KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester – I

19MMU102 **ALGEBRA** **7H – 6C**

Instruction Hours / week: L: 6 T: 1 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- The functions, relations, systems of linear equations and roots of equations.
- How to identify, evaluate and simplify algebraic expressions using the correct operations.
- The basic concepts of linear algebra.

Course Outcomes (COs)

On successful completion of this course, the students will be able to

1. Know about the basic concepts of set theory.
2. Describe the categories of functions.
3. Understand the algorithms on operation.
4. Use matrix operations to solve system of linear equations.
5. Learn how to find roots of equations.

UNIT I

BASICS OF SETS &FUNCTIONS

Sets –Finite and infinite sets-Equality sets-Subsets-Comparability -Proper subsets-Axiomatic development of set theory-Set operations. Equivalence relations- Functions- Composition of functions- Invertible functions- One to one Correspondence and cardinality of a set.

UNIT II

DIVISIBILITY AND CONGRUENCE RELATIONS

Division algorithm- Divisibility and Euclidean algorithm- Congruence relation between integers-Principles of Mathematical Induction- Statement of Fundamental Theorem of Arithmetic.

UNIT III

SYSTEM OF LINEAR EQUATIONS

Systems of linear equations - Row reduction and echelon forms - Vector equations - The matrix equation Ax=b - Solution sets of linear systems - Applications of linear systems – Linear independence.

UNIT IV

THEORY OF EQUATIONS

Roots of an equation- Relations connecting the roots and coefficients- Transformations of equations - Character and position of roots-Descarte's rule of signs-Symmetric function of roots-Reciprocal equations.

UNIT V
THEORY OF EQUATIONS (CONTINUITY)
Multiple roots-Rolle's theorem - Position of real roots of f(x) =0 – Newton's method of approximation to a root – Horner's method.

SUGGESTED READINGS

1. Edgar G. Goodaire and Michael M. Parmenter.,(2015).Discrete Mathematics with Graph Theory, 3rd Edition, Pearson Education (Singapore) P. Ltd., Indian Reprint.
2. David C. Lay., (2008). Linear Algebra and its Applications, Third Edition, Pearson Education Asia, Indian Reprint.
3. Kenneth Hoffman., Ray Kunze., (2015).Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.
4. T.K.Manicavasagom Pillai, T.Natarajan, K.S.Ganapathy., (2006), Algebra, S.Viswanatham (Printer & publishers) Private Ltd.

KARPAGAM ACADEMY OF HIGHER EDUCATION

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: V. Kuppusamy
Subject Name: Algebra **Sub.Code:19MMU102**
Semester: I **Class: I B.Sc Mathematics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material / Page Nos
		Unit – I	
1.	1	Basics of sets, finite and infinite sets	S1:Ch:2 Pg.No:37-40
2.	1	Examples of equality sets , subsets, comparability, proper subsets	S1:Ch:2 Pg.No:40-43
3.	1	Examples and theorems on set operations	S1:Ch:2 Pg.No:43-47
4.	1	Continuation of examples and theorems on set operations	S1:Ch:2 Pg.No:47-51
5.	1	Tutorial – 1	
6.	1	Theorems and examples on equivalence relation	S1:Ch:2 Pg.No:56-60
7.	1	Continuation of theorems and examples on Equivalence relation	S1:Ch:2 Pg.No:60-63
8.	1	Functions – Domain, range, one to one, onto	S1:Ch:3 Pg.No:71-73
9.	1	Continuation of functions	S1:Ch:3 Pg.No:73-77
10.	1	Tutorial – 2	
11.	1	Theorems on functions	S1:Ch:3 Pg.No:77-79
12.	1	Theorems on composition functions	S1:Ch:3 Pg.No:79-84
13.	1	Theorems on invertible functions	S1:Ch:3 Pg.No:84-87
14.	1	One-one correspondence & the cardinality of set	S1:Ch:3 Pg.No:87-95
15.	1	Tutorial-3	
16.	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned – 16 hours			
		Unit – II	
1.	1	Division algorithm	S1: Ch: 4; Pg. No :97-100
2.	1	Examples on division algorithm	S1: Ch: 4; Pg. No :100-104
3.	1	Theorems and examples on divisibility	S1: Ch: 4; Pg. No:104-107
4.	1	Continuation of theorems and examples on divisibility	S1: Ch: 4; Pg. No:107-110
5.	1	Tutorial-1	
6.	1	Euclidean algorithm	S1: Ch: 4; Pg. No:110-114
7.	1	Theorems and examples on prime numbers	S1: Ch: 4; Pg. No:114-120
8.	1	Continuation of theorems and examples on prime	S1: Ch: 4; Pg. No:120-126

		numbers	
9.	1	Congruence relation between integers	S1: Ch: 4; Pg. No:126-131
10.	1	Tutorial-2	
11.	1	Examples on congruence relation	S1: Ch: 4; Pg. No:131-136
12.	1	Application on congruence relation	S1: Ch: 4; Pg. No:136-147
13.	1	Principle of mathematical induction	S1: Ch: 4; Pg. No:147-149
14.	1	Fundamental theorem of arithmetic	S1: Ch: 5; Pg. No :152-154
15.	1	Tutorial-3	
16.	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned – 16 hours			
Unit – III			
1.	1	Examples on systems of linear equations	S2:Ch:1 Pg.No:2-7
2.	1	Continuation of examples on systems of linear equations	S2:Ch:1 Pg.No:7-12
3.	1	Examples on row reduction and echelon form	S3:Ch:1 Pg.No:11-13
4.	1	Continuation of examples on row reduction and echelon form	S3:Ch:1 Pg.No:13-16
5.	1	Examples on vector equations	S2:Ch:1 Pg.No:24-29
6.	1	Continuation of examples on vector equations	S2:Ch:1 Pg.No:29-34
7.	1	Tutorial-1	
8.	1	Examples on Matrix equation Ax=b	S2:Ch:1 Pg.No:34-39
9.	1	Continuation of Examples on Matrix equation Ax=b	S2:Ch:1 Pg.No:39-43
10.	1	Examples on solution sets of linear system	S2:Ch:1 Pg.No:43-46
11.	1	Continuation of Examples on solution sets of linear system	S2:Ch:1 Pg.No:46-49
12.	1	Applications of Linear system	S2:Ch:1 Pg.No:49-55
13.	1	Tutorial-2	
14.	1	Examples on linear independence	S2:Ch:1 Pg.No:55-59
15.	1	Continuation of Examples on linear independence	S2:Ch:1 Pg.No:59-62
16.	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned – 16 hours			
Unit – IV			
1.	1	Roots of an equation	S4:Ch:6 Pg.No:282-292
2.	1	Relations between the roots and coefficients	S4:Ch:6 Pg.No:292-303
3.	1	Continuation of relations between the roots and coefficients	S4:Ch:6 Pg.No:292-303
4.	1	Symmetric function of roots	S4:Ch:6 Pg.No:303-306
5.	1	Continuation of symmetric function of roots	S4:Ch:6 Pg.No:303-306
6.	1	Transformations of equations	S4:Ch:6 Pg.No:318-321
7.	1	Continuation of transformations of equations	S4:Ch:6 Pg.No:318-321
8.	1	Tutorial-1	
9.	1	Character and position of roots	
10.	1	Continuation of character and position of roots	
11.	1	Reciprocal equations	S4:Ch:6 Pg.No:321-327
12.	1	Continuation of reciprocal equations	S4:Ch:6 Pg.No:321-327

13.	1	Descarte's rule of signs	S4:Ch:6 Pg.No:351-354
14.	1	Continuation of descarte's rule of signs	S4:Ch:6 Pg.No:351-354
15.	1	Tutorial-2	
16.	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned – 16 hours			
Unit – V			
1.	1	Multiple roots-Rolle's theorem	S4:Ch:6 Pg.No:355-363
2.	1	Continuation of Multiple roots-Rolle's theorem	S4:Ch:6 Pg.No:355-363
3.	1	Continuation of Multiple roots-Rolle's theorem	S4:Ch:6 Pg.No:355-363
4.	1	Position of real roots of f(x) =0	S4:Ch:6 Pg.No:363-367
5.	1	Continuation of Position of real roots of f(x) =0	S4:Ch:6 Pg.No:363-367
6.	1	Continuation of Position of real roots of f(x) =0	S4:Ch:6 Pg.No:363-367
7.	1	Tutorial-1	
8.	1	Newton's method of approximation to a root	S4:Ch:6 Pg.No:370-376
9.	1	Continuation of Newton's method of approximation to a root	S4:Ch:6 Pg.No:370-376
10.	1	Continuation of Newton's method of approximation to a root	S4:Ch:6 Pg.No:370-376
11.	1	Continuation of Newton's method of approximation to a root	S4:Ch:6 Pg.No:370-376
12.	1	Horner's method	S4:Ch:6 Pg.No:376-381
13.	1	Continuation of Horner's method	S4:Ch:6 Pg.No:376-381
14.	1	Continuation of Horner's method	S4:Ch:6 Pg.No:376-381
15.	1	Continuation of Horner's method	S4:Ch:6 Pg.No:376-381
16.	1	Tutorial-2	
17.	1	Recapitulation and discussion of possible questions	
18.	1	Discussion of previous year ESE Question papers	
19.	1	Discussion of previous year ESE Question papers	
20.	1	Discussion of previous year ESE Question papers	
Total No. of Lecture hours planned – 20 hours			
Total Planned Hours			**84**

SUGGESTED READINGS

1. Edgar G. Goodaire and Michael M. Parmenter.,(2015).Discrete Mathematics with Graph Theory, 3rd Edition, Pearson Education (Singapore) P. Ltd., Indian Reprint.
2. David C. Lay., (2008). Linear Algebra and its Applications, Third Edition, Pearson Education Asia, Indian Reprint.
3. Kenneth Hoffman., Ray Kunze., (2015).Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.
4. T.K.Manicavasagom Pillai, T.Natarajan, K.S.Ganapathy., (2006), Algebra, S.Viswanatham (Printer & publishers) Private Ltd.

UNIT-I

Sets –Finite and infinite sets-Equality sets-Subsets-Comparability -Proper subsets-Axiomatic development of set theory-Set operations. Equivalence relations- Functions- Composition of functions- Invertible functions- One to one Correspondence and cardinality of a set.

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Definition 1: A ***set*** is a collection of objects together with some rule to determine whether a given object belongs to this collection. Any object of this collection is called an ***element*** of the set.

1. Each element of the set is listed within a set of brackets: { }.
2. Within the brackets, the first few elements are listed, with dots following to show that the set continues with the selection of the elements following the same rule as the first few.
3. Within the brackets, the set is described by writing out the exact rule by which elements are chosen. The name given each element is separated from the selection rule with a vertical line.

Examples:

(a) Denote by A the set of natural numbers with are greater than 25. The set could be written in the following ways:

{26,27,28….} (using the second notation listed above)

$\{x \mid x$ is a natural number and $x > 25\}$ (using the third notation above)

The above description is read as "the set of all x such that x is a natural number and $x > 25$".

Note that 32 is an element of A. We write $\mathbf{32} \in \boldsymbol{A}$, where "$\in$" denotes "*is an element of.*" Also, $\mathbf{6} \notin \boldsymbol{A}$, where "$\notin$" denotes "*is not an element of.*"

(b) Let B be the set of numbers {3,5,15,19,31,32}. Again the elements of the set are natural numbers. However, the rule is given by actually listing each element of the set (as in the first notation above). We see that $15 \in B$, but $23 \notin B$.

(c) Let C be the set of all natural numbers which are less than 1. In this set, we observe that there are no elements. Hence, C is said to be an ***empty set***. A set with no elements is denoted by $\emptyset$.

Definition: A set A is said to be a ***subset*** of a set B if *every* element of A is an element of B.

Notation: To indicate that set A is a subset of set B, we use the expression **A ⊂ B,** where "⊂" denotes "*is a subset of*". **A ⊄ B** means that A is *not* a subset of B.

Examples:

(a) Let B be the set of natural numbers. Let A be the set of even natural numbers. Clearly, A is a subset of B. However, B is not a subset of A, for $3 \in B$, but $3 \notin A$.

(b) An empty set ∅ is a subset of *any* set B. If this were not so, there would be some element $x \in \emptyset$ such that $x \notin B$. However, this would contradict with the definition of an empty set as a set with no elements.

Theorem: Properties Of Sets

Let A, B, and C be sets.

1. For any set A, $A \subset A$ (Reflexive Property)
2. If $A \subset B$ and $B \subset C$, then $A \subset C$ (Transitive Property)

Definition: Two sets, A and B, are said to be **equal** if and only if A is a subset of B and B is a subset of A. To indicate that two sets, A and B, are equal, we use the symbol **A = B**.

This means that sets A and B contain *exactly the same elements*. **A ≠ B** means that A and B are not equal sets.

Example:

Let A be the set of even natural numbers and B be the set of natural numbers which are multiples of 2. Clearly, $A \subset B$ and $B \subset A$. Therefore, since A and B contain exactly the same elements, $A = B$.

Remarks:

(a) Two equal sets always contain the same elements. However, the rules for the sets may be written differently, as in the above example.

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(b) Since any two empty sets are equal, we will refer to any empty set as ***the*** empty set.

(c) A is said to be a ***proper subset*** of B is and only if:
(i) $A \subset B$
(ii) $A \neq B$, and
(iii) $A \neq \emptyset$.

Theorem: Properties of Set Equality

(a) For any set A, $A = A$. (Reflexive Property)

(b) If $A = B$, then $B = A$. (Symmetric Property)

(c) If $A = B$ and $B = C$, then $A = C$. (Transitive Property)

Definition: Let A and B be subsets of a set X. The ***intersection*** of A and B is the set of all elements in X common to both A and B.

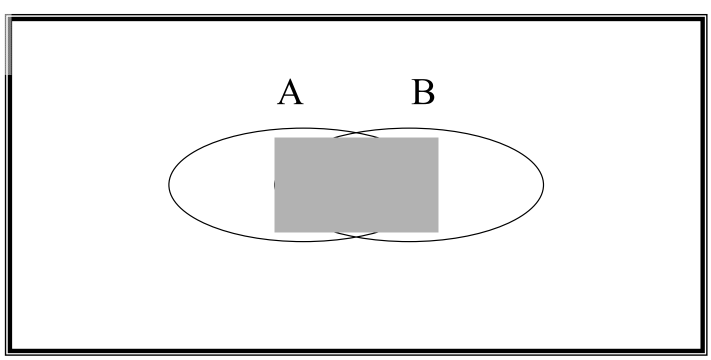
Notation: "$\boldsymbol{A \cap B}$" denotes "A intersection B" or the intersection of sets A and B.

Thus, $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$, or $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

Examples:

a. Given that the box below represents X, the shaded area represents $A \cap B$:

X:



b. Let $A = \{2,4,5\}$ and $B = \{1,4,6,8\}$ Then, $A \cap B = \{4\}$.

Note: A set that has only one element, such as $\{4\}$, is sometimes called a singleton set.

c. Let $A = \{2,4,5\}$ and $B = \{1,3\}$. Then $A \cap B = \emptyset$.

Remarks:

a. If, as in the above example 1.11c, A and B are two sets such that $A \cap B$ is the empty set, we say that A and B are ***disjoint.***

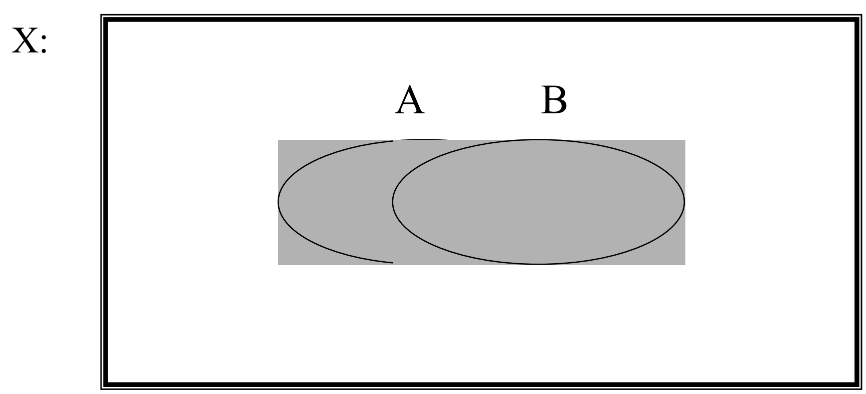
b. Given sets A and B. $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Definition: Let A and B be subsets of a set X. The ***union*** of A and B is the set of all elements belonging to A or B.

Notation: "$\boldsymbol{A \cup B}$" denotes "A union B" or the union of sets A and B.
Thus, $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$. Or $A \cup B = \{x \mid x \in A \vee x \in B\}$.

Examples:

a. Given that the box below represents X, the shaded area represents $A \cup B$:



b. Let A = {2,4,5} and B = {1,4,6,8}.
Then, A ∪ B = {1,2,4,5,6,8}

Remark:

Given sets A and B. $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

Definition: Let A and B be subsets of a set X. The set $B - A$, called the ***difference*** of B and A, is the set of all elements in B which are not in A.

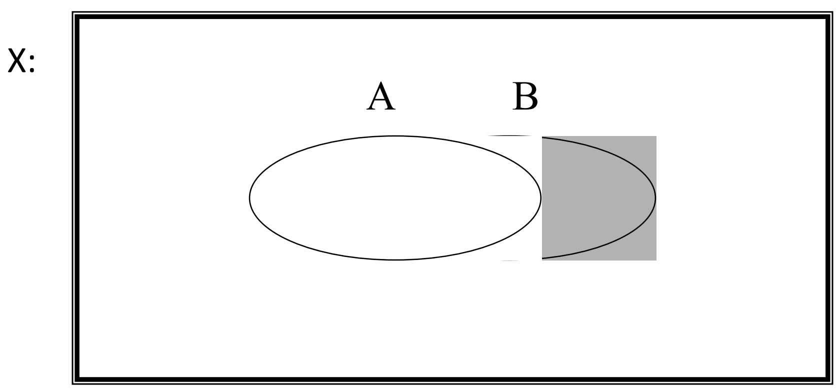
Thus, $B - A = \{x \in X \mid x \in B \text{ and } x \notin A\}$.

Examples:

a. Let $B = \{2,3,6,10,13,15\}$ and $A = \{2,10,15,21,22\}$.
Then $B - A = \{3,6,13\}$.

b. Let X be the set of natural numbers and A be the set of odd natural numbers. Then,
$X - A$ = the set of even natural numbers; or $X - A = \{x \mid x \text{ is a natural number and } x \text{ is even}\}$.

c. Given that the box below represents X, the shaded area represents $B - A$.



Definition: If $A \subset X$, then $X - A$ is sometimes called the ***complement*** of A with respect to X.

Notation: The following symbols are used to denote the complement of A with respect to X:

$$\complement_x A,\ \complement A,\ \sim A,\ \tilde{A}, \text{ and } A'$$

Thus, $\complement_x A = \{x \in X \mid x \notin A\}$.

Theorem: Let A and B be subsets of a set X.

Then, $A - B = A \cap \complement B$.

SUB- SET

Let set A be a set containing all students of your school and B be a set containing all students of class XII of the school. In this example each element of set B is also an element of set A. Such a set B is said to be subset of the set A. It is written as B Í A

Consider D ={1, 2, 3, 4,........}

E = {..... -3 -2, -1, 0, 1, 2, 3, ...}

Clearly each element of set D is an element of set E also \ D Í E

If A and B are any two sets such that each element of the set A is an element of the set B also, then A is said to be a subset of B.

Remarks

(i) Each set is a subset of itself i.e. A Í A .

(ii) Null set has no element so the condition of becoming a subset is automatically satisfied. Therefore null set is a subset of every set.

(iii) If A Í B and B Í A then A = B.

(iv) If A Í B and A ¹ B then A is said to be a proper subset of B and B is said to be a super set of A. i.e. AÌB or B É A .

Example If A = {x : x is a prime number less than 5} and

B = {y : y is an even prime number} then is B a proper subset of A ?

Solution : It is given that

A = {2, 3 }, B = {2}.

Clearly B Í A and B ¹ A

We write BÌA

and say that B is a proper subset of A.

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Example If A = {1, 2, 3, 4}, B = {2, 3, 4, 5}.

is A Í B o r B Í A ?

Solution : Here1ÎA b u t1ÏBÞAÍ/B.

Also 5Î B but 5 Ï A Þ B Í/ A .

Hence neither A is a subset of B nor B is a subset of A.

POWER SET

Let A = {a, b}

Subset of A are ϕ , {a}, {b} and {a, b}.

If we consider these subsets as elements of a new set B (say) then

B = {ϕ,{a},{b},{a,b}}

B is said to be the power set of A.

Notation : Power set of a set A is denoted by P(A).

Power set of a set A is the set of all subsets of the given set.

Example Write the power set of each of the following sets :

(i) A = {x : xÎ R and $x^2 + 7 = 0$ }.

(ii) B = {y : yÎ N and1 £ y £ 3}.

Solution :

(i) Clearly A = f (Null set)

\ f is the only subset of given set \ P (A)={f }

(ii) The set B can be written as {1, 2, 3}

\P (B) = { f , {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} } .

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UNIVERSAL SET

Consider the following sets.

A = {x : x is a student of your school}

B = {y : y is a male student of your school}

C = {z : z is a female student of your school}

D = {a : a is a student of class XII in your school}

Clearly the set B, C, D are all subsets of A.

CARTESIAN PRODUCT OF TWO SETS

Consider two sets A and B where

A={1, 2}, B= {3, 4, 5}.

Set of all ordered pairs of elements of A and B

is {(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)}

This set is denoted by A × B and is called the cartesian product of sets A and B.

i.e. A×B ={(1, 3), (1, 4),(1, 5),(2, 3),(2, 4),(2, 5)}

Cartesian product of B sets and A is denoted by B×A.

In the present example, it is given by

B×A = {(3, 1),(3, 2),(4, 1),(4, 2),(5, 1),(5, 2)}

Clearly A×B ¹ B×A.

In the set builder form :

A×B = {(a,b) : a Î A and b Î B }

B×A = {(b,a) : b Î B and a Î A }

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Note : IfA= for B= for A , B= f

then A ´ B = B ´A = f .

Example

(1) Let A={a,b,c}, B={d,e}, C={a,d}.

Find (i) A×B(ii) B×A (iii) A×(B È C) (iv) (A Ç C) ´ B

(v) (A Ç B) ´C (vi) A ´ (B - C) .

Solution : (i) A×B ={(a, d),(a, e), (b, d), (b, e), (c, d), (c, e)}.

(ii) B×A = {(d, a),(d, b), (d, c), (e, a) (e, b),(e, c)}.

(iii) A = {a, b, c}, B È C ={a,d,e}.
× (B È C) ={(a, a),(a, d),(a, e),(b, a),(b, d),(b, e), (c, a),(c, d),(c, e).

(iv) A ÇC = {a}, B={d, e}.

\(A ÇC)×B={(a, d), (a, e)}
(v) $_{AÇB}$ = f , c={a,d}, \ A Ç B ´ c = f

(vi) A = {a,b,c}, B - C = {e}.\A ´ (B - C) = {(a,e),(b,e),(c,e)}

Relations and Functions

RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim}

B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

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The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly R Í A ´ B, i.e.R = {(a,b):a Î A,b Î B and aRb}

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) R = f , R is called a void relation.

(ii) R=A×B, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) R = { (a,a) " a Î A } , is called the identity relation.

Domain and Range of a Relation

If R is a relation between two sets then the set of its first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation.

Consider previous example given above.

Domain = {Mohan, Sohan, David, Karim}

Range = {Rita, Marry, Fatima}

Example 1 Given that A = {2, 4, 5, 6, 7}, B = {2, 3}.

R is a relation from A to B defined by
R = {(a, b) : a Î A, b Î B and a is divisible by b}

find (i) R in the roster form
(ii) Domain of R
(iii) Range of R
(iv) Repersent R diagramatically.

Solution : (i) R = {(2, 2), (4, 2), (6, 2), (6, 3)}

(ii) Domain of R = {2, 4, 6}

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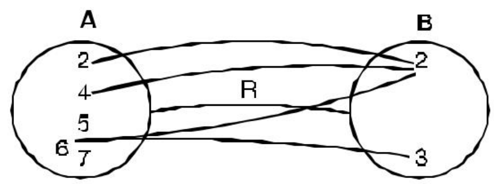
(iii) Range of R = {2, 3}

(iv)



Example 2 If R is a relation 'is greater than' from A to B, where A= {1, 2, 3, 4, 5} and B = {1,2,6}.
Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) R = {(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(ii) Domain of R = {3, 4, 5}

(iii) Range of R = {1, 2}

2.1 Overview

This chapter deals with linking pair of elements from two sets and then introduce relations between the two elements in the pair. Practically in every day of our lives, we pair the members of two sets of numbers. For example, each hour of the day is paired with the local temperature reading by T.V. Station's weatherman, a teacher often pairs each set of score with the number of students receiving that score to see more clearly how well the class has understood the lesson. Finally, we shall learn about special relations called functions.

2.1.1 *Cartesian products of sets*

Definition : Given two non-empty sets A and B, the set of all ordered pairs (x,y),where $x\in$ A and $y\in$ B is called Cartesian product of A and B; symbolically, we write

$$A \times B = \{(x, y) \mid x\in A \text{ and } y\in B\}$$

If A = {1, 2, 3} and B = {4, 5}, then

A × B = {(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)}

And B × A = {(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)}

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal, i.e. $(x, y) = (u, v)$ if and only if $x = u$, $y = v$.

(ii) If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = p \times q$.

(i) $A \times A \times A = \{(a,b,c) : a,b,c \in A\}$. Here (a,b,c) is called an ordered triplet.

2.1.2 ***Relations*** A Relation R from a non-empty set A to a non empty set B is asubset of the Cartesian product set A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B.

The set of all first elements in a relation R, is called the domain of the relation R, and the set of all second elements called images, is called the range of R.

For example, the set $R = \{(1, 2), (-2, 3), (\frac{1}{2}, 3)\}$ is a relation; the domain of $R = \{1, -2, \frac{1}{2}\}$ and the range of $R = \{2, 3\}$.

(i) A relation may be represented either by the Roster form or by the set builder form, or by an arrow diagram which is a visual representation of a relation.
(ii) If $n(A) = p$, $n(B) = q$; then the $n(A \times B) = pq$ and the total number of possible relations from the set A to set B = 2^{pq}.

2.1.3 ***Functions*** A relation *f* from a set A to a set B is said to be **function** if everyelement of set A has one and only one image in set B.

In other words, a function *f* is a relation such that no two pairs in the relation has the same first element.

The notation $f : X \to Y$ means that *f* is a function from X to Y. X is called the **domain** of *f* and Y is called the **co-domain** of *f*. Given an element $x \in X$, there is a unique element

y in Y that is related to *x*. The unique element *y* to which *f* relates *x* is denoted by $f(x)$ and is called *f* of *x*, or the **value of *f* at *x***, or the *image of x under f*.

The set of all values of $f(x)$ taken together is called the **range of *f*** or image of X under *f*. Symbolically.

$$\text{range of } f = \{ y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}$$

Definition : A function which has either **R** or one of its subsets as its range, is calleda real valued function. Further, if its domain is also either **R** or a subset of **R**, it is called a real function.

2.1.4 ***Some specific types of functions***

(i) **Identity function:**

The function f : **R**→**R** defined by $y = f(x) = x$ for each $x \in$ **R** is called the **identity function.** Domain of f = **R**

Range of f = **R**

(ii) **Constant function:** The function f: **R** → **R** defined by $y=f(x) = C, x\in$ **R**, where C is a constant ∈**R**, is a **constant function**.

Domain of f = **R**

Range of f = {C}

(iii) **Polynomial function:** A real valued function f: **R** → **R** defined by $y=f(x) = a_0 + a_1x + ...+ a_nx^n$, where $n \in$ **N**, and $a_0, a_1, a_2...a_n \in$ **R**, for each $x \in$ **R**, is called Polynomial functions.

(iv) **Rational function:** These are the real functions of the type $\frac{f(x)}{g(x)}$, where f(x) and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$. For example f : **R** – {– 2} →**R** defined by $f(x) = \frac{x+1}{x+2}$, $x \in$ **R**– {– 2 } is a rational function.

(v) **The Modulus function:** The real function f: **R** → **R** defined by $f(x) = |x| =$

$x, x \geq 0$ $-x, x < 0$

$x \in$ **R** is called the modulus function.

Domain of f = **R**

Range of f = $\mathbf{R}^+ \cup \{0\}$

(vi) **Signum function:** The real function

f :**R**→**R**defined by

$$f(x) \quad \frac{|x|}{x}, x \quad 0 \qquad 1, \text{ if } \quad x \quad 0$$

$$0, \text{ if } \quad x \quad 0$$

$$0, x \quad 0 \qquad 1, \text{ if } x \quad 0$$

is called the **signum function**. Domain of f = **R**, Range of f = {1, 0, – 1}

(vii) **Greatest integer function:** The real function$f$: **R** → **R** defined by$f(x) = [x]$, $x \in$**R**assumes the value of the greatest integer less than or equal to x, iscalled the **greatest integer function**.

Thus$f(x) = [x] = -1$ for $-1 \leq x < 0$ $f(x) = [x] = 0$ for $0 \leq x < 1$

$[x] = 1$ for $1 \leq x < 2$

$[x] = 2$ for $2 \leq x < 3$ and so on

2.1.5 ***Algebra of real functions***

(i) ***Addition of two real functions***

Let f : X →**R** and g : X →**R** be any two real functions, where X ∈**R**.

Then we define $(f + g)$: X →**R** by $(f + g)(x) = f(x) + g(x)$, for all $x \in$ X.

(ii) ***Subtraction of a real function from another***

Let f : X →**R** and g : X →**R** be any two real functions, where X ⊆**R**.

Then, we define $(f - g)$: X →**R** by $(f - g)(x) = f(x) - g(x)$, for all $x \in$ X.

(iii) ***Multiplication by a Scalar***

Let f : X →**R** be a real function and α be any scalar belonging to **R**. Then the product αf is function from X to **R** defined by $(\alpha f)(x) = \alpha f(x)$, $x \in$ X.

(iv) ***Multiplication of two real functions***

Let $f : X \to \mathbf{R}$ and $g : x \to \mathbf{R}$ be any two real functions, where $X \subseteq \mathbf{R}$. Then product of these two functions i.e. $f g : X \to \mathbf{R}$ is defined by $(f g)(x) = f(x)\, g(x)\ x \in X$.

(v) ***Quotient of two real function***

Let f and g be two real functions defined from $X \to \mathbf{R}$. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined from $X \to \mathbf{R}$ as $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in X$.

 Note Domain of sum function $f+g$, difference function $f-g$ and product function fg.
$= \{x : x \in D_f \cap D_g\}$

where Df = Domain of function f

Dg = Domain of function g

$F = \{x : x \in D_f \cap D_g$ and $g(x) \neq 0\}$

2.2 Solved Examples

Short Answer Type

Example 1 Let A = {1, 2, 3, 4} and B = {5, 7, 9}. Determine

(i) A × B (ii) B × A

(iii) Is A × B = B × A ? (iv) Is n (A × B) = n (B × A) ?

(i) A × B = {(1, 5), (1, 7), (1, 9), (2, 5), (2, 7), (2, 9), (3, 5), (3, 7), (3, 9), (4, 5), (4, 7), (4, 9)}

(ii) B × A = {(5, 1), (5, 2), (5, 3), (5, 4), (7, 1), (7, 2), (7, 3), (7, 4), (9, 1), (9, 2), (9, 3), (9, 4)}

(iii) No, A × B ≠ B × A. Since A × B and B × A do not have exactly the same ordered pairs.

(iv) n (A × B) = n (A) × n (B) = 4 × 3 = 12
n (B × A) = n (B) × n (A) = 4 × 3 = 12

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Hence n (A × B) = n (B × A)

Example 2 Find$x$and$y$if:

(i) $(4x + 3, y) = (3x + 5, -2)$ (ii) $(x - y, x + y) = (6, 10)$

Solution

(i) Since $(4x + 3, y) = (3x + 5, -2)$, so

$4x + 3 = 3x + 5$

or $x = 2$

and $y = -2$

(ii) $x - y = 6$

$x + y = 10$

$\therefore 2x = 16$

or $x = 8$

$8 - y = 6$

$\therefore y = 2$

Example 3 If A = {2, 4, 6, 9} and B = {4, 6, 18, 27, 54},a∈A,b∈B, find the set ofordered pairs such that 'a' is factor of 'b' and $a<b$.

Solution Since A = {2, 4, 6, 9}

B = {4, 6, 18, 27, 54},

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we have to find *a* set of ordered pairs (*a*, *b*) such that *a* is factor of *b* and *a*<*b*.

Since 2 is a factor of 4 and 2 < 4.

So (2, 4) is one such ordered pair.

Similarly, (2, 6), (2, 18), (2, 54) are other such ordered pairs. Thus the required set of ordered pairs is

{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54,), (9, 18), (9, 27), (9, 54)}.

FUNCTION

A **Function** assigns to each element of a set, exactly one element of a related set. Functions find their application in various fields like representation of the computational complexity of algorithms, counting objects, study of sequences and strings, to name a few. The third and final chapter of this part highlights the important aspects of functions.

Function - Definition

A function or mapping (Defined as f:X→Yf:X→Y) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.

Function 'f' is a relation on X and Y such that for each x∈Xx∈X, there exists a unique y∈Yy∈Y such that (x,y)∈R(x,y)∈R. 'x' is called pre-image and 'y' is called image of function f.

A function can be one to one or many to one but not one to many.

Injective / One-to-one function

A function f:A→Bf:A→B is injective or one-to-one function if for every b∈Bb∈B, there exists at most one a∈Aa∈A such that f(s)=tf(s)=t.

This means a function **f** is injective if a1≠a2a1≠a2 implies f(a1)≠f(a2)f(a1)≠f(a2).

Example

- $f:N\to N, f(x)=5xf:N\to N, f(x)=5x$ is injective.
- $f:N\to N, f(x)=x2f:N\to N, f(x)=x2$ is injective.
- $f:R\to R, f(x)=x2f:R\to R, f(x)=x2$ is not injective as $(-x)2=x2(-x)2=x2$

Surjective / Onto function

A function $f:A\to Bf:A\to B$ is surjective (onto) if the image of f equals its range. Equivalently, for every $b\in Bb\in B$, there exists some $a\in Aa\in A$ such that $f(a)=bf(a)=b$. This means that for any y in B, there exists some x in A such that $y=f(x)y=f(x)$.

Example

- $f:N\to N, f(x)=x+2f:N\to N, f(x)=x+2$ is surjective.
- $f:R\to R, f(x)=x2f:R\to R, f(x)=x2$ is not surjective since we cannot find a real number whose square is negative.

Bijective / One-to-one Correspondent

A function $f:A\to Bf:A\to B$ is bijective or one-to-one correspondent if and only if **f**is both injective and surjective.

Problem

Prove that a function $f:R\to Rf:R\to R$ defined by $f(x)=2x-3f(x)=2x-3$ is a bijective function.

Explanation − We have to prove this function is both injective and surjective.

If $f(x1)=f(x2)f(x1)=f(x2)$, then $2x1-3=2x2-32x1-3=2x2-3$ and it implies that $x1=x2x1=x2$.

Hence, f is **injective**.

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Here, 2x–3=y2x–3=y

So, x=(y+5)/3x=(y+5)/3 which belongs to R and f(x)=yf(x)=y.

Hence, f is **surjective**.

Since **f** is both **surjective** and **injective**, we can say **f** is **bijective**.

Inverse of a Function

The **inverse** of a one-to-one corresponding function f:A→Bf:A→B, is the function g:B→Ag:B→A, holding the following property −

f(x)=y⇔g(y)=xf(x)=y⇔g(y)=x

The function f is called **invertible**, if its inverse function g exists.

Example

- A Function f:Z→Z,f(x)=x+5f:Z→Z,f(x)=x+5, is invertible since it has the inverse function g:Z→Z,g(x)=x−5g:Z→Z,g(x)=x−5.
- A Function f:Z→Z,f(x)=x2f:Z→Z,f(x)=x2 is not invertiable since this is not one-to-one as (−x)2=x2(−x)2=x2.

Composition of Functions

Two functions f:A→Bf:A→B and g:B→Cg:B→C can be composed to give a composition gofgof. This is a function from A to C defined by (gof)(x)=g(f(x))(gof)(x)=g(f(x))

Example

Let f(x)=x+2f(x)=x+2 and g(x)=2x+1g(x)=2x+1, find (fog)(x)(fog)(x) and (gof)(x)(gof)(x).

Solution

(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3

(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+5(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+
5

Hence, (fog)(x)≠(gof)(x)(fog)(x)≠(gof)(x)

Some Facts about Composition

- If f and g are one-to-one then the function (gof)(gof) is also one-to-one.
- If f and g are onto then the function (gof)(gof) is also onto.
- Composition always holds associative property but does not hold commutative property.

The rules of mathematical logic specify methods of reasoning mathematical statements. Greek philosopher, Aristotle, was the pioneer of logical reasoning. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

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Some Discrete Examples

EXAMPLE 2 Suppose $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$ and

$$f = \{(1, x), (2, y), (3, z), (4, y)\}.$$

Then f is a function $A \to B$ with domain A and target B. Since $\text{rng}\, f = \{x, y, z\} = B$, f is onto. Since $f(2) = f(4)$ $(= y)$ but $2 \neq 4$, f is not one-to-one. [In fact, there can exist no one-to-one function $A \to B$. Why not? See Exercise 25(a).] ▲

EXAMPLE 3 Suppose $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$ and

$$f = \{(1, w), (2, y), (3, x)\}.$$

Then $f: A \to B$ is a function with domain A and range $\{w, y, x\}$. Since $\text{rng}\, f \neq B$, f is not onto. [No function $A \to B$ can be onto. Why not? See Exercise 25(b).] This function is one-to-one because $f(1)$, $f(2)$, and $f(3)$ are all different: If $f(a_1) = f(a_2)$, then $a_1 = a_2$. ▲

EXAMPLE 4 Suppose $A = \{1, 2, 3\}$, $B = \{x, y, z\}$,

$$f = \{(1, z), (2, y), (3, y)\} \quad \text{and} \quad g = \{(1, z), (2, y), (3, x)\}.$$

Then f and g are functions from A to B. The domain of f is A and $\text{dom}\, g = A$ too. The range of f is $\{z, y\}$, which is a proper subset of B, so f is not onto. On the other hand, g is onto because $\text{rng}\, g = \{z, y, x\} = B$. This function is also one-to-one because $g(1)$, $g(2)$, and $g(3)$ are all different: If $g(a_1) = g(a_2)$, then $a_1 = a_2$. Notice that f is not one-to-one: $f(2) = f(3)$ $(= y)$, yet $2 \neq 3$. ▲

EXAMPLE 5 Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x - 3$. Then $\text{dom}\, f = \mathbb{Z}$. To find $\text{rng}\, f$, note that

$$\begin{aligned} b \in \text{rng}\, f &\leftrightarrow b = 2a - 3 && \text{for some integer } a \\ &\leftrightarrow b = 2(a - 2) + 1 && \text{for some integer } a \end{aligned}$$

and this occurs if and only if b is odd. Thus, the range of f is the set of odd integers. Since $\text{rng}\, f \neq \mathbb{Z}$, f is not onto. It is one-to-one, however: If $f(x_1) = f(x_2)$, then $2x_1 - 3 = 2x_2 - 3$ and $x_1 = x_2$. ▲

EXAMPLE 6 Let $f: \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 2x - 3$. This might look like a perfectly good function, as in the last example, but actually there is a difficulty. If we try to calculate $f(1)$, we obtain $f(1) = 2(1) - 3 = -1$ and $-1 \notin \mathbb{N}$. Hence, no function has been defined. ▲

PROBLEM 7. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^2 - 5x + 5$. Determine whether or not f is one-to-one and/or onto.

Solution. To determine whether or not f is one-to-one, we consider the possibility that $f(x_1) = f(x_2)$. In this case, $x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5$, so $x_1^2 - x_2^2 = 5x_1 - 5x_2$ and $(x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$. This equation indeed has solutions with $x_1 \neq x_2$: Any x_1, x_2 satisfying $x_1 + x_2 = 5$ will do, for instance, $x_1 = 2$, $x_2 = 3$. Since $f(2) = f(3) = -1$, we conclude that f is not one-to-one.

Is f onto? Recalling that the graph of $f(x) = x^2 - 5x + 5$, $x \in \mathbb{R}$, is a parabola with vertex $(\frac{5}{2}, -\frac{5}{4})$, clearly any integer less than -1 is not in the range of f. Alternatively, it is easy to see that 0 is not in the range of f because $x^2 - 5x + 5 = 0$ has no integer solutions (by the quadratic formula). Either argument shows that f is not onto. ▮

PROBLEM 8. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 3x^3 - x$. Determine whether or not f is one-to-one and/or onto.

Solution. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{Z}$. Then $3x_1^3 - x_1 = 3x_2^3 - x_2$, so $3(x_1^3 - x_2^3) = x_1 - x_2$ and

$$3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2.$$

If $x_1 \neq x_2$, we must have $x_1^2 + x_1x_2 + x_2^2 = \frac{1}{3}$, which is impossible since x_1 and x_2 are integers. Thus, $x_1 = x_2$ and f is one-to-one.

Is f onto? If yes, then the equation $b = f(x) = 3x^3 - x$ has a solution in $\mathbb{Z}$ for every integer b. This seems unlikely and, after a moment's thought, it occurs to us that the integer $b = 1$, for example, cannot be written this way: $1 = 3x^3 - x$ for some integer x implies $x(3x^2 - 1) = 1$. But the only pairs of integers whose product is 1 are the pairs 1, 1 and $-1, -1$. So here, we would require $x = 3x^2 - 1 = 1$ or $x = 3x^2 - 1 = -1$, neither of which is possible. The integer $b = 1$ is a counterexample to the assertion that f is onto, so f is not onto. ▮

EXAMPLE

Let $g : R \to R$ be defined by $g(x) = x^2$. The domain of g is R; the range of g is the set of nonnegative real numbers. Since this is a proper subset of R, g is not onto. Neither is g one-to-one since $g(3) = g(-3)$, but $3 \neq -3$. ▲

Define $h : [0, \infty) \to R$ by $h(x) = x^2$. This function is identical to the function g of the preceding example except for its domain. By *restricting the domain* of g to the nonnegative reals we have produced a function h which is one-to-one since $h(x_1) = h(x_2)$ implies $x_1^2 = x_2^2$ and hence $x_1 = \pm x_2$. Since $x_1 \geq 0$ and $x_2 \geq 0$, we must have $x_1 = x_2$. ▲

The Identity Function

For any set A, the *identity function on* A is the function $\iota_A : A \to A$ defined by $\iota_A(a) = a$ for all $a \in A$. In terms of ordered pairs,

$$\iota_A = \{(a, a) \mid a \in A\}.$$

When there is no possibility of confusion about A, we will often write ι, rather than ι_A. (The Greek symbol ι is pronounced "yōta", so that "ι_A" is read "yota sub A."

The graph of the identity function on R is the familiar line with equation $y = x$. The identity function on a set A is indeed a function $A \to A$ since, for any $a \in A$, there is precisely one pair of the form $(a, y) \in \iota$, namely, the pair (a, a).

Relations and Functions

RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim}

B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly R Í A ´ B, i.e.R = {(a,b):a Î A,b Î B and aRb}

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) R = f , R is called a void relation.

(v) R=A×B, R is called a universal relation.

(vi) If R is a relation defined from A to A, it is called a relation defined on A.

(vii) R = { (a,a) " a Î A } , is called the identity relation.

Domain and Range of a Relation

If R is a relation between two sets then the set of its first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation.

Consider previous example given above.

Domain = {Mohan, Sohan, David, Karim}

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Range = {Rita, Marry, Fatima}

Example 1 Given that A = {2, 4, 5, 6, 7}, B = {2, 3}.

R is a relation from A to B defined by
R = {(a, b) : a Î A, b Î B and a is divisible by b}

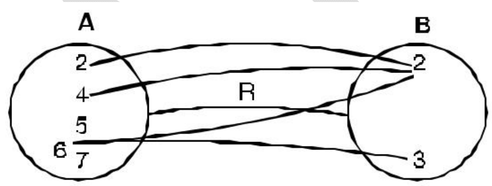
find (i) R in the roster form
(ii) Domain of R
(iii) Range of R
(iv) Repersent R diagramatically.

Solution : (i) R = {(2, 2), (4, 2), (6, 2), (6, 3)}

(ii) Domain of R = {2, 4, 6}

(iii) Range of R = {2, 3}

(iv)



Example 2 If R is a relation 'is greater than' from A to B, where A= {1, 2, 3, 4, 5} and B = {1,2,6}.
Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(iv) R = {(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(v) Domain of R = {3, 4, 5}

(vi) Range of R = {1, 2}

2.1 Overview

This chapter deals with linking pair of elements from two sets and then introduce relations between the two elements in the pair. Practically in every day of our lives, we pair the members of two sets of numbers. For example, each hour of the day is paired with the local temperature reading by T.V. Station's weatherman, a teacher often pairs each set of score with the number of students receiving that score to see more clearly how well the class has understood the lesson. Finally, we shall learn about special relations called functions.

2.1.1 ***Cartesian products of sets***

Definition : Given two non-empty sets A and B, the set of all ordered pairs (x,y),where $x \in$ A and $y \in$ B is called Cartesian product of A and B; symbolically, we write

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

If A = {1, 2, 3} and B = {4, 5}, then

$$A \times B = \{(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)\}$$

And B × A = {(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)}

(iii) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal, i.e. $(x, y) = (u, v)$ if and only if $x = u$, $y = v$.

(iv) If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = p \times q$.

(i) $A \times A \times A = \{(a,b,c) : a,b,c \in A\}$. Here (a,b,c) is called an ordered triplet.

2.1.2 ***Relations*** A Relation R from a non-empty set A to a non empty set B is asubset of the Cartesian product set A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B.

The set of all first elements in a relation R, is called the domain of the relation R, and the set of all second elements called images, is called the range of R.

For example, the set R = {(1, 2), (– 2, 3), ($\frac{1}{2}$, 3)} is a relation; the domain of

R = {1, – 2, $\frac{1}{2}$} and the range of R = {2, 3}.

(iii) A relation may be represented either by the Roster form or by the set builder form, or by an arrow diagram which is a visual representation of a relation.

(iv) If $n(A) = p$, $n(B) = q$; then the $n(A \times B) = pq$ and the total number of possible relations from the set A to set B = 2^{pq}.

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2.1.3 *Functions* A relation f from a set A to a set B is said to be **function** if every element of set A has one and only one image in set B.

In other words, a function f is a relation such that no two pairs in the relation has the same first element.

The notation $f : X \to Y$ means that f is a function from X to Y. X is called the **domain** of f and Y is called the **co-domain** of f. Given an element $x \in X$, there is a unique element

z in Y that is related to x. The unique element y to which f relates x is denoted by $f(x)$ and is called f of x, or the **value of *f* at *x***, or the *image of x under f*.

The set of all values of $f(x)$ taken together is called the **range of *f*** or image of X under f. Symbolically.

$$\text{range of } f = \{ y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}$$

Definition : A function which has either **R** or one of its subsets as its range, is calleda real valued function. Further, if its domain is also either **R** or a subset of **R**, it is called a real function.

2.1.4 *Some specific types of functions*

(ii) **Identity function:**

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = x$ for each $x \in \mathbf{R}$ is called the **identity function.**

Domain of $f = \mathbf{R}$

Range of $f = \mathbf{R}$

(iii) **Constant function:** The function $f: \mathbf{R} \to \mathbf{R}$ defined by $y=f(x) = C, x \in \mathbf{R}$, where C is a constant $\in \mathbf{R}$, is a **constant function**.

Domain of $f = \mathbf{R}$

Range of $f = \{C\}$

(v) **Polynomial function:** A real valued function $f: \mathbf{R} \to \mathbf{R}$ defined by $y=f(x) = a_0$

$+ a_1x + ... + a_nx^n$, where $n \in$ **N**, and $a_0, a_1, a_2...a_n \in$ **R**, for each $x \in$ **R**, is calledPolynomial functions.

(vi) **Rational function:** These are the real functions of the type $\frac{f(x)}{g(x)}$, where g (x) and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$. For example f : **R** – {– 2} →**R** defined by $f(x) = \frac{x+1}{x+2}$, $x \in$ **R**– {– 2 }is a rational function.

(v) **The Modulus function:** The real function$f$: **R** → **R** defined by$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$x \in$ **R**is called the modulus function.

Domain of f = **R**

Range of f = $\mathbf{R}^+ \cup \{0\}$

(vii) **Signum function:** The real function

f :**R**→**R**defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is called the **signum function**. Domain of f = **R**, Range of f = {1, 0, – 1}

(viii) **Greatest integer function:** The real function$f$: **R** → **R** defined by$f(x) = [x]$, $x \in$ **R**assumes the value of the greatest integer less than or equal to x, iscalled the **greatest integer function**.

Thus$f(x) = [x] = -1$ for $-1 \leq x < 0$ $f(x) = [x] = 0$ for $0 \leq x < 1$

$[x] = 1$ for $1 \leq x < 2$

$[x] = 2$ for $2 \leq x < 3$ and so on

2.1.5 ***Algebra of real functions***

(iv) ***Addition of two real functions***

Let f : X →**R** and g : X →**R** be any two real functions, where X ∈**R**.

Then we define ($f + g$) : X →**R** by ($f + g$) (x) = f (x) + g (x), for all x∈ X.

(v) ***Subtraction of a real function from another***

Let f : X →**R** and g : X →**R** be any two real functions, where X ⊆**R**.

Then, we define (f – g) : X →**R** by (f – g) (x) = f (x) – g (x), for all x∈ X.

(vi) ***Multiplication by a Scalar***

Let f : X →**R** be a real function and α be any scalar belonging to **R**. Then the product αf is function from X to **R** defined by (αf) (x) = αf (x), x∈ X.

(v) ***Multiplication of two real functions***

Let f : X → **R** and g : x → **R** be any two real functions, where X ⊆**R**. Then product of these two functions i.e. $f g$: X → **R** is defined by ($f g$) (x) = f (x) g (x) x∈ X.

(vi) ***Quotient of two real function***

Let f and g be two real functions defined from X → **R**. The quotient of f by g
f
denoted by g is a function defined from X → **R** as
$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided g (x) ≠ 0, x∈ X.

$gg(x)$

 Note Domain of sum function$f$+g, difference function$f$–$g$and productfunction fg.
= $\{x : x \in D_f \cap D_g\}$

where Df = Domain of function f

Dg = Domain of function g

$F = \{x : x \in D_f \cap D_g$ and $g(x) \neq 0\}$

2.2 Solved Examples

Short Answer Type

Example 1 Let A = {1, 2, 3, 4} and B = {5, 7, 9}. Determine

(i) $A \times B$ (ii) $B \times A$

(iii) Is $A \times B = B \times A$? (iv) Is $n(A \times B) = n(B \times A)$?

(v) $A \times B = \{(1, 5), (1, 7), (1, 9), (2, 5), (2, 7), (2, 9), (3, 5), (3, 7), (3, 9), (4, 5), (4, 7), (4, 9)\}$

(vi) $B \times A = \{(5, 1), (5, 2), (5, 3), (5, 4), (7, 1), (7, 2), (7, 3), (7, 4), (9, 1), (9, 2), (9, 3), (9, 4)\}$

(vii) No, $A \times B \neq B \times A$. Since $A \times B$ and $B \times A$ do not have exactly the same ordered pairs.

(viii) $n(A \times B) = n(A) \times n(B) = 4 \times 3 = 12$
$n(B \times A) = n(B) \times n(A) = 4 \times 3 = 12$

Hence $n(A \times B) = n(B \times A)$

Example 2 Find x and y if:

(i) $(4x + 3, y) = (3x + 5, -2)$ (ii) $(x - y, x + y) = (6, 10)$

Solution

(i) Since $(4x + 3, y) = (3x + 5, -2)$, so

$4x + 3 = 3x + 5$

or $x = 2$

and $y = -2$

(ii) $x - y = 6$

$x + y = 10$

$\therefore 2x = 16$

or $x = 8$

$8 - y = 6$

$\therefore y = 2$

Example 3 If A = {2, 4, 6, 9} and B = {4, 6, 18, 27, 54}, $a \in A, b \in B$, find the set ofordered pairs such that 'a' is factor of 'b' and $a<b$.

Solution Since A = {2, 4, 6, 9}

B = {4, 6, 18, 27, 54},

we have to find a set of ordered pairs (a, b) such that a is factor of b and $a<b$.

Since 2 is a factor of 4 and 2 < 4.

So (2, 4) is one such ordered pair.

Similarly, (2, 6), (2, 18), (2, 54) are other such ordered pairs. Thus the required set of ordered pairs is

{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54,), (9, 18), (9, 27), (9, 54)}.

FUNCTION

A **Function** assigns to each element of a set, exactly one element of a related set. Functions find their application in various fields like representation of the computational complexity of algorithms, counting objects, study of sequences and strings, to name a few. The third and final chapter of this part highlights the important aspects of functions.

Function - Definition

A function or mapping (Defined as $f:X\to Yf:X\to Y$) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.

Function 'f' is a relation on X and Y such that for each $x\in Xx\in X$, there exists a unique $y\in Yy\in Y$ such that $(x,y)\in R(x,y)\in R$. 'x' is called pre-image and 'y' is called image of function f.

A function can be one to one or many to one but not one to many.

Injective / One-to-one function

A function $f:A\to Bf:A\to B$ is injective or one-to-one function if for every $b\in Bb\in B$, there exists at most one $a\in Aa\in A$ such that $f(s)=tf(s)=t$.

This means a function **f** is injective if $a1\neq a2a1\neq a2$ implies $f(a1)\neq f(a2)f(a1)\neq f(a2)$.

Example

- $f:N\to N,f(x)=5xf:N\to N,f(x)=5x$ is injective.
- $f:N\to N,f(x)=x2f:N\to N,f(x)=x2$ is injective.
- $f:R\to R,f(x)=x2f:R\to R,f(x)=x2$ is not injective as $(-x)2=x2(-x)2=x2$

Surjective / Onto function

A function $f:A\to Bf:A\to B$ is surjective (onto) if the image of f equals its range. Equivalently, for every $b\in Bb\in B$, there exists some $a\in Aa\in A$ such that $f(a)=bf(a)=b$. This means that for any y in B, there exists some x in A such that $y=f(x)y=f(x)$.

Example

- $f:N\to N,f(x)=x+2f:N\to N,f(x)=x+2$ is surjective.
- $f:R\to R,f(x)=x2f:R\to R,f(x)=x2$ is not surjective since we cannot find a real number whose square is negative.

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Bijective / One-to-one Correspondent

A function $f:A \to Bf:A \to B$ is bijective or one-to-one correspondent if and only if **f**is both injective and surjective.

Problem

Prove that a function $f:R \to Rf:R \to R$ defined by $f(x)=2x–3f(x)=2x–3$ is a bijective function.

Explanation − We have to prove this function is both injective and surjective.

If $f(x1)=f(x2)f(x1)=f(x2)$, then $2x1–3=2x2–32x1–3=2x2–3$ and it implies that $x1=x2x1=x2$.

Hence, f is **injective**.

Here, $2x–3=y2x–3=y$

So, $x=(y+5)/3x=(y+5)/3$ which belongs to R and $f(x)=yf(x)=y$.

Hence, f is **surjective**.

Since **f** is both **surjective** and **injective**, we can say **f** is **bijective**.

Inverse of a Function

The **inverse** of a one-to-one corresponding function $f:A \to Bf:A \to B$, is the function $g:B \to Ag:B \to A$, holding the following property −

$f(x)=y \Leftrightarrow g(y)=xf(x)=y \Leftrightarrow g(y)=x$

The function f is called **invertible**, if its inverse function g exists.

Example

- A Function $f:Z \to Z,f(x)=x+5f:Z \to Z,f(x)=x+5$, is invertible since it has the inverse function $g:Z \to Z,g(x)=x−5g:Z \to Z,g(x)=x−5$.

- A Function f:Z→Z,f(x)=x2f:Z→Z,f(x)=x2 is not invertiable since this is not one-to-one as (−x)2=x2(−x)2=x2.

Composition of Functions

Two functions f:A→Bf:A→B and g:B→Cg:B→C can be composed to give a composition gofgof. This is a function from A to C defined by (gof)(x)=g(f(x))(gof)(x)=g(f(x))

Example

Let f(x)=x+2f(x)=x+2 and g(x)=2x+1g(x)=2x+1, find (fog)(x)(fog)(x) and (gof)(x)(gof)(x).

Solution

(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3

(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+5(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+5

Hence, (fog)(x)≠(gof)(x)(fog)(x)≠(gof)(x)

Some Facts about Composition

- If f and g are one-to-one then the function (gof)(gof) is also one-to-one.
- If f and g are onto then the function (gof)(gof) is also onto.
- Composition always holds associative property but does not hold commutative property.

The rules of mathematical logic specify methods of reasoning mathematical statements. Greek philosopher, Aristotle, was the pioneer of logical reasoning. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

Some Discrete Examples

EXAMPLE 2 Suppose $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$ and

$$f = \{(1, x), (2, y), (3, z), (4, y)\}.$$

Then f is a function $A \to B$ with domain A and target B. Since rng $f = \{x, y, z\} = B$, f is onto. Since $f(2) = f(4)$ $(= y)$ but $2 \neq 4$, f is not one-to-one. [In fact, there can exist no one-to-one function $A \to B$. Why not? See Exercise 25(a).] ▲

EXAMPLE 3 Suppose $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$ and

$$f = \{(1, w), (2, y), (3, x)\}.$$

Then $f : A \to B$ is a function with domain A and range $\{w, y, x\}$. Since rng $f \neq B$, f is not onto. [No function $A \to B$ can be onto. Why not? See Exercise 25(b).] This function is one-to-one because $f(1)$, $f(2)$, and $f(3)$ are all different: If $f(a_1) = f(a_2)$, then $a_1 = a_2$. ▲

EXAMPLE 4 Suppose $A = \{1, 2, 3\}$, $B = \{x, y, z\}$,

$$f = \{(1, z), (2, y), (3, y)\} \quad \text{and} \quad g = \{(1, z), (2, y), (3, x)\}.$$

Then f and g are functions from A to B. The domain of f is A and dom $g = A$ too. The range of f is $\{z, y\}$, which is a proper subset of B, so f is not onto. On the other hand, g is onto because rng $g = \{z, y, x\} = B$. This function is also one-to-one because $g(1)$, $g(2)$, and $g(3)$ are all different: If $g(a_1) = g(a_2)$, then $a_1 = a_2$. Notice that f is not one-to-one: $f(2) = f(3)$ $(= y)$, yet $2 \neq 3$. ▲

EXAMPLE 5 Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x - 3$. Then dom $f = \mathbb{Z}$. To find rng f, note that

$$\begin{aligned} b \in \text{rng } f &\leftrightarrow b = 2a - 3 && \text{for some integer } a \\ &\leftrightarrow b = 2(a - 2) + 1 && \text{for some integer } a \end{aligned}$$

and this occurs if and only if b is odd. Thus, the range of f is the set of odd integers. Since rng $f \neq \mathbb{Z}$, f is not onto. It is one-to-one, however: If $f(x_1) = f(x_2)$, then $2x_1 - 3 = 2x_2 - 3$ and $x_1 = x_2$. ▲

EXAMPLE 6 Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 2x - 3$. This might look like a perfectly good function, as in the last example, but actually there is a difficulty. If we try to calculate $f(1)$, we obtain $f(1) = 2(1) - 3 = -1$ and $-1 \notin \mathbb{N}$. Hence, no function has been defined. ▲

PROBLEM 7. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^2 - 5x + 5$. Determine whether or not f is one-to-one and/or onto.

Solution. To determine whether or not f is one-to-one, we consider the possibility that $f(x_1) = f(x_2)$. In this case, $x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5$, so $x_1^2 - x_2^2 = 5x_1 - 5x_2$ and $(x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$. This equation indeed has solutions with $x_1 \neq x_2$: Any x_1, x_2 satisfying $x_1 + x_2 = 5$ will do, for instance, $x_1 = 2$, $x_2 = 3$. Since $f(2) = f(3) = -1$, we conclude that f is not one-to-one.

Is f onto? Recalling that the graph of $f(x) = x^2 - 5x + 5$, $x \in \mathbb{R}$, is a parabola with vertex $(\frac{5}{2}, -\frac{5}{4})$, clearly any integer less than -1 is not in the range of f. Alternatively, it is easy to see that 0 is not in the range of f because $x^2 - 5x + 5 = 0$ has no integer solutions (by the quadratic formula). Either argument shows that f is not onto. ∎

PROBLEM 8. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 3x^3 - x$. Determine whether or not f is one-to-one and/or onto.

Solution. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{Z}$. Then $3x_1^3 - x_1 = 3x_2^3 - x_2$, so $3(x_1^3 - x_2^3) = x_1 - x_2$ and

$$3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2.$$

If $x_1 \neq x_2$, we must have $x_1^2 + x_1x_2 + x_2^2 = \frac{1}{3}$, which is impossible since x_1 and x_2 are integers. Thus, $x_1 = x_2$ and f is one-to-one.

Is f onto? If yes, then the equation $b = f(x) = 3x^3 - x$ has a solution in $\mathbb{Z}$ for every integer b. This seems unlikely and, after a moment's thought, it occurs to us that the integer $b = 1$, for example, cannot be written this way: $1 = 3x^3 - x$ for some integer x implies $x(3x^2 - 1) = 1$. But the only pairs of integers whose product is 1 are the pairs $1, 1$ and $-1, -1$. So here, we would require $x = 3x^2 - 1 = 1$ or $x = 3x^2 - 1 = -1$, neither of which is possible. The integer $b = 1$ is a counterexample to the assertion that f is onto, so f is not onto. ∎

EXAMPLE

Let $g: \mathsf{R} \to \mathsf{R}$ be defined by $g(x) = x^2$. The domain of g is R; the range of g is the set of nonnegative real numbers. Since this is a proper subset of R, g is not onto. Neither is g one-to-one since $g(3) = g(-3)$, but $3 \neq -3$. ▲

Define $h: [0, \infty) \to \mathsf{R}$ by $h(x) = x^2$. This function is identical to the function g of the preceding example except for its domain. By *restricting the domain* of g to the nonnegative reals we have produced a function h which is one-to-one since $h(x_1) = h(x_2)$ implies $x_1^2 = x_2^2$ and hence $x_1 = \pm x_2$. Since $x_1 \geq 0$ and $x_2 \geq 0$, we must have $x_1 = x_2$. ▲

The Identity Function

For any set A, the *identity function on A* is the function $\iota_A: A \to A$ defined by $\iota_A(a) = a$ for all $a \in A$. In terms of ordered pairs,

$$\iota_A = \{(a, a) \mid a \in A\}.$$

When there is no possibility of confusion about A, we will often write ι, rather than ι_A. (The Greek symbol ι is pronounced "yōta", so that "ι_A" is read "yota sub A."

The graph of the identity function on R is the familiar line with equation $y = x$. The identity function on a set A is indeed a function $A \to A$ since, for any $a \in A$, there is precisely one pair of the form $(a, y) \in \iota$, namely, the pair (a, a).

INVERSES AND COMPOSITION

The Inverse of a Function

Suppose that f is a one-to-one onto function from A to B. Given any $b \in B$, there exists $a \in A$ such that $f(a) = b$ (because f is onto) and only one such a (because f is one-to-one). Thus, for each $b \in B$, there is precisely one pair of the form $(a, b) \in f$. It follows that the set $\{(b, a) \mid (a, b) \in f\}$, obtained by reversing the ordered pairs of f, is a function from B to A (since each element of B occurs precisely once as the first coordinate of an ordered pair).

EXAMPLE 13 If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$, then

$$f = \{(1, x), (2, y), (3, z), (4, t)\}$$

is a one-to-one onto function from A to B and, reversing its pairs, we obtain a function $B \to A$: $\{(x, 1), (y, 2), (z, 3), (t, 4)\}$. ▲

DEFINITION A function $f : A \to B$ *has an inverse* if and only if the set obtained by reversing the ordered pairs of f is a function $B \to A$. If $f : A \to B$ has an inverse, the function

$$\boxed{f^{-1} = \{(b, a) \mid (a, b) \in f\}}$$

is called the *inverse* of f.

We pronounce f^{-1}, "f inverse," terminology which should not be confused with $\frac{1}{f}$: f^{-1} is simply the name of a certain function, the inverse of f.[2]

If $f : A \to B$ has an inverse $f^{-1} : B \to A$, then f^{-1} also has an inverse because reversing the pairs of f^{-1} gives a function, namely f: thus, $(f^{-1})^{-1} = f$.

EXAMPLE 14 If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$, and

$$f = \{(1, x), (2, y), (3, z), (4, t)\}$$

then

$$f^{-1} = \{(x, 1), (y, 2), (z, 3), (t, 4)\}$$

and $(f^{-1})^{-1} = \{(1, x), (2, y), (3, z), (4, t)\} = f$. ▲

PROPOSITION

A function $f : A \to B$ has an inverse $B \to A$ if and only if f is one-to-one and onto.

For any function g, remember that $(x, y) \in g$ if and only if $g(x) = y$; in particular, $(b, a) \in f^{-1}$ if and only if $a = f^{-1}(b)$. Thus,

$$a = f^{-1}(b) \leftrightarrow (b, a) \in f^{-1} \leftrightarrow (a, b) \in f \leftrightarrow f(a) = b.$$

The equivalence of the first and last equations here is very important:

(2) $$\boxed{a = f^{-1}(b) \text{ if and only if } f(a) = b.}$$

For example, if for some function f, $\pi = f^{-1}(-7)$, then we can conclude that $f(\pi) = -7$. If $f(4) = 2$, then $4 = f^{-1}(2)$.

The solution to the equation $2x = 5$ is $x = \frac{5}{2} = 2^{-1} \cdot 5$. Generally, to solve the equation $ax = b$, we ask if $a \neq 0$, and if this is the case, we multiply each side of the equation by a^{-1}, obtaining $x = a^{-1}b = \frac{b}{a}$. Since all real numbers except 0 have a multiplicative inverse, checking that $a \neq 0$ is just checking that a has an inverse.

Look again at statement (2). We solve the equation $f(x) = y$ for x in the same way we solve $ax = b$ for x. We first ask if f has an inverse, and if it does, apply f^{-1} to each side of the equation, obtaining $x = f^{-1}(y)$.

The "application" of f^{-1} to each side of the equation $y = f(x)$ is very much like multiplying each side by f^{-1}. "Multiplying by f^{-1}" may sound foolish, but there is a context (called *group theory*) in which it makes good sense. Our intent here is just to provide a good way to remember the fundamental relationship expressed in (2).

EXAMPLE

If $f : \mathrm{R} \to \mathrm{R}$ is defined by $f(x) = 2x - 3$, then f is one-to-one and onto, so an inverse function exists. According to (2), if $y = f^{-1}(x)$, then $x = f(y) = 2y - 3$. Thus, $y = \frac{1}{2}(x + 3) = f^{-1}(x)$. ▲

Let $A = \{x \in \mathrm{R} \mid x \le 0\}$, $B = \{x \in \mathrm{R} \mid x \ge 0\}$ and define $f : A \to B$ by $f(x) = x^2$. This is just the squaring function with domain restricted so that it is one-to-one as well as onto. Since f is one-to-one and onto, it has an inverse. To obtain $f^{-1}(x)$, let $y = f^{-1}(x)$, deduce [by the relationship expressed in (2)] that $f(y) = x$ and so $y^2 = x$. Solving for y, we get $y = \pm\sqrt{x}$. Since $x = f(y)$, $y \in A$, so $y \le 0$. Thus, $y = -\sqrt{x}$; $f^{-1}(x) = -\sqrt{x}$. ▲

PROBLEM 18. Let $A = \{x \mid x \neq \frac{1}{2}\}$ and define $f : A \to \mathrm{R}$ by $f(x) = \dfrac{4x}{2x - 1}$.

Is f one-to-one? Find rng f. Explain why $f : A \to \operatorname{rng} f$ has an inverse. Find dom f^{-1}, rng f^{-1}, and a formula for $f^{-1}(x)$.

Solution. Suppose $f(a_1) = f(a_2)$. Then $\dfrac{4a_1}{2a_1 - 1} = \dfrac{4a_2}{2a_2 - 1}$, so $8a_1a_2 - 4a_1 = 8a_1a_2 - 4a_2$, hence $a_1 = a_2$. Thus f is one-to-one.

Next,

$$\begin{aligned} y \in \operatorname{rng} f \leftrightarrow y = f(x) \quad & \text{for some } x \in A \\ & \leftrightarrow \text{there is an } x \in A \text{ such that } y = \frac{4x}{2x-1} \\ & \leftrightarrow \text{there is an } x \in A \text{ such that } 2xy - y = 4x \\ & \leftrightarrow \text{there is an } x \in A \text{ such that } x(2y-4) = y. \end{aligned}$$

If $y = 2$, the equation $x(2y-4) = y$ becomes $0 = 2$ and no x exists. On the other hand, if $y \neq 2$, then $2y - 4 \neq 0$ and so, dividing by $2y - 4$, we obtain $x = \dfrac{y}{2y-4}$. (It is easy to see that such x is never $\frac{1}{2}$; that is, $x \in A$.) Thus $y \in \operatorname{rng} f$ if and only if $y \neq 2$. So $\operatorname{rng} f = B = \{y \in \mathrm{R} \mid y \neq 2\}$.

Since $f : A \to B$ is one-to-one and onto, it has an inverse $f^{-1} : B \to A$. Also, $\operatorname{dom} f^{-1} = \operatorname{rng} f = B$ and $\operatorname{rng} f^{-1} = \operatorname{dom} f = A$. To find $f^{-1}(x)$, set $y = f^{-1}(x)$. Then

$$x = f(y) = \frac{4y}{2y-1}$$

and, solving for y, we get $y = \dfrac{x}{2x-4} = f^{-1}(x)$. ∎

Composition of Functions

DEFINITION If $f : A \to B$ and $g : B \to C$ are functions, then the *composition of g and f* is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

EXAMPLE 19 If $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{u, v, w\}$, and if $f : A \to B$ and $g : B \to C$ are the functions

$$f = \{(a, x), (b, y), (c, x)\}, \quad g = \{(x, u), (y, w)\},$$

then

$$(g \circ f)(a) = g(f(a)) = g(x) = u,$$
$$(g \circ f)(b) = g(f(b)) = g(y) = w,$$
$$(g \circ f)(c) = g(f(c)) = g(x) = u$$

and so $g \circ f = \{(a, u), (b, w), (c, u)\}$. ▲

EXAMPLE 20 If f and g are the functions $R \to R$ defined by

$$f(x) = 2x - 3, \quad g(x) = x^2 + 1,$$

then both $g \circ f$ and $f \circ g$ are defined and we have

$$(g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 1$$

and

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 2(x^2 + 1) - 3.$$ ▲

EXAMPLE 21 In the definition of $g \circ f$, it is required that $\operatorname{rng} f \subseteq B = \operatorname{dom} g$. If $f : R \to R$ and $g : R \setminus \{1\} \to R$ are the functions defined by

$$f(x) = 2x - 3 \quad \text{and} \quad g(x) = \frac{x}{x-1},$$

then $g \circ f$ is not defined because $\operatorname{rng} f = R \not\subseteq \operatorname{dom} g$. On the other hand, $f \circ g$ is defined and

$$(f \circ g)(x) = 2\left(\frac{x}{x-1}\right) - 3.$$ ▲

PROPOSITION Composition of functions is an associative operation.

Proof We must prove that $(f \circ g) \circ h = f \circ (g \circ h)$ whenever each of the two functions—$(f \circ g) \circ h$ and $f \circ (g \circ h)$—is defined. Thus, we assume that for certain sets A, B, C, and D, h is a function $A \to B$, g is a function $B \to C$, and f is a function $C \to D$. A direct proof is suggested.

Since the domain of $(f \circ g) \circ h$ is the domain of $f \circ (g \circ h)$ (namely, the set A), we have only to prove that $((f \circ g) \circ h)(a) = (f \circ (g \circ h))(a)$ for any $a \in A$. For this, we have

$$((f \circ g) \circ h)(a) = (f \circ g)(h(a)) = f(g(h(a)))$$

and

$$(f \circ (g \circ h))(a) = f((g \circ h)(a)) = f(g(h(a)))$$

as desired. ∎

If $f: A \to B$ has an inverse $f^{-1}: B \to A$, then, recalling (2),

$$f^{-1}(b) = a \text{ if and only if } b = f(a).$$

So for any $a \in A$,

$$a = f^{-1}(b) = f^{-1}(f(a)) = f^{-1} \circ f(a).$$

In other words, the composition $f^{-1} \circ f = \iota_A$, the identity function on A. Similarly, for any element $b \in B$,

$$b = f(a) = f(f^{-1}(b)) = f \circ f^{-1}(b).$$

Thus, the composition $f \circ f^{-1} = \iota_B$ is the identity function on B. We summarize.

PROPOSITION Functions $f: A \to B$ and $g: B \to A$ are inverses if and only if $g \circ f = \iota_A$ and $f \circ g = \iota_B$; that is, if and only if

$$g(f(a)) = a \text{ and } f(g(b)) = b \quad \text{for all } a \in A \text{ and all } b \in B.$$

PROBLEM 23. Show that the functions $f: \mathsf{R} \to (1, \infty)$ and $g: (1, \infty) \to \mathsf{R}$ defined by

$$f(x) = 3^{2x} + 1, \qquad g(x) = \tfrac{1}{2}\log_3(x-1)$$

are inverses.

Solution. For any $x \in \mathsf{R}$,

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(3^{2x} + 1)\\ &= \tfrac{1}{2}(\log_3[(3^{2x} + 1) - 1])\\ &= \tfrac{1}{2}(\log_3 3^{2x}) = \tfrac{1}{2}2x = x\end{aligned}$$

and for any $x \in (1, \infty)$,

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= f(\tfrac{1}{2}\log_3(x-1))\\ &= 3^{2(\frac{1}{2}\log_3(x-1))} + 1\\ &= 3^{\log_3(x-1)} + 1 = (x-1) + 1 = x.\end{aligned}$$

One-to-One Correspondence and the Cardinality of a Set

DEFINITIONS A *finite* set is a set which is either empty or in one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$ of the first n natural numbers, for some $n \in \mathbb{N}$. A set which is not finite is called *infinite*.

DEFINITION Sets A and B have the *same cardinality* and we write $|A| = |B|$, if and only if there is a *one-to-one correspondence* between them; that is, if and only if there exists a one-to-one onto function from A to B (or from B to A).

EXAMPLES 25

- $a \mapsto x$, $b \mapsto y$ is a one-to-one correspondence between $\{a, b\}$ and $\{x, y\}$; hence, $|\{a, b\}| = |\{x, y\}|$ $(= 2)$.
- The function $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$ defined by $f(n) = n - 1$ is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$; so $|\mathbb{N}| = |\mathbb{N} \cup \{0\}|$.
- The function $f : \mathbb{Z} \longrightarrow 2\mathbb{Z}$ defined by $f(n) = 2n$ is a one-to-one correspondence between the set $\mathbb{Z}$ of integers and the set $2\mathbb{Z}$ of even integers; thus, $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality. ▲

PROBLEM 26. Show that the set $\mathbb{R}^+$ of positive real numbers has the same cardinality as the open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

Solution. Let $f : (0, 1) \to \mathbb{R}^+$ be defined by

$$f(x) = \frac{1}{x} - 1.$$

We claim that f establishes a one-to-one correspondence between $(0, 1)$ and $\mathbb{R}^+$.

To show that f is onto, we have to show that any $y \in \mathbb{R}^+$ is $f(x)$ for some $x \in (0, 1)$. But

$$y = \frac{1}{x} - 1 \text{ implies } x = \frac{1}{1 + y}$$

which is in $(0, 1)$ since $y > 0$. Therefore,

$$y \in \mathbb{R}^+ \text{ implies } y = f\left(\frac{1}{1 + y}\right)$$

so f is indeed onto. Also, f is one-to-one because

$$f(x_1) = f(x_2) \to \frac{1}{x_1} - 1 = \frac{1}{x_2} - 1$$
$$\to \frac{1}{x_1} = \frac{1}{x_2}$$
$$\to x_1 = x_2.$$

DEFINITIONS A set A is *countably infinite* if and only if $|A| = |\mathsf{N}|$ and *countable* if and only if it is either finite or countably infinite. A set which is not countable is *uncountable*.

PROBLEM 27. Show that $|\mathsf{Z}| = \aleph_0$.

Solution. The set of integers is infinite. To show they are countably infinite, we list them: $0, 1, -1, 2, -2, 3, -3, \ldots$. This list is just $f(1), f(2), f(3), \ldots$ where $f : \mathsf{N} \to \mathsf{Z}$ is defined by

$$f(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ -\frac{1}{2}(n-1) & \text{if } n \text{ is odd,} \end{cases}$$

which is certainly both one-to-one and onto. ∎

PROBLEM 28. Show that $|\mathsf{N} \times \mathsf{N}| = |\mathsf{N}|$.

Solution. The elements of $\mathsf{N} \times \mathsf{N}$ can be listed by the scheme illustrated in Fig 3.4. The arrows indicate the order in which the elements of $\mathsf{N} \times \mathsf{N}$ should be listed—$(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), \ldots$. Wherever the arrows terminate, there is no difficulty in continuing, so each ordered pair acquires a definite position. ∎

WELL ORDERING PRINCIPLE

(Well-Ordering Principle).
Every non-empty subset of natural numbers contains its least element.
Proof:
To prove the weak form of the principle of mathematical induction. The proof is based on contradiction. That is, suppose that we need to prove that "whenever the statement P holds true, the statement Q holds true as well". A proof by contradiction starts with the assumption that "the statement P holds true and the statement Q does not hold true" and tries to arrive at a contradiction to the validity of the statement P being true

Possible Questions

2 Mark questions

1.Define finite and infinite sets
2.Define Complement of a set
3. Prove that if A and B are finite sets, then $n(AUB) = n(A) + n(B) - n(A \cap B)$
4.Define Equivalence relations.
5.Define functions with examples
6.Define composition functions with examples.
5.Define Invertible functions
6.Define one-to-one correspondence with example
7.define cardinality of a set.
8. state the two properities of composition functions
9.Write the various types of Functions.
10.Define domain & co domain of the function.
11.Define range of the function.
12.Define equality of two functions.
13.Define denumerable sets.
14.Define countable set
15.Define Identity Mapping.
16.Define constant mapping

6 Mark questions

1. State and Prove De Moivre's theorem.
2. Let A,B and C be sets then prove that i) $A \cup (B \cup C) = (A \cup B) \cap (A \cup C)$ ii) $A \cup B = A$ iff $A \leq B$ without using venn diagram.
3. State and prove De Morgan's Law
4. If ρ and σ are equivalence relations defined on a set S, Prove that $\rho \cap \sigma$ is an equivalence relation.
5. Show that the following functions are 1-1
   i) $f : R \to R$ given by $f(x)= 5x^2 - 1$
   ii) $f: Z \to E$given by $f(n)=3x^3 - x$
6. If the function $f: R\to R$ is given by $f(x)= \cos x$ and $g: R \to R$ is given by $g(x)=x^3$ find $(g \circ f)(x) and (f \circ g)(x)$ and show that they are not equal.
7. Explain about types of relation with examples.
8. Let A={1,2,3} and f,g,h and s be functions from A to A given by
   f ={ (1,2), (2,3),(3,1) } ; g = { (1,2), (2,1), (3,3) } ;
   h = { (1,1), (2,2), (3,1) } and s = { (1,1), (2,2), (3,3) }. Find $f_o g$, $g_o f$, $f_o h_o g$, $g_o s$, $s_o s$, $f_o s$.

9.Let S={1,2,3,4,5} and T={1,2,3,8,9} and define the functions f : S→ T and g: S → S by f={(1,8), (3,9),(4,3),(2,1),(5,2)} and g={(1,2),(3,1),(2,2),(4,3),(5,2)} ,then find the values of the following $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$.

10. Let f, g and h: R→ R be defined by f(x)=x+2 ,g(x)=$\frac{1}{x^2+1}$ and h(x)=3

Compute i)$h \circ g \circ f(x)$ ii)$g \circ h \circ f(x)$ iii) $g \circ f^{-1} \circ f(x)$.

11.If f: X→Y and A, B are two subsets of Y, then prove that

i)$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

12.For integers a,b define aRb if and only if a – b is divisible by m. Show that R defines an equivalence relation on Z.

13.Let A be the set A={xϵR \ x>0} and define f ,g, h :A→ R by f(x)=$\frac{x}{x+1}$,g(x)=$\frac{1}{x}$,h(x)=x+1 find $g \circ f$,$f \circ g$,$h \circ g \circ f$ and $f \circ g \circ h$.

14. Write about the types of function with example

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Subject: Algebra **Subject Code: 19MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit I

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If A = {1,2,3,4,…} then the set A is	finite	composite	infinite	equality	infinite
If a finite set S has 'n' elements then the power set of S has ____ elements	n	2^n	n-1	n+1	2^n
If A = {1,2,3,4,5}and B = {3,7,9} then A\B =	{1,2,4,5)	{1,2,3,4,5,7,9}	{7,9}	{3}	{1,2,4,5)
If A={a,b,c,d} and B ={f,b,d,g} then A∩B =	{a,b,c}	{a,b,c,d,f}	{b,d}	{f,g,d}	{b,d}
(AUB)' = ……………………	A'UB'	AUB	A'∩B'	A∩B	A'∩B'
AΔB =……………………	(A/B)U(B/A)	(A/B)∩(B/A)	(A/B)U(A/B)'	(B/A)∩(B/A)'	(A/B)U(B/A)
n(A union B)=	n(A)+n(B)	n(A)+n(B)-n(A intersection B)	n(A)-n(B)	n(A)-n(B)+n(AB)	n(A)+n(B)-n(A intersection B)
If f:A→B hence f is called a ……………..	function	form	formula	fuzzy	function
Pictorial representation of sets is called……………………	function	mapping	venn diagram	relation	venn diagram
If the function f is otherwise called as ……………………	limit	mapping	lopping	inverse	mapping
The value of n (φ) = ……………………	1	n	n+1	0	0
A set consisting of single element is called ……………………	null set	universal set	singleton set	disjoint set	singleton set
Power set is denoted by……………………	P(S)	S(A)	n(A)	φ	P(S)

A binary relation R in a set A is said to be……………………if aRb implies bRa∀a,b∈ A	anti-symmetric	transitive	symmetric	reflexive	symmetric
If f:A→B in this set A is called the ……………..of the function f.	domain	co domain	set	element	domain
If f:A→B in this set B is called the ……………..of the function f.	domain	co domain	set	element	co domain
The value of the function f for a and is denoted by ……………………………	a(f)	f(a)	a	f	f(a)
called the ………………of a	B-image	a-image	A-image	f-image	f-image
The element a may be referred to as the …………………..of f(a)	f-image	pre-image	domain	codomain	pre-image
The …………. of a function as the image of its domain	domain	range	co domain	image	range
The range of a function as the…………… of its domain	range	domain	image	preimage	image
The range of a function as the image of its ……………..	co domain	image	domain	range	domain
Let f be a mapping of A to B,Each element of A has a …………. and each element in B need not be appear as the image of an element in A.	unique preimage	unique image	unique zero	unique range	unique image
Let f be a mapping of A to B,Each element of …... has a unique image and each element in B need not be appear as the image of an element in A.	A	B	f	f(A)	A
Let f be a mapping of A to B,Each element of A has a unique image and each element in………. need not be appear as the image of an element in A.	A	B	f	f(A)	B
Let f be a mapping of A to B,Each element of A has a unique image and each element in B need not be appear as the ………… of an element in A.	domain	range	co domain	image	image

One-to-one mapping is also sometimes known as…………………	injection	bijection	surjection	imjection	injection
A mapping f:A→B is said to be ………….. if different elements in A have different f-images in B	zero	one-one	onto	into	one-one
A mapping f:A→B is said to be 1-1 if ……………….elements in A have different f-images in B	same	different	not equal	one	different
A mapping f:A→B is said to be 1-1 if different elements in A have different ……………. in B	pre images	f-images	B-images	A-images	f-images
In one-one mappings an element in B has only………………preimage in A	zero	one	two	three	one
In …………...mappings an element in B has only one preimage in A	one-one	onto	into	one-oneonto	one-one
One-one onto mapping is also sometimes known as…………………	injection	bijection	surjection	imjection	bijection
A mapping f:A→B is said to be ………….. if different elements in A have same f-images in B	one-one	onto	into	many one	many one
In many-one mappings some elements in B has more than………………preimage in A	zero	one	two	three	one
In many-one mappings some elements in B has ………….. one preimage in A	equal	more than	less than	only	more than
Two sets A and B are said to have the same number of elements iff a one-one mapping of A onto B exists, such sets are said to be ………………..	equivalent	merely equivalent	cardinally equivalent	notequivalent	cardinally equivalent
Two sets A and B are said to have the same number of elements iff a ………... mapping of A onto B exists, such sets are said to be cardinally equivalent	one-one	many one	onto	into	one-one
Two sets A and B are said to have the same number of elements iff a one-one mapping of A ………. B exists, such sets are said to be cardinally equivalent	one-one	many one	onto	into	onto

Two sets A and B are said to have the ……..number of elements iff a one-one mapping of A onto B exists, such sets are said to be cardinally equivalent	same	different	zero	finite	same
Cardinally eqivalent can be written as……………	A+B	A-B	A~B	A/B	A~B
Cardinally eqivalent sets are to have the ………... cardinal number.	zero	one	same	finite	same
Cardinally eqivalent sets are to have the same ……………. number.	rational	complex	real	cardinal	cardinal
If $f:A\rightarrow B$ is one-one onto, then $f^{-1}:B\rightarrow A$ .the mapping f^{-1} is called the …………….mapping of the mapping of f.	integral	inverse	invert	reverse	inverse
Only one-one and onto mapping posses……………..mappings.	integral	inverse	invert	reverse	inverse
Only ………….. mapping posses inverse mappings.	one-one and into	one-one	one-one and many one	one-one and onto	one-one and onto
If $f:A\rightarrow B$ is one-one onto, then $f^{-1}:B\rightarrow A$ is also …………………	one-one and into	one-one	one-one and many one	one-one and onto	one-one and onto
If $f:A\rightarrow B$ is one-one onto, then the inverse mapping of f is ……………...	zero	unique	different	same	unique
If $f:X\rightarrow Y$ and $g:Y\rightarrow Z$then the ………………..of the function f and g demoted by $(g_o f):X\rightarrow Z$.	inverse	composite	different	one-one	composite
If $f:X\rightarrow Y$ and $g:Y\rightarrow Z$then the composite of the function f and g demoted by ……………	$(f_o g):X\rightarrow Z$.	$(f_o g):X\rightarrow Y$.	$(g_o f):y\rightarrow Z$.	$(g_o f):X\rightarrow Z$.	$(g_o f):X\rightarrow Z$.
In general $g_o f$ ……………. f_o g	equal	notequal	less than	more than	notequal
If xRx ,forevery x϶A since every triangle is congruent to it self.Thus R is ……………	reflexive	symmetic	transitive	anti-symmetric	reflexive
If xRy and yRz→ x Rz,since if triangle x is congruent to y and triangley is congrugent to z then,trainglex is congruent to z.Then R is ……………	reflexive	symmetic	transitive	anti-symmetric	transitive

If xRy →y R z since if triangle x is congruent to y and triangle y is congrugent to x.Then R is ……………	reflexive	symmetic	transitive	anti-symmetric	symmetic
R is an ………………..relation	one-one	onto	equivalence	equal	equivalence
If a=16 and b=5 then find q and r in division algorithm is…………………….	q=3,r=1	q=1,r=3	q=4,r=2	q=1,r=1	q=3,r=1
In general $g_o f$ ……………$f_o g$	not equal	equal	less than	more than	not equal

UNIT-II

Division algorithm- Divisibility and Euclidean algorithm- Congruence relation between integers- Principles of Mathematical Induction- Statement of Fundamental Theorem of Arithmetic.

KAHE

THE INTEGERS

DIVISIBILITY THEORY IN THE INTEGERS

Well- Ordering Principle

Every non empty set *S* of nonnegative integers contains a least element. That is, there exists some integer *a* in *S* such that $a \le b$ for all *b* in *S*.

THE DIVISION ALGORITHM

Division Algorithm, the result is familiar to most of us roughly, it asserts that an integer *a* can be ”divided” by a positive integer *b* in such a way that the remainder is smaller than *b*. The exact statement of this fact is Theorem 1.:

Theorem 1. *Given integersaandb,withb* > 0*,there exist unique integersq and r satisfying*

$$a = qb + r \qquad 0 \le r < b$$

The integers q and r are called, respectively, the quotient and remainder in the division of a by b.

Proof. Let *a* and *b* be integers with $b > 0$ and consider the set

$$S = \{a - xb : x is an integer;\ a - xb \ge 0\}.$$

Claim: The set *S* is nonempty

It su□ces to find a value *x* which making $a-xb$ nonnegative. Since $b \ge 1$, we have $|a|b \ge |a|$ and so, $a-(-|a|)b = a + |a|b \ge a + |a| \ge 0$. For the choice $x = -|a|$, then $a - xb$ lies in *S*. Therefore *S* is nonempty, hence the claim.Therefore by Well-Ordering Principle, *S* contains a small integer, say *r*. By
the definition of *S* there exists an integer *q* satisfying

$$r = a - qb \qquad 0 \le r.$$

Claim: $r < b$

Suppose $r \ge b$. Then we have

$$a - (q + 1)b = (a - qb) - b = r - b \ge 0.$$

This implies that, $a - (q + 1)b \in S$. But $a - (q + 1)b = r - b < r$, since $b > 0$, leading to a contradiction of the choice of *r* as the smallest member of *S*. Hence, $r < b$, hence the claim.

Next we have to show that the uniqueness of q and r. Suppose that a as two representations of the desired form, say,

$$a = qb + r = q'b + r',$$

where $0 \leq r < b$ and $0 \leq r' < b$. Then $(r' - r) = b(q - q')$. Taking modulus on both sides,

$$|(r' - r)| = |b(q - q')| = |b||(q - q')| = b|(q - q')|.$$

But we have $-b < -r \leq 0$ and $0 \leq r' < b$, upon adding these inequalities we obtain $-b < r' - r < b$. This implies $b|(q - q')| < b$, which yields $0 \leq |q - q'| < 1$. Because $|q - q'|$ is a nonnegative integer, the only possibility isthat $|q - q'| = 0$, hence, $q = q'$. This implies $|r' - r| = 0$, that is, $r = r'$. Hence the proof. □

Corollary 1. *Ifaandbare integers, withb* $\neq 0$*,then there exists integersqand r such that*

$$a = qb + r \qquad 0 \leq r < |b|.$$

Proof. It is enough to consider the case in which b is negative. Then $|b| > 0$, and Theorem 1. produces unique integers q' and r for which

$$a = q'|b| + r \qquad 0 \leq r < |b|.$$

Noting that $|b| = -b$, we may take $q = -q'$ to arrive at $a = qb + r$, with $0 \leq r < |b|$.
Application of the Division Algorithm

1. Square of any integer is either of the form $4k$ or $4k + 1$. That is, the square of integer leaves the remainder 0 or 1 upon division by 4.

   Solution: Let a be any integer. If a is even, we can let $a = 2n$, n is aninteger, then $a^2 = (2n)^2 = 4n^2 = 4k$. If a is odd, we can let $a = 2n+1$, n is an integer, then $a^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1 = 4k+1$.

2. The square of any odd integer is of the form $8k + 1$.

   Solution: Let a be an integer and let $b = 4$, then by division algorithm

   a is representable as one of the four forms: $4q$, $4q + 1$, $4q + 2$, $4q + 3$. Inthis representation, only those integers of the forms $4q + 1$ and $4q + 3$ are odd. If $a = 4q + 1$, then

$$a^2 = (4q + 1)^2 = 16q^2 + 8q + 1 = 8(2q^2 + q) + 1 = 8k + 1.$$

If $a = 4q + 3$, then

$a^2 = (4q+3)^2 = 16q^2+24q+9 = 16q^2+24q+8+1 = 8(2q^2+3q+1)+1 = 8k+1$.

3. For all integer $a \geq 1$, $\frac{a(a^2+2)}{3}$ is an integer.

Solution: Let $a \geq 1$ be an integer. According to division algorithm, a is of the form $3q$, $3q + 1$ or $3q + 2$. If $a = 3q$, then

$$\frac{3q((3q)^2+2)}{3} = 9q^3 + 2q,$$

which is clearly an integer. Similarly we can prove other two cases also.

THE GREATEST COMMON DIVISOR

Definition 1. *An integer b is said to be divisible by an integer a ≠ 0, in symbols a|b, if there exists some integer c such that b = ac. We write a ∤ b to indicate that b is not divisible by a.*

Thus, for example, −22 is divisible by 11, because −22 = 11(−2). How-ever, 22 is not divisible by 3; for there is no integer c that makes the statement $22 = 3c$ true.

There is other language for expressing the divisibility relation $a|b$. We could say that a is a divisor of b, that a is a factor of b, or that b is a multiple of a. Notice that in Definition 1 there is a restriction on the divisor a: Whenever the notation $a|b$ is employed, it is understood that a is different from zero.
If a is a divisor of b, then b is also divisible by $-a$ (indeed, $b = ac$ implies that $b = (-a)(-c)$), so that the divisors of an integer always occur in pairs.
To find all the divisors of a given integer, it is sufficient to obtain the positive divisors and then adjoin to them the corresponding negative integers. For this reason, we shall usually limit ourselves to a consideration of positive divisors. It will be helpful to list some immediate consequences of Definition 1.

Theorem 2. *For integers a, b, c, the following hold:*

1. $a|0$, $1|a$, $a|a$.
2. $a|1$ *if and only if* $a = \pm1$.
3. *If* $a|b$ *and* $c|d$, *then* $ac|bd$.

4. *If* $a|b$ *and* $b|c$*, then* $a|c$.

5. $a|b$ *and* $b|a$ *if and only if* $a = \pm b$.

6. *If* $a|b$ *and* $b \neq 0$*, then* $|a| \leq |b|$.

7. *If* $a|b$ *and* $a|c$*, then* $a|(bx + cy)$ *for arbitrary integers* x *and* y.

Proof. 1. Since $0 = a.0$, $a|0$. Since $a = 1.a$, $1|a$. Since $a = a.1$, $a|a$.

2. We have $a|1$ if and only if $1 = a.c$ for some c, this is if and only if $a = \pm 1$.

3. Clear from definition.

4. Clear from definition.

5. Clear from definition.

6. If $a|b$, then there exists an integer c such that $b = ac$; also, $b \neq 0$ implies that $c \neq 0$. Upon taking absolute values, we get $|b| = |ac| = |a||c|$. Because $c \neq 0$, it follows that $|c| \geq 1$, whence $|b| = |a||c| \geq |a|$.

7. The relations $a|b$ and $a|c$ ensure that $b = ar$ and $c = as$ for suitable integers r and s. But then whatever the choice of x and y, $bx + cy = arx + asy = a(rx + sy)$. Because $rx + sy$ is an integer, this says that $a|(bx + cy)$, as desired. □

Definition 2. *Let* a *and* b *be given integers, with at least one of them different from zero. The greatest common divisor of* a *and* b*, denoted by* $gcd(a, b)$*, is the positive integer* d *satisfying the following:*

(i) $d|a$ *and* $d|b$.

(ii) *If* $c|a$ *and* $c|b$*, then* $c \leq d$.

Example: The positive divisors of -12 are $1, 2, 3, 4, 6, 12$, whereas those of 30 are $1, 2, 3, 5, 6, 10, 15, 30$; hence, the positive common divisors of -12 and 30 are $1, 2, 3, 6$. Because 6 is the largest of these integers, it follows that $gcd(-12, 30) = 6$. In the same way, we can show that $gcd(-5, 5) = 5$, $gcd(8, 17) = 1$, $gcd(-8, -36) = 4$.

Theorem 3. *Given integers* a *and* b*, not both of which are zero, there exist integers* x *and* y *such that*

$$gcd(a, b) = ax + by.$$

Proof. Consider the set S of all positive linear combinations of a and b :

$$S = \{au + bv : au + bv > 0; u, v \text{ integers}\}.$$

Since, if $a \neq 0$ then $|a| = au + b.0 \in S$, where $u = 1$, if $a > 0$; $u = -1$, if $a < 0$, S is nonempty. Therefore by the Well-Ordering Principle, S must containa smallest element, say d. Thus, from the very definition of S, there exist integers x and y for which $d = ax + by$. Claim: $d = gcd(a, b)$

By using the Division Algorithm, we can obtain integers q and r such that $a = qd + r$, where $0 \leq r < d$. Then r can be written in the form:

$$\begin{aligned} r &= a - qd \\ &= a - q(ax + by) \\ &= a(1 - qx) + b(-qy). \end{aligned}$$

If r were positive, then this representation would imply that r is a member of S, contradicting the fact that d is the least integer in S (recall that $r < d$). Therefore, $r = 0$, and so $a = qd$, or equivalently $d|a$. By similar reasoning, $d|b$, this implies d is a common divisor of a and b.

Now if c is an arbitrary positive common divisor of the integers a and b, then part (7) of Theorem 2 allows us to conclude that $c|(ax + by)$; that is, $c|d$. By part (6) of the same theorem, $c = |c| \leq |d| = d$, so that d is greater than every positive common divisor of a and b. Hence $d = gcd(a, b)$. Hence the claim. Therefore $gcd(a, b) = ax + by$. □

Corollary 2. *Ifaandbare given integers, not both zero, then the set*

$$T = ax + by : x, y \text{ are integers}$$

is precisely the set of all multiples of $d = gcd(a, b)$.

Proof. Because $d|a$ and $d|b$, we know that $d|(ax + by)$ for all integers x, y. Thus, every member of T is a multiple of d. Conversely, d may be written as $d = ax_0 + by_0$ for suitable integers x_0 and y_0, so that any multiple nd of d isof the form

$$nd = n(ax_0 + by_0) = a(nx_0) + b(ny_0).$$

Hence, nd is a linear combination of a and b, and, by definition, lies in T. □

De nition 3. *Two integersaandb,not both of which are zero, are said tobe relatively prime whenever* $gcd(a, b) = 1$.

Theorem 4. *Letaandbbe integers, not both zero. Thenaandbarerelatively prime if and only if there exist integers x and y such that* $1 = ax + by$.

Proof. If a and b are relatively prime so that $gcd(a, b) = 1$, then Theorem 3guarantees the existence of integers x and y satisfying $1 = ax+by$. Conversely, suppose that $1 = ax + by$ for some choice of x and y, and that $d = gcd(a, b)$. Because $d|a$ and $d|b$, Theorem 2 yields $d|(ax+by)$, or $d|1$. This implies $d = \pm 1$. But d is a positive integer, $d = 1$. That is a and b are relatively prime. □

Corollary 3. *Ifgcd*$(a, b) = d$,*thengcd*$(a/d, b/d) = 1$.

Proof. Since $d|a$ and $d|b$, a/d and b/d are integers. We have, if $gcd(a, b) = d$,then there exists x and y such that $d = ax + by$. Upon dividing each side of this equation by d, we obtain the expression

$$1 = (a/d)x + (b/d)y.$$

Because a/d and b/d are integers, a/d and b/d are relatively prime. Therefore $gcd(a/d, b/d) = 1$. □

Corollary 4. *Ifa*$|$*candb*$|$*c,withgcd*$(a, b) = 1$,*thenab*$|c$.

Proof. Since $a|c$ and $b|c$, we can find integers r and s such that $c = ar = bs$.
Given that $gcd(a, b) = 1$, so there exists integers x and y such that $1 = ax+by$.

Multiplying the last equation by c, we get,

$$c = c1 = c(ax + by) = acx + bcy.$$

If the appropriate substitutions are now made on the right-hand side, then

$$c = a(bs)x + b(ar)y = ab(sx + ry).$$

This implies, $ab|c$.
Theorem 5. *(Euclid's lemma.) Ifa*$|$*bc, withgcd*$(a, b) = 1$,*thena*$|c$.

Proof. Since $gcd(a, b) = 1$, we have $1 = ax + by$ for some integers x and y.
Multiplication of this equation by c produces

$$c = 1c = (ax + by)c = acx + bcy.$$

Since $a|bc$ and $a|ac$, we have $a|acx + bcy$. This implies $a|c$. □

Note: If a and b are not relatively prime, then the conclusion of Euclid's lemma may fail to hold. For example: 6|9.4 but 6 - 9 and 6 - 4.

Theorem 6. *Leta, bbe integers, not both zero. For a positive integerd,* $d = gcd(a, b)$ *if and only if*

(i) $d|a$ *and* $d|b$.

(ii) *Whenever* $c|a$ *and* $c|b$, *then* $c|d$.

Proof. Suppose that $d = gcd(a, b)$. Certainly, $d|a$ and $d|b$, so that (i) holds.By Theorem 3, d is expressible as $d = ax + by$ for some integers x, y. Thus, if $c|a$ and $c|b$, then $c|(ax + by)$, or rather $c|d$. This implies, condition (ii) holds.Conversely, let d be any positive integer satisfying the stated conditions (i) and (ii). Given any common divisor c of a and b, we have $c|d$ from hypothesis (ii). This implies that $d \geq c$, and consequently d is the greatest common divisor of a and b. □

THE EUCLIDEAN ALGORITHM

Lemma 1. *If* $a = qb + r$, *then* $gcd(a, b) = gcd(b, r)$.

Proof. If $d = gcd(a, b)$, then the relations $d|a$ and $d|b$ together imply that $d|(a - qb)$, or $d|r$. Thus, d is a common divisor of both b and r. On the otherhand, if c is an arbitrary common divisor of b and r, then $c|(qb + r)$, whence $c|a$. This makes c a common divisor of a and b, so that $c \leq d$. It now followsfrom the definition of $gcd(b, r)$ that $d = gcd(b, r)$. □

The Euclidean algorithm

The Euclidean Algorithm may be described as follows: Let a and b be two integers whose greatest common divisor is desired. Because $gcd(|a|,|b|) = gcd(a, b)$,, with out loss of generality we may assume $a \geq b > 0$. The firststep is to apply the Division Algorithm to a and b to get

$$a = q_1 b + r_1 \quad 0 \leq r_1 < b.$$

If it happens that $r_1 = 0$, then $b|a$ and $gcd(a, b) = b$. When $r_1 \neq 0$, divide b by r_1 to produce integers q_2 and r_2 satisfying

$$b = q_2 r_1 + r_2 \quad 0 \leq r_2 < r_1.$$

If $r_2 = 0$, then we stop; otherwise, proceed as before to obtain

$$r_1 = q_3 r_2 + r_3 \quad 0 \leq r_3 < r_2.$$

This division process continues until some zero remainder appears, say, at the $(n + l)^{th}$ stage where r_{n-1} is divided by r_n (a zero remainder occurs sooner or later because the decreasing sequence $b > r_1 > r_2 > \cdots \geq 0$ cannot contain more than b integers). The result is the following system of equations:

$$a = q_1 b + r_1 \qquad 0 \leq r_1 < b$$

$$b = q_2 r_1 + r_2 \qquad 0 \leq r_2 < r_1$$

$$r_1 = q_3 r_2 + r_3 \qquad 0 \leq r_3 < r_2$$

$$\vdots$$

$$r_{n-2} = q_n r_{n-1} + r_n \qquad 0 \leq r_n < r_{n-1}$$

$$r_{n-1} = q_{n+1} r_n + 0.$$

By Lemma 1,

$$gcd(a, b) = gcd(b, r_1) = = gcd(r_{n-1}, r_n) = gcd(r_n, 0) = r_n.$$

Note:Start with the next-to-last equation arising from the Euclidean Algo-rithm, we can determine x and y such that $gcd(a, b) = ax + by$.

Example: Let us see how the Euclidean Algorithm works in a concrete case by calculating, say, $gcd(12378, 3054)$. The appropriate applications of the Division Algorithm produce the equations

$$12378 = 4.3054 + 162$$

$$3054 = 18.162 + 138$$

$$162 = 1.138 + 24$$

$$138 = 5.24 + 18$$

$$24 = 1.18 + 6$$

$$18 = 3.6 + 0$$

This tells us that the last nonzero remainder appearing in these equations, namely, the integer 6, is the greatest common divisor of 12378 and 3054:

$$6 = gcd(12378, 3054).$$

To represent 6 as a linear combination of the integers 12378 and 3054, we start with the next-to-last of the displayed equations and successively eliminate the remainders 18, 24, 138, and 162:

$$\begin{aligned} 6 &= 24 - 18 \\ &= 24 - (138 - 5.24) \\ &= 6.24 - 138 \\ &= 6(162 - 138) - 138 \\ &= 6.162 - 7.138 \\ &= 6.162 - 7(3054 - 18.162) \\ &= 132.162 - 7.3054 \\ &= 132(12378 - 4.3054) - 7.3054 \\ &= 132.12378 + (-535)3054 \end{aligned}$$

Thus, we have

$$6 = gcd(12378, 3054) = 12378x + 3054y,$$

where $x = 132$ and $y = -535$. Note that this is not the only way to express the integer 6 as a linear combination of 12378 and 3054; among other possibilities, we could add and subtract 3054.12378 to get

$$6 = (132 + 3054)12378 + (-535 - 12378)3054 = 3186.12378 + (-12913)3054.$$

Theorem 7. *If* $k > 0$, *then* $gcd(ka, kb) = k\, gcd(a, b)$.

Proof. If each of the equations appearing in the Euclidean Algorithm for a and b, multiplied by k, we obtain

$$ak = q_1(bk) + r_1k \qquad 0 \le r_1k < bk$$

$$bk = q_2(r_1k) + r_2k \qquad 0 \le r_2k < r_1k$$

$$\vdots$$

$$r_{n-2}k = q_n(r_{n-1}k) + r_nk \qquad 0 \le r_nk < r_{n-1}k$$

$$r_{n-1}k = q_{n+1}(r_nk) + 0.$$

But this is clearly the Euclidean Algorithm applied to the integers ak and bk, so that their greatest common divisor is the last nonzero remainder r_nk; that is,

$$gcd(ka, kb) = r_nk = k\, gcd(a, b),$$

Hence the theorem. □

Corollary 5. *For any integer* $k \neq 0$, $gcd(ka, kb) = |k| gcd(a, b)$.

Proof. We already have, if $k > 0$, then $gcd(ka, kb) = k\, gcd(a, b)$. Thereforeit su□ces to consider the case in which $k < 0$. Then $-k = |k| > 0$ and, by Theorem 7,

$$\begin{aligned} gcd(ak, bk) &= gcd(-ak, -bk) \\ &= gcd(a|k|, b|k|) \\ &= |k|\, gcd(a, b). \end{aligned}$$

Hence the result. □

De nition 4. *The least common multiple of two nonzero integersaandb,denoted by* $lcm(a, b)$, *is the positive integer m satisfying the following:*

(i) $a|m$ *and* $b|m$.

(ii) *If* $a|c$ *and* $b|c$, *with* $c > 0$, *then* $m \le c$.

As an example, the positive common multiples of the integers -12 and 30 are 60, 120, 180, ... hence, $lcm(-12, 30) = 60$.

Theorem 8. *For positive integersaandb*

$$gcd(a, b)\, lcm(a, b) = ab.$$

Proof. Let $d = gcd(a, b)$ and let $m = ab/d$, then $m > 0$.

Claim: $m = lcm(a, b)$

Since d is the common divisor of a and b we have $a = dr$, $b = ds$ for in-tegers r and s. Then $m = as = rb$. This implies, m a (positive) common multiple of a and b.

Now let c be any positive integer that is a common multiple of a and b, then $c = au = bv$ for some integers u and v. As we know, there exist integers x and y satisfying d =ax+ by. In consequence,

$$\frac{c}{m} = \frac{cd}{ab} = \frac{c(ax + by)}{ab} = \left(\frac{c}{b}\right)x + \left(\frac{c}{a}\right)y = vx + uy.$$

This equation states that $m|c$, this implies, $m \le c$. By the definition of least common multiple, we have $m = lcm(a, b)$. Hence the claim. Therefore $gcd(a, b)\, lcm(a, b) = ab$. □

Corollary 6. *For any choice of positive integersaandb,*$lcm(a, b) = ab$ *ifand only if* $gcd(a, b) = 1$.

Definition 5. *Ifa, b, c,are three integers, not all zero,*$gcd(a, b, c)$ *is definedto be the positive integer d having the following properties:*

(i) *d is a divisor of each of a, b, c.*

(ii) *If e divides the integers a, b, c, then* $e \le d$.

For example $gcd(39, 42, 54) = 3$ and $gcd(49, 210, 350) = 7$.

Example: Consider the linear Diophantine equation

$$172x + 20y = 1000$$

Applying the Euclidean's Algorithm to the evaluation of $gcd(172, 20)$, we find that

$$\begin{aligned} 172 &= 8.20 + 12 \\ 20 &= 1.12 + 8 \\ 12 &= 1.8 + 4 \\ 8 &= 24, \end{aligned}$$

whence $gcd(172, 20) = 4$. Because $4|1000$, a solution to this equation exists. To obtain the integer 4 as a linear combination of 172 and 20, we work backward through the previous calculations, as follows:

$$\begin{aligned} 4 &= 12 - 8 \\ &= 12 - (20 - 12) \\ &= 212 - 20 \\ &= 2(172 - 8.20) - 20 \\ &= 2.172 + (-17)20 \end{aligned}$$

Upon multiplying this relation by 250, we arrive at

$$\begin{aligned} 1000 &= 250.4 \\ &= 250(2.172 + (-17)20) \\ &= 500.172 + (-4250)20, \end{aligned}$$

so that $x = 500$ and $y = -4250$ provide one solution to the Diophantine equation in question. All other solutions are expressed by

$$x = 500 + (20/4)t = 500 + 5t$$

$$y = -4250 - (172/4)t = -4250 - 43t,$$

for some integer t.

If we want to find positive solution,if any happen to exist. For this, t must be chosen to satisfy simultaneously the inequalities

$$5t + 500 > 0 \qquad -43t - 4250 > 0$$

or

$$-98\frac{36}{43} > t > -100.$$

Because t must be an integer, we are forced to conclude that $t = -99$. Thus, our Diophantine equation has a unique positive solution $x = 5$, $y = 7$ corresponding to the value $t = -99$.

THE FUNDAMENTAL THEOREM OF ARITHMETIC

Definition 6. *An integer* $p > 1$ *is called a prime number, or simply a prime, if its only positive divisors are* 1 *and* p. *An integer greater than* 1 *that is not a prime is termed composite.*

Among the first ten positive integers, 2, 3, 5, 7 are primes and 4, 6, 8, 9, 10 are composite numbers. Note that the integer 2 is the only even prime, and according to our definition the integer 1 plays a special role, being neither prime nor composite.

Theorem 1. *If* p *is a prime and* $p|ab$, *then* $p|a$ *or* $p|b$.

Proof. If $p|a$, then we need go no further, so let us assume that $p \nmid a$. Becausethe only positive divisors of p are 1 and p itself, this implies that $gcd(p, a) = 1$. Hence, by Euclid's lemma, we get $p|b$. □

Corollary 8. If p *is a prime and* $p|a_1a_2 \cdots a_n$, *then* $p|a_k$ *for some k, where* $1 \le k \le n$.

Proof. We proceed by induction on n, the number of factors. When $n = 1$,the stated conclusion obviously holds; whereas when $n = 2$, the result is the content of Theorem 10. Suppose, as the induction hypothesis, that $n > 2$ and that whenever p divides a product of less than n factors, it divides at least one of the factors. Now let $p|a_1a_2 \cdots a_n$. From Theorem 10, either $p|a_n$ or $p|a_1a_2 \cdots a_{n-1}$ If $p|a_n$, then we are through. As regards the casewhere $p|a_1a_2 \cdots a_{n-1}$, the induction hypothesis ensures that $p|a_k$ for some choice of k, with $1 \le k \le n - 1$. In any event, p divides one of the integers $a_1, a_2, \cdots, a_n$.

Theorem 2. *(Fundamental Theorem of Arithmetic.) Every positive integer* $n > 1$ *can be expressed as a product of primes; this representation is unique, apart from the order in which the factors occur.*

Proof. Either n is a prime, there is nothing to prove. If n is composite,then there exists an integer d satisfying $d|n$ and $1 < d < n$. Among all such integers d, choose p_1 to be the smallest (this is possible by the Well-Ordering Principle). Then P_1 must be a prime number. Otherwise it too would have
a divisor q with $1 < q < p_1$; but then $q|p_1$ and $p_1|n$ imply that $q|n$, which contradicts the choice of p_1 as the smallest positive divisor, not equal to 1, of n. We therefore may write $n = p_1n_1$, where p_1 is prime and $1 < n_1 < n$.If n_1 happens to be a prime, then we have our representation. In the contrarycase, the argument is repeated to produce a second prime number p_2 such that $n_1 = p_2n_2$; that is,

$$n = p_1P_2n_2 \qquad 1 < n_2 < n_1.$$

If n_2 is a prime,then it is not necessary to go further. Otherwise, write $n_2 = p_3n_3$, with p_3 a prime:

$$n = p_1P_2p_3n_3 \quad 1 < n_3 < n_2.$$

The decreasing sequence $n > n_1 > n_2 > \cdots > 1$ cannot continue indefinitely, so that after a finite number of steps n_{k-1} is a prime, call it, p_k. This leads to the prime factorization

$$n = p_1 p_2 \cdots p_k.$$

To establish the second part of the proof-the uniqueness of the prime factor-ization, let us suppose that the integer n can be represented as a product of primes in two ways, say,

$$n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s \quad r \leq s,$$

where the p_i and q_j are all primes, written in increasing magnitude so that

$$p_1 \leq p_2 \leq \cdots \leq p_r \qquad q_1 \leq q_2 \leq \cdots \leq q_s.$$

Because $p_1 | q_1 q_2 \cdots q_s$, Corollary 9 tells us that $p_1 = q_k$ for some k; but then $p_1 \geq q_1$. Similar reasoning gives $q_1 \geq p_1$, whence $p_1 = q_1$. We may cancelthis common factor and obtain

$$p_2 p_3 \cdots p_r = q_2 q_3 \cdots q_s.$$

Now repeat the process to get $p_2 = q_2$ and, in turn,

$$p_3 p_4 \cdots p_r = q_3 q_4 \cdots q_s.$$

Continue in this fashion. If the inequality $r < s$ were to hold, we would eventually arrive at

$$1 = q_{r+1} q_{r+2} \cdots q_s,$$

which is absurd, because each $q_j > 1$. Hence, $r = s$ and

$$p_1 = q_1,\ p_2 = q_2,\ \cdots,\ p_r = q_r,$$

making the two factorizations of n identical. The proof is now complete.

THE THEORY OFCONGRUENCES

Definition 1. *Letnbe a fixed positive integer. Two integersaandbaresaid to be congruent modulo n, symbolized by*

$$a \equiv b (mod n)$$

if n divides the di□erence $a - b$; that is, provided that $a - b = kn$ for some integer k.

Theorem 1. For arbitrary integers a and b, $a \equiv b(mod n)$ if and only if a and b leave the same nonnegative remainder when divided by n.

Proof. Suppose $a \equiv b(mod n)$, so that $a = b + kn$ for some integer k. Upondivision by n, b leaves a certain remainder r; that is, $b = qn + r$, where $0 \le r < n$. Therefore,

$$a = b + kn = (qn + r) + kn = (q + k)n + r$$

which indicates that a has the same remainder as b.
On the other hand, suppose we can write $a = q_1 n + r$ and $b = q_2 n + r$, with the same remainder r ($0 \le r < n$). Then

$$a - b = (q_1 n + r) - (q_2 n + r) = (q_1 - q_2)n,$$

whence $n | a-b$. That is, $a \equiv b(mod n)$. □

Theorem 2. *Let $n > 1$ be fixed and a, b, c, d be arbitrary integers. Then the following properties hold:*

1. *$a \equiv a(mod n)$.*
2. *If $a \equiv b(mod n)$, then $b \equiv a(mod n)$.*
3. *If $a \equiv b(mod n)$ and $b \equiv c(mod n)$, then $a \equiv c(mod n)$.*
4. *If $a \equiv b(mod n)$ and $c \equiv d(mod n)$, then $a + c \equiv b + d(mod n)$ and $ac \equiv bd(mod n)$.*
5. *If $a \equiv b(mod n)$, then $a + c \equiv b + c(mod n)$ and $ac \equiv bc(mod n)$.*
6. *If $a \equiv b(mod n)$, then $a^k \equiv b^k(mod n)$ for any positive integer k.*

Problem 1: Show that $41 | 2^{20} - 1$.
Solution: We have

$$2^5 \equiv -9(mod\ 41).$$

Therefore

$$(2^5)^4 \equiv (-9)^4(mod\ 41).$$

This implies that

$$2^{20} \equiv (-9)^4(mod\ 41).$$

But we have $(-9)^4 = 81.81$ and $81 \equiv -1(mod\ 41)$. Therefore

$$2^{20} \equiv (-1)(-1)(mod\ 41).$$

This implies $41 | 2^{20} - 1$.
Problem 2: Find the remainder obtained upon dividing the sum 1! + 2! +

$$3! + 4! + \cdots + 99! + 100!$$

by 12.

Solution: We have $4! \equiv 24 \equiv 0(mod\ 12)$; thus, for $k \geq 4$,

$$k! \equiv 4!.5.6 \cdots k \equiv 0.5..6 \cdots k \equiv 0(mod\ 12).$$

Therefore

$$1! + 2! + 3! + 4! + \cdots + 100! \equiv 1! + 2! + 3! + 0 + \cdots + 0 \equiv 9(mod\ 12).$$

The remainder 9.

Theorem 3. *If* $ca \equiv cb(mod\ n)$*, then* $a \equiv b(mod\ n/d)$*, where* $d = gcd(c, n)$
Proof. By hypothesis, we can write

$$c(a - b) = ca - cb = kn, \qquad (3.1)$$

for some integer k. Knowing that $gcd(c, n) = d$, there exist relatively prime integers r and s satisfying $c = dr$, $n = ds$. When these values are substituted in Eq. 3.1 and the common factor d canceled, the net result is

$$r(a - b) = ks.$$

Hence, $s|r(a-b)$ and $gcd(r, s) = 1$. Euclid's lemma yields $s|(a-b)$, which implies $a \equiv b(mod\ s)$; in other words, $a \equiv b(mod\ n/d)$. □

Corollary 12. *If* $ca \equiv cb(mod\ n)$ *and* $gcd(c, n) = 1$*, then* $a \equiv b(mod\ n)$.

Corollary 13. *If* $ca \equiv cb(mod\ p)$ *and* p - c*, where* p *is a prime number, then* $a \equiv b(mod\ p)$.

Proof. The conditions p - c and p a prime imply that $gcd(c, p) = 1$. Then byCorollary 12, $a \equiv b(mod\ p)$.

PRINCIPLE OF MATHEMATICAL INDUCTION

The principle of mathematical induction

Let P(n) be a given statement involving the natural number n such that

3. The statement is true for $n = 1$, i.e., P(1) is true (or true for any fixed natural number) and
4. If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of P(k) implies the truth of P(k + 1). Then P(n) is true for all natural numbers n.

Solved Examples

Short Answer Type

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all $n \in \mathbf{N}$, that :

Example 1 $1 + 3 + 5 + ... + (2n - 1) = n2$

Solution Let the given statement P(n) be defined as P(n) : 1 + 3 + 5 +...+ (2n– 1) =n^2, for n ∈**N**. Note that P(1) is true, since
P(1) : 1 = 1^2
Assume that P(k) is true for some k∈**N**, i.e., P(k) : 1 + 3 + 5 + ... + (2k – 1) = k 2
Now, to prove that P(k + 1) is true, we have
1 + 3 + 5 + ... + (2k – 1) + (2k + 1)
= k^2 + (2k + 1) (Why?)
= k2 + 2k + 1 = (k + 1)2

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all n∈**N**.

Example 2 2^{2n}– 1 is divisible by 3.

Solution Let the statement P(n) given as

P(n) : 22n – 1 is divisible by 3, for every natural number n.

We observe that P(1) is true, since

2^2 – 1 = 4 – 1 = 3.1 is divisible by 3.

Assume that P(n) is true for some natural number k, i.e., P(k): 2^{2k}– 1 is divisible by 3, i.e., 2^{2k} – 1 = 3q, where q∈**N** Now, to prove that P(k + 1) is true, we have
P(k + 1) : 22(k+1) – 1 = 22k + 2 – 1 = 22k . 22 – 1
= 2^{2k} . 4 – 1 = 3.2^{2k} + (2^{2k} – 1)

= 3.2^{2k} + 3q
= 3 (2^{2k} + q) = 3m, where m∈**N**
Thus P(k + 1) is true, whenever P(k) is true.
Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers n.

Example 3 2n+ 1 < 2^n, for all natual numbers$n$ε3.

Solution Let P(n) be the given statement, i.e., P(n) : (2n+ 1) < $2^n$for all naturalnumbers, nε 3. We observe that P(3) is true, since
2.3 + 1 = 7 < 8 = 23

Assume that P(n) is true for some natural number k, i.e., 2k + 1 < 2^k
To prove P(k + 1) is true, we have to show that 2(k + 1) + 1 < 2^{k+1}. Now, we have 2(k + 1) + 1 = 2 k + 3

8.2k + 1 + 2 < 2k + 2 < 2k . 2 = 2k + 1.

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers, nε 3.

Long Answer Type

Example 4 Define the sequence a_1, a_2, a_3... as follows :
$a_1 = 2$, $a_n = 5\, a_{n-1}$, for all natural numbers $n\ \varepsilon 2$.

(iii) Write the first four terms of the sequence.

(iv) Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula $a_n = 2.5^{n-1}$ for all natural numbers.

Solution

r We have $a_1 = 2$

$a_2 = 5a_{2-1} = 5a_1 = 5.2 = 10$ $a_3 = 5a_{3-1} = 5a_2 = 5.10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5.50 = 250$

r Let P(n) be the statement, i.e.,

P(n) : $a_n = 2.5\ n{-}1$ for all natural numbers. We observe that P(1) is true

Assume that P(n) is true for some natural number k, i.e., P(k) : $a_k = 2.5^{k-1}$. Now to prove that P (k + 1) is true, we have

P(k + 1) : $a_{k+1} = 5.a_k = 5\ .\ (2.5^{k-1})$

$= 2.5^k = 2.5^{(k+1)-1}$

Thus P(k + 1) is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers.

Example 5 The distributive law from algebra says that for all real numbers c, a_1 and a_2, we have $c\,(a_1 + a_2) = ca_1 + ca_2$.

Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if c, a_1, a_2, ...,a_n are any real numbers, then

$c\,(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ... + ca_n$

Solution Let P(n) be the given statement, i.e.,

P(n) : $c\,(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ...\ ca_n$ for all natural numbers $n \geq 2$, for c, a_1, a_2, ... $a_n \in \mathbf{R}$.

We observe that P(2) is true since

$c(a_1 + a_2) = ca_1 + ca_2$ (by distributive law)

Assume that P(n) is true for some natural number k, where $k > 2$, i.e.,

P(k) : $c\,(a_1 + a_2 + ... + a_k) = ca_1 + ca_2 + ... + ca_k$

Now to prove P(k + 1) is true, we have

P(k + 1) : $c\,(a_1 + a_2 + ... + a_k + a_{k+1})$

$= c\,((a_1 + a_2 + ... + a_k) + a_{k+1})$

$= c\,(a_1 + a_2 + ... + a_k) + ca_{k+1}$ (by distributive law)

$= ca_1 + ca_2 + ... + ca_k + ca_{k+1}$

Thus P(k + 1) is true, whenever P (k) is true.

Hence, by the principle of Mathematical Induction, P(n) is true for all natural numbers $n \geq 2$.

Example 7 Prove by the Principle of Mathematical Induction that

$1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! – 1$ for all natural numbers n.

Solution Let P(n) be the given statement, that is,

P(n) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! – 1$ for all natural numbers n. Note that P (1) is true, since

$$P(1) : 1 \times 1! = 1 = 2 – 1 = 2! – 1.$$

Assume that P(n) is true for some natural number k, i.e.,

P(k) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! = (k + 1)! – 1$

To prove P (k + 1) is true, we have

P (k + 1) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! + (k + 1) \times (k + 1)!$

(i) $(k + 1)! – 1 + (k + 1)! \times (k + 1)$

(ii) $(k + 1 + 1) (k + 1)! – 1$

(iii) $(k + 2) (k + 1)! – 1 = ((k + 2)! – 1$

Thus P (k + 1) is true, whenever P (k) is true. Therefore, by the Principle of Mathematical Induction, P (n) is true for all natural number n.

Example 8

Prove, by Mathematical Induction, that

$$(n+1)^2 + (n+2)^2 + (n+3)^2 + ... + (2n)^2 = \frac{n(2n+1)(7n+1)}{6}$$

is true for all natural numbers n.

Discussion

Some readers may find it difficult to write the L.H.S. in P(k + 1). Some cannot factorize the L.H.S. and are forced to expand everything.

For P(1),

L.H.S. = $2^2 = 4$, R.H.S. = $\frac{1 \times 3 \times 8}{6} = 4$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$(k+1)^2 + (k+2)^2 + (k+3)^2 + ... + (2k)^2 = \frac{k(2k+1)(7k+1)}{6} \quad .... \quad (1)$$

For P(k + 1) ,

$(k+2)^2 + (k+3)^2 + ... + (2k)^2 + (2k+1)^2 + (2k+2)^2$ (There is a missing term in front and two more terms at the back.)

$$= (k+2)^2 + (k+3)^2 + ... + (2k)^2 + (2k+1)^2 + 4(k+1)^2$$

$$= (k+1)^2 + (k+2)^2 + (k+3)^2 + ... + (2k)^2 + (2k+1)^2 + 3(k+1)^2$$

$$= \frac{k(2k+1)(7k+1)}{6} + (2k+1)^2 + 3(k+1)^2 \quad \text{, by (1)}$$

$$= \frac{(2k+1)}{6}[k(7k+1) + 6(2k+1)] + 3(k+1)^2$$ (Combine the first two terms)

$$= \frac{(2k+1)}{6}[7k^2 + 13k + 6] + 3(k+1)^2$$

$$= \frac{(2k+1)}{6}(7k+6)(k+1) + 3(k+1)^2$$

$$= \frac{(k+1)}{6}[(2k+1)(7k+6) + 18(k+1)]$$

$$= \frac{(k+1)}{6}[14k^2 + 37k + 24]$$

$$= \frac{(k+1)}{6}(2k+3)(7k+8) = \frac{(k+1)[2(k+1)+1][7(k+1)+1]}{6}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Example 9

Prove, by Mathematical Induction, that

$$1 \cdot n + 2(n-1) + 3(n-2) + \dots + (n-2) \cdot 2 + n \cdot 1 = \frac{1}{6} n(n+1)(n+2)$$

is true for all natural numbers n.

Discussion

The "up and down" of the L.H.S. makes it difficult to find the middle term, but you can avoid this.

Solution

Let P(n) be the proposition: $1 \cdot n + 2(n-1) + 3(n-2) + \dots + (n-2) \cdot 2 + n \cdot 1 = \frac{1}{6} n(n+1)(n+2)$

For P(1),

L.H.S. = 1, R.H.S. = $\frac{1}{6} \times 1 \times 2 \times 3 = 1$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$1 \cdot k + 2(k-1) + 3(k-2) + \dots + (k-2) \cdot 2 + k \cdot 1 = \frac{1}{6} k(k+1)(k+2)$$

…. (1)

For P(k + 1) ,

$$1 \cdot (k+1) + 2k + 3(k-1) + \dots + (k-1) \cdot 3 + k \cdot 2 + (k+1) \cdot 1$$

$$= 1 \cdot (k+1) + 2[(k-1)+1] + 3[(k-2)+1] + \ldots + (k-1)\cdot[2+1] + k\cdot[1+1] + (k+1)\cdot 1$$

$$= 1\cdot k + 2(k-1) + 3(k-2) + \ldots + (k-2)\cdot 2 + k\cdot 1$$

$$+1 \quad +2 \quad +3 \quad + \ldots + \quad (k-1) \quad +k \quad +(k+1)$$

(The bottom series is arithmetic)

$$= \frac{1}{6}k(k+1)(k+2) + \frac{1}{2}(k+1)(k+2) \text{ , by (1)}$$

$$= \frac{1}{6}(k+1)(k+2)[k+3] = \frac{1}{6}(k+1)[(k+1)+1]\,[(k+1)+2]$$

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Example 10

Prove, by Mathematical Induction, that n(n + 1)(n + 2)(n + 3) is divisible by 24, for all natural numbers n.

Discussion

Mathematical Induction cannot be applied directly. Here we break the proposition into three parts. Also note that $24 = 4\times3\times2\times1 = 4!$

Solution

Let P(n) be the proposition:

1. n(n + 1) is divisible by 2! =2.
2. n(n + 1)(n + 2) is divisible by 3! = 6.
3. n(n + 1)(n + 2)(n + 3) is divisible by 4! = 24.

For P(1),

1. $1\times2 = 2$ is divisible by 2.
2. $1\times2\times3 = 6$ is divisible by 3.
3. $1\times2\times3\times4 = 24$ is divisible by 24. ∴ P(1) is true.

Assume that P(k) is true for some natural number k, that is

1. k(k + 1) is divisible by 2, that is, k(k + 1) = 2a …. (1)
2. k(k + 1)(k + 2) is divisible by 6 , that is, k(k + 1)(k + 2) = 6b …. (2)
3. k(k + 1)(k + 2)(k + 3) is divisible by 24 , that is, k(k + 1)(k + 2)(k + 3) = 24c …. (3)

where a, b, c are natural numbers.

For P(k + 1) ,

1. (k + 1)(k + 2) = k(k + 1) + 2(k + 1) = 2a + 2(k + 1) , by (1) = 2 [a + k + 1]

…. (4)

, which is divisible by 2.

2. $(k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2)$

$= 6b + 3 \times 2[a + k + 1]$, by (2), (4)

$= 6 [b + a + k + 1]$

…. (5)

, which is divisible by 6.

3. $(k + 1)(k + 2)(k + 3)(k + 4) = k(k + 1)(k + 2)(k + 3) + 4(k + 1)(k + 2)(k + 3)$

$= 24c + 4 \times 6[b + a + k + 1]$, by (3) , (5)

$= 24 [c + b + a + k + 1]$

, which is divisible by 24 .

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

KAHE

Possible Questions

2 Mark questions

1. Define the divisibility over a field.
2. Define the greatest common divisor of two polynomials over a field.
3. State the Division Algorithm.
4. Define relatively prime polynomials.
5. Define quotient and remainder.
6. State the Euclidean algorithm.
7. Define reducible.
8. Define irreducible.
9. State the principles of mathematical induction.
10. State the Fundamental theorem of Arithmetic.
11. Write the any two basic properties of the Greatest Common divisor.
12. Write the any two basic properties of the Prime factors.
13. Define residue.
14. Write any two properties of congruence relation.

6 Mark Questions

1. Prove that $1^2+2^2+3^2+\ldots +n^2= n(n+1)(2n+1)/6$ by Principle of Mathematical induction.
2. Find a+b (mod n), ab (mod n) and $(a+b)^2$(mod n) if a=4003, b=-127, n=85.
3. Prove that the sum of the first n odd integers is n^2.
4. State and prove the Principles of Mathematical Induction.
5. Find the quotient q and the remainder r as defined in the Division algorithm
   i) a=500, b=17 ii)a=-500,b=17 iii)a=-500 ,b=-17
6. Define greatest common divisor& Find the greatest common divisor of a and b and express it in the form ma+nb for suitable integers m and n .
   i) a=26 ,b=118.
7. State and prove the Division Algorithm.
8. Solve the following congruence i) 3x$\equiv 1\ (mod\ 5)$ ii) i) 3x$\equiv 1\ (mod\ 6)$
9. State and prove the fundamental theorem of Arithmetic .
10. Prove that ,if a$\equiv x(modn) and b \equiv y(modn)$, then
    i) a+b$\equiv x+y(modn)$ and ii) ab$\equiv xy(modn)$.
11. State and prove Euclidean Algorthim.
12. State and prove Euclidean theorem.

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Algebra | **Subject Code: 19MMU102**

Class : I - B.Sc. Mathematics | **Semester : I**

Unit II

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Two number are said to be relatively prime if their ………………..divisor is 1.	zero	greatest common	least common	infinite	greatest common
Let m be any fixed positive integer. Then an integer a is said to be congruent to another integer b modulo m if ………………	m|(a-b)	m|(a+b)	m|(ab)	m|a	m|(a-b)
If x>0 for any integer then gcd(ax,bx)=……………………	gcd (a,b)	x gcd (a,b)	(x+1)gcd (a,b)	(x-1) gcd (a,b)	x gcd (a,b)
Let f(x),g(x)≠0 be any two polynomials of the polynmial domain F[x],over the field F. Then there exist uniquely two polynomials q(x) & r(x) in F[x] such that ………………..	f(x)= q(x)g(x)+r(x)	f(x)= q(x)+r(x)	f(x)=q(x)g(x)	f(x)=g(x)+r(x)	f(x)= q(x)g(x)+r(x)
Let f(x),g(x)≠0 be any two polynomials of the polynmial domain F[x],over the field F. Then there exist uniquely two polynomials q(x) & r(x) in F[x] such that f(x)=q(x)g(x)+r(x) where r(x)….	equal to zero	not equal to zero	less than zero	more than zero	equal to zero
Division algorithm for polynomials over a field deg r(x) ……….deg g(x)	<	>	=	≠	=

In the division algorithm, the polynomial q(x) is called the ……….on dividing f(x) by g(x)	quotient	remainder	divisior	diviend	quotient
In the division algorithm, the polynomial q(x) is called the quotient on dividing f(x) by g(x) and the polynomail r(x)is called the ……………………	quotient	remainder	divisior	diviend	remainder
A polynomial domain F[x] over a field F is a principal………………	commutatice ring	ideal ring	associative ring	division ring	ideal ring
A polynomial …………….. F[x] over a field F is a principal idea lring	domain	range	co domain	quotient	domain
A polynomial domain F[x] over a ………….. F is a principal ideal ring	ring	domain	range	field	field
In a Euclidean algorithm ,Let F be a field and f(x) and g(x) be any two polynomials in F[x], not both of which are ………………………	zero	one	two	three	zero
In a Euclidean algorithm ,Let F be a field and f(x) and g(x) be any two polynomials in F[x], not both of which are zero.Then f(x and g(x) have a ………………………d(x)	common divisor	greatest common divisor	least common divisor	equal divisor	greatest common divisor
Let F be a field and f(x) and g(x) be any two polynomials in F[x], not both of which are zero.Then f(x and g(x) have a greatest common divisor d(x),it can be expressed in the form……..	d(x)=m(x)f(x)+n(x)g(x)	d(x)=m(x)f(x)-n(x)g(x)	d(x)=f(x)+n(x)g(x)	d(x)=m(x)f(x)+n(x)	d(x)=m(x)f(x)+n(x)g(x)
In a Euclidean algorithm,the expression d(x)=m(x)f(x)+n(x)g(x) for ………………..m(x) and n(x) in F[x].	ring	field	polynomials	domain	polynomials
The greatest common divisor should be a ………………polynomial	zero	monic	double	triple	monic
If a(x)≠0 and f(x) are elements of F[x] then a(x) is a ……………..of f(x)	quotient	remainder	divisor	dividend	divisor
If a(x)≠0 and f(x) are elements of F[x] then a(x) is a divisor of f(x)iff there is a polynomial b(x) be in f[x] then ……………………	f(x)=a(x)+b(x)	f(x)=a(x)-b(x)	f(x)=a(x)b(x)	f(x)=a(x)/b(x)	f(x)=a(x)b(x)
The divisor of f(x) symbolically write …………………	a(x)/f(x)	f(x)/a(x)	b(x)/f(x)	a(x)/b(x)	a(x)/f(x)

A ……….is an element of F[x] which has a multiplicative inverse.	zero	unit	two	three	unit
A unit is an element of F[x] which has ………………. inverse.	finite	infinite	multiplicative	zero	multiplicative
A unit is an element of F[x] which has a multiplicative ……………....	ring	field	range	inverse	inverse
All the polynomials of ……………. degree belonging to F[x] are units of F[x].	1st	2nd	zero	nth	zero
All the polynomials of zero degree belonging to F[x] are……………...of F[x].	units	field	ring	range	units
The……………….. elements of F are the only units of F[x].	zero	non zero	finite	infinite	non zero
The non zero elements of F are the ……………..of F[x].	only units	not only units	double units	zero units	only units
If f(x) and g(x) are polynomials in F[x], then we call f(x) and g(x) associates if ………………..for some 0≠c ϵ F.	f(x)=g(x)	f(x)=c/g(x)	f(x)=c+g(x)	f(x)=cg(x)	f(x)=cg(x)
If f(x) and g(x) are ……………….. in F[x], then we call f(x) and g(x) associates if f(x) =c g(x) for some 0≠c ϵ F.	field	ring	polynomials	domain	polynomials
If f(x) and g(x) are polynomials in F[x], then we call f(x) and g(x) associates if f(x) =c g(x) for some ……………...	0=c ϵ F	0>c ϵ F	0<c ϵ F	0≠c ϵ F	0≠c ϵ F
Two non zero polynomials f(x) and g(x) in F[x] are associates iff …………………….. And ………………………	f(x)+g(x) & g(x)/f(x)	f(x)g(x) & g(x)f(x)	f(x)/g(x) & g(x)-f(x)	f(x)/g(x) & g(x)/f(x)	f(x)/g(x) & g(x)/f(x)
Two non zero polynomials f(x) and g(x) in F[x] are …………… iff f(x)/g(x) and g(x)/f(x)	commutates	associates	divisible	distributive	associates
The divisorsof f(x) are called its……………………divisors.	proper	improper	finite	infinite	improper
All other divisors of f(x), if there are any , are called its……………………divisors.	proper	improper	finite	infinite	proper
If f(x) be a ……………...of positive degree, then f(x) is said to be irreducible over F.	function	domain	polynomial	range	polynomial

If f(x) be a polynomial of ………….. degree, then f(x) is said to be irreducible over F.	zero	positive	negative	infinite	positive
If f(x) be a polynomial of positive degree, then f(x) is said to be ………………. over F.	irreducible	reducible	singular	non singular	irreducible
An irreducible polynomial is otherwise called as……………..	point	prime	power	degree	prime
It has ………….. proper divisors in F[x]; f(x) is irreducible over F.	no	0ne	two	infinite	no
It has no proper divisors in F[x]; f(x) is …………………. over F.	irreducible	reducible	singular	non singular	irreducible
It has a ……………... divisors in F[x]; f(x) is reducible over F.	finite	infinite	proper	improper	proper
It has a proper divisors in F[x]; f(x) is………………….. over F.	irreducible	reducible	singular	non singular	reducible
……………………. depends on the field.	irreducibility	reducibility	singularity	non singularity	irreducibility
Irreducibility depends on the ……………....	field	domain	range	ring	field
Two polynomials are said to be relatively prime if their greatest common divisor is …………....	0	1	2	3	1
………………. polynomials are said to be relatively prime if their greatest common divisor is 1.	zero	one	two	three	two
Two polynomials are said to be …………………. if their greatest common divisor is 1.	field	prime	relatively prime	uniquely prime	relatively prime
Two polynomials are said to be relatively prime if their ……………….divisor is 1.	zero	greatest common	leatest common	infinite	greatest common
Let m be any fixed positive integer.Then an integer a is said to be congruent to another integer b modulo m if ……………..	m/(ab)	m/(a-b)	m/(a+b)	m/a	m/(a-b)
Let m be any fixed ………….. integer.Then an integer a is said to be congruent to another integer b modulo m if m/(a-b).	positive	negative	zero	infinite	positive

Let m be any fixed positive integer.Then an integer a is said to be …………. to another integer b modulo m if m/(a-b).	division	range	congruent	domain	congruent
Let m be any fixed positive integer.Then an integer a is said to be congruent to another integer b ……………. m if m/(a-b).	multiplication	addition	division	modulo	modulo

UNIT-III

Systems of linear equations, row reduction and echelon forms, vector equations, the matrix equation Ax=b, solution sets of linear systems, applications of linear systems, linear independence.

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A linear equation in variables $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \cdots a_n$ and b are constant real or complex numbers. The constant a_i is called the coe□cient of x_i and b is called the constant term of theequation.

A system of linear equations (or linear system) is a finite collection of linear equations in same variables. For instance, a linear system of m equations in n variables $x_1, x_2, \cdots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n} &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn} &= b_m \end{aligned} \qquad (9.1)$$

A solution of a linear system is a n-tuple $(s_1, s_2, \cdots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \cdots, s_n$ are sub-stituted for $x_1, x_2, \cdots, x_n$, respectively. The set of all solutions of a linear system is called the solution set of the system.

Any system of linear equations has one of the following exclusive conclusions.
(a) No solution.
(b) Unique solution.
(c) Infinitely many solutions.

A linear system is said to be consistent if it has at least one solution and is said to be inconsistent if it has no solution.

The system of equations (9.1) is said to be homogeneous if all b_j are zero; otherwise, it is said to be non-homogeneous.

The system of equations (9.1) can be expressed as the single matrix equation

$$AX = B, \qquad (9.2)$$

vector (column matrix) X that satisfies the matrix equation (9.2) is also the solution of the system.

Definition 21. *The matrix* $[AB]$ *which is obtained by placing the constantcolumn matrix B to the right of the matrix A is called the augmented matrix. Thus the augmented matrix of the system* $AX = B$ *is*

$$[AB] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Theorem 35. *The system $AX = B$ is consistent if and only if A and $[AB]$ have the same rank.*

System of non-homogeneous Equations

If we are given with a system of m equations in n unknowns, proceed as follows:

1. Write down the corresponding matrix equation $AX = B$.
2. By elementary row transformations obtain row echelon matrix of the augmented matrix $[AB]$.
3. Examine whether the rank of A and the rank of $[AB]$ are the same or not.

Case 1 If rank of $A \neq$ rank of $[AB]$, then the system is inconsistent and has no solution. otherwise, it is said to be non-homogeneous.

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Theorem 35. *The system $AX = B$ is consistent if and only if A and $[AB]$ have the same rank.*

System of non-homogeneous Equations

If we are given with a system of m equations in n unknowns, proceed as follows:

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3. Examine whether the rank of A and the rank of $[AB]$ are the same or not.

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Case 1 If rank of $A \neq$ rank of $[AB]$, then the system is inconsistent and has no solution.

Case 2 If rank of A = rank of $[AB]$, then the system is consistent.

Case 2a If rank of A = rank of $[AB]$ = n = number of unknowns, then the system has unique solution.

Case 2b If rank of A = rank of $[AB]$ < n = number of unknowns, then the system has infinitely many solutions. We assign arbi-trary values to $(n-r)$ unknowns and determine the remaining r unknowns uniquely.

Solution of System of Linear Equations

Any given system of linear equations may be written in term of matrix, such that

$$AX = B \qquad \ldots(i)$$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A is known as co-efficient matrix.

If we multiply both sides of (i) by the reciprocal matrix A^{-1}, then we get $A^{-1}AX = A^{-1}B$

$$(A^{-1}A)X = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ where } \Delta \neq 0$$

$$= \frac{1}{\Delta}\begin{bmatrix} A_1d_1 + A_2d_2 + A_3d_3 \\ B_1d_1 + B_2d_2 + B_3d_3 \\ C_1d_1 + C_2d_2 + C_3d_3 \end{bmatrix} \qquad \ldots(ii)$$

Hence from (ii) equating the values of x, y and z we get the desired result.

This method is true only when (i) $\Delta \neq 0$ (ii) number of equations and number of unknowns (e.g. x, y, z etc.) are the same.

Example 1. Solve the equations with the help of determinants :
$x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$.

Sol. The co-efficient determinant is $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2 \neq 0$

$$\therefore \quad x = \frac{1}{2}\begin{bmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{bmatrix} \Rightarrow \quad x = \frac{1}{2} \times 4 = 2$$

$$y = \frac{1}{2}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{bmatrix} \Rightarrow \qquad y = \frac{1}{2}(2) = 1 \Rightarrow y = 1$$

$$z = \frac{1}{2}\begin{bmatrix} 1 & 1 & +3 \\ 1 & 2 & +4 \\ 1 & 4 & +6 \end{bmatrix} \Rightarrow \qquad z = \frac{1}{2}[-4+6+(4-6)] = 0 \Rightarrow \quad z = 0$$

$\therefore$ Solution is $x = 2$, y = 1, z = 0.

Row reduced Echelon Form:
In addition to the above three conditions, if a matrix satisfies the following conditions:
Each column which contains a leading entry of a row has all other entries zeros, then the matrix is said to be in row reduced echelon matrix.

Row Rank and Column Rank of a Matrix
Row rank of a matrix, say A is the number of non zero rows in the row echelon matrix A and is denoted by $\rho_R(A)$.
Column Rank of a matrix, say A is the number of non zero columns in the column echelon matrix A and is denoted by $\rho_C(A)$.

Note: (i) Every matrix is row equivalent to row echelon matrix.
(ii) Every matrix is column equivalent to a column echelon matrix.
(iii) If a matrix A is in row echelon form, then its transpose is in column echelon form.

Example. 1: Reduce the matrix $A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 1 & 3 & 0 & 2 & 3 \\ 0 & 2 & 6 & 1 & 3 & 9 \\ 0 & 4 & 12 & -2 & 10 & 7 \end{bmatrix}$ to the row reduced echelon form and hence find its rank.

Solution: Applying $R_2 \to R_2\text{-}R_1, R_3 \to R_3\text{-}2R_1$, and $R_4 \to R_4\text{-}4R_1$ on the matrix A,

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 & -3 & 7 \\ 0 & 0 & 0 & 2 & -2 & 3 \end{bmatrix}$$

Applying $R_1 \to R_1+R_2, R_3 \to R_3\text{-}3R_2$, and $R_4 \to R_4\text{-}2R_2$

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Applying $R_1 \to R_1\text{-}3R_3, R_2 \to R_2\text{-}2R_3$, and $R_4 \to R_4+R_3$

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required row reduced echelon form of the matrix A. Since, the number of non zero rows is 3, thus row rank of A is 3.

System of Linear Equations and Matrices

Linear Equation

$$y = m x \qquad .1$$

is an equation, in which variable y is expressed in terms of x and the constant m, is called Linear Equation. In Linear Equation exponents of the variable is always ‘ one’.

Equation 1 is also called equation of line.

Linear Equation in n variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n = b \qquad .2$$

where $x_1, x_2, x_3, ..., x_n$ are variables and
$a_1, a_2, a_{13}, ..., a_n$ and b are constants.

Linear System:

A Linear System of m linear equations and n unknowns is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n &= b_3 \\ &\cdots\cdots \qquad .3 \\ &\cdots\cdots \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n &= b_m \end{aligned}$$

where $x_1, x_2, x_3, ..., x_n$ are variables or unknowns and a’s and b’s are constants.

Solution:

Solution of the linear system (3) is a sequence of n numbers

$s_1, s_2, s_3, \ldots, s_n$, which satisfies system (3) when we substitute $x_1 = s_1, x_2 = s_2, x_3 = s_3, \ldots, x_n = s_n$.

Example.1. Solve the system of equations

$$x - 3y = -3 \qquad \rightarrow 1$$
$$2x + y = 8 \qquad \rightarrow 2$$

Solution:

$-2E_1 + E_2 \Rightarrow$

$$-2x + 6y = 6$$
$$2x + y = 8$$
$$+7y = 14 \quad \Rightarrow y = 2$$

From equation 1

$$x = -3 + 3y$$
$$x = -3 + 6 = 3$$

Solution is $x = 3$ and $y = 2$

Check Substitute the solution in Equations 1 and 2

Equation 1 $\Rightarrow$ $3 - 3(2) = 3 - 6 = -3$

Equation 2 $\Rightarrow$ $2(3) + 2 = 6 + 2 = 8$.

Example.2. Solve the system of equations

$$x - 3y = -7 \qquad \rightarrow 1$$
$$2x - 6y = 7 \qquad \rightarrow 2$$

Solution:

$2E_1 - E_2 \Rightarrow$

$$2x - 6y = -7$$
$$-2x + 6y = -14$$
$$0 + 0 = -21$$

This makes no sense as $0 \neq -21$, hence there is no solution.

NOTE: Inconsistent , the system of equations is inconsistent, if the system has no solution.

Consistent, the system of equations is consistent if the system has at least one solution.

Example: ***Inconsistent and consistent system of equations***

For the system of linear equations which is represented by straight lines:

$$a_1x - b_1y = c_1 \quad \rightarrow l_1$$
$$a_2x - b_2y = c_2 \quad \rightarrow l_2$$

There are three possibilities:

No solution	one solution	infinite many solutions
[inconsistent]	[consistent]	[consistent]

Note:1. A system will have unique solution (only one solution)when number of unknowns is equal to number of equations

Note:2. A system is over determined , if there are more equations then unknowns and it will be mostly inconsistent.

Note:3. A system is under determined if there are less equations then unknowns and it may turn inconsistent.

Augmented Matrix

System of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_{31} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_{31} = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_{31} = b_3$$

can be written in the form of matrices product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or we may write it in the form AX=b,

where $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Augmented matrix is $[A:b]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

Example: 4. Write the matrix and augmented form of the system of linear equations

3x – y + 6z = 6
x + y + z = 2
2x + y +4z = 3

Solution: Matrix form of the system is

$$\begin{bmatrix} 3 & -1 & 6 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

Augmented form is $[A:b]=\begin{bmatrix} 3 & -1 & 6 & 6 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

Elementary Row operations:

Elementary row operations are steps for solving the linear system of equations:

I. Interchange two rows
II. Multiply a row with non zero real number
III. Add a multiple of one row to another row

SYSTEM WITH NO SOLUTION

Example: 6 . Solve the system of linear equations

$$x - 2y + z - 4u = 1$$
$$x + 3y + 7z + 2u = 2$$
$$x - 12y - 11z - 16u = 5$$

Solution:

Augmented matrix is:

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix}$$

Reducing it to row echelon form (using Gaussian - elimination method)

$$\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{bmatrix} \quad R_2 - R_1, \; R_3 - R_1$$

$$\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \quad -R_3 + 2R_2$$

Last equation is

$$0x + 0y + 0z + 0u = -3$$

but $0 \neq -3$

hence there is no solution for the given system of linear equations.

<u>Conditions on Solutions</u>

Example:7. For which values of 'a' will be following system

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2 - 14)z = a + 2$$

(i) infinitely many solutions?
(ii) No solution?
(iii) Exactly one solution?

Solution:

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Augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$$

Reducing it to reduced row echelon form

$$\approx \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & -14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \quad R_2\text{-}3R_1,\ R_3\text{-}4R_1$$

$$\approx \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} -\tfrac{1}{7}R_2,\ R_3\text{-}R_2$$

writing in the equation form,

$$x+2y-3z = 4 \quad \to 1$$
$$y-2z = \tfrac{10}{7} \quad \to 2$$
$$(a^2-16)z = a-4 \quad \to 3$$

or equation 3 can be written as

$$(a+4)(a-4)z = a-4$$

CASE I .

$$a = 4 \quad \Rightarrow 0z = 0$$

$$x+2y-3z = 4$$
$$y-2z = \tfrac{10}{7}$$

as number of equations are less than number of unknowns, hence the system has infinite many solutions,

let $z = t$

$y = \frac{10}{7} + 2t$

$x = 4 + 3t - 4t - \frac{20}{7} = -t + \frac{8}{7}$

where 't' is any real number.

CASE II

$a = -4 \quad \Rightarrow \quad 0z = -8$, but $0 \neq -8$, hence, there is no solution.

CASE III

$a \neq 4, a \neq -4, \quad \text{let} \quad a = 1$

Equatins.3. $\Rightarrow \ (1-4)(1+4)z = 1-4$

$$-15z = -3$$

$$z = \tfrac{1}{5}$$

$$y = \tfrac{10}{7} + \tfrac{2}{5} = \tfrac{64}{35}$$

$$x = 4 + \tfrac{3}{5} - 2(\tfrac{64}{35}) = \tfrac{47}{35}$$

the system will have unique solution when $a \neq 4$ and $a \neq -4$
and for $a=1$ the solution is

$x = \tfrac{47}{35}, y = \tfrac{64}{35}$ and $z = \tfrac{1}{5}$.

NOTE: (i) $a=-4$, no solution,
(ii) $a=4$, infinite many solutions and
(iii) $a \neq 4$, $a \neq -4$, exactly one solution .

Example:8. What conditions must a, b, and c satisfy in order for the system of equations

$$\begin{aligned} x + y + 2z &= a \\ x \quad\ + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

to be consistent.

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}$$ reducing it to reduced row echelon form

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix} \quad R_2\text{-}R_1,\ R_3\text{-}2R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-a-b \end{bmatrix} \quad R_3\text{-}R_1$$

The system will be consistent if only if $c - a - b = 0$

or $c = a + b$

Thus the required condition for system to be consistent is

$c = a + b.$

Solution of a system *AX=b*

Let $AX = b$ be a given $m \times n$ system. The $m \times (n + 1)$ matrix $[A|b]$ is called the **augmented matrix**for the system $AX = b$. Let $[\tilde{A}|\tilde{b}]$ be the row echelon from [A|b]. The following conclusion is now obvious from the earlier discussions.

Let A**X** = **b** be a m$\times$n system of linear equation and let $[\tilde{A}|\tilde{b}]$ be the row echelon form [A|**b**], and let r be the number of nonzero rows of $[\tilde{A}|\tilde{b}]$. Note that 1 $\leq$ min {m, n}. Then the following hold:For the system A**X** = b

(i) The system is inconsistent, i.e., there is no solution if among the nonzero rows of $[\tilde{A}|\tilde{b}]$there is a row with zero everywhere except at the last place. That is (n+1)th column is not a pivot column for $[\tilde{A}|\tilde{b}]$.

(ii) The system is solvable if $[\tilde{A}|\tilde{b}]$ has r nonzero rows with r $\leq$ n. There is a unique solution if r = n i.e., $[\tilde{A}|\tilde{b}]$ has exactly n- nonzero rows, the number of variables. And, there are infinitely many solutions if $[\tilde{A}|\tilde{b}]$ has r-nonzero rows, with r < n. In fact, one can compute these solutions as follows: for 1 $\leq$ i $\leq$ r, let p_i^{th} column be the pivot column. Then, assign arbitrary values to each of the variable x_j, j $\neq p_i$ and compute the values of the variable x_{p_i}, 1 $\leq$ i $\leq$ r in terms of these (as in example 2.2.2). Thus, the general solution will have n - r variables taking arbitrary values.

Examples:

(i) Consider the system $AX = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 2 & -5 \\ 2 & 5 & -1 & -9 \\ 2 & 1 & -1 & 3 \\ 1 & 3 & 2 & 7 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ -3 \\ -11 \\ -5 \end{bmatrix}.$$

It is early to verify that the augmented matrix

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & 3 & 2 & 7 & -5 \end{array}\right]$$

is equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then by theorem 2.4.1, the system AX = b is consistent and has infinite number of solutions. In fact, if

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

Here, we can give arbitrary value to the variable x_4, and other variable can be computed by :

$$\begin{aligned} x_1 \quad\quad + 2x_4 &= -5 \\ x_2 \quad - 3x_4 &= 2 \\ x_3 - 2x_4 &= 3 \end{aligned},$$

i.e.,
$$\begin{aligned} x_3 &= 5 - 2x_4 \\ x_2 &= 2 + 3x_4 \\ x_1 &= -5 - 2x_4 \end{aligned},$$

where x_4 can be assigned any arbitrary value.

(ii) Consider the system $AX = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}.$$

The augmented matrix in this system is

$$\begin{bmatrix} 0 & 1 & -4 & | & 8 \\ 2 & -3 & 2 & | & 1 \\ 5 & -8 & 7 & | & 1 \end{bmatrix}.$$

It is easy to see that this is equivalent to

$$\begin{bmatrix} 0 & 1 & -4 & | & 1 \\ 2 & -3 & 2 & | & 8 \\ 5 & -8 & 7 & | & 5/2 \end{bmatrix}$$

Since, the last row is identically zero for the position of A and non-zero for the portion of B, the system
is inconsistent.

(iii) Consider the system $AX = b$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

The augmented matrix [A|**b**] of the system can be shown to be equivalent to

$$[\tilde{A} \mid \tilde{b}] = \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix},$$

When $\tilde{A}$ is the reduced row echelon form of A. Then, $AX = b$ has unique solution, namely

$$\tilde{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

LINEAR DEPENDENCE AND INDEPENDENCE OF ROW & COLUMN MATRICES.

Any quantity having n components is called a vector of order n. If $a_1, a_2, ...a_n$ are elements of fields (F, +, .), then these numbers written in a particular order form a vector.

Thus an n-dimensional vector X over a field (F,+, .) is written as X=$(a_1, a_2, ...a_n)$ where $a_i \in F$.

Row matrix of type $1 \times n$ is n—dimensional vector written as $X=[a_1, a_2, ...a_n]$

Column matrix of type $n \times 1$ is also n dimensional vector written as

$$X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ or} \begin{bmatrix} a_1 & a_2 & .. & a_n \end{bmatrix}$$

As the vectors are considered as either row matrix or column matrix, the operation of addition of vectors will have the same properties as the addition of matrices.

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Linear Dependence:
The set of vectors $\{v_1,v_2,.....v_n\}$ are said to be linearly dependent if there exist scalars $a_1,a_2,.....a_n$ not all zero such that $a_1v_1+a_2v_2+.....+a_nv_n=0$

Linear Independence:
The set of vectors $\{v_1,v_2,.....v_n\}$ are said to be linearly independent if there exist scalars $a_1,a_2,.....a_n$ such that $a_1v_1+a_2v_2+.....+a_nv_n=0$ gives $a_1=a_2=.....=a_n=0$.

Example1: Show that the vectors u=(1,3,2), v=(1,-7,-8) and w=(2,1,-1) are linearly independent.
Proof: The vectors are said to be linearly dependent if
au+ bv +cw=0 where a, b, c are not all zero.
means a(1,3,2)+b(1,-7,-8)+c(2,1,-1)=(0,0,0) (1)
(a+b+2c, 3a-7b+c, 2a-8b-c)= (0, 0, 0)
which gives a+b+2c=0 (2)
3a-7b+c=0 (3)
2a-8b-c=0 (4)
Adding (3) and (4), we have
5a-15b=0 $\Rightarrow$ a=3b
$\therefore$ From (3) 3(3b)-7b+c=0 $\Rightarrow$ 9b-7b+c $\Rightarrow$ c=-2b
Putting a=3b and c=-2b in (2), we get
3b+b-4b=0, which is true. Giving different real value to b we get infinite non zero real values of a and c. So a, b, c are not all zero.
Hence given vectors u, v and w are linearly independent.

Theorem 1: If two vectors are linearly dependent then one of them is scalar multiple of other.
Proof: Let u, v be the two linearly dependent set of vectors. Then there exists scalars a, b(not both zero) such that
a. u + b. v = 0 (1)
Case 1. When $a \neq 0$
From (1), au = - bv $\Rightarrow u=-\frac{b}{a}v$
Hence u is scalar multiple of v.
Case II. When $b \neq 0$
From (1), bv = - au $\Rightarrow v=-\frac{a}{b}u$
Hence v is scalar multiple of u. Thus in both cases one of them are scalar multiple of other.

Theorem 2: Every superset of a linearly dependent set is linearly dependent.
Proof: Let $S_n=\{X_1,X_1......X_n\}$ be set of n vectors which are linearly dependent.
Let $S_r=\{X_1,X_1......,X_n,X_{n+1}....,X_r\}$ where r > n be any super set of S_n.
As $\{X_1,X_1......X_n\}$ is linearly dependent set
$\therefore$ There are scalars $a_1,a_2,a_3,.......,a_n$ not all zero such that
$a_1X_1+a_2X_2+.......+a_nX_n=0$
$\Rightarrow a_1X_1+a_2X_2+.......+a_nX_n+0.X_{n+1}+0.X_{n+2}+....+0.X_r=0$
As $a_1,a_2,a_3,.......,a_n$ are not all zero
$\therefore$ Set $S_r=\{X_1,X_2......,X_n,X_{n+1}....,X_r\}$ is linearly dependent set.

Hence every set of linearly dependent set is linearly dependent.

Theorem 3: Every subset of linearly independent set is linearly independent.

Proof: Let $S_n = \{X_1, X_1 ......X_n\}$ be set of n vectors which are linearly independent.

Let $S_r = \{X_1, X_1 ......, X_r\}$ where r < n be any subset of S_n.

As $\{X_1, X_1 ......X_n\}$ is linearly independent set thus

$a_1X_1 + a_2X_2 + ... + a_nX_n = 0$ gives

$a_1 = a_2 = a_3, ... = a_n = 0$

$a_1X_1 + a_2X_2 + ... + a_rX_r = 0$ where $a_1 = a_2 = a_3, ... = a_r = 0$

$\therefore$ Set $S_r = \{X_1, X_1 ......, X_r\}$ is linearly independent set.

Hence every subset of linearly independent set is linearly independent.

Theorem 4: If vectors $X_1, X_1 ......X_n$ are linearly dependent, then at least one of them may be written as linear combination of the rest.

Proof: Since the vectors $X_1, X_1 ......X_n$, are linearly dependent, therefore there exist scalars $a_1, a_2, a_3, ..., a_n$ not all zero, such that

$a_1X_1 + a_2X_2 + ... + a_nX_n = 0$ or $a_1X_1 + a_2X_2 + ... + a_iX_i + a_{i+1}X_{i+1} ... + a_nX_n = 0$

Suppose $a_i \neq 0$

$-a_iX_i = a_1X_1 + a_2X_2 + ... a_{i-1}X_{i-1} + a_{i+1}X_{i+1} ... + a_nX_n$

or $X_i = \frac{a_1}{-a_i}X_1 + \frac{a_2}{-a_i}X_2 + ... + \frac{a_{i-1}}{-a_i}X_{i-1} + \frac{a_{i+1}}{-a_i}X_{i+1} ... + \frac{a_n}{-a_i}X_n$

Hence vector X_i is a linear combination of the rest.

Theorem 5: If the set $\{X_1, X_1 ......X_n\}$ is linearly independent and the set $\{X_1, X_1 ......X_n, Y\}$ is linearly dependent, then Y is linear combination of the vectors $X_1, X_1 ......X_n$.

Proof: Consider the relation

$$a_1X_1 + a_2X_2 + ... + a_nX_n + aY = 0 \qquad (1)$$

As set $\{X_1, X_1 ......X_n, Y\}$ is linearly dependent

$\therefore a_1, a_2, a_3, ..., a_n, a$ are not all zero

We claim that $a \neq 0$. If a=0, then (1) becomes

$$a_1X_1 + a_2X_2 + ... + a_nX_n = 0$$

As set $\{X_1, X_1 ......X_n\}$ is linearly independent

$\therefore \quad a_1 = a_2 = a_3, ... = a_n = 0$

Then from (1), the set $\{X_1, X_1 ......X_n, Y\}$ is linearly independent which a contradiction to the given condition is. Thus a = 0 is not possible. Hence $a \neq 0$

From (1), we have $-aY = a_1X_1 + a_2X_2 + ... + a_nX_n$

or $Y = \frac{a_1}{-a}X_1 + \frac{a_2}{-a}X_2 + ... + \frac{a_n}{-a}X_n$, which proves the result.

Theorem 6: The kn-vectors $A_1, A_2, ..., A_k$ are linearly dependent iff the rank of the matrix $A=[A_1, A_2, ..., A_k]$ with the given vectors as columns is less than k.

Proof: Let $x_1A_1 + x_2A_2, ... + x_kA_k = 0$

where $x_1, x_2, ..., x_k$ are scalars

$$\Rightarrow x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_1\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + ... + x_k\begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} = O$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 + ... + a_{1k}x_k = 0$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2k}x_k = 0$$
$$...$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nk}x_k = 0$$

Which can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1k} \\ a_{21} & a_{22} & ... & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nk} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow AX=O$

Let the vectors $A_1, A_2, ..., A_k$ be linearly dependent.

So, from the relation (i), scalars $x_1, x_2, ..., x_k$ are not all zero and thus the homogeneous system of equations given by (ii) has non-trivial solution. Hence $\rho(A)<k$ .Converse of this theorem is also true.

Theorem 7: A square matrix A is singular iff its columns (rows) are linearly dependent.

Proof: Let n be the order of the square matrix A and $A_1, A_2, ..., A_n$ be its columns.

$\therefore A=[A_1, A_2, ..., A_n]$

Proceed in same way as above theorem to prove $\rho(A)<n$

Since $\rho(A)<n$, thus $|A| = 0$ and hence A is singular matrix.

Conversely, the column vectors of A are linearly dependent.

Theorem 8: The kn-vectors $A_1, A_2, ..., A_k$ are linearly independent if the rank of the matrix $A=[A_1, A_2, ..., A_k]$ is equal to k.

Proof: Proceed in the same way as above theorem to obtain $AX=O$. Now suppose .

Then $|A| \neq 0$ and homogeneous system of equations given by (ii) has trivial solution only.

$\therefore x_1 = x_2 = ... = x_k = 0$

Thus, the vectors $A_1, A_2, \ldots, A_k$ are linearly independent.

Theorem 9: The number of linearly independent solution of the equation AX=O is (n-r) where r is the rank of matrix A.

Proof: Given that rank of A is r which means A has r linearly independent columns. Let first r columns are linearly independent.

Now, $A=[C_1, C_2, \ldots C_r, \ldots, C_n]$, where $C_1, C_2, \ldots, C_n$ are column vectors of A.

$$\therefore [C_1, C_2, \ldots, C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \Rightarrow C_1x_1 + C_2x_2 + \ldots + C_nx_n = 0 \qquad \ldots(i)$$

As the set $[C_1, C_2, \ldots, C_r]$ is linearly independent, thus each vector $C_r, C_{r+1}, \ldots, C_n$ can be written as linear combination of $C_1, C_2, \ldots, C_r$.

Now, $C_{r+1} = a_{11}C_1 + a_{12}C_2 + \ldots + a_{1r}C_r$

$C_{r+2} = a_{21}C_1 + a_{22}C_2 + \ldots + a_{2r}C_r$

....

$C_n = a_{k1}C_1 + a_{k2}C_2 + \ldots + a_{kr}C_r$, where k=n-r ...(ii)

From (i) and (ii), we get

$$X_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1r} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, X_{n-r} = \begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kr} \\ 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

Thus, AX=O has (n-r) solutions.

Check Your Progress

1. Find the vector p if the given vectors are linearly dependent $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ p \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Ans. p=2.

LINEAR SYSTEM OF EQUATIONS

System of Non Homogeneous Linear Equation

If

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_n \end{array}\right\} \quad \ldots(1)$$

be given system of m linear equations then (1) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}$$

$\Rightarrow$ AX = B and $C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} : b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} : b_2 \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} : b_m \end{bmatrix}$

then [A : B] or C is called augmented matrix. Sometime we also write A : B for [A : B]

Consistent Equations.

(i) If rank of A = rank of [A : B] and there is unique solution when rank of A = rank of [A : B] = n

(i) rank of A = rank of [A : B] = r < n.

Inconsistent Equations.

If rank of A ≠ rank of [A : B] i.e. have no solution.

Example 1.Discuss the consistency of the following system of equation 2x + 3y + 4z = 11, x + 5y + 7z = 15, 3x + 11y + 13z = 25, if consistent, solve.

Sol. The augmented matrix $[A : B] = \begin{bmatrix} 2 & 3 & 4:11 \\ 1 & 5 & 7:15 \\ 3 & 11 & 13:25 \end{bmatrix}$

R_{12} operation is done so $\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 2 & 3 & 4:11 \\ 3 & 11 & 13:25 \end{bmatrix}$

Next operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 0 & -7 & -10:-19 \\ 0 & -4 & -8:-20 \end{bmatrix}$$

Again, operating $R_2 \to -\frac{1}{7} R_2$ and $R_3 \to -\frac{1}{4} R_3$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 0 & 1 & \frac{10}{7}:\frac{19}{7} \\ 0 & 1 & 2:5 \end{bmatrix}$$

Next operating $R_3 \to R_3 - R_2$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7 : 15 \\ 0 & 1 & \frac{10}{7} : \frac{19}{7} \\ 0 & 0 & \frac{4}{7} : \frac{16}{7} \end{bmatrix}$$

$$x + 5y + 7z = 15$$

$$\Rightarrow \quad y + \frac{10}{7}z = \frac{19}{7} \qquad \ldots(M)$$

$$\frac{4}{7}z = \frac{16}{7}$$

From which we get rank of A = 3 as well as rank of A : B = 3. Hence the system of equations is consistent and has unique solution $\frac{4}{7}z = \frac{16}{7} \quad \Rightarrow z = 4$

And $\quad y + \frac{10}{7}z = \frac{19}{7} \quad \Rightarrow \quad y + \frac{10}{7} \times 4 = \frac{19}{7} \quad \Rightarrow y = -\frac{21}{7} = -3$

And from (M), we have $x + 5y + 7z = 15 \Rightarrow x = 2$

i.e. we have the solution $x = 2$, $y = -3$ and $z = 4$, which is the required result.

Example 2. Test the following equations for consistency and hence solve these equations $2x - 3y + 7z = 5$, $3x + y - 3z = 13$ and $2x + 19y - 47z = 32$.

Sol. The above equations may be written as AX = B.

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Operating $R_2 \to 2R_2 - 3R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11 & -27 \\ 0 & +22 & -54 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 27 \end{bmatrix}$$

Next, we operate $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11 & -27 \\ 0 & +22 & -54 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 27 \end{bmatrix}$$

This indicate the rank of A = 2 which is less than 3 (the number of variables) i.e.
$\rho(A) = 2 < 3$

So, the given equations are not consistent and so infinite number of solutions can be obtained.

Example 3. Show that if $\lambda \neq -5$, the system of equation $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solution, in each case.

Sol. The given equations are

$$\left.\begin{aligned} &3x - y + 4z = 3, \\ &x + 2y - 3z = -2 \\ \text{and} \quad &6x + 5y + \lambda z = -3 \end{aligned}\right\} \qquad \ldots(1)$$

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$ such that AX = B from (1)

Then augmented matrix $A : B = \begin{bmatrix} 3 & -1 & 4:3 \\ 1 & 2 & -3:-2 \\ 6 & 5 & \lambda:-3 \end{bmatrix}$

Operating R_{12}(i.e. interchanging R_1 and R_2)

$$A : B = \begin{bmatrix} 1 & 2 & -3:-2 \\ 3 & -1 & 4:3 \\ 6 & 5 & \lambda:-3 \end{bmatrix}$$

Now operating $R_2 - 3R_1$ [i.e. R_2, 1(−3)] and R_3, 1(−6) i.e. $R_3 - 6R_1$, we get

$$A : B \sim \begin{bmatrix} 1 & 2 & 3 \ :-2 \\ 0 & -7 & 13 \ :9 \\ 0 & -7 & \lambda+18:9 \end{bmatrix}$$

Next, $R_3 - R_2$[(i.e. R_3, 2(−1)], we get

$$\sim \begin{bmatrix} 1 & 2 & -3:-2 \\ 0 & -7 & 13:9 \\ 0 & 0 & \lambda+5:0 \end{bmatrix} \qquad \ldots(2)$$

If $\lambda = -5$, then rank of A becomes $\rho(A) = 2$ which is less than 3, (the number of unknowns) and hence the equations will be consistent and will have infinite number of solutions

Next, operating, $R_1 + \frac{2}{7}R$, we get

$$\sim \begin{bmatrix} 1 & 0 & 5:\frac{4}{7} \\ 0 & -7 & 13:9 \\ 0 & 0 & \lambda+5:0 \end{bmatrix}$$ from this matrix, if $\lambda \neq -5$

then rank is 3 and the equation will be consistent and we get

$$x + \frac{5}{7}z = \frac{4}{7} \ ; \ -7y + 13z = 9 \text{ and } (\lambda + 5)z = 0 \text{ i.e. } z = 0$$

$\Rightarrow \quad -7y = 9 \Rightarrow \quad y = -\frac{9}{7}$ and $x + 0 = \frac{4}{7}$ i.e. $x = \frac{4}{7}$.

i.e. unique solution is $x = \frac{4}{7}$, $y = -\frac{9}{7}$, $z = 0$, which is required result.

If $\lambda = -5$, then from (2), we have $x + 2y - 3z = -2$, $-7y + 13z = 9$ …(3)

If we take z = k than from (3),

$$y = \frac{13k-9}{7} \quad \text{and} \quad z = \frac{3k + 2\left(\frac{13k-9}{7}\right) - 2}{3} - \frac{4-5k}{7}$$

Example 4. Examine whether the following equations are consistent and solve them if they are consistent $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$ and

6y − 18z + 1 = 0.

Sol. The above equations may be written in the form

$$AX = B \text{ which is } \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix} \qquad \ldots(1)$$

Now the augmented matrix may be written as

$$A : B = \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \qquad \ldots(2)$$

Operating $R_2 \to R_2 - 3R_1$, we get

$$A : B \sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

Now, operating $R_3 \to R_3 - 3R_2$, we get

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix}$$

Hence rank of A = $\rho(A) = 2$ and $\rho(A : B) = 3$. So, $\rho(A) = 2 < 3$ (number of variables). This indicated that given equation are in consistent and so it has no unique solution.

Example 5. Solve the following system of equations by matrix method x + y + z = 8, x − y + 2z = 6 and 3x + 5y − 7z = 14.

Sol. The above equations written in the form AX = B.

where $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix}$$

So, we may write augmented matrix as

$$A : B = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & -1 & 2 & : & 6 \\ 3 & 5 & -7 & : & 14 \end{bmatrix} \qquad \ldots(1)$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$, we have

$$A : B \sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 2 & 10 & : & 10 \end{bmatrix} \qquad \ldots(2)$$

Again $R_3 \to R_3 + R_2$, we have

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 0 & -9 & : & -12 \end{bmatrix}$$

this implies that

$$\left.\begin{aligned} x + y + z &= 8 \\ -2y + z &= -2 \end{aligned}\right\} \qquad \ldots(3)$$

and $-9z = -12$

$\Rightarrow$ $z = \frac{4}{3}$ and $2y = z + 2 = \frac{4}{3} + 2 = \frac{10}{3}$ $\therefore$ $y = \frac{5}{3}$

Using 1st equation of (3), we get $x + y + z = 8$

$\Rightarrow$ $x + \frac{5}{3} + \frac{4}{3} = 8$ $\Rightarrow$ $x = 8 - 3 = 5$

From (2) we see that $\rho(A) = 3$ = number of variables so, the system of equations are consistent and solutions are $x = 5$, $y = \frac{5}{3}$, $z = \frac{4}{3}$.

Example 6. Determine for what values of λ and μ the following equations have (i) no solution ii) a unique solution (iii) infinite number of solution : $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$.

Sol. The above equations may be written in the form AX = B.

i.e. $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

Again operating $R_3 - R_2$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

$\Rightarrow$ we get $x + y + z = 6$, $y + 2z = 4$ and $(\lambda - 3)z = \mu - 10$.

(i) If $R(A) \neq R[A : B]$ i.e. if $\lambda - 3 = 0$ and $\mu - 10 \neq 0$, then rank of A $\neq$ rank of [A : B]. Since $\rho(A) = 2$ and $\rho(A : B) = 3$. The equation have no solution.

(ii) The equations have unique solution if rank of A = rank of [A : B] = 3, i.e. if $\lambda - 3 \neq 0$ and $\mu - 3 \neq 0$.

(iii) If $\rho(A) = \rho(A : B) = 2$ i.e. when $\lambda - 3 = 0$ and $\mu - 10 = 0$ i.e. when $\lambda = 3$ and $\mu = 10$. Then these are infinite number of solution.

System of Homogeneous Linear Equations

If

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = \end{array}\right\} \quad \ldots(1)$$

be given system of m linear equations then (1) may be written as AX=O

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ \dots \\ 0 \end{bmatrix}$$

Here A is called the coefficient matrix and the given system of equations AX=O is called linear homogeneous system of equations.

Working rule for determining solution of m homogeneous equations in n variables.

Firstly we find the rank of coefficient matrix A. Then

1. There is only a trivial solution which is $x_1=x_2=.....=x_n=0$ if $\rho(A) = n$.
2. A can be reduced to a matrix which has (n-r) zero rows and r non zero rows and if $\rho(A) < n$ so the system is consistent and has infinite number of solutions.

Thus, the given system of equations has a non- trivial solution iff $|A| = 0$

Example 1: Solve the following system of equations

$$x - y + z = 0$$
$$x + 2y - z = 0$$
$$2x + y + 3z = 0$$

Solution. Writing the given equations in the matrix form, we have

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or AX=O, where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Operating $R_2 \to R_2+(-R_1)$ and $R_3 \to R_3+(-2)R_1,$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3+(-R_2)$, $A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$

Operating $R_2 \to R_2 \times \left(\frac{1}{3}\right)$ and $R_3 \to R_3 \times \left(\frac{1}{3}\right)$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore\ \rho(A)=3$ = number of variables and hence the given system of equations has only trivial solution, x = y = z = 0.

Example: Solve the following system of equations:

x - y + 2z - 3w = 0

3x + 2y - 4z + w = 0

4x - 2y + 9w = 0

Solution: Writing the given equations in the matrix form, we have

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or AX=O, where $A=\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}$

Operating $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 4R_1$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 2 & -8 & 21 \end{bmatrix}$$

Operating $R_2 \to R_2\left(\frac{1}{5}\right)$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -8 & 21 \end{bmatrix}$$

Operating $R_3 \to R_3 - 2R_2$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \end{bmatrix}$$

$\therefore\ \rho(A)=3$, Here n = 4 (the number of unknowns)

Now $\rho(A)<4$. Thus the system of equations has infinite solutions. The solutions will contain 4 – 3=1 arbitrary constant.

Equation corresponding to the matrix are

x - y + 2z - 3w = 0 (1)

y - 2z + 2w = 0 (2)

- 4z + 17w = 0 (3)

From (3), $z=\frac{17}{4}w$

$\therefore$ From (2), $y - \frac{17}{2}w + 2w = 0 \Rightarrow y = \frac{13}{2}w$

$\therefore$ From (1), $x - \frac{13}{2}w + \frac{17}{2}w - 3w = 0 \Rightarrow x = w$

Putting w = k, we get $x = k$, $y = \frac{13}{2}k$, $z = \frac{17}{4}k$, which is the general solution, where k is an arbitrary parameter.

Check Your Progress

1. Solve the following system of liear equation

$x - y + z = 0$

$x + 2y - z = 0$

$2x + y + 3z = 0$

Ans. x = y = z = 0.

2. Find the values of a and b for which the following system of linear equations

$2x + by - z = 3$

$5x + 7y + z = 7$.

$ax + y + 3z = a$

Ans. a = 1 and b = 3.

APPLICATION OF LINEAR SYSTEM

Three by three systems of linear equations are also used to solve real-life problems. The given problem is expressed as a system of linear equations and then solved to determine the value of the variables. Sometimes, the system will consist of three equations but not every equation will have three variables. Example three is one such problem.

Example 1: Solve the following problem using your knowledge of systems of linear equations.

Jesse, Maria and Charles went to the local craft store to purchase supplies for making decorations for the upcoming dance at the high school. Jesse purchased three sheets of craft paper, four boxes of markers and five glue sticks. His bill, before taxes was $24.40. Maria spent $30.40 when she bought six sheets of craft paper, five boxes of markers and two glue sticks. Charles, purchases totaled $13.40 when he bought three sheets of craft paper, two boxes of markers and one glue stick. Determine the unit cost of each item.

Let **p** represent the number of sheets of craft paper.
Let **m** represent the number of boxes of markers.
Let **g** represent the number of glue sticks.

Express the problem as a system of linear equations:

$3p+4m+5g=\$24.40$
$6p+5m+2g=\$30.40$
$3p+2m+g=\$13.40$

Solve the system of linear equations to determine the unit cost of each item.

$$\begin{array}{l}3p+4m+5g=24.40\\3p+2m+g=13.40\end{array}\Rightarrow\begin{array}{l}3p+4m+5g=24.40\\-5(3p+2m+g=13.40)\end{array}\Rightarrow\begin{array}{l}3p+4m+5g=24.40\\\underline{-15p-10m-5g=-67.00}\end{array}$$
$-12p-6m=-42.60$

$$\begin{array}{l}6p+5m+2g=30.40\\3p+2m+g=13.40\end{array}\Rightarrow\begin{array}{l}6p+5m+2g=30.40\\-2(3p+2m+g=13.40)\end{array}\Rightarrow\begin{array}{l}6p+5m+2g=30.40\\\underline{-6p-4m-2g=-26.80}\end{array}$$
$m=3.60$

$-12p-6m=-42.60$
$-12p-6(3.60)=-42.60$
$-12p-21.60=-42.60$
$-12p-21.60+21.60=-42.60+21.60$
$-12p=-21$
$\frac{-12}{-12}p=\frac{-21}{-12}$
$p=1.75$

$3p+2m+g=13.40$
$3(1.75)+2(3.60)+g=13.40$
$5.25+7.20+g=13.40$
$12.45+g=13.40$
$12.45-12.45+g=13.40-12.45$
$g=.95$

The unit cost of each item is: 1 sheet of craft paper = \$1.75
1 box of markers = \$3.60
1 glue stick = \$0.95

Example 2: Solve the following problem using your knowledge of systems of linear equations.

Rafael, an exchange student from Brazil, made phone calls within Canada, to the United States, and to Brazil. The rates per minute for these calls vary for the different countries. Use the information in the following table to determine the rates.

Month	**Time within Canada (min)**	**Time to the U.S. (min)**	**Time to Brazil (min)**	**Charges (\$)**
September	90	120	180	\$252.00
October	70	100	120	\$184.00
November	50	110	150	\$206.00

Let **c** represent the rate for calls within Canada.
Let **u** represent the rate for calls to the United States.

Let **b** represent the rate for calls to Brazil.

Express the problem as a system of linear equations:

$$90c+120u+180b=\$252.00$$
$$70c+100u+120b=\$184.00$$
$$50c+110u+150b=\$206.00$$

$$\begin{array}{l}90c+120u+180b=252.00\\70c+100u+120b=184.00\end{array}\Rightarrow\begin{array}{l}2(90c+120u+180b=252.00)\\-3(70c+100u+120b=184.00)\end{array}$$

$$\Rightarrow\begin{array}{l}180c+240u+360b=504.00\\\underline{-210c-300u-360b=-552.00}\end{array}$$
$$-30c-60u=-48.00$$

$$\begin{array}{l}70c+100u+120b=184.00\\50c+110u+150b=206.00\end{array}\Rightarrow\begin{array}{l}-5(70c+100u+120b=184.00)\\4(50c+110u+150b=206.00)\end{array}$$

$$\Rightarrow\begin{array}{l}-350c-500u-600b=-920.00\\\underline{200c+440u+600b=824.00}\end{array}$$
$$-150c-60u=-96.00$$

$$\begin{array}{l}-30c-60u=-48.00\\-150c-60u=-96.00\end{array}\Rightarrow\begin{array}{l}-1(-30c-60u=-48.00)\\-150c-60u=-96.00\end{array}\Rightarrow\begin{array}{l}30c+60u=48.00\\\underline{-150c-60u=-96.00}\end{array}$$
$$-120c=-48.00$$
$$\frac{-120}{-120}c=\frac{-48.00}{-120}$$
$$c=.40$$

$$-30c-60u=-48.00$$
$$-30(.40)-60u=-48.00$$
$$-12.00-60u=-48.00$$
$$-12.00+12.00-60u=-48.00+12.00$$
$$-60u=-36.00$$
$$\frac{-60}{-60}u=\frac{-36.00}{-60}$$
$$u=.60$$

$$70c+100u+120b=184.00$$
$$70(.40)+100(.60)+120b=184.00$$
$$28.00+60.00+120b=184.00$$
$$88.00+120b=184.00$$
$$88.00-88.00+120b=184.00-88.00$$
$$120b=96.00$$
$$\frac{120}{120}b=\frac{96.00}{120}$$
$$b=.80$$

The cost of minutes within Canada is \$0.40/min. The cost of minutes to the United States is \$0.60/min. The cost of minutes to Brazil is \$0.80/min.

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Possible Questions

2 Mark Questions

1. Define the systems of Linear equations
2. Define the row reduction echelon matrix with example.
3. Define the row equivalent matrix.
4. What do you mean by Linear Independence?
5. When we say that the system is homogeneous.
6. In which case the linear equations are equivalent.
7. What do you mean by Linear dependence?
8. When we say that the system is Non-homogeneous.

6 Mark Questions

1.Determine if b is a linear combination of $a_1$and a_2 where $a_1=\begin{bmatrix}1\\-2\\3\end{bmatrix}$, $a_2=\begin{bmatrix}5\\-13\\-3\end{bmatrix}$ and $b=\begin{bmatrix}-3\\8\\1\end{bmatrix}$

2.Determine the system is consistent

$$x_1-6x_2=5$$
$$x_2-4x_3+x_4=0$$
$$-x_1+6x_2+x_3+5x_4=3$$
$$-x_2+5x_3+4x_4=0$$

3.Determine if the system is consistent $\begin{bmatrix}1&5&2&-6\\0&4&-7&2\\0&0&5&0\end{bmatrix}$

4.Let $A=\begin{bmatrix}2&5\\3&1\end{bmatrix}$,$u=\begin{bmatrix}4\\-1\end{bmatrix}$ and $v=\begin{bmatrix}-3\\5\end{bmatrix}$ Verify i) A(u+v)=Au+Av ii)A(5u)= 5A(u).

5.Find the general solutions of the system whose augmented matrix is $\begin{bmatrix}3&-4&2&0\\-9&12&-6&0\\-6&8&-4&0\end{bmatrix}$

6.Describe the solution of $AX = B$ where $A = \begin{bmatrix}3&5&6\\-3&-2&1\\6&1&-8\end{bmatrix}$ and $b = \begin{bmatrix}7\\-1\\-4\end{bmatrix}$

7.If $A=\begin{pmatrix}3&-1&2\\2&1&1\\1&-3&0\end{pmatrix}$ find all solutions of AX=0 by row reducing A.

8.In $V_3(R)$ the vectors (1,2,1) ,(2,1,0) and (1,-1,2) are linearly independent or not

9.Find a row reduced echelon matrix which is row equivalent to

$A = \begin{bmatrix}1&-i\\2&2\\i&1+i\end{bmatrix}$ What are the solutions of AX=0?

10.Let $v_1=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $v_2=\begin{bmatrix}4\\5\\6\end{bmatrix}$ and $v_3=\begin{bmatrix}2\\1\\0\end{bmatrix}$,

i) Determine if the set $\{v_1, v_2,v_3\}$ is linearly independent.
ii) If possible, find a linear dependence relation among v_1, v_2, and v_3

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Subject: Algebra **Subject Code: 19MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit III

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Any n-tuple of elements of F which satisfies each of the equations in linear equation is called a ………………………….of the system.	value	root	solution	function	solution
Any………..-tuple of elements of F which satisfies each of the equations in linear equation is called a solution of the system.	1	2	3	n	n
Any n-tuple of elements of F which satisfies each of the ………… in linear equation is called a solution of the system.	functions	equations	roots	solutions	equations
If y1=y2=……….=ym=0 then the system is ………………………………..	homogeneous	non homogeneous	linear	nonlinear	homogeneous
If y1=y2=……….=ym=…………..then the system is homogeneous.	0	1	2	3	0
The most fundamental technique for finding the solution of a system of linear equations is the technique of …………………………	substitution	elimination	integration by parts	differention	elimination
The most fundamental technique for finding the solution of a system of……………….equations is the technique of elimination.	integral	differential	linear	nonlinear	linear

The most fundamental technique for finding the ……………. of a system of linear equations is the technique of elimination.	function	root	solution	value	solution
………… systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.	one	Two	three	four	Two
Two systems of linear equations are …….. if each equation in each system is a linear combination of the equations in the other system.	zero	equivalent	different	division	equivalent
Two systems of linear equations are equivalent if each equation in each system is a ………………. combination of the equations in the other system.	linear	non linear	homogeneous	non homogeneous	linear
Two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the …………system.	first	same	other	finite	same
………………….systems of linear equations have exactly the same solutions.	linear	nonlinear	Equivalent	homogeneous	Equivalent
Equivalent systems of ………………...equations have exactly the same solutions.	linear	non linear	homogeneous	non homogeneous	linear
Equivalent systems of linear equations have exactly the……………. solutions.	zero	same	different	finite	same
An …………...matrix R is called a row reduced echelon matrix if R is row reduced.	mxm	nxn	mxn	nxm	mxn
An mxn matrix R is called a ………………….. matrix if R is row reduced.	row reduced echelon	column reduced echelon	echelon	null	row reduced echelon
An mxn matrix R is called a row reduced echelon matrix if R is ……………	unit	null	column reduced	row reduced	row reduced
In the row reduced echelon form every ………….. R which has all its entries 0 occurs below every row has a non zero entry.	row	column	unit	singular	row

In the row reduced echelon form every row R which has all its entries …………. occurs below every row has a non zero entry.	0	1	2	3	0
In the row reduced echelon form every row R which has all its entries 0 occurs below every row has a ……………...entry.	zero	non zero	unit	diagonal	non zero
In the ……………….. form every row R which has all its entries 0 occurs below every row has a non zero entry.	row reduced echelon	column reduced echelon	echelon	null	row reduced echelon
An ……….. matrix R is called row reduced if the first non zero entry in each non zero row of R is equal to 1	mxm	nxn	mxn	nxm	mxn
An mxn matrix R is called ……………….. if the first non zero entry in each non zero row of R is equal to 1	row reduced echelon	column reduced echelon	rowreduced	column reduced	rowreduced
An mxn matrix R is called row reduced if the first …………. entry in each non zero row of R is equal to 1	zero	non zero	diagonal	unit	non zero
An mxn matrix R is called row reduced if the first non zero entry in each non zero row of R is equal to …………..	0	1	2	3	1
In row reduced, each ……………... of R which contains the leading non zero entry of some row has all its other entries 0.	row	column	diagonal	first	column
In row reduced, each column of R which contains the…………….. non zero entry of some row has all its other entries 0.	first	second	third	leading	leading
In row reduced, each column of R which contains the leading …………….. entry of some row has all its other entries 0.	zero	non zero	diagonal	unit	non zero
In row reduced, each column of R which contains the leading non zero entry of some ……………. has all its other entries 0.	row	column	diagonal	first	row

In row reduced, each column of R which contains the leading non zero entry of some row has all its other entries………………..	0	1	2	3	0
Every ……………….. matrix A is row equivalent to a row reduced echelon matrix.	mxm	nxn	mxn	nxm	mxn
Every mxn matrix A is ………………….equivalent to a row reduced echelon matrix.	row	column	diagonal	leading	row
Every mxn matrix A is row equivalent to a ………………….. matrix.	row reduced echelon	column reduced echelon	echelon	null	row reduced echelon
If A is an mxn matrix and ……………..,then the homogeneous system of linear equations AX=0 has a non- trivial solution.	m<n	m>n	m=n	m-n	m<n
If A is an mxn matrix and m<n,then the………………...system of linear equations AX=0 has a non- trivial solution.	homogeneous	non homogeneous	linear	nonlinear	homogeneous
If A is an mxn matrix and m<n,then the homogeneous system of linear equations AX=…………. has a non- trivial solution.	0	1	2	3	0
If A is an mxn matrix and m<n,then the homogeneous system of linear equations AX=0 has a …………solution.	trivial	non- trivial	zero	non- zero	non- trivial
If A is an ………….. matrix,then A is row equivalent to the nxn identity matrix iff the system of equations AX=0 has only the trivial solution.	mxm	nxn	mxn	nxm	nxn
If A is an nxn matrix,then A is ………...to the nxn identity matrix iff the system of equations AX=0 has only the trivial solution.	row equivalent	column equivalent	diagonal	leading	row equivalent
If A is an nxn matrix,then A is row equivalent to the nxn ……………….. matrix iff the system of equations AX=0 has only the trivial solution.	zero	identity	row	column	identity
If A is an nxn matrix,then A is row equivalent to the nxn identity matrix iff the system of equations …………….. has only the trivial solution.	AX=I	AX=0	AX=R	AX=B	AX=0

If A is an nxn matrix,then A is row equivalent to the nxn identity matrix iff the system of equations AX=0 has only the ……………….solution.	trivial	non- trivial	zero	non- zero	trivial

UNIT-IV

SYLLABUS

Roots of an equation- Relations connecting the roots and coefficients- Transformations of equations - Character and position of roots-Descarte's rule of signs-Symmetric function of roots-Reciprocal equations.

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23. **Relations between the Roots and Coefficients.**—Taking for simplicity the coefficient of the highest power of x as unity, and representing, as in Art. 16, the n roots of an equation by $a_1, a_2, a_3, \ldots a_n$, we have the following identity :—

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n$$
$$= (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n). \qquad (1)$$

When the factors of the second member of this identity are multiplied together, the product will consist, as is proved in elementary works on Algebra, of a highest term x^n; plus a term x^{n-1} multiplied by the factor

$$-(a_1 + a_2 + a_3 + \ldots + a_n),$$

i. e. the sum of the roots with their signs changed; plus a term x^{n-2} multiplied by the factor

$$a_1 a_2 + a_1 a_3 + a_2 a_3 + \ldots + a_{n-1} a_n,$$

i. e. the sum of the products of the roots taken in pairs; plus a term x^{n-3} multiplied by the factor

$$-(a_1 a_2 a_3 + a_1 a_2 a_4 + \ldots + a_{n-2} a_{n-1} a_n),$$

i. e. the sum of the products of the roots with their signs changed taken three by three; and so on. It is plain that the sign of each coefficient will be negative or positive according as the number of roots in each product is odd or even. The last term is

$$\pm a_1 a_2 a_3 \ldots a_{n-1} a_n,$$

the sign being $-$ if n is odd, and $+$ if n is even. Equating the coefficients of x on each side of the identity (1), we have the following series of equations:—

$$\left.\begin{aligned} p_1 &= -(a_1 + a_2 + a_3 + \ldots + a_n), \\ p_2 &= (a_1 a_2 + a_1 a_3 + a_2 a_3 + \ldots + a_{n-1} a_n), \\ p_3 &= -(a_1 a_2 a_3 + a_1 a_3 a_4 + \ldots + a_{n-2} a_{n-1} a_n), \\ &\cdots \\ p_n &= (-1)^n a_1 a_2 a_3 \ldots a_{n-1} a_n, \end{aligned}\right\} \quad (2)$$

which furnish us with the following

Theorem.—*In every algebraic equation, the coefficient of whose highest term is unity, the coefficient p_1 of the second term with its sign changed is equal to the sum of the roots.*

The coefficient p_2 of the third term is equal to the sum of the products of the roots taken two by two.

The coefficient p_3 of the fourth term with its sign changed is equal to the sum of the products of the roots taken three by three; and so on, the signs of the coefficients being taken alternately negative and positive, and the number of roots multiplied together in each term of the corresponding function of the roots increasing by unity, till finally that function is reached which consists of the product of the n roots.

Cor. 1.—Every root of an equation is a divisor of the last term.

Cor. 2.—If the roots of an equation be all positive, the coefficients will be alternately positive and negative; and if the roots be all negative, the coefficients will be all positive. This is obvious from the equations (2) [cf. Arts. 19 and 20].

1. Solve the equation

$$x^3 - 5x^2 - 16x + 80 = 0,$$

the sum of two of its roots being equal to nothing.

Let the roots be α, β, γ. We have, then,

$$\alpha + \beta + \gamma = 5,$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -16,$$

$$\alpha\beta\gamma = -80.$$

Taking $\beta + \gamma = 0$, we have, from the first of these, $\alpha = 5$, and from either second or third we obtain $\beta\gamma = -16$. We find for β and γ the values 4 and -4. the three roots are 5, 4, -4.

2. Solve the equation

$$x^3 - 3x^2 + 4 = 0,$$

two of its roots being equal.

Let the roots be α, α, β. We have

$$2\alpha + \beta = 3,$$

$$\alpha^2 + 2\alpha\beta = 0,$$

from which we find $\alpha = 2$, and $\beta = -1$. The roots are 2, 2, -1.

3. The equation

$$x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$$

has two pairs of equal roots; find them.

Let the roots be α, α, β, β; we have

$$2\alpha + 2\beta = -4,$$

$$\alpha^2 + \beta^2 + 4\alpha\beta = -2,$$

from which we obtain for α and β the values 1 and -3.

4. Solve the equation

$$x^3 - 9x^2 + 14x + 24 = 0,$$

two of whose roots are in the ratio of 3 to 2.

25. **Depression of an Equation when a relation exists between two of its Roots.**—The examples given under the preceding Article illustrate the use of the equations connecting the roots and coefficients in determining the roots in particular cases when known relations exist among them. The object of the present Article is to show that, in general, *if a relation of the form $\beta = \phi(a)$ exist between two of the roots of an equation $f(x) = 0$, the equation may be depressed two dimensions.*

Let $\phi(x)$ be substituted for x in the identity

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n,$$

then $f(\phi(x)) = a_0(\phi(x))^n + a_1(\phi(x))^{n-1} + \ldots + a_{n-1}\phi(x) + a_n.$

We represent, for convenience, the second member of this identity by $F(x)$. Substitute a for x, then

$$F(a) = f(\phi(a)) = f(\beta) = 0;$$

hence a satisfies the equation $F(x) = 0$, and it also satisfies the equation $f(x) = 0$; hence the polynomials $f(x)$ and $F(x)$ have a common measure $x - a$; thus a can be determined, and from it $\phi(a)$ or β, and the given equation can be depressed two dimensions.

EXAMPLES.

1. The equation

$$x^3 - 5x^2 - 4x + 20 = 0$$

has two roots whose difference $= 3$: find them.

Here $\beta - a = 3$, $\beta = 3 + a$; substitute $x + 3$ for x in the given polynomial $f(x)$; it becomes $x^3 + 4x^2 - 7x - 10$; the common measure of this and $f(x)$ is $x - 2$; from which $a = 2$, $\beta = 5$; the third root is -2.

2. The equation

$$x^4 - 5x^3 + 11x^2 - 13x + 6 = 0$$

has two roots connected by the relation $2\beta + 3a = 7$: find all the roots.

Ans. $1, \quad 2, \quad 1 \pm \sqrt{-2}$.

EXAMPLES.

1. The equations

$$2x^3+5x^2-6x-9=0,$$

$$3x^3+7x^2-11x-15=0,$$

have two common roots, find them. *Ans.* $-1, \quad -3$.

2. The equations

$$x^3+px^2+qx+r=0,$$

$$x^3+p'x^2+q'x+r'=0,$$

have two common roots; find the quadratic which furnishes them, and also the 3rd root of each.

Ans. $x^2+\frac{q-q'}{p-p'}x+\frac{r-r'}{p-p'}=0, \quad \frac{-r(p-p')}{r-r'}, \quad \frac{-r'(p-p')}{r-r'}$.

26. **The Cube Roots of Unity.**—Equations of the forms

$$x^n-1=0, \quad x^n+1=0$$

are called *binomial*. The roots of the former are called the n n^{th} *roots of unity*. A general discussion of these forms will be given in a subsequent Chapter. We confine ourselves at present to the simple case of the binomial cubic, for which certain useful properties of the roots can be easily established. It has been already shown (see Ex. 5, Art. 16), that the roots of the cubic

$$x^3-1=0$$

are $$1, \quad -\frac{1}{2}+\frac{1}{2}\sqrt{-3}, \quad -\frac{1}{2}-\frac{1}{2}\sqrt{-3}.$$

If ω be a root of the cubic, ω^2 must also be a root; for, since $\omega^3 = 1$, we get, by squaring, $\omega^6 = 1$, which is $(\omega^2)^3 = 1$, thus showing that ω^2 satisfies the cubic $x^3 - 1 = 0$. We have then the identity

$$x^3 - 1 \equiv (x-1)(x-\omega)(x-\omega^2).$$

Changing x into $-x$, we get the following identity also :—

$$x^3 + 1 \equiv (x+1)(x+\omega)(x+\omega^2),$$

which furnishes the roots of

$$x^3 + 1 = 0.$$

Whenever in any product of quantities involving the imaginary cube roots of unity any power higher than the second presents itself, it can be replaced by ω, or ω^2, or by unity; for example,

$$\omega^4 = \omega^3 . \omega = \omega, \quad \omega^5 = \omega^3 . \omega^2 = \omega^2, \quad \omega^6 = \omega^3 . \omega^3 = 1, \text{ \&c.}$$

The first or second of equations (2), Art. 23, gives the following property of the imaginary cube roots :—

$$1 + \omega + \omega^2 = 0.$$

By the aid of this equation any expression involving real quantities and the imaginary cube roots can be written in either of the forms $P + \omega Q$, $P + \omega^2 Q$.

EXAMPLES.

1. Show that the product

$$(\omega m + \omega^2 n)(\omega^2 m + \omega n)$$

is rational. *Ans.* $m^2 - mn + n^2$.

2. Prove the following identities :—

$$m^3 + n^3 \equiv (m+n)(\omega m + \omega^2 n)(\omega^2 m + \omega n),$$

$$m^3 - n^3 \equiv (m-n)(\omega m - \omega^2 m)(\omega^2 m - \omega n).$$

3. Show that the product

$$(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma)$$

is rational. *Ans.* $\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta$.

4. Prove the identity

$$(\alpha + \beta + \gamma)(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta - \omega\gamma) \equiv \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma.$$

5. Prove the identity

$$(\alpha + \omega\beta + \omega^2\gamma)^3 + (\alpha + \omega^2\beta + \omega\gamma)^3 \equiv (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta).$$

[Apply Ex. 2.]

6. Prove the identity

$$(\alpha + \omega\beta + \omega^2\gamma)^3 - (\alpha + \omega^2\beta + \omega\gamma)^3 \equiv -3\sqrt{-3}\,(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

[Apply Ex. 2, and substitute for $\omega - \omega^2$ its value $\sqrt{-3}$.]

7. Prove the identity

$$\alpha'^3 + \beta'^3 + \gamma'^3 - 3\alpha'\beta'\gamma' \equiv (\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma)^2,$$

where

$$\alpha' \equiv \alpha^2 + 2\beta\gamma, \quad \beta' \equiv \beta^2 + 2\gamma\alpha, \quad \gamma' \equiv \gamma^2 + 2\alpha\beta.$$

8. Find the equation whose roots are

$$m + n, \quad \omega m + \omega^2 n, \quad \omega^2 m + \omega n.$$

Ans. $x^3 - 3mnx - (m^3 + n^3) = 0$.

9. Find the equation whose roots are

$$l + m + n, \quad l + \omega m + \omega^2 n, \quad l + \omega^2 m + \omega n.$$

Ans. $x^3 - 3lx^2 + 3(l^2 - mn)x - (l^3 + m^3 + n^3 - 3lmn) = 0$.

11 Form an equation with rational coefficients which shall have

$$\theta_1\sqrt[3]{P} + \theta_2\sqrt[3]{Q}$$

for a root, where $\theta_1^3 = 1$, and $\theta_2^3 = 1$.

Cubing the equation

$$x = \theta_1\sqrt[3]{P} + \theta_2\sqrt[3]{Q},$$

and substituting x for its value on the right-hand side, we get

$$x^3 - P - Q = 3\,\theta_1\theta_2\sqrt[3]{PQ}\,.\,x.$$

Cubing again, we have

$$(x^3 - P - Q)^3 = 27\,PQx^3.$$

Since θ_1 and θ_2 may each have any one of the values 1, ω, ω^2, the nine roots of this equation are

$$\sqrt[3]{P} + \sqrt[3]{Q}, \qquad \omega\sqrt[3]{P} + \omega\sqrt[3]{Q}, \qquad \omega^2\sqrt[3]{P} + \omega^2\sqrt[3]{Q},$$

$$\omega\sqrt[3]{P} + \omega^2\sqrt[3]{Q}, \qquad \omega^2\sqrt[3]{P} + \sqrt[3]{Q}, \qquad \omega\sqrt[3]{P} + \sqrt[3]{Q},$$

$$\omega^2\sqrt[3]{P} + \omega\sqrt[3]{Q}, \qquad \sqrt[3]{P} + \omega^2\sqrt[3]{Q}, \qquad \sqrt[3]{P} + \omega\sqrt[3]{Q}.$$

We see also that, since θ_1 and θ_2 have disappeared from the final equation, it is indifferent which of these nine roots is assumed equal to x in the first instance. The resulting equation is that which would have been obtained by multiplying together the nine factors of the form $x - \sqrt[3]{P} - \sqrt[3]{Q}$ obtained from the nine roots above written.

EXAMPLES.

1. Find the value of $\Sigma\alpha^2\beta$ of the roots of the cubic

$$x^3+px^2+qx+r=0.$$

Multiplying together the equations

$$\alpha+\beta+\gamma=-p,$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=q,$$

we obtain

$$\Sigma\alpha^2\beta+3\alpha\beta\gamma=-pq;$$

hence

$$\Sigma\alpha^2\beta=3r-pq.$$

2. Find for the same cubic the value of

$$\alpha^2+\beta^2+\gamma^2. \qquad \textit{Ans. } \Sigma\alpha^2=p^2-2q.$$

3. Find for the same cubic the value of

$$\alpha^3+\beta^3+\gamma^3.$$

Multiplying the values of $\Sigma\alpha$ and $\Sigma\alpha^2$, we obtain

$$\alpha^3+\beta^3+\gamma^3+\Sigma\alpha^2\beta=-p^3+2pq;$$

hence, by Ex. 1,

$$\Sigma\alpha^3=-p^3+3pq-3r.$$

4. Find for the same cubic the value of

$$\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2.$$

We easily obtain

$$\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)=q^2,$$

from which

$$\Sigma\alpha^2\beta^2=q^2-2pr.$$

5. Find for the same cubic the value of

$$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta).$$

This is equal to

$$2\alpha\beta\gamma+\Sigma\alpha^2\beta. \qquad \textit{Ans. } r-pq.$$

6. Find the value of the symmetric function

6. Find the value of the symmetric function

$$\alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \beta^2\gamma\delta$$
$$+ \gamma^2\alpha\beta + \gamma^2\alpha\delta + \gamma^2\beta\delta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma$$

of the roots of the biquadratic

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

Multiplying together

$$\alpha + \beta + \gamma + \delta = -p,$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r,$$

we obtain

$$\Sigma\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta = pr;$$

hence

$$\Sigma\alpha^2\beta\gamma = pr - 4s.$$

12. Find the sum of the reciprocals of the roots of the equation in the preceding example.

From the second last, and last of the equations of Art. 23, we have

$$\alpha_2\alpha_3 \ldots \alpha_n + \alpha_1\alpha_3 \ldots \alpha_n + \ldots + \alpha_1\alpha_2 \ldots \alpha_{n-1} = (-1)^{n-1}p_{n-1},$$
$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n p_n;$$

dividing the former by the latter, we have

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \ldots + \frac{1}{\alpha_n} = \frac{-p_{n-1}}{p_n},$$

or

$$\Sigma\frac{1}{\alpha_1} = \frac{-p_{n-1}}{p_n}.$$

In a similar manner the sum of the products in pairs, in threes, &c. of the reciprocals of the roots can be found by dividing the 3rd last, or 4th last, &c. coefficient by the last.

13. Find for the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

the values, in terms of the coefficients, of the following three functions of the roo α, β, γ:—

$$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2,$$

$$\alpha(\beta-\gamma)^2 + \beta\ (\gamma-\alpha)^2 + \gamma\ (\alpha-\beta)^2,$$

$$\alpha^2(\beta-\gamma)^2 + \beta^2(\gamma-\alpha)^2 + \gamma^2(\alpha-\beta)^2.$$

It will be often found convenient to write, as in the present example, an equation with *binomial coefficients*, that is, numerical coefficients corresponding to those i the expansion by the binomial theorem, in addition to the literal coefficients a_0, a &c.

We easily obtain

$$a_0^2\{(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2\} = 18\,(a_1^2 - a_0a_2),$$

$$a_0^2\{\alpha(\beta-\gamma)^2 + \beta(\gamma-\alpha)^2 + \gamma(\alpha-\beta)^2\} = 9\,(a_0a_3 - a_1a_2),$$

$$a_0^2\{\alpha^2(\beta-\gamma)^2 + \beta^2(\gamma-\alpha)^2 + \gamma^2(\alpha-\beta)^2\} = 18\,(a_2^2 - a_1a_3).$$

14. Find in terms of the coefficients of the cubic in the preceding example th quadratic

$$(x-\alpha)^2(\beta-\gamma)^2 + (x-\beta)^2(\gamma-\alpha)^2 + (x-\gamma)^2(\alpha-\beta)^2 = 0,$$

where α, β, γ are the roots of the cubic.

Ans. $(a_0a_2 - a_1^2)\,x^2 + (a_0a_3 - a_1a_2)\,x + (a_1a_3 - a_2^2) = 0.$

15. Find for the cubic of Example 13 the value of

$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta).$$

Since $$2\alpha-\beta-\gamma = 3\alpha - (\alpha+\beta+\gamma) = 3\alpha + \frac{3a_1}{a_0},$$

the required value is easily obtained by substituting $-\dfrac{a_1}{a_0}$ for x in the identity

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 \equiv a_0(x-\alpha)(x-\beta)(x-\gamma).$$

Ans. $a_0^3(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta) = -27\,(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3).$

28. *Remark.*—We close this chapter with certain observations which will be found useful in verifying the results of the calculation of symmetric functions. The first is, that *the degree of any symmetric function in the roots is always equal to the sum of the suffixes in each term of its value in terms of the coefficients.* The student will observe that this is true in the case of the results of Examples 13, 15, 16, 17, 18, 19, 21, 22; and that it must be so in general appears from the equations (2) of Art. 23, for the suffix of each coefficient in those equations is equal to the degree in the roots of the corresponding function of the roots; hence in any product of any powers of the coefficients the sum of the suffixes must be equal to the degree of the corresponding function of the roots.

1. Find the value of the symmetric function

$$\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$$

of the roots of the cubic equation

$$x^3+px^2+qx+r=0.$$

Ans. $\frac{pq}{r}-3.$

2. Find for the same cubic the value of

$$(\beta+\gamma-\alpha)^3+(\gamma+\alpha-\beta)^3+(\alpha+\beta-\gamma)^3.$$

Ans. $24r-p^3.$

3. Find the value of $\Sigma\alpha^3\beta^3$ of the roots of the same equation.

Here $\Sigma\alpha\beta\,\Sigma\alpha^2\beta^2=\Sigma\alpha^3\beta^3+\alpha\beta\gamma\,\Sigma\alpha^2\beta$; hence &c.

Ans. $q^3-3pqr+3r^2.$

4. Find for the same cubic the symmetric function

$$(\beta^3-\gamma^3)^2+(\gamma^3-\alpha^3)^2+(\alpha^3-\beta^3)^2.$$

$\Sigma\alpha^6$ is easily obtained by squaring $\Sigma\alpha^3$ (see Ex. 3, Art. 27).

Ans. $2p^6-12p^4q+12p^3r+18p^2q^2-18pqr-6q^3.$

5. Find for the same cubic the value of

$$\frac{\beta^2+\gamma^2}{\beta+\gamma}+\frac{\gamma^2+\alpha^2}{\gamma+\alpha}+\frac{\alpha^2+\beta^2}{\alpha+\beta}.$$

Ans. $\frac{2p^2q-4pr-2q^2}{r-pq}.$

6. Find for the same cubic the value of

$$\frac{\alpha^2+\beta\gamma}{\beta+\gamma}+\frac{\beta^2+\gamma\alpha}{\gamma+\alpha}+\frac{\gamma^2+\alpha\beta}{\alpha+\beta}.$$

Ans. $\dfrac{p^4-3p^2q-5pr+q^2}{r-pq}$.

7. Find for the same cubic the value of

$$\frac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}+\frac{2\gamma\alpha-\beta^2}{\gamma+\alpha-\beta}+\frac{2\alpha\beta-\gamma^2}{\alpha+\beta-\gamma}.$$

Ans. $\dfrac{p^4-2p^2q+14pr-8q^2}{4pq-p^3-8r}$.

8. Find the symmetric function $\Sigma\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)^2$ for the same cubic.

Ans. $\dfrac{-p^2q^2-4p^3r+8q^3-2pqr-9r^2}{(r-pq)^2}$.

Theorem.—*If two real quantities a and b be substituted for the unknown quantity x in any polynomial $f(x)$, and if they furnish results having different signs, one plus and the other minus; then the equation $f(x)=0$ must have at least one real root intermediate in value between a and b.*

This theorem is an immediate consequence of the property of the continuity of the function $f(x)$ established in Art. 7; for since $f(x)$ changes continuously from $f(a)$ to $f(b)$, i. e. passes through all the intermediate values, while x changes from a to b; and since one of these quantities, $f(a)$ or $f(b)$, is positive, and the other negative, it follows that for some value of x intermediate between a and b, $f(x)$ must attain the value zero which is intermediate between $f(a)$ and $f(b)$.

Corollary.—*If there exist no real quantity which, substituted for x, makes $f(x) = 0$, then $f(x)$ must be positive for every real value of x.*

13. **Theorem.**—*Every equation of an odd degree has at least one real root of a sign opposite to that of its last term.*

This is an immediate consequence of the theorem in the last Article. Substitute in succession $-\infty$, 0, ∞ for x in the polynomial $f(x)$. The results are, n being odd (see Art. 4),

$x = -\infty$, $f(x)$ is negative;

$x = 0$, sign of $f(x)$ is the same as that of a_n;

$x = +\infty$, $f(x)$ is positive.

If a_n is positive, the equation must have a real root between $-\infty$ and 0, *i.e.* a real negative root; and if a_n is negative, the equation must have a real root between 0 and ∞, *i.e.* a real positive root. The theorem is thus proved.

14. **Theorem.**—*Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other negative.*

The results of substituting $-\infty$, 0, ∞ are in this case

$$\begin{array}{rr} -\infty, & +, \\ 0, & -, \\ +\infty, & +; \end{array}$$

hence there is a real root between $-\infty$ and 0, and another between 0 and $+\infty$; *i. e.*, there exist at least one real negative, and one real positive root.

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In this simple instance we observe that, in the absence of any real values, there are two imaginary expressions which reduce the polynomial to zero. The general proposition of which this is a very particular illustration is, that *every rational integra equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

must have a root of the form

$$\alpha + \beta\sqrt{-1},$$

α and β being real finite quantities. This proposition includes both real and imaginary roots, the former corresponding to the value $\beta = 0$.

16. **Theorem.**—*Every equation of n dimensions has n roots, and no more.*

We first observe that if any quantity h is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $x - h$ without a remainder. This is evident from Art. 9; for if $f(h) = 0$, *i.e.* if h is a root of $f(x) = 0$, R must be $= 0$.

The converse of this is also obviously true.

Let, now, the given equation be

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

This equation must have a root, real or imaginary (see Art. 15), which we shall denote by the symbol a_1. Let the quotient, when $f(x)$ is divided by $x - a_1$, be $\phi_1(x)$; we have then the identical equation

$$f(x) = (x - a_1)\,\phi_1(x).$$

Again, the equation $\phi_1(x) = 0$, which is of $n-1$ dimensions, must have a root, which we represent by a_2. Let the quotient obtained by dividing $\phi_1(x)$ by $x - a_2$ be $\phi_2(x)$. Hence

$$\phi_1(x) = (x - a_2)\,\phi_2(x),$$

$$\text{and} \quad \therefore f(x) = (x - a_1)\,(x - a_2)\,\phi_2(x),$$

where $\phi_2(x)$ is of $n-2$ dimensions.

Proceeding in this manner, we prove that $f(x)$ consists of the product of n factors, each containing x in the first degree, and a numerical factor $\phi_n(x)$. Comparing the coefficients of x^n, it is plain that $\phi_n(x) = 1$. Thus we prove the identical equation

$$f(x) = (x - a_1)(x - a_2)(x - a_3)\,.\,.\,.\,.\,.\,(x - a_{n-1})\,(x - a_n).$$

It is evident that the substitution of any one of the quantities $a_1, a_2, \ldots a_n$ for x in the right-hand member of this equation will reduce that member to zero, and will therefore reduce $f(x)$ to zero; that is to say, the equation $f(x) = 0$ has for roots the n quantities $a_1, a_2, a_3 \ldots a_{n-1}, a_n$. And it can have no other roots; for if any quantity other than one of the quantities $a_1, a_2, \ldots a_n$ be substituted in the right-hand member of the above equation, the factors will be all different from zero, and therefore the product cannot vanish.

Corollary.—*Two polynomials of the n^{th} degree cannot be equal to one another for more than n values of the variable without being completely identical.*

1. Find the equation whose roots are

$$-3, \quad -1, \quad 4, \quad 5.$$

Ans. $x^4 - 5x^3 - 13x^2 + 53x + 60 = 0.$

2. The equation

$$x^4 - 6x^3 + 8x^2 - 17x + 10 = 0$$

has a root 5; find the equation containing the remaining roots.

[N. B.—Use the method of division of Art. 8.]

Ans. $x^3 - x^2 + 3x - 2 = 0.$

3. Solve the equation

$$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0,$$

two roots being 1 and 7.

Ans. The other two roots are 3, 5.

4. Form the equation whose roots are

$$-\frac{3}{2}, \quad 3, \quad \frac{1}{7}.$$

Ans. $14x^3 - 23x^2 - 60x + 9 = 0.$

5. Solve the cubic equation

$$x^3 - 1 = 0.$$

Here it is evident that $x = 1$ satisfies the equation. Divide by $x - 1$, and solve the resulting quadratic. The two roots are found to be

$$-\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \quad -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

It can be easily shown that if either of these imaginary roots is squared, the other results. It is usual to represent these roots by ω and ω^2. They are called the two *imaginary cube roots of unity*. We have the identical equation

$$x^3 - 1 \equiv (x - 1)(x - \omega)(x - \omega^2).$$

18. **Imaginary Roots enter Equations in Pairs.**— The proposition we have to prove may be stated as follows:— *If an equation $f(x) = 0$, whose coefficients are all real quantities, have for a root the imaginary expression $\alpha + \beta\sqrt{-1}$, it must also have for a root the conjugate imaginary expression $\alpha - \beta\sqrt{-1}$.*

The product

$$(x - \alpha - \beta\sqrt{-1})(x - \alpha + \beta\sqrt{-1}) = (x - \alpha)^2 + \beta^2.$$

Let the polynomial $f(x)$ be divided by the second member of this identity, and if possible let there be a remainder $Rx + R'$. We have then the identical equation

$$f(x) = \{(x - \alpha)^2 + \beta^2\} Q + Rx + R',$$

where Q is the quotient, of $n - 2$ dimensions in x. Substitute in this identity $\alpha + \beta\sqrt{-1}$ for x. This, by hypothesis, causes $f(x)$ to vanish. It also causes $(x-\alpha)^2 + \beta^2$ to vanish. Hence

$$R\,(\alpha + \beta\sqrt{-1}) + R' = 0.$$

From this we obtain the two equations

$$R\alpha + R' = 0, \qquad R\beta = 0,$$

since the real and imaginary parts cannot destroy one another; hence

$$R = 0, \qquad R' = 0.$$

Thus the remainder $Rx + R'$ vanishes; and, therefore, $f(x)$ is divisible without remainder by the product of the two factors

$$x - \alpha - \beta\sqrt{-1}, \qquad x - \alpha + \beta\sqrt{-1}.$$

The equation has, consequently, the root $\alpha - \beta\sqrt{-1}$ as well as the root $\alpha + \beta\sqrt{-1}$.

Thus the total number of imaginary roots in an equation with real coefficients will always be even; and every polynomial may be regarded as composed of real factors, each pair of imaginary roots producing a real quadratic factor, and each real root producing a real simple factor. The actual resolution of the polynomial into these factors constitutes the complete solution of the equation.

19. **Descartes' Rule of Signs—Positive Roots.**—This rule, which enables us, by the mere inspection of a given equation, to assign a superior limit to the number of its positive roots, may be enunciated as follows :—*No equation can have more positive roots than it has changes of sign from* + *to* −, *and from* − *to* +, *in the terms of its first member.*

We shall content ourselves for the present with the proof which is usually given, and which is more a verification than a general demonstration of this celebrated theorem of Descartes. It will be subsequently shown that this rule of Descartes, and other similar rules which were discovered by early investigators relative to the number of the positive, negative, and imaginary roots of equations, are immediate deductions from the more general theorems of Budan and Fourier.

Let the signs of a polynomial taken at random succeed each other in the following order :—

$$+ + - + - - - + + - + -.$$

20. **Descartes' Rule of Signs—Negative Roots.**—In order to give the most advantageous statement to Descartes' rule in the case of negative roots, we first prove that if $-x$ be substituted for x in the equation $f(x) = 0$, the resulting equation will have the same roots as the original except that their signs will be changed. This follows from the identical equation of Art. 16

$$f(x) \equiv (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n),$$

from which we derive

$$f(-x) \equiv (-1)^n (x + a_1)(x + a_2)(x + a_3) \ldots (x + a_n).$$

From this it is evident that the roots of $f(-x) = 0$ are

$$-a_1, \quad -a_2, \quad -a_3, \ldots \ldots -a_n.$$

Hence the negative roots of $f(x)$ are positive roots of $f(-x)$, and we may enunciate Descartes' rule for negative roots as follows:—*No equation can have a greater number of negative roots than there are changes of sign in the terms of the polynomial* $f(-x)$.

22. **Theorem.**—We shall close this chapter with the following theorem, which defines fully the conclusions which can be drawn as to the roots of an equation from the signs furnished by its first member when two given numbers are substituted for x:—*If two numbers a and b, substituted for x in the polynomial $f(x)$, give results with contrary signs, an odd number of real roots of the equation $f(x) = 0$ lies between them; and if they give results with the same sign, either no real root or an even number of real roots lies between them.*

We proceed to prove the first part of this proposition: the second is proved in an exactly similar manner.

Let the following m roots $\alpha_1, \alpha_2, \ldots \alpha_m$, and no others, of the equation $f(x) = 0$ lie between the quantities a and b, of which, as usual, we take a to be the lesser.

Let $\phi(x)$ be the quotient when $f(x)$ is divided by the product of the m factors $(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m)$. We have, then, the identical equation

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m)\,\phi(x).$$

Putting in this successively $x = a$, $x = b$, we obtain

$$f(a) = (a - \alpha_1)(a - \alpha_2) \ldots (a - \alpha_m)\,\phi(a),$$
$$f(b) = (b - \alpha_1)(b - \alpha_2) \ldots (b - \alpha_m)\,\phi(b).$$

Now $\phi(a)$ and $\phi(b)$ have the same sign; for if they had different signs there would be, by Art. 12, one root at least of the equation $\phi(x) = 0$ between them. By hypothesis, $f(a)$ and $f(b)$ have different signs; hence the signs of the products

$$(a - \alpha_1)(a - \alpha_2) \ldots (a - \alpha_m),$$
$$(b - \alpha_1)(b - \alpha_2) \ldots (b - \alpha_m),$$

are different; but the sign of the second is positive, since all its factors are positive; hence the sign of the first is negative; but all the factors of the first are negative; therefore their number must be odd; which proves the proposition.

Examples.

1. If the signs of the terms of an equation be all positive, it cannot have a positive root.

2. If the signs of the terms of any complete equation be alternately positive and negative, it cannot have a negative root.

3. If an equation consist of a number of terms connected by + signs followed by a number of terms connected by − signs, it has one positive root and no more.

[Apply Art. 12, substituting 0 and ∞; and Art. 19.]

4. If an equation involve only even powers of x, and if all the coefficients have positive signs, it cannot have a real root.

[Apply Arts. 19 and 20.]

5. If an equation involve only odd powers of x, and if the coefficients have all positive signs, it has the root zero and no other real root.

6. If an equation be complete, the number of continuations of sign in $f(x)$ is the same as the number of variations of sign in $f(-x)$.

7. When an equation is complete; if all its roots are real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of continuations of sign.

8. An equation having an even number of variations of sign must have its last sign positive, and one having an odd number of variations must have its last sign negative.

[N. B.—The sign + is always given to the highest power of x.]

9. Hence prove that if an equation has an even number of variations it must have an equal or less even number of positive roots; and if it has an odd number of variations it must have an equal or less odd number of positive roots; in other

20. Form an equation with rational coefficients which shall have for roots all the values of the expression

$$\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r},$$

where $$\theta_1^2=1, \quad \theta_2^2=1, \quad \theta_3^2=1.$$

There are eight different values of this expression, viz.,

$$\sqrt{p}+\sqrt{q}+\sqrt{r}, \qquad -\sqrt{p}-\sqrt{q}-\sqrt{r},$$

$$\sqrt{p}-\sqrt{q}-\sqrt{r}, \qquad -\sqrt{p}+\sqrt{q}+\sqrt{r},$$

$$-\sqrt{p}+\sqrt{q}-\sqrt{r}, \qquad \sqrt{p}-\sqrt{q}+\sqrt{r},$$

$$-\sqrt{p}-\sqrt{q}+\sqrt{r}, \qquad \sqrt{p}+\sqrt{q}-\sqrt{r}.$$

Assume

$$x=\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}.$$

Squaring this, we have

$$x^2=p+q+r+2\left(\theta_2\theta_3\sqrt{qr}+\theta_3\theta_1\sqrt{rp}+\theta_1\theta_2\sqrt{pq}\right).$$

Transposing, and squaring again,

$$(x^2-p-q-r)^2=4(qr+rp+pq)+8\theta_1\theta_2\theta_3\sqrt{pqr}\left(\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}\right). \quad (1)$$

Transposing, substituting x for $\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}$, and squaring, we obtain the final equation free from radicals

$$\{x^4-2x^2(p+q+r)+p^2+q^2+r^2-2qr-2rp-2pq\}^2=64pqrx^2.$$

This is an equation of the eighth degree, whose roots are the values above written. Since $\theta_1, \theta_2, \theta_3$ have disappeared, it is indifferent which of the eight roots $\pm\sqrt{p}\pm\sqrt{q}\pm\sqrt{r}$ is assumed equal to x in the first instance. The final equation is that which would have been obtained if each of the 8 roots had been subtracted from x, and the continued product formed, as in Ex. 6, Art. 16.

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Algebra **Subject Code: 19MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit IV

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If f(a) and f(b) are of different signs then _______ root of thequation f(x)=0 must lie between a and b.	at most one	at least one	0	exactly one	at least one
If f(x)=0 is an equation of odd degree, it has at least one _______ root whose sign is opposite to that of the last term.	complex	rational	positive	real	real
Every equation f(x)=0 of the n^thdgree has _______and no more.	n roots	(n-1) roots	(n+1) roots	zero	zero
If a function involving all the roots of an equation is unaltered value if any two of the roots are interchanged it is called a _______ function of the roots	constant	skew-symmetric	symmetric	non-symmetric	constant
Between two consecutive real roots a and b of the equation f(x)=0 where f(x) is a polynomial, there lies at least one real root of the equation _______	f ' (x)=0	f ' (x)>0	f ' (x)<0	f ' (x)≠0	f ' (x)=0
Let f(x)=x^2 - 7x - 6 . Then f(-1)=________	-1	0	1	2	2
Root of x^4 + 2x^2 + 1 = 0 is________	i	0	1	-1	i

$x-a$ is a factor of x^n-a^n if n is _________	2	4	odd	even	odd
If $f'(a)$ is ___________then $f(x)$ decreases	positive	negative	≤ 0	stable	negative
Graph of $2x^2+x-6$ meets the x axis at ___________ points	0	3	2	4	2
If $f(x)$___________ 0 has no real root then $f(x)$ is positive for every values of x	$\leq$	$>$	$<$	$\geq$	$>$
If $f(a)$ and $f(b)$ have opposite signs then $f(x)=0$ has at least ___________real root	0	1	2	3	2
What is the one root of the polynomial $x^3-1=0$ ___________	1	0	2	3	1
If $f(x)=0$ is an equation of odd degree, it has at least one________ root whose sign is opposite to that of the last term.	complex	rational	positive	real	real
Every equation $f(x)=0$ of the n^thdgree has ________and no more.	n roots	(n-1) roots	(n+1) roots	zero	zero
What is the one root of the polynomial $x^5-1=0$ ________	1	0	2	3	1
In the division algorithm, the polynomial $q(x)$ is called the ……….on dividing $f(x)$ by $g(x)$	quotient	remainder	divisior	diviend	quotient
In the division algorithm, the polynomial $q(x)$ is called the quotient on dividing $f(x)$ by $g(x)$ and the polynomail $r(x)$is called the ……………………	quotient	remainder	divisior	diviend	remainder
Let $f(x) = x^2 - 7x - 6$ then $f(-1)$ = ____________	0	1	2	3	2
Let $f(x) = x^2 - 7x - 6$ then $f(0)$ = ____________	-6	-7	-8	0	-6
Let $f(x) = x^2 - 7x + 6$ then $f(1)$ = ____________	0	1	2	3	0
Let $f(x) = x^2 + 7x + 6$ then $f(1)$ = ____________	11	12	13	14	14
Root of $x^4 + 2x^2 + 1 = 0$ is ______________	i	-1	1	0	i
Number of positive roots of $x^4 + 2x^2 + 1 = 0$ is ___________	0	1	2	3	0
Number of negative roots of $x^4 + 2x^2 + 1 = 0$ is ___________	0	1	2	3	0
Number of imaginary roots of $x^4 + 2x^2 + 1 = 0$ is ___________	1	2	3	4	4

Number of terms involving in a complete equation of degree n is _____________	n	n+1	n+2	n+2	n+1
Number of terms involving in a complete equation of degree 100 is _____________	100	101	102	103	101
First derived function of 4th degree polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	cubic
First derived function of biquadratic polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	cubic
First derived function of 5th degree polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	biquadratic
First derived function of quintic polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	biquadratic
First derived function of 3rd degree polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	quadratic
First derived function of cubic polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	quadratic
If first derived function of f(x) is positive at a then f(x) is ------	decreasing	increasing	both increasing and decreasing	neither increasing nor decreasing	increasing
If first derived function of f(x) is negative at a then f(x) is ------	decreasing	increasing	both increasing and decreasing	neither increasing nor decreasing	decreasing
second derived function of 4th degree polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	quadratic
second derived function of biquadratic polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	quadratic
second derived function of 5th degree polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	cubic
second derived function of quintic polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	cubic
second derived function of 6th degree polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	biquadratic
second derived function of cubic polynomial is a ---- polynomial	quadratic	biquadratic	cubic	quintic	biquadratic

UNIT-V

SYLLABUS

Multiple roots-Rolle's theorem - Position of real roots of f(x) =0 – Newton's method of approximation to a root – Horner's method.

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(1). *First method of solution : by resolving into factors.* Let it be required to resolve the quadratic $x^2 + Px + Q$ into its simple factors. For this purpose we put it under the form

$$x^2 + Px + Q + \theta - \theta,$$

and determine θ so that

$$x^2 + Px + Q + \theta$$

may be a perfect square, *i.e.* we make

$$\theta + Q = \frac{P^2}{4}, \quad \text{or } \theta = \frac{P^2 - 4Q}{4};$$

whence, putting for θ its value, we have

$$x^2 + Px + Q = \left(x + \frac{P}{2}\right)^2 - \left(0x + \frac{\sqrt{P^2 - 4Q}}{2}\right)^2.$$

Thus we have reduced the quadratic to the form $u^2 - v^2$; and its simple factors are $u + v$, and $u - v$.

Subsequently we shall reduce the cubic to the form

$$(lx+m)^3-(l'x+m')^3, \quad \text{or} \quad u^3-v^3,$$

and obtain its solution from the simple equations

$$u-v=0, \quad u-\omega v=0, \quad u-\omega^2 v=0.$$

It will be shown also that the biquadratic may be reduced to either of the forms

$$(lx^2+mx+n)^2-(l'x^2+m'x+n')^2,$$
$$(x^2+px+q)(x^2+p'x+q'),$$

by solving a cubic equation; and, consequently, the solution of the biquadratic completed by solving two quadratics, viz., in the first case, $lx^2+mx+n=\pm(l'x^2+m'x+n')$; and in the second case, $x^2+px+q=0$, and $x^2+p'x+q'=0$.

(2). *Second method of solution : by assuming for a root a general form involving radicals.*

Assuming $x = p + \sqrt{q}$ to be a root of the equation $x^2 + Px + Q = 0$, and rationalizing the equation $x = p + \sqrt{q}$, we have

$$x^2 - 2px + p^2 - q = 0.$$

Now, if this equation be identical with $x^2 + Px + Q = 0$, we have

$$2p = -P, \quad p^2 - q = Q,$$

giving
$$x = p + \sqrt{q} = \frac{-P \pm \sqrt{P^2 - 4Q}}{2},$$

which is the solution of the quadratic equation.

In the case of the cubic equation we shall find that

$$x = \sqrt[3]{p} + \frac{A}{\sqrt[3]{p}}$$

is the proper form to represent a root ; this formula giving precisely three values for x, in consequence of the manner in which the cube root enters into it.

In the case of the biquadratic equation we shall find that

$$\sqrt{p} + \sqrt{q} + \frac{A}{\sqrt{p}\,\sqrt{q}}, \quad \sqrt{q}\,\sqrt{r} + \sqrt{r}\,\sqrt{p} + \sqrt{p}\,\sqrt{q}$$

are forms which represent a root ; these formulas each giving four, and only four, values of x when the square roots receive their double signs.

(3). *Third method of solution : by symmetric functions of the roots.*

Consider the quadratic equation $x^2 + Px + Q = 0$, of which the roots are α, β.

Then
$$\alpha + \beta = -P,$$
$$\alpha\beta = Q.$$

If we attempt to determine α and β by these equations, we fall back on the original equation (see Art. 24); but if we could obtain a second equation between the roots and coefficients, of the form $l\alpha + m\beta = f(P, Q)$, we could easily find α and β by means of this equation and the equation $\alpha + \beta = -P$.

Now in the case of the quadratic there is no difficulty in finding the required equation; for, obviously,

$$(\alpha - \beta)^2 = P^2 - 4Q\text{; and, therefore, } \alpha - \beta = \sqrt{P^2 - 4Q}.$$

In the case of the cubic equation $x^3 + Px^2 + Qx + R = 0$, we require *two* simple equations of the form

$$l\alpha + m\beta + n\gamma = f(P, Q, R),$$

in addition to the equation $\alpha + \beta + \gamma = -P$, to determine the roots α, β, γ. It will subsequently be proved that the functions

$$(\alpha + \omega\beta + \omega^2\gamma)^3, \quad (\alpha + \omega^2\beta + \omega\gamma)^3$$

may be expressed in terms of the coefficients by solving a *quadratic* equation; and when their values are known the roots of the cubic may be easily found.

In the case of the biquadratic equation

$$x^4 + Px^3 + Qx^2 + Rx + S = 0$$

we require *three* simple equations of the form

$$l\alpha + m\beta + n\gamma + r\delta = f(P, Q, R, S),$$

in addition to the equation

$$\alpha + \beta + \gamma + \delta = -P,$$

56. The Algebraic Solution of the Cubic Equation.—Let the general cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

be put under the form

$$z^3 + 3Hz + G = 0,$$

where $z = ax + b$, $H = ac - b^2$, $G = a^2d - 3abc + 2b^3$ (see Art. 37).

To solve this equation, assume*

$$z = \sqrt[3]{p} + \sqrt[3]{q};$$

hence, cubing,

$$z^3 = p + q + 3\sqrt[3]{p}\,\sqrt[3]{q}\,(\sqrt[3]{p} + \sqrt[3]{q}),$$

therefore

$$z^3 - 3\sqrt[3]{p}\,\sqrt[3]{q}\,.\,z - (p + q) = 0.$$

Now, comparing coefficients, we have

$$\sqrt[3]{p}\,.\sqrt[3]{q} = -H, \quad p + q = -G;$$

from which equations we obtain

$$p = \tfrac{1}{2}(-G + \sqrt{G^2 + 4H^3}), \quad q = \tfrac{1}{2}(-G - \sqrt{G^2 + 4H^3});$$

58. **Expression of the Cubic as the Difference of two Cubes.**—Let the given cubic

$$ax^3 + 3bx^2 + 3cx + d = \phi(x)$$

be put under the form

$$z^3 + 3Hz + G,$$

where $z = ax + b$.

Now assume

$$z^3 + 3Hz + G \equiv \frac{1}{\mu - \nu}\{\mu(z+\nu)^3 - \nu(z+\mu)^3\}, \qquad (1)$$

where μ and ν are quantities to be determined; the second side of this identity becomes, when reduced,

$$z^3 - 3\mu\nu z - \mu\nu(\mu+\nu).$$

Comparing coefficients,

$$\mu\nu = -H, \quad \mu\nu(\mu+\nu) = -G;$$

therefore

$$\mu + \nu = \frac{G}{H}, \quad \mu - \nu = \frac{a\sqrt{\Delta}}{H};$$

where $a^2\Delta = G^2 + 4H^3$, as in Art. 41;

also $$(z+\mu)(z+\nu) = z^2 + \frac{G}{H}z - H. \qquad (2)$$

Whence, putting for z its value, $ax + b$, we have from (1)

$$a^3\phi(x) = \left(\frac{G + a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax + b + \frac{G - a\Delta^{\frac{1}{2}}}{2H}\right)^3 - \left(\frac{G - a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax + b + \frac{G + a\Delta^{\frac{1}{2}}}{2H}\right)^3,$$

which is the required expression of $\phi(x)$ as the difference of two cubes.

The function (2), when transformed and reduced, becomes

$$\frac{a^2}{H}\{(ac - b^2)x^2 + (ad - bc)x + (bd - c^2)\},$$

which contains the two factors $ax + b + \mu$, $ax + b + \nu$.

The expression of the roots of this quadratic in terms of the roots of the given cubic may be seen on referring to Ex. 23, p. 57.

60. **Homographic Relation between two Roots of a Cubic.**—Before proceeding to the discussion of the biquadratic we prove the following important proposition relative to the cubic :—

The roots of the cubic are connected in pairs by a homographic relation in terms of the coefficients.

Referring to Example 13, Art. 27, we have the relations

$$a_0^2\{(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2\}=18(a_1^2-a_0a_2),$$
$$a_0^2\{\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2\}=9(a_0a_3-a_1a_2),$$
$$a_0^2\{\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2\}=18(a_2^2-a_1a_3).$$

We adopt the notation

$$a_0a_2-a_1^2=H,\quad a_0a_3-a_1a_2=2H_1,\quad a_1a_3-a_2^2=H_2.$$

Now, multiplying the above equations by $\alpha\beta$, $-(\alpha+\beta)$, 1, respectively, and adding, since

$$\alpha^2-\alpha(\alpha+\beta)+\alpha\beta=0,\quad \beta^2-\beta(\alpha+\beta)+\alpha\beta=0,$$

we have

$$a_0^2(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)^2=18\{H\alpha\beta+H_1(\alpha+\beta)+H_2\};$$

but

$$a_0^4(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-27\Delta=108(HH_2-H_1^2)$$

(see Art. 41) ; whence

$$\pm\sqrt{-\frac{\Delta}{3}}\left(\frac{\alpha-\beta}{2}\right)=H\alpha\beta+H_1(\alpha+\beta)+H_2,$$

and, therefore,

$$H\alpha\beta+\left(H_1+\frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\alpha+\left(H_1-\frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\beta+H_2=0,$$

which is the required homographic relation (see Art. 39).

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Examples.

1. Resolve into simple factors the expression

$$(\beta-\gamma)^2(x-\alpha)^2+(\gamma-\alpha)^2(x-\beta)^2+(\alpha-\beta)^2(x-\gamma)^2.$$

Let $U=(\beta-\gamma)(x-\alpha),\quad V=(\gamma-\alpha)(x-\beta),\quad W=(\alpha-\beta)(x-\gamma).$

Ans. $\frac{2}{3}(U+\omega V+\omega^2 W)(U+\omega^2 V+\omega W)$.

2. Prove that the several equations of the system

$$(\beta-\gamma)^3(x-\alpha)^3=(\gamma-\alpha)^3(x-\beta)^3=(\alpha-\beta)^3(x-\gamma)^3$$

have two factors common.

Making use of the notation in the last Example, we have

$$U^3=V^3=W^3;$$

whence

$$U^3-V^3=(U-V)(U^2+UV+V^2)\equiv\tfrac{1}{2}(U-V)(U^2+V^2+W^2),$$

since

$$U+V+W\equiv 0;$$

therefore

$$(\beta-\gamma)^2(x-\alpha)^2+(\gamma-\alpha)^2(x-\beta)^2+(\alpha-\beta)^2(x-\gamma)^2$$

is the common quadratic factor required.

3. Resolve into simple factors the expression

$$(\beta-\gamma)^3(x-\alpha)^3+(\gamma-\alpha)^3(x-\beta)^3+(\alpha-\beta)^3(x-\gamma)^3.$$

Ans. $3(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma)$.

4. Resolve

$$(x-\alpha)(x-\beta)(x-\gamma)$$

into the difference of two cubes.

Assume

$$(x-\alpha)(x-\beta)(x-\gamma) = U_1^3 - V_1^3;$$

whence

$$U_1 - V_1 = \lambda(x-\alpha),$$

$$\omega U_1 - \omega^2 V_1 = \mu(x-\beta),$$

$$\omega^2 U_1 - \omega V_1 = \nu(x-\gamma):$$

adding these we have

$$\lambda + \mu + \nu = 0, \quad \lambda\alpha + \mu\beta + \nu\gamma = 0;$$

and, therefore,

$$\lambda = \rho(\beta-\gamma), \quad \mu = \rho(\gamma-\alpha), \quad \nu = \rho(\alpha-\beta);$$

but $\lambda\mu\nu = 1$; whence

$$\frac{1}{\rho^3} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Substituting these values of λ, μ, ν; and using the notation of Ex. 1,

$$U_1 - V_1 = \rho U, \quad \omega U_1 - \omega^2 V_1 = \rho V, \quad \omega^2 U_1 - \omega V_1 = \rho W;$$

whence

$$3U_1 = \rho(U + \omega^2 V + \omega W),$$

$$-3V_1 = \rho(U + \omega V + \omega^2 W);$$

and U_1 and V_1 are completely determined.

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Examples.

1. The functions L and M are functions of the differences of the roots.

For, $$L = \alpha + \omega\beta + \omega^2\gamma = \alpha - h + \omega(\beta - h) + \omega^2(\gamma - h)$$

for all values of h, since $1 + \omega + \omega^2 = 0$; and giving to h the values α, β, γ, in succession, we obtain three forms for L in terms of the differences $\beta - \gamma,\ \gamma - \alpha,\ \alpha - \beta$. Similarly for M.

2. To express the product of the squares of the differences of the roots in terms of the coefficients.

We have

$$L + M = 2\alpha - \beta - \gamma, \quad L + \omega^2 M = (2\beta - \gamma - \alpha)\,\omega, \quad L + \omega M = (2\gamma - \alpha - \beta)\,\omega^2;$$

and, again,

$$L - M = (\beta - \gamma)(\omega - \omega^2), \quad \omega^2 L - \omega M = (\gamma - \alpha)(\omega - \omega^2), \quad \omega L - \omega^2 M = (\alpha - \beta)(\omega - \omega^2);$$

from which we obtain, as in Art. 26,

$$L^3 + M^3 = (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta),$$

$$L^3 - M^3 = -3\sqrt{-3}\,(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta);$$

and since

$$(L^3 - M^3)^2 \equiv (L^3 + M^3)^2 - 4L^3M^3,$$

we have, substituting the previous results,

$$a_0^6(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = -27\,(G^2 + 4H^3).$$

(See Art. 41.)

3. Prove the following identities :—

$$L^3 + M^3 = \tfrac{1}{3}\{(2\alpha - \beta - \gamma)^3 + (2\beta - \gamma - \alpha)^3 + (2\gamma - \alpha - \beta)^3\},$$

$$L^3 - M^3 = \sqrt{-3}\,\{(\beta - \gamma)^3 + (\gamma - \alpha)^3 + (\alpha - \beta)^3\}.$$

These are easily obtained by cubing and adding the values of

$$L + M, \text{ \&c. }; \ L - M, \text{ \&c.}$$

in the preceding example.

hence they are the roots of the equation

$$(\phi - L)(\phi - \omega L)(\phi - \omega^2 L)(\phi - M)(\phi - \omega M)(\phi - \omega^2 M) = 0,$$

or

$$\phi^6 - (L^3 + M^3)\phi^3 + L^3 M^3 = 0.$$

Substituting for L and M from the equations

$$LM = -\frac{9H}{a^2}, \quad L^3 + M^3 = -27\frac{G}{a^3},$$

we have this equation expressed in terms of the coefficients as follows:—

$$\phi^6 + 3^3 \frac{G}{a^3}\phi^3 - 3^6 \frac{H^3}{a^6} = 0.$$

5. To obtain expressions for L^2, M^2, &c., in terms of α, β, γ.

The following forms for L^2 and M^2 are obtained by subtracting

$(\alpha^2 + \beta^2 + \gamma^2)(1 + \omega + \omega^2) \equiv 0$ from $(\alpha + \omega\beta + \omega^2\gamma)^2$, and $(\alpha + \omega^2\beta + \omega\gamma)^2$:

$$-L^2 = (\beta - \gamma)^2 + \omega^2(\gamma - \alpha)^2 + \omega(\alpha - \beta)^2,$$

$$-M^2 = (\beta - \gamma)^2 + \omega(\gamma - \alpha)^2 + \omega^2(\alpha - \beta)^2.$$

In a similar manner, we find from these formulas

$$-L^4 = (\beta - \gamma)^2(2\alpha - \beta - \gamma)^2 + \omega(\gamma - \alpha)^2(2\beta - \gamma - \alpha)^2 + \omega^2(\alpha - \beta)^2(2\gamma - \alpha - \beta)^2,$$

$$-M^4 = (\beta - \gamma)^2(2\alpha - \beta - \gamma)^2 + \omega^2(\gamma - \alpha)^2(2\beta - \gamma - \alpha)^2 + \omega(\alpha - \beta)^2(2\gamma - \alpha - \beta)^2.$$

Also, without difficulty, we have the following forms for LM, and L^2M^2:—

$$2LM = (\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2,$$

$$L^2M^2 = (\alpha - \beta)^2(\alpha - \gamma)^2 + (\beta - \gamma)^2(\beta - \alpha)^2 + (\gamma - \alpha)^2(\gamma - \beta)^2.$$

8. Form the equation whose roots are the several values of ρ, where

$$\rho = \frac{\alpha - \beta}{\beta - \gamma}.$$

Since

$$\alpha - (1 + \rho)\beta + \rho\gamma = 0,$$

substituting for α, β, γ, their values in terms of p, q; and putting

$$\lambda = 1 - (1+\rho)\,\omega + \rho\omega^2, \quad \mu = 1 - (1+\rho)\,\omega^2 + \rho\omega,$$

we have

$$\lambda\sqrt[3]{p} + \mu\sqrt[3]{q} = 0.$$

Cubing, and substituting for p, q their values,

$$G(\lambda^3 + \mu^3) + a\sqrt{\Delta}(\lambda^3 - \mu^3) = 0.$$

Squaring,

$$a^2\Delta\,\lambda^3\mu^3 = H^3(\lambda^3 + \mu^3)^2,$$

and by previous results

$$\lambda\mu = 3(1 + \rho + \rho^2), \quad \lambda^3 + \mu^3 = -27\rho(1+\rho)\;;$$

substituting these values, we have the required equation

$$a^2\Delta(1 + \rho + \rho^2)^3 - 27H^3(\rho + \rho^2)^2 = 0.$$

9. Find the relation between the coefficients of the cubics

$$ax^3 + 3bx^2 + 3cx + d = 0,$$
$$a'x'^3 + 3b'x'^2 + 3c'x' + d' = 0,$$

when the roots are connected by the equation

$$\alpha(\beta' - \gamma') + \beta(\gamma' - \alpha') + \gamma(\alpha' - \beta') = 0.$$

Multiplying by $\omega - \omega^2$, this equation becomes

$$LM' = L'M.$$

Cubing and introducing the coefficients, we find

$$G^2H'^3 = G'^2H^3,$$

the required relation.

10. Determine the condition in terms of the roots and coefficients that the cubics of Ex. 9 should become identical by the linear transformation

$$x' = px + q.$$

In this case

$$\alpha' = p\alpha + q, \quad \beta' = p\beta + q, \quad \gamma' = p\gamma + q.$$

Eliminating p and q, we have

$$\beta\gamma' - \beta'\gamma + \gamma\alpha' - \gamma'\alpha + \alpha\beta' - \alpha'\beta = 0,$$

which is the function of the roots considered in the last example. This relation, moreover, is unchanged if for α, β, γ; α', β', γ', we substitute

$$l\alpha + m, \quad l\beta + m, \quad l\gamma + m,$$
$$l'\alpha' + m', \quad l'\beta' + m', \quad l'\gamma' + m',$$

whence we may consider the cubics in the last example under the simple forms

$$z^3 + 3Hz + G = 0, \quad z'^3 + 3H'z' + G' = 0,$$

obtained by the linear transformations $z = ax + b$, $z' = a'x' + b'$; for if the condition holds for the roots of the former equations, it must hold for the roots of the latter. Now putting $z' = kz$, these equations become identical if

$$H' \equiv k^2 H, \quad G' \equiv k^3 G;$$

whence, eliminating k,

$$G^2 H'^3 = G'^2 H^3$$

is the required condition, the same as that obtained in Ex. 9. It may be observed that the reducing quadratics of the cubics necessarily become identical by the same transformation, viz.,

$$\frac{H'}{G'}(a'x' + b') = \frac{H}{G}(ax + b).$$

61. **First Solution by Radicals of the Biquadratic. Euler's Assumption :**—Let the biquadratic equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

be put under the form

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

where $z = ax + b$,

$$H = ac - b^2, \quad I = ae - 4bd + 3c^2, \quad G = a^2d - 3abc + 2b^3.$$

(See Art. 38.)

To solve this equation (a biquadratic wanting the second term) Euler assumes as the general expression for a root

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

Squaring,

$$z^2 - p - q - r = 2(\sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q}).$$

Squaring again, and reducing, we obtain the equation

$$z^4 - 2(p+q+r)z^2 - 8\sqrt{p}.\sqrt{q}.\sqrt{r}.z + (p+q+r)^2 - 4(qr+rp+pq) = 0.$$

Comparing this equation with the former equation in z, we have

$$p+q+r = -3H, \quad qr+rp+pq = 3H^2 - \frac{a^2I}{4}, \quad \sqrt{p}.\sqrt{q}.\sqrt{r} = -\frac{G}{2};$$

and consequently p, q, r are the roots of the equation

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0;$$

or, since

$$-G^2 = 4H^3 - a^2HI + a^3J \qquad \text{(see Art. 38)},$$

where

$$J = ace + 2bcd - ad^2 - eb^2 - c^3,$$

we may write this equation under the form

$$4(t+H)^3 - a^2 I(t+H) + a^3 J = 0;$$

and finally, putting $t + H \equiv a^2\theta$, we obtain the equation

$$4a^3\theta^3 - Ia\theta + J = 0,$$

which we call *the reducing cubic* of the biquadratic equation.

Also, since $t \equiv b^2 - ac + a^2\theta$; if $\theta_1, \theta_2, \theta_3$ be the roots of the reducing cubic, we have

$$p \equiv b^2 - ac + a^2\theta_1, \quad q \equiv b^2 - ac + a^2\theta_2, \quad r \equiv b^2 - ac + a^2\theta_3;$$

and, therefore,

$$z = \sqrt{b^2 - ac + a^2\theta_1} + \sqrt{b^2 - ac + a^2\theta_2} + \sqrt{b^2 - ac + a^2\theta_3}.$$

The radicals in this formula have not complete generality; for if they had, eight values of z in place of four would be given by the formula. This limitation is imposed by the relation

$$\sqrt{p} \cdot \sqrt{q} \cdot \sqrt{r} = -\frac{G}{2},$$

which (lost sight of in squaring to obtain the value of pqr) requires such signs to be attached to each of the quantities $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, that their product may maintain the sign determined by the above equation; thus,

$$\sqrt{p}\,\sqrt{q}\,\sqrt{r} = \sqrt{p}(-\sqrt{q})(-\sqrt{r}) = (-\sqrt{p})\sqrt{q}(-\sqrt{r})$$
$$= (-\sqrt{p})(-\sqrt{q})\sqrt{r}$$

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are all the possible combinations of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ fulfilling this condition, provided $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ retain the same signs throughout, whatever those signs may be. But we may avoid all ambiguity as regards sign, and express in a single algebraic formula the four values of z, by eliminating one of the quantities $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ from the formula

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r}$$

by means of the relation given above, and leaving the other two quantities unrestricted in sign. We have then

$$z = \sqrt{p} + \sqrt{q} - \frac{G}{2\sqrt{p}\sqrt{q}},$$

a formula free from all ambiguity, since it gives four, and only four, values of z when $\sqrt{p}$ and $\sqrt{q}$ receive their double signs:

the sign given to each of these in the first two terms determining that which must be attached to it in the denominator of the third term. And finally, restoring p, q, and z their values given before, we have

$$ax + b = \sqrt{b^2 - ac + a^2\theta_1} + \sqrt{b^2 - ac + a^2\theta_2} - \frac{G}{2\sqrt{(b^2 - ac + a^2\theta_1)} \cdot \sqrt{b^2 - ac + a^2\theta_2}}$$

as *the complete algebraic solution of the biquadratic equation;* θ_1 and θ_2 being roots of the equation

$$4a^3\theta^3 - Ia\theta + J = 0.$$

To assist the student in justifying Euler's apparently arbitrary assumption as to the form of solution of the biquadratic, we remark, that since the second term of the equation in z is absent, the sum of the four roots is zero, or $z_1 + z_2 + z_3 + z_4 = 0$; and consequently the functions $(z_1 + z_2)^2$, &c., of which there are in general *six* (the combinations of four quantities two and two), are in this case reduced to *three* only; so that we may assume

$$(z_2 + z_3)^2 = (z_1 + z_4)^2 = 4p,$$
$$(z_3 + z_1)^2 = (z_2 + z_4)^2 = 4q,$$
$$(z_1 + z_2)^2 = (z_3 + z_4)^2 = 4r;$$

from which we have $z_1, z_2, z_3, z_4,$ included in the formula

$$\sqrt{p} + \sqrt{q} + \sqrt{r}.$$

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EXAMPLES.

1. Show that the two biquadratic equations

$$A_0x^4 + 6A_2x^2 \pm 4A_3x + A_4 = 0$$

have the same reducing cubic.

2. Find the reducing cubic of the two biquadratic equations

$$x^4 - 6lx^2 \pm 8\sqrt{(l^3 + m^3 + n^3 - 3lmn)} \cdot x + 3(4mn - l^2) = 0.$$

Ans. $\theta^3 - 3mn\theta - (m^3 + n^3) = 0.$

3. Prove that the eight roots of the equation

$$\{x^4 - 6lx^2 + 3(4mn - l^2)\}^2 = 64(l^3 + m^3 + n^3 - 3lmn)x^2$$

are given by the formula

$$\sqrt{l + m + n} + \sqrt{l + \omega m + \omega^2 n} + \sqrt{l + \omega^2 m + \omega n}.$$

(Compare Ex. 20, p. 34.)

4. If the expression

$$\sqrt{l + m + n} + \sqrt{l + \omega m + \omega^2 n} + \sqrt{l + \omega^2 m + \omega n}$$

be a root of the equation

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

determine H, I, J in terms of l, m, n.

Ans. $H = -l$, $I = 12mn$, $J = -4(m^3 + n^3)$.

5. Prove that J vanishes for the biquadratic

$$m(x - n)^4 - n(x - m)^4.$$

6. Write down the formulas expressing the root of a biquadratic in the particular cases when $I = 0$, and $J = 0$.

7. What is the quantity under the *final* square root in the formula expressing a root?

Ans. $27J^2 - I^3$.

8. Prove that the coefficients of the equation of the squares of the differences of the roots of the biquadratic equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

may be expressed in terms a_0, H, I, and J.

Removing the second term from the equation, we obtain

$$y^4 + \frac{6H}{a_0^2}y^2 + \frac{4G}{a_0^3}y + \frac{a_0^2I - 3H^2}{a_0^4} = 0\ ;$$

and changing the signs of the roots, we have

$$y^4 + \frac{6H}{a_0^2}y^2 - \frac{4G}{a_0^3}y + \frac{a_0^2I - 3H^2}{a_0^4} = 0.$$

These transformations leave the functions $(\alpha - \beta)^2$, &c., unaltered; but G becomes $-G$, the other coefficients of the latter equation remaining unchanged; therefore G can enter the coefficients of the equation of the squares of the differences in *even* powers only. And since

$$-G^2 \equiv 4H^3 - a_0^2HI + a_0^3J,$$

G^2 may be eliminated, introducing a_0, H, I, J. In a similar manner we may prove that every even function of the differences of the roots α, β, γ, δ may be expressed in terms of a_0, H, I, J, the function G of odd degree not entering.

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Subject: Algebra **Subject Code: 19MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit V

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
In strum's function substitute -∞ and ∞ in the series and difference between the number of changes of sign will give the_______	real roots	negative roots	positive roots	zero roots	real roots
If f(x)=0 has no real root then f(x) is positive for _______values of x	every	some	no	finite	every
1+ω+ω^2=__________	0	1	2	3	0
The number of positive roots of the function f(x)=x^2-1 is ___________	0	1	2	-1	2
x^100+x^98+⋯+x^2+1=0 has no __________ root	real	positive	negative	complex	real
In strum's function substitute -∞ and 0 in the series and difference between the number of changes of sign will give the_______	constant	negative roots	positive roots	zero roots	negative roots
What is the root of the polynomial x^4-1=0 ______	±i±1	0	±2±2i	3i	±i±1
The number of roots of the function f(x)=x^5-x is _______	4	1	5	3	5
2x^2+2x+2=0 has no __________ root	Positive	negative	real	complex	real
Root of x^3-1=0 is__________	ω^2	1	0	2	1

1+ω+ω^2=__________	0	1	2	-1	0
x^100+100x=0 has at least __________positive and__________negative real root	1,2	1,3	1,1	2,1	1,1
x-a is a factor of x^n-a^n if n is ________	2	4	odd	even	odd
Roots of x^2+x+1=0 are ________positive, ________ negative and ________complex	2,0,0	0,2,0	1,1,0	1,0,1	1,1,0
Another name of polynomial is ________	quadratic	cubic	quantic	quintic	quintic
x^5+x^2+x+1 is a ________ polynomial	quadratic	cubic	quantic	quintic	quantic
The number of roots of the function f(x)=x^3-x is ________	4	1	5	3	3
If a function involving all the roots of an equation is unaltered value if any two of the roots are interchanged it is called a _______ function of the roots.	constant	skew-symmetric	symmetric	non-symmetric	constant
Between two consecutive real roots a and b of the equation f(x)=0 where f(x) is a polynomial, there lies at least one real root of the equation _______ .	f ' (x)=0	f ' (x)>0	f ' (x)<0	f ' (x)≠0	f ' (x)=0
In strum's function substitute 0 and ∞ in the series and difference between the number of changes of sign will give the_______	constant	negative roots	positive roots	zero roots	positive roots
What is the root of the polynomial x^2-1=0 ____	±1	0	±2	3	±1
The number of roots of the function f(x)=x^5-x is _______	4	1	5	3	5
If the number of points in which the curve of f(x) cuts is less than the degree of the polynomial then f(x)=0 has ------- roots	real	positive real	negative real	imaginary	imaginary
If the number of points in which the curve of f(x) cuts is equal to the degree of the polynomial then f(x)=0 has no ------- roots	real	positive real	negative real	imaginary	imaginary
If the number of points in which the curve of f(x) cuts is less than the degree of the polynomial then f(x)=0 has ------ number of imaginary roots	odd	even	3	5	even

If the number of points in which the curve of f(x) cuts is equal to the degree of the polynomial then f(x)=0 has ------- number of imaginary roots	0	1	2	3	0
If f(a) and f(b) are having opposite signs then f(x)=0 has ------ one real root between a and b	at least	at most	exactly	no	at least
If f(a) and f(b) are having opposite signs then f(x)=0 has at least ----- real root between a and b	1	2	3	4	1
If f(a) and f(b) are having ------- signs then f(x)=0 has at least one real root between a and b	opposite	same	both opposite and same	neither opposite nor same	opposite
If f(a) and f(b) are having opposite signs then f(x) must attain the value ----- between a and b	0	1	2	3	0
If f(a) and f(b) are having -------- signs then f(x) must attain the value 0 between a and b	opposite	same	both opposite and same	neither opposite nor same	opposite
If f(x)=0 has no ------ root then f (x) must be positive	real	both real and imaginary	neither real nor imaginary	imaginary	real
Every equation of an ----- degree has at least one real root of a sign opposite to that of its last term	odd	even	both odd and even	neither odd nor even	odd
Every equation of an odd degree has ----- one real root of a sign opposite to that of its last term	at least	at most	exactly	more than	at least
Every equation of an odd degree has at least ----- real root of a sign opposite to that of its last term	1	2	3	4	1
Every equation of an odd degree has at least one real root of a sign ------- to that of its last term	opposite	same	both opposite and same	neither opposite nor same	opposite
Every equation of an odd degree has at least one real root of a sign opposite to that of its ---- term	first	second	third	last	last
------ equation of an odd degree has at least one real root of a sign opposite to that of its last term	Every	No	Few	Finite	Every
Every equation of an ------- degree whose last term is negative has at least two real roots	odd	even	both odd and even	neither odd nor even	even
Every equation of an even degree whose ------ term is negative has at least two real roots	first	second	third	last	last

Every equation of an even degree whose last term is ------- has at least two real roots	negative	positive	both positive and negative	neither positive nor negative	negative
Every equation of an even degree whose last term is negative has -------- two real roots	at least	at most	exactly	more than	at least
Every equation of an even degree whose last term is negative has at least ---- real roots	2	3	4	5	2
------- equation of an even degree whose last term is negative has at least two real roots	Every	No	Few	Finite	Every
Every equation of an ------- degree whose last term is negative has at least one positive and one negative root	odd	even	both odd and even	neither odd nor even	even
Every equation of an even degree whose ------ term is negative has at least one positive and one negative root	first	second	third	last	last
Every equation of an even degree whose last term is ------- has at least one positive and one negative root	negative	positive	both positive and negative	neither positive nor negative	negative
Every equation of an even degree whose last term is negative has -------- one positive and one negative root	at least	at most	exactly	more than	at least
Every equation of an even degree whose last term is negative has at least ---- positive and one negative root	1	2	3	4	1
------- equation of an even degree whose last term is negative has at least one positive and one negative root	Every	No	Few	Finite	Every
Every equation of an even degree whose last term is negative has at least –one positive and ---- negative root	1	2	3	4	1
Every equation of n degree has ----- roots and no more	n	n+1	n+2	n+3	n
Every equation of ------- degree has n roots and no more	n	n+1	n+2	n+3	n

----- equation of n degree has n roots and no more	Every	No	Few	Finite	Every