KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

		Semester – III
18MMP301	**FUNCTIONAL ANALYSIS**	**L T P C**
		4 0 0 4

Scope: This course provides a systematic study of linear, topological or metric structures and it also deals with spaces and operators acting on them.

Objectives: To be thorough with Banach spaces, related theorems, orthonormal sets, normal and unitary operators and to be familiar with Banach algebras.

UNIT I

Banach Spaces- Normed linear space – Definitions and Examples-Theorems. Continuous Linear Transformations – Some theorems- Problems. The Hahn- Banach Theorem –Lemma and Theorems. The Natural imbedding of N in N**-Definitions and Theorems.

UNIT II

The Open Mapping Theorem- Theorem and Examples –Problems. The closed graph theorem. The conjugate of an operation- The uniform boundedness theorem- Problems.

UNIT III

Hilbert Spaces- The Definition and Some Simple Properties – Examples and Problems. Orthogonal Complements - Some theorems .Ortho-normal sets – Definitions and Examples- Bessel's inequality- The conjugate space H*.

UNIT IV

The Adjoint of an operator – Definitions and Some Properties-Problems. Self- adjoint operators – Some Theorems and Problems. Normal and Unitary operators –theorems and problems. Projections - Theorems and Problems.

UNIT V

Banach algebras: The definition and some examples of Banach algebra – Regular and singular elements – Topological divisors of zero – The spectrum – The formula for the spectral radius.

SUGGESTED READINGS

TEXT BOOK

1. Simmons. G. F., (2004). Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd, New Delhi.

REFERENCES

1. Balmohan V. and Limaye.,(2004). Functional Analysis, New Age International Pvt.Ltd, Chennai.

2. Chandrasekhara Rao, K., (2006). Functional Analysis, Narosa Publishing House, Chennai.
3. Choudhary, .B and Sundarsan Nanda. (2003). Functional Analysis with Applications, New Age International Pvt. Ltd, Chennai.
4. Ponnusamy, S., (2002). Foundations of functional analysis, Narosa Publishing House, Chennai.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: Dr. M. Santhi
SUBJECT NAME:FUNCTIONAL ANALYSIS SUB.CODE:18MMP301
SEMESTER: III CLASS: II M.SC MATHEMATICS

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1	1	Banach Spaces – Definition & Examples	S1. Chapter 9 : 211-212 S3:Chapter 3: 46
2	1	Basic definitions -Normed linear spaces	S1. Chapter 9 : 213
3	1	Definitions and Theorems on Normed linear Spaceswith examples	S1. Chapter 9 : 214-216
4	1	Continuous Linear Transformations – Definition &Theorem	S1. Chapter 9 : 219-220
5	1	Conjugate Space of N&Hahn- Banach lemma	S1. Chapter 9 : 224-226
6	1	Continuous of Hahn- Banach lemma	S1. Chapter 9 : 227-229 S2:Chapter 2:44-46
7	1	The Natural imbedding of N in N**	S1. Chapter 9 : 231 - 233
8	1	Properties of N**	S1. Chapter 9 : 231 - 233
9	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit I = 9		
		UNIT-II	
1	1	The Open Mapping –theorem	S1. Chapter 9 :235-236 S2: Chapter 2 : 52-54
2	1	Theorem – continuation of open mapping	S1. Chapter 9 : 236-237
3	1	Projection of an operator	S1. Chapter 9 : 237

4	1	Theorems on Projection	S1. Chapter 9 : 237-238
5	1	Definition of closed graph & Lemma	S1. Chapter 9 : 238
6	1	The Closed Graph Theorem	S1. Chapter 9 : 238-239
7	1	Uniform Boundedness and Conjugate of an operator	S1. Chapter 9 : 239-240 S4.Chapter 5:305-306
8	1	Theorem based on Uniform boundedness Property	S1. Chapter 9 : 240-242
9	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit II=9		
		UNIT-III	
1	1	Hilbert Spaces- Definitions, Examples and Properties.	S1. Chapter 10 : 244-246
2	1	Schwarz inequality & some problems	S1. Chapter 10 : 246-247
3	1	Theorem based on Inner product spaces	S1. Chapter 10 : 247-248
4	1	Orthogonal Complements – Theorems	S1. Chapter 10 : 249-250 S2: Chapter 3: 96-97
5	1	Continuous on Orthogonal complements and Orthonormal set.	S1. Chapter 10 : 251-252 S2: Chapter 3: 97-98
6	1	Theorems on Orthonormal set	S1. Chapter 10 : 252-253 S3. Chapter 5 :113-116
7	1	Theorem- Bessel's Inequality	S1. Chapter 10 : 253-256 S2: Chapter 3: 99-100
8	1	The conjugate space H* , Theorem on H*	S1. Chapter 10 : 260-261
9	1	Riesz Representation Theorem	S1. Chapter 10 : 261-262
10	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit III=10		
		UNIT-IV	
1	1	Introduction of Adjoint Operators	S1. Chapter 10 : 262-263
2	1	Adjoint operator – Basic Definition and Problems	S1. Chapter 10 : 263-265
3	1	Adjoint operator – Theorem	S1. Chapter 10 : 265

			T1.Chapter 7: 460-461
4	1	Some properties and problems on adjoint operator	S1. Chapter 10 : 265-266
5	1	Theorem on Self Adjoint operator	S1. Chapter 10 : 266-269
6	1	Theorem -Normal operator	S1. Chapter 10 : 269-271
7	1	Theorem -Unitary operator	S1. Chapter 10 : 271-273
8	1	Theorems on projection	S1. Chapter 10 : 273- 274 T1: Chapter 6 : 420
9	1	Continuous of theorems on projection	S1. Chapter 10 : 275-276 T1. Chapter 6 : 421
10	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit IV=10		
		UNIT-V	
1	1	Basics on Finite Dimensional Spectral theory	S1. Chapter 11 : 278-280
2	1	Examples of Banach Algebra	S1. Chapter 11 : 302-305
3	1	Regular & Singular Element –Theorem	S1. Chapter 11 : 305-306
4	1	Theorem on -continuation of regular &singular element	S1. Chapter 11 : 306-307
5	1	Topological divisors of zero	S1. Chapter 11 : 307-308 S2. Chapter 4: 143
6	1	The Spectrum –Definition, theorems on spectrum , Formula for the spectral radius	S1. Chapter 11 : 308-313
7	1	Recapitulation and discussion of possible questions	
8	1	Previous year question paper discussion	
9	1	Previous year question paper discussion	
10	1	Previous year question paper discussion	
	Total No of Hours Planned for unit V=10		
Total Planned Hours	**48**		

TEXT BOOKS:

T1: Balmohan V., and Limaye., 2004. Functional Analysis, New Age International Pvt.Ltd, Chennai.

REFERENCES:

S1: Simmons. G.F., 1963. Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd,New Delhi.

S2: Chandrasekhara Rao.K., 2006. Functional Analysis, Narosa Publishing House,Chennai.

S3: Choudhary .B, and Sundarsan Nanda., 2003. Functional Analysis with Applications,

New Age International Pvt. Ltd, Chennai.

S4: Ponnusamy.S., 2002. Foundations of functional analysis, Narosa Publishing House, Chennai.

FUNCTIONAL ANALYSIS

UNIT 1

Banach Spaces

Metric Spaces:

A metric d on a nonempty set X is a function
d : X x X → R such that for all x, y, z ε X.

i. d (x,y) $\geq$ 0 and d (x,y) = 0 iff x = y
ii. d(y, x) = d(x,y)
iii. d(x,y) $\leq$ d(x, z) + d(z, y)

A metric space is a non empty set X along with a metric on it

Normed Linear space:

A (real) complex normed space is a (real) complex vector space X together with a map : $X \to \mathrm{R}$, called the norm and denoted ||x|| such that

(i) $||x|| \geq 0$, for all $x \in X$, and $||x|| = 0$ if and only if $x = 0$.
(ii) $||\alpha(x)|| = |\alpha|\,||x||$, for all $x \in X$ and all $\alpha \in$ C (or R).
(iii) $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in X$.

Remark:

If in (i) we only require that $x \geq 0$, for all $x \in X$, then ||.|| is called a seminorm.

Remark :

If X is a normed space with norm ||.||, it is readily checked that the formula $d(x, y) = ||x - y||$, for $x, y \in X$, defines a metric d on X. Thus a normed space is naturally a metric space and all metric space concepts are meaningful. For example, convergence of sequences in X means convergence with respect to the above metric.

Definition 1.4. A complete normed space is called a Banach space.

Thus, a normed space X is a Banach space if every Cauchy sequence in X converges (where X is given the metric space structure as outlined above). One may consider real or complex Banach spaces depending, of course, on whether X is a real or complex linear space.

Problem:

Show that in a normed linear space N $| \|x\| - \|y\| | \leq \|x - y\|$

Solution:

It is enough to prove that $\|x\| - \|y\| \leq \|x - y\|$
as $\|y\| - \|x\| = -(\|x\| - \|y\|)$……………..(1)

so that ,
$\|y\| - \|x\| \leq \|y - x\|$

$= \|(-1)(x - y)\| = \|x - y\|$

Then, $-(\|x\| - \|y\|) \leq \|x - y\|$……………..(2)

Also, $\|x\| = \|x-y+y\|$,

$\leq \|x-y\| + \|y\|$

$\|x\| - \|y\| \leq \|x - y\|$, $x,y \in N$……………(3)

From (2) & (3)

Thus $| \|x\| - \|y\| | \leq \|x - y\|$

Hence shown.

Problem:

Show that norm is a continuous function i.e., $x_n \to x \Rightarrow$
$\|x_n\| \to \|x\|$.

Solution:

Suppose $x_n \to x \Rightarrow d(x_n, x) \to 0$ as $n \to \infty$ where d is the Metric In the normed linear space.

We have $\Rightarrow \|x_n - x\| \to 0$ as $n \to \infty$.

From the previous problem we have $| \|x_n\| - \|x\| | \leq \|x_n - x\|$

$\Rightarrow | \|x_n - x\| | \to 0$ as $n \to \infty$.

$\Rightarrow \|x_n\| \to \|x\|$.

Hence shown.

Theorem:

Let M be a closed linear subspace of a normed linear space N. If the norm of a coset x+M is the quotient space then N/M is defined by $\|x + M\| = \inf\{ \|x + M\| / m \in M\}$

Then N/M is a normed linear space. Also if N is a banach space , then N/M is also a banach space.

Proof:

To prove N/M is a normed linear space under the norm $\|x + M\|$.

To verify norm properties.

i) $\|x + M\| \geq 0$ as $\|x + M\| \geq 0$, for $m \in M$ now $\|x + M\| = 0$.

ii) $\|(x + M) + (y + M)\|$

$$= \|x + y + M\|$$

$$= \inf\{\|x + y + M\| / m \in M\}$$

$$= \inf\{\|(x + m_1) + (y + m_2)\| / m_1, m_2 \in M\}$$

$$\leq \inf\{\|(x + m_1)\| + \|(y + m_2)\| / m_1, m_2 \in M\}$$

$$= \inf\{\|(x + m_1)\| / m_1 \in M\} + \inf\{\|(y + m_2)\| / m_2 \in M\}$$

$$= \|x + M\| + \|y + M\|$$

iii) similarly we can prove

$$\|\alpha(x + M)\| = |\alpha|\,\|x + M\|.$$

Hence the quotient N/M is a normed linear space.

It reminds to prove that N/M is a banach space whenever N is a banach space.

Starting with the Cauchy sequence in N/M it is enough to show that this sequence has a convergent subsequence.

This will prove that the Cauchy sequence itself is convergent in N/M and hence N/M will be complete and also banach.

We can find a subsequence $\{x_n + M\}$ of the original Cauchy sequence such that $\|x_1 + M\| + \|x_2 + M\| < 1/2$,

$\|x_2 + M\| + \|x_3 + M\| < 1/2^2$ and so on .

In general we have $\|x_n + M\| + \|x_{n+1} + M\| < 1/2^n$.

We prove that the sequence $\{x_n + M\}$ is convergent in N/M. Choose a vector $y_1 \in x_1 + M$, $y_2 \in x_2 + M$, so that

$\|y_1 - y_2\| < ½$.

Having chosen in the same way $y_3 \in x_3 + M$, so that
$\|y_2 - y_3\| < 1/2^2$ and so on.

Thus we obtain a sequence $\{y_n\}$ in N , so that

$\|y_n - y_{n+1}\| < 1/2^n$.

Let m<n, consider $\|y_m - y_n\|$

$$= \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \ldots\ldots\ldots + (y_{n-1} - y_n)\|$$
$$\leq \|(y_m - y_{m+1})\| + \|(y_{m+1} - y_{m+2})\| + \ldots\ldots\ldots + \|(y_{n-1} - y_n)\|$$
$$= 1/2^{m-1}.$$

i.e., $\|y_m - y_n\| < 1/2^{m-1}$. thus y_n is a Cauchy sequence in N.

But N in a banach space is complete,
$\Rightarrow \{y_n\}$ is convergent to a vector y in N .

but , $\|(x_n + M) - (y_n + M)\| \leq \|(y_n - y)\|$ and $y_n \to y$ means that $\|(y_n - y)\| \to 0$.

$\Rightarrow \|(x_n + M) - (y_n + M)\| \to 0$
$\Rightarrow (x_n + M) \to (y_n + M)$ as $n \to \infty$.

Hence the sub sequence $(x_n + M)$ of the original Cauchy sequence is convergent. This proves that N/M is a complete normed linear space.

Hence N/M is a banach space.

Hence the proof.

Complete:

A complete metric space is a metric space in which every Cauchy sequence is convergent.

Example:

1.The space R and C are the real number and the complex number are the simplest of all normed linear spaces.The norm of a number x is of course defined by$||x|| = |x|$ and each space R and C are complete.

Hence R and C are banach.

2.The linear spaces R^n and C^n of all n-tuples

$x = (x_1, x_2, \ldots x_n)$ of real number and the complex number can be made into normed linear spaces in a infinite variety of way. If the norm is defined by

$$||x|| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

3. Let P be a real number such that $1 \leq P < \infty$. We denote by lp^n the space of all n-tuples $x = (x_1, x_2, \ldots x_n)$ of scalars with the norm defined by

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

here, p=2 so the real and complex numbers l_2^n are the n- dimensional Euclidean and unitary spaces R^n and C^n.

Then, the completeness of lp^n comes from the same reasoning of theorem.

lp^n is a banach space.

4. Let P be a real number such that $1 \leq P < \infty$. We denote by lp the space of all sequences

$x = (x_1, x_2, \ldots x_n, \ldots)$ of scalars such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$

with the norm defined by

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

here lp is actually a banach space.

5. The linear spaces of all n-tuples $x = (x_1, x_2, \ldots x_n)$ of scalars, we define the norm by

$||x|| = \max \{ |x_1|,|x_2| ,\ldots\ldots\ldots| x_n |\}\ldots\ldots(1)$

This banach space commonly denoted by
l_∞^n. i.e., $||x||_\infty = \lim ||x||_p$ as $p \to\infty$.

6. Consider the linear space of all bounded sequences
$x=(x_1, x_2,\ldots..x_n ,\ldots.)$ of scalars. We define the norm x by

$||x|| = \sup |x_n|$ .This we denote in banach space by l_∞. The set C of all convergent sequence is to be aclosed linear subspace of l_∞ and is therefore itself a banach space.

7.The C(x) of all bounde continuous scalar –valued function defined on a topological space X, with the norm given by

$$||f|| = \sup |f(x) |.$$

This norm is sometimes called Uniform norm.

Continuous Linear transformation:

Let N and N' be the normed linear space with the same scalars and let T be a linear transformation of N into N'. T is continuous if it is continuous as a mapping of the metric space N into the metric space N', $x_n \to x$ in N
$\Rightarrow T(x_n)\to T(x)$ in N'.

Theorem:
Let N & N' be a normed linear space and T be a linear transformation of N into N'. Then the following condition on T are equivalent to one another.

i. T is continuous.
ii. T is continuous at the origin in the sense that $x_n\to 0 \Rightarrow T (x_n) \to 0$;
iii. There exist a real no. $K \geq 0$ with the property that $||T(x)|| \leq K ||x||, \forall x\in N$.
iv. If $s=\{x : ||x || \leq 1\}$ is the closed unit sphere in N .then its image T(s) is a bounded set in N.

Proof:

(i)$\Rightarrow$ (ii)
If T is continuous , then since T(0)=0 it is certainly continuous at the origin .

i.e., If $x_n \to 0$ then $T(x_n) \to T(0) = 0$.

On the other hand if T is continuous at the origin , then $x_n \to x$.

$\Rightarrow x_n - x \Rightarrow T(x_n - x) \to 0$

$\Rightarrow T(x_n) - T(x) \to 0$

$\Rightarrow T(x_n) \to T(x)$

So T is continuous.

(ii)$\Rightarrow$ (iii)

It is obvious that (iii)$\Rightarrow$ (ii)

If such a K exists, then $x_n \to 0$. Clearly implies that
$T(x_n) \to 0$.

To show that (ii)$\Rightarrow$ (iii).

We assume that there is no such K. It follows from that for each positive integer n, we can find a vector x_n such that
$\|T(x)\| > n \|x_n\|$, or equivalently such that

$\|T(x) > n \|x_n\| \| > 1$.

If we put $y_n = x_n / n \| x_n \|$.

Then it is easy to see that $y_n \to 0$, but $T(y_n)$ does not tend to zero. So T is not continuous at the origin.

(iii)$\Rightarrow$ (iv)

Since a non-empty subset of a normed linear space is bounded iff it is contained in a closed sphere centered in the origin , it is evident that (iii)$\Rightarrow$ (iv), for if all $\| x \| \leq 1$, then $\| T(x) \| \leq K$ for all $x \in N$. suppose $x \in s$.

i.e., $\| x \| \leq 1$, then $\| T(x) \| \leq K$

$\Rightarrow$ T(S) is bounded.

(iv)$\Rightarrow$ (iii)

We assume that T(S) is contained in a closed sphere of radius of K centered on the origin. If x=0, then T(x) =0 and clearly, $\|T(x)\| \leq K \|x\|$ and if $x \neq 0$ then $x/ \|x\| \in S$.

$\therefore \ \| T (x / \|x\|)\| \leq K,$

Again, we have $\|T(x)\| \leq K \|x\|$.

Hence the proof.

Theorem:

The norm of a continuous linear transformation is equivalent to the following condition.

i) $\|T\| = \sup \{\|T(x)\| : \|x\| \leq 1\}$
ii) $\|T\|_0 = \sup \{\|T(x)\| : \|x\| = 1\}$
iii) $\|T\|_1 = \sup \{\|T(x)\| / \|x\| ; x \in N \ \& \ \|x\| \neq 0 \}$
iv) $\|T\|_2 = \inf \{ k: k \geq 0 \ \& \ \|T(x)\| \leq k \|x\| \ \forall x \}$

Proof:

(i) $\Leftrightarrow$ (ii)

Let us denote the norm of T in (ii) by $\|T\|_0$ and prove $\|T\| = \|T\|_0$ where $\|T\|$ is given by (i)

Let $A = \{ \|T(x)\| : \|x\| \leq 1\}$

$B = \{\|T(x)\| : \|x\| = 1\}$.

Clearly $B \subseteq A$, then $\sup B \leq \sup A$.

$\|T\|_0 \leq \|T\|$ ……….(1)

We now prove

$\|T\|_0 \geq \|T\|$

Let $x \in N$, $x \neq 0$ such that $\|x\| \leq 1$.

Define $y = x / \|x\|$, then $\|y\| = \|x\|/\|x\| = 1$.

Now, $\|T(y)\| = \| T(x/\|x\|) \| = \| T(x)/ \|x\| \|$

$= \| T(x) \| / \|x\|$

$\geq \|T(x)\|$ as $\|x\| \leq 1$.

$\Rightarrow \sup \{ \|T(y)\| : \|y\| = 1\}$
$\geq \sup \{\|T(x)\| : \|x\| \leq 1\}$

$\Rightarrow \|T\|_0 \geq \|T\|$ ……………..(2)

from (1) and (2)

$\Rightarrow \|T\|_0 = \|T\|$

(ii) $\Leftrightarrow$ (iii)

For $x \in N$, $x \neq 0$,

$\| T(x) \| / \|x\| = \| T(x/\|x\|) \|$
$= \| T(y) \|$
where $y = x/\|x\|$ and $\|y\| = 1$.

Thus $\|T\|_1 = \sup \{\|T(x)\| / \|x\| ; x \in N \;\&\; \|x\| \neq 0 \}$
$= \sup \{ \|T(y)\| : \|y\| = 1\} = \|T\|_0$

$\Rightarrow \|T\|_1 = \|T\|_0$

(i) $\Leftrightarrow$ (iv)

Let $P = \{\|T(x)\| : \|x\| \leq 1\}$ and

$Q = \{ k: k \geq 0$ and $\|T(x)\| \leq k \|x\| \; \forall x$ such that $\|x\| \leq 1\}$

Let m be the upper bound of the set P. Then
$\|T(x)\| \leq m \; \forall x$ such that $\|x\| \leq 1$.

$\therefore m \in Q$.

Conversely,

Let $k \in Q$, then $k \geq 0$ and $\|T(x)\| \leq k \|x\| \; \forall x$ such that $\|x\| \leq 1$.

$\therefore$ k is an upper bound of P.

$\therefore$ Q= the set of all upper bound of P.

sup P= lub P= the least element of Q

= inf Q.

Let $x \in N$, $\|x\| \neq 0$ and $y = x/\|x\|$ $\therefore$ $\|y\| = 1$

$\therefore$ $\|T(y)\| \in P$.

$\Rightarrow \|T(y)\| \leq \sup P = \inf Q \leq k, k \in Q$.

$\Rightarrow \|T(x)\| / \|x\| \leq k, k \in Q$ & $x \in N$ & $\|x\| \neq 0$

For $k \in Q$,

$\|T(x)\| \leq k\|x\| \ \forall \ x \in N$.

Hence $Q = \{ k : k \geq 0, \|T(x)\| \leq k\|x\| \ \forall \ x \in N , \|x\| \neq 0 \}$

Thus, $\sup \{\|T(x)\| : \|x\| \leq 1\} = \sup P$

$= \inf Q$.

$= \inf \{ k : k \geq 0, \|T(x)\| \leq k\|x\| \ \forall \ x \in N \}$.

Hence the proof.

Conjugate space of N:

Let N be an arbitrary normed linear space . The set of all continuous linear transformation of N into R or C in
B(N,R) or **B**(N, C) (as N is real or complex). It is denoted by N* is called the conjugate space of N.

The elements of N* are called continuous linear functional or functional. If f is functional

$\| f \| = \sup \{\|f(x)\| : \|x\| \leq 1\}$.

Theorem:

Let N & N' be a normed linear space the set B (N,N') of all continuous linear transformation of N into N' is a normed linear space with respect to the point wise linear operations

i)$(T+U)(x)=T(x)+U(x)$;

ii)$(\alpha T)(x)= \alpha T(x)$. and the norm defined by

$\|T\| = \sup \{\|T(x)\| : \|x\| \leq 1\}$.Also if N' is a Banach space then **B** (N,N') is also a Banach space.

Proof:

First we prove that **B** (N,N') is a linear space .

Let T_1, $T_2 \in$ **B** (N,N').

Then $(T_1+ T_2)(x + y) = T_1(x+y)+T_2(x+y)$

$= T_1(x)+ T_1(y)+ T_2(x) +T_2(y)$.

$=(T_1+ T_2)(x) + (T_1+ T_2)(y)$

Thus $(T_1+ T_2)$ is linear.

Similarly, $(T_1+ T_2)(\alpha x) =\alpha (T_1+ T_2)(x)$.

Thus $(T_1+ T_2)$ is a continuous linear transformation. Since T_1 & T_2 are continuous linear transformation. Also, αT is a continuous linear transformation.

Thus **B** (N,N') is a linear space.

To verify norm axioms:

Clearly $\|T\| \geq 0$ and as $\|T(x)\| \geq 0$

i) Also, $\|T\| =\sup \{\|T(x)\| / \|x\| \; ;x\in N \;\&\; \|x\| \neq 0 \}$

Now , $\|T\| =0$ iff $\|T(x)\| =0 \;\; \forall x$.

iff $T(x)=0 \;\; \forall x$

iff $T=0$.

ii) Also if

$\| T_1+ T_2 \| =\sup \{\| T_1+ T_2 (x)\| \; : \|x\| \leq 1\}$

$= \sup \{\| T_1(x) + T_2 (x)\| \; : \|x\| \leq 1\}$

$\leq \sup \{\| T_1(x)\| \; : \|x\| \leq 1\}+ \sup \{\| T_2 (x)\| \; : \|x\| \leq 1\}$

$=\| T_1 \| + \|T_2\|$.

iii) Similarly, $\| \alpha T\| = |\alpha| .\| T \|$.

Hence **B** (N,N') is a normed linear space. Finally we have to prove **B** (N,N') is a banach space whenever N' is a banach space.

For doing this consider a Cauchy sequence {Tn}in **B** (N,N'). If x is an arbitrary vector in N, then {Tn(x) } isa sequence in N', which is Cauchy.

But N' is a banach space which is complete. Hence $\{T_n (x)\}$ is convergent.

Let $T_n (x) \rightarrow T(x)$ this defines the mapping T of N into N'. By the joint continuity of addition and scalar multiplication T is seen to be a linear transformation .

To conclude the proof we have to show that T is continuous and $T_n \rightarrow T$ w.r.to the norm on **B** (N,N').

Since $| \| T_n (x) - T_m(x) \| | \leq \| T_n (x) - T_m(x) \|$

$< \varepsilon$ (fix ε)

Fix M so that $\| T_n (x) \| < \varepsilon + \| T_m(x) \|$.
Thus sequence $\{T_n (x)\}$ is a Cauchy sequence in N' & the norm of the terms of this Cauchy sequence form a bounde set of numbers.

$\therefore \| T (x) \| = \| \lim T_n (x) \| = \lim \| T_n (x) \|$

$\leq \sup \| T_n \| \| x \|$

$=(\sup \| T_n \|) \| (x) \|$

$= k \| x \|$ where $k = \sup \| Tn \|$.

Hence T is bounded and therefore continuous .

It remains to prove that $T_n \rightarrow T$.
i.e., To prove $\| T_n - T \| \rightarrow 0$

For a given $\varepsilon > 0$, let n_0 be a positive integer such that
$\| T_m - T_n \| < \varepsilon \ \forall \ m,n \geq n_0$ as T_n is a cauchy sequence .

$\therefore \| T_m(x) - T_n (x) \| \leq \| T_m - T_n \| \| (x) \|$
$\leq \| T_m - T_n \|$ for $\| (x) \| \leq 1$.
$< \varepsilon$

Thus, $\| T_m(x) - T(x) \| < \varepsilon \ \forall \ m,n \geq n_0$ as $T_n(x) \rightarrow T(x)$.

This shows that $T_m \rightarrow T$ and

$$\| T_m - T \| \rightarrow 0$$

Hence the proof.

Operators:

Let N be a normed linear space . A continuous linear transformation N into itself is called an operator of N. We denote the normed linear space of all operator of N by **B**(N) instead of **B** (N,N').

Note:

i) **B**(N) is a banach space when N is a banach space.

ii) **B**(N) is indeed an algebra in which multiplication of operator is given by $TT'(x) = T(T'(x))$ and
$\| T T' \| \leq \| T \| \ \| T' \|$.

iii)Addition, scalar multiplication are jointly continuous in
B(N) i.e., $T_n \rightarrow T$, $T'_n \rightarrow T' \Rightarrow T_nT'_n \rightarrow T T'$.

The identity transformation I,I(x)=x is in identity for the algebra **B**(N) and $\| I \| = 1$.

Isometrically isomorphic of N intoN':

Let N and N' be a normed linear spaces. A 1 to 1 linear transformation of N into N' such that $\| T(x) \| = \| (x) \|$
for $x \in N$, $Tx \in N'$ is called an isometrically isomorphic of N into N'. We also say that N and N' are isometrically isomorphic if it satisfies onto also.

Hahn- Banach :

Any functional defined on a linear subspace of a normed linear spaces can be extended linearly and continuously to the whole space without increasing its norm.

Lemma:
Let M be a linear subspace of a normed linear space N.let f be a functional defined on M of x_0 is a vector not in M and if $M_0 = M + x_0$. Then f can be extended to a functional f_0 defined on M_0 such that $||f_0|| = ||f||$.

Proof:
Case (i):
Let N be a real normed linear spaces. Assume $||f|| = 1$ where f is a functional defined on M, a linear subspace of N.

We may assume, without loss of generality $||f|| = 1$.

Since $x_0 \notin M$,each vector $y \in M_0$ is uniquely expressible as $y = x + \alpha x_0$ with $x \in M$.Define a mapping f_0 on M_0 as follows $f_0(y) = f_0(x + \alpha x_0) = f_0(x) + \alpha f_0(x_0)$

$= f(x) + \alpha r_0$.

Where $r_0 = f_0(x_0)$. This is an linear extension of f to M_0 and f_0 is linear for every choice of the real number x.

Clearly, f_0 is continuous as f is a functional on M. we've to choose r_0 so that $||f_0|| = 1$.

r_0 has to be chosen so that $|f_0(y)| \leq \|f_0\| \, \|y\|$
i.e., $|f_0(x+\alpha x_0)| \leq \|f_0\| \, \|x+\alpha x_0\|$
$= \|x+\alpha x_0\|$ if $\|f_0\| = 1$ were to be =1.

But $f_0(x+\alpha x_0) = f(x) + \alpha r_0$
i.e., $|f(x)+\alpha r_0| \leq \|x+\alpha x_0\|$
i.e., $-f(x/\alpha) - \|(x/\alpha)+x_0\| \leq r_0 \leq -f(x/\alpha) + \|(x/\alpha)+x_0\|$ ……….(1)

So, if we choose r_0 satisfying (1) , then $\|f_0\| = 1$.
Since f is linear and continuous, for any two vectors $x_1, x_2 \in M$, we've

$f(x_2) - f(x_1) \leq |f(x_2 - x_1)|$
$\leq \|f\| \, \|x_2 - x_1\|$
$= \|x_2 - x_1\| = \|(x_2+x_0) - (x_1+x_0)\|$
$\leq \|x_2+x_0\| + \|x_1+x_0\|$

$\therefore -f(x_1) - \|x_1 + x_0\| \leq -f(x_2) + \|x_2 + x_0\|$ ……….(2)

Define 2 real numbers a,b by

$a = \sup \{ - f(x) - \| x + x_0 \| : x \in M\}$

$b = \inf \{ - f(x) - \| x + x_0 \| : x \in M\}$

By(2) $a \leq b$

If we choose r_0 to be any real number $a \leq r_0 \leq b$, then the sequence inequality in (1) is satisfied.

Hence the proof in the case (i)

Case(ii):

Let N be a complex number in a normed linear spaces. f is a complex valued functional defined on M for which

$\|f\| =1$. A complex linear space can be regarded as a real linear space by restricting the scalars to be real.

Let g and h be the real and imaginary parts of f so that

$f(x) = g(x) + i\, h(x) \quad \forall\, x \in M$

Then both g and h are real valued functionals on the real space M.

Since $\|f\| =1$, we've $\| g \| \leq 1$.

Also, we've $f(ix) = i f(x)$ and

$i f(x) = g(ix) + ih(ix)$

$= i[g(x) + ih(x)]$

$\therefore h(x) = -g(ix)$

$\therefore f(x) = g(x) - ig(ix)$.

By case (i) we extend g to areal valued functional g_0 on the real space M_0 in such a way that

$\| g_0 \| = \| g \|$.

We define f_0 for $x \in M_0$ by $f_0(x) = g_0(x) - ig_0(ix)$.

Then f is an extension of f from M to M_0. Also, f_0 is linear , as

$f_0(x+ y) = g_0(x+y) - ig_0(i(x+y))$

$= g_0(x)+g_0(y) - ig_0(i(x))+ig_0(y)$

$= f_0(x) +f_0 (y)$ [since g_0 is linear]

Similarly,$f_0(\alpha x)=\alpha f_0(x)$forallreal$\alpha$.

This is true for complex α also as $f_0(ix) = i\, f_0(x)$.

So f_0 is linear as acomplex valued function defined on the complex space M_0. Finally to prove $\|f_0\| = 1$.

If x is avector in M_0, for which $\|x\| = 1$, then we prove , so that
$\| f_0 \| = \sup \{ | f_0 (x) | \ : \| x \| = 1 \} = 1$.

If $f_0(x)$ is complex , then we can write $f_0(x) = re^{i\theta}$ with $r>0$ so that $| f_0(x) | = r$ .It follows that $f_0(e^{-i\theta} x)$ is real.

$\therefore | f_0 (x) | \leq 1$

Hence the proof.

Hahn-Banach theorem:

Let M be a linear subspace of a normed linear space N. Let f be a functional defined on M. Then f can be extended to afunctional f_0 defined on the whole space N such that
$\| f_0 \| = \| f \|$.

Proof:

By the above lemma, for any $x \in N$ and $x \notin M$. We've an extension of f on M+[x] such that $\| f \|$ is preserved for the extension.

Consider the set G of all such extensions of f to functionals g with the same norm , defined on subspaces which contain M. This is a partially ordered set w.r.to the following relation.

$g_1 \leq g_2$ iff domain of g_1 is contained in domain of g_2 and $g_2(x) = g_1(x)$, for all x in the domain of G.

Now, every chain in G has an upper bound .
By Zorn's lemma,
" There is a maximal extension f_0. The f_0 is the required extension of the entire space n. For if f_0 is not defined on the whole of x, then there is an $x \in N$ and not in the domain M_0 of f_0 , so that f_0 can be extended to $M_0 + [x]$. But f_0 is maximal."
This is a contradiction to our assumption.
Hence the proof.

Corollaries of Hahn-Banach theorem:

Corrollary:

If N is a normed linear space and x_0 is non-zero vector in N then there exist functional f_0 in N* such that $f_0(x_0) = ||x_0||$ and $||f_0||=1$.

Proof:

Let $M=\{\alpha x_0\}$ be the linear subspace of N spanned by x_0.Define f on M by $f(\alpha x_0) = \alpha ||x_0||$. Clearly, f is afunctional on M such that $f(x_0) = ||x_0||$, taking $\alpha =1$ and $||f||=1$.

By Hahn Banach theorem f can be extended to a functional f_0 in N* such that $f_0(x_0) = f(x_0) = ||x_0||$.

And $||f_0||= ||f||=1$.

Hence the proof.

Corrollary:

If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M , then there exist a functional f_0 in N* such that $f_0(M)=0$, $f_0(x_0)\neq 0$.

Proof:

The natural mapping T of N onto N/M is a continuous linear transformation such that T(m) =0 and

$$T(x_0)= x_0 +M\neq 0.$$

By the previous corollary there exist a functional f in (N/M)* such that $f(x_0 +M)\neq 0$.Define f_0 by $f_0(x) = f(T(x))$.

Then f_0 is the desired functional with the property that $f_0(M)$.

i.e., $f_0(M)=0$, $f_0(x_0) = f(T(x_0)) = f(x_0 +M) \neq 0$.

Second Conjugate space:

The conjugate space of N* is itself a Normed linear space.We can form the conjugate space (N*)* . It is denoted by N** and is called the second conjugate space of N.

Each vector $x\in N$ gives raises to a functional F_x in N** defined by $F_x(f) = f(x)$, $x\in N$.

Properties of natural embedding on N into N:**

1. F_x is linear.
2. $\|F_x\| = \|x\|$.
3. The mapping $x \to F_x$ is a norm preserving mapping of N into N**. F_x is called an induced functional. Thus the isometric isomorphism $x \to F_x$ is a natural embedding on N into N**.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The norm of x is called as the ………… of the vector	direction	length	weight	scalar	Length
Every …………. normed linear space is a banach space	complete	metric	compact	connected	complete
A banach space is a normed linear space which is complete as a …………	complete	connected	compact	metric	metric
The metric space arise on norm as d(x,y)=……………	‖x+y ‖	‖x ‖	‖x-y ‖	‖xy ‖	‖x-y ‖
The linear operation is denoted by………..	R	N	L	K	L
The two primary operation in alinear space is called ……………	Linear operation	Arithmetic operation	Operators	Operations	Linear operation
The size of an element x is a real number denoted by ……………	norm x	real x	banach x	complex x	norm x
A linear space is called real linear space when its scalar is ……..	norm	real	banach	complex	real
A linear space is called …….. linear space when its scalar is complex	norm	real	banach	complex	complex
………………… is called the distance between x and y	c(x,y)	r(x,y)	d(x,y)	p(x,y)	d(x,y)
Every cauchy sequence has a convergent…………..	sequence	subsequence	series	serial	subsequence
The real part of Z is denoted by …………	Re(Z)	Re(x+y)	Im(Z)	Im(x+y)	Re(Z)
The imaginary part of Z is denoted by ………….	Re(Z)	Re(x+y)	Im(Z)	Im(x+y)	Im(Z)
The ……………. of A is the lub of the distance between pair of its points.	direction	distance	weight	scalar	distance
If f is a………… function if there is a real number K such that $\mid f(x) \mid \leq k$.	norm	finite	bounded	unbounded	bounded
A…………..set is one whose diameter is finite.	complete	connected	metric	bounded	bounded
Every sequentially compact metric space is…………….	complete	connected	compact	metric	compact
Every sequentially ………………. metric space is totally bounded.	complete	connected	compact	metric	compact
A mapping of a nonempty set b in to a metric space is called a………… mapping	norm	finite	bounded	unbounded	bounded
A metric space is compact iff it is ……………. and totally bounded.	complete	connected	metric	bounded	complete
A closed subspace of a complete metric space is ………. iff it is totally bounded	complete	connected	compact	metric	compact
A metric space is said to be sequentially compact if every sequence in it has a convergent …………………	sequence	subsequence	space	subspace	subsequence
The ……… is called second conjugate space of N.	N**	N	N"	N*	N**

A complete metric space is ametric space in which every cauchy sequence is…………..	complete	connected	compact	convergent	convergent
If N is a banach space then the product N/M is a…………………...	Banach space	hilbert space	Inner product space	linear space	Banach space
The elements of N* are called continuous linear functional or………………	continuous	functional	linear space	convergent	functional
The identity transformation I is an ……for the algebra B(N)	continuous	functional	linear space	identity	identity
N is said to be isometrically isomorphic to N' if ther exist an ……………of N into N'	isomorphic	isometric	isometric isomorphism	isomorphism	isometric isomorphism
If T is continuous at the origin , then Xn→0 implies ……………..	Xn→0	T(Xn)→0	Xn→1	T(Xn)→∞	T(Xn)→0
The set of all …………. for T equals the set of all radii of closed sphere centered on the origin which contain T(S)	bounds	convex set	continuous	functional	bounds
Any infinite set which is numerically equivalent to N is said to be ……………	countable	uncountable	uncountably infinite	countably finite	countably finite
A set is …………… if it is non-empty and finite	countable	uncountable	uncountably infinite	countably finite	countable
Any countable union of countable set is………..	countably finite	not countable	countable	uncountable	countable
Uncountable is otherwise called as…………………	countably finite	not countable	countable	uncountably infinite	uncountably infinite
The absolute value is the …………… between the numbers.	direction	distance	weight	scalar	distance
The triangle inequality for metric space is ………………	d(x,y) <=d(x,z)+d(y,z)	d(x,y) <d(x,z)+d(y,s)	d(x,y) <d(x,y)+d(y,z)	d(x,y) >d(x,z)+d(y,z)	d(x,y) <=d(x,z)+d(y,z)
The elements of x are called the points of………space (x,d)	Banach space	Hilbert space	Metric space	Linear space	Metric space
Let x be a metric space then it is ……. property is d(x,y)=d(y,x)	asymmetry	symmetry	abelian	commutate	symmetry
Let x be a metric space then it is symmetry Property is d(x,y)= …….	d(x,y) <d(x,z)+d(y,z	d(x,y) <d(x,z)+d(y,s	d(x,y)	d(y,x)	d(y,x)
f is said to be continuous if it is …………… at each point of x.	continuous	functional	linear space	convergent	continuous

UNIT 2

Open & Closed mapping

Lemma:

If B and B' are banach spaces and if T is a linear transformation of B onto B',then the image oech open sphere centered on the origin in B contains an open sphere centre on the origin in B'.

Proof:

We denote by Sr and S_r' the open spheres with radius r centered on the origin in B and B'.

$T(S_r) = T(r\,S_1) = r.T(S_1)$.

So it suffices to show that $T(S_1)$ contains some S_r'.

We begin by proving that $\overline{T(S_1)}$ contain some S_r'. Since T

is onto, B' = $\bigcup_{n=1}^{\infty} T(S_n)$. B' is complete, so Baire's theorem

implies that some $\overline{T(S_{n0})}$ has an interior point y_0, which may be assumed to lie in $T(S_{n0})$.

The mapping $y \rightarrow y-y_0$ is a homeomorphism of B' onto itself, so $T(S_{n0}) - y_0 \subseteq T(S_{2n0})$. From this we obtain

$\overline{T(S_{n0})} - y_0 = \overline{T(S_{n0}) - y_0} \subseteq \overline{T(S_{2n0})}$, which shows that the origin is an interior point of $\overline{T(S_{2n0})}$.

Multiplication by an nonzero scalar is a homeomorphism of B' onto itself, so

$\overline{T(S_{2n0})} = \overline{2n_0 . T(S_1)} = 2n_0 .\overline{T(S_1)}$

It follows from the fact that the origin is also an interior point of $\overline{T(S_1)}$, so $S_\varepsilon' \subseteq \overline{T(S_1)}$ for some positive number ε.

We complete the proof by showing that

$S_\varepsilon' \subseteq T(S_3)=3.T(S_1)$.

Let $y \in S_\varepsilon'$. Therefore $y \in \overline{T(S_1)}$. hence each neighbourhood of y intersects $T(S_1)$.

There is an open sphere centre on y and with radius ε/2, that intersects $T(S_1)$. There is a point on

$y_1 \in T(S_1)$ such that $||y-y_1|| < \varepsilon/2$ and there is a point

$x_1 \in B$such that $y_1 = T(x_1)$ and $||x_1|| <1$. Now

$S_\varepsilon' \subseteq \overline{T(S_1)}$. i.e., $S_{\varepsilon/2}' \subseteq \overline{T(S_{1/2})}$.

Since $||y-y_1|| < \varepsilon/2$,$y-y_1$ is a vector in $\overline{T(S_{1/2})}$.Each neighbourhood of $y-y_1$ intersects $T(S_{1/2})$.

Let $y_2 \in T(S_{1/2})$ such that $||y-y_1-y_2|| < \varepsilon/4$ where

$y_2 = T(x_2)$ for $x_2 \in B$ and $||x_2|| < 1/2$.

Continuing like this we get a sequence of vector $\{x_n\}$ in B so that $||x_n|| < 1/2^{n-1}$ and
$||y-(y_1+y_2+\ldots\ldots+y_n)|| < \varepsilon/2^n$, where $y_n=T(x_n)$.

Define $S_n = x_1+x_2+\ldots\ldots\ldots+x_n$.We find that $\{S_n\}$is a Cauchy sequence in B.

$$||S_n|| \leq ||x_1||+||x_2||+\ldots\ldots\ldots+||x_n||$$
$$< 1+1/2+\ldots\ldots\ldots+1/2^{n-1}.$$
$$<1/(1-1/2) <2.$$

Since B is complete the sequence $\{S_n\}$ converges to x in B. i.e., $S_n \to x$.

$||x|| = ||\lim S_n|| = \lim ||S_n|| \leq 2 < 3.$

$\Rightarrow x \in S_3.$

Consider $T(x) = T(\lim_{n\to\infty} S_n) = \lim_{n\to\infty} T(S_n)$.

$$= \lim [\, T(x_1+x_2+\ldots\ldots\ldots+x_n)\,]$$
$$= \lim [T(x_1)+ T(x_2) +\ldots\ldots\ldots+T(x_n)]$$
$$= y.$$

$y \in S_\varepsilon' \quad \Rightarrow S_\varepsilon' \in T(S_3)$

$\Rightarrow S'_{\varepsilon/3} \in T(S_1)$

Hence the proof.

Theorem : Open Mapping theorem

If B and B’ are banach spaces and if T is a linear transformation of B onto B’,then T is an open mapping .

Proof:

We must show that if G is open in B, then T(G) is also open set in B’. If Y is appoint in T(G) it Suffices to produce an open sphere centered on y and contained in T(G).

Let x be a point in G such that T(x)= y. Since G is open, x is the centre of an open sphere which can be written in the form x+ δr contained in G.

Our lemma now implies that T(Sr) contains some Sr'. It is clear that y+Sr' is an open sphere centered on y and the fact that it is contained in T(G) at once from

$$y+Sr' \subseteq y + T(Sr) = T(x)+ T(Sr)$$
$$\Rightarrow T(x+Sr) \subseteq T(G).$$

Hence the proof.

Interior point :

Let X be an arbitrary metric space and let A be a subset of X. A point in ' A ' is called an interior point of A if it is the center of some open sphere contained in A, and the interior of A denoted by Int(A), is the set of all interior points .

Int(A) = { x: x∈A and Sr (x) ⊆ A for some r}.

Projection:

Projection E determines a pair of linear subspace M & N such that L= M⊕ N where M= { E(x): x ∈ L } and
N={ x: E(x) =0} are the range and null space of E.

Theorem :

If P is a projection on a banach space B and if M and N are its range and null space , then M& N are closed linear subspace of B such that B = M⊕ N.

Proof:

P is an algebraic projection . So the above definition gives everything except the fact that M and N are closed.

The null space of any continuous linear transformation is closed,so N is obviously closed.The fagt that M is also closed is aconsequences of

$$M= \{ P(x): x \in B \}$$
$$= \{x: P(x) =x\}$$
$$= \{x: (I-P)(x) =0\}$$

Which exhibits M as the nullspace of the operator (I-P).

Hence the proof.

Theorem :

Let B be a Banach space and let M and N be a closed linear subspace of B such that B= M $\oplus$ N .If Z =x + y is the unique representation of a vector in B as a sum of vectors in M and N , then the mapping P defined by P(Z)=x is a projection on B whose range and null spaces are M & N.

Proof:

A pair of linear subspace M and N such that L= M $\oplus$ N determines a projection E whose range and nullspace are M and N .we want to prove that P is continuous.

If B' denotes the linear space B equipped with the norm defined by $||Z||' = ||x|| + ||y||$.

Then B' is a banch space and since

$||P(Z)|| = ||x|| \leq ||x|| + ||y|| = ||Z||'$.

P is clearly continuous as a mapping of B' into B. It suffices to prove that B' and B have the same topology.

If T denotes the identity mapping of B' onto B, then

$||T(Z)|| = ||Z|| = ||x| \leq ||x|| + ||y|| = ||Z||'$.

This shows that T is continuous as 1 to 1 linear transformation of B' onto B. Then by previous theorem implies that T is a homeomorphism.

Hence the proof.

Definition:

The graph of a linear transformation of a banach space B into another banach space B' is that subset of BXB' which consist of all ordered pairs (x, T(x)) where $x \in B$.

Lemma:

If T is continuous , then its graph is closed as a subset of the metric space B x B' With metric defined by

$d((x_1,y_1), (x_2,y_2)) = \max \{ \| x_1 - x_2 \|, \| y_1 - y_2 \| \}$.

Proof:

Let (x_0, y_0) be in the closure of the graph of T .

Then there is a sequence $\{ x_n , T(x_n)\}$ in the graph of T such that $x_n \to x_0$; $T(x_n) \to y_0$.

T is continuous , $T(x_n) \to T(x_0)$.
$\therefore T(x_0) = y_0$.

Thus the point $(x_0, T (x_0))$ belongs to the graph of T.

Hence graph of T is closed as a subset of B X B'.

Hence the proof.

Theorem : Closed graph theorem:
If B and B' are Banach and if T is a linear transformation of B into B' . Then T is continuous iff the graph of T is closed.

Proof:

T is continuous.
$\Rightarrow$ The graph T is closed.

Converse,
Let the graph of T be closed. Denote by B, the linear space 'B' with the norm defined by
$||x||_1 = ||x|| + ||T(x)||$.

We can prove that B_1 is a normed linear space under the norm, now

$||T(x)|| \leq ||x|| + ||T(x)|| = ||x||$

This shows that T is bounded and hence continuous as the linear transformation from B_1 to B'.

It is enough to prove that B and B' have the same topology.i.e., B and B' are homeomorphic.

The identity mapping of B_1 to B' is continuous for
$||T(x)||$.

We show that B_1 is a banach space to show the completeness of B_1.

Consider a Cauchy sequence $\{x_n\}$ in B_1. Thus $\{x_n\}$ is a Cauchy sequence in B and $\{T(x_n)\}$ is a cauchy sequence in B' as $||x_m - x_n|| < \varepsilon$.

$\Rightarrow ||x_m - x_n|| + ||T(x_m) - T(x_n)|| < \varepsilon$

Since B and B' are complete, there exist a sequence $x \in B$, $y \in B'$
Such that $x_n \to x \in B$ and $T(x_n) \to y \in B'$.

By hypothesis the graph of T is closed in BXB'.
This implies (x,y) lies in the graph i.e., y=T(x).

$$||x_n - x||_1 = || x_n - x || + || T(x_n - x)||$$
$$= || x_n - x || + || T(x_n) - T(x)|| \to 0.$$

$\Rightarrow x_n \to x$ in B_1. $\therefore$ B is complete and its banach. Thus T is continuous from B to B'.

Hence the proof.

Conjugate of an operator:

Each operator T on a normed linear space N induces a corresponding operator , denoted by T* and it is called the conjugate space N*.

Theorem : Uniform boundedness theorem

Let B be abanach space and N a normed linear space . If {Ti} is a non-empty set of continuous linear transformations of B into N with the property that {Ti(x)}
Is a bounded subset of N for each Vector x in B, then
{ ||Ti|| } is a bounded set of numbers , that is {Ti} is bounded as a subset of **B**(B,N).

Proof:

For each positive integer n, the set
Fn={x: $x \in B$ and $||Ti(x)|| \leq n$ for all i }is clearly a closed subset of B, and by assumption we have

$$B = \bigcup_{n=1}^{\infty} Fn$$

Since B is complete ,Baire's theorem shows that one of the Fn's, say Fn_0 has non-empty interior, and thus contains aclosed sphere S_0 with center x_0 and radius $r_0 > 0$.

It means that each vector in every set $Ti(S_0)$ has norm less than or equal to n_0:

For the sake of brevity $||Ti(S_0)|| \leq n_0$. It is clear that
$S_0 - x_0$ is the closed sphere with radius r_0 centered on the origin , S_0 $(S_0 - x_0)/r_0$ is the closed unit sphere S. Since x_0 is in S_0 , it is evident to show that $||Ti(S_0 - x_0)|| \leq 2n_0$.

This yields $||Ti(S_0)|| \leq 2n_0/r_0$, so $||Ti(S_0)|| \leq 2n_0/r_0$ for every i.

Hence the proof.

Theorem :

A non-empty subset of a normed linear space N is bounded iff f(x) is a bounded set of numbers for each f in N*.

Proof:

Since $|f(x)| \leq ||f|| \, ||x||$, it is obvious that if X is bounded , then f(x) is bounded, then f(x) is also bounded for each f.

Second part of the theorem , it is convenient to exhibit that the vectors in X by writing $X = \{x_i\}$.

We use the natural imbedding from X to the corresponding subset $\{Fx_i\}$ of N**.

Our assumption that $f(x) = \{f(x_i)\}$ is bounded for each f is clearly equivalent to $\{Fx_i(f)\}$ is bounded for each f, and since N* is complete.

By previous theorem shows that $\{Fx_i\}$ is a bounded subset of N**.

Hence the proof.

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Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The centre of some open sphere contained in A is called the ………..	closed	open	interior	exterior	interior
Each operator T on a normed linear space N induces a corresponding operator denoted by …….	T'	T**	T*	T	T*
The M is the null space for the projection on ……………..	P	I-P	Space	subspace	I-P
If P is a projection on a Banach space B and if M and N are its ………	dense sets	range and null space	subspaces	projection	range and null space
A projection on E determines a pair of linear subspace M and N then ……..	L= m+N	L=M+N	L=M-N	L= MÅ N	L= MÅ N
The image of open sphere centered on the origin in B contains an ………….. Sphere centered on the origin in B and B'	closed	open	interior	exterior	open
The …... is the null space of the operator on the projection on I-P	M	N	L=M-N	L= m+N	M
The …... is the null space of the operator on the projection on P	M	N	L=M-N	L= m+N	N
The …... is the range of the operator on the projection on I-P	M	N	L=M-N	L= m+N	N
The …... is the range of the operator on the projection on P	M	N	L=M-N	L= m+N	M
A pair of linear subspace M and N such L= MÅN determines a………. on E.	dense sets	range and null space	subspaces	projection	projection
If T is continuous , then its graph is ………. as a subset of BxB'	closed	open	interior	exterior	closed
A closed set in a topological space in a set whose compliment is………..	closed	open	interior	exterior	open
A is ………… iff A = Int(A)	closed	open	interior	exterior	open
Int(A) equals the union of all …………….. of A.	closed	open	open subset	open set	open subset
The interior of A is denoted by …………………	Int(A)	Cl(A)	Ext(A)	Im(A)	Int(A)
Int(A) is an open subset of A which contains every ……………. of A	closed	open	open subset	open set	open subset
Let x be any metric space then any union of open set in x is ……...	closed	open	open subset	open set	open
Let x be any metric space then any finite intersection of …….in x is open.	closed	open	open subset	open set	open set
In any metric space x, each open sphere is an ………………..	closed	open	open subset	open set	open set
The open sphere $Sr(x_0)$ with center x_0 and radius r is the subset of x define by ……………...	d(x,y)	d(y,x)	$d(x,x_0) < r$	$d(x,x_0) = r$	$d(x,x_0) < r$
An open sphere is always non empty for it contain its…………..	center	radius	distance	length	center
An ………… sphere with radius 1 contain only its center.	closed	open	open subset	open set	open
If the open sphere is bounded open interval $(x_0 - r, x_0 + r)$ with midpoint x_0 and total length…...	r	2r	3r	0	2r
$Sr(x_0)$ is an open sphere with radius …...centered on x_0	r	2r	3r	0	r
In the linear space the………. transformation I defined by I(x)=x	identity	linear	one to one	onto	identity
The mapping P(Z) = x is a ………………. on B.	dense sets	range and null space	subspaces	projection	projection
B and B' have same topology means they are …………….	homomorphic	homeomorphic	linear	connected	homeomorphic
B and B' have same …….. means they are homeomorphic	strong topology	nullspace	topology	weak topology	topology
The identity mapping of B' to B is ………. for $\|T(x)\|=\|x\|$.	continuous	functional	linear space	convergent	continuous

If T is continuous linear transformation of B onto B' then T is an ……. mapping.	closed	open	open subset	open set	open
A 1-1 linear transformation T of abanach space onto itself is continuous then its inverse is automatically …………	continuous	functional	linear space	convergent	continuous
The mapping T →T* is thus anorm preserving map onf B(N) into ………..	B(N)*	B(N')	B(N)	B(N)**	B(N')

UNIT 3

Hilbert Spaces

Inner Product Space:

Let X be a complex vector space over the complex scalars C. Then (x,y) is said to be an inner product of x
and y.

i)$(x, x) \geq o$ for all x in X and $(x, x) = 0$ iff $x = 0$

ii)$(y,x) = \overline{(x,y)}$ for all x and y in X

iii) $(x+y, z) = (x,z) + (y,z)$ for all x, y and z in X

iv) $(\lambda x, y) = \lambda(x,y)$ for all x,y in X and all complex number λ

A complex vector space X having the inner product is said to be an inner product space.

Hilbert Space:

A complete inner product space is said to be a Hilbert Space.

Examples:

1. Consider the spaces l_2^n where we denote l_2^n as the linear spaces of all n-tuples of scalars with the norm of a vector $x= (x_1, x_2,.....x_n)$ defined by

$$||x|| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

We know that l_2^n is a banach space .Now,we show that the inner product of 2 scalars $x= (x_1, x_2,.....x_n)$ and $y= (y_1, y_2,.....y_n)$ is defined by inner product

$$(x,y) = \sum_{i=1}^{n} x_i \overline{y_i}$$

Then l_2^n is a Hilbert space.

2.Consider the banach spaces l_2 consisting of all infinite sequence $x= (x_n)^{\infty}_{n=1}$ of a complex number with the norm of a vector defined by

$$||x|| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$$

Also, if the inner product of two vectors $x= (x_1, x_2, \ldots x_n)$
And $y= (y_1, y_2, \ldots y_n)$ is defined by inner product

$$(x,y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

Then l_2 is a Hilbert space.

Theorem:(Schwartz inequality)

If (x,y) are any two vectors in ahilbert space then

$|(x , y)| \leq ||x||.||y||$

Proof:

If y=0 then the above inequality becomes equality as both side vanishes. Now $y \neq 0$ for any scalar λ we have
$(x + \lambda y , x+ \lambda y) \geq 0.$

$\Rightarrow (x,x)+(x, \lambda y) +(\lambda y,x)+ (\lambda y, \lambda y) \geq 0.$

$\Rightarrow (x,x)+ \bar{\lambda} (x, y) +\lambda \overline{(x, y)} + |\lambda|^2.||y||^2 \geq 0.$ ....(1)

Since $y \neq 0$, $||y|| \neq 0$.

Therefore put $\lambda = -(x,y)/ ||y||^2$ in equation (1)

$\Rightarrow ||x||^2 \geq |(x,y)|^2 / ||y||^2$

$\Rightarrow ||x||^2 . ||y||^2 \geq |(x,y)|^2$

$\Rightarrow |(x , y)| \leq ||x||.||y||$

Hence the proof.

Remark:

Using these inequality we see that the inner product function is jointly continuous.

Problem:

If x and y are any two vectors in a Hilbert space H then

i) $\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2$.
(parallelogram law)

ii) $\| x + y \|^2 - \| x - y \|^2 = 2(x,y)+2(y,x)$.

iii) $\| x + y \|^2 + \| x - y \|^2 + i\| x + iy \|^2 + i \| x - iy \|^2$
$= 4(x,y)$.
(polarization identity)

Solution:

i) $\| x + y \|^2 + \| x - y \|^2 = (x+y,x+y) + (x-y,x-y)$

$= (x,x)+(y,x) + (x,y)+(y,y) +(x,x)-(y,x) -(x,y) +(y,y)$

$= 2(x,x) +2(y,y)$

$= 2 \| x \|^2 +2\| y \|^2$

Hence (i) solved .

ii) $\| x + y \|^2 - \| x - y \|^2 = (x+y,x+y) - (x-y,x-y)$

$= (x,x)+(y,x) + (x,y)+(y,y) –[(x,x)-(y,x) -(x,y) +(y,y)]$

$= 2(y,x) +2(x,y)$

Hence (ii) solved .

iii) $\| x + y \|^2 + \| x - y \|^2 + i\| x + iy \|^2 + i \| x - iy \|^2$

$= (x+y,x+y) - (x-y,x-y) + i((x+iy,x+iy) - i(x-iy , x- iy)$

$= 2(y,x) +2(x,y) +i[(x,x) +i(y,x) –i(x,y)+(y,y)] -i[(x,x) - i(y,x) +i(x,y)+(y,y)]$

$= 2(y,x) +2(x,y) -2(y,x) +2(x,y)$

$= 4(x,y)$

Hence (iii) solved

Problem:
Every inner product space is a normed linear space

Solutions:

Let V be an inner product space. In order to prove that it is a normed linear space it must satisfy the properties of normed linear space.

If $x \in V$ then $\| x \|^2 = (x,x)$

By the definition of an inner product space we know that

i) $(x,x) \geq 0$ & $(x,x) = 0 \Leftrightarrow x = 0$.

Hence $\| x \|^2 \geq 0$ & $\| x \|^2 = 0 \Leftrightarrow x = 0$.

ii) $\|\alpha x \|^2 = (\alpha x, \alpha x) = \alpha\overline{\alpha} (x,x)$
$= | \alpha |^2 (x,x)$
$= | \alpha |^2 \| x \|^2$

$\|\alpha x \|^2 = | \alpha | \| x \|$

Hence (ii)

iii) $\| x + y \|^2 = (x+y, x+y)$
$= (x,x) +(y,x) +(x,y)+(y,y)$

$= \| x \|^2 + \| y \|^2 + \overline{(x,y)}+(x,y)$

$= \| x \|^2 + \| y \|^2 +2 \operatorname{Re} (x,y)$

$=\| x \|^2 + \| y \|^2 + 2(x,y)$

$= \| x \|^2 + \| y \|^2 + 2 \| x \| \| y \|.$

$\| x + y \|^2 \leq \| x \|+ \| y \|$

Hence (iii) Solved .
It shows that every inner product space is a normed linear space.

Theorem:

A closed convex subset "C" of a Hilbert space H contains a unique vector of smallest norm.

Proof:

Let d= inf{ $\| x \|$: $x \in c$} then there exist a $\{x_n\}$ such that $\| x_n \| \to d$.

Consider two vectors x_n , $x_m \in \{x_n\}$ .Since c is aconvex subset of H . $\therefore x_n, x_m \in C$.

$\Rightarrow (x_n + x_m)/2 \in C$. By the definition of d we have
$\| (x_n + x_m)/2 \| \geq d$. so that $\| (x_n + x_m) \| \geq 2d$.

By the parallelogram law we have

$\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2$

$\Rightarrow \| x_m + x_n \|^2 + \| x_m - x_n \|^2 = 2\| x_m \|^2 + 2\| x_n \|^2$

$\Rightarrow \| x_m - x_n \|^2 = 2\| x_m \|^2 + 2\| x_n \|^2 - \| x_m + x_n \|^2$

$\leq 2\| x_m \|^2 + 2\| x_n \|^2 - 4d^2$

Now,
$2\| x_m \|^2 + 2\| x_n \|^2 - \| x_m + x_n \|^2 \to 2d^2 + 2d^2 - 4d^2 = 0$.

Hence, $\| x_m - x_n \|^2 \to 0$ as $m,n \to \infty$.

$\therefore$ $\{x_n\}$ is a Cauchy sequence in c.

Since H is complete and c is a closed subset of H. $\therefore$ c is also complete Hence the Cauchy sequence $\{x_n\}$ in c is also a Cauchy sequence in c.

Therefore there exist a vector x in c such that $x_n \to x$.
Now, $x = \lim x_n$.

$\| x \| = \| \lim x_n \| = \lim \| x_n \| = d$.

Hence x is avector in c with the smallest norm d.

To prove uniqueness of x:

Suppose x’ is a vector in c other than x, which also has norm d.Then $(x+x')/2 \in c$ & by the parallelogram law .

We have $\| (x + x')/2 \|^2 = \| x \|^2/2 + \| x' \|^2/2 - \| x - x' \|^2/2$

$< \| x \|^2/2 + \| x' \|^2/2 = d^2$

Which contradicts the definition of d.

Hence x is unique.

Hence the proof.

Theorem:

If B is a complex Banach space whose norm obeys the parallelogram law and if an inner product is defined on " B" by
$4(x,y)= \| x + y \|^2 - \| x - y \|^2 + i\| x + iy \|^2 - i \| x - iy \|^2$.

Proof:

We have to show that the inner product defined above has three properties required by the definition of a Hilbert space namely,

i) To prove $(x.x)= \| x \|^2$

$$4 (x,x) = \| x + x \|^2 - \| x - x \|^2 + i\| x + ix \|^2 - i \| x - ix \|^2.$$

$$= 4\| x \|^2 +2i \| x \|^2 – 2i\| x \|^2$$

$$= 4 \| x \|^2$$

Hence (i) proved.

ii) To prove $(x,y) = 4 \overline{(y,x)}$

$$4(x,y) = 4 \overline{(y,x)}$$

$$4(y,x) = \| y + x \|^2 - \| y - x \|^2 + i\| y + ix \|^2 - i \| y - ix \|^2$$

Then ,

$$4(y,x) = \| x + y \|^2 - \| x - y \|^2 + i\| x + iy \|^2 - i \| x - iy \|^2$$

$$4\overline{(y,x)} = \| x + y \|^2 + \| x - y \|^2 - i\| x - iy \|^2 + i \| x + iy \|^2$$

$$= 4(x,y)$$

Then $(x,y) = \overline{(y,x)}$.

Hence (ii) proved.

iii) $(x+y, z) = (x,z) + (y,z)$

$$4(x+y,z) = \| x + y + z\|^2 - \| x - y - z\|^2 + i\| x + y + iz \|^2 - i \| x + y - iz \|^2$$

$$= 4\, re(x+y,z) + i4\, im(x+y,z) \quad \ldots\ldots..(1)$$

$$4(x,z) = \| x + z \|^2 - \| x - z \|^2 + i\| x + iz \|^2 - i \| x - iz \|^2$$

$$= 4\, re(x,z) + i4\, im(x,z) \quad \ldots(2)$$

Similarly,

$$4(y,z) = 4\, re(y,z) + i4\, im(y,z) \quad \ldots(3)$$

Now, (2) +(3)

$$\Rightarrow 4(x,z) + 4(y,z) = 4\, re(x+y,z) + i4\, im(x+y,z) \quad ..(4)$$

From (1) & (4)

$$4(x+y, z) = 4(x,z) + 4(y,z)$$

Hence (iii) proved.

iv) $(\alpha x, y) = \alpha (x,y)$

$$4(\alpha x , y) = \| \alpha x + y \|^2 - \|\alpha x - y \|^2 + i\| \alpha x + iy \|^2 - i \|\alpha x - iy \|^2$$

$$= |\alpha| [\| x + y \|^2 - \| x - y \|^2 + i\| x + iy \|^2 - i \| x - iy \|^2]$$

$$= \alpha [4(x,y)]$$

Hence (iv) proved.

Then B is a Hilbert space .

Hence the proof.

Orthogonal:

Two vectors x and y in a Hilbert space H are said to be orthogonal (written $x \perp y$) if $(x,y) = 0$ i.e., $x \perp y$ [this $\perp$ symbol is read as related] .

Remark:

1. The relation of orthogonality in a Hilbert space is symmetry.
2. If x is orthogonal to y then every scalar multiple is $\perp$ y.
3. The zero vector is orthogonal to every vector.
4. The zero vector is the only vector which is orthogonal to itself.

Result:Pythogorian theorem

If x and y are any two orthogonal vectors in a Hilbert space H then we can show that

$\| x + y \|^2 = \| x - y \|^2 = \| x \|^2 + \| y \|^2$

Proof:

$\| x + y \|^2 = (x+y, x+y)$

$= (x,x) + (y,x) + (x,y) + (y,y)$

$= \| x \|^2 + \| y \|^2 + 0 + 0$

[since $x \perp y$ i.e., x,y=0]

$= \| x \|^2 + \| y \|^2$

Similarly, $\| x - y \|^2 = \| x \|^2 + \| y \|^2$

Hence proved.

Definition:

Let S be a nonempty subset of a Hilbert space H the orthogonal compliment of S written as $S^{\perp}$ is defined by

$S^{\perp} = \{ x \in H : x \perp y \quad \forall\, y \in S \}$

Theorem:

The following statement follows directly from the orthogonal compliment of the set definition.

i) $\{0\}^{\perp} = H$

ii) $H^{\perp} = \{0\}$

iii) $S \cap S^{\perp} \subseteq \{0\}$

iv) $S_1 \subseteq S_2 \Rightarrow S_1^{\perp} \supseteq S_2^{\perp}$.

v) $S^{\perp}$ is aclosed linear subspace of H, $\alpha x_1 + \beta x_1 \in S^{\perp}$.

vi) $S \subset S^{\perp\perp}$

Proof:

$S^{\perp} = \{x \in H / x \perp y\}$

i) To prove $\{0\}^{\perp} = H$

$$\{0\}^{\perp} = \{x \in H / x \perp 0\}$$
$$= \{x \in H / (x ,0) = 0\}$$
$$= H .$$

ii) To prove $H^{\perp} = \{0\}$

Let $x \in H^{\perp}$ then by definition $(x,y) = 0 \ \forall \ y \in H$.

Taking $y=x$, $(x,x) = 0$

$\Rightarrow \| x \|^2 = 0 \Rightarrow \| x \| = 0$

$\Rightarrow x \in \{0\}$

Then, $H^{\perp} = \{0\}$

iii)To prove $S \cap S^{\perp} \subseteq \{0\}$

Let $x \in S \cap S^{\perp} \Rightarrow x \in S$ &

$x \in S^{\perp} / (x,y) = 0 \ \forall \ y \in S$.

If S is orthogonal to x itself, then $(x,x) = 0 \Rightarrow \| x \|^2 = 0 \Rightarrow x \in \{0\}$.

Then, $S \cap S^{\perp} \subseteq \{0\}$

iv)To prove $S_1 \subseteq S_2 \Rightarrow S_1^{\perp} \supseteq S_2^{\perp}$

Let $x \in S_2^{\perp} \Rightarrow$ x is orthogonal to every vector in S_2.

$\Rightarrow$ x is orthogonal to every vector in S_1.

$\Rightarrow x \in S_1^{\perp}$

Then , $S_1^{\perp} \supseteq S_2^{\perp}$

v)To prove $S^{\perp}$ is aclosed linear subspace of H, $\alpha x_1 + \beta x_1 \in S^{\perp}$.

Let $x_1, x_2 \in S^{\perp}$ then $(x_1 , y) = 0$ & $(x_2, y) = 0 \quad \forall y \in S$.

$(\alpha x_1 + \beta x_2 , y) = \alpha(x_1, y) + \beta(x_2 , y)$
$= 0$
$\therefore (\alpha x_1 + \beta x_2 , y) \in S^{\perp}$.

To prove $S^{\perp}$ is closed:

Let x be a limit point of $S^{\perp}$. Then, to prove $x \in S^{\perp}$. By definition of limit point there exist $\{x_n\}$ in $S^{\perp}$ such that $\{x_n\} \rightarrow x$ .i.e., $(x_n, y) = 0 \quad \forall y \in S$.

$|(x_n - y) - (x, y)| = |(x_n - x, y)|$

$\leq \| x_n - x \| \| y \| \rightarrow 0$

$\lim(x_n , y) = (x, y)$

$\Rightarrow x \in S^{\perp}$.

vi) To prove $S \subset S^{\perp\perp}$

$S^{\perp\perp} = \{x / (x, y) = 0 \quad \forall y \in S^{\perp}\}$

If $x \in S$, then $(x, z) = 0 \ \forall z \in S^{\perp}$.

$\Rightarrow x \in S^{\perp\perp}$.

Hence the proof.

Theorem:
Let M be a closed linear subspace of a Hilbert space H. Let x be a vector not in M & let d be the distance from x to M. Then there exist a unique vector $y_0 \in M$ such that
$\| x - y_0 \| = d$.

Proof:

Since M is closed, the set c= x+M is a closed convex set.

To prove c is closed:

Let {x+y} be a limit point of x+M then to prove
{x+y} $\in$ x+M.

There exist a {x+x_n} in x+M such that {x+x_n } $\rightarrow$ x+y.

Now {x_n } is a sequence in M and {x_n }$\rightarrow$ y.
But M is closed.

Let y$\in$M . Thus {x+y} $\in$ x+M.

Since d is the distance from x to M , dis the distance from the origin to c.

By previous theorem there exist a unique vector z_0 in c such that $\| z_0 \|$ =d**.**

Now, the vector y_0 = x- z_0 is in M and
$\| x - y_0 \| = \| z_0 \|$ =d**.**

Uniqueness of y_0:

It follows from the fact that y_1 is a vector in Msuch that
$y_1 \neq y_0$ and $\| x – y_1 \|$ =d**.** then z_0 =x-y_0 is in c such that
$z_1 \neq z_0$ and $\| z_1 \|$ =d.

Which contradict the uniqueness of z_0.

Hence the proof.

Theorem:
If M is a proper closed linear subspace of a Hilbert space then there exist a non-zero vector $z_0 \in$ H, $z_0 \perp$ M.

Proof:

Let x be avector not in Mand let d be the distance from x to M.

By previous theorem there exist a unique vector $y_0 \in$ M such that $\| x - y_0 \|$ =d.

We define z_0 by z_0 =x-y_0 .

$\| z_0 \| = \| x - y_0 \| = d$.

$\Rightarrow z_0$ is anon zero vector.

We conclude the proof by showing that if y is an arbitrary vectorin M, then $z_0 \perp y$.

For any scalar α we have

$\| z_0 - \alpha y \| = \| x - (y_0 + \alpha y) \| \geq d = \| z_0 \|$.

$\Rightarrow \| z_0 - \alpha y \|^2 - \| z_0 \|^2 \geq 0$.

$\Rightarrow (z_0 - \alpha y, z_0 - \alpha y) - (z_0, z_0) \geq 0$

$\Rightarrow (z_0, z_0) - \alpha\overline{(z_0, y)} - \bar{\alpha}(z_0, y) + \alpha\bar{\alpha}(y,y) - (z_0, z_0) \geq 0$ …………(1)

It is true for every scalar α.

Let $\alpha = \beta (z_0, y)$ where β is an arbitrary real number.

Then $\bar{\alpha} = \beta \overline{(z_0, y)}$ sub the values of α and $\bar{\alpha}$ in (1) we have

$\beta \,|(z_0, y)|^2 \geq 0$. ……………..(2)

The equation (2) is true for real β.

Suppose $(z_0, y) \neq 0$. Then taking β as positive and so small that $\beta \| y \|^2 < \alpha$,

$\therefore$ We must have $(z_0, y) = 0$ which means that $z_0 \perp M$.

Hence the proof.

Theorem:

If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, Then the linear subspace M+N is closed.

Proof:

Let Z be limit point of M+N such that $Z \in M+N$ such that $Z_n \to Z$.

Since Z is alimit point of M+N there exist a {Zn} in M+N such that Zn$\rightarrow$Z.

Since M $\perp$ N, M$\cap$ N = {0}. i.e., M and N are disjoint so each Zn can be written uniquely in the form.

Zn = {x_n +y_n} where $x_n \in$M and $y_n \in$N. Consider two vectors { Zm = x_m + y_m & Zn = x_n + y_n } $\in${Zn}.

Let us consider ,
Zm-Zn =(x_m - x_n)+ (y_m - y_n)

Where x_m - $x_n \in$ M and y_m - $y_n \in$ N.

And M $\perp$ N. $\therefore$(x_m - x_n) $\perp$ (y_m - y_n) .

Then by the pythogorian theorem we have

$\|$ Zm-Zn$\|^2$ =$\|$ (x_m - x_n) $\|^2$+ $\|$ (y_m - y_n) $\|^2$.

Now {Zn } is a Cauchy sequence in H. Every convergent sequence is a cauchy sequence.

$\therefore$we have $\|$ Zm-Zn$\|^2 \rightarrow 0$ as m,n $\rightarrow \infty$.

$\Rightarrow \|$ (x_m - x_n) $\|^2 \rightarrow 0$ $\|$ (y_m - y_n) $\|^2 \rightarrow 0$.

$\Rightarrow$ {x_n } &{y_n } are the Cauchy sequence in M & N respectively.

Since, H is complete , M & N are closed subspaces of H.

$\therefore$M & N are complete . Hence the Cauchy sequence x_n & y_n in M& N are convergent sequences in M & N.

Then there exist a sequence x & y in M & N such that $x_n \rightarrow$ x & $y_n \rightarrow$ y.

Now, Z= lim Zn

= lim(x_n +y_n)

= x+y $\in$ M+N.

Thus if Z is alimit point of M+N then Z$\in$ M+N.

$\therefore$ M +N is closed.

Hence the proof.

Theorem : **(Orthogonal decomposition theorem)**

If M is a closed linear subspace of a hilbert space H then H is the direct sum of M & $M^{\perp}$ i.e., H= M+ $M^{\perp}$.

Proof:

Since M & $M^{\perp}$ are orthogonal , closed linear subspace of H, then by previous theorem shows that M and $M^{\perp}$ is also a closed subspace of H.

We must prove that M+ $M^{\perp}$ = H.If possible let we assume that M+ $M^{\perp} \neq$ H.

Then , M+ $M^{\perp}$ is a proper , closed linear subspace of H. Hence by theorem" If M is a proper closed linear subspace of a Hilbert space then there exist a non-zero vector $z_0 \in H$, $z_0 \perp M$."

There exist a vector $z_0 \neq 0$ in H, $z_0 \perp M + M^{\perp}$.
i.e., $(Z_0 , x+y) =0$ where $x \in M$ and $y \in M^{\perp}$.

(or) $(Z_0 ,x) =0$ & $(Z_0 ,y) =0$

(or) $Z_0 \in M^{\perp}$ & $Z_0 \in (M^{\perp})^{\perp} = M^{\perp\perp}$.

$\therefore$ $Z_0 \in M^{\perp} \cap M^{\perp\perp} =\{0\}$. This is not possible as Z_0 is anon-zero vector.

Thus we conclude that $M+M^{\perp}$ is not a proper closed linear subspace of H.

$\therefore$ $M + M^{\perp} = H$.

Since $M\cap M^{\perp} =\{0\}$, H is a direct sum of M & $M^{\perp}$. i.e., $H= M \oplus M^{\perp}$.

Hence the proof.

Theorem:

If M is a linear subspace of a Hilbert space , Show that it is closed iff $M=M^{\perp\perp}$.

Proof:

Let us assume that

$$M=M^{\perp\perp} = (M^{\perp})^{\perp} = S^{\perp} \quad \text{where } S=M^{\perp}.$$

$S^{\perp}$ is aclosed subspace of H. M is a closed linear subspace of H.

Conversely, M is a closed subspace of H.
Claim: $M=M^{\perp\perp}$

$M \subset M^{\perp\perp}$.

Assume that the inclusion $M \subset M^{\perp\perp}$ is proper $M \neq M^{\perp\perp}$.

M is a proper closed linear subspace of $M^{\perp\perp}$.

Hence by theorem" If M is a proper closed linear subspace of a Hilbert space then there exist a non-zero vector $z_0 \in H$, $z_0 \perp M$."

There exist a vector $z_0 \neq 0$ in $M^{\perp\perp}$, $z_0 \perp M^{\perp}$.

$\Rightarrow Z_0 \in M^{\perp} \cap M^{\perp\perp} = \{0\}$.

There exist a contradiction .

Then $M = M^{\perp\perp}$.

Hence the proof.

Orthonormal set:

A non-empty set $\{e_i\}$ of a Hilbert space H is said to be an orthonormal set if

i) $i \neq j \Rightarrow e_i \perp e_j$ (i.e.,) $(e_i, e_j) = 0 \quad \forall\, i \neq j$

$= 1 \quad \forall\, i=j$

ii) $\| e_i \| = 1 \quad \forall\, i$

Theorem:
Let $\{e_1, e_2, \ldots\ldots e_n\}$ be a finite orthonormal set in a Hilbert space H . If x is any vector in H, then

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \| x \|^2 \ldots\ldots\ldots\ldots\ldots(1)$$

Further,

$$x - \sum_{i=1}^{n} (x, e_i)e_i \perp e_j \text{ for each j.}$$

[Bessel's inequality for finite orthonormal set]

Proof:

We have

$$0 \leq \| x - \sum_{i=1}^{n} (x, e_i) e_i \|^2$$

$$= (x - \sum_{i=1}^{n} (x, e_i) e_i . x - \sum_{j=1}^{n} (x, e_j) e_j)$$

$$= \| x \|^2 - \sum_{i=1}^{n} (x, e_i)\overline{(x,e_i)} - \sum_{i=1}^{n} (x, e_j) \overline{(x,e_j)} + \sum_{i=1}^{n} (x, e_i)\overline{(x,e_i)}$$

$$0 \leq \| x \|^2 - \sum_{j=1}^{n} | (x, e_j)|^2$$

which gives

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \| x \|^2$$

To conclude the proof, we observe that

$$(x - \sum_{i=1}^{n} (x, e_i) e_{i,} e_j) = (x,e_j) - \sum_{i=1}^{n} (x, e_i)(e_i,e_j)$$

$$= (x,e_j) – (x,e_j) =0$$

$$\Rightarrow \quad x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j \text{ for each j.}$$

Hence the proof.

Theorem:

If $\{e_i\}$ is an orthonormal set in a Hilbert space, H and if X is any vector in H then the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable.

Proof:

For each positive integer n, consider the set

$Sn = \{e_i : \quad |(x, e_i)|^2 \quad > \| x \|^2 /n \}$

By Bessel's inequality Sn contain atmost n-1 vectors .For if Sn contains say n vectors $\{e_1, e_2, \ldots\ldots e_n\}$ then

$|(x, e_i)|^2 \quad > \| x \|^2 /n$ for each $i=1,2\ldots\ldots n$

Adding up we get

$\Rightarrow |(x, e_i)|^2 \quad > n \| x \|^2 /n$

$\Rightarrow |(x, e_i)|^2 \quad > \| x \|^2$

This contradicts

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \| x \|^2$$

Thus for each positive integer n the set Sn is finite.Now suppose, the set $\{e_i\} \in S$ then $(x,e_i) \neq 0$. However small be the value of $|(x, e_i)|^2$, we can take n so large that

$|(x, e_i)|^2 \quad > \| x \|^2 /n.$

If $\{e_i\} \in S$ then e_i must belong to some Sn. So we can

write $S = \bigcup_{n=1}^{\infty} Sn.$

Hence S is expressed as a countable union of finite set.

$\therefore$ S itself is a countable set.

If we have $(x,e_i) = 0$ $\forall$ I then S is empty.

Hence the proof.

Theorem: Bessel's inequality

If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then
$\Sigma |(x, e_i)|^2 \leq \|x\|^2$ for every vector x in H.

Proof:

Let $S = \{e_i : (x,e_i) \neq 0\}$ >By the previous theorem either S is empty (or) it is countable.

If S is empty, then we have $(x,e_i) = 0$ $\forall$ i. In this case we define $\Sigma |(x, e_i)|^2$ to be the number 0 and we have
$0 \leq \|x\|^2$ .Thus if S is empty then we have

$$\Sigma |(x, e_i)|^2 \leq \|x\|^2$$

Now,we assume that S is not empty,. Then either S is finite or it is countably finite. If S is finite then we can write
$S = \{e_1, e_2, \ldots\ldots\ldots e_n\}$ for some positive integer n.

In this case we define

$$\Sigma |(x, e_i)|^2 = \sum_{i=1}^{n} |(x, e_i)|^2$$ which is $\leq \|x\|^2$ by bessel's inequality.

For Finite case:

Finally, assume that S is countably infinite. Let the vectors in S be arranged in a definite order
$S = \{e_1, e_2, \ldots\ldots\ldots e_n, \ldots\ldots\ldots\}$.By the theory of absolutely convergent series if $\sum_{n=1}^{\infty} |(x, e_n)|^2$ converges then every series obtained from this by rearranging its term also converges and all series have the same sum.

We define therefore $\Sigma |(x, e_i)|^2$ to be
$\Sigma |(x, e_n)|^2$ …..(1).

Hence this sum will depend only on the set S and not on the rearrangement of its vectors.

Now by the bessels inequality in the finite case, we have

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2 \ldots\ldots\ldots\ldots..(2)$$

For various values of n, the sum on the left side of (2) are non negative . So they form a monotonic increasing sequence. Since this sequence is bounded above by $\|x\|^2$
It converges.

Since this sequence is the sequence of the partial sum of the series on the right side of (1) it converges and we have

For $e_i \in S$,

$$\sum |(x, e_i)|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 \leq \|x\|^2$$

Hence the proof.

Complete:

An orthonormal set $\{e_i\}$ in a Hilbert space H is complete if it is not possible to adjoint a vector e to $\{e_i\}$ in such a way that $\{e_i, e\}$ is an orthonormal set which properly contains $\{e_i\}$.

Theorem:
Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H then the following conditions all are equivalent to one another.

i) $\{e_i\}$ is complete.
ii) $x \perp \{e_i\} \Rightarrow x=0$
iii) If x is an arbitrary vector in H then
$x = \sum (x, e_i) e_i$.
iv) If x is an arbitrary vector in H then
$\|x\|^2 = \sum |(x, e_i)|^2$.

Proof:
(i)$\Rightarrow$ (ii)
If (ii) is not true there exist a vector $x \neq 0$ such that $x \perp \{e_i\}$. we now define e by $e = x/\|x\|$ & we observe that $\{e_i, e\}$ is an orthonormal set which properly contains $\{e_i\}$.

This contradicts the completeness of e_i.

(ii) $\Rightarrow$ (iii)

It is given that $x \perp \{e_i\} \Rightarrow x=0$. we have to show that if x is an arbitrary vector in H then $x = \sum (x, e_i) e_i$.

By the previous theorem the vector $x - \sum (x, e_i) e_i$ is orthogonal to every vector in the set e_i
i.e., $x - \sum (x, e_i) e_i \perp e_i$.

Therefore by hypothesis

$$x - \sum (x, e_i) e_i = 0 \Rightarrow x - \sum (x, e_i) e_i.$$

(iii) $\Rightarrow$ (iv)

It is given that for any vector $x \in H$. We have
$x = \sum (x, e_i) e_i$. We have to prove that $\| x \|^2 = \sum | (x, e_i)|^2$

$$\| x \|^2 = (x,x)$$
$$= (\sum (x, e_i)e_i , \sum (x, e_j)e_j)$$
$$= \sum \sum (x, e_i) \overline{(x,e_j)} (e_i, e_j)$$
$$= \sum | (x, e_i)|^2$$

(iv) $\Rightarrow$ (i)

Suppose $\{e_i\}$ is not complete. Then $\{e_i\}$ is a proper subset of an orthonormal set $\{e_i , e\}$. By hypothesis we have

$$\| e \|^2 = \sum | (e, e_i)|^2 =0$$

Since $e \perp e_i$ for each i.

Now, $\| e \|^2 = 0$ which contradicts the fact that e is a unit vector.

$\therefore$ The orthonormal set $\{e_i\}$ must be complete.

Hence the proof.

Conjugate space H*:

Let H be a Hilbert space . A continuous linear transformation from H into c is called a continuous linear functional or more briefly a functional of H.

The elements of H* are called continuous linear functional

H* ={ f/f :H$\rightarrow$ C}

Theorem:
Let y be a fixed vector in a Hilbert space H. Let f_y be a scalar valued function defined on H as $f_y(x) = (x,y)$ for every $x \in H$. Show that f_y is functional in H* and
$\| f_y \| = \| y \|$.

Proof:
Since inner product (x,y) is a scalar , clearly $f_y : H \rightarrow C$.

To prove that f_y is functional on H.we must show that f_y is linear and continuous.

i)To prove f_y is linear:
Let $x_1, x_2 \in H$ and $\alpha, \beta \in C$.
We have
$f_y(\alpha x_1+\beta x_2) = (\alpha x_1+\beta x_2 ,y)$

$= (\alpha x_1,y) +(\beta x_2 ,y)$

$= \alpha f_y(x_1) +\beta f_y(x_2)$

$\Rightarrow f_y$ is linear

ii) To prove f_y is continuous:

For every $x \in H$, $f_y(x) = (x,y)$.

$\Rightarrow | f_y(x) | =| (x, y)|$

$\leq \| x \| \| y \|$

Since y is a fixed vector in H.

Let $\| y \| =k$.

Then $| f_y(x) | \leq k \| x \| \quad \forall x \in H$.

$\Rightarrow f_y$ is bounded.

$\Rightarrow f_y$ is continuous.

f_y is norm preserving:

To prove that $\| f_y \| = \| y \|$

$\| f_y \| = \sup \{\| f_y (x)\| : \|x\| \le 1\}$

$\le \sup\{ \| x \| \| y \| : \|x\| \le 1\}$

$= \| y \| \sup\{ \| x \| : \|x\| \le 1\}$

Thus $f_y \le \| y \|$ …………..(1)

Now we show that the relation takes the form an equality. If y=0 then $\| y \| = 0$.

Also, if y=0 then $f_y (x) = (x,y) = (x,0) = 0 \ \forall \ x \in H$.

Then f_y is a zero functional & $\| f_y \| = 0$.

Thus if y=0; then $f_y = \| y \| = 0$.

Now let us take $y \ne 0$. then H is not a zero space .

We have $\| f_y \| = \sup \{\| f_y (x)\| : \|x\| \le 1\}$

Since $y \ne 0$; $y / \|y\|$ is a unit vector

Taking $x = y / \|y\|$.

we have $f_y \ \ge \ \| y \|$ ………….(2)

From (1) & (2) we have

$\| f_y \| = \| y \|$

Hence the proof.

Theorem : Riesz Representation theorem

Let H be a Hilbert space and let f be an arbitrary functional
In H then there exist a unique vector $y \in H$ such that
$f(x) = (x,y) \quad \forall$ x in H.

Proof:

First we shall show that if there exist a vector y such that $f(x) = (x,y) \quad \forall$ x in H.

Then y is necessarily unique.

Suppose y_1,y_2 are any two vectors satisfying the property we have
$f(x) = (x,y_1) \quad \forall$ x in H.
& $f(x) = (x,y_2) \quad \forall$ x in H.

$\therefore$ we have $(x,y_1) = (x,y_2)\ \forall$ x in H.

$\Rightarrow (x,y_1 - y_2) = 0 \quad \forall$ x in H .

$\Rightarrow y_1 - y_2 = 0 \Rightarrow y_1 = y_2$.

If f is a zero functional then $f(x) = 0\ \forall$ x in H.

Also, if y=0 then $f(x) = (x,y) = (x,0) = 0$.

If f is a zero functional then the vector y=0 such that

$f(x) = (x,y)\ \forall$ x in H.

Suppose f is not a zero functional. Let M be the null space of f. i.e., $M = \{x \ / \ f(x) = 0\}$. Then M is aproper subspace of H.Also the null space of any continuous linear transformation is closed.

Hence M is a proper closed linear subspace of a Hilbert space H.

We claim that for some suitably chosen scalar α, the vector $y = \alpha\, y_0$.

Case(i)

We take any value for scalar α in the vector $y = \alpha\, y_0$.
Satisfies the property (1) for every $x \in M$.

If $x \in M$ then $f(x) = 0$. Also if $x \in M$, then

$(x,y) = (x, \alpha\, y_0) = \bar{\alpha}(x,y_0) = 0$.

Thus if $x \in M$ & if $y = \alpha\, y_0$ then we have $f(x) = (x,y) = 0$.

Hence case (1) is satisfied.

Case(ii)
Let us try to choose the scalar α in such a way that

The vector $y=\alpha\, y_0$ satisfies equation(1) for $x=y_0$. Then

$f(y_0) =(y_0, \alpha\, y_0) = \bar{\alpha}\, (y_0, y_0) = \bar{\alpha}\, \| y_0 \|^2$.

Here we take $\bar{\alpha} = f(y_0) / \| y_0 \|^2$. then the veactor $y =\alpha\, y_0$ satisfies for every $x \in M$ & for every $x =y_0$ then it must satisfy (1) for every $x \in H$.

Let x be an arbitrary vector in H. Since $M \cap M^{\perp} =\{0\}$ and y_0 is anon zero vector belongs to $M^{\perp}$.

$\therefore\ y_0 \notin M$. Then

$\Rightarrow f(x) - \beta\, f(y_0) =0.$

$\Rightarrow x- \beta\, y_0 \in M$

$\Rightarrow x- \beta\, y_0 =m \in M$

Thus $x\in H \Rightarrow x = m+ \beta\, y_0$ where β is some scalar & $m \in M$. now,

$f(x) = f(m+ \beta\, y_0) = f(m) +\beta\, f(y_0) = (m,y) +\beta\, (y_0 , y)$

$= (m+ \beta\, y_0 ,y) = (x,y)$

Thus if a vector y satisfying (1) for every $x \in M$ & for every $x =y_0$ then it must satisfy (1) for every $x \in H$.

Hence $y = \alpha\, y_0$ the required vector where

$\alpha = \overline{f(y_0)} / \| y_0 \|^2$.

Hence the proof.

****

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Every inner product space is a……………	normed linear space	hilbert space	banach space	continuous	normed linear space
The ……….. is orthogonal to any vector.	product	scalar	zero vector	real value	zero vector
The relation of orthogonality in a Hilbert space is ………….	asymmetry	symmetry	abelian	commutate	symmetry
The zero vector is the only vector which is ………… to itself.	asymmetry	symmetry	orthogonal	direction	orthogonal
A complex banach space is said to be a ………….. if there is an inner product which satisfies the three conditions.	Banach space	hilbert space	Inner product space	linear space	hilbert space
For the space l_2^n we use cauchy inequality to prove …………inequality.	minkowski's	schwartz	triangle	cauchy triangle	schwartz
Two vectors x and y in a hilbert space H are said to be orthogonal if ……	(x,y)>1	(x,y)=0	(x,y)=1	(x,y)<1	(x,y)=0
If x is orthogonal to y then every scalar multiple is ………….. to y.	parallel	symmetry	orthogonal	perpendicular	perpendicular
The ………………. is orthogonal to every vector.	product	scalar	zero vector	real value	zero vector
The d is the distance from ……….. to c.	center	vertices	edges	origin	origin
If M is a closed linear subspace of ahilbert space H then H is the ………… of M and M perp	product	scalar	zero vector	direct sum	direct sum
If M and N are closed linear subspace of ahilbert space H such that M┴N then the linear subspace M+N is………	closed	open	open subset	open set	closed
The scalars in a Hilbert space are usually ……………. numbers.	Irrational	algebraic	complex	rational	complex
The distance property in inner product space is (ax+by, Z) =…………	a(x,z)+b(y,z)	a(x,x)+b(y,x)	a(x,z)-b(y,z)	a(x,z)+b(x,z)	a(x,z)+b(y,z)
The distance property in inner product space is (ax-by, Z) =…………	a(x,z)+b(y,z)	a(x,x)+b(y,x)	a(x,z)-b(y,z)	a(x,z)+b(x,z)	a(x,z)-b(y,z)
An orthonormal set cannot has an …………	product	scalar	zero vector	real value	zero vector
The set S is finite or…………………..	countable	uncountable	countably	countably	countably
The orthonormal set is either ………………. or countable.	countable	uncountable	finite	empty	empty
The orthonormal set is either empty or ……………..	countable	uncountable	finite	empty	countable
A nonempty set $\{e_i\}$ of a hilbert space H is said to be an orthonormal set if …………… for all i=j	$(e_i, e_j) >0$	$(e_i, e_j) =0$	$(e_i, e_j) =1$	$(e_i, e_j) <1$	$(e_i, e_j) =1$
A nonempty set $\{e_i\}$ of a hilbert space H is said to be an orthonormal set if ………… for all $i\neq j$	$(e_i, e_j) >0$	$(e_i, e_j) =0$	$(e_i, e_j) =1$	$(e_i, e_j) <1$	$(e_i, e_j) =0$
If H contains only the zero vector then it has no …………………	orthonormal set	orthonormal basis	Banach space	hilbert space	orthonormal set
If H contains a nonzero vector and if we normalised x then $\|e\|$=……	zero	four	five	one	one
If (x,y) are any two vectors in a Hilbert space then $\|(x,y)\|$ <=………..	$\|x\|\|y\|$	$\|x\|/\|y\|$	$\|x\|-\|y\|$	$\|x\|+\|y\|$	$\|x\|\|y\|$
The sum of Z and Z conjugate is equal to………	2 im Z	2 Re z	2 z	Re z	2 Re z
Every inner product space is expressed as a $\|x\|^2$ ……………	(x,y)>1	(x,x)	(y,y)	(y,x)	(x,x)
A close convex subset of a hilbert space H contains a unique vector of smallest ………………	metric	space	subset	norm	norm
A close ………. subset of a hilbert space H contains a unique vector of smallest norm.	concave	convex	linear	metric	convex
Parseval's equation is otherwise called as parseval's ………….	transform	fourier	identity	subscript	identity

Let x be anarbitrary vector in H the numbers (x,ei) are called the ,,,,,,,, coefficient of x.	parseval	fourier	schwartz	bessels	fourier
The set of all continuous linear functional on H is denoted by ……………	H	H**	H*	T*	H*

UNIT 4

Adjoint In Banach Spaces

Adjoint of an operator :

Let T be an operator on a Hilbert spaces H. We define the adjoint of T denoted by T* on H as follows whenever (x, y) ε H. We have

$$(Tx, y) = (x, T^*y)$$

The mapping T* is unique:

If T' is any mapping of H into itself such that

$$(Tx, y) = (x, T'y) \quad \forall (x, y) \in H.$$

Then, $(x, T'y) = (Tx, y) = (x, T^*y)$

$\Rightarrow (x, T'y) = (x, T^*y)$

$\Rightarrow (x, T'y - T^*y) = 0$

$\Rightarrow (x, (T' - T^*)y) = 0 \quad \forall\ x \in H.$

$\Rightarrow (T' - T^*)y = 0$

$\Rightarrow T' = T^*$.

Thus T is unique.

The adjoint mapping T* is linear and bounded :

Let y_1 and y_2 be any two vectors in H and let α, β be any 2 scalars . For any vector x ε H we have

$$(x, T^*(\alpha y_1 + \beta y_2)) = (Tx, (\alpha y_1 + \beta y_2))$$

$$= \bar{\alpha}(Tx, y_1) + \bar{\beta}(Tx, y_2).$$

$$= \bar{\alpha}(x, T^*y_1) + \bar{\beta}(x, T^*y_2).$$

$$= (x, \alpha T^*y_1) + (x, \beta T^*y_2)$$

$= (x, \alpha T^*y_1 + \beta T^*y_2)$

$\Rightarrow T^* (\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2$

$\Rightarrow$ T* is linear.

To prove T* is bounded:

For any vector y in H. we have

$||T^*y||^2 = (T^*y, T^*y)$

$=(TT^*y,y)$

$=| (TT^*y,y)|$

$\leq ||TT^*y|| .||y||$

$\leq ||T||. ||T^*y|| .||y||$

$\Rightarrow ||T^*y|| \leq ||T||. ||y||.$

$\therefore$ T is bounded .$||T|| \leq k$, where k is finite. Hence we

get $||T^*y|| \leq k||y|| \ \forall \ y \in H$.

$\Rightarrow$ T* is bounded.

$\Rightarrow$ T* is a bounded linear operators on H.

$\Rightarrow$ T* ε B(H) where B(H) is the set of all bounded linear operators on a Hilbert space H.

Theorem:

The adjoint operator T to T* on B(H) has the following properties.

i) $(T_1+T_2)^* = T_1^* + T_2^*$
ii) $(\alpha T)^* = \bar{\alpha}\ T^*$
iii) $(T_1 T_2)^* = T_2^* T_1^*$
iv) $T^{**} = T$
v) $\| T \| = \| T^* \|$

vi) $\| T^* T \| = \| T \|^2$

Proof:

i) $(T_1+T_2)^*$

$\Rightarrow (x, (T_1 +T_2)^* y)$

$= ((T_1+T_2) x,y)$

$= (T_1x,y) + (T_2 x,y)$

$= (x,T_1^*y) + (x,T_2^*y)$

$= (x, T_1^*y +T_2^*y)$

$= (x,(T_1^* + T_2^*)y)$

$(T_1+T_2)^* = T_1^* + T_2^*$

Hence (i) proved.

ii) $(x, (\alpha T)^* y)$

$= (\alpha Tx ,y)$

$= \alpha(Tx ,y)$

$= \alpha(x , T^*y)$

$= (x ,\bar{\alpha}T^*y)$

$(\alpha T)^* = \bar{\alpha} T^*$

Hence (ii) proved.

iii) $(x, (T_1 T_2)^* y)$

$= ((T_1T_2) x,y)$

$= (T_2 x ,T_1^* y)$

$= (xT_2^* , T_1^*y)$

$(T_1 T_2)^* = T_2^* T_1^*$

Hence (iii) proved

iv) $(x,T^{**}y) = (x, (T^*)^* y)$

$= (T^*x,y)$

$= \overline{(y, T^*x)}$

$= \overline{(Ty,x)}$

$= (x,Ty)$

$T^{**} = T$

Hence (iv) proved.

v) To prove $\| T \| = \| T^* \|$

We know that $\| T^* y \| \leq \| T \| \| y \| \ \forall \ y \ \varepsilon \ H$.

$\Rightarrow \| T^* \| \leq \| T \|$ …………(1)

Applying (1) to the operator T* we get
$\| (T^*)^* \| = \| T^{**} \| = \| T \| \leq \| T^* \|$ ………..(2)

From (1) & (2)

$\Rightarrow \| T \| = \| T^* \|$

Hence (v) proved

vi) To prove $\| T^* T \| = \| T \|^2$.

We have , $\| T^* T \| \leq \| T^* \| . \| T \|$

$= \| T \| . \| T \|$

$= \| T \|^2$

$\| T^* T \| \leq \| T \|^2$……………….(3)

Also,

$\| T(x) \|^2 = (Tx,Tx)$

$= (T^*Tx,x)$

$= |(T^*Tx,x)|$

$\leq \| T^* Tx \| . \| x \|$

$\leq \| T^* T \| . \| x \|^2$

$\Rightarrow \| T \|^2 \leq \| T^*T \|$ ………..(4)

From (3) & (4)

$\| T^* T \| = \| T \|^2$

Hence (vi) proved

Problem:

Show that the adjoint operaion on B(H) is 1 to 1 and onto.

Solution:

Let ϕ : B(H)$\rightarrow$ B(H) such that ϕ (T) =T*

ϕ is 1 $\rightarrow$ 1:

Let T_1 , T_2 $\in$B(H), then $\phi(T_1) = T_1^*$; $\phi(T_2) = T_2^*$;

$\phi(T_1) = \phi(T_2)$; $\Rightarrow T_1^* = T_2^*$

$\Rightarrow (T_1^*)^* = (T_2^*)^*$

$\Rightarrow T_1 = T_2$.

For every element T* $\in$B(H) then ϕ (T*) =(T*)* =T

Hence solved.

Problem:

Show that 0* =0 & I* =I.

Solution:

(0*x, y) =(x,0y) = (x,0) =0 = (0x,y)

$\Rightarrow$ 0*= 0.

(I*x, y) = (x,Iy) = (x, y) =(Ix,y).

$\Rightarrow$ I* = I.

Hence solved.

Problem:

If T is nonsingular operator on H then T* is also nonsingular then $(T^*)^{-1} = (T^{-1})^*$

Solution:

If T is nonsingular

Now T* is nonsingular when T is non singular, then

$TT^{-1} = T^{-1}T = I$

$(TT^{-1})^*$
$=(T^{-1}T)^*=I^*=I.$

$(T^{-1})^*T^* = T^*(T^{-1})^* =I.$

$\Rightarrow$ T* is non singular.

$(T^{-1})^* = (T^*)^{-1}.$

Hence solved.

Self Adjoint operator:

An operator T in B(H) is said to be self adjoint. If T= T*. clearly , 0 and 1 are self adjoint operators.

Theorem:
The self adjoint operators in B(H) form a closed real linear subspace of B(H) and therefore a real banach space which contains the identity transformation.

Proof:

Let S be the set of all self adjoint operators on a Hilbert space H.

To prove that S is aclosed real linear subspace of B(H). Let T_1 &T_2 $\in$ S then

$T_1^*= T_1$ and $T_2^* = T_2$.

For any α, β we have

$(\alpha T_1+ \beta T_2)^* = (\alpha T_1)^* +(\beta T_2)^*$

$= \bar{\alpha} T_1^* +\bar{\beta} T_2^*$

$= \bar{\alpha} T_1 + \bar{\beta} T_2$

$= \alpha T_1 + \beta T_2$

$\Rightarrow \alpha T_1 + \beta T_2 \in S$.
$\Rightarrow$ S is areal linear subspace of B(h) . next, we show that S is closed. Let A be a limit point of s. Then to show that
$A \in S$.

Since A is a limit point of S.there exist $\{A_n\}$ in S such that
$A_n \rightarrow A$.

We have $\| A- A^* \| = \| A- A_n +A_n -A^* \|$

$$\leq \| A- A_n \| + \| A_n -A^* \|$$

Since $A_n \in S$, $A_n^* = A_n$.

$$= \| A- A_n \| + \| A_n^* -A^* \|$$
$$= \| A- A_n \| + \| (A_n -A)^* \|$$
$$= \| A- A_n \| + \| A_n -A \|$$
$$=2 \| A- A_n \| . \rightarrow 0. \text{ as } A_n \rightarrow A.$$

$\Rightarrow A = A^*$ and so $A \in S$ .Then S is a closed real linear subspace of B(H) and hence s is a real banach space. Also
$I \in S$ as I is self adjoint .

Hence the proof.

Theorem:
If A_1 & A_2 are self adjoint operator on H then their product $A_1 A_2$ is self Adjoint iff $A_1 A_2= A_2A_1$.

Proof:
Let $A_1 A_2 = A_2 A_1$ and also it is given that $A_1^*= A_1$ &
$A_2^*= A_2$.

Now, $(A_1A_2)^* = A_2^*A_1^*$
$= A_2 A_1$.
$= A_1 A_2$.

Conersely, let $A_1 A_2$ be a self adjoint and show that they commute.

By hypothesis, $(A_1A_2)^* = A_1 A_2$ ………(1)

But $(A_1A_2)^* = A_2^*A_1^* = A_2 A_1$…………..(2)

From (1) & (2) we have

$$A_1 A_2 = A_2 A_1$$

Hence the proof.

Theorem:

If T is an operator on H for which $(T_x,x) = 0$, $\forall x$ in H iff T = 0.

Proof:

Suppose T=0 then $x \in H$.

We have, $(Tx,x) = (0x,x)$
$= (0,x)$
$= 0.$
$\therefore (Tx,x) = 0.$

Given that T is an operator on h, for which $(Tx,x)=0$, $\forall\, x \in H$.

To prove : T is zero operator on H.

If α,β be any 2 scalars and x,y are anu two vectors in H.

Then we have ,

$(T(\alpha x+\beta y), \alpha x+\beta y) = (\alpha\, Tx + \beta\, Ty, \alpha x+\beta y)$

$= (\alpha\, Tx, \alpha\, x) + (\alpha\, Tx, \beta y) + (\beta Ty, \alpha\, Tx) + (\beta Ty, \beta y)$

$= |\alpha|^2 (Tx,x) + \alpha\bar{\beta}(Tx,y) + \beta\bar{\alpha}(Ty,x) + |\beta|^2 (Ty,y)$

By hypothesis , $(Tx,x) = 0 \ \forall\, x \in H$.

$\alpha\bar{\beta}(Tx,y) + \beta\bar{\alpha}(Ty,x) = 0$ …………(1)

Put $\alpha = 1$, $\beta = 1$ in (1) we have,

$(Tx,y) + (Ty,x) = 0.$ ……………(2)

Put $\alpha = i$, $\beta = 1$ in (1) we have,

$i(Tx,y) - i(Ty,x) = 0.$ ……………..(3)

(2) x i , we get

$i(Tx,y) + i(Ty,x) = 0$ …………(4)

(3) +(4)

$2i(Tx,y) = 0 \ \forall\, x, y \in H.$

$(Tx,y) = 0\ \forall\ x, y \in H$.

$\therefore\ T = 0$

$\therefore$ T is a zero operator on H.

Hence the proof.

Theorem:

An operator T on a Hilbert space H is self adjoint if (Tx,x) is real $\forall x$.

Proof:

Let T be self adjoint then $T = T^*$.

To prove:

(Tx,x) is real.

Now, $(Tx,x) = (x,T^*x)$

$= (x,Tx)$

$(Tx,x) = (Tx,x)$

$\therefore$ (Tx,x) is real.

Conversely, Let (Tx,x) is real $\forall x$.

To prove: T on H is self adjoint.

- $(Tx,x) = (Tx,x)$

$= \overline{(x,T^*x)}$

$= \overline{(T^*x,x)}$

$(Tx,x)-(T^*x,x) = 0\ \forall\ x \in H$.

$((T-T^*)x,x) = 0$

$T-T^* = 0$

$T = T^*$

T on H is self adjoint.

Hence the proof.

Normal Operator:

An operator N on H is said to be normal if it commutes with its adjoint.

i.e., NN*=N*N

Remark:

Every self adjoint operator is normal. Since T is self adjoint then T= T*, we have TT*=T*T is true so that T is normal.

Theorem:

The set of all normal operators on H is a closed subset of B(H) which contains the set of all self adjoint operators and is closed under scalar multiplication.

Proof:

Let M be the set of all normal operators on ahilbert space H.

To prove: H is a closed subspace of B(H).

Let N be a limit point of M. We have to show that $N \in M$.Since N is a limit point of msuch that a sequence $\{N_k\}$ of disjoint points of M such that $N_k \to N$ as $K \to \infty$

Consider ,

$$\| NN^* - N^*N \|$$
$$= \| NN^* - N_kN_k^* + N_kN_k^* - N_k^*N_k + N_k^*N_k - NN^* \|$$

$$\leq \| NN^* - N_kN_k^* \| + \|N_kN_k^* - N_k^*N_k\| + \|N_k^*N_k - NN^* \|$$

Hence || NN*-N*N|| =0

$\Rightarrow$ NN*=N*N and so N$\in$M.

$\Rightarrow$M is a closed subset of B(H). we know that every self adjoint operator is normal.

$\therefore$ M is closed subset of B(H) which contains the set of all self adjoint operators.

To prove: H is closed under scalar multiplication.

i.e., If α is a scalar & N $\in$M.then αN $\in$ H.

$(\alpha N)(\alpha N)^* = (\alpha N)(\bar{\alpha} N^*) = (\bar{\alpha}\alpha)(NN^*)$

$(\alpha N)^*(\alpha N) = (\bar{\alpha} N^*)(\alpha N) = (\bar{\alpha}\alpha)(N^*N)$

Since , N is normal we get,

$(\alpha N)(\alpha N)^* = (\alpha N)^*(\alpha N)$

This proofs if N is normal , (αN) is also normal

for any scalar.

Hence M is closed under scalar multiplication.

Hence the proof.

Theorem:

If N_1 and N_2 are normal operators on H with the property that either commutates with the adjoint of the other, then

i) $N_1 + N_2$

ii) N_1 and N_2 are normal

Proof:

Given that N_1 &N_2 are normal operators.

$\therefore N_1 N_1^* = N_1^* N_1$ and

$N_2 N_2^* = N_2^* N_2$

$N_1 N_2^* = N_2^* N_1$

And $N_1^* N_2 = N_2 N_1^*$

i)$(N_1 + N_2) (N_1 + N_2)^*$

$= N_1 N_1^* + N_2 N_1^* + N_1 N_2^* + N_2 N_2^*$

$= N_1^*(N_1 + N_2) + N_2^*(N_1 + N_2)$

$= (N_1 + N_2)^* (N_1 + N_2)$

$\Rightarrow (N_1 + N_2)$ is normal.

ii) $(N_1N_2)(N_1N_2)^* = (N_1N_2)(N_2^*N_1^*)$
$= N_1(N_2N_2^*) N_1^*$
$= (N_2^* N_1^*)(N_1 N_2)$
$= (N_1N_2)^*(N_1N_2)$

$\Rightarrow (N_1N_2)$ is normal

Hence the proof.

Theorem:
An operator T on H is normal iff $\| T^*x \| = \| Tx \| \ \forall x$.

Proof.:

$\| T^* x \| = \| Tx \| \Leftrightarrow \| T^* x \|^2 = \| Tx \|^2$

$\Leftrightarrow (T^* x, T^*x) = (Tx,Tx)$

$\Leftrightarrow (TT^*x,x) = (T^*Tx,x)$

$\Leftrightarrow ((TT^* - TT^*)x,x) =0$

$\Leftrightarrow TT^* - TT^* =0$

$\Leftrightarrow TT^* = TT^*$

Hence the proof.

Theorem:
If N is a normal operator on H, then $\| N^2 \| = \| N \|^2$.

Proof:
We have $\| Tx \| = \| T^*x \| \ \forall x$

Replace T by N & Nx in place of x. we have,

$\| N(Nx) \| = \| N^*(Nx) \|$

$\Rightarrow \| N^2 x \| = \| N^*Nx \| \quad \forall x.$

Now, $\| N^2 \| = \sup\{ \| N^2x \| : \| x \| \leq 1\}$

$= \sup\{ \| N^*Nx \| : \| x \| \leq 1\}$

$= \| N^*N \|$

But we know

$\| T^* T \|^2 = \| T \|^2$

Hence $\| N^2 \| = \| N \|^2$.

Hence the proof.

Theorem:
If T is an operator on H, then T is normal iff its real and imaginary parts commutes .

Proof:

Claim : T is normal if AB=BA.

If A& B are the real and imaginary parts of T, so that
$T = A+iB$ and $T^* = A-iB$.

Then, $TT^* = (A+iB)(A-iB)$

$= A^2 + B^2 + i(BA - AB)$ …..(1)

$T^*T = (A-iB)(A+iB)$

$= A^2 + B^2 + i(AB - BA)$ …….(2)

Suppose AB=BA then from (1) & (2). We have

$TT^* = T^*T \Rightarrow$ T is normal .

Conversely , suppose that T is normal then

$TT^* = T^*T$

From (1) & (2) we have

$A^2 + B^2 + i(BA - AB) = A^2 + B^2 + i(AB - BA)$

$\Rightarrow BA - AB = AB - BA$

$\Rightarrow 2\,BA = 2\,AB$

$\Rightarrow AB = BA.$

Hence the proof.

Definition :

Let A_1, A_2 be two self adjoint operators. We say that $A_1 \le A_2$ if $(A_1x, x) \le (A_2 x, x)$ $\forall$ x.

Theorem:

The real banach space of all self adjoint operators on a Hilbert space H is a partially ordered set whose linear structure and order structure are related by the following properties.

i. If $A_1 \le A_2$ then $A_1 + A \le A_2 + A$ for every $A \in S$.

ii. If $A_1 \le A_2$ and $\alpha \ge 0$ then $\alpha A_1 \le \alpha A_2$.

Proof:

Let S be the set of all self adjoint operators on H. If $A \in S$

Then (Ax,x)

i.e., $(Ax,x) = (Ax,x)$ $\therefore$ $A \le A$.

Hence" $\le$ " is reflexive .

Suppose $A_1 \le A_2$ and $A_2 \le A_3$

Then $(A_1x,x) \le (A_2 x,x)$ & $(A_2 x,x) \le (A_3 x,x)$ $\forall$ x $\in$ H.

$\Rightarrow (A_1x,x) \le (A_3 x,x)$

$\Rightarrow A_1 \le A_3$

Hence "$\le$ " is transitive.

Suppose if $A_1 \le A_2$ and $A_2 \le A_1$. Then

$(A_1x,x) \le (A_2 x,x)$ & $(A_2 x,x) \le (A_1 x,x)$

$\Rightarrow (A_1x,x) = (A_2 x,x)$

$\Rightarrow ((A_1 - A_2) x,x) = 0$

$\Rightarrow A_1 - A_2 = 0.$

$\Rightarrow A_1 = A_2$.

Hence "$\leq$" is antisymmetric.

So $\leq$ is a partially ordered set in S.

i) Suppose $A_1 \leq A_2$ Then $(A_1x,x) \leq (A_2 x,x)$.

Hence $(A_1x,x) + (Ax,x) \leq (A_2 x,x) + (Ax,x)$.

$\Rightarrow ((A_1 + A) x,x) \leq ((A_2 + A) x,x)$

$\Rightarrow A_1 + A \leq A_2 + A$ for every $A \in S$.

ii) Given $A_1 \leq A_2$ and $\alpha \geq 0$

then $(A_1x,x) \leq (A_2 x,x)$

$\Rightarrow \alpha(A_1x,x) \leq \alpha (A_2 x,x) \ \forall \ x \in H$.

$\Rightarrow ((\alpha A_1)x,x) \leq ((\alpha A_2)x,x)$

$\Rightarrow \alpha A_1 \leq \alpha A_2$.

Hence the proof.

Positive operator:

The self adjoint operator " A" is said to be positive if $A \geq 0$ i.e., $(Ax,x) \geq 0 \ \forall x \in H$.

Theorem:

If A is a positive operator on H then If A is non-singular. In particular $I + T^*T$ and $I + TT^*$ are non-singular for any arbitrary operator T on H .

Proof:

To prove that I+A is non – singular . We have to show that I+A is 1 to 1, onto mapping of H into itself.

I+A is 1 to 1. First we show that $(I+A)x=0 \Rightarrow x=0$.

We have $(I+A)x=0 \Rightarrow x+Ax=0$.

$$Ax = -x$$
$$\Rightarrow (Ax,x) = (-x,x) = -\|x\|^2.$$

Since A is a positive operator.

i.e., $(Ax, x) \geq 0$. Hence given $-\|x\|^2 \geq 0$.which cannot be unless $\|x\| =0$.

This proves that $x=0$.

Thus $(I+A)x = 0 \Rightarrow x=0$.

Now $(I+A)x = (I+A)y \Rightarrow (I+A)(x-y) =0$

$$\Rightarrow x-y =0$$

$$\Rightarrow x = y .$$

Hence I+A is 1 to 1.

Now, we show that I+A is into.If M is the range of I+A then I+A is onto if M=H.

For every vector $x \in H$. We have

$$\|(I+A)x\|^2 = \|x+Ax\|^2 = (x+Ax, x+Ax)$$

$$= (x,x) + (x,Ax) + (Ax,x) + (Ax,Ax)$$

$$= \|x\|^2 + (Ax,x) + (Ax,x) + \|Ax\|^2$$

$$= \|x\|^2 + 2(Ax,x) + \|Ax\|^2 \geq \|x\|^2$$

Thus $\|x\| \leq \|(I+A)x\| \ \forall x \in H$. ………..(1)

Let $\{(I+A)x_n\}$ be a Cauchy sequence in M.

$\|(I+A)x_n - (I+A)x_m\| \to 0$ as $m,n \to \infty$.

$\Rightarrow (I+A)x_n \to (I+A)x$.

& $(I+A)x \in M$. Thus every Cauchy sequence $\{(I+A)x_n\}$ in M coverges to $\{(I+A)x\}$ in M.

$\therefore$ M is complete. But every complete subspace is closed. Hence M is closed.

Now M is a proper closed subspace of H. Then by an earlier theorem there exist a nonzero vector $x_0 \in H$ such that $x_0 \perp M$.

Now $\{(I+A)x_0\}$is in M & $x_0 \perp M$.

.

$\Rightarrow (x_0 ,((I+A)x_0) =0$

$\Rightarrow (x_0 , x_0) +(x_0 , Ax_0) =0$

$\Rightarrow (Ax_0 , x_0) = - (x_0 , x_0)$ ……..(2)

Since A is a positive operator.

$(Ax_0 , x_0) \geq 0$. so(2) gives $- \| x_0 \|^2 \geq 0$ which implies
$\| x_0 \|^2 \leq 0$. which cannot be unless $\| x_0 \|^2 =0$.so that
$x_0 =0$.

Contradicting the fact that x_0 is anon zero vector.

$\therefore$M= H.

$\Rightarrow$ (I+A) H= H

$\Rightarrow$ (I+A) is onto.

Hence I+A is non singular.

If T is any operator in B(H). We notice that T*T and TT*
Are both positive operators . Then by the first part of the
Theorem it follows that

I+ T*T and I+TT* are both Non singular.

Hence the proof.

Unitary operators:

An operator U on H is said to be unitary
if UU*= U*U =I.

Theorem:

If T is an operator on Hilbert space H, then the following conditions are equivalent to one another.

i. $T^*T = I$

ii. $(Tx,Ty) = (x,y) \quad \forall x,y \in H$.

iii. $\| Tx \| = \| x \| \; \forall x$.

Proof:

(i)$\Rightarrow$ (ii)

Let $T^*T = I$. Thus for any $x,y \in H$.

$(Tx,Ty) = (x, T^*Ty)$

$= (x,Iy)$

$=(x , y)$

(ii) $\Rightarrow$ (iii)

Suppose $(Tx, Ty) = (x,y) \quad \forall x,y \in H$.

In particular if we take $y = x$.

We have $(Tx,Tx) = (x,x)$

$\Rightarrow \| T x \|^2 = \| x \|^2$

$\Rightarrow \| T x \| = \| x \|$

(iii) $\Rightarrow$ (i)

Let $\| T x\| = \| x \| \quad \forall x$

$\Rightarrow \| T x \|^2 = \| x \|^2$

$\Rightarrow (Tx, Tx) = (x,x)$

$\Rightarrow (T^*Tx, x) =(x,x)$

$\Rightarrow ((T^* T - I) x,x) =0$

$\Rightarrow T^* T - I =0$

$\Rightarrow T^*T =I$

Hence the proof.

Theorem:

An operator on Hilbert space H is unitary iff T is unitary. It is an isometric isomorphism of H onto itself .

Proof:

Let T be an unitary operator then TT* =T*T=I.

Which implies that the mapping T is onto. Since T*T =I. It follows from the previous theorem that $\| T x\| = \| x \|$.

Thus T is 1 to 1, onto and preserves norm.

$\therefore$ T is an isometric isomorphism of H onto itself.

Conversely, if T is an isometric isomorphism then
$\| T x\| = \| x \| \ \forall x. \Rightarrow T^*T = I$ by the previous theorem and it is given that T is an isomorphism.

$\therefore T^{-1}$ exist. Hence T*T=I

$\Rightarrow (T^*T) T^{-1} = I T^{-1} = T^{-1}$

$\Rightarrow T^* = T^{-1}$.

Using this we can easily by premultiplying and post multiplying we have

$$TT^* = T^*T = I.$$

Which proves that T is unitary.

Hence the proof.

Projections:

A projection on abanach space B is an idempotent operator and which is continuous.T: B$\rightarrow$B is a projection if $T^2 = T$ and T is continuous.

If P is a projection on a banach space and if M & N are the range and the null space of P then M & N are closed linear subspaces of B such that $B= M\oplus N$.

M is the range of P. $M= \{ P(x) \ : x\in B\}=\{ x : p(x) =x\}$
& N is the null space of P. $N= \{ x \ / P(x) =0\}$.

Projections on a Hilbert space:

A projection P on a hilbert space H is said to perpendicular projection on H. If the range M and the nullspace N of P are orthogonal.

Theorem:
If P is a projection on Hilbert space H, with range M and null space N, then $M \perp N \Leftrightarrow$ P is self adjoint and in this case $N = M^{\perp}$.

Proof:

Suppose P is a projection on a hilbert space H with range M and nullspace N.

Then $H = M \oplus N$.

First we show that if $M \perp N$ then P is self adjoint.

Let Z be any vector in H, then z can be uniquely written as Z = x+y where $x \in M$ and $y \in N$.

We have, (Pz, z) =(x,z)

= (x,x+y)

= (x, x) +(x,y)

= (x,x)

Also, (P* z,z) =(z, Pz) = (z,x)

= (x+y, x)

= (x,x) +(y,x)

= (x,x)

$\therefore (Pz,z) = (P^*z,z) \quad \forall z \in H$.

$\Rightarrow ((P-P^*)z,z) = 0 \Rightarrow P - P^* = 0 \Rightarrow P = P^*$

$\Rightarrow$ P is self adjoint.

Conversely if P is self adjoint. Let x be any vector in M and y be any vector in N.

Then , (x,y) =(Px,y)

= (x,P*y)

= (x, Py)

=(x, 0) =0

$\Rightarrow$ M $\perp$ N

Hence the proof.

Note:

An operator P on a Hilbert space H is aprojection on H which satisfies the condition $P^2 = P$ & P* =P.

Theorem:

If P is a projection on a closed linear subspace M of H iff I-P is the projection on $M^{\perp}$.

Proof:

Suppose P is the projection on H then $P^2 = P$ & P* =P.

We have,

(I -P) * = I* -P* =I-P

& $(I-P)^2$ =(I-P) (I-P)

= (I- PI – PI+ P^2)

= I-P –P+P

= I- P

$\therefore$ (I –P) is the projection on H.

Now, we have to show that if M is the range of P then $M^{\perp}$ is the range of I –P. Let N be the range of I-P .

Then x $\in$ N $\Rightarrow$ (I –P) x =x $\Rightarrow$ x- Px = x

$\Rightarrow$ Px =0

$\Rightarrow$ x is the null space of P.

$\Rightarrow x \in M^{\perp}$

$\therefore$ $N \subseteq M^{\perp}$

Again $x \in M^{\perp} \Rightarrow Px = 0 \Rightarrow x - Px = x$
$\Rightarrow (I-P)x = x$

$\Rightarrow$ x is the range of I-P.

$\Rightarrow$ $x \in N$
$\Rightarrow$ $M^{\perp} \subseteq N$.
$\Rightarrow$ Hence $M^{\perp} = N$.

Conversely, suppose I-P is the projection on $M^{\perp}$. Then by the 1 st part of the theorem , I – (I-P) .i.e., P is the projection on $(M^{\perp})^{\perp} = M^{\perp\perp}$.

But M is closed $\Rightarrow M^{\perp\perp} = M$.

$\therefore$ P is the projection on M.

Hence the proof.

Definition:

Let T be an operator on H. A closed linear subspace M(H) is invariant under T, if $T(M) \subseteq M$.

If both M and $M^{\perp}$ is invariant under T then T is said to be reduced by M or M reduces T.

Theorem:

A closed linear subspace M(H) is invariant under an operator T iff $M^{\perp}$ is invariant under T^* .

Proof:

Let us assume that M is invariant under T. We have to show that $M^{\perp}$ is invariant under T^*.

Let y be an arbitrary vector in $M^{\perp}$, $(y,x) = 0 \ \forall \ x \in M$.It is enough to show that $T^* y \in M^{\perp}$. This is clear since
$(T^*y, x) = (y, Tx) = 0$

Thus $M^{\perp}$ is invariant under T*.

Conversely, if $M^{\perp}$ is invariant under T* then $(M^{\perp})^{\perp}$ is invariant under (T*)* .

Since M is closed.

$(M^{\perp})^{\perp} = M^{\perp\perp} = M$.

And (T*)* = T** = T.

Hence, M is invariant under T.

Hence the proof.

Theorem:
A closed linear subspace M(H) reduces an operator T iff M is invariant under both T and T* .

Proof:

If M reduces T then M and $M^{\perp}$ are invariant under T. If $M^{\perp}$ is invariant under T.

By the above theorem $(M^{\perp})^{\perp}$ is is invariant under T*. i.e., M is invariant under T*.

Conversely, If M is invariant under T* then again by above theorem $M^{\perp}$ is invariant under (T*)*= T** = T.

It is given that M is invariant under T. Thus both M and $M^{\perp}$ is invariant under T.

$\therefore$ M reduces T.

Hence the proof.

Theorem:
If T is a projection on a closed linear subspace M of H , then M is invariant under an operator T $\Leftrightarrow$ TP = PTP .

Proof:

If M is invariant under T and x is an arbitrary vector in Hthen to prove x is in M. So P(T Px) =TPx & PTP = TP.

Conversely , if TP=PTP & x is a vector in M, then

Tx= T Px = PTPx is also in Mand so M is invariant under T.

Hence the proof.

Theorem:

If P is a projection on a closed linear subspace M of H , then M reduces an operator T $\Leftrightarrow$ TP = PT .

Proof:

M reduces T iff M is invariant under T and T*.

Iff TP = PTP & T* P = PT* P

Iff TP = PTP & PT= (T*P)*

= (PT*P)*

= P* (T*)*P*

= PTP

iff TP =PT.

Hence the theorem.

Theorem:

If P and Q are the projections on a closed linear subspace M and N of H then M $\perp$ N iff PQ = 0 iff QP = 0 .

Proof:

Since P and Q are the projections on a Hilbert space H. Therefore P*=P & Q* =Q.

First we observe that

$$PQ = 0 \Leftrightarrow (PQ)^* = 0^*$$
$$\Leftrightarrow Q^*P^* = 0^*$$
$$\Leftrightarrow QP=0$$

$\therefore$ To prove the theorem it is sufficient to prove that

M $\perp$N $\Leftrightarrow$ PQ =0 .

If $M \perp N$ so that $N \subseteq M^{\perp}$. then the fact that QZ is on N for

Every Z implies that P(QZ) =0 so PQ=0.

Conversely, suppose that PQ=0 & $x \in M$, $y \in N$. Since M is the range of P then Px =x & N is the range of Q.

Then Qy=y.

We've (x,y) =(Px,Qy)

= (x, P*Qy)

= (x, PQy) = (x,0y) =(x,0) =0

$\Rightarrow M \perp N$ =0 .

Hence the proof.

Orthogonal:

Two projections P and Q on a hibert space H are said to be orthogonal if PQ =0.

Theorem:

If $P_1, P_2, \ldots P_n$ are the projections on a closed linear subspace $M_1, M_2, \ldots M_n$ of H then $P = P_1 + P_2 + \ldots + P_n$ is a projection iff the P_i's are pairwise orthogonal (In the sense that $P_iP_j = 0$ whenever $i \neq j$) and in this case P is the projection on $M = M_1 + M_2 + \ldots + M_n$.

Proof:

Given that $P_1\ P_2 \ldots\ldots\ldots.P_n$ are the projection on H. Therefore $P_i^2 = P_i = P_i^*$ for each i=1,2…….n.

Suppose that $P_i\ P_j = 0$ whenever $i \neq j$. Then to prove that P is the projection on H.

$P^* = (P_1 + P_2 + \ldots\ldots\ldots.+P_n)^* = (P_1^* + P_2^* + \ldots\ldots\ldots.+P_n^*)$

$= P_1 + P_2 \ldots\ldots\ldots.+P_n$

$= P$

And $P^2 = P.P$

$$= (P_1 + P_2 + \dots\dots\dots + P_n)(P_1 + P_2 + \dots\dots\dots + P_n)$$

$$= P_1^2 + P_2^2 + \dots\dots\dots + P_n^2$$

$$= (P_1 + P_2 + \dots\dots\dots + P_n) = P.$$

Thus, $P^* = P = P^2$. Therefore P is a projection on H.
Suppose P is a projection on H. Then $P^2 = P = P^*$. we have to prove that $P_i P_j = 0$ whenever $i \neq j$.

Let x be a vector in the range of P_i so that $P_i x = x$.

$$\text{Then} \quad \| x \|^2 = \| P_i x \|^2 \leq \sum_{j=1}^{n} \| P_j x \|^2$$

$$= \sum_{j=1}^{n} (P_j x, x)$$

$$= (P_1 x,x) + (P_2 x,x) + \dots\dots\dots + (P_n x,x)$$

$$= ((P_1 + P_2 + \dots\dots\dots + P_n) x,x)$$

$$= (Px, x)$$

$$= \| P x \|^2$$

$$\leq \| x \|^2 .$$

$$\therefore \text{ We have } \sum_{j=1}^{n} \| P_j x \|^2 = \| P_i x \|^2$$

$\Rightarrow \| P_j x \|^2 = 0$ whenever $j \neq i$.

$\Rightarrow P_j x = 0$ if $j \neq i$.

$\Rightarrow$ x is the null space of P_j for $j \neq i$.

Range space of P_i is contained in the null space of P_j for $j \neq i$.

$M_i \subseteq M_j^{\perp}$ for $j \neq i$.

i.e., $M_i \perp M_j$ for every $j \neq i$.

Then by the previous theorem we have $P_i P_j = 0$. i.e., P_i 's are pair wise orthogonal .

Finally , we have to show that P is the projection on M . i.e., Range space of P is M.

Let x be a vector in the range space of P then

$x = Px = (P_1 + P_2 + \ldots\ldots\ldots + P_n)\, x \in (M_1 + M_2 + \ldots\ldots + M_n)$
$= M.$

$\therefore$ Range of $P \subseteq M$.

Conversely, since $\| Px \| = \| x \|$ for every x in M_i , each M_i is contained in the range of P.

$\therefore M = M_1 + M_2 + \ldots\ldots + M_n$ is also contained in the range of P.

Hence M =R(P).

Hence the proof.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The null space of any continuous linear transformation is …………	closed	open	open subset	open set	closed
Let M ={x / f(x)=0} then M is the ……………….. of f.	range	linear	nullspace	open subset	nullspace
Let the adjoint of T denoted by …………. on H.	H	H**	H*	T*	T*
The adjoint of an operator is (Tx,y) =……………	(Tx,y)	(x,T*y)	(T*x,y)	(Tx,Ty)	(x,T*y)
The adjoint of operator T to T* on B(H) is (aT)* =…………….	a(T)*	Conjugate of a (T)*	T1*+a	T*	Conjugate of a (T)*
The adjoint of operator T to T* on B(H) is T** = ……………	a(T)*	T1+T2*	T	T*	T
The adjoint of operator T to T* on B(H) is ‖T‖=…………...	‖T‖*	‖T*‖	T*	T	‖T*‖
The adjoint of operator T to T* on B(H) is ‖T*T‖=…………	‖T‖*	‖T*‖	T*	$‖T‖^2$	$‖T‖^2$
If T = T* then 0 and I are ……………. operators	adjoint	commutate	self adjoint	symmetric	self adjoint
If T is an arbitrary operator on a hilbert space H then T=0 iff ………………	(Tx,y)	(x,T*y)	(T*x,y)	(Tx,y)=0	(Tx,y)=0
If T is an arbitrary operator on a hilbert space H then (Tx,x)=0 iff ………………	T=1	T=0	T=T*	T= Tx	T=0
The adjoint operator 0*= …………..	6	2	0	1	0
The adjoint operator 1*= …………..	6	2	0	1	1
If A is a positive operator on a H then I+A is ……………….	singular	nonsingular	commutate	self adjoint	nonsingular
I+T*T are…………. for any arbitrary oprator on T on H .	singular	nonsingular	commutate	self adjoint	nonsingular
The self adjoint operator A is said to be positive if ………………..	(Ax,x) =0	(Ax,x) >= 0	(A*x,y)	(Ax,y)=0	(Ax,x) >= 0
Every complete subspace of a complete space is …………….	closed	open	open subset	open set	closed
An operator N on H is said to be …………. if it commutes with its adjoint.	complete	closed	normal	open	normal
An operator N on H is said to be normal If it ……………. with its adjoint.	singular	nonsingular	commutes	self adjoint	commutes
The normal operator is NN*=……………	N*	nonsingular	N	N*N	N*N
Every ………………. operator is normal	adjoint	commutate	self adjoint	symmetric	self adjoint
An operator T on H is ……………. Iff ‖T*x‖=‖T x‖	complete	closed	normal	open	normal
If T is an operator on H then T is normal iff its real and imaginary parts ……	singular	nonsingular	commutes	self adjoint	commutes
An operator U on H is said to be ……….. If UU*= U*U= I	complete	closed	normal	unitary	unitary
An operator U on H is said to be unitary If ………………….	UU*= U*U= I	U*U=0	U=1	U=0	UU*= U*U= I
Every unitary opeartor is a ……………… operator.	complete	closed	normal	unitary	normal
………….. operators are precisely nonsingular operators.	complete	closed	normal	unitary	unitary

Unitary operator inverse equals theirs …………	adjoint	commutate	self adjoint	symmetric	adjoint
Unitary operators are precisely ………………. operators.	singular	nonsingular	commutes	self adjoint	nonsingular
A closed linear subsapce M(H) is ………… under T if T(M) Í M	invariant	commutate	self adjoint	idempotent	invariant
Two projectionP and Q on ahilbert space H are said to be……... if PQ=0	invariant	commutate	orthogonal	idempotent	orthogonal
If P is a…………… on a closed linear subspace M of H then M reduces an operator T iff TP=PT.	projection	commutate	self adjoint	idempotent	projection

UNIT 5

SPECTRAL THEORY

Finite dimensional spectral theory:

If T is an operator on a Hilbert space H ,then the simplest thing T can do to a vector is to transform it into a scalar multiple of itself.

$Tx = \lambda x$ ……………….. (1)

A non zero vector x is such that eq(1) for some scalar λ is called an eigen vector of T and for some nonzero x is called an eigen value of T.The expression (2) is called spectral resolution of T.

$T=\lambda_1 P_1+\lambda_2 P_2+\ldots\ldots\ldots\ldots..+\lambda m Pm$ ……..(2)

$T^* = \bar{\lambda}_1 P_1+\bar{\lambda}_2 P_2+\ldots\ldots\ldots\ldots..+\bar{\lambda} m Pm.$

$T^*T =|\lambda_1|^2 P_1+|\lambda_2|^2 P_2+\ldots\ldots\ldots\ldots..+|\lambda m|^2 Pm.$

Matrices:

Let $B=\{e_1,e_2,\ldots\ldots\ldots en\}$be an ordered basis for H, so that each vector in H is uniquely expressed as alinear combination of the ei's. If T is an operator on H, then for each ej, we have

$$Tej= \sum_{i=1}^{n} \alpha ij\ ej \quad .$$

$$[T] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots\ldots\ldots\ldots..\alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots\ldots\ldots\ldots..\alpha_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \alpha_{n1} & \alpha_{n2} & \ldots\ldots\ldots\ldots..\alpha_{nn} \end{pmatrix} = [\alpha ij]$$

The Spectral theorem:

Let T be an arbitrary operator on H.The distinct Eigen value of T forms a nonempty finite set of complex numbers.

Let $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda m$ be the eigen values.Let $M_1, M_2, \ldots\ldots\ldots M_m$ be their corresponding eigen spaces. Let $P_1, P_2, \ldots\ldots\ldots P_m$ be the projection on these eigen spaces.

i) The Mi's are pairwise orthogonal and span H.

ii) The Pi's are pairwise orthogonal ,

$$I = \sum_{i=1}^{m} P_i, \quad \text{and } T = \sum_{i=1}^{m} \lambda_i P_i$$

Theorem:

If T is normal then x is an eigen vector of T with eigen value λ iff x is an eigen vector of T* with eigen value $\bar{\lambda}$.

Proof:

Since T is normal , then the operator T- λ I is also normal for every scalar λ.

Then we have ,

$$\| Tx - \lambda x \| = \| T^*x - \bar{\lambda} x \|.$$

Hence the proof.

Theorem:

If T is normal then the Mi's are pairwise orthogonal.

Proof:

Let x_i and x_j be vectors in M_i and M_j for $i \neq j$, so that $Tx_i = \lambda_i x_i$ and $Tx_j = \lambda_j x_j$. The preceding theorem shows that

$$\lambda_i (x_i, x_j) = (\lambda_i x_i, x_j) = (T x_i, x_j)$$

$$= (x_i, T^*x_j) = (x_i, \bar{\lambda}_j x_j)$$

$$= \lambda_j (x_i, x_j).$$

& since $\lambda_i \neq \lambda_j$, it is clear that we have $(x_i, x_j) = 0$.

Next we say that Mi's span H when T is normal.

Hence the proof.

Theorem:
If T is normal then each Mi reduces T.

Proof:

Each Mi is invariant under T . It is enough to show that Mi is also invariant under T*.

As we know If x_i is avector in Mi, so that

$Tx_i = \lambda_i x_I$, then $T^* x_i = \bar{\lambda}_i x_i$. Finally we can say that M_i reduces T.

Hence the proof.

Theorem:
If T is normal then the Mi's span H.

Proof:

The Mi's are pair wise orthogonal .

We know that $M = M_1+M_2+\dots\dots\dots\dots+M_m$ is aclosed linear subspace of H and that its associated projection is
$P= P_1+P_2+\dots\dots\dots\dots+P_m$

Since each Mi reduces T . we see that $T P_i = P_i T \quad \forall P_i$.
It follows from the fact that TP= PT, so M reduces T.
Consequently $M^{\perp}$ is invariant under T. If
$M^{\perp} \neq \{0\}$then since all the eigen vector of T are contained in M, the restriction of T to $M^{\perp}$ is an operator on a nontrivial finite dimensional Hilbert space which has no eigen vectors and hence no eigen values.

It means that $M^{\perp} \neq \{0\}$ and so M= H and the Mi's span H.

Hence the proof.

Banach algebras:
A banach algebra is a complex banach space which is also an algebra with identity 1 and which the

structure is related to the norm by the following condition.

i) $||xy|| \le ||x||\,||y||$

ii) $||I|| = 1$.

Example

The set of all complex numbers is a Banach algebra.

Notation :

Let A be a banach algebra .we denote the set of all regular elements in A by G and its compliment the set of singular elements is denoted by S. Clearly, the identity element in A is invertible and so $I \in G$.

Theorem:
Every element x for which $||x - 1|| < 1$ is regular and the inverse of such an element is given by the formula

$$x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$$

Proof:

Put $r = ||x - 1||$ so that $r < 1$.
Consider,

$$\begin{aligned} ||(1-x)^n|| &= ||(1-x)(1-x)\ldots\ldots\ldots(1-x)|| \\ &\le ||(1-x)||\,||(1-x)||\ldots\ldots\ldots||(1-x)|| \\ &\le ||(1-x)||^n = r^n \end{aligned}$$

consider , next $\sum_{n=1}^{\infty} (1-x)^n$, then $S_n = \sum_{k=1}^{n} (1-x)^k$

Then for n>m,

$$\begin{aligned} ||S_n - S_m|| &= ||(1-x)^{m+1} + (1-x)^{m+2} + \ldots\ldots\ldots + (1-x)^n|| \\ &\le ||(1-x)||^{m+1}\,||(1-x)||^{m+2}\ldots\ldots\ldots||(1-x)||^n \\ &\le r^{m+1} + r^{m+2} + \ldots\ldots\ldots\ldots + r^n . \end{aligned}$$

Since $\sum r^n$ is convergent then there exist an integer such that $||S_n - S|| < \varepsilon \ \forall\, n,m \ge N$.

$\{S_n\}$ is a Cauchy sequence in A. But A is complete. This partial sum converges to an element of A. we denote this by

$$\sum_{n=1}^{\infty} (1-x)^n .$$

If we define y by $y = 1 + \sum_{n=1}^{\infty} (1-x)^n$ ………..(1)

Then the joint continuity of multiplication in A such that,

$$y-xy = y(1-x) = (1-x)(1+ \Sigma (1-x)^n) = \Sigma (1-x)^n = (y-1)$$

Then x has an inverse in A and so x is regular. The inverse of x is given by (1) .

Hence the proof.

Theorem:

G is an open set and therefore S is a closed set.

Proof:

Let x_0 be an element in G and x is any other element in A so that

$$\| x - x_0 \| < 1/ \| x_0^{-1} \|$$

Note that $x_0 \neq 0$. Now,

$$\| x_0^{-1}x - 1 \| = \| x_0^{-1}x - x_0^{-1}x_0 \| = \| x_0^{-1} (x - x_0) \|$$
$$\leq \| x_0^{-1} \| \| (x - x_0) \|$$
$$< \| x_0^{-1} \| .1/ \| x_0^{-1} \| =1.$$

i.e., $\| x_0^{-1}x - 1 \| < 1$. Since $x= x_0(x_0^{-1} x)$. It follows that x is also in G . So G is open. Then S= A-G, where S is the set of all singular elements. Since G is open in A. Then its complement S is closed in A.

Hence the proof.

Theorem:
The mapping $x \to x^{-1}$ of G into G is continuous and its therefore a homeomorphism of G onto itself.

Proof:

Clearly , the maspping $x \to x^{-1}$ is 1 to 1 and onto from G into itself. Let x_0 be an element of G, and x be another element of G such that,

$$\| x - x_0 \| < 1/ 2\| x_0^{-1} \|$$

Note that $x_0 \neq 0$. Now,

$$\| x_0^{-1}x - 1 \| = \| x_0^{-1}x - x_0^{-1}x_0 \| = \| x_0^{-1} (x - x_0) \|$$
$$\leq \| x_0^{-1} \| \| (x - x_0) \|$$
$$< \| x_0^{-1} \| .1/ 2 \| x_0^{-1} \| =1/2 <1.$$

i.e., $\| x_0^{-1}x - 1 \|<1$.
By the theorem $x_0^{-1} x \in G$ and

$$(x_0^{-1} x)^{-1} = 1+ \Sigma (1 - x_0^{-1}x)^n$$

$$x^{-1} x_0 = 1+ \Sigma (1 -x_0^{-1}x)^n$$

Now, $\| x^{-1}- x_0^{-1} \| = \| x^{-1}x_0x^{-1} - x_0^{-1} \|$

$$\leq \| x_0^{-1} \| \| (x^{-1} x_0 -1) \|$$
$$\leq \| x_0^{-1} \| \| \Sigma (1- x_0^{-1} x)^n \|$$
$$\leq \| x_0^{-1} \| \Sigma \| (1- x_0^{-1} x) \|^n$$
$$= \| x_0^{-1} \| 2 \| 1- x_0^{-1}x \|$$
$$= \| x_0^{-1} \| 2 \| x_0^{-1} \| \| x- x_0 \|$$

Hence when $x \to x_0$, $\| x^{-1}- x_0^{-1} \| \to 0$.

$\Rightarrow x^{-1} \to x_0^{-1}$ .i.e., The mapping is continuous . also the inverse mapping is continuous.

It is a homeomorphism of G onto Itself.

Hence the proof.

Topological divisors of Zero:

The element Z in abanach algebra A is called a topological divisors of zero.If there exist a sequence $\{z_n\}$ inA such that $\| z_n \| =1$ and either $zz_n \to 0$ or $z_n z \to 0$.
Clearly , even divisor of zero is a topological divisor of zero.There exist $z' \to zz' =0$.
Choose $z_n = z'/ \| z' \|$ such that $\| z_n \| =1$
and $zz_n = zz'/ \| z' \| \to 0$. Hence z is a topological divisor of zero. We denote theset of all topological divisor of zero by z.

Theorem:
Z is a subset of S.

Proof:
Let z is in Z. Then there exist a sequence z_n such that
$\| z_n \| =1$ and $zz_n \to 0$ .If z is in G then by joint continuity of multiplication we have,

$$z^{-1}(zz_n) = (z^{-1}z)z_n \to 0$$
$$\Rightarrow z_n \to 0.$$

Which contradicts the fact that $\| z_n \| =1$.

$\therefore z_n \in S$.

$\therefore Z \subset S$.

Hence the proof.

Theorem:
The boundary of S is a subset of Z.

Proof:

Since S is closed, its boundary consist of all points in S which are limits of convergent sequence in G.We show that if z is such a point (i.e.,) if $z \in S$ there exist $\{r_n\} \in G$ such that $r_n \to z$, then $z \in Z$.

$$\text{Now, } (r_n^{-1} z -1) = (r_n^{-1} z - r_n^{-1}r_n) = r_n^{-1}(z - r_n)$$

The sequence r_n^{-1} is unbounded. For otherwise if the sequence r_n^{-1} is bounded then there exist a number M such that $\| r_n^{-1}\| < M$. Also $r_n \to z \Rightarrow \| r_n - z \| < 1/M$.

Now, $\| r_n^{-1} z - 1 \| \le \| r_n^{-1} \| \| z - r_n \| < M.(1/M)=1$.
Also, $r_n^{-1} z \in G$. thus $z = r_n (r_n^{-1} z) \in G$.

This is a contradiction to the fact that $z \in S$.
$\therefore \{ r_n^{-1} \}$ is unbounded. We can take $\| r_n^{-1} \| \to \infty$ as $n \to \infty$.

Let $z_n = r_n^{-1}/ \| r_n^{-1}\|$, then $\| z_n \| =1$ and

$$zz_n = z r_n^{-1}/ \| r_n^{-1}\| = [1- (1- z r_n^{-1})] / \| r_n^{-1}\|$$

$$= [1+ (z - r_n)r_n^{-1}] / \| r_n^{-1}\|$$

$$= 1/ \| r_n^{-1}\| +(z- r_n) z_n \to 0 \quad \text{as } r_n \to z \text{ and}$$

$\| r_n^{-1}\| \to \infty$.

So, $zz_n \to 0$ which means that z is a topological divisor of zero.i.e., $z \in Z$.

Hence the proof.

The Spectrum:

If H is a nontrivial Hilbert space then the spectrum of t is
$\sigma(t) = \{\lambda \in c: T- \lambda I \text{ is singular } \}$ where T is an operator on H. If x is an element of an banach algebra A then the spectrum of x is given by

$\sigma(x) = \{\lambda : x- \lambda I \text{ is singular } \}$. We write $\sigma(x)$ as $\sigma_A(x)$.

Theorem:
For every element x in a banach algebra A, $\sigma(x)$ is non-empty and compact.

Proof:
Consider the function $\phi : C \to A$ defined by $\lambda \to x- \lambda I$. this function is continuous. Also, S is closed in A.

$\Rightarrow$ The inverse image of closed set is closed if the function is continuous.
$\Rightarrow \{\lambda \in c: T- \lambda I \text{ is singular } \}$ is closed.
$\Rightarrow \sigma(x)$ is closed.

To prove $\sigma(x)$ is bounded:

Claim:

If $\lambda \in \sigma(x)$ then $\lambda \leq \| x \|$.

If the claim is proved then $\sigma(x)$ is bounded .Suppose $\lambda \in C$ such that $| \lambda | > \| x \|$

Then, $\| x / \lambda \| < 1$.

$\Rightarrow$ $(1-(x/ \lambda))$ is regular , $\lambda(1-(x/ \lambda))$ is regular, $\lambda I - x$ is regular $\lambda \notin \sigma(x)$. Hence the claim is proved. Since $\sigma(x)$is closed and bounded, $\sigma(x)$ is compact.

To prove $\sigma(x)$ is non-empty:

$\sigma(x)$ is asubset of C. The compliment of $\sigma(x)$ is $c-\sigma(x) = \rho (x)$ is called the resolvement of x. Since $\sigma(x)$ is closed, $\rho (x)$ is an open subset of the complex plane and it contains the set $\{Z: |Z | >\|x\|\}$.

Suppose $\lambda \in \rho (x) \Rightarrow \lambda \notin \sigma(x)$.

$\Rightarrow x- \lambda I$ is regular

$\Rightarrow (x- \lambda I)^{-1}$ exists.

Define the resolvement of x is the function $\rho (x) \rightarrow A$ given by $x(\lambda) = (x- \lambda I)^{-1}$.

This is a continouous function .Also,

$x(\lambda) = \lambda^{-1}(x/\lambda - 1)^{-1}, \lambda \neq 0$.

$\Rightarrow x(\lambda)\rightarrow 0$ as $\lambda\rightarrow \infty$.

If λ and μ are both in $\rho (x)$. then ,

$x(\lambda) = x(\lambda)(x-\mu I) (x-\mu I)^{-1}$

$\therefore x(\lambda) - x(\mu) = (\lambda -\mu) .x(\lambda).x(\mu)$

This relation is called the resolvement equation . Let f be functional on A. i.e., $f \in$ conjugate space A^* .

Define $f(\lambda) = f(x(\lambda))$, $x \in A, \lambda \in \rho (x)$.

It has a derivative at each point of ρ (x).
Also, $|f(\lambda)| = |f(x(\lambda))| = ||f|| \, ||x(\lambda)||$. As $\lambda \to \infty$, $|f(\lambda)| \to 0$.

Assume now $\sigma(x)$ is empty.

Then ρ (x) = C- $\sigma(x)$= C (whole complex plane)

By Liouville's theorem, we conclude that $f(\lambda)$ =0 for all
$\lambda \in \rho$ (x). Since f is an arbitrary functional on A.

$\Rightarrow x(\lambda) = 0 \quad \forall \lambda$.

This is impossible , for no inverse can equal to zero and therefore it cannot be true that $\sigma(x)$ is empty.

Hence the proof.

Regular:

A division algebra is an algebra with identity in which nonzero element is regular.

Theorem: Gelfand Mazur theorem:
If A is a division algebra then it equals the set of all scalar multiples of the identity.

Proof:
We have to prove that if x is an element of A then x equals λ I.Suppose on the contrary that $x \neq \lambda I$ for every λ, then

$x - \lambda I \neq 0$ for every λ.

$\Rightarrow x - \lambda I$ is regular for every λ and therefore $\sigma(x)$ is empty.

This contradicts that $\sigma(n) \neq \phi$.

$\therefore x = \lambda I$ for some λ.

Hence the proof.

Theorem:
If '0' is the only topological divisor of zero in abanach algebra A then A=C.
Proof:
Let $x \in A$, $\sigma(x)$ is non-empty and closedset it has a boundary point in λ, then $x- \lambda I$ is a boundary point of th
a singular elements.

Since the boundary of S is a subset of Z. It follows that $x- \lambda I \in Z$. i.e., $x- \lambda I$ is a topological divisor

Then $x- \lambda I =0 \Rightarrow x = \lambda I$.

Every element $x \in A$ is of the form λI where $\lambda \in C$.

$\therefore$ A= C.

Hence the proof.

Theorem:
If the norm in a banach algebra A satisfies $\|x y\| \geq K\|x\| \|y\|$
for some positive constant, then A=C.
Proof:

If Z is a topological divisor of zero then there exist a sequence z_n such that $\| z_n \| =1$ and $zz_n \rightarrow 0$.

By hypothesis $\| z z_n \| \geq K\|z\| \|z_n\| \geq K\| z \|$.

Since $K>0$, $\Rightarrow \| z \| = 0$.

$\Rightarrow$ 0 is the only topological divisor of zero.
$\Rightarrow$A=C.

Hence the proof.

****

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
A non zero vector x such that Tx=l x is true for some scalar l is called an ………….. of T.	eigen value	eigen vector	scalar	idempotent	eigen value
A scalar l such that Tx=lx holds for some nonzero x is called an……… of T.	eigen value	eigen vector	scalar	idempotent	eigen vector
Each eigen vector corresponds precisely to one…………………..	eigen value	eigen vector	scalar	idempotent	eigen value
Each eigen value has one or more ……………. associated with it.	eigen value	eigen vector	scalar	idempotent	eigen vector
Eigen value are otherwise called as ………………………..	characterestic value	characterestic vector	eigen vector	scalar	characterestic value
Eigen vector are otherwise called as ………………………..	characterestic value	characterestic vector	eigen value	scalar	characterestic vector
If T is an operator on hilbert space H, then T to a vector x is to transform it into a scalar multiple ………………	Tx=lx	Tx =0	Tx=1	l x=1	Tx=lx
If T has different Eigen values then each one is ……….. to one another	corresponding	same	distinct	identity	distinct
The image of the identity operator is the ………… matrix.	singular	identity	nonsingular	null	identity
The ……….. Matrix is 1's down the main diagonal and zero elsewhere.	singular	identity	nonsingular	null	identity
Two matrices in An are ………… iff they are the matrices of a single operator . on H relative to different bases	similar	asimilar	vary	distinct	similar
The …………. of S is a subset of Z.	boundary	resolvement	spectral	distinct	boundary
The set of all …………… divisor of zero by z.	identical	topological	boundary	resolvement	topological
The set of all complex number is a…………….. Algebra.	Ring	hardy	banach	functional	banach
The regular element is the compliment of …………… element.	singular	identity	nonsingular	null	singular
A banach algebra is acomplex ………….. which is also an algebra with identity 1.	Banach space	Hilbert space	Inner product space	Linear space	Banach space
Let A be a …………. algebra then the set of all reular elements in A by G.	Ring	hardy	Banach	functional	Banach
Let A be a …………. algebra then the set of all reular elements in A by S.	singular	identity	nonsingular	null	singular
The set of all values in a banach algebra is ……….. Number.	complex	real	inverse	scalar	complex
G is an open set and therefore s is an …………. set.	closed	open	open subset	open set	closed
The compliment of spectrum is called the ……………… of x.	resolvement	spectral	distinct	identity	resolvement
For every element x in a banach algebra A the ……….. of x is nonempty and compact.	resolvement	spectrum	distinct	identity	spectrum
A division algebra is an algebra with identity in which each non zero element is ……………..	singular	nonsingular	commutate	regular	regular
0 is the only …………… divisor of zero in a banach algebra then A=C.	identical	topological	boundary	resolvement	topological
0 is the only topological divisor of zero in a banach algebra then ………..	A=C	A=1	A=0	A=V	A=C
A banach algebra is called a banach* algebra if it has an …………	involution	topological	boundary	resolvement	involution
The element x* is called the ………… of x and so asubalgebra of A is said to be self adjoint if it contains the adjoint of each of its elements.	adjoint	commutate	self adjoint	idempotent	adjoint
An element xÎA is ………… if there exist an element y such that xy=yx=1.	singular	left regular	right regular	regular	regular
An element xÎA is ………… if there exist an element y such that yx=1.	singular	left regular	right regular	regular	left regular

An element xÎA is ………… if there exist an element y such that xy=1.	singular	left regular	right regular	regular	right regular
Every maximal left ideal in A is ……………..	closed	open	open subset	open set	closed
If x is not right regular then it is called……………..	right singular	left regular	right regular	regular	right singular
If x is not left regular then it is called……………..	left singular	left regular	right regular	regular	left singular
If x is both right and left regular then it is called …………	left singular	left regular	right regular	regular	regular
A is the intersection of all its ……….. left ideal.	maximal	minimal	right regular	regular	maximal
A maximal left ideal in A is a proper left ideal which is not properly contained if their …………… left ideal.	maximal	minimal	proper	regular	proper