KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester - IV

		L T P C
17MMP402	Stochastic Process	**3 0 0 3**

Objectives: This course aims to give essential ideas on stochastic process which have wide applications with differential equations.

UNIT -I

Definition of Stochastic Processes – Markov chains: definition, order of a Markov Chain – Higher transition probabilities – classification of states and chains.

UNIT -II

Markov Process with discrete state space: Poisson process – and related distributions – properties of Poisson process, Generalizations of Poisson Processes – Birth and death Processes – continuous time Markov Chains.

UNIT -III

Markov processes with continuous state space: Introduction, Brownian motion – Weiner Process and differential equations for Weiner process, Kolmogrov equations – first passage time distribution for Weiner process – Ornstein – Uhlenbech process.

UNIT -IV

Branching Processes: Introduction – properties of generating functions of Branching process– Distribution of the total number of progeny, Continuous- Time Markov Branching Process, Age dependent branching process: Bellman-Harris process.

UNIT -V

Stochastic Processes in Queuing Systems: Concepts – Queuing model M/M1 – transient behavior of M/M/1 model – Birth and death process in Queuing theory: M/M/1 – Model related distributions – M/M/1 - M/M/S/S – loss system - M/M/S/M – Non birth and death Queuing process: Bulk queues – M(x)/M/1.

SUGGESTED READINGS

TEXT BOOK

T .Medhi, J., (2006). Stochastic Processes, 2nd Edition, New age international Private limited, New Delhi.

REFERENCES

R1. Basu, K., (2003). Introduction to Stochastic Process, Narosa Publishing House, New Delhi.

R2.Goswami and Rao, B. V., (2006). A Course in Applied Stochastic Processes, Hindustan Book Agency, New Delhi.

R3.Grimmett, G. and Stirzaker D., (2001). Probability and Random Processes, 3rd Ed., Oxford University Press, New York.

R4. Papoulis.A and Unnikrishna Pillai,(2002). Probability, Random variables and Stochastic Processes, Fourth Edition, McGraw-Hill, New Delhi.

-KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: Dr. K.KALIDASS
SUBJECT NAME: STOCHASTIC PROCESS SUB.CODE:17MMP402
SEMESTER: IV CLASS: II M. Sc. MATHEMATICS

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
UNIT-I			
1	1	Introduction to stochastic processes	T: Ch 1, 49
2	1	Definitions and examples on stochastic processes	T: Ch 1, 50-51
3	1	Definitions and Examples on Markov chains	T: Ch 2, 63-65
4	1	Theorems on order of a Markov Chain	T: Ch 2, 69-70
5	1	Theorems on higher transition probabilities	T: Ch 2, 70-72
6	1	Problems on higher transition probabilities	T: Ch 2, 73-74
7	1	Theorems on classification of states and chains	T: Ch 2, 78-80
8	1	Problems on classification of states and chains	T: Ch 2, 82-88
9	1	Recapitulation and Discussion of possible questions	
Total number of hours planed for unit I 9 hours			
UNIT-II			
1	1	Markov process with discrete state space	T: Ch 2, 101-103
2	1	Continuation of Markov process with discrete state space	T: Ch 2, 104-106
3	1	Theorems on Poisson process	T: Ch 3, 138-144
4	1	Continuation of theorems on Poisson process	T: Ch 3, 144-149
5	1	Properties of Poisson process	T: Ch 3, 150-155
6	1	Generalizations of Poisson processes	T: Ch 3, 155-160
7	1	Theorems on birth and death processes	T: Ch 3, 165-170
8	1	Theorems on continuous time Markov Chains	T: Ch 3, 171-175
9	1	Recapitulation and Discussion of possible questions	
Total number of hours planed for unit II 9 hours			
UNIT-III			
1	1	Markov processes with continuous state space	T: Ch 2, 122-124
2	1	Introduction to Brownian motion	T: Ch 4, 197-198

3	1	Some simple properties of Weiner process	T: Ch 4, 198-200
4	1	Differential equations for Weiner process	T: Ch 4, 200-201
5	1	Kolmogrov equations	T: Ch 4, 201-202
6	1	First passage time distribution for Weiner process	T: Ch 4, 202-203
7	1	Examples and Problems on Ornstein	T: Ch 4, 203-205
8	1	Examples and Problems on Uhlenbech process	T: Ch 4, 205-207
9	1	Recapitulation and Discussion of possible questions	
Total number of hours planed for unit III 9 hours			
UNIT-IV			
1	1	Introduction to Branching Processes	T: Ch 9, 347-350
2	1	Properties of generating functions of Branching process	T: Ch 9, 350-359
3	1	Distribution of the total number of progeny	T: Ch 9, 359-361
4	1	Continuous- Time Markov Branching Process	T: Ch 9, 371-377
5	1	Age dependent branching process	T: Ch 9, 377
6	1	Bellman-Harris process	T: Ch 9, 378
7	1	Examples and Problems on Bellman-Harris process	T: Ch 9, 379-380
8	1	Examples and Problems on Bellman-Harris process	T: Ch 9, 381
9	1	Recapitulation and Discussion of possible questions	
Total number of hours planed for unit IV 9 hours			
UNIT-V			
1	1	Introduction of Queuing Systems	T: Ch 10,388-390
2	1	Queuing model M/M1	T: Ch 10,391
3	1	transient behavior of M/M/1 model	T: Ch 10,392
4	1	Birth and death process in Queuing theory	T: Ch 10,392
5	1	M/M/S/S – loss system	T: Ch 10,393-394
6	1	Non birth and death Queuing process	T: Ch 10,395
7	1	Bulk queues- M(x)/M/1	T: Ch 10,396-398
8	1	Recapitulation and Discussion of possible questions	
9	1	Discussion of possible questions	
Total number of hours planed for unit V 12 Hours			

TEXT BOOK

T Medhi, J., (2006). Stochastic Processes, 2nd Edition, New age international Private limited, New Delhi

REFERENCES

R1 Basu, K., (2003). Introduction to Stochastic Process, Narosa Publishing House, New Delhi
R2 Goswami and Rao, B. V., (2006). A Course in Applied Stochastic Processes, Hindustan Book Agency, New Delhi
R3 Grimmett, G. and Stirzaker D., (2001). Probability and Random Processes, 3rd Ed., Oxford University Press, New York
R4 Papoulis.A and Unnikrishna Pillai,(2002). Probability, Random variables and Stochastic Processes, Fourth Edition, McGraw-Hill, New Delhi.

Total no. of Hours for the Course: 45 hours

UNIT-I

SYLLABUS

Markov Process with discrete state space: Poisson process – and related distributions – properties of Poisson process, Generalizations of Poisson Processes – Birth and death Processes – continuous time Markov Chains.

section (Section 1.4) can be used as quick reference for the various mathematical concepts involved in the text.

- The moment generating function (shown at the end of Section 1.3) and Taylor-series expansion (given at the end of Section 1.4) are introduced here but will only be needed in Chapter 7.

1.0 Overview

This chapter introduces the subject of stochastic processes, reviews transform techniques to facilitate problem solving and analysis in applied probability, and presents some mathematical background needed in the sequel. In the first section, we define what is meant by a stochastic process and the ideas of stationary and independent increments. The section also gives an overview of the text. The next two sections review generating functions and Laplace transforms. They are quite useful in handling discrete and continuous random variables that we will encounter in the study of stochastic processes. In addition to inversion by algebraic means (manageable only for problems of small size and simple structure), we also present approaches for inverting probability generating functions and Laplace transforms numerically. In this age of computers, numerical inversion enlarges the domain of applicability of transform methods. Readers who have experiences in using generating functions and Laplace transforms in other contexts can go through Sections 1.2 and 1.3 rather quickly. The last section lists a minimal set of results in mathematical analysis that are needed for the text. The section is written primarily for readers who do not have training in mathematics beyond calculus. For others, the section can serve as a source for quick reference. Those who already have had a course in advanced calculus or elementary analysis (say at a level of Rudin [1976] or Bartle [1976]) can skip the last section and go directly to the next chapter.

1.1 Introduction

Let $X(t)$ denote the state of a system at time t. For example, the state $X(t)$ can be the closing price of an IBM stock on day t. The collection of the random variables $\boldsymbol{X} = \{X(t), t \in \boldsymbol{T}\}$ is called a *stochastic process*, in which the set $\boldsymbol{T}$ is called the *index set*. When the index set is countable, $\boldsymbol{X}$ is called a discrete-time process. Thus the daily closing prices of an IBM stock form a discrete-time stochastic process, in which $\boldsymbol{T} = \{0, 1, \dots\}$. When the index set is an interval of the real line, the stochastic process is called a continuous-time process. If $X(t)$ denotes the price of an IBM stock at time t on a given day, then the process $\boldsymbol{X} = \{X(t), t \in \boldsymbol{T}\}$ is a continuous-time process, in which $\boldsymbol{T}$ is the interval covering a trading day.

If we assume that $X(t)$ takes values in a set $\boldsymbol{S}$ for every $t \in \boldsymbol{T}$, then $\boldsymbol{S}$ is called the state space of the process $\boldsymbol{X}$. When $\boldsymbol{S}$ is countable, we say that the process has a discrete state space. The two stochastic processes involving the price of an IBM stock both have discrete state spaces whose elements are dollars in increment of

1/8. When S is an interval of a real line, the process has a continuous state space. As an example, if $X(t)$ denotes the temperature at Houston Intercontinental Airport at time t, then, in principle, $X(t)$ can assume any value in an interval S.

A realization of a stochastic process X is called a *sample path* of the process. In Figure 1.1, we depict a sample path associated with a discrete-time process with a discrete state space—namely, the daily closing prices of an IBM stock. In Figure 1.2, we do the same for a continuous-time process with a discrete state space—namely, the price at any time t on a given day. Similarly, in Figure 1.3 we plot a sample path for a continuous-time process with a continuous state space representing the uninterrupted temperature readings at Houston Intercontinental Airport over a given period. If these temperature readings are taken at a set of

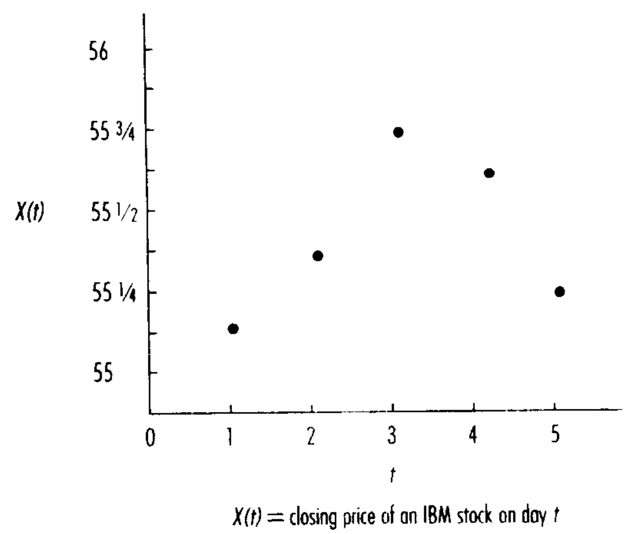

FIGURE 1.1 A sample path of a discrete-time process with a discrete state space.

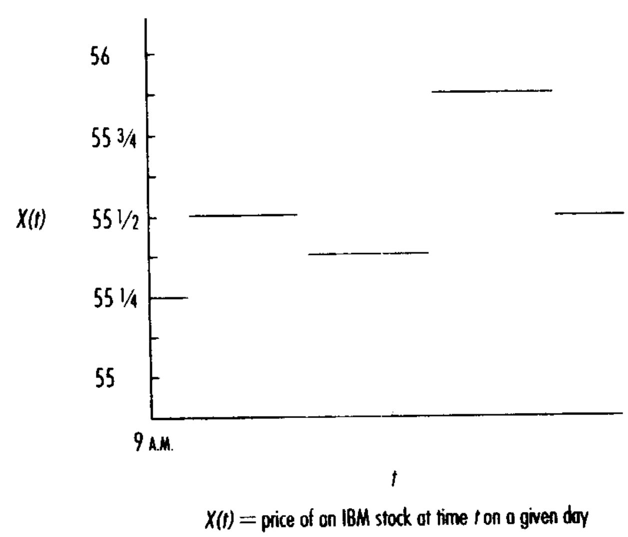

FIGURE 1.2 A sample path of a continuous-time process with a discrete state space.

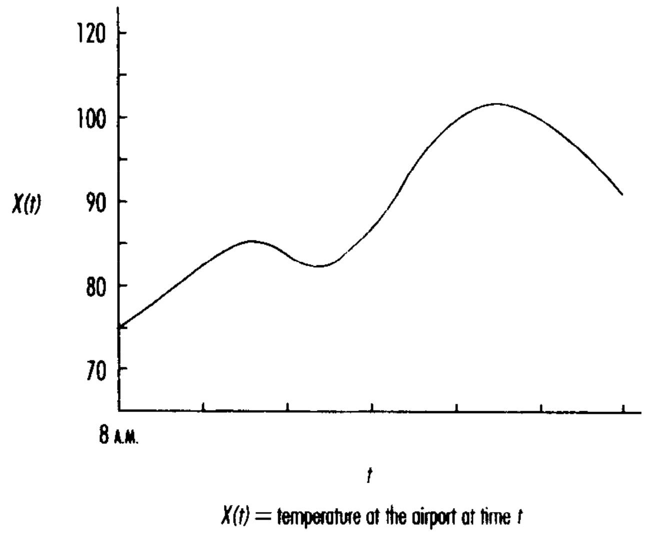

FIGURE 1.3 A sample path of a continuous-time process with a continuous state space.

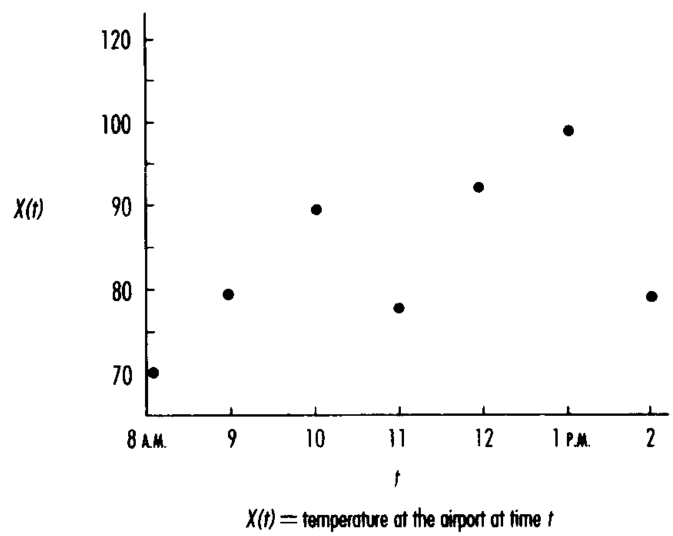

FIGURE 1.4 A sample path of a discrete-time process with a continuous state space.

distinct epochs, say every hour on the hour, then the process becomes a discrete-time process with a continuous state space. The latter is depicted in Figure 1.4.

Without structural properties, little can be said or done about a stochastic process. Two important properties are the independent-increment and stationary-increment properties of a stochastic process. A process $X = \{X(t), t \geq 0\}$ possesses the *independent-increment* property if for all $t_0 < t_1 < \cdots < t_n$, random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent (the time indices will either be discrete or continuous depending on the context). Hence in a process with independent increments, the magnitudes of state change over nonoverlapping intervals are mutually independent. A process possesses the *stationary-increment* property if the random variable $X(t + s) - X(t)$ possesses the same probability distribution for all t and any $s > 0$. In other words, the probability distribution governing the magnitude of state change depends only on the difference in the lengths of the time indices and is independent of the time origin used for the indexing variable.

Let $N(t)$ denote the number of arrivals of a given event by time t (e.g., car arrivals to a toll booth). The stochastic process $N = \{N(t), t \geq 0\}$ is called a *counting* process. The Poisson process studied in Chapter 2 is a counting process in which interarrival times of successive events are independently and identically distributed (i.i.d.) exponential random variables. The process possesses both the independent-increment and stationary-increment properties. Poisson processes are used extensively in modeling arrival processes to service systems and demand processes in inventory systems. There are many useful variants of Poisson processes. An important extension is the nonhomogeneous Poisson process in which we assume that the arrival rate is time dependent. This extension makes Poisson a versatile process for real-world applications.

In a counting process, when interarrival times of successive events follow a probability distribution other than the exponential and yet these times are mutually independent, the resulting process is called a *renewal process*. Chapter 3 is devoted to the study of renewal and related processes. The theory of a renewal process forms a cornerstone for the development of other more complicated stochastic processes, which is accomplished by use of its extension known as the *regenerative process*. At the arrival epoch of a renewal event, the future of the process becomes independent of the past. Therefore the interval between two successive renewals forms a regeneration cycle. The regeneration cycles are probabilistic replica to one another. When we are interested in a long-run property of a stochastic process, studying it over one regeneration cycle will enable us to ascertain its asymptotic value.

In Chapter 4, we introduce Markov chains. In a Markov chain, both the state space and index set are discrete. A change of state depends probabilistically only on the current state of the system and is independent of the past given that the present state is known. A process possessing this property is known to have the *Markovian* property. The successes one can have in employing Markov chains for modeling in applications depend on proper state definitions at selected epochs to maintain the Markovian property at these epochs. When there are rewards associated with state occupancy, the resulting process is called a Markov reward process. Markov chains and Markov reward processes have been used extensively in modeling and analyses of many systems in production, inventory, computers, and communication.

In a Markov chain, we are interested in the state changes over the state space and unconcerned about the sojourn times in each state before a state change takes place. For such a chain, when sojourn times in each state follow exponential distributions with state-dependent parameters, the resulting stochastic process is called a continuous-time Markov process with a discrete state space. For the special case when transitions from a given state will only be made to states other than itself, the resulting process is called the *continuous-time Markov chain*. Various subjects relating to continuous-time Markov chains will be examined in detail in Chapter 5.

A generalization of a Markov chain allows sojourn times in each state to follow probability distributions that depend on the starting and ending states associated with each transition. Stochastic processes resulting from such a generalization are called *Markov renewal processes*. This generalization makes renewal processes and discrete-time and continuous-time Markov chains all special cases of Markov renewal processes. Subjects relating to Markov renewal processes are covered in Chapter 6.

Stochastic processes presented in Chapters 2–6 all have discrete state space. In the last chapter, we will study processes with continuous state space—particularly the Brownian motion process. The mathematics needed to handle Brownian motion and related processes is more demanding. Our coverage of the subjects involved will be relatively limited.

1.2 Discrete Random Variables and Generating Functions

Let $\{a_n\}$ denote a sequence of numbers. We define the *generating function* for the sequence $\{a_n\}$ as

$$a^g(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.2.1}$$

where the power series $a^g(z)$ converges in some interval $|z| < R$. $a^g(z)$ is also called the Z-transform or geometric transform for the sequence $\{a_n\}$. To illustrate, consider the case in which $a_n = \alpha^n$, $n = 0, 1, \ldots$. Then we see that $a^g(z) = 1/(1 - \alpha z)$ when $|\alpha z| < 1$. In Table 1.1, we present an abbreviated listing relating some sequences $\{a_n\}$ and their respective generating functions. For the ith pair shown in the table, we use the notation Z-i. The pairs Z-1 and Z-2 imply that the generating function is a linear operator in the sense that if a sequence is a linear combination of two sequences, the linear relation is preserved under the transform by using the generating function. The pair Z-3 implies that the convolution operation of two sequences becomes a multiplication operation if we work with the respective generating functions instead. The sequence $\{b_n\}$ in Z-6 is the sequence $\{a_n\}$ "delayed" by k units, whereas the sequence $\{b_n\}$ in Z-7 is the sequence $\{a_n\}$ "advanced" by k units. The sequences in Z-8 and Z-9 perform respectively the "summing" and "differencing" operations. They are the discrete analogs of integration and differentiation. The two pairs enable us to do these operations when the functions have first been transformed. If A_n is a square matrix with elements $\{a_{ij}(n)\}$, then the (i, j)th element of the matrix generating function $A^g(z)$ is defined as $\sum_{n=0}^{\infty} z^n a_{ij}(n)$. When the elements of matrix A are $\{a_{ij}\}$, Z-10 gives the corresponding matrix generating function.

When $\lim_{n\to\infty} a_n$ exists, we can evaluate this limit by working with the generating function using the *final value property:*

$$\lim_{n\to\infty} a_n = \lim_{z\to 1}(1 - z)a^g(z).$$

TABLE 1.1 A Table of Generating Functions

The Sequence $\{a_n\}$	Generating Function $a^g(z) = \sum_{n=0}^{\infty} a_n z^n$
1. $\{\alpha a_n\}$	$\alpha a^g(z)$
2. $\{\alpha a_n + \beta b_n\}$	$\alpha a^g(z) + \beta b^g(z)$, where $b^g(z) = \sum_{n=0}^{\infty} b_n z^n$
3. $\left\{\sum_{m=0}^{n} a_m b_{n-m}\right\}$ Convolution	$a^g(z) b^g(z)$
4. $\{a^n\}$	$\dfrac{1}{1-az}$
5. $\left\{\dfrac{1}{k!}(n+1)(n+2)\cdots(n+k)a^n\right\}$	$\dfrac{1}{(1-az)^{k+1}}$
6. $\{b_n\}$, where $b_n = 0$ if $n < k$; $= a_{n-k}$ if $n \geq k$; and k is a positive integer	$z^k a^g(z)$
7. $\{b_n\}$, where $b_n = 0$ if $n < 0$; $= a_{n+k}$ if $n \geq 0$; and k is a positive integer	$\dfrac{1}{z^k}\left[a^g(z) - a_0 - a_1 z - \cdots - a_{k-1} z^{k-1}\right]$
8. $\left\{\sum_{m=0}^{n} a_m\right\}$	$\dfrac{1}{1-z} a^g(z)$
9. $\{b_n\}$, where $b_n = a_0$ if $n = 0$; $= a_n - a_{n-1}$ if $n \geq 1$	$(1-z)a^g(z)$
10. $\{A^n\}$, where A is a square matrix	$\sum_{n=0}^{\infty} (zA)^n = [I - Az]^{-1}$, where I is an identity matrix

A formal proof of the property can be found in a reference cited in the Bibliographic Notes. We leave an alternate proof based on Z-6 as an exercise.

Problem manipulations involving transforms are sometimes referred to as operations in the transform domain. When we invert a transform to its corresponding sequence $\{a_n\}$, we call the procedure an inversion of the transform to the time domain. Generating functions are quite useful in solving systems of difference equations; however, we shall focus our attention on their applications in stochastic modeling.

Let X denote a discrete random variable and $a_n = \text{Prob}\{X = n\}$. Then $P_X(z) = a^g(z) = E[z^X]$ is called the probability generating function for the random variable X. Here we impose the condition $|z| \leq 1$ so as to ensure the uniform convergence of the power series $a^g(z)$. If we know the probability generating function of X, the coefficients of the power series expansion of $a^g(z)$ give the probabilities that X assumes various values. Many times problem solving is somewhat messy in the time domain. We do our manipulations in the transform domain and then make an inversion to obtain the desired result.

We can obtain moments of a random variable X from its probability generating function $P_X(z)$. Define the kth derivative of $P_X(z)$ by

$$P_X^{(k)}(z) = \frac{d^k}{dz^k} P_X(z).$$

Then we see that

$$P_X^{(1)}(z) = \frac{d}{dz}\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{d}{dz} a_n z^n = \sum_{n=0}^{\infty} n a_n z^{n-1} \qquad \text{and} \qquad E[X] = P_X^{(1)}(1).$$

Similarly, we have

$$P_X^{(2)}(z) = \frac{d}{dz} P_X^{(1)}(z) = \frac{d}{dz}\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=1}^{\infty} n(n-1) a_n z^{n-2},$$

and $P_X^{(2)}(1) = E[X(X-1)] = E[X^2] - E[X]$. So the second derivative of $P_X(z)$ with respect to z evaluated at 1 gives the *second factorial moment* of X. The second moment of X is given by

$$E[X^2] = P_X^{(2)}(1) + P_X^{(1)}(1). \tag{1.2.2}$$

Other higher moments of X can be found analogously.

EXAMPLE 1.2.1 **The Binomial Random Variable** Let X be a binomial random variable with parameters n and p and

$$P\{X = j\} = a_j = \binom{n}{j} p^j q^{n-j} \qquad j = 0, 1, \ldots, n,$$

where $q = 1 - p$. The probability generating function is given by

$$P_X(z) = \sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} z^j = \sum_{j=0}^{n} \binom{n}{j} (pz)^j q^{n-j} = (pz+q)^n.$$

With $P_X^{(1)}(z) = n(pz+q)^{n-1}p$ and $P_X^{(2)}(z) = n(n-1)(pz+q)^{n-2}p^2$, we obtain $E[X] = P_X^{(1)}(1) = np$ and $E[X^2] = n(n-1)p^2 + np$ by Equation 1.2.2. This gives $Var[X] = npq$. ▪

EXAMPLE 1.2.2 **The Poisson Random Variable** Let X be a Poisson random variable with parameter $\lambda > 0$ and

$$P\{X = n\} = a_n = e^{-\lambda}\frac{\lambda^n}{n!} \qquad n = 0, 1, \ldots.$$

The probability generating function is given by

$$P_X(z) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} z^n = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda z)^n}{n!} = e^{\lambda(z-1)}.$$

Differentiating $P_X(z)$ with respect to z twice, we obtain

$$P_X^{(1)} = \lambda e^{\lambda(z-1)} \qquad \text{and} \qquad P_X^{(2)} = \lambda^2 e^{\lambda(z-1)}.$$

So, $E[X] = P_X^{(1)}(1) = \lambda$, and $P_X^{(2)}(1) = \lambda^2$. This gives $E[X^2] = \lambda^2 + \lambda$ and $Var[X] = \lambda$. ▪

EXAMPLE 1.2.3 **The Geometric Random Variable** Let X be a geometric random variable with parameter p and

$$P\{X = n\} = a_n = pq^n \qquad n = 0, 1, \ldots,$$

where $q = 1 - p$. We can interpret X as the number of failures needed to obtain the first success in a sequence of independent Bernoulli trials with probability of p of finding a success in a single trial. The probability generating function of X is given by

$$P_X(z) = \sum_{n=0}^{\infty} pq^n z^n = \frac{p}{1-qz}.$$

Finding the first two moments of X will be left as an exercise. ▪

Let $X_1, \ldots, X_k$ denote k independent, nonnegative, and integer-valued random variables where X_i follows probability generating function $P_i(z)$. Let S be the sum of these k random variables. Since S is the convolution of the k independent random variables, by Z-3 we conclude that the probability generation function of S is given by

$$P_S(z) = P_1(z) \cdots P_k(z) \tag{1.2.3}$$

Hence if we work in the transform domain the convolution operations are reduced to multiplication operations.

EXAMPLE 1.2.4 **The Negative Binomial Random Variable** For a negative binomial random variable S, we have $S = X_1 + \cdots + X_k$, where $X_1, \ldots, X_k$ are i.i.d. random variables with a common geometric distribution $\{pq^n\}$. We note that S can be interpreted as the number of failures needed to obtain the kth success for the first time in a sequence of independent Bernoulli trials with probability p of finding a success in a single trial. In Example 1.2.3, we recall that the probability generating function $P_i(z)$ for X_i is given by $p/(1 - qz)$. Using Equation 1.2.3, we find the probability generating function

$$P_S(z) = \left(\frac{p}{1-qz}\right)^k = \sum_{n=0}^{\infty} p_n z^n.$$

To invert the previous equation to the time domain, we see that

$$\frac{1}{(1-qz)^k} = \frac{1}{(1-qz)^{(k-1)+1}},$$

and an application of Z-5 shows that the sequence in the time domain reads

$$\frac{1}{(k-1)!}(n+1)\cdots(n+(k-1))q^n.$$

Using Z-1, we conclude that

$$p_n = \frac{p^k}{(k-1)!}(n+1)(n+2)\cdots(n+k-1)q^n = \frac{(n+k-1)\cdots(n+1)n!}{(k-1)!n!}p^k q^n$$
$$= \binom{n+k-1}{n} p^k q^n \quad n = 0, 1, \ldots . \ \blacksquare$$

We now introduce the notion of a *compound* random variable. Let $\{X_i\}$ be a sequence of i.i.d., nonnegative, and integer-valued random variables with a common probability generating function $P_X(z)$. Let N be a nonnegative and integer-valued random variable with a probability generating function $\pi_N(z)$. Assume that N is independent of $\{X_i\}$. The compound random variable S_N is defined as the sum of $X_1, \ldots, X_N$. This random variable is often called the *random sum*. We let $H_S(z)$ denote the probability generating function of S_N. Now we see that

$$\begin{aligned} H_S(z) &= E[z^S] = E_N\left[E[z^S|N]\right] = E_N\left[E[z^{X_1+\cdots+X_N}|N]\right] \\ &= E_N\left[E[z^{X_1+\cdots+X_N}]\right] && \text{(by independence of } N \text{ and } \{X_i\}) \\ &= E_N\left[E[z^{X_1}]\cdots E[z^{X_N}]\right] && \text{(by independence of } X_1, \ldots, X_N) \\ &= E_N\left[(P_X(z))^N\right] = \pi_N(P_X(z)). && \text{(1.2.4)} \end{aligned}$$

Therefore, the probability generating function $H_S(z)$ is obtained by simply using the probability generating function of X (a function in z) as the argument of the probability generating function of N. One way to find the first two moments of the random sum S_N is by using the approach involving differentiation of $H_S(z)$. We leave the application of this approach as an exercise. Another way is by use of conditional expectations. First, we see that $E[S_N|N] = E[X_1 + \cdots + X_N|N] = E[NX_1|N] = NE[X_1]$. This implies that

$$E[S_N] = E\left[E[S_N|N]\right] = E\left[NE[X_1]\right] = E[X_1]E[N]. \quad \text{(1.2.5)}$$

For any random variables Y and N, we recall from probability theory that the conditional variance formula is given by

$$Var[Y] = E\left[Var[Y|N]\right] + Var\left[E[Y|N]\right]. \quad \text{(1.2.6)}$$

With $Y = S_N$ in Equation 1.2.6 and $Var[S_N|N] = NVar[X_1]$ by independence of $\{X_i\}$, we conclude

$$Var[S_N] = E\big[NVar[X_1]\big] + Var\big[NE[X_1]\big] = Var[X_1]E[N] + E^2[X_1]\mathrm{var}[N]. \quad (1.2.7)$$

EXAMPLE 1.2.5 Let N be the number of times a person will visit a store in a year. Assume that N follows the geometric distribution $P\{N = n\} = (1-\theta)\theta^n$, $n = 0, 1, \ldots$. From Example 1.2.3, we find the probability generating function $\pi_N(z) = (1-\theta)/(1-\theta z)$. During each visit with probability p the person buys something. Purchases over successive visits are probabilistically independent and whether a purchase will be made during a visit is independent of number of times the person visits the store in a year. We are interested in deriving the probability distribution for S, the *number of times* the person will buy something from the store in a year. We let $X_i = 1$ if the person buys something during the ith visit and 0 otherwise. Then we have $S = X_1 + \cdots + X_N$. The probability generating function of X_i is $P_X(z) = q + pz$. Using Equation 1.2.4, we obtain

$$H_S(z) = \pi_N\big(P_X(z)\big) = \frac{1-\theta}{1-\theta P_X(z)} = \frac{1-\theta}{1-\theta[q+pz]} = \frac{1-\theta}{(1-q\theta)-p\theta z}$$

$$= \frac{\dfrac{1-\theta}{1-q\theta}}{1-\left(\dfrac{p\theta}{1-q\theta}\right)z} = \frac{1-Q}{1-Qz},$$

where we let $Q = p\theta/(1-q\theta)$. Noting that the previous equation is actually the probability generating function of a geometric distribution, we conclude that $P\{S = k\} = (1-Q)Q^k$, $k = 0, 1, \ldots$. ▪

Let $B_1, \ldots, B_k$ be mutually exclusive and collectively exhaustive events. For any event A, we recall from probability theory that

$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i).$$

Moreover, if X and Y are two discrete random variables, we have

$$P(Y = y) = \sum_x P(Y = y|X = x)P(X = x).$$

The preceding formulas are commonly known as the *laws of total probability*. In problem solving in applied probability, sometimes the following three-step approach can be useful: (i) by conditioning on the outcomes of the initial trials and using the law of total probability, write a system of difference equations; (ii) rewrite the system in the transform domain; and (iii) derive desired results from the transform. The next example illustrates the use of such an approach.

EXAMPLE 1.2.6 Consider a biased coin with probability p of obtaining heads and $q = 1 - p$ of getting tails. The coin is tossed repeatedly and stopped when two heads occur in succession for the first time. Let X denote the number of such trials needed and let $a_n = P\{X = n\}$. We want to find the probability generating function and the first two moments of X.

Following the first step of the approach, we note that $a_0 = 0$, $a_1 = 0$, $a_2 = p^2$, and $a_3 = p^2 q$. For $n = 4, 5, \ldots$, the probabilities associated with the three mutually exclusive and collectively exhaustive outcomes $B_1 = \{T\}$, $B_2 = \{H, T\}$, and $B_3 = \{H, H\}$ are q, pq, and p^2. Let H^2 denote the event that two heads occur in succession and A_n denote the event that H^2 occurs for the first time at the nth trial. For $n > 3$, we apply the law of total probability and find

$$P(A_n) = \sum_{i=1}^{3} P(A_n | B_i)P(B_i) = P(A_{n-1})q + P(A_{n-2})pq + (0)p^2.$$

In the previous derivation, to see that $P(A_n|B_1) = P(A_{n-1})$ we observe that given $\{T\}$ occurs at the first trial, then H^2 must occur for the first time at the $(n-1)$st remaining trials so that H^2 indeed occurs at the nth trial of the whole experiment. The term $P(A_n|B_2) = P(A_{n-2})$ can be interpreted similarly. Finally, if $\{H, H\}$ occurs initially, then it is impossible for H^2 to occur for the first time at trial n for $n > 3$. This gives $P(A_n|B_3) = 0$. Since $a_n = P(A_n)$, we obtain

$$a_n = qa_{n-1} + pqa_{n-2} \qquad n = 4, 5, \ldots. \tag{1.2.8}$$

We now move to the second step of the approach. We rewrite Equation 1.2.8 as

$$a_{n+2} = qa_{n+1} + pqa_n \qquad n = 2, 3, \ldots. \tag{1.2.9}$$

We multiply the nth equation of Equation 1.2.9 by z^n and add the resulting equations. This gives

$$\sum_{n=2}^{\infty} z^n a_{n+2} = \sum_{n=2}^{\infty} qz^n a_{n+1} + \sum_{n=2}^{\infty} pqz^n a_n$$

$$\frac{1}{z^2}\sum_{n=2}^{\infty} z^{n+2} a_{n+2} = \frac{1}{z}\sum_{n=2}^{\infty} qz^{n+1} a_{n+1} + pqP_X(z)$$

$$\frac{1}{z^2}\left[P_X(z) - z^3 a_3 - z^2 a_2\right] = \frac{q}{z}\left[P_X(z) - z^2 a_2\right] + pqP_X(z)$$

$$P_X(z) - z^3 a_3 - z^2 a_2 = qz\left[P_X(z) - z^2 a_2\right] + pqz^2 P_X(z).$$

Using the initial conditions a_2 and a_3 and rearranging the terms, we obtain

$$P_X(z) = \frac{p^2 z^2}{1 - qz - pqz^2}. \tag{1.2.10}$$

We now give a slightly simpler way of obtaining Equation 1.2.10. Define the indicator variable $I\{A\} = 1$ if A is true, and 0 otherwise, and define $a_n = 0$ if $n < 0$. Then Equation 1.2.8 and the initial conditions can be combined in a single expression:

$$a_n = qa_{n-1} + pqa_{n-2} + p^2 I\{n = 2\} \qquad n = 0, 1, \ldots.$$

It is easy to verify that the above holds for all n. Multiplying the nth equation by z^n and adding the resulting equations give

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} qa_{n-1} z^n + \sum_{n=0}^{\infty} pqa_{n-2} z^n + p^2 z^2.$$

Changes of indexing variables will produce

$$P_X(z) = \sum_{n=0}^{\infty} qa_n z^{n+1} + \sum_{n=0}^{\infty} pqa_n z^{n+2} + p^2 z^2 = qzP_X(z) + pqz^2 P_X(z) + p^2 z^2$$

and hence Equation 1.2.10.

Having obtained the probability generating function, we are now at the last step. By differentiating Equation 1.2.10 twice with respect to z and proceeding methodically, we will find the first two moments of X—after some cumbersome algebra. A somewhat intriguing alternative is to look at Equation 1.2.10 in the following manner:

$$P_X(z) = \frac{z^2}{\left(\dfrac{1 - qz - pqz^2}{p^2}\right)} \equiv \frac{P_W(z)}{P_Y(z)}$$

and *assume* that W and Y are two legitimate random variables with respective probability generating functions $P_W(z)$ and $P_Y(z)$. If this were the case, then we would have concluded without hesitation that $W = X + Y$ and X and Y are independent. For W, we see that $P\{W = 2\} = 1$ and therefore is a legitimate random variable. While $P_Y(1) = 1$, we see that the coefficients of the power series expansion associated with the terms z and z^2 are both negative. This implies that the "probabilities" that $Y = 1$ and $Y = 2$ are both negative. Fortunately, we can proceed with our computation by assuming that having negative probabilities is acceptable. The reason that this transgression is acceptable in the present context is that the results pertaining to convolution and moments are actually derived for generating functions whose coefficients are not restricted to be numbers in a unit interval as long as generating functions equal 1 when their arguments are set to 1. It is easy to verify that

$$P_Y^{(1)}(1) = \frac{1}{p^2}(-q - 2pq) = 2 - \frac{1}{p} - \frac{1}{p^2} = E[Y] \text{ and } P_Y^{(2)}(1) = \frac{1}{p^2}(-2pq) = 2 - \frac{2}{p}.$$

The variance of Y is then

$$Var[Y] = P_Y^{(2)}(1) + P_Y^{(1)}(1) - \left[P_Y^{(1)}(1)\right]^2 = -\frac{1}{p^4} - \frac{2}{p^3} + \frac{2}{p^2} + \frac{1}{p}.$$

Since $E[W] = 2$ and $Var[W] = 0$, we conclude that

$$E[X] = 2 - E[Y] = \frac{1}{p} + \frac{1}{p^2} \quad \text{and} \quad Var[X] = -Var[Y] = \frac{1}{p^4} + \frac{2}{p^3} - \frac{2}{p^2} - \frac{1}{p}.$$

(The expression $Var[X] = -Var[Y]$ might raise some eyebrows if one forgets that Y is not a legitimate random variable.) ▪

If we have a generating function, we can invert it to the corresponding discrete function in the time domain—namely the sequence $\{a_n\}$—either algebraically or numerically. To do it algebraically, we use *partial fraction expansion.* Typically the transform $A^g(z)$ is written as the ratio of two polynomials. For the method to work, the degree of the numerator polynomial must be at least one less than that of the denominator. If this is not so, we can either factor enough z out of the numerator or divide the two polynomials so that the mentioned condition is met. We then do a partial fraction expansion of the remainder. There are two types of ratio and their respective expansions to consider. To illustrate, one type reads

$$\frac{a + bz}{(1 - cz)(1 - dz)} = \frac{A}{(1 - cz)} + \frac{B}{(1 - dz)},$$

where $c \neq d$. The preceding equation is equivalent to $a + bz = A(1 - dz) + B(1 - cz)$. Setting $z = 1/c$, we find the value of A. The value of B can be found similarly. Another option is to set z at any two distinct values and solve the resulting system of linear equations. Inverting the transform is done by invoking Z-1, Z-2, and Z-4. The other type of ratio is one in which the denominator factors are not distinct. This is illustrated by

$$\frac{a + bz + cz^2}{(1 - dz)^2(1 - ez)} = \frac{A}{(1 - dz)^2} + \frac{B}{(1 - dz)} + \frac{C}{(1 - ez)},$$

where $d \neq e$. Finding the coefficients A, B, and C can be done in a manner similar to the first case. MATLAB function **`residue`** will do the aforementioned partial fraction expansion. Since there is a one-to-one correspondence between the probability generating function and the respective probability distribution, the inversion enables us to uncover the functional form of the latter. This is illustrated in the following two examples.

EXAMPLE 1.2.7 Assume that we are given the following probability generating function of random variable X

$$P_X(z) = \frac{4}{(2 - z)(3 - z)^2}.$$

What is the probability distribution $\{p_n\}$ of X? We need to invert $P_X(z)$. First, we write

$$\frac{4}{(2-z)(3-z)^2} = \frac{A}{(2-z)} + \frac{B}{(3-z)} + \frac{C}{(3-z)^2}.$$

Multiplying that by the denominator on the left side gives

$$A(3-z)^2 + B(2-z)(3-z) + C(2-z) = 4.$$

Setting z at 2, 3, and 0 in succession, we find respectively $A = 4$, $C = -4$, and $B = -4$. The partial fraction expansion of $P_X(z)$ is then given by

$$P_X(z) = \frac{4}{(2-z)} - \frac{4}{(3-z)} - \frac{4}{(3-z)^2} = \frac{2}{\left(1-\frac{1}{2}z\right)} - \frac{\frac{4}{3}}{\left(1-\frac{1}{3}z\right)} - \frac{\frac{4}{9}}{\left(1-\frac{1}{3}z\right)^2}.$$

In the Appendix, we illustrate the use of MATLAB to do the partial fraction expansion. Using Z-2, Z-4, and Z-5, we invert the previous equation to the time domain. This gives the probability distribution

$$\begin{aligned} p_n &= 2\left(\frac{1}{2}\right)^n - \left(\frac{4}{3}\right)\left(\frac{1}{3}\right)^n - \left(\frac{4}{9}\right)\left(\frac{1}{3}\right)^n (n+1) \\ &= 2^{-n+1} - 4(3)^{-(n+1)} - 4(n+1)(3)^{-(n+2)} \qquad n = 0, 1, 2, \ldots \end{aligned}$$ ▪

EXAMPLE 1.2.8 We return to Example 1.2.6 with the goal of obtaining a closed-form expression for the distribution of X—the number of trials needed to obtain two heads in succession in a sequence of Bernoulli trials. Recall the probability generating function is given by Equation 1.2.10. Note that

$$\frac{1}{1-qz-pqz^2} = \frac{-1}{pq\left(z^2 + \frac{q}{pq}z - \frac{1}{pq}\right)}.$$

The term in the last parentheses can be factored as $(z-z_1)(z-z_2)$, where

$$z_1 = \frac{-q+\sqrt{q^2+4pq}}{2pq} \quad \text{and} \quad z_2 = \frac{-q-\sqrt{q^2+4pq}}{2pq}.$$

Using partial fraction expansion and $z_1 - z_2 = (\sqrt{q^2+4pq})/pq$, we find

$$\frac{1}{1-qz-pqz^2} = \frac{1}{\sqrt{q^2+4pq}}\left[\frac{-1}{z-z_1} + \frac{1}{z-z_2}\right].$$

UNIT-II

SYLLABUS

Markov Process with discrete state space: Poisson process – and related distributions – properties of Poisson process, Generalizations of Poisson Processes – Birth and death Processes – continuous time Markov Chains

In some homes the use of the telephone can become quite a sensitive issue. Suppose that if the phone is free during some period of time, say the nth minute, then with probability p, where $0 < p < 1$, it will be busy during the next minute. If the phone has been busy during the nth minute, it will become free during the next minute with probability q, where $0 < q < 1$. Assume that the phone is free in the 0th minute. We would like to answer the following two questions.

1) What is the probability x_n that the telephone will be free in the nth minute?
2) What is $\lim_{n\to\infty} x_n$, if it exists?

Suppose that S is a finite or a countable set. Suppose also that a probability space $(\Omega, \mathcal{F}, P)$ is given. An S-valued sequence of random variables ξ_n, $n \in \mathbb{N}$, is called an S-valued *Markov chain* or a Markov chain on S if for all $n \in \mathbb{N}$ and all $s \in S$

$$P(\xi_{n+1} = s|\xi_0, \dots, \xi_n) = P(\xi_{n+1} = s|\xi_n). \tag{5.10}$$

Here $P(\xi_{n+1} = s|\xi_n)$ is the conditional probability of the event $\{\xi_{n+1} = s\}$ with respect to random variable ξ_n, or equivalently, with respect to the σ-field $\sigma(\xi_n)$ generated by ξ_n. Similarly, $P(\xi_{n+1} = s|\xi_0, \dots, \xi_n)$ is the conditional probability of $\{\xi_{n+1} = s\}$ with respect to the σ-field $\sigma(\xi_0, \cdots, \xi_n)$ generated by the random variables $\xi_0, \cdots, \xi_n$.

Property (5.10) will usually be referred to as the *Markov property* of the Markov chain ξ_n, $n \in \mathbb{N}$. The set S is called the *state space* and the elements of S are called *states*.

Show that a stochastic matrix is doubly stochastic if and only if the sum of the entries in each row is 1, i.e. $\sum_{i \in S} a_{ji} = 1$ for any $j \in S$.

Proof

Put $A^t = [b_{ij}]$. Then, by the definition of the transposed matrix, $b_{ij} = a_{ji}$. Therefore, A^t is a stochastic matrix if and only if

$$\sum_i a_{ji} = \sum_i b_{ij} = 1,$$

completing the proof. □

Show that if $P = [p_{ji}]_{j,i \in S}$ is a stochastic matrix, then any natural power P^n of P is a stochastic matrix. Is the corresponding result true for a double stochastic matrix?

Definition 5.4

The *n-step transition matrix* of a Markov chain ξ_n with transition probabilities $p(j|i)$, $j, i \in S$ is the matrix P_n with entries

$$p_n(j|i) = P(\xi_n = j|\xi_0 = i). \tag{5.17}$$

Exercise 5.5

Find an exact formula for P_n for the matrix P from Exercise 5.4.

Hint Put $x_n = P(\xi_n = 0|\xi_0 = 0)$ and $y_n = P(\xi_n = 1|\xi_0 = 1)$. Is it correct to suppose that $p_n(0|0) = x_n$ and $p_n(1|1) = y_n$? If yes, you may be able use Example 5.1 and Exercise 5.1.

Exercise 5.6

You may suspect that P_n equals P^n, the nth power of the matrix P. This holds for $n = 1$. Check if it is true for $n = 2$. If this is the case, try to prove that $P_n = P^n$ for all $n \in \mathbb{N}$.

Hint Once again, this is an exercise in matrix multiplication.

The following is a generalization of Exercise 5.6.

Proposition 5.3 (Chapman–Kolmogorov equation)

Suppose that ξ_n, $n \in \mathbb{N}$, is an S-valued Markov chain with n-step transition probabilities $p_n(j|i)$. Then for all $k, n \in \mathbb{N}$

$$p_{n+k}(j|i) = \sum_{s \in S} p_n(j|s) p_k(s|i), \quad i, j \in S. \tag{5.18}$$

Exercise 5.7

Proof (of Proposition 5.3)

Let P and P_n be, respectively, the transition probability matrix and the n-step transition probability matrix. Since $p_n(j|i)$ are the entries of P_n, we only need to show that $P_n = P^n$ for all $n \in \mathbb{N}$. This can be done by induction. The assertion is clearly true for $n = 1$. Suppose that $P_n = P^n$. Then, for $i, j \in S$, by the total probability formula and the Markov property (5.10)

$$\begin{aligned} p_{n+1}(j|i) &= P(\xi_{n+1} = j|\xi_0 = i) \\ &= \sum_{s \in S} P(\xi_{n+1} = j|\xi_0 = i, \xi_n = s) P(\xi_n = s|\xi_0 = i) \\ &= \sum_{s \in S} P(\xi_{n+1} = j|\xi_n = s) P(\xi_n = s|\xi_0 = i) \end{aligned}$$

Proposition 5.4

For all $p \in (0,1)$

$$P(\xi_n = i|\xi_0 = i) \to 0, \quad \text{as } n \to \infty. \tag{5.20}$$

Proof

To begin with, we shall consider the case $p \neq \frac{1}{2}$. When $j = i$, formula (5.19) becomes

$$P(\xi_n = i|\xi_0 = i) = \begin{cases} \frac{(2k)!}{(k!)^2}(pq)^k, & \text{if } n = 2k, \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{5.21}$$

Then, denoting $a_k = \frac{(2k)!}{(k!)^2}(pq)^k$, we have

$$\frac{a_{k+1}}{a_k} = pq\frac{(2k+1)(2k+2)}{(k+1)^2} \to 4pq < 1.$$

Hence, $a_k \to 0$. Thus, $P(\xi_{2k} = i|\xi_0 = i) \to 0$. The result follows, since $P(\xi_{2k+1} = i|\xi_0 = i) = 0 \to 0$.

This argument does not work for $p = \frac{1}{2}$ because $4pq = 1$. In this case we shall need the Stirling formula[1]

$$k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k, \text{ as } k \to \infty. \tag{5.22}$$

Here we use the standard convention: $a_n \sim b_n$ whenever $\frac{a_n}{b_n} \to 1$ as $n \to \infty$. By (5.22)

$$\begin{aligned} a_k &\sim \frac{\sqrt{4\pi k}}{2\pi k}\left(\frac{2k}{e}\right)^{2k}\left(\frac{e}{k}\right)^{2k}(pq)^k \\ &= \frac{1}{\sqrt{\pi k}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

Let us note that the second method works in the first case too. However, in the first case there is no need for anything as sophisticated as the Stirling formula. □

Proposition 5.5

The probability that the random walk ξ_n ever returns to the starting point is

$$1 - |p - q|.$$

Proof

Suppose that $\xi_0 = 0$ and denote by $f_0(n)$ the probability that the process returns to 0 at time n for the first time, i.e.

$$f_0(n) = P(\xi_n = 0, \xi_i \neq 0, i = 1, \cdots, n-1).$$

If also $p_0(n) = P(\xi_n = 0)$ for any $n \in \mathbb{N}$, then we can prove that

$$\sum_{n=1}^{\infty} p_0(n) = \sum_{n=0}^{\infty} p_0(n) \sum_{n=1}^{\infty} f_0(n). \tag{5.23}$$

Since all the numbers involved are non-negative, in order to prove (5.23) we need only to show that

$$p_0(n) = \sum_{k=1}^{n} f_0(k) p_0(n-k) \text{ for } n \geq 1.$$

The total probability formula and the Markov property (5.10) yield

$$\begin{aligned} p_0(n) &= \sum_{k=1}^{n} P(\xi_n = 0, \xi_k = 0, \xi_i \neq 0, i = 1, \cdots, k-1) \\ &= \sum_{k=1}^{n} P(\xi_k = 0, \xi_i \neq 0, i = 1, \cdots, k-1) \\ &\qquad \times P(\xi_n = 0 | \xi_k = 0, \xi_i \neq 0, i = 1, \cdots, k-1) \\ &= \sum_{k=1}^{n} P(\xi_k = 0, \xi_i \neq 0, i = 1, \cdots, k-1) P(\xi_n = 0 | \xi_k = 0) \\ &= \sum_{k=1}^{n} f_0(k) p_0(n-k). \end{aligned}$$

Having proved (5.23), we are going to make use of it. First we notice that the probability that the process will ever return to 0 equals $\sum_{n=1}^{\infty} f_0(n)$. Next, from (5.23) we infer that

$$\begin{aligned} P(\exists n \geq 1 : \xi_n = 0) &= \sum_{n=1}^{\infty} f_0(n) \\ &= 1 - \left(\sum_{n=0}^{\infty} p_0(n) \right)^{-1} = 1 - \left(\sum_{k=0}^{\infty} p_0(2k) \right)^{-1}. \end{aligned}$$

Since $p_0(2k) = \frac{(2k)!}{(k!)^2} (pq)^k$ and

$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = (1-4x)^{-1/2}, \ |x| < \frac{1}{4}, \tag{5.24}$$

it follows, that for $p \neq 1/2$

$$P(\exists n \geq 1 : \xi_n = 0) = 1 - (1-4pq)^{1/2} = 1 - |p-q|, \tag{5.25}$$

since, recalling that $q = 1-p$, we have $1 - 4pq = 1 - 4p + 4p^2 = (1-2p)^2 = (q-p)^2$.

The case $p = 1/2$ is more delicate and we shall not pursue this topic here. Let us only remark that the case $p = 1/2$ needs a special treatment as in Proposition 5.4. □

Proposition 5.6

The probability of survival in Exercise 5.12, part 2) equals 0 if $\lambda \leq 1$, and $1-\hat{r}^k$ if $\lambda > 1$, where k is the initial Vugiel population and $\hat{r} \in (0,1)$ is a solution to

$$r = e^{(r-1)\lambda}. \tag{5.28}$$

Proof

We denote by $\phi(i)$, $i \in \mathbb{N}$ the probability of dying out subject to the condition $\xi_0 = i$. Hence, if $A = \{\xi_n = 0 \text{ for } n \in \mathbb{N}\}$, then

$$\phi(i) = P\left(A|\xi_0 = i\right). \tag{5.29}$$

Obviously, $\phi(0) = 1$ and the total probability formula together with the Markov property (5.10) imply that for each $i \in \mathbb{N}$

$$\begin{aligned}\phi(i) &= \sum_{j=0}^{\infty} P\left(A|\xi_0 = i, \xi_1 = j\right) P\left(\xi_1 = j|\xi_0 = i\right)\\ &= \sum_{j=0}^{\infty} P\left(A|\xi_1 = j\right) P\left(\xi_1 = j|\xi_0 = i\right)\\ &= \sum_{j=0}^{\infty} \phi(j)p(j|i).\end{aligned}$$

Therefore, the sequence $\phi(i)$, $i \in \mathbb{N}$ is bounded (by 1 from above and by 0 from below) and satisfies the following system of equations

$$\begin{aligned}\phi(i) &= \sum_{j=0}^{\infty} \phi(j)p(j|i), \quad i \in \mathbb{N}, \qquad (5.30)\\ \phi(0) &= 1.\end{aligned}$$

UNIT-III

SYLLABUS

.

Markov processes with continuous state space: Introduction, Brownian motion – Weiner Process and differential equations for Weiner process, Kolmogrov equations – first passage time distribution for Weiner process – Ornstein – Uhlenbech process

Consider a system starting in state x_0 at time 0. We suppose that the system remains in state x_0 until some positive time τ_1, at which time the system jumps to a new state $x_1 \neq x_0$. We allow the possibility that the system remains permanently in state x_0, in which case we set $\tau_1 = \infty$. If τ_1 is finite, upon reaching x_1 the system remains there until some time $\tau_2 > \tau_1$ when it jumps to state $x_2 \neq x_1$. If the system never leaves x_1, we set $\tau_2 = \infty$. This procedure is repeated indefinitely. If some $\tau_m = \infty$, we set $\tau_n = \infty$ for $n > m$.

Let $X(t)$ denote the state of the system at time t, defined by

$$X(t) = \begin{cases} x_0, & 0 \leq t < \tau_1, \\ x_1, & \tau_1 \leq t < \tau_2, \\ x_2, & \tau_2 \leq t < \tau_3, \\ \vdots \end{cases} \tag{1}$$

The process defined by (1) is called a *jump process*. At first glance it might appear that (1) defines $X(t)$ for all $t \geq 0$. But this is not necessarily the case.

Consider, for example, a ball bouncing on the floor. Let the state of the system be the number of bounces it has made. We make the physically reasonable assumption that the time in seconds between the nth bounce and the $(n + 1)$th bounce is 2^{-n}. Then $x_n = n$ and

$$\tau_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}.$$

We see that $\tau_n < 2$ and $\tau_n \to 2$ as $n \to \infty$. Thus (1) defines $X(t)$ only fo $0 \le t < 2$. By the time $t = 2$ the ball will have made an infinite numbe of bounces. In this case it would be appropriate to define $X(t) = \infty$ fo $t \ge 2$.

In general, if

$$\lim_{n\to\infty} \tau_n < \infty, \tag{2}$$

we say that the $X(t)$ process *explodes*. If the $X(t)$ process does no explode, i.e., if

$$\lim_{n\to\infty} \tau_n = \infty, \tag{3}$$

then (1) *does* define $X(t)$ for all $t \ge 0$.

We will now specify a probability structure for such a jump process We suppose that all states are of one of two types, *absorbing* or *non-absorbing*. Once the process reaches an absorbing state, it remains there permanently. With each non-absorbing state x, there is associated a distribution function $F_x(t)$, $-\infty < t < \infty$, which vanishes for $t \le 0$, and *transition probabilities* Q_{xy}, $y \in \mathscr{S}$, which are nonnegative and such tha $Q_{xx} = 0$ and

$$\sum_y Q_{xy} = 1. \tag{4}$$

A process starting at x remains there for a random length of time τ_1 having distribution function F_x and then jumps to state $X(\tau_1) = y$ with probability Q_{xy}, $y \in \mathscr{S}$. We assume that τ_1 and $X(\tau_1)$ are chosen independently of each other, i.e., that

$$P_x(\tau_1 \le t, X(\tau_1) = y) = F_x(t) Q_{xy}.$$

The *transition function* $P_{xy}(t)$ cannot be used directly to obtain such probabilities as

$$P(X(t_1) = x_1, \ldots, X(t_n) = x_n)$$

unless the jump process satisfies the *Markov property*, which states that for $0 \le s_1 \le \cdots \le s_n \le s \le t$ and $x_1, \ldots, x_n, x, y \in \mathscr{S}$,

$$P(X(t) = y \mid X(s_1) = x_1, \ldots, X(s_n) = x_n, X(s) = x) = P_{xy}(t - s).$$

By a *Markov pure jump process* we mean a pure jump process that satisfies the Markov property. It can be shown, although not at the level of this book, that a pure jump process is Markovian if and only if all non-absorbing states x are such that

$$P_x(\tau_1 > t + s \mid \tau_1 > s) = P_x(\tau_1 > t), \qquad s, t \ge 0,$$

i.e., such that

$$(5) \qquad \frac{1 - F_x(t + s)}{1 - F_x(s)} = 1 - F_x(t), \qquad s, t \ge 0.$$

Now a distribution function F_x satisfies (5) if and only if it is an exponential distribution function (see Chapter 5 of *Introduction to Probability Theory*). We conclude that a pure jump process is Markovian if and only if F_x is an exponential distribution for all non-absorbing states x.

Let $X(t)$, $0 \leq t < \infty$, be a Markov pure jump process. If x is a non-absorbing state, then F_x has an exponential density f_x. Let q_x denote the parameter of this density. Then $q_x = 1/E_x(\tau_1) > 0$ and

$$f_x(t) = \begin{cases} q_x e^{-q_x t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Observe that

$$P_x(\tau_1 \geq t) = \int_t^\infty q_x e^{-q_x s}\, ds = e^{-q_x t}, \qquad t \geq 0.$$

If x is an absorbing state, we set $q_x = 0$.

It follows from the Markov property that for $0 \leq t_1 \leq \cdots \leq t_n$ and $x_1, \ldots, x_n$ in $\mathscr{S}$,

$$\text{(6)} \qquad P(X(t_1) = x_1, \ldots, X(t_n) = x_n)$$
$$= P(X(t_1) = x_1) P_{x_1 x_2}(t_2 - t_1) \cdots P_{x_{n-1} x_n}(t_n - t_{n-1}).$$

In particular, for $s \geq 0$ and $t \geq 0$

$$P_x(X(t) = z, X(t+s) = y) = P_{xz}(t) P_{zy}(s).$$

Since

$$P_{xy}(t+s) = \sum_z P_x(X(t) = z, X(t+s) = y),$$

we conclude that

$$\text{(7)} \qquad P_{xy}(t+s) = \sum_z P_{xz}(t) P_{zy}(s), \qquad s \geq 0 \text{ and } t \geq 0.$$

Equation (7) is known as the *Chapman-Kolmogorov* equation.

The quantities q_{xy}, $x \in \mathscr{S}$ and $y \in \mathscr{S}$, are called the *infinitesimal parameters* of the process. These parameters determine q_x and Q_{xy}, and thus by our construction determine a unique Markov pure jump process. We can rewrite (10) in terms of the infinitesimal parameters as

(14) $$P'_{xy}(t) = \sum_z q_{xz} P_{zy}(t), \qquad t \geq 0.$$

This equation is known as the *backward equation*.

If $\mathscr{S}$ is finite, we can differentiate the Chapman-Kolmogorov equation with respect to s, obtaining

(15) $$P'_{xy}(t + s) = \sum_z P_{xz}(t) P'_{zy}(s), \qquad s \geq 0 \text{ and } t \geq 0.$$

In particular,

$$P'_{xy}(t) = \sum_z P_{xz}(t) P'_{zy}(0), \qquad t \geq 0,$$

or equivalently,

(16) $$P'_{xy}(t) = \sum_z P_{xz}(t) q_{zy}, \qquad t \geq 0.$$

Formula (16) is known as the *forward equation*. It can be shown that (15) and (16) hold even if $\mathscr{S}$ is infinite, but the proofs are not easy and will be omitted.

In Section 3.2 we will describe some examples in which the backward or forward equation can be used to find explicit formulas for $P_{xy}(t)$.

UNIT-IV

SYLLABUS

Branching Processes: Introduction – properties of generating functions of Branching process– Distribution of the total number of progeny, Continuous- Time Markov Branching Process, Age dependent branching process: Bellman-Harris process.

Let $\mathscr{S} = \{0, 1, \ldots, d\}$ or $\mathscr{S} = \{0, 1, 2, \ldots\}$. By a *birth and death process* on $\mathscr{S}$ we mean a Markov pure jump process on $\mathscr{S}$ having infinitesimal parameters q_{xy} such that

$$q_{xy} = 0, \qquad |y - x| > 1.$$

Thus a birth and death process starting at x can in one jump go only to the states $x - 1$ or $x + 1$.

The parameters $\lambda_x = q_{x,x+1}$, $x \in \mathscr{S}$, and $\mu_x = q_{x,x-1}$, $x \in \mathscr{S}$, are called respectively the *birth rates* and *death rates* of the process. The parameters q_x and Q_{xy} of the process can be expressed simply in terms of the birth and death rates. By (13)

$$-q_{xx} = q_x = q_{x,x+1} + q_{x,x-1},$$

so that

(17) $$q_{xx} = -(\lambda_x + \mu_x) \quad \text{and} \quad q_x = \lambda_x + \mu_x.$$

Thus x is an absorbing state if and only if $\lambda_x = \mu_x = 0$. If x is a non-absorbing state, then by (12)

(18) $$Q_{xy} = \begin{cases} \dfrac{\mu_x}{\lambda_x + \mu_x}, & y = x - 1, \\ \dfrac{\lambda_x}{\lambda_x + \mu_x}, & y = x + 1, \\ 0, & \text{elsewhere.} \end{cases}$$

A birth and death process is called a *pure birth process* if $\mu_x = 0$, $x \in \mathscr{S}$, and a *pure death process* if $\lambda_x = 0$, $x \in \mathscr{S}$. A pure birth process can move only to the right, and a pure death process can move only to the left.

Example 1. Branching process. Consider a collection of particles which act independently in giving rise to succeeding generations of particles. Suppose that each particle, from the time it appears, waits a random length of time having an exponential distribution with parameter q and then splits into two identical particles with probability p and disappears with probability $1 - p$. Let $X(t)$, $0 \leq t < \infty$, denote the number of particles present at time t. This branching process is a birth and death process. Find the birth and death rates.

Consider a branching process starting out with x particles. Let $\xi_1, \ldots, \xi_x$ be the times until these particles split apart or disappear. Then $\xi_1, \ldots, \xi_x$ each has an exponential distribution with parameter q, and hence $\tau_1 = \min(\xi_1, \ldots, \xi_x)$ has an exponential distribution with parameter $q_x = xq$. Whichever particle acts first has probability p of splitting into two particles and probability $1 - p$ of disappearing. Thus for $x \geq 1$

$$Q_{x,x+1} = p \qquad \text{and} \qquad Q_{x,x-1} = 1 - p.$$

State 0 is an absorbing state. Since $\lambda_x = q_x Q_{x,x+1}$ and $\mu_x = q_x Q_{x,x-1}$, we conclude that

$$\lambda_x = xqp \qquad \text{and} \qquad \mu_x = xq(1-p), \qquad x \geq 0.$$

In the preceding example we did not actually prove that the process is a birth and death process, i.e., that it "starts from scratch" after making a jump. This intuitively reasonable property basically depends on the fact that an exponentially distributed random variable ξ satisfies the formula

$$P(\xi > t + s \mid \xi > s) = P(\xi > t), \qquad s, t \geq 0,$$

but a rigorous proof is complicated.

By (17) and the definition of λ_x and μ_x, the backward and forward equations for a birth and death process can be written respectively as

$$(20) \qquad P'_{xy}(t) = \mu_x P_{x-1,y}(t) - (\lambda_x + \mu_x)P_{xy}(t) + \lambda_x P_{x+1,y}(t), \qquad t \geq 0,$$

and

$$(21) \qquad P'_{xy}(t) = \lambda_{y-1} P_{x,y-1}(t) - (\lambda_y + \mu_y)P_{xy}(t) + \mu_{y+1} P_{x,y+1}(t), \qquad t \geq 0.$$

In (21) we set $\lambda_{-1} = 0$, and if $\mathscr{S} = \{0, \ldots, d\}$ for $d < \infty$, we set $\mu_{d+1} = 0$.

We will solve the backward and forward equations for a birth and death process in some special cases. To do so we will use the result that if

$$(22) \qquad f'(t) = -\alpha f(t) + g(t), \qquad t \geq 0,$$

then

$$(23) \qquad f(t) = f(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} g(s)\, ds, \qquad t \geq 0.$$

The proof of this standard result is very easy. We multiply (22) through by $e^{\alpha t}$ and rewrite the resulting equation as

$$\frac{d}{dt}(e^{\alpha t} f(t)) = e^{\alpha t} g(t).$$

Integrating from 0 to t we find that

$$e^{\alpha t} f(t) - f(0) = \int_0^t e^{\alpha s} g(s)\, ds,$$

3.2.1. Two-state birth and death process. Consider a birth and death process having state space $\mathscr{S} = \{0, 1\}$, and suppose that 0 and 1 are both non-absorbing states. Since $\mu_0 = \lambda_1 = 0$, the process is determined by the parameters λ_0 and μ_1. For simplicity in notation we set $\lambda = \lambda_0$ and $\mu = \mu_1$. We can interpret such a process by thinking of state 1 as the system (e.g., telephone or machine) operating and state 0 as the system being idle. We suppose that starting from an idle state the system remains idle for a random length of time which is exponentially distributed with parameter λ, and that starting in an operating state the system continues operating for a random length of time which is exponentially distributed with parameter μ.

We will find the transition function of the process by solving the backward equation. It is left as an exercise for the reader to obtain the same results by solving the forward equation.

Setting $y = 0$ in (20), we see that

$$P'_{00}(t) = -\lambda P_{00}(t) + \lambda P_{10}(t), \qquad t \geq 0, \tag{24}$$

and

$$P'_{10}(t) = \mu P_{00}(t) - \mu P_{10}(t), \qquad t \geq 0. \tag{25}$$

Subtracting the second equation from the first,

$$\frac{d}{dt}(P_{00}(t) - P_{10}(t)) = -(\lambda + \mu)(P_{00}(t) - P_{10}(t)).$$

Applying (23),

$$\begin{aligned} P_{00}(t) - P_{10}(t) &= (P_{00}(0) - P_{10}(0))e^{-(\lambda+\mu)t} \\ &= e^{-(\lambda+\mu)t}. \end{aligned} \tag{26}$$

Here we have used the formulas $P_{00}(0) = 1$ and $P_{10}(0) = 0$. It now follows from (24) that

$$\begin{aligned} P'_{00}(t) &= -\lambda(P_{00}(t) - P_{10}(t)) \\ &= -\lambda e^{-(\lambda+\mu)t}. \end{aligned}$$

Thus

$$\begin{aligned} P_{00}(t) &= P_{00}(0) + \int_0^t P'_{00}(s)\, ds \\ &= 1 - \int_0^t \lambda e^{-(\lambda+\mu)s}\, ds \\ &= 1 - \frac{\lambda}{\lambda + \mu}(1 - e^{-(\lambda+\mu)t}), \end{aligned}$$

3.2.2. Poisson process. Consider a pure birth process $X(t)$, $0 \le t < \infty$, on the nonnegative integers such that

$$\lambda_x = \lambda > 0, \qquad x \ge 0.$$

Since a pure birth process can move only to the right,

(33) $$P_{xy}(t) = 0, \qquad y < x \text{ and } t \ge 0.$$

Also $P_{xx}(t) = P_x(\tau_1 > t)$ and hence

(34) $$P_{xx}(t) = e^{-\lambda t}, \qquad t \ge 0.$$

UNIT-V

SYLLABUS

Stochastic Processes in Queuing Systems: Concepts – Queuing model M/M1 – transient behavior of M/M/1 model – Birth and death process in Queuing theory: M/M/1 – Model related distributions – M/M/1 - M/M/S/S – loss system - M/M/S/M – Non birth and death Queuing process: Bulk queues – M(x)/M/1.

The analysis of a queueing system with fixed (deterministic) interarrival and service times does not present much difficulty. We shall be concerned with models or systems where one or both (interarrival and service times) are stochastic. Their analyses will involve a stochastic description of the system and related performance measures, as discussed below.

(1) Distribution of the number $N(t)$ in the system at time t (the number in the queue and the one being served, if any). $N(t)$ is also called the queue length of the system at time t. By the *number* in the system (queue), we will always mean the *number of customers* in the system (queue).

(2) Distribution of the waiting time in the queue (in the system), the time that an arrival has to wait in the queue (remain in the system). If W_n denotes the waiting time of the nth arrival, then of interest is the distribution of W_n.

(3) Distribution of the virtual waiting time $W(t)$—the length of time an arrival has to wait had he arrived at time t.

(4) Distribution of the busy period being the length (or duration) of time during which the server remains busy. The busy period is the interval from the moment of arrival of a unit at an empty system to the moment that the channel becomes free for the first time. The busy period is a random variable.

From a complete description of the above distributions, various performance measures of interest are obtained.

The problems studied in queueing theory may be grouped as:

(i) Stochastic behavior of various random variables, or stochastic processes that arise, and evaluation of the related performance measures;

(ii) Method of solution—exact, transform, algorithmic, asymptotic, numerical, approximations, etc.;

(iii) Nature of solution—time dependent, limiting form, etc.;

(iv) Control and design of queues—comparison of behavior and performances under various situations, as well as queue disciplines, service rules, strategies, etc.; and

(v) Optimization of specific objective functions involving performance measures, associated cost functions, etc.

(3) Distribution of the virtual waiting time $W(t)$—the length of time an arrival has to wait had he arrived at time t.

(4) Distribution of the busy period being the length (or duration) of time during which the server remains busy. The busy period is the interval from the moment of arrival of a unit at an empty system to the moment that the channel becomes free for the first time. The busy period is a random variable.

From a complete description of the above distributions, various performance measures of interest are obtained.

The problems studied in queueing theory may be grouped as:

(i) Stochastic behavior of various random variables, or stochastic processes that arise, and evaluation of the related performance measures;

(ii) Method of solution—exact, transform, algorithmic, asymptotic, numerical, approximations, etc.;

(iii) Nature of solution—time dependent, limiting form, etc.;

(iv) Control and design of queues—comparison of behavior and performances under various situations, as well as queue disciplines, service rules, strategies, etc.; and

(v) Optimization of specific objective functions involving performance measures, associated cost functions, etc.

The notation introduced by Kendall (1951) is generally adopted to denote a queueing model. It consists of the specifications of three basic characteristics: the input, the service time, and the number of (parallel) servers. Symbols used to denote some of the common formulations are as follows:

M	Exponential interarrival (Poisson input) and service time distribution (having Markov property)
E_k	Erlang-k distribution
H	Hyperexponential distribution
PH	Phase-type distribution
D	Deterministic (constant)(interarrival or service time)
G	Arbitrary (general) distribution

Denote by $N(t)$ the number in the system (the number in the queue plus the number being served, if any) at time t measured from a fixed initial moment $(t = 0)$ and its probability distribution by

$$p_n(t) = Pr\{N(t) = n\}, \quad n = 0, 1, 2, \ldots. \text{ Then}$$
$$p_i(0) = 1, \quad (p_j(0) = 0,\ j \neq i)$$

implies that the number of customers at the initial moment was i (where i could be $0, 1, 2, \ldots.$). For a complete description of the stochastic behavior of the queue-length processes $\{N(t), t \geq 0\}$, we need to find a time-dependent solution $p_n(t), n \geq 0$. It is often difficult to obtain such solutions. Or even when found, these may be too complicated to handle. For many practical situations, however, one needs the equilibrium behavior—that is, the behavior when the system reaches an equilibrium state after being in operation for a sufficiently long time. In other words, one is often interested in the limiting behavior of $p_n(t)$ as $t \to \infty$. Denote

$$p_n = \lim_{t \to \infty} p_n(t), \quad n = 0, 1, 2, \ldots$$

Theorem 2.1. Burke's Theorem

In any queueing system in which arrivals and departures occur one by one and that has reached equilibrium state,

$$a_n = d_n \quad \text{for all } n \geq 0.$$

Proof: Consider that an arrival will see, on arrival, n in the system; then the number in the system will increase by 1 and will go from n to $n + 1$. Again, a departure will leave n in the system, implying that the number in the system will decrease by 1 and will go from $n + 1$ to n. In any interval of time T, the number of transitions A from n to $n + 1$ and the number of transitions B from $n + 1$ to n will differ at most by 1; in other words, either $A = B$ or $A \sim B = 1$. Then for large T, the rates of transitions A/T and B/T will be equal. Thus, on the average, arrivals and departures always see the same number of customers, which means that $a_n = d_n$ always and for every $n \geq 0$. ■

Assume that steady state exists and let

$$p_n = \lim_{t\to\infty} Pr\{N(t) = n\}, \quad n = 0, 1, 2, \ldots, \tag{3.2.1}$$

$N(t)$ being the number in the system (in the service channel and in queue, if any) at instant t; p_n is also the proportion of time the process is in state n.

We proceed to derive the equations involving p_n by using the rate-equality principle; then we proceed to solve the equations to find p_n.

Consider state n $(n \geq 0)$. The system can go to the next state $(n+1)$ at rate λp_n, and it can come down from state $(n+1)$ to the original state n at rate μp_{n+1}.

For equilibrium these two rates—that is, the rate up from a particular state n to the next state $(n+1)$—and the rate down—that is, from the state $(n+1)$ to the original state n—must be equal. (In equilibrium, rate up = rate down.) This implies that

$$\lambda p_n = \mu p_{n+1} \quad (n \geq 0)$$

$$\text{or} \quad p_{n+1} = \frac{\lambda}{\mu} p_n = a p_n = a^2 p_{n-1} \tag{3.2.2}$$

$$\ldots \quad \ldots$$

$$= a^{n+1} p_0$$

$$\text{or} \quad p_n = a^n p_0, \quad n \geq 0$$

Using $\sum_{n=0}^{\infty} p_n = 1$, one gets, for $a < 1$,

$$p_n = (1-a)a^n, \quad n = 0, 1, 2, \ldots$$

Since $a = \rho$, we get

$$p_0 = (1-a) = 1 - \rho$$

$$p_n = (1-\rho)\rho^n, \quad n = 1, 2, \ldots \tag{3.2.3}$$

The distribution is geometric and is memoryless.

Let N be the number and W the waiting time in the system in steady state. We have

$$E\{N\} = \sum_{n=0}^{\infty} n p_n = \sum_{n=1}^{\infty} n(1-\rho)\rho^n$$

$$= \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1} = \frac{\rho(1-\rho)}{(1-\rho)^2} = \frac{\rho}{1-\rho} \qquad (3.2.4)$$

and

$$E\{N^2\} = \sum_{n=0}^{\infty} n^2 p_n = \sum_{n=1}^{\infty} n^2(1-\rho)\rho^n$$

$$= (1-\rho)\sum\{(n^2-n)+n\}\rho^n$$

$$= (1-\rho)\frac{2\rho^2}{(1-\rho)^3} + \frac{(1-\rho)\rho}{(1-\rho)^2} = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{1-\rho}$$

$$= \frac{\rho+\rho^2}{(1-\rho)^2}$$

so that $\operatorname{var}\{N\} = E\{N^2\} - [E\{N\}]^2$

$$= \frac{\rho}{(1-\rho)^2}. \qquad (3.2.5)$$

Using Little's formula $L = \lambda W$, we get that the expected waiting time in the system, $E\{W\}$, equals

$$E\{W\} = \frac{E\{N\}}{\lambda} = \frac{1}{\lambda}\frac{\rho}{(1-\rho)} = \frac{1}{\mu(1-\rho)} \qquad (3.2.6)$$

Reg. No..................

17MMP402

Karpagam Academy of Higher Education
Coimbatore-21
Department of Mathematics
Fourth Semester- I Internal test
Stochastic Process

Date: 04.02.2019(AN) Time: 2 hours
Class: II M.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. There are ——- types of stochastic processes
a. 1 b. 2
c. 3 d. 4

2. A ——- state Markov process is called a *Markov chain*
a. discrete b. continuous
c. both a and b d. neither a nor b

3. In a process, future state depends only on the present state is
a. random process b. stochastic process
c. Markov process d. non Markov process

4. In a Markov chain $\{X_n : n \geq 0\}$,
$$P(X_{n+1} = j|X_n = i_n X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) =$$
a. $P(X_{n+1} = j|X_n = i_n)$
b. $P(X_{n+1} = j|X_n = i_n X_{n-1} = i_{n-1})$
c. neither a nor b
d. both a and b

5. In a transition probability matrix, —— equal to 1
a. row b. column
c. neither a nor b d. both a and b

6. In a doubly stochastic matrix, —— equal to 1
a. row b. column
c. neither a nor b d. both a and b

7. If $d(j) ----1$, then state j is called periodic
a. = b. >
c. < d. neither a nor b

8. State j is said to be an absorbing state if $p_{jj} =$
a. 1 b. 0
c. neither a nor b d. both a and b

9. The collection of r.v.'s $\{X_n : n \geq 0\}$ with $P(X_n = 0) = p$ and $P(X_n = 1) = 1 - p,\ 0 \leq p \leq 1$, is
a. random process b. stochastic process
c. Bernoulli process d. all the above

10. Consider patients coming to a doctor's office at random points in time. Let X_n, denote the time (in hours) that the nth patient has to wait in the office before being admitted to see the doctor. The process $\{X_n\}$ is ——- time and —— state space
a. discrete, discrete b. continuous, continuous
c. discrete, continuous d. continuous, discrete

11. State j is absorbing iff
a. $p_{jj} = 1$ b. $p_{jk} = 0$ for all $k \neq j$
c. both a and b d. neither a nor b

12. $i \to j$ if $p_{ij} > -----$ for some $n \geq 1$
a. 1 b. 2
c. -1 d. 0

13. A state j is aperiodic iff
a. $p_{jj} \neq 1$ b. $p_{jj} \neq 0$
c. both a and b d. neither a nor b

14. A state j is persistent if $F_{jj} =$
a. 1 b. 2
c. 3 d. 4

15. A state j is transient if $F_{jj} <$
a. 1 b. 2
c. 3 d. 4

16. A state j is ergodic if j is
a. persistent b. non-null
c. ergodic d. all the above

17. If $\mu_{jj} < \infty$ then j is
a. persistent b. non-null
c. ergodic d. all the above

18. If $\sum_{n=0}^{\infty} p_{jj}^{n} = \infty$ then j is
a. persistent b. non-null
c. ergodic d. all the above

19. If $P = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$, then P^{103}
a. P b. P^3
c. both a and b d. neither a nor b

20. Which of the following is true?
a. $p_{jk}^{m+n} \geq p_{jr}^{n} p_{rk}^{m}$ b. $p_{jk}^{m+n} \geq p_{jr}^{m} p_{rk}^{n}$
c. both a and b d. neither a nor b

Part B-($3 \times 2 = 6$ **marks)**

21. Define Markov process

22. Define aperiodic state

23. Draw the transition diagram of a Markov chain with 4 recurrent states, each with periodicity 4.

Part C-($3 \times 8 = 24$ **marks)**

24. a) Describe about Polya's urn model

OR

b) Consider a two-state Markov chain with the transition probability matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

with $0 < a, b < 1$. Find P^n when $n \to \infty$

25. a) Draw the state transition diagram and classify the states of the Markov chain

$$P = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

OR

b) Consider a Markov chain with state space $\{0, 1\}$ and transition probability matrix

$$P = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Classify the states 0 and 1.

26. a) Consider a Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Classify the state 0.

OR

b) Show that if P is a Markov matrix, then P^n is also a Markov matrix for any positive integer n.

Reg. No..................

17MMP402

Karpagam Academy of Higher Education
Coimbatore-21
Department of Mathematics
Fourth Semester- II Internal test
Stochastic Process

Date:11.03.2019(FN) Time: 2 hours
Class: II M.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. $P_n(h) = 0(h)$ if $n \geq ---$
   a. 1 b. 2
   c. 0 d. all the above

2. $P(z)$ converges for
   a. $|z| = 1$ b. $|z| < 1$
   c. either a or b d. neither a nor b

3. For a Poisson distribution, $P''(1) =$
   a. λ b. λ^2
   c. λ^3 d. λ^4

4. $P'(1) =$
   a. $E[X]$ b. $E[X^2]$
   c. neither a nor b d. both a and b

5. Suppose X is a geometric random variable. Then
$$P(X = s + r|X \geq s) =$$
   a. $P(X = s)$ b. $P(X = r)$
   c. neither a nor b d. both a and b

6. Suppose X follows logarithmic series distribution. Then $E[X]$
   a. $\frac{\alpha q}{1-q}$ b. $\frac{\alpha(1-p)}{q}$
   c. neither a nor b d. both a and b

7. Recurssive formula for Poisson distribution is
   a. $p_{k+1} = \frac{\lambda}{k+1}p_k$ b. $p_{k+1} = \frac{\lambda}{k}p_k$
   c. $p_{k+1} = \frac{\lambda^k}{k}p_k$ d. $p_{k+1} = \frac{\lambda^k}{k+1}p_k$

8. $P''(1) =$
   a. $E[X]$ b. $E[X^2]$
   c. $E[X(X-1)]$ d. $\text{Var}(X)$

9. In *Poisson* distribution X and Y are ——variables
   a. dependent b. indendent
   c. neither a or b d. none

10. The Poisson distribution is a —- probability distribution
    a. discrete continuous b. continuous
    c. discrete d. continuous discrete

11. λ is a————
    a. variable b. constant
    c. parameter d. none

12. In Poisson distribution λ——μ
    a. $\geq$ b. $=$
    c. $\leq$ d. $\neq$

13. $\lambda >$ —— is called the rate of the distribution
    a. 1 b. 0
    c. 2 d none

14. Exponential distribution is calculated using ——— by parts
    a. differentiating b integration
    c both a and b d. neither a and b

15. Markov process is a ———-property
a. memory less b memorable
c. neither a or b d. none

16. The outcomes are called the ———of Markov chain
a. states b. trails
c. both a and b d. none

17. The states of Markov chain can be described by———
a. only graph b. only matrix
c. neither a or b d. matrix and graph

18. In M/M/1 queue the first M denote———
a. arrival b. srevice
c. server d.none

19. The traffic intensity is denoted by———
a. λ b. μ
c. ρ d. none

20. Steady state is ——- on time
a. dependent b. independent
c. neither a or b c. none

Part B-($3 \times 2 = 6$ **marks)**

21. Define binomial distribution

22. Find the mean value of geometric distribution

23. State two properties of exponential distribution

Part C-($3 \times 8 = 24$ **marks)**

24. a) Derive Var(X) in terms of P.G.F of the random variable X

OR

b) Describe about logarithmic series distribution

25. a) If X is an exponential distribution, show that $E[X^r] = \frac{r!}{\lambda^r}$

OR

b) State and prove two properties of Poisson distribution

26. a) Descibe about Poisson process

OR

b) Find the average number of customers in $M/M/1$ queue.