Semester – V

17MMU303	MULTIVARIATE CALCULUS	L	T	P	C
		4	2	0	6

Scope: On successful completion of course the learners gain about the functions of several variables, limit and continuity functions of two variables.

Objectives: To enable the students to learn and gain knowledge about line integrals and its geometrical applications.

UNIT I

Functions of several variables: Limit and continuity of functions of two variables, partial differentiation, total differentiability and differentiability, sufficient condition for differentiability. Chain rule for one and two independent parameters, directional derivatives, the gradient, maximal and normal property of the gradient, tangent planes.

UNIT II

Extrema of functions of two variables: Method of Lagrange multipliers, constrained optimization problems, Definition of vector field, divergence and curl.

UNIT III

Double integration over rectangular region: Double integration over non-rectangular region, double integrals in polar co-ordinates, Triple integrals, Triple integral over a parallelepiped and solid regions. Volume by triple integrals, cylindrical and spherical co-ordinates. Change of variables in double integrals and triple integrals

UNIT IV

Line integrals: Applications of line integrals, Mass and Work. Fundamental theorem for line integrals, conservative vector fields, independence of path.

UNITV

Green's theorem: Surface integrals, integrals over parametrically defined surfaces. Stoke's theorem, The Divergence theorem.

SUGGESTED READINGS

TEXT BOOK

1. Strauss M.J., Bradley G.L. and Smith K. J., (2007). Calculus, Third Edition, Dorling Kindersley (India) Pvt.Ltd. (Pearson Education), Delhi.

REFERENCES

1. Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.

2. Marsden E., Tromba A.J. and Weinstein A., (2005). Basic Multivariable Calculus, Springer (SIE), Indian reprint, New Delhi.

3. James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks Cole, Thomson Learning, USA.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Subject Name: Multivariate Calculus Subject Code: 17MMU503A

UNIT - I			
S.NO	**DURATION HOURS**	**TOPICS TO BE COVERED**	**SUPPORT MATERIAL**
1	1	Limit and continuity of functions of two variables	T1: chap-1 Pg.No:75-80
2	1	Partial differentiation, total differentiability	R3: chap-15 Pg.No:914-918
3	1	Tutorial	
4	1	sufficient condition for differentiability	R3: chap-15 Pg.No:921-928
5	1	Chain rule for one and two independent parameters	R3: chap-15 Pg.No:937-940
6	1	Tutorial	
7	1	Gradient and directional derivatives-Problems	R3: chap-15 Pg.No:946-960
8	1	Tutorial	
9	1	maximal and normal property of the gradient	R3: chap-15 Pg.No:960-963
10	1	Tutorial	
11	1	Problems about tangent planes.	R3: chap-15 Pg.No:963-966
12	1	Tutorial	
13	1	Recapitulation and Discussion of possible questions	
Total	**13 Hours**		
UNIT-II			
1	1	Introduction to Extrema of functions of two variables	R3: chap-15 Pg.No:970-971
2	1	Method of Lagrange multipliers	R3: chap-15 Pg.No:971-975
3	1	Tutorial	
4	1	Problems of Method of Lagrange multipliers	R3: chap-15 Pg.No:975-976
5	1	Continuation of problems on Method of Lagrange multipliers	R3: chap-15 Pg.No:977-979
6	1	Tutorial	

7	1	Constrained optimization problems	R3: chap-17 Pg.No:1063-1067
8	1	Definition, Examples and Problems over vector field	R3: chap-17 Pg.No:1063-1075
9	1	Tutorial	
10	1	Problems of divergence	R1: chap-10 Pg.No:806-808
11	1	Tutorial	
12	1	Problems of Curl	R1: chap-10 Pg.No:815-818
13	1	Tutorial	
14	1	Recapitulation and Discussion of possible questions	
Total	**14 Hours**		
UNIT-III			
1	1	Introduction to Double integration over rectangular region	R3: chap-16 Pg.No:987-990
2	1	Double integration over non-rectangular region	R3: chap-16 Pg.No:990-994
3	1	Tutorial	
4	1	Double integrals in polar co-ordinates	R3: chap-16 Pg.No:1010-1015
5	1	Triple integrals-Problems	R3: chap-16 Pg.No:1026-1030
6	1	Tutorial	
8	1	Triple integral over a parallelepiped and solid regions	R3: chap-16 Pg.No:1030-1032
9	1	Volume by triple integrals	R3: chap-16 Pg.No:1032-1035
10	1	Tutorial	
11	1	Cylindrical and spherical co-ordinates	R3: chap-16 Pg.No:1041-1048
12	1	Change of variables in double and triple integrals	R3: chap-16 Pg.No:1041-1048
13	1	Tutorial	
14	1	Problems based on Change of variables in double integrals	R3: chap-16 Pg.No:1048-1055
15	1	Continuous of problems based on Change of variables in double integrals	R3: chap-16 Pg.No:1050-1055
16	1	Tutorial	
17	1	Recapitulation and Discussion of possible questions	
Total	**17 Hours**		
UNIT-IV			
1	1	Introduction to Line integrals	R3: chap-17 Pg.No:1081-

			1083
2	1	Applications and Problems on line integrals	R3: chap-17 Pg.No:1070-1080
3	1	Tutorial	
4	1	Concept about Mass and Work	R1:Chap-13 Pg.No:1034-1035
5	1	Problems on Mass and Work	R1:Chap- 13 Pg.No:1035-1037
6	1	Tutorial	
7	1	Fundamental theorem for line integrals	R3: chap-17 Pg.No:1083-1090
8	1	Problems on Fundamental theorem for line integrals	R3: chap-17 Pg.No:1083-1090
9	1	Tutorial	
10	1	Problems on Conservative vector fields and independence of path	R2: chap-10 Pg.No:1061-1090
11	1	Tutorial	
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		
UNIT-V			
1	1	Introduction of Green's theorem	R3: chap-17 Pg.No:1091-1097
2	1	Surface integrals-Problems	R3: chap-17 Pg.No:1117-1128
3	1	Tutorial	
4	1	Integrals over parametrically defined surfaces	R1: chap-13 Pg.No:1045-1050
5	1	Problems on integrals over parametrically defined surfaces	R1: chap-13 Pg.No:1051-1056
6	1	Tutorial	
7	1	Stoke's theorem- Problems	R3: chap-17 Pg.No:1128-1133
8	1	Tutorial	
9	1	Divergence theorem	R3: chap-17 Pg.No:1135-1136
10	1	Tutorial	
11	1	Problems on Divergence theorem	R3: chap-17 Pg.No:1136-1138
12	1	Tutorial	
13	1	Recapitulation and Discussion of possible questions	
14	1	Discussion on Previous ESE Question Papers	
15	1	Discussion on Previous ESE Question Papers	
16	1	Discussion on Previous ESE Question Papers	

Total	16 Hours		
S. No	**Lecture Duration Hour**	**Topics To Be Covered**	**Support Materials**
		UNIT-I	
S.NO	**DURATION HOURS**	**TOPICS TO BE COVERED**	**SUPPORT MATERIAL**
1	1	Limit and continuity of functions of two variables	T1: chap-1 Pg.No:75-80
2	1	Partial differentiation, total differentiability	R3: chap-15 Pg.No:914-918
3	1	Tutorial	
4	1	sufficient condition for differentiability	R3: chap-15 Pg.No:921-925
5	1	Continuous on sufficient condition for differentiability-Problems	R3: chap-15 Pg.No:925-928
6	1	Tutorial	
7	1	Chain rule for one and two independent parameters	R3: chap-15 Pg.No:937-940
8	1	directional derivatives-Problems	R3: chap-15 Pg.No:946-958
9	1	Problems for the gradient	R3: chap-15 Pg.No:958-960
10	1	Tutorial	
11	1	maximal and normal property of the gradient	R3: chap-15 Pg.No:960-961
12	1	Continuous on maximal and normal property of the gradient	R3: chap-15 Pg.No:962-962
13	1	Continuous on maximal and normal property of the gradient	R3: chap-15 Pg.No:962-963
14	1	Tutorial	
15	1	Problems about tangent planes.	R3: chap-15 Pg.No:963-966
16	1	Continuous the problems on tangent planes.	R3: chap-15 Pg.No:963-966
17	1	Tutorial	
18	1	Recapitulation and Discussion of possible questions	
Total	**18 Hours**		

Text Book:
T1: Strauss M.J., Bradley G.L. and Smith K. J., (2007). Calculus, Third Edition, Dorling Kindersley (India) Pvt.Ltd. (Pearson Education), Delhi.
Reference Book:
R3: James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks

Cole, Thomson Learning, USA.			
UNIT-II			
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4	1	Tutorial	
5	1	Continuous of problems on Method of Lagrange multipliers	R3: chap-15 Pg.No:975-979
6	1	Constrained optimization problems	R3: chap-17 Pg.No:1063
7	1	Continuous of problems on Constrained optimization	R3: chap-17 Pg.No:1064-1067
8	1	Tutorial	
9	1	Definition of vector field	R3: chap-17 Pg.No:1063-1070
10	1	Examples and Problems over vector field	R3: chap-17 Pg.No:1070-1075
11	1	Continuous of problems over vector field	R3: chap-17 Pg.No:1070-1075
12	1	Tutorial	
13	1	Problems of divergence	R1: chap-10 Pg.No:806-808
14	1	Continuous problem for divergence	R1: chap-10 Pg.No:806-808
15	1	Tutorial	
16	1	Problems of Curl	R1: chap-10 Pg.No:815-818
17	1	Continuous problem for Curl	R1: chap-10 Pg.No:815-818
18	1	Tutorial	
19	1	Recapitulation and Discussion of possible questions	
Total	**19 Hours**		
Reference Book: **R1:** Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi. **R3:** James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks Cole, Thomson Learning, USA.			
UNIT-III			
1	1	Introduction to Double integration over rectangular region	R3: chap-16 Pg.No:987-990
2	1	Double integration over non-rectangular region	R3: chap-16 Pg.No:990-994
3	1	Continuous on Double integration over non-	R3: chap-16 Pg.No:990-

		rectangular region	994
4	1	Tutorial	
5	1	Double integrals in polar co-ordinates	R3: chap-16 Pg.No:1010-1015
6	1	Triple integrals-Problems	R3: chap-16 Pg.No:1026-1030
8	1	Tutorial	
9	1	Triple integral over a parallelepiped and solid regions	R3: chap-16 Pg.No:1030-1032
10	1	Volume by triple integrals	R3: chap-16 Pg.No:1032-1035
11	1	Cylindrical and spherical co-ordinates	R3: chap-16 Pg.No:1041-1048
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15	1	Tutorial	
16	1	Continuous of problems based on Change of variables in double integrals	R3: chap-16 Pg.No:1050-1055
17	1	Continuous of problems based on Change of variables in double integrals	R3: chap-16 Pg.No:1050-1055
18	1	Tutorial	
19	1	Recapitulation and Discussion of possible questions	
Total	**19 Hours**		

Reference Book:

R3: James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, BrooksCole,Thomson Learning, USA.

UNIT-IV

1	1	Introduction to Line integrals	R3: chap-17 Pg.No:1081-1083
2	1	Applications of line integrals	R3: chap-17 Pg.No:1070-1080
3	1	Problems on Line integral	R3: chap-17 Pg.No:1070-1080
4	1	Tutorial	
5	1	Concept about Mass and Work	R1:Chap-13 Pg.No:1034-1035
6	1	Problems on Mass and Work	R1:Chap- Pg.No:1035-1037
7	1	Fundamental theorem for line integrals	R3: chap-17 Pg.No:1083-1090

8	1	Tutorial	
9	1	Problems on Fundamental theorem for line integrals	R3: chap-17 Pg.No:1083-1090
10	1	Continuous of problems on Fundamental theorem	R3: chap-17 Pg.No:1083-1090
11	1	Conservative vector fields	R2: chap-10 Pg.No:1061-1090
12	1	Tutorial	
13	1	Problems about Conservative vector fields	
14	1	Problems on independence of path	
15	1	Tutorial	
16	1	Recapitulation and Discussion of possible questions	
Total	**16 Hours**		

Reference Book:

R1: Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.

R2: Marsden E., Tromba A.J. and Weinstein A., (2005). Basic Multivariable Calculus, Springer (SIE),Indian reprint, New Delhi.

R3: James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks Cole, Thomson Learning, USA.

UNIT-V

1	1	Introduction of Green's theorem	R3: chap-17 Pg.No:1091-1097
2	1	Surface integrals	R3: chap-17 Pg.No:1117-1120
3	1	Problems on Surface integrals	R3: chap-17 Pg.No:1120-1128
4	1	Tutorial	
5	1	Integrals over parametrically defined surfaces	R1: chap- Pg.No:
6	1	Problems on integrals over parametrically defined surfaces	R1: chap- Pg.No:
7	1	Tutorial	
8	1	Stoke's theorem	R3: chap-17 Pg.No:1128-1129
9	1	Problems on Stoke's theorem	R3: chap-17 Pg.No:1129-1133
10	1	Tutorial	
11	1	Divergence theorem	R3: chap-17 Pg.No:1135-1136
12	1	Problems on Divergence theorem	R3: chap-17 Pg.No:1136-1138
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15	1	Recapitulation and Discussion of possible	

		questions	
16	1	Discussion on Previous ESE Question Papers	
17	1	Discussion on Previous ESE Question Papers	
18	1	Discussion on Previous ESE Question Papers	
Total	**18 Hours**		

Reference Book:
R1: Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.
R3: Kenneth Hoffman., Ray Kunze., (2003). Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.

Total no. of Hours for the Course: 90 hours

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DEPARTMENT OF MATHEMATICS

Subject: Multivariate Calculus	**Semester: V**	**L T P C**
Subject Code: 17MMU503A	**Class: III-B.Sc Mathematics**	**4 2 0 6**

UNIT I

Functions of several variables: Limit and continuity of functions of two variables, partial differentiation, total differentiability and differentiability, sufficient condition for differentiability. Chain rule for one and two independent parameters, directional derivatives, the gradient, maximal and normal property of the gradient, tangent planes.

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Reference Book:

R3: James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks Cole,Thomson Learning, USA.

Functions of several variables

Limit and continuity of several variables:

Limit:

DEFINITION A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

EXAMPLE 4 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

which is the disk with center (0, 0) and radius 3. (See Figure 4.) The range of g is

$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

Since z is a positive square root, $z \geq 0$. Also

$$9 - x^2 - y^2 \leq 9 \quad \Rightarrow \quad \sqrt{9 - x^2 - y^2} \leq 3$$

So the range is

$$\{z \mid 0 \leq z \leq 3\} = [0, 3]$$

□

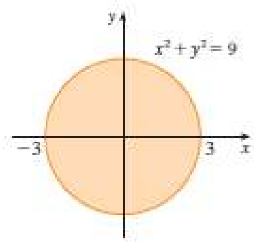
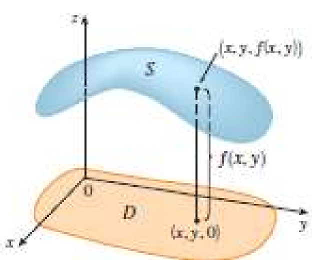


FIGURE 4
Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$



DEFINITION If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane. (See Figure 5.)

LEVEL CURVES

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

DEFINITION The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

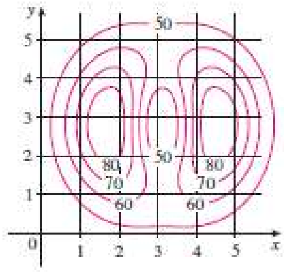


FIGURE 14

EXAMPLE 9 A contour map for a function f is shown in Figure 14. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

SOLUTION The point (1, 3) lies partway between the level curves with z-values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that

$$f(4, 5) \approx 56$$

□

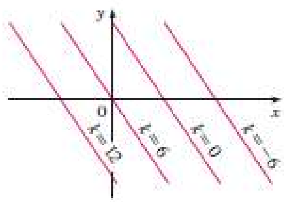


FIGURE 15
Contour map of
$f(x, y) = 6 - 3x - 2y$

EXAMPLE 10 Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

SOLUTION The level curves are

$$6 - 3x - 2y = k \qquad \text{or} \qquad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$, and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 15. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6). □

EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \qquad \text{or} \qquad x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center (0, 0) and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown in Figure 16. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7. (See TEC Visual 15.1A.)

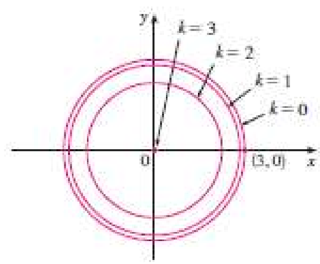


FIGURE 16
Contour map of $g(x, y) = \sqrt{9 - x^2 - y^2}$

□

EXAMPLE 12 Sketch some level curves of the function $h(x, y) = 4x^2 + y^2$.

SOLUTION The level curves are

$$4x^2 + y^2 = k \qquad \text{or} \qquad \frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

which, for $k > 0$, describes a family of ellipses with semiaxes $\sqrt{k}/2$ and $\sqrt{k}$. Figure 17(a) shows a contour map of h drawn by a computer with level curves corresponding to $k = 0.25, 0.5, 0.75, \ldots, 4$. Figure 17(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure 17 how the graph of h is put together from the level curves.

TEC Visual 15.1B demonstrates the connection between surfaces and their contour maps.

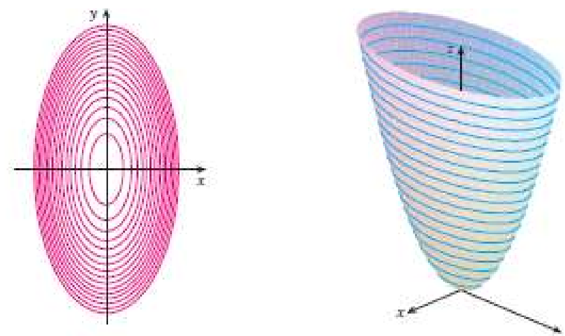


FIGURE 17
The graph of $h(x, y) = 4x^2 + y^2$ is formed by lifting the level curves.

(a) Contour map (b) Horizontal traces are raised level curves □

FUNCTIONS OF THREE OR MORE VARIABLES

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

EXAMPLE 14 Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

SOLUTION The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$. □

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

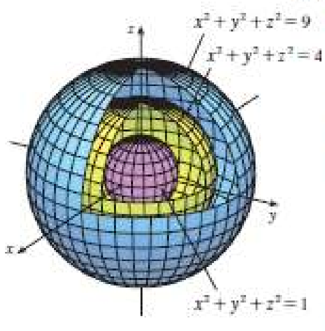


FIGURE 20

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure 20.) Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed. □

Functions of any number of variables can be considered. A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$\boxed{3} \qquad C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

LIMITS AND CONTINUITY

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \qquad \text{and} \qquad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

TABLE 1 Values of $f(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

TABLE 2 Values of $g(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.) It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y) \to (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \qquad \text{and} \qquad \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

In general, we use the notation

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f. In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

1 DEFINITION Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

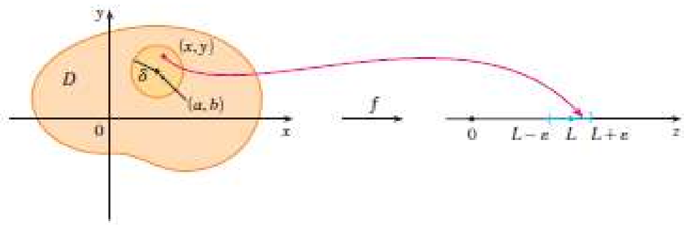
if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

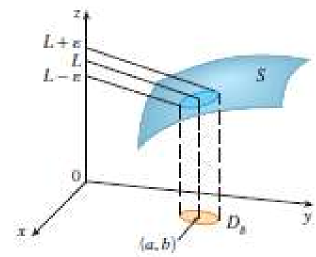
$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.





Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right. We recall from Chapter 2 that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.

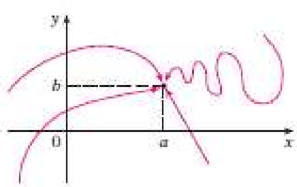


FIGURE 3

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

EXAMPLE 1 Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

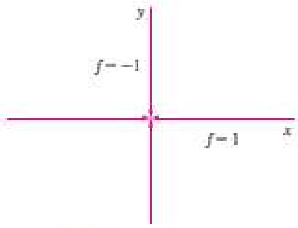


FIGURE 4

(See Figure 4.) Since f has two different limits along two different lines, the given limit does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.) □

EXAMPLE 2 If $f(x, y) = \frac{xy}{x^2 + y^2}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\quad f(x, y) \to \frac{1}{2} \quad$ as $\quad (x, y) \to (0, 0)$ along $y = x$

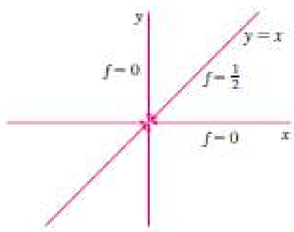


FIGURE 5

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. □

Figure 6 sheds some light on Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

TEC In Visual 15.2 a rotating line on the surface in Figure 6 shows different limits at the origin from different directions.

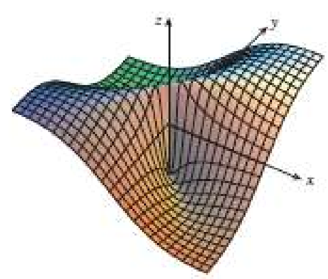


FIGURE 6 $f(x, y) = \frac{xy}{x^2 + y^2}$

EXAMPLE 3 If $f(x, y) = \frac{xy^2}{x^2 + y^4}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any nonvertical line through the origin. Then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

■ Figure 7 shows the graph of the function in Example 3. Notice the ridge above the parabola $x = y^2$.

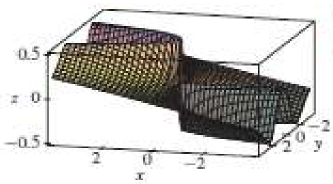


FIGURE 7

So $$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } y = mx$$

Thus f has the same limiting value along every nonvertical line through the origin. But that does not show that the given limit is 0, for if we now let $(x, y) \to (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $$f(x, y) \to \tfrac{1}{2} \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } x = y^2$$

Since different paths lead to different limiting values, the given limit does not exist. □

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 2.3 can be extended to functions of two variables: The limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

2 $$\lim_{(x, y) \to (a, b)} x = a \qquad \lim_{(x, y) \to (a, b)} y = b \qquad \lim_{(x, y) \to (a, b)} c = c$$

The Squeeze Theorem also holds.

EXAMPLE 4 Find $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

SOLUTION As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

that is, $$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \frac{3x^2|y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2 + y^2) \leqslant 1$ and therefore

3 $$\frac{3x^2|y|}{x^2 + y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leqslant 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

■ Another way to do Example 4 is to use the Squeeze Theorem instead of Definition 1. From (2) it follows that

$$\lim_{(x, y) \to (0, 0)} 3|y| = 0$$

and so the first inequality in (3) shows that the given limit is 0.

Hence, by Definition 1,

$$\lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0$$ □

CONTINUITY

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x\to a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

4 DEFINITION A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains. Let's use this fact to give examples of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in (2) show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that *all polynomials are continuous on* $\mathbb{R}^2$. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

EXAMPLE 5 Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$ □

EXAMPLE 6 Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. □

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist (see Example 1). □

Figure 8 shows the graph of the continuous function in Example 8.

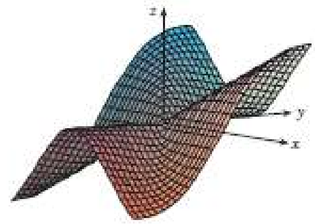


FIGURE 8

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on $\mathbb{R}^2$. □

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

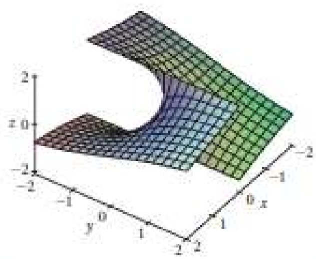


FIGURE 9
The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

EXAMPLE 9 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where $x = 0$. The graph in Figure 9 shows the break in the graph of h above the y-axis. □

FUNCTIONS OF THREE OR MORE VARIABLES

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f. Because the distance between two points (x, y, z) and (a, b, c) in $\mathbb{R}^3$ is given by $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, we can write the precise definition as follows: For every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y, z) \text{ is in the domain of } f \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$$

$$\text{then } |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 15.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

> **5** If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that
>
> $$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (5) is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\mathbf{x} = \langle x, y\rangle$, $\mathbf{a} = \langle a, b\rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x-a)^2 + (y-b)^2}$, so (5) becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z\rangle$, $\mathbf{a} = \langle a, b, c\rangle$, and (5) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

PARTIAL DERIVATIVES

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

1 $$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

and so Equation 1 becomes

2 $$f_x(a, b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of f with respect to y at (a, b)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed $(x = a)$ and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3 $$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

> **4** If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by
>
> $$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
>
> $$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f/\partial x$. But here $\partial f/\partial x$ can't be interpreted as a ratio of differentials.

NOTATIONS FOR PARTIAL DERIVATIVES If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed. Thus we have the following rule.

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

□

INTERPRETATIONS OF PARTIAL DERIVATIVES

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

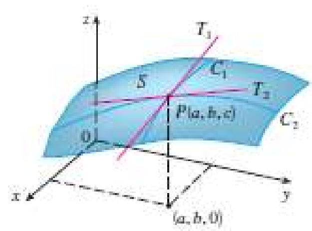


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*. If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

SOLUTION We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.) The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)

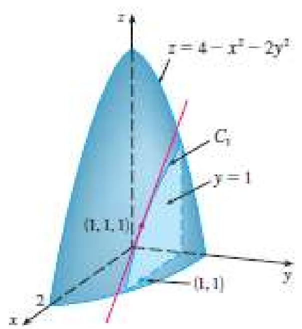


FIGURE 2

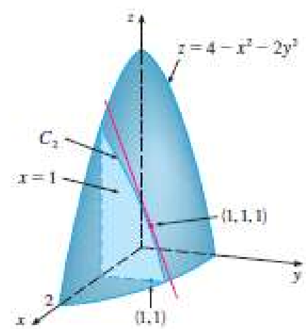


FIGURE 3

Figure 4 is a computer-drawn counterpart to Figure 2. Part (a) shows the plane $y = 1$ intersecting the surface to form the curve C_1 and part (b) shows C_1 and T_1. [We have used the vector equations $\mathbf{r}(t) = \langle t, 1, 2 - t^2 \rangle$ for C_1 and $\mathbf{r}(t) = \langle 1 + t, 1, 1 - 2t \rangle$ for T_1.] Similarly, Figure 5 corresponds to Figure 3.

FIGURE 4

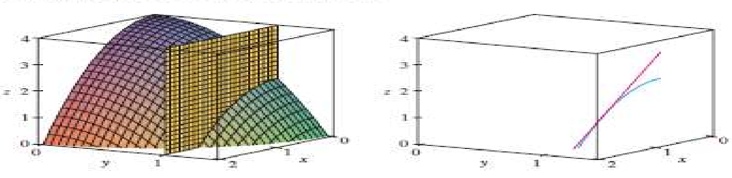
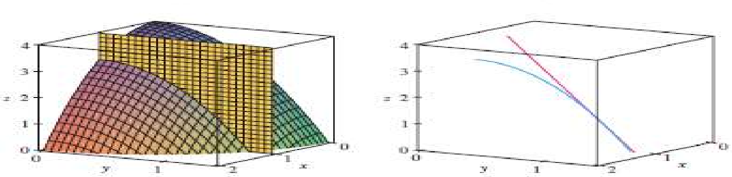


FIGURE 5



EXAMPLE 3 If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$ □

■ Some computer algebra systems can plot surfaces defined by implicit equations in three variables. Figure 6 shows such a plot of the surface defined by the equation in Example 4.

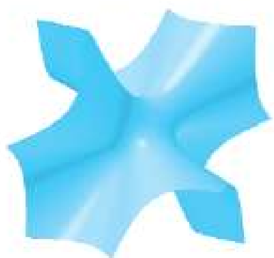


FIGURE 6

EXAMPLE 4 Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

SOLUTION To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$ □

FUNCTIONS OF MORE THAN TWO VARIABLES

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x. If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

EXAMPLE 5 Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy}\ln z$.

SOLUTION Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy}\ln z$$

Similarly, $\quad f_y = xe^{xy}\ln z \quad$ and $\quad f_z = \frac{e^{xy}}{z}$ □

HIGHER DERIVATIVES

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\,\partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

EXAMPLE 6 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

SOLUTION In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

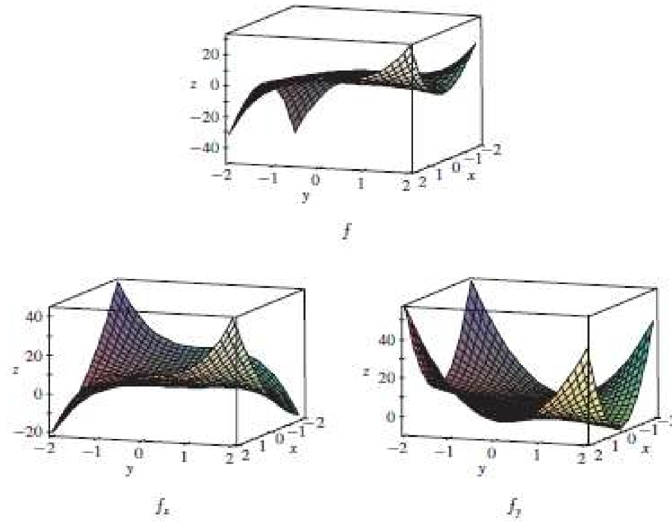
$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

□

Notice that $f_{xy} = f_{yx}$ in Example 6. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice. The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$. The proof is given in Appendix F.

■ Figure 7 shows the graph of the function f in Example 6 and the graphs of its first- and second-order partial derivatives for $-2 \le x \le 2$, $-2 \le y \le 2$. Notice that these graphs are consistent with our interpretations of f_x and f_y as slopes of tangent lines to traces of the graph of f. For instance, the graph of f decreases if we start at $(0, -2)$ and move in the positive x-direction. This is reflected in the negative values of f_x. You should compare the graphs of f_{yx} and f_{yy} with the graph of f_y to see the relationships.



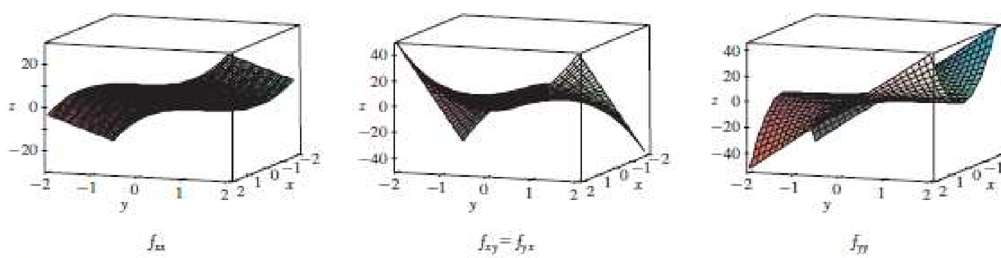


FIGURE 7

■ Alexis Clairaut was a child prodigy in mathematics: he read l'Hospital's textbook on calculus when he was ten and presented a paper on geometry to the French Academy of Sciences when he was 13. At the age of 18, Clairaut published *Recherches sur les courbes à double courbure*, which was the first systematic treatise on three-dimensional analytic geometry and included the calculus of space curves.

CLAIRAUT'S THEOREM Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial y^2\, \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

V EXAMPLE 7 Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

SOLUTION

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

□

PARTIAL DIFFERENTIAL EQUATIONS

Partial derivatives occur in *partial differential equations* that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827). Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

EXAMPLE 8 Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

SOLUTION

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation. □

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

FIGURE 8

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string. For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in Figure 8), then $u(x, t)$ satisfies the wave equation. Here the constant a depends on the density of the string and on the tension in the string.

EXAMPLE 9 Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

SOLUTION

$$u_x = \cos(x - at) \qquad u_{xx} = -\sin(x - at)$$

$$u_t = -a\cos(x - at) \qquad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation. □

EXAMPLE 9 Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

SOLUTION

$$u_x = \cos(x - at) \qquad u_{xx} = -\sin(x - at)$$

$$u_t = -a\cos(x - at) \qquad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation. □

TANGENT PLANES

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)

■ Note the similarity between the equation of a tangent plane and the equation of a tangent line: $y - y_0 = f'(x_0)(x - x_0)$

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

□

EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

SOLUTION The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$
$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$
$$= 1 + 1(x - 1) + 1 \cdot y = x + y$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$.

■ Figure 5 shows the graphs of the function f and its linearization L in Example 2.

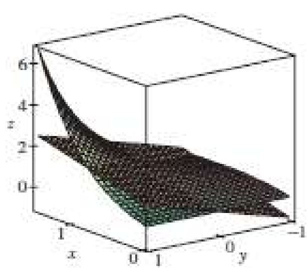


FIGURE 5

EXAMPLE 3 At the beginning of Section 14.3 we discussed the heat index (perceived temperature) I as a function of the actual temperature T and the relative humidity H and gave the following table of values from the National Weather Service.

Relative humidity (%)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Actual temperature (°F)

Find a linear approximation for the heat index $I = f(T, H)$ when T is near 96°F and H is near 70%. Use it to estimate the heat index when the temperature is 97°F and the relative humidity is 72%.

SOLUTION We read from the table that $f(96, 70) = 125$. In Section 14.3 we used the tabular values to estimate that $f_T(96, 70) \approx 3.75$ and $f_H(96, 70) \approx 0.9$. (See pages 878–79.) So the linear approximation is

$$f(T, H) \approx f(96, 70) + f_T(96, 70)(T - 96) + f_H(96, 70)(H - 70)$$
$$\approx 125 + 3.75(T - 96) + 0.9(H - 70)$$

In particular,

$$f(97, 72) \approx 125 + 3.75(1) + 0.9(2) = 130.55$$

Therefore, when $T = 97$°F and $H = 72\%$, the heat index is

$$I \approx 131°\text{F}$$

□

DIFFERENTIALS

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number. The differential of y is then defined as

9 $$dy = f'(x)\,dx$$

(See Section 3.10.) Figure 6 shows the relationship between the increment Δy and the differential dy: Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

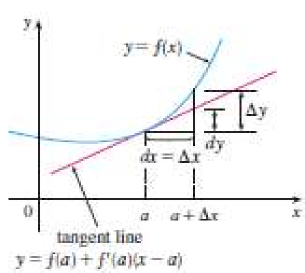


FIGURE 6

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

10 $$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

(Compare with Equation 9.) Sometimes the notation df is used in place of dz.

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation (4) can be written as

$$f(x, y) \approx f(a, b) + dz$$

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz: dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

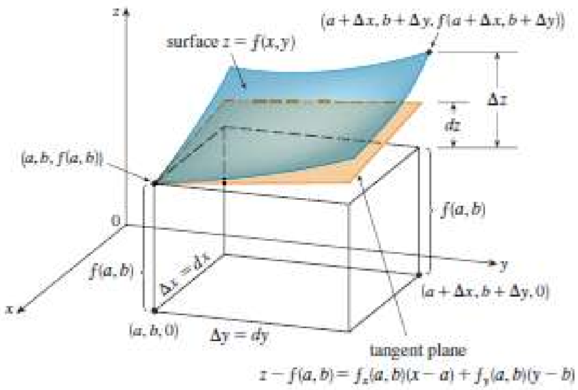


FIGURE 7

EXAMPLE 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

SOLUTION

(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (2x + 3y)\, dx + (3x - 2y)\, dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\Delta z = f(2.05, 2.96) - f(2, 3)$$
$$= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]$$
$$= 0.6449$$

Notice that $\Delta z \approx dz$ but dz is easier to compute. □

■ In Example 4, dz is close to Δz because the tangent plane is a good approximation to the surface $z = x^2 + 3xy - y^2$ near (2, 3, 13). (See Figure 8.)

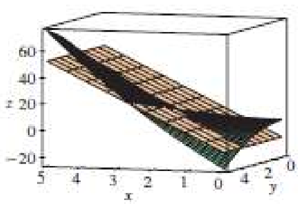


FIGURE 8

EXAMPLE 5 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

SOLUTION The volume V of a cone with base radius r and height h is $V = \pi r^2 h/3$. So the differential of V is

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = \frac{2\pi rh}{3} dr + \frac{\pi r^2}{3} dh$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leq 0.1$, $|\Delta h| \leq 0.1$. To find the largest error in the volume we take the largest error in the measurement of r and of h. Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about $20\pi\ \text{cm}^3 \approx 63\ \text{cm}^3$. □

FUNCTIONS OF THREE OR MORE VARIABLES

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

EXAMPLE 6 The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

SOLUTION If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + xz\,dy + xy\,dz$$

We are given that $|\Delta x| \leq 0.2$, $|\Delta y| \leq 0.2$, and $|\Delta z| \leq 0.2$. To find the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box. □

THE CHAIN RULE

2 THE CHAIN RULE (CASE 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

SOLUTION The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$ □

FIGURE 1
The curve $x = \sin 2t$, $y = \cos t$

EXAMPLE 2 The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

SOLUTION If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since

$$P = 8.31\frac{T}{V}$$

the Chain Rule gives

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s. □

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$. Then z is indirectly a function of s and t and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$. Recall that in computing $\partial z/\partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t. Therefore we can apply Theorem 2 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

A similar argument holds for $\partial z/\partial s$ and so we have proved the following version of the Chain Rule.

3 THE CHAIN RULE (CASE 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

SOLUTION Applying Case 2 of the Chain Rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t)$$ □

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable. Notice that Theorem 3 has one term for each intermediate variable and each of these terms

4 THE CHAIN RULE (GENERAL VERSION) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

EXAMPLE 4 Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

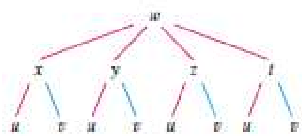


FIGURE 3

SOLUTION We apply Theorem 4 with $n = 4$ and $m = 2$. Figure 3 shows the tree diagram. Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is $\partial y/\partial u$. With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$ □

EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

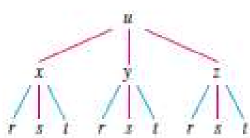


FIGURE 4

SOLUTION With the help of the tree diagram in Figure 4, we have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$ □

EXAMPLE 6 If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

SOLUTION Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0$$ □

EXAMPLE 7 If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

SOLUTION

(a) The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

(b) Applying the Product Rule to the expression in part (a), we get

5 $$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right) = 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

But, using the Chain Rule again (see Figure 5), we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\,\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

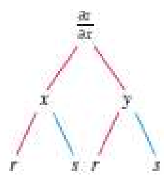


FIGURE 5

Putting these expressions into Equation 5 and using the equality of the mixed second-order derivatives, we obtain

$$\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y\,\partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x\,\partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)$$

$$= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x\,\partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}$$

IMPLICIT DIFFERENTIATION

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections 3.5 and 14.3. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we solve for dy/dx and obtain

6 $$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x. The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

SOLUTION The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

■ The solution to Example 8 should be compared to the one in Example 2 in Section 3.5.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$ □

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $$\frac{\partial}{\partial x}(x) = 1 \quad \text{and} \quad \frac{\partial}{\partial x}(y) = 0$$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in Equations 7. The formula for $\partial z/\partial y$ is obtained in a similar manner.

7 $$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid: If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by (7).

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

■ The solution to Example 9 should be compared to the one in Example 4 in Section 14.3.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

DIRECTIONAL DERIVATIVES

2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

3 THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

PROOF If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\boxed{4} \qquad g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$
$$= D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\, a + f_y(x, y)\, b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\boxed{5} \qquad g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b \qquad \square$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

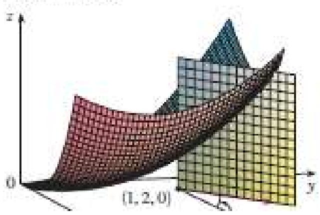
$$\boxed{6} \qquad D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of $\mathbf{u}$. This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through (1, 2, 0) in the direction of $\mathbf{u}$ shown in Figure 5.



SOLUTION Formula 6 gives

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\frac{\pi}{6} + f_y(x, y) \sin\frac{\pi}{6}$$
$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$
$$= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$

THE GRADIENT VECTOR

THE GRADIENT VECTOR

8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $$\nabla f(0, 1) = \langle 2, 0 \rangle$$ □

With this notation for the gradient vector, we can rewrite the expression (7) for the directional derivative as

9 $$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

■ The gradient vector $\nabla f(2, -1)$ in Example 4 is shown in Figure 6 with initial point $(2, -1)$. Also shown is the vector **v** that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f.

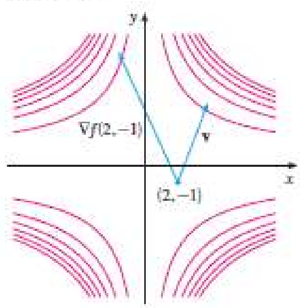


FIGURE 6

EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}$$

Note that **v** is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Therefore, by Equation 9, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$ □

FUNCTIONS OF THREE VARIABLES

10 DEFINITION The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

11 $$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

or, for short,

13 $$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

14 $$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

EXAMPLE 5 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$
$$= \langle \sin yz, xz \cos yz, xy \cos yz\rangle$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3\rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore Equation 14 gives

$$D_{\mathbf{u}} f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \mathbf{u}$$
$$= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right)$$
$$= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}$$

EXAMPLE 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle$$
$$\nabla f(2, 0) = \langle 1, 2\rangle$$

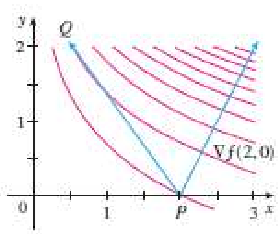


FIGURE 7

At $(2, 0)$ the function in Example 6 increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2\rangle$. Notice from Figure 7 that this vector appears to be perpendicular to the level curve through $(2, 0)$. Figure 8 shows the graph of f and the gradient vector.

The unit vector in the direction of $\vec{PQ} = \langle -1.5, 2\rangle$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2\rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5}\rangle$$
$$= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1$$

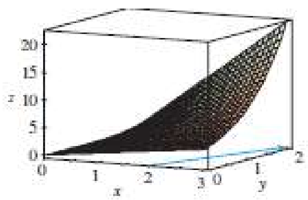


FIGURE 8

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

□

EXAMPLE 7 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\begin{aligned}\nabla T &= \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\,\mathbf{i} - 2y\,\mathbf{j} - 3z\,\mathbf{k})\end{aligned}$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4°\text{C/m}$. □

Possible Questions

PART-B (2 Mark)

1. What is Chain Rule.
2. Define Maximum Principle.
3. Define Limit of a function of three variable.
4. Write the Difference between total and partial derivative.
5. If $f(x,y)=x^3+x^2y^3-2y^2$, find $f_x(1,2)$ and $f_y(1,2)$.
6. Define Laplace equation.
7. Calculate f_{xxyz} if $f(x,y,z)=\sin(3x+yz)$.

PART-C (8 Mark)

1. Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exist.

2. Find the directional derivative of the function $f(x,y)=x^2y^3-4y$ at the point $(2,-1)$ in the direction of the vector $v=2\vec{i}+5\vec{j}$.

3.Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is implicitly as a function of x and y by the equation $x^3+y^3+z^3+6xyz=1$.

4. If $f(x,y,z)=x\sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1,3,0)$ in the direction of $v=\vec{i}+2\vec{j}-\vec{k}$.

5. Show that $f(x,y)=xe^{xy}$ is differentiable at $(1,0)$ and find its Linearization and use it to approximate $f(1.1,-0.1)$.

6. suppose that the temperature at the point (x,y,z) in space is given by $T(x,y,z)=\frac{80}{(1+x^2+2y^2+3z^2)}$, where T is measured degree celsiusand x, y, z in meters. In which direction does the temperature increases fastest at the point $(1,1,-2)$ .What is the maximum rate of increases.

7. If $f(x,y)=\frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

8. If $g(s,t)=f(s^2-t^2,t^2-s^2)$ and f is differentiable, show that g satisfies the equation $t\frac{\partial g}{\partial s}+s\frac{\partial g}{\partial t}=0$.

9. Find the second partial derivatives of $f(x,y)=\sin\left(\frac{x}{1+y}\right)$.

10. If $z=f(x,y)$ has continuous partial second order derivative and $x=r^2+s^2$ and $y=2rs$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Multivariate Calculus **Subject Code: 17MMU503A**

UNIT-I

Question	Option-1	Option-2	Option-3	Option-4	Answer
Every codomain element has a preimage then f(x,y) is_____________	Bijective	1-1	Onto	Reflexive	Bijective
Relation is a subset of ____________	Function	1-1 function	Bijective	Cartesian product	Cartesian product
If z= f(x, y) then the variable x and y are _____________ .	Independent	Dependent	Image	Function	Independent
Differentiation of sinhx = ______	(- coshx)	sinh2x	coshx	(-sinhx)	coshx
If the partial derivatives are continuous then_________	FxyFyx	Fxy = Fyx	Fx = Fy	FxFy	Fxy = Fyx
In a polar coordinate r denotes a ____	distance	area	angle	radius	radius
The Clairaut's theorem is ____________ if partial derivatives are continuous.	FxyFyx	FxFy	Fx = Fy	Fxy = Fyx	Fxy = Fyx
$\lim_{x\to 0}$ (sinx / x) =	0	-1	1	2	1
The equation $U_{xx}+U_{yy}=0$ is called________	Laplace	Heat	Wave	Function	Laplace
The Linearization process gives __________	Second degree to linear	Linear to linear	Second degree to second	Linear to second degree	Second degree to linear
The notation▼F is denoted by ______	Function	Divergence	Curl	Gradient	Gradient
The equation 5x+3 = 0 is gives _____	Straight line	Circle	Parabola	Elliptic	Straight line
If z= f(x, y) then z is _____________ variable .	Dependent	Independent	Image	Isolated	Dependent
$\cosh^2x - \sinh^2x$ = ______	1	0	cosh2x	sinh2x	1
The element of R*R*R is___________	(1, 2)	(2, 1)	(x, y)	(x, y, z)	(x, y, z)
The Level curve of f(x, y) is __________	x	2x	x^2	15	15
In a polar coordinates θ denotes a ____	distance	area	angle	radius	angle
If $U_{xx} + U_{yy}$ =0 then U is Called _______________	Laplace	Circle	Heat equation	Harmonic	Harmonic
The Level curve of f(x, y, z) is _________	f(x, y, z) = 1	f(x, y, z) = x	f(x, y, z) = x+y	f(x, y, z) = x+y+z	f(x, y, z) = 1
The curve of a function f(x, y, z) = k is called __________	Identity function	Identity function	Level curves	1-1	Level curves
The range of f(x) = 2x for every x in N is _________	2Z	2R	2N	N	2N
If f(x, y) = L_1 and f(x, y) = L_2 as (x, y)→(a ,b) along C_1 and C_2 then f has ____	Limit for $L_{1\neq}L_2$	a Limit for $L_{2\neq}L_1$	**a Limit for $L_1 = L_2$**	Continuous for L1 = L2	**a Limit for $L_1 = L_2$**
From the below the functions of two variable is ____	f(x, y, z)	z = f(x)	y = f(x)	**z = f(x,y)**	**z = f(x,y)**
(F x G) (t) =_____________	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) / G(t	F(t) x G(t)
▼ x (▼ f) =	1	0	2	-1	0
▼x (fA) is equal to	(Ñ .f).A + f(Ñ . A)	(Ñ x f).A + f(Ñ x A)	(Ñ .f x A + f x (Ñ . A)	(Ñf) x A + f(Ñ x A)	(Ñ x f).A + f(Ñ x A)
▼ (r^m) is equal to	mr^{m-1}	$m^2 r^{m-2}$	$m(m+1) r^{m-2}$	$(m+1) m r^{m-1}$	$m(m+1) r^{m-2}$
The divergence theorem enables to convert a surface integral on a closed surface into a ------	line integral	volume integral	surface integral	None	volume integral
If A is solenoidal , then	div A=0	curl A =0	|A| = 0	div (curlA) =0	div A=0
If r is position vector , then ▼x r =	3	2	1	0	0
If f=4xi+yj-2k then ▼.f=?	1	0	3	2	3
The function f is said to satisfify the laplace equation if	$▼^2$ f	▼f	$▼^4$f	$▼^3$f	$▼^2$ f
If u , v and w are vectors in R then u x (v + w) =	(u x v) + (u x w)	u.v + u. w	uv +(u +w)	u + w	(u x v) + (u x w)
(F + G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) / G(t)	F(t) + G(t)
(F - G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) / G(t)	F(t) - G(t)
(F x G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) / G(t)	F(t) x G(t)
(*f* F)(t)	*f*(t)F	*f*(t)F(t)	*f*(t)	F(t)	*f*(t)F(t)
(F . G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) . G(t)	F(t) . G(t)
The square of the time of one complete revolution of a planet about its orbit is proportional to the cube of the length of the _______ of its orbit.	minor axis	semi major axis	major axis	semi minor axis	semi major axis
lim [F(t) + G(t)] =	lim F(t) - G(t)	lim F(t) - lim G(t)	[lim F(t)] [lim G(t)]	lim F(t) + lim G(t)	lim F(t) + lim G(t)
lim [F(t) - G(t)] =	lim F(t) - G(t)	lim F(t) - lim G(t)	[lim F(t)] [lim G(t)]	lim F(t) + lim G(t)	lim F(t) - lim G(t)
lim [F(t) . G(t)] =	lim F(t) - lim G(t)	[lim F(t)] x [lim G(t)]	lim F(t) + lim G(t)	[lim F(t)] [lim G(t)]	[lim F(t)] [lim G(t)]
lim [F(t) x G(t)] =	lim F(t) - lim G(t)	[lim F(t)] x [lim G(t)]	lim F(t) + lim G(t)	[lim F(t)] [lim G(t)]	[lim F(t)] x [lim G(t)]
A vector function F(t) is said to be _______ at t if t is in the domain of F	bounded	continuous	differentiable	derivative	continuous
lim F(t) + lim G(t) =	lim [F(t) + G(t)]	lim [F(t) x G(t)]	lim [F(t) . G(t)]	lim [F(t) - G(t)]	lim [F(t) + G(t)]
sinh(2x) = ____________	2sinhxcoshx	sinhx + coshx	coshxcoshx	sinhxcoshx	2sinhxcoshx
$\cosh^2x + \sinh^2x$ = ______	tanhx	cosh2x	1	sinh2x	cosh2x
differentiation of sinhx = ______	(- coshx)	sinh2x	coshx	(-sinhx)	coshx
$\cosh^2x - \sinh^2x$ = ______	1	0	cosh2x	sinh2x	1
The slope of a graph ____ on an interval where the graph is concave up	behind	increases	zero	decreases	increases
If the curve $y = x^4$ has no _______ at x = 0	hyperbolic	inflection point	concavity	saddle point	inflection point
The slope of a graph ____ on an interval where the graph is concave down	increases	zero	decreases	behind	decreases
The graph of the function f is concave up on any open interval I where	f''(x) >0	f''(x) < 1	f''(x) <0	f''(x) = 0	f''(x) >0
The graph of the function f is concave down on any open interval I where	f''(x) >0	f''(x) < 1	f''(x) <0	f''(x) = 0	f''(x) <0
A point P(c , f (c)) on a curve is called _______	hyperbolic	inflection point	concavity	saddle point	inflection point
sinh(-x) = _____	(- coshx)	sinh2x	coshx	(- sinhx)	(- sinhx)
coshx coshy + sinhx sinhy = ________	cosh(x +y)	sin(x - y)	cosh(x - y)	sinh (x + y)	cosh(x +y)
differentiation of y = In (sinhx)	sinhx	cothx	tanhx	coshx	cothx
$\lim_{x\to 0}$ (sinx / x) =	0	(-1)	1	2	1
$\lim_{x\to 0}$ ((3x - sinx) / x) =	0	(-1)	1	2	2
[lim F(t)] [lim G(t)] =	lim [F(t) + G(t)]	lim [F(t) x G(t)]	lim [F(t) . G(t)]	lim [F(t) - G(t)]	lim [F(t) . G(t)]
_____of the projectile is $v^2 \sin\alpha/g$	speed	Range	Distance	Time of flight	Range
_____of the projectile is 2vsinα/g	speed	Range	Distance	Time of flight	Time of flight

F(t) . G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F . G) (t)
F(t) x G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F x G) (t)
F(t) - G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F - G) (t)
F(t) + G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F + G) (t)
(F . G)' (t) =	(F' . G)(t) - (F . G')(t)	F'(t) . G'(t)	F' G' (t)	(F' . G)(t) + (F . G')(t)	(F' . G)(t) + (F . G')(t)
(F x G)' (t) =	(F' x G)(t) + (F x G')(t)	F'(t) x G'(t)	F'x G' (t)	(F' . G)(t) + (F . G')(t)	(F' x G)(t) + (F x G')(t)
If f:A→B hence f is called a ……………..	function	form	formula	fuzzy	function
If the function f is otherwise called as ……………………	limit	mapping	lopping	inverse	mapping
If f:A→B in this set A is called the ……………..of the function f.	domain	co domain	set	element	domain
If f:A→B in this set B is called the ……………..of the function f.	domain	co domain	set	element	co domain
The value of the function f for a and is denoted by ……………………………	a(f)	f(a)	a	f	f(a)
If aϵA then the element in B which is assignedto ais called the ………………of a	B-image	a-image	A-image	f-image	f-image
The element a may be referred to as the …………………..of f(a)	f-image	pre-image	domain	codomain	pre-image
The ………….. of a function as the image of its domain	domain	range	co domain	image	range
The range of a function as the……………. of its domain	range	domain	image	preimage	image
The range of a function as the image of its ……………..	co domain	image	domain	range	domain
Let f be a mapping of A to B,Each element of A has a ………… and each element in B need not be appear as the image of an element in A.	unique preimage	unique image	unique zero	unique range	unique image

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DEPARTMENT OF MATHEMATICS

Subject: Multivariate Calculus	**Semester: V**	**L T P C**
Subject Code: 17MMU503A	**Class: III-B.Sc Mathematics**	**4 2 0 6**

UNIT II

Extrema of functions of two variables: Method of Lagrange multipliers, constrained optimization problems, Definition of vector field, divergence and curl.

Reference Book:

R1: Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.

R3: James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks Cole, Thomson Learning, USA.

TANGENT PLANES TO LEVEL SURFACES

19 $$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 13.5.3, its symmetric equations are

20 $$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

EXAMPLE 8 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Figure 10 shows the ellipsoid, tangent plane, and normal line in Example 8.

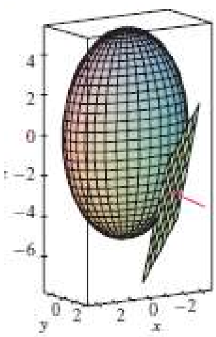


FIGURE 10

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$ □

MAXIMUM AND MINIMUM VALUES

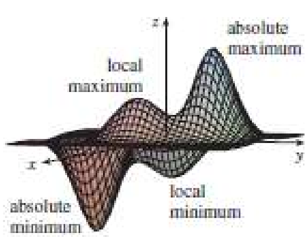


FIGURE 1

Look at the hills and valleys in the graph of f shown in Figure 1. There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the *absolute maximum*. Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values. The smaller of these two values is the *absolute minimum*.

1 DEFINITION A function of two variables has a **local maximum** at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \le f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \ge f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

■ Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

2 THEOREM If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

PROOF Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem (see Theorem 4.1.4). But $g'(a) = f_x(a, b)$ (see Equation 15.3.1) and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. □

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane (Equation 15.4.2), we get $z = z_0$. Thus the geometric interpretation of Theorem 2 is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2. □

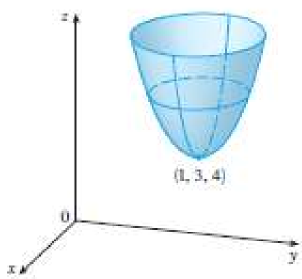


FIGURE 2
$z = x^2 + y^2 - 2x - 6y + 14$

EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

SOLUTION Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f, so f has no extreme value. □

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

NOTE 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

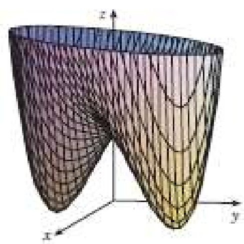


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure 4. □

A contour map of the function f in Example 3 is shown in Figure 5. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$, on the other hand, resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1), the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and saddle point that we found in Example 3.

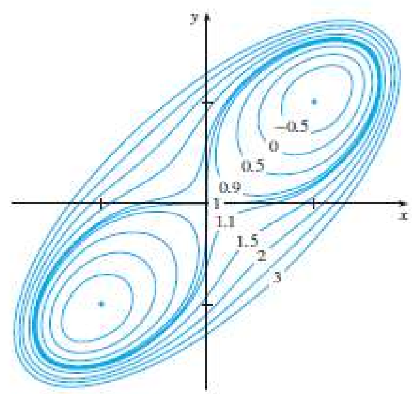


FIGURE 5

TEC In Module 15.7 you can use contour maps to estimate the locations of critical points.

EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

SOLUTION The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0 \qquad (4)$$

$$5x^2 - 4y - 4y^3 = 0 \qquad (5)$$

From Equation 4 we see that either

$$x = 0 \quad \text{or} \quad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case ($10y - 5 - 2x^2 = 0$), we get

$$\boxed{6} \qquad x^2 = 5y - 2.5$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$\boxed{7} \qquad 4y^3 - 21y + 12.5 = 0$$

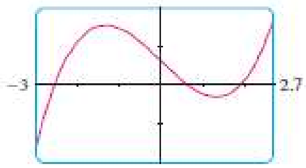


FIGURE 6

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

as in Figure 6, we see that Equation 7 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \qquad y \approx 0.6468 \qquad y \approx 1.8984$$

(Alternatively, we could have used Newton's method or a rootfinder to locate these roots.) From Equation 6, the corresponding x-values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	−10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	−55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	−1.48	−5.87	−187.64	saddle point

Figures 7 and 8 give two views of the graph of f and we see that the surface opens downward. [This can also be seen from the expression for $f(x, y)$: The dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.] Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is $f(\pm 2.64, 1.90) \approx 8.50$. In other words, the highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.

TEC Visual 15.7 shows several families of surfaces. The surface in Figures 7 and 8 is a member of one of these families.

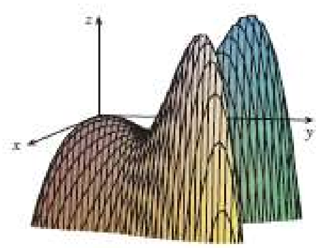


FIGURE 7

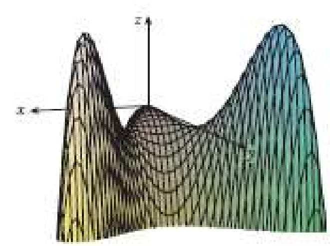


FIGURE 8

□

■ The five critical points of the function f in Example 4 are shown in red in the contour map of f in Figure 9.

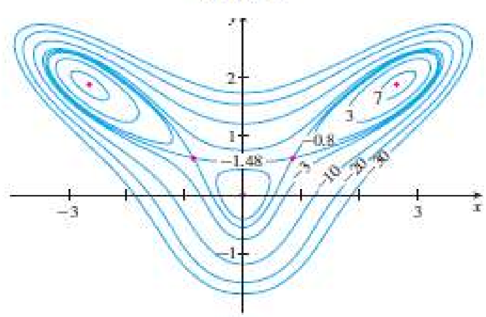


FIGURE 9

EXAMPLE 5 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

SOLUTION The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $(\frac{11}{6}, \frac{5}{3})$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2} = \sqrt{(\tfrac{5}{6})^2 + (\tfrac{5}{3})^2 + (\tfrac{5}{6})^2} = \tfrac{5}{6}\sqrt{6}$$

Example 5 could also be solved using vectors. Compare with the methods of Section 13.5.

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$. □

EXAMPLE 6 A rectangular box without a lid is to be made from 12 m² of cardboard. Find the maximum volume of such a box.

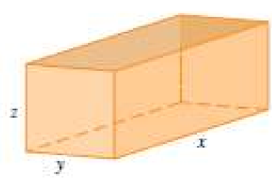


FIGURE 10

SOLUTION Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\frac{12 - xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m³. ■

8 EXTREME VALUE THEOREM FOR FUNCTIONS OF TWO VARIABLES If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D. Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

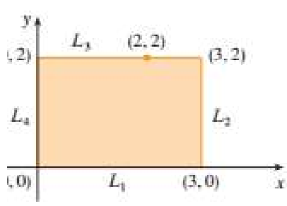
EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

SOLUTION Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and



URE 12

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f.

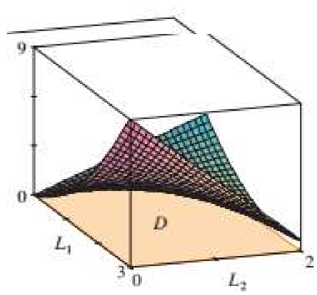


FIGURE 13
$f(x, y) = x^2 - 2xy + 2y$

PROOF OF THEOREM 3, PART (A) We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 15.6.3:

$$D_{\mathbf{u}} f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f = D_{\mathbf{u}}(D_{\mathbf{u}} f) &= \frac{\partial}{\partial x}(D_{\mathbf{u}} f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

$$\boxed{10} \qquad D_{\mathbf{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 10, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B. This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus $f(x, y) \geqslant f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum. ■

LAGRANGE MULTIPLIERS

In Example 6 in Section 15.7 we maximized a volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$, which expressed the side condition that the surface area was 12 m². In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

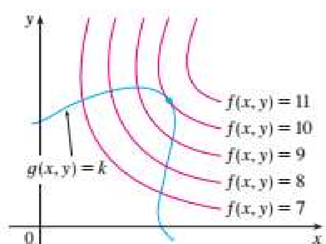


FIGURE 1

TEC Visual 15.8 animates Figure 1 for both level curves and level surfaces.

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 1 shows this curve together with several level curves of f. These have the equations $f(x, y) = c$, where $c = 7$, 8, 9, 10, 11. To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. But we already know from Section 15.6 that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve. (See Equation 15.6.18.) This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

agrange multipliers are named after the nch-Italian mathematician Joseph-Louis range (1736–1813). See page 217 for a graphical sketch of Lagrange.

1 $$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The number λ in Equation 1 is called a **Lagrange multiplier**. The procedure based on Equation 1 is as follows.

METHOD OF LAGRANGE MULTIPLIERS To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and $$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

■ In deriving Lagrange's method we assumed that $\nabla g \neq \mathbf{0}$. In each of our examples you can check that $\nabla g \neq \mathbf{0}$ at all points where $g(x, y, z) = k$. See Exercise 21 for what can go wrong if $\nabla g = \mathbf{0}$.

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of its components, then the equations in step (a) become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Our first illustration of Lagrange's method is to reconsider the problem given in Example 6 in Section 15.7.

EXAMPLE 1 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 6 in Section 15.7, we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x \qquad V_y = \lambda g_y \qquad V_z = \lambda g_z \qquad 2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \qquad yz = \lambda(2z + y)$$

$$\boxed{3} \qquad xz = \lambda(2z + x)$$

$$\boxed{4} \qquad xy = \lambda(2x + 2y)$$

$$\boxed{5} \qquad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x, (3) by y, and (4) by z, then the left sides of these equations will be identical. Doing this, we have

■ Another method for solving the system of equations (2–5) is to solve each of Equations 2, 3, and 4 for λ and then to equate the resulting expressions.

$$\boxed{6} \qquad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \qquad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \qquad xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (2), (3), and (4) and this would contradict (5). Therefore, from (6) and (7), we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (7) and (8) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (5), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 15.7. ■

■ In geometric terms, Example 2 asks for the highest and lowest points on the curve C in Figure 2 that lies on the paraboloid $z = x^2 + 2y^2$ and directly above the constraint circle $x^2 + y^2 = 1$.

EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

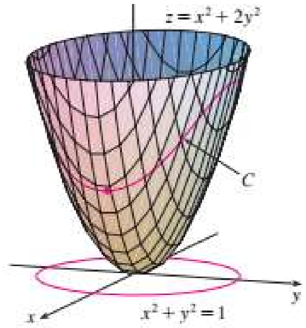
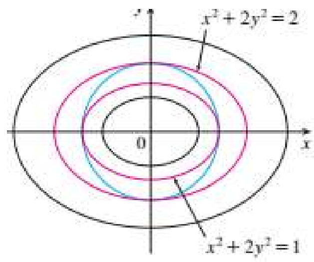


FIGURE 2

■ The geometry behind the use of Lagrange multipliers in Example 2 is shown in Figure 3. The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$.



IGURE 3

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

9 $$2x = 2x\lambda$$

10 $$4y = 2y\lambda$$

11 $$x^2 + y^2 = 1$$

From (9) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (11) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (10), so then (11) gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. Checking with Figure 2, we see that these values look reasonable. ■

EXAMPLE 3 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

SOLUTION According to the procedure in (15.7.9), we compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$. We compare the value of f at that point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

Therefore the maximum value of f on the disk $x^2 + y^2 \leq 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$. ■

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x - 3)^2 + (y - 1)^2 + (z + 1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

12 $$2(x - 3) = 2x\lambda$$

13 $$2(y - 1) = 2y\lambda$$

14 $$2(z + 1) = 2z\lambda$$

15 $$x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From (12) we have

$$x - 3 = x\lambda \quad \text{or} \quad x(1 - \lambda) = 3 \quad \text{or} \quad x = \frac{3}{1 - \lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from (12).] Similarly, (13) and (14) give

$$y = \frac{1}{1-\lambda} \qquad z = -\frac{1}{1-\lambda}$$

Therefore, from (15), we have

$$\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$$

which gives $(1-\lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \quad \text{and} \quad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11})$ and the farthest is $(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11})$. ■

■ Figure 4 shows the sphere and the nearest point P in Example 4. Can you see how to find the coordinates of P without using calculus?

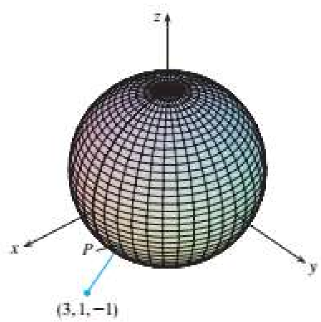


FIGURE 4

TWO CONSTRAINTS

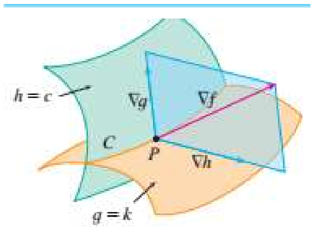


FIGURE 5

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.) Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P. But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C. This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.) So there are numbers λ and μ

16 $$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$
$$f_y = \lambda g_y + \mu h_y$$
$$f_z = \lambda g_z + \mu h_z$$
$$g(x, y, z) = k$$
$$h(x, y, z) = c$$

The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse (Figure 6). Example 5 asks for the maximum value of f when (x, y, z) is restricted to lie on the ellipse.

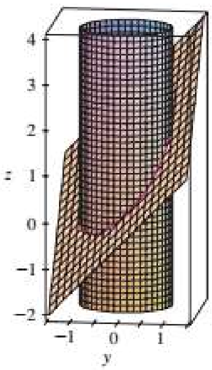


FIGURE 6

EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

SOLUTION We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

17 $$1 = \lambda + 2x\mu$$

18 $$2 = -\lambda + 2y\mu$$

19 $$3 = \lambda$$

20 $$x - y + z = 1$$

21 $$x^2 + y^2 = 1$$

Putting $\lambda = 3$ [from (19)] in (17), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (18) gives $y = 5/(2\mu)$. Substitution in (21) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from (20), $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$.

Possible Questions

PART-B (2Mark)

1. Define Vector Field.
2. What is Lagrange's Multiplier.
3. Define Limit of a function $f(x,y)$ $\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y)$
4. Find f_x, f_y and f_z if $f(x,y,z) = e^{xy} In\, z$.
5. Verify that the function $u(x,t) = \sin(x - at)$ satisfies the wave equation.
6. Define Gradient of the function f.
7. What is Level curve.

PART-C (8 Mark)

1. Find the Local maximum, minimum and saddle point of $f(x,y) = x^4 + y^4 - 4xy + 1$.
2. Find the maximum value of the function $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.
3. Find and classify the critical points of the function $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$
4. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to farthest from the point $(3,1,-1)$.
5. Find the shortest distant form the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.
6. Find the Extreme value of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.
7. A rectangular box without a lid is to be made from 12m^2 of cardboard. Find the maximum value of the box.
8. Find the absolute maximum and minimum of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) / 0 \le x \le 3, 0 \le y \le 2\}$.
9. (i)Write about Local maximum, minimum and saddle point.
   (ii) Find the local minimum and saddle point of the function
   $f(x,y) = x^2 + y^2 - 2x - 6y + 14$
10. A rectangular box without a lid is to be made from 12m^2 of cardboard. Use Lagrange multiplier method to Find the maximum value of the box.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Multivariate Calculus **Subject Code: 17MMU503A**

UNIT-II

Question	Option-1	Option-2	Option-3	Option-4	Answer
The volume of rectangular box is V=____________	x+y+z	xy+yz+xz	xyz	(xyz)^2	xyz
The equation $U_{tt}=a^2U_{xx}$ is called ______________	Laplace	Heat	Quadratic	Wave	Laplace
In this equation $\nabla f = \lambda \nabla g$,is called __________	Euler multiplier	Lagrange multiplier	Legendre multiplier	Laplace multiplier	Lagrange multiplier
The function f is Local minimum at (a, b) if _________	f(x, y) ≤ f(a, b)	f(x, y)>f(a, b)	f(x, y) ≥ f(a, b)	f(x, y)<f(a, b)	f(x, y) ≥ f(a, b)
The function is _________ at every(x, y) in D.	Local maximum	absolute maximum	Local minimum	absolute minimum	absolute maximum
The value of ∫∫(4-x-y) dx dy x,y in (0,1) is__________	1	2	3	4	3
If λ and ϵ are lagrange's multiplier and then the equation is ___________	$\nabla F = \lambda \nabla g$	$\nabla F = (\lambda+\epsilon) \nabla g$	$\nabla F = (\lambda-\epsilon) \nabla g$	$\nabla F = \lambda \nabla g + \epsilon \nabla h$	$\nabla F = \lambda \nabla g + \epsilon \nabla h$
Divergence of a function f is denoted by _______	∇. F	∇* F	∇F	(∇.∇)F	∇. F
The operator ∇ is called _________	integral operator	Matrix operator	Differential operator	Laplace operator	Differential operator
The equation of circle is ______________	**$(x-a)^2+(x-b)^2 = r^2$**	$x^2-y^2 = r^2$	$(x+y)^2 = r^2$	$(x-y)^2 = r^2$	**$(x-a)^2+(x-b)^2 = r^2$**
If $F_{xx} F_{yy} - (F_{xy})^2 > 0$ and $F_{xx} > 0$ then f(a, b) is _________	**Local minimum**	Local maximum	absolute maximum	absolute minimum	**Local minimum**
If f has a local maximum or minimum at (a, b) and the first order partial derivatives of f exist then _________	$f_x (a, b) = 0$	$f_x (a, b) = 0$ or $f_x (a, b) = 0$	$f_x (a, b) = 0$	**$f_x (a, b) = 0$ and $f_x (a, b) = 0$**	**$f_x (a, b) = 0$ and $f_x (a, b) = 0$**
Curl of f is denoted by ______________	∇. F	∇* F	∇F	∇.∇f	∇* F
If r is position vector , then ∇.r =	0	1	2	3	1
If A is irrotational , then	| A | = 1	∇xA =0	|A| =0	∇.A=0	∇.A=0
The divergence of the position vector r is	1	2	r	3	1
If r = xi +yj +zk , then ∇.(ar) is equal to	a	r	0	3a	0
Which of the following is a scalar function ?	∇.A	∇f	∇(∇.A)	∇xA	∇.A
Given that $f = x^2 +y^2 + z^2$, then ∇^2 f is	1	3	6	0	6
If i, j and k are the unit vectors along the coordinate axes , then (i .i) is	0	1	p	j	1
If x = a(θ - sinθ) and y = a (1- cosθ) is called a equation of ______	hyperbola	parabola	cycloid	solid	cycloid
The parametric equation of _______ is x = acost and y = asint	ellipse	circle	hyperbola	parabola	circle
The parametric equation of _______ is x = acost and y = bsint	hyperbola	parabola	ellipse	circle	ellipse
If x = sect and y = tan t the find dy/dx	1/tan t	sect / tant	tant / sect	1/sect	sect / tant
The parametric equation of _______ is x = asect and y = btant	hyperbola	parabola	ellipse	circle	hyperbola
If f (x) = x + sin x , then f ′(x) =	sin x − x cos x	1+ cos x	cos x	1− cos x	1+ cos x
The curve represented by the parametric equations $x = t^2$ and $y = t^3$ is called	ellipse	semicubical parabola	hyperbola	parabola	semicubical parabola
The volume of the cylinder is _____	base - height	base x height	2(base + height)	(base x height) / 2	base x height
A function with a continuous first derivative is said to be smooth and its graph is called ______	smooth curve	length	smooth plane	smooth derivative	smooth curve
If a right cylinder is generated by translating a region of area A through a distance h, then h is called	circumference	base	height	length	height
A function with a continuous first derivative is said to be	length	smooth derivative	smooth	smooth curve	smooth
A piece of cone is called a	frustum	surface	area	radil	frustum
(base circumference x slant height) / 2 =	volume of cone	lateral surface area	volume of solid	area of revolution	lateral surface area
Volume of a right circular cylinder is	πr2	2πr2h	2πr	πr2h	πr2h
_______ is a solid that generayte when a plane region is translated along a line or axis that is perpendicular to the region	sphere	right cylinder	cone	pyramid	right cylinder
A right cylinder is a solid that generayte when a plane region is translated along a line or axis that is ______ to the region	perpendicular	bounded	parallel	linear	perpendicular
The volume of a solid can be obtained by integrating the ________ from one end of the solid to the other .	length	height	cross sectional area	surface area	cross sectional area
volume of a sphere is	4/3 πr3	1/2 πr2h	πr2h	2πr	4/3 πr3
_______ is a solid enclosed by two concentric right circular cylinders	right cylinder	surface area	cylindrical shell	cone	cylindrical shell
volume of a cylindrical shell = ________	2π	πr2	2πr2h	2πr	2π
A_________ is a surface that is generated by revolving a plane curve about an axis thatb lies in the same plane as the curve.	lateral surface area	surface of revolution	area of revolution	cross sectional area	surface of revolution

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DEPARTMENT OF MATHEMATICS

Subject: Multivariate Calculus	**Semester: V**	**L T P C**
Subject Code: 17MMU503A	**Class: III-B.Sc Mathematics**	**4 2 0 6**

UNIT III

Double integration over rectangular region: Double integration over non-rectangular region, double integrals in polar co-ordinates, Triple integrals, Triple integral over a parallelepiped and solid regions. Volume by triple integrals, cylindrical and spherical co-ordinates. Change of variables in double integrals and triple integrals

Reference Book:

R3: James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks Cole,Thomson Learning, USA.

DOUBLE INTEGRALS OVER RECTANGLES

■ Notice the similarity between Definition 5 and the definition of a single integral in Equation 2.

5 DEFINITION The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

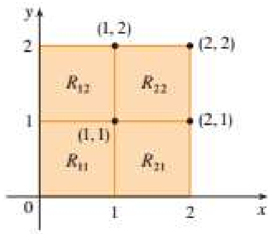
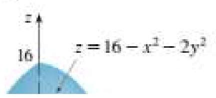


FIGURE 6



EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is 1. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A \\ &= f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

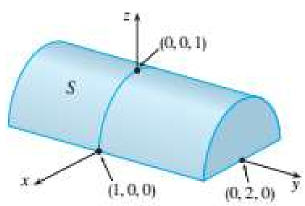


FIGURE 9

SOLUTION It would be very difficult to evaluate this integral directly from Definition 5 but, because $\sqrt{1 - x^2} \geq 0$, we can compute the integral by interpreting it as a volume. If $z = \sqrt{1 - x^2}$, then $x^2 + z^2 = 1$ and $z \geq 0$, so the given double integral represents the volume of the solid S that lies below the circular cylinder $x^2 + z^2 = 1$ and above the rectangle R. (See Figure 9.) The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus

$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$ ■

MIDPOINT RULE FOR DOUBLE INTEGRALS

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

EXAMPLE 3 Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

SOLUTION In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 10. So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

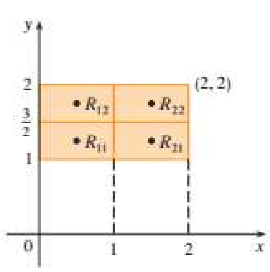


FIGURE 10

$$\begin{aligned} \iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\ &= f(\bar{x}_1, \bar{y}_1)\, \Delta A + f(\bar{x}_1, \bar{y}_2)\, \Delta A + f(\bar{x}_2, \bar{y}_1)\, \Delta A + f(\bar{x}_2, \bar{y}_2)\, \Delta A \\ &= f(\tfrac{1}{2}, \tfrac{5}{4})\, \Delta A + f(\tfrac{1}{2}, \tfrac{7}{4})\, \Delta A + f(\tfrac{3}{2}, \tfrac{5}{4})\, \Delta A + f(\tfrac{3}{2}, \tfrac{7}{4})\, \Delta A \\ &= (-\tfrac{67}{16})\tfrac{1}{2} + (-\tfrac{139}{16})\tfrac{1}{2} + (-\tfrac{51}{16})\tfrac{1}{2} + (-\tfrac{123}{16})\tfrac{1}{2} \\ &= -\tfrac{95}{8} = -11.875 \end{aligned}$$

Thus we have

$$\iint_R (x - 3y^2)\, dA \approx -11.875$$ ■

PROPERTIES OF DOUBLE INTEGRALS

We list here three properties of double integrals that can be proved in the same manner as in Section 5.2. We assume that all of the integrals exist. Properties 7 and 8 are referred to as the *linearity* of the integral.

Double integrals behave this way because the double sums that define them behave this way.

7 $$\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

8 $$\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA \qquad \text{where } c \text{ is a constant}$$

If $f(x,y) \geqslant g(x,y)$ for all (x,y) in R, then

9 $$\iint_R f(x,y)\,dA \geqslant \iint_R g(x,y)\,dA$$

EXAMPLE 1 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ (b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

SOLUTION

(a) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y\,dy = \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2} = x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \tfrac{3}{2}x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y\,dy\,dx = \int_0^3 \left[\int_1^2 x^2 y\,dy\right] dx$$
$$= \int_0^3 \tfrac{3}{2}x^2\,dx = \frac{x^3}{2}\bigg]_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 \left[\int_0^3 x^2 y\,dx\right] dy = \int_1^2 \left[\frac{x^3}{3}y\right]_{x=0}^{x=3} dy$$
$$= \int_1^2 9y\,dy = 9\frac{y^2}{2}\bigg]_1^2 = \frac{27}{2}$$

Theorem 4 is named after the Italian mathematician Guido Fubini (1879–1943), who proved a very general version of this theorem in 1907. But the version for continuous functions was known to the French mathematician Augustin-Louis Cauchy almost a century earlier.

4 FUBINI'S THEOREM If f is continuous on the rectangle $R = \{(x,y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

EXAMPLE 2 Evaluate the double integral $\iint_R (x - 3y^2)\,dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$. (Compare with Example 3 in Section 15.1.)

SOLUTION 1 Fubini's Theorem gives

$$\iint_R (x - 3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx = \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\,dx = \frac{x^2}{2} - 7x\Big]_0^2 = -12$$

SOLUTION 2 Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3\Big]_1^2 = -12$$

■ Notice the negative answer in Example 2; nothing is wrong with that. The function f in that example is not a positive function, so its integral doesn't represent a volume. From Figure 3 we see that f is always negative on R, so the value of the integral is the *negative* of the volume that lies *above* the graph of f and *below* R.

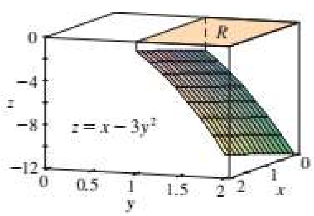


FIGURE 3

EXAMPLE 3 Evaluate $\iint_R y\sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.

SOLUTION 1 If we first integrate with respect to x, we get

$$\iint_R y\sin(xy)\,dA = \int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy = \int_0^\pi \left[-\cos(xy)\right]_{x=1}^{x=2} dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0$$

SOLUTION 2 If we reverse the order of integration, we get

$$\iint_R y\sin(xy)\,dA = \int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$$

To evaluate the inner integral, we use integration by parts with

$$u = y \qquad dv = \sin(xy)\,dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

and so
$$\int_0^\pi y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Big]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

■ For a function f that takes on both positive and negative values, $\iint_R f(x, y)\,dA$ is a difference of volumes: $V_1 - V_2$, where V_1 is the volume above R and below the graph of f and V_2 is the volume below R and above the graph. The fact that the integral in Example 3 is 0 means that these two volumes V_1 and V_2 are equal. (See Figure 4.)

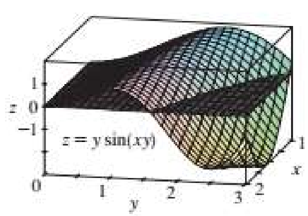


FIGURE 4

If we now integrate the first term by parts with $u = -1/x$ and $dv = \pi\cos\pi x\,dx$, we get $du = dx/x^2$, $v = \sin\pi x$, and

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

Therefore
$$\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$$

In Example 2, Solutions 1 and 2 are equally straightforward, but in Example 3 the first solution is much easier than the second one. Therefore, when we evaluate double integrals, it is wise to choose the order of integration that gives simpler integrals.

and so $$\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx = \left[-\frac{\sin \pi x}{x}\right]_1^2 = -\frac{\sin 2\pi}{2} + \sin \pi = 0$$

EXAMPLE 4 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

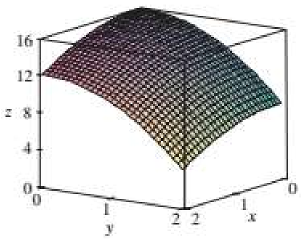


FIGURE 5

SOLUTION We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$. (See Figure 5.) This solid was considered in Example 1 in Section 15.1, but we are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

$$\begin{aligned} V &= \iint_R (16 - x^2 - 2y^2)\,dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy \\ &= \int_0^2 \left[16x - \tfrac{1}{3}x^3 - 2y^2x\right]_{x=0}^{x=2} dy \\ &= \int_0^2 \left(\tfrac{88}{3} - 4y^2\right) dy = \left[\tfrac{88}{3}y - \tfrac{4}{3}y^3\right]_0^2 = 48 \end{aligned}$$

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$. Then Fubini's Theorem gives

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx\right)\right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant. Therefore, in this case, the double integral of f can be written as the product of two single integrals:

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx\right)\right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant. Therefore, in this case, the double integral of f can be written as the product of two single integrals:

5 $$\iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

EXAMPLE 5 If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation 5,

$$\begin{aligned} \iint_R \sin x \cos y\,dA &= \int_0^{\pi/2} \sin x\,dx \int_0^{\pi/2} \cos y\,dy \\ &= \left[-\cos x\right]_0^{\pi/2} \left[\sin y\right]_0^{\pi/2} = 1 \cdot 1 = 1 \end{aligned}$$

DOUBLE INTEGRALS OVER GENERAL REGIONS

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

1 $$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

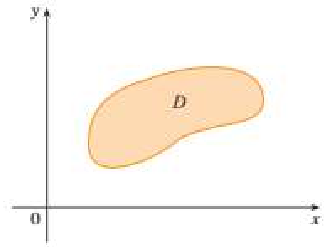


FIGURE 1

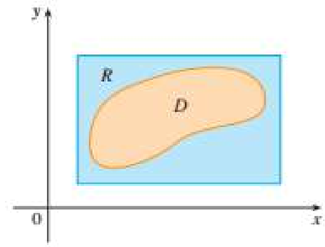
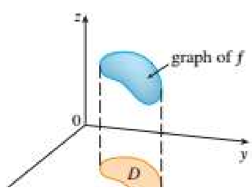


FIGURE 2



If F is integrable over R, then we define the **double integral of f over D** by

2 $$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

then $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

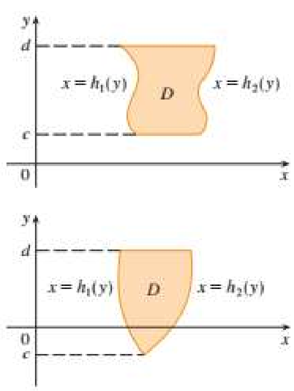


FIGURE 7
Some type II regions

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

4 $$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

Using the same methods that were used in establishing (3), we can show that

5 $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

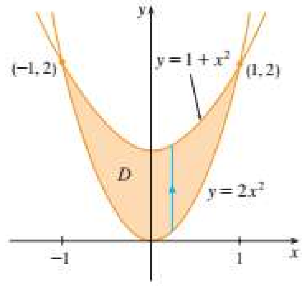


FIGURE 8

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Big]_{-1}^{1} = \frac{32}{15}$$

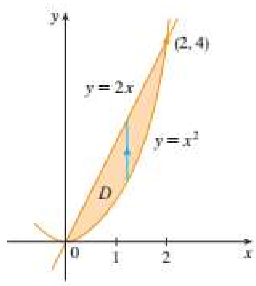


FIGURE 9
D as a type I region

NOTE When we set up a double integral as in Example 1, it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in Figure 8. Then the limits of integration for the *inner* integral can be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration. For a type II region the arrow is drawn horizontally from the left boundary to the right boundary.

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

SOLUTION 1 From Figure 9 we see that D is a type I region and

$$D = \{(x, y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$$

Therefore the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3}\right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Big]_0^2 = \frac{216}{35}$$

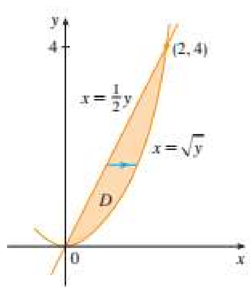
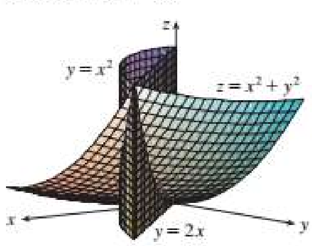


FIGURE 10

SOLUTION 2 From Figure 10 we see that D can also be written as a type II region:

$$D = \{(x, y) \mid 0 \le y \le 4,\ \tfrac{1}{2}y \le x \le \sqrt{y}\}$$

■ Figure 11 shows the solid whose volume is calculated in Example 2. It lies above the xy-plane, below the paraboloid $z = x^2 + y^2$, and between the plane $y = 2x$ and the parabolic cylinder $y = x^2$.



Therefore another expression for V is

$$V = \iint_D (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy$$

$$= \tfrac{2}{15}y^{5/2} + \tfrac{2}{7}y^{7/2} - \tfrac{13}{96}y^4\Big]_0^4 = \tfrac{216}{35}$$

FIGURE 11

EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \{(x, y) \mid -2 \leq y \leq 4,\ \tfrac{1}{2}y^2 - 3 \leq x \leq y + 1\}$$

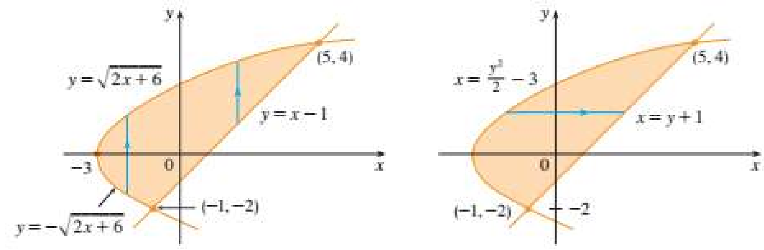


FIGURE 12 (a) D as a type I region (b) D as a type II region

Then (5) gives

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

If we had expressed D as a type I region using Figure 12(a), then we would have obtained

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

but this would have involved more work than the other method. ■

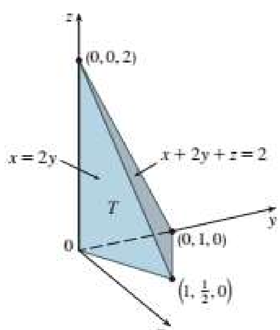


FIGURE 13

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION In a question such as this, it's wise to draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies. Figure 13 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$. Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. (See Figure 14.)

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

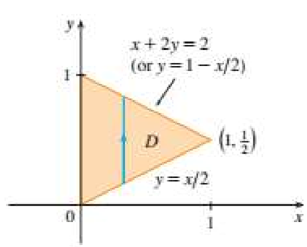


FIGURE 14

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ x/2 \leq y \leq 1 - x/2\}$$

Therefore

$$V = \iint_D (2 - x - 2y)\, dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx$$

$$= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx$$

$$= \int_0^1 (x^2 - 2x + 1)\, dx = \frac{x^3}{3} - x^2 + x\Big]_0^1 = \frac{1}{3}$$ ■

V EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\, dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\, dy$ is not an elementary function. (See the end of Section 7.5.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

where $$D = \{(x, y) \mid 0 \leq x \leq 1,\ x \leq y \leq 1\}$$

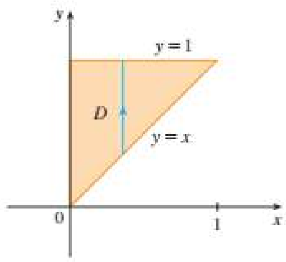


FIGURE 15
D as a type I region

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$$

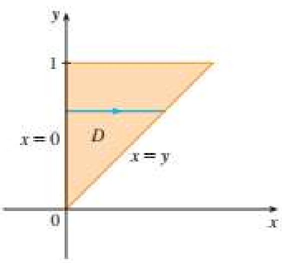


FIGURE 16
D as a type II region

This enables us to use (5) to express the double integral as an iterated integral in the reverse order:

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

$$= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin(y^2)\, dy = -\tfrac{1}{2} \cos(y^2)\Big]_0^1$$

$$= \tfrac{1}{2}(1 - \cos 1)$$ ■

Note:

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10 $$\iint_D 1\, dA = A(D)$$

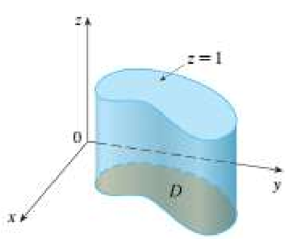


FIGURE 19
Cylinder with base *D* and height 1

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 7, 8, and 10 to prove the following property. (See Exercise 57.)

11 If $m \leq f(x, y) \leq M$ for all (x, y) in D, then

$$mA(D) \leq \iint_D f(x, y)\, dA \leq MA(D)$$

EXAMPLE 6 Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and therefore

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$

TRIPLE INTEGRALS

3 DEFINITION The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

4 FUBINI'S THEOREM FOR TRIPLE INTEGRALS If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\bigg]_0^3 = \frac{27}{4}$$

FIGURE 4
A type 1 solid region with a type II projection

EXAMPLE 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure 5) and one of its projection D on the xy-plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 6, and we have

9 $$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

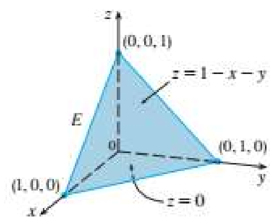


FIGURE 5

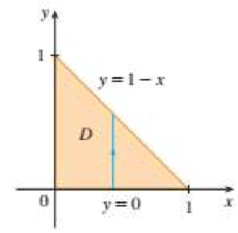


FIGURE 6

9 $$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx = \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$ ■

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 7). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

10 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA$$

tions 7 and 8).

V EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

SOLUTION The solid E is shown in Figure 9. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region in Figure 10. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

TEC Visual 15.6 illustrates how solid regions (including the one in Figure 9) project onto coordinate planes.

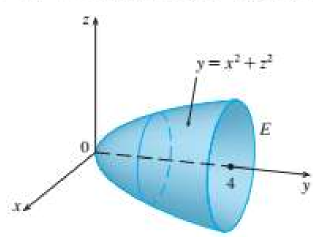
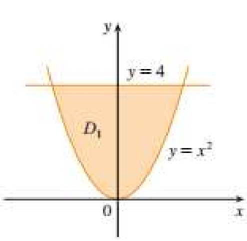


FIGURE 9
Region of integration



From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore the description of E as a type 1 region is

$$E = \{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\}$$

and so we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

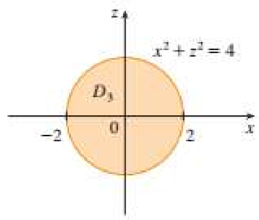


FIGURE 11
Projection on *xz*-plane

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy \right] dA = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

Although this integral could be written as

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx$$

it's easier to convert to polar coordinates in the xz-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr$$

$$= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15}$$

The most difficult step in evaluating a triple integral is setting up an expression for the region of integration (such as Equation 9 in Example 2). Remember that the limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.

EXAMPLE 4 Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$.

SOLUTION This iterated integral is a triple integral over the solid region

$$E = \left\{(x, y, z) \mid -2 \le x \le 2,\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2},\ \sqrt{x^2 + y^2} \le z \le 2\right\}$$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \le 4$. The lower surface of E is the cone $z = \sqrt{x^2 + y^2}$ and its upper surface is the plane $z = 2$. (See Figure 9.) This region has a much simpler description in cylindrical coordinates:

$$E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r \le z \le 2\right\}$$

Therefore, we have

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx = \iiint_E (x^2 + y^2)\, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^3(2 - r)\, dr$$

$$= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \tfrac{16}{5}\pi$$

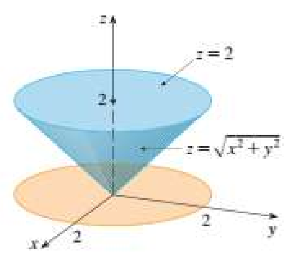


FIGURE 9

EVALUATING TRIPLE INTEGRALS WITH CYLINDRICAL COORDINATES

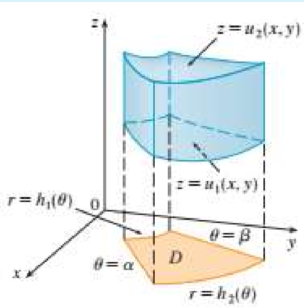


FIGURE 6

Suppose that E is a type 1 region whose projection D on the xy-plane is conveniently described in polar coordinates (see Figure 6). In particular, suppose that f is continuous and

$$E = \left\{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\right\}$$

where D is given in polar coordinates by

$$D = \left\{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\right\}$$

We know from Equation 15.6.6 that

3
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

But we also know how to evaluate double integrals in polar coordinates. In fact, combining Equation 3 with Equation 15.4.3, we obtain

$$\boxed{4} \quad \iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

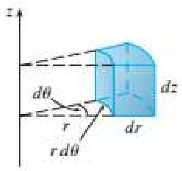


FIGURE 7
Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

Formula 4 is the **formula for triple integration in cylindrical coordinates**. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$. (Figure 7 shows how to remember this.) It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

EXAMPLE 3 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$ and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

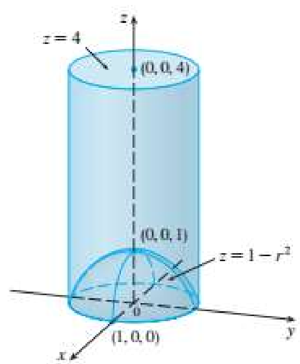


FIGURE 8

SOLUTION In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 1 - r^2 \le z \le 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant. Therefore, from Formula 15.6.13, the mass of E is

$$\begin{aligned} m &= \iiint_E K\sqrt{x^2 + y^2}\, dV \\ &= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta \\ &= K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr \\ &= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5} \end{aligned}$$

EXAMPLE 4 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION The tetrahedron T and its projection D on the xy-plane are shown in Figures 12 and 13. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

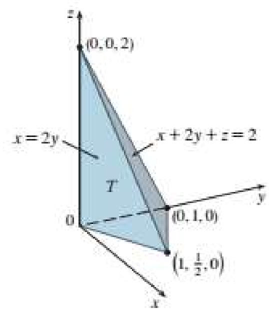


FIGURE 12

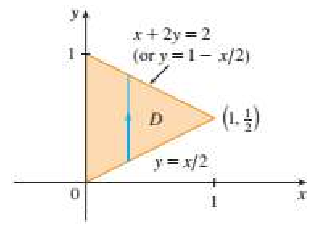


FIGURE 13

Therefore we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\,dy\,dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\,dy\,dx = \tfrac{1}{3}$$

TRIPLE INTEGRALS IN SPHERICAL COORDINATES

SPHERICAL COORDINATES

The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi$$

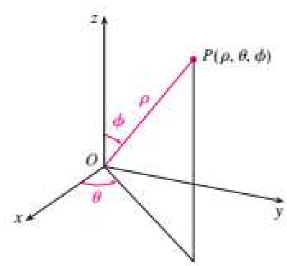


FIGURE 1
The spherical coordinates of a point

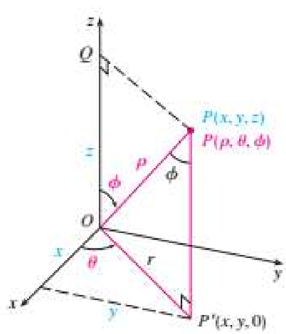


FIGURE 5

The relationship between rectangular and spherical coordinates can be seen from Figure 5. From triangles OPQ and OPP' we have

$$z = \rho \cos \phi \qquad r = \rho \sin \phi$$

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

1
$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

Also, the distance formula shows that

2
$$\rho^2 = x^2 + y^2 + z^2$$

We use this equation in converting from rectangular to spherical coordinates.

EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

SOLUTION We plot the point in Figure 6. From Equations 1 we have

$$x = \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi = 2 \cos\frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

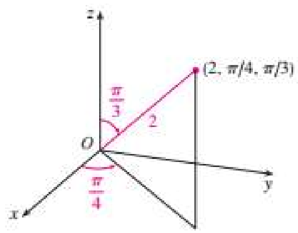


FIGURE 6

Thus the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates.

WARNING There is not universal agreement on the notation for spherical coordinates. Most books on physics reverse the meanings of θ and ϕ and use r in place of ρ.

EXAMPLE 2 The point $\left(0, 2\sqrt{3}, -2\right)$ is given in rectangular coordinates. Find spherical coordinates for this point.

SOLUTION From Equation 2 we have

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$

and so Equations 1 give

$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho \sin\phi} = 0 \qquad \theta = \frac{\pi}{2}$$

TEC In Module 15.8 you can investigate families of surfaces in cylindrical and spherical coordinates.

(Note that $\theta \neq 3\pi/2$ because $y = 2\sqrt{3} > 0$.) Therefore spherical coordinates of the given point are $(4, \pi/2, 2\pi/3)$.

EVALUATING TRIPLE INTEGRALS WITH SPHERICAL COORDINATES

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

3
$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

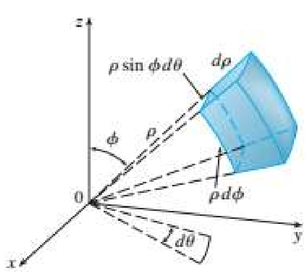


FIGURE 8

Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

SOLUTION Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

Thus (3) gives

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(\rho^2)^{3/2}}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^1\rho^2 e^{\rho^3}\,d\rho$$

$$= \Big[-\cos\phi\Big]_0^{\pi}(2\pi)\Big[\tfrac{1}{3}e^{\rho^3}\Big]_0^1 = \tfrac{4}{3}\pi(e-1)$$

NOTE It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates. In rectangular coordinates the iterated integral would have been

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$. (See Figure 9.)

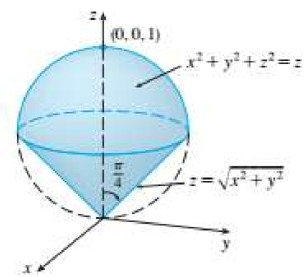


FIGURE 9

Figure 10 gives another look (this time drawn by Maple) at the solid of Example 4.

FIGURE 10

SOLUTION Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho\cos\phi \qquad \text{or} \qquad \rho = \cos\phi$$

The equation of the cone can be written as

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\,\cos^2\theta + \rho^2\sin^2\phi\,\sin^2\theta} = \rho\sin\phi$$

This gives $\sin\phi = \cos\phi$, or $\phi = \pi/4$. Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4,\ 0 \le \rho \le \cos\phi\}$$

Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi}d\phi$$

TEC Visual 15.8 shows an animation of Figure 11.

$$= \frac{2\pi}{3}\int_0^{\pi/4}\sin\phi\,\cos^3\phi\,d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$$

CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of x and u, we can write the Substitution Rule (5.5.6) as

$$\boxed{1} \qquad \int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

$$\boxed{2} \qquad \int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula (15.4.2) can be written as

$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$\boxed{3} \qquad x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If no two points have the same image, T is called **one-to-one**. Figure 1 shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the **image of S**, consisting of the images of all points in S.

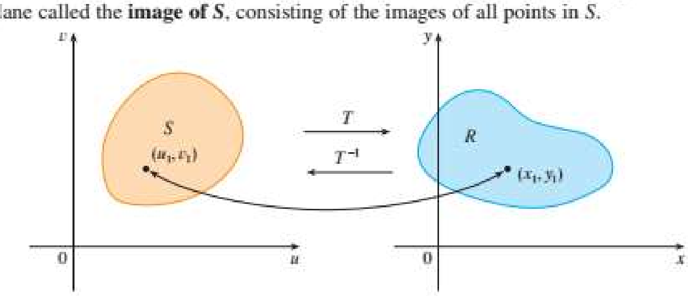


FIGURE 1

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations 3 for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y)$$

■ The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). Although the French mathematician Cauchy first used these special determinants involving partial derivatives, Jacobi developed them into a method for evaluating multiple integrals.

7 DEFINITION The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

9 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

SOLUTION The region R is pictured in Figure 2 (on page 1014). In Example 1 we discovered that $T(S) = R$, where S is the square $[0, 1] \times [0, 1]$. Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R. First we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Theorem 9,

$$\iint_R y\, dA = \iint_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\, du\, dv$$

$$= 8 \int_0^1 \int_0^1 (u^3 v + uv^3)\, du\, dv = 8 \int_0^1 \left[\tfrac{1}{4}u^4 v + \tfrac{1}{2}u^2 v^3 \right]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v + 4v^3)\, dv = \left[v^2 + v^4 \right]_0^1 = 2$$

NOTE Example 2 was not a very difficult problem to solve because we were given a suitable change of variables. If we are not supplied with a transformation, then the first step is to think of an appropriate change of variables. If $f(x, y)$ is difficult to integrate, then the form of $f(x, y)$ may suggest a transformation. If the region of integration R is awkward, then the transformation should be chosen so that the corresponding region S in the uv-plane has a convenient description.

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

SOLUTION Since it isn't easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of this function:

10 $$u = x + y \qquad v = x - y$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane. Theorem 9 talks about a transformation T from the uv-plane to the xy-plane. It is obtained by solving Equations 10 for x and y:

11 $$x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v)$$

The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\tfrac{1}{2}$$

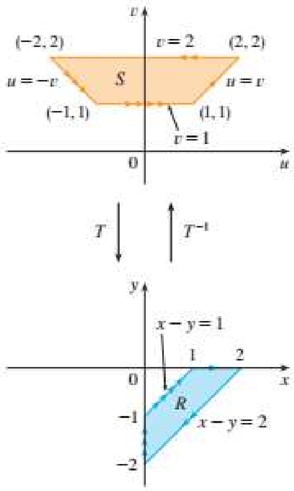


FIGURE 8

To find the region S in the uv-plane corresponding to R, we note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

and, from either Equations 10 or Equations 11, the image lines in the uv-plane are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus the region S is the trapezoidal region with vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$ shown in Figure 8. Since

$$S = \{(u, v) \mid 1 \leqslant v \leqslant 2,\ -v \leqslant u \leqslant v\}$$

Theorem 9 gives

$$\iint_R e^{(x+y)/(x-y)}\, dA = \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right) du\, dv = \tfrac{1}{2}\int_1^2 \left[v e^{u/v}\right]_{u=-v}^{u=v} dv$$

$$= \tfrac{1}{2}\int_1^2 (e - e^{-1}) v\, dv = \tfrac{3}{4}(e - e^{-1})$$

TRIPLE INTEGRALS

There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

12
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

13
$$\iiint_R f(x, y, z)\, dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

EXAMPLE 4 Use Formula 13 to derive the formula for triple integration in spherical coordinates.

SOLUTION Here the change of variables is given by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\,(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta)$$

$$\quad - \rho\sin\phi\,(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi = -\rho^2\sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin \phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \left| -\rho^2 \sin \phi \right| = \rho^2 \sin \phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

which is equivalent to Formula 15.8.3.

Possible Questions

PART-B (2 Mark)

1. What is Double Integral.
2. Define Change of Variables.
3. Define continuous function with three variable.
4. How to define Triple integral.
5. State Clairaut's theorem.
6. Define tangent plane to the surface $z = f(x, y)$.
7. Define Directional derivative.
8. Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

PART-C (8 Mark)

1. (i) Evaluate the double integral $\iint_R (x - 3y^2) dA$, where $R = \{(x, y) / 0 \le x \le 2, 1 \le y \le 2\}$.

(ii) Evaluate $\iint_R y \sin(xy) dA$, where $R = [1,2] \times [0, \pi]$.

2. Find the area of the surface generated by revolving the curve about the x-axis. Evaluate $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

3. Discuss about the application of line integral.

4. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the $xy-$plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

5. Write the change of variable example for triple integral.

6. Evaluate $\iint_R (3x + 4y^2) dA$, where R is the region in the upper half -plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

7. Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} \left(x^2 + y^2\right) dz\, dy\, dx.$

8. Find the mass and center of mass of the triangular lamina with vertices $(0,0), (1,0)$ and $(0,2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

9. Evaluate the triple integral $\iiint_B xyz^2 dV$, where B is the rectangular box given by $B = \{(x, y, z) / 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Multivariate Calculus **Subject Code: 17MMU503A**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4
Double Riemann sum is used for _________	Double integral	Single integral	Triple integral	Scalar value
The ordinates of Spherical is___________	(x, y, z)	(x, y)	(r,θ)	(r, θ, ϕ)
The vector 0 is denoted __________	**i + j + k**	Div f	Curl f	0i + 0j +0k
∫secx tanx dx = ____	tanx	sinx	**secx**	cos x
The graph of f crosses it tangent plan for D <0 then (a, b) is _______	Max	Min	Saddle	Max and min
volume of a sphere is____________	4/3 πr3	1/2 πr2h	**πr2h**	2πr
The circumference of rectangular box is __________	xy+yz+xz	**2(xy+yz+xz)**	2xy+2yz+xz	xy+2yz+2xz
The area of (x^2/a^2) + (y^2+b^2) = 1is___________	πab	πxy	π	2π
The value of ▼× f gives _______	**Vector**	Scalar	0	**0**
The integral is __________ theorem.	Stock's	Red's theorem	Green's	Convergence
If i , j and k are the unit vectors along the coordinate axes , then (jxj) is	1	k	0	p
If i and j are the unit vectors along x and y projections, then (i.j) is	0	1	k	3
If r = xi +yj +zk, then ▼.r =?	1	2	0	3
If i , j and k are the unit vectors along the x, y, z axes , then jxk is equal to	0	1	I	3
If r = 2xi -yj +2zk, then ▼.r =?	0	4	2	3
If A = 3i-5j+2k and B = 4i+3j then A.B is equal to	-3	19	-14	11
If A is irrotational , then	▼.A =0	▼xA=0	▼.A≠0	▼xA ≠0
A vector A is said to be solenoidal if	▼xA ≠0	▼xA=0	▼.A≠0	▼.A =0
If F is solinoidal ,then	▼.F =0	▼xF=0	▼2F=0	▼ . ▼XF =0
In a polar cooridinates r denotes a ____	distance	area	angle	radius
An Rectangular coordinates means	pole	cartesian coordinate	polar plane	polar coordinate
In a polar cooridinates θ denotes a ____	distance	area	angle	radius
The polar coordinates is denoted by	S(r , θ)	P(r , θ)	R(r , θ)	Q(r , θ)
The polar angle is denoted by	θ	O	r	P
If the polar equation is r cosθ = 2 then the cartesian equation is	x = -1	x = 2	x = -2	x = 0
The slope of the polar curve = f(θ) is given by ____	2(dy"/dx")	dy'/dx'	dy/dx	dx/dy
Only …………... mapping posses inverse mappings.	one-one and into	one-one	one-one and many one	one-one and onto
If f:A→B is one-one onto, then f^{-1}:B→A is also …………………	one-one and into	one-one	one-one and many one	one-one and onto
If f:A→B is one-one onto, then the inverse mapping of f is ……………...	zero	unique	different	same
If f:X→Y and g:Y→Zthen the ………………..of the function f and g demc	inverse	composite	different	one-one
If f:X→Y and g:Y→Zthen the composite of the function f and g demoted b	$(f_o g)$:X→Z.	$(f_o g)$:X→Y.	$(g_o f)$:y→Z.	$(g_o f)$:X→Z.

Answer

Double integral
(r, θ, ϕ)
0i + 0j +0k
secx
Max and min
πr2h
2(xy+yz+xz)
πab
Vector
Convergence
0
0
3
I
3
-3
▼xA=0
▼.A =0
▼.F =0
distance
cartesian coordinate
angle
P(r , θ)
θ
x = 2
dy/dx
one-one and onto
one-one and onto
unique
composite
$(g_o f)$:X→Z.

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DEPARTMENT OF MATHEMATICS

Subject: Multivariate Calculus	**Semester: V**	**L T P C**
Subject Code: 17MMU503A	**Class: III-B.Sc Mathematics**	**4 2 0 6**

UNIT IV

Line integrals: Applications of line integrals, Mass and Work. Fundamental theorem for line integrals, conservative vector fields, independence of path.

Reference Book:

R1: Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.

R2: Marsden E., Tromba A.J. and Weinstein A., (2005). Basic Multivariable Calculus, Springer (SIE), Indian reprint, New Delhi.

R3: James Stewart., (2001). Multivariable Calculus, Concepts and Contexts, Second Edition, Brooks Cole,Thomson Learning, USA.

LINE INTEGRALS

2 DEFINITION If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

$$\boxed{3} \qquad \int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

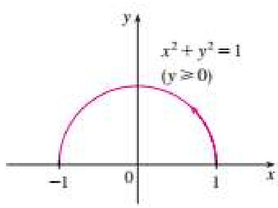


FIGURE 3

SOLUTION In order to use Formula 3, we first need parametric equations to represent C. Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \qquad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leq t \leq \pi$. (See Figure 3.) Therefore Formula 3 gives

$$\int_C (2 + x^2 y)\, ds = \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\, dt = \left[2t - \frac{\cos^3 t}{3} \right]_0^\pi$$

$$= 2\pi + \tfrac{2}{3}$$

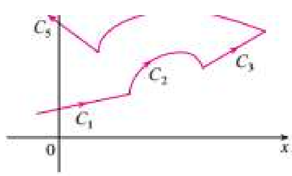


FIGURE 4
A piecewise-smooth curve

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

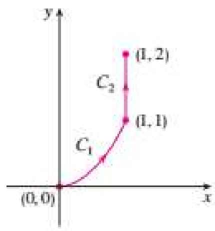


FIGURE 5
$C = C_1 \cup C_2$

EXAMPLE 2 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

SOLUTION The curve C is shown in Figure 5. C_1 is the graph of a function of x, so we can choose x as the parameter and the equations for C_1 become

$$x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$$

Therefore

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2x\sqrt{1 + 4x^2}\,dx$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5\sqrt{5} - 1}{6}$$

On C_2 we choose y as the parameter, so the equations of C_2 are

$$x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$$

and
$$\int_{C_2} 2x\,ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \int_1^2 2\,dy = 2$$

Thus
$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5} - 1}{6} + 2$$

Any physical interpretation of a line integral $\int_C f(x, y)\,ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C. Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*)\,\Delta s_i$ and so the total mass of the wire is approximately $\Sigma\,\rho(x_i^*, y_i^*)\,\Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\,\Delta s_i = \int_C \rho(x, y)\,ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

4
$$\bar{x} = \frac{1}{m}\int_C x\,\rho(x, y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\,\rho(x, y)\,ds$$

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

SOLUTION As in Example 1 we use the parametrization $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant \pi$, and find that $ds = dt$. The linear density is

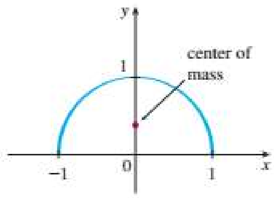
$$\rho(x, y) = k(1 - y)$$

where k is a constant, and so the mass of the wire is

$$m = \int_C k(1 - y)\,ds = \int_0^\pi k(1 - \sin t)\,dt = k\Big[t + \cos t\Big]_0^\pi = k(\pi - 2)$$

From Equations 4 we have

$$\bar{y} = \frac{1}{m}\int_C y\,\rho(x, y)\,ds = \frac{1}{k(\pi - 2)}\int_C y\,k(1 - y)\,ds$$



$$= \frac{1}{\pi - 2}\int_0^{\pi} (\sin t - \sin^2 t)\, dt = \frac{1}{\pi - 2}\left[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{\pi}$$

$$= \frac{4 - \pi}{2(\pi - 2)}$$

By symmetry we see that $\bar{x} = 0$, so the center of mass is

$$\left(0, \frac{4 - \pi}{2(\pi - 2)}\right) \approx (0, 0.38)$$

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2. They are called the **line integrals of f along C with respect to x and y**:

5 $$\int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

6 $$\int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

When we want to distinguish the original line integral $\int_C f(x, y)\, ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\, dt$, $dy = y'(t)\, dt$.

7 $$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

resentation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

8 $$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

EXAMPLE 4 Evaluate $\int_C y^2\, dx + x\, dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

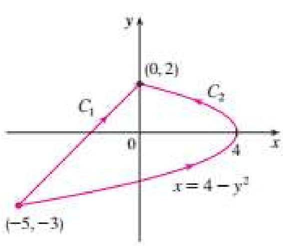


FIGURE 7

SOLUTION

(a) A parametric representation for the line segment is

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \leqslant t \leqslant 1$$

(Use Equation 8 with $\mathbf{r}_0 = \langle -5, -3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.) Then $dx = 5\, dt$, $dy = 5\, dt$, and Formulas 7 give

$$\int_{C_1} y^2\, dx + x\, dy = \int_0^1 (5t - 3)^2(5\, dt) + (5t - 5)(5\, dt)$$

$$= 5\int_0^1 (25t^2 - 25t + 4)\, dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

(b) Since the parabola is given as a function of y, let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \qquad y = y \qquad -3 \leqslant y \leqslant 2$$

Then $dx = -2y\,dy$ and by Formulas 7 we have

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4 - y^2)\,dy$$

$$= \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^{2} = 40\tfrac{5}{6}$$

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 16.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If $-C_1$ denotes the line segment from $(0, 2)$ to $(-5, -3)$, you can verify, using the parametrization

$$x = -5t \qquad y = 2 - 5t \qquad 0 \leqslant t \leqslant 1$$

that

$$\int_{-C_1} y^2\,dx + x\,dy = \tfrac{5}{6}$$

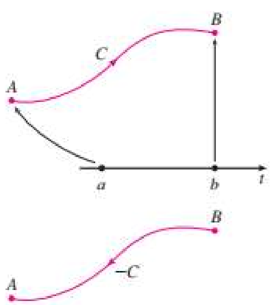


FIGURE 8

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. (See Figure 8, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.)

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y)\,dx = -\int_C f(x, y)\,dx \qquad \int_{-C} f(x, y)\,dy = -\int_C f(x, y)\,dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\,ds = \int_C f(x, y)\,ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

LINE INTEGRALS IN SPACE

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \leqslant t \leqslant b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

We evaluate it using a formula similar to Formula 3:

9 $$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

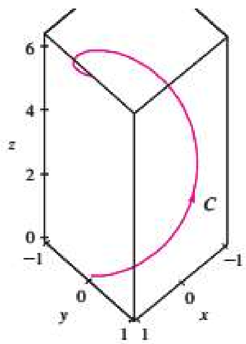


FIGURE 9

EXAMPLE 5 Evaluate $\int_C y \sin z\, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

SOLUTION Formula 9 gives

$$\int_C y \sin z\, ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \sqrt{2} \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t)\, dt$$

$$= \frac{\sqrt{2}}{2} \left[t - \tfrac{1}{2} \sin 2t\right]_0^{2\pi} = \sqrt{2}\,\pi$$

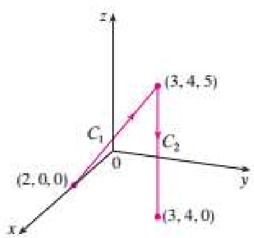


FIGURE 10

EXAMPLE 6 Evaluate $\int_C y\, dx + z\, dy + x\, dz$, where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).

SOLUTION The curve C is shown in Figure 10. Using Equation 8, we write C_1 as

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2 + t, 4t, 5t\rangle$$

or, in parametric form, as

$$x = 2 + t \qquad y = 4t \qquad z = 5t \qquad 0 \leqslant t \leqslant 1$$

Thus

$$\int_{C_1} y\, dx + z\, dy + x\, dz = \int_0^1 (4t)\, dt + (5t)4\, dt + (2 + t)5\, dt$$

$$= \int_0^1 (10 + 29t)\, dt = 10t + 29 \frac{t^2}{2}\Big]_0^1 = 24.5$$

Likewise, C_2 can be written in the form

$$\mathbf{r}(t) = (1 - t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle = \langle 3, 4, 5 - 5t\rangle$$

or $$x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \leqslant t \leqslant 1$$

Then $dx = 0 = dy$, so

$$\int_{C_2} y\, dx + z\, dy + x\, dz = \int_0^1 3(-5)\, dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y\, dx + z\, dy + x\, dz = 24.5 - 15 = 9.5$$

LINE INTEGRALS OF VECTOR FIELDS

13 DEFINITION Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

■ Figure 12 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

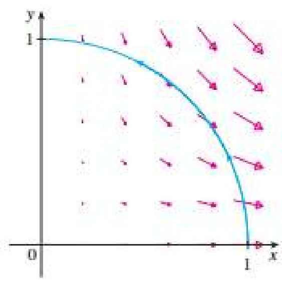


FIGURE 12

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leq t \leq \pi/2$.

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t \sin t\,\mathbf{j}$$

and $$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Therefore the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{\pi/2} (-2\cos^2 t \sin t)\,dt$$

$$= 2\,\frac{\cos^3 t}{3}\Big]_0^{\pi/2} = -\frac{2}{3}$$ ■

NOTE Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector $\mathbf{T}$ is replaced by its negative when C is replaced by $-C$.

■ Figure 13 shows the twisted cubic C in Example 8 and some typical vectors acting at three points on C.

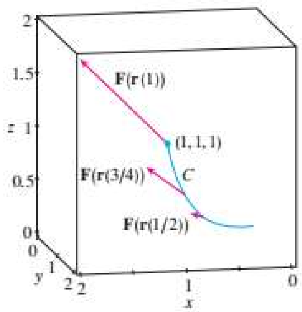


FIGURE 13

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leq t \leq 1$$

SOLUTION We have

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

Thus $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^1 (t^3 + 5t^6)\,dt = \frac{t^4}{4} + \frac{5t^7}{7}\Big]_0^1 = \frac{27}{28}$$ ■

Note:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

2 THEOREM Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Let's prove Theorem 2 for this case.

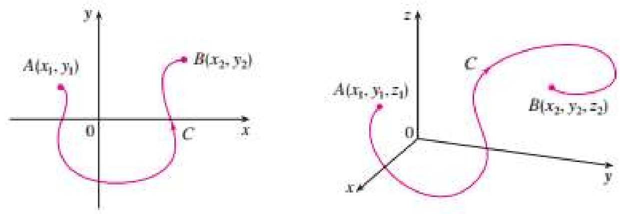


FIGURE 1

PROOF OF THEOREM 2 Using Definition 16.2.13, we have

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad \text{(by the Chain Rule)} \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

3 THEOREM $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

4 THEOREM Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

5 THEOREM If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

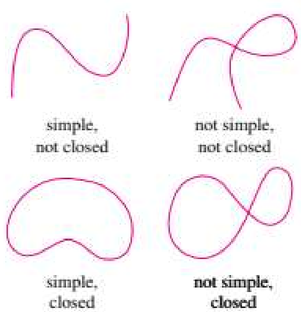
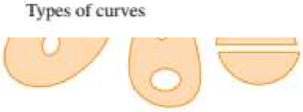


FIGURE 6
Types of curves

The converse of Theorem 5 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region D such



regions that are not simply-connected

FIGURE 7

6 THEOREM Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

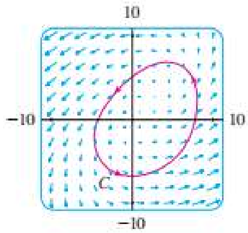


FIGURE 8

Figures 8 and 9 show the vector fields in Examples 2 and 3, respectively. The vectors in Figure 8 that start on the closed curve C all appear to point in roughly the same direction as C. So it looks as if $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$ and therefore $\mathbf{F}$ is not conservative. The calculation in Example 2 confirms this impression. Some of the vectors near the curves C_1 and C_2 in Figure 9 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that line integrals around all closed paths are 0. Example 3 shows that $\mathbf{F}$ is indeed conservative.

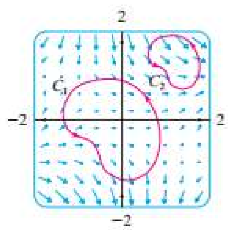


FIGURE 9

EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

Since $\partial P/\partial y \neq \partial Q/\partial x$, $\mathbf{F}$ is not conservative by Theorem 5. ■

EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

Also, the domain of $\mathbf{F}$ is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected. Therefore we can apply Theorem 6 and conclude that $\mathbf{F}$ is conservative. ■

In Example 3, Theorem 6 told us that $\mathbf{F}$ is conservative, but it did not tell us how to find the (potential) function f such that $\mathbf{F} = \nabla f$. The proof of Theorem 4 gives us a clue as to how to find f. We use "partial integration" as in the following example.

EXAMPLE 4

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leq t \leq \pi$$

SOLUTION

(a) From Example 3 we know that $\mathbf{F}$ is conservative and so there exists a function f with $\nabla f = \mathbf{F}$, that is,

7 $$f_x(x, y) = 3 + 2xy$$

8 $$f_y(x, y) = x^2 - 3y^2$$

Integrating (7) with respect to x, we obtain

9 $$f(x, y) = 3x + x^2 y + g(y)$$

Notice that the constant of integration is a constant with respect to x, that is, a function of y, which we have called $g(y)$. Next we differentiate both sides of (9) with respect to y:

10 $$f_y(x, y) = x^2 + g'(y)$$

Comparing (8) and (10), we see that

$$g'(y) = -3y^2$$

Integrating with respect to y, we have

$$g(y) = -y^3 + K$$

where K is a constant. Putting this in (9), we have

$$f(x, y) = 3x + x^2 y - y^3 + K$$

as the desired potential function.

(b) To use Theorem 2 all we have to know are the initial and terminal points of C, namely, $\mathbf{r}(0) = (0, 1)$ and $\mathbf{r}(\pi) = (0, -e^{\pi})$. In the expression for $f(x, y)$ in part (a), any value of the constant K will do, so let's choose $K = 0$. Then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^{\pi}) - f(0, 1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f, then

11 $$f_x(x, y, z) = y^2$$

12 $$f_y(x, y, z) = 2xy + e^{3z}$$

13 $$f_z(x, y, z) = 3ye^{3z}$$

Integrating (11) with respect to x, we get

14 $$f(x, y, z) = xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x. Then differentiating (14) with respect to y, we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with (12) gives

$$g_y(y, z) = e^{3z}$$

Thus $g(y, z) = ye^{3z} + h(z)$ and we rewrite (14) as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with (13), we obtain $h'(z) = 0$ and therefore $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$. ■

Possible Questions

PART-B (2 Mark)

1. Define Mass.
2. State Fundamental theorem for line integrals.
3. Write about work.
4. Define Limit of the function $f(x, y, z)$.
5. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$
6. Write the implicit formula
7. Define chain rule.
8. Define domain and range.

PART-C (8 Mark)

1. Evaluate $\int_C y \sin z \, dS$, where C is the circular helix given by the equation $x = \cos t, y = \sin t, z = t, 0 \le t \le 2\pi$.
2. Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).
3. Evaluate $\int_C y^2\,dx + x\,dy$, where (i) C = C_1 is the line segment from (-5, -3) to (0, 2) and (ii) C = C_2 is the arc of the parabola $x = 4 - y^2$ from (-5, -3) to (0, 2).
4. Find the work done by the force field $F(x, y) = x^2 \vec{i} - xy \vec{j}$ in moving a particle along the quarter circle $r(t) = \cos t \vec{i} + \sin t \vec{j}, 0 \le t \le \pi/2$.
5. A wire takes the shape of semi circle $x^2 + y^2 = 1, y \ge 0$, and is thicker near its base then near the top. Find the center of mass of the wire if the linear density at any point is proportional to is distance the line y = 1.
6. Evaluate $\int_C F.dr$, where $F(x, y, z) = xy\vec{i} + yz\vec{j}\, zx\vec{k}$ and C is the twisted cube given by $x = t, y = t^2, z = t^3, 0 \le t \le 1$.
7. Evaluate $\int_C 2x\,dS$, where C consist of arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).
8. State and prove fundamental theorem for line integral.
9. Evaluate $\int_C (2 + x^2 y)\,dS$ where C is the upper half of the unit circle $x^2 + y^2 = 1$.
10. If $F(x, y, z) = y^2 \vec{i} + (2xy + e^{3z})\vec{j} + 3ye^{3z}\vec{k}$, find a function f such that $\nabla f = F$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Multivariate Calculus **Subject Code: 17MMU503A**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
If F is conservative then_____	Curl F = 0	Div F = 0	▼. F	▼. F = 0	Curl F = 0
The average value of f is _________	(1/(b-a)) ∫ f(x) dx as x in (a,b)	(1/(b-a)) ∫ f(x) dx as x in (b, a)	(1/(b-a)) ∫ f(x) dx as x in (c,d)	(1/(b+a)) ∫ f(x) dx as x in (a,b)	(1/(b-a)) ∫ f(x) dx as x in (a,b)
In the fundamental theorem of line integral gives ______	∫c▼f . dr = f (r(b)- f (r(a))	∫c▼f . dr = f (r(b) +f (r(a))	∫c▼f *dr = f (r(b) +f (r(a))	∫c▼f *dr = f (r(b) - f (r(a))	∫c▼f . dr = f (r(b)- f (r(a))
The vector field F is called a conservative field if ______	▼F= f	▼. F= f	▼ * F= f	▼.▼f	▼F= f
If F = ▼f then F is called _______________	Mass	Center of mass	Moment	Conservative field	Conservative field
∫c F.dr is independent of path in D iff ∫c F.dr= 0 for every_______ in D.	Open path	Closed path	Path	Open curve	Closed path
The process ∫c f(x) dx = ∫c' f(g(u)) g'(u) du is called ____________	Moment	Mass	Change of variable	Center of mass	Change of variable
A single valued function f(x,y,z) is said to be a hormonic function if its second partial derivatives exist and are continuous and if the function satisfies the ------ equation	Integral	Laplace	continuous	Differential	Laplace
∫sec²x dx = ______	tanx	sinx	(-cos x)	(-sinx)	tanx
∫(1/x) dx = ___	x	log x	2x	1 - x	log x
∫cotx dx = _______	log cosx	log tanx	log secx	logsinx	logsinx
∫sec x dx = _______	secx + tanx	log [secx + tanx]	secx + cos x	log [secx + cosec x]	log [secx + tanx]
∫log x dx = _______	x log x	log x + x	x log x - x	x log x + x	x log x - x
$\lim_{x\to 0}$ (tanx / x) =	0	(-1)	1	2	1
∫sech²x dx = _______	sinhx	cothx	tanhx	sechx	tanhx
∫sec x dx = _______	secx + tanx	log [secx + tanx]	secx + cos x	log [secx + cosec x]	log [secx + tanx]
∫cotx dx = _______	log cosx	log tanx	log secx	logsinx	logsinx
∫sechx tanhx dx = _______	(-cosh x)	(-sech x)	(-sinh x)	(-tanh x)	(-sech x)
∫cosech x coth x dx = _______	(-cosech x)	(-tanh x)	(-sech x)	(-sinh x)	(-cosech x)

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DEPARTMENT OF MATHEMATICS

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UNIT V

Green's theorem: Surface integrals, integrals over parametrically defined surfaces. Stoke's theorem, The Divergence theorem.

Reference Book:

R1: Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.

R3: Kenneth Hoffman., Ray Kunze., (2003). Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.

GREEN'S THEOREM

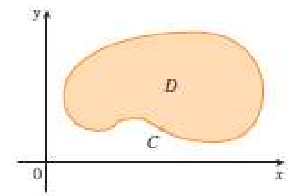


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See Figure 2.)

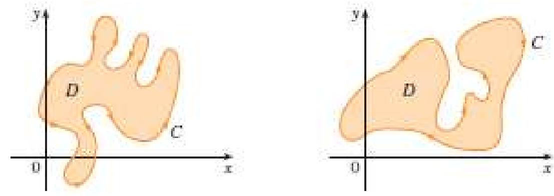


FIGURE 2 (a) Positive orientation (b) Negative orientation

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

■ Recall that the left side of this equation is another way of writing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

■ Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH D IS A SIMPLE REGION Notice that Green's Theorem will be proved if we can show that

$$\boxed{2} \qquad \int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

and

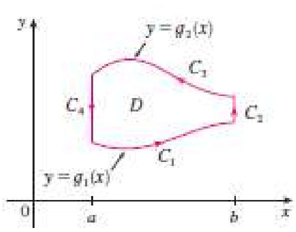
$$\boxed{3} \qquad \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\boxed{4} \qquad \iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,dy\,dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx$$



where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up C as the union of the four curves C_1, C_2, C_3, and C_4 shown in Figure 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \leq x \leq b$. Thus

$$\int_{C_1} P(x, y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \leq x \leq b$. Therefore

$$\int_{C_3} P(x, y)\,dx = -\int_{-C_3} P(x, y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\,dx = 0 = \int_{C_4} P(x, y)\,dx$$

Hence

$$\int_C P(x, y)\,dx = \int_{C_1} P(x, y)\,dx + \int_{C_2} P(x, y)\,dx + \int_{C_3} P(x, y)\,dx + \int_{C_4} P(x, y)\,dx$$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$$

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y)\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

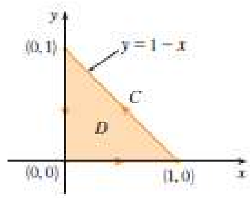


FIGURE 4

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2}\int_0^1 (1 - x)^2\,dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1 = \tfrac{1}{6}$$ □

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \le 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4 + 1}\right) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$

$$= \int_0^{2\pi} \int_0^3 (7 - 3)\,r\,dr\,d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi$$ □

■ Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4\,dA = 4 \cdot \pi(3)^2 = 36\pi$$

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \le t \le 2\pi$. Using the third formula in Equation 5, we have

$$A = \tfrac{1}{2}\int_C x\,dy - y\,dx$$

$$= \tfrac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$ □

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION Notice that although D is not simple, the y-axis divides it into two simple regions (see Figure 7). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

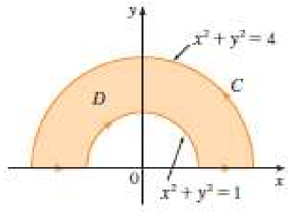


FIGURE 7

Therefore Green's Theorem gives

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2) \right] dA$$

$$= \iint_D y\,dA = \int_0^{\pi} \int_1^2 (r \sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr = \left[-\cos\theta\right]_0^{\pi} \left[\tfrac{1}{3}r^3\right]_1^2 = \frac{14}{3}$$ □

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

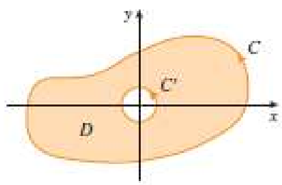


FIGURE 10

SOLUTION Since C is an *arbitrary* closed path that encloses the origin, it's difficult to compute the given integral directly. So let's consider a counterclockwise-oriented circle C' with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 10.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] dA = 0$$

Therefore $$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

that is, $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j}$, $0 \leq t \leq 2\pi$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$ □

CURL AND DIVERGENCE

CURL

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

1 $$\text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Thus the easiest way to remember Definition 1 is by means of the symbolic expression

2 $$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

EXAMPLE 1 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

SOLUTION Using Equation 2, we have

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$ □

■ Most computer algebra systems have commands that compute the curl and divergence of vector fields. If you have access to a CAS, use these commands to check the answers to the examples and exercises in this section.

Recall that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl. The following theorem says that the curl of a gradient vector field is $\mathbf{0}$.

3 THEOREM If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

PROOF We have

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\mathbf{k}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$$

by Clairaut's Theorem. □

■ Notice the similarity to what we know from Section 12.4: $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every three-dimensional vector $\mathbf{a}$.

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

■ Compare this with Exercise 27 in Section 16.3.

This gives us a way of verifying that a vector field is not conservative.

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

SOLUTION In Example 1 we showed that

$$\text{curl}\,\mathbf{F} = -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

This shows that curl $\mathbf{F} \neq \mathbf{0}$ and so, by Theorem 3, $\mathbf{F}$ is not conservative. □

4 THEOREM If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

SOLUTION
(a) We compute the curl of **F**:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= \mathbf{0}$$

Since curl $\mathbf{F} = \mathbf{0}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, $\mathbf{F}$ is a conservative vector field by Theorem 4.

(b) The technique for finding f was given in Section 16.3. We have

5 $$f_x(x, y, z) = y^2z^3$$

6 $$f_y(x, y, z) = 2xyz^3$$

7 $$f_z(x, y, z) = 3xy^2z^2$$

Integrating (5) with respect to x, we obtain

8 $$f(x, y, z) = xy^2z^3 + g(y, z)$$

Differentiating (8) with respect to y, we get $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$, so comparison with (6) gives $g_y(y, z) = 0$. Thus $g(y, z) = h(z)$ and

$$f_z(x, y, z) = 3xy^2z^2 + h'(z)$$

Then (7) gives $h'(z) = 0$. Therefore

$$f(x, y, z) = xy^2z^3 + K$$ □

DIVERGENCE

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

9 $$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Observe that curl **F** is a vector field but div **F** is a scalar field. In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

10 $$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

EXAMPLE 4 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div **F**.

SOLUTION By the definition of divergence (Equation 9 or 10) we have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$ □

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence. The next theorem shows that the result is 0.

11 **THEOREM** If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

PROOF Using the definitions of divergence and curl, we have

■ Note the analogy with the scalar triple product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

$$\begin{aligned}\text{div curl } \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\ &= \frac{\partial^2 R}{\partial x\,\partial y} - \frac{\partial^2 Q}{\partial x\,\partial z} + \frac{\partial^2 P}{\partial y\,\partial z} - \frac{\partial^2 R}{\partial y\,\partial x} + \frac{\partial^2 Q}{\partial z\,\partial x} - \frac{\partial^2 P}{\partial z\,\partial y} \\ &= 0\end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem. □

PARAMETRIC SURFACES

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that

1 $$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that

2 $$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations 2 are called **parametric equations** of S. Each choice of u and v gives a point on S; by making all choices, we get all of S. In other words, the surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D. (See Figure 1.)

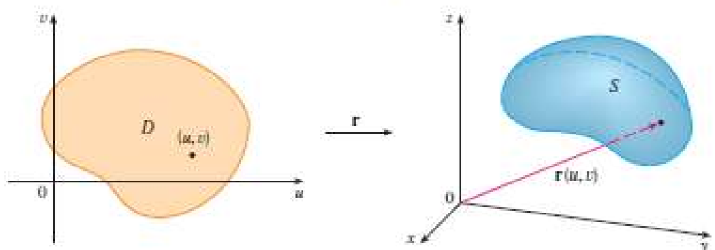


FIGURE 1
A parametric surface

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$$

SOLUTION The parametric equations for this surface are

$$x = 2\cos u \qquad y = v \qquad z = 2\sin u$$

So for any point (x, y, z) on the surface, we have

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$

This means that vertical cross-sections parallel to the xz-plane (that is, with y constant) are all circles with radius 2. Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis. (See Figure 2.) □

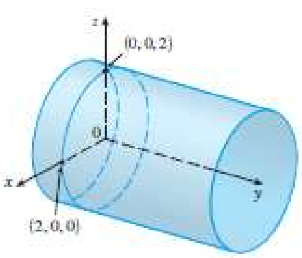


FIGURE 2

TANGENT PLANES

We now find the tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. (See Figure 11.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

4 $$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$

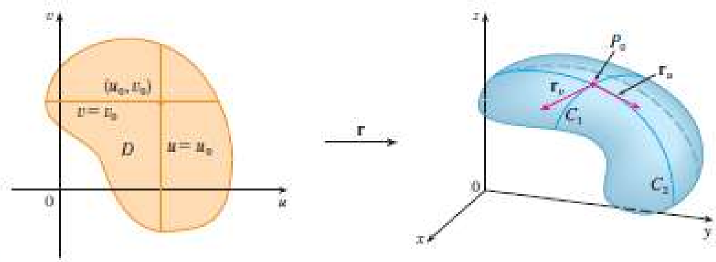


FIGURE 11

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

5 $$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth** (it has no "corners"). For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

■ Figure 12 shows the self-intersecting surface in Example 9 and its tangent plane at (1, 1, 3).

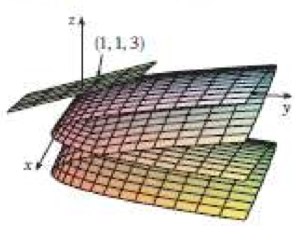


FIGURE 12

EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point (1, 1, 3).

SOLUTION We first compute the tangent vectors:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} = 2u\,\mathbf{i} + \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = 2v\,\mathbf{j} + 2\,\mathbf{k}$$

Thus a normal vector to the tangent plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\mathbf{i} - 4u\,\mathbf{j} + 4uv\,\mathbf{k}$$

Notice that the point (1, 1, 3) corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector there is

$$-2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$$

Therefore an equation of the tangent plane at (1, 1, 3) is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

or $$x + 2y - 2z + 3 = 0$$ □

FIGURE 14
Approximating a patch by a parallelogram

SURFACE AREA

6 DEFINITION If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $\quad \mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$

EXAMPLE 10 Find the surface area of a sphere of radius a.

SOLUTION In Example 4 we found the parametric representation

$$x = a \sin\phi \cos\theta \qquad y = a \sin\phi \sin\theta \qquad z = a\cos\phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$$

Thus

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2\sin\phi$$

since $\sin\phi \ge 0$ for $0 \le \phi \le \pi$. Therefore, by Definition 6, the area of the sphere is

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta$$

$$= a^2\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\, d\phi = a^2(2\pi)2 = 4\pi a^2$$ □

SURFACE AREA OF THE GRAPH OF A FUNCTION

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

so $\qquad \mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$

and

7 $$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Thus we have

8 $$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

■ Notice the similarity between the surface area formula in Equation 9 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
from Section 8.1.

and the surface area formula in Definition 6 becomes

9 $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

EXAMPLE 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. (See Figure 15.) Using Formula 9, we have

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$
$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

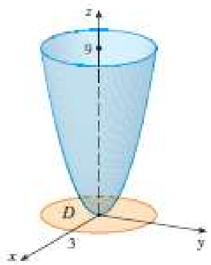


FIGURE 15

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr$$
$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right) \quad \square$$

SURFACE INTEGRALS

PARAMETRIC SURFACES

Suppose that a surface S has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

where $$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

are the tangent vectors at a corner of S_{ij}. If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown from Definition 1, even when D is not a rectangle, that

We assume that the surface is covered only once as (u, v) ranges throughout D. The value of the surface integral does not depend on the parametrization that is used.

2 $$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

EXAMPLE 1 Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 4 in Section 16.6, we use the parametric representation

$$x = \sin\phi\cos\theta \quad y = \sin\phi\sin\theta \quad z = \cos\phi \quad 0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi$$

that is, $$\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$$

As in Example 10 in Section 16.6, we can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

Therefore, by Formula 2,

$$\iint_S x^2\, dS = \iint_D (\sin\phi\cos\theta)^2\, |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA$$
$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\cos^2\theta\, d\theta \int_0^{\pi}\sin^3\phi\, d\phi$$
$$= \int_0^{2\pi}\tfrac{1}{2}(1 + \cos 2\theta)\, d\theta \int_0^{\pi}(\sin\phi - \sin\phi\cos^2\phi)\, d\phi$$
$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi} = \frac{4\pi}{3} \quad \square$$

Here we use the identities
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$
$$\sin^2\phi = 1 - \cos^2\phi$$
Instead, we could use Formulas 64 and 67 in the Table of Integrals.

The total mass of the sheet is

$$m = \iint_S \rho(x, y, z)\, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x, y, z)\, dS \qquad \bar{y} = \frac{1}{m}\iint_S y\rho(x, y, z)\, dS \qquad \bar{z} = \frac{1}{m}\iint_S z\rho(x, y, z)\, dS$$

GRAPHS

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have $\quad \mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$

Thus

3 $$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and $$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

4 $$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

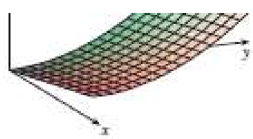


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$. (See Figure 2.)

SOLUTION Since

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y$$

Formula 4 gives

$$\iint_S y\, dS = \iint_D y\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\, dy\, dx$$

$$= \int_0^1 dx\, \sqrt{2}\int_0^2 y\sqrt{1 + 2y^2}\, dy$$

$$= \sqrt{2}\left(\tfrac{1}{4}\right)\tfrac{2}{3}(1 + 2y^2)^{3/2}\Big]_0^2 = \frac{13\sqrt{2}}{3}$$ □

V EXAMPLE 3 Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \le 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

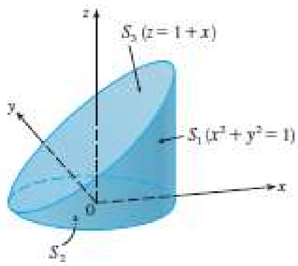


FIGURE 3

SOLUTION The surface S is shown in Figure 3. (We have changed the usual position of the axes to get a better look at S.) For S_1 we use θ and z as parameters (see Example 5 in Section 16.6) and write its parametric equations as

$$x = \cos\theta \qquad y = \sin\theta \qquad z = z$$

where

$$0 \leq \theta \leq 2\pi \qquad \text{and} \qquad 0 \leq z \leq 1 + x = 1 + \cos\theta$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and
$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Thus the surface integral over S_1 is

$$\iint_{S_1} z\,dS = \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\,dA$$
$$= \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 + \cos\theta)^2\,d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$
$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2}$$

Since S_2 lies in the plane $z = 0$, we have

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0$$

The top surface S_3 lies above the unit disk D and is part of the plane $z = 1 + x$. So, taking $g(x, y) = 1 + x$ in Formula 4 and converting to polar coordinates, we have

$$\iint_{S_3} z\,dS = \iint_D (1 + x)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{1 + 1 + 0}\;r\,dr\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (r + r^2\cos\theta)\,dr\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{3}\cos\theta\right) d\theta$$
$$= \sqrt{2}\left[\frac{\theta}{2} + \frac{\sin\theta}{3}\right]_0^{2\pi} = \sqrt{2}\,\pi$$

Therefore

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$$
$$= \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi = \left(\tfrac{3}{2} + \sqrt{2}\right)\pi$$

STOKES' THEOREM

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

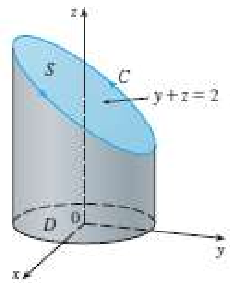


FIGURE 3

SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\mathbf{k}$$

Although there are many surfaces with boundary C, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 1$ and so using Equation 16.7.10 with $z = g(x, y) = 2 - y$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin\theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3}\sin\theta\right) d\theta$$

$$= \tfrac{1}{2}(2\pi) + 0 = \pi$$

□

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

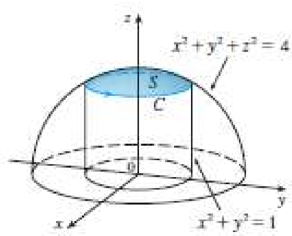


FIGURE 4

SOLUTION To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \qquad 0 \leq t \leq 2\pi$$

so

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Also, we have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3}\cos t\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \cos t \sin t\,\mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t)\, dt$$

$$= \sqrt{3} \int_0^{2\pi} 0\, dt = 0$$

□

THE DIVERGENCE THEOREM

THE DIVERGENCE THEOREM Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

PROOF Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

If $\mathbf{n}$ is the unit outward normal of S, then the surface integral on the left side of the Divergence Theorem is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three equations:

2 $$\iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV$$

3 $$\iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV$$

4 $$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$$

To prove Equation 4 we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. By Equation 15.6.6, we have

$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x, y, z)\, dz \right] dA$$

and therefore, by the Fundamental Theorem of Calculus,

5 $$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y)) \right] dA$$

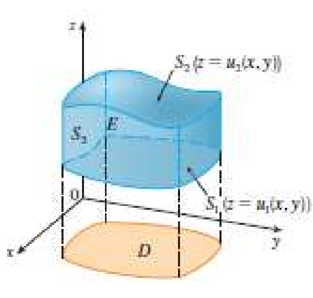


FIGURE 1

The boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and possibly a vertical surface S_3, which lies above the boundary curve of D. (See Figure 1. It might happen that S_3 doesn't appear, as in the case of a sphere.) Notice that on S_3 we have $\mathbf{k} \cdot \mathbf{n} = 0$, because $\mathbf{k}$ is vertical and $\mathbf{n}$ is horizontal, and so

$$\iint_{S_3} R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{S_3} 0\, dS = 0$$

Thus, regardless of whether there is a vertical surface, we can write

6 $$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{S_1} R\,\mathbf{k} \cdot \mathbf{n}\, dS + \iint_{S_2} R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward, so from Equation 16.7.10 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$) we have

$$\iint_{S_2} R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D R(x, y, u_2(x, y))\, dA$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal $\mathbf{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} R\,\mathbf{k} \cdot \mathbf{n}\, dS = -\iint_D R(x, y, u_1(x, y))\, dA$$

Therefore Equation 6 gives

$$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right] dA$$

Comparison with Equation 5 shows that

$$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$$

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION First we compute the divergence of $\mathbf{F}$:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leq 1$. Thus the Divergence Theorem gives the flux as

■ The solution in Example 1 should be compared with the solution in Example 4 in Section 16.7.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B \text{div}\,\mathbf{F}\, dV = \iiint_B 1\, dV = V(B) = \tfrac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$ □

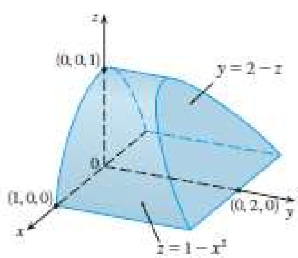


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

SOLUTION It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of S.) Furthermore, the divergence of $\mathbf{F}$ is much less complicated than $\mathbf{F}$ itself:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

Therefore we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \left\{(x, y, z) \mid -1 \leq x \leq 1,\ 0 \leq z \leq 1 - x^2,\ 0 \leq y \leq 2 - z\right\}$$

Then we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\, dV = \iiint_E 3y\, dV$$

$$= 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\, dy\, dz\, dx = 3\int_{-1}^{1}\int_0^{1-x^2} \frac{(2 - z)^2}{2}\, dz\, dx$$

$$= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2 - z)^3}{3}\right]_0^{1-x^2} dx = -\tfrac{1}{2}\int_{-1}^{1}\left[(x^2 + 1)^3 - 8\right] dx$$

$$= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7)\, dx = \tfrac{184}{35}$$ □

Possible Questions

PART-B (2 Mark)

1. What is Surface integral.
2. If $f(x,y)=\sin\left(\frac{x}{1+y}\right)$,calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
3. Write down the Linearization formula.
4. Define Define partial derivative
5. Find y' if $x^3+y^3=6xy$
6. State Green's theorem.
7. State Gauss Divergence theorem.
8. State Stoke's theorem.

PART-C (8 Mark)

1. Evaluate $\int_C F.dr$, where $F(x,y,z)=-y^2\vec{i}+x\vec{j}+z^2\vec{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$.
2. The temperature U in a metal ball is proportional to the square of the distance from center of the ball. Find the rate of heat flow across a sphere S of radius a with center at all the center of the ball.
3. Evaluate $\iint_S F.dS$, where $F(x,y,z)=y\vec{i}+x\vec{j}+z\vec{k}$ and S is the boundary of solid region E enclosed by the paraboloid $z=1-x^2-y^2$ and the plane $z=0$.
4. Find the tangent plane to the surface with parametric equations $x=u^2, y=v^2, z=u+2v$ at the point (1, 1, 3).
5. Find the surface area of the sphere with radius a.
6. Evaluate $\iint_S y\,dS$, where S is the surface $z=x+y^2, 0\le x\le 1, 0\le y\le 2$.
7. Find the area of the part of paraboloid $z=x^2+y^2$ that lies under the plane z=9.
8. Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2\le 1$ in the plane z=0 and whose top S_3 is the part of the plane $z=1+x$ that lies above S_2.
9. Find the surface integral $\iint_S x^2 dS$ where S is the unit sphere $x^2+y^2+z^2=1$.

10. Find the flux of the vector field $F(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Multivariate Calculus **Subject Code: 17MMU503A**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
∫ Pdx +Qdy = ∬ (Qx - Py) dA is ________	Green's theorem	Red theorem	Stock's theorem	Divergence theorem	Green's theorem
If Curl F = 0 then F is ________	Irrotational	Solenoidal	Divergence	Curl	Irrotational
If F(x, y) = P(x, y) **i** +Q(x, y) **j** is a conservative vector field, where P, Q have first order partial derivative then we have ________	$P_{y}+Q_{x}=0$	$P_x + Q_y = 0$	$\mathbf{P_y = Q_x}$	$P_x = Q_y$	$\mathbf{P_y = Q_x}$
The potential energy and kinetic energy is_________	Law of potential	Law of Kinetic	Law of conservation of energy	Law of vector field	Law of conservation of energy
If F(x,y) = (x-y) **i** + (x-2) **j** is ________	a vector field	not a vector field	a conservative field	not a conservative field	not a conservative field
If f is a function f that has continuous second order partial derivatives then _________	Div f =0	▼× f = 0	Div(▼f) = 0	curl(▼f) = 0	curl(▼f) = 0
If r = xi + yj + zk then, ▼ (1/ r) = -----	$-r/r^2$	r/r^3	$1/r^3$	$-r/r^3$	$-r/r^3$
The divergence of a curl of a vector is ---	one	three	zero	two	three
If A =A_1i +A_2j + A_3k, where A1, A2, A3 have continuous second partials, then ▼ . (▼ x A) =---	2	1	-1	-2	2
If ▼ . V = 0, then the vector V is said to be	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Solenoidal vector
If ▼ x V = 0, then the vector V is said to be	Irrotational vector	Position vector	Zero vector	Solenoidal vector	Irrotational vector
The vector A = $x^2z^2i + xyz^2j - xz^3k$ is	Irrotational vector	Solenoidal vector	Zero vector	Position vector	Solenoidal vector
If f is a hormonic function, then ▼f is	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Zero vector
If A and B are irrotational, then AxB is	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Zero vector
If A is irrotational, then ▼xA is	1	-1	2	0	1
If A is solenoidal, then ▼.A is	1	-1	0	-2	-2
div (curl A) = ----	0	1	-1	2	0
Curl (grad f) = ---	1	non zero	2	0	1
Curl $(\vec{A}+\vec{B})$ = -----	Curl $\vec{A}$ – Curl B	curl $\vec{A}$ + Curl B	curl $\vec{A}$ * Curl B	curl $\vec{A}$ / Curl B	curl $\vec{A}$ + Curl B
d/dt(A.B) = -----	A.dA/dt +dB/dt. B	A.dB/dt +dA/dt .B	A.dA/dt - dB/dt	A.dA/dt + dB/dt	A.dA/dt - dB/dt
If $\vec{F}$ is constant vector, then curl F = ----	1	2	non zero	0	0
▼ . (▼f) =------	$▼^2$f	0	▼f	f	$▼^2$f
Grad r^n = -----	nr^{n-1} r	nr^{n-2} r	$(n-1)r^{n-2}$ r	r^{n-1} r	nr^{n-2} r

Reg no-----------------

(17MMU503A)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore-21

DEPARTMENT OF MATHEMATICS

Fifth Semester

Multivariate Calculus

I Internal Test - July'2019

Date: **Time: 2 Hours**

Class: III B.Sc Mathematics **Maximum:50 Marks**

PART-A(20x1=20 Marks)

Answer all the Questions:

1. Every codomain element has a preimage then f(x,y) is_____
 a) Bijective b) 1-1 c) Onto d) Reflexive

2. Relation is a subset of ____________
 a)Function b) 1-1 function c) Bijective d) Cartesian product

3. If z= f(x, y) then the variable x and y are _____________ .
 a) Independent b) Dependent c) Image d) Function

4. Differentiation of sinhx = ______
 a) (- coshx) b) sinh2x c) coshx d) (-sinhx)

5. If z= f(x, y) then z is _____________ variable .
 a) Dependent b) Independent c) Image d) Isolated

6. $\cosh^2 x - \sinh^2 x$ = ______
 a) 1 b) 0 c) cosh2x d) sinh2x

7.The element of $\Re^3$ is___________
 a) (1, 2) b) (2, 1) c) (x, y) d) (x, y, z)

8.The Level curve of f(x, y) is __________
 a) x b) 2x c) x^2 d) 15

9. The curve of a function f(x, y, z) = k is called __________
 a) Identity function b) Level curves c) Family of curves d) 1-1

10.The range of f(x) = 2x for every x in N is _________
 a) 2Z b) 2R c) 2N d) N

11.If the partial derivatives are continuous then_________
 a) $F_{xy} \neq F_{yx}$ b) $F_{xy} = F_{yx}$ c) $F_x = F_y$ d) $F_x \neq F_y$

12. In a polar coordinate, r denotes a ____
 a) distance b) area c) angle d) radius

13.The Clairaut's theorem is ____________ if partial derivatives are continuous.
 a) $F_{xy} \neq F_{yx}$ b) $F_x \neq F_y$ c) $F_x = F_y$ d) $F_{xy} = F_{yx}$

14. $\lim_{x \to 0} \frac{\sin x}{x} =$
 a) 0 b) (-1) c) 1 d) 2

15. In a polar coordinates θ denotes a ____
 a) distance b) area c) angle d) radius

16.If $U_{xx} + U_{yy}$ =0 then U is Called _______________
 a) Laplace b) Circle c) Harmonic d) Heat equation

17. The function f is Local minimum at (a, b) if __________

a) $f(x, y) \leq f(a, b)$ b) $f(x, y) > f(a, b)$

c) $f(x, y) < f(a, b)$ d) $f(x, y) \geq f(a, b)$

18. If $f(x, y) \to f(a, b)$ as $(x, y) \to (a, b)$ then f is _________

a) Not continuous b) Function c) Relation d) Continuous

19. From the below the functions of two variable is ____

a) f(x, y, z) b) z = f(x) c) y = f(x) d) z = f(x,y)

20. If $F_{xx} F_{yy} - (F_{xy})^2 > 0$ and $F_{xx} > 0$ then f(a, b) is __________

a) Local minimum b) Local maximum

c) absolute maximum d) absolute minimum

PART-B (3x2=6 Marks)

Answer all the Questions:

21. Sketch the level curves of the function f(x,y) = 6-3x-2y for the values k=-6, 0, 6, 12.

22. Define continuous function for two variables.

23. State Clairaut's theorem

PART-B(3x8=24 Marks)

Answer all the Questions:

24. (a) Find $\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2}$ if it exist.

(OR)

(b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is implicitly as a function of x and y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$.

25. (a) Find the directional derivative of the function $f(x,y) = x^2 y^3 - 4y$ at the point (2,-1) in the direction of the vector $v = 2\vec{i} + 5\vec{j}$.

(OR)

(b) Find and classify the critical points of the function

$$f(x, y) = 10x^2 y - 5x^2 - 4y^2 - x^4 - 2y^4$$

26. (a) Find the shortest distance from the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

(OR)

(b) Find the Extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.