infty]$ for which $\mu(\emptyset) = 0$ and which is* **countably additive** *in the sense that for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,* *By a* **measure space** $(X, \mathcal{M}, \mu)$ *we mean a measurable space* $(X, \mathcal{M})$ *together with a measure* μ *defined on* $\mathcal{M}$.

One example of a measure space is $(\mathbf{R}, \mathcal{L}, m)$, where $\mathbf{R}$ is the set of real numbers, $\mathcal{L}$ the collection of Lebesgue measurable sets of real numbers, and m Lebesgue measure. A second example of a measure space is $(\mathbf{R}, \mathcal{B}, m)$, where $\mathcal{B}$ is the collection of Borel sets of real numbers and m is again Lebesgue measure. For any set X, we define $\mathcal{M} = 2^X$, the collection of all subsets of X, and define a measure η by defining the measure of a finite set to be the number of elements in the set and the measure of an infinite set to be ∞. We call η the **counting measure** on X. For any σ-algebra $\mathcal{M}$ of subsets of a set X and point x_0 belonging to X, the **Dirac measure** concentrated at x_0, denoted by δ_{x_0}, assigns 1 to a set in $\mathcal{M}$ that contains x_0 and 0 to a set that does not contain x_0: this defines the Dirac measure space $(X, \mathcal{M}, \delta_{x_0})$. A slightly bizarre example is the following: let X be any uncountable set and $\mathcal{C}$ the collection of those subsets of X that are either countable or the complement of a countable set. Then $\mathcal{C}$ is a σ-algebra and we can define a measure on it by setting $\mu(A) = 0$ for each countable subset of X and $\mu(B) = 1$ for each subset of X whose complement in X is countable. Then $(X, \mathcal{C}, \mu)$ is a measure space.

It is useful to observe that for any measure space $(X, \mathcal{M}, \mu)$, if X_0 belongs to $\mathcal{M}$, then $(X_0, \mathcal{M}_0, \mu_0)$ is also a measure space where $\mathcal{M}_0$ is the collection of subsets of $\mathcal{M}$ that are contained in X_0 and μ_0 is the restriction of μ to $\mathcal{M}_0$.

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

Proposition 1 *Let* $(X, \mathcal{M}, \mu)$ *be a measure space.*

(Finite Additivity) For any finite disjoint collection $\{E_k\}_{k=1}^n$ *of measurable sets,*

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

(Monotonicity) If A *and* B *are measurable sets and* $A \subseteq B$, *then*

$$\mu(A) \leq \mu(B).$$

(Excision) If, moreover, $A \subseteq B$ *and* $\mu(A) < \infty$, *then*

$$\mu(B \sim A) = \mu(B) - \mu(A),$$

so that if $\mu(A) = 0$, *then*

$$\mu(B \sim A) = \mu(B).$$

Proof Finite additivity follows from countable additivity by setting $E_k = \emptyset$, so that $\mu(E_k) = 0$, for $k > n$. By finite additivity,

$$\mu(B) = \mu(A) + \mu(B \sim A),$$

which immediately implies monotonicity and excision. To verify countable monotonicity, define $G_1 = E_1$ and then define

$$G_k = E_k \sim \left[\bigcup_{i=1}^{k-1} E_i\right] \text{ for all } k \geq 2.$$

Observe that

$$\{G_k\}_{k=1}^{\infty} \text{ is disjoint, } \bigcup_{k=1}^{\infty} G_k = \bigcup_{k=1}^{\infty} E_k \text{ and } G_k \subseteq E_k \text{ for all } k.$$

From the monotonicity and countable additivity of μ we infer that

$$\mu(E) \leq \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} G_k\right) = \sum_{k=1}^{\infty} \mu(G_k) \leq \sum_{k=1}^{\infty} \mu(E_k).$$ □

The countable monotonicity property is an amalgamation of countable additivity and monotonicity, which we name since it is invoked so frequently.

A sequence of sets $\{E_k\}_{k=1}^{\infty}$ is called **ascending** provided for each k, $E_k \subseteq E_{k+1}$, and said to be **descending** provided for each k, $E_{k+1} \subseteq E_k$.

Proposition 2 (Continuity of Measure) *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

(i) *If $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of measurable sets, then*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k). \quad (1)$$

(ii) *If $\{B_k\}_{k=1}^{\infty}$ is a descending sequence of measurable sets for which $\mu(B_1) < \infty$, then*

$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \mu(B_k). \quad (2)$$

The proof of the continuity of measure is the same, word for word, as the proof of the continuity of Lebesgue measure on the real line; see page 44.

The Borel-Cantelli Lemma *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_k\}_{k=1}^{\infty}$ a countable collection of measurable sets for which $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all x in X belong to at most a finite number of the E_k's.*

Proof For each n, by the countable monotonicity of μ, $\mu(\bigcup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k)$. Hence, by the continuity of μ,

$$\mu\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0.$$

Observe that $\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]$ is the set of all points in X that belong to an infinite number of the E_k's. □

Definition *Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is called* **finite** *provided $\mu(X) < \infty$. It is called* **σ-finite** *provided X is the union of a countable collection of measurable sets, each of which has finite measure. A measurable set E is said to be of* **finite measure** *provided $\mu(E) < \infty$, and said to be* **σ-finite** *provided E is the union of a countable collection of measurable sets, each of which has finite measure.*

Regarding the criterion for σ-finiteness, the countable cover by sets of finite measure may be taken to be disjoint. Indeed, if $\{X_k\}_{k=1}^{\infty}$ is such a cover replace, for $k \geq 2$, each X_k by $X_k \sim \bigcup_{j=1}^{k-1} X_j$ to obtain a disjoint cover by sets of finite measure. Lebesgue measure on $[0, 1]$ is an example of a finite measure, while Lebesgue measure on $(-\infty, \infty)$ is an example of a σ-finite measure. The counting measure on an uncountable set is not σ-finite.

Many familiar properties of Lebesgue measure on the real line and Lebesgue integration for functions of a single real variable hold for arbitrary σ-finite measures, and many treatments of abstract measure theory limit themselves to σ-finite measures. However, many parts of the general theory do not require the assumption of σ-finiteness, and it seems undesirable to have a development that is unnecessarily restrictive.

Definition *A measure space $(X, \mathcal{M}, \mu)$ is said to be* **complete** *provided $\mathcal{M}$ contains all subsets of sets of measure zero, that is, if E belongs to $\mathcal{M}$ and $\mu(E) = 0$, then every subset of E also belongs to $\mathcal{M}$.*

We proved that Lebesgue measure on the real line is complete. Moreover, we also showed that the Cantor set, a Borel set of Lebesgue measure zero, contains a subset that is not Borel; see page 52. Thus Lebesgue measure on the real line, when restricted to the σ-algebra of Borel sets, is not complete. The following proposition, whose proof is left to the reader (Problem 9), tells us that each measure space can be completed. The measure space $(X, \mathcal{M}_0, \mu_0)$ described in this proposition is called the **completion** of $(X, \mathcal{M}, \mu)$.

Proposition 3 *Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets E of X of the form $E = A \cup B$ where $B \in \mathcal{M}$ and $A \subseteq C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$.*

For such a set E define $\mu_0(E) = \mu(B)$. Then $\mathcal{M}_0$ is a σ-algebra that contains $\mathcal{M}$, μ_0 is a measure that extends μ, and $(X, \mathcal{M}_0, \mu_0)$ is a complete measure space.

MEASURABLE FUNCTIONS

For a measurable space $(X, \mathcal{M})$, the concept of a measurable function on X is identical with that for functions of a real variable with respect to Lebesgue measure. The proof of the following proposition is exactly the same as the proof for Lebesgue measure on the real line;

Proposition 1 *Let $(X, \mathcal{M})$ be a measurable space and f an extended real-valued function defined on X. Then the following statements are equivalent:*

(i) For each real number c, the set $\{x \in X \mid f(x) < c\}$ is measurable.
(ii) For each real number c, the set $\{x \in X \mid f(x) \leq c\}$ is measurable.
(iii) For each real number c, the set $\{x \in X \mid f(x) > c\}$ is measurable.
(iv) For each real number c, the set $\{x \in X \mid f(x) \geq c\}$ is measurable.

Each of these properties implies that for each extended real number c,

$$\text{the set } \{x \in X \mid f(x) = c\} \text{ is measurable.}$$

Definition *Let $(X, \mathcal{M})$ be a measurable space. An extended real-valued function f on X is said to be* **measurable** *(or measurable with respect to $\mathcal{M}$) provided one, and hence all, of the four statements of Proposition 1 holds.*

For a set X and the σ-algebra $\mathcal{M} = 2^X$ of all subsets of X, every extended real-valued function on X is measurable with respect to $\mathcal{M}$. At the opposite extreme, consider the σ-algebra $\mathcal{M} = \{X, \emptyset\}$, with respect to which the only measurable functions are those that are constant. If X is a topological space and $\mathcal{M}$ is a σ-algebra of subsets of X that contains the topology on X, then every continuous real-valued function on X is measurable with respect to $\mathcal{M}$. In Part 1 we studied functions of a real variable that are measurable with respect to the σ-algebra of Lebesgue measurable sets.

Since a bounded, open interval of real numbers is the intersection of two unbounded, open intervals and each open set of real numbers is the countable union of a collection of open intervals, we have the following characterizaton of real-valued measurable functions (see also Problem 1).

Proposition 2 *Let $(X, \mathcal{M})$ be a measurable space and f a real-valued function on X. Then f is measurable if and only if for each open set $\mathcal{O}$ of real numbers, $f^{-1}(\mathcal{O})$ is measurable.*

For a measurable space $(X, \mathcal{M})$ and measurable subset E of X, we call an extended real-valued function f that is defined on E measurable provided it is measurable with respect to the measurable space $(E, \mathcal{M}_E)$, where $\mathcal{M}_E$ is the collection of sets in $\mathcal{M}$ that are contained in E. The restriction of a measurable function on X to a measurable set is measurable. Moreover, for an extended real-valued function f of X and measurable subset E of X, the restriction of f to both E and $X \sim E$ are measurable if and only if f is measurable on X.

Proposition 3 *Let $(X, \mathcal{M}, \mu)$ be a complete measure space and X_0 a measurable subset of X for which $\mu(X \sim X_0) = 0$. Then an extended real-valued function f on X is measurable if and only if its restriction to X_0 is measurable. In particular, if g and h are extended real-valued functions on X for which $g = h$ a.e. on X, then g is measurable if and only if h is measurable.*

Proof Define f_0 to be the restriction of f to X_0. Let c be a real number and $E = (c, \infty)$. If f is measurable, then $f^{-1}(E)$ is measurable and hence so is $f^{-1}(E) \cap X_0 = f_0^{-1}(E)$. Therefore f_0 is measurable. Now assume f_0 is measurable. Then

$$f^{-1}(E) = f_0^{-1}(E) \cup A,$$

where A is a subset of $X \sim X_0$. Since $(X, \mathcal{M}, \mu)$ is complete, A is measurable and hence so is $f^{-1}(E)$. Therefore the function f is measurable. The second assertion follows from the first. □

This proposition is false if the measure space $(X, \mathcal{M}, \mu)$ fails to be complete (see Problem 2). The proof of the following theorem is exactly the same as the proof in the case of Lebesgue measure on the real line; see page 56.

Theorem 4 *Let $(X, \mathcal{M})$ be a measurable space and f and g measurable real-valued functions on X.*

(Linearity) For any real numbers α and β,

$$\alpha f + \beta g \text{ is measurable.}$$

(Products)

$$f \cdot g \text{ is measurable.}$$

(Maximum and Minimum) The functions $\max\{f, g\}$ and $\min\{f, g\}$ are measurable.

Remark *The sum of two extended real-valued functions is not defined at points where the functions take infinite values of opposite sign. Nevertheless, in the study of linear spaces of integrable functions it is necessary to consider linear combinations of extended real-valued measurable functions. For measurable functions that are finite almost everywhere, we proceed as we did for functions of a real variable. Indeed, for a measure space $(X, \mathcal{M}, \mu)$, consider two extended real-valued measurable functions f and g on X that are finite a.e. on X. Define X_0 to be the set of points in X at which both f and g are finite. Since f and g are measurable functions, X_0 is a measurable set. Moreover, $\mu(X \sim X_0) = 0$. For real numbers α and β, the linear combination $\alpha f + \beta g$ is a properly defined real-valued function on X_0. We say that $\alpha f + \beta g$ is measurable on X provided its restriction to X_0 is measurable with respect to the measurable space $(X_0, \mathcal{M}_0)$, where $\mathcal{M}_0$ is the σ-algebra consisting of all sets in $\mathcal{M}$ that are contained in X_0. If $(X, \mathcal{M}, \mu)$ is complete, Proposition 3 tells us that this definition is equivalent to the assertion that one, and hence any, extension of $\alpha f + \beta g$ on X_0 to an extended real-valued function on all of X is a measurable function on X. We regard the function $\alpha f + \beta g$ on X as being any measurable extended real-valued function on X that agrees with $\alpha f + \beta g$ on X_0. Similar considerations apply to the product of f and g and their maximum and minimum. With this convention, the preceding theorem holds if the extended real-valued measurable functions f and g are finite a.e. on X.*

We have already seen that the composition of Lebesgue measurable functions of a single real variable need not be measurable (see the example on page 58). However, the following composition criterion is very useful. It tells us, for instance, that if f is a measurable function and $0 < p < \infty$, then $|f|^p$ also is measurable.

Proposition 5 *Let $(X, \mathcal{M})$ be a measurable space, f a measurable real-valued function on X, and $\varphi: \mathbf{R} \to \mathbf{R}$ continuous. Then the composition $\varphi \circ f: X \to \mathbf{R}$ also is measurable.*

Proof Let $\mathcal{O}$ be an open set of real numbers. Since φ is continuous, $\varphi^{-1}(\mathcal{O})$ is open. Hence, by Proposition 2, $f^{-1}(\varphi^{-1}(\mathcal{O})) = (\varphi \circ f)^{-1}(\mathcal{O})$ is a measurable set and so $\varphi \circ f$ is a measurable function. □

A fundamentally important property of measurable functions is that, just as in the special case of Lebesgue measurable functions of a real variable, measurability of functions is preserved under the formation of pointwise limits.

Theorem 6 *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X. If either the measure space $(X, \mathcal{M}, \mu)$ is complete or the convergence is pointwise on all of X, then f is measurable.*

Proof In view of Proposition 3, possibly by excising from X a set of measure 0, we suppose the sequence converges pointwise on all of X. Fix a real number c. We must show that the set $\{x \in X \mid f(x) < c\}$ is measurable. Observe that for a point $x \in X$, since $\lim_{n\to\infty} f_n(x) = f(x)$, $f(x) < c$ if and only if there are natural numbers n and k such that for all $j \geq k$, $f_j(x) < c - 1/n$. But for any natural numbers n and j, since the function f_j is measurable, the set $\{x \in X \mid f_j(x) < c - 1/n\}$ is measurable. Since $\mathcal{M}$ is closed with respect to the formation of countable intersections, for any k,

$$\bigcap_{j=k}^{\infty} \{x \in X \mid f_j(x) < c - 1/n\}$$

also is measurable. Consequently,

$$\{x \in X \mid f(x) < c\} = \bigcup_{1 \leq k, n < \infty} \left[\bigcap_{j=k}^{\infty} \{x \in X \mid f_j(x) < c - 1/n\} \right]$$

is measurable since $\mathcal{M}$ is closed with respect to the formation of countable unions. □

This theorem is false if the measure space fails to be complete (see Problem 3).

Corollary 7 *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X. Then the following functions are measurable:*

$$\sup\{f_n\},\ \inf\{f_n\},\ \limsup\{f_n\},\ \liminf\{f_n\}.$$

Definition *Let $(X, \mathcal{M})$ be a measurable space. For a measurable set E, its* **characteristic function,** χ_E, *is the function on X that takes the value 1 on E and 0 on $X \sim E$. A real-valued function ψ on X is said to be* **simple** *provided there is a finite collection $\{E_k\}_{k=1}^n$ of measurable sets and a corresponding set of real numbers $\{c_k\}_{k=1}^n$ for which*

$$\psi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } X.$$

Observe that a simple function on X is a measurable real-valued function on X that takes a finite number of real values.

The Simple Approximation Lemma *Let $(X, \mathcal{M})$ be a measurable space and f a measurable function on X that is bounded on X, that is, there is an $M \geq 0$ for which $|f| \leq M$ on X. Then for each $\epsilon > 0$, there are simple functions φ_ϵ and ψ_ϵ defined on X that have the following approximation properties:*

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon \text{ on } X.$$

Proof Let $[c, d)$ be a bounded interval that contains the image of X, $f(X)$, and

$$c = y_0 < y_1 < \ldots < y_{n-1} < y_n = d$$

a partition of the closed, bounded interval $[c, d]$ such that $y_k - y_{k-1} < \epsilon$ for $1 \leq k \leq n$. Define

$$I_k = [y_{k-1}, y_k) \text{ and } X_k = f^{-1}(I_k) \text{ for } 1 \leq k \leq n.$$

Since each I_k is an interval and the function f is measurable, each set X_k is measurable. Define the simple functions φ_ϵ and ψ_ϵ on X by

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{X_k} \text{ and } \psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{X_k}.$$

Let x belong to X. Since $f(X) \subseteq [c, d)$, there is a unique $k, 1 \leq k \leq n$, for which $y_{k-1} \leq f(x) < y_k$ and therefore

$$\varphi_\epsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\epsilon(x).$$

But $y_k - y_{k-1} < \epsilon$, and therefore φ_ϵ and ψ_ϵ have the required approximation properties. □

The Simple Approximation Theorem *Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. Then there is a sequence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and has the property that*

$$|\psi_n| \leq |f| \text{ on } X \text{ for all } n.$$

(i) *If X is σ-finite, then we may choose the sequence $\{\psi_n\}$ so that each ψ_n vanishes outside a set of finite measure.*

(ii) *If f is nonnegative, we may choose the sequence $\{\psi_n\}$ to be increasing and each $\psi_n \geq 0$ on X.*

Proof Fix a natural number n. Define $E_n = \{x \in X \mid |f(x)| \leq n\}$. Since $|f|$ is a measurable function, E_n is a measurable set and the restriction of f to E_n is a bounded measurable function. By the Simple Approximation Lemma, applied to the restriction of f to E_n and with the choice of $\epsilon = 1/n$, we may select simple functions h_n and g_n on E_n, which have the following approximation properties:

$$h_n \leq f \leq g_n \text{ and } 0 \leq g_n - h_n < 1/n \text{ on } E_n.$$

For x in E_n, define $\psi_n(x) = 0$ if $f(x) = 0$, $\psi_n(x) = \max\{h_n(x), 0\}$ if $f(x) > 0$ and $\psi_n(x) = \min\{g_n(x), 0\}$ if $f(x) < 0$. Extend ψ_n to all of X by setting $\psi_n(x) = n$ if $f(x) > n$

and $\psi_n(x) = -n$ if $f(x) < -n$. This defines a sequence $\{\psi_n\}$ of simple functions on X. It follows, as it did in the proof for the case of Lebesgue measurable functions of a real variable (see page 62), that, for each n, $|\psi_n| \leq |f|$ on X and the sequence $\{\psi_n\}$ converges pointwise on X to f.

If X is σ-finite, express X as the union of a countable ascending collection $\{X_n\}_{n=1}^{\infty}$ of measurable subsets, each of which has finite measure. Replace each ψ_n by $\psi_n \cdot \chi_{X_n}$ and (i) is verified. If f is nonnegative, replace each ψ_n by $\max_{1 \leq i \leq n} |\psi_i|$ and (ii) is verified. □

The proof of the following general form of Egoroff's Theorem follows from the continuity and countable additivity of measure, as did the proof in the case of Lebesgue measurable functions of a real variable; see page 65.

Egoroff's Theorem *Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions on X that converges pointwise a.e. on X to a function f that is finite a.e. on X. Then for each $\epsilon > 0$, there is a measurable subset X_ϵ of X for which*

$$\{f_n\} \to f \text{ uniformly on } X_\epsilon \text{ and } \mu(X \sim X_\epsilon) < \epsilon.$$

INTEGRATION OF NONNEGATIVE MEASURABLE FUNCTIONS

Definition *Let $(X, \mathcal{M}, \mu)$ be a measure space and ψ a nonnegative simple function on X. Define the integral of ψ over X, $\int_X \psi\, d\mu$, as follows: if $\psi = 0$ on X, define $\int_E \psi\, d\mu = 0$. Otherwise, let $c_1, c_2, \ldots, c_n$ be the positive values taken by ψ on X and, for $1 \leq k \leq n$, define $E_k = \{x \in X \mid \psi(x) = c_k\}$. Define*

$$\int_X \psi\, d\mu = \sum_{k=1}^{n} c_k \cdot \mu(E_k), \tag{1}$$

using the convention that the right-hand side is ∞ if, for some k, $\mu(E_k) = \infty$. For a measurable subset E of X, the integral of ψ over E with respect to μ is defined to be $\int_X \psi \cdot \chi_E\, d\mu$ and denoted by $\int_E f\, d\mu$.

Proposition 8 *Let $(X, \mathcal{M}, \mu)$ be a measure space and φ and ψ nonnegative simple function on X. If α and β are positive real numbers, then*

$$\int_X [\alpha \cdot \psi + \beta \cdot \varphi]\, d\mu = \alpha \cdot \int_X \psi\, d\mu + \beta \cdot \int_X \psi\, d\mu. \tag{2}$$

If A and B are disjoint measurable subsets of X, then

$$\int_{A \cup B} \psi\, d\mu = \int_A \psi\, d\mu + \int_B \psi\, d\mu. \tag{3}$$

In particular, if $X_0 \subseteq X$ is measurable and $\mu(X \sim X_0) = 0$, then

$$\int_X \psi\, d\mu = \int_{X_0} \psi\, d\mu. \tag{4}$$

Furthermore, if $\psi \leq \varphi$ a.e. on X, then

$$\int_X \psi\, d\mu \leq \int_X \varphi\, d\mu. \tag{5}$$

Proof If either ψ or φ is positive on a set of infinite measure, then the linear combination $\alpha \cdot \psi + \beta \cdot \varphi$ has the same property and therefore each side of (2) is infinite. We therefore assume both ψ and φ vanish outside a set of finite measure and hence so does the linear combination $\alpha \cdot \psi + \beta \cdot \varphi$. In this case the proof of (2) is exactly the same as the proof for Lebesgue integration of functions of a real variable (see the proofs of Lemma 1 and Proposition 2 on page 72). The additivity over domains formula follows from (2) and the observation that, since A and B are disjoint,

$$\psi \cdot \chi_{A \cup B} = \psi \cdot \chi_A + \psi \cdot \chi_B \text{ on } X.$$

To verify (5), first observe that since the integral of a simple function over a set of measure zero is zero, by (3), we may assume $\psi \leq \varphi$ on X. Observe that since φ and ψ take only a finite number of real values, we may express X as $\bigcup_{k=1}^{n} X_k$, a disjoint union of measurable sets for which both φ and ψ are constant on each X_k. Therefore

$$\psi = \sum_{k=1}^{n} a_k \cdot \chi_{X_k} \text{ and } \varphi = \sum_{k=1}^{n} b_k \cdot \chi_{X_k} \text{ where } a_k \leq b_k \text{ for } 1 \leq k \leq n. \tag{6}$$

But (2) extends to finite linear combinations of nonnegative simple functions and therefore (5) follows from (6). □

Definition *Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative extended real-valued measurable function on X. The* **integral** *of f over X with respect to μ, which is denoted by $\int_X f \, d\mu$, is defined to be the supremum of the integrals $\int_X \varphi \, d\mu$ as φ ranges over all simple functions φ for which $0 \leq \varphi \leq f$ on X. For a measurable subset E of X, the integral of f over E with respect to μ is defined to be $\int_X f \cdot \chi_E \, d\mu$ and denoted by $\int_E f \, d\mu$.*

We leave it as an exercise to verify the following three properties of the integral of nonnegative measurable functions. Let $(X, \mathcal{M}, \mu)$ be a measure space, g and h nonnegative measurable functions on X, X_0 a measurable subset of X, and α a positive real number. Then

$$\int_X \alpha \cdot g\, d\mu = \alpha \cdot \int_X g\, d\mu; \tag{7}$$

$$\text{if } g \leq h \text{ a.e. on } X, \text{ then } \int_X g\, d\mu \leq \int_X h\, d\mu; \tag{8}$$

$$\int_X g\, d\mu = \int_{X_0} g\, d\mu \text{ if } \mu(X \sim X_0) = 0. \tag{9}$$

Chebychev's Inequality *Let $(X, \mathcal{M}, \mu)$ be a measure space, f a nonnegative measurable function on X, and λ a positive real number. Then*

$$\mu\{x \in X \mid f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_X f\, d\mu. \tag{10}$$

Proof Define $X_\lambda = \{x \in X \mid f(x) \geq \lambda\}$ and $\varphi = \lambda \cdot \chi_{X_\lambda}$. Observe that $0 \leq \varphi \leq f$ on X and φ is a simple function. Therefore, by definition,

$$\lambda \cdot \mu(X_\lambda) = \int_X \varphi\, d\mu \leq \int_X f\, d\mu.$$

Divide this inequality by λ to obtain Chebychev's Inequality.

Proposition 9 *Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function on X for which $\int_X f\, d\mu < \infty$. Then f is finite a.e. on X and $\{x \in X \mid f(x) > 0\}$ is σ-finite.*

Proof Define $X_\infty = \{x \in X \mid f(x) = \infty\}$ and consider the simple function $\psi = \chi_{X_\infty}$. By definition, $\int_X \psi\, d\mu = \mu(X_\infty)$ and since $0 \leq \psi \leq f$ on X, $\mu(X_\infty) \leq \int_X f\, d\mu < \infty$. Therefore f is finite a.e. on X. Let n be a natural number. Define $X_n = \{x \in X \mid f(x) \geq 1/n\}$. By Chebychev's Inequality,

$$\mu(X_n) \le n \cdot \int_X f \, d\mu < \infty.$$

Moreover,

$$\{x \in X \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$$

Therefore the set $\{x \in X \mid f(x) > 0\}$ is σ-finite.

Fatou's Lemma *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of nonnegative measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X. Assume f is measurable. Then*

$$\int_X f \, d\mu \le \liminf \int_X f_n \, d\mu. \tag{11}$$

Proof Let X_0 be a measurable subset of X for which $\mu(X \sim X_0) = 0$ and $\{f_n\} \to f$ pointwise on X_0. According to (9), each side of (11) remains unchanged if X is replaced by X_0. We therefore assume $X = X_0$. By the definition of $\int_X f \, d\mu$ as a supremum, to verify (11) it is necessary and sufficient to show that if φ is any simple function for which $0 \le \varphi \le f$ on X, then

$$\int_X \varphi \, d\mu \le \liminf \int_X f_n \, d\mu. \tag{12}$$

Let φ be such a function. This inequality clearly holds if $\int_X \varphi \, d\mu = 0$. Assume $\int_X \varphi \, d\mu > 0$.

Case 1: $\int_X \varphi \, d\mu = \infty$. Then there is a measurable set $X_\infty \subseteq X$ and $a>0$ for which $\mu(X_\infty) = \infty$ and $\varphi = a$ on X_∞. For each natural number n, define

$$A_n = \left\{x \in X \mid f_k(x) \ge a/2 \text{ for all } k \ge n\right\}.$$

Then $\{A_n\}_{n=1}^{\infty}$ is an ascending sequence of measurable subsets of X. Since $X_\infty \subseteq \bigcup_{n=1}^{\infty} A_n$, by the continuity and monotonicity of measure,

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \ge \mu(X_\infty) = \infty.$$

However, by Chebychev's Inequality, for each natural number n,

$$\mu(A_n) \le \frac{2}{a} \int_{A_n} f_n \, d\mu \le \frac{2}{a} \int_X f_n \, d\mu.$$

Therefore $\lim_{n\to\infty} \int_X f_n \, d\mu = \infty = \int_X \varphi \, d\mu$.

Case 2: $0 < \int_X \varphi \, d\mu < \infty$. By excising from X the set where φ takes the value 0, the left-hand side of (12) remains unchanged and the right-hand side does not increase. Thus we may suppose that $\varphi > 0$ on X and therefore, since φ is simple and $\int_X \varphi \, d\mu < \infty$, $\mu(X) < \infty$. To verify (12), choose $\epsilon > 0$. For each natural number n, define

$$X_n = \{x \in X \mid f_k(x) > (1-\epsilon)\varphi(x) \text{ for all } k \geq n\}.$$

Then $\{X_n\}$ is an ascending sequence of measurable subsets of X whose union equals X. Therefore $\{X \sim X_n\}$ is a descending sequence of measurable subsets of X whose intersection is empty. Since $\mu(X) < \infty$, by the continuity of measure, $\lim_{n\to\infty} \mu(X \sim X_n) = 0$. Choose an index N such that $\mu(X \sim X_n) < \epsilon$ for all $n \geq N$. Define $M > 0$ to be the maximum of the finite number of values taken by φ on X. We infer from the monotonicity and positive homogeneity properties, (8) and (7), of integration for nonnegative measurable functions, the additivity over domains and monotonicity properties, (3) and (5), of integration for nonnegative simple function and the finiteness of $\int_X \varphi \, d\mu$ that, for $n \geq N$,

$$\begin{aligned}
\int_X f_n \, d\mu \geq \int_{X_n} f_n \, d\mu &\geq (1-\epsilon) \int_{X_n} \varphi \, d\mu \\
&= (1-\epsilon)\int_X \varphi \, d\mu - (1-\epsilon)\int_{X\sim X_n} \varphi \, d\mu \\
&\geq (1-\epsilon)\int_X \varphi \, d\mu - \int_{X\sim X_n} \varphi \, d\mu \\
&\geq (1-\epsilon)\int_X \varphi \, d\mu - \epsilon \cdot M \\
&= \int_X \varphi \, d\mu - \epsilon\left[\int_X \varphi \, d\mu + M\right].
\end{aligned}$$

Hence

$$\liminf \int_X f_n \, d\mu \geq \int_X \varphi \, d\mu - \epsilon\left[\int_X \varphi \, d\mu + M\right].$$

This inequality holds for all $\epsilon > 0$ and hence, since $\int_X \varphi \, d\mu + M$ is finite, it also holds for $\epsilon = 0$. □

In Fatou's Lemma, the limit function f is assumed to be measurable. In case $\{f_n\}$ converges pointwise to f on all of X or the measure space is complete, Theorem 6 tells us that f is measurable.

We have already seen in the case of Lebesgue integration on the real line that the inequality (11) may be strict. For instance, it is strict for Lebesgue measure on $X = [0, 1]$ and $f_n = n \cdot \chi_{[0, 1/n]}$ for all n. It is also strict for Lebesgue measure on $X = \mathbf{R}$ and $f_n = \chi_{[n, n+1]}$ for all n. However, for a sequence of measurable functions $\{f_n\}$ that converges pointwise on X to f, in the case of Lebesgue integration for functions of a real variable, we established a number of criteria for justifying **passage of the limit under the integral sign**, that is,

$$\lim_{n\to\infty} \left[\int_X f_n \, d\mu \right] = \int_X \left[\lim_{n\to\infty} f_n \right] d\mu.$$

Each of these criteria has a correspondent in the general theory of integration. We first establish a general version of the Monotone Convergence Theorem.

The Monotone Convergence Theorem *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ an increasing sequence of nonnegative measurable functions on X. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. Then*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof Theorem 6 tells us that f is measurable. According to Fatou's Lemma,

$$\int_X f \, d\mu \leq \liminf \int_X f_n \, d\mu.$$

However, for each n, $f_n \leq f$ on X, and so, by (8), $\int_X f_n \, d\mu \leq \int_X f \, d\mu$. Thus

$$\limsup \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

Hence

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$ □

POSSIBLE QUESTIONS

1. If f be an extended real valued function defined on X. Then prove that the following statements are equivalent.(i) $\{x: f(x) > \alpha\} \in \mathcal{B}$ for each α
   (ii) $\{x: f(x) < \alpha\} \in \mathcal{B}$ for each α
   (iii) $\{x: f(x) \leq \alpha\} \in \mathcal{B}$ for each α
   (iv) $\{x: f(x) \geq \alpha\} \in \mathcal{B}$ for each α
2. If c is a constant and the function f and g are measurable then so are the functions f + c, cf, f . g & f v g .Moreover if $\{ f_n \}$ is a sequence of measurable functions then prove that $sup\ f_n$, $inf\ f_n$, $\overline{lim}\ f_n$ and $\underline{lim}\ f_n$ are all measurable.
3. (i) If μ is a complete measure and f is a measurable function then f = g a.e $\Longrightarrow$ g is measurable.
   ii)If $A \in \mathcal{B}$, $B \in \mathcal{B}$ and $A \subset B$ then $\mu(A) \leq \mu(B)$.
4. If $\{ f_n \}$ be a sequence of nonnegative measurable function that converges a.e on the set E to a function f then prove that $\int f \leq \underline{lim}\ f_n$.
5. If $\{ f_n \}$ be a sequence of nonnegative measurable function which converges a.e to a function f and suppose that $f_n \leq f, \forall\ n$ then prove that $\int f = \lim n \int f_n$.
6. If g be integrable over E and suppose that $\{ f_n \}$is a sequence of measurable functions such that on E $|f_n(x)| \leq g(x)$ and such that a.e on E, $f_n(x) \longrightarrow f(x)$ then prove that $\int_E f = \lim \int_E f_n$.
7. Prove that if $E_i \in B$,$\mu E_i < \infty$ and $E_i \supset E_{i+1}$ then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu E_n$
8. If f and g are non negative measurable functions and a and b non negative constants then prove that $\int af + bg = a\int f + b\int g$
9. Prove that if (X, B) be a measurable space, $<\mu_n>$ a sequence of measures that converge set wise to a measure μ and $<f_n>$ a sequence of non negative measurable functions that converge pointwise to the function then $\int f\, d\mu \leq lim \int f_n\, d\mu_n$
10. If f and g are integrable function & E is measurable then prove that
    (a) $\int_E c_1 f + c_2 g = c_1 \int_E f + c_2 \int_E g$. (b) If $|h| \leq |f|$ then h is measurable .
    (c) If $f \geq g$ a.e then $\int_E f \geq \int_E g$.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Lebesgue measure on [0,1] is an example of a ________	finite measure	countable	uncountable	infinite measure	finite measure
The counting measure on an _______ is a measure which is not.	uncountable set	countable set	complement set	disjoint set	uncountable set
A measure μ is said to be ______ if each measurable set of infinite measure contains measurable sets of arbitrary large finite	semi finite	semi circle	function	completion	semi finite
A measure space is said to be _______ if B contains all the subsets of sets of measure zero.	Addition	Subtraction	complete	sequence	complete
The measure space is called the ______	completion	equivalent	boreset	subset	completion
A measure μ is said to be absolutely continous with respect to the measure if ______	A= 0	A = 0	A= 1	A = 1	A= 0
A measure μ is said to be _______ with respect to the measure if A= 0.	Absolutely convergent	convergent	absolutely continuous	continuous	absolutely continuous
The set function is an _______	Inner measure	outer measure	measurable	integrable	outer measure

The Hahn decomposition is not ______	composite	unique	countable	convergent	unique
The ____ is not unique	Hahn decomposition	Jordan decomposition	Lebesgue decomposition	Euler decomposition	Hahn decomposition

UNIT – V

Signed measures-The Radon-Nikodym theorem-the L^P spaces.

SIGNED MEASURES:

Observe that if μ_1 and μ_2 are two measures defined on the same measurable space $(X, \mathcal{M})$, then, for positive numbers α and β, we may define a new measure μ_3 on X by setting

$$\mu_3(E) = \alpha \cdot \mu_1(E) + \beta \cdot \mu_2(E) \text{ for all } E \text{ in } \mathcal{M}.$$

It turns out to be important to consider set functions that are linear combinations of measures but with coefficients that may be negative. What happens if we try to define a set function ν on $\mathcal{M}$ by

$$\nu(E) = \mu_1(E) - \mu_2(E) \text{ for all } E \text{ in } \mathcal{M}?$$

The first thing that may occur is that ν is not always nonnegative. Moreover, $\nu(E)$ is not even defined for $E \in \mathcal{M}$ such that $\mu_1(E) = \mu_2(E) = \infty$. With these considerations in mind we make the following definition.

Definition *By a* **signed measure** *ν on the measurable space $(X, \mathcal{M})$ we mean an extended real-valued set function ν: $\mathcal{M} \to [-\infty, \infty]$ that possesses the following properties:*

(i) *ν assumes at most one of the values $+\infty, -\infty$.*

(ii) $\nu(\emptyset) = 0$.

(iii) *For any countable collection $\{E_k\}_{k=1}^{\infty}$ of disjoint measurable sets,*

$$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k),$$

where the series $\sum_{k=1}^{\infty} \nu(E_k)$ converges absolutely if $\nu\left(\bigcup_{k=1}^{\infty} E_k\right)$ is finite.

A measure is a special case of a signed measure. It is not difficult to see that the difference of two measures, one of which is finite, is a signed measure. In fact, the forthcoming Jordan Decomposition Theorem will tell us that every signed measure is the difference of two such measures.

Let ν be a signed measure. We say that a set A is **positive** (with respect to ν) provided A is measurable and for every measurable subset E of A we have $\nu(E) \geq 0$. The restriction of ν to the measurable subsets of a positive set is a measure. Similarly, a set B is called **negative** (with respect to ν) provided it is measurable and every measurable subset of B has nonpositive ν measure. The restriction of $-\nu$ to the measurable subsets of a negative set also is a measure. A measurable set is called **null** with respect to ν provided every measurable subset of it has ν measure zero. The reader should carefully note the distinction between a null set and a set of measure zero: While every null set must have measure zero, a set of measure zero may well be a union of two sets whose measures are not zero but are negatives of each other. By the monotonicity property of measures, a set is null with respect to a measure if and only if it has measure zero. Since a signed measure ν does not take the values ∞ and $-\infty$, for A and B measurable sets,

$$\text{if } A \subseteq B \text{ and } |\nu(B)| < \infty, \text{ then } |\nu(A)| < \infty. \tag{3}$$

Proposition 4 *Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then every measurable subset of a positive set is itself positive and the union of a countable collection of positive sets is positive.*

Proof The first statement is trivially true by the definition of a positive set. To prove the second statement, let A be the union of a countable collection $\{A_k\}_{k=1}^{\infty}$ of positive sets. Let E be a measurable subset of A. Define $E_1 = E \cap A_1$. For $k \geq 2$, define

$$E_k = [E \cap A_k] \sim [A_1 \cup \ldots \cup A_{k-1}].$$

Then each E_k is a measurable subset of the positive set A_k and therefore $\nu(E_k) \geq 0$. Since E is the union of the countable disjoint collection $\{E_k\}_{k=1}^{\infty}$,

$$\nu(E) = \sum_{k=1}^{\infty} \nu(E_k) \geq 0.$$

Thus A is a positive set. □

Hahn's Lemma *Let ν be a signed measure on the measurable space $(X, \mathcal{M})$ and E a measurable set for which $0 < \nu(E) < \infty$. Then there is a measurable subset A of E that is positive and of positive measure.*

Proof If E itself is a positive set, then the proof is complete. Otherwise, E contains sets of negative measure. Let m_1 be the smallest natural number for which there is a measurable set of measure less than $-1/m_1$. Choose a measurable set $E_1 \subseteq E$ with $\nu(E_1) < -1/m_1$. Let n be a natural number for which natural numbers $m_1, \ldots, m_n$ and measurable sets $E_1, \ldots, E_n$ have been chosen such that, for $1 \le k \le n$, m_k is the smallest natural number for which there is a measurable subset of $E \sim \bigcup_{j=1}^{k-1} E_j$ of measure less than $-1/m_k$ and E_k is a subset of $[E \sim \bigcup_{j=1}^{k-1} E_j]$ for which $\nu(E_k) < -1/m_k$.

If this selection process terminates, then the proof is complete. Otherwise, define

$$A = E \sim \bigcup_{k=1}^{\infty} E_k, \text{ so that } E = A \cup \left[\bigcup_{k=1}^{\infty} E_k\right] \text{ is a disjoint decomposition of } E.$$

Since $\bigcup_{k=1}^{\infty} E_k$ is a measurable subset of E and $|\nu(E)| < \infty$, by (3) and the countable additivity of ν,

$$-\infty < \nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) \le \sum_{k=1}^{\infty} -1/m_k.$$

Thus $\lim_{k\to\infty} m_k = \infty$. We claim that A is a positive set. Indeed, if B is a measurable subset of A, then, for each k,

$$B \subseteq A \subseteq E \sim \left[\bigcup_{j=1}^{k-1} E_j\right],$$

and so, by the minimal choice of m_k, $\nu(B) \ge -1/(m_k - 1)$. Since $\lim_{k\to\infty} m_k = \infty$, we have $\nu(B) \ge 0$. Thus A is a positive set. It remains only to show that $\nu(A) > 0$. But this follows from the finite additivity of ν since $\nu(E) > 0$ and $\nu(E \sim A) = \nu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \nu(E_k) < 0$. □

The Hahn Decomposition Theorem *Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there is a positive set A for ν and a negative set B for ν for which*

$$X = A \cup B \text{ and } A \cap B = \emptyset.$$

Proof Without loss of generality we assume $+\infty$ is the infinite value omitted by ν. Let $\mathcal{P}$ be the collection of positive subsets of X and define $\lambda = \sup\{\nu(E) \mid E \in \mathcal{P}\}$. Then $\lambda \ge 0$ since $\mathcal{P}$ contains the empty set. Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of positive sets for which $\lambda = \lim_{k\to\infty} \nu(A_k)$. Define $A = \bigcup_{k=1}^{\infty} A_k$. By Proposition 4, the set A is itself a positive set, and so $\lambda \ge \nu(A)$. On the other hand, for each k, $A \sim A_k \subseteq A$ and so $\nu(A \sim A_k) \ge 0$. Thus

$$\nu(A) = \nu(A_k) + \nu(A \sim A_k) \ge \nu(A_k).$$

Hence $\nu(A) \ge \lambda$. Therefore $\nu(A) = \lambda$, and $\lambda < \infty$ since ν does not take the value ∞.

Let $B = X \sim A$. We argue by contradiction to show that B is negative. Assume B is not negative. Then there is a subset E of B with positive measure and therefore, by Hahn's Lemma, a subset E_0 of B that is both positive and of positive measure. Then $A \cup E_0$ is a positive set and

$$\nu(A \cup E_0) = \nu(A) + \nu(E_0) > \lambda,$$

a contradiction to the choice of λ. □

A decomposition of X into the union of two disjoint sets A and B for which A is positive for ν and B negative is called a **Hahn decomposition** for ν. The preceding theorem tells us of the existence of a Hahn decomposition for each signed measure. Such a decomposition may not be unique. Indeed, if $\{A, B\}$ is a Hahn decomposition for ν, then by excising from A a null set E and grafting this subset onto B we obtain another Hahn decomposition $\{A \sim E, B \cup E\}$.

If $\{A, B\}$ is a Hahn decomposition for ν, then we define two measures ν^+ and ν^- with $\nu = \nu^+ - \nu^-$ by setting

$$\nu^+(E) = \nu(E \cap A) \text{ and } \nu^-(E) = -\nu(E \cap B).$$

Two measures ν_1 and ν_2 on $(X, \mathcal{M})$ are said to be **mutually singular** (in symbols $\nu_1 \perp \nu_2$) if there are disjoint measurable sets A and B with $X = A \cup B$ for which $\nu_1(A) = \nu_2(B) = 0$. The measures ν^+ and ν^- defined above are mutually singular. We have thus established the existence part of the following proposition. The uniqueness part is left to the reader (see Problem 13).

The Jordan Decomposition Theorem *Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.*

The decomposition of a signed measure ν given by this theorem is called the **Jordan decomposition** of ν. The measures ν^+ and ν^- are called the positive and negative parts (or variations) of ν. Since ν assumes at most one of the values $+\infty$ and $-\infty$, either ν^+ or ν^- must be finite. If they are both finite, we call ν a finite signed measure. The measure $|\nu|$ is defined on $\mathcal{M}$ by

$$|\nu|(E) = \nu^+(E) + \nu^-(E) \text{ for all } E \in \mathcal{M}.$$

We leave it as an exercise to show that

$$|\nu|(X) = \sup \sum_{k=1}^{n} |\nu(E_k)|, \tag{4}$$

where the supremum is taken over all finite disjoint collections $\{E_k\}_{k=1}^n$ of measurable subsets of X. For this reason $|\nu|(X)$ is called the **total variation** of ν and denoted by $\|\nu\|_{var}$.

Example Let $f: \mathbf{R} \to \mathbf{R}$ be a function that is Lebesgue integrable over $\mathbf{R}$. For a Lebesgue measurable set E, define $\nu(E) = \int_E f\,dm$. We infer from the countable additivity of integration (see page 90) that ν is a signed measure on the measurable space $(\mathbf{R}, \mathcal{L})$. Define $A = \{x \in \mathbf{R} \mid f(x) \geq 0\}$ and $B = \{x \in \mathbf{R} \mid f(x) < 0\}$ and define, for each Lebesgue measurable set E,

$$\nu^+(E) = \int_{A \cap E} f\,dm \text{ and } \nu^-(E) = -\int_{B \cap E} f\,dm.$$

Then $\{A, B\}$ is a Hahn decomposition of $\mathbf{R}$ with respect to the signed measure ν. Moreover, $\nu = \nu^+ - \nu^-$ is a Jordan decomposition of ν.

THE CARATHÉODORY MEASURE INDUCED BY AN OUTER MEASURE

We now define the general concept of an outer measure and of measurability of a set with respect to an outer measure, and show that the Carathéodory strategy for the construction of Lebesgue measure on the real line is feasible in general.

Definition *A set function $\mu: \mathcal{S} \to [0, \infty]$ defined on a collection $\mathcal{S}$ of subsets of a set X is called* **countably monotone** *provided whenever a set $E \in \mathcal{S}$ is covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$, then*

$$\mu(E) \leq \sum_{k=1}^\infty \mu(E_k).$$

As we already observed, the monotonicity and countable additivity properties of a measure tell us that a measure is countably monotone. If the countably monotone set function $\mu: \mathcal{S} \to [0, \infty]$ has the property that $\emptyset$ belongs to $\mathcal{S}$ and $\mu(\emptyset) = 0$, then μ is **finitely monotone** in the sense that whenever a set $E \in \mathcal{S}$ is covered by a finite collection $\{E_k\}_{k=1}^n$ of sets in $\mathcal{S}$, then

$$\mu(E) \leq \sum_{k=1}^n \mu(E_k).$$

To see this, set $E_k = \emptyset$ for $k > n$. In particular, such a set function μ is **monotone** in the sense that if A and B belong to $\mathcal{S}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

Definition *A set function $\mu^*: 2^X \to [0, \infty]$ is called an* **outer measure** *provided $\mu^*(\emptyset) = 0$ and μ^* is countably monotone.*

Guided by our experience in the construction of Lebesgue measure from Lebesgue outer measure on the real line, we follow Constantine Carathéodory and define the measurability of a set as follows.

Definition *For an outer measure $\mu^*: 2^X \to [0, \infty]$, we call a subset E of X* **measurable** *(with respect to μ^*) provided for every subset A of X,*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C).$$

Since μ^* is finitely monotone, to show that $E \subseteq X$ is measurable it is only necessary to show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^C) \text{ for all } A \subseteq X \text{ such that } \mu^*(A) < \infty.$$

Directly from the definition we see that a subset E of X is measurable if and only if its complement in X is measurable and, by the monotonicity of μ^*, that every set of outer measure zero is measurable. Hereafter in this section, $\mu^*: 2^X \to [0, \infty]$ is a reference outer measure and measurable means measurable with respect to μ^*.

Proposition 5 *The union of a finite collection of measurable sets is measurable.*

Proof We first show that the union of two measurable sets is measurable. Let E_1 and E_2 be measurable. Let A be any subset of X. First using the measurability of E_1, then the measurability of E_2, we have

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^C) \\ &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^C] \cap E_2) + \mu^*([A \cap E_1^C] \cap E_2^C).\end{aligned}$$

Now use the set identities

$$[A \cap E_1^C] \cap E_2^C = A \cap [E_1 \cup E_2]^C$$

and

$$[A \cap E_1] \cup [A \cap E_2 \cap E_1^C] = A \cap [E_1 \cup E_2],$$

together with the finite monotonicity of outer measure, to obtain

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^C)\\ &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^C] \cap E_2) + \mu^*([A \cap E_1^C] \cap E_2^C)\\ &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^C] \cap E_2) + \mu^*(A \cap [E_1 \cup E_2]^C)\\ &\geq \mu^*(A \cap [E_1 \cup E_2]) + \mu^*(A \cap [E_1 \cup E_2]^C).\end{aligned}$$

Thus $E_1 \cup E_2$ is measurable. Now let $\{E_k\}_{k=1}^n$ be any finite collection of measurable sets. We prove the measurability of the union $\bigcup_{k=1}^n E_k$, for general n, by induction. This is trivial for $n = 1$. Suppose it is true for $n - 1$. Thus, since

$$\bigcup_{k=1}^{n} E_k = \left[\bigcup_{k=1}^{n-1} E_k\right] \cup E_n$$

and the union of two measurable sets is measurable, the set $\bigcup_{k=1}^n E_k$ is measurable. □

Proposition 6 *Let $A \subseteq X$ and $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then*

$$\mu^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k).$$

In particular, the restriction of μ^* to the collection of measurable sets is finitely additive.

Proof The proof proceeds by induction on n. It is clearly true for $n = 1$, and we assume it is true for $n - 1$. Since the collection $\{E_k\}_{k=1}^{n}$ is disjoint,

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n = A \cap E_n$$

and

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^C = A \cap \left[\bigcup_{k=1}^{n-1} E_k\right].$$

Hence by the measurability of E_n and the induction assumption, we have

$$\mu^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \mu^*(A \cap E_n) + \mu^*\left(A \cap \left[\bigcup_{k=1}^{n-1} E_k\right]\right)$$

$$= \mu^*(A \cap E_n) + \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$$

$$= \sum_{k=1}^{n} \mu^*(A \cap E_k).$$

Proposition 7 *The union of a countable collection of measurable sets is measurable.*

Proof Let $E = \bigcup_{k=1}^{\infty} E_k$, where each E_k is measurable. Since the complement in X of a measurable set is measurable and, by Proposition 5, the union of a finite collection of measurable sets is measurable, by possibly replacing each E_k with $E_k \sim \bigcup_{i=1}^{k-1} E_i$, we may suppose that $\{E_k\}_{k=1}^{\infty}$ is disjoint. Let A be any subset of X. Fix an index n. Define $F_n = \bigcup_{k=1}^{n} E_k$. Since F_n is measurable and $F_n^C \supseteq E^C$, we have

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^C) \geq \mu^*(A \cap F_n) + \mu^*(A \cap E^C).$$

By Proposition 6,

$$\mu^*(A \cap F_n) = \sum_{k=1}^{n} \mu^*(A \cap E_k).$$

Thus

$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^C).$$

The left-hand side of this inequality is independent of n and therefore

$$\mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^C).$$

By the countable monotonicity of outer measure we infer that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^C).$$

Thus E is measurable.

Theorem 8 *Let μ^* be an outer measure on 2^X. Then the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ-algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.*

Proof We already observed that the complement in X of a measurable subset of X also is measurable. According to Proposition 7, the union of a countable collection of measurable sets is measurable. Therefore $\mathcal{M}$ is a σ-algebra. By the definition of an outer measure, $\mu^*(\emptyset) = 0$ and therefore $\emptyset$ is measurable and $\overline{\mu}(\emptyset) = 0$. To verify that $\overline{\mu}$ is a measure on $\mathcal{M}$, it remains to show it is countably additive. Since μ^* is countably monotone and μ^* is an extension of $\overline{\mu}$, the set function $\overline{\mu}$ is countably monotone. Therefore we only need show that if $\{E_k\}_{k=1}^{\infty}$ is a disjoint collection of measurable sets, then

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \mu^*(E_k). \tag{5}$$

However, μ^* is monotone and, by taking $A = X$ in Proposition 7, we see that μ^* is additive over finite disjoint unions of measurable sets. Therefore, for each n,

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k).$$

The left-hand side of this inequality is independent of n and therefore (5) holds. □

THE CONSTRUCTION OF OUTER MEASURES

We constructed Lebesgue outer measure on subsets of the real line by first defining the primitive set function that assigns length to a bounded interval. We then defined the outer measure of a set to be the infimum of sums of lengths of countable collections of bounded intervals that cover the set. This method of construction of outer measure works in general.

Theorem 9 *Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu: \mathcal{S} \to [0, \infty]$ a set function. Define $\mu^*(\emptyset) = 0$ and for $E \subseteq X$, $E \neq \emptyset$, define*

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k), \tag{6}$$

where the infimum is taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E.[1] *Then the set function $\mu^*: 2^X \to [0, \infty]$ is an outer measure called the* **outer measure induced by** μ.

Proof To verify countable monotonicity, let $\{E_k\}_{k=1}^{\infty}$ be a collection of subsets of X that covers a set E. If $\mu^*(E_k) = \infty$ for some k, then $\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k) = \infty$. Therefore we may assume each E_k has finite outer measure. Let $\epsilon > 0$. For each k, there is a countable collection $\{E_{ik}\}_{i=1}^{\infty}$ of sets in $\mathcal{S}$ that covers E_k and

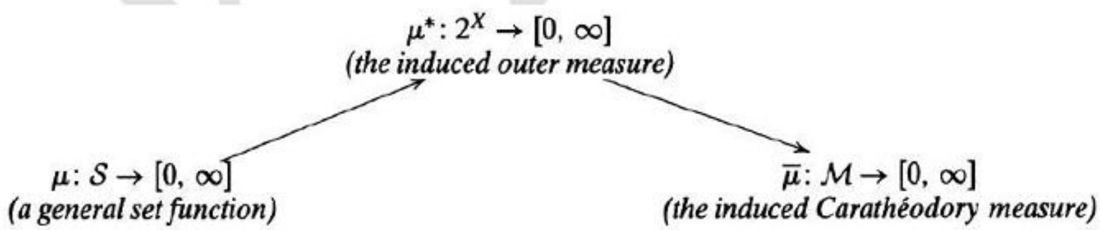
$$\sum_{i=1}^{\infty} \mu(E_{ik}) < \mu^*(E_k) + \frac{\epsilon}{2^k}.$$

Then $\{E_{ik}\}_{1\leq k,i<\infty}$ is a countable collection of sets in $\mathcal{S}$ that covers $\bigcup_{k=1}^{\infty} E_k$ and therefore also covers E. By the definition of outer measure,

$$\begin{aligned}\mu^*(E) \leq \sum_{1\leq k,i<\infty} \mu(E_{ik}) &= \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} \mu(E_{ik})\right] \\ &\leq \sum_{k=1}^{\infty} \mu^*(E_k) + \sum_{k=1}^{\infty} \epsilon/2^k \\ &= \sum_{k=1}^{\infty} \mu^*(E_k) + \epsilon.\end{aligned}$$

Since this holds for all $\epsilon > 0$, it also holds for $\epsilon = 0$.

Definition *Let $\mathcal{S}$ be a collection of subsets of X, $\mu\colon \mathcal{S} \to [0, \infty]$ a set function, and μ^* the outer measure induced by μ. The measure $\overline{\mu}$ that is the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets is called the* **Carathéodory measure induced by** μ.



For a collection $\mathcal{S}$ of subsets of X, we use $\mathcal{S}_\sigma$ to denote those sets that are countable unions of sets of $\mathcal{S}$ and use $\mathcal{S}_{\sigma\delta}$ to denote those sets that are countable intersections of sets in $\mathcal{S}_\sigma$. Observe that if $\mathcal{S}$ is the collection of open integrals of real numbers, then $\mathcal{S}_\sigma$ is the collection of open subsets of $\mathbf{R}$ and $\mathcal{S}_{\sigma\delta}$ is the collection of G_δ subsets of $\mathbf{R}$.

We proved that a set E of real numbers is Lebesgue measurable if and only if it is a subset of a G_δ set G for which $G \sim E$ has Lebesgue measure zero: see page 40. The following proposition tells us of a related property of the Carathéodory measure induced by a general set function. This property is a key ingredient in the proof of a number of important theorems, among which are the proofs of the Carathéodory-Hahn Theorem, which we prove in the following section, and the forthcoming theorems of Fubini and Tonelli.

Proposition 10 *Let $\mu: \mathcal{S} \to [0, \infty]$ be a set function defined on a collection $\mathcal{S}$ of subsets of a set X and $\overline{\mu}: \mathcal{M} \to [0, \infty]$ the Carathéodory measure induced by μ. Let E be a subset of X for which $\mu^*(E) < \infty$. Then there is a subset A of X for which*

$$A \in \mathcal{S}_{\sigma\delta},\ E \subseteq A \text{ and } \mu^*(E) = \mu^*(A).$$

Furthermore, if E and each set in $\mathcal{S}$ is measurable with respect to μ^*, then so is A and

$$\overline{\mu}(A \sim E) = 0.$$

Proof Let $\epsilon > 0$. We claim that there is a set A_ϵ for which

$$A_\epsilon \in \mathcal{S}_\sigma, E \subseteq A_\epsilon \text{ and } \mu^*(A_\epsilon) < \mu^*(E) + \epsilon. \tag{7}$$

Indeed, since $\mu^*(E) < \infty$, there is a cover of E by a collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ for which

$$\sum_{k=1}^\infty \mu(E_k) < \mu^*(E) + \epsilon.$$

Define $A_\epsilon = \bigcup_{k=1}^\infty E_k$. Then A_ϵ belongs to $\mathcal{S}_\sigma$ and $E \subseteq A_\epsilon$. Furthermore, since $\{E_k\}_{k=1}^\infty$ is a countable collection of sets in $\mathcal{S}$ that covers A_ϵ, by the definition of the outer measure μ^*,

$$\mu^*(A_\epsilon) \leq \sum_{k=1}^\infty \mu(E_k) < \mu^*(E) + \epsilon.$$

Thus (7) holds for this choice of A_ϵ.

Define $A = \bigcap_{k=1}^\infty A_{1/k}$. Then A belongs to $\mathcal{S}_{\sigma\delta}$ and E is a subset of A since E is a subset of each $A_{1/k}$. Moreover, by the monotonicity of μ^* and the estimate (7),

$$\mu^*(E) \leq \mu^*(A) \leq \mu^*(A_{1/k}) \leq \mu^*(E) + \frac{1}{k} \text{ for all } k.$$

Thus $\mu^*(E) = \mu^*(A)$.

Now assume that E is μ^*-measurable and each set in $\mathcal{S}$ is μ^*-measurable. Since the measurable sets are a σ-algebra, the set A is measurable. But μ^* is an extension of the measure $\overline{\mu}$. Therefore, by the excision property of measure,

$$\overline{\mu}(A \sim E) = \overline{\mu}(A) - \overline{\mu}(E) = \mu^*(A) - \mu^*(E) = 0. \qquad \square$$

THE RADON-NIKODYM THEOREM

Let $(X, \mathcal{M})$ be a measurable space. For μ a measure on $(X, \mathcal{M})$ and f a nonnegative function on X that is measurable with respect to $\mathcal{M}$, define the set function ν on $\mathcal{M}$ by

$$\nu(E) = \int_E f\, d\mu \text{ for all } E \in \mathcal{M}. \tag{28}$$

We infer from the linearity of integration and the Monotone Convergence Theorem that ν is a measure on the measurable space $(X, \mathcal{M})$, and it has the property that

$$\text{if } E \in \mathcal{M} \text{ and } \mu(E) = 0, \text{ then } \nu(E) = 0. \tag{29}$$

The theorem named in the title of this section asserts that if μ is σ-finite, then every σ-finite measure ν on $(X, \mathcal{M})$ that possesses property (29) is given by (28) for some nonnegative function f on X that is measurable with respect to $\mathcal{M}$. A measure ν is said to be **absolutely continuous** with respect to the measure μ provided (29) holds. We use the symbolism $\nu \ll \mu$ for ν absolutely continuous with respect to μ. The following proposition recasts absolute continuity in the form of a familiar continuity criterion.

Proposition 19 *Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite measure on the measurable space $(X, \mathcal{M})$. Then ν is absolutely continuous with respect to μ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for any set $E \in \mathcal{M}$,*

$$\textit{if } \mu(E) < \delta, \textit{ then } \nu(E) < \epsilon. \tag{30}$$

Proof It is clear that the ϵ–δ criterion (30) implies that ν is absolutely continuous with respect to μ, independently of the finiteness of ν. To prove the converse, we argue by contradiction. Suppose ν is absolutely continuous with respect to μ but the ϵ–δ criterion (30) fails. Then there is an $\epsilon_0 > 0$ and a sequence of sets in $\mathcal{M}$, $\{E_n\}$, such that for each n, $\mu(E_n) < 1/2^n$ while $\nu(E_n) \geq \epsilon_0$. For each n, define $A_n = \bigcup_{k=n}^{\infty} E_k$. Then $\{A_n\}$ is a descending sequence of sets in $\mathcal{M}$. By the monotonicity of ν and the countable subadditivity of μ,

$$\nu(A_n) \geq \epsilon_0 \text{ and } \mu(A_n) \leq 1/2^{n-1} \text{ for all } n.$$

Define $A_\infty = \bigcap_{k=1}^{\infty} A_n$. By the monotonicity of the measure μ, $\mu(A_\infty) = 0$. We infer from the continuity of the measure ν that, since $\nu(A_1) \leq \nu(X) < \infty$ and $\nu(A_n) \geq \epsilon_0$ for all n, $\nu(A_\infty) \geq \epsilon_0$. This contradicts the absolute continuity of ν with respect to μ. □

The Radon-Nikodym Theorem *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure defined on the measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to μ. Then there is a nonnegative function f on X that is measurable with respect to $\mathcal{M}$ for which*

$$\nu(E) = \int_E f\, d\mu \text{ for all } E \in \mathcal{M}. \tag{31}$$

The function f is unique in the sense that if g is any nonnegative measurable function on X that also has this property, then $g = f$ a.e. $[\mu]$.

Proof We assume that both μ and ν are finite measures and leave the extension to the σ-finite case as an exercise. If $\nu(E) = 0$, for all $E \in \mathcal{M}$, then (31) holds for $f \equiv 0$ on X. So assume ν does not vanish on all of $\mathcal{M}$. We first prove that there is a nonnegative measurable function f on X for which

$$\int_X f\, d\mu > 0 \text{ and } \int_E f\, d\mu \leq \nu(E) \text{ for all } E \in \mathcal{M}. \tag{32}$$

For $\lambda > 0$, consider the finite signed measure $\nu - \lambda\mu$. According to the Hahn Decomposition Theorem, there is a Hahn decomposition $\{P_\lambda, N_\lambda\}$ for $\nu - \lambda\mu$, that is, $X = P_\lambda \cup N_\lambda$ and $P_\lambda \cap N_\lambda = \emptyset$, where P_λ is a positive set and N_λ is a negative set for $\nu - \lambda\mu$. We claim that there is some $\lambda > 0$ for which $\mu(P_\lambda) > 0$. Assume otherwise. Let $\lambda > 0$. Then $\mu(P_\lambda) = 0$. Therefore $\mu(E) = 0$ and hence, by absolute continuity, $\nu(E) = 0$, for all measurable subsets of P_λ. Since N_λ is a negative set for $\nu - \lambda\mu$,

$$\nu(E) \leq \lambda\mu(E) \text{ for all } E \in \mathcal{M} \text{ and all } \lambda > 0. \tag{33}$$

We infer from these inequalities that $\nu(E) = 0$ if $\mu(E) > 0$ and of course, by absolute continuity, $\nu(E) = 0$ if $\mu(E) = 0$. Since $\mu(X) < \infty$, $\nu(E) = 0$ for all $E \in \mathcal{M}$. This is a contradiction. Therefore we may select $\lambda_0 > 0$ for which $\mu(P_{\lambda_0}) > 0$. Define f to be λ_0 times the characteristic function of P_{λ_0}. Observe that $\int_X f\, d\mu > 0$ and, since $\nu - \lambda_0\mu$ is positive on P_{λ_0},

$$\int_E f\, d\mu = \lambda_0\mu(P_{\lambda_0} \cap E) \leq \nu(P_{\lambda_0} \cap E) \leq \nu(E) \text{ for all } E \in \mathcal{M}.$$

Therefore (32) holds for this choice of f. Define $\mathcal{F}$ to be the collection of nonnegative measurable functions on X for which

$$\int_E f\, d\mu \leq \nu(E) \text{ for all } E \in \mathcal{M},$$

and then define

$$M = \sup_{f \in \mathcal{F}} \int_X f \, d\mu. \tag{34}$$

We show that there is an $f \in \mathcal{F}$ for which $\int_X f \, d\mu = M$ and (31) holds for any such f. If g and h belong to $\mathcal{F}$, then so does $\max\{g, h\}$. Indeed, for any measurable set E, decompose E into the disjoint union of $E_1 = \{x \in E \mid g(x) < h(x)\}$ and $E_2 = \{x \in E \mid g(x) \geq h(x)\}$ and observe that

$$\int_E \max\{g, h\} \, d\mu = \int_{E_1} h \, d\mu + \int_{E_2} g \, d\mu \leq \nu(E_1) + \nu(E_2) = \nu(E).$$

Select a sequence $\{f_n\}$ in $\mathcal{F}$ for which $\lim_{n\to\infty} \int_X f_n \, d\mu = M$. We assume $\{f_n\}$ is pointwise increasing on X, for otherwise, replace each f_n by $\max\{f_1, \ldots, f_n\}$. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. We infer from the Monotone Convergence Theorem that $\int_X f \, d\mu = M$ and also that f belongs to $\mathcal{F}$. Define

$$\eta(E) = \nu(E) - \int_E f \, d\mu \text{ for all } E \in \mathcal{M}. \tag{35}$$

By assumption, $\nu(X) < \infty$. Therefore $\int_X f \, d\mu \leq \nu(X) < \infty$, and hence, by the countable additivity of integration, η is a signed measure. It is a measure since f belongs to $\mathcal{F}$, and it is absolutely continuous with respect to μ. We claim that $\eta = 0$ on $\mathcal{M}$ and hence (31) holds for this choice of f. Indeed, otherwise, we argue as we just did, with ν now replaced by η, to conclude that there is a nonnegative measurable function $\hat{f}$ for which

$$\int_X \hat{f} \, d\mu > 0 \text{ and } \int_E \hat{f} \, d\mu \leq \eta(E) = \nu(E) - \int_E f \, d\mu \text{ for all } E \in \mathcal{M}. \tag{36}$$

Therefore $f + \hat{f}$ belongs to $\mathcal{F}$ and $\int_X [f + \hat{f}] \, d\mu > \int_X f \, d\mu = M$, a contradiction of the choice of f. It remains to establish uniqueness. But if there were two, necessarily integrable, functions f_1 and f_2 for which (31) holds, then, by the linearity of integration,

$$\int_E [f_1 - f_2] \, d\mu = 0 \text{ for all } E \in \mathcal{M}.$$

Therefore $f_1 = f_2$ a.e. $[\mu]$ on X.

In Problem 59 we outline another proof of the Radon-Nikodym Theorem due to John von Neumann: it relies on the Riesz-Fréchet Representation Theorem for the dual of a Hilbert space.

Example The assumption of σ-finiteness is necessary in the Radon-Nikodym Theorem. Indeed, consider the measurable space $(X, \mathcal{M})$, where $X = [0, 1]$ and $\mathcal{M}$ is the collection of Lebesgue measurable subsets of [0, 1]. Define μ to be the counting measure on $\mathcal{M}$, so $\mu(E)$ is the number of points in E if E is finite, and otherwise $\mu(E) = \infty$. The only set of μ measure zero is the empty-set. Thus every measure on $\mathcal{M}$ is absolutely continuous with respect to μ. Define m to be Lebesgue measure on $\mathcal{M}$. We leave it as an exercise to show that there is no nonnegative Lebesgue measurable function f on X for which

$$m(E) = \int_E f\, d\mu \text{ for all } E \in \mathcal{M}.$$

Recall that for a measurable space $(X, \mathcal{M})$ and signed measure ν on $\mathcal{M}$, there is the Jordan decomposition $\nu = \nu_1 - \nu_2$, where ν_1 and ν_2 are measures on $\mathcal{M}$, one of which is finite:

We define the measure $|\nu|$ to be $\nu_1 + \nu_2$. If μ is a measure on $\mathcal{M}$, the signed measure ν is said to be absolutely continuous with respect to μ provided $|\nu|$ is absolutely continuous with respect to μ, which is equivalent to the absolute continuity of both ν_1 and ν_2 with respect to μ. From this decomposition of signed measures and the Radon-Nikodym Theorem, we have the following version of this same theorem for finite signed measures.

Corollary 20 *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a finite signed measure on the measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to μ. Then there is a function f that is integrable over X with respect to μ and*

$$\nu(E) = \int_E f\, d\mu \text{ for all } E \in \mathcal{M}.$$

Recall that given two measures μ and ν on a measurable space $(X, \mathcal{M})$, we say that μ and ν are **mutually singular** (and write $\mu \perp \nu$) provided there are disjoint sets A and B in $\mathcal{M}$ for which $X = A \cup B$ and $\nu(A) = \mu(B) = 0$.

The Lebesgue Decomposition Theorem *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure on the measurable space $(X, \mathcal{M})$. Then there is a measure ν_0 on $\mathcal{M}$, singular with respect to μ, and a measure ν_1 on $\mathcal{M}$, absolutely continuous with respect to μ, for which $\nu = \nu_0 + \nu_1$. The measures ν_0 and ν_1 are unique.*

Proof Define $\lambda = \mu + \nu$. We leave it as an exercise to show that if g is nonnegative and measurable with respect to $\mathcal{M}$, then

$$\int_E g\,d\lambda = \int_E g\,d\mu + \int_E g\,d\nu \text{ for all } E \in \mathcal{M}.$$

Since μ and ν are σ-finite measures, so is the measure λ. Moreover, μ is absolutely continuous with respect to λ. The Radon-Nikodym Theorem tells us that there is a nonnegative measurable function f for which

$$\mu(E) = \int_E f\,d\lambda = \int_E f\,d\mu + \int_E f\,d\nu \text{ for all } E \in \mathcal{M}. \tag{37}$$

Define $X_+ = \{x \in X \mid f(x) > 0\}$ and $X_0 = \{x \in X \mid f(x) = 0\}$. Since f is a measurable function, $X = X_0 \cup X_+$ is a disjoint decomposition of X into measurable sets and thus $\nu = \nu_0 + \nu_1$ is the expression of ν as the sum of mutually singular measures, where

$$\nu_0(E) = \nu(E \cap X_0) \text{ and } \nu_1(E) = \nu(E \cap X_+) \text{ for all } E \in \mathcal{M}.$$

Now $\mu(X_0) = \int_{X_0} f\,d\lambda = 0$, since $f = 0$ on X_0, and $\nu_0(X_+) = \nu(X_+ \cap X_0) = \nu(\emptyset) = 0$. Thus μ and ν_0 are mutually singular. It remains only to show that ν_1 is absolutely continuous with respect to μ. Indeed, let $\mu(E) = 0$. We must show $\nu_1(E) = 0$. However, since $\mu(E) = 0$, $\int_E f\,d\mu = 0$. Therefore, by (37) and the additivity of integration over domains,

$$\int_E f\,d\nu = \int_{E\cap X_0} f\,d\nu + \int_{E\cap X_+} f\,d\nu = 0.$$

But $f = 0$ on $E \cap X_0$ and $f > 0$ on $E \cap X_0$ and thus $\nu(E \cap X_+) = 0$, that is, $\nu_1(E) = 0$. □

A few words are in order regarding the relationship between the concept of absolute continuity of one measure with respect to another and their integral representation and the representation of an absolutely continuous function as the indefinite integral of its derivative, which we established in Chapter 6. Let $[a, b]$ be a closed, bounded interval and the real-valued function h on $[a, b]$ be absolutely continuous. According to Theorem 10 of Chapter 6,

$$h(d) - h(c) = \int_c^d h'\,d\mu \text{ for all } [c, d] \subseteq [a, b]. \tag{38}$$

We claim that this is sufficient to establish the Radon-Nikodym Theorem in the case $X = [a, b]$, $\mathcal{M}$ is the σ-algebra of Borel subsets on $[a, b]$ and μ is Lebesgue measure on $\mathcal{M}$. Indeed, let ν be a finite measure on the measurable space $([a, b], \mathcal{M})$ that is absolutely continuous with respect to Lebesgue measure. Define the function h on $[a, b]$ by

$$h(x) = \nu([a, x]) \text{ for all } x \in [a, b]. \tag{39}$$

The function h is called the cumulative distribution function associated with ν. The function h inherits absolute continuity from the absolute continuity of the measure ν. Therefore, by (38),

$$\nu(E) = \int_E h' \, d\mu \text{ for all } E = [c, d] \subseteq [a, b].$$

However, we infer from Corollary 14 of the preceding chapter that two σ-finite measures that agree on closed, bounded subintervals of $[a, b]$ agree on the smallest σ-algebra containing these intervals, namely, the Borel sets contained in $[a, b]$. Therefore

$$\nu(E) = \int_E h' \, d\mu \text{ for all } E \in \mathcal{M}.$$

The Radon-Nikodym Theorem is a far-reaching generalization of the representation of absolutely continuous functions as indefinite integrals of their derivatives. The function f for which (31) holds is called the **Radon-Nikodym derivative** of ν with respect to μ. It is often denoted by $\frac{d\nu}{d\mu}$.

General L^p Spaces

For a measure space $(X, \mathcal{M}, \mu)$ and $1 \leq p \leq \infty$, we define the linear spaces $L^p(X, \mu)$ just as we did in Part I for the case of Lebesgue measure on the real line. Arguments very similar to those used in the case of Lebesgue measure on the real line show that the Hölder and Minkowski Inequalities hold and that $L^p(X, \mu)$ is a Banach space. We devote the first section to these and related topics. The remainder of this chapter is devoted to establishing results whose proofs lie outside the scope of ideas presented in Part I. In the second section, we use the Radon-Nikodym Theorem to prove the Riesz Representation Theorem for the dual space of $L^p(X, \mu)$, for $1 \leq p < \infty$ and μ a σ-finite measure. In the third section, we show that, for $1 < p < \infty$, the Banach space $L^p(X, \mu)$ is reflexive and therefore has the weak sequential compactness properties possessed by such spaces. In the following section, we prove the Kantorovitch Representation Theorem for the dual of $L^\infty(X, \mu)$. The final section is devoted to consideration of weak sequential compactness in the nonreflexive Banach space $L^1(X, \mu)$. We use the Vitali–Hahn–Saks Theorem to prove the Dunford-Pettis Theorem, which tells us that, if $\mu(X) < \infty$, then every bounded sequence in $L^1(X, \mu)$ that is uniformly integrable has a weakly convergent subsequence.

19.1 THE COMPLETENESS OF $L^p(X, \mu), 1 \leq p \leq \infty$

Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{F}$ to be the collection of all measurable extended real-valued functions on X that are finite almost everywhere on X. Since a function that is integrable over X is finite a.e. on X, if f is a measurable function on X and there is a p in $(0, \infty)$ for which $\int_X |f|^p \, d\mu < \infty$, then f belongs to $\mathcal{F}$. Define two functions f and g in $\mathcal{F}$ to be equivalent, and write

$$f \cong g \text{ provided } f = g \text{ a.e. on } X.$$

This is an equivalence relation, that is, it is reflexive, symmetric, and transitive. Therefore it induces a partition of $\mathcal{F}$ into a disjoint collection of equivalence classes. We denote this collection of equivalence classes by $\mathcal{F}/_{\cong}$. There is a natural linear structure on $\mathcal{F}/_{\cong}$.

Given two equivalence classes $[f]$ and $[g]$ and real numbers α and β, we define the linear combination $\alpha \cdot [f] + \beta \cdot [g]$ to be the equivalence class of the functions belonging to $\mathcal{F}$ that take the value $\alpha f(x) + \beta g(x)$ on X_0, where X_0 is the set of points in X at which both f and g are finite. Observe that linear combinations of equivalence classes are properly defined in that they are independent of the choice of representatives of the equivalence classes. The zero element of this linear space is the equivalence class of functions that vanish almost everywhere on X.

Let $L^p(X, \mu)$ be the collection of equivalence classes $[f]$ for which

$$\int_E |f|^p < \infty.$$

This is properly defined since if $f \cong f_1$, then $|f|^p$ is integrable over X if and only if $|f_1|^p$ is. We infer from the inequality

$$|a+b|^p \leq 2^p[|a|^p + |b|^p] \text{ for all } a, b \in \mathbf{R}$$

and the integral comparison test that $L^p(X, \mu)$ is a linear space. For an equivalence class $[f]$ in $L^p(X, \mu)$ we define $\|[f]\|_p$ by

$$\|[f]\|_p = \left[\int_X |f|^p \, d\mu\right]^{1/p}.$$

This is properly defined. It is clear that $\|[f]\|_p = 0$ if and only if $[f] = 0$ and $\|[\alpha \cdot f]\|_p = \alpha \cdot \|[f]\|_p$ for each real number α.

We call an equivalence class $[f]$ **essentially bounded** provided there is some $M \geq 0$, called an **essential upper bound** for $[f]$, for which

$$|f| \leq M \text{ a.e. on } X.$$

This also is properly defined, that is, independent of the choice of representative of the equivalence class. We define $L^\infty(X, \mu)$ to be the collection of equivalence classes $[f]$ for which f is essentially bounded. Then $L^\infty(X, \mu)$ also is a linear subspace of $\mathcal{F}/_\cong$. For $[f] \in L^\infty(X, \mu)$, define $\|[f]\|_\infty$ to be the infimum of the essential upper bounds for f. This is properly defined. It is easy to see that $\|[f]\|_\infty$ is the smallest essential upper bound for f. Moreover, $\|[f]\|_\infty = 0$ if and only if $[f] = 0$ and $\|[\alpha \cdot f]\|_\infty = \alpha \cdot \|[f]\|_\infty$ for each real number α. We infer from the triangle inequality for real numbers that the triangle inequality holds for $\| \cdot \|_\infty$ and hence it is a norm.

For simplicity and convenience, we refer to the equivalence classes in $\mathcal{F}/_\cong$ as functions and denote them by f rather than $[f]$. Thus to write $f = g$ means that $f(x) = g(x)$ for almost all $x \in X$.

Recall that the conjugate q of a number p in $(1, \infty)$ is defined by the relation $1/p + 1/q = 1$; we also call 1 the conjugate of ∞ and ∞ the conjugate of 1.

The proofs of the results in this section are very similar to those of the corresponding results in the case of Lebesgue integration of functions of a real variable.

Theorem 1 *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p < \infty$, and q the conjugate of p. If f belongs to $L^p(X, \mu)$ and g belongs to $L^q(X, \mu)$, then their product $f \cdot g$ belongs to $L^1(X, \mu)$ and*

Hölder's Inequality

$$\int_X |f \cdot g| \, d\mu = \|f \cdot g\|_1 \le \|f\|_p \cdot \|g\|_q.$$

Moreover, if $f \neq 0$, the function $f^* = \|f\|_p^{1-p} \cdot \operatorname{sgn}(f) \cdot |f|^{p-1}$ belongs to $L^q(X, \mu)$,

$$\int_X f f^* \, d\mu = \|f\|_p \text{ and } \|f^*\|_q = 1. \qquad (1)$$

Minkowski's Inequality For $1 \le p \le \infty$ *and* $f, g \in L^p(X, \mu)$,

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Therefore $L^p(X, \mu)$ is a normed linear space.

The Cauchy-Schwarz Inequality *Let f and g be measurable functions on X for which f^2 and g^2 are integrable over X. Then their product $f \cdot g$ also is integrable over X and, moreover,*

$$\int_X |fg| \, d\mu \le \sqrt{\int_X f^2 \, d\mu} \cdot \sqrt{\int_X g^2 \, d\mu}.$$

Proof If $p = 1$, then Hölder's Inequality follows from the monotonicity and homogeneity of integration, together with the observation that $\|g\|_\infty$ is an essential upper bound for g. Equality (1) is clear. Assume $p > 1$. Young's Inequality asserts that for nonnegative real numbers a and b,

$$ab \le \frac{1}{p} \cdot a^p + \frac{1}{q} \cdot b^q.$$

Define $\alpha = \int_X |f|^p \, d\mu$ and $\beta = \int_X |g|^q \, d\mu$. Assume α and β are positive. The functions f and g are finite a.e. on X. If $f(x)$ and $g(x)$ are finite, substitute $|f(x)|/\alpha^{1/p}$ for a and $|g(x)|/\beta^{1/b}$ for b in Young's Inequality to conclude that

$$\frac{1}{\alpha^{1/p} \cdot \beta^{1/q}} |f(x)g(x)| \le \frac{1}{p} \cdot \frac{1}{\alpha} \cdot |f(x)|^p + \frac{1}{q} \cdot \frac{1}{\beta} \cdot |g(x)|^q \text{ for almost all } x \in X.$$

Integrate across this inequality, using the monotonicity and linearity of integration, and multiply the resulting inequality by $\alpha^{1/p} \cdot \beta^{1/q}$ to obtain Hölder's Inequality. Verification of equality (1) is an exercise in the arithmetic of p's and q's. To verify Minkowski's Inequality, since we already established that $f + g$ belongs to $L^p(X, \mu)$, we may consider the associated function $(f + g)^*$ in $L^q(X, \mu)$ for which (1) holds with $f + g$ substituted for f. According to Hölder's Inequality, the functions $f \cdot (f + g)^* + g \cdot (f + g)^*$ are integrable over X. Therefore, by the linearity of integration and another employment of Hölder's Inequality,

$$\begin{aligned}\|f+g\|_p &= \int_X (f+g)\cdot(f+g)^*\,d\mu \\ &= \int_X f\cdot(f+g)^*\,d\mu + \int_X g\cdot(f+g)^*\,d\mu \\ &\leq \|f\|_p\cdot\|(f+g)^*\|_q + \|g\|_p\cdot\|(f+g)^*\|_q \\ &= \|f\|_p + \|g\|_p.\end{aligned}$$

Of course, the Cauchy-Schwarz Inequality is Minkowski's Inequality for the case $p = q = 2$. □

Corollary 2 *Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $1 \leq p_1 < p_2 \leq \infty$. Then $L^{p_2}(X, \mu) \subseteq L^{p_1}(X, \mu)$. Moreover, for*

$$c = [\mu(X)]^{\frac{p_2-p_1}{p_1p_2}} \text{ if } p_2 < \infty \text{ and } c = [\mu(X)]^{\frac{1}{p_1}} \text{ if } p_2 = \infty, \quad (2)$$

$$\|f\|_{p_1} \leq c\|f\|_{p_2} \text{ for all } f \text{ in } L^{p_2}(X). \quad (3)$$

Proof For $f \in L^{p_2}(X, \mu)$, apply Hölder's Inequality, with $p = p_2$ and $g = 1$ on X, to confirm that (3) holds for c defined by (2). □

Corollary 3 *Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 < p \leq \infty$. If $\{f_n\}$ is a bounded sequence of functions in $L^p(X, \mu)$, then $\{f_n\}$ is uniformly integrable over X.*

Proof Let $M > 0$ be such that $\|f\|_p \leq M$ for all n. Define $\gamma = 1$ if $p = \infty$ and $\gamma = (p-1)/p$ if $p < \infty$. Apply the preceding corollary, with $p_1 = 1$, $p_2 = p$, and $X = E$, a measurable subset of X of finite measure, to conclude that for any measurable subset E of X of finite measure and any natural number n,

$$\int_E |f_n|\,d\mu \leq M\cdot[\mu(E)]^\gamma.$$

Therefore $\{f_n\}$ is uniformly integrable over X. □

For a linear space V normed by $\|\cdot\|$, we call a sequence $\{v_k\}$ in V **rapidly Cauchy** provided there is a convergent series of positive numbers $\sum_{k=1}^\infty \epsilon_k$ for which

$$\|v_{k+1} - v_k\| \leq \epsilon_k^2 \text{ for all natural numbers } k.$$

Lemma 4 *Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p \le \infty$. Then every rapidly Cauchy sequence in $L^p(X, \mu)$ converges to a function in $L^p(X, \mu)$, both with respect to the $L^p(X, \mu)$ norm and pointwise almost everywhere on X.*

Proof We leave the case $p = \infty$ as an exercise. Assume $1 \le p < \infty$. Let $\sum_{k=1}^{\infty} \epsilon_k$ be a convergent series of positive numbers for which

$$\|f_{k+1} - f_k\|_p \le \epsilon_k^2 \text{ for all natural numbers } k. \tag{4}$$

Then

$$\int_X |f_{n+k} - f_n|^p \, d\mu \le \left[\sum_{j=n}^{\infty} \epsilon_j^2\right]^p \text{ for all natural numbers } n \text{ and } k. \tag{5}$$

Fix a natural number k. According to Chebychev's Inequality,

$$\mu\left\{x \in X \;\middle|\; |f_{k+1}(x) - f_k(x)|^p \ge \epsilon_k^p\right\} \le \frac{1}{\epsilon_k^p} \cdot \int_X |f_{k+1} - f_k|^p \, d\mu = \frac{1}{\epsilon_k^p} \cdot \|f_{k+1} - f_k\|_p^p, \tag{6}$$

and therefore

$$\mu\left\{x \in X \mid |f_{k+1}(x) - f_k(x)| \ge \epsilon_k\right\} \le \epsilon_k^p \text{ for all natural numbers } k.$$

Since $p \ge 1$, the series $\sum_{k=1}^{\infty} \epsilon_k^p$ converges. The Borel-Cantelli Lemma tells us that there is a subset X_0 of X for which $\mu(X \sim X_0) = 0$ and for each $x \in X_0$, there is an index $K(x)$ such that

$$|f_{k+1}(x) - f_k(x)| < \epsilon_k \text{ for all } k \ge K(x).$$

Hence, for $x \in X_0$,

$$|f_{n+k}(x) - f_n(x)| \le \sum_{j=n}^{\infty} \epsilon_j \text{ for all } n \ge K(x) \text{ and all } k. \tag{7}$$

The series $\sum_{j=1}^{\infty} \epsilon_j$ converges, and therefore the sequence of real numbers $\{f_k(x)\}$ is Cauchy. The real numbers are complete. Denote the limit of $\{f_k(x)\}$ by $f(x)$. Define $f(x) = 0$ for $x \in X \sim X_0$. Taking the limit as $k \to \infty$ in (5) we infer from Fatou's Lemma that

$$\int_X |f - f_n|^p \, d\mu \le \left[\sum_{j=n}^{\infty} \epsilon_j^2\right]^p \text{ for all } n.$$

Since the series $\sum_{k=1}^{\infty} \epsilon_k^2$ converges, f belongs to $L^p(X)$ and $\{f_n\} \to f$ in $L^p(X)$. We constructed f as the pointwise limit almost everywhere on X of $\{f_n\}$. □

The Vitali L^p Convergence Criterion *Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(X, \mu)$ that converges pointwise a.e. to f and f also belong to $L^p(X, \mu)$. Then $\{f_n\} \to f$ in $L^p(X, \mu)$ if and only if $\{|f|^p\}$ is uniformly integrable and tight.*

POSSIBLE QUESTIONS

1.State and prove Jordan decomposition theorem.

2. If E be a measurable set such that $0 < \nu E < 0$ then there is a positive set A contained in E with $\nu A > 0$.

3. State and prove Hahn decomposition theorem.

4.State and prove Riesz Representation theorem.

5. Prove that every measurable subset of a positive set is positive and the union of a countable collection of positive sets is positive.

6. If $(X, \mathcal{B}, \mu)$ be a finite measure space and g an integrable function such that for some constant M, $|g\phi \, d\mu| \leq M \|\phi\|_p$, for all simple function ϕ.Then prove that $g \in L^q$.

inf { $f_1, f_2, \ldots f_n$ }, $\sup_n f_n$, $\inf_n f_n$, $\overline{lim} f_n$ and $\underline{lim} f_n$ are all measurable.

7. State and prove Radon-Nikodym theorem.

8. Prove that if for $1 \leq p \leq \infty$ the spaces $L^p(\mu)$ are Banach spaces , and if f ε $L^p(\mu)$, with $1/p + 1/q = 1$ then fgε $L^1(\mu)$ and $\int |fg| d\mu \leq ||f||_p \; ||g||_q$

9. State and prove lebesque decomposition theorem.

10. Prove that if F be a bounded linear functional on $L^p(\mu)$ with $1 < p < \infty$ then there is a unique element g ε L^p such that $F(f) = \int fg \, d\mu$

COMPULSORY

1. State and prove Hahn decomposition theorem.

2. If { f_n } be a sequence of nonnegative measurable function that converges a.e on the set E to a function f then prove that $\int f \leq \underline{lim} f_n$.

3. If f is integrable on [a,b] and $\int_a^x f(t)dt = 0$ for all x ε [a,b] then prove that f(t)=0 a.e in [a ,b].