KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester – I

		L	T	P	C
18MMP106	**MECHANICS**	**4**	**0**	**0**	**4**

Course Objectives

To enable to dealsome of the key ideas of classical mechanics and to understand the concept of generalized coordinates.

Course Outcomes

On successful completion of this course students will be able to:

- Understand the concept of the D'Alembert's principle .
- Derive the Lagrange's equation for holomonic and non holomonic constraints.
- Attain the applications of Lagrange's formulation .
- Classify Scleronomic and Rheonomic systems .
- Solve the problems of Hamilton equations of motion .

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UNIT I

Survey of Elementary principles: Constraints - Generalized coordinates, Holonomic and non-holonomic systems, Scleronomic and Rheonomic systems. D'Alembert's principle and Lagrange's
equations – Velocity – dependent potentials and the dissipation function – some applications of the Lagrange formulation.

UNIT II

Variation principles and Lagrange's equations: Hamilton's principle – Some techniques of calculus of variations – Derivation of Lagrange's Equations from Hamilton's principle – Extension ofHamilton's principle to non-holonomic systems – Conservation theorems and symmetry properties.

UNIT III

Hamilton Equations of motion: Legendre Transformations and the Hamilton Equations of motion-canonical equations of Hamilton – Cyclic coordinates and conservation theorems – Routh'sprocedure - Derivation of Hamilton's equations from a variational principle – The principle of leastaction.

UNIT IV

Canonical transformations: The equations of canonical transformation – Examples of Canonical transformations – Poission Brackets and other Canonical invariants – integral invariants of Poincare, Lagrange brackets.

UNIT V

Hamilton Jacobi Theory: Hamilton Jacobi equations for Hamilton's principle function – Harmonic oscillator problem - Hamilton Jacobi equation for Hamilton's characteristic function – Separation of variables in the Hamilton-Jacobi equation.

SUGGESTED READINGS

TEXT BOOK

1. Goldstein. H. (2007), Classical Mechanics Third Edition, Narosa Publishing House, New Delhi.

REFERENCES

1. Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.
2. Gelfand, I. M., and Fomin, S. V., (2003), Calculus of Variations, Prentice Hall, New Delhi.
3. Loney, S. L., (2015). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: Dr.S.Sowmiya
Subject Name: Mechanics **Sub.Code:19MMP106**
Semester: I **Class: IM.Sc Mathematics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
	UNIT-I		
1	1	Survey of Elementary principles	S1: Ch1: P.No: 1-10
2	1	Constraints	S1:Ch2: P.No:11-12
3	1	Generalized coordinates	S1:Ch2: P.No: 12-13
4	1	Holonomic and non-holonomic systems, Scleronomic and Rheonomic systems	S1:Ch2: P.No: 13-15
5	1	D'Alembert's principle	S1:Ch2: P.No: 16-17
6	1	Lagrange's equations	S1:Ch3: P.No: 17-21
7	1	Velocity- dependent potentials and the dissipation function	S1:Ch3: P.No: 21-24
8	1	Some applications of the Lagrange formulation.	S1:Ch3: P.No:25-29
9	1	Recapitulation and Discussion of possible questions	
Total No. of Lecture hours planned-9 Hours			
	UNIT-II		
1	1	Variation principles and Lagrange's equations Hamilton's principle	S1:Ch2: P.No: 35-37
2	1	Some techniques of calculus of variations	S1:Ch2: P.No: 37-43
3	1	Derivation of Lagrange's Equations from Hamilton's principle	S1:Ch2: P.No: 43-45
4	1	Extension of Hamilton's principle to non-holonomic systems	S1:Ch2: P.No: 45-50
5	1	Conservation theorems and symmetry properties.	S1:Ch2: P.No: 54-56
6	1	Continuation of Conservation theorems and symmetry properties.	S1:Ch2: P.No: 56-58

7	1	Continuation of Conservation theorems and symmetry properties.	S1:Ch2: P.No: 58-60
8	1	Continuation of Conservation theorems and symmetry properties.	S1:Ch2: P.No: 61-63
9	1	Recapitulation and Discussion of possible questions	
Total No. of Lecture hours planned-9 Hours			
UNIT-III			
1	1	Legendre Transformations	S1:Ch8: P.No: 334-345
2	1	Continuation of Legendre Transformations	S1: Ch8: P.No: 335-338
3	1	Hamilton Equations of motion	S1: Ch8: P.No: 338-339
4	1	canonical equations of Hamilton	S1:Ch8 P.No: 339-343
5	1	Cyclic coordinates and conservation theorems	S1: Ch8: P.No: 343-347
6	1	Routh's procedure	S1: Ch8: P.No: 347-349
7	1	Derivation of Hamilton's equations from a variational principle	S1: Ch8: P.No: 353-356
8	1	The principle of least action.	S1: Ch8: P.No: 356-362
9	1	Recapitulation and Discussion of possible questions	
Total No. of Lecture hours planned-9 Hours			
UNIT-IV			
1	1	The equations of canonical transformation	S1:Ch9: P.No: 368-375
2	1	Examples of Canonical transformations	S1:Ch9: P.No: 375-376
3	1	Examples of Canonical transformations	S1:Ch9: P.No: 376-377
4	1	Poission Brackets and other Canonical invariants	S2:Ch7: P.No: 388-389
5	1	Continuation of Poission Brackets and other Canonical invariants	S1:Ch9: P.No: 389-391
6	1	integral invariants of Poincare	S2:Ch2: P.No: 55-56
7	1	Continuation of integral invariants of Poincare	S2:Ch2: P.No: 56-58
8	1	Lagrange brackets	S1:Ch9: P.No: 391-397
9	1	Recapitulation and Discussion of possible questions	
Total No. of Lecture hours planned-9 Hours			

UNIT-V			
1	1	Hamilton Jacobi Theory	S1:Ch10: P.No:430-431
2	1	Hamilton Jacobi equations for Hamilton's principle function	S1:Ch10: P.No: 431-433
3	1	Continuation of Hamilton Jacobi equations for Hamilton's principle function	S1:Ch10: P.No: 433-434
4	1	Harmonic oscillator problem	S2:Ch5: P.No: 76-78
5	1	Continuation of Harmonic oscillator problem	S2:Ch5: P.No: 79-81
6	1	Continuation of Harmonic oscillator problem	S1:Ch10: P.No: 434-440
7	1	Hamilton Jacobi equation for Hamilton's characteristic function	S1:Ch10: P.No: 440-444
8	1	Separation of variables in the Hamilton-Jacobi equation.	S1:Ch10: P.No: 444-445
9	1	Recapitulation and discussion of possible questions on unit V	
10	1	Discussion of Previous year ESE question paper	
11	1	Discussion of Previous year ESE question paper	
12	1	Discussion of Previous year ESE question paper	
Total No. of Lecture hours planned-12 Hours			
Total Planned Hours			**48**

SUGGESTED READINGS

1. Goldstein. H. (2007), Classical Mechanics Third Edition, Narosa Publishing House, New Delhi.
2. Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.
3. Gelfand, I. M., and Fomin, S. V., (2003), Calculus of Variations, Prentice Hall, New Delhi.
4. Loney, S. L., (2015). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

UNIT-I

Survey of Elementary principles: Constraints - Generalized coordinates, Holonomic and non-holonomic systems, Scleronomic and Rheonomic systems. D'Alembert's principle and Lagrange'sequations – Velocity – dependent potentials and the dissipation function – some applications of the Lagrange formulation.

The motion of material bodies formed the subject of some of the earliest research pursued by the pioneers of physics. From their efforts there has evolved a vast field known as analytical mechanics or dynamics, or simply, mechanics. In the present century the term "classical mechanics" has come into wide use to denote this branch of physics in contradistinction to the newer physical theories, especially quantum mechanics. We shall follow this usage, interpreting the name to include the type of mechanics arising out of the special theory of relativity. It is the purpose of this book to develop the structure of classical mechanics and to outline some of its applications of present-day interest in pure physics. Basic to any presentation of mechanics are a number of fundamental physical concepts, such as space, time, simultaneity, mass, and force. For the most part, however, these concepts will not be analyzed critically here; rather, they will be assumed as undefined terms whose meanings are familiar to the reader.

1.1 ■ MECHANICS OF A PARTICLE

Let $\mathbf{r}$ be the radius vector of a particle from some given origin and $\mathbf{v}$ its vector velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{1.1}$$

The *linear momentum* $\mathbf{p}$ of the particle is defined as the product of the particle mass and its velocity:

$$\mathbf{p} = m\mathbf{v}. \tag{1.2}$$

In consequence of interactions with external objects and fields, the particle may experience forces of various types, e.g., gravitational or electrodynamic; the vector sum of these forces exerted on the particle is the total force **F**. The mechanics of the particle is contained in *Newton's second law of motion*, which states that there exist frames of reference in which the motion of the particle is described by the differential equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \equiv \dot{\mathbf{p}}, \tag{1.3}$$

or

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}). \tag{1.4}$$

In most instances, the mass of the particle is constant and Eq. (1.4) reduces to

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}. \tag{1.5}$$

where **a** is the vector acceleration of the particle defined by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}. \tag{1.6}$$

The equation of motion is thus a differential equation of second order, assuming **F** does not depend on higher-order derivatives.

A reference frame in which Eq. (1.3) is valid is called an *inertial* or *Galilean system*. Even within classical mechanics the notion of an inertial system is something of an idealization. In practice, however, it is usually feasible to set up a coordinate system that comes as close to the desired properties as may be required. For many purposes, a reference frame fixed in Earth (the "laboratory system") is a sufficient approximation to an inertial system, while for some astronomical purposes it may be necessary to construct an inertial system by reference to distant galaxies.

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time. Equation (1.3) directly furnishes the first of these, the

Conservation Theorem for the Linear Momentum of a Particle: If the total force, $\mathbf{F}$*, is zero, then* $\dot{\mathbf{p}} = 0$ *and the linear momertum,* $\mathbf{p}$*, is conserved.*

The angular momentum of the particle about point O, denoted by $\mathbf{L}$, is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{1.7}$$

where $\mathbf{r}$ is the radius vector from O to the particle. Notice that the order of the factors is important. We now define the *moment of force* or *torque* about O as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}. \tag{1.8}$$

The equation analogous to (1.3) for $\mathbf{N}$ is obtained by forming the cross product of $\mathbf{r}$ with Eq. (1.4):

$$\mathbf{r} \times \mathbf{F} = \mathbf{N} = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}). \tag{1.9}$$

Equation (1.9) can be written in a different form by using the vector identity:

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}), \tag{1.10}$$

where the first term on the right obviously vanishes. In consequence of this identity, Eq. (1.9) takes the form

$$\mathbf{N} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{L}}{dt} \equiv \dot{\mathbf{L}}. \tag{1.11}$$

Note that both $\mathbf{N}$ and $\mathbf{L}$ depend on the point O, about which the moments are taken.

As was the case for Eq. (1.3), the torque equation, (1.11), also yields an immediate conservation theorem, this time the

Conservation Theorem for the Angular Momentum of a Particle: If the total torque, $\mathbf{N}$*, is zero then* $\dot{\mathbf{L}} = 0$*, and the angular momentum* $\mathbf{L}$ *is conserved.*

Next consider the work done by the external force $\mathbf{F}$ upon the particle in going from point 1 to point 2. By definition, this work is

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}. \tag{1.12}$$

For constant mass (as will be assumed from now on unless otherwise specified), the integral in Eq. (1.12) reduces to

$$\int \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt = \frac{m}{2} \int \frac{d}{dt}(v^2)\, dt,$$

and therefore

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2). \tag{1.13}$$

The scalar quantity $mv^2/2$ is called the kinetic energy of the particle and is denoted by T, so that the work done is equal to the change in the kinetic energy:

$$W_{12} = T_2 - T_1. \tag{1.14}$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force (and the system) is said to be *conservative*. An alternative description of a conservative system is obtained by imagining the particle being taken from point 1 to point 2 by one possible path and then being returned to point 1 by another path. The independence of W_{12} on the particular path implies that the work done around such a closed circuit is zero, i.e.

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0. \tag{1.15}$$

Physically it is clear that a system cannot be conservative if friction or other dissipation forces are present, because $F \cdot ds$ due to friction is always positive and the integral cannot vanish.

By a well-known theorem of vector analysis, a necessary and sufficient condition that the work, W_{12}, be independent of the physical path taken by the particle is that $\mathbf{F}$ be the gradient of some scalar function of position:

$$\mathbf{F} = -\nabla V(\mathbf{r}), \tag{1.16}$$

where V is called the *potential*, or *potential energy*. The existence of V can be inferred intuitively by a simple argument. If W_{12} is independent of the path of integration between the end points 1 and 2. it should be possible to express W_{12} as the change in a quantity that depends only upon the positions of the end points. This quantity may be designated by $-V$, so that for a differential path length we have the relation

$$\mathbf{F} \cdot d\mathbf{s} = -dV$$

or

$$F_s = -\frac{\partial V}{\partial s},$$

which is equivalent to Eq. (1.16). Notc that in Eq. (1.16) we can add to V any quantity constant in space, without affecting the results. Hence *the zero level of V is arbitrary*.

For a conservative system, the work done by the forces is

$$W_{12} = V_1 - V_2. \tag{1.17}$$

Combining Eq. (1.17) with Eq. (1.14), we have the result

$$T_1 + V_1 = T_2 + V_2, \tag{1.18}$$

which states in symbols the

Energy Conservation Theorem for a Particle: If the forces acting on a particle are conservative, then the total energy of the particle, $T + V$, is conserved.

The force applied to a particle may in some circumstances be given by the gradient of a scalar function that depends explicitly on both the position of the particle and the time. However, the work done on the particle when it travels a distance ds,

$$\mathbf{F} \cdot d\mathbf{s} = -\frac{\partial V}{\partial s}\, ds,$$

is then no longer the total change in $-V$ during the displacement, since V also changes explicitly with time as the particle moves. Hence, the work done as the particle goes from point 1 to point 2 is no longer the difference in the function V between those points. While a total energy $T + V$ may still be defined, it is not conserved during the course of the particle's motion.

1.2 ■ MECHANICS OF A SYSTEM OF PARTICLES

In generalizing the ideas of the previous section to systems of many particles, we must distinguish between the *external forces* acting on the particles due to sources outside the system, and *internal forces* on, say, some particle i due to all other particles in the system. Thus, the equation of motion (Newton's second law) for the ith particle is written as

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i, \tag{1 19}$$

where $\mathbf{F}_i^{(e)}$ stands for an external force, and $\mathbf{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\mathbf{F}_{ii}$, naturally, is zero). We shall assume that the $\mathbf{F}_{ij}$ (like the $\mathbf{F}_i^{(e)}$) obey Newton's third law of motion in its original form: that the forces two particles exert on each other are equal and opposite. This assumption (which does not hold for all types of forces) is sometimes referred to as the *weak law of action and reaction*

Summed over all particles, Eq. (1.19) takes the form

$$\frac{d^2}{dt^2}\sum_i m_i \mathbf{r}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i\neq j}} \mathbf{F}_{ji}. \tag{1.20}$$

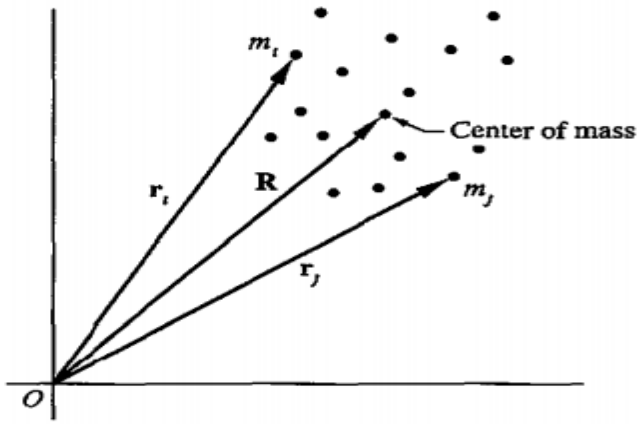
The first sum on the right is simply the total external force $\mathbf{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\mathbf{F}_{ij} + \mathbf{F}_{ji}$ is zero. To reduce the left-hand side, we define a vector $\mathbf{R}$ as the average of the radii vectors of the particles, weighted in proportion to their mass:

$$\mathbf{R} - \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}. \tag{1.21}$$

The vector $\mathbf{R}$ defines a point known as the *center of mass*, or more loosely as the center of gravity, of the system (cf. Fig. 1.1). With this definition, (1.20) reduces to

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}, \tag{1.22}$$

which states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Purely internal forces, if the obey Newton's third law, therefore have no effect on the



motion of the center of mass. An oft-quoted example is the motion of an exploding shell—the center of mass of the fragments traveling as if the shell were still in a single piece (neglecting air resistance). The same principle is involved in jet and rocket propulsion. In order that the motion of the center of mass be unaffected, the ejection of the exhaust gases at high velocity must be counterbalanced by the forward motion of the vehicle at a slower velocity.

By Eq. (1.21) the total linear momentum of the system,

$$\mathbf{P} = \sum m_i \frac{d\mathbf{r}_i}{dt} = M\frac{d\mathbf{R}}{dt}, \tag{1.23}$$

is the total mass of the system times the velocity of the center of mass. Consequently, the equation of motion for the center of mass, (1.23), can be restated as the

Conservation Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved.

We obtain the total angular momentum of the system by forming the cross product $\mathbf{r}_i \times \mathbf{p}_i$ and summing over i. If this operation is performed in Eq. (1.19), there results, with the aid of the identity, Eq. (1.10),

$$\sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i) = \dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i\neq j}} \mathbf{r}_i \times \mathbf{F}_{ji}. \tag{1.24}$$

The last term on the right in (1.24) can be considered a sum of the pairs of the form

$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}, \tag{1.25}$$

1.2 Mechanics of a System of Particles

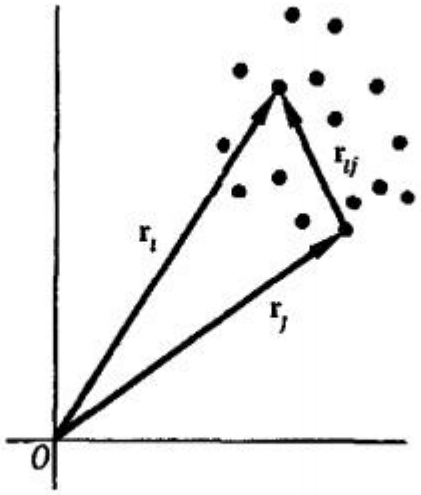


FIGURE 1.2 The vector r_{ij} between the ith and jth particles.

using the equality of action and reaction But $\mathbf{r}_i - \mathbf{r}_j$ is identical with the vector $\mathbf{r}_{ij}$ from j to i (cf. Fig. 1.2), so that the right-hand side of Eq. (1.25) can be written as

$$\mathbf{r}_{ij} \times \mathbf{F}_{ji}.$$

If the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles—a condition known as the *strong law of action and reaction*—then all of these cross products vanish. The sum over pairs is zero under this assumption and Eq. (1.24) may be written in the form

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}. \tag{1.26}$$

The time derivative of the total angular momentum is thus equal to the moment of the external force about the given point. Corresponding to Eq. (1.26) is the

Conservation Theorem for Total Angular Momentum: $\mathbf{L}$ *is constant in time if the applied (external) torque is zero.*

(It is perhaps worthwhile to emphasize that this is a *vector* theorem; i.e., L_z will be conserved if $N_z^{(e)}$ is zero, even if $N_x^{(e)}$ and $N_y^{(e)}$ are not zero.)

Note that the conservation of linear momentum in the absence of applied forces assumes that the weak law of action and reaction is valid for the internal forces. The conservation of the total angular momentum of the system in the absence of applied torques requires the validity of the strong law of action and reaction—that the internal forces in addition be *central*. Many of the familiar physical forces, such as that of gravity, satisfy the strong form of the law. But it is possible to find forces for which action and reaction are equal even though the forces are not central (see below). In a system involving moving charges, the forces between the charges predicted by the Biot-Savart law may indeed violate both forms of

the action and reaction law.* Equations (1.23) and (1.26), and their corresponding conservation theorems, are not applicable in such cases, at least in the form given here. Usually it is then possible to find some generalization of **P** or **L** that is conserved. Thus, in an isolated system of moving charges it is the sum of the mechanical angular momentum and the electromagnetic "angular momentum" of the field that is conserved.

Equation (1.23) states that the total linear momentum of the system is the same as if the entire mass were concentrated at the center of mass and moving with it. The analogous theorem for angular momentum is more complicated. With the origin O as reference point, the total angular momentum of the system is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i .$$

Let $\mathbf{R}$ be the radius vector from O to the center of mass, and let $\mathbf{r}_i'$ be the radius vector from the center of mass to the ith particle. Then we have (cf. Fig. 1.3)

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R} \tag{1.27}$$

and

$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{v}$$

where

$$\mathbf{v} = \frac{d\mathbf{R}}{dt}$$

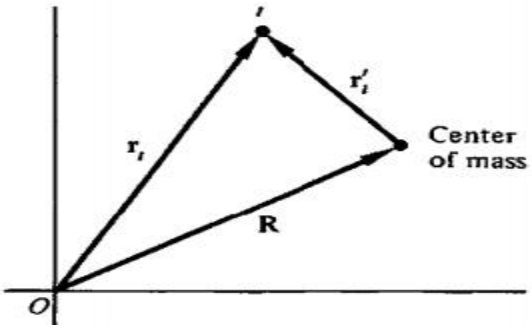


FIGURE 1.3 The vectors involved in the shift of reference point for the angular momentum.

*If two charges are moving uniformly with parallel velocity vectors that are not perpendicular to the line joining the charges, then the net mutual forces are equal and opposite but do not lie along the vector between the charges. Consider, further, two charges moving (instantaneously) so as to "cross the T," i.e., one charge moving directly at the other, which in turn is moving at right angles to the first Then the second charge exerts a nonvanishing magnetic force on the first, without experiencing any magnetic reaction force at that instant.

is the velocity of the center of mass relative to O, and

$$\mathbf{v}_i' = \frac{d\mathbf{r}_i'}{dt}$$

is the velocity of the ith particle relative to the center of mass of the system. Using Eq. (1.27), the total angular momentum takes on the form

$$\mathbf{L} = \sum_i \mathbf{R} \times m_i \mathbf{v} + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' + \left(\sum_i m_i \mathbf{r}_i' \right) \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} \sum_i m_i \mathbf{r}_i'.$$

The last two terms in this expression vanish, for both contain the factor $\sum m_i \mathbf{r}_i'$, which, it will be recognized, defines the radius vector of the center of mass in the very coordinate system whose origin is the center of mass and is therefore a null vector. Rewriting the remaining terms, the total angular momentum about O is

$$\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i'. \tag{1.28}$$

In words, Eq. (1.28) says that the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. The form of Eq. (1.28) emphasizes that in general $\mathbf{L}$ depends on the origin O, through the vector $\mathbf{R}$. Only

if the center of mass is at rest with respect to O will the angular momentum be independent of the point of reference. In this case, the first term in (1.28) vanishes, and $\mathbf{L}$ always reduces to the angular momentum taken about the center of mass.

Finally, let us consider the energy equation. As in the case of a single particle, we calculate the work done by all forces in moving the system from an initial configuration 1, to a final configuration 2:

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i. \qquad (1.29)$$

Again, the equations of motion can be used to reduce the integrals to

$$\sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s} = \sum_i \int_1^2 m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i dt = \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right).$$

Hence, the work done can still be written as the difference of the final and initial kinetic energies:

$$W_{12} = T_2 - T_1,$$

where T, the total kinetic energy of the system, is

$$T = \frac{1}{2} \sum_i m_i v_i^2. \qquad (1.30)$$

Making use of the transformations to center-of-mass coordinates, given in Eq. (1.27), we may also write T as

$$T = \frac{1}{2} \sum_i m_i (\mathbf{v} + \mathbf{v}_i') \cdot (\mathbf{v} + \mathbf{v}_i')$$

$$= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \mathbf{v} \cdot \frac{d}{dt}\left(\sum_i m_i \mathbf{r}_i'\right),$$

and by the reasoning already employed in calculating the angular momentum, the last term vanishes, leaving

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2 \tag{1.31}$$

The kinetic energy, like the angular momentum, thus also consists of two parts: the kinetic energy obtained if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Consider now the right-hand side of Eq. (1.29). In the special case that the external forces are derivable in terms of the gradient of a potential, the first term can be written as

$$\sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i = -\sum_i \int_1^2 \nabla_i V_i \cdot d\mathbf{s}_i = -\sum_i V_i \Big|_1^2,$$

where the subscript i on the del operator indicates that the derivatives are with respect to the components of $\mathbf{r}_i$. If the internal forces are also conservative, then the mutual forces between the ith and jth particles, $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, can be obtained from a potential function V_{ij}. To satisfy the strong law of action and reaction, V_{ij} can be a function only of the distance between the particles:

$$V_{ij} = V_{ij}(|\,\mathbf{r}_i - \mathbf{r}_j\,|). \tag{1.32}$$

The two forces are then automatically equal and opposite,

$$\mathbf{F}_{ji} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\mathbf{F}_{ij}, \tag{1.33}$$

and lie along the line joining the two particles,

$$\nabla V_{ij}(|\,\mathbf{r}_i - \mathbf{r}_j\,|) = (\mathbf{r}_i - \mathbf{r}_j) f, \tag{1.34}$$

where f is some scalar function. If V_{ij} were also a function of the difference of some other pair of vectors associated with the particles, such as their velocities or (to step into the domain of modern physics) their intrinsic "spin" angular momenta, then the forces would still be equal and opposite, but would not necessarily lie along the direction between the particles.

When the forces are all conservative, the second term in Eq. (1.29) can be rewritten as a sum over *pairs* of particles, the terms for each pair being of the form

$$-\int_1^2 (\nabla_i V_{ij} \cdot d\mathbf{s}_i + \nabla_j V_{ij} \cdot d\mathbf{s}_j).$$

If the difference vector $\mathbf{r}_i - \mathbf{r}_j$ is denoted by $\mathbf{r}_{ij}$, and if ∇_{ij} stands for the gradient with respect to $\mathbf{r}_{ij}$, then

$$\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij}.$$

and

$$d\mathbf{s}_i - d\mathbf{s}_j = d\mathbf{r}_i - d\mathbf{r}_j = d\mathbf{r}_{ij},$$

so that the term for the ij pair has the form

$$-\int \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij}.$$

The total work arising from internal forces then reduces to

$$-\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} \int_1^2 \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij} = -\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} V_{ij}\Big|_1^2. \tag{1.35}$$

The factor $\frac{1}{2}$ appears in Eq. (1.35) because in summing over *both* i and j each member of a given pair is included twice, first in the i summation and then in the j summation.

From these considerations, it is clear that if the external and internal forces are both derivable from potentials it is possible to define a *total potential energy*, V, of the system,

$$V = \sum_i V_i + \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} V_{ij}. \tag{1.36}$$

such that the total energy $T + V$ is conserved, the analog of the conservation theorem (1.18) for a single particle.

The second term on the right in Eq. (1.36) will be called the internal potential energy of the system. In general, it need not be zero and, more important, it may vary as the system changes with time. Only for the particular class of systems known as *rigid bodies* will the internal potential always be constant. Formally, a rigid body can be defined as a system of particles in which the distances r_{ij} are fixed and cannot vary with time. In such case, the vectors $d\mathbf{r}_{ij}$ can only be perpendicular to the corresponding $\mathbf{r}_{ij}$, and therefore to the $\mathbf{F}_{ij}$. Therefore, in a rigid body the *internal forces do no work*, and the internal potential must remain constant. Since the total potential is in any case uncertain to within an additive constant, an unvarying internal potential can be completely disregarded in discussing the motion of the system.

1.3 ■ CONSTRAINTS

From the previous sections one might obtain the impression that all problems in mechanics have been reduced to solving the set of differential equations (1.19):

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i^{(e)} - \sum_j \mathbf{F}_{ji}.$$

One merely substitutes the various forces acting upon the particles of the system, turns the mathematical crank, and grinds out the answers! Even from a purely physical standpoint, however, this view is oversimplified. For example, it may be necessary to take into account the *constraints* that limit the motion of the system. We have already met one type of system involving constraints, namely rigid bodies, where the constraints on the motions of the particles keep the distances r_{ij} unchanged. Other examples of constrained systems can easily be furnished. The beads of an abacus are constrained to one-dimensional motion by the supporting wires. Gas molecules within a container are constrained by the walls of the vessel to move only *inside* the container. A particle placed on the surface of a solid sphere is subject to the constraint that it can move only on the surface or in the region exterior to the sphere.

Constraints may be classified in various ways, and we shall use the following system. If the conditions of constraint can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, t) = 0, \tag{1.37}$$

then the constraints are said to be *holonomic*. Perhaps the simplest example of holonomic constraints is the rigid body, where the constraints are expressed by equations of the form

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0.$$

A particle constrained to move along any curve or on a given surface is another obvious example of a holonomic constraint, with the equations defining the curve or surface acting as the equations of a constraint.

Constraints not expressible in this fashion are called nonholonomic. The walls of a gas container constitute a nonholonomic constraint. The constraint involved in the example of a particle placed on the surface of a sphere is also nonholonomic, for it can be expressed as an inequality

$$r^2 - a^2 \geq 0$$

(where a is the radius of the sphere), which is not in the form of (1.37). Thus, in a gravitational field a particle placed on the top of the sphere will slide down the surface part of the way but will eventually fall off.

Constraints are further classified according to whether the equations of constraint contain the time as an explicit variable (rheonomous) or are not explicitly dependent on time (scleronomous). A bead sliding on a rigid curved wire fixed in space is obviously subject to a scleronomous constraint; if the wire is moving in some prescribed fashion, the constraint is rheonomous. Note that if the wire moves, say, as a reaction to the bead's motion, then the time dependence of the constraint enters in the equation of the constraint only through the coordinates of the curved wire (which are now part of the system coordinates). The overall constraint is then scleronomous.

In the case of holonomic constraints, the first difficulty is solved by the introduction of *generalized coordinates*. So far we have been thinking implicitly in terms of Cartesian coordinates. A system of N particles. free from constraints, has $3N$ independent coordinates or *degrees of freedom*. If there exist holonomic

constraints, expressed in k equations in the form (1.37), then we may use these equations to eliminate k of the $3N$ coordinates, and we are left with $3N - k$ independent coordinates, and the system is said to have $3N - k$ degrees of freedom. This elimination of the dependent coordinates can be expressed in another way, by the introduction of new, $3N - k$, independent variables $q_1, q_2, \ldots, q_{3N-k}$ in terms of which the old coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ are expressed by equations of the form

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_1(q_1, q_2, \ldots, q_{3N-k}, t) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N(q_1, q_2, \ldots, q_{3N-k}, t) \end{aligned} \tag{1.38}$$

containing the constraints in them implicitly. These are *transformation* equations from the set of $(\mathbf{r}_l)$ variables to the (q_l) set, or alternatively Eqs. (1.38) can be considered as parametric representations of the $(\mathbf{r}_l)$ variables. It is always assumed that we can also transform back from the (q_l) to the $(\mathbf{r}_l)$ set, i.e., that Eqs. (1.38) combined with the k equations of constraint can be inverted to obtain any q_i as a function of the $(\mathbf{r}_l)$ variable and time.

Usually the generalized coordinates, q_l, unlike the Cartesian coordinates, will not divide into convenient groups of three that can be associated together to form vectors. Thus, in the case of a particle constrained to move *on* the surface of a sphere, the two angles expressing position on the sphere, say latitude and longitude, are obvious possible generalized coordinates. Or, in the example of a double pendulum moving in a plane (two particles connected by an inextensible light rod and suspended by a similar rod fastened to one of the particles), satisfactory generalized coordinates are the two angles θ_1, θ_2. (Cf. Fig. 1.4.) Generalized coordinates, in the sense of coordinates other than Cartesian, are often useful in systems without constraints. Thus, in the problem of a particle moving in an external central force field ($V = V(r)$), there is no constraint involved, but it is clearly more convenient to use spherical polar coordinates than Cartesian coordinates. Do not, however, think of generalized coordinates in terms of conventional orthogonal position coordinates. All sorts of quantities may be impressed to serve as generalized coordinates. Thus, the amplitudes in a Fourier expansion of $\mathbf{r}_j$ may

be used as generalized coordinates, or we may find it convenient to employ quantities with the dimensions of energy or angular momentum.

If the constraint is nonholonomic, the equations expressing the constraint cannot be used to eliminate the dependent coordinates. An oft-quoted example of a nonholonomic constraint is that of an object rolling on a rough surface without slipping. The coordinates used to describe the system will generally involve angular coordinates to specify the orientation of the body, plus a set of coordinates describing the location of the point of contact on the surface. The constraint of "rolling" connects these two sets of coordinates; they are not independent. A change in the position of the point of contact inevitably means a change in its orientation. Yet we cannot reduce the number of coordinates, for the "rolling" condition is not expressible as a equation between the coordinates, in the manner of (1.37). Rather, it is a condition on the *velocities* (i e , the point of contact is stationary), a differential condition that can be given in an integrated form only *after* the problem is solved.

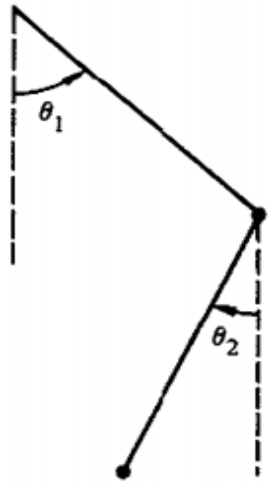


FIGURE 1.4 Double pendulum.

1.4 ■ D'ALEMBERT'S PRINCIPLE AND LAGRANGE'S EQUATIONS

A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta \mathbf{r}_i$, *consistent with the forces and constraints imposed on the system at the given instant* t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt, during which

the forces and constraints may be changing. Suppose the system is in equilibrium; i.e., the total force on each particle vanishes, $\mathbf{F}_i = 0$. Then clearly the dot product $\mathbf{F}_i \cdot \delta\mathbf{r}_i$, which is the virtual work of the force $\mathbf{F}_i$ in the displacement $\delta\mathbf{r}_i$, also vanishes. The sum of these vanishing products over all particles must likewise be zero:

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0. \tag{1.40}$$

As yet nothing has been said that has any new physical content. Decompose $\mathbf{F}_i$ into the applied force, $\mathbf{F}_i^{(a)}$, and the force of constraint, $\mathbf{f}_i$,

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i. \tag{1.41}$$

so that Eq. (1.40) becomes

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \tag{1 42}$$

We therefore have as the condition for equilibrium of a system that the virtual work of the *applied forces* vanishes:

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0. \tag{1.43}$$

To obtain such a principle, we use a device first thought of by James Bernoulli and developed by D'Alembert. The equation of motion,

$$\mathbf{F}_i = \dot{\mathbf{p}}_i,$$

Equation (1.43) is often called the *principle of virtual work.* Note that the coefficients of $\delta\mathbf{r}_i$ can no longer be set equal to zero; i.e., in general $\mathbf{F}_i^{(a)} \neq 0$, since the $\delta\mathbf{r}_i$ are not completely independent but are connected by the constraints. In order to equate the coefficients to zero, we must transform the principle into a form involving the virtual displacements of the q_i, which are independent. Equation (1.43) satisfies our needs in that it does not contain the $\mathbf{f}_i$, but it deals only with statics; we want a condition involving the general motion of the system.

can be written as

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0,$$

which states that the particles in the system will be in equilibrium under a force equal to the actual force plus a "reversed effective force" $-\dot{\mathbf{p}}_i$. Instead of (1.40), we can immediately write

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0, \tag{1.44}$$

which is often called *D'Alembert's principle*. We have achieved our aim, in that the forces of constraint no longer appear, and the superscript $^{(a)}$ can now be dropped without ambiguity. It is still not in a useful form to furnish equations of motion for the system. We must now transform the principle into an expression involving virtual displacements of the generalized coordinates, which are then independent of each other (for holonomic constraints), so that the coefficients of the δq_i can be set separately equal to zero.

The translation from $\mathbf{r}_i$ to q_j language starts from the transformation equations (1.38),

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t) \tag{1.45'}$$

(assuming n independent coordinates), and is carried out by means of the usual "chain rules" of the calculus of partial differentiation. Thus, $\mathbf{v}_i$ is expressed in terms of the $\dot{q}_k$ by the formula

$$\mathbf{v}_i \equiv \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}. \tag{1.46}$$

Similarly, the arbitrary virtual displacement $\delta \mathbf{r}_i$ can be connected with the virtual displacements δq_i by

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \tag{1.47}$$

Note that no variation of time, δt, is involved here, since a virtual displacement by definition considers only displacements of the coordinates. (Only then is the virtual displacement perpendicular to the force of constraint if the constraint itself is changing in time.)

In terms of the generalized coordinates. the virtual work of the $\mathbf{F}_i$ becomes

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_j Q_j \delta q_j. \tag{1.48}$$

where the Q_j are called the components of the *generalized force*, defined as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \tag{1.49}$$

Note that just as the q's need not have the dimensions of length, so the Q's do not necessarily have the dimensions of force, but $Q_j \delta q_j$ must always have the dimensions of work. For example, Q_j might be a torque N_j and dq_j a differential angle $d\theta_j$, which makes $N_j\, d\theta_j$ a differential of work.

We turn now to the other other term involved in Eq. (1.45), which may be written as

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i.$$

Expressing $\delta \mathbf{r}_i$ by (1.47), this becomes

$$\sum_{i,j} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

Consider now the relation

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right]. \qquad (1.50)$$

In the last term of Eq. (1.50) we can interchange the differentiation with respect to t and q_j, for, in analogy to (1.46).

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t},$$

$$- \frac{\partial \mathbf{v}_i}{\partial q_j},$$

by Eq. (1.46). Further, we also see from Eq. (1.46) that

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}. \qquad (1.51)$$

Substitution of these changes in (1.50) leads to the result that

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right],$$

and the second term on the left-hand side of Eq. (1.45) can be expanded into

$$\sum_j \left\{ \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right\} \delta q_j.$$

Identifying $\sum_i \frac{1}{2} m_i v_i^2$ with the system kinetic energy T, D'Alembert's principle (cf. Eq. (1.45)) becomes

$$\sum \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0. \qquad (1.52)$$

Note that in a system of Cartesian coordinates the partial derivative of T with respect to q_j vanishes. Thus, speaking in the language of differential geometry, this term arises from the curvature of the coordinates q_j. In polar coordinates, e.g., it is in the partial derivative of T with respect to an angle coordinate that the centripetal acceleration term appears.

Thus far, no restriction has been made on the nature of the constraints other than that they be workless in a virtual displacement. The variables q_j can be any set of coordinates used to describe the motion of the system. If, however, the constraints are holonomic, then it is possible to find sets of independent coordinates q_j that contain the constraint conditions implicitly in the transformation equations (1.38). Any virtual displacement δq_j is then independent of δq_k, and therefore the only way for (1.52) to hold is for the individual coefficients to vanish:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j. \tag{1.53}$$

There are n such equations in all.

When the forces are derivable from a scalar potential function V,

$$\mathbf{F}_i = -\nabla_i V.$$

Then the generalized forces can be written as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

which is exactly the same expression for the partial derivative of a function $-V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$ with respect to q_j:

$$Q_j = -\frac{\partial V}{\partial q_i}. \tag{1.54}$$

Equations (1.53) can then be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_i} = 0. \tag{1.55}$$

The equations of motion in the form (1.55) are not necessarily restricted to conservative systems, only if V is not an explicit function of time is the system conservative (cf. p. 4). As here defined, the potential V does not depend on the generalized velocities. Hence, we can include a term in V in the partial derivative with respect to $\dot{q}_j$:

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right) - \frac{\partial(T-V)}{\partial q_j} = 0.$$

Or, defining a new function, the *Lagrangian* L, as

$$L = T - V, \tag{1.56}$$

the Eqs. (1.53) become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{1.57}$$

expressions referred to as "Lagrange's equations."

Note that for a particular set of equations of motion there is no unique choice of Lagrangian such that Eqs. (1.57) lead to the equations of motion in the given generalized coordinates. Thus, in Derivations 8 and 10 it is shown that if $L(q, \dot{q}, t)$ is an approximate Lagrangian and $F(q, t)$ is *any* differentiable function of the generalized coordinates and time, then

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt} \tag{1.57'}$$

is a Lagrangian also resulting in the same equations of motion. It is also often possible to find alternative Lagrangians beside those constructed by this prescription (see Exercise 20). While Eq. (1.56) is always a suitable way to construct a Lagrangian for a conservative system, it does not provide the *only* Lagrangian suitable for the given system.

1.5 ■ VELOCITY-DEPENDENT POTENTIALS AND THE DISSIPATION FUNCTION

Lagrange's equations can be put in the form (1.57) even if there is no potential function, V, in the usual sense, providing the generalized forces are obtained from a function $U(q_j, \dot{q}_j)$ by the prescription

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right). \tag{1.58}$$

In such case, Eqs. (1.57) still follow from Eqs. (1.53) with the Lagrangian given by

$$L = T - U. \tag{1.59}$$

Here U may be called a "generalized potential," or "velocity-dependent potential." The possibility of using such a "potential" is not academic; it applies to one very important type of force field, namely. the electromagnetic forces on moving charges. Considering its importance, a digression on this subject is well worthwhile.

Consider an electric charge, q, of mass m moving at a velocity, $\mathbf{v}$, in an otherwise charge-free region containing both an electric field, $\mathbf{E}$, and a magnetic field, $\mathbf{B}$, which may depend upon time and position. The charge experiences a force, called the Lorentz force, given by

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]. \tag{1.60}$$

Both $\mathbf{E}(t, x, y, z)$ and $\mathbf{B}(t, x, y, z)$ are continuous functions of time and position derivable from a scalar potential $\phi(t, x, y, z)$ and a vector potential $\mathbf{A}(t, x, y, z)$ by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{1.61a}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.61b}$$

The force on the charge can be derived from the following velocity-dependent potential energy

$$U = q\phi - q\mathbf{A} \cdot \mathbf{v}, \tag{1.62}$$

so the Lagrangian, $L = T - U$, is

$$L = \tfrac{1}{2}mv^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}. \tag{1.63}$$

Considering just the x-component of Lagrange's equations gives

$$m\ddot{x} = q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) - q\left(\frac{\partial \phi}{\partial x} + \frac{dA_x}{dt}\right). \tag{1.64}$$

The total time derivative of A_x is related to the particle time derivative through

$$\begin{aligned}\frac{dA_x}{dt} &= \frac{\partial A_x}{\partial t} + \mathbf{v} \cdot \nabla A_x \\ &= \frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}.\end{aligned} \tag{1.65}$$

Equation (1.61b) gives

$$(\mathbf{v} \times \mathbf{B})_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right).$$

Combining these expressions gives the equation of motion in the x-direction

$$m\ddot{x} = q\left[\mathbf{E}_x + (\mathbf{v} \times \mathbf{B})_x\right]. \tag{1.66}$$

On a component-by-component comparison, Eqs. (1.66) and (1.60) are identical, showing that the Lorentz force equation is derivable from Eqs. (1.61) and (1.62).

Note that if not all the forces acting on the system are derivable from a potential, then Lagrange's equations can always be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j,$$

where L contains the potential of the conservative forces as before, and Q_j represents the forces *not* arising from a potential. Such a situation often occurs when frictional forces are present. It frequently happens that the frictional force is proportional to the velocity of the particle, so that its x-component has the form

where L contains the potential of the conservative forces as before, and Q_j represents the forces *not* arising from a potential. Such a situation often occurs when frictional forces are present. It frequently happens that the frictional force is proportional to the velocity of the particle, so that its x-component has the form

$$F_{fx} = -k_x v_x.$$

Frictional forces of this type may be derived in terms of a function $\mathcal{F}$, known as *Rayleigh's dissipation function*, and defined as

$$\mathcal{F} = \frac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right). \tag{1.67}$$

where the summation is over the particles of the system. From this definition it is clear that

$$F_{f_x} = -\frac{\partial \mathcal{F}}{\partial v_x}.$$

or, symbolically,

$$\mathbf{F}_f = -\nabla_v \mathcal{F}. \tag{1.68}$$

We can also give a physical interpretation to the dissipation function. The work done *by* the system *against* friction is

$$dW_f = -\mathbf{F}_f \cdot d\mathbf{r} = -\mathbf{F}_f \cdot \mathbf{v}\, dt = \left(k_x v_x^2 + k_y v_y^2 - k_z v_z^2\right) dt.$$

Hence, $2\mathcal{F}$ is the rate of energy dissipation due to friction. The component of the generalized force resulting from the force of friction is then given by

$$\begin{aligned} Q_j &= \sum_i \mathbf{F}_{fi} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum \nabla_v \mathcal{F} \cdot \frac{\partial \mathbf{r}_i}{\partial q_i} \\ &= -\sum \nabla_v \mathcal{F} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}, \qquad \text{by (1.51)}, \\ &= -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}. \end{aligned} \tag{1.69}$$

An example is Stokes' law, whereby a sphere of radius a moving at a speed v, in a medium of viscosity η experiences the frictional drag force $\mathbf{F}_f = 6\pi\eta\mathbf{a}v$. The Lagrange equations with dissipation become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0, \tag{1.70}$$

so that two scalar functions, L and $\mathcal{F}$, must be specified to obtain the equations of motion.

1.6 ■ SIMPLE APPLICATIONS OF THE LAGRANGIAN FORMULATION

The previous sections show that for systems where we can define a Lagrangian, i.e., holonomic systems with applied forces derivable from an ordinary or generalized potential and workless constraints, we have a very convenient way of setting up the equations of motion. We were led to the Lagrangian formulation by the desire to eliminate the forces of constraint from the equations of motion, and in achieving this goal we have obtained many other benefits. In setting up the original form of the equations of motion, Eqs. (1.19), it is necessary to work with many *vector* forces and accelerations. With the Lagrangian method we only deal with two *scalar* functions, T and V, which greatly simplifies the problem.

A straightforward routine procedure can now be established for all problems of mechanics to which the Lagrangian formulation is applicable. We have only to write T and V in generalized coordinates, form L from them, and substitute in (1.57) to obtain the equations of motion. The needed transformation of T and V from Cartesian coordinates to generalized coordinates is obtained by applying the transformation equations (1.38) and (1.45′). Thus, T is given in general by

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right)^2 .$$

It is clear that on carrying out the expansion, the expression for T in generalized coordinates will have the form

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k, \tag{1.71}$$

where M_0, M_j, M_{jk} are definite functions of the $\mathbf{r}$'s and t and hence of the q's and t. In fact, a comparison shows that

$$M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2 ,$$

$$M_j = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial t} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}, \tag{1.72}$$

and

$$M_{jk} = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}.$$

Thus, the kinetic energy of a system can always be written as the sum of three homogeneous functions of the generalized velocities,

$$T = T_0 + T_1 + T_2, \tag{1.73}$$

where T_0 is independent of the generalized velocities, T_1 is linear in the velocities, and T_2 is quadratic in the velocities. If the transformation equations do not contain the time explicitly, as may occur when the constraints are independent of time (scleronomous), then only the last term in Eq. (1.71) is nonvanishing, and T is always a homogeneous quadratic form in the generalized velocities.

Let us now consider simple examples of this procedure:

where T_0 is independent of the generalized velocities, T_1 is linear in the velocities, and T_2 is quadratic in the velocities. If the transformation equations do not contain the time explicitly, as may occur when the constraints are independent of time (scleronomous), then only the last term in Eq. (1.71) is nonvanishing, and T is always a homogeneous quadratic form in the generalized velocities.

Let us now consider simple examples of this procedure:

1. Single particle in space
   (a) Cartesian coordinates
   (b) Plane polar coordinates
2. Atwood's machine
3. Time-dependent constraint—bead sliding on rotating wire

1. (a) *Motion of one particle: using Cartesian coordinates.* The generalized forces needed in Eq. (1.53) are obviously F_x, F_y, and F_z. Then

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right),$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0,$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z},$$

and the equations of motion are

$$\frac{d}{dt}(m\dot{x}) = F_x, \quad \frac{d}{dt}(m\dot{y}) = F_y, \quad \frac{d}{dt}(m\dot{z}) = F_z. \tag{1.74}$$

We are thus led back to the original Newton's equations of motion.

(b) *Motion of one particle: using plane polar coordinates.* Here we must express T in terms of $\dot{r}$ and $\dot{\theta}$. The equations of transformation, i.e., Eqs. (1.38), in this case are simply

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

By analogy to (1.46), the velocities are given by

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

The kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ then reduces formally to

$$T = \tfrac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2\right]. \tag{1.75}$$

An alternative derivation of Eq. (1.75) is obtained by recognizing that the plane polar components of the velocity are $\dot{r}$ along $\mathbf{r}$, and $r\dot{\theta}$ along the direction perpendicular to r, denoted by the unit vector $\mathbf{n}$. Hence, the square of the velocity expressed in polar coordinates is simply $\dot{r}^2 + (r\dot{\theta})^2$. With the aid of the expression

$$d\mathbf{r} = \hat{\mathbf{r}}\,dr + r\hat{\boldsymbol{\theta}}\,d\theta + \hat{\mathbf{k}}\,dz$$

for the differential position vector, $d\mathbf{r}$, in cylindrical coordinates, where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the $\mathbf{r}$ and $\boldsymbol{\theta}$-directions, respectively, the components of the generalized force can be obtained from the definition, Eq. (1.49),

$$Q_r = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} = \mathbf{F} \cdot \hat{\mathbf{r}} = F_r,$$

$$Q_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{F} \cdot r\hat{\boldsymbol{\theta}} = r F_\theta,$$

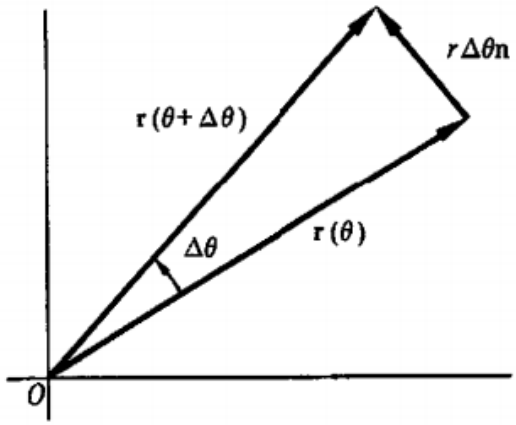


FIGURE 1.6 Derivative of r with respect to θ.

since the derivative of $\mathbf{r}$ with respect to 6 is, by the definition of a derivative, a vector in the direction of $\hat{\theta}$ (cf. Fig. 1.6). There are two generalized coordinates, and therefore two Lagrange equations. The derivatives occurring in the r equation are

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2, \qquad \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = m\ddot{r},$$

and the equation itself is

$$m\ddot{r} - mr\dot{\theta}^2 = F_r,$$

the second term being the centripetal acceleration term. For the θ equation, we have the derivatives

$$\frac{\partial T}{\partial \theta} = 0, \qquad \frac{\partial I}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta},$$

so that the equation becomes

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta.$$

Note that the left side of the equation is just the time derivative of the angular momentum, and the right side is exactly the applied torque, so that we have simply rederived the torque equation (1.26), where $L = mr^2\dot{\theta}$ and $N^{(\ell)} = rF_\theta$.

2. *Atwood's machine*—(See Fig. 1.7) an example of a conservative system with holonomic, scleronomous constraint (thc pullcy is assumed frictionless and massless). Clearly there is only one independent coordinate x, the position of the other weight being determined by the constraint that the length of the rope between them is l. The potential energy is

$$V = -M_1gx - M_2g(l - x),$$

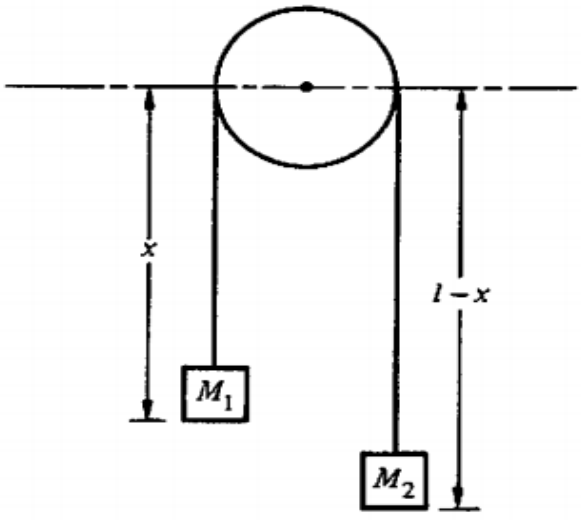


FIGURE 1.7 Atwood's machine.

while the kinetic energy is

$$T = \tfrac{1}{2}(M_1 + M_2)\dot{x}^2.$$

Combining the two, the Lagrangian has the form

$$L = T - V = \tfrac{1}{2}(M_1 + M_2)\dot{x}^2 + M_1 g x + M_2 g(l - x).$$

There is only one equation of motion, involving the derivatives

$$\frac{\delta L}{\partial x} = (M_1 - M_2)\, g,$$

$$\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\,\dot{x},$$

so that we have

$$(M_1 + M_2)\,\ddot{x} = (M_1 - M_2)\, g,$$

or

$$\ddot{x} = \frac{M_1 - M_2}{M_1 + M_2} g,$$

which is the familiar result obtained by more elementary means. This trivial problem emphasizes that the forces of constraint—here the tension in the rope—appear nowhere in the Lagrangian formulation. By the same token, neither can the tension in the rope be found directly by the Lagrangian method.

3. *A bead (or ring) sliding on a uniformly rotating wire in a force-free space.* The wire is straight, and is rotated uniformly about some fixed axis perpendicular to the wire. This example has been chosen as a simple illustration of a constraint being time dependent, with the rotation axis along z and the wire in the xy plane. The transformation equations explicitly contain the time.

$x = r\cos\omega t.$ (ω = angular velocity of rotation)

$y = r\sin\omega t.$ (r = distance along wire from rotation axis)

While we could then find T (here the same as L) by the same procedure used to obtain (1.71), it is simpler to take over (1.75) directly, expressing the constraint by the relation $\dot{\theta} = \omega$:

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right).$$

Note that T is not a homogeneous quadratic function of the generalized velocities, since there is now an additional term not involving $\dot{r}$. The equation of motion is then

$$m\ddot{r} = mr\omega^2 = 0$$

or

$$\ddot{r} = r\omega^2,$$

which is the familiar simple harmonic oscillator equation with a change of sign. The solution $r = e^{\omega t}$ shows that the bead moves exponentially outward because of the centripetal acceleration. Again, the method cannot furnish the force of constraint that keeps the bead on the wire. Equation (1.26) with the angular momentum, $\mathbf{L} = mr^2\omega^2 e^{\omega t}$, provides the force $\mathbf{F} = \mathbf{N}/r$, which produces the constraint force, $F = mr\omega^2 e^{\omega t}$, acting perpendicular to the wire and the axis of rotation.

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Mechanics **Subject Code: 19MMP106**

Class : I - M.Sc. Mathematics **Semester : I**

Unit I

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
_____ is the branch of mechanics which deals with conditions under which the bodies acted upon by forces remain at rest.	Dynamics	Statics	Kinematics	Force	Statics
_____ is that which changes the state of rest or a uniform motion in a straight line of a body.	Weight	Mass	Force	Particle	Force
_____ is defined as the line along which the force acts.	Line of action of force	Components of force	Resultant of force	Direction of force	Line of action of force
The _____ of weight of a body is due to the attraction caused by the earth.	Mass	Particle	Weight	Force	Force
If a fine string connecting two particles A and B passes over smooth surface the force exerted on the particles by the string is called _____	Force of string	Thrust of a string	Force of tension	Thrust of tension	Force of tension
If a light rod connects two particles A and B the force exerted by the rod is called _____	Tension	Thrust	Force	Particle	Thrust

When two bodies are in contact with each other the force exerted by one of the bodies upon the another is called _____	Action	Reaction	Attraction	Repulsion	Action
when two bodies are in contact with each other the force exerted by the second body on the first is called _____	Action	Attraction	Repulsion	Reaction	Reaction
Newton's third law of motion states that action and _____ are equal and opposite.	Action	Reaction	Force	Mass	Reaction
A force exerted by another particle of the same system is called _____	Equilibrium of force	Resultant of force	Internal force	External force	Internal force
A force exerted by another particle of the other system is called _____	Equilibrium of force	Resultant of force	Internal force	External force	External force
A Particle acted by a force is said to be in _____ if it doesnot move.	Tension	Thrust	Equilibrium	Line of action	Equilibrium
The force of thrust is _____ tension.	Zero	Positive	Negative	Unity	Negative
The force of _____ of a body is due to the attraction caused by the earth.	Force	Mass	Weight	Particle	Weight
Newton's third law of motion states that action and reaction are _____ and _____.	Zero and unity	One and unity	Positive and negative	Equal and opposite	Equal and opposite
A Particle acted by a force is said to be in equilibrium if is at _____	Move	Rest	Zero	Unity	Rest
When the body tend to approach each other the force is called _____	Action	Reaction	Attraction	Repulsion	Attraction
When the body tend to separate each other the force is called _____	Action	Reaction	Attraction	Repulsion	Repulsion

Every body is attracted towards the centre of the earth with a force is called _____	Weight	Mass	Particle	Force	Weight
Weight always acts at vertically _____	Upwards	Downwards	Rightwards	Leftwards	Downwards
The forces P and Q are equal and α=90 then θ= _____	90	180	0	45	45
If the forces P and Q act along the sane line in the same direction then α= _____	0	45	180	90	0
If the forces P and Q acts along the same line in the opposite direction then α= _____	0	45	180	90	180
If the forces P and Q are equal then θ= _____	0	45	180	90	90
The resultant of forces is greater when cosα is _____	Least	Zero	Greatest	Positive	Greatest
The resultant of forces is least when cosα is _____	Positive	Negative	Greatest	Least	Least
If three forces P,Q and R are in equilibrium then P+Q+R= _____	One	Zero	Infinity	None	Zero
Sine formula of triangle is _____	a/sin A=b/sin B=c/sinC	a/b=sinA	a/sinA+b/sinB=c/sinC	a/sinA-b/sinB=c/sinC	a/sin A=b/sin B=c/sinC
Types of parallel forces is _____	Five	Four	Two	Three	Two
_____ and _____ are two types of parallel forces.	Equal and opposite	Equal and unequal	Like and unlike	None	Like and unlike
Two parallel forces are said to be _____ if they act in same direction.	Equal force	Like parallel force	Unlike parallel force	Opposite force	Like parallel force
Two parallel forces are said to be _____ if they act in opposite direction.	Like parallel force	Opposite force	Unlike parallel force	Equal force	Unlike parallel force
If R=0 then the force is said to be in _____	Equilibrium	Move	Equal	Unequal	Equilibrium

_____ is the branch of mathematics which deals with the action of forces on bodies.	Statics	Dynamics	Kinematics	Mechanics	Mechanics
A force is completely known by its point of application and its magnitude and its _____.	Resultant	Direction	Moment	Mass	Direction
If the forces are in equilibrium then the resultant will be _____	One	Two	Zero	Infinity	Zero
Forces acting in a same plane is called _____	Like parallel force	Unlike parallel force	Coplanar force	Colinear force	Coplanar force
_____ law of motion states that action and reaction are equal and opposite.	Newton's first law	Newton's second law	Newton's third law	Newton's fourth law	Newton's third law
Forces acting _____ through the centre of gravity of the body.	Upwards	Downwards	Rightwards	Leftwards	Downwards
The force of weight of a body is due to the attraction caused by the _____	Mass	Force	Weight	Earth	Earth
Newton's third law of motion states that _____ and _____ are equal and opposite.	Action and Reaction	Attraction and Repulsion	Equal and Unequal forces	Like and unlike forces	Action and Reaction
If P and Q are like parallel forces then their resultant is _____	P+Q	P-Q	PQ	P/Q	P+Q
If P and Q are unlike parallel forces then their resultant is _____	P+Q	P-Q	PQ	P/Q	P-Q
Two parallel forces are said to be like parallel force if they act in _____ direction.	P	Q	Opposite	Same	Same
Two parallel forces are said to be unlike parallel force if they act in _____ direction.	Same	Opposite	P	Q	Opposite
The converse of polygon law of force is _____	True	Not true	Positive	Negative	Not true

By Parallelogram law of force the resultant will pass through the _____ of the parallelogram.	Diagonal	Sides	Angles	None	Diagonal
The maximum value of R occurs when cosα is _____	Zero	Unity	Maximum	Minimum	Maximum
The minimum value of R occurs when cosα is _____	Zero	Unity	Maximum	Minimum	Minimum
Gravitational pull of earth is _____	Attraction	Reaction	Action	Repulsion	Attraction

UNIT-II

Variation principles and Lagrange's equations: Hamilton's principle – Some techniques of calculus of variations – Derivation of Lagrange's Equations from Hamilton's principle – Extension of Hamilton's principle to non-holonomic systems – Conservation theorems and symmetry properties.

2.1 ■ HAMILTON'S PRINCIPLE

The derivation of Lagrange's equations presented in Chapter 1 started from a consideration of the instantaneous state of the system and small virtual displacements about the instantaneous state, i.e., from a "differential principle" such as D'Alembert's principle. It is also possible to obtain Lagrange's equations from a principle that considers the entire motion of the system between times t_1 and t_2, and small virtual variations of this motion from the actual motion. A principle of this nature is known as an "integral principle."

Before presenting the integral principle, the meaning attached to the phrase "motion of the system between times t_1 and t_2" must first be stated in more precise language The instantaneous configuration of a system is described by the values of the n generalized coordinates $q_1, \ldots, q_n$, and corresponds to a particular point in a Cartesian hyperspace where the q's form the n coordinate axes. This n-dimensional space is therefore known as configuration space. As time goes on, the state of the system changes and the system point moves in configuration space tracing out a curve, described as "the path of motion of the system." The "motion of the system," as used above, then refers to the motion of the system point along this path in *configuration space*. Time can be considered formally as a parameter of the curve; to each point on the path there is associated one or more values of the time. Note that configuration space has no necessary connection with the physical three-dimensional space, just as the generalized coordinates are not necessarily position coordinates. The path of motion in configuration space has no resemblance to the path in space of any actual particle; each point on the path represents the *entire* system configuration at some given instant of time.

The integral *Hamilton's principle* describes the motion of those mechanical systems for which all forces (except the forces of constraint) are derivable from a generalized scalar potential that may be a function of the coordinates, velocities, and time. Such systems will be denoted as *monogenic*. Where the potential is an explicit function of position coordinates only, then a monogenic system is also conservative (cf. Section 1.2).

For monogenic systems, Hamilton's principle can be stated as

The motion of the system from time t_1 to time t_2 is such that the line integral (called the action *or the* action integral*),*

2.1 Hamilton's Principle

$$I = \int_{t_1}^{t_2} L\, dt, \tag{2.1}$$

where $L = T - V$, has a stationary value for the actual path of the motion.

That is, out of all possible paths by which the system point could travel fron its position at time t_1 to its position at time t_2, it will actually travel along tha path for which the value of the integral (2.1) is stationary. By the term "station ary value" for a line integral, we mean that the integral along the given path ha the same value to within first-order infinitesimals as that along all neighborin paths (i.e., those that differ from it by infinitesimal displacements). (Cf. Fig. 2.1. The notion of a stationary value for a line integral thus corresponds in ordinar function theory to the vanishing of the first derivative.

We can summarize Hamilton's principle by saying that the motion is such tha the *variation* of the line integral I for fixed t_1 and t_2 is zero:

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\, dt = 0. \qquad (2.2$$

Where the system constraints are holonomic, Hamilton's principle, Eq. (2.2) is both a necessary and sufficient condition for Lagrange's equations, Eqs. (1.57) Thus, it can be shown that Hamilton's principle follows directly from Lagrange' equations. Instead, however, we shall prove the converse, namely, that Lagrange' equations follow from Hamilton's principle, as being the more important theorem That Hamilton's principle is a sufficient condition for deriving the equations o motion enables us to construct the mechanics of monogenic systems from Hamil ton's principle as the basic postulate rather than Newton's laws of motion. Suc a formulation has advantages; e g , since the integral I is obviously invariant t the system of generalized coordinates used to express L, the equations of motio must always have the Lagrangian form no matter how the generalized coordinate

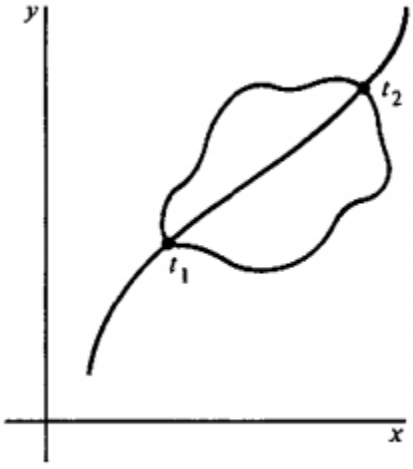

FIGURE 2.1 Path of the system point in configuration space.

are transformed. More important, the formulation in terms of a variational principle is the route that is generally followed when we try to describe apparently nonmechanical systems in the mathematical clothes of classical mechanics, as in the theory of fields.

2.2 ■ SOME TECHNIQUES OF THE CALCULUS OF VARIATIONS

Before demonstrating that Lagrange's equations do follow from (2.2), we must first examine the methods of the calculus of variations, for a chief problem of this calculus is to find the curve for which some given line integral has a stationary value.

Consider first the problem in an essentially one-dimensional form: We have a function $f(y, \dot{y}, x)$ defined on a path $y = y(x)$ between two values x_1 and x_2, where $\dot{y}$ is the derivative of y with respect to x. We wish to find a particular path $y(x)$ such that the line integral J of the function f between x_1 and x_2,

$$\dot{y} \equiv \frac{dy}{dx},$$

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx, \tag{2.3}$$

has a stationary value relative to paths differing infinitesimally from the correct function $y(x)$. The variable x here plays the role of the parameter t, and we consider only such varied paths for which $y(x_1) = y_1$, $y(x_2) = y_2$. (Cf. Fig. 2.2.) Note that Fig. 2.2 does *not* represent configuration space. In the one-dimensional configuration space, both the correct and varied paths are the segment of the straight line connecting y_1 and y_2; the paths differ only in the functional relation between y and x. The problem is one-dimensional, y is a function of x not a coordinate.

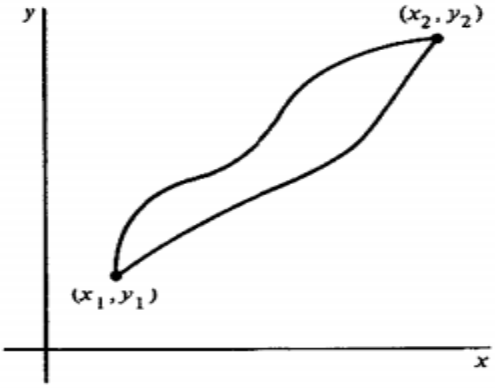

FIGURE 2.2 Varied paths of the function of $y(x)$ in the one-dimensional extremum problem.

We put the problem in a form that enables us to use the familiar apparatus of the differential calculus for finding the stationary points of a function. Since J must have a stationary value for the correct path relative to *any* neighboring path, the variation must be zero relative to *some* particular set of neighboring paths labeled by an infinitesimal parameter α. Such a set of paths might be denoted by $y(x, \alpha)$, with $y(x, 0)$ representing the correct path. For example, if we select any function $\eta(x)$ that vanishes at $x = x_1$ and $x = x_2$, then a possible set of varied paths is given by

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x). \tag{2.4}$$

For simplicity, it is assumed that both the correct path $y(x)$ and the auxiliary function $\eta(x)$ are well-behaved functions—continuous and nonsingular between x_1 and x_2, with continuous first and second derivatives in the same interval. For any such parametric family of curves, J in Eq. (2.3) is also a function of α:

$$J(\alpha) = \int_{x_1}^{x_2} f\,(y(x, \alpha), \dot{y}(x, \alpha), x)\,dx. \tag{2.5}$$

and the condition for obtaining a stationary point is the familiar one that

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0. \tag{2.6}$$

By the usual methods of differentiating under the integral sign, we find that

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right) dx. \tag{2.7}$$

Consider the second of these integrals.

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\,dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha}\,dx.$$

Integrating by parts, the integral becomes

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha}\,dx = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}\,dx. \tag{2.8}$$

The conditions on all the varied curves are that they pass through the points (x_1, y_1), (x_2, y_2), and hence the partial derivative of y with respect to α at x_1 and x_2 must vanish. Therefore, the first term of (2.8) vanishes and Eq. (2.7) reduces to

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha} \, dx.$$

The condition for a stationary value, Eq. (2.6), is therefore equivalent to the equation

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\left(\frac{\partial y}{\partial \alpha}\right)_0 dx = 0. \tag{2.9}$$

Now, the partial derivative of y with respect to α occurring in Eq. (2.9) is a function of x that is arbitrary except for continuity and end point conditions. For example, for the particular parametric family of varied paths given by Eq. (2.4), it is the arbitrary function $\eta(x)$. We can therefore apply to Eq (2.9) the so-called "fundamental lemma" of the calculus of variations, which says if

$$\int_{x_1}^{x_2} M(x)\eta(x)\,dx = 0 \tag{2.10}$$

for all arbitrary functions $\eta(x)$ continuous through the second derivative, then $M(x)$ must identically vanish in the interval (x_1, x_2). While a formal mathematical proof of the lemma can be found in texts on the calculus of variations, the validity of the lemma is easily seen intuitively. We can imagine constructing a function η that is positive in the immediate vicinity of any chosen point in the interval and zero everywhere else. Equation (2.10) can then hold only if $M(x)$ vanishes at that (arbitrarily) chosen point. which shows M must be zero throughout the interval. From Eq. (2.9) and the fundamental lemma, it therefore follows that J can have a stationary value only if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0. \tag{2 11}$$

The differential quantity,

$$\left(\frac{\partial y}{\partial \alpha}\right)_0 d\alpha \equiv \delta y, \tag{2 12}$$

represents the infinitesimal departure of the varied path from the correct path $y(x)$ at the point x and thus corresponds to the virtual displacement introduced in Chapter 1 (hence the notation δy). Similarly, the infinitesimal variation of J about the correct path can be designated

$$\left(\frac{dJ}{d\alpha}\right)_0 d\alpha \equiv \delta J. \tag{2.13}$$

The assertion that J is stationary for the correct path can thus be written

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\delta y\,dx - 0.$$

Some simple examples of the application of Eq. (2.11) (which clearly resembles a Lagrange equation) may now be considered:

1. *Shortest distance between two points in a plane.* An element of length in a plane is

$$ds = \sqrt{dx^2 + dy^2}$$

and the total length of any curve going between points 1 and 2 is

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The condition that the curve be the shortest path is that I be a minimum. This is an example of the extremum problem as expressed by Eq. (2.3), with

$$f = \sqrt{1 + \dot{y}^2}.$$

Substituting in (2.11) with

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}},$$

we have

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) = 0$$

or

$$\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = c,$$

where c is constant. This solution can be valid only if

$$\dot{y} = a,$$

where a is a constant related to c by

$$a = \frac{c}{\sqrt{1-c^2}}.$$

But this is clearly the equation of a straight line,

$$y = ax + b,$$

where b is another constant of integration. Strictly speaking, the straight line has only been proved to be an extremum path, but for this problem it is obviously also a minimum. The constants of integration, a and b, are determined by the condition that the curve pass through the two end points, (x_1, y_1), (x_2, y_2).

In a similar fashion we can obtain the shortest distance between two points on a sphere, by setting up the arc length on the surface of the sphere in terms of the angle coordinates of position on the sphere In general, curves that give the shortest distance between two points on a given surface are called the *geodesics* of the surface.

2. *Minimum surface of revolution.* Suppose we form a surface of revolution by taking some curve passing between two fixed end points (x_1, y_1) and (x_2, y_2) defining the xy plane, and revolving it about the y axis (cf. Fig. 2.3a). The problem then is to find that curve for which the surface area is a minimum. The area of a strip of the surface is $2\pi x\, ds = 2\pi x\sqrt{1+\dot{y}^2}\, dx$, and the total area is

$$2\pi \int_1^2 x\sqrt{1+\dot{y}^2}\, dx.$$

The extremum of this integral is again given by (2.11) where

$$f = x\sqrt{1+\dot{y}^2}$$

and

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}.$$

Equation (2.11) becomes in this case

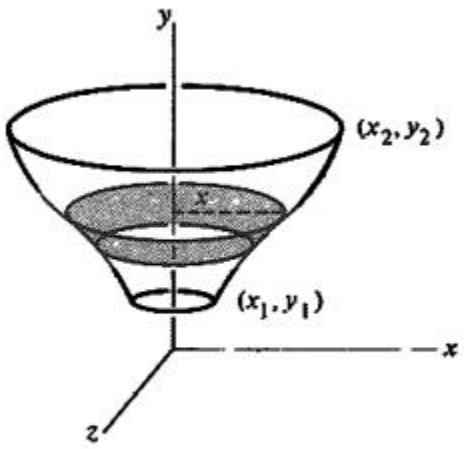

FIGURE 2.3a Minimum surface of revolution. Note that this figure is drawn for y_1 and y_2 having the same sign relative to the rotation axis. This is not assumed in the general solution.

$$\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

or

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a,$$

where a is some constant of integration clearly smaller than the minimum value of x. Squaring the above equation and factoring terms, we have

$$\dot{y}^2(x^2 - a^2) = a^2,$$

or solving,

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}.$$

The general solution of this differential equation, in light of the nature of a, is

$$y = a\int \frac{dx}{\sqrt{x^2 - a^2}} + b = a \operatorname{arc\,cosh} \frac{x}{a} + b$$

or

$$x = a \cosh \frac{y - b}{a},$$

which is the equation of a catenary. Again the two constants of integration, a and b, are determined in principle by the requirements that the curve pass through the two given end points, as shown in Fig. 2.3b.

Curves satisfying the preceding equation all scale as x/a and y/a with one independent parameter b/a. This suggests that when the solutions are examined in detail they turn out to be a great deal more complicated than these considera-

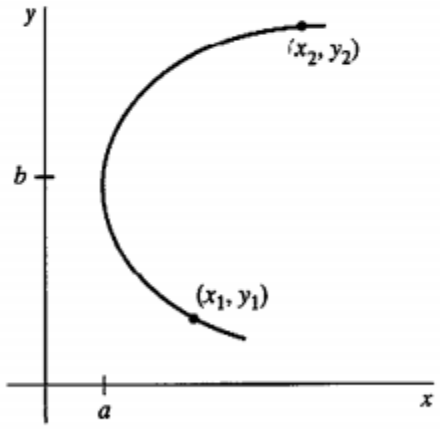

FIGURE 2.3b General catenary solution for minimum surface of revolution.

tions suggest. For some pairs of end points, unique constants of integration a and b can be found. But for other end points, it is possible to draw two different catenary curves through the end points, while for additional cases no possible values can be found for a and b. Further, recall that Eq. (2.11) represents a condition for finding curves $y(x)$ continuous through the second derivative that render the integral stationary. The catenary solutions therefore do not always represent minimum values, but may represent "inflection points" where the length of the curve is stationary but not minimum.

For certain combinations of end points (an example is x_1 and x_2 both positive and both much smaller than $y_2 - y_1$), the absolute minimum in the surface of revolution is provided (cf. Exercise 8) by a curve composed of straight line segments—from the first end point parallel to the x axis until the y axis is reached, then along the y axis until the point $(0, y_2)$ and then out in a straight line to the second end point corresponding to the area $\pi(x_1^2 + x_2^2)$. This curve results when $a = 0$, forcing either $x = 0$ or $y =$ constant. Since this curve has discontinuous first derivatives, we should not expect to find it as a solution to Eq. (2.11).

This example is valuable in emphasizing the restrictions that surround the derivation and the meaning of the stationary condition. Exercises 7 and 8 examine the conditions for the pathological behavior for a symmetric example. More information can be found in many texts on the calculus of variations.

3. *The brachistochrone problem.* (See Fig. 2.4a.) This well-known problem is to find the curve joining two points, along which a particle falling from rest under the influence of gravity travels from the higher to the lower point in the least time.

If v is the speed along the curve, then the time required to fall an arc length ds is ds/v, and the problem is to find a minimum of the integral

$$t_{12} = \int^2 \frac{ds}{v}.$$

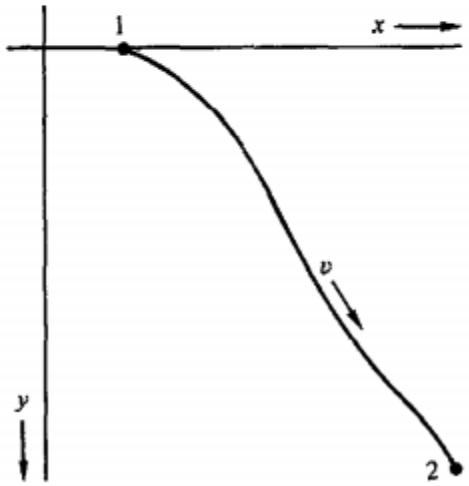

FIGURE 2.4a The brachistochrone problem.

If y is measured down from the initial point of release, the conservation theorem for the energy of the particle can be written as

$$\tfrac{1}{2}mv^2 = mgy$$

or

$$v = \sqrt{2gy}.$$

Then the expression for t_{12} becomes

$$t_{12} = \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\,dx,$$

and f is identified as

$$f = \sqrt{\frac{1+\dot{y}^2}{2gy}}.$$

The integration of Eq. (2.11) with this form for f is straightforward and is left as an exercise.

The solution in terms of its one parameter, a, given by

$$\frac{y}{a} = 1 - \cos\left[\frac{x + \sqrt{y(2a - y)}}{a}\right],$$

is sketched in Fig. 2.4b for the first cycle ($0 \leq x \leq 2\pi a$) and the beginning of the second cycle. Three cases of solutions are indicated. A power-series expansion of the solution for the limit $y \ll a$ gives

$$y = \frac{x^2}{2}a.$$

The brachistochrone problem is famous in the history of mathematics, for it was the analysis of this problem by John Bernoulli that led to the formal foundation of the calculus of variations.

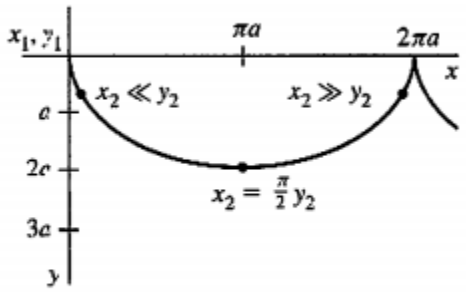

FIGURE 2.4b Catenary solution to the brachistochrone problem showing positions on the curve for the three cases $x_2 \ll y_2$, $x_2 = \frac{\pi}{2} y_2$, and $x_2 \gg y_2$

2.3 ■ DERIVATION OF LAGRANGE'S EQUATIONS FROM HAMILTON'S PRINCIPLE

The fundamental problem of the calculus of variations is easily generalized to the case where f is a function of many independent variables y_i, and their derivatives $\dot{y}_i$. (Of course, all these quantities are considered as functions of the parametric variable x.) Then a variation of the integral J,

$$\delta J = \delta \int_1^2 f(y_1(x); y_2(x), \ldots, \dot{y}_1(x); \dot{y}_2(x), \ldots, x)\, dx, \tag{2.14}$$

is obtained, as before, by considering J as a function of parameter α that labels a possible set of curves $y_1(x, \alpha)$. Thus, we may introduce α by setting

where $y_1(x, 0)$, $y_2(x, 0)$, etc., are the solutions of the extremum problem (to be obtained) and η_1, η_2, etc., are independent functions of x that vanish at the end points and that are continuous through the second derivative, but otherwise are completely arbitrary.

The calculation proceeds as before. The variation of J is given in terms of

$$\frac{\partial J}{\partial \alpha}\, d\alpha = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha}\, d\alpha + \frac{\partial f}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \alpha}\, d\alpha \right) dx. \tag{2.16}$$

Again we integrate by parts the integral involved in the second sum of Eq. (2.16):

$$\int_1^2 \frac{\delta f}{\partial \dot{y}_i} \frac{\partial^2 y_i}{\partial \alpha\, \partial x}\, dx = \frac{\partial f}{\partial \dot{y}_i} \frac{\partial y_i}{\partial \alpha}\bigg|_1^2 - \int_1^2 \frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_i} \right) dx,$$

$$\int_1^2 \frac{\delta f}{\partial \dot{y}_i} \frac{\partial^2 y_i}{\partial \alpha\, \partial x}\, dx = \frac{\partial f}{\partial \dot{y}_i} \frac{\partial y_i}{\partial \alpha}\bigg|_1^2 - \int_1^2 \frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_i} \right) dx,$$

where the first term vanishes because all curves pass through the fixed end points. Substituting in (2.16), δJ becomes

$$\delta J = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} \right) \delta y_i\, dx, \tag{2.17}$$

where, in analogy with (2.12), the variation δy_i is

$$\delta y_i = \left(\frac{\partial y_i}{\partial \alpha} \right)_0 d\alpha.$$

Since the y variables are independent, the variations δy_i are independent (e.g., the functions $\eta_i(x)$ will be independent of each other). Hence, by an obvious extension of the fundamental lemma (cf. Eq. (2.10)), the condition that δJ is zero requires that the coefficients of the δy_i separately vanish:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} = 0, \qquad i = 1, 2, \ldots, n. \tag{2.18}$$

Equations (2.18) represent the appropriate generalization of (2.11) to several variables and are known as the *Euler-Lagrange differential equations*. Their solutions represent curves for which the variation of an integral of the form given in (2.14) vanishes. Further generalizations of the fundamental variational problem are easily possible. Thus, we can take f as a function of higher derivatives $\ddot{y}$, $\dddot{y}$,

etc., leading to equations different from (2.18). Or we can extend it to cases where there are several parameters x_j and the integral is then multiple, with f also involving as variables derivatives of y_i with respect to each of the parameters x_j. Finally, it is possible to consider variations in which the end points are *not* held fixed.

For present purposes, what we have derived here suffices, for the integral in Hamilton's principle,

$$I = \int_1^2 L(q_i, \dot{q}_i, t)\, dt, \tag{2.19}$$

has just the form stipulated in (2.14) with the transformation

$$x \to t$$

$$y_i \to q_i$$

$$f(y_i, \dot{y}_i, x) \to L(q_i, \dot{q}_i, t).$$

In deriving Eqs. (2.18), we assumed that the y_i variables are independent. The corresponding condition in connection with Hamilton's principle is that the generalized coordinates q_i be independent, which requires that the constraints be holonomic. The Euler-Lagrange equations corresponding to the integral I then become the Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n,$$

2.4 ■ EXTENSION OF HAMILTON'S PRINCIPLE TO NONHOLONOMIC SYSTEMS

It is possible to extend Hamilton's principle, at least in a formal sense, to cover certain types of nonholonomic systems. In deriving Lagrange's equations from either Hamilton's or D'Alembert's principle, the requirement of holonomic constraints does not appear until the last step, when the variations q_i are considered as independent of each other. With nonholonomic systems the generalized coordinates are not independent of each other, and it is not possible to reduce them further by means of equations of constraint of the form $f(q_1, q_2, \ldots, q_n, t) = 0$. Hence, it is no longer true that the q_i's are all independent.

Another difference that must be considered in treating the variational principle is the manner in which the varied paths are constructed. In the discussion of Section 2.2, we pointed out that δy (or δq) represents a virtual displacement from a point on the actual path to some point on the neighboring varied path. But, with independent coordinates it is the final varied path that is significant, not how it is

constructed. When the coordinates are not independent, but subject to constraint relations, it becomes important whether the varied path is or is not constructed by displacements consistent with the constraints. Virtual displacements, in particular, may or may not satisfy the constraints.

It appears that a reasonably straightforward treatment of nonholonomic systems by a variational principle is possible only when the equations of constraint can be put in the form

$$f_\alpha(q_1, \ldots, q_n;\ \dot{q}_1 \ldots, \dot{q}_n) = 0, \tag{2.20}$$

when this can be done the constraints are called semi-holonomic. The index α indicates that there may be more than one such equation. We will assume there are m equations in all, i.e., $\alpha = 1, 2, \ldots, m$. Equation (2.20) commonly appears in the restricted form

$$\sum_k a_{ik}\, dq_k + a_{it}\, dt = 0. \tag{2.20'}$$

We might expect that the varied paths, or equivalently, the displacements constructing the varied path, should satisfy the constraints of Eq. (2.20). However, it has been proven that no such varied path can be constructed unless Eqs. (2.20) are integrable, in which case the constraints are actually holonomic. A variational principle leading to the correct equations of motion can nonetheless be obtained when the varied paths are constructed from the actual motion by virtual displacements.

The procedure for eliminating these extra virtual displacements is the method of *Lagrange undetermined multipliers*. If Eqs. (2.20) hold, then it is also true that

$$\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = 0, \tag{2.21}$$

where the λ_α, $\alpha = 1, 2, \ldots, m$, are some undetermined quantities, functions in general of the coordinates and of the time t. In addition, Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \tag{2.2}$$

is assumed to hold for this semiholonomic system. Following the development of Section 2.3, Hamilton's principle then implies that

$$\int_1^2 dt \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k = 0. \tag{2.22}$$

The variation cannot be taken as before since the q_k are not independent; however, combining (2.21) with (2.2) gives

$$\delta \int_{t_1}^{t_2} \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) dt = 0 \tag{2.23}$$

The variation can now be performed with the n δq_i and m λ_α for $m+n$ independent variables. For the simplifying assumption that $\lambda_\alpha = \lambda_\alpha(t)$, the resulting equations from δq_i become*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \mathrm{Q}_k, \tag{2.24}$$

where

$$Q_k = \sum_{\alpha=1}^{m} \left\{ \lambda_\alpha \left[\frac{\partial f_\alpha}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \dot{q}_k}\right)\right] - \frac{d\lambda_\alpha}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_k} \right\}, \tag{2.25}$$

while the $\delta\lambda_\alpha$ give the equations of constraint (2.20). Equations (2.24) and (2.20) together constitute $n + m$ equations for $n + m$ unknowns. The system can now be interpreted as an $m + n$ holonomic system with generalized forces Q_k. The generalization to $\lambda_\alpha = \lambda_\alpha(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ is straightforward.

As an example, let us consider a particle whose Lagrangian is

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z) \tag{2.26}$$

subject to the constraint

$$f(\dot{x}, \dot{y}, y) = \dot{x}\dot{y} + ky = 0 \tag{2.27}$$

with k a constant. The resulting equations of motion are

$$m\ddot{x} + \lambda\ddot{y} + \dot{\lambda}\dot{y} + \frac{\partial V}{\partial x} = 0, \tag{2.28}$$

$$m\ddot{y} + \lambda\ddot{x} - k\lambda + \dot{\lambda}\dot{x} + \frac{\partial V}{\partial y} = 0, \tag{2.29}$$

$$m\ddot{z} + \frac{\partial V}{\partial z} = 0, \tag{2.30}$$

and the equation of constraint, (2.20), becomes

$$\dot{y}\dot{x} + ky = 0.$$

In this process we have obtained more information than was originally sought. Not only do we get the q_k's we set out to find, but we also get $m\lambda_l$'s. What is the physical significance of the λ_l's? Suppose we remove the constraints on the system, but instead apply external forces Q'_k in such a manner as to keep the motion of the system unchanged. The equations of motion likewise remain the same. Clearly these extra applied forces must be equal to the forces of constraint, for they are the forces applied to the system so as to satisfy the condition of constraint. Under the influence of these forces Q'_k, the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q'_k. \tag{2.31}$$

But these must be identical with Eqs. (2.24). Hence, we can identify (2.25) with Q'_k, the generalized forces of constraint. In this type of problem we really do not eliminate the forces of constraint from the formulation. They are supplied as part of the answer.

Although it is not obvious, the version of Hamilton's principle adopted here for semiholonomic systems also requires that the constraints do no work in virtual displacements. This can be most easily seen by rewriting Hamilton's principle in the form

$$\delta \int_{t_1}^{t_2} L\,dt = \delta \int_{t_1}^{t_2} T\,dt - \delta \int_{t_1}^{t_2} U\,dt = 0. \tag{2.32}$$

If the variation of the integral over the generalized potential is carried out by the procedures of Section 2.3, the principle takes the form

$$\delta \int_{t_1}^{t_2} T\,dt = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial U}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right)\right] \delta q_k dt; \tag{2.33}$$

or, by Eq. (1.58),

$$\delta \int_{t_1}^{t_2} T\,dt = -\int_{t_1}^{t_2} \sum_k Q_k \delta q_k dt. \tag{2.34}$$

In this dress, Hamilton's principle says that the difference in the time integral of the kinetic energy between two neighboring paths is equal to the negative of the time integral of the work done in the virtual displacements between the paths. The work involved is that done only by the forces derivable from the generalized potential. The same Hamilton's principle holds for both holonomic and semiholonomic systems, it must be required that the additional forces of semiholonomic constraints do no work in the displacements δq_k. This restriction parallels the earlier condition that the virtual work of the forces of holonomic constraint also be

zero (cf. Section 1.4). In practice, the restriction presents little handicap to the applications, as many problems in which the semiholonomic formalism is used relate to rolling without slipping, where the constraints are obviously workless.

Note that Eq. (2.20) is not the most general type of nonholonomic constraint; e.g., it does not include equations of constraint in the form of inequalities. On the other hand, it does include holonomic constraints. A holonomic equation of constraint,

$$f(q_1, q_2, q_3, \ldots, q_n, t) = 0, \tag{2.35}$$

is equivalent to (2.20) with no dependence on $\dot{q}_k$. Thus, the Lagrange multiplier method can be used also for holonomic constraints when (1) it is inconvenient to reduce all the q's to independent coordinates or (2) we might wish to obtain the forces of constraint.

As another example of the method, let us consider the following somewhat trivial illustration—a hoop rolling, without slipping, down an inclined plane. In this example, the constraint of "rolling" is actually holonomic, but this fact will be immaterial to our discussion. On the other hand, the holonomic constraint that the hoop be on the inclined plane will be contained implicitly in our choice of generalized coordinates.

The two generalized coordinates are x, θ, as in Fig. 2.5, and the equation of rolling constraint is

$$r\,d\theta = dx.$$

The kinetic energy can be resolved into kinetic energy of motion of the center of mass plus the kinetic energy of motion about the center of mass:

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}Mr^2\dot{\theta}^2.$$

The potential energy is

$$V = Mg(l - x)\sin\phi,$$

where l is the length of the inclined plane and the Lagrangian is

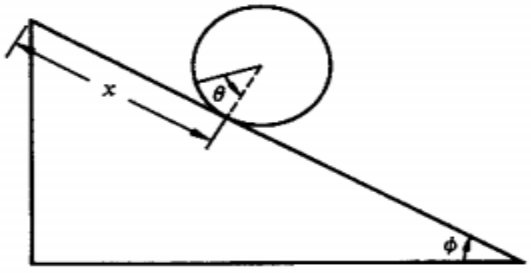

FIGURE 2.5 A hoop rolling down an inclined plane.

$$L = T - V$$
$$= \frac{M\dot{x}^2}{2} + \frac{Mr^2\dot{\theta}^2}{2} - Mg(l-x)\sin\phi. \tag{2.36}$$

Since there is one equation of constraint, only one Lagrange multiplier λ is needed. The coefficients appearing in the constraint equation are:

$$a_\theta = r,$$
$$a_x = -1.$$

The two Lagrange equations therefore are

$$M\ddot{x} - Mg\sin\phi + \lambda = 0, \tag{2.37}$$
$$Mr^2\ddot{\theta} - \lambda r = 0, \tag{2.38}$$

which along with the equation of constraint,

$$r\dot{\theta} = \dot{x}, \tag{2.39}$$

constitutes three equations for three unknowns, θ, x, λ.

Differentiating (2.39) with respect to time, we have

$$r\ddot{\theta} = \ddot{x}.$$

Hence, from (2.38)

$$M\ddot{x} = \lambda,$$

and (2.37) becomes

$$\ddot{x} = \frac{g\sin\phi}{2}.$$

along with

$$\lambda = \frac{Mg\sin\phi}{2}$$

and

$$\ddot{\theta} = \frac{g\sin\phi}{2r}.$$

Thus, the hoop rolls down the incline with only one-half the acceleration it would have slipping down a frictionless plane, and the friction force of constraint is $\lambda = Mg\sin\phi/2$.

2.5 ■ ADVANTAGES OF A VARIATIONAL PRINCIPLE FORMULATION

Although we can extend the original formulation of Hamilton's principle (2.2) to include some nonholonomic constraints, in practice this formulation of mechanics is most useful when a Lagrangian of independent coordinates can be set up for the system. The variational principle formulation has been justly described as "elegant," for in the compact Hamilton's principle is contained all of the mechanics of holonomic systems with forces derivable from potentials. The principle has the further merit that it involves only physical quantities that can be defined without reference to a particular set of generalized coordinates, namely, the kinetic and potential energies. The formulation is therefore automatically invariant with respect to the choice of coordinates for the system.

From the variational Hamilton's principle, it is also obvious why the Lagrangian is always uncertain to a total time derivative of any function of the coordinates and time, as mentioned at the end of Section 1.4. The time integral of such a total derivative between points 1 and 2 depends only on the values of the arbitrary function at the end points. As the variation at the end points is zero, the addition of the arbitrary time derivative to the Lagrangian does not affect the variational behavior of the integral.

Another advantage is that the Lagrangian formulation can be easily extended to describe systems that are not normally considered in dynamics—such as the elastic field, the electromagnetic field, and field properties of elementary particles. Some of these generalizations will be considered later, but as three simple examples of its application outside the usual framework of mechanics, let us consider the cases of an RL circuit, an LC circuit, and coupled circuits.

We consider the physical system of a battery of voltage V in series with an inductance L and a resistance of value R and choose the electric charge q as the dynamical variable. The inductor acts as the kinetic energy term since the inductive effect depends upon the time rate of change of the charge. The resistor

provides a dissipative term and the potential energy is qV. The dynamic terms in Lagrange's equation with dissipation (1.70) are

$$T = \tfrac{1}{2}L\dot{q}^2, \quad \mathcal{F} = \tfrac{1}{2}R\dot{q}^2,$$

and potential energy $= qV$. The equation of motion is

$$V = L\ddot{q} + R\dot{q} = L\dot{I} + RI. \tag{2.40}$$

where $I = \dot{q}$ is the electric current. A solution for a battery connected to the circuit at time $t = 0$ is

$$I = I_0(1 - e^{-Rt/L}),$$

where $I_0 = V/R$ is the final steady-state current flow.

The mechanical analog for this is a sphere of radius a and effective mass m' falling in a viscous fluid of constant density and viscosity η under the force of gravity. The effective mass is the difference between the actual mass and the mass of the displaced fluid, and the direction of motion is along the y axis. For this system,

$$T = \tfrac{1}{2}m'\dot{y}^2, \quad \mathcal{F} = 3\pi\eta a\dot{y}^2,$$

and potential energy $= m'gy$, where the frictional drag force, $F_f = 6\pi\eta ay$, called Stokes' law, was given at the end of Section 1.5.

The equation of motion is given by Lagrange's equations (1.70) as

$$m'g = m'\ddot{y} + 6\pi\eta a\dot{y}.$$

Using $v = \dot{y}$, the solution (if the motion starts from rest at $t = 0$), is

$$v = v_0(1 - e^{-t/\tau})$$

where $\tau = m'/(6\pi\eta a)$ is a measure of the time it takes for the sphere to reach $1/e$ of its terminal speed of $v_0 = m'g/6\pi\eta a$.

Another example from electrical circuits is an inductance, L, in series with a capacitance, C. The capacitor acts as a source of potential energy given by q^2/C where q is the electric charge. The Lagrangian produces the equation of motion,

$$L\ddot{q} + \frac{q}{C} = 0, \tag{2.41}$$

which has the solution

$$q = q_0 \cos \omega_0 t,$$

where q_0 is the charge stored in the capacitor at $t = 0$, and the assumption is that no charge is flowing at $t = 0$. The quantity

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

is the resonant frequency of the system.

The mechanical analog of this system is the simple harmonic oscillator described by the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$, which gives an equation of motion,

$$m\ddot{x} + kx = 0,$$

whose solution for the same boundary conditions is

$$x = x_0 \cos \omega_0 t \qquad \text{with} \quad \omega_0 = \sqrt{k/m}.$$

These two examples show that an inductance is an inertial term, the electrical analog of mass. Resistance is the analog of Stokes' law type of frictional drag, and the capacitance term $1/C$ represents a Hooke's law spring constant. With this

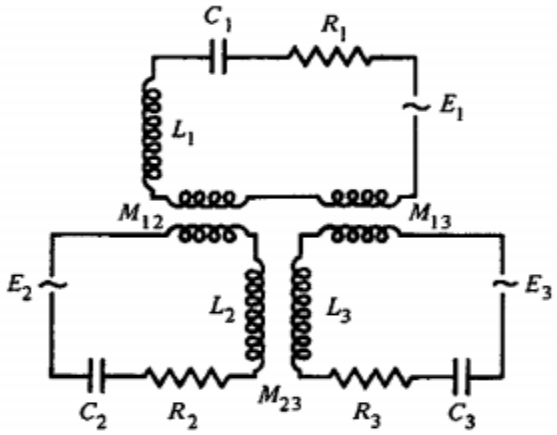

FIGURE 2.6 A system of coupled circuits to which the Lagrangian formulation can be applied.

background, a system of coupled electrical circuits of the type shown in Fig. 2.6 has a Lagrangian of the form

$$L = \frac{1}{2}\sum_j L_j \dot{q}_j^2 + \frac{1}{2}\sum_{\substack{jk \\ j \neq k}} M_{jk}\dot{q}_j\dot{q}_k - \sum_j \frac{q_j^2}{2C_j} + \sum_j e_j(t)q_j,$$

background, a system of coupled electrical circuits of the type shown in Fig. 2.6 has a Lagrangian of the form

$$L = \frac{1}{2}\sum_j L_j \dot{q}_j^2 + \frac{1}{2}\sum_{\substack{jk \\ j\neq k}} M_{jk}\dot{q}_j\dot{q}_k - \sum_j \frac{q_j^2}{2C_j} + \sum_j e_j(t)q_j,$$

and a dissipation function

$$\mathcal{F} = \frac{1}{2}\sum_j R_j \dot{q}_j^2.$$

where the mutual inductance terms, $M_{jk}\dot{q}_j\dot{q}_k$, are added to take into account the coupling between inductors. The Lagrange equations are

$$L_j\frac{d^2q_j}{dt^2} + \sum_{\substack{k \\ j\neq k}} M_{jk}\frac{d^2q_k}{dt^2} + R_j\frac{dq_j}{dt} + \frac{q_j}{C_j} = E_j(t). \qquad (2.42)$$

where the $E_j(t)$ terms are the external emf's.

This description of two different physical systems by Lagrangians of the same form means that all the results and techniques devised for investigating one of the systems can be taken over immediately and applied to the other. In this particular case, the study of the behavior of electrical circuits has been pursued intensely and some special techniques have been developed; these can be directly applied to the corresponding mechanical systems. Much work has been done in formulating equivalent electrical problems for mechanical or acoustical systems, and vice versa. Terms hitherto reserved for electrical circuits (reactance, susceptance, etc.) are now commonly found in treatises on the theory of vibrations of mechanical systems.

2.6 ■ CONSERVATION THEOREMS AND SYMMETRY PROPERTIES

Thus far, we have been concerned primarily with obtaining the equations of motion, but little has been said about how to solve them for a particular problem once they are obtained. In general, this is a question of mathematics. A system of n degrees of freedom will have n differential equations that are second order in time. The solution of each equation will require two integrations resulting, all told, in $2n$ constants of integration. In a specific problem these constants will be determined by the initial conditions, i.e., the initial values of the nq_j's and the $n\dot{q}_j$'s. Sometimes the equations of motion will be integrable in terms of known functions, but not always. In fact, the majority of problems are not completely integrable. However, even when complete solutions cannot be obtained, it is often possible to extract a large amount of information about the physical nature of the system motion. Indeed, such information may be of greater interest to the physicist than the complete solution for the generalized coordinates as a function of

cist than the complete solution for the generalized coordinates as a function of time. It is important, therefore, to see how much can be stated about the motion of a given system without requiring a complete integration of the problem.*

In many problems a number of first integrals of the equations of motion can be obtained immediately; by this we mean relations of the type

$$f(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots, t) = \text{constant}. \tag{2.43}$$

*In this and succeeding sections it will be assumed, unless otherwise specified, the system is such that its motion is completely described by a Hamilton's principle of the form (2.2).

which are first-order differential equations. These first integrals are of interest because they tell us something physically about the system. They include, in fact, the conservation laws obtained in Chapter 1.

Let us consider as an example a system of mass points under the influence of forces derived from potentials dependent on position only. Then

$$\frac{\partial L}{\partial \dot{x}_i} \equiv \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)$$
$$= m_i \dot{x}_i = p_{ix},$$

which is the x component of the linear momentum associated with the ith particle. This result suggests an obvious extension to the concept of momentum. The generalized momentum associated with the coordinate q_j shall be defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{2.44}$$

The terms *canonical momentum* and *conjugate momentum* are often also used for p_j. Notice that if q_j is not a Cartesian coordinate, p_j does not necessarily have the dimensions of a linear momentum. Further, if there is a velocity-dependent potential, then even with a Cartesian coordinate q_j the associated *generalized* momentum will not be identical with the usual *mechanical* momentum. Thus, in the case of a group of particles in an electromagnetic field, the Lagrangian is (cf. 1.63)

$$L = \sum_i \frac{1}{2} m_i \dot{r}_i^2 - \sum_i q_i \phi(x_i) + \sum_i q_i \mathbf{A}(x_i) \cdot \dot{\mathbf{r}}_i$$

(q_i here denotes charge) and the generalized momentum conjugate to x_i is

$$p_{ix} = \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i + q_i A_x, \tag{2.45}$$

i.e., mechanical momentum plus an additional term.

If the Lagrangian of a system does not contain a given coordinate q_j (although it may contain the corresponding velocity $\dot{q}_j$), then the coordinate is said to be *cyclic* or *ignorable*. This definition is not universal, but it is the customary one and will be used here. The Lagrange equation of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0,$$

reduces, for a cyclic coordinate, to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

or

$$\frac{dp_j}{dt} = 0,$$

which mean that

$$p_j = \text{constant}. \tag{2.46}$$

Hence, we can state as a general conservation theorem that *the generalized momentum conjugate to a cyclic coordinate is conserved.*

Note that the derivation of Eq. (2.46) assumes that q_j is a generalized coordinate; one that is linearly independent of all the other coordinates. When equations of constraint exist, all the coordinates are not linearly independent. For example, the angular coordinate θ is not present in the Lagrangian of a hoop rolling without slipping in a horizontal plane that was previously discussed, but the angle appears in the constraint equations $rd\theta = dx$. As a result, the angular momentum, $p_\theta = mr^2\dot{\theta}$, is not a constant of the motion.

Equation (2.46) constitutes a first integral of the form (2.43) for the equations of motion. It can be used formally to eliminate the cyclic coordinate from the problem, which can then be solved entirely in terms of the remaining generalized coordinates. Briefly, the procedure, originated by Routh, consists in modifying the Lagrangian so that it is no longer a function of the generalized velocity corresponding to the cyclic coordinate, but instead involves only its conjugate momentum. The advantage in so doing is that p_j can then be considered one of the constants of integration, and the remaining integrations involve only the noncyclic coordinates. We shall defer a detailed discussion of Routh's method until the Hamiltonian formulation (to which it is closely related) is treated.

Note that the conditions for the conservation of generalized momenta are more

general than the two momentum conservation theorems previously derived. For example, they furnish a conservation theorem for a case in which the law of action and reaction is violated, namely, when electromagnetic forces are present. Suppose we have a single particle in a field in which neither ϕ nor $\mathbf{A}$ depends on x. Then x nowhere appears in L and is therefore cyclic. The corresponding canonical momentum p_x must therefore be conserved. From (1.63) this momentum now has the form

$$p_x = m\dot{x} + qA_x = \text{constant}. \tag{2.47}$$

In this case, it is not the mechanical linear momentum $m\dot{x}$ that is conserved but rather its sum with qA_x.* Nevertheless, it should still be true that the conservation theorems of Chapter 1 are contained within the general rule for cyclic coordinates; with proper restrictions (2.46) should reduce to the theorems of Section 1.2.

*It can be shown from classical electrodynamics that under these conditions, i.e., neither $\mathbf{A}$ nor ϕ depending on x, that qA_x is exactly the x-component of the electromagnetic linear momentum of the field associated with the charge q.

We first consider a generalized coordinate q_j, for which a change dq_j represents a translation of the system as a whole in some given direction. An example would be one of the Cartesian coordinates of the center of mass of the system. Then clearly q_j cannot appear in T, for velocities are not affected by a shift in the origin, and therefore the partial derivative of T with respect to q_j must be zero. Further, we will assume conservative systems for which V is not a function of the velocities, so as to eliminate such complications as electromagnetic forces. The Lagrange equation of motion for a coordinate so defined then reduces to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} \equiv \dot{p}_j = -\frac{\partial V}{\partial q_j} \equiv Q_j. \tag{2.48}$$

We will now show that (2.48) is the equation of motion for the total linear momentum, i.e., that Q_j represents the component of the total force along the direction of translation of q_j, and p_j is the component of the total linear momentum along this direction. In general, the generalized force Q_j is given by Eq. (1.49):

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

Since dq_j corresponds to a translation of the system along some axis, the vectors $\mathbf{r}_i(q_j)$ and $\mathbf{r}_i(q_j + dq_j)$ are related as shown in Fig. 2.7. By the definition of a derivative, we have

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\mathbf{r}_i(q_j + dq_j) - \mathbf{r}_i(q_j)}{dq_j} = \frac{dq_j}{dq_j}\mathbf{n} = \mathbf{n}, \tag{2.49}$$

where $\mathbf{n}$ is the unit vector along the direction of the translation. Hence,

$$Q_j = \sum \mathbf{F}_i \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{F},$$

which (as was stated) is the component of the total force in the direction of $\mathbf{n}$. To prove the other half of the statement, note that with the kinetic energy in the form

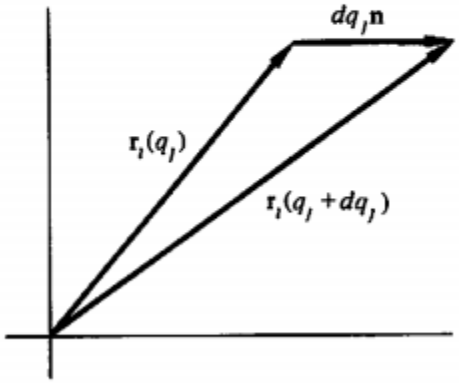

FIGURE 2.7 Change in a position vector under translation of the system.

$$T = \frac{1}{2} \sum m_i \dot{\mathbf{r}}_i^2,$$

the conjugate momentum is

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}$$

$$= \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

using Eq. (1.51). Then from Eq. (2.49)

$$p_j = \mathbf{n} \cdot \sum_i m_i \mathbf{v}_i,$$

which again, as predicted, is the component of the total system linear momentum along $\mathbf{n}$.

Suppose now that the translation coordinate q_j that we have been discussing is cyclic. Then q_j cannot appear in V and therefore

$$-\frac{\partial V}{\partial q_j} \equiv Q_j = 0.$$

But this is simply the familiar conservation theorem for linear momentum—that if a given component of the total applied force vanishes, the corresponding component of the linear momentum is conserved.

In a similar fashion, it can be shown that if a cyclic coordinate q_j is such that dq_j corresponds to a rotation of the system of particles around some axis, then the conservation of its conjugate momentum corresponds to conservation of an angular momentum. By the same argument used above, T cannot contain q_j, for a rotation of the coordinate system cannot affect the magnitude of the velocities. Hence, the partial derivative of T with respect to q_j must again be zero, and since V is independent of $\dot{q}_j$, we once more get Eq. (2.48). But now we wish to show that with q_j a rotation coordinate the generalized force is the component of the total applied torque about the axis of rotation, and p_j is the component of the total angular momentum along the same axis.

The generalized force Q_j is again given by

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

only the derivative now has a different meaning. Here the change in q_j must correspond to an infinitesimal rotation of the vector $\mathbf{r}_i$, keeping the magnitude of the vector constant. From Fig. 2.8, the magnitude of the derivative can easily be obtained:

$$|d\mathbf{r}_i| = r_i \sin\theta \, dq_j$$

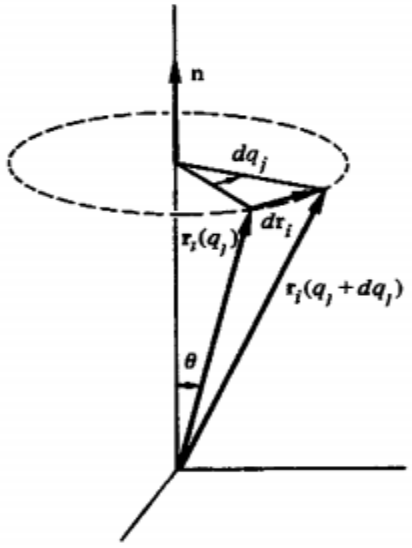

FIGURE 2.8 Change of a position vector under rotation of the system.

and

$$\left|\frac{\partial \mathbf{r}_i}{\partial q_j}\right| = r_i \sin\theta,$$

and its direction is perpendicular to both $\mathbf{r}_i$ and $\mathbf{n}$. Clearly, the derivative can be written in vector form as

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n} \times \mathbf{r}_i. \tag{2.50}$$

With this result, the generalized force becomes

$$Q_j = \sum_i \mathbf{F}_i \cdot \mathbf{n} \times \mathbf{r}_i$$
$$= \sum_i \mathbf{n} \cdot \mathbf{r}_i \times \mathbf{F}_i,$$

reducing to

$$Q_j = \mathbf{n} \cdot \sum_i \mathbf{N}_i = \mathbf{n} \cdot \mathbf{N},$$

which proves the first part. A similar manipulation of p_j with the aid of Eq. (2.50) provides proof of the second part of the statement:

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \mathbf{n} \cdot \mathbf{r}_i \times m_i \mathbf{v}_i = \mathbf{n} \cdot \sum \mathbf{L}_i = \mathbf{n} \cdot \mathbf{L}.$$

2.7 ■ ENERGY FUNCTION AND THE CONSERVATION OF ENERGY

Another conservation theorem we should expect to obtain in the Lagrangian formulation is the conservation of total energy for systems where the forces are derivable from potentials dependent only upon position. Indeed, it is possible to demonstrate a conservation theorem for which conservation of total energy represents only a special case. Consider a general Lagrangian, which will be a function of the coordinates q_j and the velocities $\dot{q}_j$ and may also depend explicitly on the time. (The explicit time dependence may arise from the time variation of external potentials, or from time-dependent constraints.) Then the total time derivative of L is

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}. \tag{2.51}$$

From Lagrange's equations,

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right),$$

and (2.51) can be rewritten as

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t}.$$

It therefore follows that

$$\frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right) + \frac{\partial L}{\partial t} = 0. \tag{2.52}$$

The quantity in parentheses is oftentimes called the *energy function** and will be denoted by h:

$$h(q_1, \ldots, q_n; \quad \dot{q}_1, \ldots, \dot{q}_n; \quad t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L, \tag{2.53}$$

and Eq. (2.52) can be looked on as giving the total time derivative of h:

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}. \tag{2.54}$$

If the Lagrangian is not an explicit function of time, i.e., if t does not appear in L explicitly but only implicitly through the time variation of q and $\dot{q}$, then Eq. (2.54) says that h is conserved. It is one of the first integrals of the motion and is sometimes referred to as Jacobi's integral.†

*The energy function h is identical in value with the Hamiltonian H (See Chapter 8) It is given a different name and symbol here to emphasize that h is considered a function of n independent variables q_j and their time derivatives $\dot{q}_j$ (along with the time), whereas the Hamiltonian will be treated as a function of $2n$ independent variables, q_j, p_j (and possibly the time)

†This designation is most often confined to a first integral in the restricted three-body problem. However, the integral there is merely a special case of the energy function h, and there is some historical precedent to apply the name Jacobi integral to the more general situation

Under certain circumstances, the function h is the total energy of the system. To determine what these circumstances are, we recall that the total kinetic energy of a system can always be written as

$$T = T_0 + T_1 + T_2, \tag{1.73}$$

where T_0 is a function of the generalized coordinates only, $T_1(q, \dot{q})$ is linear in the generalized velocities, and $T_2(q, \dot{q})$ is a quadratic function of the $\dot{q}$'s. For a very wide range of systems and sets of generalized coordinates, the Lagrangian can be similarly decomposed as regards its functional behavior in the $\dot{q}$ variables:

$$L(q, \dot{q}, t) = L_0(q, t) + L_1(q, \dot{q}, t) + L_2(q, \dot{q}, t). \tag{2.55}$$

Here L_2 is a homogeneous function of the second degree (not merely quadratic) in $\dot{q}$, while L_1 is homogeneous of the first degree in $\dot{q}$. There is no reason intrinsic to mechanics that requires the Lagrangian to conform to Eq. (2.55), but in fact it does for most problems of interest. The Lagrangian clearly has this form when the forces are derivable from a potential not involving the velocities. Even with the velocity-dependent potentials, we note that the Lagrangian for a charged particle in an electromagnetic field, Eq. (1.63), satisfies Eq. (2.55). Now, recall that Euler's theorem states that if f is a homogeneous function of degree n in the variables x_i, then

$$\sum_i x_i \frac{\partial f}{\partial x_i} = nf. \tag{2.56}$$

Applied to the function h, Eq. (2.53), for the Lagrangians of the form (2.55), this theorem implies that

$$h = 2L_2 + L_1 - L = L_2 - L_0. \tag{2.57}$$

If the transformation equations defining the generalized coordinates, Eqs. (1.38), do not involve the time explicitly, then by Eqs. (1.73) $T = T_2$. If, further, the potential does not depend on the generalized velocities, then $L_2 = T$ and $L_0 = -V$, so that

$$h = T + V = E, \tag{2.58}$$

and the energy function is indeed the total energy. Under these circumstances, if V does not involve the time explicitly, neither will L. Thus, by Eq. (2.54), h

Note that the conditions for conservation of h are in principle quite distinct from those that identify h as the total energy. We can have a set of generalized coordinates such that in a particular problem h is conserved but is not the total energy. On the other hand, h can be the total energy, in the form $T + V$, but not be conserved. Also note that whereas the Lagrangian is uniquely fixed for each

system by the prescription

$$L = T - U$$

independent of the choice of generalized coordinates, the energy function h depends in magnitude and functional form on the specific set of generalized coordinates. For one and the same system, various energy functions h of different physical content can be generated depending on how the generalized coordinates are chosen.

The most common case that occurs in classical mechanics is one in which the kinetic energy terms are all of the form $m\dot{q}_i^2/2$ or $p_i^2/2m$ and the potential energy depends only upon the coordinates. For these conditions, the energy function is both conserved and is also the total energy.

Finally, note that where the system is not conservative, but there are frictional forces derivable from a dissipation function $\mathcal{F}$, it can be easily shown that $\mathcal{F}$ is related to the decay rate of h. When the equations of motion are given by Eq. (1.70), including dissipation, then Eq. (2.52) has the form

$$\frac{dh}{dt} + \frac{\partial L}{\partial t} = \sum_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} \dot{q}_j.$$

By the definition of $\mathcal{F}$, Eq. (1.67), it is a homogeneous function of the $\dot{q}$'s of degree 2. Hence, applying Euler's theorem again, we have

$$\frac{dh}{dt} = -2\mathcal{F} - \frac{\partial L}{\partial t}. \tag{2.59}$$

If L is not an explicit function of time, *and* the system is such that h is the same as the energy, then Eq. (2.59) says that $2\mathcal{F}$ is the rate of energy dissipation,

$$\frac{dE}{dt} = -2\mathcal{F}, \tag{2.60}$$

a statement proved above (cf. Sec. 1.5) in less general circumstances.

DERIVATIONS

1. Complete the solution of the brachistochrone problem begun in Section 2.2 and show that the desired curve is a cycloid with a cusp at the initial point at which the particle is released. Show also that if the particle is projected with an initial kinetic energy $\frac{1}{2}mv_0^2$ that the brachistochrone is still a cycloid passing through the two points with a cusp at a height z above the initial point given by $v_0^2 = 2gz$.

2. Show that if the potential in the Lagrangian contains velocity-dependent terms, the canonical momentum corresponding to a coordinate of rotation θ of the entire system is no longer the mechanical angular momentum L_θ but is given by

$$p_\theta = L_\theta - \sum_i \mathbf{n} \cdot \mathbf{r}_i \times \nabla_{\mathbf{v}_i} U,$$

where $\nabla_{\mathbf{v}}$ is the gradient operator in which the derivatives are with respect to the velocity components and $\mathbf{n}$ is a unit vector in the direction of rotation. If the forces are electromagnetic in character, the canonical momentum is therefore

$$p_\theta = L_\theta + \sum_i \mathbf{n} \cdot \mathbf{r}_i \times \frac{q_i}{c} \mathbf{A}_i.$$

3. Prove that the shortest distance between two points in space is a straight line.

4. Show that the geodesics of a spherical surface are great circles, i.e., circles whose centers lie at the center of the sphere.

POSSIBLE QUESTIONS

Part B (6 Marks)

1. Define canonical momentum. Show that the generalized momentum conjugate to a cyclic co-ordinate is conserved
2. Write a short note on physical significance of the lagrangian undeterminant multiplier
3. Derive the Brachistochrome problem
4 Find the curve for which some line integral has a stationary value
5. State and prove Euler Lagrange differential equation
6. Find the equation of motion of a hoop or disc rolling without slipping down on the inclined plane.

Part C (10 Marks)

1. Find the minimum surface of revolution
2. Derive the conservation theorem for total energy of system
3. Show that the shortest curve between any two points in the plane is a straight line
4. Derive the conservation theorem for dissipation function

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Mechanics **Subject Code: 19MMP106**

Class : I - M.Sc. Mathematics **Semester : I**

Unit II

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
_____ of a force is defined to be the product of the force and perpendicular distance of the point.	Friction	Moment	Angular momentum	Mass	Moment
_____ of a force about a point is that it measures the tendency to rotate the body about that point.	Friction	Mass	Angular momentum	Momentum	Momentum
The sign is _____ if F rotates in anticlockwise direction.	Positive	Negative	Zero	Unity	Positive
The sign is _____ if F rotates in clockwise direction.	Positive	Negative	Zero	Unity	Negative
If the unit of a force is a poundal and the unit of distance be foot then the unit of moment is a _____	Poundal	Foot	Poundal-foot	Foot-poundal	Poundal-foot
If the unit of a force be dyne and the unit of distance be cm then the unit of moment is a _____	Dyne	Cm	Dyne-cm	Cm-dyne	Dyne-cm
The algebraic sum of the moment of the two forces about any point in their plane is equal to the _____ of their resultant.	Mass	Moment	Friction	Weight	Moment

The vector sum of the moment of two forces about the point in the plane is equal to the _____ of their resultant about the same point.	Friction	Mass	Weight	Moment	Moment
Resultant R=0 means there is no _____for the system.	Resultant	Resultant	Friction	Force	Resultant
If a system of a coplanar forces is in equilibrium then the algebraic sum of their moment about any point is _____	One	Zero	Two	Infinity	Zero
If ∑Pipi=0 then PR=_____	One	Two	Three	Zero	Zero
If a heavy body is dragged along the ground by means of a horizontal force a resistance called the _____	Moment	Friction	Tangential force	Coefficient of friction	Tangential force
The tangential force is always acts in the direction _____ to that of applied horizontal force.	Same	Opposite	Right	Left	Opposite
If the ground is perfectly smooth then there is no _____.	Tangential resistance	Friction	Tangential force	Moment	Tangential resistance
Smooth bodies are capable of offering _____ in the normal direction.	Force	Resistance	Moment	Coefficient of friction	Resistance
Smooth bodies are capable of offering resistance only in the _____ direction.	Same	Opposite	Normal	Downward	Normal
If two rough bodies be in contact then there is a tangential resistance is called_____	Friction	Moment	Resistance	Angular momentum	Friction
The friction always acts in the _____ direction to the direction of body which has tendency to move.	Same	Opposite	Upward	Downward	Opposite
_____ prevents the sliding of one body over the other.	Moment	Resistance	Friction	Angular momentum	Friction
The force exerted by the friction is called _____	Coefficient of friction	Cone of friction	Angle of friction	Force of friction	Force of friction
Force of friction is also called _____	Massive force	Passive force	Coplanar force	Colinear force	Passive force

The force which opposes the rolling of one body over another is called the _____	Sliding friction	Rolling friction	Coefficient of friction	Cone of friction	Rolling friction
The force which opposes the sliding of one body over another is called the _____	Sliding friction	Rolling friction	Coefficient of friction	Cone of friction	Sliding friction
When two bodies are in contact with one another are in equilibrium then the equilibrium is callled _____	Limiting friction	Cone of friction	Limiting equilibrium	Angle of friction	Limiting equilibrium
The force which opposes the _____ of one body over the other is called sliding friction.	Rolling	Sliding	Equilibrium	Existing	Sliding
The force which opposes the _____ of one body over the other is called rolling friction.	Rolling	Sliding	Equilibrium	Existing	Rolling
Friction is a _____ force.	Rolling	Sliding	Self adjusting	Resultant	Self adjusting
The maximum value of force of friction is called _____	Limiting friction	Limiting equilibrium	Force of friction	Coefficient of friction	Limiting friction
Dynamical friction is _____ than the limiting friction.	Equal	Less	Greater	None	Less
The resultant of the normal reaction and the force of friction is called _____	Limiting friction	Resultant reaction	Angle of friction	Cone of friction	Resultant reaction
The angle which the resultant reaction makes with the normal reaction is called _____	Angle of friction	Cone of friction	Coefficient of friction	Force of friction	Angle of friction
_____ is a self adjusting force.	Friction	Moment	Angular momentum	Force	Friction
_____ friction is less than the limiting friction.	Rolling friction	Sliding friction	Dynamical friction	Angle of friction	Dynamical friction
The ratio of the limiting friction and the normal reaction is called the _____	Angle of friction	Cone of friction	Coefficient of friction	Force of friction	Coefficient of friction
The coefficient of friction is defined as the tangent of the _____	Angle of friction	Cone of friction	Coefficient of friction	Force of friction	Angle of friction
Coefficient of friction is denoted by the symbol _____	α	Φ	μ	λ	μ
Angle of friction is denoted by _____	α	Φ	μ	λ	λ

μ lies between _____ and _____	One and Unity	Zero and Unity	One and Infinity	Zero and Infinity	Zero and Unity
μ= _____	tanλ	Sinλ	Cosλ	Cotλ	tanλ
For perfectly smooth bodies μ= _____	One	Zero	Two	Infinity	Zero
An ideal surface for which the coefficient of friction is Zero is said to be _____	Smooth	Rough	Perfectly smooth	Perfectly rough	Smooth
An ideal surface for which the coefficient of friction is Infinity is said to be _____	Smooth	Rough	Perfectly smooth	Perfectly rough	Perfectly rough
Surfaces which are neither perfectly rough nor perfectly smooth their coefficient of friction will lies between _____	One and Unity	Zero and Unity	One and Infinity	Zero and Infinity	Zero and Unity
If α>λ then the body will move _____ the plane.	Up	Down	Right	Left	Down
If α=λ then the body will be in _____	Up	Down	Equilibrium	Right	Equilibrium
The direction of dynamical friction is _____ to the direction of motion of the body.	Same	Opposite	Equal	None	Opposite
The coefficient of kinetic friction is denoted by the symbol _____	μ	λ	μ'	λ'	μ'
The coefficient of _____ depends only on the nature of the surfaces in contact.	Friction	Kinetic friction	Statical friction	Dynamical friction	Kinetic friction
The angle of repose is equal to the _____	Angle of friction	Cone of friction	Coefficient of friction	Kinetic friction	Angle of friction
The _____ is equal to the angle of friction.	Coefficient of friction	Kinetic friction	Dynamical friction	Angle of repose	Angle of repose

UNIT-III

Hamilton Equations of motion: Legendre Transformations and the Hamilton Equations of motion-canonical equations of Hamilton – Cyclic coordinates and conservation theorems – Routh's procedure - Derivation of Hamilton's equations from a variational principle – The principle of least action.

8.1 ■ LEGENDRE TRANSFORMATIONS AND THE HAMILTON EQUATIONS OF MOTION

In the Lagrangian formulation (nonrelativistic), a system with n degrees of freedom possesses n equations of motion of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{8.1}$$

As the equations are of second order, the motion of the system is determined for all time only when $2n$ initial values are specified, for example, the n q_i's and n $\dot{q}_i$'s at a particular time t_1, or then n q_i's at *two* times, t_1 and t_2. We represent the state of the system by a point in an n-dimensional *configuration space* whose coordinates are the n generalized coordinates q_i and follow the motion of the system point in time as it traverses its path in configuration space. Physically, in the Lagrangian viewpoint a system with n independent degrees of freedom is a problem in n independent variables $q_i(t)$, and $\dot{q}_i$ appears only as a shorthand for the time derivative of q_i. All n coordinates must be independent. In the Hamiltonian formulation there can be no constraint equations among the coordinates. If the n coordinates are not independent, a reduced set of m coordinates, with $m < n$, must be used for the formulation of the problem before proceeding with the following steps.

The Hamiltonian formulation is based on a fundamentally different picture. We seek to describe the motion in terms of *first-order* equations of motion. Since the number of initial conditions determining the motion must of course still be $2n$, there must be $2n$ independent first-order equations expressed in terms of $2n$ *independent variables*. Hence, the $2n$ equations of the motion describe the behavior of the system point in a *phase space* whose coordinates are the $2n$ independent variables. In thus doubling our set of independent quantities, it is natural (though not inevitable) to choose half of them to be the n generalized coordinates q_i. As we shall see, the formulation is nearly symmetric if we choose the other half of the set to be the generalized or *conjugate momenta* p_i already introduced by the definition (cf. Eq. (2.44)):

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i} \qquad \text{(no sum on } j\text{)} \tag{8.2}$$

We wish now to change the basis of description from x, y to a new distinct set of variables u, y, so that differential quantities are expressed in terms of the differentials du and dy. Let g be a function of u and y defined by the equation

$$g = f - ux. \tag{8.5}$$

A differential of g is then given as

$$dg = df - u\,dx - x\,du,$$

or, by (8.3), as

$$dg = v\,dy - x\,du,$$

which is exactly in the form desired. The quantities x and v are now functions of the variables u and y given by the relations

$$x = -\frac{\partial g}{\partial u}, \qquad v = \frac{\partial g}{\partial y}, \tag{8.6}$$

where the j index shows the set of q's and $\dot{q}$'s. The quantities (q, p) are known as the *canonical variables.**

From the mathematical viewpoint, it can however be claimed that the q's and $\dot{q}$'s have been treated as distinct variables. In Lagrange's equations, Eq. (8.1), the partial derivative of L with respect to q_i means a derivative taken with all other q's and all $\dot{q}$'s constant. Similarly, in the partial derivatives with respect to $\dot{q}$, the q's are kept constant. Treated strictly as a mathematical problem, the transition from Lagrangian to Hamiltonian formulation corresponds to changing the variables in our mechanical functions from $(q, \dot{q}, t)$ to (q, p, t), where p is related to q and $\dot{q}$ by Eqs. (8.2). The procedure for switching variables in this manner is provided by the *Legendre transformation*, which is tailored for just this type of change of variable.

Consider a function of only two variables $f(x, y)$, so that a differential of f has the form

$$df = u\,dx + v\,dy, \tag{8.3}$$

where

$$u = \frac{\partial f}{\partial x}, \qquad v = \frac{\partial f}{\partial y}. \tag{8.4}$$

*Unless otherwise specified, in this and subsequent chapters the symbol p will be used only for the conjugate or canonical momentum. When the forces are velocity dependent, the canonical momentum will differ from the corresponding mechanical momentum (cf. Eq. (2 47)).

We wish now to change the basis of description from x, y to a new distinct set of variables u, y, so that differential quantities are expressed in terms of the differentials du and dy. Let g be a function of u and y defined by the equation

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which is exactly in the form desired. The quantities x and v are now functions of the variables u and y given by the relations

$$x = -\frac{\partial g}{\partial u}, \qquad v = \frac{\partial g}{\partial y}, \tag{8.6}$$

which are the analogues of Eqs. (8.4).

The Legendre transformation so defined is used frequently in thermodynamics. The first law of thermodynamics relates the differential change in energy, dU, to the corresponding change in heat content, dQ, and the work done, dW:

$$dU = dQ - dW. \tag{8.7}$$

For a gas undergoing a reversible process, Eq. (8.7) can be written as

$$dU = T\,dS - P\,dV, \tag{8.8}$$

where $U(S, V)$ is written as a function of the entropy, S, and the volume, V, where the temperature, T, and the gas pressure, P, are given by

$$T = \frac{\partial U}{\partial S} \qquad \text{and} \qquad P = -\frac{\partial U}{\partial V}. \tag{8.9}$$

The enthalpy, $H(S, P)$ is generated by the Legendre transformation

$$H = U + PV, \tag{8.10}$$

which gives

$$dH = T\,dS + V\,dP. \tag{8.11}$$

where

$$T = \frac{\partial H}{\partial S} \qquad \text{and} \qquad V = \frac{\partial H}{\partial P}.$$

Additional Legendre transformations,

$$F = U - TS$$
$$G = H - TS, \tag{8.12}$$

generate the Helmholtz free energy, $F(T, V)$, and the Gibbs free energy, $G(T, P)$.

The transformation from $(q, \dot{q}, t)$ to (q, p, t) differs from the type considered in Eqs. (8.3) to (8.12) only in that more than one variable is to be transformed. We begin by writing the differential of the Lagrangian, $L(q, \dot{q}, t)$, as

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{8.13}$$

The canonical momentum was defined in Eq. (2.44) as $p_i = \partial L / \partial \dot{q}_i$; substituting this into the Lagrange equation (8.1), we obtain

$$\dot{p}_i = \frac{\partial L}{\partial q_i}, \tag{8.14}$$

so Eq. (8.13) can be written as

$$dL = \dot{p}_i \, dq_i + p_i \, d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{8.13'}$$

The Hamiltonian $H(q, p, t)$ is generated by the Legendre transformation

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t), \tag{8.15}$$

which has the differential

which has the differential

$$dH = \dot{q}_i \, dp_i - \dot{p}_i \, dq_i - \frac{\partial L}{\partial t}, \tag{8.16}$$

where the term $p_i \, d\dot{q}_i$ is removed by the Legendre transformation. Since dH can also be written as

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \tag{8.17}$$

we obtain the $2n + 1$ relations

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \end{aligned}\right\} \tag{8.18}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{8.19}$$

Equations (8.18) are known as the *canonical equations of Hamilton*; they constitute the desired set of $2n$ first-order equations of motion replacing the n second-order Lagrange equations.*

The first half of Hamilton's equations give the $\dot{q}_i$'s as functions of (q, p, t). They form therefore the inverse of the constitutive equations (8.2), which define the momenta p_i as functions of $(q, \dot{q}, t)$. It may therefore be said that they provide no new information. In terms of solving mechanical problems by means of the canonical equations, the statement is correct. But within the framework of the Hamiltonian picture, where $H(q, p, t)$ is some given function obtained no matter how, the two halves of the set of Hamiltonian equations are equally independent and meaningful. The first half says how $\dot{q}$ depends on q, p, and t; the second says the same thing for $\dot{p}$.

Of course, the Hamiltonian H is constructed in the same manner, and has identically the same value, as h, the energy function defined in Eq. (2.53). But they are functions of different variables: Like the Lagrangian, h is a function of q, $\dot{q}$ (and possibly t), while H must always be expressed as a function of q, p (and possibly t). It is to emphasize this difference in functional behavior that different symbols have been given to the quantities even though they have the same numerical values.

Nominally, the Hamiltonian for each problem must be constructed via the Lagrangian formulation. The formal procedure calls for a lengthy sequence of steps:

1. With a chosen set of generalized coordinates, q_i, the Lagrangian $L(q_i, \dot{q}_i, t) = T - V$ is constructed.
2. The conjugate momenta are defined as functions of q_i, $\dot{q}_i$, and t by Eqs. (8.2).
3. Equation (8.15) is used to form the Hamiltonian. At this stage we have some mixed function of q_i, $\dot{q}_i$, p_i, and t.
4. Equations (8.2) are then inverted to obtain $\dot{q}_i$ as functions of (q, p, t). Possible difficulties in the inversion will be discussed below.
5. The results of the previous step are then applied to eliminate $\dot{q}$ from H so as to express it solely as a function of (q, p, t).

Now we are ready to use the Hamiltonian in the canonical equations of motion.

For many physical systems it is possible to shorten this drawn-out sequence quite appreciably. As has been described in Section 2.7, in many problems the Lagrangian is the sum of functions each homogeneous in the generalized veloc-

*Canonical is used here presumably in the sense of designating a simple, general set of standard equations. It appears that the term was first introduced by C. G. J. Jacobi in 1837 (*Comptes rendus de l'Académie des Sciences de Paris*, 5, p 61) but in a slightly different context referring to an application of Hamilton's equations of motion to perturbation theory. Although the term rapidly gained common usage, the reason for its introduction apparently remained obscure even to contemporaries. By 1879, only 45 years after Hamilton explicitly introduced his equations, Thomson (Lord Kelvin) and Tait were moved by the adjective "canonical" to exclaim "Why it has been so called would be hard to say"

ities of degree 0, 1, and 2, respectively. In that case, H by the prescription of Eq. (8.15) is given by (cf. Eqs. (2.53) and (2.55))

$$H = \dot{q}_i p_i - L = \dot{q}_i p_i - [L_0(q_i, t) + L_1(q_i, t)\dot{q}_k + L_2(q_i, t)\dot{q}_k\dot{q}_m] \quad (8.20)$$

(no sum on i in the square brackets) where L_0 is the part of the Lagrangian that is independent of the generalized velocities, L_1 represents the coefficients of the part of the Lagrangian that is homogeneous in $\dot{q}_i$ in the first degree, and L_2 is the part that is homogeneous in $\dot{q}_i$ in the second degree. Further, if the equations defining the generalized coordinates don't depend on time explicitly, then $L_2\dot{q}_k\dot{q}_m = T$ (the kinetic energy), and if the forces are derivable from a conservative potential V (that is, work is independent of the path), then $L_0 = -V$. When both these conditions are satisfied, the Hamiltonian is *automatically* the total energy:

$$H = T + V = E. \quad (8.21)$$

If either Eq. (8.20) or (8.21) holds, then much of the algebra in steps 3 and 4 above is eliminated.

We can at times go further. In large classes of problems, it happens that L_2 is a quadratic function of the generalized velocities and L_1 is a linear function of the same variables with the following specific functional dependencies:

$$L(q_i, \dot{q}_i, t) = L_0(q, t) + \dot{q}_i a_i(q, t) + \dot{q}_i^2 T_i(q, t), \quad (8.22)$$

where the a_i's and the T_i's are functions of the q's and t.

The algebraic manipulations required in steps 2–5 can then be carried out, at least formally, once and for all. To show this, let us form the $\dot{q}_i$'s into a single column matrix $\dot{\mathbf{q}}$. Under the given assumptions the Lagrangian can be written as

$$L(q, \dot{q}, t) = L_0(q. t) + \tilde{\dot{\mathbf{q}}}\mathbf{a} + \tfrac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}}, \quad (8.23)$$

where the single row matrix $\tilde{\dot{\mathbf{q}}}$ has been written explicitly as the transpose of a single column matrix, $\dot{\mathbf{q}}$. Here $\mathbf{a}$ is a column matrix, and $\mathbf{T}$ is a square $n \times n$ matrix (much like the corresponding matrix introduced in Section 6.2). The elements of both are in general functions of q and t. To illustrate this formalism, let us consider the special case where $q_i = \{x, y, z\}$ and $\mathbf{T}$ is diagonal. We would then write

$$\frac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}} = \frac{1}{2}(\dot{x}\dot{y}\dot{z})\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad (8.24a)$$

and

$$\tilde{\dot{\mathbf{q}}}\mathbf{a} = (\dot{x}\dot{y}\dot{z})\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x\dot{x} + a_y\dot{y} + a_z\dot{z} = \mathbf{a} \cdot \dot{\mathbf{r}}. \quad (8.24b)$$

In this notation the Hamiltonian, $H = \tilde{\dot{\mathbf{q}}}\mathbf{p} - L$, becomes

$$H = \tilde{\dot{\mathbf{q}}}(\mathbf{p} - \mathbf{a}) - \tfrac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}} - L_0. \tag{8.24c}$$

The conjugate momenta, considered as a column matrix $\mathbf{p}$, is then, by Eq. (8.2), given as

$$\mathbf{p} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{a}, \tag{8.25}$$

which can be inverted (step 4) to the column vector $\dot{\mathbf{q}}$

$$\dot{\mathbf{q}} = \mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}). \tag{8.26a}$$

This step presupposes that $\mathbf{T}^{-1}$ exists, which it normally does by virtue of the positive definite property of kinetic energy.

The corresponding equation for $\tilde{\dot{\mathbf{q}}}$ is

$$\tilde{\dot{\mathbf{q}}} = (\tilde{\mathbf{p}} - \tilde{\mathbf{a}})\mathbf{T}^{-1}. \tag{8.26b}$$

To obtain the correct functional form for the Hamiltonian, Eqs. (8.26) must be used to replace $\dot{\mathbf{q}}$ and $\tilde{\dot{\mathbf{q}}}$, yielding the final form for the Hamiltonian:

$$H(q, p, t) = \tfrac{1}{2}(\tilde{\mathbf{p}} - \tilde{\mathbf{a}})\mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}) - L_0(q, t). \tag{8.27}$$

If the Lagrangian can be written in the form of Eq. (8.23), then we can immediately skip the intervening steps and write the Hamiltonian as Eq. (8.27). The inverse matrix $\mathbf{T}^{-1}$ can usually most easily be obtained straightforwardly as

$$\mathbf{T}^{-1} = \frac{\tilde{\mathbf{T}}_c}{|\mathbf{T}|}, \tag{8.28}$$

where $\mathbf{T}_c$ is the cofactor matrix whose elements $(\mathbf{T}_c)_{jk}$ are $(-1)^{j+k}$ times the determinant of the matrix obtained by striking out the jth row and the kth column of $\mathbf{T}$.

In the example Eq. (8.24a), these three matrices are given explicitly by

$$\mathbf{T} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \qquad \mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix}, \qquad \text{and}$$

$$\tilde{\mathbf{T}}_c = \begin{bmatrix} m^2 & 0 & 0 \\ 0 & m^2 & 0 \\ 0 & 0 & m^2 \end{bmatrix},$$

and the determinant $|\mathbf{T}| = m^3$. It is easy to see that for the usual case when $\mathbf{T}$ is diagonal, then $\mathbf{T}^{-1}$ is also diagonal with elements that are just the reciprocals of the corresponding elements of $\mathbf{T}$.

A number of exercises in applying this formalism to various mechanical systems will be found in the problems at the end of the chapter. Two very simple examples are considered here because they illustrate some important aspects of the technique. First consider the spatial motion of a particle in a central force field, using spherical polar coordinates $(r, \theta. \phi)$ for the generalized coordinates. The potential energy is some function $V(r)$ and the kinetic energy is

$$T = \frac{mv^2}{2} = \frac{m}{2}(\dot{r}^2 + r^2 \sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2). \tag{8.28'}$$

Clearly the Hamiltonian has the form of Eq. (8.21) and corresponds to the total energy $T + V$. Since $\mathbf{T}$ is diagonal the form of H is, by inspection,

$$H(r, \theta, p_r, p_\theta, p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(r). \tag{8.29}$$

Note that the Hamiltonian would have a different functional form if the generalized coordinates were chosen to be the Cartesian coordinates x_i of the particle. If we make that choice, then the kinetic energy has the form

$$T = \frac{mv^2}{2} = \frac{m\dot{x}_i\dot{x}_i}{2},$$

so that the Hamiltonian is now

$$H(x_i, p_i) = \frac{p_i p_i}{2m} + V(r). \tag{8.30}$$

It is sometimes convenient to form the canonical momenta p_i conjugate to x_i into a vector $\mathbf{p}$ such that the Hamiltonian can be written as

It is sometimes convenient to form the canonical momenta p_i conjugate to x_i into a vector $\mathbf{p}$ such that the Hamiltonian can be written as

$$H(x_i, p_i) = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + V(\sqrt{x_i x^i}). \tag{8.31}$$

We can of course take the components of $\mathbf{p}$ relative to any coordinate system we desire, curvilinear spherical coordinates, for example. But it is important not to confuse, say, p_θ with the θ component of $\mathbf{p}$, designated as $(\mathbf{p})_\theta$. The former is the canonical momentum conjugate to the coordinate θ; the latter is the θ component of the momentum vector conjugate to the Cartesian coordinates. Dimensionally, it is clear they are quite separate quantities; p_θ is an angular momentum, $(\mathbf{p})_\theta$ is a linear momentum. *Whenever a vector is used from here on to represent canonical momenta it will refer to the momenta conjugate to Cartesian position coordinates.*

For a second example, let us consider a single (nonrelativistic) particle of mass m and charge q moving in an electromagnetic field. By Eq. (1.63), the Lagrangian for this system is

$$L = T - V = \tfrac{1}{2}mv^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}.$$

where the scalar potential term, $-q\phi$, is the L_0 term of the Lagrangian as expressed in Eq. (8.22) and the vector potential term, $q\mathbf{A} \cdot \mathbf{v}$, is the L_1 term.

Using Cartesian position coordinates as generalized coordinates, the Lagrangian can also be written as

$$L = \frac{m\dot{x}_i\dot{x}_i}{2} + qA_i\dot{x}_i - q\phi, \tag{8.32}$$

where the potentials ϕ and $\mathbf{A}$ are in general functions of x_i and the time.

There is now a linear term in the generalized velocities such that the matrix $\mathbf{a}$ has the elements qA_i. Because of this linear term in V, the Hamiltonian is *not* $T + V$. However, it is still in this case the total energy since the "potential" energy in an electromagnetic field is determined by ϕ alone. The canonical momenta, either by Eq. (8.2) or Eq. (8.25), are

$$p_i = m\dot{x}_i + qA_i, \tag{8.33}$$

and the Hamiltonian (cf. Eq. (8.27)) is

$$H = \frac{(p_i - qA_i)(p_i - qA_i)}{2m} + q\phi, \tag{8.34}$$

which is the total energy of the particle. Again, the momenta p_i can be formed into a vector $\mathbf{p}$ and H written as

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi, \tag{8.35}$$

and remembering that $\mathbf{p}$ refers only to momenta conjugate to x_i.

It is clear that Hamilton's equations of motion do not treat the coordinates and momenta in a completely symmetric fashion. The equation for $\dot{p}$ has a minus sign that is absent in the equation for $\dot{q}$. Considerable ingenuity has been exercised in devising nomenclature schemes that result in entirely symmetric equations, or combine the two sets into one. Most of these schemes have only curiosity value, but one has proved to be an elegant and powerful tool for manipulating the canonical equations and allied expressions.

For a system of n degrees of freedom, we construct a column matrix $\boldsymbol{\eta}$ with $2n$ elements such that

$$\eta_i = q_i, \qquad \eta_{i+n} = p_i; \qquad i \le n. \tag{8.36}$$

Similarly, the column matrix $\partial H/\partial \boldsymbol{\eta}$ has the elements

$$\left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_i = \frac{\partial H}{\partial q_i}, \qquad \left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_{i+n} = \frac{\partial H}{\partial p_i}; \qquad i \le n. \tag{8.37}$$

Finally, let $\mathbf{J}$ be the $2n \times 2n$ square matrix composed of four $n \times n$ zero and unit matrices according to the scheme

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \tag{8.38a}$$

with the following transpose matrix, which is its inverse

$$\tilde{\mathbf{J}} = \begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \tag{8.38b}$$

which means

$$\tilde{\mathbf{J}}\mathbf{J} = \mathbf{J}\tilde{\mathbf{J}} = \mathbf{1} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \tag{8.38c}$$

so

$$\tilde{\mathbf{J}} = -\mathbf{J} = \mathbf{J}^{-1} \tag{8.38d}$$

and

$$\mathbf{J}^2 = -\mathbf{1}, \tag{8.38e}$$

and the determinant is

$$|\mathbf{J}| = +1. \tag{8.38f}$$

Here $\mathbf{0}$ is the $n \times n$ matrix all of whose elements is zero, and $\mathbf{1}$ is the standard $n \times n$ unit matrix. Hamilton's equations of motion can then be written in compact form as

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}}. \tag{8.39}$$

For two coordinate variables, this has the expanded form

For two coordinate variables, this has the expanded form

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{p}_1 \\ -\dot{p}_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \tag{8.40}$$

where use was made of Eqs. (8.37) and (8.18). This method of displaying the canonical equations of motion will be referred to as Hamilton's equations in matrix or *symplectic** notation. In subsequent chapters we shall frequently employ this matrix form of the equations.

8.2 ■ CYCLIC COORDINATES AND CONSERVATION THEOREMS

According to the definition given in Section 2.6, a cyclic coordinate q_j is one that does not appear explicitly in the Lagrangian; by virtue of Lagrange's equations

*The term *symplectic* comes from the Greek for "intertwined," particularly appropriate for Hamilton's equations where $\dot{q}$ is matched with a derivative with respect to p and $\dot{p}$ similarly with the negative of a q derivative. H. Weyl first introduced the term in 1939 in his book *The Classical Groups*.

its conjugate momentum p_j is then a constant. But comparison of Eq. (8.14) with Eq. (8.16) has already told us that

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}.$$

A coordinate that is cyclic will thus also be absent from the Hamiltonian.* Conversely if a generalized coordinate does not occur in H, the conjugate momentum is conserved. The momentum conservation theorems of Section 2.6 can thus be transferred to the Hamiltonian formulation with no more than a substitution of H for L. In particular, the connection between the invariance or symmetry properties of the physical system and the constants of the motion can also be derived in terms of the Hamiltonian. For example, if a system is completely self-contained, with only internal forces between the particles, then the system can be moved as a rigid ensemble without affecting the forces or subsequent motion. The system is said to be invariant under a rigid displacement. Hence, a generalized coordinate describing such a rigid motion will not appear explicitly in the Hamiltonian, and the corresponding conjugate momentum will be conserved. If the rigid motion is a translation along some particular direction, then the conserved momentum is the corresponding Cartesian component of the total linear (canonical) momentum of the system. Since the direction is arbitrary, the total vector linear momentum is conserved. The rigid displacement may be a rotation, from whence it follows that the total angular momentum vector is conserved. Even if the system interacts with external forces, there may be a symmetry in the situation that leads to a conserved canonical momentum. Suppose the system is symmetrical about a given axis so that H is invariant under rotation about that axis. Then H obviously cannot involve the rotation angle about the axis and the particular angle variable must be a cyclic coordinate. It follows, as in Section 2.6, that the component of the angular momentum about that axis is conserved.†

The considerations concerning h in Section 2.7 have already shown that if L (and in consequence of Eq. (8.15), also H) is not an explicit function of t, then H is a constant of motion. This can also be seen directly from the equations of motion (8.18) by writing the total time derivative of the Hamiltonian as

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}.$$

In consequence of the equations of motion (8.18), the first two sums on the right cancel each other, and it therefore follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{8.41}$$

*This conclusion also follows from the definition of Eq (8.15). for H differs from $-L$ only by $p_i\dot{q}_i$. which does not involve q_i explicitly.

†The relation between conservation laws, symmetry of the Lagrangian, (and the Hamiltonian) of the system is called Noether's theorem. The formal proof is given in Section 13 7.

Thus if t doesn't appear explicitly in L, it will also not be present in H, and H will be constant in time.

Further, it was proved in Section 2.7 that if the equations of transformation that define the generalized coordinates (1.38),

$$\mathbf{r}_m = \mathbf{r}_m(q_1, \ldots, q_n; t),$$

do not depend explicitly upon the time, and if the potential is velocity independent, then H is the total energy, $T+V$. The identification of H as a constant of the motion and as the total energy are two separate matters, and the conditions sufficient for the one are not enough for the other. It can happen that the Eqs. (1.38) do involve time explicitly but that H does not. In this case, H is a constant of the motion but it is *not* the total energy. As was also emphasized in Section (2.6), the Hamiltonian is dependent both in magnitude and in functional form upon the initial choice of generalized coordinates. For the Lagrangian, we have a specific prescription, $L = T - V$, and a change of generalized coordinates within that prescription may change the functional appearance of L but cannot alter its magnitude. On the other hand, use of a different set of generalized coordinates in the definition for the Hamiltonian, Eq. (8.15), may lead to an entirely different quantity for the Hamiltonian. It may be that for one set of generalized coordinates H is conserved, but that for another it varies in time.

To illustrate some of these points in a simple example, we may consider a somewhat artificial one-dimensional system. Suppose a point mass m is attached to a spring, of force constant k, the other end of which is fixed on a massless cart that is being moved uniformly by an external device with speed v_0 (cf. Fig. 8.1). If we take as generalized coordinate the position x of the mass particle in the stationary system, then the Lagrangian of the system is obviously

$$L(x, \dot{x}, t) = T - V = \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - v_0 t)^2. \tag{8.42}$$

(For simplicity, the origin has been chosen so that the cart passes through it at $t = 0$.) The corresponding equation of motion is clearly

$$m\ddot{x} = -k(x - v_0 t).$$

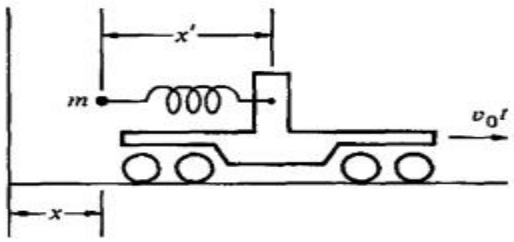

FIGURE 8.1 A harmonic oscillator fixed to a uniformly moving cart.

An obvious way of solving this equation is to change the unknown to $x'(t)$ defined as

$$x' = x - v_0 t, \tag{8.43}$$

and noting that $\ddot{x}' = \ddot{x}$, the equation of motion becomes

$$m\ddot{x}' = -kx'. \tag{8.44}$$

From Eq. (8.43), x' is the displacement of the particle relative to the cart; Eq. (8.44) says that to an observer on the cart the particle exhibits simple harmonic motion, as would be expected on the principle of equivalence in Galilean relativity.

Having looked at the nature of the motion, let us consider the Hamiltonian formulation. Since x is the Cartesian coordinate of the particle, and the potential does not involve generalized velocities, the Hamiltonian relative to x is the sum of the kinetic and potential energies, that is, the total energy. In functional form the Hamiltonian is given by

$$H(x, p, t) = T + V = \frac{p^2}{2m} + \frac{k}{2}(x - v_0 t)^2. \tag{8.45}$$

The Hamiltonian *is* the total energy of the system, but since it is explicitly a function of t, it is *not* conserved. Physically this is understandable; energy must flow into and out of the "external physical device" to keep the cart moving uniformly against the reaction of the oscillating particle.*

Suppose now we formulated the Lagrangian from the start in terms of the relative coordinate x'. The same prescription gives the Lagrangian as

$$L(x', \dot{x}') = \frac{m\dot{x}'^2}{2} + m\dot{x}'v_0 + \frac{mv_0^2}{2} - \frac{kx'^2}{2}. \tag{8.46}$$

In setting up the corresponding Hamiltonian, we note there is now a term linear in $\dot{x}'$, with the single component of $\mathbf{a}$ being mv_0. The new Hamiltonian is now

$$H'(x', p') = \frac{(p' - mv_0)^2}{2m} + \frac{kx'^2}{2} - \frac{mv_0^2}{2}. \tag{8.47}$$

Note that the last term is a constant involving neither x' nor p'; it could, if we wished, be dropped from H' without affecting the resultant equations of motion. Now H' is *not* the total energy of the system, but it *is* conserved. Except for the last term, it can be easily identified as the total energy of motion of the particle relative to the moving cart. The two Hamiltonian's are different in magnitude.

*Put another way, the moving cart constitutes a time-dependent constraint on the particle, and the force of the constraint does do work in actual (*not* virtual) displacement of the system.

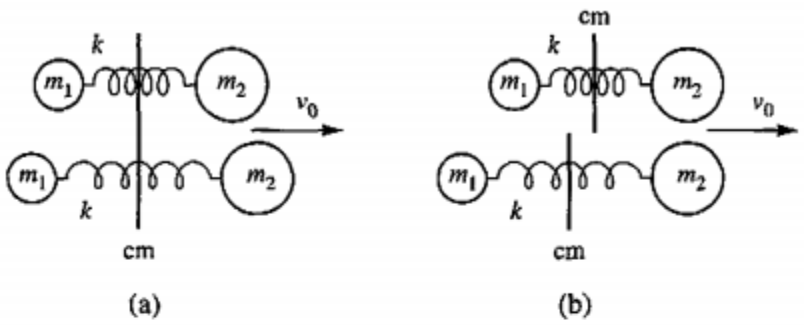

FIGURE 8.2 Vibrating dumbbell under two conditions: (a) freely oscillating, and (b) oscillating with mass m_2 kept at a constant velocity

time dependence, and functional behavior. But the reader can easily verify that both lead to the same motion for the particle.

Additional insight into the problem of the mass cart previously discussed can be gained by considering a dumbbell of two masses connected by a spring of constant k. We shall consider the case where the center of mass of the dumbbell is in constant motion at a speed v_0 along the direction determined by the spring and allow oscillations of the masses only along this direction. This is shown in Fig. 8.2, where C-O-M denotes the center of mass.

The dumbbell is made to vibrate while its center of mass has an initial velocity v_0. It will continue with this velocity with uniform translational motion. This translational motion will have no effect on the oscillations. The motion of the center of mass and the motion relative to the center of mass separate as they do in the Kepler problem. Once the motion is started, energy is conserved and the Hamiltonian is the total conserved energy. The situation is different if the mass m_2 moves at the constant speed v_0 since a periodic force is applied. The center of mass and the mass m_1 then oscillate relative to m_2. Since a changing external force must be applied to the system to keep m_2 at the constant velocity v_0, the Hamiltonian is no longer conserved, nor is the Hamiltonian the total energy.

8.3 ■ ROUTH'S PROCEDURE

It has been remarked that the Hamiltonian formulation is not particularly helpful in the direct solution of mechanical problems. Often we can solve the $2n$ first-order equations only by eliminating some of the variables, for example, the p variables, which speedily leads back to the second-order Lagrangian equations of motion. But an important exception should be noted. The Hamiltonian procedure is especially adapted to the treatment of problems involving cyclic coordinates.

Let us consider the situation in Lagrangian formulation when some coordinate, say q_n, is cyclic. The Lagrangian as a function of q and $\dot{q}$ can then be written

$$L = L(q_1, \ldots, q_{n-1};\ \dot{q}_1, \ldots, \dot{q}_n;\ t).$$

All the generalized velocities still occur in the Lagrangian and in general will be functions of the time. We still have to solve a problem of n degrees of freedom, even though one degree of freedom corresponds to a cyclic coordinate. A cyclic coordinate in the Hamiltonian formulation, on the other hand, truly deserves its alternative description as "ignorable," for in the same situation p_n is some constant α, and H has the form

$$H = H(q_1, \ldots, q_{n-1};\ p_1, \ldots, p_{n-1};\ \alpha; t).$$

In effect, the Hamiltonian now describes a problem involving only $n - 1$ coordinates, which may be solved completely ignoring the cyclic coordinate except as it is manifested in the constant of integration α, to be determined from the initial conditions. The behavior of the cyclic coordinate itself with time is then found by integrating the equation of motion

$$\dot{q}_n = \frac{\partial H}{\partial \alpha}.$$

The advantages of the Hamiltonian formulation in handling cyclic coordinates may be combined with the Lagrangian conveniences for noncyclic coordinates by a method devised by Routh. Essentially, we carry out a mathematical transformation from the $q, \dot{q}$ basis to the q, p basis only for those coordinates that are cyclic, obtaining their equations of motion in the Hamiltonian form, while the remaining coordinates are governed by Lagrange equations. If the cyclic coordinates are labeled $q_{s+1}, \ldots, q_n$, then a new function R (known as the Routhian) may be introduced, defined as

$$R(q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_s;\ p_{s+1}, \ldots, p_n;\ t) = \sum_{i=s+1}^{\cdot} p_i \dot{q}_i - L, \qquad (8.48)$$

which is equivalent to writing

$$R(q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_s;\ p_{s+1}, \ldots, p_n;\ t) =$$
$$H_{cycl}(p_{s+1}, \ldots, p_n) - L_{noncycl}(q_1, \ldots, q_s;\ \dot{q}_1, \ldots, \dot{q}_s). \qquad (8.49)$$

It is easy to show for the s nonignorable coordinates, the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0, \qquad i = 1, \ldots, s, \qquad (8.50)$$

are satisfied, while for the $n-s$ ignorable coordinates, Hamilton's equations apply as

$$\frac{\partial R}{\partial q_i} = -\dot{p}_i = 0, \qquad \text{and} \qquad \frac{\partial R}{\partial p_i} = \dot{q}_i, \qquad i = s+1, \ldots, n. \qquad (8.51)$$

gated in Section 3.7, that of a single particle moving in a plane under the influence of the inverse-square central force $f(r)$ derived from the potential $V(r) = -k/r^n$. The Lagrangian is then

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r^n}.$$

As noted before, the ignorable coordinate is θ, and if the constant conjugate momentum is denoted by p_θ, the corresponding Routhian (8.49) is

$$R(r, \dot{r}, p_\theta) = \frac{p_\theta^2}{2mr^2} - \frac{1}{2}m\dot{r}^2 - \frac{k}{r^n}.$$

Physically we see that the Routhian is the equivalent one-dimensional potential $V'(r)$ *minus* the kinetic energy of radial motion.

Applying the Lagrange equation (8.50) to the noncyclic radial coordinate r, we obtain the equation of motion (3.11)

$$\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{nk}{r^{n+1}} = 0. \tag{8.52}$$

Applying Hamilton's equation (8.51) to the cyclic variable θ, we obtain the pair of equations

$$\dot{p}_\theta = 0 \qquad \text{and} \qquad \frac{p_\theta}{mr^2} = \dot{\theta}. \tag{8.53}$$

whose solution is the same as Eq. (3.8),

$$p_\theta = mr^2\dot{\theta} = l = \text{constant}.$$

Typically, Routh's procedure does not add to the physics of the analysis presented earlier in Chapter 3, but it makes the analysis more automatic. In complicated problems with many degrees of freedom, this feature can be a considerable advantage. it is not surprising therefore that Routh's procedure finds its greatest usefulness in the direct solution of problems relating to engineering applications. But as a fundamental entity, the Routhian is a sterile hybrid, combining some of the features of both the Lagrangian and the Hamiltonian pictures. For the development of various formalisms of classical mechanics, the complete Hamiltonian formulation is more fruitful.

8.4 ■ THE HAMILTONIAN FORMULATION OF RELATIVISTIC MECHANICS

As with the Lagrangian picture in special relativity, two attitudes can be taken to the Hamiltonian formulation of relativistic mechanics. The first makes no pretense at a covariant description but instead works in some specific Lorentz or inertial frame. Time as measured in the particular Lorentz frame is then not treated on a common basis with other coordinates but serves, as in nonrelativistic mechanics, as a parameter describing the evolution of the system. Nonetheless, if the Lagrangian that leads to the Hamiltonian is itself based on a relativistically invariant physical theory (for example, Maxwell's equations and the Lorentz force), then the resultant Hamiltonian picture will be relativistically correct. The second approach of course attempts a fully covariant description of the Hamiltonian picture, but the difficulties that plagued the corresponding Lagrangian approach (cf. Section 7.9) are even fiercer here. We shall consider the noncovariant method first.

For a single-particle Lagrangian of the form of Eq. (7.136),

$$L = -mc^2\sqrt{1-\beta^2} - V,$$

we have already shown that the Hamiltonian (in the guise of the energy function h) is the total energy of the system:

$$H = T + V.$$

The energy T can be expressed in terms of the canonical momenta p_i (Eq. 7.139) through Eq. (7.38):*

$$T^2 = p^2c^2 + m^2c^4,$$

so that a suitable form for the Hamiltonian is

$$H = \sqrt{p^2c^2 + m^2c^4} + V. \tag{8.54}$$

When the system consists of a single particle moving in an electromagnetic field, the Lagrangian has been given as (cf. Eq. (7.141))

$$L = -mc^2\sqrt{1-\beta^2} + q\mathbf{A}\cdot\mathbf{v} - q\phi.$$

The term in L linear in the velocities does not appear explicitly in the Hamiltonian (cf. Eq. (8.54)), as we have seen, whereas the first term leads to the appearance of T in the Hamiltonian. Thus, the Hamiltonian is again the total particle energy:

$$H = T + q\phi. \tag{8.55}$$

For this system, the canonical momenta conjugate to the Cartesian coordinates of the particle are defined by (cf. Eq. (7.142))

$$p^i = mu^i + qA^i,$$

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2c^2 + m^2c^4} + q\phi. \tag{8.56}$$

It should be emphasized again that **p** here is the vector of the canonical momenta conjugate to the *Cartesian* position coordinates of the particle. We may also note that $(H - q\phi)/c$ is the zeroth component of the 4-vector

$$mu^v + qA^v$$

(cf. Eqs. (7.27), (7.38′), and (7.166)). While the Hamiltonian (8.56) is not expressed in covariant fashion, it does have a definite transformation behavior under a Lorentz transformation as being, in some Lorentz form, the zeroth component of a 4-vector.

In a covariant approach to the Hamiltonian formulation, time must be treated in the same fashion as the space coordinates; that is, time must be taken as one of the canonical coordinates having an associated conjugate momentum. The foundations of such an extension of the dimensionality of phase space can in fact be constructed even in nonrelativistic mechanics. Following the pattern of Section 7.10, the progress of the system point along its trajectory in phase space can be marked by some parameter θ, and t "released," so to speak, to serve as an additional coordinate. If derivatives with respect to θ are denoted by a superscript prime, the Lagrangian in the $(q_1, \ldots, q_n; t)$ configuration space is (cf. Eq. (7.159))

$$\Lambda(q, q', t, t') = t'L\left(q, \frac{q'}{t'}, t\right). \tag{8.57}$$

The momentum conjugate to t is then

$$p_t = \frac{\partial \Lambda}{\partial t'} = L + t'\frac{\partial L}{\partial t'}.$$

If we make explicit use of the connection $\dot{q} = q'/t'$, this relation becomes

$$p_t = L - \frac{q_i'}{t'}\frac{\partial L}{\partial \dot{q}_i} = L - \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} = -H. \tag{8.58}$$

The momentum conjugate to the time "coordinate" is therefore the negative of the ordinary Hamiltonian.* While the framework of this derivation is completely nonrelativistic, the result is consistent with the identification of the time component of the 4-vector momentum with E/c. As can be seen from the definition, Eq. (8.2), if q is multiplied by a constant α, then the conjugate momentum is divided by α. Hence, the canonical momentum conjugate to ct is H/c.

Thus, there seems to be a natural route available for constructing a relativistically covariant Hamiltonian. But the route turns out to be mined with booby traps. It will be recalled that the covariant Lagrangian used to start the process, Eq. (7.159) or Eq. (8.57), is homogeneous in first degree in the generalized velocities q', and for such a Lagrangian the recipe described above for constructing the Hamiltonian formulation breaks down irreparably. If L is of type L_1, the corresponding Hamiltonian, call it $H_c(q, t, p, p_t)$, is identically zero!

Fortunately, there does not seem to be any compelling reason why the covariant Lagrangian has to be homogeneous in the first degree, at least for classical relativistic mechanics. It has already been seen that for a single free particle the covariant Lagrangian

$$\Lambda(x^\mu, u^\mu) = \tfrac{1}{2} m u_\mu u^\mu$$

leads to the correct equations of motion. Of course the four-velocity components, u^μ, are still not all independent, but the constraint can be treated as a "weak condition" to be imposed only *after* all the differentiations have been carried through. There is now no difficulty in obtaining a Hamiltonian from this Lagrangian, by the same route as in nonrelativistic mechanics; the result is clearly

$$H_c = \frac{p_\mu p^\mu}{2m}. \tag{8.59}$$

For a single particle in an electromagnetic field, a covariant Lagrangian has been found previously: (cf. Eq. (7.165))*

$$\Lambda(x^\mu, u^\mu) = \tfrac{1}{2} m u_\mu u^\mu + q u^\mu A_\mu(x_\lambda), \tag{7.147}$$

with the canonical momenta (cf. Eq. (7.167)),

$$p_\mu = m u_\mu + q A_\mu. \tag{7.149}$$

In the corresponding Hamiltonian, the term linear in u_μ does not appear explicitly in the Hamiltonian, and the remaining L_2 part in terms of the canonical momenta is

$$H_c' = \frac{(p_\mu - q A_\mu)(p^\mu - q A^\mu)}{2m}. \tag{8.60}$$

Both Hamiltonians, Eqs. (8.59) and (8.60), are constant, with the same value, $-mc^2/2$, but to obtain the equations of motion it is the *functional* dependence on the 4-vectors of position and momenta that is important. With a system of one particle, the covariant Hamiltonian leads to eight first-order equations of motion

$$\frac{dx^\nu}{d\tau} = \frac{\partial H_c'}{\partial p^\nu}, \quad \frac{dp^\nu}{d\tau} = -\frac{\partial H_c}{\partial x^\nu}. \tag{8.61}$$

We know that these equations cannot be all independent. The space parts of Eqs. (8.61) obviously lead to the spatial equations of motion. We should expect therefore that the remaining two equations tell us nothing new, exactly as in the Lagrangian case. This can be verified by examining the $\nu = 0$ equations in some particular Lorentz frame. One of them is the constitutive equation for p^0:

$$u^0 = \frac{\partial H_c'}{\partial p^0} = \frac{1}{m}\left(p^0 - qA^0\right)$$

or

$$p^0 = \frac{1}{c}(T + q\phi) = \frac{H_c'}{c}, \tag{8.62}$$

a general conclusion that has been noted before. The other can be written as

$$\frac{1}{\sqrt{1-\beta^2}}\frac{dp^0}{dt} = -\frac{1}{c}\frac{\partial H_c}{\partial t}$$

or

$$\frac{dH}{dt} = \sqrt{1-\beta^2}\,\frac{\partial H_c}{\partial t}. \tag{8.63}$$

As with the covariant Lagrangian formulation, we have the problem of finding suitable covariant potential terms in the Lagrangian to describe the forces other than electromagnetic. In multiparticle systems we are confronted in full measure with the critical difficulties of including interactions other than with fields. In Hamiltonian language, the "no-interaction" theorem already referred to in Section 7.10 says that only in the absence of direct particle interactions can Lorentz invariant systems be described in terms of the usual position coordinates and corresponding canonical momenta. The scope of the relativistic Hamiltonian framework is therefore quite limited and so for the most part we shall confine ourselves to nonrelativistic mechanics.

8.5 ■ DERIVATION OF HAMILTON'S EQUATIONS FROM A VARIATIONAL PRINCIPLE

Lagrange's equations have been shown to be the consequence of a variational principle, namely, the Hamilton's principle of Section 2.1. Indeed, the variational method is often the preferable one for deriving Lagrange's equations, for it is applicable to types of systems not usually included within the scope of mechanics. It would be similarly advantageous if a variational principle could be found that

leads directly to the Hamilton's equations of motion. Hamilton's principle,

$$\delta I \equiv \delta \int_{t_1}^{t_2} L\, dt = 0, \tag{8.64}$$

lends itself to this purpose, but as formulated originally it refers to paths in configuration space. The first modification therefore is that the integral must be evaluated over the trajectory of the system point in phase space, and the varied paths must be in the neighborhood of this phase space trajectory. In the spirit of the Hamiltonian formulation, both q and p must be treated as independent coordinates of phase space, to be varied independently. To this end the integrand in the action integral, Eq. (8.64), must be expressed as a function of both q and p, and their time derivatives, through Eq. (8.15). Equation (8.64) then appears as

$$\delta I = \delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t)\right) dt = 0. \tag{8.65}$$

As a variational principle in phase space, Eq. (8.65) is sometimes referred to as the *modified Hamilton's principle*. Although it will be used most frequently in connection with transformation theory (see Chapter 9), the main interest in it here is to show that the principle leads to Hamilton's canonical equations of motion.

The modified Hamilton's principle is exactly of the form of the variational problem in a space of $2n$ dimensions considered in Section 2.3 (cf. Eq. (2.14)):

$$\delta I = \delta \int_{t_1}^{t_2} f(q, \dot{q}, p, \dot{p}, t)\, dt = 0, \tag{8.66}$$

for which the $2n$ Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j}\right) - \frac{\partial f}{\partial q_j} = 0 \qquad j = 1, \ldots, n \tag{8.67}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_j}\right) - \frac{\partial f}{\partial p_j} = 0 \qquad j = 1, \ldots, n. \tag{8.68}$$

The integrand f as given in Eq. (8.65) contains $\dot{q}_j$ only through the $p_i \dot{q}_i$ term, and q_j only in H. Hence, Eqs. (8.67) lead to

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0. \tag{8.69}$$

On the other hand, there is no explicit dependence of the integrand in Eq. (8.65) on $\dot{p}_j$. Equations (8.68) therefore reduce simply to

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0. \tag{8.70}$$

Equations (8.69) and (8.70) are exactly Hamilton's equations of motion. Eqs. (8.18). The Euler–Lagrange equations of the modified Hamilton's principle are thus the desired canonical equations of motion.

This derivation of Hamilton's equations from the variational principle is so brief as to give the appearance of a sleight-of-hand trick. One wonders whether something extra has been sneaked in while we were being misdirected by the magician's patter. Is the modified Hamilton's principle equivalent to Hamilton's principle, or does it contain some additional physics? The question is largely irrelevant; the primary justification for the modified Hamilton's principle is that it leads to the canonical equations of motion in phase space. After all, no further argument was given for the validity of Hamilton's principle than that it corresponded to the Lagrangian equations of motion. So long as Hamiltonian can be constructed, the Legendre transformation procedure shows that the Lagrangian and Hamiltonian formulations, and therefore their respective variational principles, have the same physical content.

One question that can be raised however is whether the derivation puts limitations on the variation of the trajectory that are not present in Hamilton's principle. The variational principle leading to the Euler–Lagrange equations is formulated, as in Section 2.2, such that the variations of the independent variables vanish at the end points. In phase space, that would require $\delta q_i = 0$ *and* $\delta p_i = 0$ at the end points, whereas Hamilton's principle requires only the vanishing of δq_i under the same circumstances. A look at the derivation as spelled out in Section 2.2 will show however that the variation is required to be zero at the end points only in order to get rid of the integrated terms arising from the variations in the time derivatives of the independent variables. While the f function in Eq. (8.66) that corresponds to the modified Hamilton's principle, Eq. (8.65), is indeed a function of q_j, there is no explicit appearance of p_j. Equations (8.68) and therefore (8.70) follow from Eq. (8.65) without stipulating the variations of p_j at the end points. The modified Hamilton's principle, with the integrand L defined in terms of the Hamiltonian by Eq. (8.19), leads to Hamilton's equations under the same variation conditions as those in Hamilton's principle.*

Nonetheless, there are advantages to requiring that the varied paths in the modified Hamilton's principle return to the same end points in both q and p, for we then have a more generalized condition for Hamilton's equations of motion. As with Hamilton's principle, if there is no variation at the end points we can add a total time derivative of any arbitrary (twice-differentiable) function $F(q, p, t)$ to the integrand without affecting the validity of the variational principle. Suppose, for example, we subtract from the integrand of Eq. (8.65) the quantity

The modified Hamilton's principle would then read

$$\delta \int_{t_1}^{t_2} (-\dot{p}_i q_i - H(q, p, t))\, dt = 0. \tag{8.71}$$

Here the f integrand of Eq. (8.66) is a function of $\dot{p}$, and it is easily verified that the Euler–Lagrange equations (8.67) and (8.68) with this f again correspond to Hamilton's equations of motion, Eqs. (8.18). Yet the integrand in Eq. (8.71) is not the Lagrangian nor can it in general be simply related to the Lagrangian by a point transformation in configuration space. By restricting the variation of both q and p to be zero at the end points, the modified Hamilton's principle provides an independent and general way of setting up Hamilton's equations of motion without a prior Lagrangian formulation. If you will, it does away with the necessity of a linkage between the Hamiltonian canonical variables and a corresponding Lagrangian set of generalized coordinates and velocities. This will be very important to us in the next chapter where we examine transformations of phase space variables that preserve the Hamiltonian form of the equations of motion.

The requirement of independent variation of q and p, so essential for the above derivation, highlights the fundamental difference between the Lagrangian and Hamiltonian formulations. Neither the coordinates q_i nor the momenta p_i are to be considered there as the more fundamental set of variables; both are equally independent. Only by broadening the field of independent variables from n to $2n$ quantities are we enabled to obtain equations of motion that are of first order. In a sense, the names "coordinates" and "momenta" are unfortunate, for they bring to mind pictures of spatial coordinates and linear, or at most, angular momenta. A wider meaning must now be given to the terms. The division into coordinates and momenta corresponds to no more than a separation of the independent variables describing the motion into two groups having an almost symmetrical relationship to each other through Hamilton's equations.

8.6 ■ THE PRINCIPLE OF LEAST ACTION

Another variational principle associated with the Hamiltonian formulation is known as *the principle of least action*. It involves a new type of variation, which we shall call the Δ-variation, requiring detailed explanation. In the δ-variation process used in the discussion of Hamilton's principle in Chapter 2, the varied path in configuration space always terminated at end points representing the system configuration at the same time t_1 and t_2 as the correct path. To obtain Lagrange's equations of motion, we also required that the varied path return to the same end points in configuration space, that is, $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The Δ-variation is less constrained; in general, the varied path over which an integral is evaluated may end at different times than the correct path, and there

may be a variation in the coordinates at the end points. We can however use the same parameterization of the varied path as in the δ-variation. In the notation of Section 2.3, a family of possible varied paths is defined by functions (cf. Eq. (2.15))

$$q_i(t, \alpha) = q_i(t, 0) + \alpha \eta_i(t), \tag{8.72}$$

where α is an infinitesimal parameter that goes to zero for the correct path. Here the functions η_i do not necessarily have to vanish at the end points, either the original or the varied. All that is required is that they be continuous and differentiable. Figure 8.3 illustrates the correct and varied path for a Δ-variation in configuration space.

Let us evaluate the Δ-variation of the action integral:

$$\Delta \int_{t_1}^{t_2} L\, dt \equiv \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} L(\alpha)\, dt - \int_{t_1}^{t_2} L(0)\, dt, \tag{8.73}$$

where $L(\alpha)$ means the integral is evaluated along the varied path and $L(0)$ correspondingly refers to the actual path of motion. The variation is clearly composed of two parts. One arises from the change in the limits of the integral; to first-order infinitesimals, this part is simply the integrand on the actual path times the difference in the limits in time. The second part is caused by the change in the integrand on the varied path, but now between the same time limits as the original integral. We may therefore write the Δ-variation of the action integral as

$$\Delta \int_{t_1}^{t_2} L\, dt = L(t_2)\Delta t_2 - L(t_1)\Delta t_1 + \int_{t_1}^{t_2} \delta L\, dt. \tag{8.74}$$

Here the variation in the second integral can be carried out through a parameterization of the varied path, exactly as for Hamilton's principle except that the

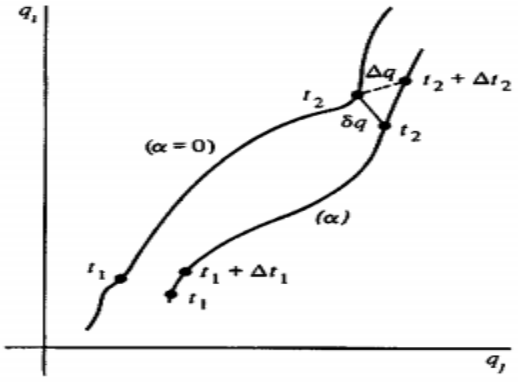

FIGURE 8.3 The Δ-variation in configuration space.

variation in q_i does not vanish at the end points. The end point terms arising in the integration by parts must be retained, and the integral term on the right appears as

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i\, dt + \frac{\partial L}{\partial \dot{q}_i}\, \delta q_i \Big|_1^2.$$

By Lagrange's equations the quantities in the square brackets vanish, and the Δ-variation therefore takes the form

$$\Delta \int_{t_1}^{t_2} L\, dt = (L\Delta t + p_i \delta q_i)\Big|_1^2. \tag{8.75}$$

In Eq. (8.75), δq_i refers to the variation in q_i at the original end point times t_1 and t_2. We would like to express the Δ-variation in terms of the change Δq_i between q_i at the end points of the actual path and q_i at the end points of the varied path, including the change in end point times. It is clear from Fig. 8.3 that these two variations are connected by the relation*

$$\Delta q_i = \delta q_i + \dot{q}_i \Delta t. \tag{8.76}$$

Hence, Eq. (8.75) can be rewritten as

$$\Delta \int_{t_1}^{t_2} L\, dt = (L\Delta t - p_i \dot{q}_i \Delta t + p_i \Delta q_i)\Big|_1^2$$

Hence, Eq. (8.75) can be rewritten as

$$\Delta \int_{t_1}^{t_2} L\,dt = (L\Delta t - p_i \dot{q}_i \Delta t + p_i \Delta q_i)\Big|_1^2$$

or

$$\Delta \int_{t_1}^{t_2} L\,dt = (p_i \Delta q_i - H\,\Delta t)\Big|_1^2. \tag{8.77}$$

To obtain the principle of least action, we restrict our further considerations by three important qualifications:

1. Only systems are considered for which L, and therefore H, are not explicit functions of time, and in consequence H is conserved.
2. The variation is such that H is conserved on the varied path as well as on the actual path.
3. The varied paths are further limited by requiring that Δq_i vanish at the end points (but not Δt).

*Equation (8 76) may be derived formally from the parameter form. Eq. (8.72), of the varied path. Thus, at the upper end point we have

$$\Delta q_i(2) = q_i(t_2 + \Delta t_2, \alpha) - q_i(t_2, 0) = q_i(t_2 + \Delta t_2, 0) - q_i(t_2, 0) + \alpha \eta_i(t + \Delta t_2),$$

which to first order in small quantities α and Δt_2 is

$$\Delta q_i(2) = \dot{q}_i(2)\Delta t_2 + \delta q_i(2),$$

which is what Eq. (8 76) predicts

The nature of the resultant variation may be illustrated by noting that the varied path satisfying these conditions might very well describe the same curve in configuration space as the actual path. The difference will be the speed with which the system point traverses this curve; that is, the functions $q_i(t)$ will be altered in the varied path. In order then to preserve the same value of the Hamiltonian at all points on the varied path, the times of the end points must be changed. With these three qualifications satisfied, the Δ-variation of the action integral, Eq. (8.77), reduces to

$$\Delta \int_{t_1}^{t_2} L\,dt = -H(\Delta t_2 - \Delta t_1). \tag{8.78}$$

But under the same conditions, the action integral itself becomes

$$\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} p_i \dot{q}_i\,dt - H(t_2 - t_1),$$

the Δ-variation of which is

$$\Delta \int_{t_1}^{t_2} L\,dt = \Delta \int_{t_1}^{t_2} p_i \dot{q}_i\,dt - H(\Delta t_2 - \Delta t_1). \tag{8.79}$$

the Δ-variation of which is

$$\Delta \int_{t_1}^{t_2} L\,dt = \Delta \int_{t_1}^{t_2} p_i \dot{q}_i\,dt - H(\Delta t_2 - \Delta t_1). \tag{8.79}$$

Comparison of Eqs. (8.78) and (8.79) finally gives the *principle of least action*:*

$$\Delta \int_{t_1}^{t_2} p_i \dot{q}_i\,dt = 0. \tag{8.80}$$

By way of caution, note that the modified Hamilton's principle can be written in a form with a superficial resemblance to Eq. (8.80). If the trajectory of the system point is described by a parameter θ, as in Sections 7.10 and 8.4, the modified Hamilton's principle appears as

$$\delta \int_{\theta_1}^{\theta_2} (p_i \dot{q}_i - H)t'\,d\theta = 0. \tag{8.81}$$

It will be recalled (cf. footnote on p. 351) that the momenta p_i do not change under the shift from t to θ, and that $\dot{q}_i t' = q_i'$. Further, the momentum conjugate to t is $-H$. Hence, Eq. (8.81) can be rewritten as

$$\delta \int_{\theta_1}^{\theta_2} \sum_{i=1}^{n+1} p_i q_i'\,d\theta = 0, \tag{8.82}$$

where t has been denoted by q_{n+1}. There should however be no confusion between Eq. (8.82) and the principle of least action. Equations (8.82) involve phase space of $(2n + 2)$ dimensions, as is indicated by the explicit summation to $i = n + 1$, whereas Eq. (8.80) is in the usual configuration space. But most important, the principle of least action is in terms of a Δ-variation for constant H, while Eq. (8.82) employs the δ-variation, and H in principle could be a function of time. Equation (8.82) is nothing more than the modified Hamilton's principle, and the absence of a Hamiltonian merely reflects the phenomenon that the Hamiltonian vanishes identically for the "homogeneous problem."

The least action principle itself can be exhibited in a variety of forms. In nonrelativistic mechanics, if the defining equations for the generalized coordinates do not involve the time explicitly, then the kinetic energy is a quadratic function of the $\dot{q}_i$'s (cf. Eq. (1.71)):

$$T = \tfrac{1}{2} M_{jk}(q)\dot{q}_j\dot{q}_k. \tag{8.83}$$

When in addition the potential is not velocity dependent, the canonical momenta are derived from T only, and in consequence

$$p_i \dot{q}_i = 2T.$$

The principle of least action for such systems can therefore be written as

$$\Delta \int_{t_1}^{t_2} T\, dt = 0. \tag{8.84}$$

If, further, there are no external forces on the system, as, for example, a rigid body with no net applied forces, then T is conserved along with the total energy H. The least action principle then takes the special form

$$\Delta(t_2 - t_1) = 0. \tag{8.85}$$

Equation (8.85) states that of all paths possible between two points, consistent with conservation of energy, the system moves along that particular path for which the time of transit is the least (more strictly, an extremum). In this form the principle of least action recalls Fermat's principle in geometrical optics that a light ray travels between two points along such a path that the time taken is the least. We discussed these considerations in Section 10–8 of the Second Edition when we considered the connection between the Hamiltonian formulation and geometrical optics.

In Section 7.4 we discussed the infinitesimal interval in a metric space giving the interval as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{7.32'}$$

where $g_{\mu\nu}$ was the metric of a possibly curvilinear space and ds^2 was the interval traversed for displacements given by dx^μ. We can do something entirely similar here whenever T is of the form of Eq. (8.83). A configuration space is therefore constructed for which the M_{jk} coefficients form the metric tensor. In general, the

space will be curvilinear and nonorthogonal. The element of path length in the space is then defined by (cf. Eq. (7.33′))

$$(d\rho)^2 = M_{jk}\, dq_j\, dq_k \tag{8.86}$$

so that the kinetic energy has the form

$$T = \frac{1}{2}\left(\frac{d\rho}{dt}\right)^2, \tag{8.87}$$

or equivalently

$$dt = \frac{d\rho}{\sqrt{2T}}. \tag{8.88}$$

Equation (8.88) enables us to change the variable in the abbreviated action integral from t to ρ, and the principle of least action becomes

$$\Delta \int_{t_1}^{t_2} T\,dt = 0 = \Delta \int_{\rho_1}^{\rho_2} \sqrt{T/2}\,d\rho,$$

or, finally

$$\Delta \int_{\rho_1}^{\rho_2} \sqrt{H - V(q)}\,d\rho = 0. \tag{8.89}$$

Equation (8.89) is often called *Jacobi's form of the least action principle*. It now refers to the path of the system point in a special curvilinear configuration space characterized by a metric tensor with elements M_{jk}. The system point traverses the path in this configuration space with a speed given by $\sqrt{2T}$. If there are no forces acting on the body, T is constant, and Jacobi's principle says the system point travels along the shortest path length in the configuration space. Equivalently stated, the motion of the system is then such that the system point travels along the geodesics of the configuration space.

Note that the Jacobi form of the principle of least action is concerned with the *path* of the system point rather than with its motion in *time*. Equation (8.89) is a statement about the element of path length $d\rho$; the time nowhere appears, since H is a constant and V depends upon q_i only. Indeed, it is possible to use the Jacobi form of the principle to furnish the differential equations for the path, by a procedure somewhat akin to that leading to Lagrange's equations. In the form of Fermat's principle, the Jacobi version of the principle of least action finds many fruitful applications in geometrical optics and in electron optics. To go into any detail here would lead us too far afield.

A host of other similar, variational principles for classical mechanics can be derived in bewildering variety. To give one example out of many, the principle of least action leads immediately to *Hertz's principle of least curvature*, which states that a particle not under the influence of external forces travels along the

POSSIBLE QUESTIONS

Part B (6 Marks)

1.Define Cyclic coordinates and Explain conservation theorems.

2.Derive Hamilton's canonical equation of motion

3.Obtain the hamilton's equation of motion considering a single non relativistic particle moving in an electromagnetic field

4.Explain the principle of least action

5.Explain the construction of Hamilton through Lagrangian.

Part C (10 Marks)

1.ExplainRouth's procedure.

2.Obtain the hamilton's equation of motion using spherical polar co-ordinates considering the spatial motion of a particle in the central force field

3.Derivation of Hamilton equation from a variational principle

4.Explain the principle of least action

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Coimbatore –641 021

Subject: Mechanics **Subject Code: 19MMP106**

Class : I - M.Sc. Mathematics **Semester : I**

Unit III

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
When a particle moves in a plane its motion is said to be _____	Colinear motion	Coplanar motion	Radial	Transverse	Coplanar motion
The radial component of radius vector is _____	$\ddot{r}$	$\dot{\theta}$	$\dot{r}$	$\theta\dot{}$	$\dot{r}$
The transverse component of radius vector is _____	$r\dot{\theta}$	$\dot{r}\theta$	$\dot{\theta}$	$\dot{r}$	$r\dot{\theta}$
The intersection of the plane and the cone is called _____	Equiangular spiral	Apse	Conic	Angular momentum	Conic
_____ is a curve in which the angle between the radius vector and the respective tangent is a constant.	Equiangular spiral	Apse	Conic	Angular momentum	Equiangular spiral
Equiangular spiral is a curve in which the angle between the radius vector and the respective tangent is a _____	Unity	Zero	Infinity	Constant	Constant
A _____ is a force whose line of action always passes through a fixed point.	Central force	Apse	Areal velocity	Central orbit	Central force
The path describe by a particle under a central force is called _____	Central force	Apse	Areal velocity	Central orbit	Central orbit

The polar equation of a conic is _____	l/r=1+ecosθ	r= 1+ecosθ	r= 1/(1+ecosθ)	r= 1- ecosθ	l/r=1+ecosθ
l/r=1+ecosθ is the polar equation of _____	Coplanar	Conic	Central force	Central orbit	Conic
The corresponding value for r at an apse point is called _____	Apse	Areal velocity	Apsidal distance	Areal distance	Apsidal distance
Each planet describes an _____ with the sun at one of the focus.	Parabola	Hyperparabola	Conic	Ellipse	Ellipse
Kepler's first law states that the orbit is an _____ orbit with the sun.	Elliptic	Hyperparabola	Parabola	Conic	Elliptic
Kepler's second law states that the _____ is constant.	Apse	Areal velocity	Central force	Central orbit	Areal velocity
Kepler's third law states that square of the _____ is proportional to the cube of the semimajor axis.	Amplitude	Periodic time	Acceleration	Areal velocity	Periodic time
_____ law states that the orbit is an elliptic orbit with the sun.	Kepler's first	Kepler's second	Kepler's third	Kepler's fourth	Kepler's first
_____ law states that the areal velocity is constant.	Kepler's first	Kepler's second	Kepler's third	Kepler's fourth	Kepler's second
_____ law states that square of the periodic time is proportional to the cube of the semimajor axis.	Kepler's first	Kepler's second	Kepler's third	Kepler's fourth	Kepler's third
Kepler's second law states that the areal velocity is _____	Constant	Zero	Unity	Infinity	Constant
The deduction of newton's law of gravitation is a consequence of _____	Kepler's law	Newton's second law	Newton's third law	Newton's first law	Kepler's law
The deduction of _____ of gravitation is a consequence of kepler's law.	Kepler's first law	Kepler's second law	Kepler's third law	Newton's law	Newton's law
Equiangular spiral is a curve in which the angle between the _____ and the respective tangent is a constant.	Acceleration	Radius	Radius vector	Acceleration vector	Radius vector
Equiangular spiral is a curve in which the angle between the radius vector and the respective _____ is a constant.	Tangent	Tangential resistance	Radius	Acceleration	Tangent

Let r be the position vector of the moving particle of mass m having a velocity v then the momentum vector is _____.	m+v	m - v	mv	m/v	mv
_____ is an orbit under a central circular forces.	Conic	Central orbit	Areal velocity	Angular momentum	Central orbit
Central orbit is an orbit under a central _____.	Colinear force	Circular force	Coplanar force	Cardiod	Circular force

UNIT-IV

Canonical transformations: The equations of canonical transformation – Examples of Canonical transformations – Poission Brackets and other Canonical invariants – integral invariants of Poincare, Lagrange brackets.

9.1 ■ THE EQUATIONS OF CANONICAL TRANSFORMATION

There is one type of problem for which the solution of the Hamilton's equations is trivial. Consider a situation in which the Hamiltonian is a constant of the motion, and where *all* coordinates q_i are cyclic. Under these conditions, the conjugate momenta p_i are all constant:

$$p_i = \alpha_i,$$

and since the Hamiltonian cannot be an explicit function of either the time or the cyclic coordinates, it may be written as

$$H = H(\alpha_1, \ldots, \alpha_n).$$

Consequently, the Hamilton's equations for $\dot{q}_i$ are simply

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i, \tag{9.1}$$

where the ω_i's are functions of the α_i's only and therefore are also constant in time. Equations (9.1) have the immediate solutions

$$q_i = \omega_i t + \beta_i, \tag{9.2}$$

where the β_i's are constants of integration, determined by the initial conditions.

It would seem that the solution to this type of problem, easy as it is, can only be of academic interest, for it rarely happens that all the generalized coordinates are cyclic. But a given system can be described by more than one set of generalized coordinates. Thus, to discuss motion of a particle in a plane, we may use as generalized coordinates either the Cartesian coordinates

$$q_1 = x, \qquad q_2 = y,$$

or the plane polar coordinates

$$q_1 = r, \qquad q_2 = \theta.$$

Both choices are equally valid, but one of the other set may be more convenient for the problem under consideration. Note that for central forces neither x nor y is cyclic, while the second set does contain a cyclic coordinate in the angle θ. The number of cyclic coordinates can thus depend upon the choice of generalized coordinates, and for each problem there may be one particular choice for which all coordinates are cyclic. If we can find this set, the remainder of the job is trivial. Since the obvious generalized coordinates suggested by the problem will not normally be cyclic, we must first derive a specific procedure for *transforming* from one set of variables to some other set that may be more suitable.

The transformations considered in the previous chapters have involved going from one set of coordinates q_i to a new set Q_i by transformation equations of the form

$$Q_i = Q_i(q, t). \tag{9.3}$$

For example, the equations of an orthogonal transformation, or of the change from Cartesian to plane polar coordinates, have the general form of Eqs. (9.3). As has been previously noted in Derivation 10 of Chapter 1, such transformations are known as *point transformations*. But in the Hamiltonian formulation the momenta are also independent variables on the same level as the generalized coordinates. The concept of transformation of coordinates must therefore be widened to include the simultaneous transformation of the independent *coordinates* and *momenta*, q_i, p_i, to a new set Q_i, P_i, with (invertible) equations of transformation:

$$\begin{aligned} Q_i &= Q_i(q, p, t), \\ P_i &= P_i(q, p, t). \end{aligned} \tag{9.4}$$

In developing Hamiltonian mechanics, only those transformations can be of interest for which the new Q, P are canonical coordinates. This requirement will be satisfied provided there exists some function $K(Q, P, t)$ such that the equations of motion in the new set are in the Hamiltonian form

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}. \tag{9.5}$$

The function K plays the role of the Hamiltonian in the new coordinate set.* It is important for future considerations that the transformations considered be problem-independent. That is to say, (Q, P) must be canonical coordinates not only for some specific mechanical systems, but for all systems of the same number of degrees of freedom. Equations (9.5) must be the form of the equations of motion in the new coordinates and momenta no matter what the particular initial form of H. We may indeed be incited to develop a particular transformation from (q, p) to (Q, P) to handle, say, a plane harmonic oscillator. But the same transformation must then also lead to Hamilton's equations of motion when applied, for example, to the two-dimensional Kepler problem.

As was seen in Section 8.5, if Q_i and P_i are to be canonical coordinates, they must satisfy a modified Hamilton's principle that can be put in the form

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t))\, dt = 0, \tag{9.6}$$

(where summation over the repeated index i is implied). At the same time the old canonical coordinates of course satisfy a similar principle:

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))\, dt = 0. \tag{9.7}$$

The simultaneous validity of Eqs. (9.6) and (9.7) does not mean of course that the integrands in both expressions are equal. Since the general form of the modified Hamilton's principle has zero variation at the end points, both statements will be satisfied if the integrands are connected by a relation of the form

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{dF}{dt}. \tag{9.8}$$

Here F is any function of the phase space coordinates with continuous second derivatives, and λ is a constant independent of the canonical coordinates and the time. The multiplicative constant λ is related to a particularly simple type of transformation of canonical coordinates known as a *scale transformation*.

Suppose we change the size of the units used to measure the coordinates and momenta so that in effect we transform them to a set (Q', P') defined by

$$Q_i' = \mu q_i, \qquad P_i' = \nu p_i. \tag{9.9}$$

Then it is clear Hamilton's equations in the form of Eqs. (9.5) will be satisfied for a transformed Hamiltonian $K'(Q', P') = \mu\nu H(q, p)$. The integrands of the corresponding modified Hamilton's principles are, also obviously, related as

$$\mu\nu(p_i \dot{q}_i - H) = P_i' \dot{Q}_i' - K', \tag{9.10}$$

which is of the form of Eq. (9.8) with $\lambda = \mu\nu$. With the aid of suitable scale transformation, it will always be possible to confine our attention to transformations of canonical coordinates for which $\lambda = 1$. Thus, if we have a transformation of canonical coordinates $(q, p) \to (Q', P')$ for some $\lambda \neq 1$, then we can always find an intermediate set of canonical coordinates (Q, P) related to (Q', P') by a simple scale transformation of the form (9.9) such that $\mu\nu$ also has the same value λ. The transformation between the two sets of canonical coordinates (q, p) and (Q, P) will satisfy Eq. (9.8), but now with $\lambda = 1$:

Since the scale transformation is basically trivial, the significant transformations to be examined are those for which Eq. (9.11) holds.

A transformation of canonical coordinates for which $\lambda \neq 1$ will be called an *extended canonical transformation*. Where $\lambda = 1$, and Eq. (9.11) holds, we will speak simply of a *canonical transformation*. The conclusion of the previous paragraph may then be stated as saying that any extended canonical transformation can be made up of a canonical transformation followed by a scale transformation. Except where otherwise stated, all future considerations of transformations between canonical coordinates will involve only canonical transformations. It is also convenient to give a specific name to canonical transformations for which the equations of transformation Eqs. (9.4) do not contain the time explicitly; they will be called *restricted canonical transformations*.

The last term on the right in Eq. (9.11) contributes to the variation of the action integral only at the end points and will therefore vanish if F is a function of (q, p, t) or (Q, P, t) or any mixture of the phase space coordinates since these have zero variation at the end points. Further, through the equations of transformation, Eqs. (9.4) and their inverses F can be expressed partly in terms of the old set of variables and partly of the new. Indeed, F is useful for specifying the exact form of the canonical transformation only when half of the variables (beside the time) are from the old set and half are from the new. It then acts, as it were, as a bridge between the two sets of canonical variables and is called the *generating function* of the transformation.

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To show how the generating function specifies the equations of transformation, suppose F were given as a function of the old and new generalized space coordinates:

$$F = F_1(q, Q, t). \tag{9.12}$$

Equation (9.11) then takes the form

$$\begin{aligned} p_i \dot{q}_i - H &= P_i \dot{Q}_i - K + \frac{dF_1}{dt} \\ &= P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i. \end{aligned} \tag{9.13}$$

Since the old and the new coordinates, q_i and Q_i, are separately independent, Eq. (9.13) can hold identically only if the coefficients of $\dot{q}_i$ and $\dot{Q}_i$ each vanish:

$$p_i = \frac{\partial F_1}{\partial q_i}, \tag{9.14a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}, \tag{9.14b}$$

leaving finally

$$K = H + \frac{\partial F_1}{\partial t}. \tag{9.14c}$$

Equations (9.14a) are n relations defining the p_i as functions of q_j, Q_j, and t. Assuming they can be inverted, they could then be solved for the n Q_i's in terms of q_j, p_j, and t, thus yielding the first half of the transformation equations (9.4). Once the relations between the Q_i's and the old canonical variables (q, p) have been established, they can be substituted into Eqs. (9.14b) so that they give the n P_i's as functions of q_j, p_j, and t, that is, the second half of the transformation equations (9.4). To complete the story, Eq. (9.14c) provides the connection between the new Hamiltonian, K, and the old one, H. We must be careful to read Eq. (9.14c) properly. First q and p in H are expressed as functions of Q and P through the inverses of Eqs. (9.4). Then the q_i in $\partial F_1/\partial t$ are expressed in terms of Q, P in a similar manner and the two functions are added to yield $K(Q, P, t)$.

The procedure described shows how, starting from a given generating function F_1, the equations of the canonical transformation can be obtained. We can usually reverse the process: Given the equations of transformation (9.4), an appropriate generating function F_1 may be derived. Equations (9.4) are first inverted to express p_i and P_i as functions of q, Q, and t. Equations (9.14a, b) then constitute a coupled set of partial differential equations than can be integrated, in principle, to find F_1 providing the transformation is indeed canonical. Thus, F_1 is always uncertain to within an additive arbitrary function of t alone (which doesn't affect the equations of transformation), and there may at times be other ambiguities.

It sometimes happens that it is not suitable to describe the canonical transformation by a generating function of the type $F_1(q, Q, t)$. For example, the transformation may be such that p_i cannot be written as functions of q, Q, and t, but rather will be functions of q, P, and t. We would then seek a generating function that is a function of the old coordinates q and the new momenta P. Clearly Eq. (9.13) must then be replaced by an equivalent relation involving $\dot{P}_i$ rather than $\dot{Q}_i$. This can be accomplished by writing F in Eq. (9.11) as

$$F = F_2(q, P, t) - Q_i P_i. \tag{9.15}$$

Substituting this F in Eq. (9.11) leads to

$$p_i\dot{q}_i - H = -Q_i\dot{P}_i - K + \frac{d}{dt}F_2(q, P, t). \tag{9.16}$$

Again, the total derivative of F_2 is expanded and the coefficients of $\dot{q}_i$ and P_i collected, leading to the equations

$$p_i = \frac{\partial F_2}{\partial q_i}, \tag{9.17a}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}, \tag{9.17b}$$

with

$$K = H + \frac{\partial F_2}{\partial t}. \tag{9.17c}$$

As before, Eqs. (9.17a) are to be solved for P_i as functions of q_j, p_j, and t to correspond to the second half of the transformation equations (9.4). The remaining half of the transformation equations is then provided by Eqs. (9.17b).

The corresponding procedures for the remaining two basic types of generating functions are obvious, and the general results are displayed in Table 9.1.

It is tempting to look upon the four basic types of generating functions as being related to each other through Legendre transformations. For example, the

TABLE 9.1 Properties of the Four Basic Canonical Transformations

Generating Function	Generating Function Derivatives	Trivial Special Case
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i, \quad Q_i = p_i, \quad P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i, \quad Q_i = q_i, \quad P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i, \quad Q_i = -q_i, \quad P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i, \quad Q_i = p_i, \quad P_i = -q_i$

transition from F_1 to F_2 is equivalent to going from the variables q, Q to q, P with the relation

$$-P_i = \frac{\partial F_1}{\partial Q_i}. \tag{9.18}$$

This is just the form required for a Legendre transformation of the basis variables, as described in Section 8.1, and in analogy to Eq. (8.5) we would set

$$F_2(q, P, t) = F_1(q, Q, t) + P_i Q_i, \tag{9.19}$$

which is equivalent to Eq. (9.15) combined with Eq. (9.12). All the other defining equations for the generating functions can similarly be looked on, in combination with Eq. (9.12) as Legendre transformations from F_1, with the last entry in Table 9.1 describing a double Legendre transformation. The only drawback to this picture is that it might erroneously lead us to believe that any given canonical transformation can be expressed in terms of the four basic types of Legendre transformations listed in Table 9.1. This is not always possible. Some transformations are just not suitable for description in terms of these or other elementary forms of generating functions, as has been noted above and as will be illustrated in the next section with specific examples. If we try to apply the Legendre transformation process, we are then led to generating functions that are identically zero or are indeterminate. For this reason, we have preferred to define each type of generating function relative to F, which is some unspecified function of $2n$ independent coordinates and momenta.

Finally, note that a suitable generating function doesn't have to conform to one of the four basic types for *all* the degrees of freedom of the system. It is possible, and for some canonical transformations necessary, to use a generating function that is a mixture of the four types. To take a simple example, it may be desirable for a particular canonical transformation with two degrees of freedom to be defined by a generating function of the form

$$F'(q_1, p_2, P_1, Q_2, t). \tag{9.20}$$

This generating function would be related to F in Eq. (9.11) by the equation

$$F = F'(q_1, p_2, P_1, Q_2, t) - Q_1 P_1 + q_2 p_2, \tag{9.21}$$

and the equations of transformation would be obtained from the relations

$$p_1 = \frac{\partial F'}{\partial q_1}, \qquad Q_1 = \frac{\partial F'}{\partial P_1},$$
$$q_2 = -\frac{\partial F'}{\partial p_2}, \qquad P_2 = -\frac{\partial F'}{\partial Q_2}, \tag{9.22}$$

with

$$K = H + \frac{\partial F'}{\partial t}. \tag{9.23}$$

Specific illustrations are given in the next section and in the exercises.

9.2 ■ EXAMPLES OF CANONICAL TRANSFORMATIONS

The nature of canonical transformations and the role played by the generating function can best be illustrated by some simple yet important examples. Let us consider, first, a generating function of the second type with the particular form

$$F_2 = q_i P_i \tag{9.24}$$

found in column 3 of Table 9.1. From Eqs. (9.17), the transformation equations are

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} = P_i, \\ Q_i &= \frac{\partial F_2}{\partial P_i} = q_i, \\ K &= H. \end{aligned} \tag{9.25}$$

The new and old coordinates are the same; hence F_2 merely generates the *identity transformation* (cf. Table 9.1). We also note, referring to Table 9.1, that the particular generating function $F_3 = p_i Q_i$ generates an identity transformation with

A more general type of transformation is described by the generating function

$$F_2 = f_i(q_1, \ldots, q_n;\ t) P_i, \tag{9.26}$$

where the f_i may be any desired set of independent functions. By Eqs. (9.17b), the new coordinates Q_i are given by

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_1, \ldots, q_n;\ t) \tag{9.27}$$

Thus, with this generating function the new coordinates depend only upon the old coordinates and the time and do not involve the old momenta. Such a transformation is therefore an example of the class of point transformations defined by Eqs. (9.3). In order to define a point transformation, the functions f_i must be independent and invertible, so that the q_j can be expressed in terms of the Q_i. Since the f_i are otherwise completely arbitrary, we may conclude that *all point transformations are canonical*. Equation (9.17c) furnishes the new Hamiltonian in terms of the old and of the time derivatives of the f_i functions.

Note that F_2 as given by Eq. (9.26) is not the only generating function leading to the point transformation specified by the f_i. Clearly the same point transformation is implicit in the more general form

$$F_2 = f_i(q_1, \ldots, q_n;\ t)P_i + g(q_1, \ldots, q_n;\ t), \tag{9.28}$$

where $g(q, t)$ is any (differentiable) function of the old coordinates and the time. Equations (9.27), the transformation equations for the coordinates, remain unaltered for this generating function. But the transformation equations of the momenta differ for the two forms. From Eqs. (9.17a), we have

$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_i}{\partial q_j}P_i + \frac{\partial g}{\partial q_j}, \tag{9.29}$$

using the form of F_2 given by Eq. (9.28). These equations may be inverted to give P as a function of (q, p), most easily by writing them in matrix notation:

$$\mathbf{p} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\mathbf{P} + \frac{\partial g}{\partial \mathbf{q}}. \tag{9.29'}$$

Here $\mathbf{p}$, $\mathbf{P}$, and $\partial g/\partial \mathbf{q}$ are n-elements of single-column matrices, and $\partial \mathbf{f}/\partial \mathbf{q}$ is a square matrix whose ijth element is $\partial f_i/\partial q_j$. In two dimensions, Eq. (9.29') can be written as

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial q_2} \end{bmatrix}.$$

It follows that $\mathbf{P}$ is a linear function of $\mathbf{p}$ given by

$$\mathbf{P} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right]^{-1}\left[\mathbf{p} - \frac{\partial g}{\partial \mathbf{q}}\right]. \tag{9.30}$$

In two dimensions, (9.30) becomes

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix}^{-1} \left[\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} - \begin{bmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial q_2} \end{bmatrix}\right]. \tag{9.31}$$

Thus, the transformation equations (9.27) for Q are independent of g and depend only upon the $f_i(q, t)$, but the transformation equations (9.29) for P do depend upon the form of g and are in general functions of both the old coordinates and momenta. The generating function given by Eq. (9.26) is only a special case of Eq. (9.28) for which $g = 0$, with correspondingly specialized transformation equations for P.

An instructive transformation is provided by the generating function of the first kind, $F_1(q, Q, t)$, of the form

$$F_1 = q_k Q_k.$$

The corresponding transformation equations, from (9.14a, b) are

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i, \tag{9.32a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i. \tag{9.32b}$$

In effect, the transformation interchanges the momenta and the coordinates; the new coordinates are the old momenta and the new momenta are essentially the old coordinates. Table 9.1 shows that the particular generating function of type $F_4 = p_i P_i$ produces the same transformation. These simple examples should emphasize the independent status of generalized coordinates and momenta. They are both needed to describe the motion of the system in the Hamiltonian formulation. The distinction between them is basically one of nomenclature. We can shift the names around with at most no more than a change in sign. There is no longer present in the theory any lingering remnant of the concept of q_i as a spatial coordinate and p_i as a mass times a velocity. Incidentally, we may see directly from Hamilton`s equations,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i},$$

that this exchange transformation is canonical. If q_i is substituted for p_i, the equations remain in the canonical form only if $-p_i$ is substituted for q_i.

A transformation that leaves some of the (q, p) pairs unchanged, and interchanges the rest (with a sign change), is obviously a canonical transformation of a "mixed" form. Thus, in a system of two degrees of freedom, the transformation

$$Q_1 = q_1, \qquad P_1 = p_1,$$
$$Q_2 = p_2, \qquad P_2 = -q_2,$$

is generated by the function

$$F = q_1 P_1 + q_2 Q_2, \tag{9.33}$$

which is a mixture of the F_1 and F_2 types.

9.3 ■ THE HARMONIC OSCILLATOR

As a final example, let us consider a canonical transformation that can be used to solve the problem of the simple harmonic oscillator in one dimension. If the force constant is k, the Hamiltonian for this problem in terms of the usual coordinates is

$$H = \frac{p^2}{2m} + \frac{kq^2}{2}. \tag{9.34a}$$

Designating the ratio k/m by ω^2, H can also be written as

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2). \tag{9.34b}$$

This form of the Hamiltonian, as the sum of two squares, suggests a transformation in which H is cyclic in the new coordinate. If we could find a canonical transformation of the form

$$p = f(P)\cos Q, \tag{9.35a}$$

$$q = \frac{f(P)}{m\omega}\sin Q, \tag{9.35b}$$

then the Hamiltonian as a function of Q and P would be simply

$$K = H = \frac{f^2(P)}{2m}(\cos^2 Q + \sin^2 Q) = \frac{f^2(P)}{2m}, \tag{9.36}$$

so that Q is cyclic. The problem is to find the form of the yet unspecified function $f(P)$ that makes the transformation canonical. If we use a generating function of the first kind given by

$$F_1 = \frac{m\omega q^2}{2}\cot Q, \tag{9.37}$$

Eqs. (9.14) then provide the equations of transformation,

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q, \tag{9.38a}$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}. \tag{9.38b}$$

Solving for q and p, we have*

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q, \tag{9.39a}$$

and comparison with Eq. (9.35a) evaluates $f(P)$:

$$f(P) = \sqrt{2m\omega P}. \tag{9.40}$$

It follows then that the Hamiltonian in the transformed variables is

$$H = \omega P. \tag{9.41}$$

Since the Hamiltonian is cyclic in Q, the conjugate momentum P is a constant. It is seen from Eq. (9.41) that P is in fact equal to the constant energy divided by ω:

$$P = \frac{E}{\omega}.$$

The equation of motion for Q reduces to the simple form

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega,$$

with the immediate solution

$$Q = \omega t + \alpha, \tag{9.42}$$

where α is a constant of integration fixed by the initial conditions. From Eqs. (9.39), the solutions for q and p are

where α is a constant of integration fixed by the initial conditions. From Eqs. (9.39), the solutions for q and p are

$$q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha), \tag{9.43a}$$

$$p = \sqrt{2mE}\cos(\omega t + \alpha). \tag{9.43b}$$

It is instructive to plot the time dependence of the old and new variables as is shown in Fig. 9.1. We see that q and p oscillate (Fig. 9.1a, b) whereas Q and P are linear plots (Fig. 9.1d, e). The figure also shows the phase space plots for p versus q (Fig. 9.1c) and for P versus Q (Fig. 9.1f). Fig. 9.1c is an ellipse with the following semimajor axes (for the q and p directions, respectively):

$$a = \sqrt{\frac{2E}{m\omega^2}} \quad \text{and} \quad b = \sqrt{2mE},$$

where m is the mass of the oscillator, ω its frequency, and E the oscillator's energy. The area, A, of this ellipse in phase space is

$$A = \pi ab = \frac{2\pi E}{\omega}.$$

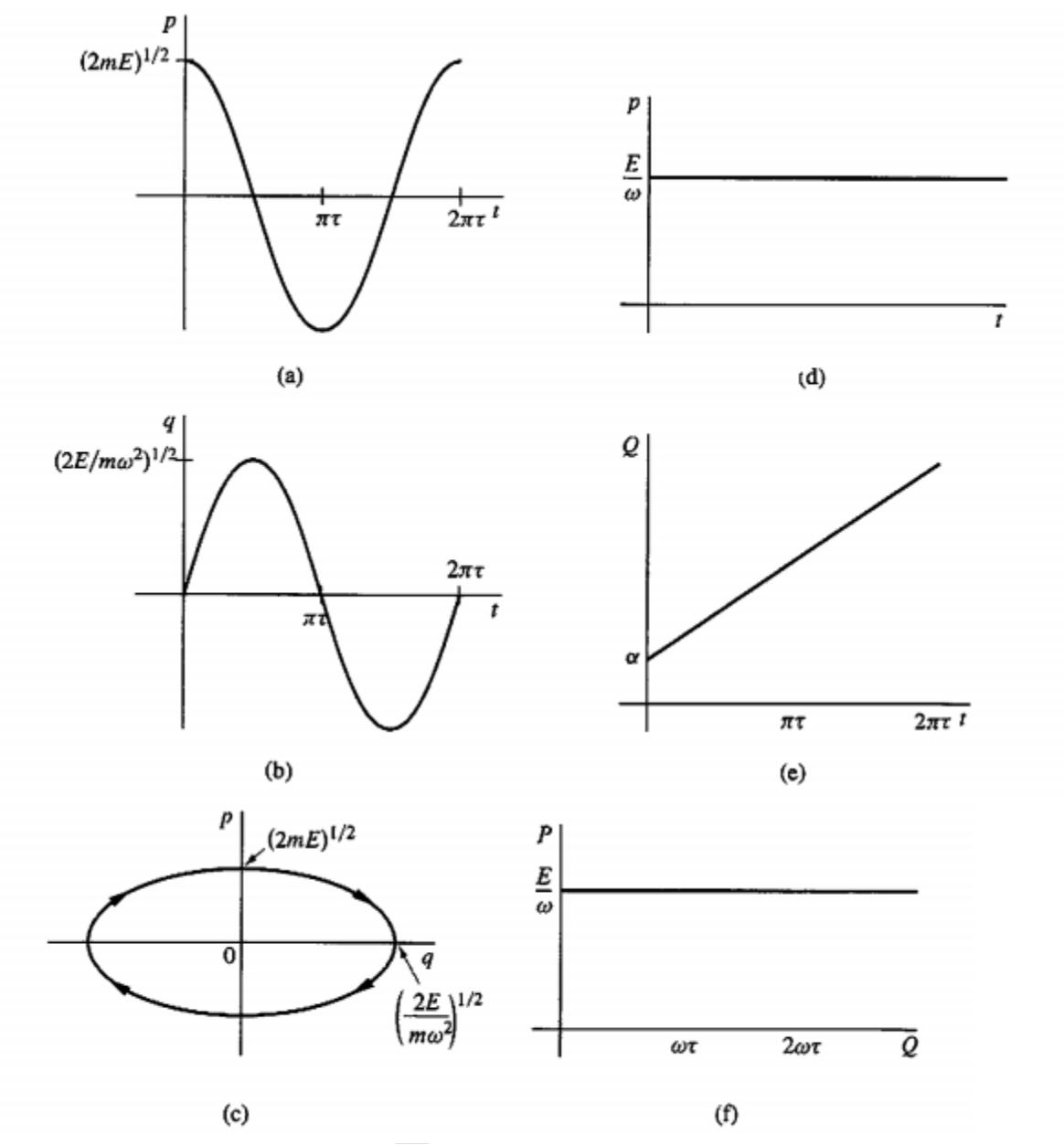

FIGURE 9.1 The harmonic oscillator in two canonical coordinate systems. Drawings (a)–(c) show the q, p system and (d)–(f) show the P, Q system.

When we invoke quantum mechanics, we write $E = \hbar\omega$, where $\hbar = h/2\pi$, and h is Planck's constant. The coordinate and momentum q and p can be normalized as

$$q' = \sqrt{\frac{m\omega^2}{2E}}q \qquad \text{and} \qquad p' = \frac{p}{\sqrt{2mE}},$$

to make the phase space plot of p' versus q' a circle of area π. This normalized form will be useful in Section 11.1 on chaos.

9.4 ■ THE SYMPLECTIC APPROACH TO CANONICAL TRANSFORMATIONS

Another method of treating canonical transformations, seemingly unrelated to the generator formalism, can be expressed in terms of the matrix or symplectic formulation of Hamilton's equations. By way of introduction to this approach, let us consider a restricted canonical transformation, that is, one in which time does not appear in the equations of transformation:

$$\begin{aligned} Q_i &= Q_i(q, p), \\ P_i &= P_i(q, p). \end{aligned} \tag{9.44}$$

We know that the Hamiltonian function does not change in such a transformation. The time derivative of Q_i, on the basis of Eqs. (9.44), is to be found as

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}. \tag{9.45}$$

On the other hand, the inverses of Eqs. (9.44),

$$\begin{aligned} q_j &= q_j(Q, P), \\ p_j &= p_j(Q, P), \end{aligned} \tag{9.46}$$

enables us to consider $H(q, p, t)$ as a function of Q and P and to form the partial derivative

$$\frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} + \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i}. \tag{9.47}$$

Comparing Eqs. (9.45) and (9.47), it can be concluded that

$$\dot{Q}_i = \frac{\partial H}{\partial P_i};$$

that is, the transformation is canonical, only if

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P}, \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}. \tag{9.48a}$$

The subscripts on the derivatives are to remind us that on the left-hand side of these equations Q_i is considered as a function of (q, p) (cf. Eqs. (9.44)), while on the right-hand side the derivatives are for q_j and p_j as functions of (Q, P) (cf. Eqs. (9.46)). A similar comparison of $\dot{P}_i$ with the partial of H with respect to Q_j leads to the conditions

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P}, \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}. \tag{9.48b}$$

The sets of Eqs. (9.48) together are sometimes known as the "direct conditions" for a (restricted) canonical transformation.

The algebraic manipulation that leads to Eqs. (9.48) can be performed in a compact and elegant manner if we make use of the symplectic notation for the Hamiltonian formulation introduced above at the end of Section 8.1. If $\boldsymbol{\eta}$ is a column matrix with the $2n$ elements q_i, p_i, then Hamilton's equations can be written, it will be remembered, as Eq. (8.39)

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}},$$

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}},$$

where $\mathbf{J}$ is the antisymmetric matrix defined in Eq. (8.38a). Similarly the new set

canonical transformation the equations of transformation (9.44) take the form

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta}).$$

Analogously to Eq. (9.45) we can seek the equations of motion for the new variables by looking at the time derivative of a typical element of $\boldsymbol{\zeta}$

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial \eta_j}\dot{\eta}_j, \qquad i, j = 1, \ldots, 2n.$$

In matrix notation, this time derivative can be written as

$$\dot{\boldsymbol{\zeta}} = \mathbf{M}\dot{\boldsymbol{\eta}},$$

where $\mathbf{M}$ is the Jacobian matrix of the transformation with elements

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}.$$

Making use of the equations of motion for $\boldsymbol{\eta}$, Eq. (9.50) becomes

$$\dot{\zeta} = \mathbf{M}\mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}}. \tag{9.52}$$

Now, by the inverse transformation H can be considered as a function of $\boldsymbol{\zeta}$, and the derivative with respect to η_i evaluated as

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \eta_i},$$

or, in matrix notation*

$$\frac{\partial H}{\partial \boldsymbol{\eta}} = \widetilde{\mathbf{M}}\frac{\partial H}{\partial \boldsymbol{\zeta}}. \tag{9.53}$$

The combination of Eqs. (9.52) and (9.53) leads to the form of the equations of motion for any set of variables $\boldsymbol{\zeta}$ transforming, independently of time, from the canonical set $\boldsymbol{\eta}$:

$$\dot{\boldsymbol{\zeta}} = \mathbf{M}\mathbf{J}\widetilde{\mathbf{M}}\frac{\partial H}{\partial \boldsymbol{\zeta}}. \tag{9.54}$$

We have the advantage of knowing from the generator formalism that for a *restricted* canonical transformation the old Hamiltonian expressed in terms of the new variables serves as the new Hamiltonian:

$$\dot{\boldsymbol{\zeta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\zeta}}. \tag{9.54'}$$

The transformation, Eq. (9.49), will therefore be canonical if $\mathbf{M}$ satisfies the condition

$$\mathbf{M}\mathbf{J}\widetilde{\mathbf{M}} = \mathbf{J}. \tag{9.55}$$

That Eq. (9.55) is also a necessary condition for a restricted canonical transformation is easily shown directly by reversing the order of the steps of the proof. Note that for an extended time-independent canonical transformation, where $K = \lambda H$, the condition of Eq. (9.55) would be replaced by

$$\mathbf{M}\mathbf{J}\widetilde{\mathbf{M}} = \lambda\mathbf{J}. \tag{9.56}$$

Equation (9.55) may be expressed in various forms. Multiplying from the right by the matrix inverse to $\widetilde{\mathbf{M}}$ leads to

$$\mathbf{M}\mathbf{J} = \mathbf{J}\widetilde{\mathbf{M}}^{-1}, \tag{9.57}$$

(since the transpose of the inverse is the inverse of the transpose). The elements of the matrix equation (9.57) will be found to be identical with Eqs. (9.48a) and (9.48b). If Eq. (9.57) is multiplied by $\mathbf{J}$ from the left and $-\mathbf{J}$ from the right, then by virtue of Eq. (8.38e) we have

$$\mathbf{JM} = \tilde{\mathbf{M}}^{-1}\mathbf{J},$$

or

$$\tilde{\mathbf{M}}\mathbf{J}\mathbf{M} = \mathbf{J}. \tag{9.58}$$

Equation (9.55), or its equivalent version, Eq. (9.58), is spoken of as the *symplectic condition* for a canonical transformation, and the matrix $\mathbf{M}$ satisfying the condition is said to be a *symplectic matrix*.

These concepts may become more obvious if we display the details of the $\mathbf{J}$ and $\mathbf{M}$ matrices corresponding to the mixed generating function $F = F_2(q_1, P_1) + F_1(q_2, Q_2)$ of Eq. (9.33). The variables $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are column vectors given by

$$\boldsymbol{\eta} = \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\zeta} = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix}.$$

The transformation $\dot{\boldsymbol{\zeta}} = \mathbf{M}\dot{\boldsymbol{\eta}}$ (cf. Eq. (9.50)) is made by the following $\mathbf{M}$ matrix:

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{p}_2 \\ \dot{p}_1 \\ -\dot{q}_2 \end{bmatrix},$$

in agreement with the expressions obtained by differentiating the results of the generating function with respect to time (cf. Column 3, Table 9.1). Hamilton's equations for the transformed variables $\dot{\zeta} = \mathbf{J}\frac{\partial H}{\partial \zeta}$ (Eq. (9.54′)) are expressed as follows independent of the generating function F

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{P}_1 \\ -\dot{P}_2 \\ \dot{Q}_1 \\ \dot{Q}_2 \end{bmatrix}$$

where $-\dot{P}_i = \partial H/\partial \zeta_i$ for ζ_1 and ζ_2 and $\dot{Q}_i = \partial H/\partial \zeta_i$ for ζ_3 and ζ_4. Note that $\mathbf{M}$ depends on F whereas J does not (cf. Eq. (8.38a)). This formalism is not applicable to all cases. For example, a simple $\mathbf{M}$ matrix cannot be written for the harmonic oscillator example discussed in Section 9.3.

A canonical transformation of the form

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta}, t) \tag{9.59}$$

evolves continuously as time increases from some initial value t_0. It is a single-parameter instance of the family of continuous transformations first studied systematically by the mathematician Sophus Lie and as such plays a distinctive role in the transformation theory of classical mechanics.

If the transformation

$$\boldsymbol{\eta} \to \boldsymbol{\zeta}(t) \tag{9.60a}$$

is canonical, then so obviously is the transformation

$$\boldsymbol{\eta} \to \boldsymbol{\zeta}(t_0). \tag{9.60b}$$

It follows then from the definition of canonical transformation that the transformation characterized by

$$\boldsymbol{\zeta}(t_0) \to \boldsymbol{\zeta}(t) \tag{9.60c}$$

is also canonical. Since t_0 in Eq. (9.60b) is a fixed constant, this canonical transformation satisfies the symplectic condition (9.58). If now the transformation of Eq. (9.60c) obeys the symplectic condition, it is easy to show (cf. Derivation 13) that the general transformation Eq. (9.60a) will also.

To demonstrate that the symplectic condition does indeed hold for canonical transformations of the type of Eq. (9.60c), we introduce the notion of an *infinitesimal canonical transformation* (abbreviated I.C.T.), a concept that will prove to be widely useful. As in the case of infinitesimal rotations, such a transformation is one in which the new variables differ from the old only by infinitesimals. Only first-order terms in these infinitesimals are to be retained in all calculations. The transformation equations can then be written as

$$Q_i = q_i + \delta q_i, \tag{9.61a}$$

$$P_i = p_i + \delta p_i, \tag{9.61b}$$

or in matrix form

$$\boldsymbol{\zeta} = \boldsymbol{\eta} + \delta\boldsymbol{\eta}. \tag{9.61c}$$

(Here δq_i and δp_i do *not* represent virtual displacements but are simply the infinitesimal changes in the coordinates and momenta.) An infinitesimal canonical transformation thus differs only infinitesimally from the identity transformation discussed in Section 9.1. In the generator formalism, a suitable generating function for an I.C.T. would therefore be

$$F_2 = q_i P_i + \epsilon G(q, P, t), \tag{9.62}$$

where ϵ is some infinitesimal parameter of the transformation, and G is any (differentiable) function of its $2n+1$ arguments. By Eq. (9.17a), the transformation equations for the momenta are to be found from

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon \frac{\partial G}{\partial q_j}$$

or

$$\delta p_j \equiv P_j - p_j = -\epsilon \frac{\partial G}{\partial q_j}. \tag{9.63a}$$

Similarly, by Eq. (9.17b), the transformation equations for Q_j are determined by the relations

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon \frac{\partial G}{\partial P_j}.$$

Since the second term is already linear in ϵ, and P differs from p only by an infinitesimal, it is consistent to first order to replace P_j in the derivative function by p_j. We may then consider G as a function of q, p only (and possibly t). Following the usual practice, we will refer to $G(q, p)$ as the *generating function of the infinitesimal canonical transformation*, although strictly speaking that designation belongs only to F. The transformation equation for Q_i can therefore be written as

$$\delta q_j = \epsilon \frac{\partial G}{\partial p_j}. \tag{9.63b}$$

Both transformation equations can be combined into one matrix equation

$$\delta \boldsymbol{\eta} = \epsilon \mathbf{J} \frac{\partial G}{\partial \boldsymbol{\eta}}. \tag{9.63c}$$

An obvious example of an infinitesimal canonical transformation would be the transformation of Eq. (9.60c) when t differs from t_0 by an infinitesimal t:

$$\boldsymbol{\zeta}(t_0) \to \boldsymbol{\zeta}(t_0 + dt), \tag{9.64}$$

with dt as the infinitesimal parameter ϵ. The continuous evolution of the transformation $\boldsymbol{\zeta}(\eta, t)$ from $\boldsymbol{\zeta}(\eta, t_0)$ means that the transformation $\boldsymbol{\zeta}(t_0) \to \boldsymbol{\zeta}(t)$ can be built up as a succession of such I.C.T.'s in steps of dt. It will therefore suffice to show that the infinitesimal transformation, Eq. (9.64), satisfies the symplectic condition (9.58). But it follows from the transformation equations (9.63) that the Jacobian matrix of any I.C.T. is a symplectic matrix. By definition the Jacobian matrix (9.51) for an infinitesimal transformation is

$$\mathbf{M} \equiv \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{\eta}} = \mathbf{1} + \frac{\partial \delta \boldsymbol{\eta}}{\partial \boldsymbol{\eta}},$$

or by Eq. (9.63c)

$$\mathbf{M} = \mathbf{1} + \epsilon \mathbf{J} \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}}. \tag{9.65}$$

The second derivative in Eq. (9.65) is a square, symmetric matrix with elements

$$\left(\frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \right)_{ij} = \frac{\partial^2 G}{\partial \eta_i\, \partial \eta_j}.$$

Because of the antisymmetrical property of $\mathbf{J}$, the transpose of $\mathbf{M}$ is

$$\tilde{\mathbf{M}} = \mathbf{1} - \epsilon \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \mathbf{J}. \tag{9.66}$$

The symplectic condition involves the value of the matrix product

$$\mathbf{M}\mathbf{J}\tilde{\mathbf{M}} = \left(\mathbf{1} + \epsilon \mathbf{J} \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \right) \mathbf{J} \left(\mathbf{1} - \epsilon \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \mathbf{J} \right).$$

The symplectic condition involves the value of the matrix product

$$\mathbf{M}\mathbf{J}\tilde{\mathbf{M}} = \left(\mathbf{1} + \epsilon \mathbf{J} \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \right) \mathbf{J} \left(\mathbf{1} - \epsilon \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \mathbf{J} \right).$$

Consistent to first order in this product is

$$\begin{aligned} \mathbf{M}\mathbf{J}\tilde{\mathbf{M}} &= \mathbf{J} + \epsilon \mathbf{J} \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \mathbf{J} - \mathbf{J} \epsilon \frac{\partial^2 G}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}} \mathbf{J} \\ &= \mathbf{J}, \end{aligned}$$

thus demonstrating that the symplectic condition holds for any infinitesima canonical transformation. By the chain of reasoning we have spun out, it there fore follows that *any* canonical transformation, whether or not it involves time a a parameter, obeys the symplectic conditions, Eqs. (9.55) and (9.58).

The symplectic approach, for the most part, has been developed independentl of the generating function method, except in the treatment of infinitesimal canon ical transformations. They are of course connected. We shall sketch later, for ex ample, a proof that the symplectic condition implies the existence of a generatin function. But the connection is largely irrelevant. Both are valid ways of looking a canonical transformations, and both encompass all of the needed properties of th transformations. For example, either the symplectic or the generator formalism can be used to prove that canonical transformations have the four properties tha

characterize a group (cf. Appendix B).

1. The identity transformation is canonical.
2. If a transformation is canonical, so is its inverse.
3. Two successive canonical transformations (the group "product" operation) define a transformation that is also canonical.
4. The product operation is associative.

9.5 ■ POISSON BRACKETS AND OTHER CANONICAL INVARIANTS

The *Poisson bracket* of two functions u, v with respect to the canonical variables (q, p) is defined as

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}. \tag{9.67}$$

In this bilinear expression we have a typical symplectic structure, as in Hamilton's equations, where q is coupled with p, and p with $-q$. The Poisson bracket thus lends itself readily to being written in matrix form, where it appears as

$$[u, v]_{\eta} = \frac{\widetilde{\partial u}}{\partial \boldsymbol{\eta}} \mathbf{J} \frac{\partial v}{\partial \boldsymbol{\eta}}. \tag{9.68}$$

The transpose sign is used on the first matrix on the right-hand side to indicate explicitly that this matrix must be treated as a single-row matrix in the multiplication. On most occasions this specific reminder will not be needed and the transpose sign may be omitted.

Suppose we choose the functions u, v out of the set of canonical variables (q, p) themselves. Then it follows trivially from the definition, either as Eq. (9.67) or (9.68), that these Poisson brackets have the values

$$[q_j, q_k]_{q,p} = 0 = [p_j, q_k]_{q,p},$$

and

$$[q_j, p_k]_{q,p} = \delta_{jk} = -[p_j, q_k]_{q,p}. \tag{9.69}$$

We can summarize the relations of Eqs. (9.69) in one equation by introducing a *square matrix Poisson bracket*, $[\boldsymbol{\eta}, \boldsymbol{\eta}]$, whose lm element is $[\eta_l, \eta_m]$. Equations (9.69) can then be written as

$$[\eta, \eta]_\eta = \mathbf{J}. \tag{9.70}$$

Now let us take for u, v the members of the transformed variables (Q, P), or ζ, defined in terms of (q, p) by the transformation equations (9.59). The set of all the Poisson brackets that can be formed out of (Q, P) comprise the matrix Poisson bracket defined as

$$[\zeta, \zeta]_\eta = \frac{\widetilde{\partial \zeta}}{\partial \eta} \mathbf{J} \frac{\partial \zeta}{\partial \eta}.$$

But we recognize the partial derivatives as defining the square Jacobian matrix of the transformation, so that the Poisson bracket relation is equivalent to

$$[\zeta, \zeta]_\eta = \widetilde{\mathbf{M}} \mathbf{J} \mathbf{M}. \tag{9.71}$$

If the transformation $\eta \rightarrow \zeta$ is canonical, then the symplectic condition holds and Eq. (9.71) reduces to (cf. Eq. (9.58))

$$[\zeta, \zeta]_\eta = \mathbf{J}, \tag{9.72}$$

and conversely, if Eq. (9.72) is valid, then the transformation is canonical.

Poisson brackets of the canonical variables themselves, such as Eqs. (9.70) or (9.72), are referred to as the *fundamental Poisson brackets*. Since we have from Eq. (9.70) that

$$[\zeta, \zeta]_\zeta = \mathbf{J}, \tag{9.73}$$

Eq. (9.72) states that the fundamental Poisson brackets of the ζ variables have the same value when evaluated with respect to *any* canonical coordinate set. In other words, the *fundamental Poisson brackets are invariant under canonical transformation*. We have seen from Eq. (9.71) that the invariance is a necessary and sufficient condition for the transformation matrix to be symplectic. The invariance of the fundamental Poisson brackets is thus in all ways equivalent to the symplectic condition for a canonical transformation.

It does not take many more steps to show that *all* Poisson brackets are invariant under canonical transformation. Consider the Poisson bracket of two functions u, v with respect to the η set of coordinates, Eq. (9.68). In analogy to Eq. (9.53), the partial derivative of v with respect to η can be expressed in terms of partial derivatives with respect to ζ as

$$\frac{\partial v}{\partial \eta} = \widetilde{\mathbf{M}} \frac{\partial v}{\partial \zeta}$$

(that is, the partial derivative transforms as a 1-form). In a similar fashion,

$$\widetilde{\frac{\partial u}{\partial \eta}} = \widetilde{\mathbf{M}} \widetilde{\frac{\partial u}{\partial \zeta}} = \widetilde{\frac{\partial u}{\partial \zeta}} \mathbf{M}.$$

Hence the Poisson bracket Eq. (9.68) can be written

$$[u, v]_{\eta} = \frac{\widetilde{\partial u}}{\partial \eta}\mathbf{J}\frac{\partial v}{\partial \eta} = \frac{\widetilde{\partial u}}{\partial \zeta}\mathbf{M}\mathbf{J}\widetilde{\mathbf{M}}\frac{\partial u}{\partial \zeta}.$$

If the transformation is canonical, the symplectic condition in the form of Eq. (9.55) holds, and we then have

$$[u, v]_{\eta} = \frac{\widetilde{\partial u}}{\partial \zeta}\mathbf{J}\frac{\partial v}{\partial \zeta} \equiv [u, v]_{\zeta}. \tag{9.74}$$

Thus, the Poisson bracket has the same value when evaluated with respect to any canonical set of variables—*all Poisson brackets are canonical invariants*. In writing the symbol for the Poisson bracket, we have so far been careful to indicate by the subscript the set of variables in terms of which the brackets are defined. So long as we use only canonical variables that practice is now seen to be unnecessary, and we shall in general drop the subscript.*

The hallmark of the canonical transformation is that Hamilton's equations of motion are invariant in form under the transformation. Similarly, the canonical invariance of Poisson brackets implies that equations expressed in terms of Poisson brackets are invariant in form under canonical transformation. As we shall see, we can develop a structure of classical mechanics, paralleling the Hamiltonian formulation, expressed solely in terms of Poisson brackets. Historically this Poisson bracket formulation, which has the same form in all canonical coordinates, was especially useful for carrying out the original transition from classical to quantum mechanics. There is a simple "correspondence principle" that says that the classical Poisson bracket is to be replaced by a suitably defined commutator of the corresponding quantum operators.

The algebraic properties of the Poisson bracket are therefore of considerable interest. We have already used the obvious properties

$$[u, u] = 0, \tag{9.75a}$$

$$[u, v] = -[v, u]. \qquad \text{(antisymmetry)} \tag{9.75b}$$

Almost equally obvious are the characteristics

$$[au + bv, w] = a[u, w] + b[v, w], \qquad \text{(linearity)} \tag{9.75c}$$

where a and b are constants, and

$$[uv, w] = [u, w]v + u[v, w]. \tag{9.75d}$$

One other property is far from obvious, but is very important in defining the nature of the Poisson bracket. It is usually given in the form of *Jacobi's identity*, which states that if u, v, and w are three functions with continuous second derivatives, then

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0; \tag{9.75e}$$

that is, the sum of the cyclic permutations of the double Poisson bracket of three functions is zero. There seems to be no simple way of proving Jacobi's identity for the Poisson bracket without lengthy algebra. However, it is possible to mitigate the complexity of the manipulations by introducing a special nomenclature. We

shall use subscripts on u, v. w (or functions of them) to denote partial derivatives by the corresponding canonical variable. Thus,

$$u_i \equiv \frac{\partial u}{\partial \eta_i}, \qquad \text{and} \qquad v_{ij} \equiv \frac{\partial v}{\partial \eta_i \, \partial \eta_j}.$$

In this notation the Poisson bracket of u and v can be expressed as

$$[u, v] = u_i J_{ij} v_j.$$

Here J_{ij}, as usual, is simply the ijth element of $\mathbf{J}$. In the proof, the only property of $\mathbf{J}$ that we shall need is its antisymmetry.

Now let us consider the first double Poisson bracket in Eq. (9.75e):

$$[u, [v, w]] = u_i J_{ij} [v, w]_j = u_i J_{ij} (v_k J_{kl} w_l)_j.$$

Because the elements J_{kl} are constants, the derivative with resect to η doesn't act on them, and we have

$$[u, [v, w]] = u_i J_{ij} (v_k J_{kl} w_{lj} + v_{kj} J_{kl} w_l). \tag{9.76}$$

The other double Poisson brackets can be obtained from Eq. (9.76) by cyclic permutation of u, v, w. There are thus six terms in all, each being a fourfold sum over dummy indices i, j, k, and l. Consider the term in Eq. (9.76) involving a second derivative of w:

$$J_{ij} J_{kl} u_i v_k w_{lj}.$$

Since the order of differentiation is immaterial, $w_{lj} = w_{jl}$, and the sum of the two terms is given by

$$(J_{ij} + J_{ji})J_{kl}u_i v_k w_{lj} = 0,$$

by virtue of the antisymmetry of J. The remaining four terms are cyclic permutations and can similarly be divided in two pairs, one involving second derivatives of u and the other of v. By the same reasoning, each of these pairs sums to zero, and Jacobi's identity is thus verified.

If the Poisson bracket of u, v is looked on as defining a "product" operation of the two functions, then Jacobi's identity is the replacement for the associative law of multiplication. Recall that the ordinary multiplication of arithmetic is associative; that is, the order of a sequence of multiplications is immaterial:

$$a(bc) = (ab)c.$$

Jacobi's identity says that the bracket "product" is not associative and gives the effect of changing the sequence of "multiplications." Brackets that satisfy Eqs. (9.75), together with the expression

$$[u_i, u_j] = \sum_k c_{ij}^k u_k. \tag{9.77}$$

constitute a generally noncommunitive algebra called a Lie algebra. For Poisson brackets in three-dimensional space, either the structure constants c_{ij}^k are all zero or only one term in the right-hand side of Eq. (9.77) exists for any pair of indices. Examples of this will be given later, and a more detailed discussion of Lie algebras is given in Appendix B.

Poisson bracket operation is not the only type of "product" familiar to physicists that satisfies the conditions for a Lie algebra. It will be left to the exercises to show that that vector product of two vectors,

$$_{\mathrm{V}}[\mathrm{A}, \mathrm{B}] \rightarrow \mathrm{A} \times \mathrm{B}, \tag{9.78a}$$

and the commutator of two matrices,

$$_{\mathrm{M}}[\mathrm{A}, \mathrm{B}] \rightarrow \mathrm{AB} - \mathrm{BA}, \tag{9.78b}$$

satisfy the same Lie algebra conditions as the Poisson bracket. It is this last that makes it feasible to replace the classical Poisson bracket by the commutator of the quantum mechanical operators. In other words, the "correspondence principle" can work only because both the Poisson bracket and commutator are representations of a Lie algebra "product."*

There are other canonical invariants besides the Poisson bracket. One, mainly of historical interest now, is the *Lagrange bracket*, denoted by $\{u, v\}$. Suppose u and v are two functions out of a set of $2n$ independent functions of the canonical variables. By inversion, the canonical variables can then be considered as functions of the set of $2n$ functions. On this basis, the Lagrange bracket of u and v with respect to the (q, p) variables is defined as

$$\{u, v\}_{q,p} = \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}, \tag{9.79}$$

or, in matrix notation,

$$\{u, v\}_{\eta} = \frac{\widetilde{\partial \eta}}{\partial u}\mathbf{J}\frac{\partial \eta}{\partial v}. \tag{9.80}$$

Proof of the canonical invariance of the Lagrange bracket parallels that for the Poisson bracket.

If for u and v we take two members of the set of canonical variables, then we obtain the *fundamental Lagrange brackets*:

$$\{q_i, q_j\}_{qp} = 0 = \{p_i, p_j\}_{qp} \qquad \{q_i, p_j\}_{qp} = \delta_{ij}, \tag{9.81}$$

or, in matrix notation,

$$\{\eta, \eta\} = \mathbf{J}. \tag{9.82}$$

The Lagrange and Poisson brackets clearly stand in some kind of inverse relationship to each other, but the precise form of this relation is somewhat complicated to express. Let u_i, $i = 1, \ldots, 2n$, be a set of $2n$ independent functions of the canonical variables, to be represented by a column (or row) matrix $\mathbf{u}$. Then $\{\mathbf{u}, \mathbf{u}\}$ is the $2n \times 2n$ matrix whose ijth element is $\{u_i, u_j\}$, with a similar description for $[\mathbf{u}, \mathbf{u}]$. The reciprocal character of the two brackets manifests itself in the relation

$$\{\mathbf{u}, \mathbf{u}\}[\mathbf{u}, \mathbf{u}] = -\mathbf{1}. \tag{9.83}$$

If for u we choose the canonical set itself, η, then Eq. (9.83) obviously follows from the fundamental bracket formulas, Eqs. (9.70) and (9.82), and the properties of $\mathbf{J}$. The proof for arbitrary u is not difficult if written in terms of the matrix definitions of the brackets and is reserved for the exercises. While the properties of the Lagrange and Poisson brackets parallel each other in many aspects, note that the Lagrange brackets do *not* obey Jacobi's identity. Lagrange brackets therefore do not qualify as a "product" operation in a Lie algebra.

Another important canonical invariant is the magnitude of a volume element in phase space. A canonical transformation $\boldsymbol{\eta} \rightarrow \boldsymbol{\zeta}$ transforms the $2n$-dimensional phase space with coordinates η_i to another phase space with coordinates ζ_i. The volume element

$$(d\eta) = dq_1 dq_2 \ldots dq_n dp_1 \ldots dp_n$$

transforms to a new volume element

$$(d\zeta) = dQ_1 dQ_2 \ldots dQ_n dP_1 \ldots dP_n.$$

As is well known, the sizes of the two volume elements are related by the absolute value of the Jacobian determinant $\|\mathbf{M}\|$;

$$(d\zeta) = \|\mathbf{M}\|(d\eta).$$

For example, in the two-dimensional transformation from $\eta_i = q, p$ to $\zeta_i = Q, P$, this expression becomes

$$dQ\,dP = \begin{vmatrix} \dfrac{\partial q}{\partial Q} & \dfrac{\partial q}{\partial P} \\ \dfrac{\partial p}{\partial Q} & \dfrac{\partial p}{\partial P} \end{vmatrix} dq\,dp = [q, p]_\zeta\, dq\,dp. \tag{9.84}$$

But, by taking the determinant of both sides of the symplectic condition, Eq. (9.58), we have

$$|\mathbf{M}|^2 |\mathbf{J}| = |\mathbf{J}|. \tag{9.85}$$

Thus, in a real canonical transformation the Jacobian determinant is ± 1, and the absolute value is always unity, proving the canonical invariance of the volume element in phase space. It follows, also, that the volume of any arbitrary region in phase space,

$$J_n = \int \cdots \int (d\eta), \tag{9.86}$$

is a canonical invariant. In our two-dimensional example, the invariant is $d\eta = dq\,dp$ and $J_1 = \int dq\,dp$.

The volume integral in Eq. (9.86) is the final member of a sequence of canonical invariants known as the *integral invariants of Poincaré*, comprising integrals over subspaces of phase space of different dimensions. The other members of the sequence cannot be stated as simply as J_n, and because they are not needed for the further development of the theory, they will not be discussed here.

Finally, the invariance of the fundamental Poisson brackets now enables us to outline a proof that the symplectic condition implies the existence of a generating function, as mentioned at the conclusion of the previous section. To simplify considerations, we shall examine only a system with one degree of freedom; the general method of the proof can be directly extended to systems with many degrees of freedom.* We suppose that the first of the equations of transformation,

$$Q = Q(q, p), \qquad P = P(q, p),$$

is invertable so as to give p as a function q and Q, say

$$p = \phi(q, Q). \tag{9.87}$$

Substitution in the second equation of transformation gives P as some function of q and Q, say

$$P = \psi(q, Q). \tag{9.88}$$

In such a case, we would expect the transformation to be generated by a generating function of the first kind,* F_1, with Eqs. (9.87) and (9.88) appearing as

$$p = \frac{\partial F_1(q, Q)}{\partial q}, \qquad P = -\frac{\partial F_1}{\partial Q}(q, Q). \tag{9.89}$$

If Eq. (9.89) holds, then it must be true that

$$\frac{\partial \phi}{\partial Q} = -\frac{\partial \psi}{\partial q}. \tag{9.90}$$

Conversely, if we can show that Eq. (9.90) is valid, then there must exist a function F_1 such that p and P are given by Eqs. (9.89).

To demonstrate the validity of Eq. (9.90), we try to look on all quantities as functions of q and Q. Thus, we of course have the identity

Conversely, if we can show that Eq. (9.90) is valid, then there must exist a function F_1 such that p and P are given by Eqs. (9.89).

To demonstrate the validity of Eq. (9.90), we try to look on all quantities as functions of q and Q. Thus, we of course have the identity

$$\frac{\partial Q}{\partial Q} = 1,$$

but if Eq. (9.87) be substituted in the first transformation equation,

$$Q = Q(q, \phi(q, Q)), \tag{9.91}$$

the partial derivative can also be written

$$\frac{\partial Q}{\partial Q} = \frac{\partial Q}{\partial p}\frac{\partial \phi}{\partial Q},$$

so that we have the relation

$$\frac{\partial Q}{\partial p}\frac{\partial \phi}{\partial Q} = 1. \tag{9.92}$$

In the same spirit we evaluate the Poisson bracket

$$[Q, P] \equiv \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1.$$

The derivatives of P are derivatives of ψ from Eq. (9.88) considered as a function of q and $Q(q, p)$. Hence, the Poisson bracket can be written

$$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\left(\frac{\partial \psi}{\partial q} + \frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial q}\right),$$

or, consolidating terms, as

$$[Q, P] = \frac{\partial \psi}{\partial Q}\left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right) - \frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q},$$

and therefore

$$1 = -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q}. \tag{9.93}$$

Combining Eqs. (9.92) and (9.93), we have

$$\frac{\partial Q}{\partial p}\frac{\partial \phi}{\partial Q} = -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q}.$$

Since the partial derivative of Q with respect to p is the same on both sides of the equation, that is, the other variable being held constant is q in both cases, and since the derivative doesn't vanish (else the Q equation could not be inverted), it follows that Eq. (9.90) must be true. Thus, from the value of the fundamental Poisson bracket $[Q, P]$, which we have seen is equivalent to the symplectic condition, we

In terms of the symplectic notation, the derivation of Eq. (9.94) would run

$$\frac{du}{dt} = \frac{\partial u}{\partial \boldsymbol{\eta}}\dot{\boldsymbol{\eta}} + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \boldsymbol{\eta}} \mathbf{J} \frac{\partial H}{\partial \boldsymbol{\eta}} + \frac{\partial u}{\partial t},$$

from whence Eq. (9.94) follows, by virtue of (9.68). Equation (9.94) may be looked on as the generalized equation of motion for an arbitrary function u in the Poisson bracket formulation. It contains Hamilton's equations as a special case when for u we substitute one of the canonical variables

$$\dot{q}_i = [q_i, H], \qquad \dot{p}_i = [p_i, H], \tag{9.95a}$$

or, in symplectic notation,

$$\dot{\boldsymbol{\eta}} = [\boldsymbol{\eta}, H]. \tag{9.95b}$$

That Eq. (9.95b) is identical with Hamilton's equations of motion may be seen directly from the observation that by the definition of the Poisson bracket, Eq. (8.39), we have

$$[\boldsymbol{\eta}, H] = \mathbf{J} \frac{\partial H}{\partial \boldsymbol{\eta}}, \tag{9.96}$$

so that Eq. (9.95b) is simply another way of writing Eq. (8.31). Another familiar property may be obtained from Eq. (9.94) by taking u as H itself. Equation (9.94) then says that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t},$$

as was obtained previously in Eq. (8.41).

Note that the generalized equation of motion is canonically invariant; it is valid in whatever set of canonical variables q, p is used to express the function u or to evaluate the Poisson bracket. However, the Hamiltonian used must be appropriate to the particular set of canonical variables. Upon transforming to another set of variables by a time-dependent canonical transformation, we must also change to the transformed Hamiltonian K.

If u is a constant of the motion, then Eq. (9.94) says it must have the property

$$[H, u] = \frac{\partial u}{\partial t}. \tag{9.97}$$

All functions that obey Eq. (9.97) are constants of the motion, and conversely the Poisson bracket of H with any constant of the motion must be equal to the explicit time derivative of the constant function. We thus have a general test for seeking and identifying the constants of the system. For those constants of the motion not

POSSIBLE QUESTIONS

Part B (6 Marks)

1. Explain the canonical transformation with an example
2. State and Prove Jacobi identity
3. Explain the simple harmonic oscillator problem
4. Explain the integral invariants of poincare
5. Derive an expression for $\delta\varphi$and $\delta\Psi$
6. Show that how the generating function specifies the equations of transformations
7. Explain the Lagrange's bracket
8. Show that the transformation P=1/2 $\times(p^2+q^2)$, Q = $\tan^{-1}$(q/p) is canonical.

Part C (10 Marks)

1. Derive Jacobi's theorem
2. Explain that the fundamental poisson brackets are invariant under canonical transformation
3. Show that the transformation $= q\ cot\ p$, $Q\ =\ log(\frac{sin\ p}{q})$ is canonical.

   Also find the generating function.

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Subject: Mechanics **Subject Code: 19MMP106**

Class : I - M.Sc. Mathematics **Semester : I**

Unit IV

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Examples for simple harmonic motion is _____.	Simple pendulum	Moving car	Throwing ball	Rope	Simple pendulum
When a particle moves in a stright line, its acceleration is directed towards a fixed point and proportional to the distance from that point, its motion is called _____	Simple harmonic	Acceleration	Angular velocity	Areal velocity	Simple harmonic motion
Equation of Simple Harmonic Motion is _____.	μx	λx	$-\mu x$	$-\lambda x$	$-\mu x$
Acceleration of Simple Harmonic Motion is _____.	μx	λx	$-\mu x$	$-\lambda x$	μx
Velocity of Simple Harmonic Motion is _____.	$\mu(a^2 + x^2)$	$\mu(a^2 - x^2)$	$\lambda(a^2 + x^2)$	$\lambda(a^2 - x^2)$	$\mu(a^2 - x^2)$
Maximum acceleration of Simple Harmonic Motion is _____.	$\mu\lambda$	$\mu\theta$	μa	λa	μa
Maximum acceleration of Simple Harmonic Motion occurs only when _____.	$\mu=\lambda$	$x=a$	$x=\lambda$	$a=\mu$	$x=a$
The symbol ξ is called _____.	Theta	Omega	Gamma	Epoch	Epoch

The composition of two simple harmonic motion with the same period along the same line is _____.	Areal velocity	Angular momen	Constant	Simple Harmonic M	Simple Harmonic Motion
If the particle possess two Simple Harmonic Motion in perpendicular direction and of the same period then the path is an _____.	Parabola	Hyperbola	Ellipse	Curve	Ellipse
If $\xi=0$ then the path of Simple Harmonic Motion is _____.	Curve	Ellipse	Straight line	Parabola	Straight line
If $\xi=\pi$ then the path of Simple Harmonic Motion is _____.	Curve	Straight line	Ellipse	Parabola	Straight line
When the velocity is zero the particle oscillates through a distance on either side of the fixed point then this distance is called _____.	Period	Simple Harmoni	Amplitude	Phase	Amplitude
The constant interval of time between two consecutive instants when it passes through the same point in the same direction is called _____.	Period	Simple Harmoni	Amplitude	Phase	Period
_____ is independent of the amplitude.	Period	Phase	Simple Harmonic	Frequency	Period
_____ is the rate of acceleration at unit distance from the origin.	Frequency	Phase	Simple Harmonic	Period	Period
_____ is the reciprocal of the perid.	Phase	Amplitude	Frequency	Velocity	Frequency
_____ is the number of oscillations made per second.	Phase	Frequency	Velocity	Acceleration	Frequency
The _____ consistts of a heavy particle attached t a fixed point by a light string.	phase of Simple I	simple pendulun	seconds pendulur	period of Simple Ha	simple pendulum
_____ motion takes place in a vertical plane.	Simple pendulum	Period of Simple	Phase of Simple I	Seconds pendulum	Simple pendulum
A _____ is one which beats seconds .	Simple pendulum	Seconds pendulu	Acceleration	Velocity	Seconds pendulum
_____ is one for which the period of one vibration is one second.	Seconds pendulu	Frequency	Simple Harmonic	Simple pendulum	Seconds pendulum

In seconds pendulum if 'g' is in foot per second units then l = _____ feet.	1.24	2.24	3.24	4.24	3.24
If g is in centigram second units then l = _____	99.5 cm	89.5 cm	79.5 cm	69.5 cm	99.5 cm
The time of _____ depends upon l and g.	Phase	Oscillation	Period	Simple Harmonic M	Oscillation
The loss or gain in the number of seconds per day may be due to a change in _____.	l	g	both	l or g	both
A seconds pendulum is said to gain 'n' seconds a day if it makes _____ half oscillations per day.	76400-n	86400-n	76400 + n	86400 + n	86400 + n
A seconds pendulum is said to lose 'n' seconds per day if it makes _____ half oscillations per day.	76400-n	86400-n	76400 + n	86400 + n	86400-n
A seconds pendulum is said to gain 'n'seconds a day hence its period _____.	Increase	Decrease	Remain same	Zero	Decrease
A seconds pendulum is said to lose 'n'seconds a day hence its period _____.	Increase	Decrease	Remain same	Zero	Increase
Period of Simple Harmonic Motion is independent of _____ .	amplitude	phase	period	oscillation	amplitude
Period of Simple Harmonic Motion is the root of _____ at unit distance from the origin.	amplitude	velocity	Acceleration	phase	Acceleration
The velocity of Simple Harmonic Motion is zero when _____.	x =a	x = -a	x =a and x = -a	x =a or x= -a	x =a or x= -a
Frequency is the reciprocal of the _____.	velocity	amplitude	period	phase	period
If ξ =_____ the path of Simple Harmonic Motion becomes circle.	π	$\pi/2$	0	∞	$\pi/2$
If ξ = _____ the path f Simple Harmonic Motion becomes a straight line.	π	$\pi/2$	0	∞	π

The resultant motion of two Simple Harmonic Motion of same period along perpendicular is an _____.	circle	ellipse	straight line	parabola	ellipse
Simple pendulum is an example of _____.	amplitude	velocity	simple harmnic n	impact	simple harmnic motion
Simple pendulum motion takes place in _____.	horizontal plane	vertical plane	along the plane	perpendicular t the p	vertical plane
In seconds pendulum if 'g' is in foot per second units then the unit f l is _____ .	feet	Second	cm	foot per second	feet

UNIT-V

Hamilton Jacobi Theory: Hamilton Jacobi equations for Hamilton's principle function – Harmonic oscillator problem - Hamilton Jacobi equation for Hamilton's characteristic function – Separation of variables in the Hamilton-Jacobi equation.

10.1 ■ THE HAMILTON–JACOBI EQUATION FOR HAMILTON'S PRINCIPAL FUNCTION

We can automatically ensure that the new variables are constant in time by requiring that the transformed Hamiltonian, K, shall be identically zero, for then the equations of motion are

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0,$$
$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0. \tag{10.1}$$

As we have seen, K must be related to the old Hamiltonian and to the generating function by the equation

$$K = H + \frac{\partial F}{\partial t},$$

Equation (10.24) can be immediately "turned inside out" to furnish q as a function of t and the two constants of integration α and $\beta = \beta'\omega$:

$$q = \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega t + \beta), \tag{10.25}$$

which is the familiar solution for a harmonic oscillator. Formally, the solution for the momentum comes from the transformation equation (10.7), which, using Eq. (10.22), can be written

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}. \tag{10.26}$$

In conjunction with the solution for q, Eq. (10.25), this becomes

$$p = \sqrt{2m\alpha(1 - \sin^2(\omega t + \beta))},$$

or

$$p = \sqrt{2m\alpha}\cos(\omega t + \beta) \tag{10.27}$$

Of course, this result checks with the simple identification of p as $m\dot{q}$.

To complete the story, the constants α and β must be connected with the initial conditions q_0 and p_0 at time $t = 0$. By squaring Eqs. (10.25) and (10.27), it is clearly seen that α is given in terms of q_0 and p_0 by the equation

$$2m\alpha = p_0^2 + m^2\omega^2 q_0^2. \tag{10.28}$$

The same result follows immediately of course from the previous identification of α as the conserved total energy E. Finally, the phase constant β is related to q_0 and p_0 by

$$\tan\beta = m\omega\frac{q_0}{p_0}. \tag{10.29}$$

The choice $q_0 = 0$ and hence $\beta = 0$ corresponds to starting the motion with the oscillator at its equilibrium position $q = 0$.

Thus, Hamilton's principle function is the generator of a canonical transformation to a new coordinate that measures the phase angle of the oscillation and to a new canonical momentum identified as the total energy.

If the solution for q is substituted into Eq. (10.23), Hamilton's principal function can be written as

$$S = 2\alpha\int\cos^2(\omega t + \beta)\,dt - \alpha t = 2\alpha\int(\cos^2(\omega t + \beta) - \tfrac{1}{2})\,dt. \tag{10.30}$$

Now, the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2m}(p^2 - m^2\omega^2 q^2) \\ &= \alpha(\cos^2(\omega t + \beta) - \sin^2(\omega t + \beta)) \\ &= 2\alpha(\cos^2(\omega t + \beta) - \tfrac{1}{2}), \end{aligned}$$

so that S is the time integral of the Lagrangian, in agreement with the general relation (10.13). Note that the identity could not be proved until *after* the solution to the problem had been obtained.

As another illustration for the Hamilton–Jacobi method, it is instructive to consider the two-dimensional anisotropic harmonic oscillator. If we let m be the mass of the oscillating body and k_x and k_y be the spring constants in the x- and y-directions, respectively, the Hamiltonian is

$$E = \frac{1}{2m}(p_x^2 + p_y^2 + m^2\omega_x^2 x^2 + m^2\omega_y^2 y^2),$$

where

$$\omega_x = \sqrt{\frac{k_x}{m}} \quad \text{and} \quad \omega_y = \sqrt{\frac{k_y}{m}}.$$

Since the coordinates and momenta separate into two distinct sets, the principal function can be written as a sum of the characteristic function for each pair. Assuming that we solve the y-functional dependency first, this means

$$S(x, y, \alpha, \alpha_y, t) = F_x(x, \alpha) + F_y(y, \alpha_y) - \alpha t, \tag{10.31}$$

and the Hamilton–Jacobi equation assumes the form

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + m^2\omega_x^2 x^2 + \left(\frac{\partial W}{\partial y}\right)^2 + m^2\omega_y^2 y^2\right] = \alpha \tag{10.32}$$

in analogy with Eq. (10.18). Since the variables are separated, the y-part of the Eq. (10.32) must be equal to a constant, which we call α_y, so

$$\frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 + \frac{1}{2}m\omega_y^2 y^2 = \alpha_y, \tag{10.33}$$

and we replace the y-term in (10.32) with α_y from (10.33), yielding

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}m\omega_x^2 x^2 = \alpha_x, \tag{10.34}$$

where we write $\alpha - \alpha_y = \alpha_x$ showing the symmetry of Eqs. (10.33) and (10.34).

Chapter 10 Hamilton–Jacobi Theory and Action-Angle Variables

Each equation has a solution analogous to Eqs. (10.25) and (10.27), so

$$\begin{aligned} x &= \sqrt{\frac{2\alpha_x}{m\omega_x^2}}\sin(\omega_x t + \beta_x) \\ p_x &= \sqrt{2m\alpha_x}\cos(\omega_x t + \beta_x) \\ y &= \sqrt{\frac{2\alpha_y}{m\omega_y^2}}\sin(\omega_y t + \beta_y) \\ p_y &= \sqrt{2m\alpha_y}\cos(\omega_y t + \beta_y), \end{aligned} \tag{10.35}$$

where the β_i's are phase constants and the total energy is given by

$$E = \alpha_x + \alpha_y = \alpha.$$

As a third example of Hamilton–Jacobi theory, we again consider the two-dimensional harmonic oscillator; only we will assume the oscillator is isotropic, so

$$k_x = k_y = k \qquad \text{and} \qquad \omega_x = \omega_y = \omega,$$

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \theta &= \tan^{-1}\frac{y}{x} \\ p_x &= m\dot{x} & p_r &= m\dot{r} \\ p_y &= m\dot{y} & p_\theta &= mr^2\dot{\theta}. \end{aligned} \tag{10.36}$$

The Hamiltonian now written as

$$E = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + m^2\omega^2 r^2\right) \tag{10.37}$$

is cyclic in the angular coordinate θ. The principle function can then be written as

$$\begin{aligned} S(r,\theta,\alpha,\alpha_\theta) &= W_r(r,\alpha) + W_\theta(\theta,\alpha_\theta) - \alpha t \\ &= W_r(r,\alpha) + \theta\alpha_\theta - \alpha t, \end{aligned} \tag{10.38}$$

where, as we show later, a cyclic coordinate q_i always has the characteristic function component $W_{q_i} = q_i\alpha_i$. The canonical momentum p_θ associated with the cyclic coordinate, θ, is calculated from the generating function

$$p_\theta = \frac{\partial F_\theta}{\partial\theta} = \alpha_\theta$$

has its expected constant value.

When this p_θ is substituted into Eqs. (10.37) and (10.38), $W_r(r,\alpha)$ satisfies

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 = \alpha. \tag{10.39}$$

Rather than solving this equation directly for W_r, we shall write the Cartesian coordinate solution for these conditions as

$$\begin{aligned} x &= \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t+\beta) & p_x &= \sqrt{2m\alpha}\cos(\omega t+\beta) \\ y &= \sqrt{\frac{2\alpha}{m\omega^2}}\sin\omega t & p_y &= \sqrt{2m\alpha}\cos\omega t, \end{aligned} \tag{10.35'}$$

and use these to get the polar counterparts,

$$r = \sqrt{\frac{2\alpha}{m\omega^2}}\sqrt{\sin^2\omega t + \sin^2(\omega t + \beta)}, \qquad p_r = m\dot{r},$$

and (10.40)

$$\theta = \tan^{-1}\left[\frac{\sin\omega t}{\sin(\omega t + \beta)}\right], \qquad p_\theta = mr^2\dot{\theta}.$$

There are two limiting cases. The linear case is when $\beta = 0$, for which

$$r = \sqrt{\frac{4\alpha}{m\omega^2}}\sin\omega t, \qquad p_r = \sqrt{2m\alpha}\cos\omega t,$$

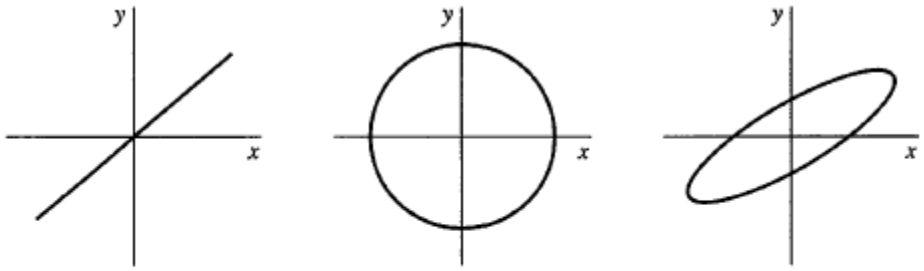
and (10.41)

$$\theta = \frac{\pi}{4}, \qquad p_\theta = 0.$$

The motion in an x-y plot will be an oscillation along a diagonal line as shown in Fig. 10.1a. The other limiting case is when $\beta = \pi/2$, for which

$$\begin{aligned} r = r_0 = \sqrt{\frac{2\alpha}{m\omega^2}}, \qquad & p_r = 0 \\ \theta = \omega t, \qquad & p_\theta = mr_0^2\omega. \end{aligned} \tag{10.42}$$

The motion in an x-y plot for this limiting case is a circle of radius r_0 as is shown in Figure 10.1b. For other values of β $(0 < \beta < \pi/2)$, the orbit in coordinate space is an ellipse. The case for $\beta = \pi/4$ is shown in Fig. 10.1c. The plots shown in Fig. 10.1 are further examples of *Lissajous figures*.



(a) $\beta = 0$ (b) $\beta = \frac{\pi}{2}$ (c) $\beta = \frac{\pi}{4}$

FIGURE 10.1 The two limiting cases (a) and (b) for the harmonic oscillator and an intermediate example (c).

10.3 ■ THE HAMILTON–JACOBI EQUATION FOR HAMILTON'S CHARACTERISTIC FUNCTION

It was possible to integrate the Hamilton–Jacobi equation for the simple harmonic oscillator primarily because S could be separated into two parts, one involving q only and the other only time. Such a separation of variables using Hamilton's characteristic function $W(q, \alpha)$ (Eq. (10.14)) is always possible *whenever the old Hamiltonian does not involve time explicitly*. This provides us with the restricted Hamilton–Jacobi equation

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1, \tag{10.43}$$

which no longer involves the time. One of the constants of integration, namely α_1, is thus equal to the constant value of H. (Normally H will be the energy, but remember that this need not always be the case, cf. Section 8.2.)

The time-independent function, Hamilton's characteristic function W, appears here merely as a part of the generating function S when H is constant. It can also be shown that W separately generates its own contact transformation with properties quite different from that generated by S. Let us consider a canonical transformation in which the new momenta are all constants of the motion α_i, and where α_1 in particular is the constant of motion H. If the generating function for this transformation be denoted by $W(q, P)$, then the equations of transformation are

$$p_i = \frac{\partial W}{\partial q_i}, \qquad Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i}. \tag{10.44}$$

While these equations resemble Eqs. (10.7) and (10.8) respectively for Hamilton's principal function S, the condition now determining W is that H is the new canonical momentum α_1:

$$H(q_i, p_i) = \alpha_1.$$

Using Eqs. (10.44), this requirement becomes the partial differential equation:

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1,$$

which is seen to be identical with Eq. (10.43). Since W does not involve the time, the new and old Hamiltonians are equal, and it follows that $K = \alpha_1$.

Hamilton's characteristic function W thus generates a canonical transformation in which all the new coordinates are cyclic. It was noted in the introduction to this chapter that when H is a constant of the motion, a transformation of this nature in effect solves the mechanical problem involved, for the integration of the new equations of motion is then trivial. The canonical equations for P_i, in fact, merely repeat the statement that the momenta conjugate to the cyclic coordinates are all constant:

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0, \qquad P_i = \alpha_i. \tag{10.45}$$

Because the new Hamiltonian depends upon only one of the momenta α_i, the equations of motion for $\dot{Q}_i$ are

$$\begin{aligned} \dot{Q}_i = \frac{\partial K}{\partial \alpha_i} &= 1, \quad i = 1, \\ &= 0, \quad i \neq 1, \end{aligned}$$

with the immediate solutions

$$\begin{aligned} Q_1 &= t + \beta_i \equiv \frac{\partial W}{\partial \alpha_1}, \\ Q_i &= \beta_i \equiv \frac{\partial W}{\partial \alpha_i} \quad i \neq 1. \end{aligned} \tag{10.46}$$

The only coordinate that is not simply a constant of the motion is Q_1, which is equal to the time plus a constant. We have here another instance of the conjugate relationship between the time as a coordinate and the Hamiltonian as its conjugate momentum.

The dependence of W on the old coordinates q_i is determined by the partial differential equation (10.43), which, like Eq. (10.3), is also referred to as the Hamilton–Jacobi equation. There will now be n constants of integration in a complete solution, but again one of them must be merely an additive constant. The $n-1$ remaining independent constants, $\alpha_2, \ldots, \alpha_n$, together with α_1 may then be taken as the new constant canonical momenta. When evaluated at t_0 the first half of Eqs. (10.44) serve to relate the n constants α_i with the initial values of q_i and p_i. Finally, Eqs. (10.45) and (10.46) can be solved for the q_i as a function of α_i, β_i, and the time t, thus completing the solution of the problem. It will be noted

that $(n-1)$ of the Eqs. (10.46) do not involve the time at all. One of the q_i's can be chosen as an independent variable, and the remaining coordinates can then be expressed in terms of it by solving only these time-independent equations. We are thus led directly to the *orbit equations* of the motion. In central force motion, for example, this technique would furnish r as a function of θ, without the need for separately finding r and θ as functions of time.

It is not always necessary to take α_1 and the constants of integration in W as the new constant canonical momenta. Occasionally it is desirable rather to use some particular set of n independent functions of the α_i's as the transformed momenta. Designating these constants by γ_i the characteristic function W can then be expressed in terms of q_i and γ_i as the independent variables. The Hamiltonian will in general depend upon more than one of the γ_i's and the equations of motion for $\dot{Q}_i$ become

$$\dot{Q}_i = \frac{\partial K}{\partial \gamma_i} = v_i,$$

where the v_i's are functions of γ_i. In this case, all the new coordinates are linear functions of time:

$$Q_i = v_i t + \beta_i. \tag{10.47}$$

The form of W cannot be found a priori without obtaining a complete integral of the Hamilton–Jacobi equation. The procedures involved in solving a mechanical problem by either Hamilton's principal or characteristic function may now by summarized in the following tabular form:

The two methods of solution are applicable when the Hamiltonian

is any general function of q, p, t: $H(q, p, t)$.	is conserved: $H(q, p) =$ constant.

We seek canonical transformations to new variables such that

all the coordinates and momenta Q_i, P_i are constants of the motion.	all the momenta P_i are constants.

To meet these requirements it is sufficient to demand that the new Hamiltonian

shall vanish identically: $K = 0$.	shall be cyclic in all the coordinates: $K = H(P_i) = \alpha_1$.

Under these conditions, the new equations of motion become

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0,$$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = v_i, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0,$$

with the immediate solutions

$$Q_i = \beta_i, \quad P_i = \gamma_i, \qquad \Big| \qquad Q_i = v_i t + \beta_i, \quad P_i = \gamma_i$$

which satisfy the stipulated requirements.

The generating function producing the desired transformation is Hamilton's

Principal Function: $S(q, P, t)$,	Characteristic Function: $W(q, P)$,

satisfying the Hamilton–Jacobi partial differential equation:

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0. \qquad \Big| \qquad H\left(q, \frac{\partial W}{\partial q}\right) - \alpha_1 = 0.$$

A complete solution to the equation contains

n nontrivial constants of integration $\alpha_1, \ldots, \alpha_n$.	$n - 1$ nontrivial constants of integration, which together with α_1 form a set of n independent constants $\alpha_1, \ldots, \alpha_n$.

The new constant momenta, $P_i = \gamma_i$, can be chosen as any n independent functions of the n constants of integration:

$$P_i = \gamma_i(\alpha_1, \ldots, \alpha_n), \qquad \Big| \qquad P_i = \gamma_i(\alpha_1, \ldots, \alpha_n),$$

so that the complete solutions to the Hamilton–Jacobi equation may be considered as functions of the new momenta:

$$S = S(q_i, \gamma_i, t). \qquad \Big| \qquad W = W(q_i, \gamma_i).$$

In particular, the γ_i's may be chosen to be the α_i's themselves. One-half of the transformations equations,

$$p_i = \frac{\partial S}{\partial q_i}, \qquad \Big| \qquad p_i = \frac{\partial W}{\partial q_i},$$

are fulfilled automatically, since they have been used in constructing the Hamilton–Jacobi equation. The other half,

$$Q_i = \frac{\partial S}{\partial \gamma_i} = \beta_i, \qquad \Big| \qquad Q_i = \frac{\partial W}{\partial \gamma_i} = v_i(\gamma_j)t + \beta_i,$$

can be solved for q_i in terms of t and the $2n$ constants β_i, γ_i. The solution to the problem is then completed by evaluating these $2n$ constants in terms of the initial values, (q_{i0}, p_{i0}), of the coordinates and momenta.

When the Hamiltonian does not involve time explicitly, both methods are suitable, and the generating functions are then related to each other according to the formula

$$S(q, P, t) = W(q, P) - \alpha_1 t.$$

10.4 ■ SEPARATION OF VARIABLES IN THE HAMILTON–JACOBI EQUATION

It might appear from the preceding section that little practical advantage has been gained through the introduction of the Hamilton–Jacobi procedure. Instead of solving the $2n$ ordinary differential equations that make up the canonical equations of motion, we now must solve the partial differential Hamilton–Jacobi equation, and partial differential equations can be notoriously complicated to solve. Under certain conditions, however, it is possible to separate the variables in the Hamilton–Jacobi equation, and the solution can then always be reduced to quadratures. In practice, the Hamilton–Jacobi technique becomes a useful computational tool only when such a separation can be effected.

A coordinate q_j is said to be separable in the Hamilton–Jacobi equation when (say) Hamilton's principal function can be split into two additive parts, one of which depends only on the coordinate q_j and the other is entirely independent of q_j. Thus, if q_1 is taken as a separable coordinate, then the Hamiltonian must be such that one can write

$$S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t) = S_1(q_1; \alpha_1, \ldots, \alpha_n; t) + S'(q_2, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t), \quad (10.48)$$

and the Hamilton–Jacobi equation can be split into two equations—one separately for S_1 and the other for S'. Similarly the Hamilton–Jacobi equation is described as *completely separable* (or simply, *separable*) if all the coordinates in the problem are separable. A solution for Hamilton's principal function of the form

$$S = \sum_i S_i(q_i; \alpha_1, \ldots, \alpha_n; t) \quad (10.49)$$

will then split the Hamilton–Jacobi equation into n equations of the type

$$H_i\left(q_j; \frac{\partial S_j}{\partial q_j}; \alpha_1, \ldots, \alpha_n; t\right) + \frac{\partial S_j}{\partial t} = 0. \quad (10.50)$$

If the Hamiltonian does not explicitly depend upon the time, then, for each S_i we have

$$S_i(q_j; \alpha_1, \ldots, \alpha_n; t) = W_i(q_j; \alpha_1, \ldots, \alpha_n; t) - \alpha_i t, \quad (10.51)$$

which provide n restricted Hamilton–Jacobi equations,

$$H_i\left(q_i;\ \frac{\partial W_i}{\partial q_i};\alpha_1,\ldots,\alpha_n\right)=\alpha_i. \tag{10.52}$$

(No summation in Eqs. (10.50) to (10.52)!)

The functions H_i in Eqs. (10.50) and (10.52) may or may not be Hamiltonians, and the α_i may be an energy, an angular momentum squared, or some other quantity depending on the nature of q_i. We shall show this by example in the Kepler problem in the next section.

The constants α_i are referred to now as the *separation constants*. Each of the Eqs. (10.52) involves only one of the coordinates q_i and the corresponding partial derivative of W_i with respect to q_i. They are therefore a set of ordinary differential equations of a particularly simple form. Since the equations are only of first order, it is always possible to reduce them to quadratures; we have only to solve for the partial derivative of W_i with respect to q_i and then integrate over q_i. In practice, each H_i will only contain one or at most a few of the α's. There will also be cases where a subset of r variables can be separated in this fashion, leaving $n-r$ variables, which will not separate. We shall also examine this eventuality in the next section.

It is possible to find examples in which the Hamilton–Jacobi equation can be solved without separating the time variable (cf. Exercise 8). Nonetheless, almost all useful applications of the Hamilton–Jacobi method involve Hamiltonians not explicitly dependent upon time, for which t is therefore a separable variable. The subsequent discussion on separability is thus restricted to such systems where H is a constant of motion, and Hamilton's characteristic function W will be used exclusively.

10.5 ■ IGNORABLE COORDINATES AND THE KEPLER PROBLEM

We can easily show that any cyclic or ignorable coordinate is separable. Suppose that the cyclic coordinate is q_1; the conjugate momentum p_1 is a constant, say γ. The Hamilton–Jacobi equation for W is then

$$H\left(q_2,\ldots,q_n;\ \gamma;\ \frac{\partial W}{\partial q_2},\ldots,\frac{\partial W}{\partial q_n}\right)=\alpha_1. \tag{10.53}$$

If we try a separated solution of the form

$$W=W_1(q_1,\alpha)+W'(q_2,\ldots,q_n;\alpha), \tag{10.54}$$

then it is obvious that Eq. (10.53) involves only the separate function W', while W_1 is the solution of the equation

$$p_1=\gamma=\frac{\partial W_1}{\partial q_1}. \tag{10.55}$$

The constant γ is thus the separation constant, and the obvious solution for W_1 (to within a trivial additive constant) is

$$W_1 = \gamma q_1,$$

and W is given by

$$W = W' + \gamma q_1. \tag{10.56}$$

There is an obvious resemblance between Eq. (10.56) and the form S assumes when H is not an explicit function of time, Eq. (10.43). Indeed, both equations can be considered as arising under similar circumstances. We have seen that t may be considered in some sense as a generalized coordinate with $-H$ as its canonical momentum (cf. Eq. (8.58)). If H is conserved, then t may be treated as a cyclic coordinate.

If S of the n coordinates are noncyclic (that is, they appear explicitly in the Hamiltonian), then the Hamiltonian is of the form $H(q_1, \ldots, q_s; \alpha_1, \ldots, \alpha_n; t)$. The characteristic function can then be written as

$$W(q_1, \ldots, q_s; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{s} W_i(q_i;\ \alpha_1, \ldots, \alpha_n) + \sum_{i=s+1}^{n} q_i \alpha_i, \tag{10.56'}$$

and there are s Hamilton–Jacobi equations to be solved:

$$H\left(q_1; \frac{\partial W_1}{\partial q_1}; \alpha_2, \ldots, \alpha_n\right) = \alpha_1. \tag{10.57}$$

Since these are ordinary first-order differential equations in the independent variable q_1, they can be immediately reduced to quadratures, and the complete solutions for W can be obtained.

In general, a coordinate q_j can be separated if q_j and the conjugate momentum p_j can be segregated in the Hamiltonian into some function $f(q_j, p_j)$ that does not contain any of the other variables. If we then seek a trial solution of the form

$$W = W_j(q_j, \alpha) + W'(q_i, \alpha),$$

where q_i represents the set of all q's *except* q_j, then the Hamilton–Jacobi equation appears as

$$H\left(q_i, \frac{\partial W'}{\partial q_i}, f\left(q_j, \frac{\partial W_j}{\partial q_j}\right)\right) = \alpha_1. \tag{10.58}$$

In principle, at least, Eq. (10.58) can be inverted so as to solve for f:

$$f\left(q_j, \frac{\partial W_j}{\partial q_j}\right) = g\left(q_i, \frac{\partial W'}{\partial q_i}, \alpha_1\right). \tag{10.59}$$

The argument used previously in connection with Eq. (10.51) holds here in slightly varied guise; f is not a function of any of the q's except q_j; g on the other hand is independent of q_j. Hence, Eq. (10.59) can hold only if both sides are equal to the same constant, independent of all q's:

$$f\left(q_j, \frac{\partial W_j}{\partial q_j}\right) = \alpha_j,$$
$$g\left(q_i, \frac{\partial W'}{\partial q_i}\right) = \alpha_j, \tag{10.60}$$

and the separation of the variable has been accomplished.

Note that the separability of the Hamilton–Jacobi equation depends not only on the physical problem involved but also on the choice of the system of generalized coordinates employed. Thus, the one-body central force problem is separable in polar coordinates, but not in Cartesian coordinates. For some problems, it is not possible to completely separate the Hamilton–Jacobi equation, the famous three-body problem being one illustration. On the other hand, in many of the basic problems of mechanics and atomic physics, separation is possible in more than one set of coordinates. In general, it is feasible to solve the Hamilton–Jacobi equation in closed form only when the variables are completely separable. Considerable ingenuity has therefore been devoted to finding the separable systems of coordinates appropriate to each problem.

No simple criterion can be given to indicate what coordinate systems lead to separable Hamilton–Jacobi equations for any particular problem. In the case of orthogonal coordinate systems, the so-called Staeckel conditions have proved useful. They provide necessary and sufficient conditions for separability under certain circumstances. A proof of the sufficiency of the conditions and references will be found in Appendix D of the second edition of this text.

The Staeckel conditions for the separation of the Hamilton–Jacobi equations are:

1. The Hamiltonian is conserved.
2. The Lagrangian is no more than a quadratic function of the generalized velocities, so the Hamiltonian takes the form:

$$H = \tfrac{1}{2}(\tilde{\mathbf{p}} - \tilde{\mathbf{a}})\mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}) + V(q). \tag{8.27}$$

3. The vector $\mathbf{a}$ has elements a_i that are functions only of the corresponding coordinate, that is $a_i = a_i(q_i)$.
4. the potential function can be written as a sum of the form

$$V(q) = \sum_i \frac{V_i(q_i)}{T_{ii}}. \tag{10.61}$$

5. Consider the matrix ϕ^{-1}, with an inverse ϕ whose elements are

$$\delta_{ij}\phi_{ij}^{-1} = \frac{1}{T_{ii}}. \qquad \text{(no summation on } i\text{)} \tag{10.62}$$

where

$$\left(\frac{\partial W_i}{\partial q_i} - a_i\right) = 2\delta_{ik}\phi_{kj}\gamma_j$$

with γ a constant unspecified vector. If the diagonal elements of both ϕ and ϕ^{-1} depend only upon the associated coordinate, that is, ${\phi^{-1}}_{ii}$ and ϕ_{ii} are constants or a function of q_i only, then provided 1–4 are true, the Hamiltonian–Jacobi equations separate.

Since we have assumed that the generalized coordinates q_i form an orthogonal coordinate system, the matrix $\mathbf{T}$ (introduced in Section 8.1) is diagonal. It follows that the inverse matrix $\mathbf{T}^{-1}$ is also diagonal and, if we are dealing with a particle in an external force field, the diagonal elements are:

$$\phi_{ii}^{-1} = \frac{1}{T_{ii}} = \frac{1}{m}, \qquad \text{(no summation)} \tag{10.63}$$

so the fifth Stackel condition is satisfied.

If the Staeckel conditions are satisfied, then Hamilton's characteristic function is completely separable:

$$W(q) = \sum_i W_i(q_i),$$

with the W_i satisfying equations of the form

$$\left(\frac{\partial W_i}{\partial q_i} - a_i\right)^2 = -2V_i(q_i) + 2\phi_{ij}\gamma_j, \tag{10.64}$$

where γ_j are constants of integration (and there is summation only over the index j).

While these conditions appear mysterious and complicated, their application usually is fairly straightforward. As an illustration of some of the ideas developed here about separability, the Hamilton–Jacobi equation for a particle moving in a central force will be discussed in polar coordinates. The problem will then be generalized to arbitrary potential laws, to furnish an application of the Staeckel conditions.

Let us first consider the central force problem in terms of the polar coordinates (r, ψ) in the plane of the orbit. The motion then involves only two degrees of freedom and the Hamiltonian has the form

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\psi^2}{r^2}\right) + V(r), \tag{10.65}$$

which is cyclic in ψ. Consequently, Hamilton's characteristic function appears as

$$W = W_1(r) + \alpha_\psi \psi, \tag{10.66}$$

where α_ψ is the constant angular momentum p_ψ conjugate to ψ. The Hamilton–Jacobi equation then becomes

$$\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{\alpha_\psi^2}{r^2} + 2mV(r) = 2m\alpha_1, \tag{10.67}$$

where α_1 is the constant identified physically as the total energy of the system. Solving Eq. (10.66) for the partial derivative of W_1 we obtain

$$\frac{\partial W_1}{\partial r} = \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\psi^2}{r^2}},$$

so that W is

$$W = \int dr \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\psi^2}{r^2}} + \alpha_\psi \psi. \tag{10.68}$$

With this form for the characteristic function, the transformation equations (10.46) appear as

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \int \frac{m\,dr}{\sqrt{2m(\alpha_1 - V) - \frac{\alpha_\psi^2}{r^2}}}, \tag{10.69a}$$

and

$$\beta_2 = \frac{\partial W}{\partial \alpha_\psi} = -\int \frac{\alpha_\psi\, dr}{r^2\sqrt{2m(\alpha_1 - V) - \frac{\alpha_\psi^2}{r^2}}} + \psi. \tag{10.69b}$$

Equation (10.69a) furnishes r as a function of t and agreees with the corresponding solution, Eq. (3.18), found in Chapter 3, with α_1 and α_ψ written explicitly as E and l, respectively. It has been remarked previously that the remaining transformation equations for Q_i, here only Eq. (10.69b), should provide the orbit equation. If the variable of integration in Eq. (10.69b) is changed to $u = 1/r$, the equation reduces to

$$\psi = \beta_2 - \int \frac{du}{\sqrt{\frac{2m}{\alpha_\psi^2}(\alpha_1 - V) - u^2}}$$

which agrees with Eq. (3.37) previously found for the orbit, identifying ψ as θ and β_2 as θ_0.

As a further example of separation of variables, we shall examine the same central force problem, but in spherical polar coordinates, that is, ignoring our a priori knowledge that the orbit lies in a plane. The appropriate Hamiltonian has been shown to be (cf. Eq. (8.29)):

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(r). \tag{10.70}$$

If the variables in the corresponding Hamilton–Jacobi equation are separable, then Hamilton's characteristic function must have the form

$$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi). \tag{10.71}$$

The coordinate ϕ is cyclic in the Hamiltonian and hence

$$W_\phi = \alpha_\phi \phi \tag{10.72}$$

where α_ϕ is a constant of integration. In terms of this form for W, the Hamilton–Jacobi equation reduces to

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta}\right] + 2mV(r) = 2mE, \tag{10.73}$$

where we have explicitly identified the constant Hamiltonian with the total energy E. Note that all dependence on θ, and on θ alone, has been segregated into the expression within the square brackets. The Hamilton–Jacobi equation then conforms to the appearance of Eq. (10.58), and following the argument given there we see that the quantity in the square brackets must be a constant:

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2. \tag{10.74}$$

Finally the dependence of W on r is given by the remainder of the Hamilton–Jacobi equation:

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m(E - V(r)). \tag{10.75}$$

The variables in the Hamilton–Jacobi equation are thus completely separated. Equations (10.74) and (10.75) may be easily reduced to quadratures providing at least a formal solution for $W_\theta(\theta)$ and $W_r(r)$, respectively.

Note that the constants of integration α_ϕ, α_θ, α_1 all have directly recognizable physical meanings. The quantity α_ϕ is of course the constant value of the angular momentum about the polar axis (cf. Eq. (10.44)):

$$\alpha_\phi = p_\phi = \frac{\partial W_\phi}{\partial \phi}. \tag{10.76}$$

To identify α_θ we use Eq. (10.44) to rewrite Eq. (10.74) as

$$p_\theta^2 + \frac{p_\theta^2}{\sin^2\theta} = \alpha_\theta^2, \tag{10.74'}$$

so that the Hamiltonian, Eq. (10.70) appears as

$$H = \frac{1}{2m}\left(p_r^2 + \frac{\alpha_\theta^2}{r^2}\right) + V(r). \tag{10.70'}$$

Comparison with Eq. (10.65) for the Hamiltonian as expressed in terms of polar coordinates in the plane of the orbit shows that α_θ is the same as p_ψ, the magnitude of the total angular momentum:

$$\alpha_\theta = p_\psi \equiv l. \tag{10.77}$$

Lastly, α_1 is of course the total energy E. Indeed, the three differential equations for the component parts of W can be looked on as statements of conservation theorems. Equation (10.75) says the z-component of the angular momentum vector, **L**, is conserved, while Eq. (10.74) states the conservation of the magnitude, l, of the angular momentum. And Eq. (10.75) is a form of the energy conservation theorem.

In this simple example, some of the power and elegance of the Hamilton–Jacobi method begins to be apparent. A few short steps suffice to obtain the dependence of r on t and the orbit equation, Eqs. (10.69a and b), results derived earlier only with considerable labor. The conserved quantities of the central force problem also appear automatically. Separation of variables for the purely central force problem can also be performed in other coordinate systems, for example, parabolic coordinates, and the conserved quantities appear there in forms appropriate to the particular coordinates.

Finally, we can employ the Staeckel conditions to find the most general form of a scalar potential V for a single particle for which the Hamilton–Jacobi equation is separable in spherical polar coordinates. The matrix $\boldsymbol{\phi}$ of the Staeckel conditions depends only on the coordinate system and not on the potential. Since the Hamilton–Jacobi equation is separable in spherical polar coordinates for at least one potential, that is, the central force potential, it follows that the matrix $\boldsymbol{\phi}$ does exist. The specific form of $\boldsymbol{\phi}$ is not needed to answer our question. Further, since **a** by hypothesis is zero, all we need do is apply Eq. (10.62) to find the most general separable form of V. From the kinetic energy (Eq. 8.28′), the diagonal elements of T are

$$T_{rr} = m, \quad T_{\theta\theta} = mr^2, \quad T_{\phi\phi} = mr^2\sin^2\theta.$$

By Eq. (10.62) it follows that the desired potential must have the form

$$V(q) = V_r(r) + \frac{V_\theta(\theta)}{r^2} + \frac{V_\phi(\phi)}{r^2\sin^2\theta}. \tag{10.78}$$

It is easy to verify directly that with this potential the Hamilton–Jacobi equation is still completely separable in spherical polar coordinates.

10.6 ■ ACTION-ANGLE VARIABLES IN SYSTEMS OF ONE DEGREE OF FREEDOM

Of especial importance in many branches of physics are systems in which the motion is periodic. Very often we are interested not so much in the details of the orbit as in the frequencies of the motion. An elegant and powerful method of handling such systems is provided by a variation of the Hamilton–Jacobi procedure. In this technique, the integration constants α_i appearing directly in the solution of the Hamilton–Jacobi equation are not themselves chosen to be the new momenta. Instead, we use suitably defined constants J_i, which form a set of n independent functions of the α_i's, and which are known as the *action variables*.

For simplicity, we shall first consider in this section systems of one degree of freedom. It is assumed the system is conservative so that the Hamiltonian can be written as

$$H(q, p) = \alpha_1.$$

Solving for the momentum, we have that

$$p = p(q, \alpha_1), \tag{10.79}$$

which can be looked on as the equation of the orbit traced out by the system point in the two-dimensional phase space, p, q when the Hamiltonian has the constant value α_1. What is meant by the term "periodic motion" is determined by the characteristics of the phase space orbit. Two types of periodic motion may be distinguished:

1. In the first type, the orbit is *closed*, as shown in Fig. 10.2(a), and the system point retraces its steps periodically. Both q and p are then periodic functions of the time with the same frequency. Periodic motion of this nature will be found when the initial position lies between two zeros of the kinetic energy. It is often designated by the astronomical name *libration*, although to a physicist it is more likely to call to mind the common oscillatory systems, such as the one-dimensional harmonic oscillator.
2. In the second type of periodic motion, the orbit in phase space is such that p is some periodic function of q, with period q_0, as illustrated in Fig. 10.2(b). Equivalently, this kind of motion implies that when a is increased by q_0, the configuration of the system remains essentially unchanged. The most familiar example is that of a rigid body constrained to rotate about a given axis, with q as the angle of rotation. Increasing q by 2π then produces no essential change in the state of the system. Indeed, the position coordinate in this type of periodicity is invariably an angle of rotation, and the motion

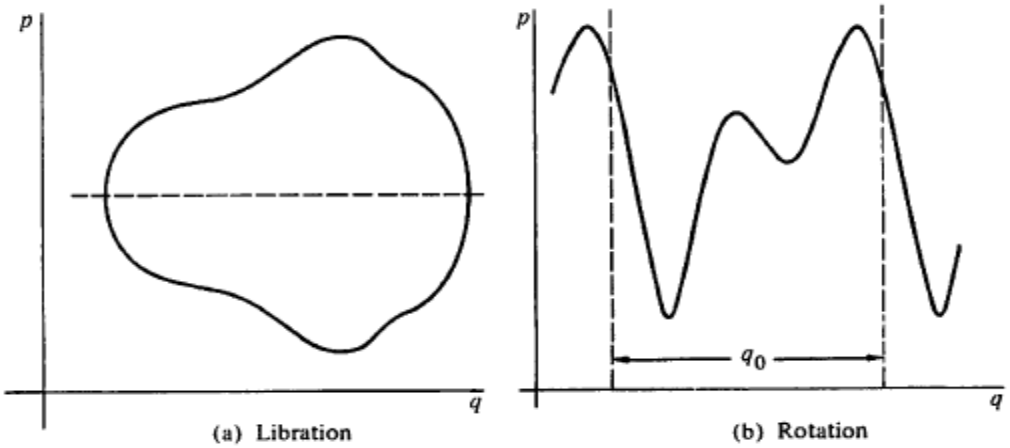


FIGURE 10.2 Orbit of the system point in phase space for periodic motion of one-dimensional systems.

will be referred to simply as *rotation*, in contrast to libration. The values of q are no longer bounded but can increase indefinitely.

It may serve to clarify these ideas to note that both types of periodicity may occur in the same physical system. The classic example is the simple pendulum where q is the angle of deflection θ. If the length of the pendulum is l and the potential energy is taken as zero at the point of suspension, then the constant energy of the system is given by

$$E = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta. \tag{10.80}$$

Solving Eq. (10.64) for p_θ, the equation of the path of the system point in phase space is

$$p_\theta = \pm\sqrt{2ml^2(E + mgl\cos\theta)}. \tag{10.81}$$

If E is less than mgl, then physical motion of the system can only occur for $|\theta|$ less than a bound, θ', defined by the equation

$$\cos\theta' = -\frac{E}{mgl}.$$

Under these conditions, the pendulum oscillates between $-\theta'$ and $+\theta'$, which is a periodic motion of the libration type. The system point then traverses some such path in phase space as the curve 1 of Fig. 10.3. However, if $E > mgl$, all values of θ correspond to physical motion and θ can increase without limit to produce a periodic motion of the rotation type. What happens physically in this case is that the pendulum has so much energy that it can swing through the vertical position

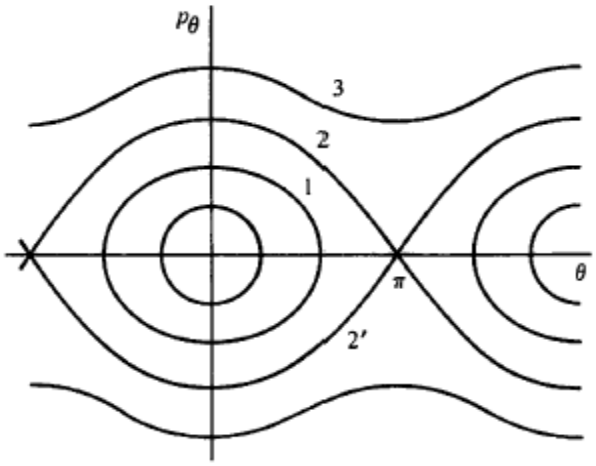


FIGURE 10.3 Phase space orbits for the simple pendulum.

$\theta = \pi$ and therefore continues rotating. Curve 3 in Fig. 10.3 corresponds to the rotation motion of the pendulum. The limiting case when $E = mgl$ is illustrated by curves 2 and 2′ in Fig. 10.3. At this energy, the pendulum arrives at $\theta = \pi$, the vertical position, with zero kinetic energy, that is, $p_\theta = 0$. It is then in unstable equilibrium and could in principle remain there indefinitely. However, if there is the slightest perturbation, it could continue its motion either along curve 2 or switch to curve 2′—it could fall down either way. The point $\theta = \pi$, $p_\theta = 0$ is a saddle point of the Hamiltonian function $H = E(p_\theta, \theta)$ and there are two paths of constant E in phase space that intersect at the saddle point. We have here an instance of what is called a *bifurcation*, a phenomenon that will be discussed extensively in the next chapter. (See also Section 6.6.)

For either type of periodic motion, we can introduce a new variable J designed to replace α_1 as the transformed (constant) momentum. The so-called action variable J is defined as (cf. Eq. (8.80))

$$J = \oint p\, dq, \tag{10.82}$$

where the integration is to be carried over a complete period of libration or of rotation, as the case may be. (The designation as action variable stems from the resemblance of Eq. (10.82) to the abbreviated action of Section 8.6. Note that J always has the dimensions of an angular momentum.) From Eq. (10.79), it follows that J is always some function of α_1 alone:

$$\alpha_1 \equiv H = H(J). \tag{10.83}$$

Hence, Hamilton's characteristic function can be written as

$$W = W(q, J). \tag{10.84}$$

The generalized coordinate conjugate to J, known as the *angle variable* w, is defined by the transformation equation:

$$w = \frac{\partial W}{\partial J}. \tag{10.85}$$

Correspondingly, the equation of motion for w is

$$\dot{w} = \frac{\partial H(J)}{\partial J} = v(J), \tag{10.86}$$

where v is a constant function of J only. Equation (10.86) has the immediate solution

$$w = vt + \beta, \tag{10.87}$$

so that w is a linear function of time, exactly as in Eq. (10.47).

So far the action-angle variables appear as no more than a particular set of the general class of transformed coordinates to which the Hamilton–Jacobi equation leads. Equation (10.85) could be solved for q as a function of w and J, which, in combination with Eq. (10.87), would give the desired solution for q as a function of time. But when employed in this fashion the variables have no significant advantage over any other set of coordinates generated by W. Their particular merit rises rather from the physical interpretation that can be given to v. Consider the change in w as q goes through a complete cycle of libration or rotation, as given by

$$\Delta w = \oint \frac{\partial w}{\partial q}\, dq. \tag{10.88}$$

By Eq. (10.85), this can also be written

$$\Delta w = \oint \frac{\partial^2 W}{\partial q \partial J}\, dq. \tag{10.89}$$

Because J is a constant, the derivative with respect to J can be taken outside the integral sign:

$$\Delta w = \frac{d}{dJ} \oint \frac{\partial W}{\partial q}\, dq = \frac{d}{dJ} \oint p\, dq = 1, \tag{10.90}$$

where the last step follows from the definition for J, Eq. (10.82).

Equation (10.90) states that w changes by unity as q goes through a complete period. But from Eq. (10.87), it follows that if τ is the period for a complete cycle of q, then

$$\Delta w = 1 = v\tau.$$

Hence, the constant v can be identified as the reciprocal of the period,

$$v = \frac{1}{\tau}, \tag{10.91}$$

and is therefore *the frequency associated with the periodic motion of* q. The use of action-angle variables thus provides a powerful technique for obtaining the frequency of periodic motion *without finding a complete solution to the motion of the system.* If it is known a priori that a system of one degree of freedom is periodic according to the definitions given above, then the frequency can be found once H is determined as a function of J. The derivative of H with respect to J, by Eq. (10.86), then directly gives the frequency v of the motion. The designation of w as an angle variable becomes obvious from the identification of v in Eq. (10.87) as a frequency. Since J has the dimensions of an angular momentum, the coordinate w conjugate to it is an angle.*

As an illustration of the application of action-angle variables to find frequencies, let us again consider the familiar linear harmonic oscillator problem. From Eqs. (10.26) and the defining equation (10.82), the constant action variable J is given by

$$J = \oint p\,dq = \oint \sqrt{2m\alpha - m^2\omega^2 q^2}\,dq, \tag{10.92}$$

where α is the constant total energy and $\omega^2 = k/m$. The substitution (10.25)

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\theta$$

reduces the integral to

$$J = \frac{2\alpha}{\omega}\int_0^{2\pi} \cos^2\theta\,d\theta, \tag{10.93}$$

where the limits are such as to correspond to a complete cycle in q. This integrates to

$$J = \frac{2\pi\alpha}{\omega}$$

or, solving for α,

$$\alpha \equiv H = \frac{J\omega}{2\pi}. \tag{10.94}$$

The frequency of oscillation is therefore

*For some applications the action variable is defined in the literature of celestial mechanics as $(2\pi)^{-1}$ times the value given in Eq. (10.82). By Eq. (10.90), the corresponding angle variable is 2π times our definition and in place of v we have ω, the angular frequency. However, we shall stick throughout to the familiar definitions used in physics, as given above.

$$\frac{\partial H}{\partial J} = v = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \tag{10.95}$$

which is the customary formula for the frequency of a linear harmonic oscillator. Although it is entirely unnecessary for obtaining the frequencies, it is nevertheless instructive (and useful for future applications) to write the solutions, Eqs. (10.25) and (10.27), in terms of J and w. It will be recognized first that the combination $(\omega t + \beta)$ is by Eqs. (10.95) and (10.87) the same as $2\pi w$, with the constant of integration suitably redefined. Hence, the solutions for q, Eq. (10.25), and p, Eq. (10.27), take on the form

$$q = \sqrt{\frac{J}{\pi m \omega}} \sin 2\pi w, \tag{10.96}$$

$$p = \sqrt{\frac{mJ\omega}{\pi}} \cos 2\pi w. \tag{10.97}$$

Note that Eqs. (10.96) and (10.97) can also be looked on as the transformation equations from the (w, J) set of canonical variables to the (q, p) canonical set.

10.7 ■ ACTION-ANGLE VARIABLES FOR COMPLETELY SEPARABLE SYSTEMS*

Action-angle variables can also be introduced for certain types of motion of systems with many degrees of freedom, providing there exists one or more sets of coordinates in which the Hamilton–Jacobi equation is completely separable. As before, only conservative systems will be considered, so that Hamilton's characteristic function will be used. Complete separability means that the equations of canonical transformation have the form

$$p_i = \frac{\partial W_i(q_i;\ \alpha_1, \ldots, \alpha_n)}{\partial q_i}, \tag{10.98}$$

which provides each p_i as a function of the q_i and the n integration constants α_j:

$$p_i = p_i(q_i;\ \alpha_1, \ldots, \alpha_n). \tag{10.99}$$

Equation (10.99) is the counterpart of Eq. (10.79), which applied to systems of one degree of freedom. It will be recognized that Eq. (10.99) here represents the orbit equation of the projection of the system point on the (p_i, q_i) plane in phase space. We can define action-angle variables for the system when the orbit equations for *all* of the (q_i, p_i) pairs describe either closed orbits (libration, as in Fig. 10.2(a)) or periodic functions of q_i (rotation, as in Fig. 10.2(b)).

Note that this characterization of the motion does not mean that each q_i and p_i will necessarily be periodic functions of the time, that is, that they repeat their

*Unless otherwise stated, the summation convention will *not* be used in this section.

values at fixed time intervals. Even when each of the separated (q_i, p_i) sets are indeed periodic in this sense, the overall system motion need not be periodic. Thus, in a three-dimensional harmonic oscillator the frequencies of motion along the three Cartesian axes may all be different. In such an example, it is clear the complete motion of the particle may not be periodic. If the separate frequencies are not rational fractions of each other, the particle will not traverse a closed curve in space but will describe an open "Lissajous figure." Such motion will be described as *multiply periodic*. It is the advantage of the action-angle variables that they lead to an evaluation of all the frequencies involved in multiply periodic motion without requiring a complete solution of the motion.

In analogy to Eq. (10.82), the action variables J_i are defined in terms of line integrals over complete periods of the orbit in the (q_i, p_i) plane:

$$J_i = \oint p_i \, dq_i. \tag{10.100}$$

If one of the separation coordinates is cyclic, its conjugate momentum is constant. The corresponding orbit in the q_i, p_i plane of phase space is then a horizontal straight line, which would not appear to be in the nature of a periodic motion. Actually the motion can be considered as a limiting case of the rotation type of periodicity, in which q_i may be assigned any arbitrary period. Since the coordinate in a rotation periodicity is invariably an angle, such a cyclic q_i always has a natural period of 2π. Accordingly, the integral in the definition of the action variable corresponding to a cyclic angle coordinate is to be evaluated from 0 to 2π, and hence

$$J_i = 2\pi p_i \tag{10.101}$$

for all cyclic variables.

By Eq. (10.98), J_i can also be written as

$$J = \oint \frac{\partial W_i(q_i;\ \alpha_1, \ldots, \alpha_n)}{\partial q_i} \, dq_i. \tag{10.102}$$

Since q_i is here merely a variable of integration, each action variable J_i is a function only of the n constants of integration appearing in the solution of the Hamilton–Jacobi equation. Further, it follows from the independence of the separate variable pairs (q_i, p_i) that the J_i's form n independent functions of the α_i's and hence are suitable for use as a set of new constant momenta. Expressing the α_i's as functions of the action variables, the characteristic function W can be written in the form

$$W = W(q_1, \ldots, q_n;\ J_1, \ldots, J_n) = \sum_j W_j(q_j;\ J_1, \ldots, J_n),$$

while the Hamiltonian appears as a function of the J_i's only:

$$H = \alpha_1 = H(J_1, \ldots, J_n). \tag{10.103}$$

As in the system of one degree of freedom, we can define conjugate angle variables w_i by the equations of transformation that here appear as

$$w_i = \frac{\partial W}{\partial J_i} = \sum_{j=1}^{n} \frac{\partial W_j(q_j;\ J_1, \ldots, J_n)}{\partial J_i}. \tag{10.104}$$

Note in general w_i could be a function of several or all of the q_i; that is, $w_i = w_i(q_i, \ldots, q_n;\ J_i, \ldots, J_n)$. The w_i's satisfy equations of motion given by

$$\dot{w}_i = \frac{\partial H(J_1, \ldots, J_n)}{\partial J_i} = v_i(J_1, \ldots, J_n). \tag{10.105}$$

Because the v_i's are constants, functions of the action variables only, the angle variables are all linear functions of time

$$w_i = v_i t + \beta_i. \tag{10.106}$$

Note that in general the separate w_i's increase in time at different rates.

The constants v_i can be identified with the frequencies of the multiply periodic motion, but the argument to demonstrate the relation is more subtle than for periodic systems of one degree of freedom. The transformation equations to the (w, J) set of variables implies that each q_j (and p_j) is a function of the constants J_i and the variables w_i. What we want to find is what sort of mathematical function the q's are of the w's. To do this, we examine the change in a particular w_i when each of the variables q_j is taken through an integral number, m_j, of cycles of libration or rotation. In carrying out this purely mathematical procedure, we are clearly *not* following the motion of the system in time. It is as if the flow of time were suspended and each of the q's were moved, manually as it were, independently through a number of cycles of their motion. In effect, we are dealing with analogues of the virtual displacements of Chapter 1, and accordingly the infinitesimal change in w_i as the q_j's are changed infinitesimally will be denoted by δw_i and is given by

$$\delta w_i = \sum_j \frac{\partial w_i}{\partial q_j}\, dq_j = \sum \frac{\partial^2 W}{\partial J_i \partial q_j}\, dq_j,$$

where use has been made of Eq. (10.104). The derivative with respect to q_i vanishes except for the W_j constituent of W, so that by Eq. (10.98) δw_i reduces to

$$\delta w_i = \frac{\partial}{\partial J_i} \sum_j p_j(q_j, J)\, dq_j. \tag{10.107}$$

Equation (10.107) represents δw_i as the sum of independent contributions each involving the q_j motion. The total change in w_i as a result of the specified ma-

neuver is therefore

$$\Delta w_i = \sum_j \frac{\partial}{\partial J_i} \oint_{m_j} p_j(q_j, J)\, dq_j. \tag{10.108}$$

the differential operator with respect to J_i can be kept outside the integral signs because throughout the cyclic motion of q_i all the J's are of course constant. Below each integral sign, the symbol m_j indicates the integration is over m_j cycles of q_j. But each of the integrals is, by the definition of the action variables, exactly $m_j J_j$. Since the J's are independent, it follows that

$$\Delta w_i = m_i. \tag{10.109}$$

Further, note that if any q_j does not go through a complete number of cycles, then in the integration over q_j there will be a remainder of an integral over a fraction of a cycle and Δw_i will not have an integral value. If the sets of w's and m's are treated as vectors $\mathbf{w}$ and $\mathbf{m}$, respectively, Eq. (10.109) can be written as

$$\Delta \mathbf{w} = \mathbf{m}. \tag{10.109'}$$

Suppose, first, that the separable motions are all of the libration type so that each q_j, as well as p_j, returns to its initial value on completion of a complete cycle. The result described by Eq. (10.109′) could now be expressed somewhat as follows: $\boldsymbol{\eta}$ (the vector of q's and p's) is such a function of $\mathbf{w}$ that a change $\Delta\boldsymbol{\eta} = 0$ corresponds to a change $\Delta\mathbf{w} = \mathbf{m}$, a vector of integer values. Since the number of cycles in the chosen motions of q_j are arbitrary, $\mathbf{m}$ can be taken as zero except for $m_i = 1$, and all the components of $\boldsymbol{\eta}$ remain unchanged or return to their original values. Hence, in the most general case the components of $\boldsymbol{\eta}$ must be periodic functions of *each* w_i with period unity; that is, the q's and p's are multiply periodic functions of the w's with unit periods. Such a multiply periodic function can always be represented by a multiple Fourier expansion, which for q_k, say, would appear as

$$q_k = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} a^{(k)}_{j_1,\ldots,j_n} \cdot e^{2\pi i(j_1 w_1 + j_2 w_2 + j_3 w_3 + \cdots + j_n w_n)}, \quad \text{(libration)} \tag{10.110}$$

where the j's are n integer indices running from $-\infty$ to ∞. By treating the set of j's also as a vector in the same n-dimensional space with $\mathbf{w}$, the expansion can be written more compactly as

$$q_k = \sum_{\mathbf{j}} a^{(k)}_{\mathbf{j}} a^{2\pi i \mathbf{j}\cdot\mathbf{w}}, \quad \text{(libration)}. \tag{10.110'}$$

If we similarly write Eq. (10.109′) as a vector equation,

$$\mathbf{w} = \mathbf{v}t + \boldsymbol{\beta}, \tag{10.106'}$$

then the time dependence of q_k appears in the form

$$q_k(t) = \sum_{\mathbf{j}} a_{\mathbf{j}}^{(k)} e^{2\pi \mathbf{j}\cdot(\nu t+\beta)}, \quad \text{(libration)}. \tag{10.111}$$

Note that in general $q_k(t)$ is *not* a periodic function of t. Unless the various ν_i's are commensurate (that is, rational multiples of each other), q_k will not repeat its values at regular intervals of time. Considered as a function of t, q_k is designated as a *quasi-periodic* function. Finally it should be remembered that the coefficients $a_j^{(k)}$ can be found by the standard procedure for Fourier coefficients; that is, they are given by the multiple integral over the unit cell in $\mathbf{w}$ space:

$$a_{\mathbf{j}}^{(k)} = \int_0^1, \ldots, \int_0^1 q_k(\mathbf{w}) e^{-2\pi i \mathbf{j}\cdot\mathbf{w}} (d\mathbf{w}). \tag{10.112}$$

Here $(d\mathbf{w})$ stands for the volume element in the n-dimensional space of the w_t's.

When the motion is in the nature of a rotation, then in a complete cycle of the separated variable pair (q_k, p_k) the coordinate q_k does not return to its original value, but instead increases by the value of its period q_{0k}. Such a rotation coordinate is therefore not itself even multiply periodic. However, during the cycle we have seen that w_k increases by unity. Hence, the function $q_k - w_k q_{0k}$ does return to its initial value and, like the librational coordinates, is a multiply periodic function of all the w's with unit periods. We can therefore expand the function in a multiple Fourier series analogous to Eq. (10.110)

$$q_k - w_k q_{0k} = \sum_{\mathbf{j}} a_{\mathbf{j}}^{(k)} e^{2\pi i \mathbf{j}\cdot\mathbf{w}}, \quad \text{(rotation)} \tag{10.113}$$

or

$$q_k = q_{0k}(\nu_k t + \beta_k) + \sum_{\mathbf{j}} a_{\mathbf{j}}^{(k)} e^{2\pi i \mathbf{j}\cdot(\nu t+\beta)}, \quad \text{(rotation)}. \tag{10.114}$$

Thus, it is always possible to derive a multiply periodic function from a rotation coordinate, which can then be handled exactly like a libration coordinate. To simplify the further discussion, we shall therefore confine ourselves primarily to the libration type of motion.

The separable momentum coordinates, p_k, are by the nature of the assumed motion also multiply periodic functions of the w's and can be expanded in a multiple Fourier series similar to Eq. (10.110). It follows then that any function of the several variable pairs (q_k, p_k) will also be multiply periodic functions of the w's and can be written in the form

$$f(q, p) = \sum_{\mathbf{j}} b_{\mathbf{j}} e^{2\pi i \mathbf{j}\cdot\mathbf{w}} = \sum_{\mathbf{j}} b_{\mathbf{j}} e^{2\pi i \mathbf{j}\cdot(\nu t+\beta)}. \tag{10.115}$$

For example, where the Cartesian coordinate of particles in the system are not themselves the separation coordinates, they can still be written as functions of time in the fashion of Eq. (10.115).

While Eqs. (10.110) and (10.111) represent the most general type of motion consistent with the assumed nature of the problem, not all systems will exhibit this full generality. In particular, for most problems simple enough to be used as illustrations of the application of action-angle variables, Eq. (10.104) simplifies to

$$w_i = \frac{\partial w_i}{\partial J_i}(q_i;\ J_1, \ldots, J_n) \tag{10.116}$$

and each separation coordinate q_i is a function only of its corresponding w_k. When this happens, q_k is then a periodic function of w_k (and therefore of time), and the multiple Fourier series reduces to a single Fourier series:

$$q_k = \sum_j a_j^{(k)} e^{2\pi i j w_k} = \sum_j a_j^{(k)} e^{2\pi i j(\nu_k t + \beta_k)}. \tag{10.117}$$

In the language of Chapter 6, in such problems the q_k's are in effect the normal coordinates of the system. However, even when the motion in the q's can be so simplified, it frequently happens that functions of all the q's, such as Cartesian coordinates, remain multiply periodic functions of the w's and must be represented as in Eq. (10.115). If the various frequencies ν_k are incommensurate, then such functions are not periodic functions of time. The motion of a two-dimensional anisotropic harmonic oscillator provides a convenient and familiar example of these considerations.

Suppose that in a particular set of Cartesian coordinates the Hamiltonian is given by

$$H = \frac{1}{2m}[(p_x^2 + 4\pi^2 m^2 \nu_x^2 x^2) + (p_y^2 + 4\pi^2 m^2 \nu_y^2 y^2)].$$

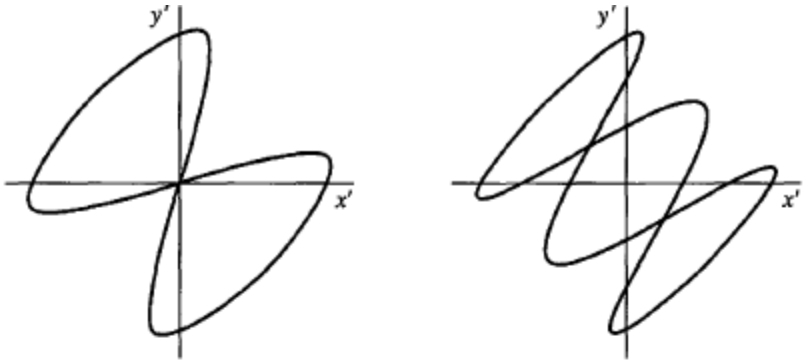
These Cartesian coordinates are therefore suitable separation variables, and each will exhibit simple harmonic motion with frequencies ν_x and ν_y, respectively. Thus, the solutions for x and y are particularly simple forms of the single Fourier expansions of Eq. (10.117). Suppose now that the coordinates are rotated 45° about the z axis; the components of the motion along the new x', y' axes will be

$$x' = \frac{1}{\sqrt{2}}[x_0 \cos 2\pi(\nu_x t + \beta_x) + y_0 \cos 2\pi(\nu_y t + \beta_y)],$$

$$y' = \frac{1}{\sqrt{2}}[y_0 \cos 2\pi(\nu_y t + \beta_y) - x_0 \cos 2\pi(\nu_x t + \beta_x)]. \tag{10.118}$$

If ν_x/ν_y is a rational number, these two expressions will be commensurate, corresponding to closed Lissajous figures of the type shown in Fig. 10.4. But if ν_x and ν_y are incommensurable, the Lissajous figure never exactly retraces its steps and Eqs. (10.118) provide simple examples of multiply periodic series expansions of the form (10.117).

Even when q_k is a multiply periodic function of all the w's, we intuitively feel there must be a special relationship between q_k and its corresponding w_k (and



(a) $\beta_x = \beta_y = \frac{1}{4}$, $\frac{v_x}{v_y} = \frac{1}{2}$ (b) $\beta_x = \frac{1}{4}$, $\beta_y = 0$, $\frac{v_x}{v_y} = \frac{1}{3}$

FIGURE 10.4 Lissajous figures for Eq. (10.118). (a) $\beta_x = \beta_y = \frac{1}{4}$, $\frac{v_x}{v_y} = \frac{1}{2}$ (b) $\beta_x = \frac{1}{4}$, $\beta_y = 0$, $\frac{v_x}{v_y} = \frac{1}{3}$.

therefore v_k). After all, the argument culminating in Eq. (10.109) says that when q_k alone goes through its complete cycle, w_k increases by unity, while the other w's return to their initial values. It was only in 1961 that J. Vinti succeeded in expressing this intuitive feeling in a precise and rigorous statement.*

Suppose that the time interval T contains m complete cycles of q_k plus a fraction of a cycle. In general, the times required for each successive cycle will be different, since q_k will not be a periodic function of t. Then Vinti showed, on the basis of a theorem in number theory, that as T increases indefinitely,

$$\lim_{t\to\infty} \frac{m}{T} = v_k. \tag{10.119}$$

The *mean* frequency of the motion of q_k is therefore always given by v_k, even when the entire motion is more complicated than a periodic function with frequency v_k.

Barring commensurability of all the frequencies, a multiply periodic function can always be formed from the generating function W. The defining equation for J_i, Eq. (10.102), in effect states that when q_i goes through a complete cycle; that is, when w_i changes by unity, the characteristic function increases by J_i. It follows that the function

$$W' = W - \sum_k w_k J_k \tag{10.120}$$

remains unchanged when *each* w_k is increased by unity, all the other angle variables remaining constant. Equation (10.120) therefore represents a multiply periodic function that can be expanded in terms of the w_i (or of the frequencies v_i) by a series of the form of Eq. (10.115). Since the transformation equations for the

angle variables are

$$w_k = \frac{\partial W}{\partial J_k},$$

it will be recognized that Eq. (10.120) defines a Legendre transformation from the q, J basis to the q, w basis. Indeed, comparison with Eq. (9.15) in combination with Eq. (9.12) shows that if $W(q, J)$ is a generating function of the form $F_2(q, P)$, then $W'(q, w)$ is the corresponding generating function of the type $F_1(q, Q)$, transforming in both cases from the (q, p) variables to the (w, J) variables. While W' thus generates the same transformation as W, it is of course *not* a solution of the Hamilton Jacobi equation.

It has been emphasized that the system configuration is multiply periodic only if the frequencies v_i are not rational fractions of each other. Otherwise, the configuration repeats after a sufficiently long time and would therefore be simply periodic. The formal condition for the commensurability of two frequencies v_i and v_j is that they satisfy the relation $j_i v_i = j_j v_j$ (no sum) where j_i and j_j are nonzero positive integers. For complete commensurability, all pairs of frequencies must satisfy relations of the form

$$j_i v_i = j_k v_k, \quad \text{(no sum)} \tag{10.121}$$

where the j_i and j_k are nonzero positive integers.

When we can express any v_i as a rational fraction of any of the other frequencies, the system is said to be *completely commensurate*. If only $m + 1$ of the n frequencies satisfy Eq. (10.121), the system is said to be m-fold commensurate. For example, consider the set of seven frequencies $v_1 = 3$ MHz, $v_2 = 5$ MHz, $v_3 = 7$ MHz, $v_4 = 2\sqrt{2}$ MHz, $v_5 = 3\sqrt{2}$ MHz, $v_6 = \sqrt{3}$ MHz, $v_7 = \sqrt{7}$ MHz. The first three v_1, v_2, and v_3 are triply commensurate, the next two v_4 and v_5 are doubly commensurate.

There is an interesting connection between commensurability and the coordinates in which the Hamilton–Jacobi equation is separable. It can be shown that the path of the system point for a noncommensurate system completely fills a limited region of both configuration and phase space. This can be seen in the Lissajous figures of incommensurate frequencies.

Suppose the problem is such that the motion in any one of the separation coordinates is simply periodic and has therefore been shown to be independent of the motion of the other coordinates. Hence, the path of the system point as a whole must be limited by the surfaces of constant q_i and p_i that mark the bounds of the oscillatory motion of the separation variables. (The argument is easily extended to rotation by limiting all angles to the region 0 to 2π.) These surfaces therefore define the volume in space that is densely filled by the system point orbit. It follows that the separation of variables in noncommensurate systems must be unique: the Hamilton–Jacobi equation cannot be separated in two different coordinate systems (aside from trivial variations such as change of scale). The possibility of separating the motion in more than one set of coordinates thus normally provides evidence that the system is commensurate.

The simplest example of being commensurate is degeneracy which occurs when two or more of the frequencies are equal. If two of the force constants in a three-dimensional harmonic oscillator are equal, then the corresponding frequencies are identical and the system is singly degenerate. In an isotropic linear oscillator, the force constants are the same along all directions, all frequencies are equal, and the system is completely degenerate.

Whenever this simple degeneracy is present, the fundamental frequencies are no longer independent, and the periodic motion of the system can be described by less than the full complement of n frequencies. Indeed, the m conditions of degeneracy can be used to reduce the number of frequencies to $n - m + 1$. The reduction of the frequencies may be most elegantly performed by means of a point transformation of the action-angle variables. The m degeneracy conditions may be written where j_{ki} are positive or negative integers

$$\sum_{i=1}^{n} j_{ki} \nu_i = 0, \quad k = 1, \ldots, m. \tag{10.122}$$

Consider now a point transformation from (w, J) to (w', J') defined by the generating function (cf. Eq. (9.26) where the summation convention is used):

$$F_2 = \sum_{k=1}^{m} \sum_{i=1}^{n} J'_k j_{ki} w_i + \sum_{k=m+1}^{n} J'_k w_k. \tag{10.123}$$

The transformed coordinates are

$$\begin{aligned} w'_k &= \sum_{i=1}^{n} j_{ki}, \qquad k = 1, \ldots, m, \\ &= w_k, \qquad k = m+1, \ldots, n. \end{aligned} \tag{10.124}$$

Correspondingly, the new frequencies are

$$\begin{aligned} \nu'_k = \dot{w}'_k &= \sum_{i=1}^{n} j_{ki} \nu_i = 0 \qquad k = 1, \ldots, m, \\ &= \nu_k \qquad\qquad\qquad k = m+1, \ldots, n. \end{aligned} \tag{10.125}$$

Thus in the transformed coordinates, m of the frequencies are zero, and we are left with a set of $n - m$ independent frequencies plus the zero frequency. It is obvious that the new w'_k may also be termed as angle variables in the sense that the system configuration is multiply periodic in the w'_k coordinates with the fundamental period unity. The corresponding constant action variables are given as the solution of the n equations of transformation

$$J_i = \sum_{k=1}^{m} J'_k j_{ki} + \sum_{k=m+1}^{n} J'_k \delta_{ki}. \tag{10.126}$$

The zero frequencies correspond to constant factors in the Fourier expansion. These are of course also present in the original Fourier series in terms of the v's, Eq. (10.110), occurring whenever the indices j_i are such that degeneracy conditions are satisfied. Since

$$v_i' = \frac{\partial H}{\partial J_i'},$$

the Hamiltonian must be independent of the action variables J_i' whose corresponding frequencies vanish. In a completely degenerate system, the Hamiltonian can therefore be made to depend upon only one of the action variables.

Note that Hamilton's characteristic function W also serves as the generating function for the transformation from the (q, p) set to the (w', J') set. Since the J' quantities are n independent constants, the original constants of integration may be expressed in terms of the J' set, and W given as $W(q, J')$. In this form, it is a generating function to a new set of canonical variables for which the J' quantities are the canonical momenta. But by virtue of the point transformation generated by the F_2 of Eq. (10.123), we know that w' is conjugate to J'. Hence, it follows that the new coordinates generated by $W(q, J')$ must be the angle variable w' set, with equations of transformation given by

$$w_i' = \frac{\partial W}{\partial J_i'}. \tag{10.127}$$

(For a more formal proof of Eq. (10.127) based on the algebraic structure of Eq. (10.123), see Derivation 3.)

The problem of the bound motion of a particle in an inverse-square law central force illustrates many of the phenomena involved in degeneracy. A discussion of this problem also affords an opportunity to show how the action-angle technique is applied to specific systems, and to indicate the connections with Bohr's quantum mechanics and with celestial mechanics. Accordingly, the next section is devoted to a detailed treatment of the Kepler problem in terms of action-angle variables.

10.8 ■ THE KEPLER PROBLEM IN ACTION-ANGLE VARIABLES*

To exhibit all of the properties of the solution, we shall examine the motion in three dimensional space, rather than make use of our a priori knowledge that the orbit lies in a plane. In terms of spherical polar coordinates, the Kepler problem becomes a special case of the general treatment given above in Section 10.5 for central force motion in space. Equations (10.70) through (10.77) can be taken over here immediately, replacing $V(r)$ wherever it occurs by its specific form

$$V(r) = -\frac{k}{r}. \tag{10.128}$$

*The summation convention will be resumed from here on.

Since the potential $V(r)$ depends only upon one of the three coordinates, it follows that the Hamilton–Jacobi equation is completely separable in spherical polar coordinates. We shall confine our discussion to the bound case, that is, $E < 0$. Hence, the motion in each of the coordinates will be periodic—libration in r and θ, and rotation in ϕ. The conditions for the application of action-angle variables are thus satisfied, and we can proceed to construct the action variables on the basis of the defining equation (10.102). From Eq. (10.72), it follows that

$$J_\phi = \oint \frac{\partial W}{\partial \phi}\, d\phi = \oint \alpha_\phi\, d\phi. \tag{10.129a}$$

Similarly, on the basis of Eq. (10.74), J_θ is given by

$$J_\theta = \oint \frac{\partial W}{\partial \theta}\, d\theta = \oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}}\, d\theta. \tag{10.129b}$$

Finally the integral for J_r from Eq. (10.75), is

$$J_r = \oint \frac{\partial W}{\partial r}\, dr = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{\alpha_\theta^2}{r^2}}\, dr. \tag{10.129c}$$

The first integral is trivial; ϕ goes through 2π radians in a complete revolution and therefore

$$J_\phi = 2\pi\alpha_\phi = 2\pi p_\phi. \tag{10.130}$$

This result could have been predicted beforehand, for ϕ is a cyclic coordinate in H, and Eq. (10.130) is merely a special case of Eq. (10.101) for the action variables corresponding to cyclic coordinates. Integration of Eq. (10.129b) can be performed in various ways; a procedure involving only elementary rules of integration will be sketched here. If the polar angle of the total angular momentum vector is denoted by i, so that

$$\cos i = \frac{\alpha_\phi}{\alpha_\theta}, \tag{10.131}$$

then Eq. (10.129b) can also be written as

$$J_\theta = \alpha_\theta \oint \sqrt{1 - \cos^2 i \csc^2\theta}\, d\theta. \tag{10.132}$$

The complete circuital path of integration is for θ going from a limit $-\theta_0$ to $+\theta_0$ and back again, where $\sin\theta_0 = \cos i$, or $\theta_0 = (\pi/2) - i$. Hence, the circuital integral can be written as 4 times the integral over from 0 to θ_0, or after some manipulation,

$$J_\theta = 4\alpha_\theta \int_0^{\theta_0} \csc\theta \sqrt{\sin^2 i - \cos^2\theta}\, d\theta.$$

The substitution

$$\cos\theta = \sin i \sin\psi$$

transforms the integral to

$$J_\theta = 4\alpha_\theta \sin^2 i \int_0^{\pi/2} \frac{\cos^2\psi\, d\psi}{1 - \sin^2 i \sin^2\psi}. \tag{10.133}$$

Finally, with the substitution

$$u = \tan\psi,$$

the integral becomes

$$\begin{aligned} J_\theta &= 4\alpha_\theta \sin^2 i \int_0^\infty \frac{du}{(1+u^2)(1+u^2\cos^2 i)} \\ &= 4\alpha_\theta \int_0^\infty du \left(\frac{1}{1+u^2} - \frac{\cos^2 i}{1+u^2\cos^2 i} \right). \end{aligned} \tag{10.134}$$

This last form involves only well-known integrals, and the final result* is

$$J_\theta = 2\pi\alpha_\theta(1 - \cos i) = 2\pi(\alpha_\theta - \alpha_\phi). \tag{10.135}$$

The last integral (Eq. (10.129c)), for J_r, can now be written as

$$J_r = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{(J_\theta + J_\phi)^2}{4\pi^2 r^2}}\, dr. \tag{10.136}$$

After performing the integration, this equation can be solved for the energy $E \equiv H$ in terms of the three action variables J_ϕ, J_θ, J_r. Note that J_ϕ and J_θ can occur in E only in the combination $J_\theta + J_\phi$, and hence the corresponding frequencies ν_ϕ and ν_θ must be equal, indicating a degeneracy. This result has not involved the inverse-square law nature of the central force; *any motion produced by a central force is at least singly degenerate*. The degeneracy is of course a consequence of the fact that the motion is confined to a plane normal to the constant angular momentum vector **L**. Motion in this plane implies that θ and ϕ are related to each other such that as ϕ goes through a complete 2π period, θ varies through a complete cycle between the limits $(\pi/2) \pm i$. Hence, the frequencies in θ and ϕ are necessarily equal.

The integral involved in Eq. (10.136) can be evaluated by elementary means, but the integration is most elegantly and quickly performed using the complex

*In evaluating the integral of the second term in the final integrand of Eq. (10.134), it has been assumed that $\cos i$ is positive. This is always possible, since there is no preferred direction for the z axis in the problem and it may be chosen at will. If $\cos i$ were negative, the sign of α_ϕ in Eq. (10.135) would be positive. For changes in the subsequent formulas, see Exercise 23.

contour integration method of residues. For the benefit of those familiar with this technique, we shall outline the steps involved in integrating Eq. (10.136). Bound motion can occur only when E is negative (cf. Section 3.3), and since the integrand is equal to $p_r = m\dot{r}$, the limits of the motion are defined by the roots r_1 and r_2 of the expression in the square root sign. If r_1 is the inner bound, as in Fig. 3.6, a complete cycle of r involves going from r_1 to r_2 and then back again to r_1. On the outward half of the journey, from r_1 to r_2, p_r is positive and we must take the positive square root. However, on the return trip to r_1, p_r is negative and the square root must likewise be negative. The integration thus involves both branches of a double-valued function, with r_1 and r_2 as the branch points. Consequently, the complex plane can be represented as one of the sheets of a Riemann surface, slit along the real axis from r_1 to r_2 (as indicated in Fig. 10.5).

Since the path of integration encloses the line between the branch points r_1 and r_2, the method of residues cannot be applied directly. However, we may also consider the path as enclosing all the rest of the complex plane, the direction of integration now being in the reverse (clockwise) direction. The integrand is single-valued in this region, and there is now no bar to the application of the method of residues. Only two singular points are present, namely, the origin and infinity, and the integration path can be distorted into two clockwise circles enclosing these two points. Now, the sign in front of the square root in the integrand must be negative for the region along the real axis below r_1, as can be seen by examining the behavior of the function in the neighborhood of r_1. If the integrand is represented as

$$-\sqrt{A + \frac{2B}{r} - \frac{C}{r^2}},$$

the residue at the origin is

$$R_0 = -\sqrt{-C}.$$

Above r_2, the sign of the square root on the real axis is found to be positive, and the residue is obtained by the standard technique of changing the variable of integration to $z = r^{-1}$:

$$-\oint \frac{1}{z^2}\sqrt{A + 2Bz - Cz^2}\, dz. \tag{10.137}$$

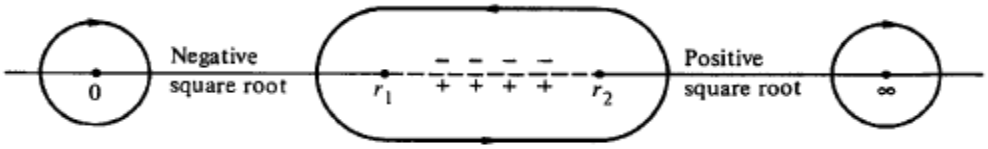


FIGURE 10.5 The complex r plane in the neighborhood of the real axis; showing the paths of integration occurring in the evaluation of J.

Expansion about $z = 0$ now furnishes the residue

$$R_\infty = -\frac{B}{\sqrt{A}}.$$

The total integral is $-2\pi i$ times the sum of the residues:

$$J_r = 2\pi i\left(\sqrt{-C} + \frac{B}{\sqrt{A}}\right), \tag{10.138}$$

or, upon substituting the coefficients A, B, and C:

$$J_r = -(J_\theta + J_\phi) + \pi k\sqrt{\frac{2m}{-E}}. \tag{10.139}$$

Equation (10.139) supplies the functional dependence of H upon the action variables; for solving for E, we have

$$H \equiv E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^2}. \tag{10.140}$$

Note that, as predicted, J_θ and J_ϕ occur only in the combination $J_\theta + J_\phi$. More than that, all three of the action variables appear only in the form $J_r + J_\theta + J_\phi$. Hence, all of the frequencies are equal; *the motion is completely degenerate*. This result could also have been predicted beforehand, for we know that with an inverse-square law of force the orbit is closed for negative energies. With a closed orbit, the motion is simply periodic and therefore, in this case, completely degenerate. If the central force contained an r^{-3} term, such as is provided by first-order relativistic corrections, then the orbit is no longer closed but is in the form of a precessing ellipse. One of the degeneracies will be removed in this case, but the motion is still singly degenerate, since $\nu_\theta = \nu_\phi$ for all central forces. The one frequency for the motion here is given by

$$\nu = \frac{\partial H}{\partial J_r} = \frac{\partial H}{\partial J_\theta} = \frac{\partial H}{\partial J_\phi} = \frac{4\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^3}. \tag{10.141}$$

If we evaluate the sum of the J's in terms of the energy from Eq. (10.140) the period of the orbit is

$$\tau = \pi k\sqrt{\frac{m}{-2E^3}}. \tag{10.142}$$

This formula for the period agrees with Kepler's third law, Eq. (3.71), if it is remembered that the semimajor axis a is equal to $-k/2E$.

The degenerate frequencies may be eliminated by canonical transformation to a new set of action-angle variables, following the procedure outlined in the previous section. Expressing the degeneracy conditions as

$$\nu_\phi - \nu_\theta = 0, \qquad \nu_\theta - \nu_r = 0,$$

the appropriate generating function is

$$F = (w_\phi - w_\theta)J_1 + (w_\theta - w_r)J_2 + w_r J_3. \tag{10.143}$$

The new angle variables are

$$\begin{aligned} w_1 &= w_\phi - w_\theta \\ w_2 &= w_\theta - w_r, \\ w_3 &= w_r, \end{aligned} \tag{10.144}$$

and, as planned, two of the new frequencies, v_1 and v_2, are zero. We can obtain the new action variables from the transformation equations

$$\begin{aligned} J_\phi &= J_1, \\ J_\theta &= J_2 - J_1, \\ J_r &= J_3 - J_2, \end{aligned}$$

which yields the relations

$$\begin{aligned} J_1 &= J_\phi, \\ J_2 &= J_\phi + J_\theta, \\ J_3 &= J_\phi + J_\theta + J_r. \end{aligned} \tag{10.145}$$

In terms of these transformed variables the Hamiltonian appears as

$$H = -\frac{2\pi^2 mk^2}{J_3^2}, \tag{10.146}$$

a form involving only that action variable for which the corresponding frequency is different from zero.

If we are willing to use, from the start, our a priori knowledge that the motion for the bound Kepler problem is a particular closed orbit in a plane, then the integrals for J_θ and J_r can be evaluated very quickly and simply. For the J_θ integral, we can apply the following procedure. It will be recalled that when the defining equations for the generalized coordinates do not involve time explicitly, then (cf. Eq. (8.20) and the material following (8.20))

$$p_i \dot{q}_i = 2L_2 \dot{q}_i \dot{q}_i = 2T.$$

Knowing that the motion is confined to a plane, we can express the kinetic energy T either in spherical polar coordinates or in the plane polar coordinates (r, ψ). It follows, then, that

$$2T = p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} = p_r \dot{r} + p\dot{\psi}, \tag{10.147}$$

where p ($\equiv l$) is the magnitude of the total angular momentum. Hence, the definition for J_θ can also be written as

$$J_\theta \equiv \oint p_\theta \, d\theta = \oint p \, d\psi - \oint p_\phi \, d\phi. \tag{10.148}$$

Because the frequencies for θ and ϕ are equal, both ϕ and ψ vary by 2π as θ goes through a complete cycle of libration, and the integrals defining J_θ reduce to

$$J_\theta = 2\pi(p - p_\phi) = 2\pi(\alpha_\theta - \alpha_\phi),$$

in agreement with Eq. (10.135).

The integral for J_r, Eq. (10.136), was evaluated in order to obtain $H \equiv E$ in terms of the three action variables. If we use the fact that the closed elliptical orbit in the bound Kepler problem is such that the frequency for r is the same as that for θ and ϕ, then the functional dependence of H on J can also be obtained from Eq. (10.147). In effect then we are evaluating J_r in a different way. The virial theorem for the bound orbits in the Kepler problem says that (cf. Eq. (3.30))

$$\overline{V} = -2\overline{T},$$

where the bar denotes an average over a single complete period of the motion. It follows that

$$H \equiv E = \overline{T} + \overline{V} = -\overline{T}. \tag{10.149}$$

Integrating Eq. (10.147) with respect to time over a complete period of motion we have

$$\frac{2\overline{T}}{\upsilon_3} = J_r + J_\theta + J_\phi = J_3, \tag{10.150}$$

where υ_3 is the frequency of the motion, that is, the reciprocal of the period. Combining Eqs. (10.149) and (10.150) leads to the relation

$$-\frac{2}{J_3} = \frac{\upsilon_3}{H} = \frac{1}{H}\frac{dH}{dJ_3}, \tag{10.151}$$

where use has been made of Eq. (10.105). Equation (10.151) is in effect a differential equation for the functional behavior of H on J_3. Integration of the equation immediately leads to the solution

$$H = \frac{D}{J_3^2}, \tag{10.152}$$

where D is a constant that cannot involve any of the J's, and must therefore depend only upon m and k. Hence, we can evaluate D by considering the elementary case of a circular orbit, of radius r_0, for which $J_r = 0$ and $J_3 = 2\pi p$. The total energy is here

$$H = -\frac{k}{2r_0} \tag{10.153}$$

(as can most immediately been seen from the virial theorem). Further, the condition for circularity, Eq. (3.40), can be written for the inverse-square force law as

$$\frac{k}{r_0^2} = \frac{p^2}{mr_0^3} = \frac{J_3}{4\pi^2 mr_0^3}. \tag{10.154}$$

Eliminating r_0 between Eqs. (10.153) and (10.154) leads to

$$H = -\frac{2\pi^2 mk^2}{J_3^2}. \tag{10.155}$$

This result has been derived only for circular orbits. But Eq. (10.152) says it must also be correct for *all* bound orbits of the Kepler problem, and indeed it is identical with Eq. (10.146). Thus, if the existence of a single period for all coordinates is taken as known beforehand, it is possible to obtain $H(J)$ without direct evaluation of the circuital integrals.

In any problem with three degrees of freedom, there must of course be six constants of motion. It has previously been pointed out that in the Kepler problem five of these are algebraic functions of the coordinates and momenta and describe the nature of the orbit in space, and only the last refers to the position of the particle in the orbit at a given time (cf. Sections 3.7 to 3.9). It is easy to see that five parameters are needed to completely specify, say, the elliptical orbit of the bound Kepler problem in space. Since the motion is in a plane, two constants are needed to describe the orientation of that plane in space. One constant is required to give the *scale* of the ellipse, for example, the semimajor axis a, and the other the *shape* of the ellipse, say, through the eccentricity e. Finally, the fifth parameter must specify the *orientation* of the ellipse relative to some arbitrary direction in the orbital plane.

The classical astronomical *elements* of the orbit provide the orbital parameters almost directly in the form given above. Two of the angles appearing in these elements have unfamiliar but time-honored names. Their definitions, and functions as orbital parameters, can best be seen from a diagram, such as is given in Fig. 10.6. Here xyz defines the chosen set of axes fixed in space, and the unit vector $\mathbf{n}$ characterizes the normal to the orbital plane. The intersection between the xy plane and the orbital plane is called the *line of nodes*. There are two points on the line of nodes at which the elliptical orbit intersects the xy plane; the point at which the particle enters from below into the upper hemisphere (or goes from the "southern" to the "northern" hemispheres) is known as the *ascending node*. In Fig. 10.6, the portion of the orbit in the southern hemisphere is shown, for clarity, as a dashed line. The dot-dashed line ON is a portion of the line of nodes containing the ascending node. We can measure the direction of ON in the xy plane by the angle xON, which is customarily denoted by Ω, and is known as the *longitude*

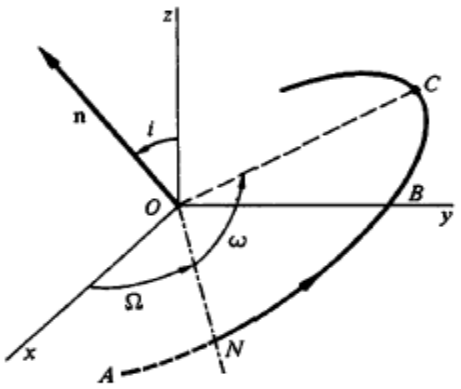


FIGURE 10.6 Angular elements of the orbit in the bound Kepler problem.

of the ascending node. Finally, if C denotes the point of periapsis in the orbit, then the angle NOC in the orbital plane is denoted by ω and is called the *argument of the perihelion*.* The more familiar angle i, introduced in Eq. (10.131), is in its astronomical usage known as the *inclination of the orbit*. One usual set of astronomical elements therefore consists of the six constants

$$i, \Omega, a, e, \omega, T,$$

where the last one, T, is the time of passage through the periapsis point. Of the remaining five, the first two define the orientation of the orbital plane in space, while a, e, and ω directly specify the scale, shape, and orientation of the elliptic orbit, respectively.

The action-angle variable treatment of the Kepler problem also leads to five algebraic constants of the motion. Three of them are obvious as the three constant action variables, J_1, J_2, and J_3. The remaining two are the angle variables w_1 and w_2, which are constants, because their corresponding frequencies are zero. It must therefore be possible to express the five constants J_1, J_2, J_3, w_1, and w_2 in terms of the classical orbital elements i, Ω, a, e, and ω, and vice versa. Some of these interrelations are immediately obvious. From Eqs. (10.145) and (10.135) it follows that

$$J_2 = 2\pi\alpha_\theta \equiv 2\pi l, \tag{10.156}$$

and hence, by Eq. (10.131),

$$\frac{J_1}{J_2} = \cos i. \tag{10.157}$$

As is well known, the semimajor axis a is a function only of the total energy E (cf. Eq. (3.61)) and therefore, by Eq. (10.146), a is given directly in terms of J_3:

*This terminology appears to be commonly used even for orbits that are not around the sun. The proper term for orbits about stars is periastra; for Earth-orbiting satellites, this term is perigee.

$$a = -\frac{k}{2E} = \frac{J_3^2}{4\pi^2 mk}. \tag{10.158}$$

In terms of J_2, Eq. (3.62) for the eccentricities can be written as

$$e = \sqrt{1 - \frac{J_2^2}{4\pi^2 mka}},$$

or

$$e = \sqrt{1 - \left(\frac{J_2}{J_3}\right)^2}. \tag{10.159}$$

It remains only to relate the angle variables w_1 and w_2 to the classic orbital elements. Obviously, they must involve Ω and ω. In fact, it can be shown that for suitable choice of additive constants of integration they are indeed proportional to Ω and ω, respectively. This will be demonstrated for w_1; the identification of w_2 will be left as an exercise.

The equation of transformation defining w_1 is, by Eq. (10.127),

$$w_1 = \frac{\partial W}{\partial J_1}.$$

It can be seen from the separated form of W, Eq. (10.71), that W can be written as the sum of indefinite integrals:

$$W = \int p_\phi \, d\phi + \int p_\theta \, d\theta + \int p_r \, dr. \tag{10.160}$$

As we have seen from the discussion on J_r, the radial momentum p_r does not involve J_1, but only J_3 (through E) and the combination $J_\theta + J_\phi = J_2$. Only the first two integrals are therefore involved in the derivative with respect to J_1. By Eq. (10.130),

$$p_\phi = \alpha_\phi = \frac{J_1}{2\pi}, \tag{10.161}$$

and by Eq. (10.74), with the help of Eqs. (10.156) and (10.161),

$$p_\theta = \pm\sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}} = \pm\frac{1}{2\pi}\sqrt{J_2^2 - \frac{J_1^2}{\sin^2\theta}}. \tag{10.162}$$

It turns out that in order to relate w_1 to the ascending node, it is necessary to choose the negative sign of the square root.* The angular variable w_1 is therefore determined by

*Note that when the particle passes through the ascending node (cf. Fig. 10.6) θ is *decreasing* and the corresponding momentum is negative. In calculating J_θ, it was not necessary to worry about the choice of sign because in going through a complete cycle both signs are encountered.

$$w_1 = \frac{\phi}{2\pi} + \frac{J_1}{2\pi}\int \frac{d\theta}{\sin^2\theta\sqrt{J_2^2 - J_1^2\csc^2\theta}},$$

or

$$2\pi w_1 = \phi + \cos i \int \frac{d\theta}{\sin^2\theta\sqrt{1-\cos^2 i\csc^2\theta}}$$
$$= \phi + \int \frac{\cot i \csc^2\theta\, d\theta}{\sqrt{1-\cot^2 i\cot^2\theta}}.$$

By a change of variable to u, defined through

$$\sin u = \cot i \cot\theta, \tag{10.163}$$

the integration can be performed trivially, and the expression for w_1 reduces to

$$2\pi w_1 = \phi - u. \tag{10.164}$$

The angle coordinate ϕ is the azimuthal angle of the projection on the xy plane measured relative to the x axis. Clearly, from Eq. (10.163) u is a function of the polar angle θ of the particle. But what is its geometrical significance? We can see what u is by reference to Napier's rules* as applied to the spherical triangle defined by the line of nodes, the radius vector, and the projection of the radius vector on the xy plane. However, it may be more satisfying to indulge in a little trigonometric manipulation and derive the relation *ab initio*. In Fig. 10.7, the line ON is the line of nodes, OR is the line of the radius vector at some time, and the dotted line OP is the projection of the radius vector on the xy plane. The angle that OP makes with the x axis is the azimuth angle ϕ. We contend that u is the angle OP makes with the line of nodes. To prove this, imagine a plane normal both to the xy plane and to the line of nodes, which intersects the radius vector at unit distance OB from the origin O. The points of intersection A, B, and C of this plane, with the three lines from the origin, define with the origin four right triangles. Since OB has unit length, it follows that $BC = \cos\theta$ and therefore $AC = \cos\theta\cot i$. On the other hand, $OC = \sin\theta$ and therefore it is also true that $AC = \sin\theta\sin u$. Hence, $\sin u = \cot i\cot\theta$, which is identical with Eq. (10.163) and proves the stipulated identification of the angle u. Figure 10.7 shows clearly that the difference between ϕ and u must be Ω, so that

$$2\pi w_1 = \Omega. \tag{10.165}$$

In a similar fashion, we can identify the physical nature of the constant w_2. Of the integrals making up W, Eq. (10.160), the two over θ and r contain J_2 and

*For an explanation of Napier's rules for spherical triangles, see handbooks such as the *Handbook of Mathematical Tables* (Chemical Rubber Publishing Co.) or *Handbook of Applied Mathematics* (Van Nostrand-Reinhold).

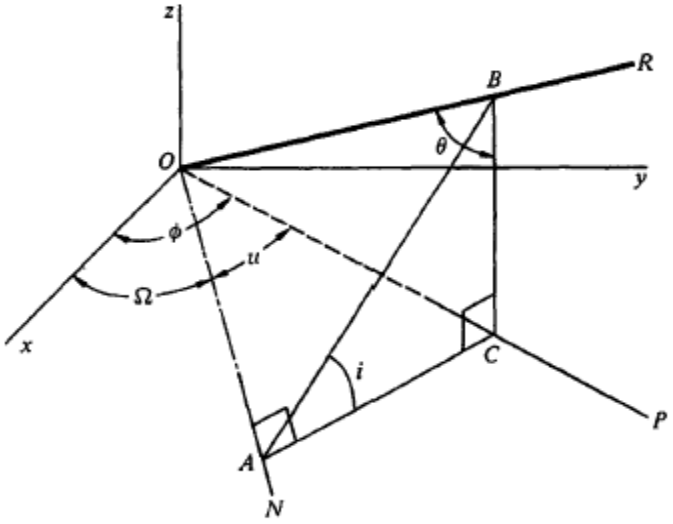


FIGURE 10.7 Diagram illustrating angles appearing in action-angle treatment of the Kepler problem.

are therefore involved in finding w_2. After differentiation with respect to J_2, the integral over θ can be performed by the same type of trigonometric substitution as employed for w_1. The corrcsponding integral over r can be carried out in a number of ways, most directly by using the orbit equation for r in terms of the polar coordinate angle in the orbital plane. By suitable choice of the arbitrary lower limit of integration, it can thus be found that $2\pi w_2$ is the difference between two angles in the orbital plane, one of which is the angle of the radius vector relative to the line of nodes and the other is the same angle but relative to the line of the periapsis. In other words, $2\pi w_2$ is the argument of the perihelion:

$$2\pi w_2 = \omega. \tag{10.166}$$

Detailed derivation is left to one of the exercises.

The method of action-angle variables is certainly not the quickest way to solve the Kepler problem, and the practical usefulness of the set of variables is not obvious. However, their value has long been demonstrated in celestial mechanics, where they appear under the guise of the *Delaunay variables*.* As will be seen in Section 12.2, they provide the natural orbital elements that can be used in perturbation theory, to describe the modifications of the nominal Kepler orbits produced by small deviations of the force from the inverse-square law. Many of the basic studies on possible perturbations of satellite orbits were carried out in terms of the action-angle variables.

*As customarily defined, the Delaunay variables differ from the (J_i, w_i) set by multiplicative constants.

POSSIBLE QUESTIONS

Part B (6 Marks)

1. Derive Hamilton Jacobi equation from Hamilton's characteristic function
2. Explain the physical significance of Hamilton Jacobi equation
3. Derive Jacobi's theorem
4. Derive kepler's problem solution by Hamilton Jacobi method
5. Derive the separation of variables in the Hamilton Jacobi equation

Part C (10 Marks)

1. Derive Harmonic oscillator problem by Hamilton Jacobi method
2. Derive Hamilton Jacobi equation from Hamilton's characteristic function
3. Derive the Lagrange's equation from Hamilton's principle for holonomic system

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Subject: Mechanics **Subject Code:19MMP106**

Class : I - M.Sc. Mathematics **Semester : I**

Unit V

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
A _____ is a position of matter occupying finite space.	Impact	Body	Momentum	Elastic body	Body
A body is said to be _____ if it remains in its original position after the removal of deforming forces.	Direct Impact	Oblique impact	impact	Elastic	Elastic
The common normal at the point of contact is called _____.	Direct Impact	Oblique impact	line of impact	Elastic	line of impact
In case of two spheres the _____ is the line joining their centre.	Direct Impact	Oblique impact	line of impact	Impulsive	line of impact
Two bodies is said to _____ if the motion after impact of both the bodies is along the common normal.	Direct Impact	Oblique impact	line of impact	Impulsive	Direct Impact
Two bodies is said to _____ if the motion after impact of atleast one of the body is not along the common normal.	Direct Impact	Oblique impact	Impulsive	impact	Oblique impact
A force of large magnitude acting for a very short interval time is called an _____.	central force	Colinear force	Impulsive force	Circular force	Impulsive force

_____ is always measured by its impulse.	Impulsive force	Colinear force	Central force	Circular force	Impulsive force
Forward momentum of the shot = _____ of the gun.	Forward momentum	Backward momentum	equal momentum	Both a and b	Backward momentum
The principle of conservation of linear momentum applies along the _____.	Direct Impact	Oblique impact	line of impact	Impulsive	line of impact
Two bodies when they strike against each other are said to _____.	Elastic	Implinge	line of impact	Pressure	Implinge
The impact of elastic bodies, the bodies are _____.	smooth	rough	perfectly smooth	perfectly rough	smooth
Elastic spheres and the mutual action between them is along the _____.	line of action	line of force	line of centres	line of impact	line of centres
Two bodies are said to implinge directly on each other then the impact is called _____.	Direct Impact	Oblique impact	line of impact	Impulsive	Direct Impact
Two bodies are said to implinge obliquely on each other then the impact is called _____.	Direct Impact	Oblique impact	line of impact	Impulsive	Oblique impact
In _____ the direction of motion is along the common normal at the point of contact.	Direct Impact	Oblique impact	line of impact	Impulsive	Direct Impact
In _____ the direction of motion is opposite to the direction of the impact.	Direct Impact	Oblique impact	Elastic body	Impact	Oblique impact
In smooth spheres the horizontal components of the velocity beffore and after impact are _____.	same	opposite	perpendicular	parallel	same
In a triangle, $\alpha + \beta + \Upsilon = 90$ then $\tan\alpha \tan\beta + \tan\beta \tan\Upsilon + \tan\Upsilon \tan\alpha =$ _____.	1	π	$\pi/2$	0	0
In a triangle, $\alpha + \beta + \Upsilon = 90$ then $\tan\alpha / \tan\beta$ is _____.	1	0	e	∞	e
If e = 1 then the sphere is _____.	smooth	perfectly smoth	Elastic	perfectly elastic	perfectly elastic

If e=0 then the sphere is in _____.	smooth	perfectly smoth	Elastic	perfectly elastic	Elastic
The formula for velocity of rebound is _____.	e x velocity of impinge	e/ velocity of impinge	e + velocity of impinge	e - velocity of impinge	e x velocity of impinge
The formula for height accended is _____.	vertical velocity x 2g	vertical velocity/ 2g	Vertical velocity + 2g	vertical veloocity - 2g	vertical velocity/ 2g
_____ of a shot = Backward mmentum of a gun.	Forward momentum	Backward momentum	equal momentum	Both a and b	Forward momentum
A body is a position of _____ occupying finite space.	weight	mass	matter	velocity	matter
Two bodies are said to implinge directly if the motion after impact of both the bodies is _____ the common normal.	along	not along	parallel to	Perpendicular	along
Two bodies are said to implinge obliquely if the motion after impact of both the bodies is _____ the common normal.	along	not along	parallel to	Perpendicular	not along
The impulse of F is given by Ft = _____.	mv x mu	mv/mu	mv+ mu	mv - mu	mv - mu
The impulse of a force is measured by the _____ change.	Force	velocity	momentum	weight	momentum
Any momentum generated to the gun in the perpendicular direction to that of recoil is neutralised by the _____.	Impulsive force	Impulsive pressure	impulsive momentum	impulse	Impulsive pressure
Forward momentum of the _____ = Backward momentum of the gun.	shot	pressure	gun	force	shot
Forward momentum of the shot = Backward momentum of the _____.	shot	pressure	gun	force	gun
The momentum destroyed per unit time is called the mean _____ on the surfaces.	Impulse	Impulsive pressure	imjpulsive force	impulsive momentum	Impulsive pressure
If the two spheres are equal and perfectly elastic then they interchange their _____ after impact.	force	pressure	velocity	momentum	velocity

There is a loss of kinetic energy due to _____.	Direct Impact	Oblique impact	Elastic body	Impact	Direct Impact
In oblique impact e= _____ for inelastic bodies.	0	1	2	∞	0
e is called the _____	coefficient of friction	coefficient of restitution	eccentricity	latus rectum	coefficient of restitution
In oblique impact e= _____ for perfectly elastic bodies.	0	1	2	∞	1
_____ is called the coefficient f restitution.	x	a	e	u	e
In oblique impact e= 0 for _____ bodies.	elastic	perfectly elastic	smooth	inelastic	inelastic
In oblique impact e =1 for _____bodies.	elastic	perfectly elastic	smooth	inelastic	perfectly elastic
There is a loss of _____ due to direct impact.	kinetic energy	Impulse	pressure	force	kinetic energy