KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University)

(Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

Semester – I

19MMP103 **NUMERICAL ANALYSIS** **4H – 4C**

Instruction Hours / week: L: 4 T: 0 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- To develop the working knowledge on different numerical techniques.
- To solve algebraic and transcendental equations.
- Appropriate numerical methods to solve differential equations.

Course Outcomes (COs)

On successful completion of this course, students will be able to

1. Identify the concept of numerical differentiation and integration.
2. Provide information on methods of iteration.
3. Solve ordinary differential equations by using euler and modified euler method.
4. Study in detail the concept of boundary value problems.
5. Attain mastery in the numerical solution of partial differential equations.

UNIT I

SOLUTIONS OF NON LINEAR EQUATIONS

Newton's method-Convergence of Newton's method- Bairstow's method for quadratic factors. Numerical Differentiation and Integration: Derivatives from difference tables – Higher order derivatives – divided difference. Trapezoidal rule– Romberg integration – Simpson's rules.

UNIT II

SOLUTIONS OF SYSTEM OF EQUATIONS

The Elimination method: Gauss Elimination and Gauss Jordan Methods – LU decomposition method. Methods of Iteration: Gauss Jacobi and Gauss Seidal iteration-Relaxation method.

UNIT III

SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

One step method: Euler and Modified Euler methods–Rungekutta methods. Multistep methods: Adams Moulton method – Milne's method

UNIT IV

BOUNDARY VALUE PROBLEMS AND CHARACTERISTIC VALUE PROBLEMS

The shooting method: The linear shooting method – The shooting method for non-linear systems. Characteristic value problems –Eigen values of a matrix by Iteration-The power method.

UNIT V

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Classification of Partial Differential Equation of the second order – Elliptic Equations. Parabolic equations: Explicit method – The Crank Nicolson difference method. Hyperbolic equations – solving wave equation by Explicit Formula.

SUGGESTED READINGS

1. Gerald, C. F., and Wheatley. P. O., (2009). Applied Numerical Analysis, Seventh edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.
2. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .
3. Burden R. L., and Douglas Faires.J,(2014). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.
4. Sastry S.S., (2009). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: Dr.M.M.SHANMUGAPRIYA
SUBJECT NAME: NUMERICAL ANALYSIS SUB.CODE:19MMP103
SEMESTER: I CLASS : I M.Sc (MATHEMATICS)

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT – I	
1.	1	Introduction and basics of nonlinear equations	S1: Ch 1: Pg: 32-33
2.	1	Newton‘s method- Introduction and Problems	S3: Ch 2: Pg: 67-69
3.	1	Convergence of Newton‘s method	S3: Ch 2: Pg: 69-72
4.	1	Bairstow’s method for quadratic factors	S2: Ch 2: Pg: 90-93
5.	1	Derivative from difference table and higher order derivatives	S4: Ch 3: Pg: 63-72
6.	1	Divided differences-Problems	S1: Ch 3: Pg: 157-160
7.	1	Trapezoidal rule and Simpson’s rule -Problems	S4: Ch 5: Pg: 197-202,205-208
8.	1	Romberg’s Integration	S4: Ch 5: Pg: 202-204,208-209
9.	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit I =09		
		UNIT – II	
1.	1	Solutions of system of Equations: Introduction and Gauss Elimination method-Procedure & problems	S4: Ch 6: Pg: 255-260
2.	1	Gauss Jordan method: Procedure & Problems	S4: Ch 6: Pg: 260-264

3.	1	LU decomposition method: Procedure & Problems	S4: Ch 6: Pg: 265-269
4.	1	Continuation of problems on LU decomposition method	S2: Ch 3: Pg: 122-127
5.	1	Gauss Jacobi method: Procedure & Problems	S2: Ch 3: Pg: 146-150
6.	1	Gauss Seidal method : Procedure & Problems	S2: Ch 3: Pg: 150-152
7.	1	Relaxation method: Procedure & Problems	S3: Ch 7: Pg: 462-466
8.	1	Continuation of problems on Relaxation method	S1:Ch 2:Pg:169-174
9.	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit II =09		
UNIT –III			
1.	1	Solution of ODE- Introduction	S4: Ch 7: Pg: 295-297
2.	1	Euler method -Derivation and Problems	S4: Ch 7: Pg: 300-303
3.	1	Modified Euler method- Derivation and Problems	S4: Ch 7: Pg: 303-304
4.	1	Runge -Kutta method- Derivation and Problems	S4: Ch 7: Pg: 304-308
5.	1	Continuation of problems on Runge- Kutta method	S2: Ch 6: Pg: 447-456
6.	1	Multistep methods: Adams Moulton method - Problems	S4: Ch 7: Pg: 309-311
7.	1	Continuation of problems on Adams Moulton method	S1: Ch 6: Pg: 351-353
8.	1	Milne's method - Problems	S4: Ch 7: Pg: 311-314
9.	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit III =09		
UNIT-IV			
1.	1	Boundary value problems	S4: Ch 7: Pg: 318-323
2.	1	Problems on linear shooting method	S3: Ch 11: Pg: 672-676 S4: Ch 7: Pg: 318-323
3.	1	Problems on shooting method for nonlinear systems	S3: Ch 11: Pg: 678-683
4.	1	Continuation of problems on shooting method for	S2: Ch 7: Pg: 567-572

		nonlinear systems	
5.	1	Characteristics value problems	S1: Ch 6: Pg: 381-383
6.	1	Problems on eigen values of a matrix by iteration	S1: Ch 6: Pg: 384-385
7.	1	Continuation of problems on eigen values of a matrix by iteration	S4: Ch 6: Pg: 279-282
8.	1	The power method-Procedure and problems	S3: Ch 9: Pg: 576-583
9.	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit IV =09		
UNIT – V			
1.	1	Classification of PDE of the second order	S4: Ch8: Pg: 333-335
2.	1	Problems on Elliptic equation	S4: Ch8: Pg: 338-345
3.	1	Problems on Parabolic equation- Explicit method	S4: Ch8: Pg: 349-351
4.	1	Problems on parabolic equation- Crank Nicolson difference method	S4: Ch8: Pg: 351-352
5.	1	Continuation of problems on Crank- Nicolson difference method	S4: Ch8: Pg: 353-355
6.	1	Hyperbolic equations	S1: Ch8: Pg: 499-506
7.	1	Continuation of problems on Hyperbolic equations	S4: Ch8: Pg: 358-362
8.	1	Problems on solving wave equation by explicit formula	S1: Ch8: Pg: 507-509
9.	1	Recapitulation and discussion of possible questions	
10.	1	Discussion on previous ESE question papers.	
11.	1	Discussion on previous ESE question papers.	
12.	1	Discussion on previous ESE question papers.	
	Total No of Hours Planned For Unit V=12		
	Total No of Hours Planned = 48		

SUGGESTED READINGS

S1: Gerald, C. F., and Wheatley. P. O., (2009). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.
S2: Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .
S3: Burden R. L., and Douglas Faires.J,(2014). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.
S4: Sastry S.S., (2009). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

UNIT-I

SYLLABUS

Solutions of Non Linear Equations: Newton's method-Convergence of Newton's method-Bairstow's method for quadratic factors. Numerical Differentiation and Integration: Derivatives from difference tables – Higher order derivatives – divided difference. Trapezoidal rule– Romberg integration – Simpson's rules.

SOLUTIONS OF NON LINEAR EQUATIONS

1.1 Introduction

In the field of Science and Engineering, the solution of equations of the form f(x) = 0 occurs in many applications. If f(x) is a polynomial of degree two or three or four, exact formulae are available. But if f(x) is a transcendental function like $a+be^x+c$, sinx +d, log x etc., the solution is not exact and we do not have formulae to get the solutions. When the co-efficients are numerical values, we can adopt various numerical approximate methods to solve such algebric and transcendental equations. We will see below some methods of solving such numerical equations. Several methods are available to find the derivative of a function f(x) or to evaluate the definite integral $\int_a^b f(x)$ dx , a, b are real finite constants, in the closed form. However, when f(x) is a complicated function or when it is given in a tabular form, we use numerical methods. In this chapter we discuss numerical methods for approximating the derivative f(x), x ≥ 1, of a given function f(x) and for the evaluation of the integral $\int_a^b f(x)$ dx where a, b may be finite or infinite. This unit focuses on the various methods of solving transcendental equations, the derivatives of a function and the evaluation of the integrals.

1.2 Transcendental And Polynomial Equations

A problem of great importance in applied mathematics and engineering is that of determining the roots of an equation of the form

$$f(x) = 0 \qquad \ldots\ldots. \quad (1.1)$$

The function f(x) may be given explicitly, for example

$f(x) = p(x)$

$$= a_0 x^x + a_1 x^{n-1} + \ldots.. + a_{n-1} x + a_n, \; a_0 \neq 0$$

A polynomial of degree n in x or f (x) may be known only implicitly as a transcendental function.

Definition : A number ξ is a solution of f (x) =0 if f (ξ) =0. Such a solution ξ is a root or a zero of f (x) = 0.

Geometrically, a root of the equation (1.1) is the value of x at which the graph y= f (x) intersects the x-axis.

Direct methods

These methods give the exact values of the roots in a finite number of steps. Further, the methods give all the root of the same time. For example, a direct method gives the root of a linear or first degree equation

$a_o x + a_1 = 0, a_0 \neq 0$ …….. (1.2) as $x = -a_1 / a_0$

Similarly, the root of the quadratic equation

$a_0 x^2 + a_1 x + a_2 = 0, a_0 \neq 0$ ……..(1.3) are given by

$$x = \frac{-a_1 \pm \sqrt{(a_1^2 - 4a_0 a_2)}}{2a_0}$$

Iterative methods

These methods are based on the idea of successive approximations, i.e., starting with one or more initial approximations to the root, we obtain a sequence of approximations or iterates $\{x_k\}$, which in the limit converges to the root. The methods give only one root at a time. For example, to solve the quadratic equation (1.3)

we may choose any one of the following iteration methods:

(a). $x_{k+1} = -\frac{a_2 + a_0 x_k^2}{a_1}, k = 0,1,2,....$

(b). $x_{k+1} = -\frac{a_2}{a_0 x_k + a_1}, k = 0,1,2,....$

(c). $x_{k+1} = -\frac{a_2 + a_1 x_k}{a_0 x_k}, k = 0,1,2,....$ (1.4)

The convergence of the sequence $\{x_k\}$ to the number ξ, the root of the equation (1.3) depends on the rearrangement (1.4) and the choice of the starting approximation x_0.

Definition: A sequence of iterates $\{x_k\}$ is said to converges to the root ξ, if

$\lim |x_k - \xi| = 0$ or $\lim x_k = \xi$.

If $x_k, x_{k-1}, \ldots x_{k-m+1}$ are m approximations to the root, then a multipoint iteration method is defined as

$$x_{K+1} = \phi(x_k, x_{k-1}, \ldots x_{k-m+1}). \qquad \ldots\ldots\ldots\ldots(1.5)$$

The function is called the multipoint iteration function.

For m =1, we get the one point iteration method

$$x_{K+1} = \phi(x_K) \qquad \ldots\ldots\ldots\ldots(1.6)$$

Then, given one or more initial approximations to the root, we require a suitable iteration function for a given function f (x), such that the sequence of iterates obtained from (1.5) or (1.6) converges to the root ξ. In practice, except in rare cases, it is not possible to find ξ which satisfies the given equation exactly. We, therefore,

attempt to find an approximate root ξ such that either $|f(\xi^*)| < \varepsilon$,

$$|x_{K+1} - x_K| < \varepsilon \qquad \ldots\ldots\ldots (1.7)$$

Where x_K and x_{K+1} are two consecutive iterates and ε is the prescribed **error tolerance.**

Initial Approximate

Initial approximations to the root are often known from the physical considerations of the problem. Otherwise, graphical methods are generally used to obtain initial approximations to the root. Since the value of x, at which the graph of y = f (x) intersects the x-axis, gives the root of f (x) =0, any value in the neighborhood of this point may be taken as an initial approximations to the root (see Fig. 1.1 a, b).if the equation f (x) =0 can be conveniently written in the form, then the point of intersection of the graphs of gives the roots of f (x) =0 and therefore any value in the neighborhood of this point can be taken as an initial approximations to the root (see Fig. 1.1 c). Another commonly used method to obtain the initial approximations to the root is based upon the **Intermediate value Theorem**, which states:

If f(x) is a continuous function on some in[illegible]l [a, b] and f (a) f (b) <0, then the equation f(x) =0 has at least one real root or a[illegible]l number of the real roots in the interval (a, b)

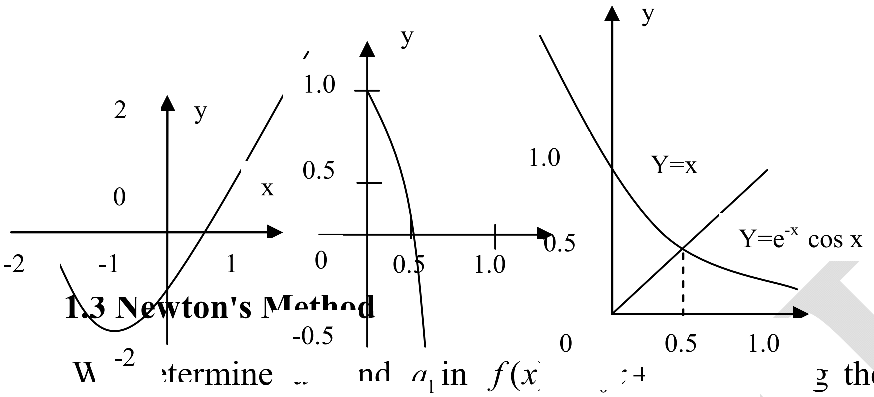


Fig 1.1 a Fig 1.1 b Fig 1.1 c

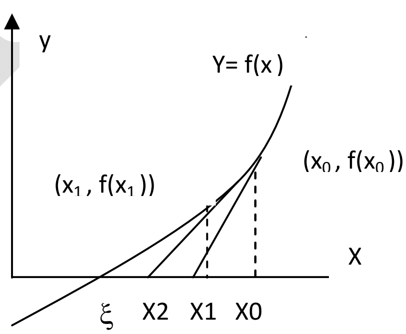
1.3 Newton's Method

W[illegible]termine [illegible] nd a_1 in $f(x$[illegible] g the conditions. $f_K = a_0 x_K + a_1$, [illegible] ……(1.[illegible] [illegible]e a prime deno[illegible]entiation with respect to x.

On substituting a_0 and a_1 from (1.8) in $x = -\frac{a_1}{a_0}$ and representing the approximate value of x by x_{K+1}, we obtain

$$x_{K+1} = x_K - \frac{f_K}{f'_K}, k = 0,1,2,... \qquad ……. \ (1.9)$$

This method is called the **Newton-Raphson** Method. The method (1.9) may also be obtained directly from $x_{K+1} = x_K - \frac{x_K - x_{K-1}}{f_K - f_{K-1}} f_K, k = 1,2,3,...$ by taking limit $x_{K-1} \to x_K$. In the limit $x_{K-1} \to x_K$ when, the chord through the points (x_K, f_K) and (x_{K-1}, f_{K-1}) becomes the tangent at the point (x_K, f_K). Thus, in this case the problem of finding the root of the equation (1.1) is equivalent to finding the point of intersection of the tangent to the curve y = f (x) at the point (x_K, f_K) with x-axis. The method is shown graphically in Fig.1.2.The Newton-Raphson method requires two evaluations for f_K, f'_K each iteration.



Alternative

Fig 1.2

Let be an approximation to the root of the equation $f(x)=0$.let Δx be an increment in x such that $x_K+\Delta x$ is an exact root. Therefore, $f(x_K+\Delta x)\equiv 0.$

Expanding in Taylor series about the point, we get

$$f(x_K)+\Delta x f'(x_K)+\frac{1}{2!}(\Delta x)^2 f''(x_K)+\ldots=0.$$

Neglecting the second and higher powers of Δx, we obtain $f(x_K)+\Delta x f'(x_K)=0$

Or $$\Delta x=-\frac{f(x_K)}{f'(x_K)}.$$

Hence, we obtain the iteration method

$$x_{K+!}=x_K+\Delta x=x_K-\frac{f(x_K)}{f'(x_K)},k=0,1,\ldots$$

Which is same as (1.9).

Rate Of Convergence

We now study the rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root.

Definition: An iterative method is said to be of **order** p or has the rate of **convergence** p, if p is the largest positive real number for which there exists a finite constant C≠0 such that

$$|\varepsilon_{K+!}|\le C|\varepsilon_K|^P \qquad (1.10)$$

Where $\varepsilon_K=x_K-\xi$ is the error in the k th iterate.

The constant C is called the asymptotic error constant and usually depends on derivatives of $f(x)$ at $x=\xi$.

Newton-Raphson Method

On substituting $x_K=\xi+\varepsilon_K$ in (1.9) and expanding $f(\xi+\varepsilon_K), f'(\xi+\varepsilon_K)$ in Taylor's series about the point ξ, we obtain

$$\varepsilon_{K+1}=\varepsilon_K-\frac{[\varepsilon_K f'(\xi)+\frac{1}{2}\varepsilon_K{}^2 f''(\xi)+\ldots]}{f'(\xi)+\varepsilon_K f''(\xi)+\ldots}$$

$$= \varepsilon_K - \left[\varepsilon_K + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_K^{\ 2} + ...\right]\left[1 + \frac{f''(\xi)}{f'(\xi)}\varepsilon_K + ...\right]^{-1}$$

$$\varepsilon_{K+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_K^{\ 2} + 0\left(\varepsilon_K^{\ 3}\right)$$

On neglecting $\varepsilon_K^{\ 3}$ and higher powers of ε_K, we get

$$\varepsilon_{K+1} = C\varepsilon_K^{\ 2}$$

Where $$C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}.$$

Thus, the Newton-Raphson Method has second order convergence.

System of Nonlinear Equations

We now extend the methods derived for the single equation $f(x) = 0$ to a system of nonlinear equations. We first consider a system of two nonlinear equations in two unknowns as

$$f(x, y) = 0 \qquad ……. \ (1.12)$$

$$g(x, y) = 0.$$

Newton-Raphson Method

Let (x_K, y_K) be a suitable approximation to the root (ξ, η) of the system (1.12)

Let Δx be an increment in x_k and Δy be an increment in y_k such that $(x_k+\Delta x, y_k+\Delta y)$ is an exact solution, that is

$$f(x_k+\Delta x, y_k+\Delta y) \equiv 0$$

$$g(x_k+\Delta x, y_k+\Delta y) \equiv 0$$

Expanding inTaylor's series about the point (x_k, y_k), we get

$$f(x_k,y_k) + \left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)f(x_k,y_k) + \frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^2 f(x_k,y_k) + .. = 0$$

$$g(x_k,y_k)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\, g(x_k,y_k)\right)+\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 g(x_k,y_k)+.. = 0$$

Neglecting second and higher powers of Δx and Δy, we obtain

$$f(x_k,y_k)+\Delta x\, f_x(x_k,y_k)+\Delta y\, f_y(x_k,y_k) = 0$$

$$g(x_k,y_k)+\Delta x\, g_x(x_k,y_k)+\Delta y\, g_y(x_k,y_k) = 0 \text{-------}(1.13)$$

where suffixes with respect to a and y represent partial differentiation.

Solving equations (1.13) for Δx and Δy, we get

$$\Delta x = \frac{-1}{Dk}[f(x_k,y_k)\, g_y(x_k,y_k) - g(x_k,y_k)\, f_y(x_k,y_k)]$$

$$\Delta y = \frac{-1}{Dk}[g(x_k,y_k)\, f_x(x_k,y_k) - f(x_k,y_k)\, g_x(x_k,y_k)]$$

Where $Dk = f_x(x_k,y_k)\, g_y(x_k,y_k) - g_x(x_k,y_k)\, f_y(x_k,y_k)$

Writing the equations (1.13) in matrix form, we get

$$\begin{pmatrix} f_x(x_k,y_k) & f_y(x_k,y_k) \\ g_x(x_k,y_k) & g_y(x_k,y_k) \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - \begin{pmatrix} f(x_k,y_k) \\ g(x_k,y_k) \end{pmatrix}$$

$$\mathbf{J_k \Delta x = -F}\,(x_k,y_k) \text{----------------------}(1.14)$$

Where $J_k = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}(x_k,y_k)$, $F = \begin{pmatrix} f \\ g \end{pmatrix}(x_k,y_k)$ and $\Delta \mathbf{x} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

The solution of (1.14) is

$$\Delta x = -Jk^{-1}F(x_k,y_k)$$

$$J_k = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}^{-1}_{(x_k,y_k)} = \frac{1}{Dk}\begin{pmatrix} g_y & -f_y \\ -g_x & f_x \end{pmatrix}_{(x_k,y_k)}$$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = -(J_k)^{-1} \begin{pmatrix} f(x_k,y_k) \\ g(x_k,y_k) \end{pmatrix}$$

And

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - (J_k)^{-1} \begin{pmatrix} f(x_k,y_k) \\ g(x_k,y_k) \end{pmatrix}$$

$$k = 0,1,2\ldots(1.15)$$

or

$$x^{(k+1)} = x^{(k)} - (J_k)^{-1} \; F(x^{(k)}) \qquad \ldots.(1.16)$$

where $x^{(k)} = [\, x^{(k)}, y^{(k)}]^T$, $F(x^{(k)}) = [f(x_k, y_k), g(x_k, y_k)]^T$

The method given by (1.16) is an extension of the Newton-Raphson method (1.9) to a system of 2x2 equations.

This method can be easily generalized for solving a system of n equations in n unknowns

$f_1(x_1,x_2,x_3,....x_n) = 0.$

$f_2(x_1,x_2,x_3,....x_n) = 0.$

……………………..

$f_n(x_1,x_2,x_3,....x_n) = 0.$ …..(1.17)

or

$F(x) = 0.$where $x = [x_1,x_2,x_3,....x_n]^T = F[f_1,f_1,f_3\ldots.f_n]^T$

If $x(0) = [x_1^{(0)},x_2^{(0)},x_3^{(0)},....x_n^{(0)}]^T$ is an initial approximation to the solution vector x , then we can write the method as

$$x^{(k+1)} = x^{(k)} - (J_k)^{-1} \; F(x^{(k)}) \;, k=0,1,2\ldots. \qquad \ldots.(1.18)$$

where

$$J_k = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \dots\dots\dots \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \dots\dots\dots \partial f_2/\partial x_n \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \dots\dots\dots \partial f_3/\partial x_n \\ \dots & \dots & \dots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \partial f_n/\partial x_3 \dots\dots .\partial f_n/\partial x_n \quad ({}_x{}^{(k)}) \end{pmatrix}$$

Is the Jacobian of the matrix of the functions $f_1, f_1, f_3 \dots f_n$ evaluated at ${}_x{}^{(k)}$.

Note that the matrix $(J_k)^{-1}$ is to be evaluated for each iteration.

The convergence of the method depends on the initial approximate vector $x^{(0)}$. A sufficient condition for convergence is that for each k

$\|(J_k)^{-1}\| < 1$

Whereas a necessary and sufficient condition for convergence is $p(J_k)^{-1} < 1$.

Where $\|.\|$ is a suitable norm and $p(J_k)^{-1}$ is the spectral radius (largest eigen value in magnitude) of the matrix $(J_k)^{-1}$.

Example 1: Perform three iterations of the Newton's method to solve the system of equations $x^2+xy+y^2 = 7$, $x^3+y^3 = 9$.Take the initial approximations as $x_0 = 1.5$, $y_0 = 0.5$.

The exact solution is x=2, y=1.

Solution :

$f(x,y) = x^2+xy+y^2 -7$

$g(x,y) = x^3+y^3 - 9$

$$(J_k)^{-1} = (1/Dk) \begin{pmatrix} g_y & -f_y \\ -g_x & f_x \end{pmatrix} = \begin{pmatrix} 3(y_k)^2 & -(x_k + 2y_k) \\ -3(x_k)^2 & 2x_k+y_k \end{pmatrix}$$

Where

$D_k = |J_k| = (3(y_k)^2)(2x_k+y_k) - (-3(x_k)^2)(-(x_k + 2y_k))$.

We can now write the methos as

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - (1/D_k) \begin{pmatrix} 3(y_k)^2 & -(x_k + 2y_k) \\ -3(x_k)^2 & 2x_k+y_k \end{pmatrix} \begin{pmatrix} x_k^2+x_k y_k+y_k^2 - 7 \\ x_k^3+y_k^3 - 9 \end{pmatrix}$$

k=0,1..

Using $(x_0,y_0) = (1.5,0.5)$, we get

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2.2675 \\ 0.9254 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2.0373 \\ 0.9645 \end{pmatrix}$$

and $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2.0013 \\ 0.9987 \end{pmatrix}$

METHODS FOR COMPLEX ROOTS

We can also obtain a root of the equation

$f(z) = 0$ ….(1.19)

in which z is a complex variable.Substituting $z = x+iy$ in equation (1.19) , we get

$f(z) = f(x+iy) = u(x,y) + iv(x,y) = 0.$ ………..(1.20)

Thus , the problem of finding the complex root of (1.19) reduces to solving a system of two nonlinear equations (1.20).The system of equations (1.20) can be solved using the method dicussed in previous section .

Example 2

Obtain the complex roots of the equation $f(z) = z^3 +1 =0$ correct to eight decimal places .Use the initial approximation to a root as $(x_0,y_0) = (0.25,0.25)$.

Compare with the exact values of the roots $(1\pm i\sqrt{3})/2$.

Solution:

Substituting $z= x+iy$ in the given equation , we get

$f(x+iy) = u(x,y) + iv(x,y) = (x+iy)^3 + 1$

$= (x^3 - 3xy^2 + 1) + i(3x^2y - y^3) = 0.$

k	z_k	$f(z_k)$	z_{k+1}
0	(0.25,0.25)	(0.9687, -0.3125)	(0.16667,2.8333)
1	(0.16667,2.8333)	(-0.3009, - 0.225)	(0.15220, 1.8937)
2	(0.15220, 1.8937)	(-0.6340, -0.6660)	(0.19264, 1.2772)
3	(0.19264, .27724)	(-0.6438,-0.1941)	(0.31932, 0.9104)
4	(0.31932, 0.9104)	(0.2385, -0.4761)	(0.4925, 0.83063)
5	(0.4925, 0.83063)	(0.1000 , -0.3140)	(0.49983, 0.8673)

Therefore,

$u(x,y) = x^3 - 3xy^2 + 1$, $v(x,y) = 3x^2y - y^3 = 0.$

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{pmatrix}$$

$D = |J| = 9(x^2 - y^2)^2 + 36x^2y^2 = 9(x^2+y^2)^2$

Using $(x_0,y_0) = (0.25,0.25)$, we get

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.1667 \\ 2.8333 \end{pmatrix}.$$

The successive iterates are given in the following table

Obviously, the approximation to the second root is

(0.5, - 0.8660).

1.4 Bairstow Method

$P_n(x)=a_0x_n+a_1x_{n-1}+a2x_{n-1}+\ldots a_{n-1}x_n+a_n=0,\ a_0\neq 0$ -----(1.21)

Where a_0, a_1,a_2……a_n are real numbers

The Bairstow method extracts a quadratic factor of the form x^2+px+q from the polynomial(1.2), which may give a pair of complex roots or a pair of real roots. If we divide the polynomial by the quadratic factor x^2+px+q, then we obtain a quotient polynomial $Q_{n-2}(x)$ of degree n-2 and a remainder term which is a polynomial of degree one, i.e., Rx+S.

Thus

$$P_n(x) = (x^2+px+q)\, Q_{n-2}(x)+Rx+S \text{---------(1.22)}$$

$$Q_{n-2}(x) = b_0x_{n-2}+b_1x_{n-3}+\ldots+b_{n-3}x+b_{n-2}$$

The problem is then to find p and q, such that

$$R(p,q) = 0,\ S(p,q) =0$$

The above equations are two simultaneous equations in two unknowns p and q. Suppose that (p_0, q_0) is an initial approximation and that ($p_0+\Delta p$, $q_0+\Delta q$) is the true solution. Following the Newton- Raphson method, we obtain

$$\Delta p= -\frac{RS_q-SR_q}{R_pS_q - R_qS_p} \qquad \Delta q = -\frac{R_pS-RS_p}{R_pS_q-R_qS_p} \text{----(1.24)}$$

Where R_p,R_q,S_p,S_q are the partial derivatives of R and S with respect to p and q respectively. These quantities and R,S are evaluated at p_0, q_0.

The coefficients b_i, R and S can be determined by comparing the like powers of x in(1.22), we obtain

$a_0 =b_0$ $\qquad$ $b_0=a_0$

$a_1=b_1+pb_0$ $\qquad$ $b_1=a_1-pb_0$

$a_2=b_2+pb_1+qb_0$ $\qquad$ $b_2=a_2-pb_1-qb_0$

. $\qquad$.

. $\qquad$.

$a_k=b_k+pb_{k-1}+qb_{k-2}$ $\qquad$ $b_k=a_k-pb_{k-1}-qb_{k-2}$---------(1.25)

. .

$a_{n-}1=R+pb_{n-2}+qb_{n-3}$ $R=a_{n-1}-pb_{n-2}-qb_{n-3}$

$a_n=S+qb_{n-2}$ $S=a_n-qb_{n-2}$.

We now introduce the recursion formula

$$b_k = a_k-pb_{k-1}-qb_{k-2} = 1, 2,\ldots\ldots n \text{---------------}(1.26)$$

Where $b_0 = a_0$, $b_{-1} = 0$

Comparing the last two equations with those of (1.25), we get

$$R = b_{n-1}$$

$$S=b_n+pb_{n-1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (1.27)$$

The partial derivatives R_p,R_q,S_p and S_q can be determined by differentiating (1.26) with respect to p and q.

We have

$$-\frac{\partial bk}{\partial p} = b_{k-1}+P\frac{\partial bk-1}{\partial p}+q\frac{\partial bk-2}{\partial p}\ ;\ \frac{\partial b0}{\partial p}=\frac{\partial b-1}{\partial p}=0$$

$$-\frac{\partial bk}{\partial p}=b_{k-2}+P\frac{\partial bk-1}{\partial p}+q\frac{\partial bk-2}{\partial p};\frac{\partial b0}{\partial p}=\frac{\partial b-1}{\partial p}=0\ \ldots(1.28)$$

Putting

$$\frac{\partial bk}{\partial p} = -c_{k-1},\ k=1, 2, \ldots..n$$

In the first equation of (1.20), we find

$$C_{k-1} = b_{k-1} - pc_{k-2}-qc_{k-3} \quad \text{---------}(1.29)$$

Furthermore, if we write $c_{k-2} = -\frac{\partial bk}{\partial p}$

Then, the second equation of (1.28) gives

$$C_{k-2} = b_{k-2}-pc_{k-3}-qc_{k-4}$$

Thus, we get a recurrence relation for the determination of c_k from b_k as

$$C_k=b_k-pc_{k-1}-qc_{k-2},\ k=1,2,\dots n,n-1$$

Where $c_{-1}=0$ and $c_0=-\frac{\partial b1}{\partial p} = -\frac{\partial}{\partial p}(a_1-pb_0) = b_0$

Where

We obtain

$$R_p=-c_{n-2},\ S_p=b_{n-1}-c_{n-1}-pc_{n-2}$$

$$R_q=c_{n-3},\ S_q=-(c_{n-2}+pc_{n-3}).$$

Substituting the above values in (1.24) and simplifying, we get

$$\Delta p = -\frac{b_nc_{n-3}-b_{n-1}c_{n-2}}{C^2_{n-2}-c_{n-3}(c_{n-1}-b_{n-1})}$$

$$\Delta q = -\frac{b_{n-1}(c_{n-1}-b_{n-1})-b_nc_{n-2}}{C^2_{n-2}-c_{n-3}(c_{n-1}-b_{n-1})}$$

The improved values of p_0 and q_0 are

$$p_1=p_0+\Delta p;\ q_1=q_0+\Delta q$$

-p	a_0	a_1	a_2	----	a_{n-2}	a_{n-1}	a_n
-q		$-pb_0$	$-pb_1$	…	$-pb_{n-3}$	$-pb_{n-2}$	$-pb_{n-1}$
			$-qb_0$	…	$-qb_{n-4}$	$-qb_{n-3}$	$-qb_{n-2}$
-p	b_0	b_1	b_2	…	b_{n-2}	b_{n-1}	b_n
-q		$-pc_0$	$-pc_1$	…	$-pc_{n-3}$	$-pc_{n-2}$	$-pc_{n-1}$
			$-qc_0$	…	$-qc_{n-4}$	$-qc_{n-3}$	$-qc_{n-2}$
	c_0	c_1	c_2	c_3	c_4	c_5	c_6

Note that the polynomial $p_n(x)$ is complete.

When p and q have been obtained to the desired accuracy, the polynomial

$$Q_{n-2}(x) = P_n(x)/ (x^2+px+q)$$

$$= b_0x_{n-2}+b_1x_{n-4}+\ldots+b_{n-2}$$

is called the deflated polynomial. The coefficients b_i, i =0, 1,2,…n-2 are known from the synthetic division procedure. The next quadratic factor is obtained using this deflated polynomial.

Example 1

Perform two iterations of the Bairstow method to extract a quadratic factor x^2+px+q from the polynomial

$$P_3(x) = x^3+x^2-x+2 = 0$$

Use the initial approximation P_0 = -0.9, q_0 = 0.9

Starting with P_0 = -0.9 and q_0= 0.9, we obtain

0.9	1	1	-1	2
-0.9		0.9	1.71	-0.171
			-0.9	-1.71
	i=b_0	1.9	-0.19=b_2	0.119=b_3
		0.9	2.52	
		-0.9		
	i=c_0	2.8=c_1	1.43=c_2	

$$\Delta p = -(b_3c_0-b_2c_1)/(c_1^2-c_0(c_2-b_2)) = -0.651/6.22 = -0.1047$$

$$\Delta q =- (b_2(c_2-b_2)-b_3c_1/(c_1^2-c_0(c_2-b_2)) = 0.6410/6.22 = 0.1031$$

$$p_1 = p_0+\Delta p = -0.9 -0.1047 = -1.0047$$

$$q_1 = q_0+\Delta q = 0.9+0.1031 = 1.0031$$

1.0047	1	1	-1	2
-1.0031		1.0047	2.0141	0.0111
			-1.0031	-1.71
	i=b_0	2.0147	-0.0110=b_2	0.0002=b_3
		1.0047	3.0235	
		-1.0031		
	i=c_0	3.0094=c_1	2.0314=c_2	

$\Delta p = -(b_3c_0-b_2c_1)/(c_1^2-c_0(c_2-b_2) = -0.0329/7.0361= 0.0047$

$\Delta q =-(b_2(c_2-b_2)-b_3c_1/(c_1^2-c_0(c_2-b_2) =0.0216/7.0361$

=0-0.0031

$p_2 = p_1+\Delta p = -1.0047-0.\ 0047 = -1.0000$

$q_2 = q_1+\Delta q = 1.0031-0.0031= 1.0000$

Hence, the extracted quadratic factor is $x_2+p_2x+q^2 = x^2-x+1$.The exact factor is x^2-x+1

Example 2

Perform one iteration of the Bairstow method to extract a quadratic factor x^2+px+q from a polynomial

$$x^4+x^3+2x^2+x+1 = 0$$

Use the initial iteration p_0 =0.5 and q_0 = 0.5

Starting with p_0 =0.5 and q_0 = 0.5, we obtain

-0.5	**1**	**1**	**2**	**1**	**1**
-0.5		-0.5	-0.25	-0.625	-0.0625
			-0.5	-0.25	-0.625
	1	0.5	1.25	0.125=b_3	0.3125=b_4
		-0.5	0.0	-0.375	

		-0.5	0.0
1	$0.0=c_1$	$0.75=c_2$	$-0.25=c_3$

$\Delta p = -(b_4c_1-b_3c_2)/(c_2^2-c_1(c_3-b_3) = 0.1667$

$\Delta q =- (b_3(c_3-b_3)-b_4c_2/(c_2^2-c_1(c_3-b_3) = 0.5$

Therefore, $p_1=p_0+\Delta p = 0.6667$, $q_1=q_0+\Delta q = 1.0$

The exact values of p and q are 1.0

1.5 Numerical differentiation

The problem of Interpolation is finding the value of y for the given value of x among (x_i , y_j) for i= 1 to n. Now we find the derivatives of the corresponding arguments . If the required value of y lies in the first half of the interval then we call it as Forward interpolation .If the required value of y (derivative value) lies in the second half of the interval we call it as Backward interpolation also if the derivative of y lies in the middle of of class interval then we solve by central difference.

Newton's forward formula for Interpolation :

$Y= y_0 +u \Delta y_0+u(u-1)/2! \Delta^2Y_0 +u (u-1)(u-2) / 3! \Delta^3Y_0+\ldots\ldots$

Where $u = (x-x_0)/h$

Differentiating with respect to x ,

$dy/dx = (dy/du). (du/dx) = (1/h) (dy / du)$

$(dy / dx)\ x \neq x_0 = (1 / h) [\Delta y_0 +(2u-1)/2\ \Delta^2y_0 +(3u^2 -6u+2)/ 6\ \Delta^3y_0 +\ldots\ldots\ldots]$

$(dy / dx)\ x = x_0 = (1 / h) [\Delta y_0 -(1/2)\ \Delta^2y_0 +(1/ 3)\ \Delta^3y_0 +\ldots\ldots\ldots]$

$(d^2y / dx^2)\ x \neq x_0= d/dx (dy / dx) = d/dx(dy / du. du / dx)$

$= (1/h^2) [\Delta^2 y_0 + 6(u-1) / 6\ \Delta^3y_0 + (12u^2 - 36 u +22) / 2\ \Delta^4y_0 +\ldots\ldots.]$

$(d^2y / dx^2)\ x= x_0=(1/h^2) [\Delta^2 y_0 - \Delta^3y_0 +(11/12) \Delta^4y_0 +\ldots\ldots.]$

Similarly,

$(d^3y / dx^3)\ x \neq x_0= (1/h^3) [\Delta^3y_0 + (2u - 3) / 2\ \Delta^4y_0 +\ldots\ldots.]$

$(d^2y / dx^2)\ x= x_0=(1/h^3)\ [\Delta^3\ y_0 -(3/2)\Delta^4 y_0 +\ldots\ldots].$

In a similar manner the derivatives using backward interpolation an also be found out.

Using backward interpolation .

$(dy / dx)\ x \neq x_n = (1 / h)\ [\nabla y_n +(2u+1)/2\ \ \nabla^2 y_n +(3u^2 +6u+2)/ 6\ \ \nabla^3 y_n +\ldots\ldots\ldots]$

$(dy / dx)\ x = x_n = (1 / h)\ [\nabla y_n -(1/2)\ \ \nabla^2 y_n +(1/ 3)\ \nabla^3 y_n +\ldots\ldots\ldots]$

$(d^2y / dx^2)\ x \neq x_0= (1/h^2)\ [\nabla^2\ y_0 + 6(u-1) / 6\ \nabla^3 y_0 + (\ 12u^2 - 36\ u +22) / 2\ \nabla^4 y_0 +\ldots\ldots.]$

$(d^2y / dx^2)\ x= x_0=(1/h^2)\ [\nabla^2\ y_0 - \nabla^3 y_0 +(11/12)\ \nabla^4 y_0 +.]$

Example 1

Find the first two derivatives of $x^{(1/3)}$ at x= 50 and x= 56, given the table below.

X: 50 51 52 53 54 55 56

Y: 3.68403.70843.73253.75633.77983.80303.8259

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	3.6840				
51	3.7084	0.0244			
52	3.7325	0.0241	-0.0003	0	
53	3.7563	0.0238	-0.0003	0	0
54	3.7798	0.0235	-0.0003	0	0
55	3.8030	0.0232	-0.0003	0	0
56	3.8259	0.0229	-0.0003		

At x= 50,

$(dy/dx)_{\ x = x0} = (1 /h)[\Delta y_0-(1/2)\ \Delta^2 y_0+(1/3)\Delta^3 y_0 +\ldots\ldots\ldots]$

$= (1/1)[0.024-(1/2)(-0.0003)+0] = 0.02455$

$(d^2y/dx^2)_{\ x= x0} =(1/h^2)\ [\Delta^2\ y_0 - \Delta^3 y_0 +(11/12)\ \Delta^4 y_0 +\ldots\ldots.]$

$= (1/1)[-0.003-0]= -.0003$

At x=56,

$(dy/dx)_{x = xn}=(1/h)[\nabla y_n +(1/2)\nabla^2 y_n+(1/3)\nabla^3 y_n +\ldots\ldots\ldots]$

$= (1/1)\ [\ 0.0229+(1/2)(-0.0003)+0] = 0.02275.$

$(d^2y/dx^2)_{x=xn}=(1/h^2)[\nabla^2 y_n+ \nabla^3 y_n +(11/12) \nabla^4 y_n +\ldots]$

$= (1/1) [-0.003-0] = -0.0003.$

For the above ptroblem let us find the first two derivatives of x when x= 52 and x= 55.

When x=52, From Newton's forward formula

$(dy / dx)\ x \neq x_0 = (1 / h) [\Delta y_0 +(2u-1)/2\ \Delta^2 y_0 +(3u^2 - 6u+2)/ 6\ \Delta^3 y_0 +\ldots\ldots],$

$= (1/1) [0.0244+(3/2)(-0.0003)+0] = 0.02395,$

Since here $u= (x-x_0) / h = (52-50)/1 = 2.$

$(d^2y / dx^2)\ x \neq x_0= (1/h^2) [\Delta^2 y_0 + 6(u-1) / 6\ \Delta^3 y_0 + (12u^2 - 36 u +22) / 2\ \Delta^4 y_0 +\ldots\ldots]$

$= (1/)m [-0.0003+0] = -0.0003.$

When x= 55,from backward interpolation

$(dy / dx)\ x \neq x_n = (1 / h) [\nabla y_n +(2v+1)/2\ \nabla^2 y_n +(3v^2 +6v+2)/ 6\ \nabla^3 y_n +\ldots\ldots]$

$= (1/1) [0.0229+(-1/2)(-0.0003)+0] = 0.02305,$

Since here $v= (x-x_n) / h = (55-56)/1 = -1.$

$(d^2y / dx^2)\ x \neq x_n= (1/h^2) [\nabla^2 y_n + 6(v+1) / 6\ \nabla^3 y_n + (12v^2 + 36 v +22) / 2\ \nabla^4 y_n +\ldots\ldots]$

$= (1/1) [0.0229+(-1/2)(-0.0003)+0] = 0.02305.$

Example 2

Given th following data, find y`(6) and maximum value of y.

X:	0	2	3	4	7	9
Y:	4	26	58	112	466	922

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0 2	4 26	(26-4)/(2-0) = 11 32	 (32-11)=7		

3	58	54	11	1	
4	112	118	16	1	0
7	466	228	22	1	0
9	922				

By Newton's divided difference formula,

$Y= f(x) = f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)\ f(x_0,x_1,x_2)+\ldots.$

$= 4 +(x-0)(11)+(x-0)(x-2)(7)+(x-0)\ (x-2)\ (x-3)\ (1) +0+\ldots$

$= x^3+2x^2+3x +4.$

$f`(x) = 3x^2+4x+3$, therefore $f`(6) = 3(36) +4(6) +3 = 135.$

$f``(x) = 6x +4 = 0$, Hence $x= (-2/3)$. so x is imaginary.

Therefore f(x) does not posses extremum.

1.6 Numerical Integration:

We know that $\int_a^b f(x)dx$ represents the area between y = f(x), x – axis and the ordinates x = a and x = b. This integration is possible only if the f(x) is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows: Given as set of (n+1) paired values (x_i,y_i), i = 0,1,2,…,n of the function y=f(x), where f(x) is not known explicitly, it is required to compute $\int_{x_0}^{x_n} y\ dx$.

As we did in the case of interpolation or numerical differentiation, we replace f(x) by an interpolating polynomial $P_n(x)$ and obtain $\int_{x_0}^{x_n} P_n(x)dx$ which is approximately taken as the value for $\int_{x_0}^{x_n} f(x)dx$.

A general quadrature formula for equidistant ordinates (or Newton – cote's formula)

For equally spaced intervals, we have Newton's forward difference formula as

$$y(x)=y(x_0+uh)=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\ldots \qquad \ldots\ldots(1)$$

Now, instead of *f(x)*, we will replace it by this interpolating formula of Newton.

Here, $u = \frac{x-x_0}{h}$ where h is interval of differencing.

Since $x_n = x_0 + nh$, and $u = \frac{x-x_0}{h}$ we have $\frac{x-x_0}{h} = n = u$.

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n+nh} f(x)dx$$

$= \int_{x_0}^{x_n+nh} P_n(x)\, dx$ where $P_n(x)$ is interpolating polynomial

$$=\int_0^n \left(y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\right)(h\,du)$$

Since $dx = hdu$, and when $x = x_0$, $u = 0$ and when $x = x_0+nh$, $u = n$.

$$= h\left[y_0(u) + \frac{u^2}{2}\Delta y_0 + \frac{\left(\frac{u^3}{3} - \frac{u^2}{2}\right)}{2}\Delta^2 y_0 + \frac{1}{6}\left(\frac{u^4}{4} - u^3 + u^2\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$\int_{x_0}^{x_n} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left[\frac{n^3}{3} - \frac{n^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right] \quad \ldots(2)$$

The equation (2), called Newton-cote's quadrature formula is a general quadrature formula. Giving various values for n, we get a number of special formula.

Trapezoidal rule

By putting n = 1, in the quadrature formula (i.e there are only two paired values and interpolating polynomial is linear).

$\int_{x_0}^{x_n+nh} f(x)dx = h\left[1.y_0 + \frac{1}{2}\Delta y_0\right]$ since other differences do not exist if n = 1.

$$=\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n+nh} f(x)dx$$

$$= \int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_n+2h} f(x)dx + \ldots\ldots\ldots +$$

$$\int_{x_0+(n-1)h}^{x_n+nh} f(x)\,dx$$

$= \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots.+y_{n-1})\right]$

$= \frac{h}{2}$ [(sum of the first and the last ordinates) + 2(sum of the remaining ordinates)]

This is known as Trapezoidal Rule and the error in the trapezoidal rule is of the order h^2.

Note:

Though this method is very simple for calculation purposes of numerical integration; the error in this case is significant. The accuracy of the result can be improved by increasing the number of intervals and decreasing the value of h.

Truncating error on Trapezoidal rule:

In the neighborhood of $x = x_0$, we can expand $y = f(x_0)$ by Taylor series in power of $x - x_0$. That is,

$y(x) = y_0 + (x-x_0)\, y'_0 + (x-x_0)2y''_0 + \ldots\ldots +$

where $y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots$ ...(1) where $y_0' = [y'(x)]\ x=x_0$

$\int_{x_0}^{x_1} y\,dx = \int_{x0}^{x1}\left[y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots..\ dx\right]$

$= \left[y_0 x + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0'' + \cdots\right]_{x_0}^{x_1}$

$= y_0 (x_1 - x_0) + \frac{(x-x_0)^2}{2!} y_0' + \frac{(x-x_0)^3}{3!} y_0'' + \ldots..$

$= h\, y_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \ldots.$ .... (2)

If h is the equal interval length.

Also $\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1)$ = area of the first trapezium = A_0.....(3)

Putting $x=x_1$ in (1)

$$y(x_1) = y_1 = y_0 + \frac{(x_1-x_0)}{1!}y_0' + \frac{(x_2-x_0)^2}{2!}y_0'' + \ldots$$

i.e., $$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \ldots \qquad \ldots\ldots\ldots (4)$$

$$A_0 = \frac{h}{2}\left[y_0 + y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots\right]$$

Using (4) in (3).

$$= h\,y_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2*2!}y_0'' + \ldots\ldots\ldots$$

Subtracting A_0 value from (2),

$$\int_{x_0}^{x_1} y\,dx - A_0 = h^3 y_0''\left[\frac{1}{3!} - \frac{1}{2*2!}\right] + \ldots\ldots$$

$$= -\frac{1}{12}h^3 y_0'' + \ldots\ldots$$

Therefore the error in the first interval (x_0, x_1) is $-\frac{1}{12}h^3 y_0''$ (neglecting other terms)

Similarly the error in the ith interval $= -\frac{1}{12}h^3 y_{i-1}''$

Therefore, the total cumulative error (approx.),

$$E = -\frac{1}{12}h^3(y_0'' + y_1'' + y_2'' + \ldots\ldots + y_{n-1}'')$$

$$|E| < \frac{nh^3}{12}(M)$$ where M is the maximum value of $|y_0''|$, $|y_1''|$, $|y_2''|$, …..

$$< \frac{(b-a)h^2}{12}(M)$$ if the interval is (a,b) and

$$h = \frac{b-a}{n}$$

Hence, the error in the trapezoidal rule is of the order h^2.

Romberg's method

For an interval of size h, let the error in the trapezoidal rule be kh^2 where k is a constant. Suppose we evaluate $I = \int_{x_0}^{x_n} y\,dx$, taking two different values of h, say h_1 and h_2, then

$$I = I_1 + E_1 = I_1 + kh_1^2 \qquad I = I_2 + E_2 = I_2 + kh_2^2$$

Where I_1, I_2 are the values of I got by two different values of h, by trapezoidal rule and E_1, E_2 are the corresponding errors.

$$I_1 + kh_1^2 = I_2 + kh_2^2$$

$$k = \frac{I_1 - I_2}{h_2^2 - h_1^2}$$

substituting in (1), $I = I_1 + \frac{I_1 - I_2}{h_2^2 - h_1^2} h_1^2$ & $I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$

This I is a better result than either I_1, I_2.

If $h_1 = h$ and $h_2 = \frac{1}{2}h$, then we get

$$I = \frac{I_1\left(\frac{1}{4}h^2\right) - I_2 h^2}{\frac{1}{4}h^2 - h^2} = \frac{4I_2 - I_1}{3} = I_2 + \frac{1}{3}(I_2 - I_1), \quad I = I_2 + \frac{1}{3}(I_2 - I_1)$$

We got this result by applying trapezoidal rule twice. By applying the trapezoidal rule many times, every time halving h, we get a sequence of results A_1, A_2, A_3,…… we apply the formula given by (3), to each of adjacent pairs and get the resultants B_1, B_2, B_3 ….. (which are improved values). Again applying the formula given by (3), to each of pairs B_1, B_2, B_3 …..we get another sequence of better results C_1, C_2, C_3 ….continuing in this way, we proceed until we get two successive values which are very close to each other. This systematic improvement of Richardson's method is called Romberg method or Romberg integration.

Simpson's one-third rule:

Setting n = 2 in Newton- cote's quadrature formula, we have $\int_{x_0}^{x_n} f(x)dx = h\left(2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right)$ (since other terms vanish)

$$= \frac{h}{3}(y_2+y_1+y_0)$$

Similarly, $\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}(y_2+4y_3+y_4)$

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}(y_i+4y_{i+1}+y_{i+2})$$

If n is an even integer, last integral will be

$$\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}(y_{n-2}+4y_{n-1}+y_n)$$

Adding all the integrals, if *n* is an even positive integer, that is, the number of ordinates $y_0, y_1, y_2....y_n$ is odd, we have

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ...+ \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}\left[(y_0+y_n) + 2(y_2+ y_4+....) + ... + 4(y_1+ y_3+.....)\right]$$

$= \frac{h}{3}$ [(sum of the first and the last ordinates) + 2(sum of remaining odd ordinates) + 2(sum of even ordinates)]

Note. Though y_2 has suffix even, it is third ordinate (odd).

Simpson's three-eighths rule:

Putting n = 3 in Newton – cotes formula

$$= \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+ y_2+y_4+y_5+....+y_{n-1})+2(y_3+y_6+y_9+....+y_n)\right] \quad .....(2)$$

Equation (2) is called *Simpson's three – eighths rule* which is applicable only when *n* is a multiple of 3.Truncation error in simpson's rule is of the order h

Note 1. In trapezoidal rule , y(x) is a linear function of x. The rule is the simplest one but it is least accurate.

2. In simpson's one – third rule, y(x) is a polynomial of degree two. To apply this rule n, the number of intervals must be even. That is, the number of ordinates must be odd.

Truncation error in simpson's rule

By taylor expansion of y=f(x) in the neighborhood of $x = x_0$ we get,

$$y = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots \qquad \ldots\ldots..(1)$$

$$\int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_2}\left[y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots\right]dx$$

$$= \left[y_0 x + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0'' + \cdots\right]_{x_0}^{x_2}$$

$$= y_0(x_2 - x_0) + \frac{(x_2-x_0)^2}{2!}y_0' + \frac{(x_2-x_0)^3}{3!}y_0'' + \ldots..$$

$$=2h\,y_0 + \frac{4h^2}{2!}y_0' + \frac{8h^3}{3!}y_0'' + \frac{16h^4}{4!}y_0''' + \ldots$$

$$=2h\,y_0 + 2h^2 y_0' + \frac{4}{3}h^3 y_0'' + \frac{2h^4}{3}y_0''' + \frac{4h^5}{15}$$

$$y_0'''' + \ldots. \quad \ldots..(2)$$

$$A_1 = \text{area} = \int_{x_0}^{x_2} y\,dx = \frac{h}{3}(y_2 + 4y_1 + y_0)$$

by simpson's rule ……….(3)

Putting $x = x_1$ in (1)

$$y_1 = y_0 + \square\, y_0' + \square\, y_0'' + \ldots.$$

$$= y_0 + \square\, y_0' + \frac{h^2}{2!}y_0'' + \ldots. \qquad \ldots\ldots\ldots (4)$$

Putting $x = x_2$ in (1)

$$y_1 = y_0 + \frac{2h}{1!}y_0' + \frac{4h^2}{2!}y_0'' + \ldots. \qquad \ldots\ldots\ldots (5)$$

substituting (4) in (5) , in (3),

$A_1 = 2hy_0 + 2h^2 y_0' + \frac{4}{3} h^3 y_0'' + \frac{2h^4}{3} y_0''' + \frac{5h^5}{18} y_0'''' + \ldots$ .....(6) equations (2) – (6) give

$$\int_{x_0}^{x_2} y\,dx - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right) h^5 y_0'''' + \ldots$$

$$= -\frac{h^5}{90} y_0'''' + \ldots$$

Leaving the remaining terms involving h^6 and higher powers of h, principal part of the error in (x_0,x_2) is

$$= -\frac{h^5}{90} y_0'''' + \ldots$$

Similarly the principal part of the error in (x_2,x_4) is

$$= -\frac{h^5}{90} y_2'''' \text{ and so far each interval.}$$

Hence the total error in all the intervals is given by

$$E = = -\frac{h^5}{90} (y_0'''' + y_2'''' + \ldots)$$

$|E| < \frac{nh^5}{90}$ (M) where M is the numerically greater value of y_0'''', y_2'''',…y_{2n-2}

since (x_{2n},x_{2n}) is the last paired value because we require odd number of ordinates to apply simpson's one – third rule. (i.e., 2n intervals).

If the interval is(a,b) then b – a = h(2n). using this, $|E| < \frac{(b-1)h^4}{180}$ (M).

Hence, the error in simpson's one – third rule is of the order h

Example 1

Evaluate $\int_{-3}^{3} x^4 dx$ by using (1) trapezoidal rule (2)simpson's rule. Verify your results by actual integration.

Solution

Here $y(x) = x^4$. Interval length$(b - a) = 6$. So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$.

We form below the table

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

(i) By trapezoidal rule:

$\int_{-3}^{3} y\,dx = \frac{h}{2}$ [(sum of the first and the last ordinates) +

2(sum of the remaining ordinates)]

$= \frac{1}{2}$ [(81+81)+2(16+1+0+1+16)]

=115

(ii) By simpson's one - third rule (since number of ordinates is odd)**:**

$\int_{-3}^{3} y\,dx = \frac{1}{3}$ [(81+81) + 2(1+1) + 4(16+0+16)]

= 98.

(iii) Since $n = 6$, (multiple of three), we can also use **simpson's three - eighths rule.** By this rule,

$\int_{-3}^{3} y\,dx = \frac{1}{3}$ [(81+81) + 3(16+1+1+16) + 2(0)]

= 99

(iv) By actual integration,

$$\int_{-3}^{3} x^4 dx = 2*\left[\frac{x^5}{5}\right]_0^3 = \frac{2\times 243}{5} = 97.2$$

From the results obtained by various methods, we see that simpson's rule gives better result than trapezoidal rule

x	0	0.2	0.4	0.6	0.8	1.0

y=1/1+x^2	1	0.96154	0.86207	0.73529	0.60976	0.50

Example 2

Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with h = 0.2. hence obtain an approximate value of π. Can you use other formulae in this case.

Solution.

Let y(x) = $\frac{1}{1+x^2}$

Interval is (1-0) = 1. Since the value of y are calculated as points taking h =0.2

(i) By Trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots.+y_{n-1})\right]$$

$=\frac{0.2}{2}$[(1+0.5)+2(0.96154+0.8620+0.73529+0.60976)]

= (0.1)[1.5+6.33732]

= 0.783732

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \frac{\pi}{4}$$

$$\frac{\pi}{4} \approx 0.783732$$

$\pi \approx$ 3.13493 (approximately).

In this case, we cannot use simpson's rule (both) and weddle's rule. (since number of intervals is 5).

Example 3

From the following table, find the areas bounded by the curve and the x-axis from x = 7.47 to x = 7.52.

x	7.47	7.48	7.49	7.50	7.51	7.52
y=f(x)	1.93	1.95	1.98	2.01	2.03	2.06

Solution.

Since only 6 ordinates (n = 5) are given, we cannot use simpson's rule. So, we will use trapezoidal rule.

Area= $\int_{7.47}^{7.52} f(x)\, dx$

$=\frac{0.01}{2}[(1.93+2.06)+2(1.95+1.98+2.01+2.3)]$

= 0.09965.

Example 4

Evaluate $\int_0^6 \frac{dx}{1+x}$, using (i) Trapezoidal rule (ii) simpson's rule (both) .Also, check up by direct integration.

Solution

Take the number of intervals as 6.

$h = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
y	1	0.5	1/3	1/4	1/5	1/6	1/7

i) By Trapezoidal rule

$\left(\frac{1}{7}\right) = \frac{1}{2}\left[\left(1+\frac{1}{7}\right)+2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\right]$

= 2.02142857

ii) By simpsons's one – third rule,

$$I = \frac{1}{3}\left(1 + \frac{1}{7}\right) + 2\left(\frac{1}{3} + \frac{1}{5}\right) + 4\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right)$$

$$= \frac{1}{3}\left(1 + \frac{1}{7} + \frac{16}{15} + \frac{22}{6}\right) = 1.95873016$$

iii) By Simpsons's three - eighths rule,

$$I = \left(\frac{3*1}{8}\left(1 + \frac{1}{7}\right) + 3\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6}\right) + 2\left(\frac{1}{4}\right)\right)$$

$= 1.96607143$

iv) By actual integration,

$$\int_0^6 \frac{1}{1+x} = [\log(1+x)]_0^6 = \log_e 7 = 1.94591015$$

Example 5

By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x dx$ by trapezoidal and Simpson's rule. Verify your answer with integration.

x	0	π/10	2π/10	3π/10	4π/10	5π/10
y=sinx	0	0.3090	0.58878	0.8090	0.9511	1.0
x	6π/10	7π/10	8π/10	9π/10	π	
y=sinx	0.9511	0.8090	0.578	0.3090	0	

Solution

Range = π - 0 = π

Hence h = $\frac{\pi}{10}$

We tabulate below the values of y at different x's

Note that the values are symmetrical about x = $\frac{\pi}{2}$

(i) By Trapezoidal rule,

$I = \frac{\pi}{20}$ [(0 + 0) + 2(0.3090+0.5878+0.8090+

0.9511+1.0+0.9511+0.8090+0.5878+0.3090)]

= 1.9843 nearly.

(ii) By Simpsons's one – third rule,

$I = \frac{1}{3}\left(\frac{\pi}{10}\right)$ [(0+0)+2(0.5878+0.9511+0.5878+0.9511) +

4(0.3090+0.8090+1+0.3090+0.8090)]

= 2.00091

Note: We cannot use simpson's three eighth's rule.

(iii) By actual integration, I = $(-\cos x)_0^{\pi}$ = 2.

Hence, Simpson's rule is more accurate than the trapezoidal rule.

POSSIBLE QUESTIONS:

Part-B(5X6 = 30 Marks)

1. Perform two iterations of the Bairstow's method to extract a quadratic x^2+px+q from the polynomial $P_4(x) = x^4- 3x^3+20x^2+44x +54 = 0$. Use the initial approximation as $p_0 = 2, q_0 = 2$.
2. Perform two iterations of the Bairstow's method to extract a quadratic x^2+px+q from the polynomial $P_4(x) = x^4+ x^3+2x^2 + x + 1 = 0$. Use the initial approximation $p_0 = 0.5, q_0 = 0.5$.
3. Write the derivation for systems of nonlinear equations using Newton's method.
4. Find the real root of the equation $x^2 – y^2 = 3$ and $x^2 + y^2 = 13$ by Newton's method correct to 4 decimal places.
5. Find a first two derivative of $x^{1/3}$ at x =50 &x =56 given the table below.

X	50	51	52	53	54	55	56
Y $=x^{1/3}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

6. The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year	1931	1941	1951	1961	1971
Population	40.62	60.80	79.95	103.56	132.65

7. Write Down the Derivative of Newton's Divided difference .
8. Find the real root of the equation $x^2+y-11=0$ and $y^2+y-7=0$ starting with the initial values $x_0=3.5$ and $y_0 = -1.5$ by Newton's method.
9. From the following table find f(x) and hence f(6) using Newton's divided difference formula.

x	1	2	7	8
f(x)	1	5	5	4

10. Use Romberg's method to compute $I = \int_0^1 \frac{dx}{1+x}$ correct to 3 decimal places.
11. Compute $\int_0^1 e^x\, dx$ by taking h =0.05 using Simpson's rule and Trapezoidal rule.
12. Evaluate $\int_{-3}^3 x^4\, dx$ using Simpson's rule.

PART C- (1 x 10 =10 Marks)
(Compulsory)

1. By dividing the range into 10 equal parts evaluate $\int_0^{\pi} sinx dx$ by Trapezoidal & Simpson's rule. Verify your answer with integration.

2. Find the real root of the equation $2x^3 - 3x - 6 = 0$ by Newton's method correct to 3 decimal places.

3. Find the value of cos (1.74) from the following table

x :	1.7	1.74	1.78	1.82	1.86
sin x :	0.9916	0.9857	0.9781	0.9691	0.9584

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Class : I M.Sc Mathematics **Semester : I**

Subject: Numerical Analysis **Subject Code:**

Unit I

Part A (20x1=20 Marks)

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The order of convergence of Newton Raphson method is ----------.	4	2	1	0	2
In Newton Raphson method, the error at any stage is proportional to	Cubic	square	square root	zero	square
----------------- Method is also called method of tangents	Gauss Seidal	Secant	Bisection	Newton Rap	Newton Rapson
If f (x) contains some functions like exponential, trigonometric	Algebraic	transcendental	numerical	polynomial	transcendental
The Newton Rapson method fails if ---------------.	f '(x) = 0	f (x) = 0	f (x) = 1	f(x) ¹0	f '(x) = 0
The order of convergence in --------------- method is t	Bisection	Regula falsi	False position	Newton Rap	Newton Rapson
In Newton Raphson method the choice of -------------- is very impo	initial value	final value	intermediate va	approximate	initial value
If f(a) and f(b) are of opposite signs, a root of f(x) = 0 lies between '	actual root	approximate roo	intermediate ro	zero	approximate root
Newton Rapson method is also called method of ---------------	Gauss Seidal	Regula Falsi	tangents	Bisection	tangents
The -------- method extracts a quadratic factor of the f	Bairstow	False Position	Newton Rapson	Regula falsi	Bairstow
The polynomial Qn-2(x) = $b_0x_{n-2} + b_1x_{n-4} + \ldots\ldots b_{n-2}$ is called the………polynomial.	trinomial	monomial	deflated	binomial	deflated
Bairstow is used to find the ------ roots of polynomial without using	real	complex valued	square root	cubic	complex valued
In Newton's forward difference formula, the value of n is cal	$n = (x-x_n) / h$	$n = (x_n-x) / h$	$n = (x-x_0) / h$	$n = (x_0-x) /$	$n = (x-x_0) / h$
In Newton's backward difference formula, the value of n is c	$n = (x-x_n) / h$	$n = (x_n-x) / h$	$n = (x-x_0) / h$	$n = (x_0-x) /$	$n = (x-x_n) / h$
In Newton's forward difference formula, the value x can be w	x_0-nh	x_n-nh	$x_n + nh$	$x_0 + nh$	$x_0 + nh$
Numerical differentiation can be used only when the differen	zero	one	costant	two	costant
Relation between Δ and E is Δ = ----------	E – 1	E + 1	E * 1	1 – E	E – 1

To find the unknown value of x for some y, which lies at the unequal intervals we use ------------------- formula.	Newton's forward	Newton's backward	Newtons divided difference	inverse interpolation	Newtons divided difference
The other name of shifting operator is ------- operator	Central	average	backward	displacement	displacement
Relation between E and ∇ is ∇ = ----------	E – 1	$1 – E^{-1}$	$1 + E^{-1}$	$1 * E^{-1}$	$1 – E^{-1}$
The divided difference operator is -----------	non-linear	normal	linear	zero	linear
The n^{th} divided difference of a polynomial of degree n are -----	zero	constant	linear	non-linear	constant
The order of error in Trapezoidal rule is -------------.	h	h^3	h^2	h^4	h^2
The order of error in Simpson's rule is -------------.	h	h^3	h^2	h^4	h^4
Numerical evaluation of a definite integral is called --------.	Integration	Differentiation	Interpolation	Triangulariz	Integration
Simpson's ⅜ rule can be applied only if the number of sub in	Equal	even	multiple of thre	unequal	multiple of three
By putting n = 2 in Newton cote's formula we get ----------- r	Simpson's 1/3	Simpson's ⅜	Trapezoidal	Romberg	Simpson's 1/3
The Newton Cote's formula is also known as --------------- fo	Simpson's 1/3	Simpson's 3/8	Trapezoidal	quadrature	quadrature
By putting n = 3 in Newton cote's formula we get -----------	Simpson's 1/3	Simpson's ⅜	Trapezoidal	Romberg	Simpson's ⅜
By putting n = 1 in Newton cote's formula we get -----------	Simpson's 1/3	Simpson's ⅜	Trapezoidal	newton's	Trapezoidal
The systematic improvement of Richardon's method is called	Simpson's 1/3	Simpson's ⅜	Trapezoidal	Romberg	Simpson's ⅜
Simpson's 1/3 rule can be applied only when the number of	Equal	even	multiple of thre	unequal	even
In Numerical integration, the length of all intervals is in ---------- distances.	Greater than the other	less than the other	equal	not equal	equal
Numerical integration is the process of computing the value of a -------- from a set of numerical values of the integrand.	indefinite integral	definite integral	expression	equation	definite integral
Numerical evaluation of a definite integral is called --------	integration	differentiation	interpolation	triangularisation	integration
What is the value of h if a=0,b=2 and n=2.	1	2	3	4	1
Integral (f(x) dx)=(h/2) [Sum of the first and last ordinates + 2(sum of the remaining ordinates)] is called -----	Constant rule	Simpsons rule	Trapezoidal rule	Rombergs rule	Trapezoidal rule
If the given integral is approximated by the sum of 'n' trapezoids, then the rule is called as ----------------.	Newton's method	Trapezoidal rule	simpson's rule	none	Trapezoidal rule
What is the formula for finding the length interval h in trapezoidal tule?	h=(b-a)/n	h=(b/a)/n	h=(b*a)/n	h=(b+a)/n	h=(b-a)/n

The accuracy of the result using the Trapezoidal rule can be improved by --------	Increasing the interval h	Decreasing the length of the interval h	Increasing the number of iterations	altering the given function	Decreasing the length of the interval h
Simpson's one-third rule on numerical integration is called a --------- formula.	closed	open	semi closed	semi opened	closed
The order of error in Simpson's formula is ---------- .	1	2	3	4	4
In two point Gaussian quadrature Formula n = --------	1	2	3	4	2
In Simpsons 1/3rd rule, the number of ordinates must be ------- .	odd	even	0	3	odd
In three point Gaussian quadrature Formula n = ---------	1	2	3	4	3
Two point Gaussian quadrature Formula requires only -------- functional evaluations and gives a good estimate of the value of the integral.	1	2	3	4	2

UNIT-II

SYLLABUS

Solutions of system of Equations: The Elimination method: Gauss Elimination and Gauss Jordan Methods – LU decomposition method. Methods of Iteration: Gauss Jacobi and Gauss Seidal iteration-Relaxation method.

SOLUTIONS OF SYSTEM OF EQUATIONS

2.1 Introduction

We come across, very often simultaneously linear algebraic equations for its solutions, especially, in the fields of science and engineering. In lower classes, we have solved such equations by Cramer's rule (determinant methods) or by matrix methods. These methods become tedious when the number of unknown in the system is large. After the availability of computers, we go to numerical methods which are suited for computer operations. These numerical methods are of two types namely: (*i*) direct and (*ii*) iterative.

We will study a few methods below deals with the solution of simultaneous Linear Algebraic Equations

Gauss Elimination Method (Direct Method)

This is a direct method based on the elimination of the unknowns by combining equations such that the *n* unknowns are reduced to an equation upper triangular system which could be solved by back substitution.

Consider the *n* linear equations in *n* unknowns, viz.

$$a_{11}x_1+a_{12}x_2+\ldots.+a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+\ldots.+a_{2n}x_n=b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

$$a_{n1}x_1+a_{n2}x_2+\ldots..+a_{nn}x_n=b_n \quad \ldots.(1)$$

Where a_{ij} and b_i are known constants and x_i's are unknowns.

The system (1) is equivalent to $AX=B$ .....*(2)*

$$\text{Where } A = \begin{pmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \ldots\ldots\ldots \\ a_{n1} & a_{n2} \ldots\ldots a_{nn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Now our aim is to reduce the augmented matrix (A,B) to upper triangular matrix.

$$(A,B) = \left(\begin{array}{ccc|c} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} & & b_1 \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} & & b_2 \\ \ldots\ldots\ldots & \ldots\ldots\ldots \ldots\ldots\ldots & & \vdots \\ a_{n1} & a_{n2} \ldots\ldots\ldots a_{nn} & & b_n \end{array}\right) \quad ..(3)$$

Now, multiply the first row of (3) (if $a_{11} \neq 0$) by $-\frac{a_{i1}}{a_{11}}$ and add to the ith row of (A,B), where i=2,3,…,n. By thia, all elements in the first column of (A,B) except a_{11} are made to zero. Now (3) is of the form

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} & & b_1 \\ 0 & b_{22} \ldots\ldots\ldots b_{2n} & & c_2 \\ \ldots\ldots\ldots & \ldots\ldots\ldots \ldots\ldots\ldots & & \vdots \\ 0 & b_{n2} \ldots\ldots\ldots b_{nn} & & c_n \end{array}\right) \quad \ldots\ldots.(4)$$

Now take the pivot b_{22}. Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second

row of (4) by $-\frac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row (i=3,4,…,n). Now all elements below b_{22} are reduced to zero. Now (4) reduces to

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13}\ldots\ldots\ldots & a_{1n} & b_1 \\ 0 & b_{22} & b_{23}\ldots\ldots\ldots & b_{2n} & c_2 \\ 0 & 0 & c_{23}\ldots\ldots\ldots & c_{3n} & d_3 \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \vdots \\ 0 & 0 & c_{n3}\ldots\ldots\ldots & c_{nn} & d_n \end{array}\right) \quad \ldots\ldots (5)$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get (A,B) after all these operations as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \ldots\ldots\ldots\ldots & & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \ldots\ldots\ldots\ldots & & b_{2n} & c_2 \\ 0 & 0 & c_{23} & c_{34}\ldots\ldots\ldots & & c_{3n} & d_3 \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots\ldots\ldots & c_{nn} & d_n \end{array}\right) \quad \ldots (6)$$

From, (6) the given system of linear equations is equivalent to

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots.+a_{1n}x_n=b_1$$

$$b_{22}x_2+b_{23}x_3+\ldots.+b_{2n}x_n=c_2$$

$$c_{33}x_3+\ldots.+c_{3n}x_n=d_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$\alpha_{nn}x_n=k_n$$

Going from the bottom of these equation, we solve for $x_n = \frac{k_n}{\alpha_{nn}}$. Using this in the penultimate equation, we get x_{n-1} and so. By this back substitution method for we solve x_n , x_{n-1}, x_{n-2}, ……x_2, x_1.

2.3 Gauss – Jordan Elimination Method (Direct Method)

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system $AX=B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making the matrix A not above the leading diagonal of A also as zeros. By this way, the system $AX=B$ will reduce to the form.

$$\left(\begin{array}{cccccc|c} a_{11} & 0 & 0 & 0 & \ldots\ldots\ldots\ldots & a_{1n} & b_1 \\ 0 & b_{22} & 0 & 0 & \ldots\ldots\ldots\ldots & b_{2n} & c_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & d_3 \\ 0 & 0 & 0 & 0 & \ldots\ldots\ldots & \alpha_{nn} & k_n \end{array}\right) \ldots(7)$$

From (7)

$$x_n = \frac{k_n}{\alpha_{nn}}, \ldots\ldots, x_2 = \frac{c_2}{b_{22}}, x_n = \frac{b_1}{a_{11}}$$

Note: By this method, the values of $x_1, x_2, \ldots. x_n$ are got immediately without using the process of back substitution.

Example 1. *Solve the system of equations by (i) Gauss elimination method (ii) Gauss – Jordan method.*

$x+2y+z=3$, $2x+3y+3z=10$, $3x-y+2z=13$.

Solution. (By Gauss method)

This given system is equivalent to

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

$$A\,X = B$$

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now, we will make the matrix A upper triangluar.

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) \quad R_2+(-2)R_1\ ,\ R_3+(-3)R_1$$

Now, take b_{22}=-1 as the pivot and make b_{32} as zero.

$$(A,B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] R_{32}(-7) \ldots\ldots\ldots..(2)$$

From this, we get

$x+2y+z = 3$, $-y+z = 4$, $-8z = -24$

$\therefore$ $z = 3,\ y = -1,\ x = 2$ by back substitution.

$x = 2,\ y = -1,\ z = 3$

Solution. (Gauss – Jordan method)

In stage 2, make the element, in the position (1,2), also zero.

$$(A,B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] R_{12}(2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_3\left(\frac{1}{8}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_{13}(3),\ R_{23}(1)$$

i.e., $x = 2, \; y = -1, \; z = 3$

Example 2 *Solve the system by Gauss- Elimination method*

$2x+3y-z = 5; \quad 4x+4y-3z = 3 \quad \text{and} \quad 2x-3y+2z = 2.$

Solution. The system is equivalent to

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$A \qquad X = B$$

$$(A,B) = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right]$$

Step 1. Taking $a_{11} = 2$ as the pivot, reduce all elements below that to zero.

$$(A,B) = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right] R_{21}(-2),\; R_{31}(-1)$$

Step 2. Taking the element -2 in the position (2,2) as pivot, reduce all elements all elements below that to zero.

$$(A,B) = \left[\begin{array}{cccc} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right] \qquad R_{32}(-3)$$

Hence $2x+3y-z = 5$

$-2y-z = -7$

$6z = 18$

$\therefore \; z = 3, \; y = 2, \; x = 1.$ By back substitution

Example 2.3 Solve the following system by Gauss - Jordan method

$5x_1 + x_2 + x_3 + x_4 = 4; \quad x_1 + 7x_2 + x_3 + x_4 = 12$

$x_1 + x_2 + 6x_3 + x_4 = -5; \quad x_1 + x_2 + x_3 + 4x_4 = -6$

solution. interchange the first and the last equation, so that coefficient of x_1 in the first equation is 1. Then we have

$$(A,B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{6} & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) R_{21}(-1),\ R_{31}(-1),\ R_{41}(-5)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{1} & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) R_2\left(\frac{1}{6}\right) \text{ to make the pivot as 1}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{5} & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) R_{12}(-1),\ R_{42}(4)$$

$$\sim \left(\begin{array}{ccccc} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{1} & -0.6 & 0.2 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) R_3\left(\frac{1}{5}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -23.4 & 46.8 \end{array}\right) \quad R_{12}(-1),\ R_{43}(4)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) \quad R_4\left(\frac{1}{23.4}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) R_{34}\left(-\frac{3}{5}\right),\ R_{24}\left(-\frac{1}{2}\right),\ R_{14}(5.1)$$

$x_1 = 1,\ \ x_2 = 2,\ \ x_3 = -1,\ \ x_4 = -2$

Example 4. *Solve the system of equations by Gauss – Jordan method:*

$$x + y + z + w = 2$$

$$2x – y + 2z – w = -5$$

$$3x + 2y + 3z + 4w = 7$$

$$x – 2y – 3z + 2w = 5$$

Solution.

$$(A,B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \quad \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & \boxed{1} & 0 & 1 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \quad R_2\left(-\frac{1}{3}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{array}\right) \quad \begin{array}{l} R_1+(-1)R_2 \\ \\ R_3+R_2 \\ R_4+3R_2 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 & -3 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3\left(\frac{1}{2}\right) \\ R_4\left(-\frac{1}{4}\right) \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \text{Interchanging } R_3 \text{ and } R_4$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad R_1+(-1)R_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1+(-1)R_4 \\ R_2+(-1)R_4 \\ R_3+R_4 \end{array}$$

$\therefore$ $x = 0, \; y = 1, z = -1, w = 2$

Example 5. *Apply Gauss – Jordan method to find the solution of the following system:*

$10x + y + z = 12; \;\; 2x + 10y + z = 13; \qquad x + y + 5z = 7.$

Solution. since the coefficient of *x* in the last equation is unity, we rewrite the equations interchanging the first and the last. Hence the augmented matrix is

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) \quad \begin{array}{l} R_2+(-2)R_1 \\ R_3+(-10)R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & -9 & -49 & -58 \end{array}\right) \quad R_2\left(\frac{1}{8}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{473}{8} & -\frac{473}{8} \end{array}\right)$$

$$\sim \left(\begin{array}{cccc} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_3\left(-\frac{8}{473}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{49}{8} & \frac{57}{8} \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1+(-1)R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} R_2+\left(\frac{9}{8}\right)R_3 \\ R_1+\left(-\frac{49}{8}\right)R_3 \end{array}$$

∴ $x = 1, \quad y = 1, z = 1$

2.4 Method Of Triangularization (Or Method Of Factorization) (*Direct Method*)

This method is also called as *decomposition* method. In this method, the coefficient matrix A of the system $AX = B$, decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U. we will explain this method in the case of three equations in three unknowns.

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system is equivalent to $AX = B$

Where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

And an upper triangular matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$ so that

$LUX = B$ Let $UX = Y$ And hence $LY = B$

That is, $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

∴ $y_1 = b, \quad l_{21}y_1 + y_2 = b_2, \quad l_{31}y_1 + l_{32}y_2 + y_3 = b_3$

By forward substitution, y_1, y_2, y_3 can be found out if L is known.

From (4), $\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$, $u_{22}x_2 + u_{23}x_3 = y_2$ and $u_{33}x_3 = y_3$

From these, x_1, x_2, x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known.Now L and U can be found from $LU = A$

i.e., $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

i.e.,

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3 l's and 6 u's.

That is, L and U re known. Hence X is found out. Going into details, we get $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$. That is the elements in the first rows of U are same as the elements in the first of A.

Also, $l_{21}u_{11} = a_{21}$ $l_{21}u_{12}+u_{22} = a_{22}$ $l_{21}u_{13}+u_{23} = a_{23}$

$$l_{21} = \frac{a_{21}}{a_{11}},\ u_{22} = a_{22} - \frac{a_{21}}{a_{11}}.a_{12} \text{ and } u_{23} = a_{23} - \frac{a_{21}}{a_{11}}.\ a_{13}$$

again, $l_{31}u_{11} = a_{31}$, $l_{31}u_{12}+l_{32}u_{22} = a_{32}$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = a_{33}$

$$\text{solving, } l_{31} = \frac{a_{31}}{a_{11}},\ l_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}}.a_{12}}{a_{22} - \frac{a_{21}}{a_{11}}.a_{12}}$$

$$u_{33} = a_{33} - \left[\frac{a_{31}}{a_{11}}.\ a_{13}\right] - \left[\frac{a_{32} - \frac{a_{31}}{a_{11}}.a_{12}}{a_{22} - \frac{a_{21}}{a_{11}}.a_{12}}\right]\ a_{33} - \frac{a_{31}}{a_{11}}.\ a_{13}$$

Therefore L and U are known.

Example 1: *By the method of triangularization, solve the following system.*

5x – 2y + z = 4, 7x + y – 5z = 8, 3x + 7y + 4z = 10.

Solution. The system is equivalent to

$$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$$A \quad X = B$$

Now, let $LU = A$

That is, $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$

Multiplying and equating coefficients,

$$u_{11} = 5, \quad u_{12} = -2, \quad u_{13} = 1$$

$$l_{21}u_{11} = 7 \quad l_{21}u_{12} + u_{22} = 1 \quad l_{21}u_{13} + u_{23} = -5$$

$$l_{21} = \frac{7}{5}, \quad u_{22} = 1 - \frac{7}{5}.(-2) = \frac{19}{5} \text{ and}$$

$$u_{23} = -5 - \frac{7}{5}.\ (1) = -\frac{32}{5}$$

Again equating elements in the third row,

$$l_{31}u_{11} = 3, \quad l_{31}u_{12} + l_{32}u_{22} = 7 \text{ and } l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$$

$$\therefore \quad l_{31} = \frac{3}{5}, \quad l_{32} = \frac{7 - \frac{3}{5}.(-2)}{\frac{19}{5}} = \frac{41}{19}$$

$$u_{33} = 4 - \frac{3}{5}.(1) - \frac{41}{19}\left(-\frac{32}{5}\right) = 4 - \frac{3}{5} + \frac{1312}{95}$$

$$= \frac{1635}{95} = \frac{327}{19}$$

Now L and U are known. Since $LUX = B$, $LY = B$ where $UX = Y$.

From $LY = B$,

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$y_1 = 4$, $\frac{7}{5}y_1 + y_2 = 8$, $\frac{3}{5}y_1 + \frac{41}{19}y_2 + y_3 = 10$

$$y_2 = 8 - \frac{28}{5} = \frac{12}{5}$$

$$y_3 = 10 - \frac{12}{5} - \frac{41}{19} \text{ X } \frac{12}{5} = 10 - \frac{12}{5} - \frac{492}{95} = \frac{46}{19}$$

$UX = Y$ gives
$$\begin{pmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{pmatrix}$$

$$5x - 2y + z = 4$$

$$\frac{19}{5}y - \frac{32}{5}z = \frac{12}{5}$$

$$\frac{327}{19}z = \frac{46}{19}$$

$$z = \frac{46}{327}$$

$$\frac{19}{5}y = \frac{12}{5} + \frac{32}{5}\left(\frac{46}{327}\right)$$

$$y = \frac{284}{327}$$

$$5x = 4+2y - z = 4 + 2\left(\frac{568}{327}\right) - \frac{46}{327}$$

$$\therefore \qquad x = \frac{366}{327}$$

$$\therefore \qquad x = \frac{366}{327},\ y = \frac{284}{327},\ z = \frac{46}{327}$$

Example 2: *Solve, by triangularization method, the following system:*

$x + 5y + z = 14$, $2x + y + 3z = 13$, $3x + y + 4z = 17$.

Solution. this is equivalent to

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$$A \qquad X = B$$

Now, let $LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$

By seeing, we can write $u_{11} = 1,\ u_{12} = 5\ \ u_{13} = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} 1 & 5 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

Hence, $l_{21} = 2,\ \ 5l_{21}+u_{22} = 1 \quad l_{21}+u_{23} = 3$

$l_{21} = 2,\ u_{22} = -9, \quad u_{23} = 1$

again, $l_{31} = 3,\ \ 5l_{31}+l_{32}u_{22} = 1$ and $l_{31}+l_{32}u_{23}+u_{33} = 4$

$$l_{32} = \frac{1-15}{-9} = \frac{14}{9};\ u_{33} = 4 - 3 - \frac{14}{9} = -\frac{5}{9}$$

$LUX = B$ implies $LY = B$ where *$UX = Y$.*

$LY = B$, gives,

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$y_1 = 14$, $2\ y_1+y_2 = 13$, $3\ y_1+\frac{14}{9}\ y_2+y_3 = 17$

$$y_1 = 14, \quad y_2 = -15, \quad y_3 = -\frac{5}{3}$$

$UX = Y$ gives $\begin{pmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{pmatrix}$

$$x + 5y + z = 14$$

$$-9y + z = -15$$

$$-\frac{5}{9}z = -\frac{5}{3}$$

$\therefore \quad x = 1, \ y = 2, \ z = 3$

2.5 Iterative Methods

This iterative methods is not always successful to all systems of equations. If this method is to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation. This condition will be satisfied if the large coefficients are along the leading diagonal of the coefficient matrix. When this condition is satisfied, the system will be solvable by the iterative method. The system,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

will be solvable by this method if

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

In other words, the solution will exist (iteration will converge) if the absolute values of the leading diagonal elements of the coefficient matrix *A* of the system *AX=B* are greater than the sum of absolute values of the other coefficients of that row. The condition is *sufficient* but not *necessary*.

2.6 Jacobi Method Of Iteration or Gauss – Jacobi Method

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations,

$$a_1x+b_1y+c_1z = d_1$$

$$a_2x+b_2y+c_2z = d_2$$

$$a_3x+b_3y+c_3z = d_3 \quad \ldots\ldots\ldots\ldots (1)$$

Let us assume $|a_1| > |b_1| + |c_1|$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

Then, iterative method can be used for the system (1). Solve for *x, y, z* (whose coefficients are the larger values) in terms of the other variables. That is,

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \ldots\ldots\ldots.. (2)$$

If x^0, y^0, z^0 are the initial values of *x, y, z* respectively, then

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)}) \ \ldots\ldots\ldots (3)$$

Again using these values $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ in (2), we get

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)}) \quad \ldots.(4)$$

Proceeding in the same way, if the rth iterates are $x^{(r)}$, $y^{(r)}$, $z^{(r)}$, the iteration scheme reduces to

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r)} - b_3 y^{(r)}) \quad \ldots..(5)$$

The procedure is continued till the convergence is assured (correct to required decimals).

Note 1: To get the (r+1)th iterates, we use the values of the rth iterates in the scheme (5).

2: In the absence of the initial values of x, y, z we take, usually, (0, 0, 0) as the initial estimate.

2.7 Gauss – Seidel Method of Iteration:

This is only a refinement of Guass – Jacobi method. As before,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial values y^0, z^0 for y and z and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

While using the second equation, we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of x^0 as in Jacobi's method, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)} and\ y^{(1)}$, use $x^{(1)}$ for x $and\ y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ are the rth iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

This process of iteration is continued until the convergence assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss – seidel method is very fast when compared to Gauss – Jacobi method. The rate of convergence in Gauss – Seidel method is roughly two times than that of Gauss – Jacobi method. As we saw the sufficient condition already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is, *the method of iteration will converge if in each equation of the given system, the absolute value of the largest* coefficient is greater than the sum of the absolute values of all the remaining coefficients. (The largest coefficients must be the coefficients for different unknowns).

Note 1: For all systems of equations, this method will not work (since convergence is not assured). It onverges only for special systems equations.

Note 2: Iteration method is self – correcting method. That is, any error made in computation, is corrected in the subsequent iterations.

Note 3: The iteration is stopped when the values of *x, y, z* start repeating with the required degree of accuracy.

Example 1. Solve the following system by Gauss – Jacobi and Gauss – Seidel methods:

10x-5y-2z = 3 ; 4x-10y+3z = -3 ; x+6y+10z = -3.

Solution: Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix}$ is diagonally dominant, since $|10| > |-5| + |-2|$,

$|-10| > |4| + |3|$,

$|10| > |1| + |6|$

Gauss – Jacobi method, solving for *x, y, z* we have

$x = \frac{1}{10}(3+5y+2z)$ .... (1)

$y = \frac{1}{10}(3+4x+3z)$ .... (2)

$z = \frac{1}{10}(-3-x-6y)$ .... (3)

First iteration: Let the initial values be (0, 0, 0).

Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}(3+5(0)+2(0)) = 0.3$$

$$y^{(1)} = \frac{1}{10}(3+4(0)+3(0)) = 0.3$$

$$z^{(1)} = \frac{1}{10}(-3-(0)-6(0)) = -0.3$$

Second iteration: using these values in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}(3+5(0.3)+2(-0.3)) = 0.39$$

$y^{(2)} = \frac{1}{10}(3 + 4(0.3) + 3(-0.3)) = 0.33$

$z^{(2)} = \frac{1}{10}(-3- (0.3) – 6(0.3)) = - 0.51$

Third iteration: using these values of $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ in (1), (2), (3), we get,

$x^{(3)} = \frac{1}{10}(3 + 5(0.33) + 2(-0.51)) = 0.363$

$y^{(3)} = \frac{1}{10}(3 + 4(0.39) + 3(-0.51)) = 0.303$

$z^{(3)} = \frac{1}{10}(-3- (0.39) – 6(0.33)) = - 0.537$

Fourth iteration:

$x^{(4)} = \frac{1}{10}(3 + 5(0.303) + 2(-0.537)) = 0.3441$

$y^{(4)} = \frac{1}{10}(3 + 4(0.363) + 3(-0.537)) = 0.2841$

$z^{(4)} = \frac{1}{10}(-3- (0.363) – 6(0.303)) = -0.5181$

Fifth iteration:

$x^{(5)} = \frac{1}{10}(3 + 5(0.2841) + 2 (-0.5181)) = 0.33843$

$y^{(5)} = \frac{1}{10}(3 + 4(0.3441) + 3(-0.5181)) = 0.2822$

$z^{(5)} = \frac{1}{10}(-3- (0.3441) – 6(0.2841)) = - 0.50487$

Sixth iteration:

$x^{(6)} = \frac{1}{10}(3 + 5(0.2822) + 2 (-0.50487)) = 0.340126$

$y^{(6)} = \frac{1}{10}(3 + 4(0.33843) + 3(-0.50487)) = 0.283911$

$z^{(6)} = \frac{1}{10}(-3- (0.33843) – 6(0.2822)) = - 0.503163$

Seventh iteration:

$x^{(7)} = \frac{1}{10}(3 + 5(0.283911) + 2(-0.503163)) = 0.3413229$

$y^{(7)} = \frac{1}{10}(3 + 4(0.340126) + 3(-0.503163)) = 0.2851015$

$z^{(7)} = \frac{1}{10}(-3 - (0.340126) - 6(0.283911)) = -0.5043592$

Eighth iteration:

$x^{(8)} = \frac{1}{10}(3 + 5(0.2851015) + 2(-0.5043592))$

$= 0.34167891$

$y^{(8)} = \frac{1}{10}(3 + 4(0.3413229) + 3(-0.5043592))$

$= 0.2852214$

$z^{(8)} = \frac{1}{10}(-3 - (0.3413229) - 6(0.2851015))$

$= -0.50519319$

Ninth iteration:

$x^{(9)} = \frac{1}{10}(3 + 5(0.2852214) + 2(-0.50519319))$

$= 0.341572062$

$y^{(9)} = \frac{1}{10}(3 + 4(0.34167891) + 3(-0.50519319))$

$= 0.285113607$

$z^{(9)} = \frac{1}{10}(-3 - (0.34167891) - 6(0.2852214)) = -0.505300731$

Hence, correct to 3 decimal places, the values are

$x = 0.342, \quad y = 0.285, \quad z = -0.505$

Gauss – Seidel method: Initial values : $y = 0, z = 0$.

First iteration: $x^{(1)} = \frac{1}{10}(3 + 5(0) + 2(0)) = 0.3$

$y^{(1)} = \frac{1}{10}(3 + 4(0.3) + 3(0)) = 0.42$

$z^{(1)} = \frac{1}{10}(-3 - (0.3) - 6(0.42)) = -0.582$

Second iteration:

$x^{(2)} = \frac{1}{10}(3 + 5(0.42) + 2(-0.582)) = 0.3936$

$y^{(2)} = \frac{1}{10}(3 + 4(0.3936) + 3(-0.582)) = 0.28284$

$z^{(2)} = \frac{1}{10}(-3 - (0.3936) - 6(0.28284)) = -0.509064$

Third iteration:

$x^{(3)} = \frac{1}{10}(3 + 5(0.28284) + 2(-0.509064)) = 0.3396072$ $y^{(3)} = \frac{1}{10}(3 + 4(0.3396072) + 3(-0.509064)) = 0.28312368$

$z^{(3)} = \frac{1}{10}(-3 - (0.3396072) - 6(0.28312368))$

$= -0.503834928$

Fourth iteration:

$x^{(4)} = \frac{1}{10}(3 + 5(0.28312368) + 2(-0.503834928))$

$= 0.34079485$

$y^{(4)} = \frac{1}{10}(3 + 4(0.34079485) + 3(-0.503834928))$

$= 0.285167464$

$z^{(4)} = \frac{1}{10}(-3 - (0.34079485) - 6(0.285167464))$

$= -0.50517996$

Fifth iteration:

$x^{(5)} = \frac{1}{10}(3 + 5(0.285167464) + 2(-0.50517996))$

$= 0.34155477$

$y^{(5)} = \frac{1}{10}(3 + 4(0.34155477) + 3(-0.50517996))$

$= 0.28506792$

$z^{(5)} = \frac{1}{10}(-3 - (0.34155477) - 6(0.28506792))$

$= -0.505196229$

Sixth iteration:

$x^{(6)} = \frac{1}{10}(3 + 5(0.28506792) + 2(-0.505196229))$

$= 0.341494714$

$y^{(6)} = \frac{1}{10}(3 + 4(0.341494714) + 3(-0.505196229))$

$= 0.285039017$

$z^{(6)} = \frac{1}{10}(-3 - (0.341494714) - 6(0.28506792))$

$= -0.5051728$

Seventh iteration:

$x^{(7)} = \frac{1}{10}(3 + 5(0.285039017) + 2(-0.5051728))$

$= 0.3414849$

$y^{(7)} = \frac{1}{10}(3 + 4(0.3414849) + 3(-0.5051728))$

$= 0.28504212$

$z^{(7)} = \frac{1}{10}(-3 - (0.3414849) - 6(0.28504212))$

= - 0.5051737

The values at each iteration by both methods are tabulated below:

Iteration	Gauss - jacobi method			Gauss – seidel method		
	x	*y*	*z*	*x*	*y*	*z*
1	0.3	0.3	-0.3	0.3	0.42	-0.582
2	0.39	0.33	-0.51	0.3936	0.2828	-0.5090
3	0.363	0.303	-0.537	0.3396	0.2831	-0.5038
4	0.3441	0.2841	-0.5181	0.3407	0.2851	-0.5051
5	0.3384	0.2822	-0.5048	0.3415	0.2850	-0.5051
6	0.3401	0.2839	-0.5031	0.3414	0.2850	-0.5051
7	0.3413	0.2851	-0.5043	0.3414	0.2850	-0.5051
8	0.3416	0.2852	-0.5051			
9	0.3411	0.2851	-0.5053			

The values correct to 3 decimal places are

$$x = 0.342,\ y = 0.285,\ z = -0.505$$

Example 2. Solve the following system of equations by using Gauss – jacobi and Gauss – Seidel methods (correct to 3 decimal places):

$$8x - 3y + 3z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35.$$

Solution: since the diagonal elements are dominant in the coefficient matrix, we write *x, y, z* as follows

$x = \frac{1}{8}(20 + 3y - 2z)$ ………………. (1)

$y = \frac{1}{11}(33 + 4x + z)$ ………………. (2)

$z = \frac{1}{12}(35 - 6x - 3y)$ ………………. (3)

Gauss – Jacobi method:

First iteration: Let the initial values be $x = 0, y = 0, z = 0$

Using the values $x = 0, y = 0, z = 0$ in (1), (2), (3) we get,

$$x^{(1)} = \frac{1}{8}(20 + 3(0) - 2(0)) = 2.5$$

$$y^{(1)} = \frac{1}{11}(33 + 4(0) + (0)) = 3.0$$

$$z^{(1)} = \frac{1}{12}(35 - 6(0) - 3(0)) = 2.916666$$

Second iteration: using these values of $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ in (1), (2), (3), we get,

$x^{(2)} = \frac{1}{8}(20 + 3(3.0) - 2(2.916666)) = 2.895833$

$y^{(2)} = \frac{1}{11}(33 + 4(2.5) + (2.916666)) = 2.356060$

$z^{(2)} = \frac{1}{12}(35 - 6(2.5) - 3(3.0)) = 0.916666$

Third iteration:

$x^{(3)} = \frac{1}{8}(20 + 3(2.356060) - 2(0.916666)) = 3.154356$

$y^{(3)} = \frac{1}{11}(33 + 4(2.895833) + (0.916666)) = 2.030303$ $z^{(3)} = \frac{1}{12}(35 - 6(2.895833) - 3(2.356060)) = 0.879735$

Fourth iteration:

$x^{(4)} = \frac{1}{8}(20 + 3(2.030303) - 2(0.879735)) = 3.041430$

$y^{(4)} = \frac{1}{11}(33 + 4(3.154356) + (0.879735)) = 2.932937$

$z^{(4)} = \frac{1}{12}(35 - 6(3.154356) - 3(2.030303)) = 0.831913$

Fifth iteration:

$x^{(5)} = \frac{1}{8}(20 + 3(2.932937) - 2(0.831913)) = 3.016873$

$y^{(5)} = \frac{1}{11}(33 + 4(3.041430) + (0.831913)) = 1.969654$

$z^{(5)} = \frac{1}{12}(35 - 6(3.041430) - 3(2.932937)) = 0.912717$

Sixth iteration:

$x^{(6)} = \frac{1}{8}(20 + 3(1.969654) - 2(0.912717)) = 3.010441$

$y^{(6)} = \frac{1}{11}(33 + 4(3.016873) + (0.912717)) = 1.985930$

$z^{(6)} = \frac{1}{12}(35 - 6(3.016873) - 3(1.969654)) = 0.915817$

Seventh iteration:

$x^{(7)} = \frac{1}{8}(20 + 3(1.985930) - 2(0.915817)) = 3.015770$

$y^{(7)} = \frac{1}{11}(33 + 4(3.010441) + (0.915817)) = 1.988550$

$z^{(7)} = \frac{1}{12}(35 - 6(3.010441) - 3(1.985930)) = 0.914964$

Eighth iteration:

$x^{(8)} = \frac{1}{8}(20 + 3(1.988550) - 2(0.914964)) = 3.016946$

$y^{(8)} = \frac{1}{11}(33 + 4(3.015770) + (0.914964)) = 1.986535$

$z^{(8)} = \frac{1}{12}\ (35 - 6(3.015770) - 3(1.988550)) = 0.911644$

Ninth iteration:

$x^{(9)} = \frac{1}{8}\ (20 + 3(1.986535) - 2(0.911696)) = 3.017039$

$y^{(9)} = \frac{1}{11}\ (33 + 4(3.016946) + (0.911696)) = 1.985805$

$z^{(9)} = \frac{1}{12}\ (35 - 6(3.016946) - 3(1.986535)) = 0.911560$

Tenth iteration:

$x^{(9)} = \frac{1}{8}\ (20 + 3(1.985805) - 2(0.911560)) = 3.016786$

$y^{(9)} = \frac{1}{11}\ (33 + 4(3.017039) + (0.911560)) = 1.985764$

$z^{(9)} = \frac{1}{12}\ (35 - 6(3.017039) - 3(1.985805)) = 0.911696$

In 8^{th} , 9^{th} and 10^{th} iterations the values of x, y, z are same correct to 3 decimal places. Hence, we stop at this level.

Gauss – Seidel method:

We take the initial values are $y = 0$, $z = 0$ and use equations (1)

First iteration:

$x^{(1)} = \frac{1}{8}\ (20 + 3(0) - 2(0)) = 2.5$

$y^{(1)} = \frac{1}{11}\ (33 + 4(2.5) + (0)) = 2.090909$

$z^{(1)} = \frac{1}{12}\ (35 - 6(2.5) - 3(2.090909)) = 1.143939$

Second iteration:

$x^{(2)} = \frac{1}{8}\ (20 + 3(2.090909) - 2(1.143939)) = 2.998106$

$y^{(2)} = \frac{1}{11}(33 + 4(2.998106) + (1.143939)) = 2.013774$ $z^{(2)} = \frac{1}{12}(35 - 6(2.998106) - 3(2.013774)) = 0.914170$

Third iteration:

$x^{(3)} = \frac{1}{8}(20 + 3(2.013774) - 2(0.914170)) = 3.026623$

$y^{(3)} = \frac{1}{11}(33 + 4(3.026623) + (0.914170)) = 1.982516$ $z^{(3)} = \frac{1}{12}(35 - 6(3.026623) - 3(1.982516)) = 0.907726$

Fourth iteration:

$x^{(4)} = \frac{1}{8}(20 + 3(1.982516) - 2(0.907726)) = 3.016512$

$y^{(4)} = \frac{1}{11}(33 + 4(3.026623) + (0.907726)) = 1.985607$

$z^{(4)} = \frac{1}{12}(35 - 6(3.016512) - 3(1.985607)) = 0.912009$

Fifth iteration:

$x^{(5)} = \frac{1}{8}(20 + 3(1.985607) - 2(0.912009)) = 3.016600$

$y^{(5)} = \frac{1}{11}(33 + 4(3.016600) + (0.912009)) = 1.985964$

$z^{(5)} = \frac{1}{12}(35 - 6(3.016600) - 3(1.985964)) = 0.911876$

Sixth iteration:

$x^{(6)} = \frac{1}{8}(20 + 3(1.985964) - 2(0.911876)) = 3.016767$

$y^{(6)} = \frac{1}{11}(33 + 4(3.016767) + (0.911876)) = 1.985892$

$z^{(6)} = \frac{1}{12}(35 - 6(3.016767) - 3(1.985892)) = 0.911810$

(The values of *x, y, z* got by jacobi method correct to 3 decimal places are got even in the 6th iteration by Gauss – seidel method.)

Seventh iteration:

$x^{(7)} = \frac{1}{8}(20 + 3(1.985892) - 2(0.911810)) = 3.016757$

$y^{(7)} = \frac{1}{11}(33 + 4(3.016757) + (0.911810)) = 1.985889$

$z^{(7)} = \frac{1}{12}(35 - 6(3.016757) - 3(1.985889)) = 0.911816$

Since the seventh and eighth iterations give the same values for *x, y, z* correct to 4 decimal places, we stop here.

$\therefore$ $x = 3.0168,\ y = 1.9859,\ z = 0.9118$

The values of *x, y, z* by both methods at each iteration are tabulated below:

Iteration	Gauss – jacobi method			Gauss – seidel method		
	x	*y*	*z*	*x*	*y*	*z*
1	2.5	3.0	2.9166	2.5	2.0909	1.1439
2	2.8958	2.3560	0.9166	2.9981	2.0137	0.9141
3	3.1543	2.0303	0.8797	3.0266	1.9825	0.9077
4	3.0414	1.9329	0.8319	3.0165	1.9856	0.9120
5	3.0168	1.9696	0.9127	3.0166	1.9859	0.9118
6	3.0104	1.9859	0.9158	3.0167	1.9858	0.9118
7	3.0157	1.9885	0.9149	3.0167	1.9858	0.9118
8	3.0169	1.9865	0.9116			
9	3.0170	1.9858	0.9115			
10	3.0167	1.9857	0.9116			

This shows that the convergence is rapid in Gauss – seidel method when compared to Gauss – Jacobi method. We see that 10 iterations are necessary in jacobi method to get the same accuracy as got by 7 iterations in Gauss – Seidel method.

Example 3. Since the diagonal elements in the coefficient matrix are not dominant, we arrange the equations, as follows, such that the elements in the coefficient matrix are dominant.

$28x + 4y - z = 32, \quad x + 3y + 10z = 24, \quad 2x + 17y + 4z = 35$

Solution: Since the diagonal elements in the coefficient matrix are not dominant, we rearrange the equations, as follows, such that the elements in the coefficient matrix are dominant.

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

Hence, $x = \frac{1}{28}(32 - 4y + z)$ ………………. (1)

$y = \frac{1}{17}(35 - 2x - 4z)$ ………………. (2)

$z = \frac{1}{10}(24 - x - 3y)$ ………………. (3)

setting $y = 0, z = 0$, we get

First iteration:

$$x^{(1)} = \frac{1}{28}(32 - 4(0) + 0) = 1.1429$$

$$y^{(1)} = \frac{1}{17}(35 - 2(1.1429) - 4(0)) = 1.9244$$

$$z^{(1)} = \frac{1}{10}(24 - 1.1429 - 3(1.9244)) = 1.8084$$

Second iteration:

$$x^{(2)} = \frac{1}{28}(32 - 4(1.9244) + 1.8084) = 0.9325$$

$y^{(2)} = \frac{1}{17}(35 - 2(0.9325) - 4(1.8084)) = 1.5236$

$z^{(2)} = \frac{1}{10}(24 – 0.9325 – 3(1.5236)) = 1.8497$

Third iteration:

$x^{(3)} = \frac{1}{28}(32 - 4(1.5236) + 1.8497) = 0.9913$

$y^{(3)} = \frac{1}{17}(35 - 2(0.9913) - 4(1.8497)) = 1.5070$

$z^{(3)} = \frac{1}{10}(24 – 0.9913 – 3(1.5070)) = 1.8488$

Fourth iteration:

$x^{(4)} = \frac{1}{28}(32 - 4(1.5070) + 1.8488) = 0.9936$

$y^{(4)} = \frac{1}{17}(35 - 2(0.9936) - 4(1.8488)) = 1.5069$

$z^{(4)} = \frac{1}{10}(24 – 0.9936 – 3(1.5069)) = 1.8486$

Fifth iteration:

$x^{(5)} = \frac{1}{28}(32 - 4(1.5069) + 1.8486) = 0.9936$

$y^{(5)} = \frac{1}{17}(35 - 2(0.9936) - 4(1.8486)) = 1.5069$

$z^{(5)} = \frac{1}{10}(24 – 0.9936 – 3(1.5069)) = 1.8486$

Since the values of x, y, z in the 4th and 5th iterations are same, we stop the process here.

Hence, $x = 0.9936$, $y = 0.5069$ and $z = 1.8486$

2.8 Relaxation Method

Consider the system of equations,

$a_1x+b_1y+c_1z = d_1$

$$a_2x+b_2y+c_2z = d_2 \qquad (1)$$

$$a_3x+b_3y+c_3z = d_3$$

we define the residuals r_1, r_2, r_3 by the relations

$$\left.\begin{aligned} r_1 &= a_1x + b_1y + c_1z - d_1 \\ r_2 &= a_2x + b_2y + c_2z - d_2 \end{aligned}\right\} \quad (2)$$

$$r_3 = a_3x + b_3y + c_3z - d_3$$

if we can find the values of x, y, z so that $r_1 = 0 = r_2 = r_3$ then those values of x, y, z are the exact values of the system. If it is not possible to make $r_1 = 0 = r_2 = r_3$, then we make simultaneously the values to r_1, r_2, r_3 to as close to zero as possible. In other words we "liquidate" the residuals r_1, r_2, r_3 by taking better approximate values of x, y, z what will be the slight change is made in the values of x, y, z what will be the corresponding changes in the residuals, r_1, r_2, r_3? We give below an 'operation table' from which we can easily know the corresponding changes in r_1, r_2, r_3 for a change of 1 unit in x, while there is no change in re is no change in y and z, for a change of 1 unit in y while there in no change in x and z for a change of 1 unit in z while there is no change in y and x.

Operation Table

Operation	*Change in (or increment in)*					
	x	y	z	r_1	r_2	r_3
R_1	1	0	0	a_1	a_2	a_3
R_2	0	1	0	b_1	b_2	b_3
R_3	0	0	1	c_1	b_3	c_3

What is the meaning of the above table ?

The operator R_1 increase the value of x by 1, y by zero, z by zero

(no change in y and z) and this operation increases the residuals r_1 by a_1, r_2 by a_2, and r_3 by a_3 (the increase in r_1, r_2, r_3 are the nothing but the coefficients of x in the equations given). Similarly R_3 increases the value of z by 1 (while x, y are kept constant) and the effect of this operation increases the values of r_1, r_2, r_3 by c_1, c_2, c_3 respectively.

One can easily see that the operation table consists of the unit matrix *I* and the transpose of the matrix *A* and *A'*, where *A* is the coefficient matrix of the system of equations.

Convergence of the relaxation method:

If the method should converge, the diagonal elements of the coefficient matrix A should be dominant; that is, A is diagonally dominant. Referring to the system of equations given above; the system can be solved by this method successfully only if

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

Where at least once the strict inequality holds.

Example 1. *Solve the following equations using relaxation method*

$$10x - 2y - 2z = 6$$

$$-x + 10y - 2z = 7$$

$$-x - y + 10z = 8$$

Solution: Since the diagonal elements are dominant, we will do by relaxation method.

The residuals r_1, r_2, r_3 are given by

$$r_1 = 10x - 2y - 2z - 6$$

$$r_2 = -x + 10y - 2z - 7$$

$$r_3 = -x - y + 10z - 8$$

Operation Table (write 1,A')

Changes in

	x	*y*	*z*	r_1	r_2	r_3
R_1	1	0	0	10	-1	-1
R_2	0	1	0	-2	10	-1
R_3	0	0	1	-2	-2	10

We will take the initial values of x, y, z as 0, 0, 0.

Setting $x=0=y=z$, we get $r_1 = -6$, $r_2 = -7$, $r_3 = -8$

We write these residuals below and *relax* these values making changes in x, y, z as shown below:

	x	y	z	r_1	r_2	r_3
	0	0	0	-6	-7	-8
$R_1 \longrightarrow$	1	0	0	-8	-9	2
$R_2 \longrightarrow$	0	1	0	-10	1	1
$R_3 \longrightarrow$	0	0	1	0	0	0
	1	1	1	0	0	0

Analysis: In line (1), for $x=0$, $y= 0$, $z = 0$ the residuals are -6,-7,-8. The numerically largest residual is -8 which is encircled.

First, we liquidate the numerically largest residual $r_3 = -8$ by a proper multiple of R_3. Since R_3 operation increases r_3 by 10, by operation $1.R_3$, we get (i.e., put $x=0, y=0, z=1$) $r_1 = -6+(-2) = -8$; $r_2 = -7+(-2) = -9$; $r_3 = -8+10 = 2$ giving line (2). Now, in line (2), numerically greatest residual is -9 which is encircled. We will liquidate this r_2 by proper multiple of R_2. An increase of 1 in y will increase r_2 by 10, r_1 by -2 and r_3 by -1. Hence doing the operation $1.R_2$ new $r_1=-8-2=-10$,$r_2=-9+10=1$, $r_3=2+(-1)=1$ and we get the line (3). Now in line (3), $r_1=-10$ is the numerically greatest value. Now, we will liquidate this $r_1=-10$ by a proper multiple of R_1. Doing the operations $R_1 \longrightarrow (1, 0, 0)$, $r_1=-10+10=0$, $r_2=1+(-1)=0$, $r_3=1+(-1)=0$. Fortunately all the residuals have become zero after the 3 operations. Adding the values of x, y, z we get $x=1$, $y= 1$, $z = 1$ as the exact solution for the system.

POSSIBLE QUESTIONS:

Part-B(5X6 = 30 Marks)

Answer all the questions:

1. Solve the following system of equations using Gauss Elimination method.
$2x+y+z=10;\ 3x+2y+3z=18;\ x+4y+9z=16$

2.Solve the following system by Gauss Elimination method
$x + y + 2z = 4$
$3x + y - 3z = -4$
$2x - 3y - 5z = -5$

3. Solve the following system by Gauss Jordan method **.**
$x + 2y + z = 3$
$2x + 3y + 3z = 10$
$3x - y + 2z = 13$

4.Explain the algorithm of LU decomposition method

5. Solve the following system by triangularization method
$x + y + z = 1,\ 4x + 3y - z = 6,\ 3x + 5y + 3z = 4$

6. Solve the following system of equations by Gauss-Jacobi method
$10x - 5y - 2z = 3$
$4x - 10y + 3z = -3$
$x + 6y + 10z = -3$

7. Solve the following system by Gauss Jacobi method **.**

$8x + y + z = 8$
$2x + 4y + z = 4$
$x + 3y + 3z = 5$

8. Solve the following system of equations by Gauss-Seidal method.
$28x + 4y - z = 32$
$x + 3y + 10z = 24$
$2x + 17y + 4z = 35$

9. Solve the system of equation by Gauss Seidel method
$10x-5y-2z=3;\ 4x-10y+3z=-3;\ x+6y+10z=-3$

10. Solve the following system by Relaxation method.
$10x-2y-2z=6;\ -x+10y-2z=7;\ -x-y+10z=8$

PART C- (1 x 10 =10 Marks)
(Compulsory)

1. Solve the following system by Gauss elimination method
$10x+y+z=12,\ x+10y+z=12,\ x+y+10z=12$

2.Solve the following system by triangularisation method.
$5x-2y+z=4$
$7x+y-5z=8$
$3x+7y+4z=10$

3. Solve the following system by Relaxation method.
$10x+y+z=31;\ 2x+8y-z=24;\ 3x+4y+10z=58$

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Class : I M.Sc Mathematics **Semester : I**

Subject: Numerical Analysis **Subject Code: 19MMP103**

Unit II

Part A (20x1=20 Marks)

Question	Opt1	Opt 2	Opt 3	Opt 4	Answer
The numerical method of solving linear equations is of two types one is direct, other is---------- method.	iterative	elimination	Newton	exact	iterative
------------ Method produces the exact solution after a finite number of steps.	Gauss Siedal	Gauss Jacobbi	Iterative method	Direct	Direct
Gauss elimination method is a -----------------.	Indirect method	direct method	iterative method	convergent	direct method
The rate of convergence in Gauss – Seidel method is roughly --------- times than that of Gauss Jacobi method.	0	3	4	2	2
Example for iterative method -------------.	Gauss elimination	Gauss Jordan	Gauss Seidal	Newton's forward	Gauss Seidal
When Gauss Jordan method is used to solve AX = B, A is transformed into -----	Scalar matrix	diagonal matrix	Upper triangular matrix	lower triangular matrix	diagonal matrix
The modification of Gauss – Elimination method is called --------------------.	Gauss Jordan	Gauss Seidal	Gauss Jacobbi	Crout	Gauss Jordan
. ------------ Method produces the exact solution after a finite number of steps.	Gauss Seidal	Gauss Jacobi	Iterative	Direct.	Direct.
In the upper triangular coefficient matrix, all the elements below the diagonal are……	positive	non zero	zero	Negative	zero
Gauss Seidal method always --------- for a special type of systems.	converges	diverges	oscillates	infinity	converges

Condition for convergence of Gauss Seidal method is ---------------.	coefficient matrix is not diagonally dominant	pivot element is Zero	coefficient matrix is diagonally dominant	pivot element is not zero	coefficient matrix is diagonally dominant
In Gauss elimination method by means of elementary row operations, from which the unknowns are found by ---------------- method.	random	back substitution	forward substitution	direct substitution	forward substitution
In Gauss elimination method the given matrix is transformed into -------------.	unit matrix	diagonal matrix	upper triangular matrix	lower triangular matrix	upper triangular matrix
Gauss Jordan method is a -----------------.	direct method	indirect method	iterative method	convergent	direct method
Gauss Jocobi method is a -----------------.	direct method	indirect method	iterative method	convergent	indirect method
The modification of Gauss – Jacobi method is called ---------------------.	Gauss Jordan	Gauss elimination	Gauss Seidal	Crout	Gauss Seidal
In Gauss Jordan method the given matrix is transformed into --------------.	unit matrix	diagonal matrix	upper triangular matrix	lower triangular matrix	upper triangular matrix
In the direct methods of solving a system of linear equations, at first the given system is written as ------------- form	An augment matrix	a triangular matrix	Constant matrix	coefficient matrix	An augment matrix
All the row operations in the direct methods can be carried out on the basis of --	all elements	pivot element	negative elements	positive elements	pivot element
In solving the system of linear equations, the system can be written as --------.	BX = A	AX=A	AX=B	AB=X	AX=B
If the coefficient matrix is diagonally dominant, then ---------- method converges quickly.	Gauss elimination	Gauss Jordan	Choleskey	Gauss Seidal	Gauss Seidal
-------------- is also a self-correction method.	direct method	Iteration method	Interpolation	extrapolation	Iteration method
In ----------- method, the coefficient matrix is transformed into diagonal matrix	Gauss elimination	Gauss Jordan	Gauss Jacobi	Gauss Seidal	Gauss Jordan
The iterative process continues till ------------- is secured	convergency	divergency	oscillation	infinity	convergency

The augment matrix is the combination of ------------------.	Coefficient matrix and constant matrix	Unknown matrix and constant matrix	Coefficient matrix and Unknown matrix	Coefficient matrix, constant matrix and Unknown matrix	Coefficient matrix and constant matrix
Method of triangularisation is also known as---------	Gauss Seidal	Gauss Jordan	Gauss Elimination	LU Decoposition	LU Decoposition
In decomposition method, the coefficient matrix is factorized into the ------- of upper and lower triangular matrix.	sum	difference	product	division	product
Method of triangularisation is also --------- method	indirect	iterative	convergent	direct	direct
Condition for convergence of Relaxation method is ---------------.	Coefficient matrix is not diagonally dominant	pivot element is Zero	Coefficient matrix is diagonally dominant	pivot element is not zero	Coefficient matrix is diagonally dominant
Jacobis method is used only when the matrix is ____	symmetric	skew-symmetric	singular	non-singular	symmetric

UNIT-III

SYLLABUS

Solutions of Ordinary Differential Equations: One step method: Euler and Modified Euler Methods –Runge-Kutta methods. Multistep methods: Adams Moulton method – Milne's method

SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

3.1 Introduction

In the fields of engineering and science we come across physical and natural phenomena which when represented by mathematical model happen to be differential equations. For example simple harmonic motion, equation motion, deflection of a beam etc.. are represented by differential equations,. Hence the solution of differential equations is a necessity in such studies. There are number of differential equations which we studied in calculus to get closed form solutions. But all differential equations do not possess closed form or finite form solutions.

Even if they possess closed form solutions, we do not know the method of getting it. In such situations depending upon the need of the hour we go in far in numerical solutions of differential equations. In researchers after advent of computer the numerical solutions of the differential equations have become easy for manipulation. Hence we present below some of the methods of numerical solutions are approximate solutions. But in many cases approximate solutions to the required accuracy are quite sufficient.

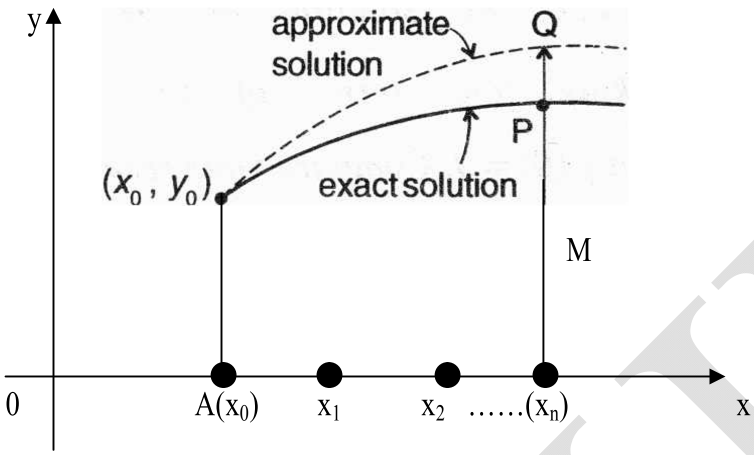
In solving a differential equation for approximate solution we find numerical values of $y_1,y_2,y_3,\ldots$ corresponding to given numerical values of independent variable values x_1,x_2,x_3, …so that the ordered pairs $(x_1y_1),(x_2y_2)$ … satisfy a particular solution, though approximately. A solution of this type is called a *point wise solution.*

Suppose we require to solve $dy/dx=f(x,y)$ with the initial condition $y(x_0)=y_0.$ By numerical solution of y at $x=x_0$, x_1, x_2,.. let $y=y(x)$be the exact solution. If we plot and draw the graph of $y=y(x)$, (exact curve) and also draw the approximate curve by plotting (x_0, y_0), (x_1, y_1), (x_2, y_2),… we get two curves.

PM= exact value, QM=approximate value at $x=x_i.$

Then

QP=MQ-MP=y_i-y (x_i) = ε is called the truncation error at x= x_i



QP=MQ-MP=y_i-y(x_i)= ε_i is called return error at x=x_i

3.2 Euler's Method

In solving a first order differential equation by numerical methods, we come across two types of solutions:

(i) A series solution of y in terms of x, which will yield the value of Y at a particular value of x by direct substitution in the series solution.

(ii) Values of y at specified values of x.

The following methods due to Euler, Runge-Kutta, Adam-Bashforth and Milne come under the second category. The methods of second category are called step-by-step methods because the values of y are calculated by short steps ahead of equal interval h of the independent variable x.

Euler's Method

AIM. To solve dy / dx = f*(x, y)* with the initial condition

y (x_0) = Y_o. -------(1)

i.e., x_i=x_0+ih, i=0,1,2……

Let the actual solution of the differential equation be denoted by the graph (continuous line graph) P_0 (x_0, Y_0) lies on the curve. We require the value of y of the curve at $x=x_1$.

The equation of tangent at (x_0, y_0) to the curve is

$y - y_0 = y'_{(x0,y0)}\,(x - x_0)$

$= f(x_0,y_0).\,(x - x_0)$

$y=y_0+ f(x_0,y_0).\,(x - x_0)$

This y is the value of y on the tangent corresponding to x = x. In the interval (x_0, x_1), the curve is approximated by the tangent. Therefore, the value of y on the curve is approximately equal to the value of y. on the tangent at (x_0,y_0) corresponding to $x=x_1$.

$y_1=y_0+ f(x_0,y_0)\,(x - x_0)$

i.e., $y_1=y_0+ hy_0'$.

Where $h=x_1-x_0$.

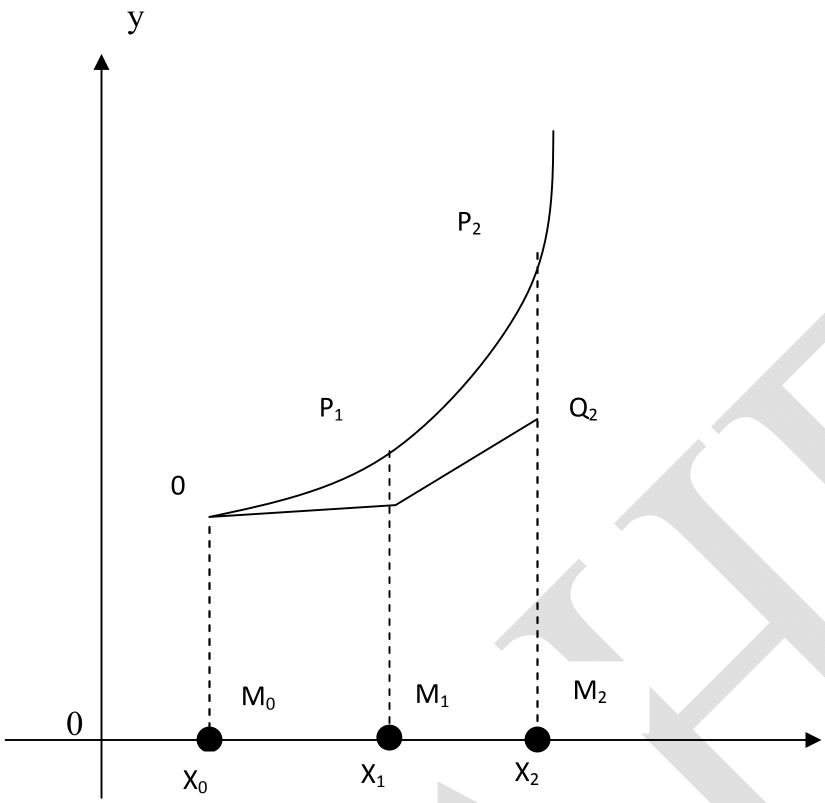
$(M_1P_1\approx M_1Q_1=y_1)$

Again, we approximate curve by the line through (x_1,y_1) and whose slope is $f(x_1,y_1)$ we get $y_2=y_1+ hf(x_1,y_1) = y_1+hy_1'$

Thus $y_{n+1}=y_n+ hf(x_n,y_n)$; n=0,1,2……

This formula is called Euler's algorithm.

In other words,



This formula is called Euler's algorithm.

$$y(x+h) = y(x)+h\, f(x,y).$$

In this method, the actual curve is approximated by a sequence of short straight lines. As the intervals increase the straight line deviates much from the actual curve. Hence the accuracy cannot be obtained as the number of intervals increase.

Q_1P_1=error at $x=x_1$

$$\frac{(x_1- x_0)^2}{2!}\, y''(x_1,y_1) = \frac{h^2}{2}\, y'' (x_1,y_1)$$

It is of order h^2.

3.3 Improved Euler Method

Let the tangent at (x0,y0) to the curve be P_0A.. In the interval (x0,x1), by previous Euler's method, we approximate the curve by the tangent P_0A.

$$y_1^{(1)}=y_0+hf(x_0,y_0) \text{ where } y_1^{(1)}=M_1Q_1$$

$Q_1(x_1,y_1^{(1)})$. Let Q_1 C be the line at Q_1 whose slope is $f\,(x_1,y_1^{(1)})$. Now take the average of the slopes at P_0 and Q_1 i.e.,

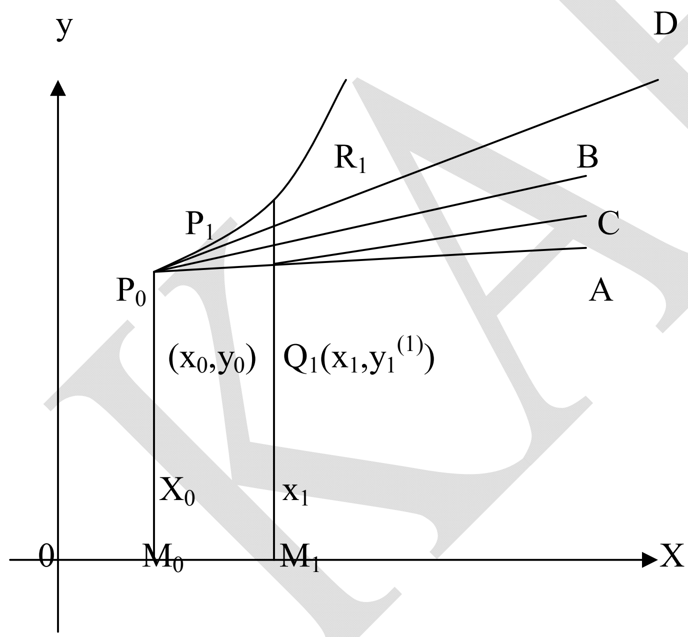
$\frac{1}{2}\,[f(x_0,y_0)+f(x_1,y_1^{(1)})]$

Now draw a line P_0D through $P_0(x_0,y_0)$ with this as the slope.

That is, $y-y_0=\frac{1}{2}\,[f(x_0,y_0)+f(x_1,y_1^{(1)})](x-x_0)$

This line intersects $x=x_1$ at

$y_1=y_0+\frac{1}{2}h\,[f(x_0,y_0)+f(x_1,y_1^{(1)})]$



$y_1=y_0+\frac{1}{2}h\,[f(x_0,y_0)+f(x_1,y_0+hf(x_0,y_0))]$ ----------- (3) writing generally,

$y_{n+1}=y_n+\frac{1}{2}h\,[f(x_n,y_n)+f(x_n,+h,y_n+hf(x_n,y_n))]$ --------- (4)

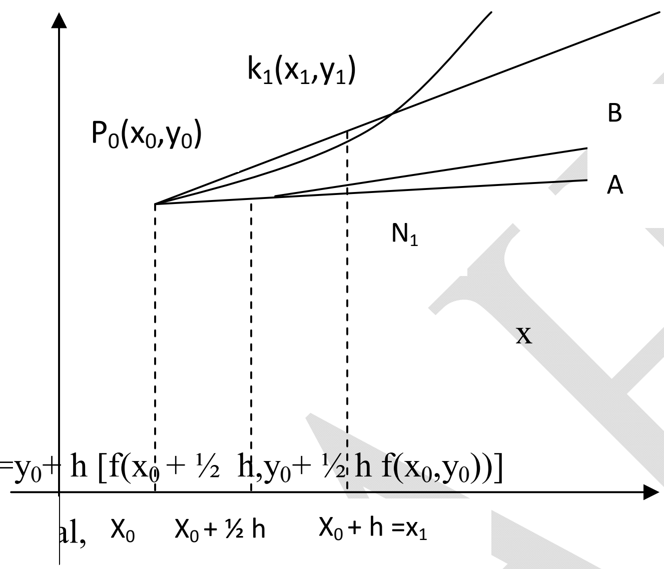
Equation (4) gives the formula for $y_{n+1.}$ This is improved Euler's method.

3.4 Modified Euler Method

Now let this tangent meet the ordinate at $x=x_0 + \frac{1}{2} h$ at N_1 y-coordinate of $N_1 = y0+ \frac{1}{2} hf (x_0,y_0)$ ------------- (1)

Calculate the slope at N_1 i.e $f(x_0+ \frac{1}{2} h, y_0+ \frac{1}{2} hf (x_0,y_0))$

Now draw the line through P (x_0,y_0) with this slope as the slope. Let this line meet $x=x_1$ at $k_1(x_1,y_1^{(1)})$. This $y_1^{(1)}$ is taken as the approximate value of y at $x=x_1$



$y_1^{(1)}=y_0+ h [f(x_0 + \frac{1}{2} h,y_0+ \frac{1}{2} h f(x_0,y_0))]$

In general,

$y_{n+1}=y_n+ h [f(x_n+ \frac{1}{2} h, y_n+ \frac{1}{2} hf(x_n,y_n))]$ --------- (2)

or

$y(x+h)=y(x)+h[f(x+ \frac{1}{2} h, y+ \frac{1}{2} hf(x,y))]$ ------------- (3)

Equations (2) or (3) is called modified Euler's formula.

Note 1: Hence the Euler predictor is

$y_{n+1}=y_n+ hy_n'$

and the corrector is

$y_{n+1}=y_n+h/2 (y_n' + y'_{n+1})$ in the improved Euler method:

Note 2: There is a lot of confusion among the authors: Some take the improved Euler method as the modified Euler method and the modified Euler method is not mentioned at all. You can see this in some books.

Example 1

Given y'=-y and y (0) =1, determine the values of y at x=(0.01) (0.01) (0.04) by Euler method.

Solution: y'=-y and y (0) =1; f(x,y)=-y.

Here, $x_0=0$, $y_0=1$, $x_1=0.01$, $x_2=0.02$, $x_3=0.03$, $x_4=0.04$

We have to find y_1,y_2,y_3,y_4. Take h=0.01.

By Euler algorithm,

$y_{n+1}=y_n+hy_n' = y_n+hf(x_n,y_n)$ --------------- (1)

$y_1=y_0+hf(x_0,y_0)=1+(0.01)(-1)=1-0.01=0.99$

$y_2=y_1+hy_1' = 0.99+(0.01)(-y_1)$

$= 0.99+(0.01)(-0.99)$

$= 0.9801$

$y_3=y_2+hf(x_2,y_2)=0.9801+(0.01)(-0.9801)$

$= 0.9703$

$y_4=y_3+hf(x_3,y_3)=0.9703+(0.01)(-0.9703)$

$= 0.9606$

Tabular values (step values) are:

X	0	0.01	0.02	0.03	0.04
Y	1	0.9900	0.9801	0.9703	0.9606
Exact y	1	0.9900	0.9802	0.9704	0.9608

Since, $y=e^{-x}$ is the exact solution.

Example 2: Using Euler's method, solve numerically the equation,

y'=x+y, y(0)=1, for x=(0.0) (0.2)(1.0)

check your answer with the exact solution.

Solution: Here h=0.2, f(x,y)=x+y, $x_0=0$, $y_0=1$

$x_1=0.2$, $x_2=0.4$, $x_3=0.6$, $x_4=0.8$, $x_5=1.0$

By Euler algorithm,

$y_1=y_0+hf(x_0,y_0)=y_0+h[x_0+y_0]$

$= 1+(0.2)(0+1)=1.2$

$y_2=y_1+h[x_1+y_1]=1.2+(0.2)(0.2+1.2)=1.48$

$y_3=y_2+h[x_2+y_2]$

$= 1.48+(0.2)(0.4+1.48)=1.856$

$y_4=1.856+(0.2)(0.6+1.856)=2.3472$

$y_5=2.3472+(0.2)(0.8+2.3472)=2.94664$

Exact Solution is $y=2e^x-x-1$. hence the tabular values are:

X	0	0.2	0.4	0.6	0.8	1.0
Euler y	1	1.2	1.48	1.856	2.3472	2.94664
Exact y	1	1.2428	1.5836	2.0442	2.6511	3.4366

The values of y deviates from the exact values as x increases. Hence we require to use either modified Euler or improved Euler method for the above problem.

Example 3: Solve numerically $y'=y+e^x$, $y(0)=0$; $f(x,y)=y+e^x$

$x_0=0$, $y_0=0$, $x_1=0.2$, $x_2=0.4$, $h=0.2$

By improved Euler method.

$y_{n+1}=y_n= ½ h[f(x_n,y_n)+f(x_n+h, y_n+hf(x_n,y_n))]$

$y_1=y_0+ ½ h[f(x_0,y_0)+f(x_1,y_0+hf(x_0,y_0))]$ -------- (1)

$=0+ (0.2/2) [y_0+e^x0+y_0+h(y_0+e^x0)+e^x0+h]$

$=(0.1)[0+1+0+0.2(0+1)+e^{0.2}]$

$y(0.2)=(0.1)[1+0.2+1.2214] = 0.24214$

$y_2=y_1+ ½ h [f(x_1,y_1)+f(x_1+h, y_1+hf(x_1,y_1))]$ -------- (2)

Here $f(x_1,y_1)=y_1+e^x{}_1=0.24214+e^{0.2}=1.46354$

$y_1+hf(x_1,y_1) = 0.24214+(0.2)\ (1.46354) = 0.53485$

$f(x_1+h,\ y_1+hf(x_1,y_1)) = f(0.4,\ 0.53485)$

$= 0.53485+e0.4 \quad = 2.02667$

Using (2),

$y_2=y(0.4)=0.24214+(0.1)\ [1.46354+2.02667]$

$= 0.59116$

$y(0.4)= 0.59116$

Example 4: Compute y at x = 0.25 by Modified Euler method given y' = 2xy, y(0) = 1.

Solution: Here , f(x,y) = 2xy : $x_0 = 0$, $y_0 = 1$.

Take h = 0.25, $x_1 = 0.25$

By Modified Euler method,

$y_{n+1} = y_n + h\ [\ f\ (\ x_n + {}_{½}h_{,}\ \ y_{n\,+\,½}\, h\ f(x_n,y_n)\ )\ ]$ ……(1)

$y_1= y_0\ + h\ [\ f\ (\ x_0 + ½\ h,\ y_0 + ½\ h\ f(\ x_0,y_0)\ )]$

$f(\ x_0,y_0) + f(\ 0,1) = 2\ (0)(1) = 0.$

$y_1= 1 + (0.25)\ [f(0.125,1)]$

$= 1 + (0.25)\ [2 \text{ x } 0.125 \text{ x } 1]$

y(0.25)= **1.0625**

By solving $\frac{dy}{dx} = 2xy$, we get $y=e^{x2}$ using y(0) = 1,

$y(0.25)= e^{(0.25)\ 2} = 1.0645$

Exact value of y(0.25) = **1.0645**

Error is only 0.002.

Note : To improve the result we can take h= 0.125 and get y(0.125) first and then get y (0.25). of course, labour is more.

3.5 Runge- Kutta Method

The use of the previous methods to solve the differential equation numerically is restricted due to either slow convergence or due to labour involved , especially in Taylor-series method. But , in Runge- Kutta methods, the derivatives of higher order are not required and we require only the given function values at different points. Since the derivation of fourth order Runge-Kutta method is tedious, we will derive Runge-Kutta method of second order.

Second order Runge-Kutta method (for first order O.D.E)

AIM : To solve $dy / dx = f(x,y)$ given $y(x_0)=y_0$ ….(1)

Proof. By Taylor series, we have,

$y(x+h) = y(x)+ hy'(x)+ h^2/2!\, y''(x)+O(h^3)$ …..…(2)

Differentiating the equation (1) w.r.t.x,

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = f_x + y' f_y = f_x + f f_y$$

Using the values of y' and y'' got from (1) and (3), in (2), we get,

$$Y(x + h)-y(x) = hf + ½ h^2 [f_x + ff_y] + O(h^3)$$

$$\Delta y=hf+ ½ h^2(f_x+ff_y)+ O(h^3)$$

Let $\Delta_1 y=k_1=f(x,y).\ \Delta x=hf(x,y),\ \Delta_2 y=k_2 =hf(x+mh,y+mk_1)$

and $\Delta y=ak_1 + bk_2$, Where a, b and m are constants to be determined to get the better accuracy of Δy. Expand k_2 and Δy in powers of h.

Expanding k_2, by Taylor series for two variables, we have

$K_2 =hf(x+mh,y+mk_1)$

$= h[f +mhf_x +mhff_y+\{(mh\partial/\partial x +mk_1\, \partial/\partial y)^2 f / 2!\} +….]$ …(8)

$= hf+mh_2(f_x+ff_y)+….$ Higher powers of h ……….. (9)

Substituting k_1,k_2 in (7),

$\Delta y=ahf+b[hf+mh^2(f_x+ff_y)+O(h^3)]$

$=(a+b)hf+bmh^2(f_x+ff_y)+O(h^3)$ …… 10)

Equating Δy from (4) and (10), we get

$=hf+mh^2(f_x+ff_y)+\ldots\ldots$ higher powers of h………. (9)

Substituting k_1, k_2 in (7),

$\Delta y = ahf+b[hf+mh^2(f_x+ff_y)+O(h^3)]$ $=(a+b)hf+bmh^2(f_x+ff_y)+O(h^3)$ …………. (10)

Equating Δy from (4) and (10), we get

$a+b=1$ and $bm= ½$ ………… (11)

Now we have only two equations given by (1) to solve for three unknowns a,b,m.

From a+b=1, a=1-b and also m= 1/2b using (7),

$\Delta y=(1-b)k_1+bk_2,$ Where $k_1=hf(x,y)$

$K_2=hf(x+h/2b, y+hf/2b)$ Now $\Delta y=y(x+h)-y(x)$

$Y(x+h)=y(x)+(1-b)hf+bhf(x+h/2b,y+hf/2b)$

i.e., $y_{n+1}=y_n+(1-b)hf(x_n,y_n) +bhf(x_n+h/2b,y_n+h/2bf(x_n,y_n))+O(h^3)$

from this general second order Runge kutta formula, setting a=0, b=1, m=1/2, we get the second order Runge kutta algorithm as

$k_1=hf(x,y)$ & $k_2=hf(x+ ½ h, y + ½ k_1)$ and $\Delta y=k_2$ where $h=\Delta x$

Since the derivation of third and fourth order Runge Kutta algorithm are tedious, we state them below for use.

The third order Runge Kutta method algorithm is given below :

$K_1=hf(x,y)$

$K_2=hf(x+1/2h, y+1/2k_1)$

$K_3=hf(x+h,y+2k_2-k_1)$

and $\Delta y=1/6 (k_1+4k_2+k_3)$

The fourth order Runge Kutta method algorithm is mostly used in problems unless otherwise mentioned. It is

$K_1=hf(x,y)$

$K_2=hf(x+1/2h, y+1/2k_1)$

$K_3=hf(x+1/2h, y+1/2k_2)$

$K_4=hf(x+h, y+k_3)$

and $\Delta y= 1/6 (k_1+2k_2+2k_3+k_4)$

$y(x+h)=y(x)+ \Delta y$

Working Rule :

To solve $dy/dx = f(x,y), y(x_0)=y_0$

Calculate $k_1=hf(x_0,y_0)$

$K_2=hf(x_0+1/2h,y_0+1/2k_1)$

$K_3= hf(x_0+1/2h,y_0+1/2k_2)$

$K_4=hf(x_0+h, y_0+k_3)$

and $\Delta y= 1/6 (k_1+2k_2+2k_3+k_4)$

where $\Delta x=h$

Now $y_1=y_0+ \Delta y$

Now starting from (x_1,y_1) and repeating the process, we get (x_2,y_2) etc.,

Note 1: In second order Runge kutta method .

$\Delta y_0=k_2=hf(x_0+h/2, y_0+ ½ k_1)$

$\Delta y_0=hf(x_0+h/2, y_0+ ½ hf(x_0,y_0))$

$y_1=y_0+ \Delta y_0=y_0+hf(x_0+h/2, y_0+ ½ hf(x_0,y_0))$

This is exactly the modified Euler method.

So, The Runge Kutta method of second order is nothing but the modified Euler method.

Note 2: If $f(x,y)=f(x)$, i.e., only a function x alone, then the fourth order Runge Kutta method reduces to

$K_1=hf(x_0)$

$\Delta y=1/6h[f(x_0)+4f(x0+h/2)+f(x_0+h)]$

$=[(h/2)/3][f(x_0)+4f(x_0+h/2)+f(x_0+h)]$

= the area of y=f(x) between $x=x_0$ and $x=x_0+h$ with 2 equal intervals of length h/2 by Simpson's one third rule.

i.e., Δy reduces to the area by Simpson's one third rule

Note 3: In all the three methods, (2^{nd} order, 3^{rd} order and 4^{th} order) the values of k_1, k_2 are same. Therefore, one need not repeat the work while doing by all the three methods.

Example 1

Apply the fourth order Runge- Kutta method to find y(0.2) given that y' = x+y, y(0) =1.

Solution: Since h is not mentioned in the question, we take h=0.1

y'= x+y, y(0) = 1. $f(x,y)=x+y$, $x_0=0$, $y_0=1$

$x_1=0.1, x_2=0.2$

By fourth order Runge-Kutta method, for the first interval,

$k_1= hf(x_{0,} y_0) = (0.1)(x_{0,} + y_0) = (0.1)(0+1) = 0.1$

$k_2 = hf(x_0 + ½ h, y_0 + ½ k_1) = (0.1) f(0.05,1.05)$

$= (0.1)(0.05 + 1.05) = 0.11$

$k_3 = hf(x_0 + ½ h, y_0 + ½ k_2)$

$= (0.1) f(0.05,1.055) = (0.1)(0.05 + 1.05) = 0.1105$

$K_4 = hf(x_0+h, y_0+ ½ k_3)$

$= (0.1)f(0.1,1.1105) = (0.1)(0.1 + 1.1105) = 0.12105$

$\Delta y = 1/6 (k_1+ 2k_2 + 2k_3 + k_4)$

$= 1/6(0.1+0.22+0.2210 +0.12105)=0.110341667.$

$y(0.1) = y_1=y_0+\Delta y=1.110341667 = 1.110342.$

Now starting from (x_1,y_1) we get (x_2,y_2). Again apply Runge-Kutta algorithm replacing (x_0,y_0) by (x_1,y_1).

$k_1=hf(x_1,y_1)=(0.1)(x_1+y_1)$

$=(0.1)(0.1+1.110342)=0.1210342$

$k_2=hf(x_1+h/2,y_1+ ½ k_1) = (0.1)f(0.15,1.170859)$

$= (0.1)(0.15+1.170859)=0.1320859$

$k_3=hf(x_1+h/2,y_1+1/2k_2)=(0.1)f(0.15,1.1763848)$

$= (0.1) (0.15+1.176348)=0.13263848$

$k_4=hf(x_1+h,y_1+k_3)=(0.1)f(0.2,1.24298048)$

$= 0.144298048$

$y(0.2) = y(0.1)+(1/6)(k_1+2k_2+2k_3+k_4)$

$= 1.110342+(1/6)(0.794781008)$

$y(0.2)=1.2428055$

Correct four decimal places, $y(0.2) = 1.2428$

Example 2

Obtain the values of y at x=0.1, 0.2 using R.K method of (i) second order (ii) third order and (iii) fourth order for the differential equation y'=-y, given y(0)=1.

Solution :Here $f(x,y)=-y, x_0=0, y_0=1, x_1=0.1, x_2=0.2$

(i) Second Order:

$k_1=hf(x_0,y_0)=(0.1)(-y_0)= - 0.1$

$k_2=hf(x_0+ ½ h, y_0+ ½ k_1) = (0.1) f(0.05,0.95)$

$= -0.1(x0.95)= - 0.095= \Delta y$

$y_1=y_0+\Delta y=1-0.095=0.905$

$y_1=y(0.1)=0.905$

Again starting from (0.1, 0.905) replacing (x_0,y_0) by (x_1,y_1)we get

$k_1=(0.1) f(x_1,y_1)=(0.1) (-0.905)= - 0.0905$

$k_2=hf(x_1+ ½ h, y_1+ ½ k_1)$

$=(0.1)[f(0.15,0.85975)]=(0.1)(-0.85975)=-0.085975$

$\Delta y=k_2 \quad y_2=y(0.2)=y_1+\Delta y=0.819025$

(ii) Third Order:

$k_1= hf(x_{0,} y_0) = -0.1$

$k_2= hf(x_0+ ½ h, y_0+ ½ k_1) = - 0.095$

$k_3=hf(x_0+h,y_0+2k_2-k_1)$

$= (0.1)f(0.1,0.9)=(0.1)(-0.9)= -0.0.9$

$\Delta y=1/6 (k_1+4k_2+k_3)$

$y(0.1)=y_1=y_0+\Delta y=1-0.09=0.91$

Again taking (x_1,y_1) has (x_0,y_0) repeat the process

$k_1=hf(x_1,y_1)=(0.1) (-0.91)=-0.091$

$k_2=hf(x_1+ ½ h, y_1+ ½ k_1)$

$= (0.1)f(0.15,0.865)=(0.1) (-0.865)= - 0.0865$

$k_3=hf(x_1+h, y_1+2k_2-k_1)$

$= (0.1)f(0.2,0.828) = -0.0828$

$y_2=y_1+\Delta y=0.91+1/6 (k_1+4k_2+k_3)$

$= 0.91+1/6 (- 0.091-0.3460 – 0.0828)$

$y(0.2)=0.823366$

(iii) Fourth order:

$k_1=hf(x_0,y_0)=(0.1)f(0.1)=-0.1$

$k_2=hf(x_0+ ½ h, y_0+1/2k_1)=(0.1)f(0.05, 0.95) = - 0.095$

$k_3=hf(x_0+ ½ h, y_0+ ½ k_2) = (0.1) f(0.05, 0.9525) = - 0.09525$

$k_4=hf(x_0+h,y_0+k_3)=(0.1)f(0.1,0.90475) = -0.090475$

$\Delta y=1/6 (k_1+2k_2+2k_3+k_4)$

$y_1=y_0+\Delta y=1+1/6\ (k_1+2k_2+2k_3+k_4)$

$y_1=y(0.1)=0.9048375$

Again start from this (x_1,y_1) and replace (x_0,y_0) and repeat

$k_1=hf(x_1,y_1)=(0.1)(-y1)= -\ 0.09048375$

$k_2=hf(x_1+1/2h,y_1+1/2k_1)$

$= (0.1)f(0.15,0.8595956) = -0.08595956$

$k_3=hf(x_1+ ½\ h,y_1+ ½\ \ k_2)$

$= (0.1)f(0.15,0.8618577)= -\ 0.08618577$

$k_4=hf(x_1+h,y_1+k_3)$

$= (0.1)f(0.2,0.8186517)= -\ 0.08186517$

$\Delta y=1/6(-0.09048375-2x\ 0.08595956- 2x\ 0.08618577 – 0.08186517) = -0.0861066067$

$y_2=y(0.2)=y_1+\Delta y=0.81873089$

Tabular values are:

x	Second order	Third order	Fourth order	Exact value Y=e-x
0.1	0.905	0.91	0.9048375	0.904837418
0.2	0.819025	0.823366	0.81873089	0.818730753

Example 3

Using Runge Kutta method of fourth order solve $dy/dx=y^2-x^2/y^2+x^2$ given y(0) = 1 at x=0.2, 0.4

Solution : $y'=f(x,y)=y^2-x^2/y^2+x^2$:

Here $x_0=0$, h=0.2, x1=0.2, x2=0.4, y0=1

$f(x_0,y_0)=f(0,1)=1-0/1+0=1$

$k_1=hf(x_0,y_0)=(0.2)x_1=0.2$

$k_2=hf(x_0+ ½\ h,\ y_0+ ½\ k_1)=(0.2)f(0.1,1.1)$

$=(0.2)[(1.1)^2-(0.1)^2/(1.1)^2+(0.1)^2]$

$=(0.2)[1.21-0.01/1.21+0.01] \quad =0.1967213$

$k_3=hf(x_0+ ½ h, y_0+ ½ k_2)$

$=(0.2) f(0.1, 1+ ½ (0.1967213))$

$= (0.2) f(0.1, 1.0983606)$

$= (0.2) [(1.0983606)^2-(0.01)/ (1.0983606)^2+(0.01)]$

$= 0.1967$

$k_4=hf(x_0+h,y_0+k_3)$

$= (0.2) f(0.2,1.1967)$

$= (0.2) [(1.1967)^2-(0.2)^2/(1.1967)^2+(0.2)^2] = 0.1891$

$\Delta y=1/6[k_1+2k_2+2k_3+k_4]$

$= 1/6[0.2+2(0.19672)+2(1.1967)+0.1891]$

$= 0.19598.$

$y(0.2)=y_1=y_0+\Delta y=1.19598$

Again to find y(0.4), start from $(x_1, y_1) = (0.2, 1.19598)$

Now, $k_1=hf(x_1,y_1)$

$=(0.2) [(1.19598)^2-(0.2)^2/(1.19598)^2+(0.2)^2] = 0.1891$

$k_2=hf(x_1+ ½ h, y_1+ ½ k_1) = (0.2) f(0.3, 1.29055)$

$=(0.2) [(1.29055)^2 – (0.3)^2/(1.29055)^2+(0.3)^2] = 0.17949$

$k_3=(0.2) f (0.4, y1+k3) = (0.2) f (0.4, 1.37528)$

$= 0.1687$

$\Delta y=1/6 (k_1+2k_2+2k_3+k_4)$

$= 1/6[0.1891+2(0.1795)+2(0.1793)+0.1687]$

$= 0.1792$

$y_2=y(0.4)=y_1+\Delta y=1.3751$.

3.6 Predictor – Corrector Methods

The methods which we have discussed so far are called single step methods because they use only the information from the last step computed. The methods of Milne's predictor and corrector , Adams-Bashforth predictor and corrector formulae are multi step methods.

In solving the equation $dy / dx = f(x,y)$, $y(x_0) = y_0$ we used Euler's formula.

$y_{i+1} = y_i + h(f`(xi , y_i)$, I = 1,2…. ……………..(1)

We proved this value by Improved Euler method

$y_{i+1} = y_i +(1/2) h[f(x_i , y_i)+ f(x_{i+1} , y_{i+1})]$,i = 1,2… …(2)

In the equation (2) , to get the value of y_{i+1} we require y_{i+1} on the RHS.To overcome this difficulty , we calculate y_{i+1} using Euler's formula(1) and then we use it on the RHS of (2) , to get the LHS of (2).This y_{i+1} can be used further to get refined y_{i+1} on the LHS .Here , we predict a value of y_{i+1} from the rough formula(1) and use in (2) to the correct value . Every time , we improve using (2).

Hence equation (1) Euler's formula is a predictor and (2) is a corrector. A predictor formula is used to predict the value of y at x_{i+1} and a corrector formula is used to correct the error and to improve that value of $y_{i+1.}$

3.7 Milne's Predictor Corrector Formulae

Suppose our aim is to solve $dy / dx = f(x,y)$, $y(x_0) = y_0$ ………………(1)

Numerically,Starting from $y_0 = y(x_0)$, we have to estimate successively

$y_1 = y(x_0+h) = y(x_1)$, $y_2 = y(x_o+2h)= y(x_2)$, $y_3 = y(x_0+3h)= y(x_3)$

Where h is a suitable accepted spacing , which is very small.

By Newton's forward interpolation formula ,

$y = y_0 + u\Delta y_0 +u (u-1) /2!\ \Delta^2 y_0+$…….

Where $u = (x- x_0) / h$.

i.e.$x=x_0 = uh$.Changing y to y`,

$y` = y_0` + u\Delta y_0` + u(u-1)/2!\ \Delta^2 y_0` + \ldots\ldots$ (2)

Integrating both sides from x_0 to x_4,

$$\int_{x_0}^{x_4} y`\,dx = \int_{x_0}^{x_0+4h} [\,y_0` + u\Delta y_0` + u(u-1)/2!\ \Delta^2 y_0` + \ldots\ldots]\,dx$$

$$(y)_{x_0}^{x_0+4h} = h\int_0^4 [y_0` + u\Delta y_0` + u(u-1)/2!\ \Delta^2 y_0` + \ldots\ldots]\,du$$

Since $x = x_0 +|\ uh$ and $dx = h\,du$

$y_4 - y_0 = h\,[\,y_0`u + \Delta y_0`\ u^2/2 + (½)\Delta^2 y_0\,(u^3/3 - u^2/2) + \ldots\ldots]$

between 0 to 4.

$= 4h / 3\ (2y_1` - y_2` + 2\ y_3`) + 14h / 45\ \Delta^4\ y_0` + \ldots\ldots\ldots\ldots(3)$

Taking into Account only up to the third order equation , (3) gives

$y_4 = y_0 + 4h/3\ (2y_1` - y_2` + 2\ y_3`)$ ……………(4)

$= y_0 + 4h / 3\ (2f_1 - f_2 + 2\ f_3)$

The error committed in (4) is $(14h / 45)\ \Delta^4\ y_0 + \ldots$.. and this can be proved to be

$(14h / 45)y^{(5)}$ (£) where $x_0 <$ £$< x_4$ since $\Delta = E-1 = hD$ for small value of h.

Therefore The error = $(14h^5 / 45)y^{(5)}$ (£) (3) becomes ,

$y_4 = y_0 + 4h / 3\ (2f_1 - f_2 + 2\ f_3) + (14h^5 / 45)y^{(5)}$ (£) …(5)

In general ,

$y_{n+1} = y_{n-3} + 4h/3(2y_{n-2}` - y_{n-1}` + 2\ y_n`) + (14h^5/ 45)y^{(5)}$ (£)…(6)

Where $x_{n-3} <$ £$_1 < x_{n+1}$.

Equation (6) is called Milne's predictor formula.

To get Milne's corrector formula , integrate (2) between the limits x_0 to $x_0 + 2h$.

Therefore

$$\int_{x_0}^{x_0+2h} y`\ dx = \int_{x_0}^{x_0+2h} \{y_0` + u\Delta y_0` + u\ (u-1)/2\ \Delta^2\ y0` + \ldots.\}\ dx$$

$= h/3\ [\ y_0` + 4y_1` + y_2] - h/90\ \Delta^4 y_0` + \ldots\ldots\ldots\ldots\ldots(7)$

Taking into account only upto third order ,

$y_2 = y_0 + h/3[\ y_0` + 4y_1` + y_{2`}]$

and the error in (8) is = $-h/90\ \Delta^4 y_0 + \ldots$..

and this can be proved to be($-h^5 / 90)y^{(5)}$ (£)) , where $x_0 <$ £$< x_2$.

(7) becomes ,

$y_2 = y_0 + h/3\ ([\ y_0` + 4y_1` + y_{2`}] - h^5 / 90)y^{(5)}$ (£) ……..(9)

In general,

$y_{n+1} = y_{n-1}+h/3\ (y_{n-1}` +4\ y_n` +y_{n+1}`)+ (14h^5/ 45\ )y^{(5)}\ (£)\ldots(10)$

Where $x_{n-1} < £_1<x_{n+1}$.

Equation (10) is called Milne's corrector formula.

Hence we predict form

$y_{n+1.p} = y_{n-3}+4h/3\ (2y_{n-2}` - y_{n-1}` +2\ y_n`)\ldots\ldots\ldots\ldots\ldots.(11)$

and correct using

$y_{n+1,c} = y_{n-1}+h/3\ (y_{n-1}` +4\ y_n` +y_{n+1}`)\ldots\ldots\ldots\ldots\ldots\ldots..(12)$

Note : Knowing 4 consecutive values of y namely , y_{n-3} , y_{n-2} , y_{n-1} and y_n we calculate y_{n+1} using predictor formula. Use this n+1 on the RHS of corrector formula to get y_{n+1} after correction. To refine the value further , we can use this latest y_{n+1} on the RHS of (12) and get a better y_{n+1}.

Example 1

Find y(2) if y(x) is the solution the solution of dy / dx = (1/2) (x+y) given y(0) = 2 , y(0.5) = 2.636 , y(1) = 3.595 and y(1.5) = 4.968.

Solution:

Here , $x_0 = 0$, $x_1= 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, h= 0.5 , $y_0 = 2$, $y_1=2.636$, $y_2 = 3.595$, $y_3 = 4.968$.

$f(x,y) = (x+y) = y`$ …………………(1)

By Milne's predictor formula ,

$y_{n+1.p} = y_{n-3}+ 4h/3\ (2y_{n-2}` - y_{n-1}` +2\ y_n`)$

therefore $y_{4,p} = y_0 +4h / 3\ (2y_1` -y_2`+2y_3`)$……………(2)

From (1),

$y_1`= ½\ (x_1 +y_1) = ½\ (0.5+2.636)=1.5680$

$y_2` = ½\ (x_2 +y_2) = ½\ (1+3.595) = 2.2975$

$y3` = ½\ (x_3+y_3) = ½\ (1.5+4.968) = 3.2340$

By (2) ,

$y_{4,p}$

$= 2 + [4 (0.5) / 3] [2(1.5680) –(2.22975) + 2(3.2340)]$

$= 6.8710$

Using Milne's corrector formula ,

$y_{n+1} = y_{n-1}+h/3 (y_{n-1}` +4 y_n` +y_{n+1}`)$

$y_{4,c} = y_2+h/3 (y_2` +4 y_3` +y_4`)$…………..(3)

$y_4` =(½) (x_4+y_4) = (1/2) (2+6.8710) = 4.4355$

Using (3), we get

$y_{4,c} = 3.595 + (0.5/3)[2.2975 + 4(3.2340)+ 4.4355]$

$= 6.8732$

Therefore corrected value of y at x=2 is 6.8732.

Example 2

Using Milne's method find y (4.4) given $5xy` +y^2 -2 = 0$, y(4) = 1 , y(4.1) = 1.0049 , y(4.2) = 1.0097 and y(4.3)= 1.0143.

Solution:

$y` = (2-y^2 / 5x)$, $x_0 = 4$, $_{x1} = 4.1$, $x_2=4.2$, $x_3 = 4.3$, $x_4 = 4.4$, $y_0 = 1$, $y_1 = 1.049$, $y_2 = 10097$, $y_3 = 1.0143$.

$y_1` =[2-(1.0049)^2 / 5(4.1)] = 0.0493$.

$y'_2 = 2-y^2_2/5x_2=2-(1.0097)^2/5(4.2)=0.0467$

$y^2_3=2-y^2_3/5x3=2-(1.0143)^2/5(4.3)=0.0452$

By Milne's Predictor formula,

$y_{4,p}= y_0+4h/3(2y_1' - y'_2 + 2y_3')$ ……….(1)

$= 1 + 4(0.1)/3 [2(0.0493)-0.0467+2(0.0452)]$

$= 1.01897$

$y'_4 = 2- y^2_4/5(x_4)= 2-(1.01897)^2 /5(4.4)= 0.0437$

Using

$y'_{4,c} = y2+h/3(y'_2 +4y'_3 +y'_4)$ ……2)

$= 1.0097 + 0.1/3 [0.0467 + 4(0.0452) + 0.0437]$

$y'_{4,c} =$ **1.101874.**

Note :

Use this corrected $y_{4,c}$ and find $y'_{4,c}$ and again use (2)

$y'_{4,c} = 2- y^2_4/5(x_4)= 2-(1.01874)^2 /5(4.4)= 0.043735$

Now using (2),

$y^{(2)}_{4,c} = 1.0097 + 0.1/3 [0.0467 + 4(0.0452) + 0.043735]$

$= 1.01874$

Since two consecutive values of $y'_{4,c}$ are equal, we take y_4= 1.01874 (correct to 5 decimals).

3.8 Adam – Bashforth (Or Adam's) Predictor – Corrector Method

We state below another predictor-corrector method, called Adam's method or Adam-Bashforth method. We give below predictor and corrector formula without proof. Here also, we require four continuous values of y to find the value of y at the fifth point similar to Milne's method.

Predictor : $y_{n+1,p} = y_n+h/24[55 y'_{n-1} + 37 y'_{n-2} – 9 y'_{n-3}]$

Corrector: $y_{n+1,c} = y_n+h/24[9 y_{n-1}' +19 y_n ' -5y_{n-1}' + y_{n-2} ']$

Example 1

Solve and get y(2) given dy/dx = ½(x+y),y(0)=2 y(0.5)=2.636, y(1)=3.595, y(1.5)=4.968 by Adam's method.

Solution: From example 1 under Milne's method,

We have $y'_0 = ½ (0+2) = 1$

$y'_1 = 1.$**5680,** $y'_2 = 2.2975$, $y'_3 = 3.2340$.

By Adam's predictor formula,

$y_{n+1},p = y_n + h/24[55\ y'n - 59\ y'_{n-1} + 37\ y'_{n-2} - 9\ y'_{n-3}]$

$y'_{4,p} = y3+h/24[55\ y'3 - 59\ y'_2 + 37\ y'_1 - 9\ y'_0]$ ….(1)

$= 4.968+ 0.5/24\ [55(3.2340)-9(2.2975)+37(1.5680)- 9(1)]$

$=6.8708$

$y'_4 = ½\ (x4+y4) = ½(2+6.8708)=4.4354$

By corrector,

$y'_{4,c} = y_3 + h/24\ [\ 9\ y'_{4\ +19}\ y'_3 + y'_1]$ ………..(2)

$= 4.968+0.5/24[9(4.4354)+19(3.234)-5(2.2975)+1.5680]$

$= 6.8731$

Note : we can further improve using this latest $y_{4,c}$ again in (2).

Example 2

Using Adam's method find y(0.4) given

dy/dx= ½ xy, y(0)=1, y(0.1)=1.01, y(0.2)=1.022, y(0.3)

= 1.023

Solution: $x_0=0,\ x_1=0.1,\ x_2=0.2,\ x_3=0.3,\ x_4=0.4$

$y_0=1, y_1=1.01, y_2=1.022,\ y_3=1.023,\ y_4=?$

By Adam's method,

Predictor: $y_{n+1},\ p=y_n+h/24[55\ y'_n – 59\ y'_{n-1} +37\ y'_{n-2} - 9\ y'_{n-3}]$

$y_{4,p} = y_3+h/24[55\ y'_3 – 59\ y_2' +37\ y'_1 - 9\ y'_0]$………… (1)

here $y_0' = ½\ x_0y_0 = 0$

$y_1' = ½\ x_1y_1 = (0.1)\ (1.01) / 2 = 0.0505$

$y_2' = ½\ x_2y_2 = (0.2)\ (1.022) / 2 = 0.1022$

$y_3' = ½\ x_3y_3 = (0.3)\ (1.023) / 2 = 0.1535$

using in (1),

y_4,p = 1.023 + 0.1/24[55(0.1535)-59(0.1022) + 37

(0.0505) – 9(0)]

= 1.0408

$y'_{4,p}$ = ½ x_4y_4 = ½ (0.4)(1.0408)=0.20816.

By Adam's corrector formula

$y_{n-1,c} = y_n + h/24[\ 9y'_{n-1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$

$y_{4,c} = y_3 + h/24[\ 9y'_4 + 19y'_3 - 5y'_2 + y'_1]$

= 1.023 + 0.1/24 [9(0.2082)+ 19(0.1535)-5(0.1022) + 0.0505]

= 1.0410

Y(0.4) = $y_{4,c}$ = 1.0410

Example 3

Find y(0.1),y(0.2),y(0.3) from dy/dx = xy + y^2, y(0) = 1 by using Runge- Kutta method and hence obtain y(0.4) using Adam's method.

Solution: f(x,y) = xy +y^2, x_0=0,x_1=0.1,x_2=0.2,x_3=0.4,x_4=0.4,y_0=1

k_1=hf(x_0, y_0) = (0.1) f(0.1)= (0.1) 1=0.1

k_2=hf(0.05, y_0 + k1/2) = (0.1)f(0.05, 1.05)

= (0.1)[(0.05)(1.05)+(1.05)^2]=0.1155

k_3= hf(0.05, y_0+k_2/2) = (0.1) f(0.05,1.0578)

= (0.1)[(0.05)(1.0578)+(1.0578)^2]

= 0.1172

k_4= hf(x_0+h,y_0+ k_3)

= (0.1)f(0.1,1.1172)

= (0.1)[(0.1)(1.1172)+(1.1172)^2]=0.13598

y_1= y_0 + 1/6[k_1 + 2k_2 + 2k_3 + 2k_4]

= 1.1169

y(0.1) = 1.1169

Again , start from y_1 :

$k_1 = hf(x_1,y_1) = (0.1)f(0.1,1.1169) = 0.1359$

$K_2= hf(x_1 +h/2, y_1 + k_1/2)=(0.1)f(0.15, 1.1849) = 0.1582$

$k_3=hf(0.15,y1+k_3/2)=(0.1)f(0.15,1.196) = 0.16098$

$k_4=(0.1)f(0.2,1.2779)=0.1889$

$y_2=1.1169+1/6[0.1359+2(0.1582+0.16098)+0.1889]$

y(0.2)=1.2774

Start from (x_2,y_2) to get y_3

$k_1= hf(x_2,y_2) = (0.1)f(0.2,1.2774) = 0.1887$

$k_2= hf(x_2 +h/2, y_2 + k_1/2)=(0.1)f(0.25, 1.3718) = 0.2225$

$k_3=hf(x_3,y_2+k_3/2)=(0.1)f(0.3,1.5048) = 0.2716$

$y_3=1.2774+1/6[0.1887+2(0.2225)+2(0.2274)+0.2716]$

= 1.5041

Now we use Adam's predictor formula

$Y_{4},p=y_3+h/24[55y_3'-59y_2'+37y_1'-9y_0']$ …………. (2)

$Y_0'=x_0y_0+y^2_0=1$

$Y_1'=x_1y_1+y^2_1=1.3592$

$Y_2'=x_2y_2+y^2_2=1.8872$

$Y_3'=x_3y_3+y^2_3=2.7135$

Using (2),

$Y_{4,p}=1.5041+0.1/2[55(2.7135)-59(1.8872)+37(1.3592)-9(1)]$

= 1.8341

$Y'_{4,p}=x_4y_4+y^2_4=(0.4)(1.8341)+(1.8341)^2 = 4.0976$

$Y_{4,c} = y_3+h/24[9y'_4+19y_3' - 5y_2'+y_1']$

$= 1.5041+0.1/24[9(4.0976) + 19 (2.7135) - 5$

$(1.8872) + 1.3592]$

$= 1.8389$

$Y(0.4)=1.8389.$

POSSIBLE QUESTIONS:

Part-B(5X6 = 30 Marks)

Answer all the questions:

1. Solve y'= –y & y(0)=1 determine the values of y at x=(0.01)(0.01)(0.04) by Euler method.

2. Compute y at x=0.25 by Modified Euler method given y'=2xy, y(0)=1.

3. Solve the equation $\frac{dy}{dx}$ = 1–y given y(0)=0 using Modified Euler method and tabulate the solutions at x=0.1,0.2.

4. Use Runge kutta method of fourth order find y for x = 0.1 and 0.2, given that
dy/dx = x + y, y(0) = 1.

5. Apply the fourth order Runge Kutta method to find y(0.1), y(0.2) given that y'=x+y,
y(0)=1.

6. Find y(2), if y(x) is the solution of $\frac{dy}{dx} = \frac{1}{2(x+y)}$ given y (0) =2, y (0.5)= 2.636 ,
y (1) = 3.595and y (1.5) = 4.968.

7. Given $\frac{dy}{dx}$=1+y^2, where y=0 when x=0, find y(0.4) using Adams Moultan method.

8. Using Milne's method find y (4.4) given 5xy' + y^2 –2 = 0 given y(4) =1,
y(4.1) =1.0049, y(4.2) =1.0097 and y(4.3) = 1.0143.

9. Derivative of Milne's Predicator and Corrector Method.

10. Determine the value of y (0.4) using Milne's Method given y' = xy+y^2, y(0)=1 and get the values of y(0.1),y(0.2) and y(0.3)

PART C- (1 x 10 =10 Marks)
(Compulsory)

1. Solve numerically the equation y' = x+y , y(0) =1 for x =0.0(0.2)(1.0) by Euler method.

2. using Adam's moulton predictor- corrector method. Find y(1.4) if y satisfies
$\frac{dy}{dx} = \frac{1-xy}{x^2}$, y(1) =1, y(1.1) = 0.996,y(1.2)= 0.986, y(1.3) = 0.972.

3. Given dy/dx = ½ (1 +x2) y2 and y(0) =1, y(0.1) =1.06, y(0.2) = 1.12, y(0.3) =1.21. evaluate y(0.4) by Milne's predicator corrector method.

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Class : I M.Sc Mathematics **Semester : I**

Subject: Numerical Analysis **Subject Code: 19MMP103**

Unit III

Part A (20x1=20 Marks)

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The order of the error in Euler method is ---------------.	h	h^2	h^3	0	h^2
. --------- method is the Runge – Kutta method of first order.	Milne’s method	Picard’s method	Simpson’s method	Euler method	Euler method
A particular case of Runge Kutta method of second order is ------------------.	Milne’s method	Picard’s method	Modified Euler method	Taylor Series	Modified Euler method
. Euler method is used for solving --------------- differential equations.	first order	fourth order	third order	second order	first order
The modified Eulers method is based on the ___________ of points	sum	multiplication	average	subratction	average
The error in modified Euler method is ---------- .	O (h^2)	O (h^4)	O (h^3)	O (h^n)	O (h^3)
Modified Euler method will provide error free solutions if the given function is ----------------	linear	parabola	polynomial	non linear	linear
The use of Runge kutta method gives ------------- to the solutions of the differential equation than Taylor's series method.	slow convergence	quick convergence	oscillation	divergence	quick convergence
.-------------------- is nothing but the modified Euler method.	Runge kutta method of second order	Runge kutta method of third order	Runge kutta method of fourth order Taylor series method		Runge kutta method of second order

In all the three methods of Rungekutta methods, the values ---------- are same.	k_4 & k_3	k_3 & k_2	k_2 & k_1	k_1, k_2, k_3 & k_4	k_2 & k_1
dy/dx is a function x alone, then fourth order Runge – Kutta method reduces to ---------------	Trapezoidal rule	Taylor series	Euler method	Simpson method	Simpson method
Milne's is a ---------- method	multistep	iterative	direct	singlestep	multistep
A particular case of Runge Kutta method of second order is ------------------.	Adam's Moulton	Milne's	Euler	Runge-Kutta	Milne's
Milne's method is simple and has a good local error of order ---------	h^2	h^4	h^4	h^5	h^5
The ------------ method is a method that does not have the same instability problem as the Milne's method	Adam's Moulton	Milne's	Euler	Runge-Kutta	Adam's Moulton
In ---------method the true values should lies between the predicted and corrected values	Milne's	Euler	Adam's Moulton	Runge-Kutta.	Adam's Moulton
In numerical methods , the boundary problems are solved by using ------- method	Finite difference	Milne's	Euler	Runge-Kutta.	Finite difference
___________is nothing but the modified Euler method.	Runge kutta method of second order	Runge kutta method of third order	Runge kutta method of fourth order	Taylor series method	Runge kutta method of second order
Runge kutta method of second order is nothing but the -----------------.	Modified Euler method	Euler method	Runge kutta method of fourth order	Taylor series method	Modified Euler method
Milne's method method is ------- method.	single step	multi-step	direct	indirect	multi-step
A predictor formula is used to ------- the values of y at x_{i+1}.	correct	predict	increase	decrease	predict
A corrector formula is used to ------- the error and to improve that value of y_{i+1}	correct	predict	increase	decrease	correct
Adams Moulton method is ------- method.	single step	multi-step	direct	indirect	multi-step
---------- method integrates over more than one interval.	Adam's Moulton	Milne's	Euler	Runge-Kutta	Milne's
Milne's method is simple and has a good local error of order ---------	h^2	h^4	h^4	h^5	h^5
The ------------ method is a method that does not have the same instability problem as the Milne's method	Adam's Moulton	Milne's	Euler	Runge-Kutta	Adam's Moulton
The ________ formula is used to predict the value y(i+1) of y at x(i+1)	Predictor	Corrector	Corrector	Picards	Predictor

The ________ formula is used to improve the value of y(i+1)	Predictor	Corrector	Taylors	Picards	Corrector
In predictor corrector methods, ____ prior values of y are needed to evaluate the value of y at x(i+1)	1	2	3	4	4
In ________ methods, 4 prior values of y are needed to evaluate the value of y at x(i+1)	Taylor's	predictor	Predictor-corrector	Euler's	Predictor-corrector

UNIT-IV

SYLLABUS

Boundary Value Problem and Characteristic value problem: The shooting method: The linear shooting method – The shooting method for non-linear systems. Characteristic value problems –Eigen values of a matrix by Iteration-The power method.

BOUNDARY VALUE PROBLEMS AND CHARACTERISTIC VALUE PROBLEM

4.1 Introduction

Consider the two point boundary value problem

$$u'' = f(x, u, u'),\ x \in (a, b) \qquad (4.1)$$

Where a prime denotes differentiation with respect to x, with one of the following three boundary conditions.

Boundary condition of the first kind:

$$u(a) = \gamma_1,\ u(b) = \gamma_2. \qquad (4.2)$$

Boundary condition of the second kind:

$$u'(a) = \gamma_1,\ u'(b) = \gamma_2. \qquad (4.3)$$

Boundary condition of the third kind(or mixed kind):

$$a_0 u(a) - a_1 u'(a) = \gamma_1 \qquad (4.4i)$$

$$b_0 u(b) + b_1 u'(b) = \gamma_2 \qquad (4.4ii)$$

Where $a_0, b_0, a_1, b_1, \gamma_1, \gamma_2$ are constant such that

$$a_0 a_1 \geq 0,\ |a_0| + |a_1| \neq 0$$

$$b_0 b_1 \geq 0,\ |b_0| + |b_1| \neq 0 \text{ and },\ |a_0| + |b_0| \neq 0.$$

In (4.1), if all the non zero terms involve only the dependent variable u and u′, then the differential equation is called homogeneous, otherwise, it is inhomogeneous. Similarly, the boundary conditions are homogeneous when γ_1 and γ_2 are zero; otherwise, they are inhomogeneous. A homogeneous boundary value problem , that is a homogeneous differential equation along with homogeneous boundary condition , possesses only a trivial solution u(x)=0. we, therefore , consider those boundary value problems in which a parameter λ occurs either in the differential equation or in the boundary condition , and we determine value of λ , called **eigenvalues** for which the boundary value problem has a nontrivial solution. Such a solution is called **eigenfunction** and the entire problem is called an **eigenvalue** or a **characteristic value problem.**

The solution of the boundary (4.1) exists and is unique if the following conditions are satisfied:

Let $u'=z$ and $-\infty< u, z <\infty$

(i) $f(x, u, z)$ is continuous,
(ii) $\partial f/\partial u$ and $\partial f/\partial z$ exist and are continuous.
(iii) $\partial f/\partial u>0$ and $|\partial f/\partial z| \le w$.

In what follows, we shall assume that the boundary value problems a unique solution and we shall attempt to determine it. The numerical methods for solving the boundary value problems may broadly be classified in to the following three types:

(i). *Shooting Methods* These are initial value problem methods. Here, we add sufficient number of conditions at one end point and adjust these conditions until the required conditions are satisfied at the other end.

(ii)*Difference methods* The differential equation is replaced by a set of difference
Equations which are solved by direct or iterative methods.

(iii) *Finite element methods* The differential equation is discretized by using approximate methods with a piece wise polynomial solution.

We shall now discuss in detail the shooting methods and for solving numerically both the linear and non linear second order boundary value problems.

4.2 Initial Value Problem Method (Shooting Method)

Consider the boundary value problem (4.1) (BVP) subject to the given boundary conditions

Since the differential equation is of second order, we require two linear independent conditions to solve the boundary value problem. one of the ways of solving the boundary value problem is the following.

(i) *Boundary conditions of the first kind* Here , we are given $u(a) = \gamma_1$.in order that an initial value method can be used, we guess the value of the slope at x=a as $u'(a)=s$.

(ii) *Boundary conditions of the second kind* Here, we are given $u'(a)= \gamma_1$.in order that an initial value method can be used, we guess the value of u(x) at x=a as u(a)=s.

(iii) *Boundary conditions of the third kind* Here, we guess the value u(a) or u′(a). if we assume that u′(a)=s, then from(4.4i), we get $u(a)=(a_1 s+ \gamma_1)/ a_0$.

The related initial value problem is solved upto x=b, by using single step or a multi-step method. If the problem is solved directly, then we use the methods for second order initial value problems. If the differential equation is reduced to a system of two first order equations, then we use the Runge-Kutta methods or the multi-step methods for a system of first order equations.

If the solution at x=b does not satisfy the given boundary condition at the other end x=b , then we take another guess value of u(a) or u′(a) and solve the initial value problem again upto x=b. these two solutions at x=b , of the initial value problems are used to obtain a better estimate of u(a) or u′(a) .A Sequence of such problems are solved, if necessary, to obtain the solution of the

given boundary value problem. For a *linear, non-homogenous boundary value problem, it is sufficient to solve two initial value problems with two linearly independent guess initial conditions.*

This technique of solving the boundary value problem by using the methods for solving the initial value problems is called the **shooting method**.

4.3 Linear Second Order Differential Equations

Consider the linear differential equations

$-u''+p(x)\, u'+q(x)\, u=r(x)\ ,\ a< x <b$ (4.5)

Subject to the given boundary conditions. We assume that the functions p(x), q(x)>0, and r(x) are continuous on [a, b], so that the boundary value problem(4.5) has a unique solution.

The general solution of (4.5) can written as

$u(x)=u_0(x)+\mu_1\, u_1(x)+ \mu_2\, u_2(x)$ (4.6)

Where (i). $u_0(x)$ is a particular solution of the non homogeneous equation (4.5), that is

$$-u_0''+p(x)\,u_0'+q(x)\,u_0=r(x) \qquad (4.7)$$

(ii) $u_1(x)$ and $u_2(x)$ are any two linearly independent , complementary solutions of the corresponding homogeneous equation of (4.5) , that is

$$-u_1''+p(x)\,u_1'+q(x)\,u_1=0 \qquad (4.8)$$

$$-u_2''+p(x)\,u_2'+q(x)\,u_2=0 \qquad (4.9)$$

We choose the initial conditions as follows:

Boundary conditions of the first kind Since $u(a) = \gamma_1$ is given, we take a guess value for $u'(a)$. We have the following two case.

Case 1: $\gamma_1 \neq 0$. We choose

$$u_0(a) = u_1(a)=u_2(a)= \gamma_1$$

$$u_0'(a)=\eta_0^* . \ u_1'(a)=\eta_1^*, \ u_2'(a)=\eta_2^* \qquad (4.10i)$$

Where η_0^* , η_1^*, η_2^* are arbitrary. Since $u_1(x)$ and $u_2(x)$ are linearly independent solutions, a suitable choice of the initial conditions is

$$\eta_0^* =0, \eta_1^*=1, \eta_2^* =0. \qquad (4.10ii)$$

Other choices of linearly independent values can also be considered.

We now solve the differential equation (4.7)-(4.9) along with the corresponding initial conditions, using value methods with the same lengths, and obtain $u_0(b)$, $u_1(b)$ and $u_2(b)$.Now since the solution (4.6) satisfies the boundary conditions at x=a and x=b , we obtain, at x=a: $u_0(a)+\mu_1 u_1(a)+ \mu_2 u_2(a)= \gamma_1$

Or $\gamma_1+\mu_1 \gamma_1+ \mu_2 \gamma_1= \gamma_1$ Or $\mu_1 + \mu_2 = 0$ x=b:

$$u_0(b)+\mu_1 u_1(b)+ \mu_2 u_2(b)= \gamma_2 \qquad (4.11i)$$

or $$\mu_2 = \frac{\gamma_2 - u_0(b)}{u_2(b) - u_1(b)}, \ u_1(b)\neq u_2(b). \qquad (4.11ii)$$

Case 2: $\gamma_1=0$. In this case, we cannot (4.10i), since $[u_1(a), u'_1(a)]^T=[0,1]^T$ and $[u_2(a), u'_2(a)]^T=[0,0]^T$ are linearly dependent . We choose the conditions as

: $u_0(a)=\eta_0$, $u_1(a)=\eta_1$, $u_2(a)=\eta_2$

$u_0'(a)=\eta_0^*$. $u_1'(a)=\eta_1^*$, $u_2'(a)=\eta_2^*$

A suitable set of values is

$\eta_0=\gamma_1=0$, $\eta_0^*=0$; $\eta_1=1$, $\eta_1^*=0$; $\eta_2=0$, $\eta_2^*=1$. (4.12)

We note that the conditions $[u_1(a), u'_1(a)]^T=[0,1]^T$ and$[u_2(a), u'_2(a)]^T=[0,0]^T$ are linearly independent . Any other linearly independent set of values can be used.

We now solve the corresponding initial values problems upto x=b.

Now, since the solution (4.6) satisfies the boundary conditions at x=a and x=b, we obtain, at

x=a: $u_0(a)+\mu_1 u_1(a)+\mu_2 u_2(a)=\gamma_1=0$.

Or $\eta_0+\mu_1\eta_1+\mu_2\eta_2=0$

Or $\mu_1=0$(using (4.12))

X=b: $u_0(b)+\mu_1 u_1(b)+\mu_2 u_2(b)=\gamma_2$ (4.13i)

Or $\mu_2=\dfrac{\gamma_2-u_0(b)}{u_2(b)}$, $u_2(b)\neq 0$

(4.13ii)

We determine μ_1 , μ_2 from (4.11) or (4.13) and obtain the solution of the given boundary value problem, using (4.6), at mesh point used in integrated the initial value problems.

Boundary conditions of the second kind Since $u'(a)=\gamma_1$ is given ,we guess the value of u(a). Again, we consider the following two cases.

Case 1: $\gamma_1\neq 0$. We choose

$u_0(a)=\eta_0$, $u_1(a)=\eta_1$, $u_2(a)=\eta_2$

$u_0'(a)=u_1'(a)=u_2'(a)=\gamma_2$ (4.14i)

A suitable set is values is

$\eta_0 = 0,\ \eta_1 = 1,\ \eta_2 = 0.$ (4.14ii)

Since the initial conditions $[u_1(a), u'_1(a)]^T = [0,1]^T$ and $[u_2(a), u'_2(a)]^T = [0,0]^T$ are linearly independent , we obtain linearly independent solutions $u_1(x)$ and $u_2(x)$. Using these initial conditions, we solve the corresponding initial value problems, with the same step lengths, upto x=b.

Now, from (4.6) , we get

$u'(x) = u'_0(x) + \mu_1 u'_1(x) + \mu_2 u'_2(x)$

Using the given condition (4.3), we get , at

x=a: $u'_0(a) + \mu_1 u'_1(a) + \mu_2 u'_2(a) = \gamma_1$

Or $\gamma_1 + \mu_1 \gamma_1 + \mu_2 \gamma_1 = \gamma_1$ ………(4.16i)

Or $\mu_1 + \mu_2 = 0$

x =b: $u'_0(b) + \mu_1 u'_1(b) + \mu_2 u'_2(b) = \gamma_2$

Or $\mu_2 = \dfrac{\gamma_2 - u_0(b)}{u_2(b) - u_1(b)}$, $u'_1(b) \neq u'_2(b)$. ……(4.16ii)

Case 2: $\gamma_1 = 0$. we cannot use the conditions as in case 1, since $[u_1(a), u'_1(a)]^T = [1,0]^T$ and $[u_2(a), u'_2(a)]^T = [0,0]^T$ are linearly dependent . In this case, we choose

: $u_0(a) = \eta_0$, $u_1(a) = \eta_1$, $u_2(a) = \eta_2$

$u'_0(a) = \eta_0^*$. $u'_1(a) = \eta_1^*$, $u'_2(a) = \eta_2^*$

A suitable set of values is

$\eta_0 = 0$, $\eta_0^* = \gamma_1 = 0$; $\eta_1 = 1$, $\eta_1^* = 0$; $\eta_2 = 0$, $\eta_2^* = 1$. (4.17)

We note that the conditions $[u_1(a), u'_1(a)]^T = [1,0]^T$ and $[u_2(a), u'_2(a)]^T = [0,0]^T$ are linearly independent . Any other linearly independent set of values can be used.

Using (4.6), (4.15) and the boundary conditions (4.3), we get, at

x=a: $u'_0(a) + \mu_1 u'_1(a) + \mu_2 u'_2(a) = \gamma_1 = 0.$

Or $\eta_0^* + \mu_1 \eta_1^* + \mu_2 \eta_2^* = 0$ ……..(4.18i)

Or $\mu_2 = 0$

X=b: $u'_0(b) + \mu_1 u'_1(b) + \mu_2 u'_2(b) = \gamma_2$

Or $\mu_2 = \dfrac{\gamma_2 - u'_0(b)}{u'_1(b)}$, $u'_1(b) \neq 0$ ………(4.18ii)

We determine μ_1, μ_2 from (4.16) or (4.18) and obtain the solution of the boundary value problem, using (4.6), at mesh point used in integrated the initial value problems.

Boundary conditions of the third kind In the case, we assume the arbitrary initial conditions as

$u_0(a) = \eta_0$, $u_1(a) = \eta_1$, $u_2(a) = \eta_2$

$u'_0(a) = \eta_0^*$. $u'_1(a) = \eta_1^*$, $u'_2(a) = \eta_2^*$ (4.19i)

A suitable set of values is

$\eta_0 = 0$, $\eta_0^* = 0$; $\eta_1 = 1$, $\eta_1^* = 0$; $\eta_2 = 0$, $\eta_2^* = 1$. (4.19ii)

Again , We note that the conditions $[u_1(a), u'_1(a)]^T = [1,0]^T$ and $[u_2(a), u'_2(a)]^T = [0,0]^T$ are linearly independent . Using these initial conditions, we solve the corresponding initial value problems, using the same step lengths, upto x=b.

Using (4.6) (4.19) and the boundary conditions (4.4), we get, at x=a:

$$a_0 [u_0(a) + \mu_1 u_1(a) + \mu_2 u_2(a)] - a_1 [u'_0(a) + \mu_1 u'_1(a) + \mu_2 u'_2(a)] = \gamma_1$$

Or $a_0[\eta_0 + \mu_1 \eta_1 + \mu_2 \eta_2] - a_1[\eta_0^* + \mu_1 \eta_1^* + \mu_2 \eta_2^*] = \gamma_1$

Or $a_0 \mu_1 - a_1 \mu_2 = \gamma_1$ ……….(4.20i)

x=b: $b_0 [u_0(b) + \mu_1 u_1(b) + \mu_2 u_2(b)] + b_1 [u'_0(b) + \mu_1 u'_1(b) + \mu_2 u'_2(b)] = \gamma_2$

Or $\mu_1 [b_0 u_1(b) + b_1 u'_1(b)] + \mu_2 [b_0 u_2(b) + b_1 u'_2(b)]$

$= \gamma_2 - [b_0 u_0(b) + b_1 u'_0(b)]$ …….. (4.20ii)

We determine μ_1 , μ_2 from (4.20) and obtain the solution of the boundary value problem, using (4.6), at mesh point used in integrated the initial value problems.

Boundary value problem of the first kind we solve the initial value problems (4.21i)(4.21ii) using the initial conditions

$$u_1(a) = \gamma_1 \ , u'_1(a) = 0$$

$$u_2(a) = \gamma_1 \ , u'_2(a) = 1 \qquad \text{……….(4.23i)}$$

up to x =b. Any other value for $u'_2(a)$ can also be used. Since the general solution (4.22) satisfies the boundary condition at x =b, we get

$$u(b) = \gamma_2 = \lambda u_1(b)+(1-\lambda) u_2(b)$$

or $$\lambda = \frac{\gamma_2 - u_2(b)}{u_1(b) - u_2(b)}, \ u_1(b) \neq u_2(b). \qquad \text{……...(4.23ii)}$$

Boundary value problem of the second kind we solve the initial value problem (4.21i) , (4.21ii) using the initial conditions

$$u_1(a) = 0 \ , u'_1(a) = \gamma_1 \ \ u_2(a) = 1 \ , u'_2(a) = \gamma_1 \qquad \text{………(4.23iii)}$$

upto x=b. since the general solution (4.22) satisfies the boundary condition at x=b, we have $$u'(b) = \gamma_2 = \lambda u'_1(b)+(1+\lambda) u'_2(b)$$

or $$\lambda = \frac{\gamma_2 - u'_2(b)}{u'_1(b) - u'_2(b)}, \ u'_1(b) \neq u'_2(b). \qquad \text{……(4.23iv)}$$

Boundary value problem of the third kind we solve the initial value problem (4.21i) , (4.21ii) using the initial conditions $u_1(a) = 0$, $u'_1(a) = -\gamma_1 / a_1$

$$u_2(a) = 1 \ , u'_2(a) = (a_0 - \gamma_1)/a_1 \qquad \text{……………(4.23v)}$$

upto x=b. the general solution (4.22) satisfies the boundary condition at x=b, we get

$$\gamma_2 = b_0 \ u(b) + b_1 \ u'(b) = b_0 \ [\lambda u_1(b)+(1-\lambda) u_2(b)] + b_1[\lambda u'_1(b)+(1-\lambda) u'_2(b)]$$

$$= \lambda [b_0 \ u_1(b) + b_1 \ u'_1(b)] + (1-\lambda) [b_0 u_2(b) + b_1 \ u'_2(b)]$$

Or $$\lambda = \frac{\gamma_2 - b_0 u_2(b) + b_1 u_3(b)}{[b_0 u_1(b) + b_1 u'_1(b)] - [b_0 u_2(b) + b_1 u'_2(b)]} \qquad \text{………..(4.23vi)}$$

The results obtained are identical in both the approaches.

Example 1

Using the shooting method, solve the first boundary value problem

$$u''=u+1,\ 0< x <1$$

$$u(0)=0\ ,\ u(1)= e-1.$$

Use the Euler-Cauchy method with h=0.25 to solve the resulting system of first order initial problems. Compare the solution with the exact solution $u(x) = e^{x} -1$.

Since boundary value problem in linear and non-homogeneous, we assume the solution in the form

$$u(x)=u_0(x)+\mu_1\ u_1(x)+ \mu_2\ u_2(x) \qquad \ldots\ldots\ldots(4.24i)$$

Where $u_0(x)$ satisfies the non-homogeneous differential equation and $u_1(x)$, $u_2(x)$ satisfy the homogeneous differential equation. Therefore, we have

$$u''_0-u_0(x)=1\ ,\ u''_1-u_1(x)=0 \quad \text{and} \quad u''_2-u_2(x)=1$$

We assume the initial conditions as given in (4.12), that is

$$u_0(0)=0\ ,\ u'_0(0)=0\ ;\ u_1(0)=1\ ,\ u'_1(0)=0\ ;\ u_2(0)=0\ ,\ u'_2(0)=1\ .$$

For the sake of illustration, we shall follow the steps in the method and obtain the analytical solution also.

Solving the differential equations and using the initial conditions, we obtain

$$u_0(x)=(1/2)(e^{x}+e^{-x})-1,\ \ u_1(x)=(1/2)(e^{x}+e^{-x}),$$

$$u_2(x)=(1/2)(e^{x}-e^{-x}) \qquad \ldots\ldots\ldots(4.24ii)$$

Now from (4.24i) we obtain

$$u(0)=u_0(0)+\mu_1\ u_1(0)+ \mu_2\ u_2(0)$$

$$u(1)=u_0(1)+\mu_1\ u_1(1)+ \mu_2\ u_2(1)$$

$$= u_0(1)+ \mu_2\ u_2(1)=e-1. \qquad \ldots\ldots\ldots(4.24iii)$$

Now from (4.24ii) we obtain

$u_0(1)=(1/2)(e-e^{-1})-1$ and $u_2(1)=(1/2)(e-e^{-1})$

Hence, from (4.24iii), we get

$$\mu_2=\frac{(e-1)-u_0(1)}{u_2(1)}=\frac{2(e-1)-(e+e^{-1}-2)}{(e-e^{-1})}$$

$$=\frac{e-e^{-1}}{e-e^{-1}}=1$$

Therefore, the analytical of the problem is

$u(x)=u_0(x)+\mu_1\, u_1(x)+\mu_2\, u_2(x)$

$=(1/2)(e^{x}+e^{-x})-1+(1/2)(e^{x}-e^{-x})=e^{x}-1.$

The illustrates the general of implementation of the method.

We now determine the solution of the initial value problems, using the Euler –Cauchy method with h=0.25.

We need to solve the following three, second order initial problems in $0< x <1$.

$u''_0-u_0(x)=1$, $u_0(0)=0$, $u'_0(0)=0$.

$u''_1-u_1(x)=0$, $u_1(0)=1$, $u'_1(0)=0$.

$u''_2-u_2(x)=1$, $u_2(0)=0$, $u'_2(0)=1$. ……(4.24iv)

We write these problems as equivalent first order systems.

Denote $u_0(x)=Y_0(x)$, $u'_0(x)=Y'_0(x) = Z_0(x)$,

$u_1(x)=Y_1(x)$, $u'_1(x)=Y'_1(x) = Z_1(x)$,

$u_2(x)=Y_2(x)$, $u'_2(x)=Y'_2(x) = Z_2(x)$.

then , we can write (4.24iv) as the following systems

$$\begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}' = \begin{pmatrix} Z_0 \\ 1+Y_0 \end{pmatrix}, \begin{pmatrix} Y_0(0) \\ Z_0(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix}' = \begin{pmatrix} Z_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} Y_1(0) \\ Z_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} Y_2 \\ Z_2 \end{pmatrix}' = \begin{pmatrix} Z_2 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Y_2(0) \\ Z_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Applying the Euler-Cauchy method

$$u_{j+1} = u_j + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h\ f(t_j, u_j),\ k_2 = h\ f\ (t_j + h, u_j + k_1)$$

We obtain the following systems:

System 1 we have $f_1 = Z_0$ and $f_2 = 1 + Y_0$

$$\begin{pmatrix} Y_{0,j+1} \\ Z_{0,j+1} \end{pmatrix} = \begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \frac{h}{2}\begin{pmatrix} Z_{0,j} \\ 1+Y_{0,j} \end{pmatrix} + \frac{h}{2}\begin{pmatrix} Z_{0,j} + h(1+Y_{0,j}) \\ 1+Y_{0,j} + hZ_{0,j} \end{pmatrix}$$

$$= \begin{pmatrix} 1+(h^2/2) & h \\ h & 1+(h^2/2) \end{pmatrix}\begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \begin{pmatrix} h^2/2 \\ h \end{pmatrix}$$

$$= \mathbf{B}(h)\begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \begin{pmatrix} h^2/2 \\ h \end{pmatrix}$$

Where $\mathbf{B}(h) = \begin{pmatrix} 1+(h^2/2) & h \\ h & 1+(h^2/2) \end{pmatrix}$.

The initial conditions are $Y_{0,0}=0$, $Z_{0,0}=0$.

The system 2 and 3 can be immediately written as

$$\begin{pmatrix} Y_{1,j+1} \\ Z_{1,j+1} \end{pmatrix} = \mathbf{B}(h)\begin{pmatrix} Y_{1,j} \\ Z_{1,j} \end{pmatrix},\ Y_{1,j}=1,\quad Z_{1,0}=0.$$

And $\begin{pmatrix} Y_{2,j+1} \\ Z_{2.j+1} \end{pmatrix} = \mathbf{B}(h)\begin{pmatrix} Y_{2,j} \\ Z_{2,j} \end{pmatrix}$, $Y_{2,0}=0$, $Z_{2,0}=1$.

Where **B**(h) is same as above.

Using h=0.25 . We obtain

$$\begin{pmatrix} Y_{0,j+1} \\ Z_{0,j+1} \end{pmatrix} = \begin{pmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{pmatrix}\begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \begin{pmatrix} 0.03125 \\ 0.25 \end{pmatrix}$$

With $Y_{0,0}=0$, $Z_{0,0}=0$ for j=0,1,2,3, we get

$u_0(0.25)\approx Y_{0,1}=0.03125$ $u'_0(0.25)\approx Z_{0,1}=0.025$

$u_0(0.50)\approx Y_{0,2}=0.12598$ $u'_0(0.50)\approx Z_{0,2}=0.51563$

$u_0(0.75)\approx Y_{0,3}=0.29007$ $u'_0(0.75)\approx Z_{0,3}=0.81324$

$u_0(1.00)\approx Y\ \ =0.53369$ $u'_0(1.00)\approx Z_{0,4}=1.16117$

we have $\begin{pmatrix} Y_{1,j+1} \\ Z_{1,j+1} \end{pmatrix} = \begin{pmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{pmatrix} \begin{pmatrix} Y_{1,j} \\ Z_{1,j} \end{pmatrix}$, $Y_{1,j}=1$, $Z_{1,0}=0$.

$u_1(0.25)\approx Y_{1,1}=1.03125$ $u'_1(0.25)\approx Z_{1,1}=0.025$

$u_1(0.50)\approx Y_{1,2}=1.12598$ $u'_1(0.50)\approx Z_{1,2}=0.51563$

$u_1(0.75)\approx Y_{1,3}=1.29007$ $u'_1(0.75)\approx Z_{1,3}=0.81324$

$u_1(1.00)\approx Y_{1,4}=1.53369$ $u'_1(1.00)\approx Z_{1,4}=1.16117$

$\begin{pmatrix} Y_{2,j+1} \\ Z_{2.j+1} \end{pmatrix} = \begin{pmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{pmatrix} \begin{pmatrix} Y_{2,j} \\ Z_{2,j} \end{pmatrix}$, $Y_{2,0}=0$, $Z_{2,0}=1$.

$u_2(0.25)\approx Y_{2,1}=0.025$ $u'_2(0.25)\approx Z_{2,1}=1.03125$

$u_2(0.50)\approx Y_{2,2}=0.51563$ $u'_2(0.50)\approx Z_{2,2}=1.12598$

$u_2(0.75)\approx Y_{2,3}=0.81324$ $u'_2(0.75)\approx Z_{2,3}=1.129007$

$u_2(1.00)\approx Y_{2,4}=1.16117$ $u'_2(1.00)\approx Z_{2,4}=1.53369$

From (4.13) , we get

$\mu_1=0$, $\mu_2=\dfrac{\gamma_2-u_0(1)}{u_2(1)}=\dfrac{e-1-0.53369}{1.16117}=1.02017$

we obtain the solution of the boundary value problem from

$$u(x)=u_0(x)+1.02017\, u_2(x).$$

the solution at the model points are given in table 4.1 . The maximum absolute error which

Occurs at x=0.50 is given by

max.abs.error=0.00329

TABLE 1 SOLUTION OF EXAMPLE 1

x_j	Exact: $u(x_j)$	u_j
0.25	0.28403	0.28629
0.50	0.64872	0.65201
0.75	1.11700	1.11971
1.00	1.71828	1.71828

More accurate results can be obtained by using smaller step length h.

Alternative Method

To apply alternative method, we solve the two initial value problems

$$u''=u_1+1\ ,\ u_1(0)=0\ ,\ u'_1(0)=0$$

and $$u''=u_2+1\ ,\ u_2(0)=0\ ,\ u'_2(0)=1.$$

We can also take the initial condition $u'_1(0)$ as $u'_1(0)=\alpha$, $\alpha\neq 0,1$. Therefore, we obtain the equation

$$\begin{bmatrix} Y_{i,j+1} \\ Z_{i,j+1} \end{bmatrix} = \begin{bmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{bmatrix} \begin{bmatrix} Y_{i,j} \\ Z_{i,j} \end{bmatrix} + \begin{bmatrix} 0.03125 \\ 0.25 \end{bmatrix}$$

Where $Y_1=u_1$ and $Z = u_2$.

Using the condition $Y_{1,0}=0$, $Z_{1,0}=0$, we obtain

$$\begin{bmatrix} Y_{1,1} \\ Z_{1,1} \end{bmatrix} = \begin{bmatrix} 0.03125 \\ 0.25 \end{bmatrix} ,\ \begin{bmatrix} Y_{1,2} \\ Z_{1,2} \end{bmatrix} = \begin{bmatrix} 0.12598 \\ 0.51563 \end{bmatrix}$$

$$\begin{bmatrix} Y_{1,3} \\ Z_{1,3} \end{bmatrix} = \begin{bmatrix} 0.29007 \\ 0.81324 \end{bmatrix} ,\ \begin{bmatrix} Y_{1,4} \\ Z_{1,4} \end{bmatrix} = \begin{bmatrix} 0.53369 \\ 1.16117 \end{bmatrix}$$

Using the condition $Y_{2.0} = 0$, $Z_{2.0} = 1$, we obtain

$$\begin{pmatrix} Y_{2,1} \\ Z_{2,1} \end{pmatrix} = \begin{pmatrix} 0.28125 \\ 1.28125 \end{pmatrix}, \begin{pmatrix} Y_{2,2} \\ Z_{2.2} \end{pmatrix} = \begin{pmatrix} 0.64160 \\ 1.64160 \end{pmatrix}$$

$$\begin{pmatrix} Y_{2,3} \\ Z_{2,3} \end{pmatrix} = \begin{pmatrix} 1.10330 \\ 2.10330 \end{pmatrix}, \begin{pmatrix} Y_{2,4} \\ Z_{2,4} \end{pmatrix} = \begin{pmatrix} 1.69485 \\ 2.69485 \end{pmatrix}$$

From (4.23ii), we get

$$\lambda = \frac{(e-1) - Y_{2,4}}{Y_{1,4} - Y_{2,4}} = \frac{e - 1 - 1.69485}{0.53369 - 1.69485} = -0.02019$$

Hence, $u(x) = -0.02019\, Y_1(x) + 1.02019\, Y_2(x)$.

Substituting x= 0.25, 0.5, 0.75 and 1.0, we get

$u(0.25) = 0.28630$, $u(0.50) \approx 0.65201$, $u(0.75) \approx 1.11972$, $u(1.0) \approx 1.71829$.

These values are same as given in Table 4.1, except for the round-off error in the last digit.

Example 2

Use the shooting method to solve the mixed boundary value problem.

$u'' = u - 4x\, e^x$, $0 < x < 1$.

$u(0) - u'(0) = -1$, $u(1) + u'(1) = -e$.

Use the Taylor series method

$$u_{j+1} = u_j + h\, u_j' + \frac{h^2}{2} u_j'' + \frac{h^3}{6} u_j'''$$

$$u_{j+1}' = u_j' + h\, u_j'' + \frac{h^2}{2}\, u_j'''$$

to solve the initial value problems. Assume h = 0.25. Compare with the exact solution

$u(x) = x(1 - x)\, e^x$.

We assume the solution in the form

$u(x) = u_0(x) + \mu_1 u_1(x) + \mu_2 u_2(x)$

where $u_0(x)$, $u_1(x)$ and $u_2(x)$ satisfy the differential equations

$u_0'' - u_0 = -4x\, e^x$, $u_1'' - u_1 = 0$

$u_2'' - u_2 = 0$

The initial conditions may be assumed as given in(4.19ii)

$u_0(0) = 0,\ u_0'(0) = 0.$

$u_1(0) = 1,\ u_1'(0) = 0.$

$u_2(0) = 0,\ u_2'(0) = 0.$

To illustrate the solution procedure, we solve analytically the initial value problems. The analytical solutions of the above initial value problems are given by

$u_0(x) = (1/2)\, e^{-x} - e^x\, (x^2 - x + (1/2))$

$u_1(x) = (1/2)(e^x + e^{-x})$, $u_2(x) = (1/2)(e^x - e^{-x})$.

We also have

$u(0) = u_0(0) + \mu_1 u_1(0) + \mu_2 u_2(0) = \mu_1$

$u'(0) = u_0'(0) + \mu_1 u_1'(0) + \mu_2 u_2'(0) = \mu_2$

$u_0(1) = -(1/2)(e - e^{-1})$, $u_1(1) = (1/2)(e + e^{-1})$, $u_2(1) = (1/2)(e - e^{-1})$.

$u_0'(1) = -(1/2)(3e + e^{-1})$, $u_1'(1) = (1/2)(e - e^{-1})$, $u_2'(1) = (1/2)(e + e^{-1})$,

$u(1) = u_0(1) + \mu_1 u_1(1) + \mu_2 u_2(1)$

$= (1/2)(e^{-1} - e) + (1/2)\mu_1(e + e^{-1}) + (1/2)\mu_2(e - e^{-1})$

$u'(1) = u_0'(1) + \mu_1 u_1'(1) + \mu_2 u_2'(1)$

$= -(1/2)(3e + e^{-1}) + (1/2)\mu_1(e - e^{-1}) + (1/2)\mu_2(e + e^{-1})$

Substituting into the boundary condition we get the relations

$$\mu_2 - \mu_1 = 1,\ \mu_2 + \mu_1 = 1, \text{ or } \mu_1 = 0,\ \mu_2 = 1.$$

Thus, the initial conditions are given by $u(0) = 0$, $u'(0) = 1$.

The required solution is $u(x) = u_0(x) + u_2(x) = x(1-x)e^x$. We now solve the three, second order initial value problems

$$u_0'' = u_0 - 4xe^x, u_0(0) = 0, u_0(0) = 0$$

$$u''_1 = u_1 \quad , u_1(0)=1 , u'_1(0)=0$$

$$u''_2 = u_2 \quad , u_2(0)=1 , u'_2(0)=0$$

by using the given Taylor series method with h=0.25. we have the following results.

(i) $i=0$, $u_{0,0}=0$, $u'_{0,0}=0$.

$u'_{0,j} = u_{0,j} - 4x_j e^{x_j}$, $u'''_{0,j} = u_{0,j} - 4(x_j+1)e^{x_j}$, j=1, 2, 3.

Hence $u_{0,j+1} = u_{0,j} + h u'_{0,j} + \frac{h^2}{2}(u_{0,j} - 4x_j e^{x_j}) +$

$$\frac{h^3}{6}[u_{0,j} - 4(x_j+1)e^{x_j}].$$

$$= \left(1+\frac{h^2}{2}\right) u_{0,j} + \left(h+\frac{h^3}{6}\right) u'_{0,j} - \left[\frac{2}{3}h^3(1+x_j) + 2h^2 x_j\right] e^{x_j}$$

$$= 1.03125\, u_{0,j} + 0.25260\, u'_{0,j} - (0.13542\, x_j + 0.0625)\, e^{x_j}$$

Hence,

$u_0(0.25) \approx u_{0,1} = -0.01042$, $u'_0(0.25) \approx u'_{0,1} = -0.12500$,

$u_0(0.50) \approx u_{0,2} = -0.09917$, $u'_0(0.50) \approx u'_{0,2} = -0.65315$,

$u_0(0.75) \approx u_{0,3} = -0.39606$, $u'_0(0.75) \approx u'_{0,3} = -1.83185$,

$u_0(1.00) \approx u_{0,4} = -1.10823$, $u'_0(1.00) \approx u'_{0,4} = -4.03895$.

(ii) i=1, $u_{1,0} = 1$,

$u''_{1,j} = u_{1,j}$, $u'''_{1,j} = u'_{1,j}$, j=1 ,2,3.

$$u_{1,j+1} = u_{1,j} + h\,u'_{1,j} + \frac{h^2}{2}\,u_{1,j} + \frac{h^3}{6}\,u'_{1,j}$$

$$= \left(1 + \frac{h^2}{2}\right) u_{1,j} + \left(h + \frac{h^3}{6}\right) u'_{1,j}$$

$$= 1.03125\,u_{1,j} + 0.2560\,u'_{1,j}$$

$$u'_{1,j+1} = u'_{1,j} + h\,u''_{1,j} + \frac{h^2}{2}\,u'_{1,j}$$

$$= h\,u_{1,j} + \frac{h^2}{2}\,u'_{1,j} = 0.25\,u_{1,j} + 1.03125\,u'_{1,j}.$$

Hence,

$u_1(0.25) \approx u_{1,1} = 1.03125$, $u'_1(0.25) \approx u'_{1,1} = 0.25$,

$u_1(0.50) \approx u_{1,2} = 1.12663$, $u'_1(0.50) \approx u'_{1,2} = 0.51563$,

$u_1(0.75) \approx u_{1,3} = 1.29209$, $u'_1(0.75) \approx u'_{1,3} = 0.81340$,

$u_1(1.00) \approx u_{1,4} = 1.53794$, $u'_1(1.00) \approx u'_{1,4} = 1.16184$

(iii). i=2 , $u_{2,0} = 0$, $u'_{2,0} = 1$.

$u''_{2,j} = u_{2,j}$, $u''_{2,j} = u'_{2,j}$, j=1, 2, 3.

Since the differential equation is same as for u_1, we get

$$u_{2,j+1} = 1.03125u_{2,j} + 0.25260u'_{2,j}$$

$$u'_{2,j+1} = 0.25u_{2,j} + 1.03125u'_{2,j}$$

Hence,

$u_2(0.25) \approx u_{2,1} = 0.25260$, $u'_2(0.25) \approx u'_{2,1} = 1.03125$,

$u_2(0.50) \approx u_{2,2} = 0.52099$, $u'_2(0.50) \approx u'_{2,2} = 1.12663$,

$u_2(0.75) \approx u_{2,3} = 0.82186$, $u'_2(0.75) \approx u'_{2,3} = 1.29208$,

$u_2(1.00) \approx u_{2,4} = 1.17393$, $u'_2(1.00) \approx u'_{2,4} = 1.53792$.

From (4.20) and the given boundary conditions, we have

$a_0 = a_1 = 1,\ b_0 = b_1 = 1,\ \gamma_1 = -1,\ \gamma_2 = -e.$

$\mu_1 + \mu_2 = -1$

$[u_1(1) + u'_1(1)]\,\mu_1 + [u_2(1) + u_2(1)]\,\mu_2 =$

$-e - [u_0(1) + u'_0(1)]$

Or $2.69978\,\mu_1 + 2.71185\,\mu_2 = 2.42890.$

Solving these equations, we obtain $\mu_1 = -0.05229$, $\mu_2 = 0.94771$.

We obtain the solution of the boundary value problem from

$u(x) = u_0(x) - 0.05229u_1(x) + 0.94771u_2(x).$

the solution at the nodal points are given table 4.2. The maximum absolute error which occurs at x=0.75, is given by

max. abs. error = 0.08168.

TABLE 2 :SOLUTION OF EXAMPLE 2

x_j	Exact: $u(x_j)$	u_j
0.25	0.24075	0.17505
0.50	0.41218	0.33567
0.75	0.39694	0.31526
1.00	0.0	-0.07610

Alternative Method

Here, we solve the initial value problems

$u''_1 - u_1 = -4x e^x, u_1(0)=0, u'_1(0)=(-\gamma_1 / a_1)=1$

$u''_2 - u_2 = -4x e^x, u_2(0)=1, u'_2(0)=[(a_0 - \gamma_1)/a_1]=2$

(See (4.4i), (4.4ii), and (4.23v)).

Using the given Taylor's method with h=0.25, we obtain (as done earlier)

$u'_{i,j+1} = 0.025u_{i,j} + 1.03125u'_{i,j} - 2(0.5625x_j + 0.0625)e^{x_j}$

i=1, 2 and j=0, 1, 2, 3.

Using the initial conditions, we obtain

$u_1(0.25) \approx u_{1,1} = 0.24218$, $u'_1(0.25) \approx u'_{1,1} = 0.90625$,

$u_1(0.50) \approx u_{1,2} = 0.42182$, $u'_1(0.50) \approx u'_{1,2} = 0.47348$,

$u_1(0.75) \approx u_{1,3} = 0.42579$, $u'_1(0.75) \approx u'_{1,3} = -0.53976$,

$u_1(1.00) \approx u_{1,4} = 0.06568$, $u'_1(1.00) \approx u'_{1,4} = -2.50102$.

$u_2(0.25) \approx u_{2,1} = 1.52603$, $u'_2(0.25) \approx u'_{2,1} = 2.18750$,

$u_2(0.50) \approx u_{2,2} = 2.06943$, $u'_2(0.50) \approx u'_{2,2} = 211573$,

$u_2(0.75) \approx u_{2,3} = 2.53972$, $u'_2(0.75) \approx u'_{2,3} = 1.56571$,

$u_2(1.00) \approx u_{2,4} = 2.77751$, $u'_2(1.00) \approx u'_{2,4} = 0.19872$.

Using (4.23vi), we get

$$\lambda = \frac{-e - [u_2(1) + u'(1)]}{[u_1(1) + u'_1(1)] - [u_2(1) + u'_2(1)]}$$

$$= \frac{-5.69451}{-2.43534 - 2.97623} = 1.05228$$

Hence, we have

$$u(x) = \lambda u_1(x) + (1-\lambda)u_2(x)$$

$$=1.05228u_1(x) - 0.05228u_2(x)$$

Substituting x=0.25, 0.5, 0.75 and 1.0, we get

$u(0.25) \approx 0.17506, u(0.50) \approx 0.33568,$
$u(0.75) \approx 0.31527, u(1.00) \approx -0.07609.$

These values are same as given in table 4.2, except for the round-off error in the last digit.

4.4 Nonlinear Second Order Differential Equation

We now consider the nonlinear differential equation

$$u'' = f(x,u,u'),\ a< x <b$$

Subject to one of the boundary conditions (4.2) to (4.4). Since the differential equation is nonlinear , we cannot write the solution in the form (4.6). in this case we proceed as follows.

We assume $u'(a) = s$ and solve the initial value problem

$$u'' = f(x,u,u')$$

$$u(a) = \gamma_1,\ u'(a) = s \qquad (4.25)$$

Up to x=b using any numerical method. The solution of the initial value problem denoted by $u(b,s)$ should satisfy the boundary conditions at x=b. let

$$\phi(s) = u'(b,s) - \gamma_2. \qquad (4.26)$$

Hence the problem is to find s, such that $\phi(s) = 0.$

Boundary conditions of the second kind: The boundary conditions are $u'(a) = \gamma_1$ and $u'(b) = \gamma_2$.

We assume u(a)=s and solve the initial value problem

$$u'' = f(x,u,u')$$

$$u(a) = s,\ u'(a) = \gamma_1 \qquad (4.27)$$

Upto x=b using any numerical method. The solution of the initial value problem denoted by $u(b,s)$ should satisfy the boundary conditions at x=b. let

$$\phi(s) = u'(b,s) - \gamma_2. \qquad (4.28)$$

Hence the problem is to find s, such that $\phi(s) = 0.$

Boundary conditions of the third kind: we have the boundary conditions as $a_0u(a) - a_1u'(a) = \gamma_1$ and $b_0u(b) - b_1u'(b) = \gamma_2$. Here, we assume the initial value of $u(a)$ or $u'(a)$. Let $u'(a) = s$, then from

$a_0u(a) - a_1u'(a) = \gamma_1$, we get $u(a) = (a_1s + \gamma_1)/a_0$.

We now solve the initial value problem

$$u'' = f(x,u,u')$$

$$u(a) = \frac{1}{a_0}(a_1s + \gamma_1), u'(a) = s \qquad (4.29)$$

Up to x=b using the numerical method. The solution of this initial value problem denoted by $u(b,s) = s$ should satisfy the boundary condition at x=b. let

$$\phi(s) = b_0u(b,s) + b_1u'(b,s) - \gamma_2 \qquad \ldots\ldots\ldots\ldots\ldots(4.30)$$

Hence, the problem is to find s, such that $\phi(s) = 0.$

The function $\phi(s)$ in (4.26) or (4.28) or (4.30) is a nonlinear function in s.

We solve the equation

$$\phi(s) = 0. \qquad \ldots\ldots\ldots\ldots\ldots(4.31)$$

By using iterative method.

Secant Method

The iterative procedure for solving (4.31) is given by

$$s^{(K+1)} = s^{(K)} - \left[\frac{s^{(K)} - s^{(K-1)}}{\phi(s^{(K)}) - \phi(s^{(K-1)})}\right]\phi(s^{(K)}), k = 1,2,\ldots \quad (4.32)$$

Where $s^{(0)}$ and $s^{(1)}$ are two initial approximations to s. To start the application of the secant method, we need to solve the initial value problem (4.25) or (4.27) or (4.29) for two values of s, that is for $s^{(0)}, s^{(1)}$. The iteration may be stopped when $\left|\phi(s^{(K+1)})\right| < ($ given error tolerance).

Newton-Raphson Method

The iterative procedure for solving (4.31) is given by

$$s^{(K+1)} = s^{(K)} - \frac{\phi(s^{(K)})}{\phi'(s^{(K)})}, k = 0,1,2,.... \qquad \text{……….(4.33)}$$

To determine $\phi'(s^{(K)})$, we use the following method. Denote

$$u_s = u(x,s), \quad u'_s = u'(x,s), \; u''_s = u''(x,s).$$

Then (4.29) can be written as

$$u'_s = f(x, u_s, u'_s) \qquad \text{……….(4.34i)}$$

$$u_s(a) = \frac{1}{a_0}(a_1 s + \gamma_1), u'_s(a) = s. \qquad \text{……….(4.34ii)}$$

Differentiating (4.34i) partially with respect to s, we get

$$\frac{\partial}{\partial s}(u''_s) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial u_s}\frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u'_s}\frac{\partial u'_s}{\partial s}$$

$$= \frac{\partial f}{\partial u_s}\frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u'_s}\frac{\partial u'_s}{\partial s} \qquad \text{…………(4.35)}$$

Since x is independent of s. differentiating (4.34ii), partially with respect to s, we get

$$\frac{\partial}{\partial s}\left[u_s(a)\right] = \frac{a_1}{a_0}, \frac{\partial}{\partial s}\left[u'_s(a)\right] = 1. \qquad \text{…………(4.36)}$$

Let, $v = \frac{\partial u_s}{\partial s}$. Then,

$$v' = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u_s}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial u_s}{\partial x}\right) = \frac{\partial}{\partial s}(u_s')$$

$$v'' = \frac{\partial v'}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial s}\left(\frac{\partial u_s}{\partial x}\right)\right) = \frac{\partial}{\partial s}\left(\frac{\partial^2 u_s}{\partial^2 x}\right) = \frac{\partial}{\partial s}(u_s'')$$

From (4.35) and (4.36), we obtain

$$v'' = \frac{\partial f}{\partial u_s}(x, u_s, u_s')v + \frac{\partial f}{\partial u_s'}(x, u_s, u_s')v' \qquad \text{………(4.37i)}$$

$$v(a) = a_1 / a_0, v'(a) = 1. \qquad \text{………..(4.37ii)}$$

The differential equation (4.37i) is called the first variational equation. It can be solved step-by-step along (4.34), that is, (4.34) and (4.37) can be solved together as a single system. When the computation of one cycle is completed, $v(b)$ and $v'(b)$ are available.

Now, from (4.30), at x=b, we have of $\frac{d\varphi}{ds} = b_0 \frac{\partial u_s}{\partial s} + b_1 \frac{\partial u_s'}{\partial s} = b_0 v(b) + b_1 v'(b)$ (4.38)

Thus, we have the value of $\phi'(s^{(K)})$ to be used in (4.33).

If the boundary conditions of the first kind are given, then we have

$a_0 = 1, a_1 = 0, b_0 = 1, b_1 = 0$ And $\phi(s) = u_s(b) - \gamma_2$. (4.39)

The initial conditions (4.36), on v become

V(a)=0 , $v'(a) = 1$. ………….(4.40)

Then, we have from (4.38)

$$\frac{d\phi}{ds} = v(b). \qquad \text{…………..(4.41)}$$

Example 3

Use the shooting method to solve the boundary value problem

$$u'' = 2uu', \quad 0 < x < 1$$

$$u(0) = 0.5, u(1) = 1.$$

Use the Taylor series method

$$u_{j+1} = u_j + hu'_j + \frac{h^2}{2}u''_j + \frac{h^3}{6}u'''_j$$
$$u'_{j+1} = u'_j + hu''_j + \frac{h^2}{2}u'''_j \qquad (4.42)$$

To solve the corresponding initial value problems and the secant method for the iteration. Iterate until tolerance is less than 0.005. Assume h=0.25. Compare with the exact solution $u(x) = 1/(2-x)$.

Let the starting value of the slope at x=0 be taken as $u'(0) = s^{(0)} = 0.5.$ therefore, we need to solve initial value problem

$$u'' = 2uu'$$
$$u(0) = 0.5, u'(0) = s^{(0)} = 0.5.$$

Using the given taylor series method and substituting

$u''_j = 2u_j u'''_j = 2[(u'_j)^2 + u_j u''_j]$ With h=0.25, $u_0 = 0.5.$

$u'_0 = 0.5.$ in (4.42), we obtain,

$$u_{j+1} = u_j + hu'_j + \frac{h^2}{2}(2u_j u'_j) + \frac{h^3}{3}[(u'_j)^2 + u_j u''_j] =$$
$$u_j + 0.25u'_j + 0.0625u_j u'_j + 0.00521[(u'_j)^2 + u_j u''_j]$$
$$= u_j + 0.25u'_j + 0.0625u_j u'_j + 0.00521[(u'_j)^2 + 2(u_j)^2 u'_j] \qquad (4.43)$$

$$u'_{j+1} = u'_j + h(2u_j u'_j) + h^2[(u'_j)^2 + u_j(2u_j u'_j)]$$
$$= u'_j + 0.5u_j u'_j + 0.0625[(u'_j)^2 + 2(u_j)^2 u'_j)] \qquad (4.44)$$

We obtain from (4.43) and (4.44), j=1, 2, 3

$u(0.25) \approx u_1 = 0.64323$, $u'(0.25) \approx u'_1 = 0.65625$

$u(0.50) \approx u_2 = 0.83875$, $u(0.50) \approx u_2' = 0.92817$

$u(0.75) \approx u_3 = 1.13074$, $u(0.75) \approx u_3' = 1.45289$

$u(1.00) \approx u_4 = 1.62699$, $u(1.00) \approx u_4' = 2.63844$

From (4.39), we get $\phi(s^{(0)}) = u(1, s^{(0)}) - 1.0 = 0.62699$

We now take another guess value of the slope at x=0 as $u'(0) = s^{(1)} = 0.1$. Therefore, we need to solve the equations (4.44) with $u_0 = 0.5$ and $u_0' = 0.1$. we obtain, for j=0, 1, 2, 3.

$u(0.25) \approx u_1 = 0.52844$, $u'(0.25) \approx u_1' = 0.12875$.

$u(0.50) \approx u_2 = 0.56534$, $u(0.50) \approx u_2' = 0.16830$.

$u(0.75) \approx u_3 = 0.61407$, $u(0.75) \approx u_3' = 0.30698$.

$u(1.00) \approx u_4 = 0.67991$, $u(1.00) \approx u_4' = -0.32009$.

From (4.39), we get $\phi(s^{(1)}) = u(1, s^{(1)}) - 1.0 = -0.32009$.

Using the secant method (4.32), we obtain

$$s^{(2)} = s^{(1)} - \left[\frac{s^{(1)} - s^{(0)}}{\phi(s^{(1)}) - \phi(s^{(0)})}\right]\phi(s^{(1)}),$$

$$= 0.1 - \left[\frac{0.1 - 0.5}{-0.32009 - 0.62699}\right](-0.32009) = 0.23519.$$

Now we solve the equation $u_0 = 0.5$ and $u_0' = 0.23519$. we obtain, for j=0, 1, 2, 3.

$u(0.25) \approx u_1 = 0.56705$, $u'(0.25) \approx u_1' = 0.30479$.

$u(0.50) \approx u_2 = 0.65555$, $u(0.50) \approx u_2' = 0.40926$.

$u(0.75) \approx u_3 = 0.77734$, $u(0.75) \approx u_3' = 0.57586$.

$u(1.00) \approx u_4 = 0.95464$, $u(1.00) \approx u_4' = 0.86390$.

From (4.39), we get $\phi(s^{(2)}) = u(1, s^{(2)}) - 1.0 = -0.04536.$

Using the secant method (4.32), we obtain

$$s^{(3)} = s^{(2)} - \left[\frac{s^{(2)} - s^{(1)}}{\phi(s^{(2)}) - \phi(s^{(1)})}\right]\phi(s^{(2)}),$$

$$= 0.23519 - \left[\frac{0.23519 - 0.1}{-0.04536 + 0.32009}\right](-0.04536) = 0.25751.$$

Now we solve the equation $u_0 = 0.5$ and $u_0' = 0.25751.$ we obtain, for j=0, 1, 2, 3.

$$u(0.25) \approx u_1 = 0.57344, \qquad u'(0.25) \approx u_1' = 0.33408.$$

$$u(0.50) \approx u_2 = 0.67066, \qquad u(0.50) \approx u_2' = 0.45058.$$

$$u(0.75) \approx u_3 = 0.80536, \qquad u(0.75) \approx u_3' = 0.63969.$$

$$u(1.00) \approx u_4 = 1.00394, \qquad u(1.00) \approx u_4' = 0.97472.$$

From (4.39), we get $\phi(s^{(3)}) = u(1, s^{(3)}) - 1.0 = 0.00394 < 0.05.$ the iteration is now stopped. Solutions occur at x=0.75 and its value is

max. abs. error = 0.00536.

TABLE 3 SOLUTION OF EXAMPLE 3

x_j	Exact: u(x)	u_j
0.25	0.57143	0.17505
0.50	0.66667	0.33567
0.75	0.80000	0.31526
1.00	1.00000	1.00394

4.5 Iterative Method For Eigen Values

Power method

Power method is used to determine numerically largest eigen value and corresponding eigen vector of a matrix A.

Let A be a n×n square matrix and let $\lambda_1, \lambda_2 \ldots \lambda_n$ be distinct eigen value of so that

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots |\lambda_n| \qquad (1)$$

Let $v_1, v_2, \ldots v_n$ be their corresponding eigen vectors

$$\therefore \quad Av_i = \lambda v_i, i = 1,2,3,\ldots\ldots n \qquad (2)$$

This method is applicable only if the vectors $v_1, v_2, \ldots v_n$ are linearly independent. This may be true even if the eigen value $\lambda_1, \lambda_2 \ldots \lambda_n$ are not distinct.

These n vectors constitute a vector space of which these vectors from a basis.

Let Y_0 be any vector of this space.

Then $Y_0 = C_1 v_1 + C_2 v_2 + C_3 v_3 + \ldots + C_n v_n$

Where C_i's are constants (scalars).

Pre-multiplying by A, we get

$$Y_1 = AY_0 = C_1 Av_1 + C_2 Av_2 + C_3 Av_3 + \ldots + C_n Av_n$$

$$= C_1 \lambda_1 v_1 + C_2 \lambda_2 v_2 + C_3 \lambda_3 v_3 + \ldots + C_n \lambda_n v_n$$

Similarly $Y_2 = C_1 \lambda^2{}_1 v_1 + C_2 \lambda^2{}_2 v_2 + C_3 \lambda^2{}_3 v_3 + \ldots + C_n \lambda^2{}_n v_n$

Continuing this process

$$Y_r = AY_{r-1} = C_1 \lambda^r{}_1 v_1 + C_2 \lambda^r{}_2 v_2 + C_3 \lambda^r{}_3 v_3 + \ldots + C_n \lambda^r{}_n v_n$$

$$= \lambda^r{}_1 \left[C_1 v_1 + C_2 \left(\frac{\lambda_2}{\lambda_1}\right)^r v_2 + C_3 \left(\frac{\lambda_3}{\lambda_1}\right)^r v_3 + \ldots + C_n \left(\frac{\lambda_n}{\lambda_1}\right)^r v_n \right]$$

Similarly,

$$Y_{r+1} = A^{r+1} Y_0 = \lambda^{r+1}{}_1 \left[C_1 v_1 + C_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{r+1} v_2 + C_3 \left(\frac{\lambda_3}{\lambda_1}\right)^{r+1} v_3 + \ldots + C_n \left(\frac{\lambda_n}{\lambda_1}\right)^{r+1} v_n \right]$$

As $r \to \infty, \left(\frac{\lambda_i}{\lambda_1}\right)^r \to 0, i = 2,3,\ldots n$

In the limit as $r \to \infty$

$Y_r \to \lambda_1^{\ r} C_1 v_1$

$Y_{r+1} \to \lambda_1^{\ r+1} C_1 v_1$

$\therefore \lambda_i = \lim_{r \to \infty} \dfrac{\left(A^{r+1} Y_0\right)}{\left(A^r Y_0\right)}$, I =1, 2….n.

Where the suffix i denotes ith component of the vector.

To get the convergence quicker, we normalize the vector before multiplication by A.

Method: Let v_0 be an arbitrary vector and find

Example 1: Find the dominant eigen value of A=$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$by power method and hence find the other eigen value also. Verify your results by any other matrix theory.

Solution

Let an initial arbitrary vector be $X_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$A\,X_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = 4X_2.$$

$$A\,X_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 7.5 \end{pmatrix} = 4\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} = 4X_3$$

$$A\,X_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ 5 \end{pmatrix} = 5\begin{pmatrix} \frac{7}{15} \\ 1 \end{pmatrix} = bX_4$$

$$A\,X_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{7}{15} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{37}{15} \\ \frac{81}{15} \end{pmatrix} = \frac{81}{15}\begin{pmatrix} \frac{37}{81} \\ 1 \end{pmatrix} = \frac{81}{15}X_5$$

$$A\,X_5 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{37}{81} \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4568 \\ 5.3704 \end{pmatrix} = 5.3704\begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = 5.3704X_6$$

$$A X_6 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4575 \\ 5.3724 \end{pmatrix} = 5.3704\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 54.3724 X_7$$

$$A X_7 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3723 \end{pmatrix} = 5.3723\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 5.3723 X_8$$

$$A X_8 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3723 \end{pmatrix} = 5.3723\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$$

Hence $\lambda_1 = 5.3723$ and eigen vector $X_1 = \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$.

Since $\lambda_1 + \lambda_2 =$ Trace of A=1+4=5

Second eigen value= $\lambda_2 = -0.3723$

Characterstic equation is $\lambda^2 - (1+4)\lambda + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 0$

I.e., $\lambda^2 - 5\lambda - 2 = 0 \therefore \lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2} = 5.3723, -0.3723.$

The values got by power method exactly coinside with the solution from analytical method.

Example 2: Find the dominant eigen value and the corresponding eigen vector of A= $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Find also the least root and hence the third eigen value also.

Solution

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary initial eigen vector.

$$A X_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1.X_2$$

$$A X_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7.X_3$$

$$A X_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.5714 X_4$$

$$A X_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12.X_5$$

$$A X_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706 X_5$$

$$A X_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 4.0072 X_6$$

$$A X_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix} = 3.9982 X_7$$

$$A X_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4.X_9$$

$\therefore$ Dominant eigen value = 4; corresponding eigen vector is (1,0.5,0).

To find the least eigen value, let B=A-4I since λ_1=4.

$$\therefore B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 04 & \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

We will find the dominant eigen value of B.

Let $Y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary initial eigen vector.

$$\text{B}\,Y_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -3Y_2$$

$$\text{B}\,Y_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5Y_3$$

$$\text{B}\,Y_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix}$$

dominant eigen value of B is =-5.

Adding 4, smallest value of A=-5+4=-1

Sum of eigen value =Trace of A=1+2+3=6

$$4+(-1)+\lambda_3=6 \;\therefore\; \lambda_3=3.$$

All the three eigen value are 4,3,-2.

Example 3

Find the numerically largest eigen value of A= $\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ and the corresponding eigen vector.

Solution:

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary initial eigen vector.

$$A X_1 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = 25X_2$$

$$A X_2 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = \begin{pmatrix} 25.2 \\ 1.12 \\ 1.68 \end{pmatrix} = 25.2\begin{pmatrix} 1 \\ 0.0444 \\ 0.0667 \end{pmatrix} = 25X_3$$

$$A X_3 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0444 \\ 0.0667 \end{pmatrix} = \begin{pmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{pmatrix} = 25.1778\begin{pmatrix} 1 \\ 0.0450 \\ 0.0688 \end{pmatrix} = 25.1778X_4$$

$$A X_4 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0450 \\ 0.0688 \end{pmatrix} = 25.1826\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = 25.1826X_5$$

$$A X_5 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = 25.1821\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = 25.1821X_6$$

We have reached the limit.

$\therefore \lambda_1 = 25.1821$ and the corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix}$.

Example 4: Using power method , find all eigen values of A=$\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Solution.

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an approximation eigen vector.

$$\text{A}\,X_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = 5X_2$$

$$\text{A}\,X_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2\begin{bmatrix} 1 \\ 0 \\ 03846 \end{bmatrix} = 5.2X_3$$

$$\text{A}\,X_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 03846 \end{bmatrix} = \begin{bmatrix} 5.3846 \\ 0 \\ 2.9231 \end{bmatrix} = 5.3846\begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = 5.3846X_4$$

$$\text{A}\,X_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = \begin{bmatrix} 5.5429 \\ 0 \\ 3.7143 \end{bmatrix} = 5.5429\begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = 5.5429X_5$$

$$A X_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = \begin{bmatrix} 5.6701 \\ 0 \\ 4.3505 \end{bmatrix} = 5.6701 \begin{bmatrix} 1 \\ 0 \\ 0.7672 \end{bmatrix} = 5.6701 X_6$$

$$A X_6 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.7672 \end{bmatrix} = \begin{bmatrix} 5.7672 \\ 0 \\ 4.8360 \end{bmatrix} = 5.7672 \begin{bmatrix} 1 \\ 0 \\ 0.8385 \end{bmatrix} = 5.7672 X_7$$

$$A X_7 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8385 \end{bmatrix} = \begin{bmatrix} 5.8385 \\ 0 \\ 5.1927 \end{bmatrix} = 5.8385 \begin{bmatrix} 1 \\ 0 \\ 0.8894 \end{bmatrix} = 5.8385 X_8$$

$$A X_8 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8894 \end{bmatrix} = \begin{bmatrix} 5.8894 \\ 0 \\ 5.4470 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 0.9249 \end{bmatrix} = 5.8894 X_9$$

$$A X_9 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9249 \end{bmatrix} = \begin{bmatrix} 5.9249 \\ 0 \\ 5.6244 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 0.9493 \end{bmatrix} = 5.9249 X_{10}$$

$$A X_{10} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9493 \end{bmatrix} = \begin{bmatrix} 5.9493 \\ 0 \\ 5.7465 \end{bmatrix} = 5.9493 \begin{bmatrix} 1 \\ 0 \\ 0.9659 \end{bmatrix} = 5.9493 X_{11}$$

$$A X_{11} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9659 \end{bmatrix} = \begin{bmatrix} 5.9659 \\ 0 \\ 5.8296 \end{bmatrix} = 5.9659 \begin{bmatrix} 1 \\ 0 \\ 0.9771 \end{bmatrix} = 5.9659 X_{12}$$

$$X_{12} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9771 \end{bmatrix} = \begin{bmatrix} 5.9771 \\ 0 \\ 5.8857 \end{bmatrix} = 5.9771 \begin{bmatrix} 1 \\ 0 \\ 0.9847 \end{bmatrix} = 5.9771 X_{13}$$

$$A X_{13} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9847 \end{bmatrix} = \begin{bmatrix} 5.9847 \\ 0 \\ 5.9236 \end{bmatrix} = 5.9847 \begin{bmatrix} 1 \\ 0 \\ 0.9898 \end{bmatrix} = 5.9847 X_{14}$$

$$A X_{14} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9898 \end{bmatrix} = \begin{bmatrix} 5.9898 \\ 0 \\ 5.9489 \end{bmatrix} = 5.9898 \begin{bmatrix} 1 \\ 0 \\ 0.9932 \end{bmatrix} = 5.9898 X_{15}$$

$$A X_{15} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9932 \end{bmatrix} = \begin{bmatrix} 5.9932 \\ 0 \\ 5.9659 \end{bmatrix} = 5.9932 \begin{bmatrix} 1 \\ 0 \\ 0.9954 \end{bmatrix} = 5.9932 X_{16}$$

$$A X_{16} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9954 \end{bmatrix} = \begin{bmatrix} 5.9954 \\ 0 \\ 5.9772 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 0.9970 \end{bmatrix} = 5.9954 X_{17}$$

$$A X_{17} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9954 \end{bmatrix} = 5.9970 \begin{bmatrix} 1 \\ 0 \\ 0.9980 \end{bmatrix}$$

$\therefore \lambda_1 = 6;$ **eigen vector=** $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

B=A-6I= $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} take Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$BY_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -1Y_2$$

$$BY_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -2Y_3$$

$$BY_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Greatest eigen value of B=-2

Smallest eigen value of A=-2+6=4

$\lambda_1 + \lambda_2 + \lambda_3$ Trace =5-2+5=8

6+4+ λ_3 =8. λ_3 =-2.

The eigen values are 6, 4, -2

POSSIBLE QUESTIONS:

Part-B(5X6 = 30 Marks)

Answer all the questions:

1. Explain the types for solving boundary value problem
2. Solve the boundary value problem $\frac{d^2y}{dx^2} - y = 0$, with y(0) =0 and y(2) =3.62686.
3. Write the derivation of shooting method.
4. Solve the boundary value problem $y''(x) = y(x)$; y(0) = 0, y(1) = 1 by shooting method, taking m_0=0.7 and m_1=0.8

5. Solve the boundary value problem $y''(x) = y(x)$; y(0) = 0; y(1) = 1.1752 by shooting method, taking m_0=0.7 and m_1=0.8. .
6. Write the Derivative of Characteristic value Problems

7. Using Jacobi method , find the eigen value of $A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$.

8. Using Power method find all the eigen values are A=$\begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$

9. Using Power method find all the eigen values are
A=$\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

10. Using Jacobi method , find the eigen value of
$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

PART C- (1 x 10 =10 Marks)
(Compulsory)

1. Find the dominant eigen value and the corresponding eigen vector of A= $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
2. Solve the boundary value problem $\frac{d^2y}{dx^2}$ –y = 0 with y(0) = 0 and y(2) = 3.62686
3. Using power method find eigen value and eigen vector of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$

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Class : I M.Sc Mathematics **Semester : I**

Subject: Numerical Analysis **Subject Code: 19MMP103**

Unit IV

Part A (20x1=20 Marks)

Question	Opt 1	OPt 2	Opt 3	Opt 4	Answer
----------- method is used to determine numerically largest eigen value and the corresponding eigen vector of matrix A	Gauss Jordan	Power	Choleskey	Gauss Seidal	Power
Sum of the eigen values of a matrix is equal to the -------- of the diagonal element of the matrix.	sum	product	divide	square	sum
The power method, will work satisfactorily only if A has a --------- eigen value.	dominant	smallest	greatest	zero	dominant
If the coefficient matrix is diagonally dominant, then ---------- method converges quickly.	Gauss elimination	Gauss Jordan	Choleskey	Gauss Seidal	Gauss Seidal
If the eigen values of A are 1,3,4 then the dominant eigen value of A is --------	0	4	1	3	4
The iterative process continues till ------------- is secured	convergency	divergency	oscillation	infinite	convergency
------method is used to find the eigen values of a real symmetric matrix.	Gauss elimination	Gauss Jordan	Choleskey	Jacobi	Jacobi
A square matrix A is said to be orthogonal if ---------	$AA^1 = I$	AA^1 1 I	$AA^1 = 0$	$AA^1 = 1$	$AA^1 = I$
For an orthogonal matrix , if det A = ---------	0	1	±1	I	±1
For a real symmetric matrix,A all the eigen values are -------	real	imaginary	zero	one	real
------------ method is initial value problem methods.	Milne's	Euler	Shooting	Runge-Kutta	Shooting
----------------- methods are the implicit (or) explicit relation between the derivatives and the function values at the adjacent nodal points.	Shooting	Euler	Runge-Kutta	Finite difference	Finite difference

In numerical methods , the boundary problems are solved by using ------- method	Finite difference	Milne's	Euler	Runge-Kutta.	Finite difference	
In Finite difference method, the nodes x_{-1} and x_{n-2} are called --------- nodes	fictitious	normal	isolated	zero	fictitious	
In numerical methods, the boundary problems are solved by using --------- method.	Finite difference	Milne's	Euler	Runge-Kutta.	Finite difference	
----------------- method is initial value problem methods	Milne's	Euler	Shooting	Runge-Kutta	Shooting	
If all the non zero terms involve only the dependent variable u and u' then the differential equation is called ---------------------.	homogeneous	non homogeneous	linear	non linear	homogeneous	
In power method the element in vector in each iteration will become very large, to avoid this we divide each vector by its __________ component	smallest	largest	positive	negative	largest	
Power method generally gives the largest Eigen value of A provided the Eigen values are____.	equal	negative	positive	real and distinct	real and distinct	
If the eigen values of A are -3,3,1 then the dominant eigen value of A is________.	3	1	-3	No dominant eigen value	No dominant eigen value	
The smallest eigen value of A is the reciprocal of the dominant eigen value of____________	A^(-1)	det A	A^T	A	A^(-1)	
If the Eigen values of A are -6, 2, 4 then ______ is dominant.	2	4	-6	-2	-6	
If the eigen values of A are 4,3,1 then the dominant eigen value of A is________.	3	1	4	none	4	
The Power method is used for finding ________ eigen value	dominant	least	central	positive	dominant	

UNIT-V

SYLLABUS

Numerical Solution of Partial Differential Equations: Classification of Partial Differential Equation of the second order – Elliptic Equations. Parabolic equations: Explicit method – The Crank Nicolson difference method. Hyperbolic equations – solving wave equation by Explicit Formula.

NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

5.1 Introduction

Partial differential equations occur very frequently in science, engineering and applied mathematics. Many Partial differential equations cannot be solved by analytical methods in closed form solution, in most of the research work.

In fields like applied elasticity, theory of plates and shells, hydrodynamics, quantum mechanism etc., the research problems reduce to Partial differential equation. Since analytical solutions are available, we go in for numerical solutions of the Partial differential equations by various methods. Certain types of boundary value problems can be solved by replacing the differential equation by the corresponding differential equation and then solving the latter by a process of iteration. This method was devised and first used by I.F.Richardson and it was later improved by H.Liebmann.

5.2 Difference Quotients

A difference quotient is the Quotient obtained by dividing the difference between two values of a function by the difference between two corresponding values of the independent variable.

We know $$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$$

If h is small we approximate

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} = \frac{y(x+h) - y(x)}{(x+h) - x}$$

The right side is a difference quotient. Therefore the derivative is replaced by a difference quotient. In the case of partial derivatives, we have two independent variables and hence we consider the differences in both variables.

If y_0 is fixed and x is a variable, be Taylor series,

and $u_{yy}(x_0,y_0) = \dfrac{U(x_0,y_0+k) - 2U(x_0,y_0) + U(x_0,y_0-k)}{k^2}$

And the truncation error is $k_2/12\ u_{yyy}(x_0,\eta)$ where $y_0 - k < \eta < y_0 = k$.

5.3 Graphical representation of partial quotients

The xy plane is dived into a series of rectangles whose sides are parallel to x and y- axes such that $\Delta x=y$ and $\Delta y=k$. the gried points or mesh points lattice points are

(x,y),(x+h,y),(x+2h,y)…(x-h),y),(x-2h,y))…
If (x_i,y_i) is any grid point
$x_i=x_0+ih, y_i=y_0+jk$. If we take one corner as origin,
$x_i=ih, y_j, i,j=0,1,2\ldots$
y

				(x,y+2k)			
				(x,y+k)			
	(x-2hy)	(x-h,y)		(x,y)	(x+h,y)	(x+2h,y)	
				(x,y-k)			
				(x,y-2k)			

0

$\Delta x=y$ h

Coordinates of grid points

				(i,j+2)			
				(i,j+1)			
	(i-2,j)	(i-1,j)		(i,j)	(i,+1,j)	(i, +2,j)	
				(i,j-1)			

Mesh points denoted by suffices.

Here (x=ih, y=jk) is denoted by (i,j).

From the figures,

$$u_x = \frac{u_{i+1,j}-u_{i,j}}{h} \quad \text{(forward difference)} \quad \ldots (1)$$

$$u_x = \frac{u_{i,j}-u_{i-1,j}}{h} \quad \text{(back ward difference)} \quad \ldots (2)$$

$$u_x = \frac{u_{ij+1,j}-u_{i,j}}{k} \quad \text{(forward difference)} \quad \ldots (3)$$

$$u_x = \frac{u_{i,j}-u_{i,j,-1}}{k} \quad \text{(back ward difference)} \quad \ldots (4)$$

$$u_{xx} = \frac{u_{i+1,j}-2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{yy} = \frac{u_{ij+1}-2u_{i,j} + u_{i,j+1}}{k^2}$$

We can also write

$$u_x = \frac{u_{i+1,j}-u_{i-1,j}}{2h}$$

$$u_y = \frac{u_{i,j+1}-u_{i,j-1}}{2k}$$

5.4 Classification Of Partial Differential Equations Of The Second Order

The most general liner Partial differential equations of the second order can be write as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial y} + E\frac{\partial u}{\partial x} + F\, u=0$$

i.e.. $A\, u_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$

where A,B,C,D,E,F are in general functions of x and y.

the above equation of second order (liner) (1) is said to

(i) Elliptic at a point (x,y) in the plane if $B^2-4AC<0$

(ii) Parabolic if $B^2-4AC=0$

(iii) Hyperbolic if $B^2-4AC>0$

Examples:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \text{(Laplace equation in two dimension)}$$

Parabolic type:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \text{(one dimensional heat equation)}$$

Hyperbolic type:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2} \quad \text{(one dimensional wave equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \text{ (Poisson's equation)}$$

Example 1: Classify the following equations:

(i) $$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x^2} = 0$$

(ii) $x^2 f_{xx} + (1-y^2) f_{yy} = 0$

(i) Here A=1, B=2, C=1

$$B^2-4AC = -4x^2(1-y^2)$$
$$= 4x^2(y^2-1)$$

For all x except x=o, x^2 is +ve.

If $-1 < y<1, y^2 - 1$ is negative.

$\therefore B^2 - 4AC$ is –ve if $-1<y<1, x\neq 0$

$\therefore$ For $-\infty < x < \infty$ $(x \neq 0)$, $1<y<1$, the equation is elliptic;
For $-\infty < x < \infty, x \neq 0, y>1$, the equation is hyperbolic;

For x=0 for all y of for all x,y = ±1 the equation is parabolic.

$$B^2 - 4AC = 4\,(x + 2)^2 - 4\,(x + 1)(x + 3)$$
$$= 4\,[1] = 4 > 0$$

$\therefore$ the equation is hyperbolic at all points of the region.

Example 2: classify the following partial differential equations:

(i) $U_{xx} = 4u_{xy} + (x^2 + 4y^2)\,u_{yy} = \sin(x+y)$

(ii) $(x + 1)\,u_{xx} - 2\,(x + 2)\,u_{xy} + (x + 3)\,u_{yy} = 0.$

(iii) $X\,f_{xx} = y\,f_{yy} = 0,\ x> 0,\ y>0.$

Solution

(i) Here, $A = 1$, $B=4$, $C=(x^2 + 4y^2)$

$$B^2 - 4AC = 16\text{-}4\,(x^2 + 4y^2)$$
$$= 4[4 - x^2 - ay^2]$$

The equation is elliptic if $4 - x^2\ 4y^2<0$

i.e., $x^2 + 4y^2 > 4$

i.e., $\frac{x^2}{4} + \frac{y^2}{1} > 1$

$\therefore$ It is elliptic in the region outside the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} > 1.$$

It is hyperbolic inside the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} > 1.$$

It is parabolic on the ellipse

$$\frac{x2}{4} + \frac{y^2}{1} > 4$$

(ii) here, A = x + 1, B = - 2 (x + 2), C = x + 3

$B^2 - 4AC = 4(x + 2)^2 - 4(x + 1)(x + 3)$
$= 4 [1] = 4 > 0$

∴ the equation is hyperbolic at all points of the region.

(iii) A = x, B = 0, C =y
$B^2 - 4AC = -4xy$, (x>0, y>0 given)
= -ve
∴ It is elliptic for all x> 0, y > 0.

5.5 Elliptic equations

An important and frequently occurring elliptic Equations] is Laplace's Equation, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ i.e., } \Delta^2 u=0 \text{ or } u_{xx} + u_{yy} = 0:$$

Replacing the derivatives by difference equations we get,

$$\frac{u_{i+1,j}-2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{ij+1}-2u_{i,j} + u_{i,j+1}}{k^2} = 0$$

Taking k = h, (square mesh) in the above equation,

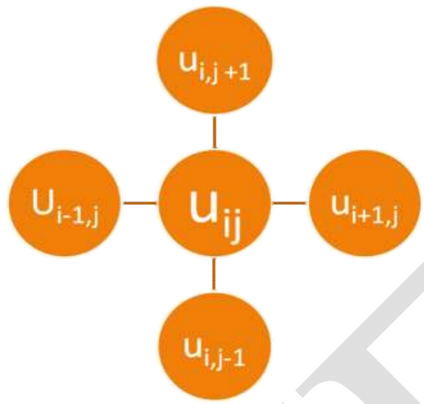
$$4u_{i,j} = u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}$$

$$\therefore u_{ij} = \tfrac{1}{4} [u_{i,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$$

That is the value of u at any interior point is the arithmetic mean of the values of u at the four lattice (Two of them are vertically just above and below and the other two in the horizontal line just after and below this point).

	h		
h		$u_{i,j+1}$	
	$U_{i-1,j}$	u_{ij}	$u_{i+1,j}$
		$u_{i,j-1}$	

Or



Schematic diagram

Central value = average of the other four values.

Diagonal five point formula

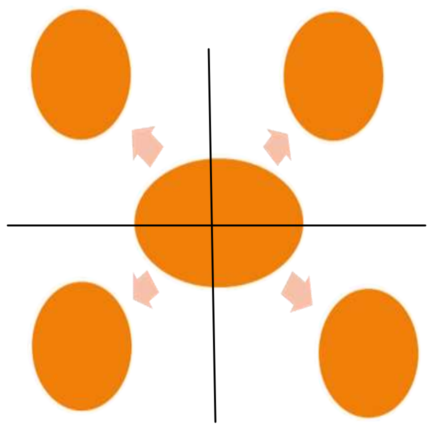
Instead of the formula (1) we can also used the formula

$u_{ij} = \frac{1}{4} [\ u_{i-1,j-1} + u_{i-1,j=1,} + u_{i,+1,j-1} + u_{i,+1,j+1} \ ]$ ...(2)

Which is called the Diagonal five point formula since this formula involves the values on Diagonals u_{ij}. Since the Laplace equation is invariant in any coordinate system, the formula remains same when the coordinate axes are rotated through 45 degree. But the error in the Diagonals formula is four times the error in the standard formula. Therefore, we always prefer the standard formula to the diagonals formula.

$u_{i,j+1}$		h	
		$h\sqrt{2}$	$u_{i+1,j}$
	$h\sqrt{2}$	$u_{i,j}$	
	$u_{i-,1,j-1}$		$u_{i+1,j-1}$

$u_{i+1,j+1}$ $u_{i-,1,j+1}$



$u_{i-,1,j-1}$ $u_{i+1,j-1}$

5.6 Solution Of Laplace's Equation :(By Liebmann's Iteration Process)

AIM: To solve the Laplace's Equation $u_{xx} + u_{yy} = 0$ (i) in bounded square region R with a boundary C when the boundary values of u are given on the bound ary(or at least at the grid points in the boundary).

Let us divide the square region into a network of sub- squares of side h

b_1 b_2 b_3 b_4 b_5

b_{16}		u_1	u_2	u_3
		u_4	u_5	u_6
b_{15}		u_7	u_8	u_9
b_{14}				

b_{13} b_{12} b_{11} b_{10} b_9

The values of u at the interior lattice of grid points are assumed to be u_1 u_2 $u_{3......}u_{9.}$ To start the iteration process, initially we find rough values at interior points and then we improved them by iterative process mostly using standard five point formula.

Find u_5 first: $u_5 = ¼\ (b_3 + b_7 + b_{11} + b_{15})$ (by standard five point formula – SFPF)

Knowing u_5 we find $u_{1,}$ $u_{3,}$ $u_{7,} u_{9,}$ that is the values at the centers of the four larger inner squares by using diagonal five point formula DFPF.

That is $u_1 = ¼\ (b_3+b_{15}+b_1+u_5)$

$u_3 = ¼\ (b_5+u_5+b_3+b_7)$

$u_7 = ¼\ (u_5+b_{13}+b_{11}+b_{15})$

$u_1 = ¼\ (b_7+b_{11}+b_9+u_5)$

the remaining 4 values u_2,u_4,u_6,u_8 can be got by using SFPF.

That is $u_2 = ¼\ (b_3+u_5+u_1+u_3)$

$u_4 = ¼\ (u_1+u_7+u_5+b_{15})$

$u_6 = ¼\ (u_3+u_9+ u_5+ b_7)$

$u_8= ¼\ (u_5+b_{11}+u_7+u_9)$

Now we know all the boundary values of u and rough values of u at every grid point in the interior of the region R. Now we iterate the process and improve the values of u with accuracy. Start with u_5 and proceed to get the values of $u_1,u_3….u_9$ always using SFPF taking into account the latest available values of u to use in the formula. The iterative formula is

$u_{i.j}^{(n+1)} = ¼\ (u_{ij}+_1^{(n)}\ u_{i-j}^{(n+1)}+u_{i,j-1}^{(n)}+u_{i,j+1}^{(n+1)})$ …I

Let the interior values of u at the grid points be u_1, u_2,…u_9. We will find the values of u at the interior mash as explained in the previous article. We will first the rough values of u and then proceed to refine them.

Example 1

Solve the equation $\nabla^2 u = 0$ for the following mesh, with boundary values as shown, using Leibmann's iteration procedure.

0	500	100 0	500	0
1000	u_1	u_2	u_3	1000
2000	u_4	u_5	u_5	2000
1000	u_7	u_8	u_9	1000
0	500	100	500	

Solution:

Take the central horizontal and vertical lines as AB and CD

Let $u_1, u_2 \dots u_9$ be the values of u at the interior grid points of the mesh.

The values of u on the boundary are symmetrical w.r.t. the lines AB and CD.

Hence the values of u inside the mesh will also be symmetrical about AB and CD.

$\therefore u_1=u_3=u_7=u_9; u_3=u_8; u_4=u_6$ and u_5 is not equal to any value.

$\therefore$ it is enough if we find u_1, u_2, u_4 and u_5.

Rough values of u's:

$u_5 = \frac{1}{4}(2000+2000+1000+1000)=1500$ (SFPF)

$u_1 = \frac{1}{4}(0+1500+1000+2000)=1125$ (DFPF)

$u_2 = \frac{1}{4}(1000+1500+1125+1125)=1187.5$ (SFPF)

$u_4 = \frac{1}{4}(u_1+u_5+u_7+2000) = 1437.5$ (SFPF)

$u_5 = \frac{1}{4}(2u_2+2u_4) = 1312.5$ (SFPF)

Here after we use only SFPF.

First iteration

$u_1^{(1)} = \frac{1}{4}(1000+500+1187.5+1437.5) = 1031.5$

$u_2^{(1)} = \frac{1}{4}(1000+1031.25+1031.25+1.341.25) = 1093.75$

$u_4^{(1)} = \frac{1}{4}(2000+2(1031.25)+1312.5) = 1343.75$

$u_5^{(1)} = \frac{1}{4}(2u_2 + 2u_4)$

$= \frac{1}{2}(1093.75+1343.75)=1218.75$

Now we go to second iteration

$u_1^{(1)} = 984.38$

$u_2^{(1)} = 1046.88$

$u_4^{(1)} = 1296.88$

$u_5^{(1)} = 1171.88$

<table>
<tr><td>0</td><td>500</td><td>1000</td><td>500</td><td>0</td></tr>
<tr><td>1000</td><td>u_1
1125
1031.25
984.38
960.94
949.22
943.36
940.43
939.1
938.3
937.7
937.6
937.6</td><td>u_2
1187.5
1093.75
1046.88
1023.44
1011.72
1005.86
1002.93
1001.6
1000.4
1000.2
1000.1
1000.1</td><td>u_3</td><td>1000</td></tr>
<tr><td>2000</td><td>u_4
1437.5
1343.75
1296.88
1273.44
1261.72
1255.86
1252.93
1250.8
1250.2
1250.1
1250.1</td><td>u_5
1500
1312.5
1218.75
1171.88
1148.44
1136.72
1130.86
1127.93
1126.6
1125.8
1125.2
1125.1
1125.1</td><td>u_5</td><td>2000</td></tr>
</table>

1000	u_7	u_8	u_9	1000
0	500	1000	500	0

Hence solution is

$u_1 = 937.6$, $u_2 = 1000.1$, $u_4 = 1250.1$, $u_5 = 1125.1$

Example 2 : Evaluate the function u (x,y) satisfying $\nabla^2 u = 0$, at the lattice points given the boundary values as follows.

D1000	1000	1000	1000C
2000	u_1	u_2	500
2000	u_3	u_4	0
A1000	500	0	0B

Solution Method 1:

We have

$4u_1 = 1000+2000+ u_3+ u_2= 3000+ u_2+ u_3$ ...(1)

$4u_2 = 1500+ u_1+ u_4$ ....(2)

$4u_3 = 2500+ u_4+ u_1$ ….. (3)

$4u_4 = u_2+ u_3$ …. (4)

i.e., $4u_1 - u_2- u_3 = 3000$ ….(5)

$u_1 - 4u_2 + u_4 = - 1500$ ....(6)

$u_1 - 4u_3+ u_4 = - 2500$ ….(7)

$u_2+ u_3 - 4 u_4 = 0$ …..(8)

We eliminate u_1 from (5) and (6) and (7)

$15 u_2 - u_3 - 4u_4 = 9000$ ....(9)

$4u_2- 4u_3 = - 1000$ ....(10)

We eliminate u_4 from (8) and (9)

$4u_2 - 2u_3 = 9000$ ….(11)

From (10) and (11), $u_2 = 791.7$, $u_3 = 1041.7$

From (5), $u_1 = 1208.4$ and $u_4 = 458.4$

Method 2:

Instead getting 4 equations in u_1, u_2,u_3 and u_4, and solving them for u's , we can assume some value for u_4 (or any other u)and proceed iterative procedure; we can take $u_4 = 0$ and proceed or take a value of $u_4 = 400$ (guess this seeing the values of u on the vertical line through u_2, u_4).

Rough values:

$u_1 = (1000+2000+1000+400+)/4=1100$ (DFPF)

$u_2 = ¼ (u_1+,u_4 + 1500) = 750$ (SFPF)

$u_3 = ¼ (u_1+,u_4 + 2500) = 1000$ (SFPF)

$u_4 = ¼ (u_2 + u_3) = 437.$ (SFPF)

First iteration: here after we adopt only SFPF.

$u_1^{(1)} = ¼ (750+1000+3000) =1187.5$

$u_2^{(1)} = ¼ (1187.5+437.5+1500) =781.25$

$u_3^{(1) = ¼}(1187.5+437.5+2500) = 1031.25$

$u_4^{(1) = ¼}(781.25+1032.25)= 453.25$

Second iteration

$u_1^{(2)} = ¼ (781.25+1031.25+3000)=1203.125$

$u_2^{(2)} = ¼ (1203.125+453.125+1500)=789.1$

$u_3^{(2) =} ¼ (1203.125+453.125+2500)=1039.1$

$u_4^{(2) = ¼}(789.1+1039.1)=457.1$

Third iteration

$u_1^{(3)} = ¼ (789.1+1039.1+3000)=1207.1$

$u_2^{(3)} = ¼ (1207.1+457.1+1500)=791.1$

$u_3^{(3) =} ¼ (1207.1+457.1+2500)=1041.1$

$u_4^{(3)} = \frac{1}{4}(791.1+1041.1)=458.1$

Fourth iteration

$u_1^{(4)} = \frac{1}{4}(791.1+1041.1+3000)=1208.1$

$u_2^{(4)} = \frac{1}{4}(1208.1+458.1+1500)=791.6$

$u_3^{(4)} = \frac{1}{4}(1208.1+458.1+2500)=1041.6$

$u_4^{(4)} = \frac{1}{4}(791.1+1041.1)=458.3$

Fifth iteration

$u_1^{(5)} = \frac{1}{4}(791.6+1041.6+3000)=1208.3$

$u_2^{(5)} = \frac{1}{4}(1208.3+458.3+1500)=791.7$

$u_3^{(5)} = \frac{1}{4}(1208.3+458.3+2500)=1041.7$

$u_4^{(5)} = \frac{1}{4}(791.7+1041.7)=458.4$

We are getting result correct to one decimal place. Further the increase in the value is <0.1.

We stop here. One more iteration will give you the decision to make.

$\therefore u_1 = 1208.3, \ u_2 = 791.7 \ u_3 = 1041.7, u_4 = 458.4$

Note : instead of taking $u_4 = 400$, if we have started with $u_4 = 0$, we require more iteration. So avoid this excess labor, judiciously assume the value.

Example 3

Solve $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary conditions as shown below. Iterate until the maximum difference between successive values at any grid point is less than. 0.001

A	1	2	B
1	u_1	u_2	2
2	u_3	u_4	1
D	2	1	C

Solution:

From the above figure, we see that it is symmetrical about the diagonals AB and BD.

Let u_1, u_2, u_3, u_4 be the values at interior grid points.

By symmetrical $u_1 = u_4$, $u_2 = u_3$.

Therefore, we need to find only two values u_1 and u_2.

A	1	2	B
1	u_1	u_2	2
2	u_3	u_4	1
D	2	1	C

Since the corner values are not known, assuming u_2 , we will get u_1 but assume u_2. Judiciously seeing the values of u in the vertical line through u_2. Therefore let $u_2 = 1.6$ (please note u_2 is 1/3 distance of the side length from the volue 2)

Rough values estimation:

$u_2 = 1.6$

$u_1 = ¼ (1+1+1.6+1.6) = 1.3$

u_2 ¼ $(2+2+1.3+1.3) = 1.65$

Method 1

First iteration

$u_1=1/4(2+2u_2)=1/2(1+u_2)=1.325$

$u_2=1/4(4+2u_1)=1/2(2+u_1)=1..6625$

Second iteration

$u_1=1/2(1+2u_2) =1/2(1+1.6625) =1.33125$

$u_2=1/2(2+u_1)=1/2(3.33125)=1.6656$

Third iteration

$u_1=1/2(1+1.6656)=1.3328$

$u_2=1/2(3.3328)=1.6664$

Fourth iteration

$u_1=1/2(1+1.6664)=1.3332$

$u_2=1/2(3.3332)=1.6666$

Method 2

$u_1=1/2(1+u_2)$

$u_2=1/2(2+u_1)$

Solving

$u_1=4/3=1.3333$

and $u_2=5/3=1.6666$

The difference between 2 consecutive values of u_1 is 0.0004 and that between 2 consecutive values of u_2 is 0.0002 which are less than 0.001. Hence, u_1 = 1.3332 and u_2 = 1.666.

5.7 Poisson's Equation

An Equation of the form $\nabla^2 u = f(x,y)$

(i.e) $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = f(x,y)$ …………..(1)

is called as Poisson equation where f(x,y) is a function of x and y only.

We will solve the above equation numerically at the points of the square mesh , replacing the derivativ by difference coefficients . Taking x=ih,y=jk=jh (here) the differential equation reduces to

$(u_{i+1,j}-2u_{i,j}+u_{i-1,j})/(h^2) + (u_{i,j-1}-2u_{i,j}+u_{i,j+1})/(h^2) = f(ih,jh)$

(i.e) $u_{i+1,j}+u_{i-1,j}+u_{i,j-1}-4u_{i,j}+u_{i,j+1} = h^2 f(ih,jh)$ ……(2)

By applying the above formula at each mesh point , we get a system of linear equation in the pivotal values i,j.

We can follow this method easily by working out the following example.

Example 1 Solve $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides x=0, x=3, y=3 with u=0 on the boundary and mesh length 1 unit.

Solution u=0

A				B
	D	u_1E	u_2	u=0
	F	u_3G	u_4	C

The P.D.E is $\nabla^2 u = -10(x^2+y^2+10)$

using the theory, (here h=1)

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4\,u_{i,j} = -10(i^2+j^2+10)$$

Applying the formula at D(i=1,j=2)

$0+0+ u_2 + u_2 - 4\,u_1 = -10(15) = -150$

$u_2 + u_3 - 4\,u_1 = 150$ ……..(3)

Applying at E(i=2,j=2)

$u_1 + u_4 - 4\,u_2 = -180$ ………(4)

Applying at F(i=1,j=1)

$u_1 + u_4 - 4u_3 = -120$ ……….(5)

Applying at G(i=2,j=1)

$u_2 + u_3 - 4\,u_4 = 10(2^2+1^2+10) = -150$ ……….(6)

We can solve the equation 3,4,5,6 either by elimination or by Gauss-Seidel method.

Method 1.

(5)-(4) gives (Eliminate u_1)

$4(u_2 + u_3) = 60$

$u_2 + u_3 = 15$

Eliminate u_1 from (3) and (4); (3)+ 4(4) gives,

$-15u_2 + u_3 + 4\,u_4 = -870$

Adding (6) and (8)

$-7u_2 + u_3 = -510$

From (7),(9) adding $u_2 = 82.5$

Using (7), $u_3 = u_2 - 15 = 82.5 - 15 = 67.5$

put in (3), $4u_1 = 300$

Therefore $u_1 = 75$

$4u_4 = 150+150;$

$u_4 = 75.$

$u_1 = u_4 = 75$, $u_2 = 82.5$, $u_3 = 67.5$

Note:

Since the differential equation is unchanged when x,y are interchanged and boundary conditions are also same after interchange x and y, the result will be symmetrical about the line y=x

Therefore $u_4 = u_1$

If we use this idea the 4 equations would have reduced to 3 equations namely,

$u_2 + u_3 - 4u_1 = 150$

$2u_1 - 4u_2 = -180$

$2u_1 - 4u_3 = -120$

$u_2 + u_3 - 4u_1 = 150$

Solving will be easier now.

Method 2

We can use Gauss-Seidel method to solve.

$u_1 = 1/2(150 + u_2 + u_3)$

$u_2 = 1/4(2u_1 + 180)$

$u_3 = 1/4(2u_1 + 120)$

The tabulated values are:

	1	2	3	4	5	6	7	8	9	10
$u_4 = u_1$	**-**	**37.5**	**65.56**	**72.64**	**74.41**	**74.85**	**74.96**	**74.99**	**75**	**75**
u_2	0	63.75	77.79	81.32	82.21	82.43	82.48	82.5	82.5	82.5
u_3	0	48.75	62.78	66.32	67.21	67.43	67.48	67.5	67.5	67.5

We get the values after 9 iteration.

Example 2

Solve $\nabla^2 u = 8x^2y^2$ for the square mesh given u=0 on the 4boundaries dividing the square into 16 sub-squares of length 1 unit.

Solution

u=0

	u_1	u_2	u_3	
u=0	u_4	u_5	u_6	u=0
	u_7	u_8	u_9	
		u=0		

Take the coordinate system with origin at centre of the square. Since the P.D.E and boundary conditions are symmetrical about x,y axes and y=x we have, $u_1=u_3=u_7 = u_9$

$u_2=u_5=u_6 = u_8.$

We need to find u_1 ,u_2 , u_5 only.(here h=1)

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4\, u_{i,j}=h^2\, f(ih\,,jh) = f(i,j)$ …1

At(i=-j , j=-1), we have, $u_2 + u_4 - 4u_1 = 8(-1)^2(-1)^2 = 8$

$u_2 - 2u_1 = 4$ …2

At(i=0, j=1) $u_1 + u_3 + u_5 - 4u_2 = 0$

$2u_2 + u_5 - 4u_2 = 0$

At(i=0, j=0) $u_2 + u_4 + u_{6} + u_8 - 4u_5 = 0$

$4u_2 - 4u_5 = 0$

$u_2 - u_5 = 0$ …4

From (2),

$u_1 = 1/2(u_2 - 4)$

From (4)

$u_5 = u_2$

Using in (3), $u_2 - 4 - 4u_2 - u_2 = 0.$

$u_2 = -2$; $u_5 = -2$; $u_1 = -3$

$u_1 = -3$, $u_2 = -2 = u_5$

5.8 Parabolic Equations

Bender-Schmidt Method

The one dimensional heat equation , namely

$\partial u/ \partial t = \alpha^2 \partial^2 u/ \partial t^2$, where $\alpha^2 = k/ pc$ is an example of parabolic equation.

Setting $\alpha^2 = 1/a$, the equation becomes ,

$\partial^2 u/ \partial x^2 - a\, \partial u/ \partial t = 0.$

Here A=1, B=0 , C=0 .Therefore $B^2 - 4AC = 0$, it is parabolic at all the points.

AIM: Our aim is to solve this by the method of finite differences .To solve $u_{xx} = a\, u_t$ …..(1)

With boundary conditions

$u(0,t) = T_0$ ……….(2)

$u(l,t) = T_1$ ……….(3)

and with initial condition $u(x,0) = f(x),\ 0<x<l$ ….(4)

We a spacing h for the variable a x and a spacing k for the time

variable t.

$u_{xx} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2$ and $u_t = (u_{i,j+1} - u_{i,j})/k$

Hence (1) becomes

$(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2 = a\,(u_{i,j+1} - u_{i,j})/k$

Therefore, $u_{i,j+1} - u_{i,j} = k/ah^2\,(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$

$= \lambda\,(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$, where $\lambda = k/ah^2$.

(i.e) $\mathbf{u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i-1,j}}$ **…….(5)**

writing the boundary conditions as

$u_{0,j} = T_0$ ………...(6)

$u_{n,j} = T_l$ ……...(7)

where nh = l and the initial condition as

$u_{i,0} = f(ih)$, i= 0,1,…. …..(8)

U is known at t= 0.

Equation (5) facilitates to get the value of u at x= ih and time t_{j+k}.

Equation (5) is called explicit formula.

It is valid if $0 < \lambda \le ½$.

If we take , $\lambda = ½$, the coefficient of $u_{i,j}$ vanishes.

Hen $\mathbf{u_{i,j+1} = (1/2)\,[u_{i-1,j} + u_{i+1,j}]}$ **………..(9)**

when $\lambda = ½ = k / ah^2$ (i.e) $k = ah^2 / 2$

(i.e) the value of u at $x=x_i$ at $t= t_{j+1}$ is equal to the average of the values of u the surrounding points x_{i-1} and x_{i+1} at the previous time tj.

Equation (9) is called Bender-Schmidt recurrence equation.

This is valid only if $k = ah^2 / 2$.(so, select k like this)

Example 1

Solve $(\partial^2 u / \partial x^2) - 2(\partial u/\partial t)=0$ given u(0,t)=0 , u(4,t)=0 , u(x,0)=x(4-x). Assume h=1. Find the values of u upto t=5.

Solution.

$u_{xx} = a\, u_t$ Therefore a=2

To use Bender-Schmidt equation, $k=a/2\ h^2 = 1$.

Step size in time=k=1. The values of $u_{i,j}$ are tabulated below.

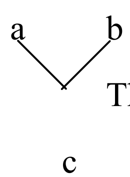
j \ i	0	1	2	3	4
0	0	3	4	3	0 1 ←u(x, 0) =x(4-x)
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

Analysis: Range for x: (0,4); for t: (0,5)

U(x,0)=x(4-x). This gives u(0,0) = 0 , u(1,0) = 3 , u(2,0) = 4 , u(3,0)=3 , u(4,0) = 0

For all t, at x=0, u=0 and for all t at x=4 , u=0.

Using these values we fill up coloum under x=90 , x= 4 and row against t=0.



This means c=(a+b)/2

The values of u at t = 1 are written by seeing the values of u at t=0 and using the average formula.

Example 2

Solve $(\partial^2 u / \partial x^2) =(\partial u/\partial t)=0$ given $u(0,t)=0$, $u(4,t)=0$, $u(x,0)=x(4-x)$ assuming h=k=1. Find the values of u upto t=5.

Solution

If we want to use Bender-Schmidt formula, we should have $k=a/2\ h^2$.

Here h=k=1 , a=1. These values do not satisfy the condition.. hence we cannot employ Bender-Schmidt formula.

Hence we go to the basic equation,

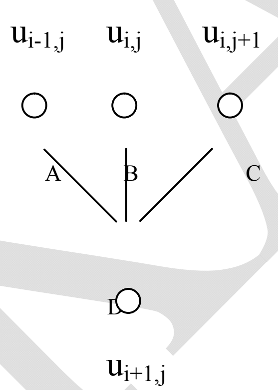
$u_{i,j+1}= \lambda\ u_{i+1,j} + (1-2\lambda)\ u_{i,j} + \lambda\ u_{i-1,j}$ …(1)

Now $\lambda= k/ah^2 = 1/\ 1\times1 = 1$

Hence (1) reduce to,

$u_{i,j+1}= u_{i+1,j} - u_{i,j} + u_{i-1,j}$

That is,



Value of u at D= value of u at A+ value of u at C – value of u at B.

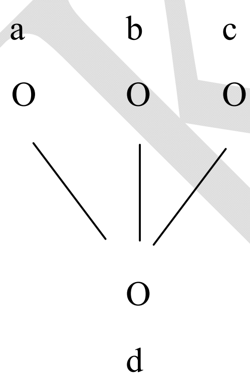
Now we are ready to create the table values.

→ x director

j	0	1	2	3	4
0	0	3	4	3	0
1	0	1	2	1	0
2	0	1	0	1	0
3	0	-1	2	-1	0
4	0	3	-4	3	0
5	0	7	10	-7	0

↓ t direction

Here



This figure means d=a+c-b

Note: Since λ=1 is used in the working , it violates the condition for use of Explicit formula. So the solution is not stable and it is not a practical problem. Such question should be avoided, since unstable solution do not exist.

Example 3

Solve $u_t = u_{xx}$ subject to u(0,t)=0 , u(1,t)=0 and u(x,0)=sinπx, 0<x<1.

Solution.

Since h and k are not given we will select them properly and use Bender-Schmidt Method.

$k=a/2\ h^2 = ½\ h^2$

Therefore a=1.

Since range of x is (0,1) , take h=0.2.

Hence $k=(0.2)^2 / 2 = 0.02$.

The formula is $u_{i,j+1} = ½ (u_{i-1,j} , u_{i+1,j})$

u(0,0) =0 , u(0.2,0) = sinπ/5 = 0.5878

u(0.4,0) = sin 2π/5 = 0.9511; u(0.6,0) = sin 3π/5 = 0.9511; u(0.8,0) = sin 4π/5 = 0.5878

We form the tablex → direction h=0.2

j \ x	0	0.2	0.4	0.6	0.8	1.0
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.1	0	0.2037	0.3296	0.3296	0.2037	0

t direction

k=0.02

Example 4

Given $(\partial^2 f / \partial x^2) + (\partial f/\partial t)=0$ given f(0,t)=0 , f(5,t)=0 , $f(x,0)=x^2(25-x^2)$ Find f in the range taking h=1and upto 5 seconds.

Solution.

To use Bender-Schmidt Method.

$k=a/2\ h^2$

Therefore a=1, h=1.

Therefore k= ½

Step time = ½ = t

Step size =1 = h

f(1,0) = 24; f(2,0) = 84; f(3,0) = 144; f(4,0)=144;f(5,0)=0.

The formula is $u_{i,j+1}= \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$,

j \ i	0	1	2	3	4	5
0	0	24	84	144	144	0
1/2	0	42	84	144	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0
4.5	0	11.4843	17.0625	1805859	10.5469	0
5	0	8.5312	15.03	13.804	9.2929	0

			51	7		

Crank-Nicholson Difference Method

To solve this by the method of finite differences.

To solve $u_{xx} = a\, u_t$ …..(1)

With boundary conditions

$u(0,t) = T_0$ ……….(2)

$u(l,t) = T_1$ ……….(3)

and with initial condition $u(x,0) = f(x), 0<x<l$….(4)

We a spacing h for the variable a x and a spacing k

for the time variable t.

At $u_{i,j}$, $u_{xx} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2$

and at $u_{i,j+1}$, $u_{xx} = (u_{i+1,j+1} - 2u_{i,j} + u_{i-1,j+1})/h^2$

Taking the average of these two values,

$u_{xx} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j} + u_{i-1,j+1})/2h^2$

Using $u_t = (u_{i,j+1} - u_{i,j}) / k$, equation (1) reduces to

$(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j} + u_{i-1,j+1})/2h^2 = a\,(u_{i,j+1} - u_{i,j}) / k$

Setting $k / ah^2 = \lambda$, the above equation reduces to

$\frac{1}{2}\lambda\, u_{i+1,j+1} + \frac{1}{2}\lambda\, u_{i-1,j+1} - (\lambda+1)\, u_{i,j+1} =$

$- (\frac{1}{2})\lambda\, u_{i+1,j} - (1/2)\lambda\, u_{i-1,j} + (\lambda-1)\, u_{i,j}$ ………(I)

Equation (I) is called Crank – Nicholson difference scheme or method.

Note 1: A convenient choice of λ makes the scheme simple. Setting $\lambda = 1$ (i.e) $k= ah^2$ the Crank – Nicholson method

$u_{i,j+1} = (1/4)\, [u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}]$ ….(II)

In problems , we will use this simplified formula subject to $k = a\, h^2$.

Note 2:

The Crank – Nicholson scheme converges for all values of λ.

Example 1

Solve by Crank-Nicholson Method the equation $u_{xx} = u_t$ subject to u(x,0)=0 , u(0,t)=0 and u(1,t)=t for two time steps.

Solution.

X ranges from 0 to 1. Take h=1/4; here a=1

$K=ah^2$ to use simple form

$K=1(1/4)^2 = 1/16$

$$u_{i,j+1}=1/4 \quad \{u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i-1,j}\}$$

x → direction

↓t

j \ i	0	0.25	0.5	0.75	1
0 ↓	0	0	0	0	0
1/16	0	u_1	u_2	u_3	1/16
2/16	0	u_4	u_5	u_6	2/16
3/16	0				3/16

Let the unknown represents by u_1 , u_2 , u_3 , …

The boundary conditions are marked in the table against t=0 , x=0 and x=1.

Using the scheme(1),

u_1 = ¼ (0+0+0+ u_2)

$u_2 = ¼ (0+0+ u_1 + u_3)$

$u_3 = ¼ (0+0+ u_2 +1/16)$

That is

$u_1 = ¼ u_2$

$u_2 = ¼ (u_1 + u_3)$

$u_3 = ¼ (u_2 +1/16)$

Solving the three equations we get u_1 , u_2 , u_3.

Substituting u_3 , u_1 values in u_2

$u_2 = ¼ (1/4 u_2 +1/4(u_2+1/16))$

u_2=1/224 (0.0045) , u_1=1/896 (0.0011) , u_3 = 0.0168

Similarly u_4 , u_5 , u_6 can be got again getting 3 equations in 3 unknown u_4 , u_5 , u_6.

We get u_4 = 0.005899 , u_5 = 0.01913 , u_6 = 0.05277.

Example 2

Using Crank-Nicholson's scheme, solve $u_{xx} = 16u_t$, $0<x<1$, $t>0$ given u(x,0)=0 , u(0,t)=0 and u(1,t)=100t.

Solution.

Here h=1/4 ; a=16,K=ah^2 to use simple form

K=16(1/4)^2 =1.

$u_{i,j+1}=1/4 \{u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i-1,j} \}$

x → direction

j	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

↓t

$u_1 = ¼ (0+0+0+ u_2)$

$u_2 = ¼ (0+0+ u_1 + u_3)$

$u_3 = ¼ (0+0+ u_2 +100)$

That is

$u_1 = ¼ u_2$

$u_2 = ¼ (u_1 + u_3)$

$u_3 = ¼ (u_2 +100)$

Solving the three equations we get $u_1 , u_2 , u_3.$

Substituting u_3 , u_1 values in u_2

$u_2 = ¼ (1/4 (2u_2 +100) = 1/8 u_2 + 25/4$

$u_2=50/7 = 7.1429$, $u_1=1.7857$, $u_3 = 26.7857$.

The values are

$u_1=1.7857$; $u_2= 7.1429$; $u_3 = 26.7857$

5.9 Hyperbolic Equations

The wave equation in one dimension (vibration of strings) is

$a^2 \partial^2u/ \partial x^2 - \partial^2u/ \partial t^2 =0$, (i.e) $a^2 u_{xx} – u_{tt} =0$

Here $A = a^2$, $B = 0$, $C = -1$. Therefore $B^2 – 4 AC = +ve$.

Hence the equation is hyperbolic.

Let us solve this equation by reducing it to difference equation.

AIM : Solve $a^2 u_{xx} – u_{tt} =0$ …………….(1)

together with the boundary conditions u (0,t) = 0……(2)

u (l,t) = 0 ………(3)

and the initial conditions

u(x,0) = f(x)…….(4)

$u_t(x,0) = 0$ ……..(5)

Assuming $\Delta x = h$, $\Delta t = k$, we have

$u_{xx} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) / h^2$

$u_{tt} = (u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) / k^2$.

substituting these values in (1) ,

$$[a^2 / h^2](u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - (1/k^2)(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = 0.$$

(i.e)

$$\lambda^2 a^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - (u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = 0$$

Where $\lambda = k / h$.

$$u_{i,j+1} = 2(1 - \lambda^2 a^2) u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \ldots\ldots.(6)$$

To make the equation simpler , select **λ** such that

$1 - \lambda^2 a^2 = 0$, (i.e) $\lambda^2 = 1/a^2 = k^2/h^2$, (i.e) $k = h/a$.

Under this selection of $\lambda^2 = 1/a^2$ the equation (6) reduces to the simplest form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \ldots\ldots\ldots\ldots..(7)$$

Equation (6) is called an Explicit scheme or explicit formula to solve the wave equation.

Equation (7) gives a simpler form under the condition k= h/a.

Note 1: The boundary condition $u(0,t) = 0$ gives the values of u along the line x=0 , that all u =0.

The boundary condition $u(l,t) = 0$ gives the values of u along the line x=l , i.e. all u=0 along this line.

Note 2: Initial condition $u(x,0) = f(x)$ becomes

$u(i,0) = f(ih)$, i=1,2…

This gives the value of u along t=0 for various values of i.

$u(i,0) = f(ih) = f_i$.

Note 3: The initial condition $u_t(x,0) = 0$ gives $u_{i,1} = u_{i,-1}$. which implies

$$u_{i,1} = (1/2)\,(u_{i-1,0} + u_{i+1,0})$$

Note 4 : If $1-\lambda^2 a^2 < 0$, $\lambda a > 1$, (i.e) $a\,k / h > 1$, the solution is unstable. If $ka / h = 1$, it is stable and if $ka / h < 1$, it is stable but the accuracy of the solution decreases as ak / h decreases.

That is , for $\lambda = 1/a$ the solution is stable.

Example 1

Solve numerically, $4u_{xx}=u_{tt}$ with the boundary conditions $u(0,t)=0$, $u(4,t)=0$ and the initial conditions $u_t(x,0) = 0$ and $u(x,0)=x(4-x)$, taking $h=1$.(for 4 time steps)

Solution.

Since $a^2 =4$, $h=1$, $k=h/a = ½$

Taking $k=1/2$, we use the formula

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

From $u(0,t)=0 \Rightarrow u$ along $x=0$ are all zero.

From $u(4,t)=0 \Rightarrow u$ along $x=4$ are all zero.

$u(x,0)=x(4-x)$ implies that

$u(0,0) = 0$, $u(1,0)=3$, $u(2,0)=4$, $u(3,0) = 3$.

Now, we fill up the row $t=0$ using the above values

$u_i(x,0) = 0$, implies $u_{i,1} = (u_{i+1,0} + u_{i-1,0})/ 2$

Now we draw the table; for that we require

$u_{1,1}=(u_{2,0} + u_{0,0}) / 2 = (4+0)/2 = 2$

$u_{2,1}=(u_{3,0} + u_{1,0}) / 2 = (3+3)/2 =3$

$u_{3,1}=(u_{4,0} + u_{2,0}) / 2 = 2$

$u_{4,1}= 0$.

t	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
		(3+0-3)	(2+2-4)	(3+0-3)	
1.5	0	-2	-3	-2	0
2	0	-3	-4	-3	0
2.5	0	-2	-3	-2	0
3	0	0	0	0	0
3.5	0	2	3	2	0
4	0	3	4	3	0

Period is 4 seconds or 8(k) = 8(1/2) = 4 secs.

Example 2

Solve numerically, $25u_{xx} - u_{tt} = 0$ for u at a pivotal points, given u(0,t)=0, u(5,t)=0 and the initial conditions $u_t(x,0) = 0$ and u(x,0)=2x for $0 \le x \le 2.5$

$= 10 - 2x$ for $2.5 \le x \le 5$.

for one half period of viberation.

Solution.

Since $a^2 = 25$

Period of viberation = 2l/a = (2×5)/ 5 = 2 seconds, Half period = 1 second.

therefore we want the values upto t=1 second

k=h/a = 1/5 , taking h=1

step size in t-direction = 1/5.

The explicit scheme is

$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ ………(1)

Boundary conditions are

$u(0,t)=0$ or $u_{0,j}$=o.

$u(5,t)=0$ $u_{5,j}=0$ for all j.

$u_i(x,0) = 0$, implies $u_{i,1} = (u_{i+1,0} + u_{i-1,0})/ 2$

$u(x,0)=2x$ for $0\leq x \leq 2.5$

$= 10 - 2x$ for $2.5 \leq x \leq 5$.

u(0,0) = 0, u(1,0)=2 , u(2,0)=4 , u(3,0) = 4, u(4,0)=2, u(5,0)=0.

$u_{1,1}=(u_{2,0} + u_{0,0}) / 2 = (4+0)/2 = 2$

$u_{2,1}=(u_{3,0} + u_{1,0}) / 2 = (3+3)/2 =3$

$u_{3,1}=(u_{4,0} + u_{2,0}) / 2 = 3$

$u_{4,1}= (u_{5,0} + u_{3,0}) / 2 = 2.$

t	0	1	2	3	4	5
0 (j=0)	0	2	4	4	2	0
t=1/5 (j=1)	0	2	3	3	2	0
t=2/5	0	1	1	1	1	0

(j=2)						
t=3/5 (j=3)	0	-1	-1	-1	-1	0
t=4/5 (j=4)	0	-2	-3	-3	-2	0
2.5	0	-2	-3	-2		0
3	0	0	0	0		0
3.5	0	2	3	2		0
4	0	3	4	3		0

Note 1.

First fill up all value against j=0 and j=1 and then go for filling up other rows using formula(1)

Note 2.

In using $u_t(x,0)=0$ we used central difference approximation for first derivative

$$u_t = (u_{i,j+1} - u_{i,j-1}) / 2k$$

But instead, we could also use

$u_t = (u_{i,j+1} - u_{i,j}) / k$ in which case $u_t(x,0)=0 \Rightarrow u_{i,1} = u_{i,0}$

In other words the value of u corresponding to j=0 and j=1 are same. If this is adopted, then the value of u against t=0 and t=0.5 in the table of worked will be same.

t \ x	0	1	2	3	4
0	0	3	4	3	0
0.5	0	3	4	3	0

This will make all the entries of the table different from the one given.

This assumption of $u_i(x,0)$ makes the value of u same at t=0; and t=0.5 which is not acceptable in practice.

Hence, we do not adopt this definition $u_t(t,0)$ and so we accepted the central difference approximation which is more reasonable.

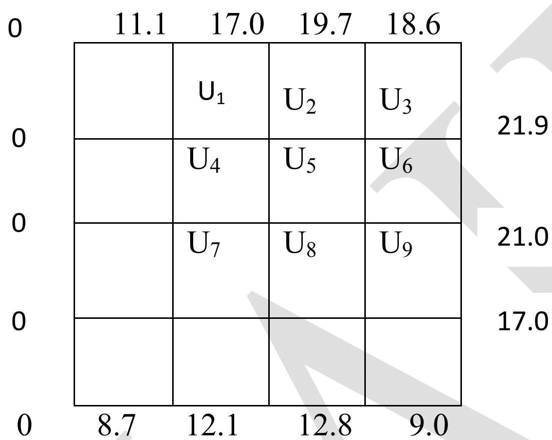
POSSIBLE QUESTIONS:

Part-B(5X6 = 30 Marks)

Answer all the questions:

1. Explain the classification of Partial differential Equations.

2. Find by Libmann's method the values at the interior points of the square region of the harmonic function u whose boundary values are as shown in the following figure.



3. Solve $\nabla^2 u = 8x^2y^2$ for square mesh given u=0 on the 4 boundaries dividing the square into 16 sub-squares of length 1 unit.

4. Write the Derivative for Crank Nicholson method.

5. Using Crank-Nicholson's scheme, solve $u_{xx} = 16u_t$, 0<x<1, t>0 given u(x,0) = 0, u(0,t) = 0, u(1,t) = 100t. Compute u for one step in t direction taking h= /4.

6. Solve by Crank Nicholson method the equation $u_{xx} = u_t$ subject to u(x, 0)=0, u(0, t)=0 & u(1, t)=t for two time steps.

7. Solve $u_t = u_{xx}$ subject to u(0,t) = 0, u(1,t) =0 and u(x,0) = sin πx, 0<x<1.

8. Use Bender Schmidt recurrence relation to solve the equation $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$ with the conditions u(x, 0)=4x-x^2, u(0, t)=u(4, t)=0. Assume h=0.1. find the values of u upto t=5.

9. Write the derivative of Bender Schmidt method to solve parabolic equations.

10. Solve the poisson equation $u_{xx} + u_{yy} = -10(x^2+y^2+10)$.

(

PART C- (1 x 10 =10 Marks)
(Compulsory)

1. Solve numerically $4u_{xx} = u_{tt}$ with the boundary condition, u(0, t)=u(4, t)=0 and the initial condition $u_t(x, 0)=0$ & u(x, 0)=x(4-x), taking h=1(for 4 time steps).

2. Solve $u_{xx}+ u_{yy} = 0$ over the square mesh of side 4 units;satisfying the following Boundary conditions:
   i)$u(0,y) = 0$ for $0 \le y \le 4$
   ii)$u(4,y) = 12+y$ for $0 \le y \le 4$
   iii) $u(x,0) = 3x$ for $0 \le x \le 4$
   iv) $u(x,4) = x^2$ for $0 \le x \le 4$

3. Solve $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides x =0, y =0,x =3, y =3 with u = 0 on the boundary and mesh length one unit.

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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Class : I M.Sc Mathematics | **Semester : I**

Subject: Numerical Analysis | **Subject Code: 19MMP103**

Unit V

Part A (20x1=20 Marks)

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
A -------quotient is obtained by dividing the difference between two values of a function by the difference between two corresponding values of the independent variable	difference	partial	normal	binomial	difference
If B^2-4AC = 0, then the differential equation is said to be______.	parabolic	elliptic	hyperbolic	equally spaced	parabolic
If B^2-4AC > 0, then the differential equation is said to be______.	parabolic	elliptic	hyperbolic	equally spaced	hyperbolic
If B^2-4AC < 0, then the differential equation is said to be______.	parabolic	elliptic	hyperbolic	equally spaced	elliptic
The linear partial differential equation of second order can be written as ---------	$Au_{xx} + Bu_{xy} + Cu_{xy} + Du_x + Eu_y + Fu=0$	$Au_{xx} + Bu_{xy} + Cu_{xy} + Du_x =1$	$Au_{xx} + Bu_{xy} + Cu_{xy} = 0$	$Du_x + Eu_y + Fu=0$	$Au_{xx} + Bu_{xy} + Cu_{xy} + Du_x + Eu_y + Fu=0$
The linear partial differential equation of second order is said to be elliptic at a point (x,y) in the plane if -------	$B^2 – 4AC =0$	$B^2 – 4AC <0$	$B^2 – 4AC>0$	$B^2 – 4AC\ ^1\ 0.$	$B^2 – 4AC <0$
The linear partial differential equation of second order is said to be parabolic at a point (x,y) in the plane if -------	$B^2 – 4AC =0$	$B^2 – 4AC <0$	$B^2 – 4AC>0$	$B^2 – 4AC\ ^1\ 0.$	$B^2 – 4AC =0$
The linear partial differential equation of second order is said to be hyperbolic at a point (x,y) in the plane if -------	$B^2 – 4AC =0$	$B^2 – 4AC <0$	$B^2 – 4AC>0$	$B^2 – 4AC\ ^1\ 0.$	$B^2 – 4AC>0$
The differential equation xuxx + uyy = 0 is said to elliptic if --------	x < 0	x = 0	$x\ ^1\ 0$	x > 0	x > 0

The differential equation xuxx + uyy = 0 is said to hyperpolic if --------	x < 0	x = 0	x ¹ 0	x > 0	x < 0
. The differential equation $xu_{xx} + u_{yy} = 0$ is said to parapolic if --------	x < 0	x = 0	x ¹ 0	x > 0	x = 0
The error in the diagonal formula is ------ times the error in the standard formula	3	2	5	4	4
------- method is used to solve the Laplace's equation.	Crank-Nicholson difference	Liebmann's iteration	Bender-Schmidt	Laplace	Liebmann's iteration
An equation of the form Ñ²u = f(x,y) is called as ---------equation.	laplace	parabolic	poisson	elliptic	poisson
------- method is used to solve the parabolic equation.	Crank-Nicholson difference	Liebmann's iteration	Bender-Schmidt	Explicit scheme	Crank-Nicholson difference
The ----------- scheme converges for all values of l. .	Liebmann's iteration	Bender-Schmidt	Explicit scheme	Crank-Nicholson difference	Crank-Nicholson difference
The wave equation in one dimension is----------	hyperbolic	parabolic	poisson	elliptic	hyperbolic
-------method is used to solve the wave equation	Liebmann's iteration	Bender-Schmidt	Crank-Nicholson difference	Explicit scheme	Explicit scheme
Liebmann's iteration process is used to solve laplace equation in ------- dimension	one	two	three	zero	two
Classify the equation $u_{xx} + 2u_{xy} + 4u_{yy} = 0$ is ----------	hyperbolic	parabolic	poisson	elliptic	elliptic
If u is harmonic, then it satisfies D^2u =-------	0	1	2	3	0
An important and frequently occurring elliptic equation is --------- equation	laplace	parabolic	hyperbolic	elliptic	laplace
Classift the equation $f_{xx} – 2f_{xy} = 0$ as-------	laplace	parabolic	hyperbolic	elliptic	hyperbolic
Classift the equation $f_{xy} – f_x = 0$ as-------	hyperbolic	parabolic	poisson	elliptic	hyperbolic
Classift the equation $u_{xx}=u_t$ as-------	laplace	parabolic	hyperbolic	elliptic	parabolic
The number of condition required to solve the Laplace equation is --------	4	5	3	1	4

Crank-Nickolson's method is used to solve the -------- equation of the form $u_{xx}= au_t$	laplace	parabolic	hyperbolic	elliptic	parabolic
Explicit method is used to solve the ---------- equation	one dimensional	poisson	laplace	wave	wave
One dimensional heat equation is the example of _______ equation.	Laplace	Poisson	Parabolic	Hyperbolic	Parabolic
One dimensional wave equation is the example of _______ equation.	elliptic	rectangular hyperbolic	Parabolic	Hyperbolic	Hyperbolic
Two dimensional heat equation is the example of _______ equation.	elliptic	rectangular hyperbolic	Parabolic	Hyperbolic	elliptic
Poisson equation is an example of ________equation.	Parabolic	elliptic	hyperbolic	rectangular hyperbolic	elliptic
_____________ equation is an example of parabalic equation.	One dimensional heat	One dimensional wave	Poisson	Laplace	One dimensional heat
__________equation is an example of hyperbolic equation.	One dimensional heat	One dimensional wave	Poisson	Laplace	One dimensional wave
_________ equation is an example of elliptic equation.	One dimensional heat	One dimensional wave	Poisson	Laplace	Poisson
____________ process is used to solve two dimensional heat equations	Explicit	Bender-Schmidt	Crank-Nicolson	Liebmanns iteration	Liebmanns iteration
One dimensional heat equation can be solved using _______ method.	Newtons	Crank-Nicolson	elimination	Liebmanns iteration	Crank-Nicolson
One dimensional heat equation can be solved using _______ method.	Newtons	Bender-Schmidt	elimination	Liebmanns iteration	Bender-Schmidt
One dimensional wave equationcan be solved using _______ method.	Explicit	Bender-Schmidt	Crank-Nicolson	Liebmanns iteration	Explicit
Poisson equationcan be solved using _______ method.	Explicit	Bender-Schmidt	Crank-Nicolson	Liebmanns iteration	Liebmanns iteration
Liebmanns iteration process is used to solve ----------- equations.		One dimensional heat	two dimensional heat		two dimensional heat

____________ equation can be solved using Crank-Nicolson method.			One dimensional heat		One dimensional heat
_________equation can be solved using Bender-Schmidt method.			One dimensional heat		One dimensional heat
__________ equationcan be solved using Explicit method.		One dimensional heat	One dimensional wave		One dimensional wave
_________ equationcan be solved using Liebmanns iteration method.		One dimensional heat	Poisson		Poisson
Crank-Nicolson method is also called as ______ method.	Explicit	Implicit	elimination	reduction	Implicit
Bender-Schmidt method is also called as ______ method.	Explicit	Implicit	elimination	reduction	Explicit

Reg. No..................................

[17MMP103]

KARPAGAM UNIVERSITY

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COIMBATORE – 641 021
(For the candidates admitted from 2017 onwards)

M.Sc., DEGREE EXAMINATION, NOVEMBER 2017

First Semester

MATHEMATICS

NUMERICAL ANALYSIS

Time: 3 hours   Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a. By dividing the range into 10 equal parts. Evaluate $\int_0^{\pi} \sin x \, dx$ by Trapezoidal Rule

Or

b. Find the 1st two derivation of x and y for x=50 using newton forward method

x	50	51	52	53	54	55	56
y	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

22. a. Solve $x+y+z=1; 4x+3y-z=6; \ 3x+5y+3z=4$ by factorization method

Or

b. Solve $27x+6y-z=85; 6x+15y+2z=72; \ x+y+54z=110$ by Gauss Jacobi method

23. a. Determine the value of y when x=0.1 given that y(0)=1 and $y'=x^2+y$.

Or

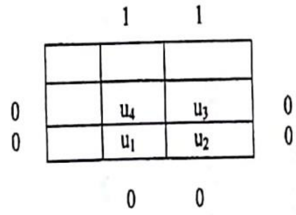
b. Solve the initial value problem $y'=3x+\frac{y}{2}$ with the condition y(0)=1 and y(0.2) using Runge-kutta IVth order with h=0.05

24. a. Solve the boundary value problem $\frac{d^2y}{dx^2}-y=0$ with y(0)=0, y(2)=3.62686. The exact solution of this problem is y = sinhx.

Or

b. Using power method, find all eigen values of $A=\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

25. a. Solve the equation $u_{xx}+u_{yy}=0$ in the domain of figure below by Jacobi's method.



Or

b. Solve $\nabla^2 u = 8x^2y^2$ for square mesh given u=0 on the 4 boundaries dividing the square into 16 sub squares of length 1 unit.

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Using Stirlings formula, find y(1.22) from the following table

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749	0.99957	0.99385	0.97385

Reg. No................................

[14MMP103]

KARPAGAM UNIVERSITY
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(For the candidates admitted from 2014 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2014

First Semester

MATHEMATICS

NUMERICAL ANALYSIS

Time: 3 hours

Maximum : 60 marks

PART – A (10 x 2 = 20 Marks)
Answer any TEN Questions

1. Write down the Newton raphson's method formula.
2. Write down the formula for $f''(x)$ at $x = x_n$ in Newton's backward difference formula.
3. Define Deflated polynomial.
4. Solve the following system by Gauss Elimination method.
   $4x - 3y = 11$
   $3x + 2y = 4$
5. What is the condition for convergence of Gauss Seidal Method?
6. What do you mean by diagonally dominant?
7. What are the two types of Euler's method?
8. Write down the Adam's corrector formula.
9. Write down the third order Runge-kutta method.
10. What are the three kinds of boundary conditions?
11. What is mean by homogeneous?
12. Define shooting method in the boundary value problem.
13. Write down the general linear partial differential equation of second order.
14. Write down the diagonal five point formula for u_9.
15. Write down the hyperbolic equation.

PART B (5 X 8= 40 Marks)
Answer ALL the Questions

16. a. Find the real root of the equation $x^3 - 3x^2 + 7x - 8 = 0$. correct to 3 decimal places by Newton Raphson method.

Or

b. Perform two iterations of the Bairstow's method to extract a quadratic x^2+px+q from the polynomial $P_3(x) = x^3+x^2-x+2 = 0$. Use the initial approximation $P_0 = -0.9$, $q_0 = 0.9$.

17. a. Solve the following system of equations by Gauss-seidal method

$$5x - 2y + z = -4$$
$$x + 6y - 2z = -1$$
$$3x + y + 5z = 13$$

Or

b. Solve the following equations using LU decomposition method.

$$x + y + z = 1$$
$$4x + 3y - z = 6$$
$$3x + 5y + 3z = 4.$$

18. a. Given $y' = -y$, $y(0) = 0$. Determine the value of y at x = (0.01)(0.01)(0.04) by Euler method.

Or

b. using Adam's moulton predictor- corrector method. Find y(1.4) if y satisfies $\frac{dy}{dx} = \frac{1-xy}{x^2}$, $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$.

19. a. Solve by finite difference method the boundary value problem $y''(x) - y(x) = 2$. where $y(0) = 0$ and $y(1) = 1$ taking $h = 1/4$.

Or

b. Using Power method find all the eigenvalues are $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

20. **Compulsory : -**

Solve by crank-Nicholson method the equation $u_{xx} = u_t$ subject to $u(x, 0) = 0$ $u(0, t) = 0$ and $u(1, t) = t$ for two time steps.

Reg. No.................................

[15MMP103]

KARPAGAM UNIVERSITY

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COIMBATORE – 641 021
(For the candidates admitted from 2015 onwards)

M.Sc., DEGREE EXAMINATION, NOVEMBER 2015

First Semester

MATHEMATICS

NUMERICAL ANALYSIS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a) Find the real root of $x^3-2x-5=0$ using Newton's method and correct to four decimal places.

Or

b) Using trapezoidal rule, evaluate $\int_{-1}^{1}\frac{dx}{1+x^2}$ taking 8 intervals.

22. a) Solve $x+3y+3z=16$
$x+4y+3z=18$
$x+3y+4z=19$ by Gauss elimination method.

Or

b) Solve the following equations by Gauss-Sidel method
$4x+2y+z=14$
$x+5y-z=10$
$x+y+8z=20$

23. a) Evaluate y (1.2) correct to 3 decimal places by modified Euler method given that $\frac{dy}{dx}=(y-x^2)^1$ y(1)=0 taking h=0.2

Or

b) Apply the fourth order Runge - Kutta method, to find an approximate value of y when x=0.2 given that $y^1=x+y$, y(0) = 1with h=0.2

24. a) Explain briefly boundary value problems with an example.

Or

b) Find the Eigen values of matrix A, $A=\begin{pmatrix}3 & -1 & 0\\ -2 & 4 & -3\\ 0 & -1 & 1\end{pmatrix}$

25. a) Explain types of partial differential equations.

Or

b) Explain the text : PARABOLIC EQUATIONS

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ in $0<x<t, t\geq 0$ given that u(x , 0) =20 , u (5 , t) = 100
Compute u for the time step with h=1 by Crank – Nicholson method.

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[18MMP103]

KARPAGAM ACADEMY OF HIGHER EDUCATION

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M.Sc., DEGREE EXAMINATION, NOVEMBER 2018

First Semester

MATHEMATICS

NUMERICAL ANALYSIS

Time: 3 hours — Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a. Solve $f(x,y)=x^2+y^2-4=0$ and $g(x,y)=y+e^x-1=0$ starting with an approximate solution (1,-1.7) by Newton's method.

Or

b. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Simpson's 1/3rd rule with h=1.

22. a. Solve by Gauss Elimination method
$3x+4y+5z=18,\ 2x-y+8z=13,\ 5x-2y+7z=20$

Or

b. Solve the following system by Gauss-seidel method correct to four decimal places.
$x+y+54z=110,\ 27x+6y-z=85,\ 6x+15y+2z=72$

23. a. Obtain the values of y at x=0.1, 0.2 using Runge Kutta method of second order for the differential equation $y'=-y$, given that $y(0)=1$.

Or

b. Using Milne's method, find y(4.4), given that $xy'+y^2-2=0$, given $y(4)=1$, $y(4.1)=1.0049$, $y(4.2)=1.0093$ & $y(4)=1.0143$.

24. a. Explain shooting method.

Or

b. Using power method, find the dominant eigen value of $A=\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

25. a. Evaluate the pivotal values of the following equation taking h=1 and up to one half of the period of the oscillation if $u_{xx}=u_{tt}$, given that $u(0,t)=u(5,t)=0$, $u(x,0)=x^2(5-x)$ and $u(x,0)=1$.

Or

b. Solve $u_{xx}+u_{yy}=0$ over the square mesh of side 4 units, satisfying boundary conditions
i. $u(0,y)=0$ for $0\le y\le 4$ — iii. $u(x,0)=3x$ for $0\le x\le 4$
ii. $u(4,y)=12+y$ for $0\le y\le 4$ — iv. $u(x,4)=x^2$ for $0\le x\le 4$

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. By LU decomposition method, solve
$5x-2y+z=4,\ 7x+y-5z=8,\ 3x+7y+4z=10.$
