KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021

SYLLABUS

18MMU303A	**ANALYTICAL GEOMETRY**	**Semester III** **6H – 4C**

Instruction Hours / week: L: 4 T: 2 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- Geometry and its applications in the real world
- Geometric ideas in the language of the mathematician.

Course Outcomes (COs)

On successful completion of the course, students will be able to:

1. Expertise on fundamental theorems of isomorphism.
2. Know about automorphism and its developments.
3. Understand the concept of internal and external direct product.
4. Acquire the knowledge on basic concepts of group actions and their applications.
5. Apply Sylow's theorems to determine the structure of certain groups of small order.

UNIT I

Coordinates- Lengths of straight lines and areas of triangle- Polar coordinates. Locus- Equation to a locus.Straight line: Equation of a straight line- angle between two straight line. Length of a perpendicular techniques for sketching parabola- ellipse and hyperbola. Reflection properties of parabola.

UNIT II

Parabola and Ellipse: Classification of quadratic equations representing lines.Parabola : Loci Connected with the parabola -Three normals passing through a given points - Parabola referred to two tangent as axes. Ellipse: Auxiliary circle and eccentric angle - Equation to a tangent - Some properties of Ellipse - Poles and polar - Conjugate diameters - Four normals through any points.

UNIT III

Hyperbola: Asymptotes – equations referred to the asymptotes axes-One variables examples. Spheres: The Equation of a sphere - Tangents and tangent plane to a sphere - The radical plane of two spheres cylindrical surfaces. Illustrations of graphing standard quadric surfaces like cone, Ellipsoid.

UNIT IV

The angles between two directed lines- The projection of a segment - Relation between a segment and its projection - The projection of a broken line - the angle between two planes - Relation between areas of a triangle and its projection - Relation between areas of a polygon.

UNIT V

Polar equation to a conic: General Equations Tracing of Curves- Particular cases of Conic sections- Transformation of equations to center as origin- Equations to asymptotes - Tracing a parabola - Tracing a central conic - Eccentricity and foci of general conic.

SUGGESTED READINGS

1. Loney S.L.,(2005). The Elements of Coordinate Geometry, McMillan and Company, London.
   (For Unit I , II, III & V)
2. Bill R.J.T., (1994). Elementary Treatise on Coordinate Geometry of Three Dimensions,
   McMillan India Ltd. New Delhi. **(For Unit IV)**
3. Anton H., Bivens I. and Davis S., (2002). Calculus, John Wiley and Sons (Asia) Pvt. Ltd.
4. Thomas G.B., and Finney R.L., (2005).Calculus, Ninth Edition, Pearson Education, Delhi.
5. Fuller, Gordon.,(2000). Analytic Geometry, Addison Wesley Publishing Company Inc. Cambridge.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: P. VICTOR
SUBJECT NAME: ANALYTICAL GEOMETRY SUB.CODE:18MMU303A
SEMESTER: III CLASS: II B.Sc. Mathematics

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page No
		UNIT-I	
1	1	Introduction to Coordinates	S1:Ch: 1: Pg.No:1-2
2	1	Lengths of straight lines and areas of triangle	S1: Ch: 1: Pg.No:2-8
3	1	Polar coordinates-Problems	S1: Ch: 1: Pg.No:9-19
4	1	Tutorial – 1	
5	1	Locus, equation to a locus-Problems	S1: Ch: 1: Pg.No:20-24
6	1	Definition of straight line and properties	S5: Ch: 2: Pg.No:21-22
7	1	Tutorial – 2	
8	1	Equation of a straight line-Problems	S5: Ch: 2: Pg.No:23-34
9	1	Angle between two straight lines-Problems	S1: Ch: 1: Pg.No:40-42
10	1	Tutorial – 3	
11	1	Length of a perpendicular techniques for sketching parabola, ellipse and hyperbola.	S3: Ch: 1: Pg.No:40-42
12	1	Reflection properties of parabola.	S3: Ch: 1: Pg.No:40-42
13	1	Tutorial – 4	
14	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit 1=14		

UNIT-II			
1	1	Parabola and Ellipse: Classification of quadratic equations representing lines. Parabola	S1: Ch: 11: Pg.No:198-200
2	1	Problems on loci Connected with the parabola	S1: Ch: 11: Pg.No:201-206
3	1	Tutorial – 1	
4	1	Three normals passing through a given points parabola referred to two tangent as axes.	S1: Ch: 11: Pg.No:206-217
5	1	Ellipse: Auxiliary circle and eccentric angle , equation to a tangent ,	S1: Ch: 11: Pg.No:217-237
6	1	Tutorial – 2	
7	1	Some properties of Ellipse and problems	S1: Ch: 12: Pg.No:237-242
8	1	Definition of Poles and polar Problems	S3: Ch: 10: Pg.No:692-705
9	1	Tutorial – 3	
10	1	Continuation of Poles and polar problems	S3: Ch: 10: Pg.No:692-705
11	1	Problems on conjugate diameters	S1: Ch: 12: Pg.No:249-254
12	1	Tutorial – 4	
13	1	Problems on four normals through any points	S1: Ch: 12: Pg.No:254-265
14	1	Tutorial – 5	
15	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit II=15		
UNIT-III			
1	1	Introduction to Hyperbola	S1: Ch: 13;Pg.No:266-271
2	1	Continuation of Hyperbola and its problems	S1: Ch: 13;Pg.No:266-271
3	1	Tutorial – 1	
4	1	Definition of asymptotes and properties	S1: Ch: 13;Pg.No:271-284
5	1	Continuation of Asymptotes problems	S1: Ch: 13;Pg.No:271-284
6	1	Tutorial – 2	
7	1	Spheres: The Equation of a sphere	S2: Ch: 5;Pg.No:76-81
8	1	Tutorial – 3	

9	1	Equation of tangents and tangent plane to a sphere	S2: Ch: 5;Pg.No:81-82
10	1	The radical plane of two spheres Cylindrical surfaces	S2: Ch: 5;Pg.No:82-83
11	1	Tutorial – 4	
12	1	Illustrations of graphing standard quadric surfaces like cone, ellipsoid	S2: Ch: 5;Pg.No:85-88
13	1	Tutorial – 5	
14	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit III=14		
UNIT-IV			
1	1	The angle between two directed lines and problems.	S2: Ch: 2; Pg.No:13-14
2		Tutorial – 1	
3	1	The projection of a segment and its problems	S2: Ch: 2; Pg.No:14-15
4	1	Continuation of The projection of a segment	S2: Ch: 2; Pg.No:14-15
5	1	Tutorial –2	
6	1	Relation between a segment and its projection	S2: Ch: 2; Pg.No:15-16
7	1	The projection of a broken line-Problems	S2: Ch: 2; Pg.No:16-17
8	1	Tutorial – 3	
9	1	Problems on angle between two planes	S2: Ch: 2; Pg.No:17-18
10	1	Relation between areas of a triangle and its projection	S2: Ch: 2; Pg.No:18-19
11	1	Tutorial – 4	
12	1	Relation between areas of a polygon.	S2: Ch: 2; Pg.No:19-20
13	1	Tutorial – 5	
14	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit IV=14		
UNIT-V			
1	1	Polar equation to a conic	S4: Ch: 10; Pg.No:714-723
2	1	Tutorial – 1	

3	1	General Equations Tracing of Curves	S4: Ch: 10; Pg.No:723-730
4	1	Tutorial – 2	
5	1	Particular cases of Conic sections	S1: Ch: 14; Pg.No:322-323
6	1	Tutorial – 3	
7	1	Equations to asymptotes	S1: Ch: 14; Pg.No:327-329
8	1	Tutorial – 4	
9	1	Tracing a central conic and problems	S1: Ch: 14; Pg.No:333-338
10	1	Eccentricity and foci of general conic.	S1: Ch: 14; Pg.No:339-342
11	1	Tutorial – 5	
12	1	Recapitulation and discussion of possible questions	
13	1	Discussion on Previous ESE Question Papers	
14	1	Discussion on Previous ESE Question Papers	
15	1	Discussion on Previous ESE Question Papers	
	Total No of Hours Planned for unit V=15		
Total Planned Hours			**72**

SUGGESTED READINGS

1. Loney S.L.,(2005). The Elements of Coordinate Geometry, McMillan and Company, London.(For Unit I , II, III & V)

2. Bill R.J.T., (1994). Elementary Treatise on Coordinate Geometry of Three Dimensions, McMillan India Ltd. New Delhi. (For Unit IV)

3. Anton H., Bivens I. and Davis S., (2002). Calculus, John Wiley and Sons (Asia) Pvt. Ltd.

4. Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.

5. Fuller, Gordon.,(2000). Analytic Geometry, Addison Wesley Publishing Company Inc. Cambridge.

UNIT-I

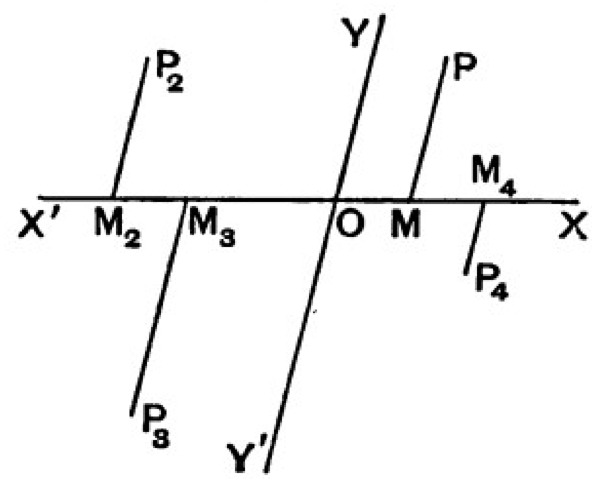
Coordinates, Lengths of straight lines and areas of triangle, polar coordinates. Locus, equation to a locus.Straight line: Equation of a straight line, angle between two straight line. Length of a perpendicular techniques for sketching parabola, ellipse and hyperbola. Reflection properties of parabola

COORDINATES. LENGTHS OF STRAIGHT LINES AND AREAS OF TRIANGLES.

Coordinates. Let OX and OY be two fixed straight lines in the plane of the paper. The line OX is called the axis of x, the line OY the axis of y, whilst the two together are called the axes of coordinates.

The point O is called the origin of coordinates or, more shortly, the origin.

From any point P in the plane draw a straight line parallel to OY to meet OX in M.



The distance OM is called the Abscissa, and the distance MP the Ordinate of the point P, whilst the abscissa and the ordinate together are called its Coordinates.

Distances measured parallel to OX are called x, with or without a suffix, (*e.g.* x_1, x_2... x', x'',...), and distances measured parallel to OY are called y, with or without a suffix, (*e.g.* y_1, y_2,... y', y'',...).

If the distances OM and MP be respectively x and y, the coordinates of P are, for brevity, denoted by the symbol (x, y).

Conversely, when we are given that the coordinates of a point P are (x, y) we know its position. For from O we have only to measure a distance OM $(=x)$ along OX and

then from M measure a distance MP $(=y)$ parallel to OY and we arrive at the position of the point P. For example in the figure, if OM be equal to the unit of length and $MP=2OM$, then P is the point (1, 2).

Ex. *Lay down on paper the position of the points*

(i) $(2, -1)$, (ii) $(-3, 2)$, and (iii) $(-2, -3)$.

To get the first point we measure a distance 2 along OX and then a distance 1 parallel to OY'; we thus arrive at the required point.

To get the second point, we measure a distance 3 along OX', and then 2 parallel to OY.

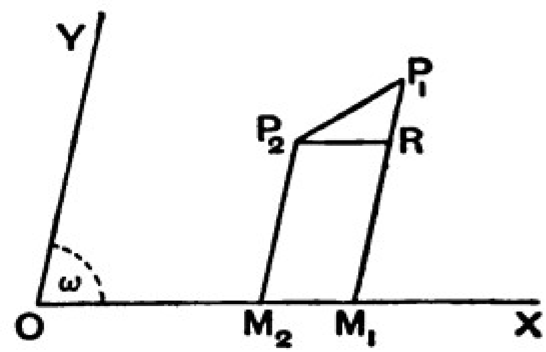
To get the third point, we measure 2 along OX' and then 3 parallel to OY'.

THEOREM

To find the distance between two points whose coordinates are given.

Let P_1 and P_2 be the two given points, and let their coordinates be respectively (x_1, y_1) and (x_2, y_2).

Draw P_1M_1 and P_2M_2 parallel to OY, to meet OX in M_1 and M_2. Draw P_2R parallel to OX to meet M_1P_1 in R.



Then

$$P_2R = M_2M_1 = OM_1 - OM_2 = x_1 - x_2,$$

$$RP_1 = M_1P_1 - M_2P_2 = y_1 - y_2,$$

and $\angle P_2RP_1 = \angle OM_1P_1 = 180° - P_1M_1X = 180° - \omega.$

We therefore have [*Trigonometry*, Art. 164]

$$P_1P_2^2 = P_2R^2 + RP_1^2 - 2P_2R \,.\, RP_1 \cos P_2RP_1$$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 - 2\,(x_1 - x_2)\,(y_1 - y_2) \cos\,(180° - \omega)$$

$$= \mathbf{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2\,(x_1 - x_2)\,(y_1 - y_2) \cos \omega} \ldots(1).$$

If the axes be, as is generally the case, at right angles, we have $\omega = 90°$ and hence $\cos \omega = 0$.

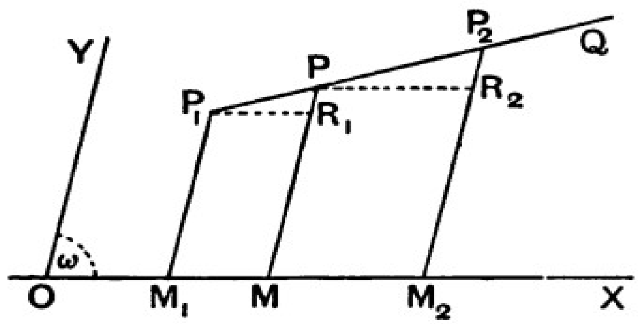
The formula (1) then becomes

$$P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

so that in rectangular coordinates the distance between the two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

To find the coordinates of the point which divides in a given ratio $(m_1 : m_2)$ the line joining two given points (x_1, y_1) and (x_2, y_2).



Let P_1 be the point (x_1, y_1), P_2 the point (x_2, y_2), and P the required point, so that we have

$$P_1P : PP_2 :: m_1 : m_2.$$

Let P be the point (x, y) so that if P_1M_1, PM, and P_2M_2 be drawn parallel to the axis of y to meet the axis of x in M_1, M, and M_2, we have

$$OM_1 = x_1, \quad M_1P_1 = y_1, \quad OM = x, \quad MP = y, \quad OM_2 = x_2,$$

and
$$M_2P_2 = y_2.$$

Draw P_1R_1 and PR_2, parallel to OX, to meet MP and M_2P_2 in R_1 and R_2 respectively.

Then
$$P_1R_1 = M_1M = OM - OM_1 = x - x_1,$$
$$PR_2 = MM_2 = OM_2 - OM = x_2 - x,$$
$$R_1P = MP - M_1P_1 = y - y_1,$$

and
$$R_2P_2 = M_2P_2 - MP = y_2 - y.$$

From the similar triangles P_1R_1P and PR_2P_2 we have

$$\frac{m_1}{m_2} = \frac{P_1P}{PP_2} = \frac{P_1R_1}{PR_2} = \frac{x - x_1}{x_2 - x}.$$

$$\therefore \quad m_1(x_2 - x) = m_2(x - x_1),$$

i.e.

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}.$$

Again

$$\frac{m_1}{m_2} = \frac{P_1P}{PP_2} = \frac{R_1P}{R_2P_2} = \frac{y - y_1}{y_2 - y},$$

so that

$$m_1(y_2 - y) = m_2(y - y_1),$$

and hence

$$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}.$$

The coordinates of the point which divides P_1P_2 internally in the given ratio $m_1 : m_2$ are therefore

$$\frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } \frac{m_1y_2 + m_2y_1}{m_1 + m_2}.$$

If the point Q divide the line P_1P_2 *externally* in the same ratio, *i.e.* so that $P_1Q : QP_2 :: m_1 : m_2$, its coordinates would be found to be

$$\frac{m_1x_2 - m_2x_1}{m_1 - m_2} \text{ and } \frac{m_1y_2 - m_2y_1}{m_1 - m_2}.$$

The proof of this statement is similar to that of the preceding article and is left as an exercise for the student.

Ex. *In any triangle ABC prove that*

$$AB^2+AC^2=2\,(AD^2+DC^2),$$

where D is the middle point of BC.

Take B as origin, BC as the axis of x, and a line through B perpendicular to BC as the axis of y.

Let $BC=a$, so that C is the point (a, O), and let A be the point (x_1, y_1).

Then D is the point $\left(\frac{a}{2}, O\right)$.

Hence $$AD^2=\left(x_1-\frac{a}{2}\right)^2+y_1^2, \text{ and } DC^2=\left(\frac{a}{2}\right)^2.$$

Hence $$2\,(AD^2+DC^2)=2\left[x_1^2+y_1^2-ax_1+\frac{a^2}{2}\right]$$
$$=2x_1^2+2y_1^2-2ax_1+a^2.$$

Also $$AC^2=(x_1-a)^2+y_1^2,$$
and $$AB^2=x_1^2+y_1^2.$$

Therefore $$AB^2+AC^2=2x_1^2+2y_1^2-2ax_1+a^2.$$

Hence $$AB^2+AC^2=2\,(AD^2+DC^2).$$

Ex. *ABC is a triangle and D, E, and F are the middle points of the sides BC, CA, and AB; prove that the point which divides AD internally in the ratio 2 : 1 also divides the lines BE and CF in the same ratio.*

Hence prove that the medians of a triangle meet in a point.

Let the coordinates of the vertices A, B, and C be (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively.

The coordinates of D are therefore $\frac{x_2+x_3}{2}$ and $\frac{y_2+y_3}{2}$.

Let G be the point that divides internally AD in the ratio $2:1$, and let its coordinates be $\bar{x}$ and $\bar{y}$.

By the last article

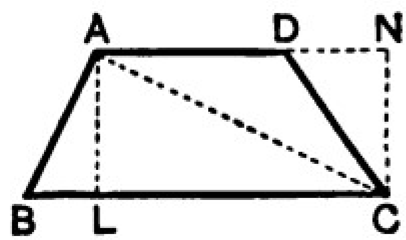
$$\bar{x}=\frac{2\times\frac{x_2+x_3}{2}+1\times x_1}{2+1}=\frac{x_1+x_2+x_3}{3}.$$

So
$$\bar{y}=\frac{y_1+y_2+y_3}{3}.$$

To prove that the area of a trapezium, i.e. *a quadrilateral having two sides parallel, is one half the sum of the two parallel sides multiplied by the perpendicular distance between them.*

Let $ABCD$ be the trapezium having the sides AD and BC parallel.

Join AC and draw AL perpendicular to BC and CN perpendicular to AD, produced if necessary.



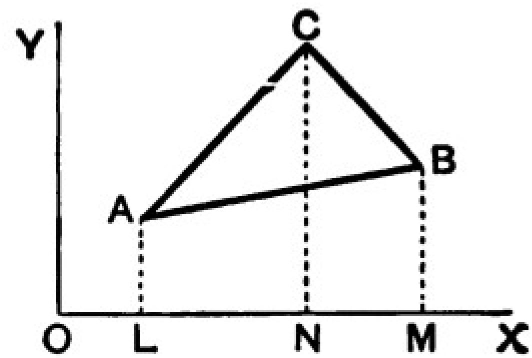
Since the area of a triangle is one half the product of any side and the perpendicular drawn from the opposite angle, we have

$$\begin{aligned}\text{area } ABCD &= \Delta ABC + \Delta ACD\\ &= \tfrac{1}{2}.BC.AL+\tfrac{1}{2}.AD.CN\\ &= \tfrac{1}{2}(BC+AD)\times AL.\end{aligned}$$

To find the area of the triangle, the coordinates of whose angular points are given, the axes being rectangular.

Let ABC be the triangle and let the coordinates of its angular points A, B and C be (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Draw AL, BM, and CN perpendicular to the axis of x, and let Δ denote the required area.



Then

$$\Delta = \text{trapezium } ALNC + \text{trapezium } CNMB - \text{trapezium } ALMB$$

$$= \tfrac{1}{2}LN\,(LA + NC) + \tfrac{1}{2}NM\,(NC + MB) - \tfrac{1}{2}LM\,(LA + MB),$$

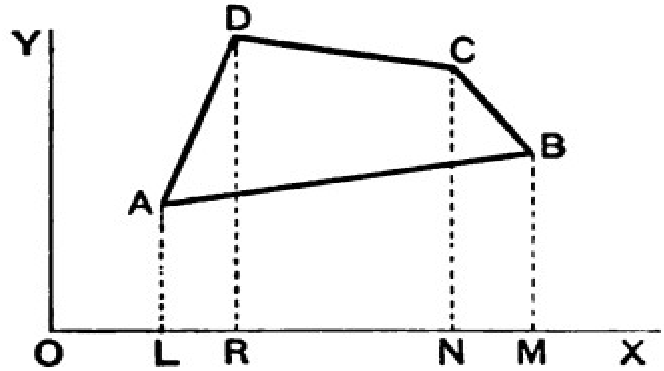
by the last article,

$$= \tfrac{1}{2}\left[(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)\right].$$

On simplifying we easily have

$$\Delta = \tfrac{1}{2}\left(\mathbf{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3}\right),$$

To find the area of a quadrilateral the coordinates of whose angular points are given.



Let the angular points of the quadrilateral, taken in order, be A, B, C, and D, and let their coordinates be respectively (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4).

Draw AL, BM, CN, and DR perpendicular to the axis of x.

Then the area of the quadrilateral

$$= \text{trapezium } ALRD + \text{trapezium } DRNC + \text{trapezium } CNMB - \text{trapezium } ALMB$$

$$= \tfrac{1}{2}LR\,(LA + RD) + \tfrac{1}{2}RN\,(RD + NC) + \tfrac{1}{2}NM\,(NC + MB) - \tfrac{1}{2}LM\,(LA + MB)$$

$$= \tfrac{1}{2}\{(x_4 - x_1)(y_1 + y_4) + (x_3 - x_4)(y_3 + y_4) + (x_2 - x_3)(y_3 + y_2) - (x_2 - x_1)(y_1 + y_2)\}$$

$$= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\}.$$

Polar Coordinates. There is another method, which is often used, for determining the position of a point in a plane.

Suppose O to be a fixed point, called the **origin** or **pole**, and OX a fixed line, called the **initial line**.

Take any other point P in the plane of the paper and join OP. The position of P is clearly known when the angle XOP and the length OP are given.

[For giving the angle XOP shews the direction in which OP is drawn, and giving the distance OP tells the distance of P along this direction.]

The angle XOP which would be traced out by the line OP in revolving from the initial line OX is called the vectorial angle of P and the length OP is called its radius vector. The two taken together are called the polar coordinates of P.

If the vectorial angle be θ and the radius vector be r, the position of P is denoted by the symbol (r, θ).

The radius vector is positive if it be measured from the origin O along the line bounding the vectorial angle; if measured in the opposite direction it is negative.

To find the length of the straight line joining two points whose polar coordinates are given.

Let A and B be the two points and let their polar coordinates be (r_1, θ_1) and (r_2, θ_2) respectively, so that

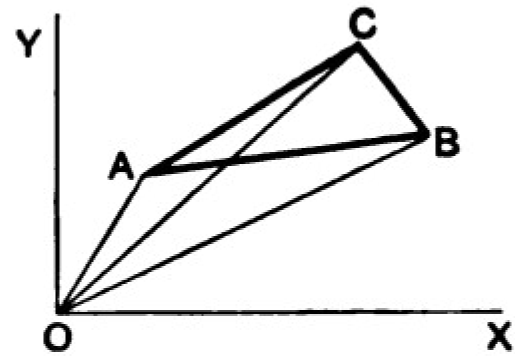
$$OA = r_1,\ OB = r_2,\ \angle XOA = \theta_1, \text{ and } \angle XOB = \theta_2.$$

Then

$$AB^2 = OA^2 + OB^2 - 2OA \,.\, OB \cos AOB$$
$$= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2).$$

To find the area of a triangle the coordinates of whose angular points are given.

Let ABC be the triangle and let (r_1, θ_1), (r_2, θ_2), and (r_3, θ_3) be the polar coordinates of its angular points.



We have

$$\triangle ABC = \triangle OBC + \triangle OCA - \triangle OBA \quad \ldots\ldots(1).$$

Now

$$\triangle OBC = \tfrac{1}{2} OB \,.\, OC \sin BOC$$

$$= \tfrac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2).$$

So $\triangle OCA = \frac{1}{2} OC \,.\, OA \sin COA = \frac{1}{2} r_3 r_1 \sin(\theta_1 - \theta_3)$,

and $\triangle OAB = \frac{1}{2} OA \,.\, OB \sin AOB = \frac{1}{2} r_1 r_2 \sin(\theta_1 - \theta_2)$

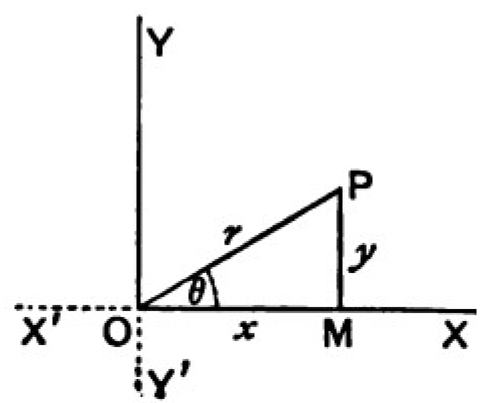
$$= -\tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1).$$

Hence (1) gives

$$\triangle ABC = \tfrac{1}{2} [r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3) + r_1 r_2 \sin(\theta_2 - \theta_1)].$$

To change from Cartesian Coordinates to Pola Coordinates, and conversely.

Let P be any point whose Cartesian coordinates, referre to rectangular axes, are x and y, and whose polar coordinates, referred to O as pole and OX as initial line, are (r, θ).



Draw PM perpendicular to OX so that we have

$OM = x$, $MP = y$, $\angle MOP = \theta$,

and $OP = r$.

From the triangle MOP we have

$$x = OM = OP \cos MOP = r \cos\theta \;\ldots\ldots\ldots(1),$$

$$y = MP = OP \sin MOP = r \sin\theta \;\ldots\ldots\ldots(2),$$

$$r = OP = \sqrt{OM^2 + MP^2} = \sqrt{x^2 + y^2} \;\ldots\ldots(3),$$

and

$$\tan\theta = \frac{MP}{OM} = \frac{y}{x} \text{ ....(4).}$$

Equations (1) and (2) express the Cartesian coordinates in terms of the polar coordinates.

Equations (3) and (4) express the polar in terms of the Cartesian coordinates.

The same relations will be found to hold if P be in any other of the quadrants into which the plane is divided by XOX' and YOY'.

Ex. *Change to Cartesian coordinates the equations*

$$(1)\ r = a\sin\theta, \text{ and } (2)\ r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{\theta}{2}.$$

(1) Multiplying the equation by r, it becomes $r^2 = ar\sin\theta$, *i.e.* by equations (2) and (3), $x^2 + y^2 = ay$.

(2) Squaring the equation (2), it becomes

$$r = a\cos^2\frac{\theta}{2} = \frac{a}{2}(1 + \cos\theta),$$

i.e. $$2r^2 = ar + ar\cos\theta,$$

i.e. $$2(x^2 + y^2) = a\sqrt{x^2 + y^2} + ax,$$

i.e. $$(2x^2 + 2y^2 - ax)^2 = a^2(x^2 + y^2).$$

LOCUS. EQUATION TO A LOCUS.

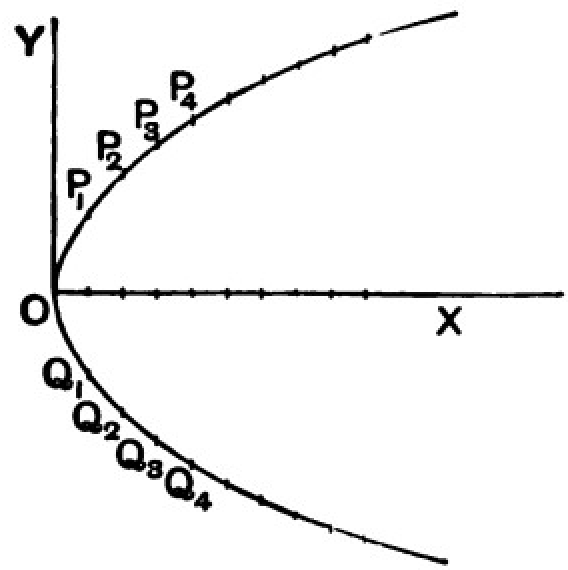
When a point moves so as always to satisfy a given condition, or conditions, the path it traces out is called its Locus under these conditions.

For example, suppose O to be a given point in the plane of the paper and that a point P is to move on the paper so that its distance from O shall be constant and equal to a. It is clear that all the positions of the moving point must lie on the circumference of a circle whose centre is O and whose radius is a. The circumference of this circle is therefore the "Locus" of P when it moves subject to the condition that its distance from O shall be equal to the constant distance a.

example let us trace the locus of the point whose coordinates satisfy the equation

$$y^2 = 4x \qquad \ldots\ldots(1).$$

If we give x a negative value we see that y is impossible; for the square of a real quantity cannot be negative.



We see therefore that there are no points lying to the left of OY.

If we give x any positive value we see that y has two real corresponding values which are equal and of opposite signs.

The following values, amongst an infinite number of others, satisfy (1), viz.

$$\left.\begin{matrix} x=0, \\ y=0 \end{matrix}\right\},\quad \left.\begin{matrix} x=1, \\ y=+2 \text{ or } -2 \end{matrix}\right\},\quad \left.\begin{matrix} x=2, \\ y=2\sqrt{2} \text{ or } -2\sqrt{2} \end{matrix}\right\},$$

$$\left.\begin{matrix} x=4 \\ y=+4 \text{ or } -4 \end{matrix}\right\},\ \ldots\ \left.\begin{matrix} x=16, \\ y=8 \text{ or } -8 \end{matrix}\right\},\ \ldots\ \left.\begin{matrix} x=+\infty, \\ y=+\infty \text{ or } -\infty \end{matrix}\right\}.$$

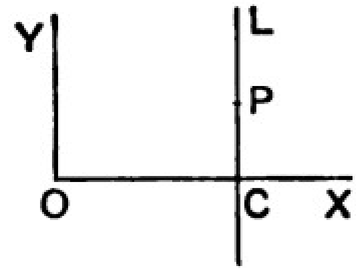
The origin is the first of these points and P_1 and Q_1, P_2 and Q_2, P_3 and Q_3, ... represent the next pairs of points.

THE STRAIGHT LINE. RECTANGULAR COORDINATES.

To find the equation to a straight line which is parallel to one of the coordinate axes.

Let CL be any line parallel to the axis of y and passing through a point C on the axis of x such that $OC = c$.

Let P be any point on this line whose coordinates are x and y.



Then the abscissa of the point P is always c, so that

$$x = c \quad \ldots\ldots\ldots\ldots\ldots (1).$$

This being true for every point on the line CL (produced indefinitely both ways), and for no other point, is, by Art. 42, the equation to the line.

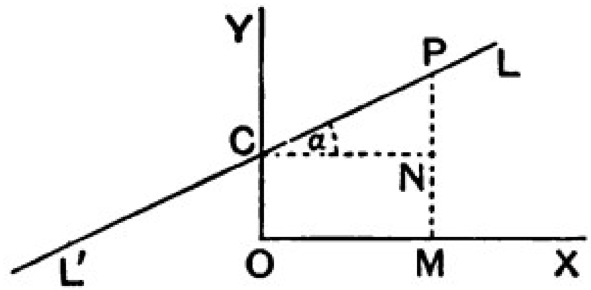
It will be noted that the equation does not contain the coordinate y.

Similarly the equation to a straight line parallel to the axis of x is $y = d$.

To find the equation to a straight line which cuts off a given intercept on the axis of y and is inclined at a given angle to the axis of x.

Let the given intercept be c and let the given angle be α.

Let C be a point on the axis of y such that OC is c. Through C draw a straight line LCL' inclined at an angle α $(=\tan^{-1} m)$ to the axis of x, so that $\tan\alpha = m$.



The straight line LCL' is therefore the straight line required, and we have to find the relation between the coordinates of any point P lying on it.

Draw PM perpendicular to OX to meet in N a line through C parallel to OX.

Let the coordinates of P be x and y, so that $OM = x$ and $MP = y$.

Then $MP = NP + MN = CN \tan \alpha + OC = m . x + c$,

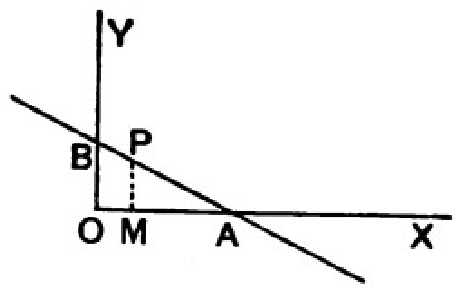
i.e. $$\mathbf{y = mx + c.}$$

This relation being true for *any* point on the given straight line is, by Art. 42, the equation to the straight line.

To find the equation to the straight line which cuts off given intercepts a and b from the axes.

Let A and B be on OX and OY respectively, and be such that $OA = a$ and $OB = b$.

Join AB and produce it indefinitely both ways. Let P be any point (x, y) on this straight line, and draw PM perpendicular to OX.



We require the relation that always holds between x and y, so long as P lies on AB.

$$\frac{OM}{OA} = \frac{PB}{AB}, \text{ and } \frac{MP}{OB} = \frac{AP}{AB}.$$

$$\therefore \frac{OM}{OA} + \frac{MP}{OB} = \frac{PB + AP}{AB} = 1,$$

i.e.
$$\frac{x}{a} + \frac{y}{b} = 1.$$

Find the equation to the straight line which passes through the point $(-5, 4)$ and is such that the portion of it between the axes is divided by the point in the ratio of $1 : 2$.

Let the required straight line be $\frac{x}{a} + \frac{y}{b} = 1$. This meets the axes in the points whose coordinates are $(a, 0)$ and $(0, b)$.

The coordinates of the point dividing the line joining these points in the ratio $1 : 2$, are (Art. 22)

$$\frac{2.a+1.0}{2+1} \text{ and } \frac{2.0+1.b}{2+1}, \text{ i.e. } \frac{2a}{3} \text{ and } \frac{b}{3}.$$

If this be the point $(-5, 4)$ we have

$$-5=\frac{2a}{3} \text{ and } 4=\frac{b}{3},$$

so that $$a=-\frac{15}{2} \text{ and } b=12.$$

The required straight line is therefore

$$\frac{x}{-\frac{15}{2}}+\frac{y}{12}=1,$$

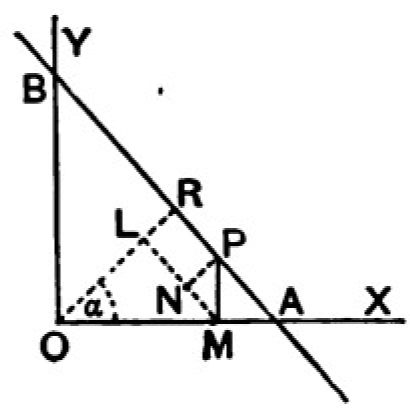
i.e. $$5y-8x=60.$$

To find the equation to a straight line in terms of the perpendicular let fall upon it from the origin and the angle that this perpendicular makes with the axis of x.

Let OR be the perpendicular from O and let its length be p.

Let α be the angle that OR makes with OX.

Let P be any point, whose coordinates are x and y, lying on AB; draw the ordinate PM, and also ML perpendicular to OR and PN perpendicular to ML.



Then $OL = OM \cos \alpha$ ....(1),

and $LR = NP = MP \sin NMP$.

But $\angle NMP = 90° - \angle NMO = \angle MOL = \alpha$.

$\therefore$ $LR = MP \sin \alpha$ ....(2).

Hence, adding (1) and (2), we have

$$OM \cos \alpha + MP \sin \alpha = OL + LR = OR = p,$$

i.e. $$\mathbf{x \cos \alpha + y \sin \alpha = p.}$$

This is the required equation.

Any equation of the first degree in x and y always represents a straight line.

For the most general form of such an equation is

$$Ax + By + C = 0 \quad \text{.................... (1)},$$

where A, B, and C are constants, *i.e.* quantities which do not contain x and y and which remain the same for all points on the locus.

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be *any* three points on the locus of the equation (1).

Since the point (x_1, y_1) lies on the locus, its coordinates when substituted for x and y in (1) must satisfy it.

Hence $$Ax_1 + By_1 + C = 0 \quad \text{.................. (2)}.$$

So $$Ax_2 + By_2 + C = 0 \quad \text{.................. (3)},$$

and $$Ax_3 + By_3 + C = 0 \quad \text{.................. (4)}.$$

Since these three equations hold between the three quantities A, B, and C, we can, as in Art. 12, eliminate them.

The result is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0 \quad \text{.................. (5)}.$$

But, by Art. 25, the relation (5) states that the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is zero.

Also these are any three points on the locus.

To find the equation to the straight line which passes through the two given points (x', y') and (x'', y'').

By Art. 47, the equation to **any** straight line is

$$y = mx + c \qquad \text{........................(1).}$$

By properly determining the quantities m and c we can make (1) represent any straight line we please.

If (1) pass through the point (x', y'), we have

$$y' = mx' + c \quad \text{.......................(2).}$$

Substituting for c from (2), the equation (1) becomes

$$\mathbf{y - y' = m\,(x - x')} \quad \text{.................(3).}$$

This is the equation to the line going through (x', y') making an angle $\tan^{-1} m$ with OX. If in addition (3) passes through the point (x'', y''), then

$$y'' - y' = m\,(x'' - x'),$$

giving
$$m = \frac{y'' - y'}{x'' - x'}.$$

Substituting this value in (3), we get as the required equation

$$\mathbf{y - y' = \frac{y'' - y'}{x'' - x'}\,(x - x')}.$$

Ex. *Find the equation to the straight line which passes through the points* $(-1, 3)$ *and* $(4, -2)$.

Let the required equation be

$$y = mx + c \quad \dots\dots(1).$$

Since (1) goes through the first point, we have

$$3 = -m + c, \text{ so that } c = m + 3.$$

Hence (1) becomes

$$y = mx + m + 3 \quad \dots\dots(2).$$

If in addition the line goes through the second point, we have

$$-2 = 4m + m + 3, \text{ so that } m = -1.$$

Hence (2) becomes

$$y = -x + 2, \text{ i.e. } x + y = 2.$$

Or, again, using the result of the last article the equation is

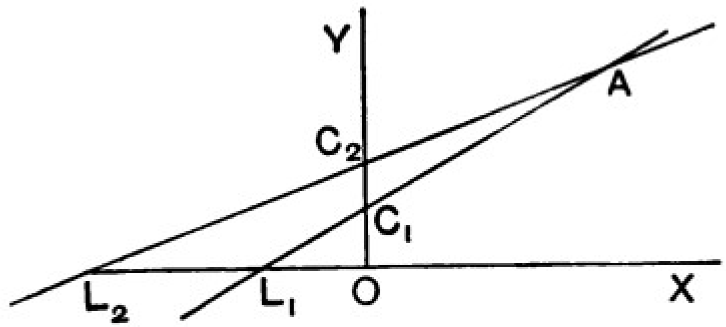
$$y - 3 = \frac{-2-3}{4-(-1)}(x+1) = -x - 1,$$

i.e. $$y + x = 2.$$

Angles between straight lines.

To find the angle between two given straight lines.

Let the two straight lines be AL_1 and AL_2, meeting the axis of x in L_1 and L_2.



I. Let their equations be

$$y = m_1 x + c_1 \text{ and } y = m_2 x + c_2 \ldots\ldots\ldots\ldots (1).$$

By Art. 47 we therefore have

$$\tan AL_1X = m_1, \text{ and } \tan AL_2X = m_2.$$

Now $$\angle L_1AL_2 = \angle AL_1X - \angle AL_2X.$$

$\therefore$ $$\tan L_1AL_2 = \tan [AL_1X - AL_2X]$$

$$= \frac{\tan AL_1X - \tan AL_2X}{1 + \tan AL_1X \,.\, \tan AL_2X} = \frac{m_1 - m_2}{1 + m_1m_2}.$$

Hence the required angle $= \angle L_1AL_2$

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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Subject: Analytical Geometry **Subject Code: 18MMU303A**
Class : II - B.Sc. Mathematics **Semester : III**

Unit I

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Questions	Opt1	Opt2	Opt3	Opt4	Answer
Two straight lines in a space …………………… intersect	must	may be	need not	necessarily	need not
The three mutually perpendicular lines x'ox, y'oy is called ………	cartesian coordinates	eulerian coordinates	coordinates	triangular cartesian coordinates	cartesian coordinates
The co-ordinate of a vertices of a triangle are called ……………	circumference	origin	centroid	ortho center	centroid
If a point lies in the YOZ plane its X coordinate is ……………………..	xz plane	YZ plane	Xy plane	0	0
Co planar Straight lines will always……………………	0	unique	perpendicular	intersect	intersect
The path traced by a moving point under certain geometricalconditions is called the ……………of the point.	line	circle	coplanar	locus	locus
The equation of y-axis is …………….	x=0	y=0	x>0	x>0	x=0
The equation of x-axis is …………….	x=0	y=0	x>0	x>0	y=0
The line parallel to x-axis have …………….slope.	[90]	0	∞	[45]	0
The line parallel to y-axis have …………….slope.	[90]	0	∞	[45]	∞
The two lines have same slope, they make the same angle with x-axis and hence they are …………….	parallel	straight line	circle	perpendicular	parallel
The lines with slopes m1,m2 are perpendicular to each other than………	m1m2=-1	m1m2=1	m1m2>1	m1m2<1	m1m2=-1
The normal form of equation ……………..	xcosα-ysinα=p	xcosα+ysinα=p	xcosα*ysinα=p	xcosα-ysinα=-p	xcosα+ysinα=p
The equation of the in joining the origin to the point(x1,y1) is ……….	xy1+y1=0	xy1-yx1=1	xy1-yx1>0	xy1-yx1=0	xy1-yx1=0
The lines y=m1x+c1 and y=m2+c2 are parallel if …………..	m1/m2	m1m2	m1=m2	m1-m2	m1=m2
The slope of the line ax+by+cz=d is ……………	m=b/a	m=c/b	m=-c/a	m=-a/b	m=-a/b
The area of triangle so formed by taking the points as vertices is zero then the three points are called……………	perpendicular	orthogonal	collinear	concurrent	collinear
If the equation on line making intercept a and b on the axis is	x/a-y/b=1	x/a+y/b=1	a/x-b/y=1	x/a-y/b=-1	x/a-y/b=1

The ellipse is a conic section in which the eccentric e is ……………	equal to 1	0	less than 1	greater than 1	less than 1
The parabola is a conic section in which the eccentric e is ……………	equal to 1	0	less than 1	greater than 1	0
The hyperbola is a conic section in which the eccentric e is ……………	equal to 1	0	less than 1	greater than 1	greater than 1
The equation of parabola for parallel to x-axis is……………	x^2=4ax	y^2=4ax	x^2=-4ax	y^2=4ax	y^2=4ax
The straight line passing through the Focus and perpendicular to the Directrix is called the ……………	line	axis	eccentricity	vertex	axis
If a point lies in the XOY plane its Z coordinate is ………………..	xz plane	YZ plane	Xy plane	0	0
The three mutually perpendicular lines x'ox,y'oy z'oz is called ………………..	rectangular cartesian coordinates	rectangular Eulerian coordinates	coordinates	triangular cartesian coordinates	rectangular cartesian coordi
The co ordinate point of perbola y^2=4ax are ……………..	(at^2,2at)	(at,2at)	(2at,at^2)	(at,-2at)	(at^2,2at)
The angle between two lines tan θ………………………	m2 -m1 / 1+ m1m2	m2 +m1 / 1+ m1m2	m2 -m1 / 1- m1m2	m2 -m1 / m1m2	m2 -m1 / 1+ m1m2
The ---------- of a point which moves in such a wat that distance from a fixed straight line is parabola	origin	distance	locus	center	locus
The study of points are defined by means of frame of reference and co-ordinates	Geometry	geodics	Anatomy	Analytical Geometry	Analytical Geometry
The co ordinate of a vertices of a triangle are called ------------------	circumference	origin	centroid	ortho center	centroid
The ---------- of a point on the line is the foot of the perpendicular drawn from the point on the line	conjucate	bijection	projection	projectile	projection
Write the formula for finding midpoint--------	x1 + x2 / 2	x1 - x2 / 2	x1 * x2 / 2	x1 + x2 * 2	x1 + x2 / 2
The intercept form of the equation of a plane is -----------	x/a + y /b + z /c = -1	x/a + y /b + z /c = 0	x/a + y /b + z /c = 1	x/a + y /b + z /c > 1	x/a + y /b + z /c = 1
The latusrectum of parabola y2=4ax is ………..	2a	3a	4a	a	4a
The latusrectum of ellipse is ………..	2b2/a	b2/a	b/a	a/b	2b2/a

nates

UNIT-II

Parabola and Ellipse: Classification of quadratic equations representing lines.Parabola : Loci Connected with the parabola ,three normals passing through a given points , parabola referred to two tangent as axes. Ellipse: Auxiliary circle and eccentric angle , equation to a tangent , some properties of Ellipse , poles and polar , conjugate diameters , four normals through any points.

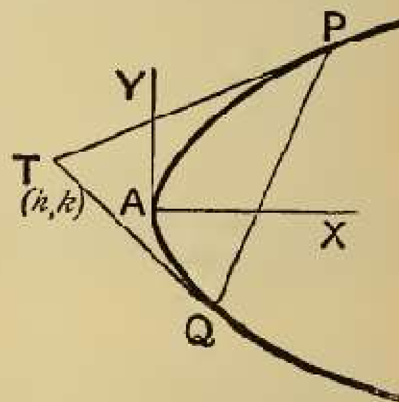
Some examples of Loci connected with the Parabola.

235. Ex. 1. *Find the locus of the intersection of tangents to the parabola $y^2 = 4ax$, the angle between them being always a given angle α.*

The straight line $y = mx + \frac{a}{m}$ is always a tangent to the parabola.

If it pass through the point $T(h, k)$ we have

$$m^2h - mk + a = 0 \quad \dots\dots\dots\dots(1).$$



If m_1 and m_2 be the roots of this equation we have (by Art. 2)

$$m_1 + m_2 = \frac{k}{h} \quad \dots\dots\dots\dots(2),$$

and

$$m_1 m_2 = \frac{a}{h} \quad \dots\dots\dots\dots(3),$$

and the equations to TP and TQ are then

$$y = m_1 x + \frac{a}{m_1} \text{ and } y = m_2 x + \frac{a}{m_2}.$$

Hence, by Art. 66, we have

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\frac{k^2}{h^2} - \frac{4a}{h}}}{1 + \frac{a}{h}} = \frac{\sqrt{k^2 - 4ah}}{a + h}, \text{ by (2) and (3).}$$

$$\therefore\ k^2 - 4ah = (a+h)^2 \tan^2 \alpha.$$

Hence the coordinates of the point T always satisfy the equation

$$y^2 - 4ax = (a+x)^2 \tan^2 \alpha.$$

We shall find in a later chapter that this curve is a hyperbola.

As a particular case let the tangents intersect at right angles, so that $m_1 m_2 = -1$.

From (3) we then have $h = -a$, so that in this case the point T lies on the straight line $x = -a$, which is the directrix.

Hence the locus of the point of intersection of tangents, which cut at right angles, is the directrix.

Ex. 2. *Prove that the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$ is the curve*

$$y^2(x+2a) + 4a^3 = 0.$$

Let PQ be a chord which is normal at P. Its equation is then

$$y = mx - 2am - am^3 \qquad \text{.....(1).}$$

Let the tangents at P and Q intersect in T, whose coordinates are h and k, so that we require the locus of T.

Since PQ is the polar of the point (h, k) its equation is

$$yk = 2a(x+h) \qquad \text{.....(2).}$$

Now the equations (1) and (2) represent the same straight line, so that they must be equivalent. Hence

$$m = \frac{2a}{k}, \text{ and } -2am - am^3 = \frac{2ah}{k}.$$

Eliminating m, *i.e.* substituting the value of m from the first of these equations in the second, we have

$$-\frac{4a^2}{k} - \frac{8a^4}{k^3} = \frac{2ah}{k},$$

i.e.
$$k^2(h+2a)+4a^3=0,$$

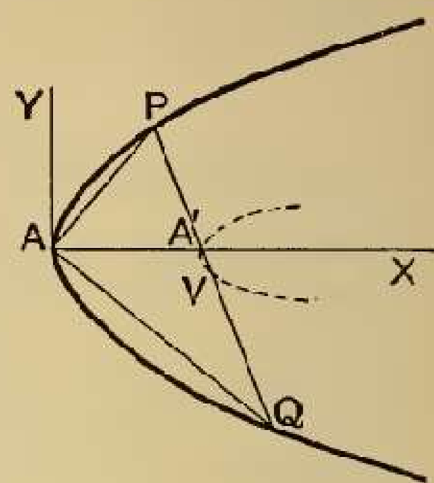
The locus of the point T is therefore

$$y^2(x+2a)+4a^3=0.$$

Ex. 3. *Find the locus of the middle points of chords of a parabola which subtend a right angle at the vertex, and prove that these chords all pass through a fixed point on the axis of the curve.*

First Method. Let PQ be any such chord, and let its equation be

$$y=mx+c \text{..............} (1).$$



The lines joining the vertex with the points of intersection of this straight line with the parabola

$$y^2=4ax \text{..............} (2),$$

are given by the equation

$$y^2c=4ax(y-mx). \quad \text{(Art. 122)}$$

These straight lines are at right angles if

$$c+4am=0. \quad \text{(Art. 111)}$$

Substituting this value of c in (1), the equation to PQ is

$$y=m(x-4a) \text{..........................} (3).$$

$$y=m(x-4a) \text{..........................} (3).$$

This straight line cuts the axis of x at a constant distance $4a$ from the vertex, *i.e.* $AA'=4a$.

If the middle point of PQ be (h, k) we have, by Art. 220,

$$k = \frac{2a}{m} \quad \text{.............................. (4).}$$

Also the point (h, k) lies on (3), so that we have

$$k = m(h - 4a) \quad \text{.............................. (5).}$$

If between (4) and (5) we eliminate m, we have

$$k = \frac{2a}{k}(h - 4a),$$

i.e. $$k^2 = 2a(h - 4a),$$

so that (h, k) always lies on the parabola

$$y^2 = 2a(x - 4a).$$

This is a parabola one half the size of the original, and whose vertex is at the point A' through which all the chords pass.

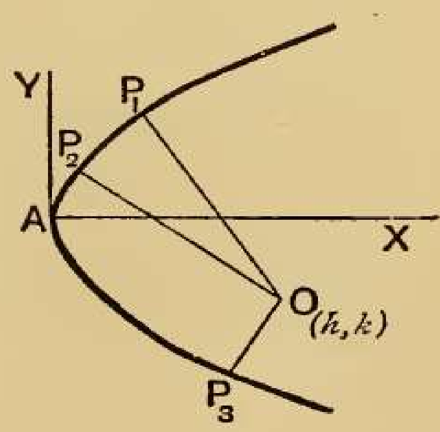
236. *To prove that, in general, three normals can be drawn from any point to the parabola and that the algebraic sum of the ordinates of the feet of these three normals is zero.*

The straight line

$$y = mx - 2am - am^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is, by Art. 208, a normal to the parabola at the points whose coordinates are

$$am^2 \text{ and } -2am \ldots\ldots(2).$$



If this normal passes through the fixed point O, whose coordinates are h and k, we have

$$k = mh - 2am - am^3,$$

i.e.

$$am^3 + (2a - h)m + k = 0 \quad \ldots\ldots\ldots\ldots\ldots(3),$$

This equation, being of the third degree, has three roots, real or imaginary. Corresponding to each of these roots, we have, on substitution in (1), the equation to a normal which passes through the point O.

Hence three normals, real or imaginary, pass through any point O.

If m_1, m_2, and m_3 be the roots of the equation (3), we have

$$m_1 + m_2 + m_3 = 0.$$

If the ordinates of the feet of these normals be y_1, y_2, and y_3, we then have, by (2),

$$y_1 + y_2 + y_3 = -2a(m_1 + m_2 + m_3) = 0.$$

Hence the second part of the proposition.

237. Ex. *Find the locus of a point which is such that (α) two of the normals drawn from it to the parabola are at right angles, (β) the three normals through it cut the axis in points whose distances from the vertex are in arithmetical progression.*

Any normal is $y = mx - 2am - am^3$, and this passes through the point (h, k), if

$$am^3 + (2a - h)m + k = 0 \quad \text{.........................(1)}.$$

If then m_1, m_2, and m_3 be the roots, we have, by Art. 2,

$$m_1 + m_2 + m_3 = 0, \quad \text{.........................(2)},$$

$$m_2m_3 + m_3m_1 + m_1m_2 = \frac{2a - h}{a}, \quad \text{.................(3)},$$

and

$$m_1m_2m_3 = -\frac{k}{a} \quad \text{.........................(4)}.$$

(α) If two of the normals, say m_1 and m_2, be at right angles, we have $m_1m_2 = -1$, and hence, from (4), $m_3 = \frac{k}{a}$.

The quantity $\frac{k}{a}$ is therefore a root of (1) and hence, by substitution, we have

$$\frac{k^3}{a^2} + (2a - h)\frac{k}{a} + k = 0,$$

i.e.

$$k^2 = a(h - 3a).$$

The locus of the point (h, k) is therefore the parabola $y^2 = a(x - 3a$ whose vertex is the point $(3a, 0)$ and whose latus rectum is one-quarte that of the given parabola.

The student should draw the figure of both parabolas.

(β) The normal $y = mx - 2am - am^3$ meets the axis of x at a poin whose distance from the vertex is $2a + am^2$. The conditions of th question then give

$$(2a + am_1^2) + (2a + am_3^2) = 2(2a + am_2^2),$$

i.e.

$$m_1^2 + m_3^2 = 2m_2^2 \qquad \text{.............................(5).}$$

If we eliminate m_1, m_2, and m_3 from the equations (2), (3), (4) and (5) we shall have a relation between h and k.

From (2) and (3), we have

$$\frac{2a - h}{a} = m_1m_3 + m_2(m_1 + m_3) = m_1m_3 - m_2^2 \qquad \text{............ (6).}$$

Also, (5) and (2) give

$$2m_2^2 = (m_1 + m_3)^2 - 2m_1m_3 = m_2^2 - 2m_1m_3,$$

i.e.

$$m_2^2 + 2m_1m_3 = 0 \qquad \text{.............................(7).}$$

Solving (6) and (7), we have

$$m_1m_3 = \frac{2a - h}{3a}, \text{ and } m_2^2 = -2 \times \frac{2a - h}{3a}.$$

Substituting these values in (4), we have

$$\frac{2a-h}{3a}\sqrt{-2\,\frac{2a-h}{3a}}=-\frac{k}{a},$$

i.e. $$27ak^2=2(h-2a)^3,$$

so that the required locus is

$$27ay^2=2(x-2a)^3.$$

238. Ex. *If the normals at three points P, Q, and R meet in a point O and S be the focus, prove that* $SP\,.\,SQ\,.\,SR=a\,.\,SO^2$.

As in the previous question we know that the normals at the points $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$ meet in the point (h, k) if

$$m_1+m_2+m_3=0 \quad\ldots\ldots(1),$$

$$m_2m_3+m_3m_1+m_1m_2=\frac{2a-h}{a} \quad\ldots\ldots(2),$$

and $$m_1m_2m_3=-\frac{k}{a} \quad\ldots\ldots(3).$$

By Art. 202 we have

$$SP=a(1+m_1^2),\ SQ=a(1+m_2^2),\ \text{and}\ SR=a(1+m_3^2).$$

Hence $$\frac{SP\,.\,SQ\,.\,SR}{a^3}=(1+m_1^2)(1+m_2^2)(1+m_3^2)$$

$$=1+(m_1^2+m_2^2+m_3^2)+(m_2^2m_3^2+m_3^2m_1^2+m_1^2m_2^2)+m_1^2m_2^2m_3^2.$$

Also, from (1) and (2), we have

$$m_1^2+m_2^2+m_3^2=(m_1+m_2+m_3)^2-2(m_2m_3+m_3m_1+m_1m_2)$$

$$=2\,\frac{h-2a}{a},$$

and

$$m_2^2m_3^2 + m_3^2m_1^2 + m_1^2m_2^2 = (m_2m_3 + m_3m_1 + m_1m_2)^2 - 2m_1m_2m_3(m_1+m_2+m_3)$$

$$= \left(\frac{h-2a}{a}\right)^2, \text{ by (1) and (2).}$$

Hence $$\frac{SP \,.\, SQ \,.\, SR}{a^3} = 1 + 2\frac{h-2a}{a} + \left(\frac{h-2a}{a}\right)^2 + \frac{k^2}{a^2}$$

$$= \frac{(h-a)^2 + k^2}{a^2} = \frac{SO^2}{a^2},$$

i.e. $$SP \,.\, SQ \,.\, SR = SO^2 \,.\, a.$$

239. In Art. 197 we obtained the simplest possible form of the equation to a parabola.

We shall now transform the origin and axes in the most general manner.

Let the new origin have as coordinates (h, k), and let the new axis of x be inclined at θ to the original axis, and let the new angle between the axes be ω'.

By Art. 133 we have for x and y to substitute

$$x \cos\theta + y\cos(\omega' + \theta) + h,$$

and $$x\sin\theta + y\sin(\omega' + \theta) + k$$

respectively.

The equation of Art. 197 then becomes

$$\{x\sin\theta + y\sin(\omega'+\theta) + k\}^2 = 4a\{x\cos\theta + y\cos(\omega'+\theta) + h\},$$

i.e.

$$\{x\sin\theta + y\sin(\omega'+\theta)\}^2 + 2x\{k\sin\theta - 2a\cos\theta\}$$
$$+ 2y\{k\sin(\omega'+\theta) - 2a\cos(\omega'+\theta)\} + k^2 - 4ah = 0 \qquad \text{.............(1).}$$

This equation is therefore the most general form of the equation to a parabola.

We notice that in it the terms of the second degree always form a perfect square.

This equation is therefore the most general form of the equation to a parabola.

We notice that in it the terms of the second degree always form a perfect square.

240. *To find the equation to a parabola, any two tangents to it being the axes of coordinates and the points of contact being distant a and b from the origin.*

By the last article the most general form of the equation to any parabola is

$$(Ax + By)^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots(1).$$

This meets the axis of x in points whose abscissae are given by

$$A^2x^2 + 2gx + c = 0 \quad \ldots\ldots\ldots\ldots(2).$$

If the parabola touch the axis of x at a distance a from the origin, this equation must be equivalent to

$$A^2 (x - a)^2 = 0 \quad \ldots\ldots\ldots\ldots(3).$$

Comparing equations (2) and (3), we have

$$g = - A^2a, \text{ and } c = A^2a^2 \quad \ldots\ldots\ldots\ldots(4).$$

Similarly, since the parabola is to touch the axis of y at a distance b from the origin, we have

$$f = -B^2 b, \text{ and } c = B^2 b^2 \quad \ldots\ldots\ldots\ldots(5).$$

From (4) and (5), equating the values of c, we have

$$B^2 b^2 = A^2 a^2,$$

so that

$$B = \pm A \frac{a}{b} \quad \ldots\ldots\ldots\ldots (6).$$

Taking the negative sign, we have

$$B = -A\frac{a}{b}, \quad g = -A^2 a, \quad f = -A^2 \frac{a^2}{b}, \text{ and } c = A^2 a^2.$$

Substituting these values in (1) we have, as the require equation,

$$\left(x - \frac{a}{b}y\right)^2 - 2ax - 2\frac{a^2}{b}y + a^2 = 0,$$

i.e.

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 - \frac{2x}{a} - \frac{2y}{b} + 1 = 0 \quad \ldots\ldots\ldots (7).$$

This equation can be written in the form

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 - 2\left(\frac{x}{a} + \frac{y}{b}\right) + 1 = \frac{4xy}{ab},$$

i.e.

$$\frac{x}{a} + \frac{y}{b} - 1 = \pm 2\sqrt{\frac{xy}{ab}},$$

i.e.

$$\left(\sqrt{\frac{x}{a}} \mp \sqrt{\frac{y}{b}}\right)^2 = 1,$$

i.e.

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \quad \ldots\ldots\ldots\ldots (8).$$

241. If in the previous article we took the positive sign in (6), the equation would reduce to

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2-2\frac{x}{a}-\frac{2y}{b}+1=0,$$

i.e.
$$\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=0.$$

This gives us (Fig., Art. 243) the pair of coincident straight lines PQ. This pair of coincident straight lines is also a conic meeting the axes in two coincident points at P and Q, but is not the parabola required.

242. *To find the equation to the tangent at any point (x', y') of the parabola*

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1.$$

Let (x'', y'') be any point on the curve close to (x', y'). The equation to the line joining these two points is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad \ldots\ldots\ldots\ldots\ldots (1).$$

But, since these points lie on the curve, we have

$$\sqrt{\frac{x'}{a}} + \sqrt{\frac{y'}{b}} = 1 = \sqrt{\frac{x''}{a}} + \sqrt{\frac{y''}{b}} \quad \ldots\ldots\ldots (2),$$

so that

$$\frac{\sqrt{y''} - \sqrt{y'}}{\sqrt{x''} - \sqrt{x'}} = -\frac{\sqrt{b}}{\sqrt{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The equation (1) is therefore

$$y - y' = \frac{\sqrt{y''} - \sqrt{y'}}{\sqrt{x''} - \sqrt{x'}} \frac{\sqrt{y''} + \sqrt{y'}}{\sqrt{x''} + \sqrt{x'}}(x - x'),$$

or, by (3),

$$y - y' = -\frac{\sqrt{b}}{\sqrt{a}} \frac{\sqrt{y''} + \sqrt{y'}}{\sqrt{x''} + \sqrt{x'}}(x - x') \quad \ldots\ldots\ldots (4).$$

The equation to the tangent at (x', y') is then obtained by putting $x'' = x'$ and $y'' = y'$, and is

$$y - y' = -\frac{\sqrt{b}}{\sqrt{a}} \frac{\sqrt{y'}}{\sqrt{x'}} (x - x'),$$

i.e.
$$\frac{x}{\sqrt{ax'}} + \frac{y}{\sqrt{by'}} = \sqrt{\frac{x'}{a}} + \sqrt{\frac{y'}{b}} = 1 \; \dots\dots\dots (5).$$

This is the required equation.

Ex. *To find the condition that the straight line* $\frac{x}{f} + \frac{y}{g} = 1$ *may be a tangent.*

This line will be the same as (5), if

$$f = \sqrt{ax'} \text{ and } g = \sqrt{by'},$$

so that
$$\sqrt{\frac{x'}{a}} = \frac{f}{a}, \text{ and } \sqrt{\frac{y'}{b}} = \frac{g}{b}.$$

Hence
$$\frac{f}{a} + \frac{g}{b} = 1.$$

This is the required condition; also, since $x' = \frac{f^2}{a}$ and $y' = \frac{g^2}{b}$, the point of contact of the given line is $\left(\frac{f^2}{a}, \frac{g^2}{b}\right)$.

Similarly, the straight line $lx + my = n$ will touch the parabola if $\frac{n}{al} + \frac{n}{bm} = 1$.

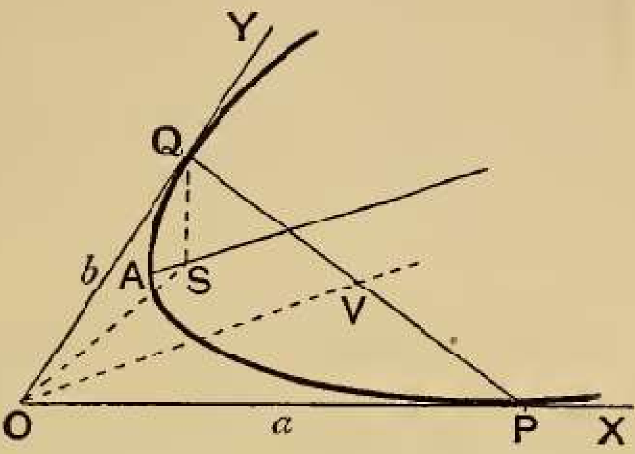
243. *To find the* **focus** *of the parabola*

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1.$$

Let S be the focus, O the origin, and P and Q the points of contact of the parabola with the axes.

Since, by Art. 230, the triangles OSP and QSO are similar, the angle SOP = angle SQO.

Hence S lies on the circle passing through the origin O, the point Q, $(0, b)$, and touching the axis of x at the origin.



The equation to this circle is

$$x^2 + 2xy \cos \omega + y^2 = by \quad \text{..............} (1).$$

Similarly, since $\angle SOQ = \angle SPO$, S will lie on the circle through O and P and touching the axis of y at the origin, *i.e.* on the circle

$$x^2 + 2xy \cos \omega + y^2 = ax \quad \text{..............} (2).$$

The intersections of (1) and (2) give the point required.

On solving (1) and (2), we have as the focus the point

$$\left(\frac{ab^2}{a^2 + 2ab \cos \omega + b^2}, \frac{a^2 b}{a^2 + 2ab \cos \omega + b^2}\right).$$

244. *To find the equation to the* **axis**.

If V be the middle point of PQ, we know, by Art. 223, that OV is parallel to the axis.

Now V is the point $\left(\frac{a}{2}, \frac{b}{2}\right)$.

Hence the equation to OV is $y = \frac{b}{a}x$.

The equation to the axis (a line through S parallel to OV) is therefore

$$y - \frac{a^2b}{a^2 + 2ab\cos\omega + b^2} = \frac{b}{a}\left(x - \frac{ab^2}{a^2 + 2ab\cos\omega + b^2}\right).$$

i.e.

$$ay - bx = \frac{ab(a^2 - b^2)}{a^2 + 2ab\cos\omega + b^2}.$$

245. *To find the equation to the* **directrix**.

If we find the point of intersection of OP and a tangent perpendicular to OP, this point will (Art. 211, γ) be on the directrix.

Similarly we can obtain the point on OQ which is on the directrix.

A straight line through the point $(f, 0)$ perpendicular to OX is

$$y = m(x - f), \text{ where (Art. 93) } 1 + m\cos\omega = 0.$$

The equation to this perpendicular straight line is then

$$x + y\cos\omega = f \quad \text{.....................(1).}$$

This straight line touches the parabola if (Art. 242)

$$\frac{f}{a} + \frac{f}{b\cos\omega} = 1, \quad i.e. \text{ if } f = \frac{ab\cos\omega}{a + b\cos\omega}.$$

The point $\left(\frac{ab\cos\omega}{a + b\cos\omega}, 0\right)$ therefore lies on the directrix.

Similarly the point $\left(0, \dfrac{ab\cos\omega}{b + a\cos\omega}\right)$ is on it.

The equation to the directrix is therefore

$$x(a + b\cos\omega) + y(b + a\cos\omega) = ab\cos\omega \quad \ldots\ldots(2).$$

The latus rectum being twice the perpendicular distance of the focus from the directrix = twice the distance of the point

$$\left(\frac{ab^2}{a^2 + 2ab\cos\omega + b^2}, \frac{a^2 b}{a^2 + 2ab\cos\omega + b^2}\right)$$

from the straight line (2)

$$= \frac{4a^2b^2\sin^2\omega}{(a^2 + 2ab\cos\omega + b^2)^{\frac{3}{2}}},$$

by Art. 96, after some reduction.

246. *To find the coordinates of the* **vertex** *and the equation to the tangent at the vertex.*

The vertex is the intersection of the axis and the curve, *i.e.* its coordinates are given by

$$\frac{y}{b} - \frac{x}{a} = \frac{a^2 - b^2}{a^2 + 2ab\cos\omega + b^2} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

and by $$\left(\frac{x}{a} - \frac{y}{b}\right)^2 - \frac{2x}{a} - \frac{2y}{b} + 1 = 0 \quad \ldots\ldots(\text{Art. } 240),$$

i.e. by $$\left(\frac{x}{a} - \frac{y}{b} + 1\right)^2 = \frac{4x}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2), we have

$$x = \frac{a}{4}\left[1 - \frac{a^2 - b^2}{a^2 + 2ab\cos\omega + b^2}\right]^2 = \frac{ab^2(b + a\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}.$$

Similarly $$y = \frac{a^2 b(a + b\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}.$$

These are the coordinates of the vertex.

The tangent at the vertex being parallel to the directrix, its equation is

$$(a + b\cos\omega)\left[x - \frac{ab^2(b + a\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}\right]$$

$$+ (b + a\cos\omega)\left[y - \frac{a^2 b(a + b\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}\right] = 0,$$

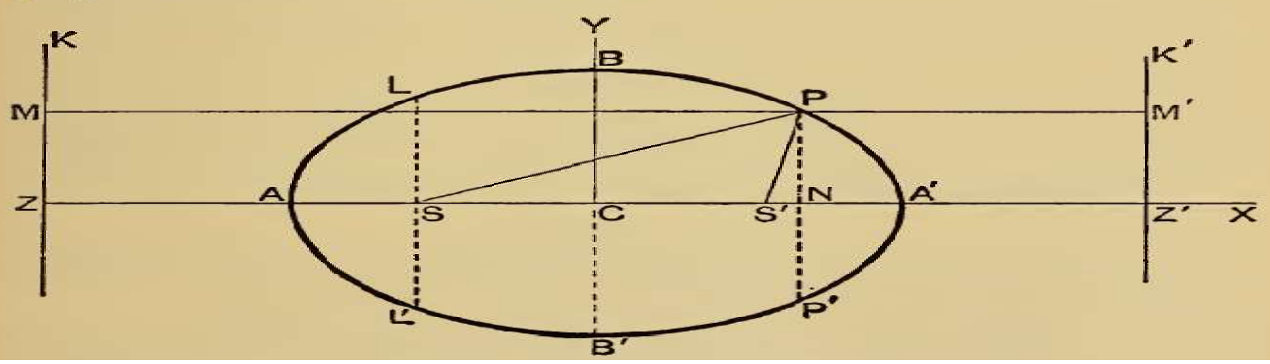
i.e. $$\frac{x}{b + a\cos\omega} + \frac{y}{a + b\cos\omega} = \frac{ab}{a^2 + 2ab\cos\omega + b^2}.$$

THE ELLIPSE.

247. The ellipse is a conic section in which the eccentricity e is less than unity.

To find the equation to an ellipse.

Let ZK be the directrix, S the focus, and let SZ be perpendicular to the directrix.



There will be a point A on SZ, such that

$$SA = e \, . \, AZ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since $e < 1$, there will be another point A', on ZS produced, such that

$$SA' = e \, . \, A'Z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let the length AA' be called $2a$, and let C be the middl point of AA'. Adding (1) and (2), we have

$$2a = AA' = e\,(AZ + A'Z) = 2 \, . \, e \, . \, CZ,$$

i.e. $$CZ = \frac{a}{e} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Subtracting (1) from (2), we have

$$e\,(A'Z - AZ) = SA' - SA = (SC + CA') - (CA - CS),$$

i.e. $$e \, . \, AA' = 2CS,$$

and hence $$CS = a \, . \, e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let C be the origin, CA' the axis of x, and a line through C perpendicular to AA' the axis of y.

Let P be any point on the curve, whose coordinates are x and y, and let PM be the perpendicular upon the directrix, and PN the perpendicular upon AA'.

The focus S is the point $(-ae, 0)$.

The relation $SP^2 = e^2 \, . \, PM^2 = e^2 \, . \, ZN^2$ then gives

$$(x + ae)^2 + y^2 = e^2 \left(x + \frac{a}{e}\right)^2, \text{ (Art. 20)},$$

i.e $$x^2(1-e^2)+y^2=a^2(1-e^2),$$

i.e. $$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If in this equation we put $x=0$, we have

$$y=\pm a\sqrt{1-e^2},$$

shewing that the curve meets the axis of y in two points, B and B', lying on opposite sides of C, such that

$$B'C=CB=a\sqrt{1-e^2}, \text{ i.e. } CB^2=CA^2-CS^2.$$

Let the length CB be called b, so that

$$b=a\sqrt{1-e^2}.$$

The equation (5) then becomes

$$\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

257. Auxiliary circle. Def. The circle which is described on the major axis, AA', of an ellipse as diameter, is called the auxiliary circle of the ellipse.

Let NP be any ordinate of the ellipse, and let it be produced to meet the auxiliary circle in Q.

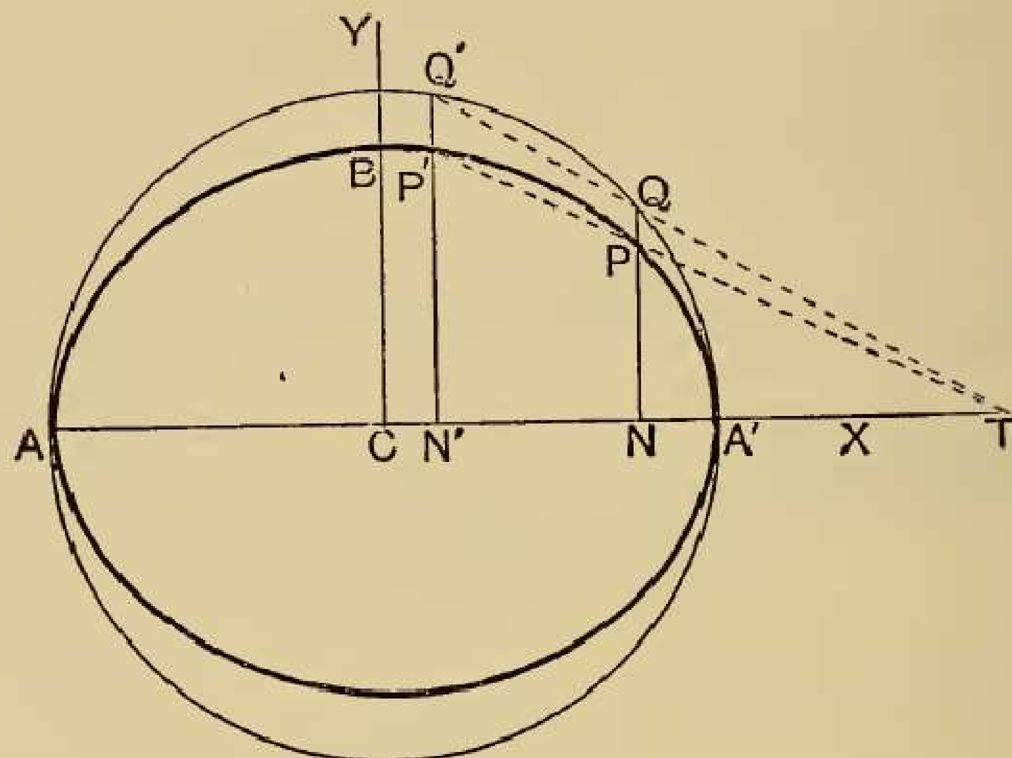
Since the angle AQA' is a right angle, being the angle in a semicircle, we have, by Euc. VI. 8, $QN^2=AN.NA'$.

Hence Art. 248 gives

$$PN^2 : QN^2 :: BC^2 : AC^2,$$

so that

$$\frac{PN}{QN} = \frac{BC}{AC} = \frac{b}{a}.$$



The point Q in which the ordinate NP meets the auxiliary circle is called the corresponding point to P.

The ordinates of any point on the ellipse and the corresponding point on the auxiliary circle are therefore to one another in the ratio $b : a$, *i.e.* in the ratio of the semi-minor to the semi-major axis of the ellipse.

The ellipse might therefore have been defined as follows :

Take a circle and from each point of it draw perpendiculars upon a diameter ; the locus of the points dividing these perpendiculars in a given ratio is an ellipse, of which the given circle is the auxiliary circle.

258. Eccentric Angle. Def. The eccentric angle of any point P on the ellipse is the angle NCQ made with the major axis by the straight line CQ joining the centre C to the point Q on the auxiliary circle which corresponds to the point P.

This angle is generally called ϕ.

We have $CN = CQ \,.\, \cos\phi = a\cos\phi$,

and $NQ = CQ\sin\phi = a\sin\phi$.

Hence, by the last article,

$$NP = \frac{b}{a} \,.\, NQ = b\sin\phi.$$

The coordinates of any point P on the ellipse are therefore $a\cos\phi$ and $b\sin\phi$.

Since P is known when ϕ is given, it is often called "the point ϕ."

259. *To obtain the equation of the straight line joining two points on the ellipse whose eccentric angles are given.*

Let the eccentric angles of the two points, P and P', be ϕ and ϕ', so that the points have as coordinates

$$(a\cos\phi,\ b\sin\phi) \text{ and } (a\cos\phi',\ b\sin\phi').$$

The equation of the straight line joining them is

$$y - b\sin\phi = \frac{b\sin\phi' - b\sin\phi}{a\cos\phi' - a\cos\phi}(x - a\cos\phi)$$

$$= \frac{b}{a}\cdot\frac{2\cos\frac{1}{2}(\phi+\phi')\sin\frac{1}{2}(\phi'-\phi)}{2\sin\frac{1}{2}(\phi+\phi')\sin\frac{1}{2}(\phi-\phi')}(x - a\cos\phi)$$

$$= -\frac{b}{a}\cdot\frac{\cos\frac{1}{2}(\phi+\phi')}{\sin\frac{1}{2}(\phi'+\phi)}(x - a\cos\phi),$$

i.e.

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2} + \frac{y}{b}\sin\frac{\phi+\phi'}{2} = \cos\phi\cos\frac{\phi+\phi'}{2} + \sin\phi\sin\frac{\phi+\phi'}{2}$$

$$= \cos\left[\phi - \frac{\phi+\phi'}{2}\right] = \cos\frac{\phi-\phi'}{2} \quad \ldots\ldots\ldots\ldots(1).$$

This is the required equation.

Cor. The points on the auxiliary circle, corresponding to P and P', have as coordinates $(a\cos\phi,\ a\sin\phi)$ and $(a\cos\phi',\ a\sin\phi')$.

The equation to the line joining them is therefore (Art. 178)

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2} + \frac{y}{a}\sin\frac{\phi+\phi'}{2} = \cos\frac{\phi-\phi'}{2}.$$

This straight line and (1) clearly make the same intercept on the major axis.

Hence the straight line joining any two points on an ellipse, and the straight line joining the corresponding points on the auxiliary circle, meet the major axis in the same point.

260. *To find the intersections of any straight line with the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{.......................(1).}$$

Let the equation of the straight line be

$$y = mx + c \quad \text{.......................(2).}$$

The coordinates of the points of intersection of (1) and (2) satisfy both equations and are therefore obtained by solving them as simultaneous equations.

Substituting for y in (1) from (2), the abscissae of the points of intersection are given by the equation

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

i.e. $$x^2(a^2m^2+b^2) + 2a^2mcx + a^2(c^2-b^2) = 0 \quad \ldots\ldots(3).$$

This is a quadratic equation and hence has two roots, real, coincident, or imaginary.

Also corresponding to each value of x we have from (2) one value of y.

The straight line therefore meets the curve in two points real, coincident, or imaginary.

The roots of the equation (3) are real, coincident, or imaginary according as

$(2a^2mc)^2 - 4(b^2+a^2m^2) \times a^2(c^2-b^2)$ is positive, zero, or negative,

i.e. according as $b^2(b^2+a^2m^2) - b^2c^2$ is positive, zero, or negative,

i.e. according as c^2 is $<=$ or $> a^2m^2+b^2$.

261. *To find the length of the chord intercepted by the ellipse on the straight line $y = mx + c$.*

As in Art. 204, we have

$$x_1 + x_2 = -\frac{2a^2mc}{a^2m^2+b^2}, \text{ and } x_1x_2 = \frac{a^2(c^2-b^2)}{a^2m^2+b^2},$$

so that $$x_1 - x_2 = \frac{2ab\sqrt{a^2m^2+b^2-c^2}}{a^2m^2+b^2}.$$

The length of the required chord therefore

$$=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=(x_1-x_2)\sqrt{1+m^2}$$

$$=\frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2+b^2-c^2}}{a^2m^2+b^2}.$$

262. *To find the equation to the* **tangent** *at any point* (x', y') *of the ellipse.*

Let P and Q be two points on the ellipse, whose coordinates are (x', y') and (x'', y'').

The equation to the straight line PQ is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x') \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Since both P and Q lie on the ellipse, we have

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1 \quad \ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{and } \frac{x''^2}{a^2}+\frac{y''^2}{b^2}=1 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Hence, by subtraction,

$$\frac{x''^2-x'^2}{a^2}+\frac{y''^2-y'^2}{b^2}=0,$$

i.e.
$$\frac{(y''-y')(y''+y')}{b^2}=-\frac{(x''-x')(x''+x')}{a^2},$$

i.e.
$$\frac{y''-y'}{x''-x'}=-\frac{b^2}{a^2}\frac{x''+x'}{y''+y'}.$$

On substituting in (1) the equation to any secant PQ becomes

$$y - y' = -\frac{b^2}{a^2}\frac{x'' + x'}{y'' + y'}(x - x') \;\dots\dots\dots\dots (4).$$

To obtain the equation to the tangent we take Q indefinitely close to P, and hence, in the limit, we put $x'' = x'$ and $y'' = y'$.

The equation (4) then becomes

$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x'),$$

i.e.

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \text{ by equation (2).}$$

The required equation is therefore

$$\mathbf{\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.}$$

263. *To find the equation to a tangent in terms of the tangent of its inclination to the major axis.*

As in Art. 260, the straight line

$$y = mx + c \;\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

meets the ellipse in points whose abscissae are given by

$$x^2(b^2 + a^2m^2) + 2mca^2x + a^2(c^2 - b^2) = 0,$$

and, by the same article, the roots of this equation are coincident if

$$c = \sqrt{a^2m^2 + b^2}.$$

In this case the straight line (1) is a tangent, and it becomes

$$\mathbf{y = mx + \sqrt{a^2m^2 + b^2}} \;\dots\dots\dots\dots\dots (2).$$

This is the required equation.

264. By a proof similar to that of the last article, it may be shewn that the straight line

$$x \cos \alpha + y \sin \alpha = p$$

touches the ellipse, if

$$\mathbf{p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha.}$$

Similarly, it may be shewn that the straight line

$$lx + my = n$$

touches the ellipse, if $a^2l^2 + b^2m^2 = n^2$.

265. *Equation to the tangent at the point whose eccentric angle is ϕ.*

The coordinates of the point are $(a \cos \phi, b \sin \phi)$.

Substituting $x' = a \cos \phi$ and $y' = b \sin \phi$ in the equation of Art. 262, we have, as the required equation,

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1 \quad \ldots\ldots\ldots\ldots (1).$$

This equation may also be deduced from Art. 259.

For the equation of the tangent at the point "ϕ" is obtained by making $\phi' = \phi$ in the result of that article.

Ex. *Find the intersection of the tangents at the points ϕ and ϕ'.*

The equations to the tangents are

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1=0,$$

and

$$\frac{x}{a}\cos\phi'+\frac{y}{b}\sin\phi'-1=0.$$

The required point is found by solving these equations.

We obtain

$$\frac{\frac{x}{a}}{\sin\phi - \sin\phi'} = \frac{\frac{y}{b}}{\cos\phi' - \cos\phi} = \frac{-1}{\sin\phi'\cos\phi - \cos\phi'\sin\phi} = \frac{1}{\sin(\phi-\phi')},$$

i.e.

$$\frac{x}{2a\cos\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2}} = \frac{y}{2b\sin\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi}{2}} = \frac{1}{2\sin\frac{\phi-\phi'}{2}\cos\frac{\phi-\phi}{2}}.$$

Hence $\quad x = a\dfrac{\cos\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}$, and $y = b\dfrac{\sin\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}$.

266. *Equation to the* **normal** *at the point* (x', y').

The required normal is the straight line which passes through the point (x', y') and is perpendicular to the tangent, *i.e.* to the straight line

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}.$$

Its equation is therefore

$$y - y' = m(x - x'),$$

where $m\left(-\frac{b^2x'}{a^2y'}\right) = -1$, *i.e.* $m = \frac{a^2y'}{b^2x'}$, (Art. 69).

The equation to the normal is therefore $y - y' = \frac{a^2y'}{b^2x'}(x - x')$,

i.e.

$$\frac{x - x'}{\frac{x'}{a^2}} = \frac{y - y'}{\frac{y'}{b^2}}.$$

267. *Equation to the normal at the point whose eccentric angle is* ϕ.

The coordinates of the point are $a\cos\phi$ and $b\sin\phi$.

Hence, in the result of the last article putting

$$x' = a\cos\phi \text{ and } y' = b\sin\phi,$$

it becomes
$$\frac{x - a\cos\phi}{\frac{\cos\phi}{a}} = \frac{y - b\sin\phi}{\frac{\sin\phi}{b}},$$

i.e.
$$\frac{ax}{\cos\phi} - a^2 = \frac{by}{\sin\phi} - b^2.$$

The required normal is therefore

$$\mathbf{ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2.}$$

270. Some properties of the ellipse.

(a) $SG = e.SP$, *and the tangent and normal at P bisect the external and internal angles between the focal distances of P.*

By Art. 269, we have $CG = e^2x'$.

Hence $SG = SC + CG = ae + e^2x' = e.SP$, by Art. 251.

Also $S'G = CS' - CG = e(a - ex') = e.S'P$.

Hence $SG : S'G :: SP : S'P$.

Therefore, by Euc. VI, 3, PG bisects the angle SPS'.

It follows that the tangent bisects the exterior angle between SP and $S'P$.

(β) *If SY and $S'Y'$ be the perpendiculars from the foci upon the tangent at any point P of the ellipse, then Y and Y' lie on the auxiliary circle, and $SY \,.\, S'Y' = b^2$. Also CY and $S'P$ are parallel.*

The equation to any tangent is

$$x \cos \alpha + y \sin \alpha = p \quad \ldots\ldots\ldots\ldots (1),$$

where $$p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} \text{ (Art. 264).}$$

The perpendicular SY to (1) passes through the point $(-ae, 0)$ and its equation, by Art. 70, is therefore

$$(x + ae) \sin \alpha - y \cos \alpha = 0 \quad \ldots\ldots\ldots\ldots (2).$$

If Y be the point (h, k) then, since Y lies on both (1) and (2), we have

$$h \cos \alpha + k \sin \alpha = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha},$$

and $$h \sin \alpha - k \cos \alpha = -ae \sin \alpha = -\sqrt{a^2 - b^2} \sin^2 \alpha.$$

Squaring and adding these equations, we have $h^2 + k^2 = a^2$, so that Y lies on the auxiliary circle $x^2 + y^2 = a^2$.

Similarly it may be proved that Y' lies on this circle.

Again S is the point $(-ae, 0)$ and S' is $(ae, 0)$.

Hence, from (1),

$$SY = p + ae \cos \alpha, \text{ and } S'Y' = p - ae \cos \alpha. \quad \text{(Art. 75.)}$$

Thus $$SY \,.\, S'Y' = p^2 - a^2e^2 \cos^2 \alpha$$
$$= a^2 \cos^2 \alpha + b^2 \sin^2 \alpha - (a^2 - b^2) \cos^2 \alpha$$
$$= b^2.$$

Also $$CT = \frac{a^2}{CN},$$

and therefore $$S'T = \frac{a^2}{CN} - ae = \frac{a(a - eCN)}{CN}.$$

$$\therefore \frac{CT}{S'T} = \frac{a}{a - e \,.\, CN} = \frac{CY}{S'P}.$$

Hence CY and $S'P$ are parallel. Similarly CY' and SP are parallel.

(γ) *If the normal at any point P meet the major and minor axes in G and g, and if CF be the perpendicular upon this normal, then* $PF . PG = b^2$ *and* $PF . Pg = a^2$.

The tangent at any point P (the point "ϕ") is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

Hence PF = perpendicular from C upon this tangent

$$= \frac{1}{\sqrt{\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}}} = \frac{ab}{\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}} \quad \ldots\ldots\ldots (1).$$

The normal at P is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If we put $y=0$, we have $CG = \frac{a^2-b^2}{a}\cos\phi$.

$$\therefore PG^2 = \left(a\cos\phi - \frac{a^2-b^2}{a}\cos\phi\right)^2 + b^2\sin^2\phi$$

$$= \frac{b^4}{a^2}\cos^2\phi + b^2\sin^2\phi,$$

i.e. $$PG = \frac{b}{a}\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}.$$

From this and (1), we have $PF . PG = b^2$.

If we put $x=0$ in (2), we see that g is the point

$$\left(0, -\frac{a^2-b^2}{b}\sin\phi\right).$$

Hence $$Pg^2 = a^2\cos^2\phi + \left(b\sin\phi + \frac{a^2-b^2}{b}\sin\phi\right)^2,$$

so that $$Pg = \frac{a}{b}\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}.$$

From this result and (1) we therefore have

$$PF . Pg = a^2.$$

271. *To find the locus of the point of intersection of tangents which meet at right angles.*

Any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2},$$

and a perpendicular tangent is

$$y = -\frac{1}{m}x + \sqrt{a^2\left(-\frac{1}{m}\right)^2 + b^2}.$$

Hence, if (h, k) be their point of intersection, we have

$$k - mh = \sqrt{a^2m^2 + b^2} \quad \text{..................(1)},$$

and

$$mk + h = \sqrt{a^2 + b^2m^2} \quad \text{..................(2)}.$$

If between (1) and (2) we eliminate m, we shall have a relation between h and k. Squaring and adding these equations, we have

$$(k^2 + h^2)(1 + m^2) = (a^2 + b^2)(1 + m^2),$$

i.e.

$$h^2 + k^2 = a^2 + b^2.$$

Hence the locus of the point (h, k) is the circle

$$x^2 + y^2 = a^2 + b^2,$$

i.e. a circle, whose centre is the centre of the ellipse, and whose radius is the length of the line joining the ends of the major and minor axis. This circle is called the **Director Circle.**

272. *To prove that through any given point (x_1, y_1) there pass, in general, two tangents to an ellipse.*

The equation to any tangent is (by Art. 263)

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \text{..................(1)}.$$

If this pass through the fixed point (x_1, y_1), we have

$$y_1 - mx_1 = \sqrt{a^2m^2 + b^2},$$

i.e. $$y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2m^2 + b^2,$$

i.e. $$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - b^2) = 0 \text{.........}(2).$$

For any given values of x_1 and y_1 this equation is in general a quadratic equation and gives two values of m (real or imaginary).

Corresponding to each value of m we have, by substituting in (1), a different tangent.

The roots of (2) are real and different, if

$$(-2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - b^2) \text{ be positive,}$$

i.e. if $$b^2x_1^2 + a^2y_1^2 - a^2b^2 \text{ be positive,}$$

i.e. if $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \text{ be positive,}$$

i.e. if the point (x_1, y_1) be outside the curve.

The roots are equal, if

$$b^2x_1^2 + a^2y_1^2 - a^2b^2$$

be zero, *i.e.* if the point (x_1, y_1) lie on the curve.

The roots are imaginary, if

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

be negative, *i.e.* if the point (x_1, y_1) lie within the curve (Art. 255).

273. *Equation to the chord of contact of tangents drawn from a point* (x_1, y_1).

The equation to the tangent at any point Q, whose coordinates are x' and y', is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

Also the tangent at the point R, whose coordinates are x'' and y'', is

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} = 1.$$

If these tangents meet at the point T, whose coordinates are x_1 and y_1, we have

$$\frac{x_1x'}{a^2} + \frac{y_1y'}{b^2} = 1 \qquad \text{(1)},$$

and

$$\frac{x_1x''}{a^2} + \frac{y_1y''}{b^2} = 1 \qquad \text{(2)}.$$

The equation to QR is then

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \text{(3)}.$$

274. *To find the equation of the polar of the point* (x_1, y_1) *with respect to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{[Art. 162.]}$$

Let Q and R be the points in which any chord drawn through the point (x_1, y_1) meets the ellipse [Fig. Art. 214].

Let the tangents at Q and R meet in the point whose coordinates are (h, k).

We require the locus of (h, k).

Since QR is the chord of contact of tangents from (h, k), its equation (Art. 273) is

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1.$$

Since this straight line passes through the point (x_1, y_1), we have

$$\frac{hx_1}{a^2} + \frac{ky_1}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the relation (1) is true, it follows that the point (h, k) lies on the straight line

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence (2) is the equation to the polar of the point (x_1, y_1).

277. *To find the coordinates of the pole of any given line*

$$Ax + By + C = 0 \quad \text{.................. (1).}$$

Let (x_1, y_1) be its pole. Then (1) must be the same as the polar of (x_1, y_1), *i.e.*

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0 \quad \text{.................. (2).}$$

Comparing (1) and (2), as in Art. 218, the required pole is easily seen to be

$$\left(-\frac{Aa^2}{C}, -\frac{Bb^2}{C}\right).$$

278. *To find the equation to the pair of tangents that can be drawn to the ellipse from the point* (x_1, y_1).

Let (h, k) be any point on either of the tangents that can be drawn to the ellipse.

The equation of the straight line joining (h, k) to (x_1, y_1) is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1),$$

i.e.

$$y = \frac{k - y_1}{h - x_1}x + \frac{hy_1 - kx_1}{h - x_1}.$$

If this straight line touch the ellipse, it must be of the form

$$y = mx + \sqrt{a^2m^2 + b^2}. \qquad \text{(Art. 263.)}$$

Hence

$$m = \frac{k - y_1}{h - x_1}, \quad \text{and} \quad \left(\frac{hy_1 - kx_1}{h - x_1}\right)^2 = a^2m^2 + b^2.$$

Hence

$$\left(\frac{hy_1 - kx_1}{h - x_1}\right)^2 = a^2\left(\frac{k - y_1}{h - x_1}\right)^2 + b^2.$$

But this is the condition that the point (h, k) may lie on the locus

$$(xy_1 - x_1y)^2 = a^2(y - y_1)^2 + b^2(x - x_1)^2 \;\ldots\ldots\; (1).$$

This equation is therefore the equation to the required tangents.

It would be found that (1) is equivalent to

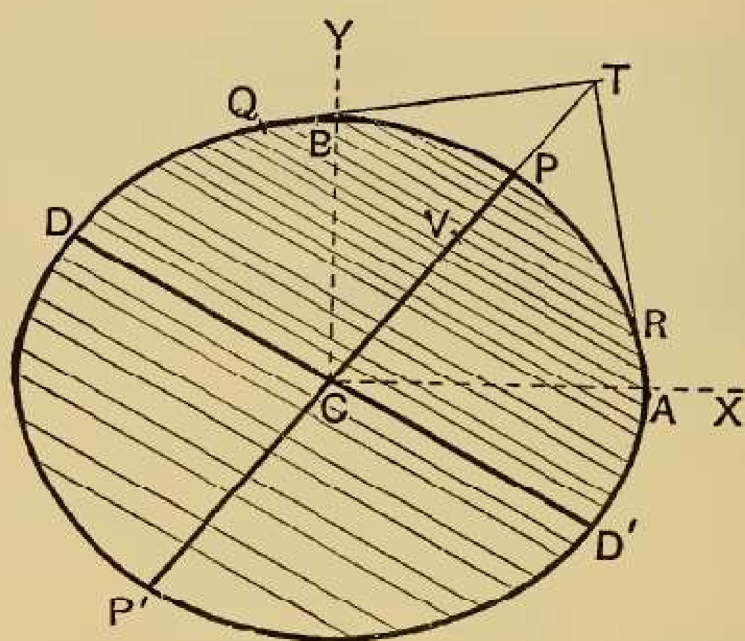
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2.$$

279. *To find the locus of the middle points of paralle[l] chords of the ellipse.*

Let the chords make with the axis an angle whos[e] tangent is m, so that the equation to any one of them [,] QR, is

$$y = mx + c \text{.......................} (1),$$

where c is different for the different chords.



This straight line meets the ellipse in points whose abscissae are given by the equation

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

i.e. $$x^2(a^2m^2 + b^2) + 2a^2mcx + a^2(c^2 - b^2) = 0 \text{ ......} (2).$$

Let the roots of this equation, *i.e.* the abscissae of Q and R, be x_1 and x_2, and let V, the middle point of QR, be the point (h, k).

Then, by Arts. 22 and 1, we have

$$h = \frac{x_1 + x_2}{2} = -\frac{a^2mc}{a^2m^2 + b^2} \ldots\ldots\ldots\ldots\ldots (3).$$

Also V lies on the straight line (1), so that

$$k = mh + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If between (3) and (4) we eliminate c, we have

$$h = -\frac{a^2m\,(k - mh)}{a^2m^2 + b^2},$$

i.e.

$$b^2h = -a^2mk \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Hence the point (h, k) always lies on the straight line

$$y = -\frac{b^2}{a^2m}x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The required locus is therefore the straight line

$$y = m_1 x, \text{ where } m_1 = -\frac{b^2}{a^2 m},$$

i.e.

$$mm_1 = -\frac{b^2}{a^2} \qquad \text{.....................} \qquad (7).$$

280. *Equation to the chord whose middle point is (h, k).*

The required equation is (1) of the foregoing article, where m and c are given by equations (4) and (5), so that

$$m = -\frac{b^2 h}{a^2 k}, \text{ and } c = \frac{a^2k^2 + b^2h^2}{a^2 k}.$$

The required equation is therefore

$$m = -\frac{b^2h}{a^2k}, \text{ and } c = \frac{a^2k^2 + b^2h^2}{a^2k}.$$

The required equation is therefore

$$y = -\frac{b^2h}{a^2k}x + \frac{a^2k^2 + b^2h^2}{a^2k},$$

i.e.

$$\frac{k}{b^2}(y-k) + \frac{h}{a^2}(x-h) = 0.$$

It is therefore parallel to the polar of (h, k).

281. Diameter. Def. The locus of the middle points of parallel chords of an ellipse is called a diameter, and the chords are called its double ordinates.

By equation (6) of Art. 279 we see that any diameter passes through the centre C.

Also, by equation (7), we see that the diameter $y = m_1x$ bisects all chords parallel to the diameter $y = mx$, if

$$mm_1 = -\frac{b^2}{a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But the symmetry of the result (1) shows that, in this case, the diameter $y = mx$ bisects all chords parallel to the diameter $y = m_1x$.

Such a pair of diameters are called Conjugate Diameters. Hence

Conjugate Diameters. Def. Two diameters are said to be conjugate when each bisects all chords parallel to the other.

Two diameters $y = mx$ and $y = m_1x$ are therefore conjugate, if

$$mm_1 = -\frac{b^2}{a^2}.$$

282. *The tangent at the extremity of any diameter is parallel to the chords which it bisects.*

In the Figure of Art. 279 let (x', y') be the point P on the ellipse, the tangent at which is parallel to the chord QR, whose equation is

$$y = mx + c \qquad \text{......(1).}$$

The tangent at the point (x', y') is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \qquad \text{......(2).}$$

Since (1) and (2) are parallel, we have

$$m = -\frac{b^2x'}{a^2y'},$$

i.e. the point (x', y') lies on the straight line

$$y = -\frac{b^2}{a^2m}x.$$

But, by Art. 279, this is the diameter which bisects QR and all chords which are parallel to it.

283. *The tangents at the ends of any chord meet on the diameter which bisects the chord.*

Let the equation to the chord QR (Art. 279) be

$$y = mx + c \qquad \text{......(1).}$$

Let T be the point of intersection of the tangents at Q and R, and let its coordinates be x_1 and y_1.

Since QR is the chord of contact of tangents from T, its equation is, by Art. 273,

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1 \qquad \text{......(2).}$$

The equations (1) and (2) therefore represent the same straight line, so that

$$m = -\frac{b^2 h}{a^2 k},$$

i.e. (h, k) lies on the straight line

$$y = -\frac{b^2}{a^2 m} x,$$

which, by Art. 279, is the equation to the diameter bisecting the chord QR. Hence T lies on the straight line CP.

284. *If the eccentric angles of the ends, P and D, of a pair of conjugate diameters be ϕ and ϕ', then ϕ and ϕ' differ by a right angle.*

Since P is the point $(a\cos\phi, b\sin\phi)$, the equation to CP is

$$y = x \cdot \frac{b}{a}\tan\phi \qquad \text{.....................(1).}$$

So the equation to CD is

$$y = x \cdot \frac{b}{a}\tan\phi' \qquad \text{.....................(2).}$$

These diameters are (Art. 281) conjugate if

$$\frac{b^2}{a^2}\tan\phi\tan\phi' = -\frac{b^2}{a^2},$$

i.e. if $\tan\phi = -\cot\phi' = \tan(\phi' \pm 90^\circ)$,

i.e. if $\boldsymbol{\phi - \phi' = \pm 90^\circ.}$

Cor. 1. The points on the auxiliary circle corresponding to P and D subtend a right angle at the centre.

For if p and d be these points then, by Art. 258, we have

$$\angle pCA' = \phi \text{ and } \angle dCA' = \phi'.$$

Hence

$$\angle pCd = \angle dCA' - \angle pCA' = \phi - \phi' = 90°.$$

Cor. 2. In the figure of Art. 286 if P be the point ϕ, then D is the point $\phi + 90°$ and D' is the point $\phi - 90°$.

285. From the previous article it follows that if P be the point $(a \cos \phi, b \sin \phi)$, then D is the point

$$\{a \cos (90° + \phi), \quad b \sin (90° + \phi)\} \text{ i.e. } (-a \sin \phi, b \cos \phi).$$

Hence, if PN and DM be the ordinates of P and D, we have

$$\frac{NP}{b} = -\frac{CM}{a}, \text{ and } \frac{CN}{a} = \frac{MD}{b}.$$

286. *If PCP′ and DCD′ be a pair of conjugate diameters, then (1) $CP^2 + CD^2$ is constant, and (2) the area of the parallelogram formed by the tangents at the ends of these diameters is constant.*

Let P be the point ϕ, so that its coordinates are $a \cos \phi$ and $b \sin \phi$. Then D is the point $90° + \phi$, so that its coordinates are

$$a \cos (90° + \phi) \text{ and } b \sin (90° + \phi),$$

i.e.
$$-a \sin \phi \text{ and } b \cos \phi.$$

(1) We therefore have

$$CP^2 = a^2 \cos^2 \phi + b^2 \sin^2 \phi,$$

and $$CD^2 = a^2 \sin^2 \phi + b^2 \cos^2 \phi.$$

Hence $$CP^2 + CD^2 = a^2 + b^2$$

= the sum of the squares of the semi-axes of the ellipse.

(2) Let $KLMN$ be the parallelogram formed by the tangents at P, D, P', and D'.

By Euc. I. 36, we have

$$\text{area } KLMN = 4 \,.\, \text{area } CPKD$$
$$= 4 \,.\, CU \,.\, PK = 4\, CU \,.\, CD,$$

where CU is the perpendicular from C upon the tangent at P.

Now the equation to the tangent at P is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi - 1 = 0,$$

so that (Art. 75) we have

$$CU = \frac{1}{\sqrt{\dfrac{\cos^2 \phi}{a^2} + \dfrac{\sin^2 \phi}{b^2}}} = \frac{ab}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} = \frac{ab}{CD}.$$

Hence $$CU \,.\, CD = ab.$$

Thus the area of the parallelogram $KLMN = 4ab$, which is equal to the rectangle formed by the tangents at the ends of the major and minor axes.

287. *The product of the focal distances of a point P is equal to the square on the semidiameter parallel to the tangent at P.*

If P be the point ϕ, then, by Art. 251, we have

$$SP = a + ae\cos\phi, \text{ and } S'P = a - ae\cos\phi.$$

Hence $$SP \,.\, S'P = a^2 - a^2e^2\cos^2\phi$$
$$= a^2 - (a^2 - b^2)\cos^2\phi$$
$$= a^2\sin^2\phi + b^2\cos^2\phi$$
$$= CD^2.$$

288. Ex. *If P and D be the ends of conjugate diameters, find the locus of*

(1) *the middle point of PD,*

(2) *the intersection of the tangents at P and D,*

and (3) *the foot of the perpendicular from the centre upon PD.*

P is the point $(a\cos\phi, b\sin\phi)$ and D is $(-a\sin\phi, b\cos\phi)$.

(1) If (x, y) be the middle point of PD, we have

$$x = \frac{a\cos\phi - a\sin\phi}{2}, \text{ and } y = \frac{b\sin\phi + b\cos\phi}{2}.$$

If we eliminate ϕ we shall get the required locus. We obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \tfrac{1}{4}[(\cos\phi - \sin\phi)^2 + (\sin\phi + \cos\phi)^2] = \tfrac{1}{2}.$$

The locus is therefore a concentric and similar ellipse.

[N.B. Two ellipses are similar if the ratios of their axes are the same, so that they have the same eccentricity.]

(2) The tangents are

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1,$$

and $$-\frac{x}{a}\sin\phi + \frac{y}{b}\cos\phi = 1.$$

Both of these equations hold at the intersection of the tangents. If we eliminate ϕ we shall have the equation of the locus of their intersections.

By squaring and adding, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2,$$

so that the locus is another similar and concentric ellipse.

(3) By Art. 259, on putting $\phi'=90^\circ+\phi$, the equation to PD is

$$\frac{x}{a}\cos(45^\circ+\phi)+\frac{y}{b}\sin(45^\circ+\phi)=\cos 45^\circ.$$

Let the length of the perpendicular from the centre be p and let it make an angle ω with the axis. Then this line must be equivalent to

$$x\cos\omega+y\sin\omega=p.$$

Comparing the equations, we have

$$\cos(45^\circ+\phi)=\frac{a\cos\omega\cos 45^\circ}{p}, \text{ and } \sin(45^\circ+\phi)=\frac{b\sin\omega\cos 45^\circ}{p}.$$

Hence, by squaring and adding, $2p^2=a^2\cos^2\omega+b^2\sin^2\omega$, *i.e.* the locus required is the curve

$$2r^2=a^2\cos^2\theta+b^2\sin^2\theta,\ \textit{i.e.}\ \ 2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

293. *To prove that, in general, four normals can be drawn from any point to an ellipse, and that the sum of the eccentric angles of their feet is equal to an odd multiple of two right angles.*

The normal at any point, whose eccentric angle is ϕ, is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2 = a^2e^2.$$

If this normal pass through the point (h, k), we have

$$\frac{ah}{\cos\phi} - \frac{bk}{\sin\phi} = a^2e^2 \qquad \text{.................} (1).$$

For a given point (h, k) this equation gives the eccentric angles of the feet of the normals which pass through (h, k).

Let $\tan\frac{\phi}{2} = t$, so that

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{1-t^2}{1+t^2}, \text{ and } \sin\phi = \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{2t}{1+t^2}.$$

Substituting these values in (1), we have

$$ah\frac{1+t^2}{1-t^2} - bk\frac{1+t^2}{2t} = a^2e^2,$$

i.e. $$bkt^4 + 2t^3(ah + a^2e^2) + 2t(ah - a^2e^2) - bk = 0 \ldots (2).$$

Let t_1, t_2, t_3, and t_4 be the roots of this equation, so that, by Art. 2,

$$t_1 + t_2 + t_3 + t_4 = -2\frac{ah + a^2e^2}{bk} \quad \ldots\ldots\ldots\ldots (3),$$

$$t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = 0 \quad \ldots\ldots\ldots (4),$$

$$t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3 = -2\frac{ah - a^2e^2}{bk} \quad \ldots\ldots (5),$$

and $$t_1t_2t_3t_4 = -1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Hence (*Trigonometry*, Art. 125), we have

$$\tan\left(\frac{\phi_1}{2} + \frac{\phi_2}{2} + \frac{\phi_3}{2} + \frac{\phi_4}{2}\right) = \frac{s_1 - s_3}{1 - s_2 + s_4} = \frac{s_1 - s_3}{0} = \infty.$$

$$\therefore \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{2} = n\pi + \frac{\pi}{2},$$

and hence $$\phi_1 + \phi_2 + \phi_3 + \phi_4 = (2n+1)\pi$$

= an odd multiple of two right angles.

294. We shall conclude the chapter with some examples of loci connected with the ellipse.

Ex. 1. *Find the locus of the intersection of tangents at the ends of chords of an ellipse, which are of constant length 2c.*

Let QR be any such chord, and let the tangents at Q and R meet in a point P, whose coordinates are (h, k).

Since QR is the polar of P, its equation is

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The abscissæ of the points in which this straight line meets the ellipse are given by

$$\left(1-\frac{xh}{a^2}\right)^2=\frac{k^2}{b^2}\left(1-\frac{x^2}{a^2}\right),$$

i.e.

$$\frac{x^2}{a^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)-\frac{2xh}{a^2}+1-\frac{k^2}{b^2}=0.$$

If x_1 and x_2 be the roots of this equation, *i.e.* the abscissæ of Q and R, we have

$$x_1+x_2=\frac{2a^2b^2h}{b^2h^2+a^2k^2}, \text{ and } x_1x_2=\frac{a^4(b^2-k^2)}{b^2h^2+a^2k^2}.$$

$$\therefore (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=\frac{4a^4[b^2h^2+a^2k^2-a^2b^2]k^2}{(b^2h^2+a^2k^2)^2} \quad \ldots(2).$$

If y_1 and y_2 be the ordinates of Q and R, we have from (1)

$$\frac{x_1h}{a^2}+\frac{y_1k}{b^2}=1,$$

and

$$\frac{x_2h}{a^2}+\frac{y_2k}{b^2}=1,$$

so that, by subtraction,

$$y_2-y_1=-\frac{b^2h}{a^2k}(x_2-x_1).$$

The condition of the question therefore gives

$$4c^2=(x_2-x_1)^2+(y_2-y_1)^2=\left(1+\frac{b^4h^2}{a^4k^2}\right)(x_2-x_1)^2$$

$$=\frac{4(a^4k^2+b^4h^2)(b^2h^2+a^2k^2-a^2b^2)}{(b^2h^2+a^2k^2)^2}, \text{ by (2)}.$$

Hence the point (h, k) always lies on the curve

$$c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\left(\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right),$$

which is therefore the locus of P.

Ex. 2. *Find the locus (1) of the middle points, and (2) of the poles, of normal chords of the ellipse.*

The chord, whose middle point is (h, k), is parallel to the polar of (h, k), and is therefore

$$(x-h)\frac{h}{a^2}+(y-k)\frac{k}{b^2}=0 \quad\text{.......................(1).}$$

If this be a normal, it must be the same as

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2 \quad\text{.....................(2).}$$

We therefore have

$$\frac{a\sec\theta}{\frac{h}{a^2}}=\frac{-b\operatorname{cosec}\theta}{\frac{k}{b^2}}=\frac{a^2-b^2}{\frac{h^2}{a^2}+\frac{k^2}{b^2}},$$

so that
$$\cos\theta=\frac{a^3}{h(a^2-b^2)}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right),$$

and
$$\sin\theta=-\frac{b^3}{k(a^2-b^2)}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right).$$

Hence, by the elimination of θ,

$$\left(\frac{a^6}{h^2}+\frac{b^6}{k^2}\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2=(a^2-b^2)^2.$$

The equation to the required locus is therefore

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2.$$

Again, if (x_1, y_1) be the pole of the normal chord (2), the latter equation must be equivalent to the equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad\text{............................(3).}$$

Comparing (2) and (3), we have

$$\frac{a^3\sec\theta}{x_1}=-\frac{b^3\operatorname{cosec}\theta}{y_1}=a^2-b^2,$$

so that
$$1=\cos^2\theta+\sin^2\theta=\left(\frac{a^6}{x_1{}^2}+\frac{b^6}{y_1{}^2}\right)\frac{1}{(a^2-b^2)^2},$$

and hence the required locus is

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Analytical Geometry **Subject Code: 18MMU303A**
Class : II - B.Sc. Mathematics **Semester : III**

Unit II

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Questions	Opt1	Opt2	Opt3	Opt4	Answer
The normal equation of parabola at the point (am^2,2am)………..	y=mx-2a	y=mx-2am-am^3	y=ax+am^3	y=ax-am	y=mx-2am-am^3
The equation to the tangent at any point (x', y') of the parabola…………..	root of (x/a-y/b)=1	root of (x/a+y/b)=1	root of (x/a*y/b)=1	root of (x/a-y/b)=-1	root of (x/a-y/b)=1
latus-rectum of the ellipse is…………….	a/b	a^2/b	a/b^2	b^2/a	b^2/a
The circle which is described on the major axis,AA' of an ellipse as diameter,is called……………...	straight line	circle	auxilary circle	ellipse	auxilary circle
A circle, whose centre is the centre of the ellipse, and whose radius is the length of the line joining the ends of the major and minor axis. This circle is called ……………	director Circle.	circle	ellipse	hyperbola	director Circle.
Two diameters y =mx and y =- mx are therefore conjugate, if………	m1m2=b/a	m1m2=-b/a	m1m2=-b^2/a^2	m1m2=-b/a^2	m1m2=-b^2/a^2
If the eccentric angles of the ends of a pair of conjugate diameters be and then and differ by…….. degree	0	45	60	90	90
Four normals can be drawn from any point to an ellipse, and that the sum of the eccentric angles of their feet is equal to …..……..mlultiple of two right angles.	even	odd	zero	positive	odd
The locus of the intersection of tangents at the ends of chords of an ellipse, which are of constant length ………..	⟦90⟧ C	2C	0	4C ⟦45⟧	2C
The normals at the points(h,k) if……………..	m1-m2-m3=0	m1m2m3=0	m1=m2=m3=0	m1+m2+m3=0	m1+m2+m3=0
The sum of the focal distances of any point on the curve is equal to the………………...	⟦90⟧ minor axis	major axis	axis	⟦45⟧ parallel axis	major axis
Whose length is the major axis of the required ellipse, and fasten its ends at the points S and S' which are to be the .	foci	axis	vertex	latus	foci
Equation to the tangent at the point whose eccentric angle is α..........	x/a cosα+y/bsinα=1	x/a cosα-y/bsinα=1	x/a cosα*y/bsinα=1	x/a cosα+y/bsinα=-1	x/a cosα+y/bsinα=1
The straight line which passes through the point (x,y) and is perpendicular to the tangent is called the……….	straight line	tangent	normal	ellipse	normal

The equation of the directed circle………………..	x^2+y^2=a+b	x^2+y^2=a^2+b^2	x^2+y^2=a^2-b^2	x^2+y^2=a-b	x^2+y^2=a^2+b^2
Any two lines is said to be coplanar if it must be either ------- or intersecting	parallel	perpendicular	perpendicular	tangent	parallel
If the line is -----------to the plane, then it must be perpendicular to the normal tothe plane	intersect	perpendicular	parallel	tangent	parallel
If the line is parallel to the plane, then the angle between them is --------	45^o	90^o	0^o	30^o	0^o
Any straight line is parallel to the plane, then it must be perpendicular to the -------- to the plane.	tangent	perpendicular	normal	radius	normal
Any --------will lie in the plane if it is parallel and, in addition, any one point of the line lies in the plane.	Straight line	normal	point	tangent	Straight line
Two lines which do not intersect is called………………..	Skew lines	Sphere	Plane	perpendicular	Skew lines
Any linear equation in x,y,z represents ………………………	Skew lines	Sphere	a Plane	perpendicular	a Plane
Any…………….…. in x,y,z represents a Plane	equation	non linear equation	linear equation	quadratic equation	linear equation
The straight line joining two points P and Q on a surface is called a ------- of the surface	tangent	diameter	chord	normal	chord

UNIT-III

Hyperbola: Asymptotes – equations referred to the asymptotes an axes – one variables examples.
Spheres: The Equation of a sphere - Tangents and tangent plane to a sphere - The radical plane of two spheres cylindrical surfaces. Illustrations of graphing standard quadric surfaces like cone, ellipsoid.

THE HYPERBOLA.

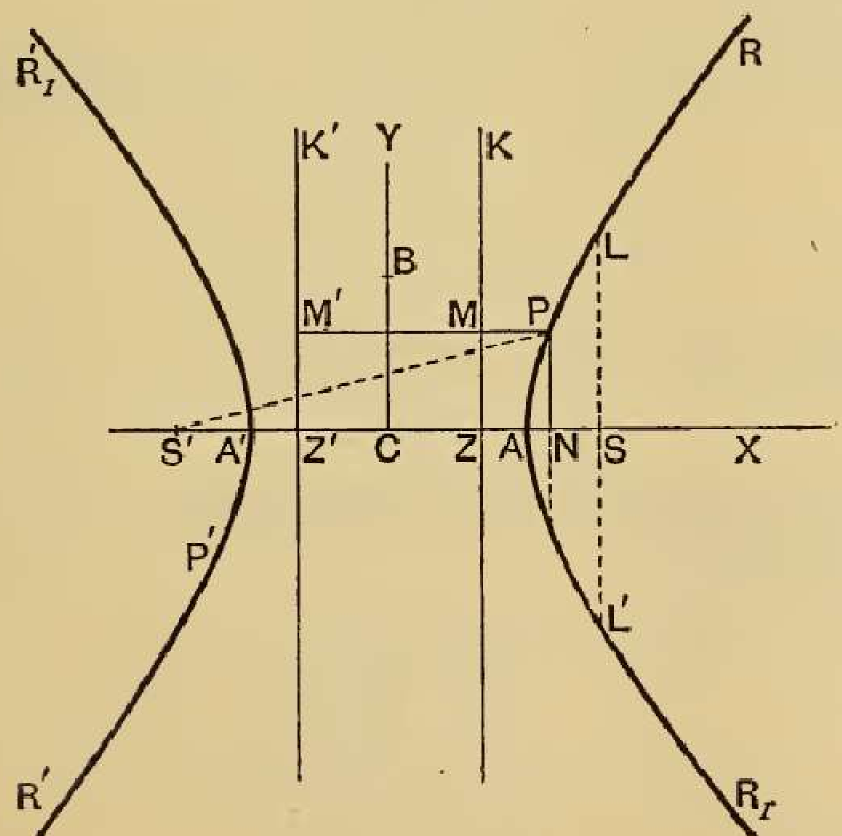
295. THE hyperbola is a Conic Section in which the eccentricity e is greater than unity.

To find the equation to a hyperbola.

Let ZK be the directrix, S the focus, and let SZ be perpendicular to the directrix.

There will be a point A on AZ, such that

$$SA = e \,.\, AZ \quad \ldots\ldots\ldots\ldots (1).$$



Since $e > 1$, there will be another point A', on SZ produced, such that

$$SA' = e \,.\, A'Z \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let the length AA' be called $2a$, and let C be the middle point of AA'.

Subtracting (1) from (2), we have

$$2a = AA' = e \,.\, A'Z - e \,.\, AZ$$

$$= e\left[CA' + CZ\right] - e\left[CA - CZ\right] = e \,.\, 2CZ,$$

i.e.

$$CZ = \frac{a}{e} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Adding (1) and (2), we have

$$e(AZ + A'Z) = SA' + SA = 2CS,$$

i.e. $$e \, . \, AA' = 2 \, . \, CS,$$

and hence $$CS = ae \text{ .... } (4).$$

Let C be the origin, CSX the axis of x, and a straight line CY, through C perpendicular to CX, the axis of y.

Let P be any point on the curve, whose coordinates are x and y, and let PM be the perpendicular upon the directrix, and PN the perpendicular on AA'.

The focus S is the point $(ae, 0)$.

The relation $SP^2 = e^2 \, . \, PM^2 = e^2 \, . \, ZN^2$ then gives

$$(x - ae)^2 + y^2 = e^2 \left[x - \frac{a}{e} \right]^2,$$

i.e. $$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2.$$

Hence $$x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1),$$

i.e. $$\frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} = 1 \text{ .... } (5).$$

Since, in the case of the hyperbola, $e > 1$, the quantity $a^2 (e^2 - 1)$ is positive. Let it be called b^2, so that the equation (5) becomes

$$\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \text{ .... } (6),$$

where $$b^2 = a^2e^2 - a^2 = CS^2 - CA^2 \text{ .... } (7),$$

and therefore $$CS^2 = a^2 + b^2 \text{ .... } (8).$$

296. The equation (6) may be written

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{x^2 - a^2}{a^2} = \frac{(x-a)(x+a)}{a^2},$$

i.e.

$$\frac{PN^2}{b^2} = \frac{AN \cdot NA'}{a^2},$$

so that

$$PN^2 : AN \cdot NA' :: b^2 : a^2.$$

If we put $x = 0$ in equation (6), we have $y^2 = -b^2$, shewing that the curve meets the axis CY in imaginary points.

Def. The points A and A' are called the vertices of the hyperbola, C is the centre, AA' is the transverse axis of the curve, whilst the line BB' is called the conjugate axis, where B and B' are two points on the axis of y equidistant from C, as in the figure of Art. 315, and such that

$$B'C = CB = b.$$

297. Since S is the point $(ae, 0)$, the equation referred to the focus as origin is, by Art. 128,

$$\frac{(x+ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

i.e.

$$\frac{x^2}{a^2} + 2\frac{ex}{a} - \frac{y^2}{b^2} + e^2 - 1 = 0.$$

Similarly, the equations, referred to the vertex A and foot of the directrix Z respectively as origins, will be found to be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2x}{a} = 0,$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2x}{ae} = 1 - \frac{1}{e^2}.$$

The equation to the hyperbola, whose focus, directrix, and eccentricity are any given quantities, may be written down as in the case of the ellipse (Art. 249).

298. *There exist a second focus and a second directrix to the curve.*

On SC produced take a point S', such that

$$SC = CS' = ae,$$

and another point Z', such that

$$ZC = CZ' = \frac{a}{e}.$$

Draw $Z'M'$ perpendicular to AA', and let PM be produced to meet it in M'.

The equation (5) of Art. 295 may be written in the form

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2,$$

i.e. $$(x + ae)^2 + y^2 = e^2\left(\frac{a}{e} + x\right)^2,$$

i.e. $$S'P^2 = e^2(Z'C + CN)^2 = e^2 . PM'^2.$$

Hence any point P of the curve is such that its distance from S' is e times its distance from $Z'K'$, so that we should have obtained the same curve if we had started with S' as focus, $Z'K'$ as directrix, and the same eccentricity e.

299. *The difference of the focal distances of any point on the hyperbola is equal to the transverse axis.*

For (Fig., Art 295) we have

$$SP = e \,.\, PM, \text{ and } S'P = e \,.\, PM'.$$

Hence $S'P - SP = e\,(PM' - PM) = e \,.\, MM'$

$$= e \,.\, ZZ' = 2e \,.\, CZ = 2a$$

$$= \text{the transverse axis } AA'.$$

Also $SP = e \,.\, PM = e \,.\, ZN = e \,.\, CN - e \,.\, CZ = \mathbf{ex' - a},$

and $S'P = e \,.\, PM' = e \,.\, Z'N = e \,.\, CN + e \,.\, Z'C = \mathbf{ex' + a},$

where x' is the abscissa of the point P referred to the centre as origin.

300. *Latus-rectum of the Hyperbola.*

Let LSL' be the latus-rectum, *i.e.* the double ordinate of the curve drawn through S.

By the definition of the curve, the semi-latus-rectum SL

$= e$ times the distance of L from the directrix

$$= e \,.\, SZ = e\,(CS - CZ)$$

$$= e \,.\, CS - eCZ = ae^2 - a = \frac{b^2}{a},$$

by equations (3), (4), and (7) of Art. 295.

312. Asymptote. Def. An asymptote is a straight line, which meets the conic in two points both of which are situated at an infinite distance, but which is itself not altogether at infinity.

313. *To find the asymptotes of the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

As in Art. 260, the straight line

$$y = mx + c \quad \text{.......................(1)}$$

meets the hyperbola in points, whose abscissae are given by the equation

$$x^2(b^2 - a^2m^2) - 2a^2mcx - a^2(c^2 + b^2) = 0 \quad \text{...... (2).}$$

If the straight line (1) be an asymptote, both roots of (2) must be infinite.

Hence (C. Smith's Algebra, Art. 123), the coefficients of x^2 and x in it must both be zero.

We therefore have

$$b^2 - a^2m^2 = 0, \text{ and } a^2mc = 0.$$

Hence $$m = \pm\frac{b}{a}, \text{ and } c = 0.$$

Substituting these values in (1), we have, as the required equation,

$$y = \pm\frac{b}{a}x.$$

There are therefore two asymptotes both passing through the centre and equally inclined to the axis of x, the inclination being

$$\tan^{-1}\frac{b}{a}.$$

The equation to the asymptotes, written as one equation, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Cor. For all values of c one root of equation (2) is infinite if $m = \pm\frac{b}{a}$. Hence any straight line, which is parallel to an asymptote, meets the curve in one point at infinity and in one finite point.

314. That the asymptote passes through two coincident points at infinity, *i.e.* touches the curve at infinity, may be seen by finding the equations to the tangents to the curve which pass through any point $\left(x_1, \frac{b}{a}x_1\right)$ on the asymptote $y=\frac{b}{a}x$.

As in Art. 305 the equation to either tangent through this point is

$$y=mx+\sqrt{a^2m^2-b^2},$$

where $$\frac{b}{a}x_1=mx_1+\sqrt{a^2m^2-b^2},$$

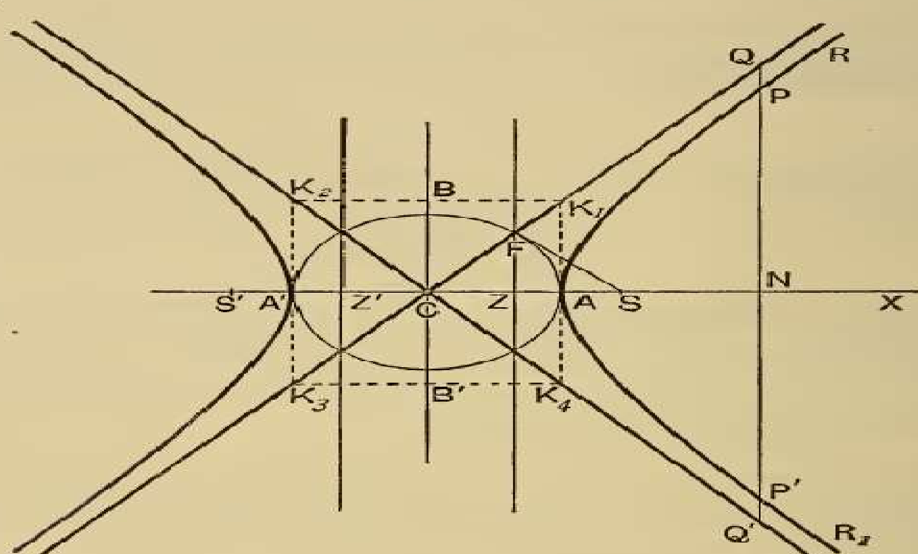
i.e. on clearing of surds,

$$m^2(x_1{}^2-a^2)-2m\frac{b}{a}x_1{}^2+(x_1{}^2+a^2)\frac{b^2}{a^2}=0.$$

One root of this equation is $m=\frac{b}{a}$, so that one tangent through the given point is $y=\frac{b}{a}x$, *i.e.* the asymptote itself.

315. *Geometrical construction for the asymptotes.*

Let $A'A$ be the transverse axis, and along the conjugate axis measure off CB and CB', each equal to b. Through B and B' draw parallels to the transverse axis and through A and A' parallels to the conjugate axis, and let these meet respectively in K_1, K_2, K_3, and K_4, as in the figure.



Clearly the equations of K_1CK_3 and K_2CK_4 are

$$y = \frac{b}{a}x, \text{ and } y = -\frac{b}{a}x,$$

and these are therefore the equations of the asymptotes.

316. Let any double ordinate PNP' of the hyperbola be produced both ways to meet the asymptotes in Q and Q', and let the abscissa CN be x'.

Since P lies on the curve, we have, by Art. 302,

$$NP = \frac{b}{a}\sqrt{x'^2 - a^2}.$$

Since Q is on the asymptote whose equation is $y = \frac{b}{a}x$,

we have $$NQ = \frac{b}{a}x$$

Hence $$PQ = NQ - NP = \frac{b}{a}(x' - \sqrt{x'^2 - a^2}),$$

and $$QP' = \frac{b}{a}(x' + \sqrt{x'^2 - a^2}).$$

Therefore $$PQ \, . \, QP' = \frac{b^2}{a^2}\{x'^2 - (x'^2 - a^2)\} = b^2.$$

Hence, if from any point on an asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve, is always equal to the square on the semi-conjugate axis.

Again,

$$PQ = \frac{b}{a}(x' - \sqrt{x'^2 - a^2}) = \frac{b}{a}\frac{a^2}{x' + \sqrt{x'^2 - a^2}}$$

$$= \frac{ab}{x' + \sqrt{x'^2 - a^2}}.$$

PQ is therefore always positive, and therefore the part of the curve, for which the coordinates are positive, is altogether between the asymptote and the transverse axis.

Also as x' increases, *i.e.* as the point P is taken further and further from the centre C, it is clear that PQ continually decreases; finally, when x' is infinitely great, PQ is infinitely small.

The curve therefore continually approaches the asymptote but never actually reaches it, although, at a very great distance, the curve would not be distinguishable from the asymptote.

This property is sometimes taken as the definition of an asymptote.

317. If SF be the perpendicular from S upon an asymptote, the point F lies on the auxiliary circle. This follows from the fact that the asymptote is a tangent, whose point of contact happens to lie at infinity, or it may be proved directly.

For

$$CF = CS \cos FCS = CS \cdot \frac{CA}{CK} = \sqrt{a^2+b^2} \cdot \frac{a}{\sqrt{a^2+b^2}} = a.$$

Also Z being the foot of the directrix, we have

$$CA^2 = CS \cdot CZ, \qquad \text{(Art. 295)}$$

and hence $CF^2 = CS \cdot CZ$, *i.e.* $CS : CF :: CF : CZ$.

By Euc. VI. 6, it follows that $\angle CZF = \angle CFS =$ a right angle, and hence that F lies on the directrix.

Hence *the perpendiculars from the foci on either asymptote meet it in the same points as the corresponding directrix, and the common points of intersection lie on the auxiliary circle.*

THE SPHERE.

56. Equation to a sphere. If the axes are rectangular the square of the distance between the points P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2) is given by $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$, and therefore the equation to the sphere whose centre is P and whose radius is of length r, is

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2.$$

Any equation of the form

$$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$$

can be written

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2},$$

and therefore represents a sphere whose centre is

$$\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right) \text{ and radius } \frac{\sqrt{u^2+v^2+w^2-ad}}{a}.$$

57. Tangents and tangent planes. If P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2) are points on the sphere $x^2+y^2+z^2=a^2$, then

$$x_1^2+y_1^2+z_1^2=a^2=x_2^2+y_2^2+z_2^2,$$

and therefore

$$(x_1-x_2)(x_1+x_2)+(y_1-y_2)(y_1+y_2)+(z_1-z_2)(z_1+z_2)=0.$$

Now the direction-cosines of PQ are proportional to $x_1-x_2, y_1-y_2, z_1-z_2$; and if M is the mid-point of PQ and O is the origin, the direction-cosines of OM are proportional to $x_1+x_2, y_1+y_2, z_1+z_2$. Therefore PQ is at right angles

to OM. Suppose that OM meets the sphere in A and that PQ moves parallel to itself with its mid-point, M, on OA. Then when M is at A, PQ is a **tangent** to the sphere at A, and hence a tangent at A is at right angles to OA, and the locus of the tangents at A is the plane through A at right angles to OA. This plane is the **tangent plane** at A. The equation to the tangent plane at A, (α, β, γ), is

$$(x-\alpha)\alpha+(y-\beta)\beta+(z-\gamma)\gamma=0,$$

or $$x\alpha+y\beta+z\gamma=\alpha^2+\beta^2+\gamma^2=a^2.$$

***58. Radical plane of two spheres.** *If any secant through a given point* O *meets a given sphere in* P *and* Q, OP . OQ *is constant.*

The equations to the line through O, (α, β, γ), whose direction-cosines are l, m, n, are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad (=r).$$

The point on this line, whose distance from O is r, has coordinates $\alpha+lr, \beta+mr, \gamma+nr$, and lies on the sphere

$$F(xyz)=a(x^2+y^2+z^2)+2ux+2vy+2wz+d=0$$

if $$ar^2+r\left(l\frac{\partial F}{\partial\alpha}+m\frac{\partial F}{\partial\beta}+n\frac{\partial F}{\partial\gamma}\right)+F(\alpha, \beta, \gamma)=0.$$

This equation gives the lengths of OP and OQ, and hence OP . OQ is given by $F(\alpha, \beta, \gamma)/a$, which is the same for all secants through O.

Definition. The measure of OP . OQ is **the power** of O with respect to the sphere.

If
$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$
are the equations to two spheres, the locus of points whose powers with respect to the spheres are equal is the plane given by

$$S_1=S_2, \text{ or } 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0.$$

This plane is called the **radical plane** of the two spheres. It is evidently at right angles to the line joining the centres.

The radical planes of three spheres taken two by two pass through one line.

(The equations to the line are $S_1=S_2=S_3$.)

The radical planes of four spheres taken two by two pass through one point.

(The point is given by $S_1=S_2=S_3=S_4$.)

The equations to any two spheres can be put in the form

$$x^2+y^2+z^2+2\lambda_1x+d=0, \quad x^2+y^2+z^2+2\lambda_2x+d=0.$$

(Take the line joining the centres as x-axis and the radical plane as $x=0$.)

The equation $x^2+y^2+z^2+2\lambda x+d=0$, where λ is a parameter, represents a system of spheres any two of which have the same radical plane. The spheres are said to be *coaxal.*

THE CONE.

59. Equation to a cone. A cone is a surface generated by a straight line which passes through a fixed point and intersects a given curve. If the given point O, say, be chosen as origin, the equation to the cone is homogeneous. For if P, (x', y', z') is any point on the cone, x', y', z' satisfy the equation. And since any point on OP is on the cone, and has coordinates (kx', ky', kz'), the equation is also satisfied by kx', ky', kz' for all values of k, and therefore must be homogeneous.

Cor. If $x/l = y/m = z/n$ is a generator of the cone represented by the homogeneous equation $f(x, y, z) = 0$, then $f(l, m, n) = 0$. Conversely, if the direction-ratios of a straight line which always passes through a fixed point satisfy a homogeneous equation, the line is a generator of a cone whose vertex is at the point.

60. Angle between lines in which a plane cuts a cone. We find it convenient to introduce here the following notation, to which we shall adhere throughout the book.

$$D \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

$$A \equiv \frac{\partial D}{\partial a} = bc - f^2, \quad B \equiv \frac{\partial D}{\partial b} = ca - g^2, \quad C \equiv \frac{\partial D}{\partial c} = ab - h^2;$$

$$F \equiv \frac{1}{2}\frac{\partial D}{\partial f} = gh - af, \quad G \equiv \frac{1}{2}\frac{\partial D}{\partial g} = hf - bg, \quad H \equiv \frac{1}{2}\frac{\partial D}{\partial h} = fg - ch.$$

The student can easily verify that

$$BC - F^2 = aD, \quad CA - G^2 = bD, \quad AB - H^2 = cD$$

$$GH - AF = fD, \quad HF - BG = gD, \quad FG - CH = hD$$

In what follows we use P^2 to denote

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & 0 \end{vmatrix},$$

or $$-(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv).$$

The axes being rectangular to find the angle between the lines in which the plane $ux + vy + wz = 0$ cuts the cone

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

If the line $x/l = y/m = z/n$ lies in the plane,

$$ul + vm + wn = 0; \quad \text{.....................}(1)$$

if it lies on the cone,

$$f(l, m, n) = 0. \quad \text{.......................}(2)$$

Eliminate n between (1) and (2), and we obtain

$$l^2(cu^2+aw^2-2gwu)+2lm(hw^2+cuv-fuw-gvw)$$
$$+m^2(cv^2+bw^2-2fvw)=0. \quad \ldots\ldots\ldots\ldots(3)$$

Now the direction-cosines of the two lines of section satisfy the equations (1) and (2), and therefore they satisfy equation (3). Therefore if they are l_1, m_1, n_1; l_2, m_2, n_2;

$$\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}$$
$$=\frac{l_1m_2+l_2m_1}{-2(hw^2+cuv-fuw-gvw)}$$
$$=\frac{l_1m_2-l_2m_1}{\pm 2\{(hw^2+cuv\ldots)^2-(bw^2\ldots)(cu^2\ldots)\}^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots(4)$$
$$=\frac{l_1m_2-l_2m_1}{\pm 2wP}.$$

From the symmetry, each of the expressions in (4) is seen to be equal to

$$\frac{n_1 n_2}{av^2+bu^2-2huv}=\frac{m_1n_2-m_2n_1}{\pm 2u\mathsf{P}}=\frac{n_1l_2-n_2l_1}{\pm 2v\mathsf{P}}$$

But if θ is the angle between the lines,

$$\frac{\cos\theta}{l_1l_2+m_1m_2+n_1n_2}=\frac{\sin\theta}{\{\Sigma(m_1n_2-m_2n_1)^2\}^{\frac{1}{2}}}.$$

$$\therefore \frac{\cos\theta}{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)}=\frac{\sin\theta}{\pm 2(u^2+v^2+w^2)^{\frac{1}{2}}\mathsf{P}}.$$

61. Condition of tangency of plane and cone. If $\mathsf{P}=0$, or

$$\mathsf{A}u^2+\mathsf{B}v^2+\mathsf{C}w^2+2\mathsf{F}vw+2\mathsf{G}wu+2\mathsf{H}uv=0, \quad \ldots\ldots(1)$$

then $\sin\theta=0$, and therefore the lines of section coincide, or the plane touches the cone. Equation (1) shews that the line $\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$, *i.e.* the normal through O to the plane, is a generator of the cone

$$\mathsf{A}x^2+\mathsf{B}y^2+\mathsf{C}z^2+2\mathsf{F}yz+2\mathsf{G}zx+2\mathsf{H}xy=0. \quad \ldots\ldots\ldots(2)$$

Similarly, since we have $\mathsf{BC}-\mathsf{F}^2=a\mathsf{D}$, and the corresponding equations at the head of paragraph 60, it follows that a normal through the origin to a tangent plane to the cone (2) is a generator of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

i.e. of the given cone. The two cones are therefore such that each is the locus of the normals drawn through the origin to the tangent planes to the other, and they are on that account said to be *reciprocal*.

62. Condition that the cone has three mutually perpendicular generators. The condition that the plane should cut the cone in perpendicular generators is

$$(a+b+c)(u^2+v^2+w^2)=f(u, v, w). \qquad \text{.........(1)}$$

If also the normal to the plane lies on the cone, we have

$$f(u, v, w)=0,$$

and therefore $$a+b+c=0.$$

In this case the cone has three mutually perpendicular generators, viz., the normal to the plane and the two perpendicular lines in which the plane cuts the cone.

If $a+b+c=0$, the cone has an infinite number of sets of mutually perpendicular generators. For if $ux+vy+wz=0$ be any plane whose normal lies on the cone, then

$$f(u, v, w)=0,$$

and therefore $(a+b+c)(u^2+v^2+w^2)=f(u, v, w)$,

since $$a+b+c=0.$$

Hence, by (1), the plane cuts the cone in perpendicular generators. Thus any plane through the origin which is normal to a generator of the cone cuts the cone in perpendicular lines, or there are two generators of the cone at right angles to one another, and at right angles to any given generator.

63. Equation to cone with given conic for base. *To find the equation to the cone whose vertex is the point (α, β, γ) and base the conic*

$$f(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0, \quad z=0.$$

The equations to any line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=z-\gamma,$$

and the line meets the plane $z=0$ in the point

$$(\alpha-l\gamma,\ \beta-m\gamma,\ 0).$$

This point is on the given conic if $f(\alpha-l\gamma, \beta-m\gamma)=0$,

i.e. if $$f(\alpha, \beta)-\gamma\left(l\frac{\partial f}{\partial \alpha}+m\frac{\partial f}{\partial \beta}\right)+\gamma^2\phi(l, m)=0, \quad \text{.........(1)}$$

where $\phi(x; y)\equiv ax^2+2hxy+by^2$. If we eliminate l and m between the equations to the line and (1), we obtain the equation to the locus of lines which pass through (α, β, γ) and intersect the conic, i.e. the equation to the cone. The result is

$$f(\alpha, \beta)-\gamma\left(\frac{x-\alpha}{z-\gamma}\frac{\partial f}{\partial \alpha}+\frac{y-\beta}{z-\gamma}\frac{\partial f}{\partial \beta}\right)+\gamma^2\phi\left(\frac{x-\alpha}{z-\gamma}, \frac{y-\beta}{z-\gamma}\right)=0,$$

i.e. $$(z-\gamma)^2 f(\alpha, \beta)-\gamma(z-\gamma)\left(\overline{x-\alpha}\frac{\partial f}{\partial \alpha}+\overline{y-\beta}\frac{\partial f}{\partial \beta}\right)$$
$$+\gamma^2\phi(x-\alpha, y-\beta)=0.$$

This equation may be transformed as follows:

The coefficient of γ^2 is

$$f(\alpha, \beta)+(x-\alpha)\frac{\partial f}{\partial \alpha}+(y-\beta)\frac{\partial f}{\partial \beta}+\phi(x-\alpha, y-\beta)$$
$$=f(\alpha+\overline{x-\alpha},\ \beta+\overline{y-\beta})=f(x, y);$$

and the coefficient of $-z\gamma$ is

$$(x-\alpha)\frac{\partial f}{\partial \alpha}+(y-\beta)\frac{\partial f}{\partial \beta}+2f(\alpha, \beta).$$

If $f(x, y)$ be made homogeneous by means of an auxiliary variable t which is equated to unity after differentiation, we have, by Euler's theorem,

$$\alpha\frac{\partial f}{\partial \alpha}+\beta\frac{\partial f}{\partial \beta}+t\frac{\partial f}{\partial t}=2f(\alpha, \beta, t).$$

Therefore the coefficient of $-z\gamma$ becomes

$$x\frac{\partial f}{\partial \alpha}+y\frac{\partial f}{\partial \beta}+t\frac{\partial f}{\partial t}.$$

Hence the equation to the cone is

$$z^2f(\alpha, \beta)-z\gamma\left(x\frac{\partial f}{\partial \alpha}+y\frac{\partial f}{\partial \beta}+t\frac{\partial f}{\partial t}\right)+\gamma^2f(x, y)=0.$$

It is to be noted that by equating to zero the coefficient of $z\gamma$, we obtain the equation to the polar of $(\alpha, \beta, 0)$ with respect to the given conic.

64. The locus of the equation

(1) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

(2) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

(3) $$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

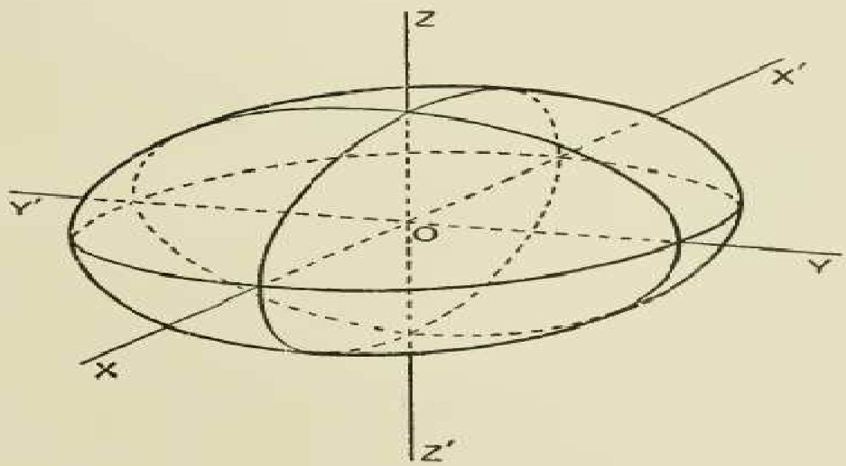


FIG. 29

We have shewn in §9 that the equation (1) represents the surface generated by the variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}, \quad z=k,$$

whose centre moves along **Z′OZ**, and passes in turn through every point between (0, 0, $-c$) and (0, 0, $+c$). The surface is the **ellipsoid**, and is represented in fig. 29. The section by any plane parallel to a coordinate plane is an ellipse.

Similarly, we might shew that the surface represented by equation (2) is generated by a variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}, \quad z=k,$$

whose centre moves on **Z′OZ**, passing in turn through every point on it. The surface is the **hyperboloid of one sheet**, and is represented in fig. 30. The section by any plane parallel to one of the coordinate planes **YOZ** or **ZOX** is a hyperbola.

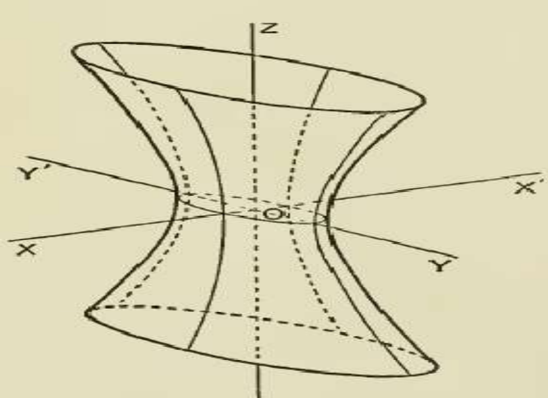


FIG. 30.

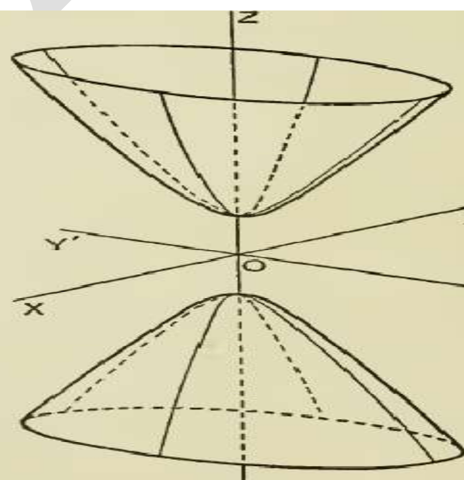


FIG. 31.

The surface given by equation (3) is also generated by a variable ellipse whose centre moves on Z′OZ. The ellipse is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}-1, \quad z=k,$$

and is imaginary if $-c<k<c$; hence no part of the surface lies between the planes $z=\pm c$.

The surface is the **hyperboloid of two sheets**, and is represented in fig. 31. The section by any plane parallel to one of the coordinate planes YOZ, ZOX is a hyperbola.

If (x', y', z') is any point on one of these surfaces, $(-x', -y', -z')$ is also on it; hence the origin bisects all chords of the surface which pass through it. The origin is the only point which possesses this property, and is called **the centre.** The surfaces are called **the central conicoids.**

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Subject: Analytical Geomentry **Subject Code: 17MMU401**
Class : II - B.Sc. Mathematics **Semester : IV**

Unit III

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Questions	Opt1	Opt2	Opt3	Opt4	Answer
The condition for the line spheres to cut orthogonally is ------	$2u_1u_2 + 2v_1v_2 = d_1+d_2$	$2u_1u_2 - 2v_1v_2 - 2w_1w_2 = d_1+d_2$	$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1+d_2$	$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1-d_2$	$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1-d_2$
The radius of the sphere Ö($u^2+ v^2+ w^2 – d$) is imaginary , then that sphere is called -------sphere.	real	imaginary	equal	point	imaginary
The radius of the sphere Ö($u^2+ v^2+ w^2 – d$) is equal to zero, then that sphere is called ------ sphere.	real	imaginary	equal	point	point
A -------is the locus of a point which moves such that its distance from a fixed point is always equal to a constant.	sphere	cone	cylinder	right circular cone	sphere
In a sphere the constant distance is known as ---------- of the sphere.	diameter	radius	chord	normal	chord
The equation of a sphere whose centre(a, b, c) and radius r is ---------	$(x – a)^2 +(y – b)^2 +(z – c)^2 = r^2$	$(x + a)^2 +(y +b)^2 +(z + c)^2 = r^2$	$(x – a)^2 +(y – b)^2 +(z – c)^2 = r$	$(x – a) +(y – b) +(z – c) = r$	$(x – a)^2 +(y – b)^2 +(z – c)^2 = r^2$
The equation of a sphere whose centre is the origin and radius 'r' is ---------	$(x – a)^2 +(y – b)^2 +(z – c)^2 = r^2$	$(x + a)^2 +(y +b)^2 +(z + c)^2 = r^2$	$x^2 +y^2 +z^2 = r^2$	$x +y +z = r$	$x^2 +y^2 +z^2 = r^2$
A sphere is the locus of a point which moves such that its distance from a fixed point is always ------ to a constant.	90^0 not equal	equal	less than	45^0 greater than	equal
In a --------- the constant distance is known as radius of the sphere.	90^0 sphere	cone	cylinder	45^0 line	sphere
The equation of a……………whose centre(a, b, c) and radius r is $(x – a)^2 +(y – b)^2 +(z – c)^2 = r^2$	sphere	cone	cylinder	line	sphere
The equation of a ------- whose centre is the origin and radius 'a' is $x^2 +y^2 +z^2 = r^2$	cylinder	cone	sphere	line	sphere
In a sphere $S = x^2 +y^2 +z^2 + 2ux + 2vy + 2wz + d = 0$, the centre is given by -------	xcosα-ysinα=p (-u, -v, -w)	(u, v, w)	(u^2, v^2, w^2)	$(-u^2, -v^2, -w^2)$	(-u, -v, -w)
In a sphere $S = x^2 +y^2 +z^2 + 2ux + 2vy + 2wz + d = 0$, the radius is given by -------	Ö($u^2+ v^2+ w^2 – d$)	Ö(u+ v+ w – d)	Ö($u^2+ v^2+ w^2$)	Ö($u^2+ v^2+ – d$)	Ö($u^2+ v^2+ w^2 – d$)

A sphere is of the ----- degree in x, y, z.	first	third	second	fourth	second
In a sphere equation, the coefficients of x^2, y^2, z^2 are all ------.	2	1	3	4	1
In a sphere $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, the ------- is given by Ö($u^2+ v^2+ w^2 – d$)	chord	centre	radius	diameter	radius
A ------ is of the second degree in x, y, z.	sphere	cone	cylinder	line	sphere
In a ---------- equation, the coefficients of x^2, y^2, z^2 are all equal.	cylinder	cone	sphere	line	sphere
The radius of the sphere Ö($u^2+ v^2+ w^2 – d$) is real, then that sphere is called ------- sphere.	real	imaginary	equal	positive	positive
The general equation of the sphere is………………..	x+y+z=0	$x^2+y^2+z^2=0$	$(x-a)^2+(y-a)^2+(z-a)^2=r^2$	$(x-a)^2+(y-a)^2+(z-a)^2=0$	$(x-a)^2+(y-a)^2+(z-a)^2=r^2$
The equation of the sphere, whose centre is the origin and radius is r is………………..	x+y+z=0	$x^2+y^2+z^2=r^2$	$(x-a)^2+(y-a)^2+(z-a)^2=r^2$	$(x-a)^2+(y-a)^2+(z-a)^2=0$	$x^2+y^2+z^2=r^2$
In a sphere ,the fixed point is called………………..	centre	radius	origin	circumference	centre
In a sphere,the constant distance is called --------of the sphere.	centre	centre	radius	origin	radius
The centre of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is,…………………..	–u,-v,-w	u,v,w	u,v,-w	–u,v,w	–u,-v,-w
The radius of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is…………………….	$u^2+v^2+w^2-d$	$\sqrt{u^2+v^2+w^2-d}$	$\sqrt{u^2+v^2+w^2+d}$	$\sqrt{u^2+v^2+w^2}$	$\sqrt{u^2+v^2+w^2-d}$
If the centre is (2,-3,1) and radius is 5,then the equation of the sphere is………………	$x^2+y^2+z^2-4x+6y-2z-11=0$	$x^2+y^2+z^2+2x+2y+2z+1=0$	$x^2+y^2+z^2+2ux+2vy+2wz+d=0$	$x^2+y^2+z^2$	$x^2+y^2+z^2-4x+6y-2z-11=0$
The centre of the sphere $x^2+y^2+z^2+2x-4y-6z+5=0$ is……………..	1,2,3	1,3,2	1,-3,2	-1,2,3	-1,2,3
The radius of the sphere $x^2+y^2+z^2+2x-4y-6z+5=0$ is…………………	4	2	3	5	3
The plane section of a sphere is……………………	circle	cone	cyclinder	sphere	circle
The curve of intersection of two spheres is a ………………	cone	circle	cyclinder	sphere	circle
The section of a sphere by a plane passing through its centre is called a…………….	circle	cone	great circle	cylinder	great circle
The equation of the tangent plane is ……………………..	$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$	$x^2+y^2+z^2=0$	$x^2+y^2+z^2+2ux+2vy+2wz+d=0$	$x^2 + y^2 +z^2 = r^2$	$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$
In the standard equation of the sphere, the coefficients of x^2,y^2 ,$z^2$are ---	parallel	perpendicular	equal	not equal	equal
Two spheres S1 and S2 whose radii are r1 and r2 touch externally if the distance between their centers is equal to the--------	difference of their radii	sum of their radii	multiple of their radii	division of their radii	sum of their radii
The locus of a point which moves so that it's distance from a fixed point remains constant …………………..	Parapola	Coplanar	Sphere	Stright line	Sphere

If the center of the sphere is at the origin then the equation of the sphere is………………	$x^2 + y^2 + z^2 = r^2$	$x^2 + y^2 + z^2 = 0$	$x^2 + y^2 + z^2 = 1$	$x^2 + y^2 + z^2 \neq 1$	$x^2 + y^2 + z^2 = r^2$
The sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ the center is ……………………	C (-u,v.w)	C (-u,-v.-w)	C (-u,v.-w)	C (u,-v.w)	C (-u,-v.-w)
The condition that the plane ax+by+cz+d =0 may touch the sphere is length of the Perpendicular from the center of the sphere to the plane is equal to………………….	radius of the sphere	center of the sphere	one	diameter of the sphere	radius of the sphere
That the plane section of a sphere is a ………………………	radius of the sphere	center of the sphere	circle	sphere	circle
In ------------the moving straight line in any position is called generator.	curve	Sphere	circle	cone	cone
The equation of the cone with vertex at the origin is ----------	$ax^2 - by^2 - cz^2 - 2fyz + 2gzx + 2hxy = 0$	$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$	$ax^2 + by^2 + cz^2 - 2fyz - 2gzx - 2hxy = 0$	$ax^2 + by^2 + cz^2 = 0$	$ax^2 + by^2 + cz^2 - 2fyz - 2gzx - 2hxy = 0$

UNIT-IV

The angles between two directed lines, the projection of a segment , relation between a segment and its projection , the projection of a broken line , the angle between two planes , relation between areas of a triangle and its projection , relation between areas of a polygon.

1. Segments. Two segments AB and CD are said to have the same direction when they are collinear or parallel, and when B is on the same side of A as D is of C. If AB and CD have the same direction, BA and CD have opposite directions. If AB and CD are of the same length and in the same direction they are said to be **equivalent segments.**

2. If A, B, C, ... N, P are any points on a straight line X′OX, and the convention is made that a segment of the straight line is positive or negative according as its direction is that of OX or OX′, then we have the following relations:

$$AB = -BA; \quad OA + AB = OB, \quad \text{or} \quad AB = OB - OA,$$
$$\text{or} \quad OA + AB + BO = 0;$$
$$OA + AB + BC + \ldots NP = OP.$$

3. Coordinates. Let X′OX, Y′OY, Z′OZ be any three fixed intersecting lines which are not coplanar, and whose positive directions are chosen to be X′OX, Y′OY, Z′OZ; and let planes through any point in space, P, parallel respectively to the planes YOZ, ZOX, XOY, cut X′X, Y′Y, Z′Z in A, B, C, (fig. 1), then the position of P is known when the segments

B.G. A C

OA, OB, OC are given in magnitude and sign. A construction for P would be: cut off from OX the segment OA, draw AN, through A, equivalent to the segment OB, and draw NP, through N, equivalent to the segment OC. OA, OB, OC are known when their measures are known, and

these measures are called the **Cartesian coordinates** of P with reference to the coordinate axes X′OX, Y′OY, Z′OZ. The point O is called the origin and the planes YOZ, ZOX, XOY, the coordinate planes. The measure of OA, the segment cut off from OX or OX′ by the plane through P

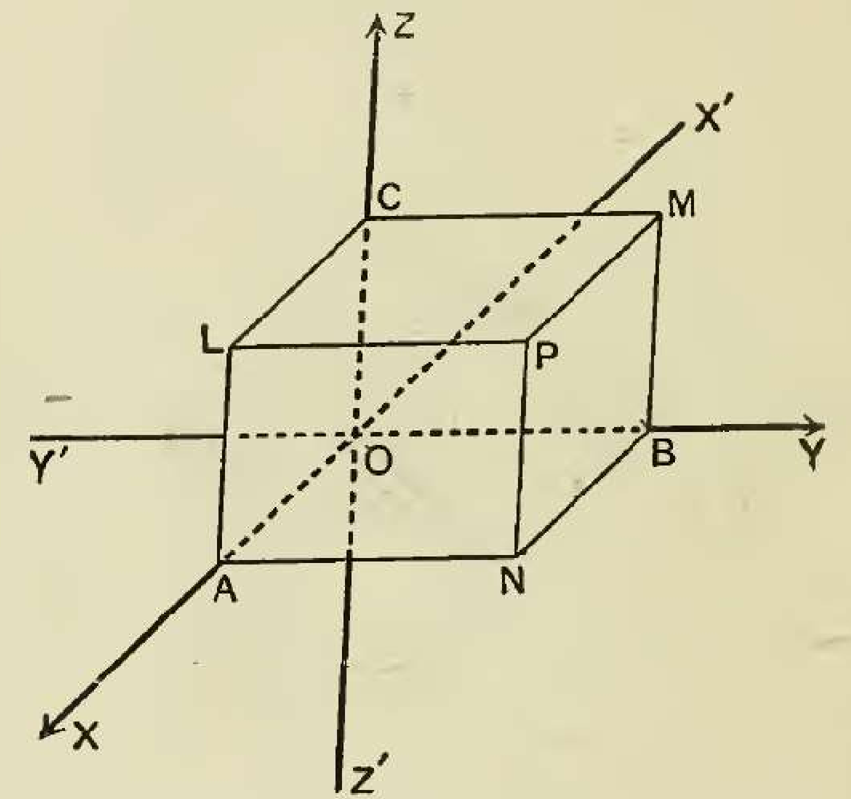


FIG. 1.

parallel to YOZ, is called the x-coordinate of P; the measures of OB and OC are the y and z-coordinates, and the symbol P, (x, y, z) is used to denote, "the point P whose coordinates are x, y, z." The coordinate planes divide space into eight parts called octants, and the signs of the coordinates of a point determine the octant in which it lies. The following table shews the signs for the eight octants:

Octant	OXYZ	OX′YZ	OX′Y′Z	OXY′Z	OXYZ′	OX′YZ′	OX′Y′Z′	OXY′Z′
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

It is generally most convenient to choose mutually perpendicular lines as coordinate axes. The axes are then "rectangular," otherwise they are "oblique."

4. Sign of direction of rotation. By assigning positive directions to a system of rectangular axes X′X, Y′Y, Z′Z, we have fixed the positive directions of the normals to the coordinate planes YOZ, ZOX, XOY. Retaining the usual convention made in plane geometry, the positive direction of rotation for a ray revolving about O in the plane XOY is that given by XYX′Y′, that is, is counter-clockwise, if the clock dial be supposed to coincide with the plane and front in the positive direction of the normal. Hence to fix the positive direction of rotation for a ray in *any* plane, we have the rule: *if a clock dial is considered to coincide with the plane and front in the positive direction of the normal to the plane, the positive direction of rotation for a ray revolving in the plane is counter-clockwise.* Applying this rule to the other coordinate planes the positive directions of rotation for the planes YOZ, ZOX are seen to be YZY′Z′, ZXZ′X′.

5. Cylindrical coordinates. If X′OX, Y′OY, Z′OZ, are rectangular axes, and PN is the perpendicular from any point P to the plane XOY, the position of P is determined if ON, the angle XON, and NP are known. The measures of these quantities, u, ϕ, z, are the **cylindrical coordinates** of P. The positive direction of rotation for the plane XOY has been defined, and the direction of a ray originally coincident with OX, and then turned through the given angle ϕ, is the positive direction of ON. In the figure, u, ϕ, z are all positive.

If the Cartesian coordinates of P are x, y, z, those of N are x, y, 0. If we consider only points in the plane XOY, the Cartesian coordinates of N are x, y, and the polar, u, ϕ. Therefore

$$x = u\cos\phi,\ \ y = u\sin\phi;\quad u^2 = x^2 + y^2,\ \ \tan\phi = y/x.$$

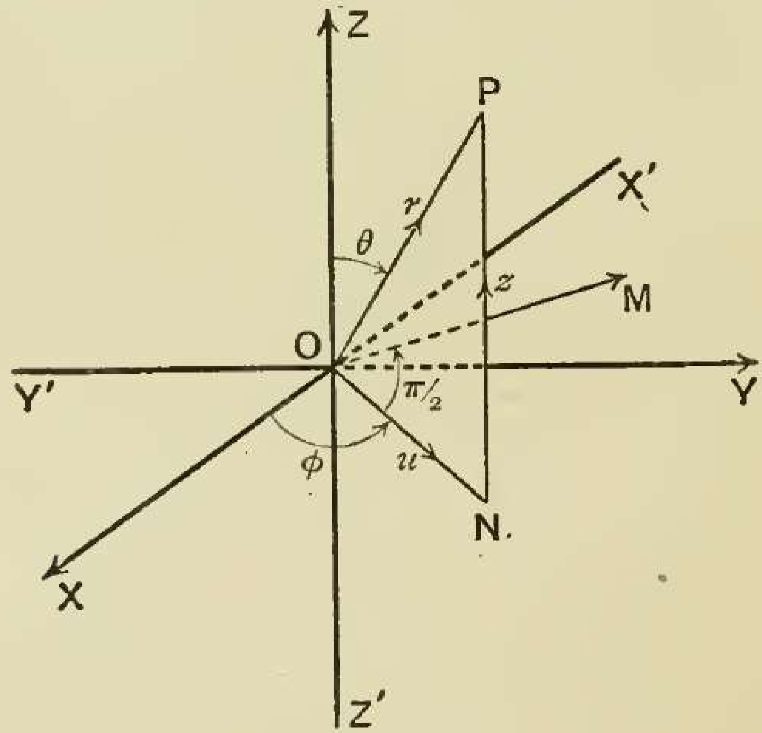


FIG. 2.

6. Polar coordinates. Suppose that the position of the plane OZPN, (fig. 2), has been determined by a given value of ϕ, then we may define the positive direction of the normal through O to the plane to be that which makes an angle $\phi + \pi/2$ with X'OX. Our convention, (§ 4), then fixes the positive direction of rotation for a ray revolving in the plane OZPN. The position of P is evidently determined when, in addition to ϕ, we are given r and θ, the measures

of OP and $\angle$ZOP. The quantities r, θ, ϕ are the **polar coordinates** of P. The positive direction of OP is that of a ray originally coincident with OZ and then turned in the plane OZPN through the given angle θ. In the figure, OM is the positive direction of the normal to the plane OZPN, and r, θ, ϕ are all positive.

If we consider P as belonging to the plane OZPN and OZ and ON as rectangular axes in that plane, P has Cartesian coordinates z, u, and polar coordinates r, θ. Therefore

$$z = r\cos\theta,\ \ u = r\sin\theta; \quad r^2 = z^2 + u^2,\ \tan\theta = \frac{u}{z}.$$

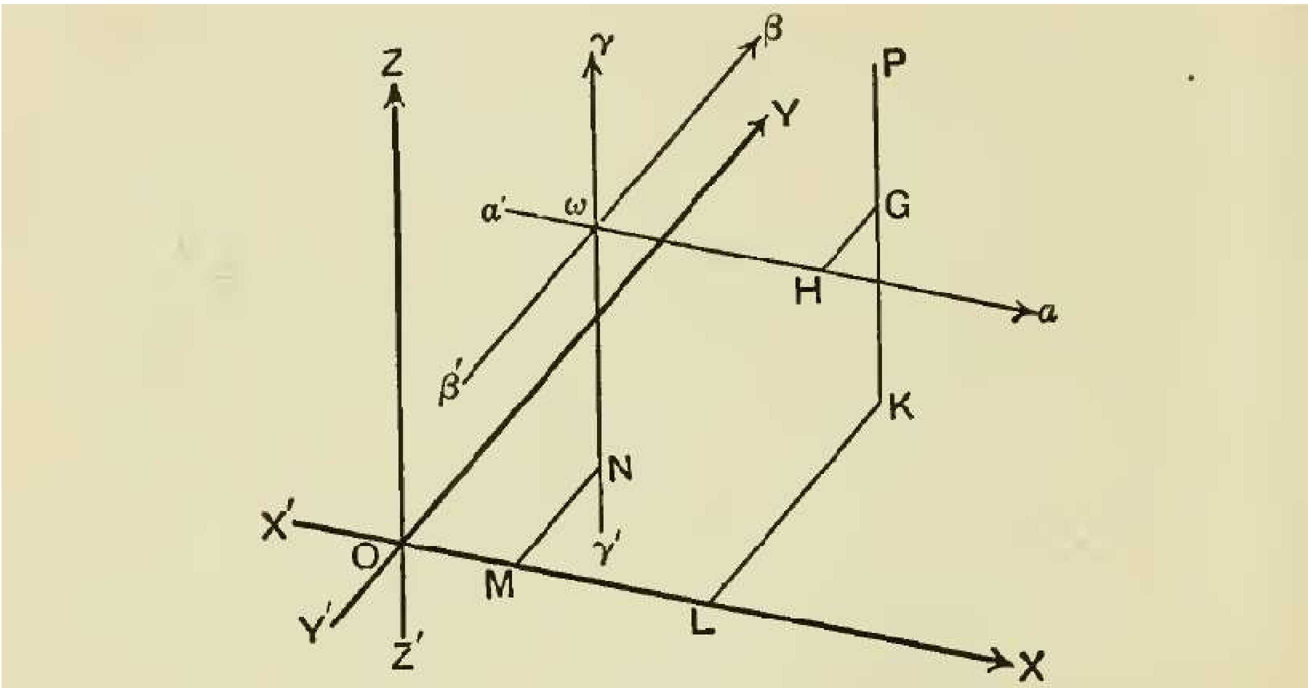
But if P is (x, y, z), $x = u\cos\phi$, $y = u\sin\phi$.

Whence $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$;

$$r^2 = x^2 + y^2 + z^2,\ \tan\theta = \frac{\pm\sqrt{x^2+y^2}}{z},\ \tan\phi = \frac{y}{x}.$$

Cor. If the axes are rectangular the distance of (x, y, z) from the origin is given by $\sqrt{x^2+y^2+z^2}$.

7. Change of origin. Let X′OX, Y′OY, Z′OZ; $\alpha'\omega\alpha$, $\beta'\omega\beta$, $\gamma'\omega\gamma$, (fig. 3), be two sets of parallel axes, and let any point P be (x, y, z) referred to the first and (ξ, η, ζ) referred to the second set. Let ω have coordinates a, b, c, referred to OX, OY, OZ. NM is the line of intersection of the planes $\beta\omega\gamma$, XOY, and the plane through P parallel to $\beta\omega\gamma$ cuts $\alpha\omega\beta$ in GH and XOY in KL.



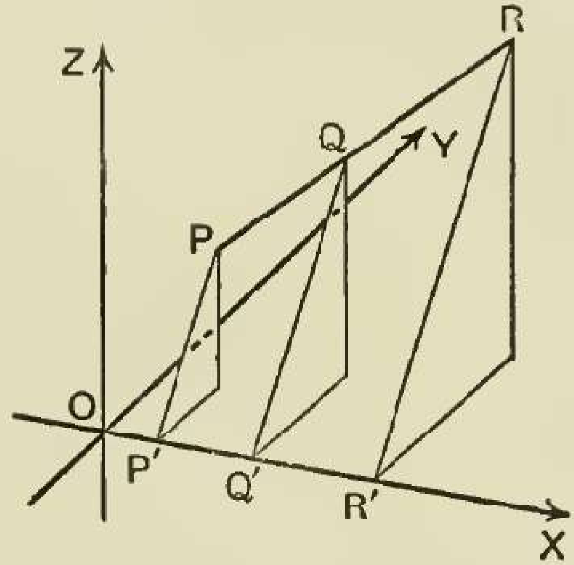
Then $\mathsf{OL}=\mathsf{OM}+\mathsf{ML}=\mathsf{OM}+\omega\mathsf{H}$,

therefore $x=a+\xi$. Similarly, $y=b+\eta$, $z=c+\zeta$;

whence $\xi=x-a$, $\eta=y-b$, $\zeta=z-c$.

8. *To find the coordinates of the point which divides the join of* P, (x_1, y_1, z_1) *and* Q, (x_2, y_2, z_2) *in a given ratio,* $\lambda:1$.

Let R, (x, y, z), (fig. 4), be the point, and let planes through P, Q, R, parallel to the plane YOZ, meet OX in P′, Q′, R′.



Then, since three parallel planes divide any two straight lines proportionally, $P'R' : P'Q' = PR : PQ = \lambda : \lambda + 1$. Therefore

$$\frac{x - x_1}{x_2 - x_1} = \frac{\lambda}{\lambda + 1}, \quad \text{and} \quad x = \frac{\lambda x_2 + x_1}{\lambda + 1}.$$

Similarly, $$y = \frac{\lambda y_2 + y_1}{\lambda + 1}, \quad z = \frac{\lambda z_2 + z_1}{\lambda + 1}.$$

These give the coordinates of R for all real values of λ, positive or negative. If λ is positive, R lies between P and Q; if negative, R is on the same side of both P and Q.

9. The equation to a surface. *Any equation involving one or more of the current coordinates of a variable point represents a surface or system of surfaces which is the locus of the variable point.*

The locus of all points whose x-coordinates are equal to a constant α, is a plane parallel to the plane YOZ, and the equation $x=\alpha$ represents that plane. If the equation $f(x)=0$ has roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$, it is equivalent to the equations $x=\alpha_1, x=\alpha_2, \ldots x=\alpha_n$, and therefore represents a system of planes, real or imaginary, parallel to the plane YOZ.

Similarly, $f(y)=0$, $f(z)=0$ represent systems of planes parallel to ZOX, XOY. In the same way, if polar coordinates be taken, $f(r)=0$ represents a system of spheres with a common centre at the origin, $f(\theta)=0$, a system of coaxal right circular cones whose axis is OZ, $f(\phi)=0$, a system of planes passing through OZ.

Consider now the equation $f(x, y)=0$. This equation is satisfied by the coordinates of all points of the curve in the plane XOY whose two-dimensional equation is $f(x, y)=0$.

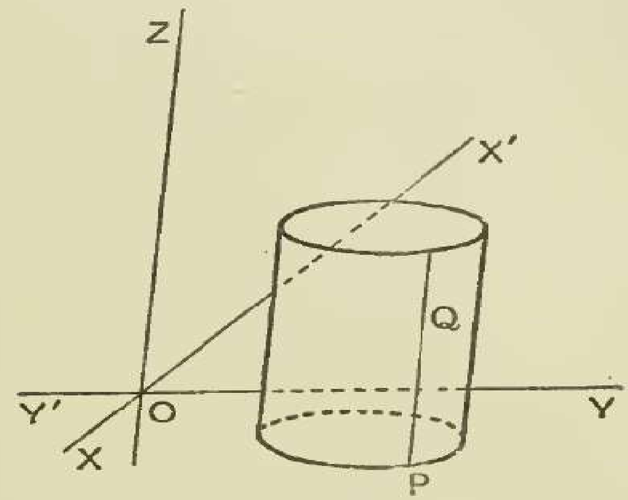


FIG. 5.

Let P, (fig. 5), any point of the curve, have coordinates $x_0, y_0, 0$. Draw through P a parallel to OZ, and let Q be any point on it. Then the coordinates of Q are x_0, y_0, z_0, and since P is on the curve, $f(x_0, y_0)=0$, thus the coordinates of Q satisfy the equation $f(x, y)=0$. Therefore the coordinates of every point on PQ satisfy the equation and every point on PQ lies on the locus of the equation. But P is any point of the curve, therefore the locus of the equation is the cylinder generated by straight lines drawn

parallel to OZ through points of the curve. Similarly, $f(y, z)=0$, $f(z, x)=0$ represent cylinders generated by parallels to OX and OY respectively.

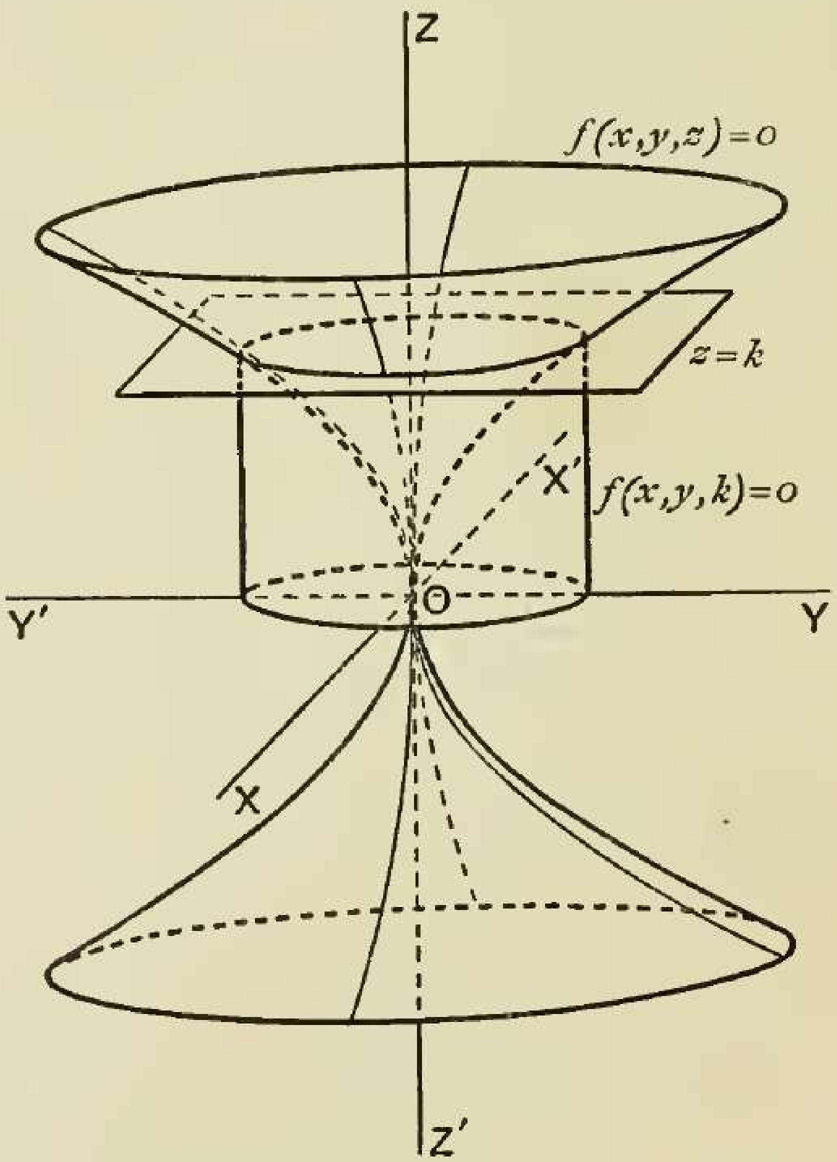
Ex. What surfaces are represented by (i) $x^2+y^2=a^2$, (ii) $y^2=4ax$, the axes being rectangular?

Two equations are necessary to determine the curve in the plane XOY. The curve is on the cylinder whose equation is $f(x, y)=0$ and on the plane whose equation is $z=0$, and hence "the equations to the curve" are $f(x, y)=0$, $z=0$.

Consider now the equation $f(x, y, z)=0$. The equation $z=k$ represents a plane parallel to XOY, and the equation $f(x, y, k)=0$ represents, as we have just proved, a cylinder

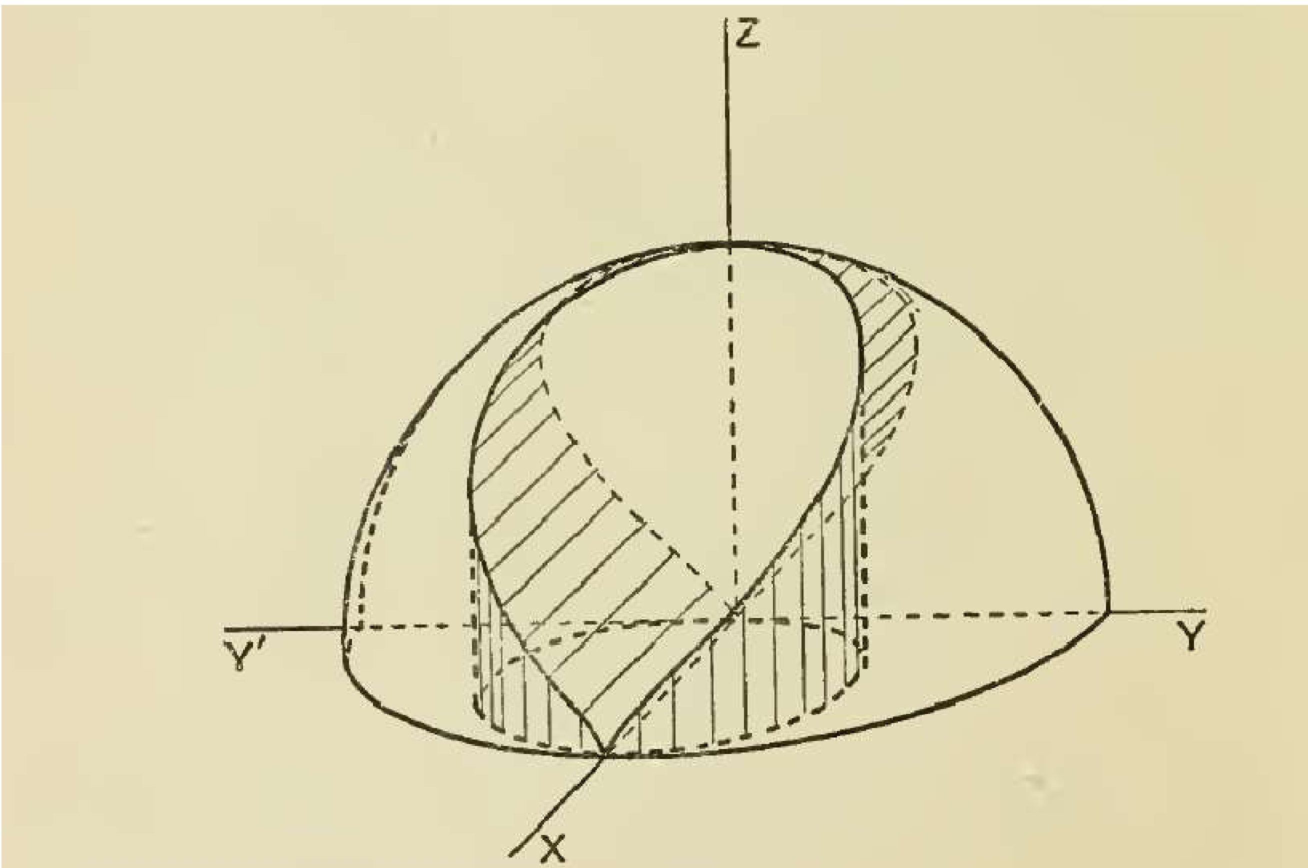
Ex. What curves are represented by

(i) $x^2+y^2=a^2,\ z=0$; (ii) $x^2+y^2=a^2,\ z=b$; (iii) $z^2=4ax,\ y=c$?



generated by lines parallel to OZ. The equation $f(x, y, k)=0$ is satisfied at all points where $f(x, y, z)=0$ and $z=k$ are simultaneously satisfied, *i.e.* at all points common to the plane and the locus of the equation $f(x, y, z)=0$, and hence $f(x, y, k)=0$ represents the cylinder generated by lines parallel to OZ which pass through the common points, (fig. 6). The two equations $f(x, y, k)=0$, $z=k$ represent the curve of section of the cylinder by the plane $z=k$, which is the curve of section of the locus by the plane $z=k$. If, now, all real values from $-\infty$ to $+\infty$ be given to k, the curve $f(x, y, k)=0$, $z=k$, varies continuously and generates a surface. The coordinates of every point on this surface satisfy the equation $f(x, y, z)=0$, for they satisfy, for some value of k, $f(x, y, k)=0$, $z=k$; and any point (x_1, y_1, z_1) whose coordinates satisfy $f(x, y, z)=0$ lies on the surface, for the coordinates satisfy $f(x, y, z_1)=0$, $z=z_1$, and therefore the point is on one of the curves which generate the surface. Hence the equation $f(x, y, z)=0$ represents a surface, and the surface is the locus of a variable point whose coordinates satisfy the equation.

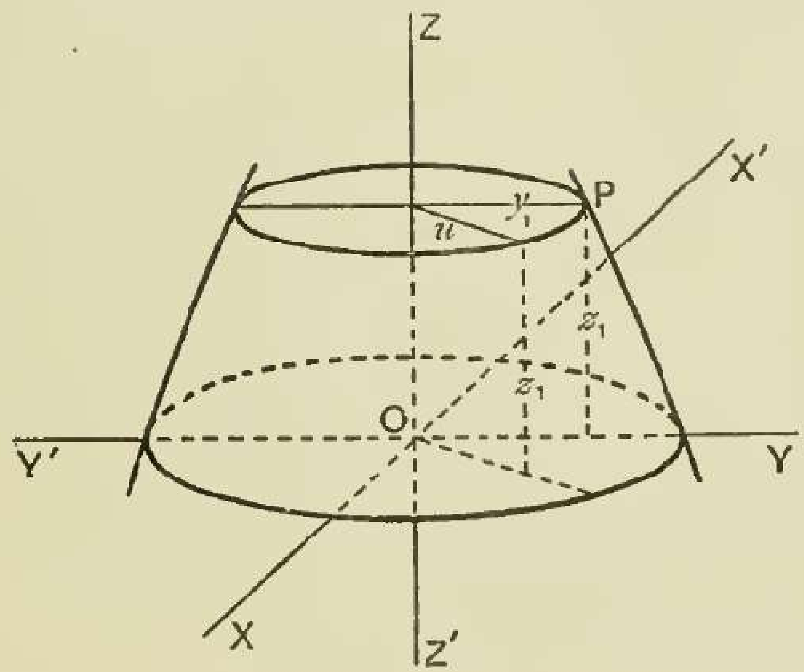
10. The equations to a curve. The two equations $f_1(x, y, z)=0$, $f_2(x, y, z)=0$ represent the curve of intersection of the two surfaces given by $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$. If we eliminate one of the variables, z,



say, between the two equations, we obtain an equation. $\phi(x, y)=0$, which represents a cylinder whose generators are parallel to OZ. If any values of x, y, z satisfy $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$, they satisfy $\phi(x, y)=0$, and hence the cylinder passes through the curve of intersection of the surfaces. If the axes are rectangular $\phi(x, y)=0$ represents the cylinder which projects orthogonally the curve of intersection on the plane XOY, and the equations to the projection are $\phi(x, y)=0$, $z=0$.

11. Surfaces of revolution. Let P, $(0, y_1, z_1)$, (fig. 8), be any point on the curve in the plane YOZ whose Cartesian equation is $f(y, z)=0$. Then

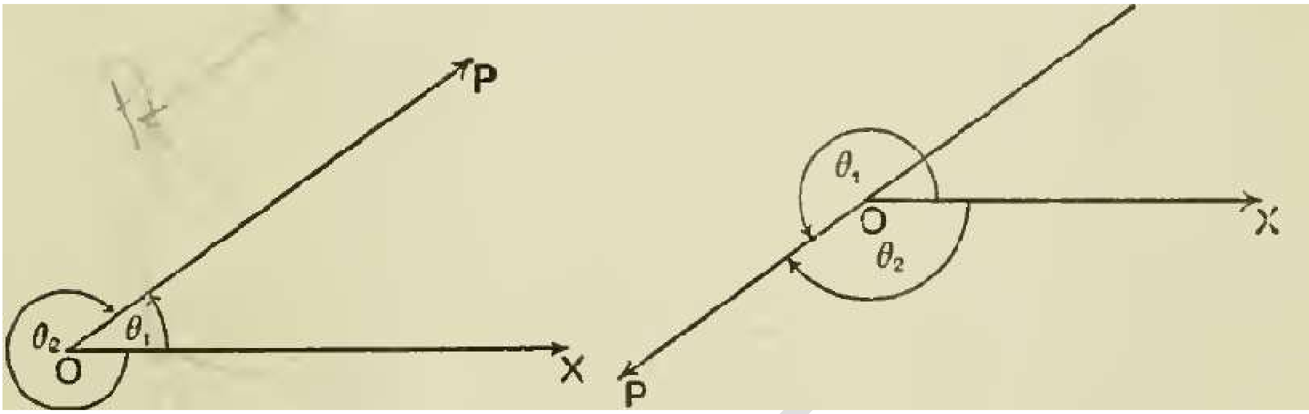
$$f(y_1, z_1)=0 \quad \text{.........................(1)}$$



The rotation of the curve about OZ produces a surface of revolution. As P moves round the surface, z_1, the z-coordinate of P remains unaltered, and u, the distance of P from the z-axis, is always equal to y_1. Therefore, by (1), the cylindrical coordinates of P satisfy the equation $f(u, z)=0$. But P is any point on the curve, or surface, and therefore the cylindrical equation to the surface is $f(u, z)=0$. Hence the Cartesian equation to the surface is $f(\sqrt{x^2+y^2}, z)=0$.

PROJECTIONS.

12. The angle that a given directed line OP makes with a second directed line OX we shall take to be the smallest angle generated by a variable radius turning in the plane XOP from the position OX to the position OP. The sign of the angle is determined by the usual convention. Thus, in figures 9 and 10, θ_1 is the positive angle, and θ_2 the negative angle that OP makes with OX.



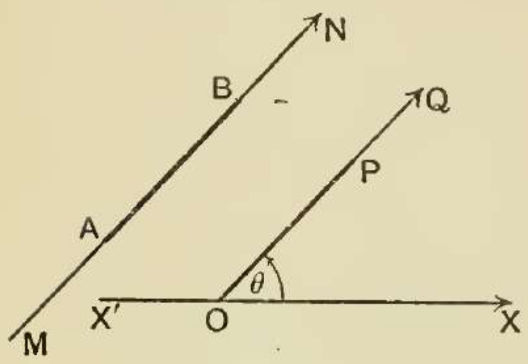
13. Projection of a segment. *If* AB *is a given segment and* A′, B′ *are the feet of the perpendiculars from* A, B *to a given line* X′X, *the segment* A′B′ *is the projection of the segment* AB *on* X′X.

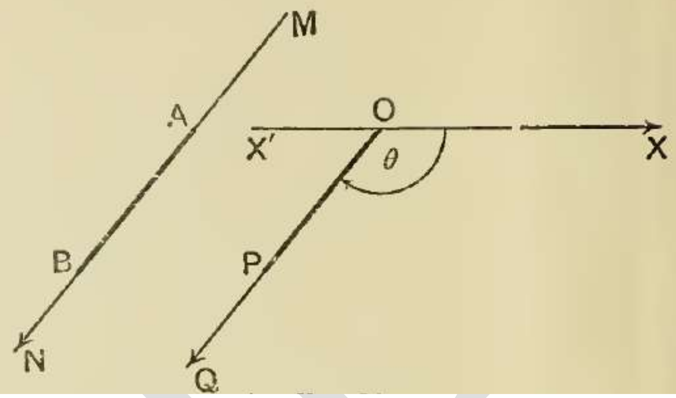
From the definition it follows that the projection of BA is B′A′, and therefore that the projections of AB and BA differ only in sign.

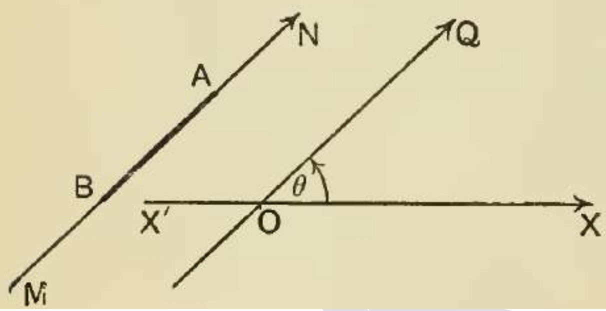
It is evident that A′B′ is the intercept made on X′X by the planes through A and B normal to X′X, and hence *the projections of equivalent segments are equivalent segments.*

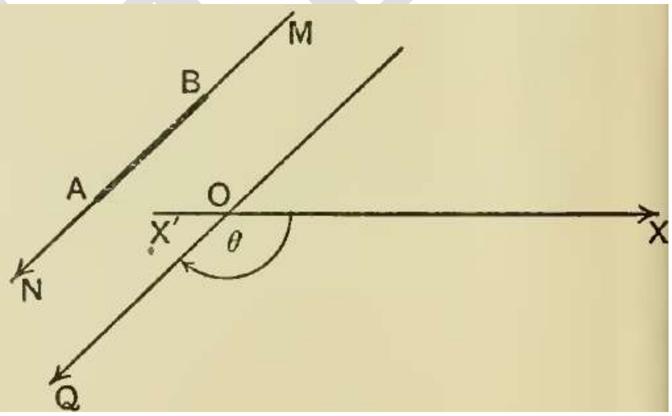
14. *If* AB *is a given segment of a directed line* MN *whose positive direction,* MN, *makes an angle* θ *with a* *given line* X′X, *the projection of* AB *on* X′X *is equal to* AB . cos θ.

In figures 11 and 12, AB is positive, in figures 13 and 14, AB is negative.









Draw OQ from O in the same direction as MN. *If* AB *is positive*, cut off OP, the segment equivalent to AB; then

the projection of AB = the projection of OP,

$= OP . \cos\theta$, (by the definition

$= AB . \cos\theta$. of cosine),

If AB *is negative*, BA is positive, and therefore

the projection of $BA = BA . \cos\theta$,

i.e. −(the projection of AB) $= -AB . \cos\theta$,

i.e. the projection of $AB = AB . \cos\theta$.

15. *If* A, B, C, ... M, N *are any n points in space, the sum of the projections of* AB, BC, ... MN, *on any given line* X′X *is equal to the projection of the straight line* AN *on* X′X.

Let the feet of the perpendiculars from A, B, ... M, N, to X′X be A′, B′, ... M′, N′. Then, (§ 2),

$$A'B' + B'C' + \ldots M'N' = A'N',$$

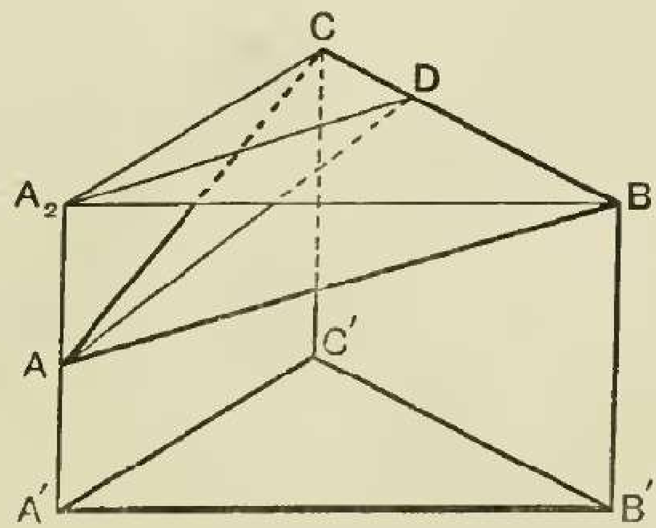
which proves the proposition.

17. Projection of a closed plane figure. *If the projections of three points* A, B, C *on a given plane are* A′, B′, C′, *then* $\triangle A'B'C' = \cos\theta \triangle ABC$, *where* θ *is the angle between the planes* ABC, A′B′C′.

Consider first the areas ABC, A′B′C′ without regard to sign.

(i) If the planes ABC, A′B′C′ are parallel, the equation $\triangle A'B'C' = \cos\theta \triangle ABC$ is obviously true.

(ii) If one side of the triangle ABC, say BC, is parallel to the plane A′B′C′, let AA′ meet the plane through BC parallel to the plane A′B′C′ in A_2, (fig. 15). Draw A_2D at right angles to BC, and join AD. Then BC is at right angles to



A_2D and AA_2, and therefore BC is normal to the plane AA_2D, and therefore at right angles to AD. Hence the angle A_2DA is equal to θ, or its supplement.

But $\triangle A'B'C' = \triangle A_2BC$,

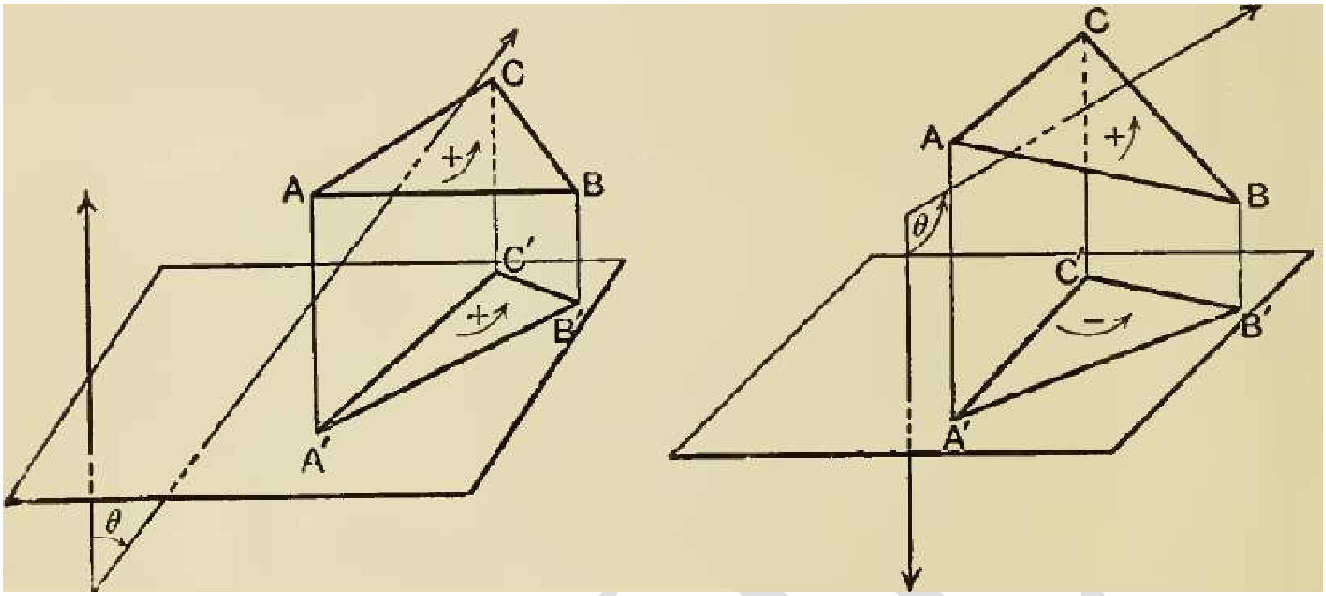
and $\triangle A_2BC : \triangle ABC = A_2D : AD = \cos \angle A_2DA$;

therefore $\triangle A'B'C' = \cos\theta \, \triangle ABC$.

(iii) If none of the sides of the triangle ABC is parallel to the plane A′B′C′, draw lines through A, B, C parallel to the line of intersection of the planes ABC, A′B′C′. These lines lie in the plane ABC and are parallel to the plane A′B′C′, and one of them, that through A, say, will cut the

opposite side, BC, of the triangle ABC, internally. And therefore the triangle ABC can always be divided by a line through a vertex into two triangles, with a common side parallel to the given plane A′B′C′, and hence, by (ii), $\triangle A'B'C' = \cos\theta \, \triangle ABC$

Suppose now that the areas ABC, A′B′C′ are considered positive or negative according as the directions of rotation given by ABC, A′B′C′ are positive or negative. Then, applying the convention of §4 to figures 16 and 17, we



see that if $\cos\theta$ is positive, the directions of rotation ABC, A′B′C′ have the same sign, and that if $\cos\theta$ is negative, they have opposite signs. That is, the areas have the same sign if $\cos\theta$ is positive, and opposite signs if $\cos\theta$ is negative. Hence the equation $\triangle A'B'C' = \cos\theta \triangle ABC$ is true for the signs as well as the magnitudes of the areas.

18. *If* A, B, C, ... N *are any coplanar points and* A′, B′, C′, ... N′ *are their projections on any given plane, then*

$$\text{area } A'B'C' \dots N' : \text{area } ABC \dots N = \cos\theta,$$

where θ *is the angle between the planes.*

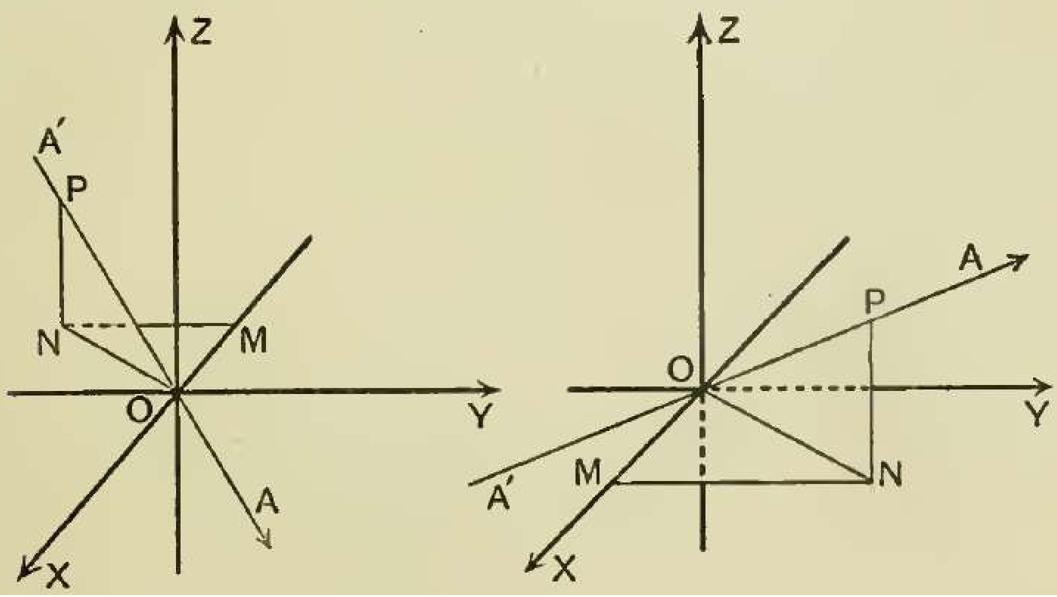
Let O be any point of the plane ABC ... N, and O′ be its projection on the plane A′B′C′ .. N′.

Then area ABC ... N = △ OAB + △ OBC + ... △ ONA,

and area A′B′C′ ... N′ = △ O′A′B′ + △ O′B′C′ + ... △ O′N′A′.

But △ O′A′B′ = cos θ △OAB, etc., and therefore the result follows.

20. If α, β, γ are the angles that a given directed line makes with the positive directions X′OX, Y′OY, Z′OZ of the coordinate axes, cos α, cos β, cos γ are the **direction-cosines** of the line.



21. Direction-cosines referred to rectangular axes. Let A′OA be the line through O which has direction-cosines cos α, cos β, cos γ. Let P, (x, y, z) be any point on A′OA, and OP have measure r. In fig. 18, r is positive; in fig. 19, r is negative. Draw PN perpendicular to the plane XOY, and NM in the plane XOY, perpendicular to OX. Then the measures of OM, MN, NP are x, y, z respectively. Since OM is the projection of OP on OX,

$$x = r\cos\alpha, \text{ and similarly, } y = r\cos\beta,\ z = r\cos\gamma. \quad\ldots\ldots(1)$$

Again the projection of OP on any line is equal to the sum of the projections of OM, MN, NP, and therefore, projecting on OP, we obtain

$$r = x\cos\alpha + y\cos\beta + z\cos\gamma. \quad\ldots\ldots(2)$$

But $x/r = \cos\alpha$, $y/r = \cos\beta$, $z/r = \cos\gamma$; therefore

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma. \quad\ldots\ldots(3)$$

This is the formula in three dimensions which corresponds to $\cos^2\theta + \sin^2\theta = 1$ in plane trigonometry.

23. The angle between two lines. *If* OP *and* OQ *have direction-cosines* $\cos\alpha$, $\cos\beta$, $\cos\gamma$; $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, *and* θ *is the angle that* OP *makes with* OQ,

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

If, as in §21, P is (x, y, z) and the measure of OP is r, projecting OP and OM, MN, NP on OQ, we obtain

$$r\cos\theta = x\cos\alpha' + y\cos\beta' + z\cos\gamma'.$$

But $x = r\cos\alpha,\ y = r\cos\beta,\ z = r\cos\gamma;$

therefore $\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$

Cor. 1. We have the identity

$$(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$$
$$\equiv (mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2.$$

(This identity is known as *Lagrange's identity.* We shall frequently find it advantageous to apply it.)

Hence

$$\sin^2\theta = (\cos^2\alpha+\cos^2\beta+\cos^2\gamma)(\cos^2\alpha'+\cos^2\beta'+\cos^2\gamma')$$
$$-(\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma')^2,$$
$$=(\cos\beta\cos\gamma'-\cos\gamma\cos\beta')^2+(\cos\gamma\cos\alpha'-\cos\alpha\cos\gamma')^2$$
$$+(\cos\alpha\cos\beta'-\cos\beta\cos\alpha')^2.$$

24. Distance of a point from a line. *To find the distance of* P, (x', y', z') *from the line through* A, (a, b, c), *whose direction-cosines are* $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Let PN, the perpendicular from P to the line, have measure δ. Then AN is the projection of AP on the line, and its measure is, (Ex. 3, § 21),

$$(x'-a)\cos\alpha+(y'-b)\cos\beta+(z'-c)\cos\gamma.$$

But $$PN^2=AP^2-AN^2,$$

therefore

$$\delta^2=\{(x'-a)^2+(y'-b)^2+(z'-c)^2\}(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$$
$$-\{(x'-a)\cos\alpha+(y'-b)\cos\beta+(z'-c)\cos\gamma\}^2,$$

which, by Lagrange's identity, gives

$$\delta^2=\{(y'-b)\cos\gamma-(z'-c)\cos\beta\}^2$$
$$+\{(z'-c)\cos\alpha-(x'-a)\cos\gamma\}^2$$
$$+\{(x'-a)\cos\beta-(y'-b)\cos\alpha\}^2.$$

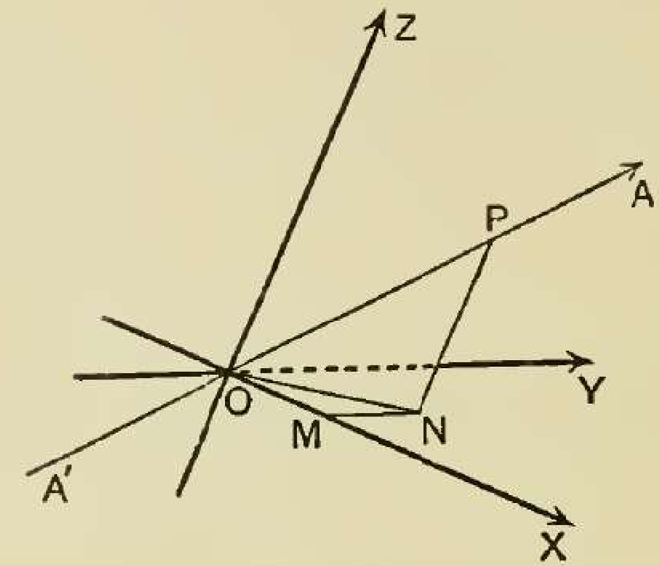
sum of the projections of OM, MN, NP, projecting on OX, OY, OZ, OP in turn, we obtain

$$r\cos\alpha=x+y\cos\nu+z\cos\mu, \quad\text{.................(1)}$$

$$r\cos\beta=x\cos\nu+y+z\cos\lambda, \quad\text{.................(2)}$$

$$r\cos\gamma=x\cos\mu+y\cos\lambda+z, \quad\text{.................(3)}$$

$$r=x\cos\alpha+y\cos\beta+z\cos\gamma. \quad\text{...........(4)}$$



Therefore, eliminating r, x, y, z, we have the relation satisfied by the direction-cosines of any line

$$\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & \cos\alpha \\ \cos\nu, & 1, & \cos\lambda, & \cos\beta \\ \cos\mu, & \cos\lambda, & 1, & \cos\gamma \\ \cos\alpha, & \cos\beta, & \cos\gamma, & 1 \end{vmatrix} = 0,$$

which may be written,

$$\Sigma \sin^2\lambda \cos^2\alpha - 2\Sigma(\cos\lambda - \cos\mu\cos\nu)\cos\beta\cos\gamma$$
$$= 1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu.$$

*27. **The angle between two lines.** If OQ has direction-cosines $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, and makes an angle θ with OP, projecting on OQ, we obtain

$$r\cos\theta = x\cos\alpha' + y\cos\beta' + z\cos\gamma'. \quad\ldots\ldots\ldots\ldots(5)$$

Therefore eliminating x, y, z, r between equations (1), (2), (3) of § 25, and (5), we have

$$\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & \cos\alpha \\ \cos\nu, & 1, & \cos\lambda, & \cos\beta \\ \cos\mu, & \cos\lambda, & 1, & \cos\gamma \\ \cos\alpha', & \cos\beta', & \cos\gamma', & \cos\theta \end{vmatrix} = 0, \quad \text{or}$$

$$\Sigma(\sin^2\lambda\cos\alpha\cos\alpha') - \Sigma\{(\cos\lambda - \cos\mu\cos\nu) \times (\cos\beta\cos\gamma' + \cos\beta'\cos\gamma)\}$$
$$= \cos\theta(1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu).$$

Cor. The angles between the lines whose direction-cosines are proportional to a, b, c; a', b', c' are given by

$$\cos\theta = \frac{\pm\{\Sigma(aa'\sin^2\lambda) - \Sigma(bc' + b'c)(\cos\lambda - \cos\mu\cos\nu)\}}{\{\Sigma a^2\sin^2\lambda - 2\Sigma bc(\cos\lambda - \cos\mu\cos\nu)\}^{\frac{1}{2}} \times \{\Sigma a'^2\sin^2\lambda - 2\Sigma b'c'(\cos\lambda - \cos\mu\cos\nu)\}^{\frac{1}{2}}}.$$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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Coimbatore –641 021

Subject: Analytical Geometry **Subject Code: 18MMU303A**

Class : II - B.Sc. Mathematics **Semester : III**

Unit IV

Part A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Questions	Opt1	Opt2	Opt3	Opt4	Answer
The general equationof the……………degree in x, y, z represents a plane.	first	second	third	zero	first
The same length and in the same direction they are said to be…………..	equivalent segments.	segments	straight lines	parallel axis	equivalent segments.
Be any three fixed intersecting lines which are…………………	coplanar	not coplanar	right angle	tangent	not coplanar
If a clock dial is considered to coincide- with the plane and front in the positive direction of the normed to the plane, the positive direction of rotation for a ray revolving in the plane is…………	counter-clockwise	clockwise	parallel	normal	counter-clockwise
If AB is a given segment of a directed line MN whose positive direction, MN, makes an angle θ with a given line X'X, the projection of AB on X'X is equal to……………	$AB\cos\theta$	$AB\cos 2\theta$	$AB\sin\theta$	$AB\sin 2\theta$	$AB\cos\theta$
The direction-cosines of the line ………………..	$\cos\alpha,\cos\beta,\cos\gamma$	$\cos\alpha,\sin\beta,\cos\gamma$	$\cos\alpha,\cos\beta,\sin\gamma$	$\sin\alpha,\sin\beta,\sin\gamma$	$\cos\alpha,\cos\beta,\cos\gamma$
The general equation ax2 +by2 + 2fy + 2gx +c= 0, the axis of the coordinates are being rectangular the curve is ellipse………………	h^2=ab	h^2>ab	h^2<ab	h^2=-ab	h^2=ab
The polar coordinates points in two dimension………..	$x=\cos\theta, y=\sin\theta$	$x=r\cos\theta, y=r\sin\theta$	$x=r\cos\theta, y=\sin\theta$	$x=\cos\theta, y=r\sin\theta$	$x=r\cos\theta, y=r\sin\theta$
This equation is satisfied by the coordinates of all points of the curve in the plane XOY whose two-dimensional equation is…………….	〚90〛 f(x,y)=0	f(x,y)>0	f(x,y)<0	〚45〛 f(x,y)=1	f(x,y)=0
A line makes angles $\propto, \beta, \delta$ with four diagonals of a cube , Prove that $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta =$	〚90〛 2/3	3/4	1 1/2	〚45〛 1 1/3	1 1/3
Find the direction-cosines of a line that makes equal angles with the axes…………..	$\cos\alpha$=1/3	$\cos\alpha$=1/root 3	$\cos\alpha$=1/2	$\cos\alpha$=1	$\cos\alpha$=1/root 3
The general equation of the first degree in x, y, z represents a plane…………….	ax^2-by-cz=d	ax+by+cz=d	ax+by^2+cz=d	ax+by+cz^2=d	ax+by+cz=d

The direction-cosines of the line whose direction-ratios are l,m,n ……………..	xcosα-ysinαβp lcosα-mcosβ-ncosγ=1	lcosα-mcosβ+ncosγ=1	lcosα+mcosβ-ncosγ=1	lcosα+mcosβ+ncosγ=	lcosα+mcosβ+ncosγ=1
Any point on the line through origin whose direction-ratios are l,m,n,then……….	l/x=m/y=n/z	x/l=y/m=z/n	x/l+y/m+n/z	l/x+m/y+n/z	x/l=y/m=z/n
The ------- joining two points P and Q on a surface is called a chord of the surface	tangent	normal	diameter	straight line	straight line
Let OL be drawn from O in the same direction as a given directed line PQ and of unit lengtli. Then the coordinates of L evidently depend only on tiie direction of PQ, and when given, determine that direction. They are therefore called the ………………. of PQ.	perpenticular	direction-ratios	tangent	normal	direction-ratios
plane can be found to pass tli rough any three non ……….points.	equal	colinear	perpenticular	normal	colinear
Two segments are said to have the same direction when they are……….	parallel	equal	perpendiculr	normal	parallel
All plane sections of a surface represented by an equation of the second degree are………….	cylnder	sphere	cone	conies	conies
The ---------- of a point on the line is the foot of the perpendicular drawn from the point on the line	conjucate	bijection	projection	projectile	projection

UNIT-V

Polar equation to a conic: General Equations Tracing of Curves, particular cases of Conic sections, transformation of equations to center as origin, equations to asymptotes, tracing a parabola , tracing a central conic , eccentricity and foci of general conic.

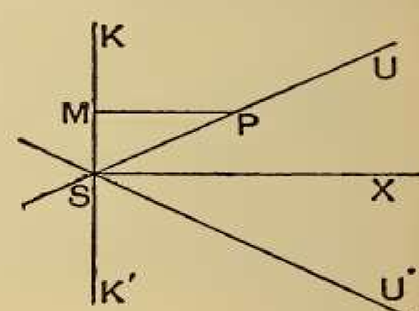
GENERAL EQUATION OF THE SECOND DEGREE. TRACING OF CURVES.

348. Particular cases of Conic Sections. The general definition of a Conic Section in Art. 196 was that it is the locus of a point P which moves so that its distance from a given point S is in a constant ratio to its perpendicular distance PM from a given straight line ZK.

When S does not lie on the straight line ZK, we have found that the locus is an ellipse, a parabola, or a hyperbola according as the eccentricity e is $<=$ or >1.

The Circle is a sub-case of the Ellipse. For the equation of Art. 139 is the same as the equation (6) of Art. 247 when $b^2=a^2$, *i.e.* when $e=0$. In this case $CS=0$, and $SZ=\frac{a}{e}-ae=\infty$. The Circle is therefore a Conic Section, whose eccentricity is zero, and whose directrix is at an infinite distance.

Next, let S lie on the straight line ZK, so that S and Z coincide.



In this case, since

$$SP=e\,.\,PM,$$

we have

$$\sin PSM=\frac{PM}{SP}=\frac{1}{e}.$$

If $e>1$, then P lies on one or other of the two straight lines SU and SU' inclined to KK' at an angle

$$\sin^{-1}\left(\frac{1}{e}\right).$$

If $e=1$, then PSM is a right angle, and the locus becomes two coincident straight lines coinciding with SX.

If $e<1$, the $\angle PSM$ is imaginary, and the locus consists of two imaginary straight lines.

If, again, both KK' and S be at infinity and S be on KK', the lines SU and SU' of the previous figure will be two straight lines meeting at infinity, *i.e.* will be two parallel straight lines.

Finally, it may happen that the axes of an ellipse may both be zero, so that it reduces to a point.

Under the head of a conic section we must therefore include:

(1) An Ellipse (including a circle and a point).

(2) A Parabola.

(3) A Hyperbola.

(4) Two straight lines, real or imaginary, intersecting, coincident, or parallel.

349. *To shew that the general equation of the second degree*

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots\ldots\ldots(1)$$

always represents a conic section.

Let the axes of coordinates be turned through an angle θ, so that, as in Art. 129, we substitute for x and y the quantities $x\cos\theta - y\sin\theta$ and $x\sin\theta + y\cos\theta$ respectively.

The equation (1) then becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta)$$
$$+ 2f(x\sin\theta + y\cos\theta) + c = 0,$$

i.e.

$$x^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$$
$$+ 2xy\{h(\cos^2\theta - \sin^2\theta) - (a-b)\cos\theta\sin\theta\}$$
$$+ y^2(a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta) + 2x(g\cos\theta + f\sin\theta)$$
$$+ 2y(f\cos\theta - g\sin\theta) + c = 0 \ldots\ldots\ldots\ldots(2).$$

Now choose the angle θ so that the coefficient of xy in this equation may vanish,

i.e. so that $h(\cos^2\theta - \sin^2\theta) = (a-b)\sin\theta\cos\theta$,

i.e. $2h\cos 2\theta = (a-b)\sin 2\theta$,

i.e. so that $\tan 2\theta = \dfrac{2h}{a-b}$.

Whatever be the values of a, b, and h, there is always a value of θ satisfying this equation and such that it lies between $-45°$ and $+45°$. The values of $\sin\theta$ and $\cos\theta$ are therefore known.

On substituting their values in (2), let it become

$$Ax^2 + By^2 + 2Gx + 2Fy + c = 0 \qquad (3).$$

First, let neither A nor B be zero.

The equation (3) may then be written in the form

$$A\left(x+\frac{G}{A}\right)^2 + B\left(y+\frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - c.$$

Transform the origin to the point $\left(-\dfrac{G}{A},\ -\dfrac{F}{B}\right)$.

The equation becomes

$$Ax^2 + By^2 = \frac{G^2}{A} + \frac{F^2}{B} - c = K \text{ (say)} \quad \ldots\ldots\ldots(4),$$

i.e.

$$\frac{x^2}{\frac{K}{A}} + \frac{y^2}{\frac{K}{B}} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If $\frac{K}{A}$ and $\frac{K}{B}$ be both positive, the equation represents an ellipse. (Art. 247.)

If $\frac{K}{A}$ and $\frac{K}{B}$ be one positive and the other negative, it represents a hyperbola (Art. 295). If they be both negative, the locus is an imaginary ellipse.

If K be zero, then (4) represents two straight lines, which are real or imaginary according as A and B have opposite or the same signs.

350. Centre of a Conic Section. Def. The centre of a conic section is a point such that all chords of the conic which pass through it are bisected there.

When the equation to the conic is in the form

$$ax^2 + 2hxy + by^2 + c = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

the origin is the centre.

For let (x', y') be *any* point on (1), so that we have

$$ax'^2 + 2hx'y' + by'^2 + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

This equation may be written in the form

$$a(-x')^2 + 2h(-x')(-y') + b(-y')^2 + c = 0,$$

and hence shews that the point $(-x', -y')$ also lies on (1).

But the points (x', y') and $(-x', -y')$ lie on the same straight line through the origin, and are at equal distances from the origin.

The chord of the conic which passes through the origin and any point (x', y') of the curve is therefore bisected at the origin.

The origin is therefore the centre.

351. When the equation to the conic is given in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots(1),$$

the origin is the centre only when both f and g are zero.

For, if the origin be the centre, then corresponding to *each* point (x', y') on (1), there must be also a point $(-x', -y')$ lying on the curve.

Hence we must have

$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0 \ldots\ldots(2),$$

and

$$ax'^2 + 2hx'y' + by'^2 - 2gx' - 2fy' + c = 0 \ldots\ldots(3).$$

Subtracting (3) from (2), we have

$$gx' + fy' = 0.$$

This relation is to be true for *all* the points (x', y') which lie on the curve (1). But this can only be the case when $g = 0$ and $f = 0$.

352. *To obtain the coordinates of the centre of the conic given by the general equation, and to obtain the equation to the curve referred to axes through the centre parallel to the original axes.*

Transform the origin to the point $(\bar{x}, \bar{y})$, so that for x and y we have to substitute $x + \bar{x}$ and $y + \bar{y}$. The equation then becomes

$$a(x+\bar{x})^2 + 2h(x+\bar{x})(y+\bar{y}) + b(y+\bar{y})^2 + 2g(x+\bar{x}) + 2f(y+\bar{y}) + c = 0,$$

i.e.

$$ax^2 + 2hxy + by^2 + 2x(a\bar{x} + h\bar{y} + g) + 2y(h\bar{x} + b\bar{y} + f) + a\bar{x}^2 + 2h\bar{x}\bar{y} + b\bar{y}^2 + 2g\bar{x} + 2f\bar{y} + c = 0 \quad \ldots\ldots\ldots (2).$$

If the point $(\bar{x}, \bar{y})$ be the centre of the conic section, the coefficients of x and y in the equation (2) must vanish, so that we have

$$a\bar{x} + h\bar{y} + g = 0 \quad \text{......(3)},$$

and

$$h\bar{x} + b\bar{y} + f = 0 \quad \text{......(4)}.$$

Solving (3) and (4), we have, in general,

$$\bar{x} = \frac{fh - bg}{ab - h^2}, \quad \text{and} \quad \bar{y} = \frac{gh - af}{ab - h^2} \quad \text{......(5)}.$$

With these values the constant term in (2)

$$= a\bar{x}^2 + 2h\bar{x}\bar{y} + b\bar{y}^2 + 2g\bar{x} + 2f\bar{y} + c$$

$$= \bar{x}(a\bar{x} + h\bar{y} + g) + \bar{y}(h\bar{x} + b\bar{y} + f) + g\bar{x} + f\bar{y} + c$$

$$= g\bar{x} + f\bar{y} + c \quad \text{......(6)},$$

by equations (3) and (4),

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}, \text{ by equations (5)},$$

$$= \frac{\Delta}{ab - h^2},$$

where Δ is the discriminant of the given general equation (Art. 118).

The equation (2) can therefore be written in the form

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0.$$

This is the required equation referred to the new axes through the centre.

Ex. *Find the centre of the conic section*

$$2x^2 - 5xy - 3y^2 - x - 4y + 6 = 0,$$

and its equation when transformed to the centre.

The centre is given by the equations $2\bar{x} - \frac{5}{2}\bar{y} - \frac{1}{2} = 0$, and $-\frac{5}{2}\bar{x} - 3\bar{y} - 2 = 0$, so that $\bar{x} = -\frac{2}{7}$, and $\bar{y} = -\frac{3}{7}$.

The equation referred to the centre is then

$$2x^2 - 5xy - 3y^2 + c' = 0,$$

where $c' = -\frac{1}{2}.\bar{x} - 2.\bar{y} + 6 = \frac{1}{7} + \frac{6}{7} + 6 = 7.$ (Art. 352.)

The required equation is thus

$$2x^2 - 5xy - 3y^2 + 7 = 0.$$

353. Sometimes the equations (3) and (4) of the last article do not give suitable values for $\bar{x}$ and $\bar{y}$.

For, if $ab - h^2$ be zero, the values of $\bar{x}$ and $\bar{y}$ in (5) are both infinite. When $ab - h^2$ is zero, the conic section is a parabola.

The centre of a parabola is therefore at infinity.

Again, if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$, the result (5) of the last article is of the form $\frac{0}{0}$ and the equations (3) and (4) reduce to the same equation, viz.,

$$a\bar{x} + h\bar{y} + g = 0.$$

We then have only one equation to determine the centre, and there is therefore an infinite number of centres all lying on the straight line

$$ax + hy + g = 0.$$

In this case the conic section consists of a pair of parallel straight lines, both parallel to the line of centres.

354. The student who is acquainted with the Differential Calculus will observe, from equations (3) and (4) of Art. 352, that the coordinates of the centre satisfy the equations that are obtained by differentiating, with regard to x and y, the original equation of the conic section.

It will also be observed that the coefficients of $\bar{x}$, $\bar{y}$, and unity in the equations (3), (4), and (6) of Art. 352 are the quantities (in the order in which they occur) which make up the determinant of Art. 118.

This determinant being easy to write down, the student may thence recollect the equations for the centre and the value of c.

The reason why this relation holds will appear from the next article.

355. Ex. *Find the condition that the general equation of the second degree may represent two straight lines.*

The centre $(\bar{x}, \bar{y})$ of the conic is given by

$$a\bar{x}+h\bar{y}+g=0 \quad \text{.........................} \quad (1),$$

and

$$h\bar{x}+b\bar{y}+f=0 \quad \text{.........................} \quad (2).$$

Also, if it be transformed to the centre as origin, the equation becomes

$$ax^2 + 2hxy + by^2 + c' = 0 \ldots\ldots\ldots\ldots(3),$$

where $c' = g\bar{x} + f\bar{y} + c$.

Now the equation (3) represents two straight lines if c' be zero, *i.e.* if

$$g\bar{x} + f\bar{y} + c = 0 \ldots\ldots\ldots\ldots(4).$$

The equation therefore represents two straight lines if the relations (1), (2), and (4) be simultaneously true.

Eliminating the quantities $\bar{x}$ and $\bar{y}$ from these equations, we have, by Art. 12,

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

This is the condition found in Art. 118.

356. *To find the equation to the asymptotes of the conic section given by the general equation of the second degree.*

Let the equation be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots(1).$$

Since the equation to the asymptotes has been shewn to differ from the equation to the curve only in its constant term, the required equation must be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0 \ldots\ldots(2).$$

Also (2) is to be a pair of straight lines.

Hence

$$ab(c+\lambda) + 2fgh - af^2 - bg^2 - (c+\lambda)h^2 = 0. \quad \text{(Art. 116.)}$$

Therefore $$\lambda = -\frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = -\frac{\Delta}{ab - h^2}.$$

The required equation to the asymptotes is therefore

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2} = 0 \ldots(2).$$

357. *To determine by an examination of the general equation what kind of conic section it represents.*

[On applying the method of Art. 313 to the ellipse and parabola, it would be found that the asymptotes of the ellipse are imaginary, and that a parabola only has one asymptote, which is at an infinite distance and perpendicular to its axis.]

The straight lines $ax^2 + 2hxy + by^2 = 0$ ....(1)

are parallel to the lines (2) of the last article, and hence represent straight lines parallel to the asymptotes.

Now the equation (1) represents real, coincident, or imaginary straight lines according as h^2 is $>=$ or $< ab$, *i.e.* the asymptotes are real, coincident, or imaginary, according as $h^2 >=$ or $< ab$, *i.e.* the conic section is a hyperbola, parabola, or ellipse, according as $h^2 >=$ or $< ab$.

Again, the lines (1) are at right angles, *i.e.* the curve is a rectangular hyperbola, if $a + b = 0$.

Also, by Art. 143, the general equation represents a circle if $a = b$, and $h = 0$.

Finally, by Art. 116, the equation represents a pair of straight lines if $\Delta = 0$; also these straight lines are parallel if the terms of the second degree form a perfect square, *i.e.* if

$$h^2 = ab.$$

358. The results for the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are collected in the following table, the axes of coordinates being rectangular.

Curve.	**Condition.**
Ellipse.	$h^2 < ab$.
Parabola.	$h^2 = ab$.
Hyperbola.	$h^2 > ab$.
Circle.	$a = b$, and $h = 0$.
Rectangular hyperbola.	$a + b = 0$.
Two straight lines, real or imaginary.	$\Delta = 0$, i.e. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
Two parallel straight lines.	$\Delta = 0$, and $h^2 = ab$.

359. *To trace the parabola given by the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots(1),$$

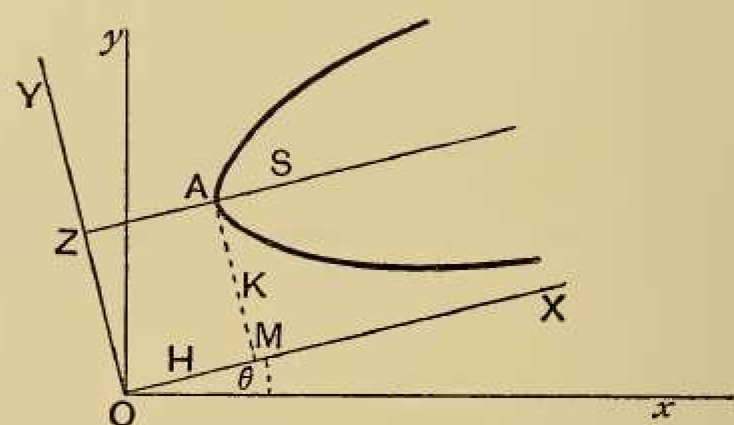
and to find its latus rectum.

First Method. Since the curve is a parabola we have $h^2 = ab$, so that the terms of the second degree form a perfect square.

Put then $a = \alpha^2$ and $b = \beta^2$, so that $h = \alpha\beta$, and the equation (1) becomes

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots(2).$$

Let the direction of the axes be changed so that the straight line $\alpha x + \beta y = 0$, *i.e.* $y = -\frac{\alpha}{\beta}x$, may be the new axis of X.



We have therefore to turn the axes through an angle θ such that $\tan\theta = -\frac{\alpha}{\beta}$, and therefore

$$\sin\theta = -\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \text{ and } \cos\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}.$$

For x we have to substitute

$$X\cos\theta - Y\sin\theta,\ i.e.\ \frac{\beta X + \alpha Y}{\sqrt{\alpha^2+\beta^2}},$$

and for y the quantity

$$X\sin\theta + Y\cos\theta,\ i.e.\ \frac{-\alpha X + \beta Y}{\sqrt{\alpha^2+\beta^2}}. \quad \text{(Art. 129.)}$$

For $\alpha x + \beta y$ we therefore substitute $Y\sqrt{(\alpha^2+\beta^2)}$.

The equation (2) then becomes

$$Y^2(\alpha^2+\beta^2) + \frac{2}{\sqrt{\alpha^2+\beta^2}}[g(\beta X + \alpha Y) + f(\beta Y - \alpha X)] + c = 0,$$

i.e.
$$Y^2 + 2Y\frac{\alpha g + \beta f}{(\alpha^2+\beta^2)^{\frac{3}{2}}} = 2X\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}} - \frac{c}{\alpha^2+\beta^2},$$

i.e.
$$(Y-K)^2 = 2\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}}[X-H] \quad \ldots\ldots\ldots (3),$$

where
$$K = -\frac{\alpha g + \beta f}{(\alpha^2+\beta^2)^{\frac{3}{2}}} \quad \ldots\ldots\ldots (4),$$

and
$$-2\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}} \times H = K^2 - \frac{c}{\alpha^2+\beta^2},$$

i.e.
$$H = \frac{\sqrt{\alpha^2+\beta^2}}{2(\alpha f - \beta g)}\left[c - \frac{(\alpha g + \beta f)^2}{(\alpha^2+\beta^2)^2}\right] \quad \ldots\ldots\ldots (5).$$

The equation (3) represents a parabola whose latus rectum is $2\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}}$, whose axis is parallel to the new axis of X, and whose vertex referred to the new axes is the point (H, K).

360. *Equation of the axis, and coordinates of the vertex, referred to the original axes.*

Since the axis of the curve is parallel to the new axis of X, it makes an angle θ with the old axis of x, and hence the perpendicular on it from the origin makes an angle $90° + \theta$.

Also the length of this perpendicular is K.

The equation to the axis of the parabola is therefore

$$x \cos(90° + \theta) + y \sin(90° + \theta) = K,$$

i.e.

$$-x \sin\theta + y \cos\theta = K,$$

i.e.

$$\alpha x + \beta y = K\sqrt{\alpha^2 + \beta^2} = -\frac{\alpha g + \beta f}{\alpha^2 + \beta^2} \quad \text{.........(6)}.$$

Again, the vertex is the point in which the axis (6) meets the curve (2).

We have therefore to solve (6) and (2), *i.e.* (6) and

$$\frac{(\alpha g + \beta f)^2}{(\alpha^2 + \beta^2)^2} + 2gx + 2fy + c = 0 \quad \text{............(7)}.$$

The solution of (6) and (7) therefore gives the required coordinates of the vertex.

363. Ex. *Trace the parabola*

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0.$$

The equation is

$$(3x - 4y)^2 - 18x - 101y + 19 = 0 \quad \text{................(1)}.$$

First Method. Take $3x-4y=0$ as the new axis of x, *i.e.* turn the axes through an angle θ, where $\tan\theta=\frac{3}{4}$, and therefore $\sin\theta=\frac{3}{5}$ and $\cos\theta=\frac{4}{5}$.

For x we therefore substitute $X\cos\theta-Y\sin\theta$, *i.e.* $\dfrac{4X-3Y}{5}$; for y we put $X\sin\theta+Y\cos\theta$, *i.e.* $\dfrac{3X+4Y}{5}$, and hence for $3x-4y$ the quantity $-5Y$.

The equation (1) therefore becomes

$$25Y^2-\tfrac{1}{5}[72X-54Y]-\tfrac{1}{5}[303X+404Y]+19=0,$$

i.e.
$$25Y^2-75X-70Y+19=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

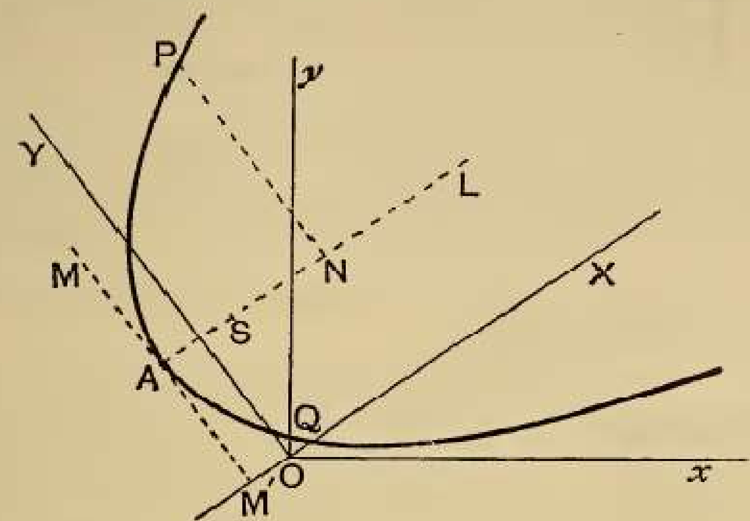
This is the equation to the curve referred to the axes OX and OY.

But (2) can be written in the form

$$Y^2-\frac{14Y}{5}=3X-\tfrac{19}{25},$$

i.e.
$$(Y-\tfrac{7}{5})^2=3X-\tfrac{19}{25}+\tfrac{49}{25}=3\,(X+\tfrac{2}{5}).$$

Take a point A whose coordinates referred to OX and OY are $-\frac{2}{5}$ and $\frac{7}{5}$, and draw AL and AM parallel to OX and OY respectively.



Referred to AL and AM the equation to the parabola is $Y^2 = 3X$. It is therefore a parabola, whose vertex is A, whose latus rectum is 3, and whose axis is AL.

364. *To find the direction and magnitude of the axes of the central conic section*

$$ax^2 + 2hxy + by^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

First Method. We know that, when the equation to a central conic section has no term containing xy and the axes are rectangular, the axes of coordinates are the axes of the curve.

Now in Art. 349 we shewed that, to get rid of the term involving xy, we must turn the axes through an angle θ given by

$$\tan 2\theta = \frac{2h}{a-b} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The axes of the curve are therefore inclined to the axes of coordinates at an angle θ given by (2).

Now (2) can be written

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a-b} = \frac{1}{\lambda} \text{ (say)},$$

$$\therefore \tan^2\theta + 2\lambda\tan\theta - 1 = 0 \quad \ldots\ldots\ldots\ldots(3).$$

This, being a quadratic equation, gives two values for θ, which differ by a right angle, since the product of the two values of $\tan\theta$ is -1. Let these values be θ_1 and θ_2, which are therefore the inclinations of the required axes of the curve to the axis of x.

Again, in polar coordinates, equation (1) may be written

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) = 1 = \cos^2\theta + \sin^2\theta,$$

i.e.

$$r^2 = \frac{\cos^2\theta + \sin^2\theta}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta} = \frac{1+\tan^2\theta}{a + 2h\tan\theta + b\tan^2\theta} \quad \ldots\ldots\ldots\ldots(4).$$

365. Ex. 1. *Trace the curve*

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad\ldots\ldots\ldots\ldots(1).$$

Since $(-2)^2 - 14 . 11$ is negative, the curve is an ellipse. [Art. 35

By Art. 352 the centre $(\bar{x}, \bar{y})$ of the curve is given by the equatio

$$14\bar{x} - 2\bar{y} - 22 = 0, \text{ and } -2\bar{x} + 11\bar{y} - 29 = 0.$$

Hence $\bar{x} = 2$, and $\bar{y} = 3$.

The equation referred to parallel axes through the centre is

therefore $14x^2 - 4xy + 11y^2 + c' = 0,$

where $c' = -22\bar{x} - 29\bar{y} + 71 = -60,$

so that the equation is

$$14x^2 - 4xy + 11y^2 = 60 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The directions of the axes are given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{14-11} = -\tfrac{4}{3},$$

so that $\dfrac{2\tan\theta}{1-\tan^2\theta} = -\tfrac{4}{3},$

and hence $2\tan^2\theta - 3\tan\theta - 2 = 0.$

Therefore $\tan\theta_1 = 2$, and $\tan\theta_2 = -\tfrac{1}{2}$.

Referred to polar coordinates the equation (2) is

$$r^2(14\cos^2\theta - 4\cos\theta\sin\theta + 11\sin^2\theta) = 60(\cos^2\theta + \sin^2\theta),$$

i.e. $r^2 = 60\,\dfrac{1+\tan^2\theta}{14 - 4\tan\theta + 11\tan^2\theta}.$

When $\tan\theta_1 = 2$, $r_1{}^2 = 60 \times \dfrac{1+4}{14-8+44} = 6.$

When $\tan\theta_2 = -\tfrac{1}{2}$, $r_2{}^2 = 60 \times \dfrac{1+\frac{1}{4}}{14+2+\frac{11}{4}} = 4.$

Ex. 2. *Trace the curve*

$$x^2-3xy+y^2+10x-10y+21=0 \ldots\ldots\ldots\ldots(1).$$

Since $\left(\frac{-3}{2}\right)^2-1.1$ is positive, the curve is a hyperbola.

[Art. 358.]

The centre $(\bar{x}, \bar{y})$ is given by

$$\bar{x}-\frac{3}{2}\bar{y}+5=0,$$

and

$$\frac{-3}{2}\bar{x}+\bar{y}-5=0,$$

so that $\bar{x}=-2$, and $\bar{y}=2$.

The equation to the curve, referred to parallel axes through the centre, is then

$$x^2-3xy+y^2+5(-2)-5\times 2+21=0,$$

i.e.

$$x^2-3xy+y^2=-1 \ldots\ldots\ldots\ldots(2).$$

The direction of the axes is given by

$$\tan 2\theta=\frac{2h}{a-b}=\frac{-3}{1-1}=\infty,$$

so that $2\theta=90^\circ$ or 270°,

and hence $\theta_1=45^\circ$ and $\theta_2=135^\circ$.

The equation (2) in polar coordinates is

$$r^2(\cos^2\theta-3\cos\theta\sin\theta+\sin^2\theta)=-(\sin^2\theta+\cos^2\theta),$$

i.e.

$$r^2=-\frac{1+\tan^2\theta}{1-3\tan\theta+\tan^2\theta}.$$

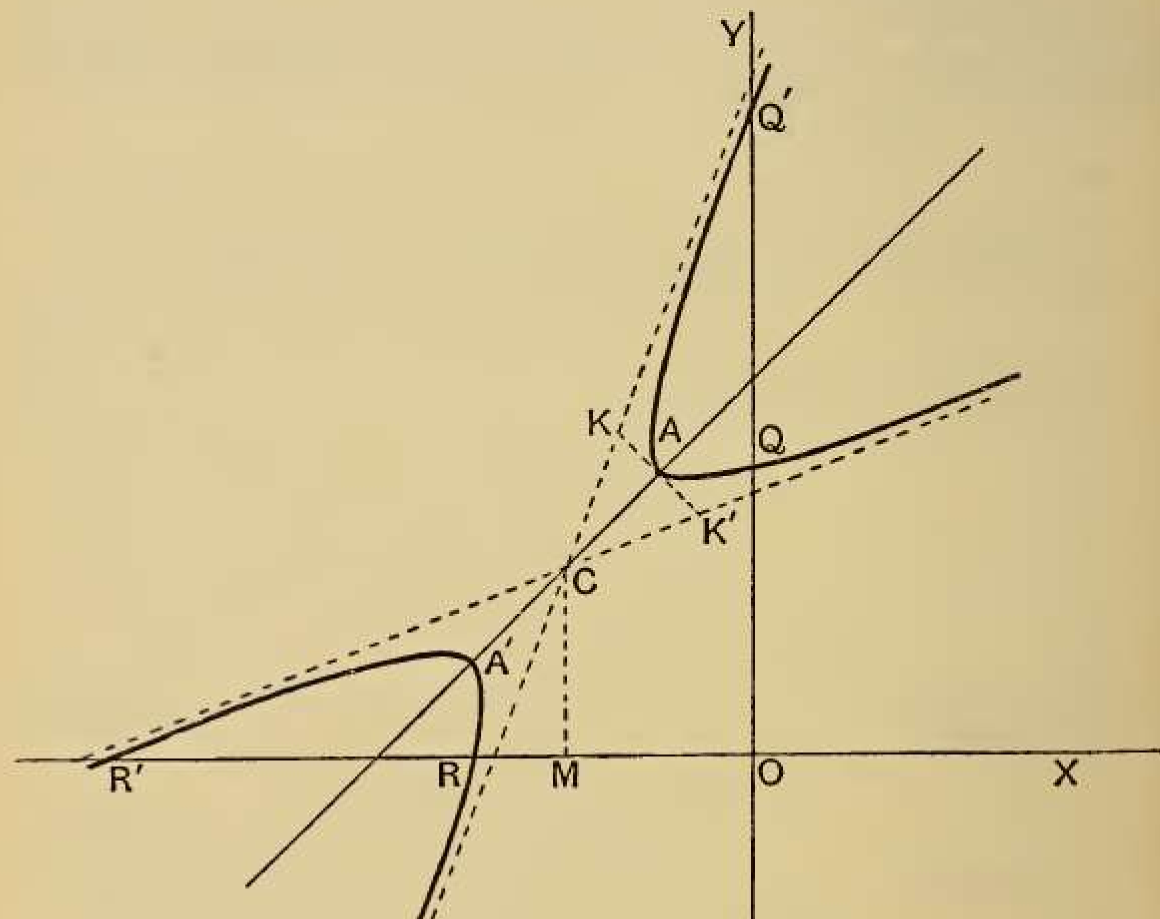
When $\theta_1 = 45°$, $r_1^2 = -\dfrac{2}{1-3+1} = 2$, so that $r_1 = \sqrt{2}$.

When $\theta_2 = 135°$, $r_2^2 = -\dfrac{2}{1+3+1} = \dfrac{-2}{5}$, so that $r_2 = \sqrt{\dfrac{-2}{5}}$.

To construct the curve take the point C whose coordinates are -2 and 2. Through C draw a straight line ACA' inclined at 45° to the axis of x and mark off $A'C = CA = \sqrt{2}$.

Also through A draw a straight line KAK' perpendicular to CA and take $AK = K'A = \sqrt{\frac{2}{5}}$. By Art. 315, CK and CK' are then the asymptotes.

The curve is therefore a hyperbola whose centre is C, whose transverse axis is $A'A$, and whose asymptotes are CK and CK'.



On putting $x=0$ it will be found that the curve meets the axis of y where $y=3$ or 7, and, on putting $y=0$, that it meets the axis of x where $x=-3$ or -7.

Hence $OQ=3$, $OQ'=7$, $OR=3$, and $OR'=7$.

366. *To find the eccentricity of the central conic section*

$$ax^2+2hxy+by^2=1 \text{..................(1).}$$

First, let h^2-ab be negative, so that the curve is an ellipse, and let the equation to the ellipse, referred to its axes, be

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1.$$

By the theory of Invariants (Art. 135) we have

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=a+b \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and

$$\frac{1}{\alpha^2\beta^2}=ab-h^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also, if e be the eccentricity, we have, if α be $>\beta$,

$$e^2=\frac{\alpha^2-\beta^2}{\alpha^2}.$$

$$\therefore\ \frac{e^2}{2-e^2}=\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}.$$

But, from (2) and (3), we have

$$\alpha^2+\beta^2=\frac{a+b}{ab-h^2} \text{ and } \alpha^2\beta^2=\frac{1}{ab-h^2}.$$

Hence

$$\alpha^2-\beta^2=+\sqrt{(\alpha^2+\beta^2)^2-4\alpha^2\beta^2}=+\frac{\sqrt{(a-b)^2+4h^2}}{ab-h^2}.$$

$$\therefore\ \frac{e^2}{2-e^2}=+\frac{\sqrt{(a-b)^2+4h^2}}{a+b}\quad\ldots\ldots\ldots\ldots(4).$$

This equation at once gives e^2.

Secondly, let $h^2 - ab$ be positive, so that the curve is a hyperbola, and let the equation referred to its principal axes be

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1,$$

so that in this case

$$\frac{1}{\alpha^2} - \frac{1}{\beta^2} = a + b, \text{ and } -\frac{1}{\alpha^2\beta^2} = ab - h^2 = -(h^2 - ab).$$

Hence $\alpha^2 - \beta^2 = -\dfrac{a+b}{h^2 - ab}$ and $\alpha^2\beta^2 = \dfrac{1}{h^2 - ab}$,

so that $\alpha^2 + \beta^2 = +\sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2} = +\dfrac{\sqrt{(a-b)^2 + 4h^2}}{h^2 - ab}$.

In this case, if e be the eccentricity, we have

$$e^2 = \frac{\alpha^2 + \beta^2}{\alpha^2},$$

i.e. $$\frac{e^2}{2 - e^2} = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} = -\frac{\sqrt{(a-b)^2 + 4h^2}}{a+b} \quad \ldots\ldots\ldots(5).$$

This equation gives e^2.

In each case we see that e is a root of the equation

$$\left(\frac{e^2}{2 - e^2}\right)^2 = \frac{(a-b)^2 + 4h^2}{(a+b)^2},$$

i.e. of the equation

$$e^4(ab - h^2) + \{(a-b)^2 + 4h^2\}(e^2 - 1) = 0.$$

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Analytical Geometry **Subject Code: 18MMU303A**
Class : II - B.Sc. Mathematics **Semester : III**

Unit V

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Possible Questions

Questions	Opt1	Opt2	Opt3	Opt4	Answer
The general equation ax2 +by2 + 2fy + 2gx +c= 0, the axis of the coordinates are being rectangular the curve is ellipse………………	h2<ab	h2>ab	h2=ab	h2=-ab	h2<ab
The center of a conic section is a point such that all chords of the conic which pass through it are……….………	bisected	perpendicular	orthogonal	equal	bisected
When ab-h^2 is zero, the conic section is a parabola. The centre of a parabola is therefore at …………	infinity	normal	point	tangent	infinity
The lines ax^2+2hxy+by^2=0 are at right angles, that is the curve is a rectangular hyperbola………..	a/b=0	ab=0	a+b=0	a-b=0	a+b=0
The lines ax^2+2hxy+by^2=0 are at right angles, that is the curve is a rectangular circle………..	a/b=0	ab=0	a+b=0	a-b=0	a-b=0
When the equation to the conic is given in the form ax2 +by2 + 2fy + 2gx +c= 0 the origin is the center only when both……………	f and g are not zero	f and g are zero	f and g are one	f and g greater than ze	f and g are zero
The general equation ax2 +by2 + 2fy + 2gx +c= 0, the axis of the coordinates are being rectangular the curve is two straight lines real or imaginary………………	delta=0	delta>0	delta<0	delta not equal to zero	delta=0
The general equation ax2 +by2 + 2fy + 2gx +c= 0, the axis of the coordinates are being rectangular the curve is circle………………	a=b,and h<0	a=b,and h>0	a=b,and h=0	a<b,and h=1	a=b,and h=0
The distance of any point on the right circular cylinder from its axis is equal to the radius of the -------------.	⟦90⟧ origin	guiding vertex	guiding curve	⟦45⟧ guiding circle	guiding circle
Two hyperbolas with the same eccentricity are said to be ……..	similar ⟦90⟧	different	zero	one ⟦45⟧	similar
A conic section is the curve described by a point which moves in a plane in such a manner that it's distance from a fixed point in the plane (a focus) is in a constant ratio to it's distance from a fixed line (a directrix) in the plane. This ratio is known as the....................	orgin	eccentricity	centre	radius	eccentricity

The angle between the line of eccentricity and the axis will always be…………………… for an ellipse	greater than 45°	equal to 45°	less than 45°	none of these	less than 45°
The angle between the line of eccentricity and the axis will always be…………………… for a hyperbola	xcosα+ysinα=p greater than 45°	equal to 45°	less than 45°	none of these	greater than 45°
The angle between the line of eccentricity and the axis will always be…………………… for a parabola	greater than 45°	equal to 45°	less than 45°	none of these	equal to 45°
For every point on the curve the distance to the focal point over the distance to the directrix is in a ratio of ……………	0.8	1.333333333	1.5	0.5	1.333333333
The shortest distance of the vertex from any ordinate of the parabola, is known as the…………………	double ordinate	latus rectum	abscissa	vertex	abscissa
When a conic touches a second conic at each of two points, the two conies are said to have ………………..with one another.	double contact	single contact	multiple contact	zero contact	double contact
All conies tvhich pass through the intersections of two rectangular hyperbolas are themselves………………..	hyperbola	rectangular hyperbola	parabola	ellipes	rectangular hyperbolas
The general equation to a conic is…………..	pi(x,y)=0	pi(x,y)>0	pi(x,y)<0	pi(x,y) not equal to ze	pi(x,y)=0
conic sections which are given by the general equationof the ……………..	first degree	second degree	third degree	zero degree	second degree
If a rectangular hyperbola circumscribe a triangle, it also passes through the …………….. of the triangle.	semi center	center	orthocentre	not center	orthocentre
If a circle and the rectangular hyperbola xy =c^2 meet in the four points t1,t2,t3 and t4 then……………	t1t2t3t4=0	t1t2t3t4=1	t1t2t3t4=-1	t1t2t3t4>0	t1t2t3t4=1
The equation to any hyperbola whose asymptotes are x = 0 and y=0 is……………constant.	x+y	x-y	xy	x^y	xy