KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act1956)

Coimbatore – 641 021.

DEPARTMENT OF MATHEMATICS

19MMU101	CALCULUS	Semester – I 4H – 4C
Instruction Hours / week: L: 4 T: 0 P: 0	**Marks:** Internal: **40**	External: **60** Total: **100** **End Semester Exam:** 3 Hours

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Course Objectives

This course enables the students to learn

- The concepts of essentials of concavity, inflection points and its geometrical applications.
- The Higher order derivatives and its applications in business, economics and life sciences.

Course Outcomes (COs)

On successful completion of this course, the students will be able to

1. Understand the concepts of Linear, quadratic, power, polynomial, algebraic, rational, trigonometric, exponential, hyperbolic and logarithmic functions.
2. Explore the concept of reduction formula and calculate limits in indeterminate forms by a repeated use of L'Hospital rule.
3. Use single and multiple integration to calculate the arc length, area and volume.
4. Understand the techniques of sketching conics and properties of conics.
5. Acquire the knowledge on application of vector functions.

UNIT – I

DIFFENTIAL CALCULUS

Hyperbolic functions, higher order derivatives, Leibniz rule and its applications to problems of type $e^{ax+b}\sin x$, $e^{ax+b}\cos x$, $(ax+b)^n\sin x$, and $(ax+b)^n\cos x$.

UNIT II

INTEGRAL CALCULUS

Reduction formulae, derivations and illustrations of reduction formulae of the type $\int \sin nx\, dx$, $\int \cos nx\, dx$, $\int \tan nx\, dx$, $\int \sec nx\, dx$, $\int \log x^n\, dx$, $\int \sin^n x \sin^m x\, dx$. Curve tracing in Cartesian coordinates, tracing in polar coordinates of standard curves, L'Hospital's rule, applications in business, economics and life sciences.

UNIT III

APPLICATIONS OF INTEGRATION

Volumes by slicing, disks and washers methods, volumes by cylindrical shells, parametric equations, parameterizing a curve, arc length, arc length of parametric curves, area of surface of revolution.

UNIT IV
CURVE SKETCHING
Concavity and Inflection points, asymptotes. Techniques of sketching conics, reflection properties of conics, rotation of axes and second degree equations, classification into conics using the discriminant, polar equations of conics.

UNIT V
VECTOR FUCTIONS
Introduction to vector functions, operations with vector-valued functions, limits and continuity of vector functions, differentiation and integration of vector functions, tangent and normal components of acceleration, modeling ballistics and planetary motion, Kepler's second law.

SUGGESTED READINGS

1. Thomas G.B., and Finney R.L., (2008).Calculus, Ninth Edition, Pearson Education, Delhi.
2. Anton H., Bivens I., and Davis S.,(2017). Calculus, Tenth Edition, John Wiley and Sons (Asia) P. Ltd., Singapore.
3. Strauss M.J., Bradley G.L.,and Smith K. J., (2007). Calculus, Third Edition, Dorling Kindersley (India) Pvt. Ltd. (Pearson Education), Delhi.
4. Courant R., and John F., (2000). Introduction to Calculus and Analysis (Volumes I & II), Springer- Verlag, New York.

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Instruction Hours / week: L: 4 T: 0 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**
End Semester Exam: 3 Hours

Name of the Faculty : M.Sangeetha

S.No.	Lecture Duration (Hr)	Topics to be covered	Support Materials
UNIT – I			
1.	1	Introduction to Hyperbolic function	S3:Ch 7;Pg:350-353
2.	1	Inverse hyperbolic function	S3:Ch 7;Pg:353-356
3.	1	Higher order derivatives	S4:Ch 4;Pg:156-159
4.	1	Leibiniz rule and its applications	S4:Ch 4;Pg:169-177
5.	1	Problems on type e^{ax+b} sinx, e^{ax+b}cos x,$(ax+b)^n$sinx , $(ax+b)^n$cosx	S4:Ch 4;Pg:178-179
6.	1	Continuation on on type e^{ax+b} sinx, e^{ax+b}cos x,$(ax+b)^n$sinx , $(ax+b)^n$cosx	S4:Ch 4;Pg:179-180
7.	1	Finding Inflection point	S3:Ch 4;Pg:124-129
8.	1	Curve Sketching with Asymptotes	S4:Ch 12;Pg:389-409
9.	1	Recapitulation and discussion of possible question	
Total : 9 hrs			
UNIT – II			
1.	1	Reduction formula – derivation and illustration	S2:Ch 7;Pg:497-498
2.	1	Problems based on reduction formula	S2:Ch 7;Pg:500-503
3.	1	Continuation of problems on reduction formula	S2:Ch 7;Pg:503-506
4.	1	Curve tracing in Cartesian Coordinates	S2:Ch 11;Pg:767-770
5.	1	Tracing in polar coordinate for standard curves	S4:Ch 1;Pg:101-103
6.	1	Theorm on L'Hospital's Rule	S3:Ch 4;Pg:148-150

7.	1	Problems based on L'Hospital's Rule	S3:Ch 4;Pg:151-155
8.	1	Application in business, economics and life sciences.	S3:Ch 6;Pg:287-294
9.	1	Recapitulation and discussion of possible question	
Total : 9 hrs			
UNIT – III			
1.	1	Volume by slicing	S2:Ch 6;Pg:421-424
2.	1	Volume by Disks methods	S1:Ch 5;Pg:397-399
3.	1	Volume by washers methods	S1:Ch 5;Pg:400-403
4.	1	Volumes by cylindrical shells	S2:Ch 6;Pg:432-436
5.	1	Area of a surface of revolution	S2:Ch 6;Pg:444-447
6.	1	Parametric Equations	S2:Ch 10;Pg:692-695
7.	1	Tangent Lines to Parametric Curves	S2:Ch 10;Pg:695-697
8.	1	Arc Length of Parametric Curves	S2:Ch 10;Pg:697-700
9.	1	Recapitulation and discussion of possible question	
Total : 9 hrs			
UNIT – IV			
1.	1	Introduction to conic section	S2:Ch 10;Pg:730-732
2.	1	Techniques of sketching conics	S1:Ch 9;Pg:727-730
3.	1	Equations of conics in standard position	S2:Ch 10;Pg:732-740
4.	1	Translated conics	S2:Ch 10;Pg:740-742
5.	1	Reflection properties of the conic sections	S2:Ch 10;Pg:742-744
6.	1	Rotation of axes with examples	S2:Ch 10;Pg:748-752
7.	1	Classification of conics using discriminant	S1:Ch 9;Pg:748-750
8.	1	Polar equation in conics	S2:Ch 10;Pg:755-759
9.	1	Recapitulation and discussion of possible question	
Total : 9 hrs			
UNIT – V			
1.	1	The Triple product	S1:Ch 10;Pg:824-835
2.	1	Introduction to Vector functions	S3:Ch 10;Pg:494-496
3.	1	Operation with Vector-valued functions	S3:Ch 10;Pg:496-497

4.	1	Limits and continuity of vector functions	S3:Ch 10;Pg:498-500
5.	1	Differentiation and integration of vector functions	S3:Ch 10;Pg:502-511
6.	1	Tangent and normal components of acceleration	S3:Ch 10;Pg:522-525
7.	1	Modeling ballistics and planetary motion	S3:Ch 10;Pg:512-516
8.	1	Kepler's second law	S3:Ch 10;Pg:516-519
9.	1	Recapitulation and discussion of possible question	
10.	1	Discussion of pervious ESE question papers	
11.	1	Discussion of pervious ESE question papers	
12.	1	Discussion of pervious ESE question papers	
Total : 12 hrs			

SUGGESTED READINGS
Reference Book

1.Thomas G.B., and Finney R.L ., (2008). Calculus , Ninth Edition, Pearson Education, Delhi.
2.Anton H., Bivens I ., and Davis S., (2017). Calculus , Tenth Edition , John Wiley and sons (Asia) Pvt Ltd, Singapore.
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4.Courant R and John F (2000). Introduction to Calculus and Analysis (Volume I & II), Springer verlag, NewYork.

UNIT – I

DIFFENTIAL CALCULUS

Hyperbolic functions, higher order derivatives, Leibniz rule and its applications to problems of type $e^{ax+b}\sin x$, $e^{ax+b}\cos x$, $(ax+b)^n\sin x$, and $(ax+b)^n\cos x$.

Hyperbolic functions

1. Introduction

The three hyperolic functions f(x) = sinhx, f(x) = coshx and f(x) = tanhx. We shall look at the graphs of these functions, investigate some of their properties.

2. Defining $f(x) = \cosh x$

The hyperbolic functions $\cosh x$ and $\sinh x$ are defined using the exponential function e^x. We shall start with $\cosh x$. This is defined by the formula

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

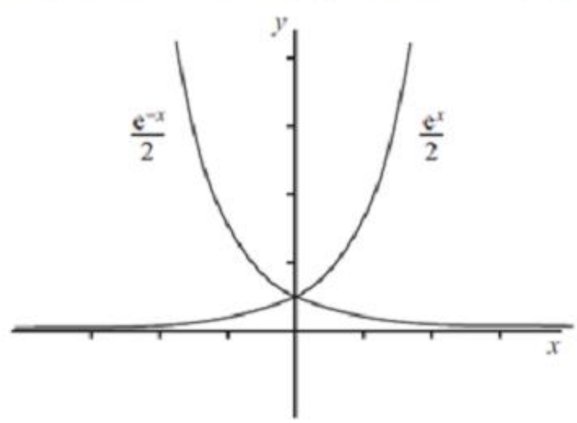
We can use our knowledge of the graphs of e^x and e^{-x} to sketch the graph of $\cosh x$. First, let us calculate the value of $\cosh 0$. When $x = 0$, $e^x = 1$ and $e^{-x} = 1$. So

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1.$$

Next, let us see what happens as x gets large. We shall rewrite $\cosh x$ as

$$\cosh x = \frac{e^x}{2} + \frac{e^{-x}}{2}.$$

1 To see how this behaves as x gets large, recall the graphs of the two exponential functions.



As x gets larger, e^x increases quickly, but e^{-x} decreases quickly. So the second part of the sum $e^x/2 + e^{-x}/2$ gets very small as x gets large. Therefore, as x gets larger, $\cosh x$ gets closer and closer to $e^x/2$. We write this as

$$\cosh x \approx \frac{e^x}{2} \qquad \text{for large } x.$$

But the graph of $\cosh x$ will always stay above the graph of $e^x/2$. This is because, even though $e^{-x}/2$ (the second part of the sum) gets very small, it is always greater than zero. As x gets larger and larger the difference between the two graphs gets smaller and smaller.

3. Defining $f(x) = \sinh x$

We shall now look at the hyperbolic function $\sinh x$. In speech, this function is pronounced as 'shine', or sometimes as 'sinch'. The function is defined by the formula

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

4. Defining $f(x) = \tanh x$

We shall now look at the hyperbolic function $\tanh x$. In speech, this function is pronounced as 'tansh', or sometimes as 'than'. The function is defined by the formula

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

We can work out $\tanh x$ out in terms of exponential functions. We know how $\sinh x$ and $\cosh x$ are defined, so we can write $\tanh x$ as

$$\tanh x = \frac{e^x - e^{-x}}{2} \div \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

5. Identities for hyperbolic functions

Hyperbolic functions have identities which are similar to, but not the same as, the identities for trigonometric functions. In this section we shall prove two of these identities, and list some others.

The first identity is

$$\cosh^2 x - \sinh^2 x = 1\,.$$

To prove this, we start by substituting the definitions for $\sinh x$ and $\cosh x$:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2.$$

If we expand the two squares in the numerators, we obtain

$$\begin{aligned}(e^x + e^{-x})^2 &= e^{2x} + 2(e^x)(e^{-x}) + e^{-2x} \\ &= e^{2x} + 2 + e^{-2x}\end{aligned}$$

and

$$\begin{aligned}(e^x - e^{-x})^2 &= e^{2x} - 2(e^x)(e^{-x}) + e^{-2x} \\ &= e^{2x} - 2 + e^{-2x},\end{aligned}$$

where in each case we use the fact that $(e^x)(e^{-x}) = e^{x+(-x)} = e^0 = 1$. Using these expansions in our formula, we obtain

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}.$$

Now we can move the factor of $\frac{1}{4}$ out to the front, so that

$$\cosh^2 x - \sinh^2 x = \tfrac{1}{4}\left((e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right).$$

If, finally, we remove the inner brackets and simplify, we obtain

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) \\ &= \tfrac{1}{4} \times 4 \\ &= 1,\end{aligned}$$

Here is another identity involving hyperbolic functions:

$$\sinh 2x = 2 \sinh x \cosh x.$$

On the left-hand side we have $\sinh 2x$ so, from the definition,

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}.$$

We want to manipulate the right-hand side to achieve this. So we shall start by substituting the definitions of $\sinh x$ and $\cosh x$ into the right-hand side:

$$2\sinh x\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right).$$

We can cancel the 2 at the start with one of the 2's in the denominator, and then we can take the remaining factor of $\frac{1}{2}$ out to the front. We get

$$2\sinh x\cosh x = \tfrac{1}{2}(e^x - e^{-x})(e^x + e^{-x}).$$

Now we can multiply the two brackets together. This gives us

$$2\sinh x\cosh x = \tfrac{1}{2}(e^{2x} + 1 - 1 - e^{-2x}).$$

Cancelling the ones finally gives us

$$2\sinh x\cosh x = \tfrac{1}{2}(e^{2x} - e^{-2x}) = \sinh 2x,$$

which is what we wanted to achieve.

There are several more identities involving hyperbolic functions:

$$\begin{aligned}
\cosh 2x &= (\cosh x)^2 + (\sinh x)^2 \\
\sinh(x+y) &= \sinh x\cosh y + \sinh y\cosh x \\
\cosh(x+y) &= \cosh x\cosh y + \sinh x\sinh y \\
\cosh^2\frac{x}{2} &= \frac{1+\cosh x}{2} \\
\sinh^2\frac{x}{2} &= \frac{\cosh x - 1}{2}
\end{aligned}$$

Remark : The word 'hyperbolic' stems from the fact that x = a cosh t, y = b sinh t are the general co-ordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ = 1, inasmuch as

$$\frac{(a\cosh t)^2}{a^2} - \frac{(b\cosh t)^2}{b^2} = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$$

$$= 1.$$

Just as we have standard trigonometric identities $\cos^2 x + \sin^2 x = 1$ etc., there are identities involving hyperbolic functions sinh x, cosh x etc. For instance,

$$\cosh^2 x - \sinh^2 x = 1$$

$$\text{sech}^2 x = 1 - \tanh^2 x$$

$$\text{cosech}^2 x = \coth^2 x - 1$$

All the above identities can be verified easily by substituting the values of the functions in terms of the exponential function. Similarly, you may verify the following relations :

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh(x - y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x + \tanh y}$$

$$\tanh(x - y) = \frac{\tanh x + \tanh y}{1 - \tanh x + \tanh y}$$

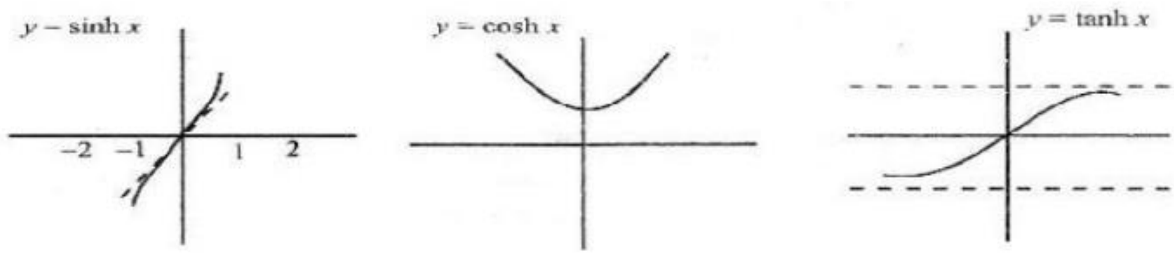
Graphs of these hyperbolic functions are given in next page (Figure 14) :

Since the six hyperbolic functions are defined in terms of e^x, and $\frac{d}{dx}(e^x) = e^x$ therefore it is very easy to write down the derivatives of sinh x, tanh x, coth x, sech x, cosech x etc.

We have,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$



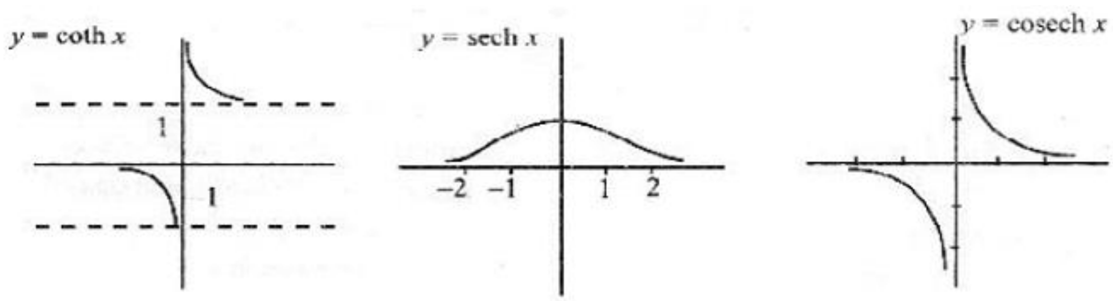


Figure 14

Similarly,
$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$
$$= \frac{e^{-x} - e^{-x}}{2} = \sinh x$$

Also,
$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$
$$= \frac{(e^{-x} + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$
$$= \frac{(e^x + e^{-x}) - (e^x - e^{-x})^2}{(e^x + e^x)^2}$$
$$= 1 - \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]^2$$
$$= 1 - \tanh^2 x = \operatorname{sech}^2 x$$

The derivative of coth x, sech x, and cosech x can be similarly obtained. The results are summarised in the following table.

$f\ (x)$	$f`(x)$
Sinh x	cosh x
cosh x	sinh x
tanh x	sech^2 x
coth x	–cosech^2 x
sech x	–sech x tanh x
cosech x	–cosech x tanh x

6. Inverse Hyperbolic Functions and Their Derivatives

(i) The inverse hyperbolic sine

The inverse of sinh x exists for all values of x.

An expression for $\sinh^{-1} x$ in terms of logarithms can be obtained as follows:

We have,

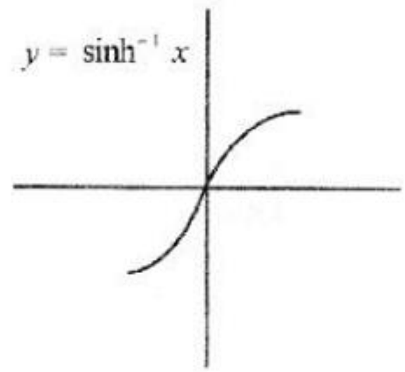


Figure 15

$$y = \sinh^{-1} x \Leftrightarrow x = \sinh\ y = \frac{e^y - e^{-y}}{2}$$

$$\Leftrightarrow 2x = e^y - e^{-y}$$

$$\Leftrightarrow e^{2y} - 2x\ e^y - 1 = 0$$

[the value $x - \sqrt{1+x^2}$ is discarded since this would give a negative value for e^y]

$$\Leftrightarrow e^y = x + \sqrt{1+x^2}$$

$$\Leftrightarrow y = \log\left(x + \sqrt{1+x^2}\right)$$

Thus, $$\sinh^{1} x = \log\left(x + \sqrt{1+x^2}\right) \quad x \in \left]-\infty, \infty\right[$$

Now, if we want to the derivative of $\sinh^{-1} x$, we must write the derivative of $\log\left(x + \sqrt{1+x^2}\right)$.

Let $$y = \log\left(x + \sqrt{1+x^2}\right).$$

Then, $$\frac{dy}{dx} = \frac{1}{x + \sqrt{1+x^2}} \cdot \left[1 + \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{\sqrt{x^2+1}}$$

Hence, $$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$$

(ii) The Inverse Hyperbolic co.sine

The inverse of cosh x exists in [1, ∞[. To obtain an expression for $\cosh^{-1} x$, we proceed as follows:

$$y = \cosh^{-1} x \Leftrightarrow x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$\Leftrightarrow e^{2Y} - 2x\, e^y + 1 = 0$$

$$\Leftrightarrow e^y = x + \sqrt{x^2 - 1}$$

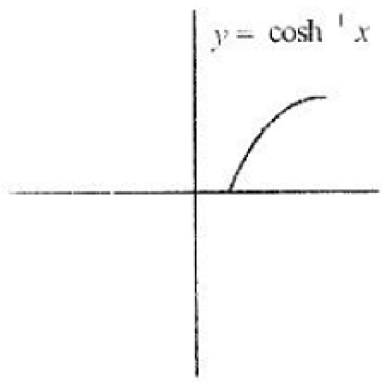


Figure 16

[Notice that the value $x - \sqrt{x^2 - 1}$ would render $e^y < 1$ and so $y < 0$, which is not possible.]

Thus,

$$\cosh^{-1} x = \log\left[x + \sqrt{x^2 - 1}\right], x \geq 1.$$

Now to obtain the derivative of $\cosh^{-1} x$, we have to write the derivative of $\log\left[x + \sqrt{x^2 - 1}\right], x \geq 1$

We have,

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{d}{dx}\left[\log\left(x + \sqrt{x^2 - 1}\right)\right], x \geq 1.$$

$$= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left[1 + \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2 - 1}}\right], x > 1$$

$$= \frac{1}{\sqrt{x^2 - 1}}, \; x > 1$$

[Notice that although $\cosh^{-1} x$ is defined for $x = 1$, its derivative does not exist for $x = 1$.]

Hence, $$\frac{d}{dx}\left(\cosh^{-1}x\right) = \frac{1}{\sqrt{x^2-1}}, x > 1$$

(iii) The Inverse Hyperbolic Tangent

The inverse hyperbolic tangent $\tanh^{-1} x$ exist for all $x \in [-1, 1]$. To obtain an expression for $\tanh^{-1} x$ in terms of logarithms, we proceed as follows :

$$y = \tanh^{-1} x \Leftrightarrow x = \tanh y$$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\Leftrightarrow \frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}}$$

$$\Leftrightarrow \quad e^{2y} = \frac{1+x}{1-x}$$

$$\Leftrightarrow \quad y = \frac{1}{2}\log\frac{1+x}{1-x}, |x|<1.$$

Thus, $$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}, |x|<1$$

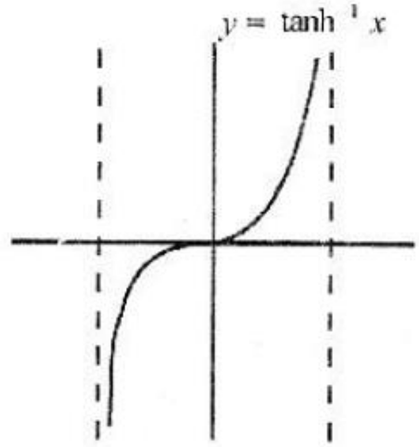


Figure 17

To write down the derivative of $\tanh^{-1} x$, we must obtain the derivative of $\frac{1}{2}\log\frac{1+x}{1-x}|x|<1$.

Let $$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}, |x|<1.$$

Then $$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)\right], |x|<1$$

$$= \frac{1}{2(1+x)} - \frac{-1}{2(1-x)}$$

$$= \frac{1}{1-x^2}, |x|<1$$

Hence, $$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$$

(iv) The inverse hyperbolic cotangent

The inverse hyperbolic cotangent $\coth^{-1} x$ exist for all x such that $|x| > 1$.

Now, $$y = \coth^{-1} x$$

$$\Leftrightarrow \quad x = \coth y = \frac{e^y + e^{-y}}{e^y - e^{-y}}$$

$$\Leftrightarrow \quad \frac{x+1}{x-1} = \frac{2e^y}{2e^{-y}}$$

$$\Leftrightarrow \quad \frac{x+1}{x-1} = e^{2y}$$

$$\Leftrightarrow \quad y = \frac{1}{2}\log\frac{x+1}{x-1}$$

Thus, $$\coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1}, \quad |x| > 1.$$

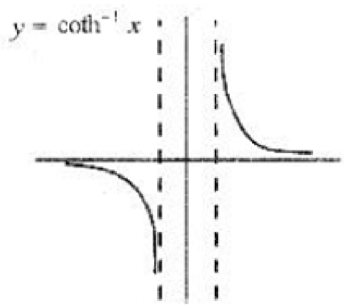


Figure 18

Therefore, to write down the derivative of $\coth^{-1} x$, we must obtain the derivative of $\frac{1}{2}\log\frac{x+1}{x-1}, \ |x| > 1.$

We have,

$$\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{d}{dx}\left[\frac{1}{2}\left(\log\frac{x+1}{x-1}\right)\right]$$

$$= \frac{d}{dx}\left[\frac{1}{2}\{\log(x+1) - \log(x-1)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right]$$

$$= \frac{-1}{x^2-1}$$

Hence,

$$\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{-1}{x^2-1}, |x| > 1$$

(v) The Inverse Hyperbolic secant

The inverse hyperbolic secant $\text{sech}^{-1}\, x$ exists for all x in [0, 1].

We have,

$$y = \text{sech}^{-1}\, x \Leftrightarrow x = \text{sech}\, y$$

$$\Leftrightarrow x = \frac{2}{e^{y}+e^{-y}}$$

$$\Leftrightarrow y = \log\left[\frac{1+\sqrt{1-x^2}}{x}\right], 0 < x \le 1$$

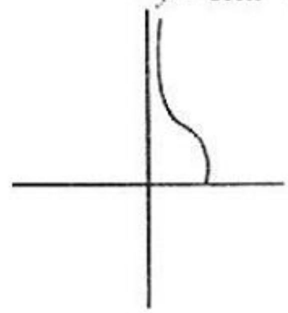


Figure 19

Thus,

$$\text{Sech}^{-1}\, x = \log\left[\frac{1+\sqrt{1-x^2}}{x}\right], 0 < x \le 1$$

To write down the derivative $\text{sech}^{-1}\, x$ we must differentiate $\log\left[\frac{1+\sqrt{1-x^2}}{x}\right],\ 0 < x \le 1$.

We have,

$$\frac{d}{dx}\left\{\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right\}$$

$$= \frac{d}{dx}\left(\log\,(1+\sqrt{1-x^2})\right) - \frac{d}{dx}(\log x)$$

$$= \frac{1}{1+\sqrt{1-x^2}}\cdot\frac{1}{2}\cdot(-2x)(1-x^2)^{-1/2} - \frac{1}{x}$$

$$= \frac{-x}{\sqrt{1-x^2}\left(1+\sqrt{1-x2}\right)} - \frac{1}{x}$$

$$= -\left[\frac{x^2+\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)}{x\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)}\right]$$

$$= -\left[\frac{x^2+\sqrt{1-x^2}+1-x^2}{x\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)}\right]$$

$$= -\left[\frac{\left(1+\sqrt{1-x^2}\right)}{x\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)}\right]$$

$$= \frac{-1}{x\sqrt{1-x^2}},\ 0<x<1$$

[Notice that although $\operatorname{sech}^{-1} x$ is defined for x =1, its derivative does not exist for x = 1.]

Hence, $$\frac{d}{dx}(\sec h^{-1} x) = \frac{-1}{x\sqrt{1-x^2}},\ 0<x<1$$

(vi) The Inverse Hyperbolic Cosecant

The inverse hyperbolic cosecant cosech–1 x exists for all $x : x \neq 0$.

We have,

$$y = \operatorname{cosech}^{-1} x \Leftrightarrow x = \operatorname{cosech} y$$

$$\Leftrightarrow x = \frac{2}{e^y - e^{-y}}$$

$$\Leftrightarrow y = \log\left[\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right]$$

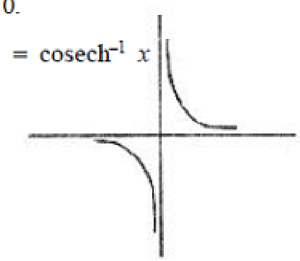


Figure 20

To write the derivative of $\operatorname{cosech}^{-1} x$ we have to differentiate $\log\left[\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right]$. It is easily varified that

$$\frac{d}{dx}(\operatorname{cosec} h^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}},\ x \neq 0$$

The expression for the inverse hyperbolic functions in terms of logarithms and their derivatives are summarised in the following two tables :

$\text{Sinh}^{-1} x$	$= \log\left[x+\sqrt{1+x^2}\right], \quad \forall\ x$
$\cosh^{-1} x$	$= \log\left[x+\sqrt{1-x^2}\right], \quad x \geq 1$
$\tanh^{-1} x$	$= \log\frac{1+x}{1-x}, \quad \lvert x\rvert < 1$
$\coth^{-1} x$	$= \frac{1}{2}\log\frac{x+1}{x-1}, \lvert x\rvert > 1$
$\text{sech}^{-1} x$	$= \log\frac{1+\sqrt{1-x^2}}{x}, 0 < x < 1$
$\text{cosech}^{-1} x$	$= \log\left[\frac{1}{x}+\frac{\sqrt{1+x^2}}{\lvert x\rvert}\right], x \neq 0$

$f(x)$	$f'(x)$
$\text{Sinh}^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$

$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}, x>1$
$\tanh^{-1}x$	$\frac{1}{1-x^2},\ \lvert x\rvert<1$
$\coth^{-1}x$	$\frac{-1}{x^2-1},\ \lvert x\rvert>1$
$\text{Sech}^{-1}x$	$\frac{-1}{x\sqrt{1-x^2}}, 0<x<1$
$\text{cosech}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}, x\neq 0$

Higher-Order Derivatives

Let f be a function that is differentiable at some points belonging to dom(f). Then f' is a function.

- If, in addition, f' is differentiable at some points belonging to dom(f'), then the derivative of f' exists and is denoted by f''; it is the function given by $f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$ and is called the *second derivative* of f.
- If, in addition, f'' is differentiable at some points belonging to dom(f''), then the derivative of f'' is denoted by f''', called the *third derivative* of f.
- In general, the n-th derivative of f (where n is a positive integer), denoted by $f^{(n)}$, is defined to be the derivative of the $(n-1)$-th derivative of f (where the 0-th derivative of f means f). For $n = 1$, the first derivative of f is simply the derivative f' of f. For $n > 1$, $f^{(n)}$ is called a *higher-order derivative* of f.

Notation Similar to first order derivative, we have different notations for second order derivative of f.

$$y', \quad f'', \quad \frac{d^2y}{dx^2}, \quad D^2f, \quad D^2y, \quad f''(x) \quad \text{and} \quad \frac{d^2}{dx^2}f(x).$$

Readers may compare these with that on page 109. Similarly, we also have different notations for other higher-order derivatives.

Example Let $f(x) = 5x^3 - 2x^2 + 6x + 1$. Find the derivative and all the higher-order derivatives of f.

Explanation The question is to find for each positive integer n, the domain of the n-th derivative of f and a formula for $f^{(n)}(x)$. To find $f'(x)$, we can apply differentiation term by term. To find $f''(x)$, by definition, we have $f''(x) = \frac{d}{dx}f'(x)$ which can be simplified using the result for $f'(x)$ and rules for differentiation.

Solution

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(5x^3 - 2x^2 + 6x + 1) \\
&= 15x^2 - 4x + 6 && \text{Derivative of Polynomial} \\
f''(x) &= \frac{d}{dx}(15x^2 - 4x + 6) \\
&= 30x - 4 && \text{Derivative of Polynomial} \\
f'''(x) &= \frac{d}{dx}(30x - 4) \\
&= 30 && \text{Derivative of Polynomial} \\
f^{(4)}(x) &= 0 && \text{Derivative of Constant}
\end{aligned}$$

From this we see that for $n \geq 4$, $f^{(n)}(x) = 0$. Moreover, for every positive integer n, the domain of $f^{(n)}$ is $\mathbb{R}$.

Example Let $f(x) = \frac{x^3 - 1}{x}$. Find $f'(3)$ and $f''(-4)$.

Explanation

- To find $f'(3)$, we find $f'(x)$ first and then substitute $x = 3$. Although $f(x)$ is written as a quotient of functions, it is better to find $f'(x)$ by expanding $(x^3 - 1)x^{-1}$.
- To find $f''(-4)$, we find $f''(x)$ first and then substitute $x = -4$. To find $f''(x)$, we differentiate the re obtained for $f'(x)$.

Solution

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left((x^3 - 1)x^{-1}\right) \\
&= \frac{d}{dx}\left(x^2 - x^{-1}\right) \\
&= 2x - (-1)x^{-2} && \text{Term by Term Differentiation \& Power Rule} \\
&= 2x + x^{-2} \\
f'(3) &= 2 \cdot (3) + 3^{-2} \\
&= \frac{55}{9} \\
f''(x) &= \frac{d}{dx}\left(2x + x^{-2}\right) && \text{By result for } f'(x) \\
&= 2 + (-2)x^{-3} && \text{Term by Term Differentiation \& Power Rule} \\
&= 2 - 2x^{-3} \\
f''(-4) &= 2 - 2 \cdot (-4)^{-3} \\
&= \frac{65}{32}
\end{aligned}$$

Meaning of Second Derivative

- The graph of $y = f(x)$ is a curve. Note that $f'(x) = \frac{dy}{dx}$ is the slope function; it is the rate of change of slope and is related to a concept called *convexity* (*bending*) of a curve. More details can be found in Chapter 5.
- If $x = t$ is time and if $y = s(t)$ is the displacement function of a moving object, then $s'(t) = \frac{ds}{dt}$ is the velocity function. The derivative of velocity is $s''(t)$ or $\frac{d^2s}{dt^2}$; it is the rate of change of the velocity (function), that is, the *acceleration* (function).

Example 1. Given $y = \frac{1}{x}$, Find $\frac{d^n y}{dx^n}$.

Solution : Here $y = \frac{1}{x} = x^{-1}$

$$\therefore \quad \frac{dy}{dx} = (-1)x^{-2}$$

$$\frac{d^2y}{dx^2} = (-1)(-2)x^{-3} = \frac{(-1)^2 2!}{x^3}$$

$$\frac{d^3y}{dx^3} = (-1)(-2)(-3)x^{-4} = \frac{(-1)^3 3!}{x^4}$$

Hence by induction,

$$\frac{d^n y}{dx^n} = (-1)(-2)(-3)\ldots\ldots\ldots(-n)x^{-(n+1)}$$

$$= \frac{(-1)^n n!}{x^{(n+1)}}$$

We shall now obtain expression for the n^{th} derivative of some standard functions.

1. The n^{th} derivative f e^{ax} is $a^n e^{ax}$

If $y = e^{ax}$

$$\frac{dy}{dx} = ae^{ax}$$

and $\frac{d^2y}{dx^2} = a.ae^{ax} = a^2e^{ax}$

By induction $\frac{d^n y}{dx^n} = a^n e^{ax}$, where n is a positive integer.

2. The n^{th} derivative of a^{mx}

Let $y = a^{mx}, a > 0$

$$\therefore \quad \frac{dy}{dx} = ma^{mx} \log a \qquad \left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$$\frac{d^2y}{dx^2} = m^2 a^{mx} (\log a)^2$$

$$\frac{d^3y}{dx^3} = m^3 a^{mx} (\log a)^3$$

....

....

$$\therefore \quad \frac{d^ny}{dx^n} = m^n a^{mx} (\log a)^n \qquad \text{where } a > 0.$$

Cor. : If $y = e^{mx}$, then $a = e \Rightarrow \log a = \log e = 1$

$$\therefore \quad \frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx}$$

3. The n^{th} derivative of $(ax + b)^m$

Let $y = (ax + b)^m$

Then $y_1 = ma(ax + b)^{m-1}$

$$y_2 = m(m-1)\, a^2(ax + b)^{m-2}$$

$$y_3 = m(m-1)(m-2)a^3(ax + b)^{m-3}$$

....

....

$$y_n = m(m-1)(m-2)............\left(m-\overline{n-1}\right)a^n(ax+b)^{m-n}$$

$$= m(m-1)(m-2)............(m-n+1)a^n (ax+b)^{m-n}$$

$$= \frac{m(m-1)(m-2)..........(m-n+1).(m-n)!}{(m-n)!}a^n(ax+b)^{m-n}$$

Cor. I. In case m is a postive integer

$$y = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$$

The m^{th} derivative of $(ax + b)^m$ can be obtained by putting n = m, we get

$$y_m = \frac{m!}{0!} a^m (ax+b)^{m-n}$$

$$= m!\, a^m \qquad (\because\ 0! = 1)$$

m^{th} derivative of $(ax + b)^m$ is constant viz.$m!\ a^m$ and hence the $(m + 1)^{th}$ derivative of $(ax + b)^m$ is zero.

Cor. 2. For $m = -1$, we get

$$y = \frac{1}{ax+b}$$

$$y_n = (-1)(-2)(-3)\ldots\ldots\ldots(-n)a^n(ax+b)^{-1-n}$$

$$= \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

Cor. 3. Let y = log $(ax + b)$, then

$$y_1 = \frac{a}{ax+b}$$

$\therefore$ $$y_n = (n-1)^{th} \text{ derivative of } y_1 \text{ or } \frac{a}{ax+b}$$

$$\frac{d^n}{dx^n}[\log(ax+b)] = a.\frac{(-1)^{n-1}(n-1)!a^{n-1}}{(ax+b)^n} = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

4.The n^{th} derivative of sin (ax + b) and cos (ax + b).

Let $y = \cos(ax+b)$

$$\therefore \quad \frac{dy}{dx} = -a\sin(ax+b)$$

$$= a\cos\left(ax+b+\frac{\pi}{2}\right) \qquad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$$

$$\frac{d^2y}{dx^2} = -a^2\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$= a^2\cos\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$\frac{d^3y}{dx^3} = -a^3\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$= a^3\cos\left(ax+b+\frac{3\pi}{2}\right) \text{etc.}$$

In general $$\frac{d^ny}{dx^n} = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly we can show that:

$$\frac{d^n}{dx^n}[\sin(ax+b)] = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

6. The n^{th} derivative of e^{ax} sin bx and e^{ax} cos bx.

Let $$y = e^{ax}\sin bx$$

$$y_1 = \frac{dy}{dx} = a\, e^{ax}\sin bx + e^{ax}\, b\cos bx$$

$$= e^{ax}(a\sin bx + b\cos bx)$$

Put $$a = r\cos\theta,\ b = r\sin\theta$$

$$\Rightarrow \quad a^2 + b^2 = r^2, \quad \tan\theta = \frac{b}{a} \text{ or } \theta = \tan^{-1}\frac{b}{a}$$

$$\therefore \quad y_1 = e^{ax}[r\sin bx.\cos + r\sin\theta.\cos bx]$$
$$= r\, e^{ax}\sin(bx + \theta)$$

Thus we notice that y_1 can be obtained from y by multiplying it by the constant r and increasing bx by the constant angle θ

Similarly $$y_2 = \frac{d^2y}{dx^2} = r^2 e^{ax}\sin(bx + 2\theta)$$

In general

$$\frac{d^ny}{dx^n} = r^n e^{ax}\sin(bx + n\theta)$$

$$= \left(a^2+b^2\right)^{n/2} e^{ax}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right)$$

$$\left[\because r^2 = a^2 + b^2 and\tan\theta = \frac{b}{a} i.e.,\ r = \sqrt{a^2+b^2}\ and\theta = \tan^{-1}\frac{b}{a}\right]$$

Similarly if $y = e^{ax}\cos bx$

then $$\frac{d^ny}{dx^n} = r^n e^{ax}\cos(bx + n\theta)$$

$$\left[Where\ r^2 = a^2 + b^2\ \tan\theta = \frac{b}{a} i.e., r = \sqrt{a^2+b^2}\ and\ \theta = \tan^{-1}\frac{b}{a}\right]$$

$$\therefore \quad \frac{d^ny}{dx^n} = e^{ax}(a^2+b^2)^{n-2}\cos\left(bx + n\tan^{-1}\frac{b}{a}\right)$$

7. The n^{th} derivative of $(x + a)^{-m}$ where $x \neq -a$.

Let $$y = (x + a)^{-m},\ x \neq -a$$

$$\therefore \quad \frac{dy}{dx} = -m.(x + a)^{-m-1}$$

$$\frac{d^2y}{dx^2} = (-m)(-m-1)(x+a)^{-m-2}$$

$$\frac{d^3y}{dx^3} = (-m)(-m-1)(-m-2)(x+a)^{-m-3}$$

....

....

By induction,

$$\frac{d^ny}{dx^n} = (-m)(-m-1)(-m-2)\ldots\ldots\ldots(-m-n+1)\,(x+a)^{-m-n}$$

$$= (-1)^n.m\,(m+1)(m+2)\ldots\ldots\ldots(m+n-1)(x+a)^{-m-n}$$

$$= \frac{(-1)1.2\ldots\ldots\ldots(m-1)m(m+1)\ldots\ldots\ldots(m+n-1)(x+a)^{-m-n}}{1.2\ldots\ldots\ldots(m-1)}$$

$$= \frac{(-1)(m+n-1)!}{(m-1)!}.(x+a)^{-m-n} = \frac{(-1)^n(M+n-1)!}{(m-1)!}\frac{1}{(x+a)^{m+n}}$$

Where $x \neq -a$

Cor. : If $m=1$, we have $\frac{d^n}{dx^n}\left(\frac{1}{x+a}\right) = \frac{(-1)^n n!(x+a)^{-n-1}}{0!}$

$$= \frac{(-1)^n n!}{(x+a)^{n+1}}$$

Similarly we can show that

$$\frac{d^n}{dx^n}\left((x-a)^{-m}\right) = \frac{(-1)^n(m+n-1)!}{(m-1)}\frac{1}{(x-a)^{m+n}}$$

8. Application of De Moivre's Therorem and Partial fraction in finding the n^th derivative

In order to determine the nth derivative of rational functional, we resolve it into partial fractions and then use the standard results. Sometimes we can use trigonometric transformations or methods like the application of De Moivres theorem in finding the n^{th} derivatives. The following examples will illustrate the procedure.

Example 4. Find the nth derivative of $y = \frac{1}{(x-1)^3(x-2)}$

Solution : We resolve $y = \frac{1}{(x-1)^3(x-2)}$ into partial fractions

$$\frac{1}{(x-1)^3(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{D}{x-2}$$

$1 = A(x-1)^2(x-2) + B(x-1)(x-2) + C(x-2) + D(x-1)^3$

Putting $x = 1$, we get $1 = -C \Rightarrow C = -1$

Putting $x = 2$, we get $1 = D$ i.e., $D = 1$

Equating the coefficients of x^3, x^2 on both sides we have

$$0 = A + D \Rightarrow A = -D = -1 \qquad (\because D = 1)$$

$$0 = 4A + B - 3D \Rightarrow B = 4A + 3D$$

$$\therefore \quad B = 4A + 3D = -4 + 3 = -1 \qquad (\because A = -1, D = 1)$$

Hence $$\frac{1}{(x-1)^3(x-2)} = \frac{-1}{x-1} - \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3} + \frac{1}{x-2}$$

We have the standard resutls

$$\frac{d^n}{dx^n}\left[\frac{1}{x-a}\right] = \frac{(-1)^n n!}{(x-2)^{n+1}}, \quad \text{where } x \neq a$$

$$\frac{d^n}{dx^n}\left[\frac{1}{(x-a)^m}\right] = \frac{(-1)^n (m+n-1)!}{(m-1)!(m-1)^{m+n}} \quad \text{where } x \neq a$$

$$\therefore \quad \frac{d^n}{dx^n}\left[\frac{1}{(x-1)^3(x-2)}\right] = \frac{d^n}{dx^n}\left[\frac{-1}{x-1}\right] + \frac{d^n}{dx^n}\left[\frac{-1}{(x-1)^2}\right] + \frac{d^n}{dx^n}\left[\frac{-1}{(x-1)^3}\right] + \frac{d^n}{dx^n}\left[\frac{-1}{x-2}\right]$$

$$= \frac{(-1)^n n!}{(x-1)^{n+1}} - \frac{(-1)^n (n+1)!}{1!(x-1)^{n+2}}$$

$$= \frac{(-1)^n (n+2)!}{2!(x-1)^{n+3}} + \frac{(-1)^n n!}{(x-2)^{n+1}}$$

Hence we get

$$\frac{d^n y}{dx^n} = (-1)^{n+1} n!\left(\frac{1}{(x-1)^{n+1}} - \frac{1}{(x-2)^{n+1}} + \frac{n+1}{(x-1)^{n+2}} + \frac{(n+2)(n+1)}{2(x-1)^{n+3}}\right)$$

Example 5. If $y = \dfrac{1}{x^2+a^2}$ find $\dfrac{d^n y}{dx^n}$

Solution : We have $y = \dfrac{1}{x^2+a^2} = \dfrac{1}{(x+ai)(x-ai)}$ where $i = \sqrt{-1}$

$$= \frac{1}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$\therefore \quad \frac{d^n y}{dx^n} = \frac{1}{2ai}\left[\frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$$

Now put $x = r\cos\theta$, $a = r\sin\theta$ so that

$$x - ai = r(\cos\theta - i\sin\theta)$$
$$x + ai = r(\cos\theta + i\sin\theta)$$

Now by De Moivre's theorem we obtain.

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{[r(\cos\theta + i\sin\theta)]^{n+1}} = \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}$$
$$= r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)}$$
$$= r^{-(n+1)}[\cos(n+1)\theta + i\sin(n+1)\theta]$$

Similarly $$\frac{1}{(x+ai)^{n+1}} = r^{-(n+1)}(\cos(n+1)i\sin(n+1)\theta)$$

Substituting the values we get :

$$\frac{d^n y}{dx^n} = \frac{(-1)^n n!}{2ai} r^{-(n+1)}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]$$
$$r^{-(n+1)}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$$
$$= \frac{(-1)^n n!}{r^{(n+1)}2ai}(2i\sin(n+1)\theta)$$
$$= \frac{(-1)^n n!}{r^{(n+1)}a}\sin(n+1)\theta$$

Also $$a = r\sin\theta \Rightarrow \frac{1}{r} = \frac{\sin\theta}{a}$$

$\therefore$ $$\frac{1}{r^{n+1}} = \frac{\sin^{n+1}\theta}{a^{n+1}}$$

$\therefore$ $$\frac{d^n y}{dx^n} = \frac{(-1)^n n!\sin(n+1)\theta\sin^{n+1}\theta}{a^{n+2}}$$

where θ is given by $\tan\theta = \frac{a}{x}$ or $\theta = \tan^{-1}\left(\frac{a}{x}\right)$

Leibnitz Theorem : If $y = uv$ where u and v are functions of x possessing derivatives of the n^{th} order, then $D^n y = D^2(uv) = (D^n u)v + nC_1 D^{n-1} u Dv + {}^nC_2 D^{n-2} u D^2 v + \ldots\ldots + {}^nC_r D^{n-r} u D^r v \ldots\ldots + u D^n v.$

We notice that the coefficients on the R.H.S. are the same as in the expression of $(a+b)^n$ by Binomial Theorem, where n is a positive integer.

Proof : We shall prove the theorem by mathematical induction.

Step I. By direct differentiation it is easy to see that the theorem is true for $n = 1$, $n = 2$

$$\begin{aligned} D(uv) &= (Du)v + u(Dv) \\ D^2(uv) &= (D^2u)v + DuDv + DuDv + uD^2v \\ &= (D^2u)v + {}^2C_1 Du\,Dv + {}^2C_2 u\,D^2 v \end{aligned}$$

Step II. Let us assume that the Theorem is true for a particular value of n say $n = m$

i.e. $D^m(uv) = (D^m u)v + {}^mC_1 D^{m-1} u.Dv + {}^mC_2 D^{m-2} u.D^2 v + \ldots\ldots$

$+ {}^mC_{r-1} D^{m-r+1} u D^{r-1} v + {}^mC_r D^{m-r} u.D^r v + \ldots\ldots + u\,D^m v.$

Differentiating both sides w.r.t. x we have

$$\begin{aligned} D^{m+1}(uv) &= \left[(D^{m+1}u)v + D^m u\,Dv\right] + {}^mC_1\left[D^m u.Dv + D^{m-1}u.D^2 v\right] \\ &\quad + {}^mC_2\left[D^{m-1}u.D^2v + D^{m-2}u.D^3v\right]\ldots\ldots \\ &\quad + \ldots\ldots + {}^mC_{r-1}\left[D^{m-r+1}u\,D^{r-1}u + D^{m-r+1}u.D^r v\right] \\ &\quad + {}^mC_r\left[D^{m-r+1}u.D^r v + D^{m-r}u\,D^{r+1}u\right] + \ldots\ldots + uD^{m+1}v. \\ &= (D^{m+1}u)v + (1 + {}^mC_1)D^m u\,Dv + ({}^mC_1 + {}^mC_2)D^{m-1}u.D^2v + \ldots\ldots \\ &\quad + ({}^mC_{r-1} + {}^mC_r)D^{m-r+1}u.D^r v + \ldots\ldots + uD^{m+1}v. \end{aligned}$$

From the theory of permutations and combinations we know that ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

Also ${}^mC_n = ,1 + {}^mC_1 = {}^mC_0 + {}^mC_1 = {}^{m+1}C_1$ and ${}^mC_m = {}^{m+1}C_{m+1}$

$D^{m+1}(uv) = (D^{m+1}u)v + {}^{m+1}C_1 D^m u.Dv + {}^{m+2}C_2 D^{m-1}u.D^2v + \ldots\ldots$

$+ {}^{m+1}C_r D^{m+1-r}u.D^r v + \ldots\ldots + uD^{m+1}v.$

Thus if the theorem is true for any value m of n, then it is also true for the next higher value $(m+1)$ of n.

In step I we have seen that the theorem is true for $n = 1$, $n = 2$, then it must be true for $n = 2 + 1 = 3$ and so $n = 3 + 1 = 4$ and so on. Thus by the principle of mathematical induction it follows that the Theorem is true for all positive integral values of n.

Example 8. Find the n^{th} derivative of x^3e^{ax}.

Solution : We write $y = x^3e^{ax}$ where $u = e^{ax}$. $v = x^3$

We have
$$D^n u = D^n\left(e^{ax}\right) = a^n e^{ax}$$
$$Dv = 3x^2, D^2v = 6x, D^3v = 6, D^4v = 0, D^5v = 0$$

Using Leibnitz Rule, we get

$$\frac{d^n y}{dx^n} = \left(D^n u\right)v + {}^nC_1 D^{n-1}uDv + {}^nC_2 D^{n-2}uD^2v + \ldots\ldots\ldots + uD^n v$$
$$= x^3a^ne^{ax} + {}^nC_1 a^{n-1}e^{ax}.3x^2 + {}^nC_2 a^{n-2}e^{ax}.6x + {}^nC_3 a^{n-3}e^{ax}.6$$
$$= x^3a^ne^{ax} + 3n\,a^{n-1}x^2e^{ax} + 3n(n-1)a^{n-2}xe^{ax} + n(n-1)(n-2)a^{n-3}e^{ax}$$

POSSIBLE QUESTIONS

TWO MARKS

1. Find the derivative of f(x)=$5x^3-2x^2+6x+1$.

2. Find $\frac{dy}{dx}$ when y=e^{3x}cosh4x.

3. Find the third derivative of x^3-3x^2+4x-1.

4. Find $\frac{dy}{dx}$ $when$ y=e^{3x}cosh4x

5. State Leibniz theorem.

SIX MARKS

1.Prove that $\sinh(x+y) = sinhxcoshy + coshxsinhy$

2.Prove that $\cosh(x+y) = coshxcoshy + sinhxsinhy$

3.Find the n^{th} derivative of y=e^{ax}cos(bx+c)

4.Find the n^{th} derivative of e^{2x}sin3xsin4x

5.If xy=ae^x+be^{-x} prove that x$\frac{\partial^2 y}{\partial x^2}$ +2$\frac{\partial y}{\partial x}$ - xy = 0.

6.If x=a$\cos^3\theta$, y=a$\sin^3\theta$ find $\frac{\partial^2 y}{\partial x^2}$.

7.Show that prove that $sech^2 x = 1 - tanh^2 x$

8. $\cosh^2 x - \sinh^2 x = 1$.

9.Find the n^{th} derivative of $\frac{x+1}{x^2-4}$

10.If $y = \sin(sinx)\ prove\ that\ \frac{d^2y}{dx^2} +\ tanx\ \frac{dy}{dx} + y\cos^2 x = 0$.

11.Find the n^{th} derivative of $cosx\ cos2x\cos 3x$.

12.Evaluate $i)\ \int x^4 \cosh(x^5)\,dx \quad ii)\ \int 4^{sinh2x}\, cosh2xdx$

13.Find the n^{th} derivative of y=e^{ax}.sin(bx+c).

UNIT – II

INTEGRAL CALCULUS

Reduction formulae, derivations and illustrations of reduction formulae of the type $\int \sin nx\, dx$, $\int \cos nx\, dx$, $\int \tan nx\, dx$, $\int \sec nx\, dx$, $\int \log x^n\, dx$, $\int \sin^n x \sin^m x\, dx$. Curve tracing in Cartesian coordinates, tracing in polar coordinates of standard curves, L'Hospital's rule, applications in business, economics and life sciences.

Reduction formula for $\int \sin^n x\, dx$

$$\begin{aligned}
\int \sin^n x\, dx &= \int \sin x \sin^{n-1} x\, dx \\
&= -\cos x \sin^{n-1} x - \int (-\cos x).(n-1)\sin^{n-2} x \cos x\, dx \\
&\qquad \text{(integrating by parts)} \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\, dx \\
&\qquad \text{(since } \cos^2 x = 1-\sin^2 x\text{)} \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx. \qquad (1)
\end{aligned}$$

There is now a term in $\int \sin^n x\, dx$ on the right-hand side as well as on the left-hand side. Bringing these terms together on the left-hand side, (1) becomes

$$\begin{aligned}
n\int \sin^n x\, dx &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx \\
\therefore \int \sin^n x\, dx &= -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x\, dx. \qquad (2)
\end{aligned}$$

The use of the reduction formula (2) to integrate a power of $\sin x$ is demonstrated in worked example no. 2.

Reduction formula for $\int \cos^n x\,dx$

$$\begin{aligned}
\int \cos^n x\,dx &= \int \cos x\,\cos^{n-1}x\,dx \\
&= \sin x\,\cos^{n-1}x - \int (\sin x).(n-1)\cos^{n-2}x\,(-\sin x)\,dx \\
&\qquad \text{(integrating by parts)} \\
&= \sin x\,\cos^{n-1}x + (n-1)\int \cos^{n-2}x\,(1-\cos^2 x)\,dx \\
&\qquad (\text{since} \quad \sin^2 x = 1-\cos^2 x\,) \\
&= \sin x\,\cos^{n-1}x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx \\
\therefore n\int \cos^n x\,dx &= \sin x\,\cos^{n-1}x + (n-1)\int \cos^{n-2}x\,dx \\
\therefore \int \cos^n x\,dx &= \frac{1}{n}\sin x\,\cos^{n-1}x + \frac{(n-1)}{n}\int \cos^{n-2}x\,dx. \qquad (3)
\end{aligned}$$

REDUCTION FORMULAS

Integration by parts can be used to derive *reduction formulas* for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \qquad (9)$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx \qquad (10)$$

To illustrate how such formulas can be obtained, let us derive (10). We begin by writing $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$ and letting

$$u = \cos^{n-1} x \qquad dv = \cos x\, dx$$
$$du = (n-1)\cos^{n-2} x(-\sin x)\, dx \qquad v = \sin x$$
$$= -(n-1)\cos^{n-2} x \sin x\, dx$$

so that

$$\int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx = \int u\, dv = uv - \int v\, du$$
$$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx$$

Moving the last term on the right to the left side yields

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx$$

from which (10) follows. The derivation of reduction formula (9) is similar (Exercise 63).

Reduction formulas (9) and (10) reduce the exponent of sine (or cosine) by 2. Thus, if the formulas are applied repeatedly, the exponent can eventually be reduced to 0 if n is even or 1 if n is odd, at which point the integration can be completed. We will discuss this method in more detail in the next section, but for now, here is an example that illustrates how reduction formulas work.

Example 8 Evaluate $\int \cos^4 x\, dx$.

Solution. From (10) with $n = 4$

$$\int \cos^4 x\, dx = \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{4}\int \cos^2 x\, dx \qquad \boxed{\text{Now apply (10) with } n = 2.}$$
$$= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{4}\left(\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx\right)$$
$$= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x + C$$

Polar Coordinates

Up until now we have been dealing with the coordinates x and y, which are known as **Cartesian coordinates** or **rectangular coordinates**. We can equally describe a plane using any pair of coordinates so that each position on the plane is uniquely described by the pair. In particular a useful set of coordinates are **polar coordinates**.

Defining Polar Coordinates

To define polar coordinates, we begin by fixing a point O, called the **origin** or **pole**. We then define a half-line (or ray) which begins at 0 and continues to infinity in a given direction, called the **polar axis**. This is shown in

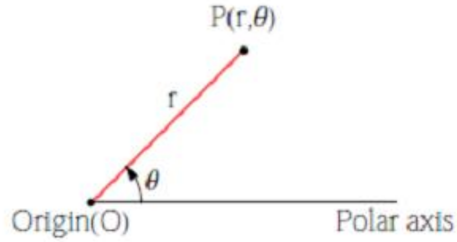


Figure 5: Polar coordinates

Figure 5. The distance from the origin to the point P is called the **radial coordinate**, r, and the angle that the line $|OP|$ makes with the polar axis is called the **angular coordinate**, θ, more commonly known as the **polar angle**. It is important to notice that the polar angle returns to its original position when the angle is 2π. Therefore, any angle greater than or equal to 2π is equivalent to an angle in the range $(0 \leq \theta \leq 2\pi)$. In fact, more generally, the angles $\theta - 2\pi\, n$, θ and $\theta + 2\pi\, n$ are equivalent if n is an integer.

Relationship to Cartesian Coordinates

It is quite easy to make a link between polar coordinates and Cartesian coordinates. The trick is to make the polar axis coincide with the x-axis. Then, we can see from Figure 6 that the following relationship holds

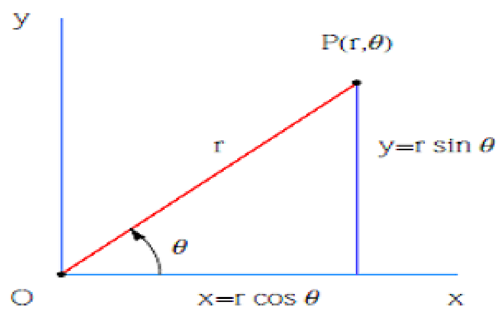


Figure 6: Polar coordinates in terms of Cartesian coordinates

$$x = r\cos\theta\,, \quad y = r\sin\theta\,, \tag{10}$$

which can also be written in the form

$$r^2 = x^2 + y^2\,, \quad \tan\theta = \frac{y}{x}\,. \tag{11}$$

Example: Change the Cartesian coordinates $(4, 4\sqrt{3})$ into polar coordinates.

Solution:

$$r^2 = 4^2 + (4\sqrt{3})^2 = 64 \Rightarrow r = 8\,,$$

and

$$\tan\theta = \frac{4\sqrt{3}}{4} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}\,.$$

The polar coordinates are therefore $(r, \theta) = (8, \pi/3)$.

Example: Change the polar coordinates $(3, 3\pi/4)$ into Cartesian coordinates.

Solution:

$$x = 3\cos\frac{3\pi}{4} = -\frac{3}{\sqrt{2}}\,,$$

and

$$y = 3\sin\frac{3\pi}{4} = \frac{3}{\sqrt{2}}\,,$$

The Cartesian coordinates are therefore $(x, y) = (-3/\sqrt{2}, 3/\sqrt{2})$.

If you are asked to graph an equation in polar coordinates, the easiest thing

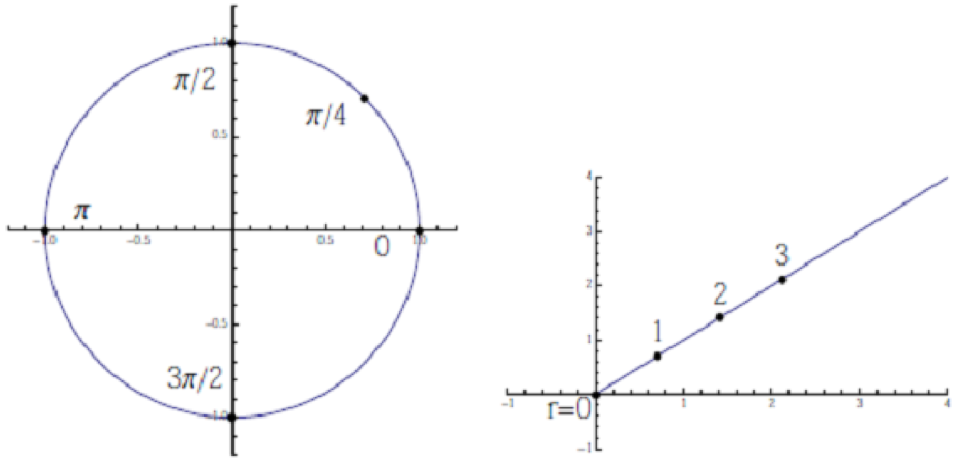


Figure 7: Unit circle and the half-line $\theta = \frac{\pi}{4}$.

to do is to simply plot some points and see what happens. Later on, you may begin to recognise some of the graphs and be able to plot them from memory. For now, let's look at some simple examples. If we fix r, say to $r = 1$, we get a circle, see Figure 7. Note also that once we hit $\theta = 2\pi$, the graph repeats, and so we do not need to continue. Also in the same figure we see the plot of $\theta = \pi/4$. We can either imagine that we can take negative values of r,

or as we would normally expect we could impose $r \geq 0$ and plot a half-line, as is done here. Finally, let's look at an example where both coordinates vary, say $r = \sin\theta$. Then we get the plot in Figure 8, where we see that the plot repeats values after $\theta = \pi$, because the Cartesian coordinates are $x = \sin t\ \cos\theta$, $y = \sin\theta \sin\theta = \sin^2\theta \geq 0$.

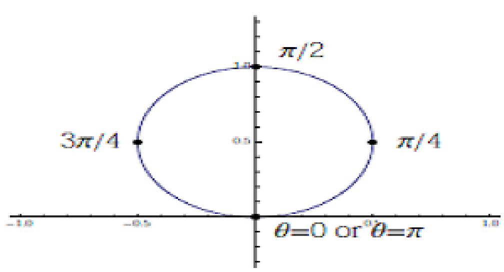


Figure 8: Plot of $r = \sin\theta$.

EXAMPLE: Express the equation $x^2 = 4y$ in polar coordinates.

Solution: We use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$x^2 = 4y$$

$$(r\cos\theta)^2 = 4r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

$$r = 4\frac{\sin\theta}{\cos^2\theta} = 4\sec\theta\tan\theta$$

Polar to Cartesian Conversion Formulas

$x = r\cos\theta$	$y = r\sin\theta$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 \end{aligned}$$

This is a very useful formula that we should remember, however we are after an equation for r so let's take the square root of both sides. This gives,

$$r = \sqrt{x^2 + y^2}$$

Note that technically we should have a plus or minus in front of the root since we know that r can be either positive or negative. We will run with the convention of positive r here.

Getting an equation for θ is almost as simple. We'll start with,

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

Taking the inverse tangent of both sides gives,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

We will need to be careful with this because inverse tangents only return values in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Recall that there is a second possible angle and that the second angle is given by $\theta + \pi$.

Cartesian to Polar Conversion Formulas

$$r^2 = x^2 + y^2 \qquad r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example 1 Convert each of the following points into the given coordinate system.

(a) $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates. [Solution]

(b) (-1,-1) into polar coordinates. [Solution]

Solution

(a) Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

This conversion is easy enough. All we need to do is plug the points into the formulas.

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$
$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$.

(b) Convert (-1,-1) into polar coordinates.

Let's first get r.

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Now, let's get θ.

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

This is not the correct angle however. This value of θ is in the first quadrant and the point we've been given is in the third quadrant. As noted above we can get the correct angle by adding π onto this. Therefore, the actual angle is,

$$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

So, in polar coordinates the point is $\left(\sqrt{2}, \frac{5\pi}{4}\right)$. Note as well that we could have used the first θ that we got by using a negative r. In this case the point could also be written in polar coordinates as $\left(-\sqrt{2}, \frac{\pi}{4}\right)$.

Example 2 Convert each of the following into an equation in the given coordinate system.

(a) Convert $2x-5x^3=1+xy$ into polar coordinates. [Solution]

(b) Convert $r=-8\cos\theta$ into Cartesian coordinates. [Solution]

Solution

(a) Convert $2x-5x^3=1+xy$ into polar coordinates.

In this case there really isn't much to do other than plugging in the formulas for x and y (*i.e.* the Cartesian coordinates) in terms of r and θ (*i.e.* the polar coordinates).

$$2(r\cos\theta)-5(r\cos\theta)^3=1+(r\cos\theta)(r\sin\theta)$$

$$2r\cos\theta-5r^3\cos^3\theta=1+r^2\cos\theta\sin\theta$$

[Return to Problems]

(b) Convert $r=-8\cos\theta$ into Cartesian coordinates.

This one is a little trickier, but not by much. First notice that we could substitute straight for the r. However, there is no straight substitution for the cosine that will give us only Cartesian coordinates. If we had an r on the right along with the cosine then we could do a direct substitution. So, if an r on the right side would be convenient let's put one there, just don't forget to put one on the left side as well.

$$r^2=-8r\cos\theta$$

We can now make some substitutions that will convert this into Cartesian coordinates.

$$x^2+y^2=-8x$$

Lines

Some lines have fairly simple equations in polar coordinates.

1. $\theta=\beta$.

   We can see that this is a line by converting to Cartesian coordinates as follows

$$\theta=\beta$$

$$\tan^{-1}\left(\frac{y}{x}\right)=\beta$$

$$\frac{y}{x}=\tan\beta$$

$$y=(\tan\beta)x$$

   This is a line that goes through the origin and makes an angle of β with the positive x-axis. Or, in other words it is a line through the origin with slope of $\tan\beta$.

2. $r\cos\theta=a$

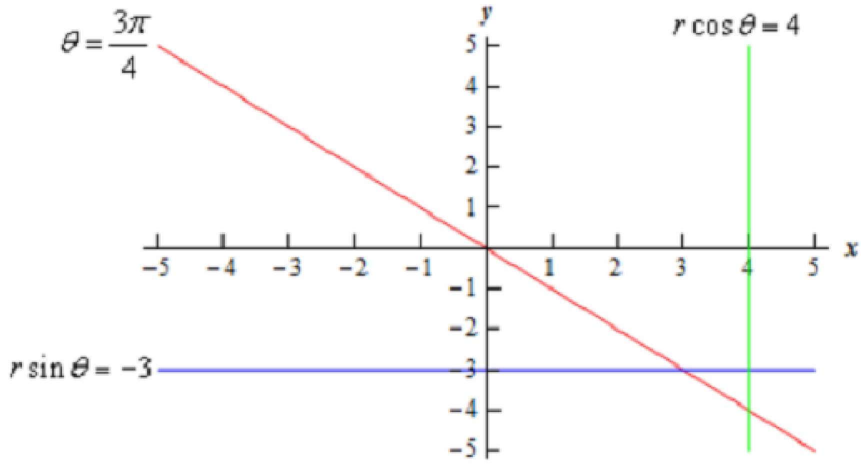
   This is easy enough to convert to Cartesian coordinates to $x=a$. So, this is a vertical line.

3. $r\sin\theta=b$

   Likewise, this converts to $y=b$ and so is a horizontal line.

Example 3 Graph $\theta = \frac{3\pi}{4}$, $r\cos\theta = 4$ and $r\sin\theta = -3$ on the same axis system.

Solution
There really isn't too much to this one other than doing the graph so here it is.



Circles
Let's take a look at the equations of circles in polar coordinates.

1. $r = a$.
   This equation is saying that no matter what angle we've got the distance from the origin must be a. If you think about it that is exactly the definition of a circle of radius a centered at the origin.

   So, this is a circle of radius a centered at the origin. This is also one of the reasons why we might want to work in polar coordinates. The equation of a circle centered at the origin has a very nice equation, unlike the corresponding equation in Cartesian coordinates.

2. $r = 2a\cos\theta$.
   We looked at a specific example of one of these when we were converting equations to Cartesian coordinates.

   This is a circle of radius $|a|$ and center $(a, 0)$. Note that a might be negative (as it was in our example above) and so the absolute value bars are required on the radius. They should not be used however on the center.

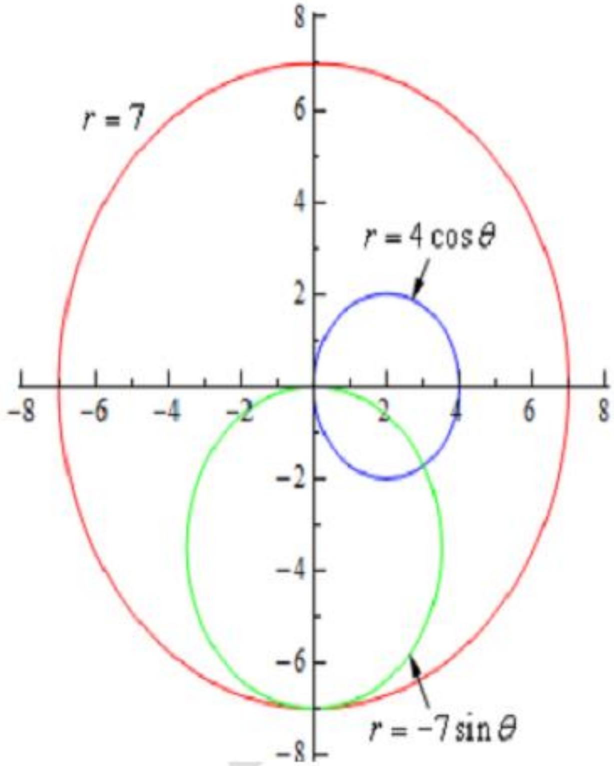
3. $r = 2b\sin\theta$.
   This is similar to the previous one. It is a circle of radius $|b|$ and center $(0, b)$.

4. $r = 2a\cos\theta + 2b\sin\theta$.
   This is a combination of the previous two and by completing the square twice it can be shown that this is a circle of radius $\sqrt{a^2 + b^2}$ and center (a, b). In other words, this is the general equation of a circle that isn't centered at the origin.

Example 4 Graph $r=7$, $r=4\cos\theta$, and $r=-7\sin\theta$ on the same axis system.

Solution

The first one is a circle of radius 7 centered at the origin. The second is a circle of radius 2 centered at (2,0). The third is a circle of radius $\frac{7}{2}$ centered at $\left(0,-\frac{7}{2}\right)$. Here is the graph of the three equations.



► **Example 1** Find the rectangular coordinates of the point P whose polar coordinates are $(r, \theta) = (6, 2\pi/3)$ (Figure 10.2.6).

Solution. Substituting the polar coordinates $r = 6$ and $\theta = 2\pi/3$ in (1) yields

$$x = 6\cos\frac{2\pi}{3} = 6\left(-\frac{1}{2}\right) = -3$$

$$y = 6\sin\frac{2\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

Thus, the rectangular coordinates of P are $(x, y) = (-3, 3\sqrt{3})$. ◄

► **Example 2** Find polar coordinates of the point P whose rectangular coordinates are $(-2, -2\sqrt{3})$ (Figure 10.2.7).

Solution. We will find the polar coordinates (r, θ) of P that satisfy the conditions $r > 0$ and $0 \le \theta < 2\pi$. From the first equation in (2),

$$r^2 = x^2 + y^2 = (-2)^2 + (-2\sqrt{3})^2 = 4 + 12 = 16$$

so $r = 4$. From the second equation in (2),

$$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$$

From this and the fact that $(-2, -2\sqrt{3})$ lies in the third quadrant, it follows that the angle satisfying the requirement $0 \le \theta < 2\pi$ is $\theta = 4\pi/3$. Thus, $(r, \theta) = (4, 4\pi/3)$ are polar coordinates of P. All other polar coordinates of P are expressible in the form

$$\left(4, \frac{4\pi}{3} + 2n\pi\right) \quad \text{or} \quad \left(-4, \frac{\pi}{3} + 2n\pi\right)$$

where n is an integer. ◄

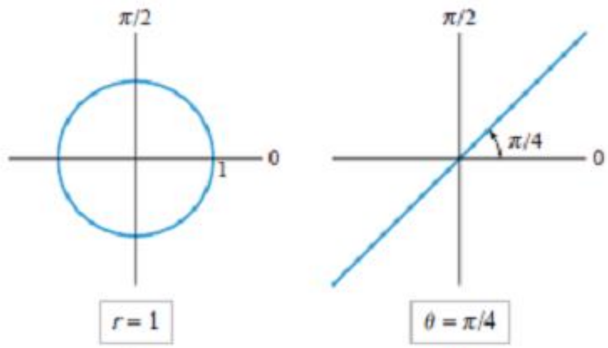
► **Example 3** Sketch the graphs of

$$\text{(a) } r = 1 \qquad \text{(b) } \theta = \frac{\pi}{4}$$

in polar coordinates.

Solution (a). For all values of θ, the point $(1, \theta)$ is 1 unit away from the pole. Since θ is arbitrary, the graph is the circle of radius 1 centered at the pole (Figure 10.2.8*a*).

Solution (b). For all values of r, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Figure 10.2.8*b*). Positive values of r correspond to points on the line in the first quadrant and negative values of r to points on the line in the third quadrant. Thus, in absence of any restriction on r, the graph is the entire line. Observe, however, that had we imposed the restriction $r \geq 0$, the graph would have been just the ray in the first quadrant. ◄



► **Figure 10.2.8** (*a*) (*b*)

Equations $r = f(\theta)$ that express r as a function of θ are especially important. One way to graph such an equation is to choose some typical values of θ, calculate the corresponding values of r, and then plot the resulting pairs (r, θ) in a polar coordinate system. The next two examples illustrate this process.

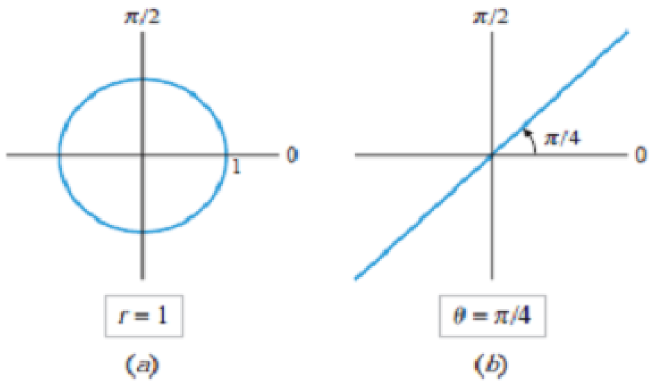
► **Example 3** Sketch the graphs of

$$\text{(a) } r = 1 \qquad \text{(b) } \theta = \frac{\pi}{4}$$

in polar coordinates.

Solution (a). For all values of θ, the point $(1, \theta)$ is 1 unit away from the pole. Since θ is arbitrary, the graph is the circle of radius 1 centered at the pole (Figure 10.2.8*a*).

Solution (b). For all values of r, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Figure 10.2.8*b*). Positive values of r correspond to points on the line in the first quadrant and negative values of r to points on the line in the third quadrant. Thus, in absence of any restriction on r, the graph is the entire line. Observe, however, that had we imposed the restriction $r \geq 0$, the graph would have been just the ray in the first quadrant. ◄

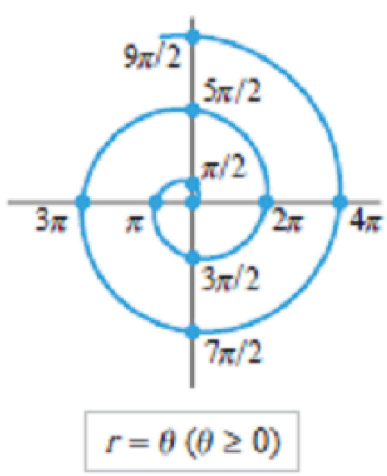


► **Figure 10.2.8**

Equations $r = f(\theta)$ that express r as a function of θ are especially important. One way to graph such an equation is to choose some typical values of θ, calculate the corresponding values of r, and then plot the resulting pairs (r, θ) in a polar coordinate system. The next two examples illustrate this process.

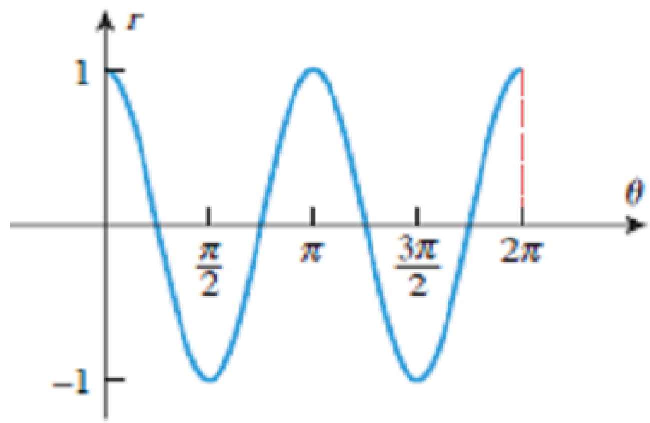
► **Example 4** Sketch the graph of $r = \theta$ $(\theta \geq 0)$ in polar coordinates by plotting points.

Solution. Observe that as θ increases, so does r; thus, the graph is a curve that spirals out from the pole as θ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the points that correspond to values of θ that are integer multiples of $\pi/2$, keeping in mind that the value of r is always equal to the value of θ (Figure 10.2.9). ◄



▲ **Figure 10.2.9**

Example 6 Sketch the graph of $r = \cos 2\theta$ in polar coordinates.



$r = \cos 2\theta$

L'HÔPITAL'S RULE

INDETERMINATE FORMS OF TYPE 0/0

Recall that a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \tag{1}$$

in which $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an ***indeterminate form of type 0/0***. Some examples encountered earlier in the text are

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

The first limit was obtained algebraically by factoring the numerator and canceling the common factor of $x - 1$, and the second two limits were obtained using geometric methods. However, there are many indeterminate forms for which neither algebraic nor geometric methods will produce the limit, so we need to develop a more general method.

To motivate such a method, suppose that (1) is an indeterminate form of type $0/0$ in which f' and g' are continuous at $x = a$ and $g'(a) \neq 0$. Since f and g can be closely approximated by their local linear approximations near a, it is reasonable to expect that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} \tag{2}$$

Since we are assuming that f' and g' are continuous at $x = a$, we have

$$\lim_{x \to a} f'(x) = f'(a) \quad \text{and} \quad \lim_{x \to a} g'(x) = g'(a)$$

and since the differentiability of f and g at $x = a$ implies the continuity of f and g at $x = a$, we have

$$f(a) = \lim_{x \to a} f(x) = 0 \quad \text{and} \quad g(a) = \lim_{x \to a} g(x) = 0$$

Thus, we can rewrite (2) as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x - a)}{g'(a)(x - a)} = \lim_{x \to a} \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{3}$$

This result, called ***L'Hôpital's rule***, converts the given indeterminate form into a limit involving derivatives that is often easier to evaluate.

Although we motivated (3) by assuming that f and g have continuous derivatives at $x = a$ and that $g'(a) \neq 0$, the result is true under less stringent conditions and is also valid for one-sided limits and limits at $+\infty$ and $-\infty$. The proof of the following precise statement of L'Hôpital's rule is omitted.

3.6.1 THEOREM (*L'Hôpital's Rule for Form 0/0*) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

In the examples that follow we will apply L'Hôpital's rule using the following three-step process:

Applying L'Hôpital's Rule

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

► **Example 1** Find the limit

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution. The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\frac{d}{dx}[x^2-4]}{\frac{d}{dx}[x-2]}=\lim_{x\to 2}\frac{2x}{1}=4$$

This agrees with the computation

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$ ◄

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or a one-sided limit) of a function is $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=+\infty \quad \text{or} \quad \lim_{x\to a^+} f(x)=-\infty$$

$$\lim_{x\to +\infty} f(x)=\infty \quad \text{means} \quad \lim_{x\to +\infty} f(x)=+\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x)=-\infty$$

$$\lim_{x\to a} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=\pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x)=\pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an *indeterminate form of type* ∞/∞. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

3.6.2 THEOREM (L'Hôpital's Rule for Form ∞/∞) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x\to a} f(x) = \infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \infty$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

Example 3 In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)} \quad \lim_{x\to+\infty} \frac{x}{e^x} \qquad \text{(b)} \quad \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} \tag{4}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (4) can be rewritten as

$$\lim_{x\to 0^+}\left(-\frac{\sin x}{x}\tan x\right) = -\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\tan x = -(1)(0) = 0$$

Thus,

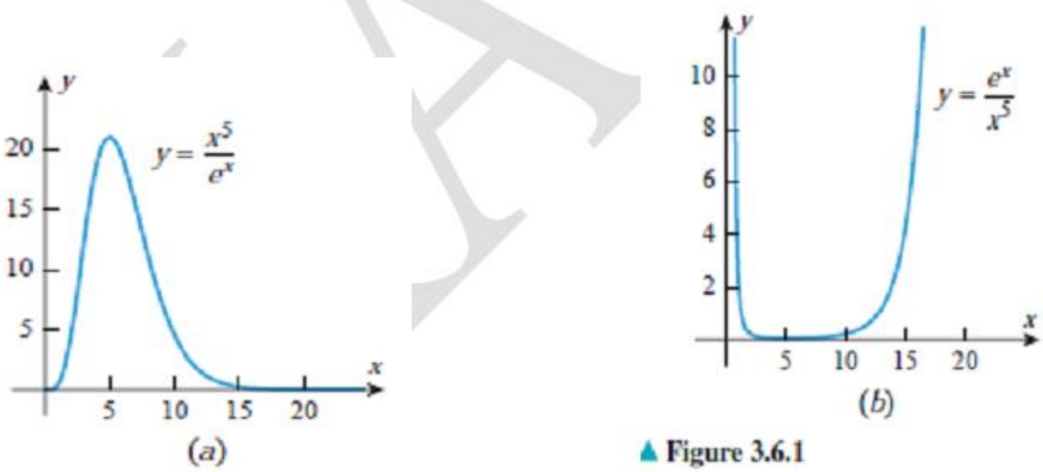
$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = 0 \blacktriangleleft$$

ANALYZING THE GROWTH OF EXPONENTIAL FUNCTIONS USING L'HÔPITAL'S RULE

If n is any positive integer, then $x^n \to +\infty$ as $x \to +\infty$. Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example, we know that $e^x \to +\infty$ as $x \to +\infty$ and that the growth of e^x is very rapid (Table 0.5.5); however, the growth of x^n is also rapid when n is a high power, so it is reasonable to ask whether high powers of x grow more or less rapidly than e^x. One way to investigate this is to examine the behavior of the ratio x^n/e^x as $x \to +\infty$. For example, Figure 3.6.1a shows the graph of $y = x^5/e^x$. This graph suggests that $x^5/e^x \to 0$ as $x \to +\infty$, and this implies that the growth of the function e^x is sufficiently rapid that its values eventually overtake those of x^5 and force the ratio toward zero. Stated informally, "e^x eventually grows more rapidly than x^5." The same conclusion could have been reached by putting e^x on top and examining the behavior of e^x/x^5 as $x \to +\infty$ (Figure 3.6.1b). In this case the values of e^x eventually overtake those of x^5 and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that e^x *eventually grows more rapidly than any positive integer power of* x, that is,

$$\lim_{x\to+\infty} \frac{x^n}{e^x} = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty \tag{5–6}$$

Both limits are indeterminate forms of type ∞/∞ that can be evaluated using L'Hôpital's rule. For example, to establish (5), we will need to apply L'Hôpital's rule n times. For this purpose, observe that successive differentiations of x^n reduce the exponent by 1 each time, thus producing a constant for the nth derivative. For example, the successive derivatives



▲ Figure 3.6.1

of x^3 are $3x^2$, $6x$, and 6. In general, the nth derivative of x^n is $n(n-1)(n-2)\cdots 1 = n!$ (verify).* Thus, applying L'Hôpital's rule n times to (5) yields

$$\lim_{x\to+\infty} \frac{x^n}{e^x} = \lim_{x\to+\infty} \frac{n!}{e^x} = 0$$

Limit (6) can be established similarly.

INDETERMINATE FORMS OF TYPE $0 \cdot \infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x)\cdot g(x), \quad f(x)^{g(x)}, \quad f(x)-g(x), \quad f(x)+g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x\to 0^+} x\ln x$$

is an ***indeterminate form of type*** $\mathbf{0\cdot\infty}$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x\to+\infty} [\sqrt{x}(1-x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

Indeterminate forms of type $0\cdot\infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

► **Example 4** Evaluate

$$\text{(a)} \lim_{x\to 0^+} x\ln x \qquad \text{(b)} \lim_{x\to\pi/4} (1-\tan x)\sec 2x$$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0\cdot\infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \text{or} \quad \lim_{x\to 0^+} \frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to\pi/4}(1-\tan x)\sec 2x = \lim_{x\to\pi/4}\frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}$$

$$= \lim_{x\to\pi/4}\frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1 ◀$$

INDETERMINATE FORMS OF TYPE $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty)-(+\infty), \quad (-\infty)-(-\infty),$$
$$(+\infty)+(-\infty), \quad (-\infty)+(+\infty)$$

is called an ***indeterminate form of type*** $\infty - \infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty)+(+\infty), \quad (+\infty)-(-\infty),$$
$$(-\infty)+(-\infty), \quad (-\infty)-(+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty-\infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

► **Example 5** Evaluate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty-\infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0^+}\frac{\sin x - x}{x\sin x}$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0 ◀$$

INDETERMINATE FORMS OF TYPE 0^0, ∞^0, 1^∞

Limits of the form

$$\lim f(x)^{g(x)}$$

can give rise to ***indeterminate forms of the types* 0^0, ∞^0, *and* 1^∞**. (The interpretations of these symbols should be clear.) For example, the limit

$$\lim_{x \to 0^+} (1+x)^{1/x}$$

whose value we know to be e [see Formula (1) of Section 3.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then computing the limit of $\ln y$. Since

$$\ln y = \ln[f(x)^{g(x)}] = g(x) \cdot \ln[f(x)]$$

the limit of $\ln y$ will be an indeterminate form of type $0 \cdot \infty$ (verify), which can be evaluated by methods we have already studied. Once the limit of $\ln y$ is known, it is a straightforward matter to determine the limit of $y = f(x)^{g(x)}$, as we will illustrate in the next example.

Example 6 Find $\lim_{x \to 0} (1 + \sin x)^{1/x}$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1 + \sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1+\sin x)^{1/x} = \frac{1}{x}\ln(1+\sin x) = \frac{\ln(1+\sin x)}{x}$$

Thus,

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+\sin x)}{x}$$

which is an indeterminate form of type $0/0$, so by L'Hôpital's rule

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+\sin x)}{x} = \lim_{x \to 0} \frac{(\cos x)/(1+\sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus,

$$\lim_{x \to 0} (1 + \sin x)^{1/x} = e$$

POSSIBLE QUESTIONS

TWO MARKS

1.Convert the polar equation to Cartesian equation for $r = \frac{5}{sin\theta - 2cos\theta}$

2.Find the value of $\int_0^{\frac{\pi}{2}} cos^{12}\, xdx$.

3. State a L'Hospital's Rule.

4.Evaluate $\lim_{x\to 2} \frac{x^7 - 128}{x^3 - 8}$

5.Evaluate $\int_0^{\infty} e^{-x} x^5 dx$.

SIX MARKS

1.Show that $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

2.Find the Reduction form of $\int cos^m x sin^n x dx$

3.Find i) $\lim_{x\to 0^+} x^{sinx}$

4.Evaluate $\int x^4 (logx)^3 dx$.

5.Find a polar equation for the circle $x^2 + (y-3)^2 = 9$

6.Find the Cartesian equation for $r^2 = 4r\, cos\theta\ \ and\ rcos\theta = -4$

7.Derive the reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\, dx$.

8.Derive the reduction formula for $\int_0^{\infty} e^{-x} x^n\, dx$ and also find the value of $\int_0^{\infty} e^{-x} x^8\, dx$

9.Derive the reduction formula for $\int_0^{\frac{\pi}{2}} x^n\, sinx\, dx$

10.Evaluate $\lim_{x\to 0} \frac{x - sinx}{x^3}$

11.Evaluate $\lim_{x\to +\infty} \frac{2x^2 - 3x + 1}{3x^2 + 5x - 2}$

11.Derive the reduction formula for $\int_0^{\frac{\pi}{2}} cos^n x dx$

12.Evaluate $\lim_{x\to +\infty} x^{\frac{1}{x}}$

UNIT – III

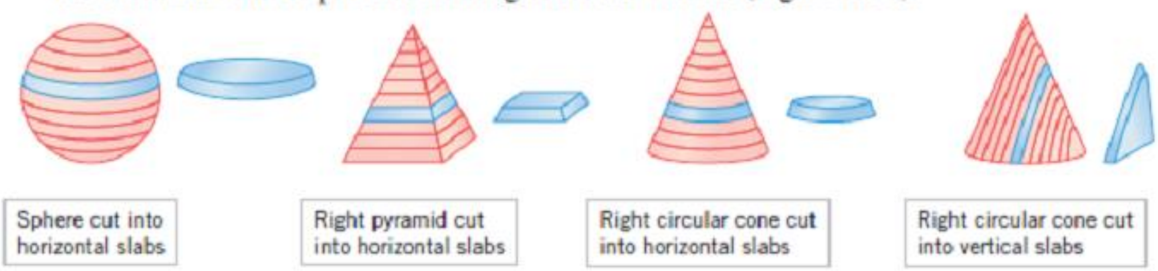
APPLICATIONS OF INTEGRATION

Volumes by slicing, disks and washers methods, volumes by cylindrical shells, parametric equations, parameterizing a curve, arc length, arc length of parametric curves, area of surface of revolution.

VOLUMES BY SLICING; DISKS AND WASHERS

VOLUMES BY SLICING

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).

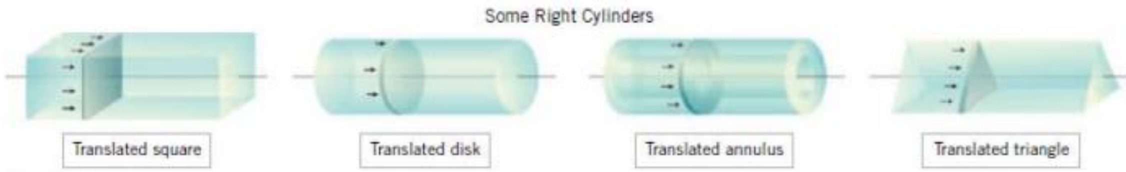


▲ **Figure 6.2.1**

One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius r, since all cross sections taken perpendicular to the central axis are circular regions of radius r. The volume V of a right circular cylinder of radius r and height h can be expressed in terms of the height and the area of a cross section as

$$V = \pi r^2 h = [\text{area of a cross section}] \times [\text{height}] \qquad (1)$$

This is a special case of a more general volume formula that applies to solids called right cylinders. A ***right cylinder*** is a solid that is generated when a plane region is translated along a line or ***axis*** that is perpendicular to the region (Figure 6.2.3).



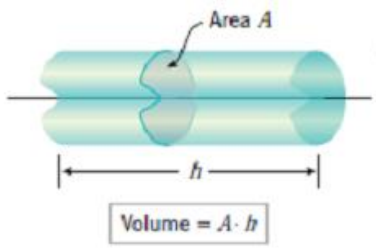
▲ Figure 6.2.3

If a right cylinder is generated by translating a region of area A through a distance h, then h is called the ***height*** (or sometimes the ***width***) of the cylinder, and the volume V of the cylinder is defined to be

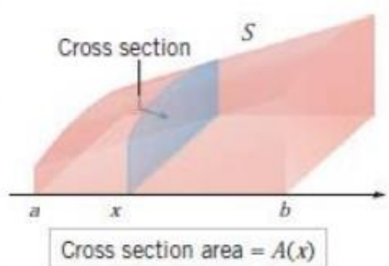
$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}] \tag{2}$$

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right *circular* cylinder.

We now have all of the tools required to solve the following problem.



▲ Figure 6.2.4



▲ Figure 6.2.5

If a right cylinder is generated by translating a region of area A through a distance h, then h is called the ***height*** (or sometimes the ***width***) of the cylinder, and the volume V of the cylinder is defined to be

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}] \tag{2}$$

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right *circular* cylinder.

We now have all of the tools required to solve the following problem.

6.2.1 PROBLEM Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at $x = a$ and $x = b$ (Figure 6.2.5). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each x in the interval $[a, b]$.

To solve this problem we begin by dividing the interval $[a, b]$ into n subintervals, thereby dividing the solid into n slabs as shown in the left part of Figure 6.2.6. If we assume that the width of the kth subinterval is Δx_k, then the volume of the kth slab can be approximated by the volume $A(x_k^*)\Delta x_k$ of a right cylinder of width (height) Δx_k and cross-sectional area $A(x_k^*)$, where x_k^* is a point in the kth subinterval (see the right part of Figure 6.2.6).

Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

In summary, we have the following result.

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

▸ **Example 1** Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

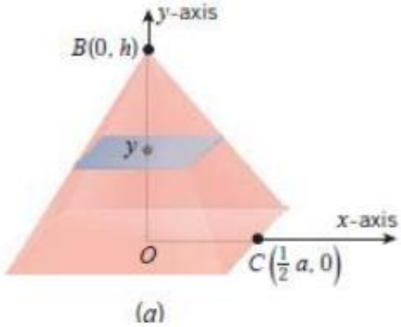
Solution. As illustrated in Figure 6.2.7a, we introduce a rectangular coordinate system in which the y-axis passes through the apex and is perpendicular to the base, and the x-axis passes through the base and is parallel to a side of the base.

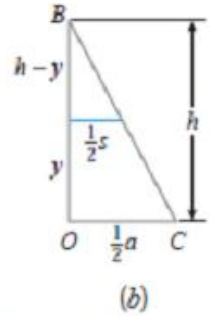
At any y in the interval $[0, h]$ on the y-axis, the cross section perpendicular to the y-axis is a square. If s denotes the length of a side of this square, then by similar triangles (Figure 6.2.7b)

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \text{or} \quad s = \frac{a}{h}(h-y)$$

Thus, the area $A(y)$ of the cross section at y is

$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2$$





▲ **Figure 6.2.7**

and by (4) the volume is

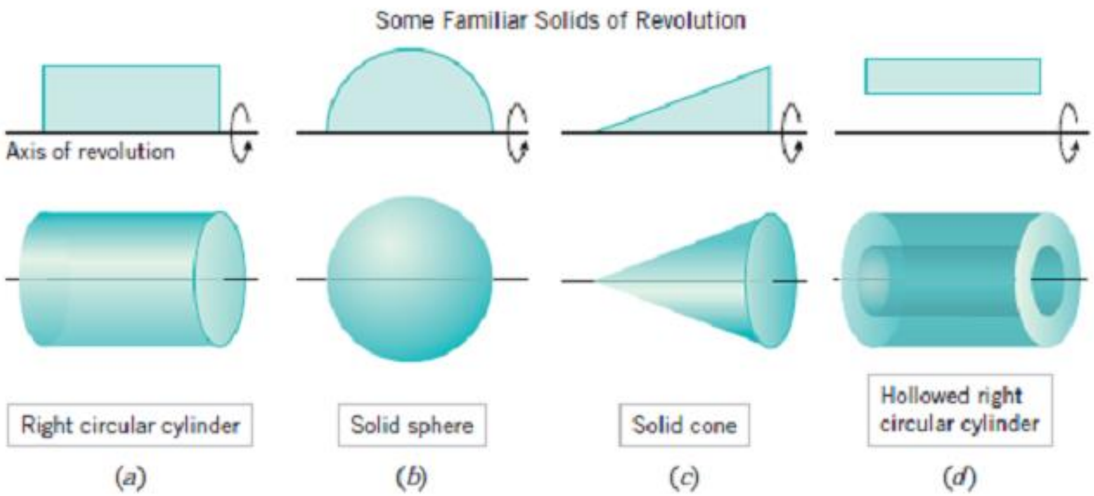
$$V = \int_0^h A(y)\,dy = \int_0^h \frac{a^2}{h^2}(h-y)^2\,dy = \frac{a^2}{h^2}\int_0^h (h-y)^2\,dy$$

$$= \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_{y=0}^h = \frac{a^2}{h^2}\left[0+\frac{1}{3}h^3\right] = \frac{1}{3}a^2h$$

That is, the volume is $\frac{1}{3}$ of the area of the base times the altitude. ◀

SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type (Figure 6.2.8).

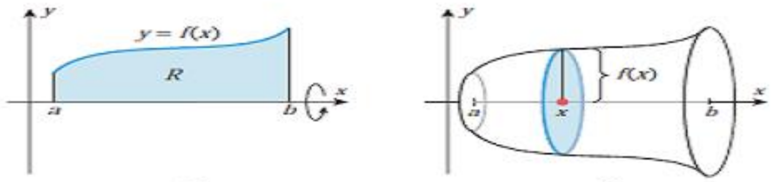


6.2.8

VOLUMES BY DISKS PERPENDICULAR TO THE x-AXIS

We will be interested in the following general problem.

6.2.4 PROBLEM Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.



► **Figure 6.2.9**

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is a circular disk of radius $f(x)$ (Figure 6.2.9b). The area of this region is

$$A(x) = \pi[f(x)]^2$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2\,dx \tag{5}$$

Because the cross sections are disk shaped, the application of this formula is called the ***method of disks***.

► **Example 2** Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 6.2.10).

Solution. From (5), the volume is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_1^4 \pi x\,dx = \left.\frac{\pi x^2}{2}\right]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$ ◄

Example 3 Derive the formula for the volume of a sphere of radius r.

Solution. As indicated in Figure 6.2.11, a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x-axis and

$$x^2 + y^2 = r^2$$

about the x-axis. Since the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, it follows from (5) that the volume of the sphere is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3$$

VOLUMES BY WASHERS PERPENDICULAR TO THE x-AXIS

6.2.5 PROBLEM Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis (Figure 6.2.12b).

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is the annular or "washer-shaped" region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 6.2.12b); its area is

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi([f(x)]^2 - [g(x)]^2)$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx \tag{6}$$

Because the cross sections are washer shaped, the application of this formula is called the ***method of washers***.

Example 4 Find the volume of the solid generated when the region between the grap of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about t x-axis.

Solution. First sketch the region (Figure 6.2.13*a*); then imagine revolving it about t x-axis (Figure 6.2.13*b*). From (6) the volume is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx = \int_0^2 \pi\left(\left[\tfrac{1}{2} + x^2\right]^2 - x^2\right)dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right)dx = \pi\left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2 = \frac{69\pi}{10}$$

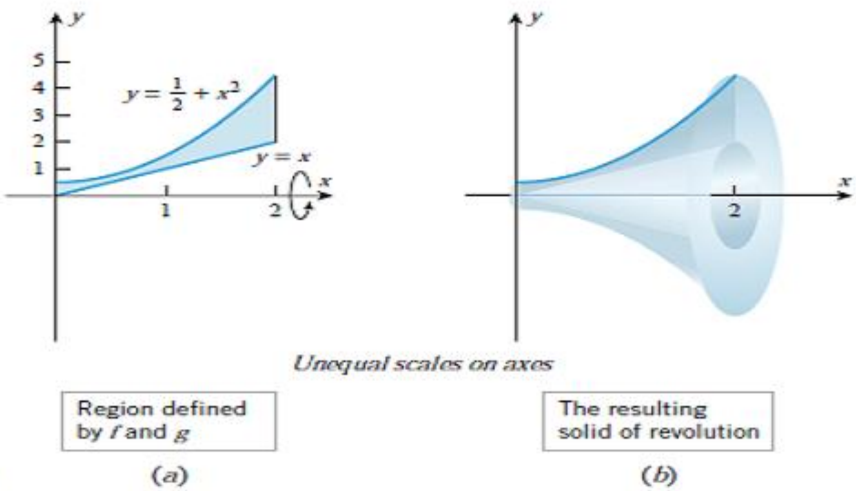


Figure 6.2.13

VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE *y*-AXIS

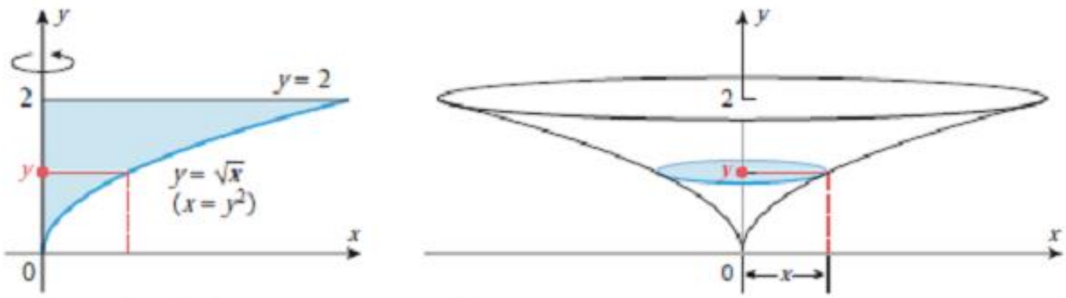
The methods of disks and washers have analogs for regions that are revolved about the y-axis (Figures 6.2.14 and 6.2.15). Using the method of slicing and Formula (4), you should be able to deduce the following formulas for the volumes of the solids in the figures.

$$V = \int_c^d \pi[u(y)]^2\,dy \quad \text{(Disks)} \qquad V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy \quad \text{(Washers)} \qquad (7\text{–}8)$$

► **Example 5** Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.

Solution. First sketch the region and the solid (Figure 6.2.16). The cross sections taken perpendicular to the y-axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

$$V = \int_c^d \pi[u(y)]^2\,dy = \int_0^2 \pi y^4\,dy = \frac{\pi y^5}{5}\Bigg]_0^2 = \frac{32\pi}{5}$$ ◄

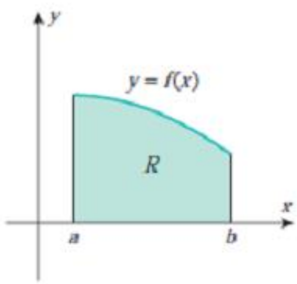


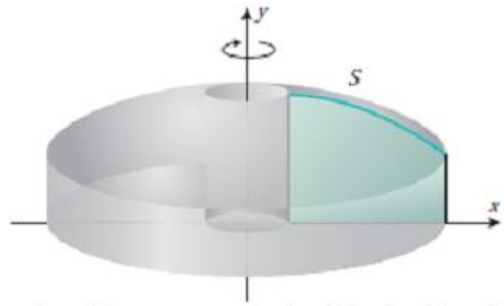
VOLUMES BY CYLINDRICAL SHELLS

CYLINDRICAL SHELLS

In this section we will be interested in the following problem.

6.3.1 PROBLEM Let f be continuous and nonnegative on $[a, b]$ $(0 \le a < b)$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume V of the solid of revolution S that is generated by revolving the region R about the y-axis (Figure 6.3.1).





A *cylindrical shell* is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2). The volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h can be written as

$$\begin{aligned} V &= [\text{area of cross section}] \cdot [\text{height}] \\ &= (\pi r_2^2 - \pi r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \cdot \left[\tfrac{1}{2}(r_1 + r_2)\right] \cdot h \cdot (r_2 - r_1) \end{aligned}$$

But $\frac{1}{2}(r_1 + r_2)$ is the average radius of the shell and $r_2 - r_1$ is its thickness, so

$$V = 2\pi \cdot [\text{average radius}] \cdot [\text{height}] \cdot [\text{thickness}] \tag{1}$$

6.3.2 VOLUME BY CYLINDRICAL SHELLS ABOUT THE y-AXIS Let f be continuous and nonnegative on $[a, b]$ $(0 \le a < b)$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the y-axis is given by

$$V = \int_a^b 2\pi x f(x)\,dx \tag{2}$$

► **Example 1** Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis is revolved about the y-axis.

Solution. First sketch the region (Figure 6.3.6*a*); then imagine revolving it about the y-axis (Figure 6.3.6*b*). Since $f(x) = \sqrt{x}$, $a = 1$, and $b = 4$, Formula (2) yields

$$V = \int_1^4 2\pi x\sqrt{x}\,dx = 2\pi \int_1^4 x^{3/2}\,dx = \left[2\pi \cdot \frac{2}{5}x^{5/2}\right]_1^4 = \frac{4\pi}{5}[32 - 1] = \frac{124\pi}{5}$$ ◄

► **Example 2** Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the y-axis (Figure 6.3.8*a*).

Solution. As illustrated in part (*b*) of Figure 6.3.8, at each x in $[0, 1]$ the cross section of R parallel to the y-axis generates a cylindrical surface of height $x - x^2$ and radius x. Since the area of this surface is

$$2\pi x(x - x^2)$$

the volume of the solid is

$$V = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\pi}{6}$$

Example 3 Use cylindrical shells to find the volume of the solid generated when the region R under $y = x^2$ over the interval $[0, 2]$ is revolved about the line $y = -1$.

Solution. First draw the axis of revolution; then imagine revolving the region about the axis (Figure 6.3.9*a*). As illustrated in Figure 6.3.9*b*, at each y in the interval $0 \le y \le 4$, the cross section of R parallel to the x-axis generates a cylindrical surface of height $2 - \sqrt{y}$ and radius $y + 1$. Since the area of this surface is

$$2\pi(y + 1)(2 - \sqrt{y})$$

it follows that the volume of the solid is

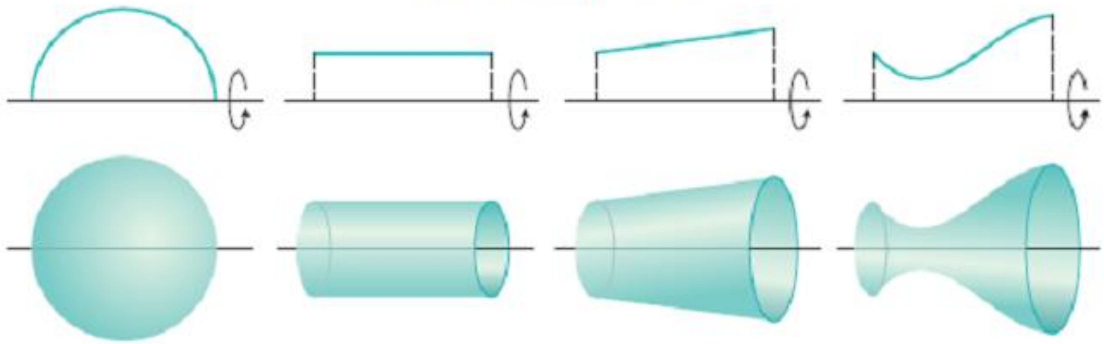
$$\int_0^4 2\pi(y + 1)(2 - \sqrt{y})\,dy = 2\pi \int_0^4 (2y - y^{3/2} + 2 - y^{1/2})\,dy$$

$$= 2\pi\left[y^2 - \frac{2}{5}y^{5/2} + 2y - \frac{2}{3}y^{3/2}\right]_0^4 = \frac{176\pi}{15}$$

AREA OF A SURFACE OF REVOLUTION

SURFACE AREA

A ***surface of revolution*** is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).



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6.5.1 SURFACE AREA PROBLEM Suppose that f is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis (Figure 6.5.2). Define what is meant by the *area* S of the surface, and find a formula for computing it.

6.5.2 DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Moreover, if g is nonnegative and $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis can be expressed as

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad (5)$$

Example 1 Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

Solution. First sketch the curve; then imagine revolving it about the x-axis (Figure 6.5.6). Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

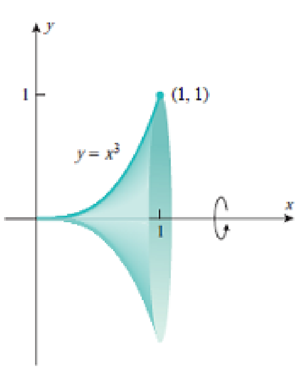
$$S = \int_0^1 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}\, dx$$

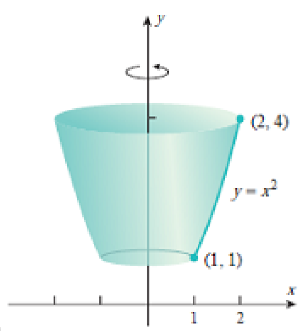
$$= 2\pi \int_0^1 x^3(1 + 9x^4)^{1/2}\, dx$$

$$= \frac{2\pi}{36}\int_1^{10} u^{1/2}\, du \qquad \boxed{\begin{array}{c} u = 1 + 9x^4 \\ du = 36x^3\, dx \end{array}}$$

$$= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2} - 1) \approx 3.56$$



▲ Figure 6.5.6



▲ Figure 6.5.7

Example 2 Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.

Solution. First sketch the curve; then imagine revolving it about the y-axis (Figure 6.5.7). Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and

$x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$S = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^4 2\pi \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \pi \int_1^4 \sqrt{4y + 1}\, dy$$

$$= \frac{\pi}{4} \int_5^{17} u^{1/2}\, du \qquad \boxed{\begin{array}{c} u = 4y + 1 \\ du = 4\,dy \end{array}}$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big]_{u=5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85 \blacktriangleleft$$

PARAMETRIC EQUATIONS; TANGENT LINES AND ARC LENGTH FOR PARAMETRIC CURVES

PARAMETRIC EQUATIONS

Suppose that a particle moves along a curve C in the xy-plane in such a way that its x- and y-coordinates, as functions of time, are

$$x = f(t), \quad y = g(t)$$

We call these the ***parametric equations*** of motion for the particle and refer to C as the ***trajectory*** of the particle or the ***graph*** of the equations (Figure 10.1.1). The variable t is called the ***parameter*** for the equations.

► **Example 1** Sketch the trajectory over the time interval $0 \le t \le 10$ of the particle whose parametric equations of motion are

$$x = t - 3\sin t, \quad y = 4 - 3\cos t \qquad (1)$$

Solution. One way to sketch the trajectory is to choose a representative succession of times, plot the (x, y) coordinates of points on the trajectory at those times, and connect the points with a smooth curve. The trajectory in Figure 10.1.2 was obtained in this way from the data in Table 10.1.1 in which the approximate coordinates of the particle are given at time increments of 1 unit. Observe that there is no t-axis in the picture; the values of t appear only as labels on the plotted points, and even these are usually omitted unless it is important to emphasize the locations of the particle at specific times. ◄

► **Example 2** Find the graph of the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \leq t \leq 2\pi) \tag{2}$$

Solution. One way to find the graph is to eliminate the parameter t by noting that

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

Thus, the graph is contained in the unit circle $x^2 + y^2 = 1$. Geometrically, the parameter t can be interpreted as the angle swept out by the radial line from the origin to the point $(x, y) = (\cos t, \sin t)$ on the unit circle (Figure 10.1.3). As t increases from 0 to 2π, the point traces the circle counterclockwise, starting at $(1, 0)$ when $t = 0$ and completing one full revolution when $t = 2\pi$. One can obtain different portions of the circle by varying the interval over which the parameter varies. For example,

$$x = \cos t, \quad y = \sin t \qquad (0 \leq t \leq \pi) \tag{3}$$

represents just the upper semicircle in Figure 10.1.3. ◄

► **Example 3** Graph the parametric curve

$$x = 2t - 3, \quad y = 6t - 7$$

by eliminating the parameter, and indicate the orientation on the graph.

Solution. To eliminate the parameter we will solve the first equation for t as a function of x, and then substitute this expression for t into the second equation:

$$t = \left(\tfrac{1}{2}\right)(x + 3)$$

$$y = 6\left(\tfrac{1}{2}\right)(x + 3) - 7$$

$$y = 3x + 2$$

Thus, the graph is a line of slope 3 and y-intercept 2. To find the orientation we must look to the original equations; the direction of increasing t can be deduced by observing that x increases as t increases *or* by observing that y increases as t increases. Either piece of information tells us that the line is traced left to right as shown in Figure 10.1.5. ◄

TANGENT LINES TO PARAMETRIC CURVES

We will be concerned with curves that are given by parametric equations

$$x = f(t), \quad y = g(t)$$

in which $f(t)$ and $g(t)$ have continuous first derivatives with respect to t. It can be proved that if $dx/dt \neq 0$, then y is a differentiable function of x, in which case the chain rule implies that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{4}$$

This formula makes it possible to find dy/dx directly from the parametric equations without eliminating the parameter.

Example 4 Find the slope of the tangent line to the unit circle

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

at the point where $t = \pi/6$ (Figure 10.1.9).

Solution. From (4), the slope at a general point on the circle is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

Thus, the slope at $t = \pi/6$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\cot\frac{\pi}{6} = -\sqrt{3}$$

ARC LENGTH OF PARAMETRIC CURVES

10.1.1 ARC LENGTH FORMULA FOR PARAMETRIC CURVES If no segment of the curve represented by the parametric equations

$$x = x(t), \quad y = y(t) \quad (a \leq t \leq b)$$

is traced more than once as t increases from a to b, and if dx/dt and dy/dt are continuous functions for $a \leq t \leq b$, then the arc length L of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{9}$$

Example 8 Use (9) to find the circumference of a circle of radius a from the parametric equations

$$x = a\cos t, \quad y = a\sin t \qquad (0 \le t \le 2\pi)$$

Solution.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$= \int_0^{2\pi} a\, dt = at\Big]_0^{2\pi} = 2\pi a$$

Arc Length

Symbolically $L = \int_C ds$

$\langle 0 \ldots a \ldots b \ldots x\text{-},\ 0 \ldots y\text{-},$ curve C over $[a, b]$, triangle $dx, dy, ds \rangle$

$L = \int_c \sqrt{dx^2 + dy^2}$

Suppose C is described by parametric equations

$x = f(t), \quad y = g(t)$

$dx = \frac{dx}{dt}dt, \quad dy = \frac{dy}{dt}dt$

then

$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

where $a = f(\alpha)$ and $b = f(\beta)$.

Example Find the length of the curve

$x = e^t - t,\ y = 4e^{\frac{t}{2}},\quad 0 \le t \le 1$

$\frac{dx}{dt} = e^t - 1, \qquad \frac{dy}{dt} = 2e^{\frac{t}{2}}$

$\left(\frac{dx}{dt}\right)^2 = e^{2t} - 2e^t + 1,\ \left(\frac{dy}{dt}\right)^2 = 4e^t$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2\,e^t + 1 = (e^t + 1)^2$

$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^1 (e^t + 1)dt$

$= e^t + t|_0^1$

$= (e + 1) - (1 + 0)$

$= e \quad \blacksquare$

POSSIBLE QUESTIONS

TWO MARKS

1. Define length of a parameterized curve .

2. Define Arc length.

3. Find arc length for the circumference of a circle of radius a form the parametric equations $x = a\cos t \ , y = asint \quad (0 \le t \ \le 2\pi\,)$

4. Write down the surface area formula for the Revolution about the X-axis.

5.Define Arc length.

SIX MARKS

1.Use Cylindrical shells to find the volume of the solid generate when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x – axis is revolved about the y - axis.

2.Find the Volume of the solid by revolving the region bounded by the line and the curve about the x –axis where $y=4-x^2$,$y=2-x$.

3.Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$ about the x-axis .

4.The region bounded by the curve $y=x^2+1$ and the line $y=-x+3$ is revolved about x-axis to generate a solid.Find the Volume of a solid.

5.Find the Volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines y = 1 , x = 4 about the line y = 1.

6.Find the area of the surface generated by revolving the curve $y = x^3$, $0 \le x \le \frac{1}{2}$ about the x-axis.

7.The region bounded by the curve $y = x^2$ and the line $y = 2x$ in the first quadrant is resolved about the y-axis to generate a soild.Find the Volume of the solid.

8.Find the length of the asteroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$.

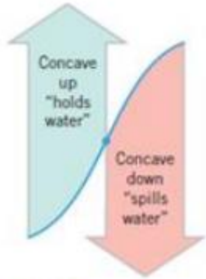
9.Find the Volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line x = 3 about the line x = 3.

10.Find the length of asteroid $x = cos^3 t$, $y = sin^3 t$,$0 \le t \le 2\pi$.

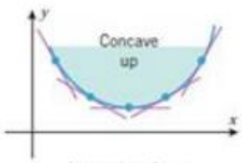
UNIT – IV

CURVE SKETCHING

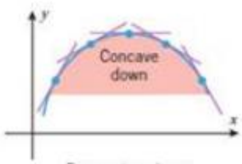
Concavity and Inflection points, asymptotes. Techniques of sketching conics, reflection properties of conics, rotation of axes and second degree equations, classification into conics using the discriminant, polar equations of conics.



► Figure 4.1.7



Increasing slopes



Decreasing slopes

CONCAVITY

Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of *curvature*. For example, the graph is increasing on both sides of the point in Figure 4.1.7, but on the left side it has an upward curvature ("holds water") and on the right side it has a downward curvature ("spills water"). On intervals where the graph of f has upward curvature we say that f is *concave up*, and on intervals where the graph has downward curvature we say that f is *concave down*.

Figure 4.1.8 suggests two ways to characterize the concavity of a differentiable function f on an open interval:

- f is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.
- f is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.

Our formal definition for "concave up" and "concave down" corresponds to the first of these characterizations.

4.1.3 DEFINITION If f is differentiable on an open interval, then f is said to be ***concave up*** on the open interval if f' is increasing on that interval, and f is said to be ***concave down*** on the open interval if f' is decreasing on that interval.

Since the slopes of the tangent lines to the graph of a differentiable function f are the values of its derivative f', it follows from Theorem 4.1.2 (applied to f' rather than f) that f' will be increasing on intervals where f'' is positive and that f' will be decreasing on intervals where f'' is negative. Thus, we have the following theorem.

4.1.4 THEOREM *Let f be twice differentiable on an open interval.*

(*a*) *If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.*

(*b*) *If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.*

► **Example 4** Figure 4.1.4 suggests that the function $f(x) = x^2 - 4x + 3$ is concave up on the interval $(-\infty, +\infty)$. This is consistent with Theorem 4.1.4, since $f'(x) = 2x - 4$ and $f''(x) = 2$, so

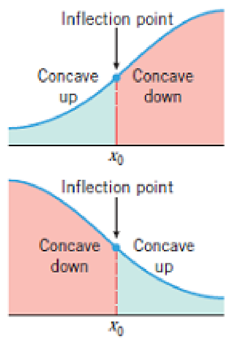
$$f''(x) > 0 \quad \text{on the interval } (-\infty, +\infty)$$

Also, Figure 4.1.5 suggests that $f(x) = x^3$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, +\infty)$. This agrees with Theorem 4.1.4, since $f'(x) = 3x^2$ and $f''(x) = 6x$, so

$$f''(x) < 0 \quad \text{if } x < 0 \quad \text{and} \quad f''(x) > 0 \quad \text{if } x > 0$$ ◄

INFLECTION POINTS

We see from Example 4 and Figure 4.1.5 that the graph of $f(x) = x^3$ changes from concave down to concave up at $x = 0$. Points where a curve changes from concave up to concave down or vice versa are of special interest, so there is some terminology associated with them.



▶ Figure 4.1.9

4.1.5 DEFINITION If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point at x_0***, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f (Figure 4.1.9).

▶ **Example 5** Figure 4.1.10 shows the graph of the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

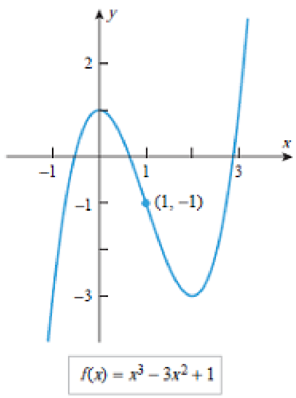
$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

The sign analysis of these derivatives is shown in the following tables:

INTERVAL	$(3x)(x-2)$	$f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	+	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	−	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	+	f is increasing on $[2, +\infty)$

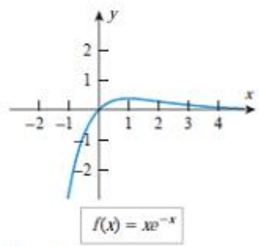
INTERVAL	$6(x-1)$	$f''(x)$	CONCLUSION
$x < 1$	$(-)$	−	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	+	f is concave up on $(1, +\infty)$



▶ Figure 4.1.10

The second table shows that there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. The inflection point is $(1, f(1)) = (1, -1)$. All of these conclusions are consistent with the graph of f. ◀

One can correctly guess from Figure 4.1.10 that the function $f(x) = x^3 - 3x^2 + 1$ has an inflection point at $x = 1$ without actually computing derivatives. However, sometimes changes in concavity are so subtle that calculus is essential to confirm their existence and identify their location. Here is an example.



► Figure 4.1.11

► **Example 6** Figure 4.1.11 suggests that the function $f(x) = xe^{-x}$ has an inflection point but its exact location is not evident from the graph in this figure. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points.

Solution. Calculating the first two derivatives of f we obtain (verify)

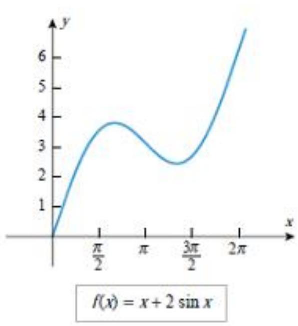
$$f'(x) = (1 - x)e^{-x}$$
$$f''(x) = (x - 2)e^{-x}$$

Keeping in mind that e^{-x} is positive for all x, the sign analysis of these derivatives is easily determined:

INTERVAL	$(1-x)(e^{-x})$	$f'(x)$	CONCLUSION
$x < 1$	(+)(+)	+	f is increasing on $(-\infty, 1]$
$x > 1$	(−)(+)	−	f is decreasing on $[1, +\infty)$

INTERVAL	$(x-2)(e^{-x})$	$f''(x)$	CONCLUSION
$x < 2$	(−)(+)	−	f is concave down on $(-\infty, 2)$
$x > 2$	(+)(+)	+	f is concave up on $(2, +\infty)$

The second table shows that there is an inflection point at $x = 2$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f. ◄



► Figure 4.1.12

► **Example 7** Figure 4.1.12 shows the graph of the function $f(x) = x + 2\sin x$ over the interval $[0, 2\pi]$. Use the first and second derivatives of f to determine where f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 1 + 2\cos x$$
$$f''(x) = -2\sin x$$

Since f' is a continuous function, it changes sign on the interval $(0, 2\pi)$ only at points where $f'(x) = 0$ (why?). These values are solutions of the equation

$$1 + 2\cos x = 0 \quad \text{or equivalently} \quad \cos x = -\tfrac{1}{2}$$

There are two solutions of this equation in the interval $(0, 2\pi)$, namely, $x = 2\pi/3$ and $x = 4\pi/3$ (verify). Similarly, f'' is a continuous function, so its sign changes in the interval $(0, 2\pi)$ will occur only at values of x for which $f''(x) = 0$. These values are solutions of the equation

$$-2\sin x = 0$$

There is one solution of this equation in the interval $(0, 2\pi)$, namely, $x = \pi$. With the help of these "sign transition points" we obtain the sign analysis shown in the following tables:

INTERVAL	$f'(x) = 1 + 2\cos x$	CONCLUSION
$0 < x < 2\pi/3$	+	f is increasing on $[0, 2\pi/3]$
$2\pi/3 < x < 4\pi/3$	–	f is decreasing on $[2\pi/3, 4\pi/3]$
$4\pi/3 < x < 2\pi$	+	f is increasing on $[4\pi/3, 2\pi]$

INTERVAL	$f''(x) = -2\sin x$	CONCLUSION
$0 < x < \pi$	–	f is concave down on $(0, \pi)$
$\pi < x < 2\pi$	+	f is concave up on $(\pi, 2\pi)$

The second table shows that there is an inflection point at $x = \pi$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f. ◄

In the preceding examples the inflection points of f occurred wherever $f''(x) = 0$. However, this is not always the case. Here is a specific example.

► **Example 8** Find the inflection points, if any, of $f(x) = x^4$.

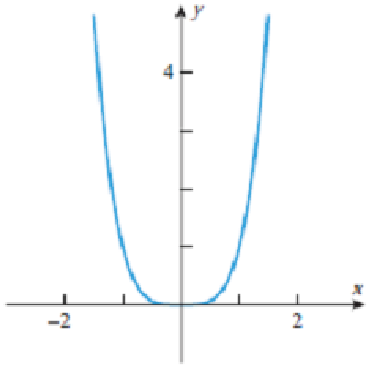
Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

Since $f''(x)$ is positive for $x < 0$ and for $x > 0$, the function f is concave up on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$. Thus, there is no change in concavity and hence no inflection point at $x = 0$, even though $f''(0) = 0$ (Figure 4.1.13). ◄

We will see later that if a function f has an inflection point at $x = x_0$ and $f''(x_0)$ exists, then $f''(x_0) = 0$. Also, we will see in Section 4.3 that an inflection point may also occur where $f''(x)$ is not defined.



$f(x) = x^4$

▶ Figure 4.1.13

Asymptotes

Before continuing with asymptotes, it is recommended that you review the vertical asymptote and infinite limits section of the limits tutorial at the link below.

Vertical Asymptotes and Infinite Limits

In order to properly sketch a curve, we need to determine how the curve behaves as *x* approaches positive and negative infinity. We must find the limit of the function as x approaches infinity.

For a function f defined on (a, ∞),

$$\lim_{x\to\infty} f(x) = L$$

means that the values of f(*x*) approach the value *L* when *x* is taken to be sufficiently large.

For a function f defined on $(-\infty, a)$,

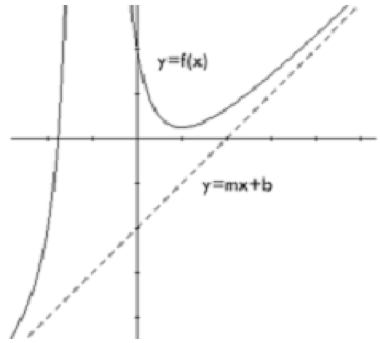
In order to find the horizontal asymptotes of a function, we use the following theorem. If n is a positive number, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

If n is a positive, rational number such that xn is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Functions do not always approach a value as x approaches positive or negative infinity. Often there is no horizontal asymptote and the functions have infinite limits at infinity. For example, the function f(x) = x2 approaches infinity when x is taken to be sufficiently large, positively or negatively.



Slant Asymptotes

Some curves may have an asymptote that is neither vertical nor horizontal. These curves approach a line as *x* approaches positive or negative infinity. This line is called the slant asymptote of the function. The graph to the right illustrates the concept of slant asymptotes.

$$\lim_{x\to\infty}[f(x) - (mx+b)] = 0 \quad \text{or} \quad \lim_{x\to-\infty}[f(x) - (mx+b)] = 0$$

then the function f(*x*) has a slant asymptote of y = m*x* + b.

Rational functions will have a slant asymptote when the degree of the numerator is one more than the degree of the denominator. To find the equation of the slant asymptote, we divide the numberator by the denominator using long division. The quotient will be the equation of the slant asymptote. The remainder is the quantity f(*x*) - (m*x* + b). We must show that the remainder approaches 0, as *x* approaches positive or negative infinity. The example below will give you a better idea of how to find the slant asymptote of a function.

Example 1: Sketch a curve for f(x) = $\frac{2}{x^2+3}$

Step 1: Find the y-intercepts, when x = 0

y = $\frac{2}{3}$

Therefore (0, $\frac{2}{3}$) is the y-intercept

Step 2: We cannot find the x-intercepts, since $y \neq 0$

Step 3: Check if the curve is symmetric, i.e. is the function odd or even.

f(x) = $\frac{2}{x^2+3}$

$f(-x) = \dfrac{2}{x^2 + 3}$

$f(x) = f(-x)$

So this is an even function, and is symmetric about y-axis.

Step 4: Check for any discontinuities, and find the asymptotes, if any, or the limits

$$\lim_{x \to \infty} \frac{2}{x^2 + 3} = \lim_{x \to \infty} \frac{\frac{2}{x^2}}{1 + \frac{3}{x^2}}$$

$$= \frac{0}{1 + 0} = 0+$$

$$\lim_{x \to -\infty} \frac{2}{x^2 + 3} = \lim_{x \to -\infty} \frac{\frac{2}{x^2}}{1 + \frac{3}{x^2}}$$

$= 0-$

Step 5: Find stationary points (put y' = f'(x) = 0)

$$\frac{dy}{dx} = -\frac{2}{\left(x^2 + 3\right)^2} .2x = \frac{-4x}{\left(x^2 + 3\right)^2}$$

$\frac{dy}{dx} = 0$ So $\frac{-4x}{\left(x^2+3\right)^2} = 0$

$\Rightarrow x = 0$

$\left(0, \frac{2}{3}\right)$ is a stationary point.

Step 6: Find the point of inflection

$$\frac{d^2y}{dx^2} = \frac{(-4)\left(x^2+3\right)^2 + 4x(2)\left(x^3+3\right).2x}{\left(x^2+3\right)^4}$$

$$= \frac{4\left(x^2+3\right)\left(-\left(x^3+3\right)+16x^2\right)}{\left(x^2+3\right)^4}$$

$$= \frac{4\left(15x^2-3\right)}{\left(x^2+3\right)^3}$$

For point of inflection, $\frac{d^2y}{dx^2} = 0$.
Therefore $15x^2 - 3 = 0$

$x^2 = \frac{1}{5}$, or $x = \pm\frac{1}{\sqrt{5}}$

When x = 0, $\frac{d^2y}{dx^2} = \frac{\frac{4\left(15(0)^2 - 3\right)}{\left((0)^2+3\right)^3}}{} = \frac{4(-3)}{3^3} = -\frac{4}{9} < 0$

Hence $\left(0, \frac{2}{3}\right)$ is a maximum point.

Test for point of inflection at x = $\frac{1}{\sqrt{5}}$ = 0.447

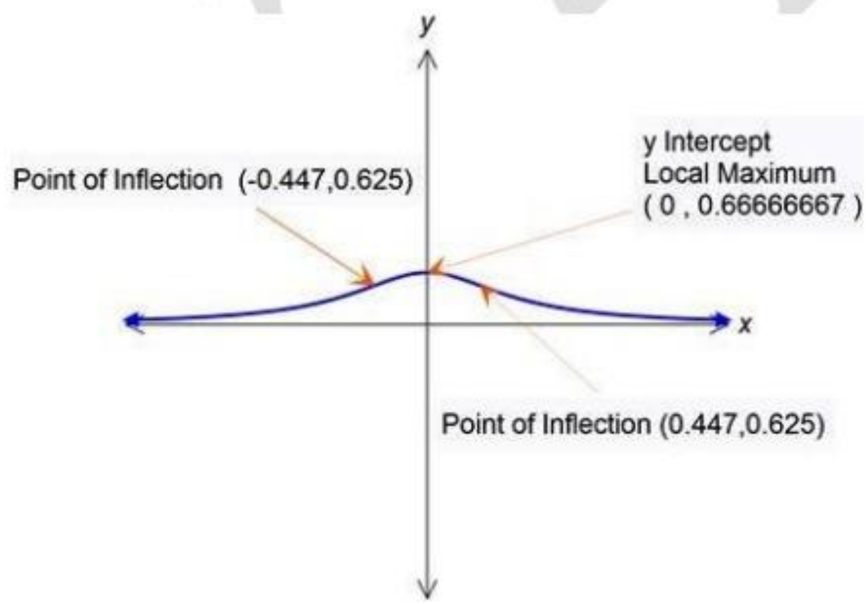
x	0.4	$\frac{1}{\sqrt{5}}$ = 0.447	0.5
f"(x)	-0.07 < 0	0	0.087 > 0

When x = 0.447, y = 0.625
So (0.447, 0.625) is a point of inflection.

Test for point of inflection at x = $-\frac{1}{\sqrt{5}}$ = -0.447

x	-0.5	$-\frac{1}{\sqrt{5}}$ = -0.447	-0.4
f"(x)	0.087 > 0	0	-0.07 < 0

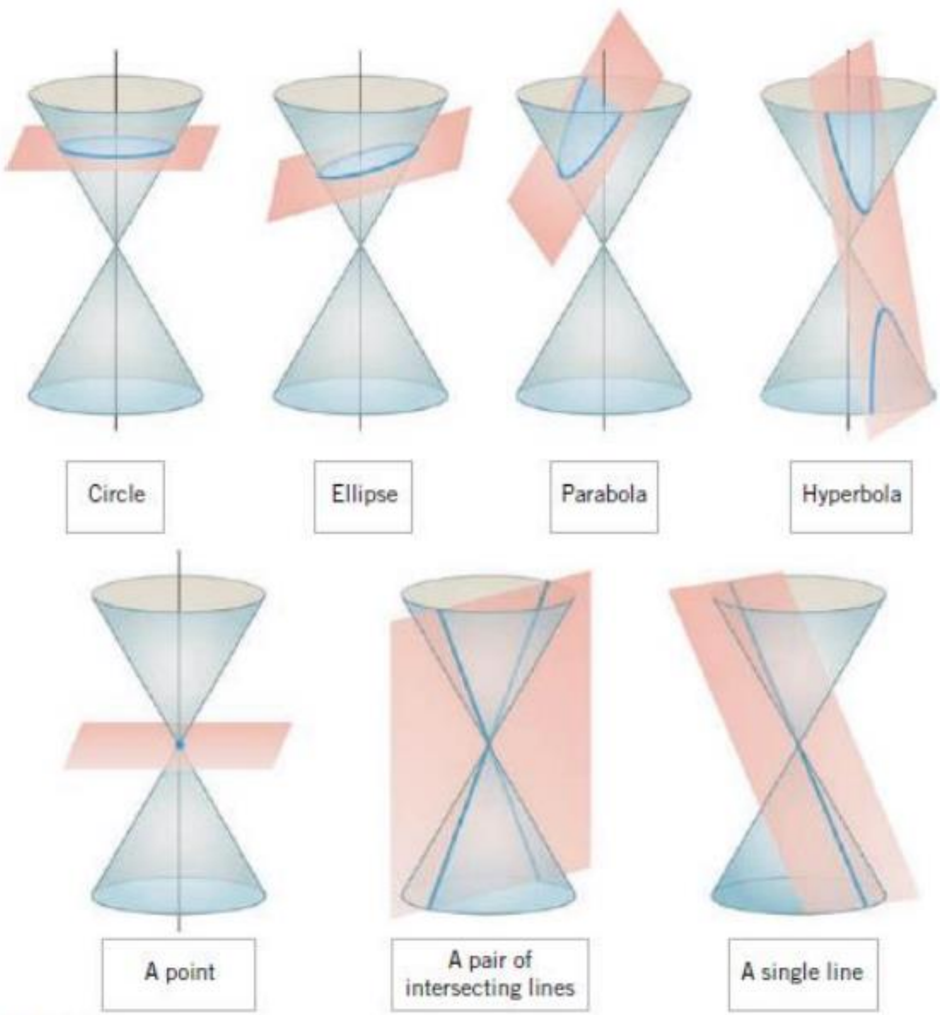
When x = -0.447, y = 0.625
So (-0.447, 0.625) is a point of inflection.



CONIC SECTIONS

CONIC SECTIONS

Circles, ellipses, parabolas, and hyperbolas are called ***conic sections*** or ***conics*** because they can be obtained as intersections of a plane with a double-napped circular cone (Figure 10.4.1). If the plane passes through the vertex of the double-napped cone, then the intersection is a point, a pair of intersecting lines, or a single line. These are called ***degenerate conic sections***.



▲ Figure 10.4.1

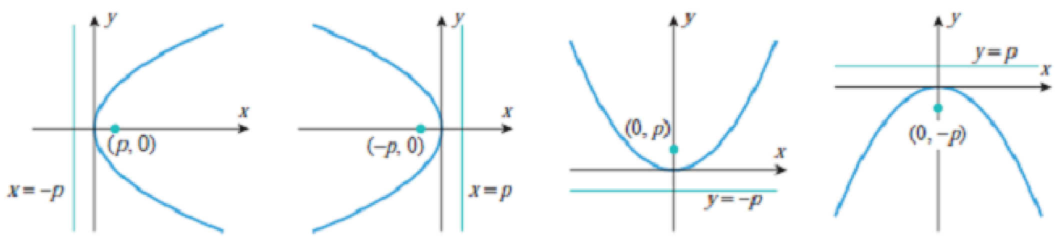
10.4.1 DEFINITION A *parabola* is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.

10.4.2 DEFINITION An *ellipse* is the set of all points in the plane, the sum of whose distances from two fixed points is a given positive constant that is greater than the distance between the fixed points.

10.4.3 DEFINITION A *hyperbola* is the set of all points in the plane, the difference of whose distances from two fixed distinct points is a given positive constant that is less than the distance between the fixed points.

EQUATIONS OF PARABOLAS IN STANDARD POSITION

PARABOLAS IN STANDARD POSITION



$y^2 = 4px$ $y^2 = -4px$ $x^2 = 4py$ $x^2 = -4py$

A TECHNIQUE FOR SKETCHING PARABOLAS

Parabolas can be sketched from their *standard equations* using four basic steps:

Sketching a Parabola from Its Standard Equation

Step 1. Determine whether the axis of symmetry is along the x-axis or the y-axis. Referring to Figure 10.4.6, the axis of symmetry is along the x-axis if the equation has a y^2-term, and it is along the y-axis if it has an x^2-term.

Step 2. Determine which way the parabola opens. If the axis of symmetry is along the x-axis, then the parabola opens to the right if the coefficient of x is positive, and it opens to the left if the coefficient is negative. If the axis of symmetry is along the y-axis, then the parabola opens up if the coefficient of y is positive, and it opens down if the coefficient is negative.

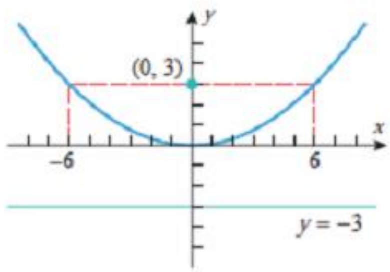
Step 3. Determine the value of p and draw a box extending p units from the origin along the axis of symmetry in the direction in which the parabola opens and extending $2p$ units on each side of the axis of symmetry.

Step 4. Using the box as a guide, sketch the parabola so that its vertex is at the origin and it passes through the corners of the box (Figure 10.4.8).

Example 1 Sketch the graphs of the parabolas

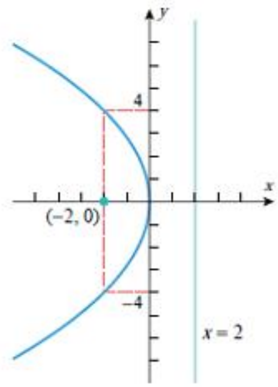
$$\text{(a)}\ x^2 = 12y \qquad \text{(b)}\ y^2 + 8x = 0$$

and show the focus and directrix of each.



$x^2 = 12y$

▲ Figure 10.4.9



$y^2 = -8x$

▲ Figure 10.4.10

Solution (a). This equation involves x^2, so the axis of symmetry is along the y-axis, and the coefficient of y is positive, so the parabola opens upward. From the coefficient of y, we obtain $4p = 12$ or $p = 3$. Drawing a box extending $p = 3$ units up from the origin and $2p = 6$ units to the left and $2p = 6$ units to the right of the y-axis, then using corners of the box as a guide, yields the graph in Figure 10.4.9.

The focus is $p = 3$ units from the vertex along the axis of symmetry in the direction in which the parabola opens, so its coordinates are $(0, 3)$. The directrix is perpendicular to the axis of symmetry at a distance of $p = 3$ units from the vertex on the opposite side from the focus, so its equation is $y = -3$.

Solution (b). We first rewrite the equation in the standard form

$$y^2 = -8x$$

This equation involves y^2, so the axis of symmetry is along the x-axis, and the coefficient of x is negative, so the parabola opens to the left. From the coefficient of x we obtain $4p = 8$, so $p = 2$. Drawing a box extending $p = 2$ units left from the origin and $2p = 4$ units above and $2p = 4$ units below the x-axis, then using corners of the box as a guide, yields the graph in Figure 10.4.10. ◄

Example 2 Find an equation of the parabola that is symmetric about the y-axis, has its vertex at the origin, and passes through the point $(5, 2)$.

Solution. Since the parabola is symmetric about the y-axis and has its vertex at the origin, the equation is of the form

$$x^2 = 4py \quad \text{or} \quad x^2 = -4py$$

where the sign depends on whether the parabola opens up or down. But the parabola must open up since it passes through the point $(5, 2)$, which lies in the first quadrant. Thus, the equation is of the form

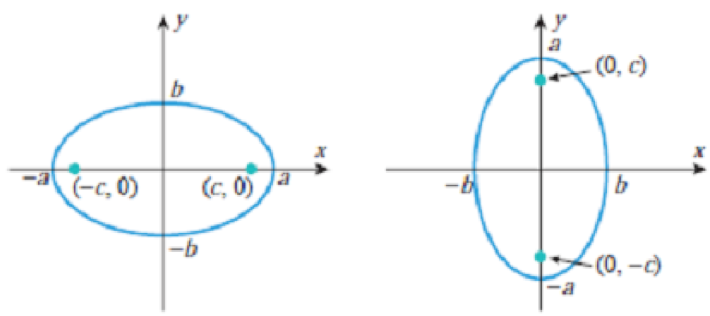
$$x^2 = 4py \tag{5}$$

Since the parabola passes through $(5, 2)$, we must have $5^2 = 4p \cdot 2$ or $4p = \frac{25}{2}$. Therefore, (5) becomes

$$x^2 = \frac{25}{2}y$$

EQUATIONS OF ELLIPSES IN STANDARD POSITION

ELLIPSES IN STANDARD POSITION



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

A TECHNIQUE FOR SKETCHING ELLIPSES

Ellipses can be sketched from their *standard equations* using three basic steps:

Sketching an Ellipse from Its Standard Equation

Step 1. Determine whether the major axis is on the x-axis or the y-axis. This can be ascertained from the sizes of the denominators in the equation. Referring to Figure 10.4.14, and keeping in mind that $a^2 > b^2$ (since $a > b$), the major axis is along the x-axis if x^2 has the larger denominator, and it is along the y-axis if y^2 has the larger denominator. If the denominators are equal, the ellipse is a circle.

Step 2. Determine the values of a and b and draw a box extending a units on each side of the center along the major axis and b units on each side of the center along the minor axis.

Step 3. Using the box as a guide, sketch the ellipse so that its center is at the origin and it touches the sides of the box where the sides intersect the coordinate axes (Figure 10.4.16).

Example 3 Sketch the graphs of the ellipses

$$\text{(a)}\ \frac{x^2}{9}+\frac{y^2}{16}=1 \qquad \text{(b)}\ x^2+2y^2=4$$

showing the foci of each.

Solution (a). Since y^2 has the larger denominator, the major axis is along the y-axis. Moreover, since $a^2 > b^2$, we must have $a^2 = 16$ and $b^2 = 9$, so

$$a = 4 \quad \text{and} \quad b = 3$$

Drawing a box extending 4 units on each side of the origin along the y-axis and 3 units on each side of the origin along the x-axis as a guide yields the graph in Figure 10.4.17.

The foci lie c units on each side of the center along the major axis, where c is given by (7). From the values of a^2 and b^2 above, we obtain

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7} \approx 2.6$$

Thus, the coordinates of the foci are $(0, \sqrt{7})$ and $(0, -\sqrt{7})$, since they lie on the y-axis.

Solution (b). We first rewrite the equation in the standard form

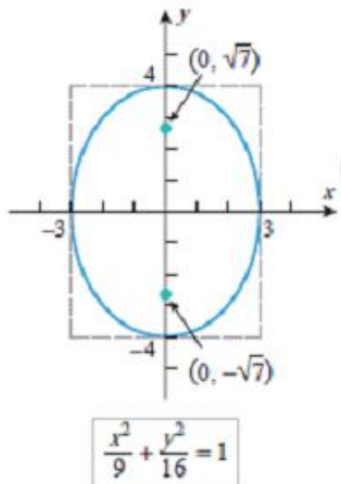
$$\frac{x^2}{4} + \frac{y^2}{2} = 1$$

Since x^2 has the larger denominator, the major axis lies along the x-axis, and we have $a^2 = 4$ and $b^2 = 2$. Drawing a box extending $a = 2$ units on each side of the origin along the x-axis and extending $b = \sqrt{2} \approx 1.4$ units on each side of the origin along the y-axis as a guide yields the graph in Figure 10.4.18.

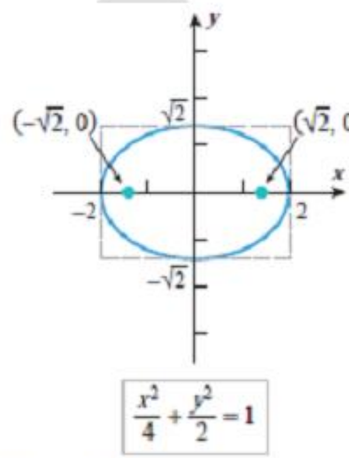
From (7), we obtain

$$c = \sqrt{a^2 - b^2} = \sqrt{2} \approx 1.4$$

Thus, the coordinates of the foci are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$, since they lie on the x-axis. ◄



▲ **Figure 10.4.17**



▲ **Figure 10.4.18**

Example 4 Find an equation for the ellipse with foci $(0, \pm 2)$ and major axis with end-points $(0, \pm 4)$.

Solution. From Figure 10.4.14, the equation has the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

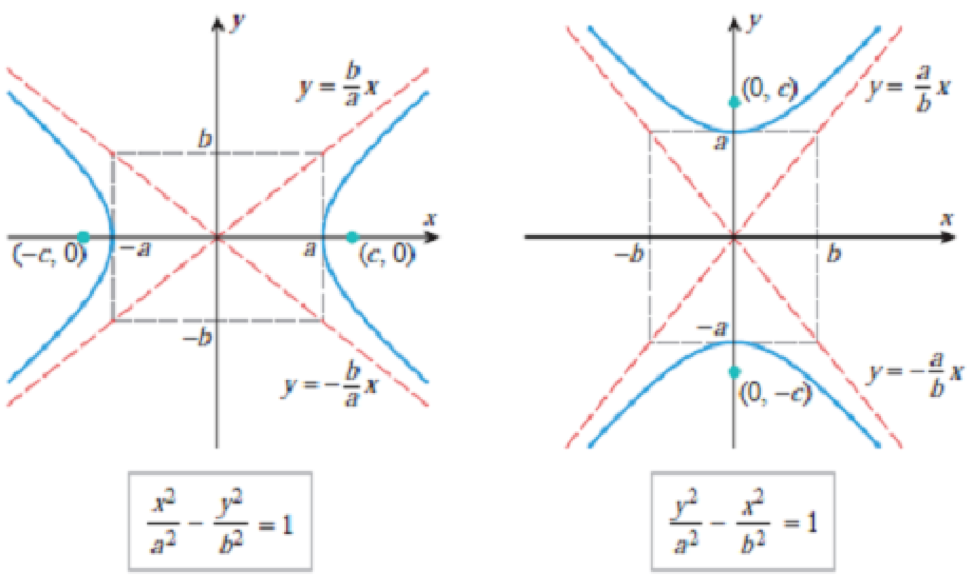
and from the given information, $a = 4$ and $c = 2$. It follows from (6) that

$$b^2 = a^2 - c^2 = 16 - 4 = 12$$

so the equation of the ellipse is
$$\frac{x^2}{12} + \frac{y^2}{16} = 1$$

EQUATIONS OF HYPERBOLAS IN STANDARD POSITION

HYPERBOLAS IN STANDARD POSITION



A TECHNIQUE FOR SKETCHING HYPERBOLAS

Hyperbolas can be sketched from their *standard equations* using four basic steps:

Sketching a Hyperbola from Its Standard Equation

Step 1. Determine whether the focal axis is on the x-axis or the y-axis. This can be ascertained from the location of the minus sign in the equation. Referring to Figure 10.4.22, the focal axis is along the x-axis when the minus sign precedes the y^2-term, and it is along the y-axis when the minus sign precedes the x^2-term.

Step 2. Determine the values of a and b and draw a box extending a units on either side of the center along the focal axis and b units on either side of the center along the conjugate axis. (The squares of a and b can be read directly from the equation.)

Step 3. Draw the asymptotes along the diagonals of the box.

Step 4. Using the box and the asymptotes as a guide, sketch the graph of the hyperbola (Figure 10.4.24).

► **Example 5** Sketch the graphs of the hyperbolas

$$\text{(a)}\ \frac{x^2}{4} - \frac{y^2}{9} = 1 \qquad \text{(b)}\ y^2 - x^2 = 1$$

showing their vertices, foci, and asymptotes.

Solution (a). The minus sign precedes the y^2-term, so the focal axis is along the x-axis. From the denominators in the equation we obtain

$$a^2 = 4 \quad \text{and} \quad b^2 = 9$$

Since a and b are positive, we must have $a = 2$ and $b = 3$. Recalling that the vertices lie a units on each side of the center on the focal axis, it follows that their coordinates in this case are $(2, 0)$ and $(-2, 0)$. Drawing a box extending $a = 2$ units along the x-axis on each side of the origin and $b = 3$ units on each side of the origin along the y-axis, then drawing the asymptotes along the diagonals of the box as a guide, yields the graph in Figure 10.4.25.

To obtain equations for the asymptotes, we replace 1 by 0 in the given equation; this yields

$$\frac{x^2}{4} - \frac{y^2}{9} = 0 \quad \text{or} \quad y = \pm\frac{3}{2}x$$

The foci lie c units on each side of the center along the focal axis, where c is given by (11). From the values of a^2 and b^2 above we obtain

$$c = \sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13} \approx 3.6$$

Since the foci lie on the x-axis in this case, their coordinates are $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$.

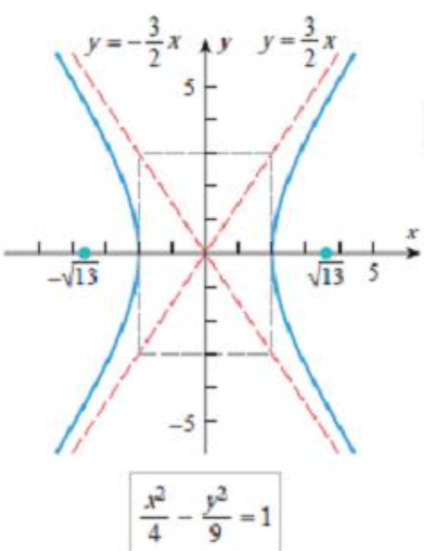
Solution (b). The minus sign precedes the x^2-term, so the focal axis is along the y-axis. From the denominators in the equation we obtain $a^2 = 1$ and $b^2 = 1$, from which it follows that

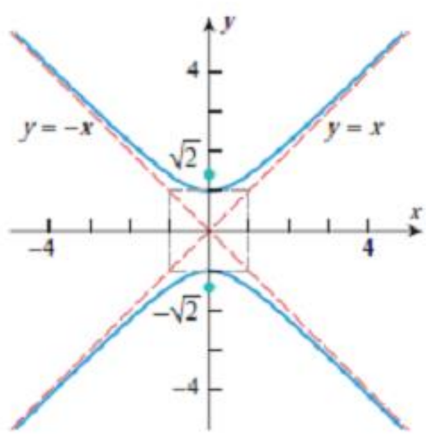
$$a = 1 \quad \text{and} \quad b = 1$$

Thus, the vertices are at $(0, -1)$ and $(0, 1)$. Drawing a box extending $a = 1$ unit on either side of the origin along the y-axis and $b = 1$ unit on either side of the origin along the x-axis, then drawing the asymptotes, yields the graph in Figure 10.4.26. Since the box is actually a square, the asymptotes are perpendicular and have equations $y = \pm x$. This can also be seen by replacing 1 by 0 in the given equation, which yields $y^2 - x^2 = 0$ or $y = \pm x$. Also,

$$c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2}$$

so the foci, which lie on the y-axis, are $(0, -\sqrt{2})$ and $(0, \sqrt{2})$. ◂





Example 6 Find the equation of the hyperbola with vertices $(0, \pm 8)$ and asymptotes $y = \pm\frac{4}{3}x$.

Solution. Since the vertices are on the y-axis, the equation of the hyperbola has the form $(y^2/a^2) - (x^2/b^2) = 1$ and the asymptotes are

$$y = \pm\frac{a}{b}x$$

From the locations of the vertices we have $a = 8$, so the given equations of the asymptotes yield

$$y = \pm\frac{a}{b}x = \pm\frac{8}{b}x = \pm\frac{4}{3}x$$

from which it follows that $b = 6$. Thus, the hyperbola has the equation

$$\frac{y^2}{64} - \frac{x^2}{36} = 1 \blacktriangleleft$$

Example 7 Find an equation for the parabola that has its vertex at $(1, 2)$ and its focus at $(4, 2)$.

Solution. Since the focus and vertex are on a horizontal line, and since the focus is to the right of the vertex, the parabola opens to the right and its equation has the form

$$(y-k)^2 = 4p(x-h)$$

Since the vertex and focus are 3 units apart, we have $p = 3$, and since the vertex is at $(h, k) = (1, 2)$, we obtain

$$(y-2)^2 = 12(x-1) \blacktriangleleft$$

Example 8 Describe the graph of the equation

$$y^2 - 8x - 6y - 23 = 0$$

Solution. The equation involves quadratic terms in y but none in x, so we first take all of the y-terms to one side:

$$y^2 - 6y = 8x + 23$$

Next, we complete the square on the y-terms by adding 9 to both sides:

$$(y-3)^2 = 8x + 32$$

Finally, we factor out the coefficient of the x-term to obtain

$$(y-3)^2 = 8(x+4)$$

This equation is of form (12) with $h = -4$, $k = 3$, and $p = 2$, so the graph is a parabola with vertex $(-4, 3)$ opening to the right. Since $p = 2$, the focus is 2 units to the right of the vertex, which places it at the point $(-2, 3)$; and the directrix is 2 units to the left of the vertex, which means that its equation is $x = -6$. The parabola is shown in Figure 10.4.27. ◄

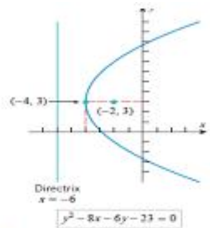


Figure 10.4.27

► **Example 9** Describe the graph of the equation

$$16x^2 + 9y^2 - 64x - 54y + 1 = 0$$

Solution. This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(16x^2 - 64x) + (9y^2 - 54y) = -1$$

Next, factor out the coefficients of x^2 and y^2 and complete the squares:

$$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$$

or

$$16(x-2)^2 + 9(y-3)^2 = 144$$

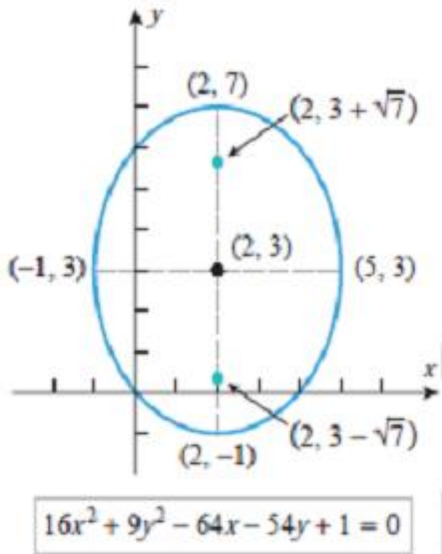
Finally, divide through by 144 to introduce a 1 on the right side:

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$

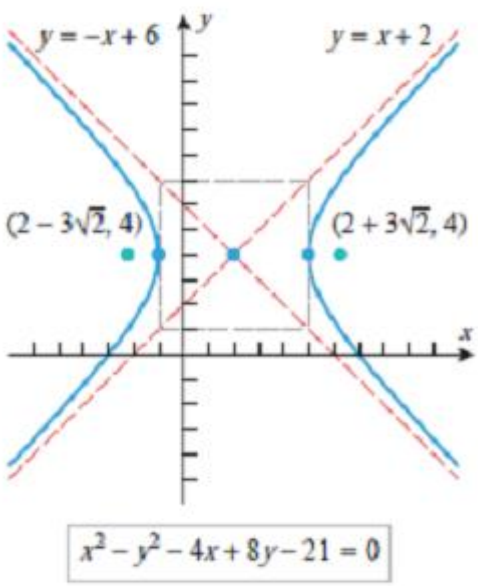
This is an equation of form (17), with $h = 2$, $k = 3$, $a^2 = 16$, and $b^2 = 9$. Thus, the graph of the equation is an ellipse with center $(2, 3)$ and major axis parallel to the y-axis. Since $a = 4$, the major axis extends 4 units above and 4 units below the center, so its endpoints are $(2, 7)$ and $(2, -1)$ (Figure 10.4.28). Since $b = 3$, the minor axis extends 3 units to the left and 3 units to the right of the center, so its endpoints are $(-1, 3)$ and $(5, 3)$. Since

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

the foci lie $\sqrt{7}$ units above and below the center, placing them at the points $(2, 3 + \sqrt{7})$ and $(2, 3 - \sqrt{7})$. ◄



▲ **Figure 10.4.28**



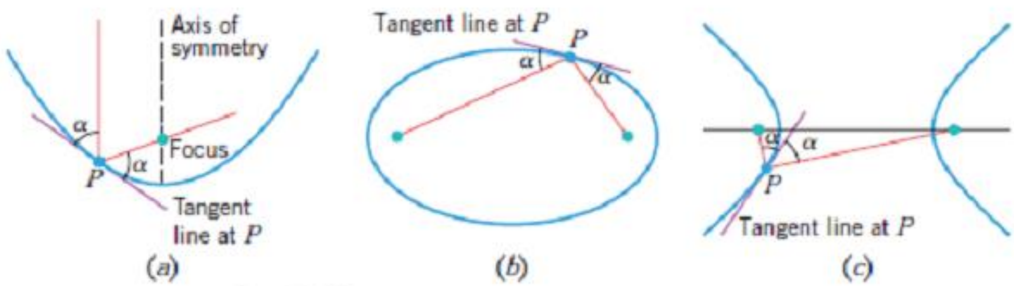
REFLECTION PROPERTIES OF THE CONIC SECTIONS

Parabolas, ellipses, and hyperbolas have certain reflection properties that make them extremely valuable in various applications. In the exercises we will ask you to prove the following results.

10.4.4 THEOREM (***Reflection Property of Parabolas***) *The tangent line at a point P on a parabola makes equal angles with the line through P parallel to the axis of symmetry and the line through P and the focus (Figure 10.4.30a).*

10.4.5 THEOREM (***Reflection Property of Ellipses***) *A line tangent to an ellipse at a point P makes equal angles with the lines joining P to the foci (Figure 10.4.30b).*

10.4.6 THEOREM (***Reflection Property of Hyperbolas***) *A line tangent to a hyperbola at a point P makes equal angles with the lines joining P to the foci (Figure 10.4.30c).*



ROTATION OF AXES; SECOND-DEGREE EQUATIONS

ROTATION OF AXES

To study conics that are tilted relative to the coordinate axes it is frequently helpful to rotate the coordinate axes, so that the rotated coordinate axes are parallel to the axes of the conic. Before we can discuss the details, we need to develop some ideas about rotation of coordinate axes.

In Figure 10.5.2*a* the axes of an xy-coordinate system have been rotated about the origin through an angle θ to produce a new $x'y'$-coordinate system. As shown in the figure, each point P in the plane has coordinates (x', y') as well as coordinates (x, y). To see how the two are related, let r be the distance from the common origin to the point P, and let α be the angle shown in Figure 10.5.2*b*. It follows that

$$x = r\cos(\theta + \alpha), \quad y = r\sin(\theta + \alpha) \tag{3}$$

and

$$x' = r\cos\alpha, \quad y' = r\sin\alpha \tag{4}$$

Using familiar trigonometric identities, the relationships in (3) can be written as

$$x = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$
$$y = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$$

and on substituting (4) in these equations we obtain the following relationships called the ***rotation equations***:

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \tag{5}$$

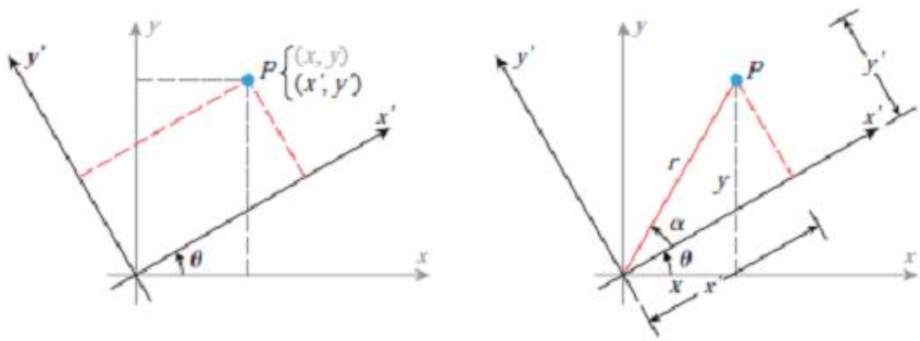


Figure 10.5.2 (*a*) (*b*)

▶ **Example 1** Suppose that the axes of an xy-coordinate system are rotated through an angle of $\theta = 45°$ to obtain an $x'y'$-coordinate system. Find the equation of the curve

$$x^2 - xy + y^2 - 6 = 0$$

in $x'y'$-coordinates.

Solution. Substituting $\sin\theta = \sin 45° = 1/\sqrt{2}$ and $\cos\theta = \cos 45° = 1/\sqrt{2}$ in (5) yields the rotation equations

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

Substituting these into the given equation yields

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - 6 = 0$$

or

$$\frac{x'^2 - 2x'y' + y'^2 - x'^2 + y'^2 + x'^2 + 2x'y' + y'^2}{2} = 6$$

or

$$\frac{x'^2}{12} + \frac{y'^2}{4} = 1$$

which is the equation of an ellipse (Figure 10.5.3). ◀

If the rotation equations (5) are solved for x' and y' in terms of x and y, one obtains (Exercise 16):

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \qquad (6)$$

▶ **Example 2** Find the new coordinates of the point $(2, 4)$ if the coordinate axes are rotated through an angle of $\theta = 30°$.

Solution. Using the rotation equations in (6) with $x = 2$, $y = 4$, $\cos\theta = \cos 30° = \sqrt{3}/2$, and $\sin\theta = \sin 30° = 1/2$, we obtain

$$\begin{aligned} x' &= 2(\sqrt{3}/2) + 4(1/2) = \sqrt{3} + 2 \\ y' &= -2(1/2) + 4(\sqrt{3}/2) = -1 + 2\sqrt{3} \end{aligned}$$

Thus, the new coordinates are $(\sqrt{3} + 2, -1 + 2\sqrt{3})$. ◀

ELIMINATING THE CROSS-PRODUCT TERM

10.5.1 THEOREM *If the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{7}$$

is such that $B \neq 0$, and if an $x'y'$-coordinate system is obtained by rotating the xy-axes through an angle θ satisfying

$$\cot 2\theta = \frac{A - C}{B} \tag{8}$$

then, in $x'y'$-coordinates, Equation (7) will have the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

CONIC SECTIONS IN POLAR COORDINATES

► **Example 1** Sketch the graph of $r = \dfrac{2}{1 - \cos\theta}$ in polar coordinates.

Solution. The equation is an exact match to (4) with $d = 2$ and $e = 1$. Thus, the graph is a parabola with the focus at the pole and the directrix 2 units to the left of the pole. This tells us that the parabola opens to the right along the polar axis and $p = 1$. Thus, the parabola looks roughly like that sketched in Figure 10.6.4. ◄

► **Example 2** Find the constants a, b, and c for the ellipse $r = \dfrac{6}{2 + \cos\theta}$.

Solution. This equation does not match any of the forms in Theorem 10.6.2 because they all require a constant term of 1 in the denominator. However, we can put the equation into one of these forms by dividing the numerator and denominator by 2 to obtain

$$r = \frac{3}{1 + \frac{1}{2}\cos\theta}$$

This is an exact match to (3) with $d = 6$ and $e = \frac{1}{2}$, so the graph is an ellipse with the directrix 6 units to the right of the pole. The distance r_0 from the focus to the closest vertex can be obtained by setting $\theta = 0$ in this equation, and the distance r_1 to the farthest vertex can be obtained by setting $\theta = \pi$. This yields

$$r_0 = \frac{3}{1 + \frac{1}{2}\cos 0} = \frac{3}{\frac{3}{2}} = 2, \quad r_1 = \frac{3}{1 + \frac{1}{2}\cos \pi} = \frac{3}{\frac{1}{2}} = 6$$

Thus, from Formulas (8), (10), and (9), respectively, we obtain

$$a = \tfrac{1}{2}(r_1 + r_0) = 4, \quad b = \sqrt{r_0 r_1} = 2\sqrt{3}, \quad c = \tfrac{1}{2}(r_1 - r_0) = 2$$

The ellipse looks roughly like that sketched in Figure 10.6.6. ◂

POSSIBLE QUESTIONS

TWO MARKS

1. Define discriminate test.

2. Define an ellipse.

3. Define eccentricity of the ellipse.

4. Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144.$

5. Find an equation for the hyperbola with eccentricity 3/2 and directrix x = 2.

SIX MARKS

1.Describe the graph of the equation $16x^2 + 9y^2 - 64x - 54y + 1 = 0.$

2.Identify and sketch the curve xy=1.

3.Determine the open interval on which the graph of f(x)=$\frac{6}{x^2+3}$ is concave up or concave down

4.Find the graph of the hyperbola (i) $\frac{x^2}{4} - \frac{y^2}{5} = 1$ (ii) $\frac{y^2}{4} - \frac{x^2}{5} = 1$

5.Sketch the graph of the parabolas $i)\ x^2 = 12y$ $ii)\ y^2 + 8x = 0$ and show that focus and directrix of each.

6.Determine the point of inflection and discuss the concavity of the curve f(x)=$\frac{x^3}{3x^2+1}$

7.Find the equation of the curve $x^2 - xy + y^2 - 6 = 0$ in $x'y'$ - coordinates. if the coordinate axes are rotated through an angle of $\theta = 45°$

8.Determine the graph of the equation $16x^2+9y^2-64x-54y+1=0$.

9.Sketch the graph of the ellipse $i) \frac{x^2}{9} + \frac{y^2}{16} = 1$ $ii)\ x^2 + 2y^2 = 4.$ and showing the foci of each.

10.Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3 , 0) and the line x = 1 as the corresponding directrix.

UNIT – V

VECTOR FUCTIONS

Introduction to vector functions, operations with vector-valued functions, limits and continuity of vector functions, differentiation and integration of vector functions, tangent and normal components of acceleration, modeling ballistics and planetary motion, Kepler’s second law.

VECTOR-VALUED FUNCTIONS

The twisted cubic defined by the equations in (3) is the set of points of the form (t, t^2, t^3) for real values of t. If we view each of these points as a terminal point for a vector $\mathbf{r}$ whose initial point is at the origin,

$$\mathbf{r} = \langle x, y, z \rangle = \langle t, t^2, t^3 \rangle = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

then we obtain $\mathbf{r}$ as a function of the parameter t, that is, $\mathbf{r} = \mathbf{r}(t)$. Since this function produces a *vector*, we say that $\mathbf{r} = \mathbf{r}(t)$ defines $\mathbf{r}$ as a ***vector-valued function of a real variable***, or more simply, a ***vector-valued function***. The vectors that we will consider in this text are either in 2-space or 3-space, so we will say that a vector-valued function is in 2-space or in 3-space according to the kind of vectors that it produces.

If $\mathbf{r}(t)$ is a vector-valued function in 3-space, then for each allowable value of t the vector $\mathbf{r} = \mathbf{r}(t)$ can be represented in terms of components as

$$\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{4}$$

The functions $x(t)$, $y(t)$, and $z(t)$ are called the ***component functions*** or the ***components*** of $\mathbf{r}(t)$.

► **Example 3** The component functions of

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

are

$$x(t) = t, \quad y(t) = t^2, \quad z(t) = t^3 \blacktriangleleft$$

► **Example 4** Find the natural domain of

$$\mathbf{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t} \rangle = (\ln|t-1|)\mathbf{i} + e^t\mathbf{j} + \sqrt{t}\mathbf{k}$$

Solution. The natural domains of the component functions

$$x(t) = \ln|t-1|, \quad y(t) = e^t, \quad z(t) = \sqrt{t}$$

are

$$(-\infty, 1) \cup (1, +\infty), \quad (-\infty, +\infty), \quad [0, +\infty)$$

respectively. The intersection of these sets is

$$[0, 1) \cup (1, +\infty)$$

(verify), so the natural domain of $\mathbf{r}(t)$ consists of all values of t such that

$$0 \le t < 1 \quad \text{or} \quad t > 1 \blacktriangleleft$$

LIMITS AND CONTINUITY

Our first goal in this section is to develop a notion of what it means for a vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space to approach a limiting vector $\mathbf{L}$ as t approaches a number a. That is, we want to define

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L} \tag{1}$$

One way to motivate a reasonable definition of (1) is to position $\mathbf{r}(t)$ and $\mathbf{L}$ with their initial points at the origin and interpret this limit to mean that the terminal point of $\mathbf{r}(t)$ approaches the terminal point of $\mathbf{L}$ as t approaches a or, equivalently, that the vector $\mathbf{r}(t)$ approaches the vector $\mathbf{L}$ in both length and direction at t approaches a (Figure 12.2.1). Algebraically, this is equivalent to stating that

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0 \tag{2}$$

12.2.1 DEFINITION Let $\mathbf{r}(t)$ be a vector-valued function that is defined for all t in some open interval containing the number a, except that $\mathbf{r}(t)$ need not be defined at a. We will write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

if and only if

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0$$

► **Example 1** Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\lim_{t \to 0} \mathbf{r}(t) = \left(\lim_{t \to 0} t^2\right)\mathbf{i} + \left(\lim_{t \to 0} e^t\right)\mathbf{j} - \left(\lim_{t \to 0} 2\cos \pi t\right)\mathbf{k} = \mathbf{j} - 2\mathbf{k}$$

Alternatively, using the angle bracket notation for vectors,

$$\lim_{t \to 0} \mathbf{r}(t) = \lim_{t \to 0} \langle t^2, e^t, -2\cos \pi t\rangle = \left\langle \lim_{t \to 0} t^2, \lim_{t \to 0} e^t, \lim_{t \to 0}(-2\cos \pi t)\right\rangle = \langle 0, 1, -2\rangle$$ ◄

DERIVATIVES

The derivative of a vector-valued function is defined by a limit similar to that for the derivative of a real-valued function.

12.2.3 DEFINITION If $\mathbf{r}(t)$ is a vector-valued function, we define the ***derivative of*** $\mathbf{r}$ ***with respect to*** t to be the vector-valued function $\mathbf{r}'$ given by

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{4}$$

The domain of $\mathbf{r}'$ consists of all values of t in the domain of $\mathbf{r}(t)$ for which the limit exists.

Example 2 Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(e^t)\mathbf{j} - \frac{d}{dt}(2\cos \pi t)\mathbf{k} \\ &= 2t\mathbf{i} + e^t\mathbf{j} + (2\pi \sin \pi t)\mathbf{k}\end{aligned}$$

12.2.4 GEOMETRIC INTERPRETATION OF THE DERIVATIVE Suppose that C is the graph of a vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space and that $\mathbf{r}'(t)$ exists and is nonzero for a given value of t. If the vector $\mathbf{r}'(t)$ is positioned with its initial point at the terminal point of the radius vector $\mathbf{r}(t)$, then $\mathbf{r}'(t)$ is tangent to C and points in the direction of increasing parameter.

DERIVATIVE RULES

(a) $\dfrac{d}{dt}[\mathbf{c}] = \mathbf{0}$

(b) $\dfrac{d}{dt}[k\mathbf{r}(t)] = k\dfrac{d}{dt}[\mathbf{r}(t)]$

(c) $\dfrac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] + \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(d) $\dfrac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] - \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(e) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\dfrac{d}{dt}[\mathbf{r}(t)] + \dfrac{d}{dt}[f(t)]\mathbf{r}(t)$

TANGENT LINES TO GRAPHS OF VECTOR-VALUED FUNCTIONS

Motivated by the discussion of the geometric interpretation of the derivative of a vector-valued function, we make the following definition.

12.2.7 DEFINITION Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to P (Figure 12.2.4). If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then we call $\mathbf{r}'(t_0)$ a ***tangent vector*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$, and we call the line through P that is parallel to the tangent vector the ***tangent line*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$.

▶ **Example 3** Find parametric equations of the tangent line to the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

where $t = t_0$, and use that result to find parametric equations for the tangent line at the point where $t = \pi$.

Solution. The vector equation of the helix is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$$

so we have

$$\mathbf{r}_0 = \mathbf{r}(t_0) = \cos t_0\mathbf{i} + \sin t_0\mathbf{j} + t_0\mathbf{k}$$

$$\mathbf{v}_0 = \mathbf{r}'(t_0) = (-\sin t_0)\mathbf{i} + \cos t_0\mathbf{j} + \mathbf{k}$$

It follows from (5) that the vector equation of the tangent line at $t = t_0$ is

$$\mathbf{r} = \cos t_0\mathbf{i} + \sin t_0\mathbf{j} + t_0\mathbf{k} + t[(-\sin t_0)\mathbf{i} + \cos t_0\mathbf{j} + \mathbf{k}]$$
$$= (\cos t_0 - t\sin t_0)\mathbf{i} + (\sin t_0 + t\cos t_0)\mathbf{j} + (t_0 + t)\mathbf{k}$$

Thus, the parametric equations of the tangent line at $t = t_0$ are

$$x = \cos t_0 - t\sin t_0, \quad y = \sin t_0 + t\cos t_0, \quad z = t_0 + t$$

In particular, the tangent line at $t = \pi$ has parametric equations

$$x = -1, \quad y = -t, \quad z = \pi + t$$

The graph of the helix and this tangent line are shown in Figure 12.2.5. ◀

Example 4 Let

$$\mathbf{r}_1(t) = (\tan^{-1} t)\mathbf{i} + (\sin t)\mathbf{j} + t^2\mathbf{k}$$

and

$$\mathbf{r}_2(t) = (t^2 - t)\mathbf{i} + (2t - 2)\mathbf{j} + (\ln t)\mathbf{k}$$

The graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

Solution. The graph of $\mathbf{r}_1(t)$ passes through the origin at $t = 0$, where its tangent vector is

$$\mathbf{r}_1'(0) = \left\langle \frac{1}{1+t^2}, \cos t, 2t \right\rangle \bigg|_{t=0} = \langle 1, 1, 0 \rangle$$

The graph of $\mathbf{r}_2(t)$ passes through the origin at $t = 1$ (verify), where its tangent vector is

$$\mathbf{r}_2'(1) = \left\langle 2t - 1, 2, \frac{1}{t} \right\rangle \bigg|_{t=1} = \langle 1, 2, 1 \rangle$$

By Theorem 11.3.3, the angle θ between these two tangent vectors satisfies

$$\cos\theta = \frac{\langle 1, 1, 0 \rangle \cdot \langle 1, 2, 1 \rangle}{\|\langle 1, 1, 0 \rangle\| \, \|\langle 1, 2, 1 \rangle\|} = \frac{1+2+0}{\sqrt{2}\sqrt{6}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

It follows that $\theta = \pi/6$ radians, or $30°$. ◄

DEFINITE INTEGRALS OF VECTOR-VALUED FUNCTIONS

If $\mathbf{r}(t)$ is a vector-valued function that is continuous on the interval $a \le t \le b$, then we define the ***definite integral*** of $\mathbf{r}(t)$ over this interval as a limit of Riemann sums, just as in Definition 5.5.1, except here the integrand is a vector-valued function. Specifically, we define

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \mathbf{r}(t_k^*)\Delta t_k \qquad (10)$$

Example 6 Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\int_0^1 \mathbf{r}(t)\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 e^t\,dt\right)\mathbf{j} - \left(\int_0^1 2\cos \pi t\,dt\right)\mathbf{k}$$

$$= \frac{t^3}{3}\bigg]_0^1 \mathbf{i} + e^t\bigg]_0^1 \mathbf{j} - \frac{2}{\pi}\sin \pi t\bigg]_0^1 \mathbf{k} = \frac{1}{3}\mathbf{i} + (e-1)\mathbf{j} \text{ ◄}$$

RULES OF INTEGRATION

As with differentiation, many of the rules for integrating real-valued functions have analogs for vector-valued functions.

(a) $\displaystyle\int_a^b k\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt$

(b) $\displaystyle\int_a^b [\mathbf{r}_1(t) + \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt + \int_a^b \mathbf{r}_2(t)\,dt$

(c) $\displaystyle\int_a^b [\mathbf{r}_1(t) - \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt - \int_a^b \mathbf{r}_2(t)\,dt$

Example 7

$$\int (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = \left(\int 2t\,dt\right)\mathbf{i} + \left(\int 3t^2\,dt\right)\mathbf{j}$$
$$= (t^2 + C_1)\mathbf{i} + (t^3 + C_2)\mathbf{j}$$
$$= (t^2\mathbf{i} + t^3\mathbf{j}) + (C_1\mathbf{i} + C_2\mathbf{j}) = (t^2\mathbf{i} + t^3\mathbf{j}) + \mathbf{C}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j}$ is an arbitrary vector constant of integration. ◄

Example 8 Evaluate the definite integral $\displaystyle\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt$.

Solution. Integrating the components yields

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = t^2\Big]_0^2\mathbf{i} + t^3\Big]_0^2\mathbf{j} = 4\mathbf{i} + 8\mathbf{j}$$

Example 9 Find $\mathbf{r}(t)$ given that $\mathbf{r}'(t) = \langle 3, 2t\rangle$ and $\mathbf{r}(1) = \langle 2, 5\rangle$.

Solution. Integrating $\mathbf{r}'(t)$ to obtain $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle 3, 2t\rangle\,dt = \langle 3t, t^2\rangle + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration. To find the value of $\mathbf{C}$ we substitute $t = 1$ and use the given value of $\mathbf{r}(1)$ to obtain

$$\mathbf{r}(1) = \langle 3, 1\rangle + \mathbf{C} = \langle 2, 5\rangle$$

so that $\mathbf{C} = \langle -1, 4\rangle$. Thus,

$$\mathbf{r}(t) = \langle 3t, t^2\rangle + \langle -1, 4\rangle = \langle 3t - 1, t^2 + 4\rangle$$ ◄

NORMAL AND TANGENTIAL COMPONENTS OF ACCELERATION

12.6.2 THEOREM *If a particle moves along a smooth curve C in 2-space or 3-space, then at each point on the curve velocity and acceleration vectors can be written as*

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T} \qquad \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \tag{10–11}$$

where s is an arc length parameter for the curve, and **T**, **N**, *and* κ *denote the unit tangent vector, unit normal vector, and curvature at the point (Figure 12.6.4).*

Example 4 Suppose that a particle moves through 3-space so that its position vector at time t is

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

(a) Find the scalar tangential and normal components of acceleration at time t.

(b) Find the scalar tangential and normal components of acceleration at time $t = 1$.

(c) Find the vector tangential and normal components of acceleration at time $t = 1$.

(d) Find the curvature of the path at the point where the particle is located at time $t = 1$.

Solution (a). We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = 2\mathbf{j} + 6t\mathbf{k}$$
$$\|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$
$$\mathbf{v}(t)\cdot\mathbf{a}(t) = 4t + 18t^3$$
$$\mathbf{v}(t)\times\mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}$$

Thus, from (15) and (16)

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$$

$$a_N = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\sqrt{36t^4+36t^2+4}}{\sqrt{1+4t^2+9t^4}} = 2\sqrt{\frac{9t^4+9t^2+1}{9t^4+4t^2+1}}$$

Solution (b). At time $t=1$, the components a_T and a_N in part (a) are

$$a_T = \frac{22}{\sqrt{14}} \approx 5.88 \quad \text{and} \quad a_N = 2\sqrt{\frac{19}{14}} \approx 2.33$$

Solution (c). Since $\mathbf{T}$ and $\mathbf{v}$ have the same direction, $\mathbf{T}$ can be obtained by normalizing $\mathbf{v}$, that is,

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$$

At time $t=1$ we have

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\|\mathbf{i}+2\mathbf{j}+3\mathbf{k}\|} = \frac{1}{\sqrt{14}}(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$$

From this and part (b) we obtain the vector tangential component of acceleration:

$$a_T(1)\mathbf{T}(1) = \frac{22}{\sqrt{14}}\mathbf{T}(1) = \frac{11}{7}(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \frac{11}{7}\mathbf{i}+\frac{22}{7}\mathbf{j}+\frac{33}{7}\mathbf{k}$$

To find the normal vector component of acceleration, we rewrite $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ as

$$a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T}$$

Thus, at time $t=1$ the normal vector component of acceleration is

$$\begin{aligned} a_N(1)\mathbf{N}(1) &= \mathbf{a}(1) - a_T(1)\mathbf{T}(1) \\ &= (2\mathbf{j}+6\mathbf{k}) - \left(\frac{11}{7}\mathbf{i}+\frac{22}{7}\mathbf{j}+\frac{33}{7}\mathbf{k}\right) \\ &= -\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k} \end{aligned}$$

Solution (d). We will apply Formula (17) with $t = 1$. From part (a)

$$\|\mathbf{v}(1)\| = \sqrt{14} \quad \text{and} \quad \mathbf{v}(1) \times \mathbf{a}(1) = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

Thus, at time $t = 1$

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{1}{14}\sqrt{\frac{38}{7}} \approx 0.17$$

In the case where $\|\mathbf{a}\|$ and a_T are known, there is a useful alternative to Formula (16) for a_N that does not require the calculation of a cross product. It follows algebraically from Formula (14) (see Exercise 51) or geometrically from Figure 12.6.6 and the Theorem of Pythagoras that

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} \tag{18}$$

KEPLER'S LAWS OF PLANETARY MOTION

- First law (***Law of Orbits***). Each planet moves in an elliptical orbit with the Sun at a focus.
- Second law (***Law of Areas***). Equal areas are swept out in equal times by the line from the Sun to a planet.
- Third law (***Law of Periods***). The square of a planet's period (the time it takes the planet to complete one orbit about the Sun) is proportional to the cube of the semimajor axis of its orbit.

KEPLER'S FIRST AND SECOND LAWS

It follows from our general discussion of central force fields that the planets have elliptical orbits with the Sun at the focus, which is Kepler's first law. To derive Kepler's second law, we begin by equating (10) and (13) to obtain

$$r^2\frac{d\theta}{dt} = r_0 v_0 \tag{24}$$

To prove that the radial line from the center of the Sun to the center of a planet sweeps out equal areas in equal times, let $r = f(\theta)$ denote the polar equation of the planet, and let A denote the area swept out by the radial line as it varies from any fixed angle θ_0 to an angle θ. It follows from the area formula in 10.3.4 that A can be expressed as

$$A = \int_{\theta_0}^{\theta} \frac{1}{2}[f(\phi)]^2\, d\phi$$

where the dummy variable ϕ is introduced for the integration to reserve θ for the upper limit. It now follows from Part 2 of the Fundamental Theorem of Calculus and the chain rule that

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}[f(\theta)]^2\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

Thus, it follows from (24) that

$$\frac{dA}{dt} = \frac{1}{2}r_0 v_0 \tag{25}$$

which shows that A changes at a constant rate. This implies that equal areas are swept out in equal times.

KEPLER'S THIRD LAW

To derive Kepler's third law, we let a and b be the semimajor and semiminor axes of the elliptical orbit, and we recall that the area of this ellipse is πab. It follows by integrating (25) that in t units of time the radial line will sweep out an area of $A = \frac{1}{2}r_0 v_0 t$. Thus, if T denotes the time required for the planet to make one revolution around the Sun (the period), then the radial line will sweep out the area of the entire ellipse during that time and hence

$$\pi ab = \frac{1}{2}r_0 v_0 T$$

from which we obtain

$$T^2 = \frac{4\pi^2 a^2 b^2}{r_0^2 v_0^2} \tag{26}$$

However, it follows from Formula (1) of Section 10.6 and the relationship $c^2 = a^2 - b^2$ for an ellipse that

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

Thus, $b^2 = a^2(1 - e^2)$ and hence (26) can be written as

$$T^2 = \frac{4\pi^2 a^4(1-e^2)}{r_0^2 v_0^2} \tag{27}$$

But comparing Equation (20) to Equation (17) of Section 10.6 shows that

$$k = a(1 - e^2)$$

Finally, substituting this expression and (21) in (27) yields

$$T^2 = \frac{4\pi^2 a^3}{r_0^2 v_0^2} k = \frac{4\pi^2 a^3}{r_0^2 v_0^2} \frac{r_0^2 v_0^2}{GM} = \frac{4\pi^2}{GM} a^3 \tag{28}$$

Thus, we have proved that T^2 is proportional to a^3, which is Kepler's third law. When convenient, Formula (28) can also be expressed as

$$T = \frac{2\pi}{\sqrt{GM}} a^{3/2} \tag{29}$$

► **Example 1** A geosynchronous orbit for a satellite is a circular orbit about the equator of the Earth in which the satellite stays fixed over a point on the equator. Use the fact that the Earth makes one revolution about its axis every 24 hours to find the altitude in miles of a communications satellite in geosynchronous orbit. Assume the Earth to be a sphere of radius 4000 mi.

Solution. To remain fixed over a point on the equator, the satellite must have a period of $T = 24$ h. It follows from (28) or (29) and the Earth value of $GM = 1.24 \times 10^{12}$ mi^3/h^2 from Table 12.7.1 that

$$a = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \sqrt[3]{\frac{(1.24 \times 10^{12})(24)^2}{4\pi^2}} \approx 26{,}250 \text{ mi}$$

and hence the altitude h of the satellite is

$$h \approx 26{,}250 - 4000 = 22{,}250 \text{ mi} \blacktriangleleft$$

POSSIBLE QUESTIONS

TWO MARKS

1. Write down the polar formulas for velocity and acceleration.

2. Write down the polar formulas for velocity and acceleration.

3. Find $\lim_{t\to 2} F(t)$, where $F(t) = (t^2 - 3)i + e^t j + (sin\pi t)k$.

4. For what values of t is $G(t) = |t|i + (cost)j + (t-5)k$ differentiable.

5. Write down the tangential and normal components of acceleration.

SIX MARKS

1.Find the second and third derivative of the vector function

$$i)\ F(t) = e^t i + (sint)j + (t^3 + 5t)k.$$
$$ii)\ F(t) = e^{2t} i + (1 - t^2)j + (cos2t)k.$$

2.A boy standing at the edge of a cliff throws a ball upwards at a $30°$ angle with an initial speed of 64 ft/s. suppose that when the ball leaves the boy's hand, it is 48 ft above the ground as the base of the cliff.

i) what are the time of flight of the ball and its range?
ii) what are the velocity of the ball and its speed at impact?

3.Let $F(t) = i + tj + t^2 k\ \ anf\ G(t) = ti + e^t j + 3k.$

Verify that $(FXG)'(t) = (F'XG)(t) + (FXG')(t)$

4.If the velocity of a particle moving in space is $V(t) = e^t i + t^2 j + (cos2t)k$. Find the particle's position as a function of t if the position at time t=0 is $R(0) = 2i + j - k$.

5.Find the Volume of the Parallelepiped determined by the vectors $u = i - 2j + 3k$,

$v = -4i + 7j - 11k$,$w = 5i + 9j - k$

6.If the position vector of a moving body is $R(t) = 2ti - t^2 j\ \ for\ t \geq 0$. Express R and the velocity vector V(t) in terms of $u_r\ \ and\ u_\theta$.

7.Show that $\lim_{t\to 1}[F(t)XG(t)] = \left(\lim_{t\to 1} F(t)\right) X(\lim_{t\to 1} G(t))$ for the vector

functions $F(t) = ti + (1-t)j + t^2k \text{ and } G(t) = e^t i - (3+e^t)k$

8.Find the tangential and normal components of the acceleration of an object the moves

with position vector $R(t) = \langle t^3, t^2, t\rangle$.

9.Let $F(t) = t^2\bar{i} + t\bar{j} - (sint)\bar{k} \ \ and \ \ G(t) = t\bar{i} + \frac{1}{t}\bar{j} + 5\bar{k}.$ find

i) (F+G)(t) ii) (F X G)(t) iii) (F.G)(t)

10.State and prove Kepler's second law of motion.