KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester – III

18MMP303 **MATHEMATICAL MODELING** **4H – 4C**

Instruction Hours / week: L: 4 T: 0 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- To enable the students to enrich the fundamental of mathematical modeling skills.
- The construction and analysis of mathematical models inspired by real life problems
- Several modeling techniques and the means to analyze the resulting systems.

Course Outcomes (COs)

On successful completion of this course the student will be able to

1. Solve problems involving dynamic models, and probabilistic models.
2. Understand the use of modern technology in solving real-world.
3. Problems through ordinary differential equations, probability theory, graphs.
4. Formulate a mathematical model given a clear statement of the underlying scientific principles.
5. Solve basic linear equations and solve application problems.

UNIT I

MATHEMATICAL MODELING THROUGH ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER:

Linear Growth and Decay Models – Non-Linear Growth and Decay Models – Compartment Models – Dynamics problems – Geometrical problems.

UNIT II

MATHEMATICAL MODELING THROUGH SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER:

Population Dynamics – Epidemics – Compartment Models – Economics – Medicine, Arms Race, Battles and International Trade – Dynamics.

UNIT III

MATHEMATICAL MODELING THROUGH ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER

Planetary Motions – Circular Motion and Motion of Satellites – Mathematical Modelling through Linear Differential Equations of Second Order – Miscellaneous Mathematical Models.

UNIT IV

MATHEMATICAL MODELING THROUGH DIFFERENCE EQUATIONS

Simple Models – Basic Theory of Linear Difference Equations with Constant Coefficients – Economics and Finance – Population Dynamics and Genetics – Probability Theory.

UNIT V

MATHEMATICAL MODELING THROUGH GRAPHS

Solutions that can be modeled through Graphs – Mathematical Modeling in Terms of Directed Graphs, Signed Graphs, Weighted Digraphs and Unoriented Graphs.

SUGGESTED READINGS

1. Kapur J.N., (2015). Mathematical Modeling, Wiley Eastern Limited, New Delhi.
2. Kapur, J. N., (1985). Mathematical Models in Biology and Medicine, Affiliated East –West Press Pvt Limited, New Delhi.
3. Brain Albright, (2010). Mathematical Modeling with Excel, Jones and Bartlett Publishers, New Delhi.
4. Frank. R. Giordano, Maurice. D.Weir, WilliamP. Fox, (2003). A first course in Mathematical Modelling, Vikash Publishing House, UK.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: V. Kuppusamy
Subject Name: Mathematical Modeling **Sub.Code:18MMP303**
Semester: III **Class: II M.Sc Mathematics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material/ Page Nos
UNIT-I			
1	1	Introduction and simple illustrations	S1:Chapter 1, Sec 1.1 Pg.No : 1-15
2	1	Mathematical modeling through differential equations	S1 : Chapter 2,Sec 2.1 Pg.No : 30-35
3	1	Linear growth and decay models problems	S1 : Chapter 2,Sec 2.1 Pg.No : 30-35
4	1	Non-linear growth and decay models problems	S1 : Chapter 2,Sec 2.3 Pg.No :35-39
5	1	Compartment models problems	S1 : Chapter 2,Sec 2.4 Pg.No :39-43
6	1	Mathematical modeling in dynamics through ordinary differential equations of first order	S1 : Chapter 2,Sec 2.5 Pg.No :43-45
7	1	Continuation on mathematical modeling in dynamics through ordinary differential equations of first order	S1 : Chapter 2,Sec 2.5 Pg.No :45-48
8	1	Mathematical modeling of geometrical problems through ordinary differential equations of first order	S4 : Chapter 2, Pg.No :75 - 74
9	1	Continuation on mathematical modeling of geometrical problems through ordinary differential equations of first order	S4 : Chapter 2, Pg.No :74 - 79
10	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-10 Hours			
UNIT-II			
1	1	Continuation on mathematical modeling in population dynamics	S1 : Chapter 3,Sec 3.1 Pg.No :53-60
2	1	Mathematical modeling in population dynamics	S1 : Chapter 3,Sec 3.1 Pg.No :53-60
3	1	Mathematical modeling of epidemics through systems of ordinary differential equations of first order	S3 : Chapter 4, Pg.No :118-124

4	1	Compartment models through systems of ordinary differential Equations	S1 : Chapter 3,Sec 3.3 Pg.No :63-64
5	1	Mathematical modeling in Economics through systems of ordinary differential equations of first order	S1 : Chapter 3,Sec 3.4 Pg.No :64 - 69
6	1	Mathematical models in medicine, Arms Race, Battles and International trade in terms of systems of ordinary differential equations	S4 : Chapter 9, Pg.No :350 – 355
7	1	Mathematical modeling in dynamics through systems of ordinary differential equations of first order	S1 : Chapter 3,Sec 3.6 Pg.No :72-76
8	1	Continuation on mathematical modeling in dynamics through systems of ordinary differential equations of first order	S1 : Chapter 3,Sec 3.6 Pg.No :72-76
9	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-9 Hours			
UNIT-III			
1	1	Mathematical modeling of planetary motions	S1 : Chapter 4,Sec 4.1 Pg.No :76-82
2	1	Continuation on Mathematical modeling of planetary motions	S1 : Chapter 4,Sec 4.1 Pg.No :76-82
3	1	Mathematical modeling of circular motion and motion of satellites	S1 : Chapter 4,Sec 4.2 Pg.No :82-88
4	1	Continuation on mathematical modeling of circular motion and motion of satellites	S1 : Chapter 4,Sec 4.2 Pg.No :82-88
5	1	Mathematical modeling through linear differential equations of second order	S1 : Chapter 4,Sec 4.3 Pg.No :88-93
6	1	Continuation of problems on mathematical modeling through linear differential equations of second order	S3 : Chapter 7, Pg.No :238-244
7	1	Problems on mathematical modeling through linear differential equations of second order	S3 : Chapter 7, Pg.No :238-244
8	1	Miscellaneous mathematical model through ordinary differential equations of the second order	S4: Chapter 11, Pg.No : 437-452
9	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-9 Hours			
UNIT-IV			
1	1	The need for mathematical modeling through difference equations : some simple models	S1 : Chapter 5,Sec 5.1 Pg.No :96-98
2	1	Basic theory of linear difference equations with constant coefficients	S1 : Chapter 5,Sec 5.2 Pg.No :98-101
3	1	Continuation of types of basic theory of linear difference equations with constant coefficients	S1 : Chapter 5,Sec 5.2 Pg.No :101-105
4	1	Mathematical modeling through difference equations in economics and finance	S1 : Chapter 5,Sec 5.3 Pg.No :105-110
5	1	Mathematical modeling through difference equations in population dynamics and genetics	S1 : Chapter 5,Sec 5.4 Pg.No :110 - 113
6	1	Continuation of types of mathematical modeling	S1 : Chapter 5,Sec 5.4

		through difference equations in population dynamics and genetics	Pg.No :113 - 117
7	1	Mathematical modeling through difference equations in probability theory	S4: Chapter 6, Pg.No :217-223
8	1	Miscellaneous examples of mathematical modeling through difference equations	S1 : Chapter 5,Sec 5.6 Pg.No :121-122
9	1	Continuation of miscellaneous examples of mathematical modeling through difference equations	S1 : Chapter 5,Sec 5.6 Pg.No :122-124
10	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-10Hours			
UNIT-V			
1	1	Situations that can be modelled through graphs	S1 : Chapter 7,Sec 7.1 Pg.No :151-154
2	1	Mathematical models in terms of directed graphs	S1 : Chapter 7,Sec 7.2 Pg.No :154-156
3	1	Continuation of types of mathematical models in terms of directed graphs	S1 : Chapter 7,Sec 7.2 Pg.No :156-161
4	1	Mathematical models in terms of signed graphs	S4 : Chapter 3, Pg.No : 101-107
5	1	Mathematical modeling in terms of weighted digraphs	S1 : Chapter 7,Sec 7.4 Pg.No :164-170
6	1	Mathematical modeling in terms of unoriented graphs	S1 : Chapter 7,Sec 7.5 Pg.No :170-177
7	1	Recapitulation and discussion of possible questions	
8	1	Discussion of previous year ESE question papers	
9	1	Discussion of previous year ESE question papers	
10	1	Discussion of previous year ESE question papers	
Total No. of Lecture hours planned-10 Hours			
Total Planned Hours			**48**

SUGGESTED READINGS

TEXT BOOK

1. Kapur J.N., (2015). Mathematical Modeling, Wiley Eastern Limited, New Delhi.
2. Kapur, J. N., (1985). Mathematical Models in Biology and Medicine, Affiliated East –West Press Pvt Limited, New Delhi.
3. Brain Albright, (2010). Mathematical Modeling with Excel, Jones and Bartlett Publishers, New Delhi.
4. Frank. R. Giordano, Maurice. D.Weir, WilliamP. Fox, (2003). A first course in Mathematical Modelling, Vikash Publishing House, UK.

UNIT-I

Mathematical Modeling through Ordinary Differential Equations of First order: Linear Growth and Decay Models – Non-Linear Growth and Decay Models – Compartment Models – Dynamics problems – Geometrical problems.

2 Mathematical Modelling Through Ordinary Differential Equations of First Order

2.1 MATHEMATICAL MODELLING THROUGH DIFFERENTIAL EQUATIONS

Mathematical Modelling in terms of differential equations arises when the situation modelled involves some *continuous* variable(s) varying with respect to some other continuous variable(s) and we have some reasonable hypotheses about the *rates of change* of dependent variable(s) with respect to independent variable(s).

When we have one dependent variable x (say population size) depending on one independent variable (say time t), we get a mathematical model in terms of an *ordinary differential equation* of the *first order*, if the hypothesis is about the rate of change dx/dt. The model will be in terms of an *ordinary differential equation of the second order* if the hypothesis involves the rate of change of dx/dt.

If there are a number of dependent continuous variables and only one independent variable, the hypothesis may give a mathematical model in terms of a *system of first or higher order ordinary differential equations.*

If there is one dependent continuous variable (say velocity of fluid u) and a number of independent continuous variables (say space coordinates x, y, z and time t), we get a mathematical model in terms of a *partial differential equation.* If there are a number of dependent continuous variables and a number of independent continuous variables, we can get a mathematical model in terms of systems of *partial differential equations.*

Mathematical models in terms of ordinary differential equations will be studied in this and the next two chapters. Mathematical models in terms of partial differential equations will be studied in Chapter 7.

2.2 LINEAR GROWTH AND DECAY MODELS

2.2.1 Populational Growth Models

Let $x(t)$ be the population size at time t and let b and d be the birth and death rates, i.e. the number of individuals born or dying per individual

2.2.2 Growth of Science and Scientists

Let $S(t)$ denote the number of scientists at time t, $bS(t)\Delta t + 0(\Delta t)$ be the number of new scientists trained in time interval $(t \;\; t + \Delta t)$ and let $dS(t)\Delta t \;\; + 0(\Delta t)$ be the number of scientists who retire from science in the same period, then the above model applies and the number of scientists should grow exponentially.

The same model applies to the growth of Science, Mathematics and Technology. Thus if $M(t)$ is the amount of Mathematics at time t, then the rate of growth of Mathematics is proportional to the amount of Mathematics, so that

$$dM/dt = aM \quad \text{or} \quad M(t) = M(0) \exp{(at)} \tag{6}$$

Thus according to this model, Mathematics, Science and Technology grow at an exponential rate and double themselves in a certain period of time. During the last two centuries this doubling period has been about ten years. This implies that if in 1900, we had one unit of Mathematics, then in 1910, 1920, 1930, 1940, . . . 1980 we have 2, 4, 8, 16, 32, 64, 128, 256 unit of Mathematics and in 2000 AD we shall have about 1000 units of Mathematics. This implies that 99.9% of Mathematics that would exist at the end of the present century would have been created in this century and 99.9% of all mathematicians who ever lived, would have lived in this century.

The doubling period of mathematics is 10 years and the doubling period of the human population is 30-35 years. These doubling periods cannot obviously be maintained indefinitely because then at some point of time, we shall have more mathematicians than human beings. Ultimately the doubling period of both will be the same, but hopefully this is a long way away.

This model also shows that the doubling period can be shortened by having more intensive training programmes for mathematicians and scientists and by creating conditions in which they continue to do creative work for longer durations in life.

2.2.3 Effects of Immigration and Emigration on Population Size

If there is immigration into the population from outside at a rate proportional to the population size, the effect is equivalent to increasing the birth rate. Similarly if there is emigration from the population at a rate proportional to the population size, the effect is the same as that of increase in the death rate.

If however immigration and emigration take place at constant rate i and e respectively, equation (3) is modified to

$$\frac{dx}{dt} = bx - dx + i - e = ax + k \tag{7}$$

Integrating (7) we get

$$x(t) + \frac{k}{a} = \left(x(0) + \frac{k}{a}\right)e^{at} \tag{8}$$

The model also applies to growth of populations of bacteria and micro-organisms, to the increase of volume of timber in forest, to the growth of malignant cells etc. In the case of forests, planting of new plants will correspond to immigration and cutting of trees will correspond to emigration.

2.2.4 Interest Compounded Continuously

Let the amount at time t be $x(t)$ and let interest at rate r per unit amount per unit time be compounded continuously then

$$x(t + \Delta t) = x(t) + rx(t)\Delta t + 0(\Delta t),$$

giving

$$\frac{dx}{dt} = xr; \quad x(t) = x(0)e^{rt} \tag{9}$$

This formula can also be derived from the formula for compound interest

$$x(t) = x(0)\left(1 + \frac{r}{n}\right)^{nt}, \tag{10}$$

when interest is payable n times per unit time, by taking the limit as $n \to \infty$. In fact comparison of (9) and (10) gives us two definitions of the trancendental number e viz.

(i) e is the amount of an initial capital of one unit invested for one unit of time when the interest at unit rate is compounded continuously

(ii) $$e = \operatorname*{Lt}_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \tag{11}$$

Also from (9) if $x(t) = 1$, then

$$x(0) = e^{-rt}, \tag{12}$$

so that e^{-rt} is the present value of a unit amount due one period hence when interest at the rate r per unit amount per unit time is compounded continuously.

2.2.5 Radio-Active Decay

Many substances undergo radio-active decay at a rate proportional to the amount of the radioactive substance present at any time and each of them has a half-life period. For uranium 238 it is 4.55 billion years. For potassium it is 1.3 billion years. For thorium it is 13.9 billion years. For rubidium it is 50 billion years while for carbon 14, it is only 5568 years and for white lead it is only 22 years.

In radiogeology, these results are used for radioactive dating. Thus the ratio of radio-carbon to ordinary carbon (carbon 12) in dead plants and animals enables us to estimate their time of death. Radioactive dating has also been used to estimate the age of the solar system and of earth as 45 billion years.

2.2.6 Decrease of Temperature

According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the difference between the temperature T of the body and temperature T_s of the surrounding medium, so that

$$\frac{dT}{dt} = k(T - T_s),\ k < 0 \tag{13}$$

and

$$T(t) - T_s = (T(0) - T_s)e^{kt} \tag{14}$$

and the excess of the temperature of the body over that of the surrounding medium decays exponentially.

2.2.7 Diffusion

According to Fick's law of diffusion, the time rate of movement of a solute across a thin membrane is proportional of the area of the membrane and to the difference in concentrations of the solute on the two sides of the membrane.

If the area of the membrane is constant and the concentration of solute on one side is kept fixed at a and the concentration of the solution on the other side initially is $c_0 < a$, then Fick's law gives

$$\frac{dc}{dt} = k(a - c), \quad c(0) = c_0, \tag{15}$$

so that

$$a - c(t) = (a - c(0))e^{-kt} \tag{16}$$

and $c(t) \to a$ as $t \to \infty$, whatever be the value of c_0.

2.2.8 Change of Price of a Commodity

Let $p(t)$ be the price of a commodity at time t, then its rate of change is proportional to the difference between the demand $d(t)$ and the supply $s(t)$ of the commodity in the market so that

$$\frac{dp}{dt} = k(d(t) - s(t)), \tag{17}$$

where $k > 0$, since if demand is more than the supply, the price increases. If $d(t)$ and $s(t)$ are assumed linear functions of $p(t)$, i.e. if

$$d(t) = d_1 + d_2p(t), \quad s(t) = s_1 + s_2p(t), \quad d_2 < 0, s_2 > 0 \tag{18}$$

we get

$$\frac{dp}{dt} = k(d_1 - s_1 + (d_2 - s_2)p(t)) = k(a - \beta p(t)), \quad \beta > 0 \tag{19}$$

or

$$\frac{dp}{dt} = K(p_e - p(t)), \tag{20}$$

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or

$$\frac{dp}{dt} = K(p_e - p(t)), \tag{20}$$

where p_e is the equilibrium price, so that

$$p_e - p(t) = (p_e - p(0))e^{-kt} \tag{21}$$

and

$$p(t) \to p_e \quad \text{as} \quad t \to \infty$$

EXERCISE 2.2

1. Suppose the population of the world now is 4 billion and its doubling period is 35 years, what will be the population of the world after 350 years, 700 years, 1050 years? If the surface area of the earth is 1,860,000 billion square feet, how much space would each person get after 1050 years? →4,096 billion
2. Find the relation between doubling, tripling and quadrupling times for a population.
3. In an archeological wooden specimen, only 25% of original radio carbon 12 is present. When was it made?
4. The rate of change of atmospheric pressure p with respect to height h is assumed proportional to p. If $p = 14.7$ psi at $h = 0$ and $p = 7.35$ at $h = 17{,}500$ feet, what is p at $h = 10{,}000$ feet?
5. What is the rate of interest compounded continuously if a bank's rate of interest is 10% per annum?
6. A body where temperature T is initially 300°C is placed in a large block of ice. Find its temperature at the end of 2 and 3 minutes?
7. The concentration of potassium in kidney is 0.0025 milligrammes per cubic centimetre. The kidney is placed in a large vessel in which the potassium concentration is 0.0040 mg/cm³. In 1 hour the concentration in the kidney increases to 0.0027 mg/cm³. After how much time will the concentration be 0.0035 mg/cm³?
8. A population is decaying exponentially. Can this decay be stopped or reversed by an immigration at a large constant rate into the population?

2.3 NON-LINEAR GROWTH AND DECAY MODELS

2.3.1 Logistic Law of Population Growth

As population increases, due to overcrowding and limitations of resources, the birth rate b decreases and the death rate d increases with the population size x. The simplest assumption is to take

$$b = b_1 - b_2x,\; d = d_1 + d_2x, \quad b_1, b_2, d_1, d_2 > 0, \tag{22}$$

so that (2) becomes

$$\frac{dx}{dt} = ((b_1 - d_1) - (b_2 + d_2)x) = x(a - bx),\; a > 0,\, b > 0 \tag{23}$$

Integrating (23), we get

$$\frac{x(t)}{a - bx(t)} = \frac{x(0)}{a - bx(0)}\, e^{at} \tag{24}$$

Equations (23) and (24) show that

where p_e is the equilibrium price, so that

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so that (2) becomes

$$\frac{dx}{dt} = ((b_1 - d_1) - (b_2 + d_2)x) = x(a - bx), \; a > 0, \, b > 0 \tag{23}$$

Integrating (23), we get

$$\frac{x(t)}{a - bx(t)} = \frac{x(0)}{a - bx(0)} e^{at} \tag{24}$$

Equations (23) and (24) show that

(i) $x(0) < a/b \Rightarrow x(t) < a/b \Rightarrow dx/dt > 0 \Rightarrow x(t)$ is a monotonic increasing function of t which approaches a/b as $t \to \infty$.

(ii) $x(0) > a/b \Rightarrow x(t) > a/b \Rightarrow dx/dt < 0 \Rightarrow x(t)$ is a monotonic decreasing function of t which approaches a/b as $t \to \infty$.

Now from (23)

$$\frac{d^2x}{dt^2} = a - 2bx, \tag{25}$$

so that $d^2x/dt^2 \gtreqless 0$ according as $x \lesseqgtr a/2b$. Thus in case (i) the growth curve is convex if $x < a/2b$ and is concave if $x > a/2b$ and it has a point of inflexion at $x = a/2b$. Thus the graph of $x(t)$ against t is as given in Figure 2.2.

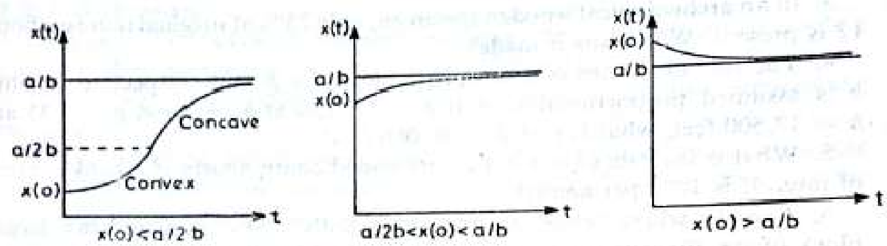

Figure 2.2

—If $x(0) < a/2b$, $x(t)$ increases at an increasing rate till $x(t)$ reaches $a/2b$ and then it increases at a decreasing rate and approaches a/b at $t \to \infty$

—If $a/2b < x(0) < a/b$, $x(t)$ increases at a decreasing rate and approaches a/b as $t \to \infty$

—If $x(0) = a/b$, $x(t)$ is always equal to a/b

—If $x(0) > a/b$, $x(t)$ decreases at a decreasing absolute rate and approaches a/b as $t \to \infty$

2.3.2 Spread of Technological Innovations and Infectious Diseases

Let $N(t)$ be the number of companies which have adopted a technological innovation till time t, then the rate of change of the number of these companies depends both on the number of companies which have adopted this innovation and on the number of those which have not yet adopted it, so that if R is the total number of companies in the region

$$\frac{dN}{dt} = kN(R - N), \tag{26}$$

which is the logistic law and shows that ultimately all companies will adopt this innovation.

Similarly if $N(t)$ is the number of infected persons, the rate at which the number of infected persons increases depends on the product of the numbers of infected and susceptible persons. As such we again get (26), where R is the total number of persons in the system.

It may be noted that in both the examples, while $N(t)$ is essentially an integer-valued variable, we have treated it as a continuous variable. This can be regarded as an idealisation of the situation or as an approximation to reality.

2.3.3 Rate of Dissolution

Let $x(t)$ be the amount of undissolved solute in a solvent at time t and let c_0 be the maximum concentration or saturation concentration, i.e. the maximum amount of the solute that can be dissolved in a unit volume of the solvent. Let V be the volume of the solvent. It is found that the rate at which the solute is dissolved is proportional to the amount of undissolved solute and to the difference between the concentration of the solute at time t and the maximum possible concentration, so that we get

$$\frac{dx}{dt} = kx(t)\left(\frac{x(0) - x(t)}{V} - c_0\right) = \frac{kx(t)}{V}\left((x_0 - c_0V) - x(t)\right) \quad (27)$$

2.3.4 Law of Mass Action: Chemical Reactions

Two chemical substances combine in the ratio $a : b$ to form a third substance Z. If $z(t)$ is the amount of the third substance at time t, then a proportion $az(t)/(a + b)$ of it consists of the first substance and a proportion $bz(t)/(a + b)$ of it consists of the second substance. The rate of formation of the third substance is proportional to the product of the amount of the two component substances which have not yet combined together. If A and B are the initial amounts of the two substances, then we get

$$\frac{dz}{dt} = k\left(A - \frac{az}{a+b}\right)\left(B - \frac{bz}{a+b}\right) \quad (28)$$

This is the non-linear differential equation for a second order reaction. Similarly for an nth order reaction, we get the non-linear equation

$$\frac{dz}{dt} = k(A_1 - a_1z)(A_2 - a_2z) \ldots (A_n - a_nz), \quad (29)$$

where $a_1 + a_2 + \ldots + a_n = 1$.

EXERCISE 2.3

1. If in (24), $a = 0.03134$, $b = (1.5887)(10)^{-10}$, $x(0) = 39 \times 10^6$, show that

$$x(t) = \frac{313,400,000}{1.5887 + 78,7703^{-0.03134t}}$$

This is Verhulst model for the population of USA when time zero corresponds to 1790. Estimate the population of USA in 1800, 1850, 1900 and 1950. Show that the point of inflexion should have occurred in about 1914. Find also the limiting population of USA on the basis of this model.

2. In (26) $k = 0.007$, $R = 1000$, $N(0) = 50$, find $N(10)$ and find when $N(t) = 500$.

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3. Obtain the solution of (27) when $x_0 > c_0 V$ and $x_0 < c_0 V$ and interpret your results.

4. Obtain the solutions of (28) and (29).

5. Substances X and Y combine in the ratio 2 : 3 to form Z. When 45 grams of X and 60 grams of Y are mixed together, 50 gms of Z are formed in 5 minutes. How many grams of Z will be found in 210 minutes? How much time will it like to get 70 gms of Z?

6. Cigarette consumption in a country increased from 50 per capita in 1900 A.D to 3900 per capita in 1960 AD. Assuming that the growth in consumption follows a logistic law with a limiting consumption of 4000 per capita, estimate the consumption per capita in 1950.

7. One possible weakness of the logistic model is that the average growth rate $1/x\, dx/dt$ is largest when x is small. Actually some species may become extinct if this population becomes very small. Suppose m is the minimum viable population for such a species, then show that

$$dx/dt = rx\left(1 - \frac{x}{k}\right)\left(1 - \frac{m}{x}\right)$$

has the desired property that x becomes extinct if $x_0 < m$. Also solve the differential equations in the two cases when $x_0 > m$ and $x_0 < m$.

8. Show that the logistic model can be written as

$$\frac{1}{N}\frac{dN}{dt} = r\left(\frac{K-N}{K}\right)$$

Deduce that K is the limiting size of the population and the average rate of growth is proportional to the fraction by which the population is unsaturated.

9. If $F(t)$ is the food consumed by population $N(t)$ and S is the food consumed by the population K, Smith replaced $(K-N)/N$ in Ex. 8 by $(S-F)/S$. He also argued that since a growing population consumes food faster than a saturated population, we should take $F(t) = c_1 N + c_2\, dN/dt$, $c_1, c_2 > 0$. Use this assumption to modify the logistic model and solve the resulting differential equation.

9. A generalisation of the logistic model is

$$\frac{1}{N}\frac{dN}{dt} = \frac{r}{\alpha}\left(1 - \left(\frac{N}{K}\right)^{\alpha}\right), \quad \alpha > 0$$

Solve this differential equation. Show that the limiting population is still K and the point of inflexion occurs when the population in $K(\alpha+1)^{1/2\alpha}$. Show that this increases monotonically from $K/2$ to K as α increases from unity to ∞. What is the model if $\alpha \to 0$? What happens if $\alpha \to -1$?

10. A fish population which is growing according to logistic law is harvested at a constant rate H. Show that

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$$

Show that if $D = kH/r - K^2/4 = a^2 > 0$, $N(t)$ approaches a constant limit as $t \to \pi/2\, K/r^2$, but is discontinuous there and cannot predict beyond this

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3. Obtain the solution of (27) when $x_0 > c_0V$ and $x_0 < c_0V$ and interpret your results.

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Solve this differential equation. Show that the limiting population is still K and the point of inflexion occurs when the population in $K(\alpha+1)^{1/2\alpha}$. Show that this increases monotonically from $K/2$ to K as α increases from unity to ∞. What is the model if $\alpha \to 0$? What happens if $\alpha \to -1$?

10. A fish population which is growing according to logistic law is harvested at a constant rate H. Show that

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$$

Show that if $D = kH/r - K^2/4 = a^2 > 0$, $N(t)$ approaches a constant limit as $t \to \pi/2\, K/r^2$, but is discontinuous there and cannot predict beyond this value of t. If $D = 0$, show that the limiting population is $K/2$. If $D < 0$, show that the ultimate population size is $K/2(1 + \sqrt{1 - 4H/rK})$.

11. For each of the models discussed in this subsection, state explicitly the assumptions made. Try to extend the model when one or more of these assumptions are given up or modified. Obtain some critical results which may be different between the original and modified models and which may be capable of being tested through observations and experiments.

2.4 COMPARTMENT MODELS

In the last two sections, we got mathematical models in terms of ordinary differential equations of the first order, in all of which variables were separable. In the present section, we get models in terms of linear differential equations of first order.

We also use here the principle of continuity i.e. that the gain in amount of a substance in a medium in any time is equal to the excess of the amount that has entered the medium in the time over the amount that has left the medium in this time.

2.4.1 A Simple Compartment Model

Let a vessel contain a volume V of a solution with concentration $c(t)$ of a substance at time t (Figure 2.3) Let a solution with constant concentration C in an overhead tank enter the vessel at a constant rate R and after mixing thoroughly with the solution in the vessel, let the mixture with concentration $c(t)$ leave the vessel at the same rate R so that the volume of the solution in the vessel remains V.

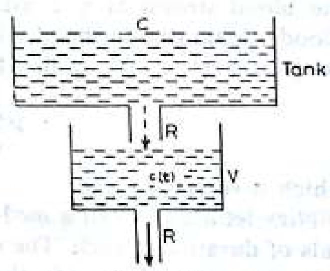

Figure 2.3

Using the principle of continuity, we get

$$V(c(t + \Delta t) - c(t)) = RC\Delta t - Rc(t)\Delta t + 0(\Delta t)$$

giving

$$V\frac{dc}{dt} + Rc = RC \tag{30}$$

Integrating

$$c(t) = c(0)\exp\left(-\frac{R}{V}t\right) + C\left(1 - \exp\left(-\frac{R}{V}t\right)\right) \tag{31}$$

As $t \to \infty$, $c(t) \to C$, so that ultimately the vessel has the same concentration as the overhead tank. Since

$$c(t) = C - (C - c_0)\exp\left(-\frac{R}{V}t\right), \tag{32}$$

if $C > c_0$, the concentration in the vessel increases to C; on the other hand if $C < c_0$, the concentration in the vessel decreases to C (Figure 2.4).

If the rate R' at which the solution leaves the vessel is less than R, the equations of continuity gives

$$\frac{d}{dt}[(V_0 + (R - R')t)c(t)] = RC - R'(ct) \tag{33}$$

where V is the initial volume of the solution in the vessel. This is also a linear differential equation of the first order.

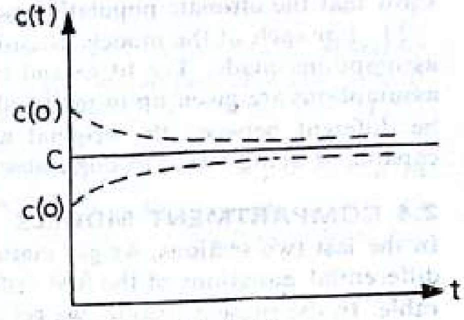

Figure 2.4

2.4.2 Diffusion of Glucose or a Medicine in the Blood Stream

Let the volume of blood in the human body be V and let the initial concentration of glucose in the blood stream be $c(0)$. Let glucose be introduced in the blood stream at a constant rate I. Glucose is also removed from the blood stream due to the physiological needs of the human body at a rate proportional to $c(t)$, so that the continuity principle gives

$$V\frac{dc}{dt} = I - kc \tag{34}$$

which is similar to (30).

Now let a dose D of a medicine be given to a patient at regular intervals of duration T each. The medicine also disappears from the system at a rate proportional to $c(t)$, the concentration of the medicine in the blood stream, then the differential equation given by the continuity principle is

$$V\frac{dc}{dt} = -kc \tag{35}$$

Integrating

$$c(t) = D\exp\left(-\frac{k}{V}t\right), \quad 0 \leqslant t < T \tag{36}$$

At time T, the residue of the first dose is $D\exp\left(-\frac{k}{V}T\right)$ and now another dose D is given so that we get

$$c(t) = \left(D\exp\left(-\frac{k}{V}T\right) + D\right)\exp\left(-\frac{k}{V}(t-T)\right), \tag{37}$$

$$= D\exp\left(-\frac{k}{V}t\right) + D\exp\left(-\frac{k}{V}(t-T)\right), \tag{38}$$

$$T \leqslant t < 2T$$

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if $C > c_0$, the concentration in the vessel increases to C; on the other hand if $C < c_0$, the concentration in the vessel decreases to C (Figure 2.4).

If the rate R' at which the solution leaves the vessel is less than R, the equations of continuity gives

$$\frac{d}{dt}[(V_0 + (R - R')t)c(t)]$$

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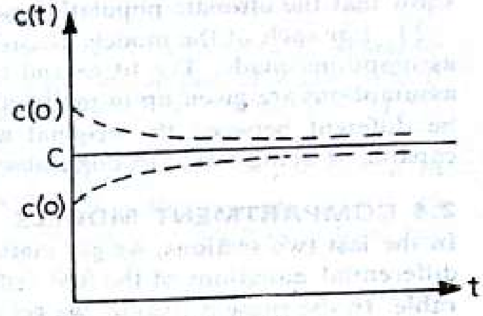

Figure 2.4

2.4.2 Diffusion of Glucose or a Medicine in the Blood Stream

Let the volume of blood in the human body be V and let the initial concentration of glucose in the blood stream be $c(0)$. Let glucose be introduced in the blood stream at a constant rate I. Glucose is also removed from the blood stream due to the physiological needs of the human body at a rate proportional to $c(t)$, so that the continuity principle gives

$$V\frac{dc}{dt} = I - kc \tag{34}$$

which is similar to (30).

Now let a dose D of a medicine be given to a patient at regular intervals of duration T each. The medicine also disappears from the system at a rate proportional to $c(t)$, the concentration of the medicine in the blood stream, then the differential equation given by the continuity principle is

$$V\frac{dc}{dt} = -kc \tag{35}$$

Integrating

$$c(t) = D\exp\left(-\frac{k}{V}t\right), \quad 0 \leqslant t < T \tag{36}$$

At time T, the residue of the first dose is $D\exp\left(-\frac{k}{V}T\right)$ and now another dose D is given so that we get

$$c(t) = \left(D\exp\left(-\frac{k}{V}T\right) + D\right)\exp\left(-\frac{k}{V}(t - T)\right), \tag{37}$$

$$= D\exp\left(-\frac{k}{V}t\right) + D\exp\left(-\frac{k}{V}(t - T)\right), \tag{38}$$

$$T \leqslant t < 2T$$

The first term gives the residual of the first dose and the second term gives the residual of the second dose. Proceeding in the same way, we get after n doses have been given

$$c(t) = D\exp\left(-\frac{k}{V}t\right) + D\exp\left(-\frac{k}{V}(t-T)\right)$$
$$+ D\exp\left(-\frac{k}{V}(t-2T)\right) + \ldots + D\exp\left(-\frac{k}{V}(t-\overline{n-1}T)\right) \quad (39)$$
$$= D\exp\left(-\frac{k}{V}t\right)\left(1+\exp\left(\frac{k}{V}T\right)+\exp\left(\frac{2k}{V}T\right)\right.$$
$$\left.+\ldots+\exp\left((n-1)\frac{k}{V}T\right)\right)$$
$$= D\exp\left(-\frac{k}{V}t\right)\frac{\exp\left(n\frac{k}{V}T\right)-1}{\exp\left(\frac{k}{V}T\right)-1}, \quad (n-1)T \leqslant t < nT \quad (40)$$

$$c(nT-0) = D\frac{1-\exp\left(-\frac{k}{V}nT\right)}{\exp\left(\frac{kT}{V}\right)-1} \quad (41)$$

$$c(nT+0) = D\frac{\exp\left(\frac{kT}{V}\right)-\exp\left(-\frac{k}{V}nT\right)}{\exp\left(\frac{kT}{V}\right)-1} \quad (42)$$

Thus the concentration never exceeds $D/\left(1-\exp\left(-\frac{kT}{V}\right)\right)$. The graph of $c(t)$ is shown in Figure 2.5.

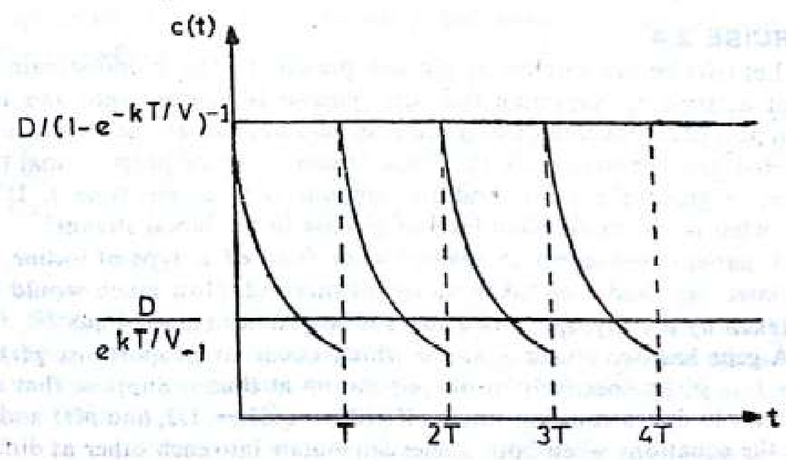

Figure 2.5

Thus in each interval, concentration decreases. In any interval, the concentration is maximum at the beginning of this interval and thus maximum concentration at the beginning of an interval goes on increasing as the number of intervals increases, but the maximum value is always below $D/(1 - e^{-kT/V})$. The minimum value in an interval occurs at the end of each interval. This also increases, but it lies below $D/(\exp(kT/V) - 1)$.

The concentration curve is piecewise continuous and has points of discontinuity at $T, 2T, 3T, \ldots$

By injecting glucose or penicillin in blood and fitting curve (36) to the data, we can estimate the value of k and V. In particular this gives a method for finding the volume of blood in the human body.

2.4.3 The Case of a Succession of Compartments

Let a solution with concentration $c(t)$ of a solute pass successively into n tanks in which the initial concentrations of the solution are $c_1(0), c_2(0), \ldots, c_n(0)$. The rates of inflow in each tank is the same as the rate of outflow from the tank. We have to find the concentrations $c_1(t), c_2(t) \ldots c_n(t)$ at time t. We get the equations

$$\begin{aligned} V\frac{dc_1}{dt} &= Rc - Rc_1 \\ V\frac{dc_2}{dt} &= Rc_1 - Rc_2 \\ &\cdots\cdots\cdots\cdots \\ V\frac{dc_n}{dt} &= Rc_{n-1} - Rc_n \end{aligned} \tag{43}$$

By solving the first of these equations, we get $c_1(t)$. Substituting the value of $c_1(t)$ and proceeding in the same way, we can find $c_3(t), \ldots, c_n(t)$.

EXERCISE 2.4

1. Let $G(t)$ be the amount of glucose present in the blood-stream of a patient at time t. Assuming that the glucose is injected into the blood stream at a constant rate of C grames per minute, and at the same time is converted and removed from the blood stream at a rate proportional to the amount of glucose present, find the amount $G(t)$ at any time t. If $G(0) = G_0$, what is the equilibrium level of glucose in the blood stream?

2. A patient was given .5 micro-Curies (μc_i) of a type of iodine. Two hours later $.5\mu c_i$ had been taken up by his thryroid. How much would have been taken by the thyroid in two hours if he had been given $15\mu c_i$?

3. A gene has two alleles A and a which occur in proportions $p(t)$ and $q(t) = 1 - p(t)$ respectively in the population at time t. Suppose that allele A mutates to a at a constant rate μ. If $p(0) = q(0) = 1/2$, find $p(t)$ and $q(t)$. Write the equations when both alleles can mutate into each other at different rates.

Thus in each interval, concentration decreases. In any interval, the concentration is maximum at the beginning of this interval and thus maximum concentration at the beginning of an interval goes on increasing as the number of intervals increases, but the maximum value is always below $D/(1 - e^{-kT/V})$. The minimum value in an interval occurs at the end of each interval. This also increases, but it lies below $D/(\exp(kT/V) - 1)$.

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$$
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V\frac{dc_1}{dt} &= Rc - Rc_1 \\
V\frac{dc_2}{dt} &= Rc_1 - Rc_2 \qquad (43)\\
&\cdots\cdots\cdots\cdots \\
V\frac{dc_n}{dt} &= Rc_{n-1} - Rc_n
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Integrating

$$v^2 = \mu(a^2 - x^2), \tag{45}$$

where the particle is initially at rest at $x = a$. Equation (44) gives

$$\frac{dx}{dt} = -\sqrt{\mu}\,\sqrt{a^2 - x^2} \tag{46}$$

We take the negative sign since velocity increases as x decreases (Figure 2.6).

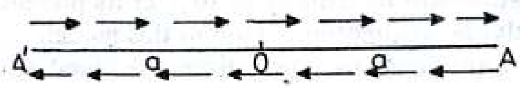

Figure 2.6

Integrating again and using the condition that at $t = 0$, $x = a$

$$x(t) = a \cos \sqrt{\mu}\, t \tag{47}$$

so that

$$v(t) = -a\sqrt{\mu}\, \sin \sqrt{\mu}\, t, \tag{48}$$

Thus in simple harmonic motion, both displacement and velocity are periodic functions with period $2\pi/\sqrt{\mu}$.

The particle starts from A with zero velocity and moves towards 0 with increasing velocity and reaches 0 at time $\pi/2\sqrt{\mu}$ with velocity $\sqrt{\mu}a$. It continue to move in the same direction, but now with decreasing velocity till it reaches $A'(0A' = a)$ where its velocity is again zero. It then begins moving towards 0 with increasing velocity and reaches 0 with velocity $\sqrt{\mu}a$ and again comes to rest at A after a total time period $2\pi/\sqrt{\mu}$. The periodic motion then repeats itself.

As one example of SHM, consider a particle of mass m attached to one end of a perfectly elastic string, the other end of which is attached to a fixed point 0 (Figure 2.7). The particle moves under gravity in vacuum.

Let l_0 be the natural length of the string and let a be its extension when the particle is in equilibrium so that by Hooke's law

$$mg = T_0 = \lambda \frac{a}{l_0} \tag{49}$$

where λ is the coefficient of elasticity. Now let the string be further stretched a distance c and then the mass be left free. The equation of motion which states that mass $\times$ acceleration in any direction = force

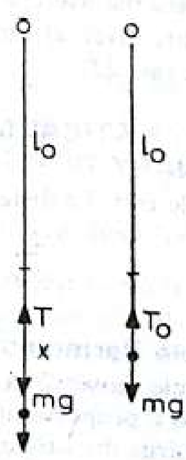

Figure 2.7

on the particle in that direction, gives

$$mv\frac{dv}{dx} = mg - T = mg - \lambda\frac{a + x}{l_0} = -\frac{\lambda x}{l_0} \qquad (50)$$

or

$$v\frac{dv}{dx} = -\frac{\lambda}{m}\frac{x}{l_0} = -\frac{gx}{a}, \qquad (51)$$

which gives a simple harmonic motion with time period $2\pi\sqrt{\frac{a}{g}}$.

2.5.2 Motion Under Gravity in a Resisting Medium

A particle falls under gravity in a medium in which the resistance is proportional to the velocity. The equation of motion is

$$m\frac{dv}{di} = mg - mkv$$

or

$$\frac{dv}{V - v} = k\,dt; \quad V = \frac{g}{k} \qquad (52)$$

Integrating

$$V - v = Ve^{-kt} \qquad (53)$$

If the particle starts from rest with zero velocity. Equation (50) gives

$$v = V(1 - e^{-kt}), \qquad (54)$$

so that the velocity goes on increasing and approaches the limiting velocity g/k as $t \to \infty$. Replacing v by dx/dt, we get

$$\frac{dx}{dt} = V(1 - e^{-kt}) \qquad (55)$$

Integrating and using $x = 0$ when $t = 0$, we get

$$x = Vt + \frac{Ve^{-kt}}{k} - \frac{V}{k}. \qquad (56)$$

2.5.3 Motion of a Rocket

As a first idealisation, we neglect both gravity and air resistance. A rocket moves forward because of the large supersonic velocity with which gases produced by the burning of the fuel inside the rocket come out of the converging-diverging nozzle of the rocket (Figure 2.8).

Let $m(t)$ be the mass of the rocket at time t and let it move forward with velocity $v(t)$ so that the momentum at time t is $m(t)v(t)$.

In the interval of time $(t, t + \Delta t)$, the mass of the rocket becomes

$$m(t + \Delta t) = m(t) + \frac{dm}{dt}\Delta t + 0(\Delta t)$$

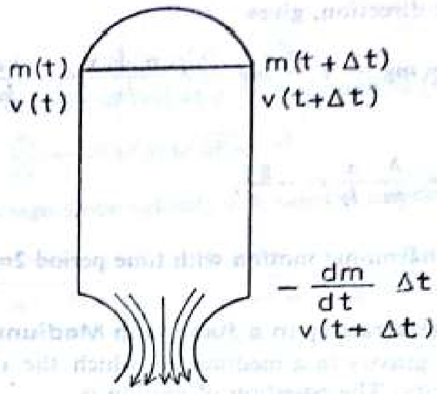

Figure 2.8

Since the rocket is losing mass, dm/dt is negative and the mass of gases $-dm/dt\ \Delta t$ moves with velocity u relative to the rocket, i.e. with a velocity $v(t + \Delta t) - u$ relative to the earth so that the total momentum of the rocket and the gases at time $t + \Delta t$ is

$$m(t + \Delta t)v(t + \Delta t) - \frac{dm}{dt}\Delta t(v(t + \Delta t) - u) \tag{57}$$

Since we are neglecting air resistance and gravity, there is no external force on the rocket and as such the momentum is conserved, giving the equation

$$m(t)v(t) = \left(m(t) + \frac{dm}{dt}\Delta t\right)\left(v(t) + \frac{dv}{dt}\Delta t\right) - \frac{dm}{dt}\Delta t(v - u) + 0(\Delta t)^2 \tag{58}$$

Dividing by Δt and proceeding to the limit as $\Delta t \to 0$, we get

$$m(t)\frac{dv}{dt} = -u\frac{dm}{dt} \tag{59}$$

or

$$\frac{dm}{m} = -\frac{1}{u}dv \tag{60}$$

or

$$\ln\frac{m(t)}{m(0)} = -\frac{v(t)}{u} \tag{61}$$

assuming that the rocket starts with zero velocity.

As the fuel burns, the mass of the rocket decreases. Initially the mass of the rocket $= m_P + m_F + m_S$ when m_P is the mass of the pay-load, m_F is the mass of the fuel and m_S is the mass of the structure. When the fuel is completely burnt out, m_F becomes zero and if v_B is the velocity of the rocket at this stage, when the fuel is all burnt, then (60) gives

$$v_B = u \ln \frac{m_P + m_F + m_S}{m_P + m_S} = u \ln \left(1 + \frac{m_F}{m_P + m_S}\right) \tag{62}$$

This is the maximum velocity that the rocket can attain and it depends on the velocity u of efflux of gases and the ratio $m_F/(m_P + m_S)$. The larger the values of u and $m_F/(m_P + m_S)$, the larger will be the maximum velocity attained.

For the best modern fuels and structural materials, the maximum velocity this gives is abount 7 km/sec. In practice it would be much less since we have neglected air resistance and gravity, both of which tend to reduce the velocity. However if a rocket is to place a satellite in orbit, we require a velocity of more than 7 km/sec.

The problem can be overcome by using the concept of multi-stage rockets.

The fuel may be carried in a number of containers and when the fuel of a container is burnt up, the container is thrown away, so that the rocket has not to carry any dead weight.

Thus in a three-stage rocket, let $m_{F_1}, m_{F_2}, m_{F_3}$ be the masses of the fuels and $m_{S_1}, m_{S_2}, m_{S_3}$ be the three corresponding masses of containers, then velocity at the end of the first stage is

$$v_1 = u \ln \frac{m_P + m_{F_1} + m_{S_1} + m_{F_2} + m_{S_2} + m_{F_3} + m_{S_3}}{m_P + m_{F_2} + m_{S_2} + m_{F_3} + m_{S_3}} \tag{63}$$

At the end the second stage, the velocity is

$$v_2 = v_1 + u \ln \frac{m_P + m_{F_2} + m_{F_3} + m_{S_3}}{m_P + m_{F_3} + m_{S_3}} \tag{64}$$

and at the end of the third stage, the velocity

$$v_3 = v_2 + u \ln \frac{m_P + m_{F_3}}{m_P} \tag{65}$$

In this way, a much larger velocity is obtained than can be obtained by a single-stage rocket.

EXERCISE 2.5

1. Discuss the problem of Section 2.5.1 when the particle start from A with velocity v_0 away from the origin.
2. Draw the graphs of $v(t)$ and $x(t)$ against t for two complete oscillations.
3. Discuss the motion of the particle in Section 2.5.2 when $c > a$
4. Show that for the same pay-load, same total fuel mass and some total structure mass, the final velocity of a multistage rocket is more than that of a single-stage rocket.
5. Discuss the motion of a rocket when gravity is taken into account.

6. If the particle attached to the elastic string in Figure 2.7 moves in a resisting medium, discuss its motion when the resistance is proportional to (i) velocity (ii) square of the velocity.

7. Discuss the motion of a particle projected vertically upwards under gravity with initial velocity U when the air resistance is proportional to the square of the velocity. With what velocity will the particle return to the Earth?

8. Assuming that a particle projected vertically upwards from the surface of the earth moves in vacuum under a force ga^2/x^2 directed toward the centre of earth, where x is the distance of the particle from the centre of the earth, find the initial velocity of projection so that the particle never return to earth.

2.6 MATHEMATICAL MODELLING OF GEOMETRICAL PROBLEMS THROUGH ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

2.6.1 Simple Geometrical Problems

Many geometrical entities can be expressed in terms of derivatives and as such relations between these entities can give rise to differential equations whose solution will give us a family of curves for which the given relation between geometrical entities is satisfied.

(i) Find curves for which tangent at a point is always perpendicular to the line joining the point to the origin.

The slope of the tangent is dy/dx and the slope of line joining the point (x, y) to the origin is y/x and since these lines are given to be orthogonal

$$\frac{dy}{dx} = -\frac{x}{y} \tag{66}$$

Integrating

$$x^2 + y^2 = a^2 \tag{67}$$

which represents a family of concentric circle.

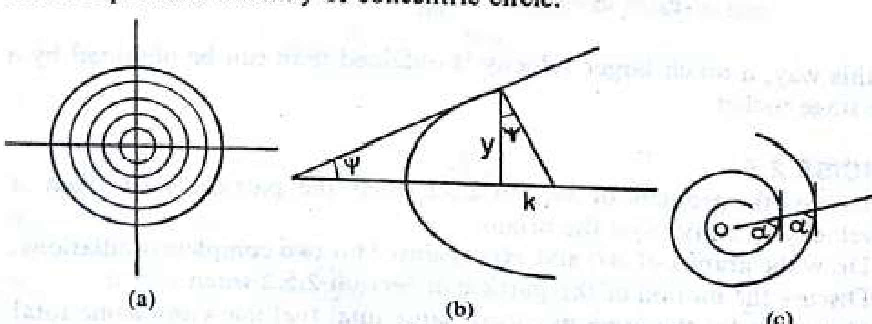

Figure 2.9

(ii) Find curves for which the projection of the normal on the x-axis is of constant length.

This condition gives

$$y\frac{dy}{dx} = k \tag{68}$$

Integrating

$$y^2 = 2kx + A, \tag{69}$$

which represents a family of parabolas, all with the same axis and same length of latus rectum.

(iii) Find curves for which tangent makes a constant angle with the radius vector.

Here it is convenient to use polar coordinates and the conditions of the problem gives

$$r\frac{d\theta}{dr} = \tan\alpha \tag{70}$$

Integrating

$$r = Ae^{\theta\cot\alpha}, \tag{71}$$

which represents a family of equiangular spirals.

2.6.2 Orthogonal Trajectories

Let

$$f(x, y, a) = 0 \tag{72}$$

represent a family of curves, one curve for each value of the parameter a. Differentiating (72), we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \tag{73}$$

Eliminating a between (72) and (73), we get a differential equation of the first order

$$\varphi\left(x, y, \frac{dy}{dx}\right) = 0, \tag{74}$$

of which (72) is the general solution.

Now we want a family of curves cutting every member of (72) at right angle at all points of intersection.

At a point of intersection of the two curves, x, y are the same but the slope of the second curve is negative reciprocal of the slope of the first curve. As such differential equation of the family of orthogonal trajectories is

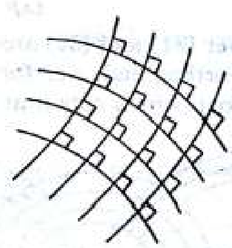

Figure 2.10

$$\varphi\left(x, y, -\frac{1}{dy/dx}\right) = 0 \tag{75}$$

Integrating (75), we get

$$g(x, y, b) = 0, \tag{76}$$

which give the orthogonal trajectories of the family (72).

(i) Let the original family be $y = mx$, when m is a parameter then

$$dy/dx = m$$

and eliminating m, we get the differential equation of this concurrent family of straight lines as

$$\frac{y}{x} = \frac{dy}{dx} \tag{77}$$

To get the orthogonal trajectories, we replace dy/dx by $-1/(dy/dx)$ to get

$$\frac{y}{x} = -\frac{1}{dy/dx}$$

Integrating

$$x^2 + y^2 = a^2 \tag{78}$$

which gives the orthogonal trajectories as concentric circles (Figure 2.9a).

(i) Find the orthogonal trajectories of the family of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \tag{79}$$

where λ is a parameter. Differentiating, we get

$$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0 \tag{80}$$

Eliminating λ between (79) and (80), we get

$$(xp - y)(x + py) = p(a^2 - b^2); \quad p = \frac{dy}{dx} \tag{81}$$

To get the orthogonal trajectories, we replace p by $-\frac{1}{p}$ to get

$$\left(-\frac{x}{p} - y\right)\left(x - \frac{y}{p}\right) = -\frac{1}{p}(a^2 - b^2)$$

or

$$(xp - y)(x + py) = p(a^2 - b^2) \tag{82}$$

However (81) and (82) are identical. As such the family of confocal conics is self-orthogonal, i.e. for every conic of the family, there is another with same focii which cuts it at right angles.

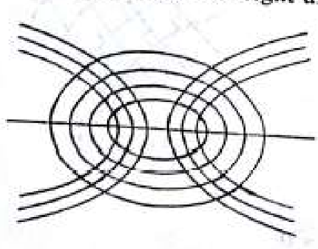

Figure 2.11

One family consists of confocal ellipses and the other consists of confocal hyperbolas with the same focii (Figure 2.11).

(iii) In polar coordinates after getting the differential equation of the family of curves, we have to replace $r\frac{d\theta}{dr}$ by $-1\Big/\left(r\frac{d\theta}{dr}\right)$ and then integrate the resulting differential equation.

Then if the original family is

$$r = 2a\cos\theta, \tag{83}$$

with $a > 0$ as a parameter, its differential equation is obtained by eliminating a between (83) and

$$\frac{dr}{d\theta} = -2a\sin\theta \tag{84}$$

to get

$$r\frac{d\theta}{dr} = -\cot\theta \tag{85}$$

Replacing $r\dfrac{d\theta}{dr}$ by $-\left(r\dfrac{d\theta}{dr}\right)^{-1}$, we get

$$r\frac{d\theta}{dr} = \tan\theta \tag{86}$$

Integrating we get

$$r = 2b\sin\theta \tag{87}$$

The orthogonal trajectories are shown in Figure 2.12.

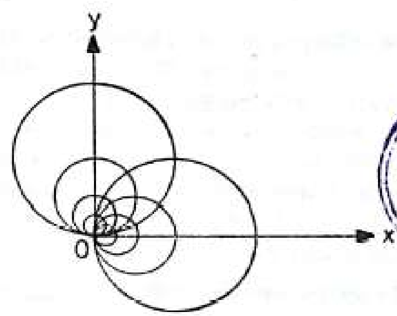

Figure 2.12

The circles of both families pass through the origin, but while the centre of one family lie on x-axis, the centres of the orthogonal family lie on y-axis.

EXERCISE 2.6

1. Find a family of curves such that for each curve, the length of the tangent intercepted between the axes is of constant length. Draw the curves.
2. Find a family of curves such that for each curve, the length of tangent intercepted between the point (x, y) and the axis of y is of constant length.
3. Find a curve such that all rays of light starting from the origin are reflected from points of the curve in the direction of the y-axis.
4. Find a curve such that all rays emanating from a given point $(-a, 0)$ after being reflected from points on the curve pass through the point $(a, 0)$.

5. Find the orthogonal trajectories of the families of curves

(i) $y^2 = 4cx$ (ii) $x^2 + y^2 - 2ax = 0$

(iii) $r = ae^{\theta \cot \alpha}$ (iv) $y^2 = 4cx + 4c^2$

(v) $r = a(1 + \cos \theta)$

5. In electrostatics, lines of force always cut equipotential curves (surfaces) at right angles. Find lines of force and equipotential surfaces for (i) one charge (ii) for two charges, and verify the result stated.

POSSIBLE QUESTIONS

Part B (6 Marks)

1. Write a note on Radio – Active Decay.

2. Discuss a simple Compartment Model.

3. Give a brief note on diffusion or a medicine in the blood stream.

4. Suppose the population of the world now is 4 billion and its doubling period is 35 years,what will be the population of the world after 350 years?

5. Design a mathematical model for motion of a rocket.

6. Give an explanatory note on simple compartment models

7. Explain about simple harmonic motion.

8. Discuss in detail about motion under gravity in a resisting medium.

9. Find the relation between doubling, tripling and quadrupling times a population.

Part C (10 Marks)

1. Discuss about logistic law of population growth.
2. Discuss a simple Compartment Model.
3. Give an explanatory note on simple compartment models.
4. Explain about simple geometric problems.

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Coimbatore –641 021

Subject: Mathematical Modelling **Subject Code: 18MMP303**

Class : II - M.Sc. Mathematics **Semester : III**

Unit I

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If there is one dependent continuous variable and a number of independent continuous variables then system is called	ODE	PDE	LDE	None	PDE
If there is one dependent variable and a one independent variable then system is called	ODE	PDE	LDE	None	ODE
In populatioanal growth model b and d denotes	birth and death	Business and death	birth and decrease	None	birth and death
In populatioanal growth T is called	Doubling period	Half life period	Total period	Time period	Doubling period
In growth if sciences and scientists the no of scientist should grow	Logically	exponentially	inversely	proportionally	Exponentially
If there is immigration into the population from outside at a rate __________ to the population size	Logically	exponentially	inversely	proportionally	Proportionally
e is the amount of an initial capital of 1 unit invested for 1 unit of time then the interest at unit rate is ________ continuously	simple	compounded	cumulative	principle	compounded
e is the amount of an initial capital of 1 unit invested for _____unit of time then the interest at unit rate is compounded continuously	1	2	3	4	1

In radio geology the of age solar system is used to estimate __________	Radio active	Diffusion	Decay	immigration	Radio active
The ratio of radio carbon to ordinary carbon in dead plants and animals enables to estimate their ____________	Time of birth	time of death	Time of dating	None	Time of death
________ law is used in the model' decrease of temperature'	Fick's	Hooke's	Newton's	Gauss	Newton's
________ law is used in the model 'Diffusion'	Fick's	Hooke's	Newton's	Gauss	Fick's
If P(t) price of commodity and its rate of change is proportional to the ___________ between demand and supply	Addition	Difference	Division	Multiplication	Difference
If P(t) price of commodity and its rate of change is ____________ to the difference between demand and supply	Logically	exponentially	inversely	Proportional	Proportional
In the model 'change of price of commodity S(t) denotes	System	Supply	Size	None	Supply
In the model 'change of price of commodity d(t) denotes	Demand	Death	Decrease	Diffusion	Demand
In the model 'change of price of commodity p_e denotes	Equilibrium price	Eligible price	Essential price	Evaluation price	Equilibrium price
As population increases the birth rate be decrease and death rate be _____________	Increases	stable	decreases	None	Increases
As population increases the birth rate be ____________ and death rate be increases	Increases	stable	decreases	None	Decreases
In the model spread of technological innovation and infestious diseases kN(R-N), R denotes	Total no of companies	companies adopted technological innovation	region	rate	Total no of companies
In rate of dissonution C0 be ______________ concentration	Maximum	Minimum	Both	None	Maximum
Two chemical substances combined in the ratio __________ to form the third substances Z	a:b	a:2b	2a:b	a:3b	a:b
The gain in amount of a substance in a medium in any time is ___________ to the excess of the amount that has entered the medium	Equal	Proportional	Linear	Exponential	Equal

The gain in amount of a substance in a medium in any time is equal to the excess of the amount that has______ the medium	exit	entered	outer	None	entered
A particle moves in a straight line then its acceleration is _________ to its distance from the origin	Logically	exponentially	inversely	Proportional	proportional
A particle moves in a straight line then its acceleration is proportional to its distance from the origin states that	SHM	MOR	MUG	None	SHM
A particle falls under __________ in a medium in which resistance is proportional to the velocity	Gravity	Sense	Force	Mass	Gravity
A particle falls under gravity in a medium in which resistance is proportional to the _________	Velocity	Sense	Force	Mass	Velocity
The equation of motion which states that mass x acceleration in any direction is _________ on the particle	Velocity	Sense	Force	Mass	Force
A rocket moves forward because of the large ____________ velocity	Ultra	Supersonic	Infrared	None	Supersonic
m(t) be the mass of rocket at time t with velocity v(t) then momentum is	m(t)+v(t)	m(t)v(t)	m(t)-v(t)	m(t)/v(t)	m(t)v(t)
mass of the rocket = mF+mP+mS then F is	Fuel	pay load	structure	ferrocity	Fuel
mass of the rocket = mF+mP+mS then P is	Pressure	pay load	structure	ferrocity	pay load
mass of the rocket = mF+mP+mS then S is	System	pay load	structure	ferrocity	structure
Curves for which tangent at a point is ___________ to the line joining the point to the origin.	Equal	Proportional	Perpendicular	Exponential	Perpendicular
Curves for which tangent at a point is perpendicular to the line joining the point to the _____________.	centre	point	origin	parallel	origin
Curves for which the projection of the normal on the x axis is of ______ length.	Variable	Constant	unit	y axis	Constant
Curves for which the _________ of the normal on the x axis is of constant length.	Projection	Property	Process	parameter	Projection
Curves for which the projection of the ________ on the x axis is of constant length.	Normal	Proportional	Perpendicular	Exponential	Normal

Curves for which __________ makes a constant angle with the radius vector.	tangent	Secant	Cosecant	Cot	tangent
Curves for which tangent makes a constant angle with the ______ vector.	diameter	radius	unit	scalar	radius
Curves for which tangent makes a _________ angle with the radius vector.	Variable	Constant	unit	y axis	Constant
The point of intersection of two curves the slope of second curve is _________ reciprocal of the first curve.	Positive	negative	unity	trajective	negative
The point of intersection of two curves the slope of second curve is negative ________ of the first curve.	Proportional	reciprocal	Exponential	Logically	reciprocal
The circles of both families pass through _________	Point	centre	Origin	None	Origin
The centres of one family lie on x axis the centres of orthogonal family lie on _______________	X axis	Y axis	Both axes	None	Y axis
The centres of one family lie on x axis the centres of ________ family lie on Y axis.	Proportional	linear	unit	orthogonal	orthogonal
The centres of one family lie on _______ the centres of orthogonal family lie on Y axis.	X axis	Y axis	Both axes	None	X axis
The family of confocal conics are________	self orthogonal	proportional	orthogonal	linear	self orthogonal

UNIT-II

Mathematical Modeling through Systems of Ordinary Differential Equations of First Order: Population Dynamics – Epidemics – Compartment Models – Economics – Medicine, Arms Race, Battles and International Trade – Dynamics.

3

Mathematical Modelling Through Systems of Ordinary Differential Equations of the First Order

3.1 MATHEMATICAL MODELLING IN POPULATION DYNAMICS

3.1.1 Prey-Predator Models

Let $x(t)$, $y(t)$ be the populations of the prey and predator species at time t. We assume that

(i) if there are no predators, the prey species will grow at a rate proportional to the population of the prey species,

(ii) if there are no prey, the predator species will decline at a rate proportional to the population of the predator species,

(iii) the presence of both predators and preys is beneficial to growth of predator species and is harmful to growth of prey species. More specifically the predator species increases and the prey species decreases at rates proportional to the product of the two populations.

These assumptions give the systems of non-linear first order ordinary differential equations

$$\frac{dx}{dt} = ax - bxy = x(a - by), \qquad a, b > 0 \tag{1}$$

$$\frac{dy}{dt} = -py + qxy = -y(p - qx), \quad p, q > 0 \tag{2}$$

Now dx/dt, dy/dt both vanish if

$$x = x_e = \frac{p}{q}, \qquad y = y_e = \frac{a}{b}. \tag{3}$$

If the initial populations of prey and predator species are p/q and a/b respectively, the populations will not change with time. These are the equilibrium sizes of the populations of the two species. Of course $x = 0$, $y = 0$ also gives another equilibrium position.

From (1) and (2)

$$\frac{dy}{dx} = -\frac{y(p - qx)}{x(a - by)} \quad \frac{dy/dt}{dx/dt} \tag{4}$$

or

$$\frac{a - by}{y}\,dy = -\frac{p - qx}{x}dx; \quad x_0 = x(0), \qquad y_0 = y(0) \tag{5}$$

Integrating

$$a \ln \frac{y}{y_0} + p \ln \frac{x}{x_0} = b(y - y_0) + q(x - x_0) \tag{6}$$

Thus through every point of the first quadrant of the x–y plane, there is a unique trajectory. No two trajectories can intersect, since intersection will imply two different slopes at the same point.

If we start with (0, 0) or $(p/q, a/b)$, we get point trajectories. If we start with $x = x_0, y = 0$, from (1) and (2), we find that x increases while y remains zero. Similarly if we start with $x = 0$, $y = y_0$, we find that x remains zero while y decreases. Thus positive axes of x and y give two line trajectories (Figure 3.1).

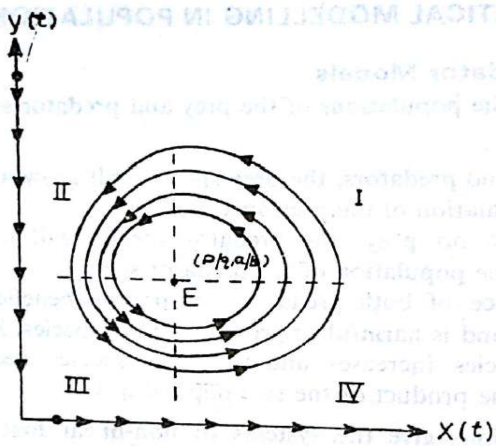

Figure 3.1

Since no two trajectories intersect, no trajectory starting from a point situated within the first quadrant will intersect the x-axis and y-axis trajectories. Thus all trajectories corresponding to positive initial populations will lie strictly within the first quadrant. Thus if the initial populations are positive, the populations will be always positive. If the population of one (or both) species is initially zero, it will always remain zero.

The lines through $(p/q, a/b)$ parallel to the axes of coordinates divide the first quadrant into four parts I, II, III and IV. Using (1), (2), we find that

in I,	$dx/dt < 0$,	$dy/dt > 0$,	$dy/dx < 0$
in II,	$dx/dt < 0$,	$dy/dt < 0$,	$dy/dx > 0$
in III,	$dx/dt > 0$,	$dy/dt < 0$,	$dy/dx < 0$
in IV,	$dx/dt > 0$,	$dy/dt > 0$,	$dy/dx > 0$

This give the direction field at all points as shown in Figure 3.1. Each trajectory is a closed convex curve. These trajectories appear relatively cramped near the axes.

In I and II, prey species decreases and in III and IV, it increases. Similarly in IV and I, predator species increases and in II and III, it decreases. After a certain period, both species return to their original sizes and thus both species sizes vary periodically with time.

3.1.2 Competition Models

Let $x(t)$ and $y(t)$ be the populations of two species competing for the same resources, then each species grows in the absence of the other species, and the rate of growth of each species decreases due to the presence of the other species. This gives the system of differential equations

$$\frac{dx}{dt} = ax - bxy = bx\left(\frac{a}{b} - y\right); \quad a > 0, \quad b > 0 \tag{7}$$

$$\frac{dy}{dt} = py - qxy = y(p - qx) = qy\left(\frac{p}{q} - x\right); \quad p > 0, \quad q > 0 \tag{8}$$

There are two equilibrium positions viz. $(0, 0)$ and $(p/q,\ a/b)$. There are two point trajectories viz. $(0, 0)$ and $(p/q,\ a/b)$ and there are two line trajectories viz. $x = 0$ and $y = 0$.

$$\begin{array}{llll} \text{In I} & dx/dt < 0, & dy/dt < 0, & dy/dx > 0 \quad (9) \\ \text{In II} & dx/dt < 0, & dy/dt > 0, & dy/dx < 0 \\ \text{In III} & dx/dt > 0, & dy/dt > 0, & dy/dx > 0 \\ \text{In IV} & dx/dt > 0, & dy/dt < 0, & dy/dx < 0 \quad (10) \end{array}$$

This gives the direction field as shown in Figure 3.2. From (7) and (8)

$$\frac{dy}{dx} = \frac{y(p - qx)}{x(a - by)} \quad \text{or} \quad \frac{a - by}{y}\, dy = \frac{p - qx}{x}\, dx \tag{11}$$

Integrating

$$a \ln \frac{y}{y_0} - b(y - y_0) = p \ln \frac{x}{x_0} - q(x - x_0) \tag{12}$$

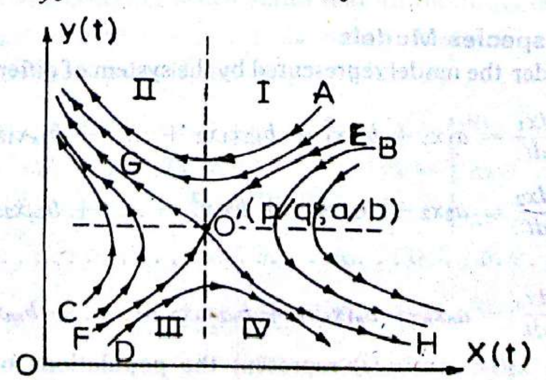

Figure 3.2

The trajectory which passes through $(p/q, a/b)$ is

$$a \ln \frac{by}{a} - by + a = p \ln \frac{qx}{p} - qx + p \tag{13}$$

If the initial populations correspond to the point A, ultimately the first species dies but and the second species increases in size to infinity. If the initial populations correspond to the point B, then ultimately the second species dies out and the first species tends to infinity. Similarly if the initial populations correspond to point C, the first species dies out and the second species goes to infinity and if the initial populations correspond to point D, the second species dies out and the first species goes to infinity.

If the initial populations correspond to point E or F, the species populations converge to equilibrium populations p/q, a/b and if the initial population correspond to point G, H, the first and second species die out respectively.

Thus except when the initial populations correspond to points on curves $O'E$ and $O'F$, only one species will survive in the competition process and the species can coexist only when the initial population sizes correspond to points on the curve EF.

It is also interesting to note that while the initial populations corresponding to A, E, B are quite close to one another, the ultimate behaviour of these populations are drastically different. For populations starting at A, the second species alone survives, for populations starting at B, the first species alone survives, while for population starting at E, both species can coexist. Thus a slight change in the initial population sizes can have a catastrophic effect on the ultimate behaviour.

It may also be noted that for both prey-predator and competition models, we have obtained a great deal of insight into the models without using the solution of these equations (1), (2) or (7), (8). By using numerical methods of integration with the help of computers, we can draw some typical trajectories in both cases and can get additional insight into the behaviour of these models.

3.1.3 Multi-species Models

We can consider the model represented by the system of differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= a_1 x_1 + b_{11} x_1^2 + b_{12} x_1 x_2 + \ldots + b_{1n} x_1 x_n \\
\frac{dx_2}{dt} &= a_2 x_2 + b_{21} x_2 x_1 + b_{22} x_2^2 + \ldots + b_{2n} x_2 x_n \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{dx_n}{dt} &= a_n x_n + b_{n1} x_n x_1 + b_{n2} x_n x_2 + \ldots + b_{nn} x_n^2
\end{aligned} \tag{14}$$

Here $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ represent the populations of the n species. Also a_i is positive or negative according as the ith species grows or decays in the absence of other species and b_{ij} is positive or negative according as the ith species benefits or is harmed by the presence of the jth species. In general b_{ii} is negative since members of the ith species also compete among themselves for limited resources.

We can find the positions of equilibrium by putting

$$dx_i/dt = 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

and solving the n algebraic equations for $x_1, x_2, \ldots, x_n$. We can also obtain all degenerate solutions in which one or more x_i's are zero, i.e. in which one or more species have disappeared and finally we have the equilibrium position in which all species can disappear.

If $x_{10}, x_{20}, \ldots, x_{n0}$ is an equilibrium position, we can discuss its local stability by substituting

$$x_1 = x_{10} + u_1, \quad x_2 = x_{20} + u_2, \ldots, \quad x_n = x_{n0} + u_n \tag{15}$$

14) and getting a system of linear differential equations

$$\begin{aligned} \frac{du_1}{dt} &= c_{11}u_1 + c_{12}u_2 + \ldots + c_{1n}u_n \\ \frac{du_2}{dt} &= c_{21}u_1 + c_{22}u_2 + \ldots + c_{2n}u_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{du_n}{dt} &= c_{n1}u_1 + c_{n2}u_2 + \ldots + c_{nn}u_n, \end{aligned} \tag{16}$$

by neglecting squares, products and higher powers of u_i's. We can try the solutions $u_1 = A_1e^{\lambda t}$, $u_2 = A_2e^{\lambda t}, \ldots, u_n = A_ne^{\lambda t}$ to get

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & c_{13} & \ldots & c_{1n} \\ c_{21} & c_{22}-\lambda & c_{23} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n1} & c_{n2} & c_{n3} & \ldots & c_{nn}-\lambda \end{vmatrix} = 0 \tag{17}$$

Thus the equilibrium position would be stable if the real parts of all the eigenvalues of the matrix $[c_{ij}]$ are negative. The conditions for this are given by Routh-Hurwitz criterion which states that all the roots of

$$a_0x^n + a_1x^{n-1} + \ldots + a_n = 0, \quad a_0 > 0 \tag{18}$$

will have negative real parts if and only if $T_0, T_1, T_2, \ldots$ are positive where

$$T_0 = a_0, \quad T_1 = a_1, \quad T_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad T_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$

$$T_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} \tag{19}$$

This is true if and only if $a_i > 0$ and either all even-numbered T_k or all odd-numbered T_k are positive. Alternatively (18) will have all roots with negative real parts iff this is true for the $(n-1)$th degree equation

$$a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \dots - \frac{a_0}{a_1} a_3 x^{n-2} - \frac{a_0}{a_1} a_5 x^{n-4} - \dots = 0 \quad (20)$$

The above method will enable us to discuss only local stability of a position of equilibrium, i.e. this will decide that if the populations of different species are changed slightly from these equilibrium values, whether the population sizes will return to their original equilibrium values or not. The problem of discussing the global stability i.e. of discussing whether the populations will return to these equilibrium values, whatever be the magnitudes of the disturbances, is a more difficult problem and it is possible to solve this problem in special cases only.

3.1.4 Age-Structured Population Models

Let $x_1(t), x_2(t), \dots, x_p(t)$ be the populations of the p pre-reproductive age-groups; let $x_{p+1}(t), \dots, x_{p+q}(t)$ be the populations of q reproductive age-groups and let $x_{p+q+1}(t), \dots, x_{p+q+r}(t)$ be the populations of the r post-reproductive age-groups. Let $b_{p+1}, b_{p+2}, \dots, b_{p+q}$ be the birth rates in the q reproductive age-groups, let d_i be the death rates in the ith age-group $(i = 1, 2, \dots, p+q+r)$ and let m_j be the rate of migration from the jth age-group to the $(j+1)$th age-group $(j = 1, 2, \dots, p+q+r-1)$, then we get the system of differential equations

$$\frac{dx_1}{dt} = b_{p+1}x_{p+1} + \dots + b_{p+q}x_{p+q} - (d_1 + m_1)x_1 \quad (21)$$

$$\frac{dx_2}{dt} = m_1 x_1 - (d_2 + m_2)x_2$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{dx_n}{dt} = m_{n-1}x_{n-1} - d_n x_n; \quad n = p + q + r$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

$$= \begin{bmatrix} -(d_1+m_1) & 0 & \dots & b_{p+1} & \dots & b_{p+q} & \dots & 0 & 0 \\ m_1 & -(d_2+m_2) & \dots & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & m_2 & \dots & 0 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & \dots & m_{n-1} & -d_n \end{bmatrix}$$

$$\times \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} \tag{22}$$

or

$$\frac{dX}{dt} = AX(t), \tag{23}$$

where A is a matrix, all of whose diagonal elements are negative, all of whose main subdiagonal elements are positive, q other elements of the first row are positive and all other elements are zero. Equation (22) has the solution

$$X(t) = \exp(At)X(0) \tag{24}$$

EXERCISES 3.1

1. Draw some trajectories for the model

$$\frac{dx}{dt} = x(1 - 0.1y), \quad \frac{dy}{dt} = -y(1 - 0.1x)$$

2 Discuss the stability of the equilibrium positions (0, 0) and $(p/q,\ a/b)$ for the prey-predator model represented by equations (1) and (2) and the competition model represented by equations (7) and (8).

3. Draw some trajectories for the competition model

$$\frac{dx}{dt} = x(1 - 0.1y), \quad \frac{dy}{dt} = y(1 - 0.1x).$$

4. By integrating (1), (2) round a closed trajectory, show that

$$0 = a\bar{x} - b\overline{xy}, \quad 0 = -p\bar{y} + q\overline{xy}$$
$$0 = a - b\bar{y}, \quad 0 = -p + q\bar{x},$$

where $\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt, \quad \bar{y} = \frac{1}{T}\int_0^T y(t)\,dt, \quad \overline{xy} = \frac{1}{T}\int_0^T x(t)y(t)\,dt,$

and T is the time for the populations to return to original values.

5. Write the basic equations for the wolf-goat-cabbage model in which wolves eat goats, goats eat cabbages, but wolves do not eat cabbages.

6. Show that the model represented by

$$\frac{dx}{dt} = x(4 - x - y), \quad \frac{dy}{dt} = y(15 - 5x - 3y), \ x \geq 0, \quad y \geq 0$$

has a position of equilibrium, this position is stable and two species can coexist.

7. Show that the model represented by

$$\frac{dx}{dt} = x(15 - 5x - 3y), \quad \frac{dy}{dt} = y(4 - x - y), \ x \geq 0, \quad y \geq 0$$

has a position of equilibrium, this position is unstable, only one species will survive and which species survives depends on initial conditions:

8. Show that the model represented by

$$\frac{dx}{dt} = x(30 - 60 - 5y), \frac{dy}{dt} = y(12 - 40 - 3y), \quad x \geqslant 0, \quad y \geqslant 0$$

has no position of equilibrium and that only the first species will survive.

9. Show that the model represented by

$$\frac{dx}{dt} = x(12 - 4x - 3y), \ \frac{dy}{dt} = y(30 - 6x - 5y), \ x \geqslant 0, \ y \geqslant 0$$

has no position of equilibrium and that only the second species will survive.

10. For the model representing competition between two species, each of which can exist and grow without the other and contact between which inhibits the growth of both, the differential equations are given by

$$\frac{dx}{dt} = x(A_1 - B_1x - C_1y), \ \frac{dy}{dt} = y(A_2 - B_2y - C_2x),$$

where A_1, B_1, C_1, A_2, B_2, C_2 are all positive.

Show that

(i) the equilibrium will be biologically meaningful, i.e. the equilibrium position will be in the first quadrant if

$$B_2/C_2 > A_2/A_1 > C_2/B_1 \text{ or } C_2/B_1 > A_2/A_1 > B_2/C_1.$$

(ii) if a biologically meaningful equilibrium exists, it will be stable iff $B_1B_2 > C_1C_2$, i.e. if the product of self-restraint coefficients is greater than the product of the other restraint coefficients.

(iii) if the equilibrium does not exist, the first species will survive if

$$A_1/C_2 > A_2/B_2 \quad \text{and} \quad A_1/B_1 > A_2/C_2.$$

11. Discuss the modification of the prey-predator model when
(i) the predator population is harvested at a constant rate h_1 or
(ii) the prey population is harvested at a constant rate h_2 or
(iii) both species are harvested at constant rates.

12. Discuss the possibility of the existence of a stable age-structure i.e. age-structure which does not change with time in the model of Section 3.1.4.

3.2 MATHEMATICAL MODELLING OF EPIDEMICS THROUGH SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

3.2.1 A Simple Epidemic Model

Let $S(t)$ and $I(t)$ be the number of susceptibles (i.e. those who can get a disease) and infected persons (i.e. those who have already got the disease).

Initially let there be n susceptible and one infected person in the system so that

$$S(t) + I(t) = n + 1, \quad S(0) = n, \quad I(0) = 1 \tag{25}$$

The number of infected persons grows at a rate proportional to the product of susceptible and infected persons and the number of susceptible persons decreases at the same rate so that we get the system of differential equations

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI, \tag{26}$$

so that

$$\frac{dS}{dt} + \frac{dI}{dt} = 0, \quad S(t) + I(t) = \text{constant} = n + 1 \tag{27}$$

and

$$\frac{dS}{dt} = -\beta S(n + 1 - S),$$

$$\frac{dI}{dt} = \beta I(n + 1 - I). \tag{28}$$

Integrating

$$S(t) = \frac{n(n+1)}{n + e^{(n+1)\beta t}}, \quad I(t) = \frac{(n+1)e^{(n+1)\beta t}}{n + e^{(n+1)\beta t}}, \tag{29}$$

so that

$$\operatorname*{Lt}_{t\to\infty} S(t) = 0, \quad \operatorname*{Lt}_{t\to\infty} I(t) = n + 1 \tag{30}$$

3.2.2 A Susceptible-Infected-Susceptible (SIS) Model

Here, a susceptible person can become infected at a rate proportional to SI and an infected person can recover and become susceptible again at a rate γI, so that

$$\frac{dS}{dt} = -\beta SI + \gamma I, \quad \frac{dI}{dt} = \beta SI - \gamma I, \tag{31}$$

which gives

$$\frac{dI}{dt} = (\beta(n+1) - \gamma)I - \beta I^2 \tag{32}$$

3.2.3 SIS Model with Constant Number of Carriers

Here infection is spread both by infectives and a constant number C of carriers, so that (30) becomes

$$\begin{aligned}\frac{dI}{dt} &= \beta(I + C)S - \gamma I \\ &= \beta C(n+1) + \beta(n + 1 - C - \gamma/\beta)I - \beta I^2. \end{aligned} \tag{33}$$

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3.2.4 Simple Epidence Model with Carriers

In this model, only carriers spread the disease and their number decreases exponentially with time as these are identified and eliminated, so that we get

$$\frac{dS}{dt} = -\beta S(t)C(t) + \gamma I(t), \quad \frac{dI}{dt} = \beta C(t)S(t) - \gamma I(t),$$
$$\frac{dC}{dt} = -\alpha C \tag{34}$$

so that

$$S(t) + I(t) = S_0 + I_0 = N \text{ (say)}, \ C(t) = C_0 \exp(-\alpha t) \tag{35}$$

and

$$\frac{dI}{dt} = \beta C_0 N \exp(-\alpha t) - [\beta C_0 \exp(-\alpha t) + \gamma] I \tag{36}$$

3.2.3 Model with Removal

Here infected persons are removed by death or hospitalisation at a rate proportional to the number of infectives, so that the model is

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I = \beta I\left(S - \frac{\gamma}{\beta}\right)$$
$$= \beta I(S - \rho); \quad \rho = \frac{\gamma}{\beta} \tag{37}$$

with initial conditions

$$S(0) = S_0 > 0, \ I(0) = I_0 > 0, \ R(0) = R_0 = 0,$$
$$S_0 + I_0 = N. \tag{38}$$

3.2.6 Model with Removal and Immigration

We modify the above model to allow for the increase of susceptibles at a constant rate μ so that the model is

$$\frac{dS}{dt} = -\beta SI + \mu, \ \frac{dI}{dt} = \beta SI - \gamma I, \ \frac{dR}{dt} = \gamma I. \tag{39}$$

EXERCISE 3.2

1. Verify (29) and (30).
2. Integrate (32) and show that

$$\underset{t\to\infty}{\text{Lt}}\, I(t) = n + 1 - \rho \quad \text{if} \quad n + 1 > \rho = \gamma/\beta$$
$$= 0 \quad \text{if} \quad n + 1 \leqslant \rho = \gamma/\beta$$

3. Solve SIS model when β is a known function of t.
4. Integrate (36) and find limit of $I(t)$ as $t \to \infty$.
5. Discuss integration of models given by (37) and (39) and interpret your results.

3.3 COMPARTMENT MODELS THROUGH SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Pharmokinetics (also called drug kinetics or tracer kinetics or multi-compartment analysis) deals with the distribution of drugs, chemicals, tracers or radio-active substances among various compartments of the body where compartments are real or fictitious spaces for drugs.

Let $x_i(t)$ be the amount of the drug in the ith compartment at time t. We shall assume that the amount that can be transferred from the ith to the jth compartment $(j \neq i)$ in the time interval $(t, t + \Delta t)$ is $k_{ij}x_i(t)\Delta t + 0(\Delta t)$ where k_{ij} is called the transfer coefficient from the ith to the jth compartment. The total change Δx_i in time Δt is given by the amount entering the ith compartment from other compartments which is reduced by the amount leaving the ith compartment for other compartments including the zeroeth compartment that denotes the outside system.

Thus we get

$$\Delta x_i = -\sum_{\substack{j=0\\ j\neq i}}^{n} k_{ij}x_i\Delta t + \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ji}x_j\Delta t + 0(\Delta t) \tag{40}$$

Dividing by Δt and proceeding to the limit as $\Delta t \to 0$, we get

$$\frac{dx_i}{dt} = -x_i \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij} + \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ji}x_j \tag{41}$$

$$= \sum_{j=1}^{n} k_{ji}x_j, \quad (i = 1, 2, \ldots, n), \tag{42}$$

where we define

$$k_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}, \quad (i = 1, 2, \ldots, n) \tag{43}$$

In matrix notation, we have

$$dX/dt = KX, \tag{44}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ k_{1n} & k_{2n} & \cdots & k_{nn} \end{bmatrix} \tag{45}$$

If $X = Be^{\lambda t}$, when B is a column matrix, (44) gives

$$\lambda Be^{\lambda t} = KBe^{\lambda t} \tag{46}$$

This gives a consistant system of equations to determine B if

$$| K - \lambda I | = 0 \tag{47}$$

where I is $n \times n$ unit matrix. Thus λ has to be an eigenvalue of the matrix K. We note that all the diagonal elements of K are negative, all the non-diagonal elements are non-negative and the sum of element of every column is greater than or equal to zero. For such a matrix, it can be shown that the real parts of the eigenvalues are always less than or equal to zero, and the imaginary part is non-zero only when the real part is strictly less than zero. Thus if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues then

$$\text{Re}\,(\lambda_i) \leqslant 0$$
$$\text{Im}\,(\lambda_i) \neq 0 \text{ only if Rl}\,(\lambda_i) < 0 \tag{48}$$

If the drug is injected at a constant rate given by the column vector D with components $D_1, D_2, \ldots, D_n$, (44) becomes

$$dX/dt = KX + D \tag{49}$$

Equations (44) and (49) constitute the basic equations for the analysis of drug distribution in the n-compartment system.

EXERCISE 3.3

1. Solve (44) and (49) for given initial conditions.
2. Let dose D be given at time 0, T, $2T$, $3T, \ldots$, Find

$$X(nT-0),\ X(nT+0),\quad X(nT+t),\quad (0 < t < T)$$

3. Discuss the special cases when $n = 1, n = 2$.

3.4 MATHEMATICAL MODELLING IN ECONOMICS BASED ON SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

3.4.1 Domar Macro Model

Let $S(t)$, $I(t)$, $Y(t)$ be the Savings, Investment and National Income at time t, then it is assumed that

(i) Savings are proportional to national income, so that

$$S(t) = \alpha Y(t),\quad \alpha > 0$$

(ii) Investment is proportional to the rate of increase of national income so that

$$I(t) = \beta Y'(t),\quad \beta > 0 \tag{51}$$

(iii) All savings are invested, so that

$$S(t) = I(t) \tag{52}$$

We get a system of three ordinary differential equations of first order for determining $S(t)$, $Y(t)$, $I(t)$. Solving we get

$$Y(t) = Y(0)\, e^{\alpha t/\beta},\qquad I(t) = \alpha Y(0) e^{\alpha t/\beta} = S(t), \tag{53}$$

so that the national income, investment and savings all increase exponentially.

3.4.2 Domar First Debt Model

Let $D(t)$, $Y(t)$ denote the total national debt and total national income respectively, then we assume that

(i) Rate at which national debt changes is proportional to national income so that

$$D'(t) = \alpha Y(t) \tag{54}$$

(ii) National income increases at a constant rate, so that

$$Y'(t) = \beta \tag{55}$$

Solving
$$D(t) = D(0) + \alpha Y(0)t + \frac{1}{2}\alpha\beta t^2 \tag{56}$$

$$Y(t) = Y(0) + \beta t \tag{57}$$

so that
$$\frac{D(t)}{Y(t)} = \frac{D(0) + \alpha Y(0)t + 1/2\alpha\beta t^2}{Y(0) + \beta t} \tag{58}$$

In this model, the ratio of national debt to national income tends to increase without limit.

3.4.3 Domar's Second Debt Model

In this model, the first assumption remains the same, but the second assumption is replaced by the assumption that the rate of increase of national income is proportional to the national income so that

$$Y'(t) = \beta Y(t) \tag{59}$$

Solving (54) and (59)

$$Y(t) = Y(0)e^{\beta t} \tag{60}$$

$$D(t) = D(0) + \frac{\alpha}{\beta}Y(0)(e^{\beta t} - 1) \tag{61}$$

$$\frac{D(t)}{Y(t)} = \frac{D(0)}{Y(0)e^{\beta t}} + \frac{\alpha}{\beta}(1 - e^{-\beta t}) \tag{62}$$

In this case $D(t)/Y(t) \to \alpha/\beta$ as $t \to \infty$. Thus when debt increases at a rate proportional to income, then if the ratio of debt to income is not to increase indefinitely, income must increase exponentially.

3.4.4 Allen's Speculative Model

Let $d(t)$, $s(t)$, $p(t)$ denote the demand, supply and price of a commodity, then this model is given by

$$d(t) = \alpha_0 + \alpha_1 p(t) + \alpha_2 p'(t), \quad \alpha_0 > 0, \alpha_1 < 0, \alpha_2 > 0 \tag{63}$$

$$s(t) = \beta_0 + \beta_1 p(t) + \beta_2 p'(t), \quad \beta_0 > 0, \beta_1 > 0, \beta_2 < 0 \tag{64}$$

If $\alpha_2 = 0$, $\beta_2 = 0$ this gives Evan's price-adjustment model in which $\alpha_1 < 0$ since when price increases, demand decreases and $\beta_1 > 0$ since when price increases, supply increases. In Allen's model, coefficients α_2, β_2 account for the effect of speculation. If the price is increasing, demand increases in the expectation of the further increase in prices and supply decreases for the same reason.

For dynamic equilibrium

$$d(t) = s(t), \tag{65}$$

so that (63), (64) and (65) give

$$(\beta_2 - \alpha_2)\frac{dp}{dt} + (\beta_1 - \alpha_1)\,p(t) = \alpha_0 - \beta_0 \tag{66}$$

Solving

$$p(t) = p_e + (p(0) - p_e)e^{\lambda t}, \tag{67}$$

where

$$p_e = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1}, \quad \lambda = \frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2} \tag{68}$$

The behaviour of $p(t)$ depends on whether $p(\infty)$ or p_e is large and whether $\lambda < 0$ or $\lambda > 0$. The speculative model is highly unstable.

3.4.5 Samuelson's Investment Model

Let $K(t)$ represent the capital and $I(t)$ the investment at time t, then we assume that

(i) the investment gives the rate of increase of capital so that

$$\frac{dK}{dt} = I(t) \tag{69}$$

(ii) the deficiency of capital below a certain equilibrium level leads to an acceleration of the rate of investment proportional to this deficiency and a surplus of capital above this equilibrium level leads to a declaration of the rate of investment, again proportional to the surplus, so that

$$\frac{dI}{dt} = -m(K(t) - K_e), \tag{70}$$

where K_e is the capital equilibrium level. If $k(t) = K(t) - K_e$, we get

$$\frac{dk}{dt} = I(t), \quad \frac{dI}{dt} = -mk(t), \tag{71}$$

so that

$$-mk(t) = \frac{dI}{dt} = \frac{dI}{dk}\frac{dk}{dt} = I\frac{dI}{dk} \tag{72}$$

Integrating

$$I^2 = m(k_0^2 - k^2); \quad k_0 = k(0);\ I(0) = 0, \tag{73}$$

so that

$$\frac{dk}{dt} = -\sqrt{m}\sqrt{k_0^2 - k^2} \tag{74}$$

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and
$$k(t) = k(0) \cos \sqrt{m}\, t \tag{75}$$
$$I(t) = -k(0) \sqrt{m} \sin \sqrt{m}\, t \tag{76}$$
so that both $k(t)$ and $I(t)$ oscillate with a time period $2\pi/\sqrt{m}$.

It will be noted that if we put $k(t) = x(t)$, $I(t) = v(t)$, equations (71) are the equations for simple harmonic motion. Thus the mathematical models for the oscillation of a particle in a simple harmonic motion and for the oscillation of capital about its equilibrium value are the same.

3.4.6 Samuelson's Modified Investment Model

In this case, the rate of investment is slowed not only by excess capital as before, but it is also slowed by a high investment level so that (71) become
$$\frac{dk}{dt} = I(t), \quad \frac{dI}{dt} = -mk(t) - nI(t), \tag{77}$$
so that
$$I \frac{dI}{dk} + mk(t) + nI(t) = 0, \tag{78}$$
or
$$\frac{d^2k}{dt^2} + n\frac{dk}{dt} + mk = 0, \tag{79}$$
which are the equations for damped harmonic motion corresponding to the case when a particle performing SHM is acted as by a resistance force proportional to the velocity.

3.4.7 Stability of Market Equilibrium

Let $p_r(t)$, $s_r(t)$ and $d_r(t)$ be the price, supply and demand of a commodity in the rth market, so that Evan's price adjustment model mechanism suggests
$$\frac{dp_r}{dt} = -\mu_r(s_r - d_r), \quad r = 1, 2, \ldots, n \tag{80}$$
Now we assume that the supply and demand of the commodity in the rth market depends upon its price in all the markets, so that
$$s_r - d_r = c_r + \sum_{s=1}^{n} d_{rs}\, p_s \tag{81}$$
where c_r's and d_{rs}'s are constants. From (80) and (81), we get
$$\frac{dp_r}{dt} = -\mu_r(c_r + \sum_{s=1}^{n} d_{rs}p_s), \quad r = 1, 2, \ldots, n \tag{82}$$
If $p_{1e}, p_{2e}, \ldots p_{ne}$ are the equilibrium prices in the n markets and
$$P_r = p_r - p_{re},$$
we get
$$\frac{dP_r}{dt} = -\mu_r \sum_{s=1}^{n} d_{rs}P_s = \sum_{s=1}^{n} e_{rs}P_s, \quad r = 1, 2, \ldots, n \tag{83}$$

where
$$e_{rs} = -\mu_r d_{rs} \tag{84}$$
Substituting $P_r = A_r e^{\lambda t}$ and eliminating $A_1, A_2, \ldots, A_n$, we get
$$|\lambda I - E| = 0, \quad E = [e_{rs}] \tag{85}$$
Thus the equilibrium will be stable if all the eigen-values of the matrix E have negative real parts.

If $d_{rs} = 0$ when $r \neq s$, the markets are independent so that non-zero value of some or all of these d_{rs}'s introduce dependence among markets.

3.4.8 Leontief's Open and Closed Dynamical Systems for Inter-industry Relations

We consider n industries. Let

x_{rs} = contribution from the rth industry to the sth industry per unit time
x_r = contribution from the rth industry to consumers per unit time
X_r = total output of the rth industry per unit time
ξ_r = input of labour in the rth industry
p_r = price per unit of the product of the rth industry
w = wage per unit of labour per unit time
Y = total labour input into the system
S_{rs} = stock of the product of the rth industry held by the sth industry
S_r = stock of the rth industry.

Thus we get the following equations:

(i) From the principle of continuity, the rate of change of stock of the rth industry = excess of the total output of the rth industry per unit time over the contribution of the rth industry to consumers and other industries per unit time, so that
$$\frac{d}{dt} S_r = X_r - x_r - \sum_{s=1}^{n} x_{rs} \tag{86}$$
and since
$$S_r = \sum_{s=1}^{n} S_{rs} \tag{87}$$
$$\frac{d}{dt} \sum_{s=1}^{n} S_{rs} = X_r - x_r - \sum_{s=1}^{n} x_{rs}, \quad (r = 1, 2, \ldots, n) \tag{88}$$
(ii) Since the total labour input into the system = sum of labour inputs into all industries, we get
$$Y = \sum_{r=1}^{n} \xi_r \tag{89}$$
(iii) Assuming the condition of perfect competition and no profit in each industry, we should have for each industry the value of input equal to the value of output so that
$$p_r X_r = \sum_{s=1}^{n} p_s x_{sr} + w\, \xi_r \quad (r = 1, 2, \ldots, n) \tag{90}$$

(iv) We further assume that the input coefficients

$$a_{rs} = \frac{x_{rs}}{X_s},\ b_{rs} = \frac{S_{rs}}{X_s},\ b_r = \frac{\xi_r}{X_r}\ (r, s = 1, 2, \ldots, n) \tag{91}$$

are constants.

We then get the equations

$$\frac{d}{dt}\sum_{s=1}^{n} b_{rs}X_s = X_r - x_r - \sum_{s=1}^{n} a_{rs}X_s,\quad (r = 1, 2, \ldots, n) \tag{92}$$

$$Y = \sum_{s=1}^{n} b_s X_s \tag{93}$$

$$p_r = \sum_{s=1}^{n} p_s a_{sr} + wb_r,\quad (r = 1, 2, \ldots, n) \tag{94}$$

We assume that the constants a_{rs}, b_{rs} b_s, are known. We also assume that $x_1, x_2, \ldots, x_n$ and w are given to us as function of time, then equations (92) determine $X_1, X_2, \ldots, X_n$ and then (93) determines Y and finally (94) determine $p_1, p_2, \ldots, p_n$.

Thus if the final consumer's demands from all industries are known as functions of time, we can find the output which each industry must give and the total labour force required at any time. Knowing the wage rate at any time, we can find the prices of products of different industries.

EXERCISE 3.4

1. Solve Domer debt model when $Y'(t) = \beta Y^n(t)$ and deduce the two models of subsections 3.4.2 and 3.4.3 by letting $n \to 0$ and $n \to 1$. Discuss the behaviour of $D((t)/Y(t)$ as $t \to \infty$ for a general value of n.
2. Discuss the solution of Allen's speculative model when (i) $\lambda > 0$ (ii) $\lambda < 0$ (iii) $p_e > p(0)$ (iv) $p_e < p(0)$ and interpret the solution in each case.
3. Discuss the solution of Samuelson's modified investment models, when

$$\frac{dk}{dt} = I(t),\quad \frac{dI}{dt} = -mk^n(t)$$

$$\frac{dk}{dt} = I(t),\quad \frac{dI}{dt} = -mk(t) - nI^2(t)$$

4. Discuss in detail the particular case of 3.4.7 when $n = 2$.
5. Obtain the steady-state solution of Leontief's model.

3.5 MATHEMATICAL MODELS IN MEDICINE, ARMS RACE BATTLES AND INTERNATIONAL TRADE IN TERMS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

3.5.1 A Model for Diabetes Mellitus

Let $x(t)$, $y(t)$ be the blood sugar and insulin levels in the blood stream at time t. The rate of change dy/dt of insulin level is proportional to (i) the excess $x(t)-x_0$ of sugar in blood over its fasting level, since this exces makes the pancreas secrete insulin into the blood stream (ii) the amoun $y(t)$ of insulin since insulin left to itself tends to decay at a rate proportiona to its amount and (iii) the insulin dose $d(t)$ injected per unit time. This give

$$\frac{dy}{dt} = a_1 (x - x_0) H(x - x_0) - a_2 y + a_3 d(t), \quad (95$$

where a_1, a_2, a_3 are positive constants and $H(x)$ is a step function whict takes the value unity when $x > 0$ and taken the value zero otherwise. This occurs in (95) because if blood sugar level is less than x_0, there is no secretion of insulin from the pancreas.

Again the rate of change dx/dt of sugar level is proportional to (i) the product xy since the higher the levels of sugar and insulin, the higher is the metabolism of sugar (ii) $x_0 - x$ since if sugar level falls below fasting level, sugar is released from the level stores to raise the sugar level to normal (iii) $x - x_0$ since if $x > x_0$, there is a natural decay in sugar level proportional to its excess over fasting level (iv) function of $t - t_0$ where t_0 is the time at which food is taken

$$\frac{dx}{dt} = -b_1 xy + b_2(x_0 - x) H(x_0 - x) - b_3(x - x_0) H(x - x_0) + b_4 z(t - t_0), \quad (96)$$

where a suitable form for $z(t - t_0)$ can be

$$\begin{aligned} z(t - t_0) &= 0, \quad t < t_0 \\ &= Qe^{-\alpha(t-t_0)}, \quad t > t_0 \end{aligned} \quad (97)$$

Equations (95) and (96) give two simultaneous differential equations to determine $x(t)$ and $y(t)$. These equation can be numerically integrated.

3.5.2 Richardson's Model for Arms Race

Let $x(t)$, $y(t)$ be the expenditures on arms by two countries A and B, then the rate of change dx/dt of the expenditure by the country A has a term proportional to y, since the larger the expenditure in arms by B, the larger will be the rate of expenditure on arms by A. Similarly it has a term proportional to $(-x)$ since its own arms expenditure has an inhibiting effect on the rate of expenditure on arms by A. It may also contain a term independent of the expenditures depending on mutual suspicions or mutual goodwill. With these considerations, Richardson gave the model

$$\frac{dx}{dt} = ay - mx + r, \quad \frac{dy}{dt} = bx - ny + s \quad (98)$$

Here a, b, m, n are all > 0. r and s will be positive in the case of mutual suspicions and negative in the case of mutual goodwill.

A position of equilibrium x_0, y_0, if it exists, will be given by

$$\begin{aligned} mx_0 - ay_0 - r &= 0 \\ bx_0 - ny_0 + s &= 0 \end{aligned} \quad \text{or} \quad \frac{x_0}{-as - nr} = \frac{y_0}{-br - ms} = \frac{1}{-mn + ab}$$

$(ms+br)x_0 = (as+nr)y_0$

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or
$$x_0 = \frac{as + nr}{mn - ab},\ y_0 = \frac{ms + br}{mn - ab}. \tag{99}$$

If r, s are positive, a position of equilibrium exists if $ab < mn$. If $X = x - x_0$, $Y = y - y_0$, we get

$$\frac{dX}{dt} = aY - mX,\ \frac{dY}{dt} = bX - nY \tag{100}$$

$X = Ae^{\lambda t}$, $Y = Be^{\lambda t}$ will satisfy these equations if

$$\begin{vmatrix} \lambda + m & -a \\ -b & \lambda + n \end{vmatrix} = 0,\quad \lambda^2 + \lambda(m + n) + mn - ab = 0 \tag{101}$$

Now the following cases arise:

(i) $mn - ab > 0$, $r > 0$, $s > 0$. In this case $x_0 > 0$, $y_0 > 0$ and from (101) $\lambda_1 < 0$, $\lambda_2 < 0$. As such there is a position of equilibrium and it is stable.

(ii) $mn - ab > 0$, $r < 0$, $s < 0$, there is no position of equilibrium since $x_0 < 0$, $y_0 < 0$. However since $\lambda_1 < 0$, $\lambda_2 < 0$, $X(t) \to 0$, $Y(t) \to 0$ as $t \to \infty$, so that $x(t) \to x_0$, $y(t) \to y_0$. However x_0 and y_0 are negative and populations cannot become negative. In any case to become negative, they have to pass through zero values. As such, as $x(t)$ becomes zero, (98) is modified to

$$\frac{dy}{dt} = -ny + s \tag{102}$$

and since $s < 0$, $y(t)$ decreases till it reaches zero. Similarly if $y(t)$ becomes zero first, (98) is modified to

$$\frac{dx}{dt} = -mx + r, \tag{103}$$

and since $r < 0$, $x(t)$ decreases till it reaches zero. Thus if $mn - ab > 0$, $r < 0$, $s < 0$, there will ultimately be complete disarmament.

(iii) $ma - ab < 0$, $r > 0$, $s > 0$. These give $x_0 < 0$, $y_0 < 0$, one of λ_1, λ_2 is positive and the other is negative. In this case there will be a run-away arms race.

(iv) $ma - ab < 0$, $r < 0$, $s < 0$. These give $x_0 > 0$, $y_0 > 0$ one of λ_1, λ_2 is positive and the other is negative. In this case there will be a run-away arms race or disarmament depending on the initial expenditure on arms.

3.5.3 Lanchester's Combat Model

Let $x(t)$ and $y(t)$ be the strengths of the two forces engaged in combat and let M and N be the fighting powers of individuals depending on physical fitness, types of arms and training, then Lanchester postulated that the reduction in strength of each force is proportional to the effective fighting strength of the opposite force, so that

$$\frac{dx}{dt} = -ayN,\quad \frac{dy}{dt} = -axM \tag{104}$$

giving
$$\frac{dx}{yN} = \frac{dy}{xM} \quad \text{or} \quad Mx^2 - Ny^2 = \text{constant} \tag{105}$$

If the proportional reduction of strengths in the two forces are the same
$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt} \quad \text{or} \quad \frac{Ny}{x} = \frac{Mx}{y} \quad \text{or} \quad Mx^2 = Ny^2 \tag{106}$$

This is the square law. The fighting strength of an army depends on the square of its numerical strength and directly on the fighting quality of individuals.

3.5.4 International Trade Model

Since international trade is beneficial to all parties, we can consider the model
$$\begin{aligned}
\frac{dx_1}{dt} &= a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots + a_{1n}x_1x_n \\
\frac{dx_2}{dt} &= a_{21}x_2x_1 + a_{23}x_2x_3 + \ldots + a_{2n}x_2x_n \\
&\ldots \\
\frac{dx_n}{dt} &= a_{n1}x_nx_1 + a_{n2}x_nx_2 + \ldots + a_{nn-1}x_nx_{n-1}
\end{aligned} \tag{107}$$

where all a_{ij}'s are positive. An equilibrium position is $(0, 0, \ldots, 0)$ and this is stable.

EXERCISE 3.5

1. For the Richardson's model, draw the lines $ay - mx + r = 0$, $bx - ny + s = 0$ in the four cases discussed in section 3.5.2. Draw the direction fields and possible trajectories in each case and verify the results obtained in that section.
2. For the model
$$\frac{dN_1}{dt} = N_1(a_1 - b_1N_1 - b_2N_2),\ \frac{dN_2}{dt} = N_2(a_2 - c_1N_1 - c_2N_n),\quad \begin{matrix} a_1, a_2 > 0 \\ b_1, b_2 > 0 \\ c_1, c_2 > 0 \end{matrix}$$
find the positions of equilibrium and discuss their stability. Draw also the direction fields and possible trajectories.
3. Show that for the Lanchester model, the trajectories are hyperbolas, all of which have the same asymptotes.
4. Show that for the international trade model (107), the origin represents a position of stable equilibrium.

3.6 MATHEMATICAL MODELLING IN DYNAMICS THROUGH SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

3.6.1 Modelling in Dynamics

If a particle moves in two dimensional space, we want to determine $x(t)$,

$y(t)$, its coordinates at any time t and $u(t)$, $v(t)$ its velocity components at the same time. Similarly for the motion of a particle in three dimensions, we have to determine $x(t)$, $y(t)$, $z(t)$, $u(t)$, $v(t)$, $w(t)$. For motion of a rigid body in three dimensional space, we require twelve quantities at time t viz. six coordinates and velocities of its centre of gravity and six angles and angular velocities about the centre of gravity.

Since equation of motion are based on the principle: mass $\times$ acceleration in any direction = force in that direction, we get systems of second order differential equations. However since acceleration is the rate of change of velocity and velocity is the rate of change of displacement, we can decompose one ordinary differential equation of the second order into two ordinary differential equations of the first order.

We discuss below the motion of a particle in a plane under gravity. More general dynamical motions will be discussed in the next chapter.

3.6.2 Motion of a Projectile

A particle of mass m is projected from the origin in vacuum with velocity V inclined at an angle α to the horizontal. Suppose at time t, it is at position $x(t)$, $y(t)$ and its horizontal and vertical velocity components are $u(t)$, $v(t)$ respectively, then the equations of motion are:

$$m\frac{du}{dt} = 0 \qquad m\frac{dv}{dt} = -mg \tag{108}$$

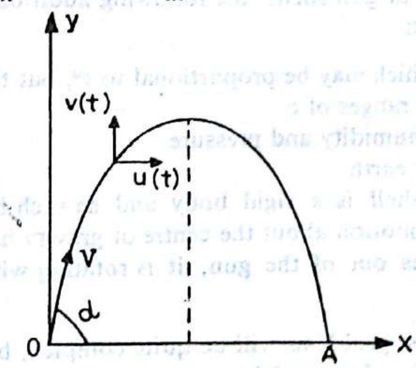

Figure 3.3

Integrating $\quad u = V\cos\alpha, \quad v = V\sin\alpha - gt, \qquad (109)$

so that
$$\frac{dx}{dt} = V\cos\alpha, \quad \frac{dy}{dt} = V\sin\alpha - gt \tag{110}$$

Integrating again

$$x = V\cos\alpha t, \quad y = V\sin\alpha t - \frac{1}{2}gt^2 \tag{111}$$

Eliminating t between these two equations, we get

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{V^2\cos^2\alpha} \tag{112}$$

which is a parabola, since the terms of the second degree form a perfect square. The parabola cuts $y = 0$, when

$$x = 0 \quad \text{or} \quad x = \frac{V^2 \sin 2\alpha}{g} \tag{113}$$

corresponding to position 0 and A in Figure 3.3 so that the range of the particle is given by

$$R = \frac{V^2 \sin 2\alpha}{g} \tag{114}$$

Putting $y = 0$ in (111) we get

$$t = 0 \quad \text{or} \quad t = \frac{2V \sin \alpha}{g} \tag{115}$$

This gives the time T of flight. Since the horizontal velocity is constant and equal to $V \cos \alpha$, the total horizontal distance travelled is

$$V \cos \alpha(2V \sin \alpha)/(g) = V^2 \sin 2\alpha/g$$

which gives us the same range.

3.6.3 External Ballistics of Gun Shells

To study the motion of gun shells, the following additional factors have to be taken into account:

(i) air resistance which may be proportional to v^n, but the power n can be different for different ranges of v

(ii) wind velocity, humidity and pressure

(iii) rotation of the earth

(iv) the fact that shell is a rigid body and as such both motion of its centre of gravity and motion about the centre of gravity have to be studied. When the shell comes out of the gun, it is rotating with a large angular velocity.

It is obvious that the problems will be quite complex, but all these problems have been solved and powerful computers have been developed to solve these problems because of their importance to defence.

In the case of intercontinental ballistic missiles, heating and aerodynamic effects have also to be considered.

EXERCISE 3.6

1. Show that the projectile attains the maximum height $V^2 \sin^2 \alpha/2g$ at time $V \sin/g$.
2. If the projectile is projected on a plane inclined at an angle β to the horizontal, find the range and time of flight.
3. Write the system of differential equations if there is air-resistance proportional to the nth power of the velocity. Solve the system when $n=1$.

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3. Write the system of differential equations if there is air-resistance proportional to the nth power of the velocity. Solve the system when $n = 1$.

Unit

POSSIBLE QUESTIONS

Part B (6 Marks)

1.Discuss in detail prey prey-predator models.

2.Discuss in detail on Samuelson's investment model.

3.Derive a Simple Epidemic Model.

4.Show that national income , investment and savings increase exponentially.

5.Design any two mathematical models in economics based on ordinary differential equations of first order give by Domar.

6.Give a detailed note on multi .multi-species models.

7.Explain about motion of a projectile.

.Part C (10 Marks)

1.Discuss in detail on Samuelson's investment model.

2.Explain a simple epidemic model.

3.Discuss in detail Domar Macro model.

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KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Mathematical Modelling **Subject Code: 18MMP303**

Class : II - M.Sc. Mathematics **Semester : III**

Unit II

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If there are no predators the ________ species will grow at a rate proportional to the population.	Prey	trajectory	permanent	persuieng	Prey
If there are no predators the prey species will grow at a rate ____________-l to the population.	Proportional	reciprocal	Exponential	Logically	Proportional
The predator species _____ and the prey species decreases at a rate proportional to the product of two populations.	increases	decreases	uniformly	stable	increases
The predator species increases and the prey species ____________ at a rate proportional to the product of two populations.	increases	decreases	uniformly	stable	decreases
The predator species increases and the prey species decreases at a rate ______ to the product of two populations.	Proportional	linear	unit	orthogonal	Proportional
The predator species increases and the prey species decreases at a rate proportional to the ______ of two populations.	Addition	subtraction	product	division	product
If there are no prey the ______ species will decline at a rate proportional to the population.	Prey	Predator	permanent	persuieng	Predator

If there are no prey the predator species will _______at a rate proportional to the population.	decline	denied	different	decrease	decline
If there are no prey the predator species will decline at a rate ________ to the population.	Proportional	reciprocal	Exponential	Logically	Proportional
The initial populations of prey and preador species are ________	p/q and a/b	a/b and p/q	a/b	p/q	p/q and a/b
The population of x=0 and y=0 is called ________ position.	zero	equilibrium	unit	none	equilibrium
x(t) and y(t) are the populations of two species competing for the same resources stands________ model	epidemic	population	dynamic	competition	competition
The rate of growth of each species _______ due to the presence of the other.	increases	decreases	uniformly	stable	decreases
The rate of growth of each species decreases due to the ________ of the other.	presence	absence	both	none	presence
x1(t(),x2(t)…..xn(t) represent the populations of n species states_______ model.	multi-species	single-species	prey	predator	multi-species
The real parts of all the eigenvalues of the matrix[cij] is negative are called	Rout-Herwitz	fick's	newtyons	gauss	Rout-Herwitz
Age structured population model deals _______________ age groups	productive	reproductive	decline	increase	reproductive
In simple epidemic mode S(t) denotes	susceptible	system	synopsis	success	susceptible
In simple epidemic mode I(t) denotes	Infected	increase	innovation	intensity	Infected
In simple epidemic mode S(t)+I(t) =	n	n+1	n-1	2n	n+1
In simple epidemic mode limit t tends to infinity of S(t) denotes	0	1	2	3	0
In simple epidemic mode limit t tends to infinity of I(t) denotes	n	n+1	n-1	2n	n+1
A susceptible person can infected at a rate proportional to	SI	SIS	SHM	MOC	SI
In SIS Infected person can recover and become susceptible at a rate	Gamma I	SI	SHM	SIS	Gamma I
A susceptible person can infected at a rate _______ to SI	Proportional	linear	unit	orthogonal	Proportional

Only carriers spread the disease deals _________ model	Simple epidence	epidemic	SIS	SI	Simple epidence
The model with removal deals the infected persons are removed by___________	Death	moving	migration	none	Death
The model with removal deals the infected persons are removed by___________	Hospitalisation	moving	migration	none	Hospitalisation
Model with removal and immigration allows the ________ of susceptible.	Increases	decreases	decline	equate	Increases
Model with removal and immigration allows the increase of susceptible.	infected	susceptible	preys	predators	susceptible
_________ deals the distribution of drugs , chemicals tracers or radio active.	Pharmokinetics	kinetics	medicine	diffusion	Pharmokinetics
Parmokinetics deals the distribution of _________	Drugs	blood	Glucose	Rice	Drugs
Parmokinetics deals the distribution of _________	Chemicals	blood	Glucose	Rice	Chemicals
Parmokinetics deals the distribution of _________	Tracers	blood	Glucose	Rice	Tracers
Parmokinetics deals the distribution of _________	Radio active	blood	Glucose	Rice	Radio active
In Domar Macro Model S(t) denotes	Savings	Success	Susceptible	System	Savings
In Domar Macro Model I(t) denotes	Increases	Investment	innovation	Instalment	Investment
In Domar Macro Model Y(t) denotes	Income	National Income	Debt	National debt	National Income
In Domar Macro Model savings are proportional to	Income	National Income	Debt	National debt	National Income
In Domar Macro Model Investment is proportional to the rate of increase of_________	Income	National Income	Debt	National debt	National Income
In Domar Macro Model all savings are Investment so that	$S(t) = I(t)$	$S(t) = 1/2\, I(t)$	$2S(t) = I(t)$	None	$S(t) = I(t)$
In Domar first Debt model D(t) denotes	debt	total national debt	income	national income	total national debt
In Domar first Debt model Y(t) denotes	income	total income	national income	total national income	total national income

In Domar first Debt model rate at which national debt changes is __________ to the national income.	Proportional	linear	unit	orthogonal	Proportional
In Domar first Debt model rate at which national debt changes is proportional to the ___________.	income	total income	national income	total national income	national income
In Domar first Debt model national income increases at a ________ rate.	Variable	Constant	unit	orthogonal	Constant
In Domar first Debt model national income ________ at constant rate.	Increases	decreases	decline	equate	Increases
In Domar's Second Debt model the ratio of debt to on come is not to increase indefinitely income must increase ________	Proportional	reciprocal	Exponential	Logically	Exponential
In Allen's Speculative Model d(t) denotes	demand	supply	price of a commodity	debt	demand
In Allen's Speculative Model s(t) denotes	demand	supply	price of a commodity	System	supply
In Allen's Speculative Model p(t) denotes	demand	supply	price of a commodity	Prey	price of a commodity
In Samuelson's Investment model K(t) denotes	Capital	investment	savings	debt	Capital
In Samuelson's Investment model the investment gives rate of increase of ___________	Capital	investment	savings	debt	Capital
In Samuelson's Investment model the investment gives rate of _________ of capital.	Increases	decreases	investment	decline	Increases
In Samuelson's Modified Investment model a particle performing ______ is acted by a resistance force proportional to velocity.	SHM	MOC	SIS	none	SHM
In Samuelson's Modified Investment model a particle performing SHM is acted by a resistance force _______ to velocity.	Proportional	linear	unit	orthogonal	Proportional
In a model for Diabetes Mellitus x(t) denotes	blood sugar	salt	urea	fat	blood sugar
In a model for Diabetes Mellitus y(t) denotes	insulin	thyroid	salt	urea	insulin

In Leontief's Inter - Industries relation model, the notation of contribution from the rth industry to sth industry per unit time is	xrs	xr	Xr	xr	xrs
In Leontief's Inter - Industries relation model,the notation of contribution from the rth industry to consumers per unit time is	xrs	xr	Xr	xr	xr
In Leontief's Inter - Industries relation model,the notation of total output of the rth industry per unit time is	xrs	xr	Xr	xr	Xr
In Leontief's Inter - Industries relation model,the notation of input of the labour in the rth industry is	xrs	xr	Xr	xr	xr
In Leontief's Inter - Industries relation model,the notation of price per unit of the product of the rth industry is	pr	xr	Xr	xr	pr
In Leontief's Inter - Industries relation model,the notation of wage per unit of labour per unit time is	w	xr	Xr	xr	w
In Leontief's Inter - Industries relation model,the notation of total labour input into the system is	Y	xr	Xr	xr	Y
In Leontief's Inter - Industries relation model,the notation of stock of the product of the rt industry held by the sth industry is	Srs	xr	Xr	xr	Srs
In Leontief's Inter - Industries relation model, the notation of stock of the rt industry is	Sr	Srs	Xr	xr	Sr
The excess of sugar in blood over its fasting level makes _______ secrete insulin into the blood stream.	thyroid	harmone	pancreas	none	pancreas
The fighting strength of an army depends on the ______ of its numerical strength and directly on the fighting quality of individuals.	square	circle	rectangle	ellipse	square
A particle of mass m is projected from the origin in vacuum with velocity inclined at an angle proportional to the_______	vertical	slope	horizontal	equal	horizontal

A particle of mass m is projected from the origin in vacuum with velocity inclined at an angle ________ to the horizontal.	Proportional	reciprocal	Exponential	Logically	proportional
In the case of intercontinental ballistic missiles eating and_______ have to be considered.	aerodynamics	dynamics	mechanics	aeromechanics	aerodynamics
Both range and maximum eight of projectile are reduced by ______ resistance.	air	water	liquid	solid	air

UNIT-III

Mathematical Modeling through Ordinary Differential Equations of Second Order: Planetary Motions – Circular Motion and Motion of Satellites – Mathematical Modelling through Linear Differential Equations of Second Order – Miscellaneous Mathematical Models.

4

Mathematical Modelling Through Ordinary Differential Equations of Second Order

4.1 MATHEMATICAL MODELLING OF PLANETARY MOTIONS

4.1.1 Need for the Study of Motion Under Central Forces

Every planet moves mainly under the gravitational attractive force exerted by the Sun. If S and P are masses of the Sun and the planet and G is the universal constant of gravitation, then the forces of gravitational attraction on the Sun and planet are both GSP/r^2, where r is the distance between the Sun and the planet. Accordingly the acceleration (Fig. 4.1) of the Sun towards the planet is GP/r^2 and the acceleration of the planet towards the Sun is GS/r^2. The acceleration of the planet relative to the Sun is

$$G(S + P)/r^2 = \mu/r^2.$$

Now we take the Sun as fixed, then the planet can be said to move under a central force μ/r^2 per unit mass i.e. under a force which is always directed towards a fixed centre S.

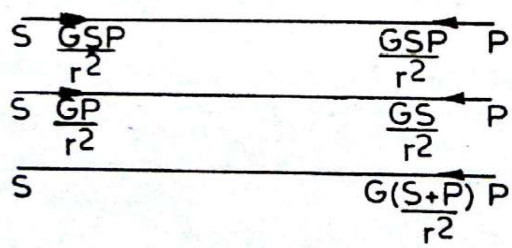

Figure 4.1

We shall for the present also regard P as a particle so that to study the motion of the planet, we have to study the motion of a particle moving under a central force. We can take S as origin so that the central force is always along the radius vector. To study this motion, it is convenient to use polar coordinates and to find the components of the velocity and acceleration along and perpendicular to the radius vector.

4.1.2 Components of Velocity and Acceleration Vectors along Radial and Transverse Directions

As the particle moves from P to Q, the displacement along the radius vector

$$= ON - OP = (r + \Delta r)\cos\Delta\theta - r \tag{1}$$

and the radial component u of velocity is

$$u = \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{(r+\Delta r)\cos\Delta\theta - r}{\Delta t}$$

$$= \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta r}{\Delta t} = \frac{dr}{dt} \tag{2}$$

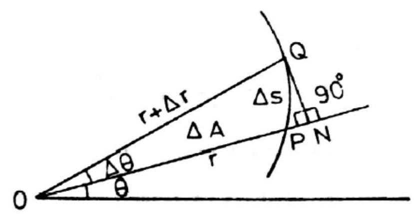

Figure 4.2

Similarly the displacement perpendicular to the radius vector

$$= (r + \Delta r)\sin\Delta\theta \tag{3}$$

and the transverse component v of the velocity is given by

$$v = \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{(r+\Delta r)\sin\Delta\theta}{\Delta t} = \underset{\Delta t \to 0}{\mathrm{Lt}}\, r\frac{\sin\Delta\theta}{\Delta\theta}\frac{\Delta\theta}{\Delta t} = r\frac{d\theta}{dt} \tag{4}$$

As such the velocity components in polar coordinates are

$$u = \frac{dr}{dt} = r' \quad \text{and} \quad v = r\frac{d\theta}{dt} = r\theta' \tag{5}$$

Now the change in the velocity along the radius vector

$$= (u + \Delta u)\cos\Delta\theta - (v + \Delta v)\sin\Delta\theta - u \tag{6}$$

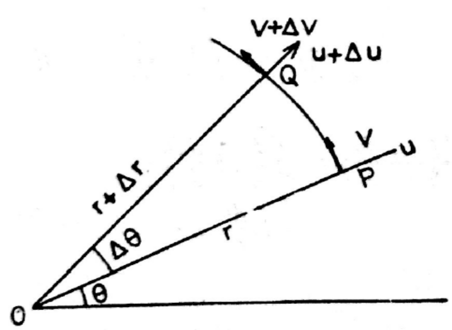

Figure 4.3

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and the radial component of acceleration

$$= \operatorname*{Lt}_{\Delta t \to 0} \frac{(u + \Delta u)\cos\Delta\theta - (v + \Delta v)\sin\Delta\theta - u}{\Delta t}$$

$$= \operatorname*{Lt}_{\Delta t \to 0} \frac{\Delta u - v\,\Delta\theta}{\Delta t} = \frac{du}{dt} - v\frac{d\theta}{dt} = \frac{d}{dt}(r') - r\theta'\,\theta'$$

$$= r'' - r\theta'^2 \tag{7}$$

Similarly the transverse component of acceleration

$$= \operatorname*{Lt}_{\Delta t \to 0} \frac{(u + \Delta u)\sin\Delta\theta + (v + \Delta v)\cos\Delta\theta - v}{\Delta t}$$

$$= \operatorname*{Lt}_{\Delta t \to 0} \frac{u\Delta\theta + \Delta v}{\Delta t}$$

$$= u\frac{d\theta}{dt} + \frac{dv}{dt} = r'\theta' + \frac{d}{dt}(r\theta') = \frac{1}{r}\frac{d}{dt}(r^2\theta') \tag{8}$$

Thus the radial and transverse components of acceleration are

$$r'' - r\theta'^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\theta') \tag{9}$$

4.1.3 Motion Under a Central Force

Let the force acting on a particle of mass m be $mF(r)$ and let it be directed towards the origin, then the equations of motion are

$$m(r'' - r\theta'^2) = -mF(r) \tag{10}$$

$$\frac{m}{r}\frac{d}{dt}(r^2\theta') = 0 \tag{11}$$

From (11)

$$r^2\theta' = \text{constant} = h \text{ (say)}, \tag{12}$$

then (10) gives

$$r'' - r\theta'^2 = -F(r) \tag{13}$$

We can eliminate t between (12) and (13) to get a differential equation between r and θ. We find it convenient to use $u = 1/r$ instead of r, so that making use of (12), we get

$$r' = \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta} \tag{14}$$

and

$$r'' = \frac{d}{dt}\left(-h\frac{du}{dt}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}hu^2 = -h^2u^2\frac{d^2u}{d\theta^2} \tag{15}$$

From (12), (13) and (15)

$$-F(r) = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

or
$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}, \tag{16}$$

where F can be easily expressed as a function of u. This is the differential equation of the second order whose integration will give the relation between u and θ or between r and θ i.e. the equation of the path described by a particle moving under a central force F per unit mass.

4.1.4 Motion Under the Inverse Square Law

If the central force per unit mass is μ/r^2 or μu^2, Equation (16) gives

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{17}$$

Integrating this linear equation with constant coefficients, we get

$$u = A\cos(\theta - \alpha) + \frac{\mu}{h^2}$$

or
$$\frac{h^2/u}{r} = \frac{L}{r} = 1 + e\cos(\theta - \alpha);\ h^2 = \mu L, \tag{18}$$

which represents a conic with a focus at the centre of force. Thus if a particle moves under a central force μ/r^2 per unit mass, the path is a conic section with a focus at the centre. The conic can be an ellipse, parabola, or hyperbola according as $e \lesseqgtr 1$.

Now the velocity V of the particle is given by

$$V^2 = r'^2 + r^2\theta'^2 = \left(\frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}\right)^2 + \frac{1}{u^2}(hu^2)^2$$

$$V^2 = h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2 \tag{19}$$

Using (18)

$$L\frac{du}{d\theta} = -e\sin(\theta - \alpha) \tag{20}$$

From (19) and (20)

$$V^2 = \mu L\left(\frac{e^2\sin^2(\theta - \alpha)}{L^2} + \frac{(1 + e\cos(\theta - \alpha))^2}{L^2}\right)$$

$$= \frac{\mu}{L}(1 + e^2 + 2e\cos(\theta - \alpha))$$

$$= \frac{\mu}{L}(e^2 - 1 + 2(1 + e\cos(\theta - \alpha))$$

$$= \frac{\mu}{L}(e^2 - 1) + \frac{2\mu}{r} \tag{21}$$

If the path is an ellipse $\quad L = a(1 - e^2)$
If the path is a parabola $\quad e = 1$ (22)

If the path is a hyperbola $L = a(e^2 - 1)$,

so that $V^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$ in the case of a hyperbola

$$= \mu\left(\frac{2}{r}\right) \text{ in the case of a parabola} \tag{23}$$

$$= \mu\left(\frac{2}{r} - \frac{1}{a}\right) \text{ in the case of an ellipse.}$$

Thus if the particle is projected with velocity V from a point at a distance r from the centre of force, the path will be a hyperbola, parabola or ellipse according as

$$V^2 - \frac{2\mu}{r} \gtreqless 0 \tag{24}$$

We have proved that if the central force is μ/r^2 per unit mass, the path is a conic section with the centre of forces at one focus. Conversely if we know that the path is a conic section

$$\frac{L}{r} = Lu = 1 + e\cos(\theta - \alpha), \tag{25}$$

with a focus at the centre of force, then the force per unit mass is given by

$$F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= h^2u^2\left(\frac{-e\cos(\theta - \alpha)}{L} + \frac{1 + \cos(\theta - \alpha)}{L}\right)$$

$$= \frac{h^2}{L}u^2 = \frac{\mu}{r^2}, \tag{26}$$

so that the central force follows the inverse square law.

Since all planets are observed to move in elliptic orbits with the Sun at one focus, it follows that the law of attraction between different planets and Sun must be the inverse square law.

4.1.5 Kepler's Laws of Planetory Motions

On the basis of the long period of observations of planetory motions by his predecessors and by Kepler himself, Kepler deduced the following three laws of motion empirically

(i) Every planet describes an ellipse with the Sun at one focus

(ii) The radius vector from the Sun to a planet describes equal areas in equal intervals of time.

(iii) The squares of periodic time of planets are proportional to the cubes of the semimajor axes of the orbits of the planets

We can deduce all these three laws from the mathematical modelling of planetory motion discussed above, when the law of attraction is the inverse square law.

(i) We have already seen that under the inverse square law, the path has to be a conic section and this includes elliptic orbits.

(ii) Since $r^2\theta' = h$, we get

$$\underset{\Delta t \to 0}{\mathrm{Lt}} \frac{1}{2} \frac{r^2 \Delta\theta}{\Delta t} = \frac{1}{2}h \tag{27}$$

From Figure 4.2, the area ΔA bounded by radius vectors OP and OQ and the arc PQ is $1/2r^2 \sin \Delta\theta$ so that (27) gives

$$\frac{dA}{dt} = \frac{1}{2}h, \tag{28}$$

and the rate of description of sectorical area is constant and equal areas are described in equal intervals of time. This is Kepler's second law.

(iii) The total area of the ellipse is πab and since the areal velocity is $\frac{1}{2}h$, the periodic time T is given by

$$T = \frac{\pi\, ab}{\frac{1}{2}h} = \frac{2\pi\, ab}{\sqrt{\mu L}} = \frac{2\pi\, ab}{\sqrt{\mu}\sqrt{b^2/a}} = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \tag{29}$$

For two different planets of masses P_1, P_2, and semiaxes of orbits a_1, a_2, this gives

$$\frac{T_1}{T_2} = \frac{\sqrt{\mu_2}}{\sqrt{\mu_1}} \frac{a_1^{3/2}}{a_2^{3/2}} = \frac{\sqrt{G(S+P_2)}}{\sqrt{G(S+P_1)}} \frac{a_1^{3/2}}{a_2^{3/2}} \tag{30}$$

or

$$\frac{T_1^2}{T_2^2} = \frac{S+P_2}{S+P_1} \frac{a_1^3}{a_2^3} = \frac{1+\frac{P_2}{S}}{1+\frac{P_1}{S}} \frac{a_1^3}{a_2^3} \tag{31}$$

Since P_1, P_2 are very small compared with S, this gives, as a very good approximation

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3} \tag{32}$$

which is Kepler's third law of planetory motion.

Deduction of Kepler's three laws of planetory motion from the universal law of gravitation was an important success of mathematical modelling. Results which took hundreds of years to obtain by observation could be obtained in a very short time by using mathematical modelling.

Here we have neglected the forces of attraction of other planets on the given planet. These are very small as compared with the attractive force of the Sun. However these can be taken into account. In fact possibly the most sensational achievement of mathematical modelling was achieved when the discrepancies from the above theory observed in the motion of planets were explained as possibly due to the existence of another small planet. The position of this planet, not observed till that time, was calculated, and when the telescope was pointed out to that position in the sky, the planet was there!

Again the occurrence of many of the fundamental particles in physics has been theoretically predicted on the basis of mathematical modelling.

The advantages of developing a successful theoretical model over relying on purely observational and empirical models are that (i) this development can suggest development of mathematical models for similar situations elsewhere and those new models can later be validated and (ii) the theoretical models, unlike empirical models, can be generalised. Thus the model developed by Newton for planetory motion could be easily extended to apply to motion of artificial satellites. Similarly in urban transportation, a gravity model was developed by trial and error and ad hoc empirical methods extending over a period of thirty to forty years. When the same model was obtained theoretically from the principle of maximum entropy, it could be easily generalised for many more complex situations than could ever be handled by the empirical methods.

EXERCISE 4.1

1. You are given the following data on orbits of major planets

Planet	Mean distance a from the Sun in millions of miles	Eccentricity e	Period T
Mercury	36.0	0.2056234	87.967 days
Venus	67.3	0.0067992	224.701 days
Earth	93.0	0.0167322	365.256 days
Mars	141.7	0.0935543	1.881 years
Jupiter	483.9	0.0484108	11.862 years
Saturn	857.1	0.0557337	29.458 years
Uranus	1785.0	0.0471703	84.015 years
Neptune	2797.0	0.0085646	164.788 years
Pluto	3670.0	0.2485200	247.697 years

(i) Show that the periods T verify Kepler's third law quite closely.
(ii) Given mass of the Sun is 2×10^{33} gms, find G
(iii) Given $G = 6.673\times 10^{-8}$ cm^3/gm sec^2, estimate the mass of the Sun.
(iv) Find the velocity of each planet at perihelion and apehilion.

2. Find the central force $F(r)$ if the orbit is an ellipse with the centre of force coinciding with the centre of the ellipse.

3. For a particle moving in a circular orbit of radius a, find expressions for its velocity and acceleration components.

4. Find the value of g at the surface of the Sun.

4.2 MATHEMATICAL MODELLING OF CIRCULAR MOTION AND MOTION OF SATELLITES

4.2.1 Circular Motion

When a particle moves in a circle of radius a so that $r = a$, the radial component of velocity $= r' = 0$, the transverse component of velocity =

$r\theta' = a\theta'$ the radial component of acceleration $= r'' - r\theta'^2 = -a\theta'^2$, the transverse component of acceleration $= \frac{1}{r}\frac{d}{dt}(r^2\theta') = \frac{1}{a}\frac{d}{dt}(a^2\theta') = a\theta''$. Thus the velocity is $a\theta'$ along the tangent and the acceleration has two components $a\theta''$ along the tangent and $a\theta'^2$ along the normal.

If a particle moves in a circle of radius a, its equations of motion are

$ma\theta''$ = external force in the direction of the tangent

$ma\theta'^2$ = external force in the direction of the inward normal.

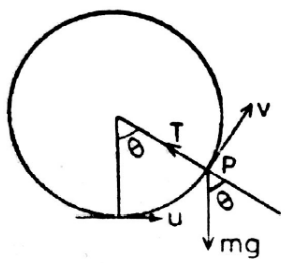


Figure 4.4

Thus if a particle is attached to one end of a string, the other end of which is fixed and the particle moves in a vertical circle, the equations of motion are (Figure 4.4)

$$ma\theta'' = -mg\sin\theta \qquad (33)$$

$$ma\theta'^2 = T - mg\cos\theta \qquad (34)$$

If θ is small, (33) gives

$$\theta'' = -\frac{g}{a}\theta, \qquad (35)$$

which is the equation for a simple harmonic motion. Thus for small oscillations of a simple pendulum, the time period is

$$T = 2\pi\sqrt{a/g} \qquad (36)$$

If θ is not necessarily small, integration of (33) gives

$$a\theta'^2 = 2g\cos\theta + \text{constant} \qquad (37)$$

If the particle is projected from the lowest point with velocity u, then $a\theta' = u$ when $\theta = 0$, so that

$$a\theta'^2 = \frac{v^2}{a} = \frac{u^2}{a} - 2g(1 - \cos\theta), \qquad (38)$$

where v is the velocity of the particle, so that

$$v^2 = u^2 - 2ga(1 - \cos\theta) \qquad (39)$$

or $$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mga(1 - \cos\theta) = \frac{1}{2}mu^2 - mgh \qquad (40)$$

where h is the vertical distance travelled by the particle. Equation (40) can be obtained directly from the principle of conservation of energy. Equation (34) then gives

$$T = m\frac{v^2}{a} + mg\cos\theta = m\frac{u^2}{a} - 2mg + 3\,mg\cos\theta \qquad (41)$$

At the highest point $\theta = \pi$ and $T = m\frac{u^2}{a} - 5mg$. If $u^2 \geqslant 5ag$, the particle will move in the complete vertical circle again and again. However if

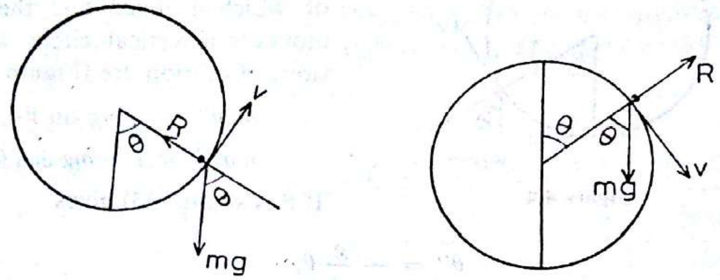
$u^2 < 5ag$, tension will vanish before the particle reaches the highest point. When the tension vanishes, the particle begins to move freely under gravity and describes a parabolic path till the string again becomes tight and the circular motion is started again.

4.2.2. Motion of a Particle on a Smooth or Rough Vertical Wire

(a) If the particle moves on the inside of a smooth wire, the equations of motion (Fig. 4.5a) are:

$$ma\theta'' = -mg \sin \theta \tag{42}$$

$$ma\theta'^2 = R - mg \cos \theta \tag{43}$$



(a) (b)

Figure 4.5

These are the same as (33) and (34) when T is replaced by the normal reaction R. As such if $u^2 \geqslant 5ag$, the particle makes an indefinite number of complete rounds of the circular wire. If $u^2 < 5ag$, the reaction vanishes before the particle reaches the highest point, the particle leaves the curve, describes a parabolic path till it meets the circular wire again and it again describes a circular path. This motion is repeated again and again.

(b) If the particle moves on the outside of the smooth vertical wire (Fig. 4.5b), the equations of motion are

$$ma\theta'' = mg \sin \theta. \tag{44}$$

$$ma\theta'' = -R + mg \cos \theta \tag{45}$$

Integrating (44) $\qquad \theta'^2 = u^2 + 2ga(1 - \cos \theta) \qquad (46)$

Using (45) $\qquad R = 3mg \cos \theta - \dfrac{mu^2}{a} - 2mg \qquad (47)$

At the highets point $\qquad \theta = 0,\ R = mg - \dfrac{mu^2}{a} \qquad (48)$

At the point A, $\qquad \theta = \pi/2,\ R = -\dfrac{mu^2}{a} - 2mg \qquad (49)$

If $u^2 > ag$, the particle leaves contact with the wire immediately and describes a parabolic path.

If $u^2 < ga$, the particle remains in contact for some distance, but leaves contact when R vanishes i.e. before it reaches A and then it describes a parabolic path.

(c) If the particle moves on the inside of rough vertical circular wire, then there is an additional frictional force μR along the tangent opposing the motion. As such equations (42) and (43) are modified to

$$ma\theta'' = -mg \sin\theta - \mu R \tag{50}$$

$$ma\theta'^2 = -mg\cos\theta + R \tag{51}$$

Eliminating R between these equations, we get a non-linear differential equation

$$a\theta'' = -g\sin\theta - \mu(-g\cos\theta - a\theta'^2) \tag{52}$$

which can be integrated by substituting $\theta' = w$, $\theta'' = w\,dw/d\theta$.

Similarly (44) and (45) are modified to

$$ma\theta'' = mg\sin\theta - \mu R \tag{53}$$

$$ma\theta'^2 = -R + mg\cos\theta \tag{54}$$

We can again eliminate R, solve for θ' and θ and find the value of θ when R vanishes.

4.2.3 Circular Motion of Satellites

Just as planets move in elliptic orbits with the Sun in one focus, the man-made artificial satellites move in elliptic (or circular) orbits with the Earth (or rather its centre) at one focus.

If the Earth is of mass M and radius a and a satellite of mass m ($\ll M$) is projected from a point P at a height h above the Earth with velocity V at right angles to OP (Figure 4.6) it will move under a central force $Gm\,M/r^2$. Since the central force of a circular orbits is mV^2/r, we get, if the path is to be circular,

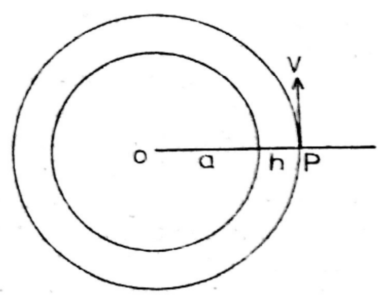


Figure 4.6

$$\frac{mV^2}{a+h} = \frac{Gm\,M}{(a+h)^2} \quad \text{or} \quad V^2 = \frac{GM}{a+h} \tag{55}$$

If g is the acceleration due to gravity, then the gravitational force on a particle of mass m on the surface of the Earth is mg. Alternatively from Newton's inverse square law, it is GMm/a^2 so that

$$\frac{GMm}{a^2} = mg \quad \text{or} \quad Gm = ga^2 \tag{56}$$

From (55) and (56)

$$V^2 = \frac{ga^2}{a+h} \tag{57}$$

This gives the velocity of a satellite describing a circular orbit at a height h above the surface of the Earth. Its time period is given by

$$T = \frac{2\pi(a+h)}{V} = \frac{2\pi(a+h)}{\sqrt{ga}}(a+h)^{1/2} = \frac{2\pi}{\sqrt{ga}}(a+h)^{3/2} \quad (58)$$

The earth completes one revolution about its axis in twenty-four hours. As such if T is 24 hours, the satellite would have the same period as the Earth and would appear stationary, to an observer on the Earth. Now taking $g = 32$ ft/sec^2, $a = 4000$ miles, $T = 24$ hours, we get if h is measured in miles

$$((4000+h)\times 1760\times 3)^{3/2} = \frac{24\times 60\times 60\sqrt{32}\times 4000\times 1760\times 3\times 7}{2\times 22}$$

$$= 1642607.416\times 10^6$$

$$(4000+h)\times 5280 = 13919.3408\times 10^4$$

$$4000+h = 26.36238788\times 10^3 = 26362.38788$$

$$h = 22362.38788 \text{ miles}$$

This gives the height of the synchronous or synchron satellite, which is very useful for communication purposes.

4.2.4 Elliptic Motion of Satellites

If a satellite is projected at a height $a+h$ above the centre of the Earth with a velocity different from $\sqrt{g}\,a/\sqrt{a+h}$ or if it is not projected at right angles to the radius vector, the orbit will not be circular, but can be elliptic, parabolic or hyperbolic depending on V and the angle of projection.

If the angle of projection is 90° and the orbit is an elliptic with semi major axis a' and eccentricity e, then there are two possibilities depending on whether the point of projection is the apogoee or the perigee

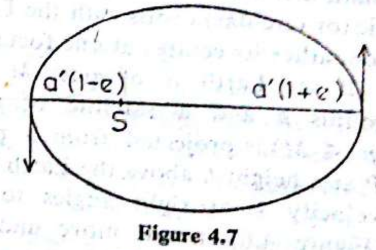


Figure 4.7

Using equation (23)

$$V^2 = \mu\left(\frac{2}{a'(1+e)} - \frac{1}{a'}\right),\quad a'(1+e) = a+h \quad (59)$$

or $$V^2 = \mu\left(\frac{2}{a'(1-e)} - \frac{1}{a'}\right),\quad a'(1-e) = a+h \quad (60)$$

i.e. $$V^2 = \frac{ga^2}{a+h}(1-e) \quad \text{or} \quad V^2 = \frac{ga^2}{a+h}(1+e)$$

i.e, $$V^2 = V_0^2(1-e) \quad \text{or} \quad V^2 = V_0^2(1+e), \quad (61)$$

where V_0 is the velocity required for a circular orbit for which $e = 0$. Thus if $V > V_0$, the point of projection is nearest point of the orbit to the centre of the Earth and if $V < V_0$, this point is the furthest point.

For the elliptic orbit, the time period is

$$T = \frac{2\pi}{\sqrt{ga}} a'^{3/2} \tag{62}$$

where if $V < V_0$, $e = \sqrt{1 - \frac{V^2}{V_0^2}}$, $a' = \dfrac{a+h}{1+\sqrt{1 - V^2/V_0^2}}$ (63)

and if $V > V_0$, $e = \sqrt{\frac{V^2}{V_0^2} - 1}$, $a' = \dfrac{a+h}{1-\sqrt{V^2/V_0^2 - 1}}$ (64)

If h_{max} and h_{min} are the maximum and minimum heights of a satellite above the Earth's surface and a is the radius of the Earth, we get

$$\frac{a'(1+e)}{a'(1-e)} = \frac{a+h_{max}}{a+h_{min}} \text{ or } \frac{1+e}{a+h_{max}} = \frac{1-e}{a+h_{min}}$$

$$= \frac{2}{2a + h_{max} + h_{min}}$$

or

$$\frac{1+e}{a+h_{max}} = \frac{1}{a + \dfrac{h_{max}+h_{min}}{2}} = \frac{e}{\dfrac{h_{max}-h_{min}}{2}}$$

or

$$e = \frac{h_{max} - h_{min}}{2a + h_{max} - h_{min}} \tag{65}$$

EXERCISE 4.2

1. Show that the force required to make a particle of mass move in a circular orbit of radius a with velocity v is mv^2/a directed towards the centre.
2. A particle of mass m is attached to the end of string, of length L, the other end of which is attached to a fixed point. The particle now moves in a horizontal circle of radius $a(< L)$. Discuss the motion of this conical pendulum.
3. Integrate (38) when $\theta' = 0$ when $\theta = \alpha$ and α is small.
4. Complete the discussion of section 4.1.1 when $u^2 = 4ag$.
5. Complete the discussion of motion of a particle on the inside of a smooth vertical circular wire when it is projected from the lowest period with horizontal velocity $2\sqrt{ag}$.
6. Complete the discussion of motion of a particle on the outside of a smooth vertical circular wire when it is projected from the highest point with velocity $3\sqrt{ag}$.
7. The following table gives data on some earth satellites

Name	max ht. (miles)	min ht. miles	weight 1bs	orbit time mts
Sputnik I	560	145	184.00	96.2
Sputnik II	1056	150	1120.00	103.7
Explorer I	1567	219	30.80	114.5

(*contd.*)

Vanguard	2466	405	3.25	134.0
Explorer III	1741	117	31.00	115.7
Sputnik III	1168	150	2920.00	106.0
Explorer IV	1386	178	38.43	110.0

Find the semi-major axis, semi-minor axis, eccentricity and the orbit time of each orbit and verify that the given values of the orbit times are what you expect on theoretical considerations.

8. Given $g = 981$ cm/s^2, $a = 6440 \times 10^5$ cm, $G = 6.670 \times 10^{-8}$ cm^3/(g s^2), find the mass of the Earth.

9. Find V so that the orbit may be a parabola or a hyperbola.

4.3 MATHEMATICAL MODELLING THROUGH LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

4.3.1 Rectilinear Motion

Let one end 0 of an elastic string of natural length $L(= 0A)$ be fixed (Figure 4.8) and let the other end to which a particle of mass m is attached

Figure 4.8

be stretched a distance a and then released. At any time t, let $x(t)$ be the extension, then the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -\lambda\frac{x}{L} = -kx, \tag{66}$$

where k is the elastic constant. If the particle moves in a resisting medium with resistance proportional to the velocity x', (66) becomes

$$mx'' + cx' + kx = 0, \tag{67}$$

which is a linear differential equation of the second order. Its solution is

$$x(t) = A_1e^{\lambda_1 t} + Ae^{\lambda_2 t} \tag{68}$$

where λ_1, λ_2 are the roots of

$$m\lambda^2 + c\lambda + k = 0 \tag{69}$$

Here $\lambda_1 + \lambda_2 = -\frac{c}{m}$, $\lambda_1\lambda_2 = \frac{k}{m}$. The sum of the roots is negative and the product of the roots is positive.

Case (i) $c^2 > 4km$, the roots are real and distinct and are negative. As such $x(t) \to 0$ as $t \to \infty$. The motion is said be *overdamped*.

Case (ii) $c^2 = 4km$, the roots are real and equal and

$$x(t) = (A_1 + A_2t)\exp\left(-\frac{c}{2m}t\right) \tag{70}$$

and again $x(t) \to 0$ as $t \to \infty$. In this case the motion is said to be *critically damped*.

Case (iii) $c^2 < 4$ km, the roots are complex conjugate with the real parts of the roots negative. $x(t)$ always oscillates but oscillations are damped out and tend to zero. In this case, the motion is said to be *under damped.*

Next we consider the case when there is an external force $m \cdot F(t)$ acting on the particle. In this case (67) becomes

$$mx'' + cx' + kx = mF(t) \tag{71}$$

A particular case of interest is given by the model

$$x'' + w_0^2 x = F \cos wt \tag{72}$$

i.e., when in the absence of the external force, the motion is simple harmonic with period $2\pi/w_0$ and the external force is periodic with period $2\pi/w$. The solution of (72) is given by

$$x(t) = A \cos(w_0 t - \alpha) + F \cos wt/(w_0^2 - w^2) \qquad w \neq w_0 \tag{73}$$

$$= A \cos(w_0 t - \alpha) + \frac{F}{2w_0}\, t \sin w_0 t \qquad w = w_0 \tag{74}$$

When $w = w_0$, the first term is periodic and its amplitude never exceeds $|A|$. However as $t \to \infty$ along a sequence for which $\sin w_0 t = \pm 1$, the magnitude of the second term approaches infinity.

The phenomenon we have discussed here is known as of *pure* or *undamped resonance*. It occurs when $c = 0$ and the input and natural frequencies are equal. We shall get a similar phenomenon when c is small. The forcing function $F \cos wt$ is then said to be in resonance with the system.

Bridges, cars, planes, ships are vibrating systems and an external periodic force with the same frequency as their natural frequency can damage them. This is the reason why soldiers crossing a bridge are not allowed to march in step. However resonance phenomenon can also be used to advantage e.g. in uprooting trees or in getting a car out of a ditch.

When w and w_0 differ only slightly, the solution represents superposition of two sinusoidal waves whose periods differ only slightly and this leads to the occurrence of beats.

4.3.2 Electrical Circuits

Figure 4.9 shows an electrical circuit. The current $i(t)$ amperes represents the time rate of change of charge q flowing in the circuits, so that

$$\frac{dq}{dt} = i(t) \tag{75}$$

(i) There is a resistance of R Ohms in the circuit. This may be provided by a light bulb, an electric heater or any other electrical device opposing the motion of the charge and causing a potential drop of magnitude $E_R = Ri$ volts.

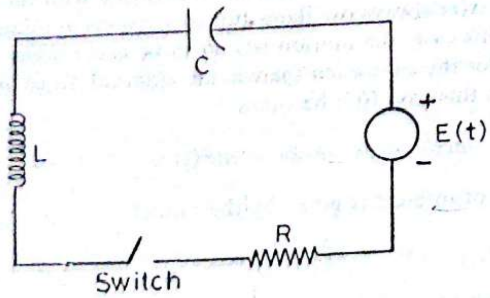


Figure 4.9

(ii) There is an induction of inductance L henrys which produces a potential drop $E_L = L\,di/dt$.

(iii) There is a capacitance C which produces a potential drop

$$E_c = \frac{1}{C} q.$$

All these potential drops are balanced by the battery which produces a voltage E volts. Now according to Kirchhoff's second law, the algebraic sum of the voltage drops round a closed circuit is zero so that

$$Ri + L\frac{di}{dt} + \frac{1}{C} q = E(t) \tag{76}$$

Differentiating and using (75), we get

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = \frac{dE}{dt} \tag{77}$$

Also substituting for (75) in (76) we get

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C} q = E(t) \tag{78}$$

Both (77) and (78) represent linear differential equations with constant coefficients and their solutions will determine $i(t)$ and $q(t)$.

Comparing (71) and (78), we get the correspondences

mass $m \leftrightarrow$ inductance L
friction coefficient $c \leftrightarrow$ resistance R
spring constant $k \leftrightarrow$ inverse capacitance $1/C$
impressed force $F \leftrightarrow$ impressed voltage E
displacement $x \leftrightarrow$ charge q
velocity $v = dx/dt \leftrightarrow$ current $i = \dfrac{dq}{dt}$.

This shows the correspondence between mechanical and electrical systems. This forms the basis of analogue computers. A linear differential equation of the second order can be solved by forming an electrical circuit and measuring the electric current in it. Similar analogues exist between hydrodynamical and electrical systems. Mathematical modelling brings out the isomorphisms between mathematical structures of quite different systems and gives a method for solving all these models in terms of the simplest of these models.

We can have analogues of (71), (78) in economic system when $k(t)$ represents the excess of the capital invested over the equilibrium capital and $E(t)$ can represent external investments.

4.3.3 Phillip's Stabilization Model for a Closed Economy

The assumptions of the model are:

(i) The producers adjust the national production Y of a product according to the aggregate demand D. If $D > Y$, they increase production and if $D < Y$, they decrease production so that we get

$$dY/dt = \alpha(D - Y), \alpha > 0, \tag{79}$$

where α is a reaction coefficient representing the velocity of adjustment.

(ii) Aggregate demand D is the sum of private demand, government demand G and an exogenous disturbance u. The private demand is proportional to the national income or output so that

$$D = (1 - L)\, Y + G - u \tag{80}$$

where $1 - L$ is the marginal propensity to spend i.e. it is the marginal propensity to consume plus the marginal propensity to invest. We assume that $0 < L < 1$.

(iii) The government adjusts its demand to bring the national out-put to a desired level, which without loss of generality may be taken as zero.

The Government decides its demand according to one of the following policies:

(*a*) *proportionate stabilization policy* according to which

$$G^* = -f_p\, Y \tag{81}$$

where $f_p > 0$ is the coefficient of proportionality and we use the negative sign on the right hand side since if the output is less than the described level, government will come out with a positive demand.

(*b*) *derivative stabilization policy* according to which

$$G^* = -f_d\, Y', \tag{82}$$

where $f_d > 0$ and the government demand is proportional to Y'.

(*c*) *mixed proportionate derivative policy* according to which

$$G^* = -f_p\, Y - f_d\, Y' \tag{83}$$

(*d*) *integral stabilization policy* according to which

$$G^* = -f_I \int_0^t Y\, dt, \quad f_I > 0 \tag{84}$$

(iv) G^* is the potential demand which the Government may like to make, but the actual demand G will be gradually adjusted so that

$$G' = \beta(G^* - G), \tag{85}$$

where β is the reaction coefficient. $\beta > 0$ since if $G < G^*$, the government tends to increase the demand to reach G^*.

Now from (79) and (80)

$$dY/dt = \alpha((1 - L)\, Y + G - u - Y), \tag{86}$$

so that

$$d^2Y/dt^2 = -\alpha L\, dY/dt + \alpha\, dG/dt \tag{87}$$

Eliminating G between (85), (86) and (87)

$$\frac{d^2Y/dt^2}{\alpha} + L\, dY/dt = \beta\left(G^* - \frac{dY/dt}{\alpha} - (Ly + u)\right) \tag{88}$$

or
$$d^2Y/dt^2 + dY/dt\,(\alpha L + \beta) + \alpha BLY + \alpha\beta u = \alpha\beta\, G^* \tag{89}$$

If we substitute for G^* from (81), (82) or (83), we get a linear differential equation of the second order with constant coefficients. If however the government uses integral stabilization policy, we use (84) to get the third order differential equation

$$d^3Y/dt^3 + (\alpha 1 + \beta)\, d^2Y/dt^2 + \alpha\beta\, dY/dt + \alpha\beta\, f_I Y = 0 \tag{90}$$

The equations (89) and (90) can be easily solved. Even without solving these, the stability of the solutions and their behaviour as $t \to \infty$ can be easily obtained.

EXERCISE 4.3

1. Solve $x'' + 13x' + 36x = 0$; $x(0) = 1$, $x'(0) = 0$ and plot $x(t)$ against t.
2. Solve $x'' + 8x' + 36x = 24 \cos 6t$ and discuss the behaviour of the solution as t approaches infinity.
3. Solve $x'' + 25x = 25 \cos 5t$ and plot $x(t)$. Discuss the nature of the motion.
4. Solve (89) for the proportionate stabilization policy. Show that the solution is

$$Y(t) = A\, e^{\lambda_1 t} + B\, e^{\lambda_2 t} + \frac{1}{1 + f_p}$$

where both λ_1, λ_2 are real and negative if $\Delta > 0$ where

$$\Delta = (\alpha L - \beta)^2 - 4\alpha\beta f_p$$

and these are complex with negative real parts of $\Delta < 0$.

5. Solve (89) for mixed proportionate-derivative stabilization policy and discuss the stability of the solution.

6. Show that all the roots of $a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ have negative real parts of

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_1 a_2 - a_0 a_3 > 0$$

7. Show that if (89) is solved subject to (84) and $u = 1$, the characteristic equation is

$$\lambda^3 + ((\alpha L + \beta)\lambda^2 + \alpha\beta(L + f_I)\lambda + \alpha\beta f_I = 0$$

and deduce that the stability condition is

$$f_I < (\alpha L + \beta)(L + f_p).$$

4.4 MISCELLANEOUS MATHEMATICAL MODELS THROUGH ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

4.4.1 The Catenary

A perfectly inflexible string is suspended under gravity from two fixed points A and B (Fig. 4.10).

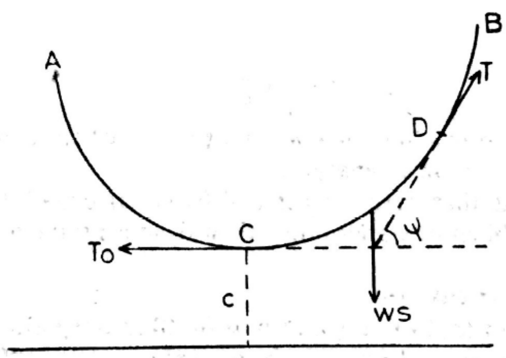


Figure 4.10

Consider the equilibrium of the part CD of the string of length s where C is the lowest point of the string at which the tangent is horizontal.

The forces acting on this part of the string are (i) tension T_0 at C (ii) tension T at point D along tangent at D (iii) weight ws of the string.

Equating the horizontal and vertical components of forces, we get

$$T\cos\psi = T_0, \qquad T\sin\psi = ws \tag{91}$$

Let T_0 be equal to weight of length c of the string, then (91) give

$$\tan\psi = \frac{ws}{To} = \frac{ws}{wc} = \frac{s}{c} \tag{92}$$

$$\frac{ds}{d\psi} = \rho = c\sec^2\psi, \tag{93}$$

where ρ is radius of curvature of the string at D; so that

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} = c\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

or
$$c\left(\frac{d^2y}{dx^2}\right) = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}, \tag{94}$$

which is a non-linear differential equation of second order. If $\frac{dy}{dx} = p$, then (94) gives

$$c\frac{dp}{\sqrt{1+p^2}} = dx \tag{95}$$

Integrating
$$\sinh^{-1} p = \frac{x}{c} + A \tag{96}$$

When $x = 0, p = 0$, so that $A = 0$ and

$$\frac{dy}{dx} = \sinh\frac{x}{c} \tag{97}$$

Integrating

$$y = c\cosh\frac{x}{c}, \tag{98}$$

where we choose x-axis in such a way that $y = c$ when $x = 0$. This is the equation of the common catenary.

It may be noted that here we get a differential equation of the second order from a problem of statics rather than from a problem of dynamics.

4.4.2 A Curve of Pursuit

A ship at the point $(a, 0)$ sights a ship at $(0, 0)$ moving along y-axis with a uniform velocity $ku(0 < k < 1)$. It begins to pursue ship B with a velocity u always moving in the direction of the ship B so that at any time AB is along the tangent to the path of A.

From Figure 4.11

$$\tan(\pi - \psi) = \frac{kut - y}{x}$$

or
$$-\frac{dy}{dx} = -\frac{y}{x} + \frac{kut}{x}$$

or
$$x\frac{dy}{dx} - y = -kut \tag{99}$$

Differentiating with respect to x, we get

$$x\frac{d^2y}{dx^2} = -ku\frac{dt}{dx} \tag{100}$$

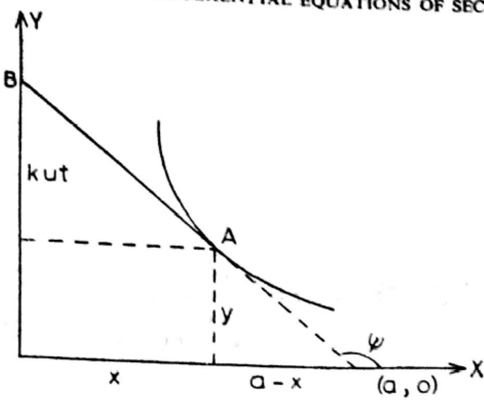


Figure 4.11

Now dx/dt = horizontal component of velocity of $A = u\cos(\pi - \psi)$

$$= -u\cos\psi = -\frac{u}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \tag{101}$$

so that from (99) and (100)

$$x\frac{d^2y}{dx^2} = k\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{102}$$

Putting $\frac{dy}{dx} = p$, we get

$$\frac{dp}{\sqrt{1+p^2}} = k\frac{dx}{x} \tag{103}$$

Integrating $$\frac{dy}{dx} = k\left(\sinh^{-1}\left(\ln\frac{x}{a}\right)\right) \tag{104}$$

Integrating once again, we get y as a function of x.

$k \ln x = \sinh^{-1}(p)$

EXERCISE 4.4

1. Prove that for the common catenary

$y^2 = c^2 + s^2,$ $\quad s = c\tan\psi,$

$y = c\sec\psi$ $\quad s = c\sinh\frac{x}{c}$

$$x = c\ln\frac{y+\sqrt{y^2-c^2}}{a} = c\ln(\sec\psi + \tan\psi)$$

$$= c\ln\frac{s+\sqrt{s^2+c^2}}{c}$$

$k \log x = \sinh^{-1} p$

$p = k \sinh^{-1}$

2. Integrate (104) and find y as a function of x.
3. Obtain the curves of pursuit when $k = 1$, $k > 1$.
4. When $k < 1$, when and where does A intercept B?

POSSIBLE QUESTIONS

Part B (6 Marks)

1. Derive the components of velocity and acceleration vectors along radial and transverse directions.
2. Find the height of synchronous from the circular motion of satellites.
3. Explain about the catenary.
4. Design a mathematical model for motion of a projectile.
5. Explain on elliptic motion of satellites.
6. Discussion detail on a curve of pursuit.
7. Discuss motion of a particle on a rough vertical wire.

Part C (10 Marks)

1. Explain in detail Kepler's law of planetary motion.
2. Explain on circular motion of satellites.
3. Discuss motion of a particle on a rough vertical wire.

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Subject: Mathematical Modelling **Subject Code: 18MMP303**

Class : II - M.Sc. Mathematics **Semester : III**

Unit III

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Every planet moves mainly under the gravitational attractive force exerted by_____	Jupiter	earth	sun	star	sun
If S and P are masses of sun and planet then G is the_____________	gravitation	generation	gauss	glimpse	gravitation
The acceleration of the Sun towards the planet is _______	Gp/r2	Gpr2	r2/Gp	GPS	Gp/r2
The acceleration of the planet towards the sun is _______	GS/r2	GSr2	r2/GS	GPS	GS/r2
In a model Motion under central force the equation of the path described by a particle moving under _____ force F.	central	corner	linear	nonlinear	central
In a model Motion under central force the equation of the path described by a particle moving under central force ____.	A	B	C	F	F
A particle moves under a central force per unit mass the path is ________	conic	circle	cube	square	conic
In a model Motion under inverse square law, the conic is ellipse then	e<1	e=1	e>1	e=0	e<1

In a model Motion under inverse square law, the conic is parabola then	e<1	e=1	e>1	e=0	e=1
In a model Motion under inverse square law, the conic is hyperbola then	e<1	e=1	e>1	e=0	e>1
The law of attraction between different planets and sun must be _______ law.	Inverse square	inverse cube	inverse rectangle	inverse circle	Inverse square
Every planets describes an ellipsis with sun at one focus states _________ law	Inverse square	Kepler's	Gauss	Newton's	Kepler's
The radius vector from the sun to a planet describes equal areas in equal interval of time states _______ law	Inverse square	Kepler's	Gauss	Newton's	Kepler's
The squares of periodic time of planets are proportional to the cubes of the semi major axes of the orbits of the planets states ______ law	Inverse square	Kepler's	Gauss	Newton's	Kepler's
The squares of periodic time of planets are ___________ to the cubes of the semi major axes of the orbits of the planets.	Proportional	linear	unit	orthogonal	Proportional
____________ no of kepler's law are in planetary motion	1	2	3	4	3
Detection of Kepler's 3 laws of planetary motion from universal law of gravitation was success of _______ modelling	Mathematical	Physical	Chemical	Biological	Mathematical
Detection of Kepler's 3 laws of planetary motion from universal law of ____________ was success of mathematical modelling	Gravitation	generation	gauss	glimpse	gravitation
The partial begins to move freely under gravity and describe _____ path till the string again becomes tight and the circular motion is started again	Parabola	Hyperbola	Ellipse	Circle	Parabola
The particle begins to move freely under gravity and describes a parabolic path till the string again becomes tight . States ______	Circular motion	SHM	MOC	SIS	Circular motion

In a model of circular motion satellites the man made artificial satellites move in ___________ orbit with earth	Parabola	Hyperbola	Elliptic	Circle	Elliptic
In rectilinear motion model the roots are real and distinct and are negative as such x(t) tends to 0 as t tends infinity then motion is _________	Overdampped	Critically damped	under damped	undamped resonance	over damped
In rectilinear motion model the roots are real and equal as such x(t) tends to 0 as t tends infinity then motion is _________	Overdampped	Critically damped	under damped	undamped resonance	Critically damped
In rectilinear motion model the roots are complex and the roots are negative then the motion said to be	Overdampped	Critically damped	under damped	undamped resonance	under damped
In rectilinear motion model the roots are complex and the roots are negative and also external force acting on the particle then the motion _______	Overdampped	Critically damped	under damped	undamped resonance	undamped resonance
In the model of catenary differential equation of the second order from a problem of statics rather than ___________	Dynamics	Mechanics	Statistics	Analysis	Dynamics

UNIT-IV

Mathematical Modeling through Difference Equations: Simple Models – Basic Theory of Linear Difference Equations with Constant Coefficients – Economics and Finance – Population Dynamics and Genetics – Probability Theory.

5

Mathematical Modelling Through Difference Equations

5.1 THE NEED FOR MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS: SOME SIMPLE MODELS

We need difference equation models when either the independent variable is discrete or it is mathematically convenient to treat it as a discrete variable.

Thus in Genetics, the genetic characteristics change from generation to generation and the variable representing a generation is a discrete variable.

In Economics, the price changes are considered from year to year or from month to month or from week to week or from day to day. In every case, the time variable is discretized.

In Population Dynamics, we consider the changes in population from one age-group to another and the variable representing the age-group is a discrete variable.

In finding the probability of n persons in a queue or the probability of n persons in a state or the probability of n successes in a certain number of trials, the independent variable is discrete.

For mathematical modelling through differential equations, we give an increment Δx to independent variable x, find the change Δy in y and let $\Delta x \rightarrow 0$ to get differential equations. In most cases, we cannot justify the limiting process rigorously. Thus for modelling fluid motion, making $\Delta x \rightarrow 0$ has no meaning since a fluid consists of a large number of particles and the distance between two neighbouring particles cannot be made arbitrary small. Continuum mechanics is only an approximation (through fortunately a very good one) to reality.

Even if the limiting process can be justified e.g. when the independent variable is time, the resulting differential equation may not be solvable analytically. We then solve it numerically and for this purpose, we again replace the differential equation by a system of difference equations. Numerical methods of solving differential equations essentially mean solving difference equations.

It is even argued that since in most cases, we have to ultimately solve difference equations, we may avoid modelling through differential equations altogether. This is of course going too far since as we have seen in earlier chapters, mathematical modelling through differential equations is of immense importance to science and technology. Another argument in favour of difference equation models is that those biological and social scientists who do not know calculus and transcendental numbers like e can still work with difference equation models and some important consequences of these models can be deduced with the help of even pocket calculators by even high school students.

We now give simple difference equation models parallel to the differential equation models studied in earlier chapters.

(i) *Population Growth Model*: If the population at time t is $x(t)$, then assuming that the number of births and deaths in the next unit interval of time are proportional to the populations at time t, we get the model:

$$x(t+1) - x(t) = bx(t) - dx(t) \quad \text{or} \quad x(t+1) = ax(t), \tag{1}$$

so that

$$x(t) = ax(t-1) = a^2x(t-2) = a^3x(t-3) = \dots = a^tx(0) \tag{2}$$

This may be compared with the differential equation model:

$$\frac{dx}{di} = ax \text{ with the solution } x(t) = x(0)e^{at} \tag{3}$$

For solving the difference equation model, we require only simple algebra, but for solving the differential equation model, we require knowledge of calculus, differential equation and exponential functions.

(ii) *Logistic Growth Model*: This is given by

$$x(t+1) - x(t) = ax(t) - bx^2(t) \tag{4}$$

This is not easy to solve, but given $x(0)$, we can find $x(1)$, $x(2)$, $x(3)$, . . . in succession and we can get a fairly good idea of the behaviour of the model with the help of a pocket calculator.

(iii) *Prey-Predator Model*: This is given by

$$\left.\begin{aligned} x(t+1) - x(t) &= -ax(t) + bx(t)y(t) \\ y(t+1) - y(t) &= py(t) - qx(t)y(t) \end{aligned}\right] \begin{aligned} a, b > 0 \\ p, q > 0 \end{aligned} \tag{5}$$

and again given $x(0)$, $y(0)$, we can find $x(1)$, $y(1)$; $x(2)$, $y(2)$; $x(3)$, $y(3)$, . . . , in succession.

(iv) *Competition Model*: This is given by

$$\left.\begin{aligned} x(t+1) - x(t) &= ax(t) - bx(t)y(t) \\ y(t+1) - y(t) &= px(t) - qx(t)y(t) \end{aligned}\right] \begin{aligned} a, b > 0 \\ p, q > 0 \end{aligned} \tag{6}$$

(v) *Simple Epidemics Model*: This is given by

$$\left.\begin{aligned} x(t+1) - x(t) &= -\beta x(t)y(t) \\ y(t+1) - y(t) &= \beta x(t)y(t) \end{aligned}\right], \quad \beta > 0 \tag{7}$$

EXERCISE 5.1

1. For model (i), let $x(0) = 100$, $a = 0.5$ or 1 or 2; find $x(t)$ for $t = 1$ to 50 and plot $x(t)$ as a function of t in each case.
2. For model (ii) let $x(0) = 100$, $a = 0.1$, $b = 0.001$, find $x(t)$ for $t = 1$ to 100 and plot $x(t)$ as a function of t.
3. In models (iii) and (iv) let $x(0) = 40$, $y(0) = 10$, $a = 0.01$, $b = 0.001$; $p = 0.005$, $q = 0.0001$. Plot points $x(t)$, $y(t)$ for $t = 0$ to 50.
4. In model (v), let $x(0) = 100$, $y(0) = 1$, $\beta = 0.5$, plot $x(t)$, $y(t)$ in the x-y plane for $t = 0$ to 100.

5.2 BASIC THEORY OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

This theory is parallel to the corresponding theory of linear differential equations with constant coefficients, but is not usually taught in many places. We are therefore including a brief account here.

Note

5.2.1 The Linear Difference Equation

An equation of the form

$$f(x_{t+n}, x_{t+n-1}, \ldots, x_t, t) = 0 \tag{8}$$

is called a difference equations of nth order. The equation

$$f_0(t)x_{t+n} + f_1(t)x_{t+n-1} + \ldots + f_n(t)x_t = \varphi(t) \tag{9}$$

is called a linear difference equation, since it involves $x_t, x_{t+1}, \ldots, x_{t+n}$ only in the first degree. The equation

$$a_0x_{t+n} + a_1x_{t+n-1} + \ldots + a_nx_t = \varphi(t) \tag{10}$$

is called a linear difference equation with constant coefficients. The equation

$$a_0x_{t+r} + a_1x_{t+n-1} + \ldots + a_nx_t = 0 \tag{11}$$

is called a homogeneous linear difference equations with constant coefficients. Let $x_t = g_1(t), g_2(t), \ldots, g_n(t)$ be n linearly independent solutions of (11), then it is easily seen that

$$x_t = A_1g_1(t) + A_2g_2(t) + \ldots + A_ng_n(t) \tag{12}$$

is also a solution of (11) where $A_1, A_2, \ldots, A_n$ are n arbitrary constants. This is the most general solution of (11).

Again it can be shown that if $G_1(t)$ is the solution of (11) containing n arbitrary constants and $G_2(t)$ is any particular solution of (10) containing no arbitrary constant, then $G_1(t) + G_2(t)$ is the most general solution of (10). $G_1(t)$ is called the complementary function and G_2 is called a particular solution.

Note

5.2.2 The Complementary Function

We try the solution $x_t = a\lambda^t$. If this satisfies (11), we get

$$g(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0 \tag{13}$$

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This algebraic equation of nth degree has n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, real or complex. The complementary function is then given by

$$G_1(t) = c_1\lambda_1^t + c_2\lambda_2^t + \ldots + c_n\lambda_n^t \tag{14}$$

Case (i): If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all real and distinct, (14) gives us the complementary function when $c_1, c_2, \ldots, c_n$ are any n arbitrary real constants.

Case (ii): If two of the roots λ_1, λ_2 are equal, then (14) contains only $n - 1$ arbitrary constants and as such it cannot be the most general solution. We try the solution $ct\lambda_1^t$. We get

$$a_0(t+n)\lambda_1^n + a_1(t+n-1)\lambda_1^{n-1} + \ldots + a_n = 0$$

or $$tg(\lambda_1) + g'(\lambda_1) = 0, \tag{15}$$

which is identically satisfied since both $g(\lambda_1) = 0$ and $g'(\lambda_1) = 0$ as λ_1 is a repeated root. In this case

$$G_1(t) = (c_1 + c_2t)\lambda_1^t + c_3\lambda_3^t + c_4\lambda_4^t + \ldots + c_n\lambda_n^t \tag{16}$$

Case (iii): If a root λ_1 is repeated k times, the complementary function is

$$G_1(t) = (c_1 + c_2t + c_3t^2 + \ldots + c_kt^{k-1})\lambda_1^t + c_{k+1}\lambda_{k+1}^t + \ldots + c_n\lambda_n^t \tag{17}$$

Case (iv): Let $g(\lambda) = 0$ have two complex roots $\alpha \pm i\beta$, then their contribution to complementary function is

$$c_1(\alpha + i\beta)^t + c_2(\alpha - i\beta)^t \tag{18}$$

Putting $\alpha = r\cos\theta$, $\beta = r\sin\theta$ and using De Moivre's theorem, this reduces to

$$\begin{aligned} &c_1r^t(\cos\theta + i\sin\theta)^t + c_2r^t(\cos\theta - i\sin\theta)^t \\ &= r^t\cos(\theta t)(c_1 + c_2) + r^t\sin(\theta t)(ic_1 - ic_2) \\ &= r^t(d_1\cos(\theta t) + d_2\sin(\theta t)) \\ &= (\alpha^2 + \beta^2)^{t/2}(d_1\cos(\theta t) + d_2\sin(\theta t)), \end{aligned} \tag{19}$$

$$\text{where } \tan\theta = \frac{\beta}{\alpha} \tag{20}$$

and d_1, d_2 are arbitrary constants.

Case (v): If the complex roots $\alpha \pm i\beta$ are repeated k times, then contribution to the complementary function is

$$\begin{aligned} &(\alpha^2 + \beta^2)^{t/2}((d_0 + d_1t + \ldots + d_{k-1}t^{k-1}\cos(\theta t) \\ &\quad + (f_0 + f_1t + \ldots + f_{k-1}t^{k-1})\sin(\theta t) \end{aligned} \tag{21}$$

where $d_0, d_1, \ldots, d_{k-1}, f_0, \ldots, f_{k-1}$ are $2k$ arbitrary constants.

5.2.3 The Particular Solution

Here we want a solution of (10) not containing any arbitrary constant.

Case (i): Let $\varphi(t) = AB^t$, B is not a root of $g(\lambda) = 0$ (22)

We try the solution CB^t. Substituting in (10), we get

$$CB^t(a_0B^n + a_1B^{n-1} + \ldots + a_n) = AB^t \tag{23}$$

If $B \neq \lambda_1, \lambda_2, \ldots, \lambda_n$, we get

$$C = \frac{A}{a_0B^n + a_1B^{n-1} + \ldots + a_n} \tag{24}$$

and the particular solution is

$$\frac{AB^t}{a_0B^n + a_1B^{n-1} + \ldots + a_n} \tag{25}$$

Case (ii): Let

$$\varphi(t) = AB^t, \ B \text{ is a non-repeated root of } g(\lambda) = 0 \tag{26}$$

We try the solution CtB^t. Substituting in (10), we get

$$B^t(Ct\, g(B) + Cg'(B)) = AB^t \tag{27}$$

Since $g(B) = 0$, $g'(B) \neq 0$

$$C = \frac{A}{g'(B)}, \tag{28}$$

so that the particular solution is

$$\frac{AtB^t}{a_0nB^{n-1} + a_1(n-1)B^{n-2} + \ldots + a_{n-1}} \tag{29}$$

Case (iii): Let

$$\varphi(t) = AB^t, \quad g(B) = 0, \quad g'(B) = 0, \ldots,$$
$$g^{(k-1)}(B) = 0, \qquad g^{(k)}(B) \neq 0, \tag{30}$$

then the particular solution is

$$\frac{At^{k-1}B^t}{g^{(k)}(B)} \tag{31}$$

Case (iv): Let $\qquad \varphi(t) = At^k \qquad (32)$

We try the solution

$$d_0t^k + d_1t^{k-1} + d_2t^{k-2} + \ldots + d_k \tag{33}$$

Substituting in (10) we get

$$\begin{aligned} &a_0(d_0(t+n)^k + d_1(t+n)^{k-1} + d_2(t+n)^{k-2} + \ldots + d_k) \\ &+ a_1(d_0(t+n-1) + d_1(t+n-1)^{k-1} + d_2(t+n-1)^{k-2} \\ &+ \ldots + d_k) + \ldots + a_n(d_0t^k + d_1t^{k-1} + d_2t^{k-2} + \ldots + d_k) \\ &= 0 \end{aligned} \tag{34}$$

Equating the coefficients of $t^k, t^{k-1}, \ldots, t^0$, on both sides, we get $(k+1)$ equations which in general will enable us to determine $d_0, d_1, d_2, \ldots, d_k$ and thus the particular solution will be determined.

5.2.4 Obtaining Complementary Function by Use of Matrices

Let

$$\begin{aligned} x_t &= x_1(t) \\ x_{t+1} &= x_2(t) = x_1(t+1) \\ x_{t+2} &= x_3(t) = x_2(t+1) \\ \dots &\quad \dots \quad \dots \\ x_{t+n} &= x_{n+1}(t) = x_n(t+1), \end{aligned} \tag{35}$$

so that (11) becomes

$$a_0 x_n(t+1) = -a_1 x_n(t) - a_2 x_{n-1}(t) - \dots - a_n x_1(t) \tag{36}$$

Equations (35) and (36) give

$$\begin{aligned} x_1(t+1) &= x_2(t) \\ x_2(t+1) &= x_3(t) \\ \dots &\quad \dots \\ x_{n-1}(t+1) &= x_n(t) \\ x_n(t+1) &= -\frac{a_1}{a_0}x_n(t) - \frac{a_2}{a_0}x_{n-1}(t) - \dots - \frac{a_n}{a_0}x_1(t), \end{aligned} \tag{37}$$

which can be written in the matrix form

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \dots & -\frac{a_1}{a_0} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{38}$$

or

$$X(t+1) = AX(t), \tag{39}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \dots & -\frac{a_1}{a_0} \end{bmatrix} \tag{40}$$

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Applying (39) repeatedly

$$X(k) = A^k X(0), \tag{41}$$

where

$$X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \\ x_n(0) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(n-1) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \tag{42}$$

Thus knowing the values of x_1 at times $0, 1, 2, \ldots, n-1$, we can find its value at all subsequent times.

5.2.5 Solution of a System of Linear Homogeneous Difference Equations with Constant Coefficients

Let the system be given by

$$\begin{aligned} x_1(t+1) &= a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) \\ x_2(t+1) &= a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t) \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n(t+1) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t) \end{aligned} \tag{43}$$

This can be written in the matrix form

$$X(t+1) = AX(t), \tag{44}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \tag{45}$$

Applying (44) repeatedly, we wet

$$X(k) = A^k X(0) \tag{46}$$

5.2.6 Solution of Linear Difference Equations by Using Laplace Transform

Let the linear difference equation be

$$\begin{aligned} a_0 f(t) + a_1 f(t-1) + \ldots + a_n f(t-n) &= \varphi(t), \\ f(t) = 0 \quad \text{when } t &< 0 \end{aligned} \tag{47}$$

Let $\bar{f}(\lambda)$ be the Laplace transform of $f(t)$ so that

$$\bar{f}(\lambda) = L(f(t)) = \int_0^\infty e^{-\lambda t} f(t)\, dt \tag{48}$$

then $L(f(t-1)) = \int_1^\infty e^{-\lambda t} f(t-1)\,dt$

$$= e^{-\lambda} \int_0^\infty e^{-\lambda t} f(t)\,dt = e^{-\lambda} \bar{f}(\lambda)$$

$$L(f(t-2)) = \int_2^\infty e^{-\lambda t} f(t-2)\,dt$$

$$= e^{-2\lambda} \int_0^\infty e^{-\lambda t} f(t)\,dt = e^{-2\lambda}\bar{f}(\lambda) \qquad (49)$$

and so on so that taking Laplace transform of both sides of (49), we get

$$(a_0 + a_1 e^{-\lambda} + a_2 e^{-2\lambda} + \ldots + a_n e^{-n\lambda})\bar{f}(\lambda) = L(\varphi(t)) = \bar{\varphi}(\lambda), \qquad (50)$$

so that $\bar{f}(\lambda)$ is known. Inverting the Laplace transform, we get $f(t)$. In this case t is regarded as a continuous variate such that $f(t) = 0$ when $t < 0$. If t is a discrete variate, it is better to use the z-transform.

5.2.7 Solution of Linear Difference Equations by Using z-Transform

Note

Let $\{u_n\}$ be an infinite sequence, then its z-transform is defined by

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}, \qquad (51)$$

whenever this infinite series converges. If $\{u_n\}$ is a probability distribution and $z = 1/s$, it will be the same as the probability generating function. The following results can be easily established

(i) If $k > 0$, $Z(u_{n-k}) = z^{-k} Z(u_n)$ (52)

(ii) If $k > 0$, $Z(u_{n+k}) = z^k[Z(u_n) - \sum_{m=0}^{k-1} u_m z^{-m}]$ (53)

(iii)

u_n :	1	a^n	e^{an}
$Z(u_n)$:	$z/(z-1)$	$z/(z-a)$	$z/(z-e^a)$

(54)

Taking z-transform of both sides of a linear difference equation, we can find $Z(u_n)$ and expanding it in powers of $1/z$ and finding the coefficient of z^{-n}, we can get u_n.

5.2.8 Solution of non-Linear Difference Equations Reducible to Linear Equations

Note

Thus equations

$$y_{n+1} = \sqrt{y_n} \qquad (55)$$

$$y_n y_{n+2} = y_{n+1}^2 \qquad (56)$$

become linear on substitution $u_n = \ln y_n$. Also

$$y_{n+2} = \frac{y_n y_{n+1}}{y_n + y_{n+1}} \qquad (57)$$

becomes linear on substitution $u_n = 1/y_n$.

5.2.9 Stability Theory for Difference Equations

If $x_t = K$ satisfies

$$f(x_t, x_{t+1}, x_{t+2}, \ldots, x_{t+n}) = 0 \tag{58}$$

then this gives an equilibrium position. To find its stability, we substitute $x_t = K + u_t$ in (58) and simplify neglecting squares and products and higher powers of u_t's to get a linear equation

$$a_1 u_{t+n} + a_2 u_{t+n-1} + \ldots + a_n u_t = 0 \tag{59}$$

We try the solution $u_t = A\lambda^t$ and get the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \tag{60}$$

If the absolute value of each of the n roots of this equation is less than unity, then u_t would tend to zero as $t \to \infty$ for all small initial disturbances and the equilibrium position would be locally asymptotically stable.

The conditions for all the roots of (60) having magnitude less than unity are given by Schur's criterion viz. that all the following determinants should be positive.

$$\Delta_1 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_0 & 0 & \vdots & a_n & a_{n-1} \\ a_1 & a_0 & \vdots & 0 & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & 0 & \vdots & a_0 & a_1 \\ a_{n-1} & a_n & \vdots & 0 & a_0 \end{vmatrix}$$

$$\Delta_n = \begin{vmatrix} a_0 & 0 & \cdots & 0 & \vdots & a_n & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & 0 & \vdots & 0 & a_n & \cdots & a_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & a_n & \cdots & a_0 & \vdots & 0 & 0 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & 0 & \cdots & 0 & \vdots & a_0 & a_1 & \cdots & a_{n-1} \\ a_n & a_n & & 0 & \vdots & 0 & a_0 & & a_{n-2} \\ a_1 & a_2 & \cdots & a_n & \vdots & 0 & 0 & \cdots & a_0 \end{vmatrix} \tag{61}$$

EXERCISE 5.2

1. Solve the following and discuss the behaviour of each solution as $t \to \infty$:

(i) $x_{t+2} - 7x_{t+1} + 12x_t = 0$

(ii) $x_{t+3} - 5x_{t+2} + 7x_{t+1} - 3x_t = 0$

(iii) $x_{t+2} - 2x_{t+1} + 2x_t = 0$

(iv) $8x_{t+3} - 12x_{t+2} + 6x_{t+1} - x_t = 0$

(v) $x_{t+2} + 2x_{t+1} + x_t = 0$

(vi) $2x_{t+2} - 2x_{t+1} + x_t = 0$

(vii) $x_{t+2} - x_{t+1} + x_t = 0$

2. Solve the following difference equations

(i) $x_{t+2} - 4x_{t+1} + 4x_t = 2^t$

(ii) $x_{t+2} - 4x_{t+1} + 3x_t = t$

(iii) $x_{t+2} - 7x_{t+1} + 12x_t = 3^t + t^4 + 4^t t^3$.

3. Solve the following simultaneous equations

(i) $x_{n+1} - x_n + 2y_{n+1} = 0$

$y_{n+1} - y_n - 2x_n = 2^n$

(ii) $x_{n+1} - 2x_n - y_n = n$

$y_{n+1} - 2x_n - 3y_n = -n$

4. Solve difference equations in Exercises 1 and 2 by using

(i) Laplace transform method

(ii) Z-transform method

(iii) Transforming to a matrix equation.

5. Prove results (52), (53), (54) and solve equations (55), (56), (57).

6. Show that the system (44) will be stable if all the eigenvalue of this matrix have magnitude less than unity.

7. Prove that for (44) to be stable, it is necessary that

$$|A| < 1, \ -n < \text{trace } A < n$$

8. Prove that if the sum of the elements of each column of a square matrix with non-negative elements is less than unity, then all the characteristic roots of this matrix have magnitude less than unity.

9. Discuss the stability of the following systems

(i) $x_{t+3} + 9x_{t+2} - 5x_{t+1} - 2x_t = 0$

(ii) $2x_{t+2} - 2x_{t+1} + x_t = 0$

(iii) $$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

10. Write explicitly the conditions that all roots of

(i) $a_0\lambda^2 + a_1\lambda + a_2 = 0$ (ii) $a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$

are less than unity in magnitude.

5.3 MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS IN ECONOMICS AND FINANCE

5.3.1 The Harrod Model

Note

Let $S(t)$, $Y(t)$, $I(t)$ denote the savings, national income and investment respectively. We make now the following assumptions:

(i) Savings made by the people in a country depend on the national income i.e.

$$S(t) = \alpha Y(t), \quad \alpha > 0 \tag{62}$$

(ii) The investment depends on the difference between the income of the current year and the last year i.e.

$$I(t) = \beta(Y(t) - Y(t-1)), \quad \beta > 0 \tag{63}$$

(iii) All the savings made are invested, so that

$$S(t) = I(t) \tag{64}$$

From (62), (63) and (64), we get the difference equation

$$Y(t) = \frac{\beta}{\beta - \alpha} Y(t-1), \tag{65}$$

which has the solution

$$Y(t) = A\left(\frac{\beta}{\beta - \alpha}\right)^t = Y(0)\left(\frac{\beta}{\beta - \alpha}\right)^t \tag{66}$$

Assuming that $Y(t)$ is always positive,

$$\beta > \alpha, \ \beta/(\beta - \alpha) > 1, \tag{67}$$

so that the national income increases with t. The national incomes at different times 0, 1, 2, 3, ... form a geometrical progression.

Thus if all savings are invested, savings are proportional to national income and the investment is proportional to the excess of the current years income over the preceding years income, then the national income increases geometrically.

5.3.2 The Cobweb Model

Let p_t = price of a commodity in the year t and

q_t = amount of the commodity available in the market in year t,

then we make the following assumptions

(i) Amount of the commodity produced this year and available for sale is a linear function of the price of the commodity in the last year, i.e.

$$q_t = \alpha + \beta p_{t-1}, \tag{68}$$

where $\beta > 0$ since if the last year's price was high, the amount available this year would also be high.

(ii) The price of the commodity this year is a linear function of the amount available this year i.e.

$$p_t = \gamma + \delta q_t, \tag{69}$$

where $\delta < 0$, since if q_t is large, the price would be low. From (68) and (69)

$$p_t - \beta\delta\, p_{t-1} = \gamma + \alpha\delta, \tag{70}$$

which has the solution

$$\left(p_t - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right) = \left(p_0 - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right)(\beta\delta)^t, \tag{71}$$

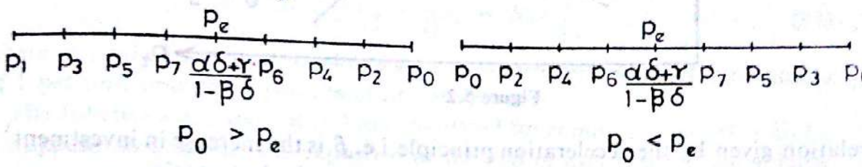
so that

$$\left(p_t - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right) = \left(p_{t-1} - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right)(\beta\delta) \qquad (72)$$

Since $\beta\delta$ is negative $p_0, p_1, p_2, p_3, \ldots$ are alternatively greater and less than $(\alpha\delta + \gamma)/(1 - \beta\delta)$.

If $|\beta\delta| > 1$, the deviation of p_t from $(\alpha\delta + \gamma)/(1 - \beta\delta)$ goes on increasing. On the other hand if $|\beta\delta| < 1$, this deviation goes on decreasing and ultimately $p_t \to (\alpha\delta + \gamma)/(1 - \beta\delta)$ as $t \to \infty$.

Figures 5.1a and 5.1b show how the price approaches the equilibrium price $p_e = (\alpha\delta + \gamma)/(1 - \beta\delta)$ as t increases in the two cases when $p_0 > p_e$ and $p_0 < p_e$ respectively.



(a) (b)

Figure 5.1

In the same way, eliminating p_t from (67), (68) we get

$$q_t = \alpha + \beta\gamma + \beta\delta\, q_{t-1}, \qquad (73)$$

which has the solution

$$\left(q_t - \frac{\alpha + \beta\gamma}{1 - \beta\delta}\right) = \left(q_e - \frac{\alpha + \beta\gamma}{1 - \beta\delta}\right)(\beta\delta)^t, \qquad (74)$$

so that q_t also oscillates about the equilibrium quantity level

$$q_t = (a + \beta\gamma)/(1 - \beta\delta) \quad \text{if} \quad |\beta\delta| < 1$$

The variation of both prices and quantities is shown simultaneously in Figure 5.2.

Suppose we start in the year zero with price p_0, and quantity q_0 represented by the point A. In year 1, the quantity q_1 is given by $\alpha + \beta\, p_0$ and the price is given by $p_1 = \gamma + \delta q_1$. This brings us to the point C in two steps via B. The path of prices and quantities is thus given by the Cobweb path $ABCDEFGHI, \ldots$ and the equilibrium price and quantity are given by the intersection of the two straight lines.

5.3.3 Samuelson's Interaction Models

The basic equations for the first interaction model are:

$$Y(t) = C(t) + I(t),\ C(t) = \alpha Y(t-1),\ I(t) = \beta[C(t) - C(t-1)] \qquad (75)$$

Here the positive constant α is the marginal propensity to consume with respect to income of the previous year and the positive constant β is the

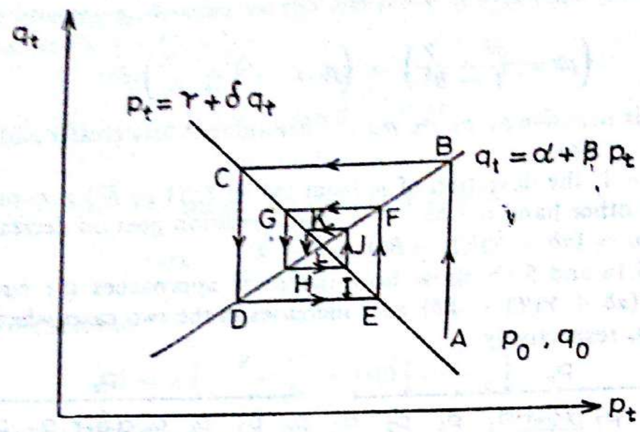


Figure 5.2

relation given by the acceleration principle i.e. β is the increase in investment per unit of excess of this years consumption over the last year's.

From (75), we get the second order difference equation

$$Y(t) - \alpha(1 + \beta)Y(t - 1) + \alpha\beta Y(t - 2) = 0 \tag{76}$$

In the second interaction model, there is an additional investment by the government and this investment is assumed to be a constant γ. In this case (76) is modified to

$$Y(t) - \alpha(1 + \beta)Y(t - 1) + \alpha\beta Y(t - 2) - \gamma = 0 \tag{77}$$

The solution of (76) and (77) can show either an increasing trend in $Y(t)$ or a decreasing trend in $Y(t)$ or an oscillating trend in it.

5.3.4 Application to Actuarial Science

Note

One important aspect of actuarial science is what is called mathematics of finance or mathematics of investment.

If a sum S_0 is invested at compound interest of i per unit amount per unit time and S_t is the amount at the end of time t, then we get the difference equation

$$S_{t+1} = S_t + iS_t = (1 + i)S_t, \tag{78}$$

which has the solution

$$S_t = S_0(1 + i)^t, \tag{79}$$

which is the well-known formula for compound interest.

Suppose a person borrows a sum S_0 at compound interest i and wants to amortize his debt, i.e. he wants to pay the amount and interest back by payment of n equal instalments, say R, the first payment to be made at the end of the first year.

Let S_t be the amount due at the end of t years, then we have the difference equation

$$S_{t+1} = S_t + iS_t - R = (1 + i)S_t - R \tag{80}$$

Its solution is

$$S_t = \left(S_0 - \frac{R}{i}\right)(1 + i)^t + \frac{R}{i} \tag{81}$$

$$= S_0(1 + t)^t - R\,\frac{(1 + i)^t - 1}{i} \tag{82}$$

If the amount is paid back in n years, $S_n = 0$, so that

$$R = S_0\frac{i}{1 - (1 + i)^{-n}} = S_0\,\frac{1}{a_{\overline{n}|i}}, \tag{83}$$

where $a_{\overline{n}|i}$ called the amortization factor is the present value of an annuity of 1 per unit time for n periods at an interest rate i.

The functions $a_{\overline{n}|i}$ and $(a_{\overline{n}|i})^{-1}$ are tabulated for common values of n and i.

Suppose an amount R is deposited at the end of every period in a bank and let S_t be the amount at the end of t periods, then

$$S_{t+1} = S_t(1 + i) + R, \tag{84}$$

so that (since $S_0 = 0$)

$$S_n = R\,\frac{(1 + i)^n - 1}{i} = RS_{\overline{n}|i} \tag{85}$$

From (83) and (85)

$$S_{\overline{n}|i} = (1 + i)^n a_{\overline{n}|i} \tag{86}$$

or

$$\frac{1}{S_{\overline{n}|i}} = \frac{(1 + i)^{-n}}{a_{\overline{n}|i}} \tag{87}$$

If a person has to pay an amount S at the end of n years, he can do it by paying into a sinking find an amount R per period where

$$R = S\,\frac{1}{S_{\overline{n}|i}} \tag{88}$$

where $\frac{1}{S_{\overline{n}|i}}$ is the sinking fund factor and can be tabulated by using (87).

EXERCISE 5.3

1. Show that the necessary and sufficient conditions for both roots of

$$m^2 + a_1 m + a_2 = 0$$

to be less than unity in absolute magnitude are

$$1 + a_1 + a_2 > 0, \quad 1 - a_1 + a_2 > 0, \quad 1 - a_2 > 0$$

2. Use the condition of Ex. 1 to show that the model of equation (76) is stable if

$$1 - \alpha > 0, \quad 1 - \alpha\beta > 0$$

i.e. if both the marginal propensity to consume and its product with the relation must be less than unity.

3. Show that if the condition of Ex. 2 are satisfied, then for the model of equation (77), the national income will tend to its equilibrium value $Y/(1-\alpha)$. Show also that the approach to equilibrium value will be oscillatory if

$$\alpha(1+\beta)^2 < 4\alpha\beta$$

4. For the model

$$Y_t = I_t + C_t, \quad C_t = C + mY_t, \quad rI_t = Y_{t+1} - Y_t$$

find C_t, I_t, Y_t and discuss stability of equilibrium position.

5. Let S_t denote the amount due at the end of t periods when the amounts being paid are $R, 2R, 3R, \ldots$. Show that

$$S_{t+1} = S_t(1+i) + (t+1)R$$

Show that the solution is

$$S_t = \frac{R}{i}[(1+i)S_{\overline{t|}} - t]$$

6. Discuss the extended Cohweb model for which

$$p_t - p_e = c(1-\rho)(p_{t-1} - p_e) + c\rho(p_{t-2} - p_e),$$

where c is the ratio of slopes of supply and demand curves and ρ (usually $0 \leqslant \rho \leqslant 1$) represents the expectation of suppliers about price reversal, in the case when the roots of the auxiliary equation are complex.

7. Discuss the nature of the solution of (76) when the roots of the auxiliary equation are real and distinct, real and coincident or complex conjugate.

8. Discuss the Harrod-Domar growth model

$$Y_t = (1+v)Y_{t-1} - (v+s)Y_{t-2}$$

where $s = 1 - c$ = marginal propensity to save and v is the power of the accelerator. Discuss also all possible solutions of

$$Y_t = \left(v + \frac{v+s}{v}\right)Y_{t-1} - (v+s)Y_{t-2}$$

5.4 MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS IN POPULATION DYNAMICS AND GENETICS

5.4.1 Non-Linear Difference Equations Model for Population Growth: Non-Linear Difference Equations

Let x_t be the population at time t and let births and deaths in time-interval $(t, t+1)$ be proportional to x_t, then the population x_{t+1} at time $t+1$ is given by

$$x_{t+1} = x_t + bx_t - d\,x_t = x_t(1+a) \tag{89}$$

This has the solution

$$x_t = x_0(1 + a)^t, \tag{90}$$

so that the population increases or decreases exponentially according as $a > 0$ or $a < 0$. We now consider the generalisation when births and deaths b and d per unit population depend linearly on x_t so that

$$x_{t+1} = x_t + (b_0 - b_1x_t)x_t - (d_0 + d_1x_t)x_t$$
$$= mx_t - rx_t^2 = mx_t\left(1 - \frac{r}{m}x_t\right) \tag{91}$$

This is the simplest non-linear generalisation of (90) and gives the discrete version of the logistic law of population growth. However this model shows many new features not present in the continuous version of the logistic model. Let $rx_t/m = y_t$, then (91) becomes

$$y_{t+1} = my_t(1 - y_t) \tag{92}$$

One-Period Fixed Points and Their Stability

A one-period fixed point of this equation is that value of y_t for which $y_{t+1} = y_t$ i.e. for which

$$y_t = my_t(1 - y_t), \tag{93}$$

so that there are two one-period fixed points 0 and $(m - 1)/m$. If $y_0 = 0$, then $y_1, y_2, y_3, \ldots$ are all zero and the population remains fixed at zero value:

If $y_0 = (m - 1)/m$, then $y_1, y_2, y_3, \ldots$ are all equal to $(m - 1)/m$. The second fixed point exists only if $m > 1$.

We now discuss the stability of equilibrium of each of these equilibrium positions.

Putting $y_t = 0 + u_t$ in (92) and neglecting squares and higher powers of u_t, we get $u_{t+1} = mu_t$ and since $m > 0$, the first equilibrium position is one of unstable equilibrium.

Again putting $y_t = (m - 1)/m + u_t$ in (92) and neglecting squares and higher powers of u_t, we get

$$u_{t+1} = (2 - m)u_t, \tag{94}$$

so that the second position of equilibrium is stable only if $-1 < 2 - m < 1$ or if $1 > m - 2 > -1$ or if $1 < m < 3$.

Thus if $0 < m < 1$, there is only one one-period fixed point and it is unstable. If $1 < m < 3$, there are two one-period fixed points, the first is unstable and the second is stable. If $m > 3$, there are two one-period fixed points, both of which are unstable.

Two-Period Fixed Points and Their Stability

A point is called a two-period fixed point if it repeats itself after two periods i.e. if $y_{t+2} = y_t$ i.e. if

$$y_{t+2} = my_{t+1}(1 - y_{t+1}) = m^2y_t(1 - y_t)(1 - my_t + my_t^2) = y_t \tag{95}$$

or

$$y_t(my_t - (m-1))(m^2y_t^2 - m(1+m)y_t + (1-m)) = 0 \tag{96}$$

This is a fourth degree equation and as such there can be four two-period fixed points. Two of these are the same as the one-period fixed points. This is obvious from the consideration that every one-period fixed point is also a two-period fixed point. The genuine two-period fixed points are obtained by solving the equation

$$m^2y_t^2 - m(1+m)y_t + (1+m) = 0 \tag{97}$$

Its roots are real if $m > 3$. Thus if $m > 3$, the two one-period fixed points become unstable, but two new two-period fixed points exist and we can discuss their stability as before.

It can be shown that if $m_2 < m < m_4$, where $m_2 = 3$ and m_4 is a number slightly greater than 3, then the two two-period fixed points are stable but if $m > m_4$, all the four one- and two-periods become unstable, but four new four-period fixed points exist which are stable if $m_4 < m < m_8$ and become unstable if $m > m_8$.

2^n-Period Fixed Points and Their Stability

It can be shown that there exists an increasing infinite sequence of real numbers $m_2, m_4, m_8, \ldots, m_{2n}, m_{2^{n+1}}, \ldots$ such that when $m_{2n} < m < m_{2^{n+1}}$ there are $2^{n+1}2^{n+1}$-period fixed points, out of which 2^n fixed points are also fixed points of lower order time periods and all these are unstable and the remaining 2^n points are genuine 2^{n+1} period fixed points and are stable.

From 5.3 represents the stable fixed period points.

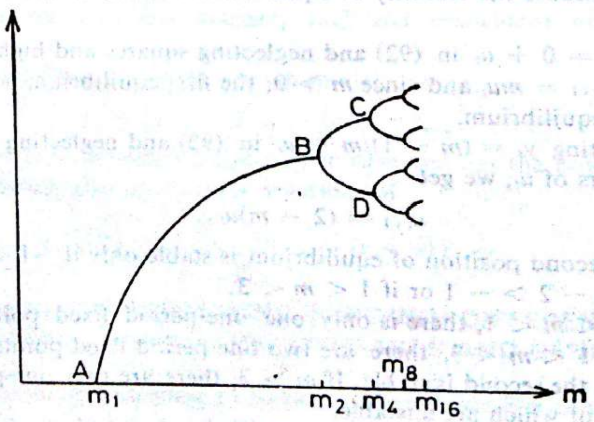


Figure 5.3

When m lies between m_1 and m_2, there is one stable one-period fixed point. When m lies between m_2 and m_4 there are two stable two-period fixed points.

When m lies between m_4 and m_8, there are four stable four-period fixed points, and so on.

Fixed Points of other Periods

The sequence $m_2, m_4, m_8, \ldots$ is bounded above by a fixed number m^*. If $m > m^*$, there can be a three-period fixed point and if there is a three-period fixed point, there will also be fixed points of periods,

$$\begin{gathered} 3, 5, 7, 9, \ldots \\ 2\cdot 3, 2\cdot 5, 2\cdot 7, 2\cdot 9, \ldots \\ 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, \ldots \end{gathered} \tag{98}$$

This is expressed by saying that Period Three Means Chaos.

Chaotic Behaviour of the Non-linear Model

If m lies between m_8 and m_{16}, there will be eight 16-period stable fixed points. If a population size starts from any one of these values, it will oscillate through fifteen other values to return to the original value and this pattern will go on repeating itself. If we draw the graph, it will show rapid oscillations and will look like the graph representing a random phenomenon. Our model is perfectly deterministic, though its behaviour may *appear* to be random and stochastic.

Special Features of Non-linear Difference Equation Models

The simple model illustrates the differences in behaviour between difference and differential equation models. The problems of existence and uniqueness of solutions, of the stability of equilibrium positions are all different due to the basic fact that inspite of similarities, the Discrete and the Continuous are really different.

5.4.2 Age-Structured Population Models

Let $x_1(t), x_2(t), \ldots, x_p(t)$ be the population sizes of p pre-reproductive age-groups at time t;

Let $x_{p+1}(t), x_{p+2}(t), \ldots, x_{p+q}(t)$ be the population sizes of q reproductive age-groups at time t, and

Let $x_{p+q+1}(t), x_{p+q+2}(t), \ldots, x_{p+q+r}(t)$ be the population sizes of r post-reproductive age-groups at time t.

Let $b_{p+1}, b_{p+2}, \ldots, b_{p+q}$ be the birth rates i.e. the number of births per unit time per individual in the reproductive age groups.

In other age-groups, the birth rates are zero.

Let $d_1, d_2, \ldots, d_{p+q+r}$ be the death rates in the $p + q + r$ age-groups.

Let $m_1, m_2, \ldots, m_{p+q+r}$, be the rates of migration to the next age-groups, then we get the system of difference equations

$$\begin{aligned} x_1(t+1) &= b_{p+1}x_{p+1}(t) + \ldots + b_{p+q}x_{p+q}(t) - (d_1 + m_1)x_1(t) \\ x_2(t+1) &= m_1x_1(t) - (d_2 + m_2)x_2(t) \\ \ldots & \quad \ldots \quad \ldots \\ x_{p+q+r-1}(t+1) &= m_{p+q+r-2}(t) - (d_{p+q+r-1} + m_{p+q+r-1})x_{p+q+r-1}(t) \\ x_{p+q+r}(t+1) &= m_{p+q+r-1}x_{p+q+r-1}(t) - (d_{p+q+r})x_{p+q+r}(t) \end{aligned} \tag{99}$$

which can be written in the matrix form

$$X(t+1) = LX(t), \tag{100}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_{p+q+r}(t) \end{bmatrix},$$

$$L = \begin{bmatrix} -(d_1+m_1) & 0 & 0\ldots0 & b_{p+1} & b_{p+2}\ \ldots & b_{p+q} & 0 & \ldots & 0 & 0 \\ m_1 & -(d_2+m_2) & 0\ldots & 0 & 0\ \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & m_2 & -(d_3+m_3) & 0 & 0\ \ldots & 0 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot & & & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot & & & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0\ldots & 0 & 0 & 0 & 0 & \ldots & m_{n-1} & -d_n \end{bmatrix} \tag{101}$$

where $p+q+r=n$.

L is called the Leslie matrix. All the elements of its main diagonal are negative and all the elements of its main subdiagonal are positive. In addition q elements in the first row are positive and the rest of the elements are all zero. The solution of (100) can be written as

$$X(t) = L^t X(0) \tag{102}$$

Now the Leslie matrix has the property that it has a dominant eigenvalue which is real and positive, which is greater in absolute value than any other eigenvalue and for which the corresponding eigenvector has all its components positive. If this dominant eigenvalue is greater than unity, then the populations of all age-groups will increase exponentially and if it is less than unity the population of all age-groups will die out. If this dominant eigenvalue is unity, the population can have a stable age structure.

The Leslie model is in terms of a system of linear difference equations. If we take the effects of overcrowding and density dependence into account, the equations are nonlinear.

5.4.3 Mathematical Modelling through Difference Equations in Genetics

(a) *Hardy-Weinberg Law*

Every characteristic of an individual, like height or colour of the hair, is determined by a pair of genes, one obtained from the father and the other obtained from the mother. Every gene occurs in two forms, a dominant (denoted by a capital letter say G) and a recessive (denoted by the corresponding small letter say g). Thus with respect to a characteristic, an individual may be a dominant (GG), a hybrid (Gg or gG) or a recessive (gg).

In the nth generation, let the proportions of dominants, hybrids and recessives be p_n, q_n, r_n so that

$$p_n + q_n + r_n = 1, \qquad p_n \geqslant 0,\ q_n \geqslant 0,\ r_n \geqslant 0 \tag{103}$$

We assume that individuals, in this generation mate at random. Now p_{n+1} = the probability that an individual in the $(n+1)$th generation is a dominant (GG) = (probability that this individual gets a G from the father) $\times$(probability that the individual gets a G from the mother)

$$= \left(p_n + \frac{1}{2}q_n\right)\left(p_n + \frac{1}{2}q_n\right) = \left(p_n + \frac{1}{2}q_n\right)^2$$

or
$$p_{n+1} = \left(p_n + \frac{1}{2}q_n\right)^2 \tag{104}$$

Similarly
$$q_{n+1} = 2\left(p_n + \frac{1}{2}q_n\right)\left(r_n + \frac{1}{2}q_n\right) \tag{105}$$

$$r_{n+1} = \left(r_n + \frac{1}{2}q_n\right)^2, \tag{106}$$

so that
$$p_{n+1} + q_{n+1} + r_{n+1} = \left(p_n + \frac{1}{2}q_n + \frac{1}{2}q_n + r_n\right)^2 = 1, \tag{107}$$

as expected. Similarly

$$\begin{aligned} p_{n+2} &= \left(p_{n+1} + \frac{1}{2}q_{n+1}\right)^2 \\ &= \left(\left(p_n + \frac{1}{2}q_n\right)^2 + \left(p_n + \frac{1}{2}q_n\right)\left(r_n + \frac{1}{2}q_n\right)\right)^2 \\ &= \left(p_n + \frac{1}{2}q_n\right)^2\left(p_n + \frac{1}{2}q_n + \frac{1}{2}q_n + r_n\right)^2 \\ &= \left(p_n + \frac{1}{2}q_n\right)^2 = p_{n+1} \end{aligned} \tag{108}$$

and
$$q_{n+2} = q_{n+1}, \qquad r_{n+2} = r_{n+1}, \tag{109}$$

so that the proportions of dominants, hybrids and recessives in the $(n+2)$th generation are same as in the $(n+1)$th generation.

Thus in any population in which random mating takes place with respect to a characteristic, the proportions of dominants, hybrids and recessive do not change after the first generation. This is known as Hardy-Weinberg law after the mathematician Hardy and geneticist Weinberg who jointly discovered it.

The equations (104)–(107) is a set of difference equations of the first order.

(b) *Improvement of Plants through Elimination of Recessives*

Suppose the recessives are undesirable and as such we do not allow the recessives in any generation to breed.

Let p_n, q_n, r_n be the proportions of dominants, hybrids and recessives before elimination of recessives and let p_n', q_n', 0 be the populations after the elimination, then

$$\frac{p_n'}{p_n} = \frac{q_n'}{q_n} = \frac{p_n' + q_n'}{p_n + q_n} = \frac{1}{1 - r_n} \tag{110}$$

Now we allow random mating and let p_{n+1}, q_{n+1}, r_{n+1} be the proportions in the next generation before elimination of recessives, then using (104)–(108)

$$p_{n+1} = \left(p_n' + \frac{1}{2}q_n'\right)^2 \tag{111}$$

$$q_{n+1} = 2\left(p_n' + \frac{1}{2}q_n'\right)\left(\frac{1}{2}q_n'\right) = q_n'\left(p_n' + \frac{1}{2}q_n'\right) \tag{112}$$

$$r_{n+1} = \left(\frac{1}{2}q_n'\right)^2 = \frac{1}{4}q_n'^2 \tag{113}$$

After elimination of recessives, let the new proportions be p_{n+1}', q_{n+1}', so that

$$\frac{p_{n+1}'}{p_{n+1}} = \frac{q_{n+1}'}{q_{n+1}} = \frac{1}{p_{n+1} + q_{n+1}} = \frac{1}{1 - \frac{1}{4}q_n'^2} \tag{114}$$

so that
$$q_{n+1}' = \frac{q_n'(p_n' + \frac{1}{2}q_n')}{1 - \frac{1}{4}q_n'^2} = \frac{q_n'(1 - \frac{1}{2}q_n')}{1 - \frac{1}{4}q_n'^2}$$

$$= \frac{q_n'}{1 + \frac{1}{2}q_n'} \tag{115}$$

This is a non-linear difference equation of the first order. To solve it we substitute

$$q_n' = 1/u_n$$

to get
$$u_{n+1} = u_n + \frac{1}{2} \tag{116}$$

which has the solution
$$u_n = A + \frac{1}{2}n \tag{117}$$

or
$$q_n' = \frac{1}{A + \frac{1}{2}n} \tag{118}$$

so that $q_n' \to 0$ and $p_n' \to 1$ as $n \to \infty$. Thus ultimately we should be left with all dominants. Equation (118) determines the rate at which hybrids disappear.

EXERCISE 5.4

1. Show that in Figure 5.3, AB is the arc of a rectangular hyperbola.
2. Find m_4 and draw the curves BC and BD.

3. Find the four stable eight-period fixed points.
4. For the condition for the existence of a three-period fixed point.
5. Find the characteristic equation for the Leslie matrix and show that it always has a positive real root. Find the condition that this root is less than unity.
6. Let $y_{t+1} = 3.1(1 - y_t)$. Draw the graph of its solution for $y_0 = 0.5$.
7. Draw the graphs of $\ln x_1(t)$, $\ln x_2(t)$, $\ln x_3(t)$ for the system

$$X(t+1) = AX(t) \quad \text{when}$$

$$A = \begin{bmatrix} 0 & 10 & 8 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 2 & 2 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

when $x_1(0) = 10, \quad x_2(0) = 10, \quad x_3(0) = 10$
and interpret the graphs.

8. Discuss the probiem of Section 5.4.3(b) when only a fraction k of the recessives are eliminated at each stage.

5.5 MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS IN PROBABILITY THEORY

5.5.1 Markov Chains

Let a system be capable of being in n possible states $1, 2, \ldots, n$ and let the probability of transition from state i to state j in time interval t to $t+1$ be p_{ij}. Let $p_j(t)$ denote the probability that the system is in state j at time t $(j = 1, 2, \ldots, n)$, then at time $t+1$ it can be in any one of the states $1, 2, \ldots, n$.

It can be in the ith state at time $t+1$ in n exclusive ways since it could have been in any one of the n states $1, 2, \ldots, n$ at time t and it could have transited from that state to ith state in time interval $(t, t+1)$. By using the theorems of total and compound probability, we get

$$p_i(t+1) = \sum_{j=1}^{n} p_{ji}p_j(t), \qquad i = 1, 2, \ldots, n \tag{119}$$

or

$$\begin{aligned} p_1(t+1) &= p_{11}p_1(t) + p_{21}p_2(t) + \ldots + p_{n1}p_n(t) \\ p_2(t+1) &= p_{12}p_1(t) + p_{22}p_2(t) + \ldots + p_{n2}p_n(t) \\ \ldots \quad & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ p_n(t+1) &= p_{1n}p_1(t) + p_{2n}p_2(t) + \ldots + p_{nn}p_n(t) \end{aligned} \tag{120}$$

or

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \\ \cdot \\ \cdot \\ p_n(t+1) \end{bmatrix} = \begin{bmatrix} p_{11} & p_{21} & \ldots & p_{n1} \\ p_{12} & p_{22} & \ldots & p_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ p_{1n} & p_{2n} & \ldots & p_{nn} \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ \cdot \\ \cdot \\ p_n(t) \end{bmatrix} \tag{121}$$

or
$$P(t+1) = AP(t), \tag{122}$$
where $P(t)$ is a probability vector and A is a matrix, all of whose elements lie between zero and unity (since these are all probabilities). Further the sum of elements of every column is unity, since the sum of elements of the ith column is $\sum_{j=1}^{n} p_{ij}$ as this denotes the sum of the probabilities of the system going from the ith state to any other state and this sum must be unity.

The solution of the matrix difference equation (122) is
$$P(t) = A^t P(0) \tag{123}$$
If all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are distinct, we can write
$$A = S\Lambda S^{-1} \tag{124}$$
where
$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \tag{125}$$
so that
$$A^t = (S\Lambda S^{-1})(S\Lambda S^{-1}) \ldots (S\Lambda S^{-1})$$
$$= S\Lambda^t S^{-1}$$
$$= S \begin{bmatrix} \lambda_1^t & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2^t & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n^t \end{bmatrix} S^{-1} \tag{126}$$
The probability vector will not change if $P(t+1) = P(t)$ so that from (122)
$$(I - A)P(t) = 0 \tag{127}$$
Thus if P is the eigenvector of the matrix A corresponding to unit eigenvalue, then P does not change i.e. if the system start with probability vector P at time 0, it will always remain in this state. Even if the system starts from any other probability vector, it will ultimately be described by the probability vector P as $t \to \infty$.

As a special case, suppose we have a machine which can be in two states, working or non-working. Let the probability of its transition from working to non-working be α, of its transition from non-working to working be β, then the transition probability matrix A is obtained from
$$\begin{array}{ll} & \begin{array}{cc} \text{working} & \text{non-working} \end{array} \\ \begin{array}{l} \text{working} \\ \text{non-working} \end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \end{array} \tag{128}$$

The system of difference equations is

$$p_1(t+1) = p_1(t)(1-\alpha) + p_2(t)\beta$$
$$p_2(t+1) = p_1(t)\alpha + p_2(t)(1-\beta) \tag{129}$$

or

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \end{bmatrix} = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \tag{130}$$

The eigenvalues of the matrix A is given by

$$\begin{vmatrix} 1-\alpha-\lambda & \beta \\ \alpha & 1-\beta-\lambda \end{vmatrix} = 0 \text{ or } (\lambda-1)(\lambda - \overline{1-\alpha-\beta}) = 0 \tag{131}$$

The eigenvector corresponding to the unit eigenvalue is $\beta/(\alpha+\beta)$, $\alpha/(\alpha+\beta)$ and as such ultimately the probability of the machines being found in working order is $\beta/(\alpha+\beta)$ and the probability of its being found in a non-working state is $\alpha/(\alpha+\beta)$.

5.5.2 Gambler's Ruin Problems

Let a gambler with capital n dollars play against an infinitely rich adversary. Let the probability of his winning or losing a unit dollar in any game be p and q respectively where $p+q=1$ and let p_n be the probability of his being ultimately ruined. At the next game, the probability of his winning is p and if he wins, his capital would become $n+1$ and the probability of his ultimate ruin would be p_{n+1}. On the other hand if he loses at the next game, the probability for which is q, his capital would become $n-1$ and the probability of his ultimate ruin would be p_{n-1}, so that we get the linear difference equation of the second order

$$p_n = pp_{n+1} + qp_{n-1} \tag{132}$$

The auxiliary equation for this is

$$p\lambda^2 - \lambda + (1-p) = 0$$

or

$$p(\lambda-1)\left(\lambda - \frac{1-p}{p}\right) = 0 \tag{133}$$

As such the solution of (132) is

$$p_n = A + B\left(\frac{q}{p}\right)^n \tag{134}$$

Now let the gambler decide to stop this game when his capital becomes a dollars so that the probability of his being ruined when his starting capital is a dollars is zero i.e. $p_a = 0$. In the same way when his starting capital is zero, he is already ruined, so we put $p_0 = 1$. Using

$$p_0 = 1, \quad p_a = 0 \tag{135}$$

(134) gives

$$p_n = \frac{(q/p)^a - (q/p)^n}{(q/p)^a - 1} \tag{136}$$

Now let D_n denote the expected number of games before the gambler is ruined. If he wins at the next game, his capital becomes $n+1$ and the expected number of games would then be D_{n+1} and if he loses, his capital becomes $n-1$ and the expected number of games would be only D_{n-1}. As such, we get

$$D_n = pD_{n+1} + qD_{n-1} + 1 \tag{137}$$

with boundary conditions

$$D_0 = 0, \quad D_a = 0 \tag{138}$$

This gives the solution

$$D_n = \frac{n}{q-p} - \frac{a}{q-p}\,\frac{1-(q/p)^n}{1-(q/p)^a}$$

EXERCISE 5.5

1. Show that the solution of (129) is

$$p_1(t) = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^t\left(p_1(0) - \frac{\beta}{\alpha+\beta}\right)$$

$$p_2(t) = \frac{\alpha}{\alpha+\beta} + (1-\alpha-\beta)^t\left(p_2(0) - \frac{\alpha}{\alpha+\beta}\right)$$

2. Show that $-1 < 1-\alpha-\beta < 1$ and deduce that $p_1(t) \to \dfrac{\beta}{\alpha+\beta}$ and $p_2(t) \to \dfrac{\alpha}{\alpha+\beta}$ as $t \to \infty$. Show also that $\beta/(\alpha+\beta)$, $\alpha/(\alpha+\beta)$ give the components of the eigenvector of the matrix A corresponding to the unit eigenvalue.

3. In a panel survey, a person gives an answer 'yes' or 'no'. The probability of his changing from 'yes' to 'no' in the next survey is α and that of changing from 'no' to 'yes' is β. Find the probability that ultimately he will answer 'yes'.

4. In a game of chance, the probability of a person winning a second game after losing the first game is α and the probability of his losing a second game after winning the first game is β. Find the ultimate chance of winning.

5. Show that if $p = q = 1/2$, the solution of (132) is

$$p_n = 1 - n/a$$

Show also that this is the limiting value of p_n given by (136) when p and q both approach 1/2.

6. Show that if $p = q = \frac{1}{2}$, the solution of (137) is

$$D_n = n(a-n)$$

Show also that this is the limiting value of D_n given by (139) when p and q both approach 1/2.

7. In gambler's ruin problem, discuss the special cases when

$$n = 1 \quad \text{or} \quad n = a - 1.$$

8. A particle is at the point n on the positive real axis where n is a non-negative integer. At every unit interval of time it can move unit distance towards the right or towards the left with probability p and $q(p + q = 1)$ respectively. If the particle reaches 0 or a, it is absorbed there. Find the probabilities of the particle being ultimately absorbed at 0 or at a. Find also the expected duration before absorption in either case.

9. n letters to each of which corresponds an envelope are placed in the envelopes at random. If u_n is the number of ways in which all letters go wrong, show that

$$u_n = (n - 1)(u_{n-1} + u_{n-2})$$

Prove that $\quad u_n - nu_{n-1} = (-1)^{n-2}(u_2 - 2u_1) = (-1)^n$

and
$$u_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$$

Deduce that the probability that all n letters go wrong is given by the first $(n - 1)$ terms in expression of $1 - e^{-1}$.

10. A player tosses a coin and is to score one point for every head turned up and two for every tail. He is to play on until his score reaches or passes n. If p_n is the probability of attaining exactly n, show that

$$p_n = \frac{1}{2}(p_{n-1} + p_{n-2}), \quad p_n = \frac{1}{2}\left[2 + (-1)^n \frac{1}{2^n}\right].$$

5.6 MISCELLANEOUS EXAMPLES OF MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS

Difference equations arise in economics since values of prices, quantities, national income, savings, investments at discrete intervals of time are related. These arise in genetics because proportions of dominants, hybrids and recessives in different generations are related by genetic laws. These arise in population dynamics because population sizes at discrete instants of time are related by births, deaths, immigration and emigration. These arise in finance because amounts at discrete instants of time are related by rates of interest. These arise in gambler's ruin problem because the probability of ruin (or duration of game) when gambler's capital is n is related to the probability of ruin (or duration of game) when his capital is $n + 1$.

Similarly in geometry, difference-equations can arise because the number of compartments in which n lines or curves divide a plane or surface is related to the number of components determined by $(n + 1)$ lines or curves; in dynamics the ranges after successive rebounds of an elastic ball from a horizontal or inclined place are related; in electrical currents, the potential at neighbouring nodes and currents in neighbouring circuits are related by Kirchhoff's laws and so on.

EXERCISE 5.6

1. If u_n is the number of compartment^ formed by n straight lines drawn in a plane such that no two are parallel and no three are concurrent, show that

$$u_{n+1} = u_n + (n + 1), \qquad u_n = \frac{1}{2}(n^2 + n + 2).$$

2. Show that if u_n is the number of compartments formed when n closed curves are drawn on a closed surface in such a way that no three intersect at the same point and every pair crosses at two points and only at two points then

$$u_n = u_n + 2n, \qquad u_n = n^2 - n + 2$$

3. If $I_n = \int_0^\pi \frac{\cos n\theta \, d\theta}{\cos\theta - \cos\alpha}$, show that $I_n + I_{n-2} = 2\cos\alpha \, I_{n+1}$ and hence show that $I_n = \pi \sin n\alpha/\sin\alpha$.

4. Using the difference equation

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0$$

valid for Legender's polynomials, evaluate

$$I_n = \int_{-1}^{1} \frac{P_n(x)P_{n-1}(x)}{x}\, dx$$

by first showing that

$$(n + 1)I_{n+1} + nI_n = 2.$$

5. N equal uniform rods, smoothly jointed together and at rest in a straight line on a horizontal table, have an impulse J applied to the free end of the first rod, J being horizontal and perpendicular to the line or rods. Denoting the equal and opposite reactions at the ith joint by R_i, and adopting the convention that the impulse R_i acting on the $(i + 1)$th rod is measured in the same sense as J, prove that

$$R_{i-1} + 4R_i + R_{i+1} = 0$$

and explain what values have to be given to R_0 and R_N in order to make the equation hold for $i = 1, 2, \ldots, N - 1$.

6. Fibonacci's numbers are defined by $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$, find F_n and an asymptotic formula for it when n is large.

7. Generalised Fibonacci's numbers are defined by

$$F_{n,r} = F_{n-1,r} + F_{n-2,r} + \ldots + F_{n-r,r}$$

Find formula for $F_{n,r}$ and discuss its properties.

8. In the steady-state, the probability of there being n persons in a queue is given by

$$(\lambda + \mu)p_n = \lambda p_{n-1} + \mu p_{n+1}, \qquad n = 0, 1, 2, 3, \ldots$$

show that $p_n = (1-\rho)\rho^n; \quad \rho = \lambda/\mu.$

9. Show that the number of transformation of n points into themselves in which $n - r$ points remain fixed is given by

$$^nc_r\, r!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots + \frac{(-1)^r}{r!}\right)$$

10. Show that the number of transformations in which no point remains fixed and in which just one point remain fixed differ always by unity

POSSIBLE QUESTIONS

Part B (6 Marks)

1. Give any two disciplines that difference equation arises.

2. Write about Hardy-Weinberg law.

3. Write an explanatory note on complementary function.

4. Discuss about application to actuarial science.

5. Find a solution of linear difference equation by Laplace transform.

6. Explain in detail Harrod model.

Part C (10 Marks)

1. Explain in detail markov chains.

2. Write about Hardy-Weinberg law.

3. Discuss in detail on particular solution.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Mathematical Modelling **Subject Code: 18MMP303**

Class : II - M.Sc. Mathematics **Semester : III**

Unit IV

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
In _______ the genetic characteristics will change generation to generation and the variable representing generation is discrete variable.	Economics	Genetics	Population dynamics	None	Genetics
In genetics, the genetic characteristics will change generation to generation and the variable representing generation is ________ variable.	Discrete	Numeric	Feasible	Optimum	Discrete
In ________ the price changes are consider from year to year or month to month or week to week or day to day	Economics	Genetics	Population dynamics	None	Economics
In _ the changes are consider in population from one age group to another and the variable representing the age group is discrete	Economics	Genetics	Population dynamics	None	Population dynamics
In population dynamics _ the changes are consider in population from one age group to another and the variable representing the age group is __________	Discrete	Numeric	Feasible	Optimum	Discrete
No of birth and deaths are proportional to the population then the model is ____________	PGM	LGM	PTM	CM	PGM

The solution of linear differential equation is of the form	CF+PI	CF-PI	CF*PI	CF/PI	CF+PI
CF denotes	Complementary function	convergen function	Conditional function	None	Complementary function
the sort form of particular integral is	PI	Par-Ing	Ping	None	PI
Complementary function can be obtained by ___________	Matrix	Determinate	Eigen value	None	Matrix
the solution of linear differences equation can be obtained by _________ transform if t is continuous	Laplace	Z	Fourier	Gauss	Laplace
the___is solution of linear differences equation can be obtained by _________ transform if t is discrete	Laplace	Z	Fourier	Gauss	Z
the non linear difference equations reducible to linear equation by ___ method	Substitution	Direct	Indirect	Normal	Substitution
In difference equation ________ theory is applied	Stability	Non stability	Uniformity	Non uniformity	Stability
The Horrod model is used in the field of ______________	Economics	Genetics	Population dynamics	None	Economics
The investment depends on ___________ between the income of current year and last year	Addition	Difference	product	division	Difference
All the saving made are invested in the Horrod model then	S(t) = I(t)	S(t) =1/2 I(t)	2S(t) = I(t)	None	S(t) = I(t)
In the cobweb model price of the commodity in the year denotes	Pt	qt	rt	st	pt
In the cobweb model amount of the commodity available in the market in year t denotes	Pt	qt	rt	st	qt
Amount of the commodity produced this year available for sale is a ____________ function of the price of commodity	Linear	Non linear	Stable	Non stable	Linear
In the cobweb path ABCEFGI, .. And the equilibrium price and quantity are given by __________ of two straight lines	Intersection	Union	Disjunction	Conjunction	Intersection
In the cobweb path ABCEFGI, .. And the equilibrium price and quantity are given by intersection of two ____________	Straight lines	Circles	Squares	Cubes	Straight lines

the actuarial science is called ________________	Mathematics of finance	Mathematics of economics	Dynamics	Statics	Mathematics of finance
the actuarial science is called ________________	Mathematics of investment	Mathematics of economics	Dynamics	Statics	Mathematics of investment
One-period fixed points and their stability	Yt+1=yt	Yt+2=y2t	Yt-1=yt	yt+2=yt	Yt+1=yt
Two-period fixed points and their stability	Yt+2=yt	Yt+2=y2t	Yt-1=yt	yt+2=y2t	Yt+2=yt
Any population in which random meeting take place with respect to a characteristic , the proportion of dominants hybrids and recessive do not change after the first generation states_____________ law	Gauss	Hardy-weinberg	Fick's	Routhwelt	Hardy-weinberg
The probability of transition from state I to state j is	Markov chain	Hardy-weinberg	Fick's	Routhwelt	Markov chain

UNIT-V

Mathematical Modeling through Graphs: Solutions that can be modeled through Graphs – Mathematical Modeling in Terms of Directed Graphs, Signed Graphs, Weighted Digraphs and Unoriented Graphs.

Mathematical Modelling Through Graphs

7.1 SITUATIONS THAT CAN BE MODELLED THROUGH GRAPHS

7.1.1 Qualitative Relations in Applied Mathematics

It has been stated that "Applied Mathematics is nothing but solution of differential equations". This statement is wrong on many counts (i) Applied Mathematics also deals with solutions of difference, differential-difference, integral, integro-differential, functional and algebraic equations (ii) Applied Mathematics is equally concerned with inequations of all types (iii) Applied Mathematics is also concerned with mathematical modelling; in fact mathematical modelling has to precede solution of equations (iv) Applied Mathematics also deals with situations which cannot be modelled in terms of equations or inequations; one such set of situations is concerned with qualitative relations.

Mathematics deals with both quantitative and qualitative relationships. Typical qualitative relations are: y likes x, y hates x, y is superior to x, y is subordinate to x, y belongs to same political party as x, set y has a non-null intersection with set x; point y is joined to point x by a road, state y can be tansformed into state x, team y has defeated team x, y is father of x, course y is a prerequisite for course x, operation y has to be done before operation x, species y eats species x, y and x are connected by an airline, y has a healthy influence on x, any increase of y leads to a decrease in x, y belongs to same caste as x, y and x have different nationalities and so on.

Such relationships are very conveniently represented by graphs where a graph consists of a set of vertices and edges joining some or all pairs of these vertices. To motivate the typical problem situations which can be modelled through graphs, we consider the first problem so historically modelled viz. the problem of seven bridges of Königsberg.

7.1.2 The Seven Bridges Problem

There are four land masses A, B, C, D which are connected by seven bridges numbered 1 to 7 across a river (Figure 7.1). The problem is to start from any point in one of the land masses, cover each of the seven bridges once and once only and return to the starting point.

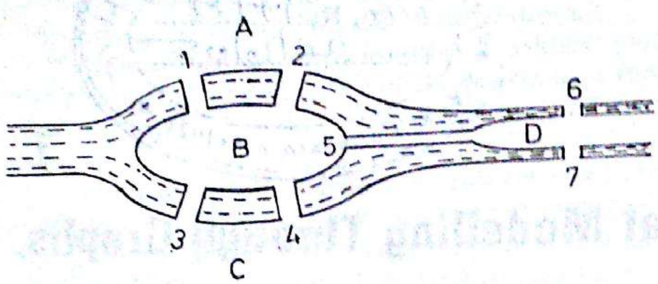


Figure 7.1

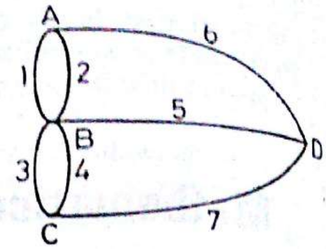


Figure 7.2

There are two ways of attacking this problem. One method is to try to solve the problem by walking over the bridges. Hundreds of people tried to do so in their evening walks and failed to find a path satisfying the conditions of the problem. A second method is to draw a scale map of the bridges on paper and try to find a path by using a pencil.

It is at this stage that concepts of mathematical modelling are useful. It is obvious that the sizes of the land masses are unimportant, the lengths of the bridges or even whether these are straight or curved are irrelevant. What is relevant information is that A and B are connected by two bridges 1 and 2, B and C are connected by two bridges 3 and 4, B and D are connected by one bridge number 5, A and D are connected by bridge number 6 and C and D are connected by bridge number 7. All these facts are represented by the graph with four vertices and seven edges in Figure 7.2. If we can trace this graph in such a way that we start with any vertex and return to the same vertex and trace every edge once and once only without lifting the pencil from the paper, the problem can be solved. Again trial and error method cannot be satisfactorily used to show that no solution is possible.

The number of edges meeting at a vertex is called the degree of that vertex. We note that the degrees of A, B, C, D-are 3, 5, 3, 3 respectively and each of these is an odd number. If we have to start from a vertex and return to it, we need an even number of edges at that vertex. Thus it is easily seen that Königsberg bridges problem cannot be solved.

This example also illustrates the power of mathematical modelling. We have not only disposed of the seven-bridges problem, but we have discovered a technique for solving many problems of the same type.

7.1.3 Some Types of Graphs

A graph is called *complete* if every pair of its vertices is joined by an edge (Figure 7.3(a)).

A graph is called a *directed graph* or a *digraph* if every edge is directed with an arrow. The edge joining A and B may be directed from A to B or from B to A. If an edge is left undirected in a digraph, it will be assumed to be directed both ways (Figure 7.3(b)).

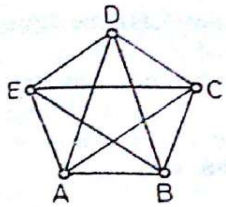


Figure 7.3a

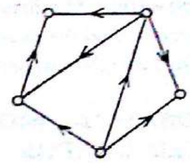


Figure 7.3b

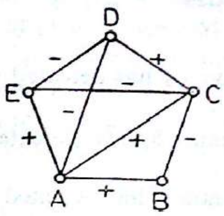


Figure 7.3c

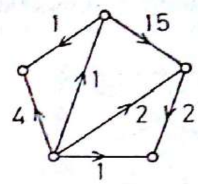


Figure 7.3d

A graph is called a *signed graph* if every edge has either a plus or minus sign associated with it (Figure 7.3(c)).

A digraph is called a *weighted digraph* if every directed edge has a weight (giving the importance of the edge) associated with it (Figure 7.3(d)). We may also have digraphs with positive and negative numbers associated with edges. These will be called *weighted signed digraphs*.

7.1.4 Nature of Models in Terms of Graphs

In all the applications we shall consider, the length of the edge joining two vertices will not be relevant. It will not also be relevant whether the edge is straight or curved. The relevant facts would be (a) which edges are joined; (b) which edges are directed and in which direction(s); (c) which edges have positive or negative signs associated with them; (d) which edges have weights associated with them and what these weights are.

EXERCISE 7.1

1. In the Königsberg problem suggest deletion or addition of minimum number of bridges which may lead to a solution of the problem.
2. Show that in any graph, the sum of local degrees of all the vertices is an even number. Deduce that a graph has an even number of odd vertices.
3. Three houses A, B, C have to be connected with three utilities a, b, c by separate wires lying in the same plane and not crossing one another. Explain why this is not possible.
4. Each of the four neighbours has connected his house with the other three houses by paths which do not cross. A fifth man builds a house nearby. Prove that (a) he cannot connect his house with all others by non-intersecting paths (b) he can however connect with three of the houses.

5. A graph is called regular if each of its vertices has same degree r. Draw regular graphs with 6 vertices and degree 5, 4 and 3.

6. Show that in Königsberg, four one-way bridges will be enough to connect the four land masses.

7.2 MATHEMATICAL MODELS IN TERMS OF DIRECTED GRAPHS

7.2.1 Representing Results of Tournaments

The graph (Figure 7.4) shows that

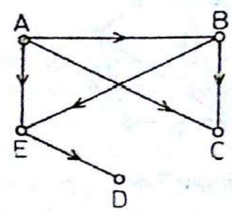


Figure 7.4

(i) Team A has defeated teams B, C, E.

(ii) Team B has defeated teams C, E.

(iii) Team E has defeated D.

(iv) Matches between A and D, B and D, C and D and C and E have yet to be played.

7.2.2 One-Way Traffic Problems

The road map of a city can be represented by a directed graph. If only one-way traffic is allowed from point a to point b, we draw an edge directed from a to b. If traffic is allowed both ways, we can either draw two edges, one directed from a to b and the other directed from b to a or simply draw an undirected edge between a and b. The problem is to find whether we can introduce one-way traffic on some or all of the roads without preventing persons from going from any point of the city to any other point. In other words, we have to find when the edges of a graph can be given direction in such a way that there is a directed path from any vertex to every other. It is easily seen that one-way traffic on the road DE cannot be introduced without disconnecting the vertices of the graph (Figure 7.5).

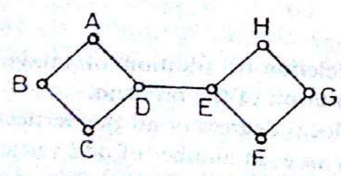


Figure 7.5(a)

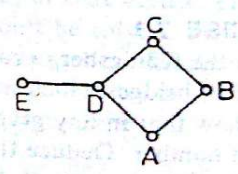


Figure 7.5(b)

In Figure 7.5(a), DE can be regarded as a bridge connecting two regions of the town. In Figure 7.5(b) DE can be regarded as a blind street on which a two-way traffic is necessary. Edges like DE are called *separating edges*, while other edges are called *circuit edges*. It is necessary that on separating edges, two-way traffic should be permitted. It can also be shown that this is sufficient. In other words, the following theorem can be established:

If G is an undirected connected graph, then one can always direct the circuit edges of G and leave the separating edges undirected (or both way directed) so that there is a directed path from any given vertex to any other vertex.

7.2.3 Genetic Graphs

In a genetic graph, we draw a directed edge from A to B to indicate that B is the child of A. In general each vertex will have two incoming edges, one from the vertex representing the father and the other from the vertex representing the mother. If the father or mother is unknown, there may be less then two incoming edges. Thus in a genetic graph, the local degree of incoming edges at each vertex must be less than or equal to two. This is a necessary condition for a directed graph to be a genetic graph, but it is not a sufficient condition. Thus Figure 7.6 does not give a genetic graph inspite of the fact that the number of incoming edges at each vertex does not exceed two. Suppose A_1 is male, then A_2 must be female, since A_1, A_2 have a child B_1. Then A_3 must be male, since A_2, A_3 have a child B_2. Now A_1, A_3 being both males cannot have a child B_3.

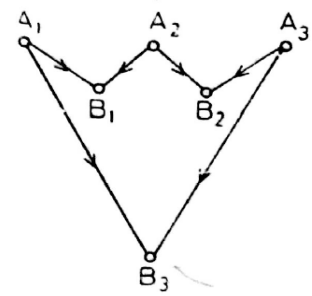


Figure 7.6

7.2.4 Senior-Subordinate Relationship

If a is senior to b, we write aSb and draw a directed edge from a to b. Thus the organisational structure of a group may be represented by a graph like the following [Figure 7.7].

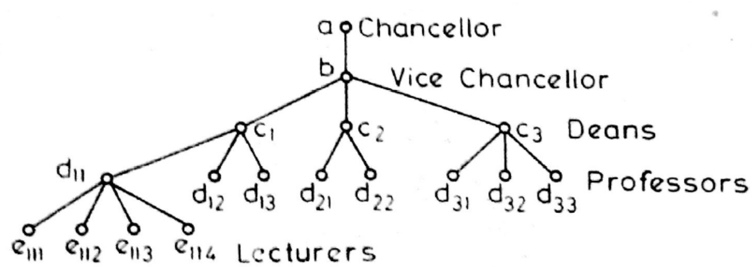


Figure 7.7

The relationship S satisfies the following properties:

(i) $\sim(aSa)$ i.e. no one is his own senior

(ii) $aSb \Rightarrow \sim(bSa)$ i.e. a is senior to b implies that b is not senior to a

(iii) $aSb, bSc \Rightarrow aSc$ i.e. if a is senior to b and b is senior to c, then a is senior to c.

The following theorem can easily be proved: "The necessary and sufficient condition that the above three requirements hold is that the graph of an organisation should be free of cycles"

We want now to develop a *measure for the status* of each person. The status $m(x)$ of the individual should satisfy the following reasonable requirements.

(i) $m(x)$ is always a whole number

(ii) If x has no subordinate, $m(x) = 0$

(iii) If, without otherwise changing the structure, we add a new individual subordinate to x, then $m(x)$ increases

(iv) If, without otherwise changing the structure, we move a subordinate of a to a lower level relative to x, then $m(x)$ increases.

A measure satisfying all these criteria was proposed by Harary. We define the level of seniority of x over y as the length of the shortest path from x to y. To find the measure of status of x, we find n_1, the number of individuals who are one level below x, n_2 the number of individuals who are two levels below x and in general, we find n_k the number of individuals who are k levels below x. Then the Harary measure $h(x)$ is defined by

$$h(x) = \sum_k k n_k \tag{1}$$

It can be shown that among all the measure which satisfy the four requirements given above, Harary measure is the least.

If however, we define the level of senority of x over y as the length of the longest path from x to y, and then find $H(x) = \sum_k k n_k$, we get another measure which will be the largest among all measures satisfying the four requirements. For Figure 7.8, we get

$h(a) = 1.2 + 4.2 + 2.3 = 16$ $\quad H(a) = 1.1 + 3.2 + 2.3 + 2.4 = 21$

$h(b) = 1.3 + 2.4 = 11$ $\quad H(b) = 2.1 + 2.2 + 2.3 + 1.4 = 16$

$h(c) = 1.2 + 1.2 = 4$ $\quad H(c) = 1.1 + 1.2 + 1.3 = 6$

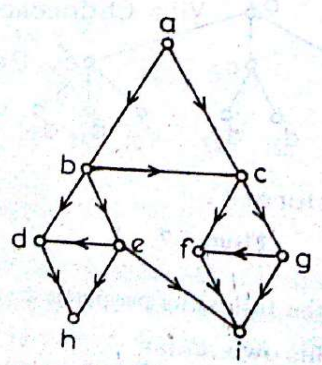


Figure 7.8

$h(d) = 1.1$	$= 1$	$H(d) = 1.1$	$= 1$
$h(e) = 1.3$	$= 3$	$H(e) = 1.2 + 2.1$	$= 4$
$h(f) = 1.1$	$= 1$	$H(f) = 1.1$	$= 1$
$h(g) = 1.2$	$= 2$	$H(g) = 1.2$	$= 2$
$h(k)$	$= 0$	$H(k)$	$= 0$
$h(l)$	$= 0$	$H(l)$	$= 0$

7.2.5 Food Webs

Here aSb if a eats b and we draw a directed edge from a to b. Here also $\sim(aSa)$ and $aSb \Rightarrow \sim(bSa)$. However the transitive law need not hold. Thus consider the food web in Fig. 7.9. Here fox eats bird, bird eats grass, but fox does not eat grass.

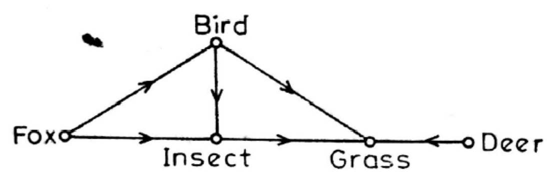


Figure 7.9

We can however calculate measure of the status of each species in this food web by using (1) h(bird) = 2, h(fox) = 4, h(insect) = 1, h(grass) = 0, h(deer) = 1.

7.2.6 Communication Networks

A directed graph can serve as a model for a communication network. Thus consider the network given in Figure 7.10. If an edge is directed from a to b, it means that a can communicate with b. In the given network e can communicate directly with b, but b can communicate with e only indirectly through c and d. However every individual can communicate with every other individual.

Our problem is to determine the importance of each individual in this network. The importance can be measured by the fraction of the messages on an average that pass through him. In the absence of any other knowledge, we can assume that if an individual can send message direct to n individuals, he will send a message to any one of them with probability $1/n$. In the present example, the communication probability matrix is

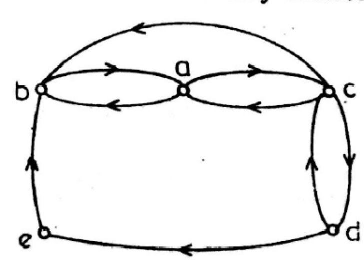


Figure 7.10

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (2)$$

No individual is to send a message to himself and so all diagonal elements are zero. Since all elements of the matrix are non-negative and the sum of elements of every row is unity, the matrix is a stochastic matrix and one of its eigenvalues is unity. The corresponding normalised eigenvector is [11/45, 13/45, 3/10, 1/10, 1/15]. In the long run, these fractions of messages will pass through a, b, c, d, e respectively. Thus we can conclude that in this network, c is the most important person.

If in a network, an individual cannot communicate with every other individual either directly or indirectly, the Markov chain is not ergodic and the process of finding the importance of each individual breaks down.

7.2.7 Matrices Associated with a Directed Graph

For a directed graph with n vertices, we define the $n \times n$ matrix $A = (a_{ij})$ by $a_{ij} = 1$ if there is an edge directed from i to j and $a_{ij} = 0$ if there is no edge directed from i to j. Thus the matrix associated with the graph of Figure 7.11 is given by

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad (3)$$

We note that (i) the diagonal elements of the matrix are all zero (ii) the number of non-zero elements is equal to the number of edges (iii) the number of non-zero elements in any row is equal to the local outward degree of the vertex corresponding to the row (iv) the number of non-zero elements in a column is equal to the local inward degree of the vertex corresponding to the column. Now

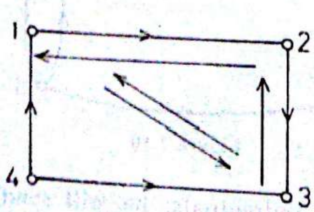


Figure 7.11

Scanned by CamScanner

$$A^2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array} = (a_{ij}^{(2)}) \qquad (4)$$

The element $a_{ij}^{(2)}$ gives the number of 2-chains from i to j. Thus from vertex 2 to vertex 1, there are two 2-chains viz. via vertex 3 and vertex 4. We can generalise this result in the form of a theorem viz. "The element $a_{ij}^{(2)}$ of A^2 gives the number of 2-chains i.e. the number of paths with two-edges from vertex i to vertex j".

The theorem can be further generalised to "The element $a_{ij}^{(m)}$ of A^m gives the number of m-chains i.e. the number of paths with m edges from vertex i to vertex j". It is also easily seen that "The ith diagonal element of A^2 gives the number of vertices with which i has symmetric relationship".

From the matrix A of a graph, a symmetric matrix S can be generated by taking the elementwise product of A with its transpose so that in our case

$$S = A \times A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5)$$

S obviously is the matrix of the graph from which all unreciprocated connections have been eliminated. In the matrix S (as well as in $S^2, S^3, \ldots$) the elements in the row and column corresponding to a vertex which has no symmetric relation with any other vertex are all zero.

7.2.8 Application of Directed Graphs to Detection of Cliques

A subset of persons in a socio-psychological group will be said to form a clique if (i) every member of this subset has a symmetrical relation with every other member of this subset (ii) no other group member has a symmetric relation with all the members of the subset (otherwise it will be included in the clique) (iii) the subset has at least three members.

If other words a clique can be defined as a maximal completely connected subset of the original group, containing at least three persons. This subset should not be properly contained in any larger completely connected subset.

It the group consists of n persons, we can represent the group by n vertices of a graph. The structure is provided by persons knowing or being connected to other persons. If a person i knows j, we can draw a directed edge from i to j. If i knows j and j knows i, then we have a symmetrical relation between i and j.

With this interpretation, the graph of Figure 7.11 shows that persons 1, 2, 3 form a clique. With very small groups, we can find cliques by carefully observing the corresponding graphs. For larger groups analytical methods based on the following results are useful: (i) i is a member of a clique if the ith diagonal element of S^3 is different from zero. (ii) If there is only one clique of k members in the group, the corresponding k elements of S^3 will be $(k-1)(k-2)/2$ and the rest of the diagonal elements will be zero. (iii) If there are only two cliques with k and m members respectively and there is no element common to these cliques, then k elements of S^3 will be $(k-1)(k-2)/2$, m elements of S^3 will be $(m-1)(m-2)/2$ and the rest of the elements will be zero. (iv) If there are m disjoint cliques with $k_1, k_2, \ldots, k_m$ members, then the trace of S^3 is $\frac{1}{2}\sum_{i=1}^{m} k_i(k_i-1)(k_i-2)$. (v) A member is non-cliquical if only if the corresponding row and column of $S^2 \times S$ consists entirely of zeros.

EXERCISE 7.2

1. Show that the graph of Figure 7.12 is a possible genetic graph if and only if n is even.

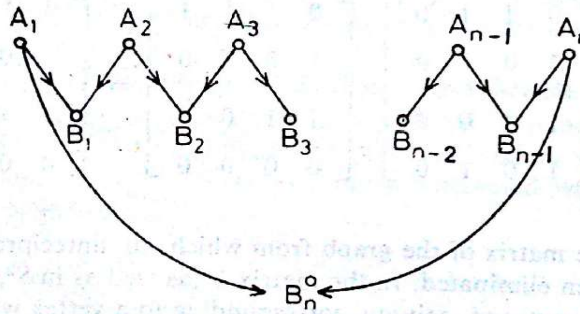


Figure 7.12

2. For each of the following communication networks, set up the corresponding transition probability matrix and find the importance of each member in the network.

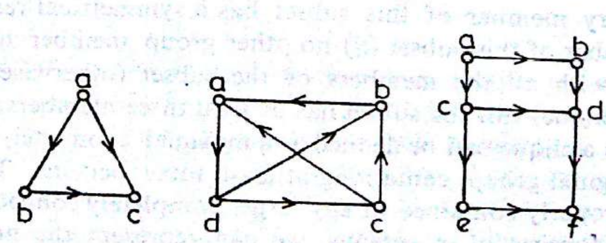


Figure 7.13

3. An intelligence officer can communicate with each of his n subordinates and each subordinate can communicate with him, but the subordinates cannot

communicate among themselves. Draw the graph and find the importance of each subordinate relative to the officer.

4. Find the Harary measure for each individual in the organisational graphs of Figure 7.14.

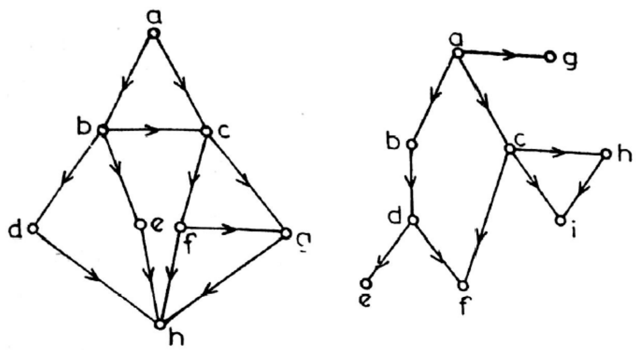


Figure 7.14

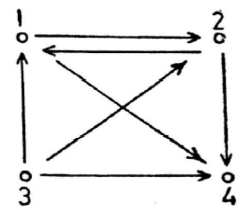


Figure 7.15

5. In Exercise 4, find the measure if the definition of level is based (i) in the longest number of steps between two persons (ii) on the average of the shortest and longest number of steps between two persons.

6. Find the eigenvector corresponding to the unit eigenvalue of matrix (2).

7. Prove all the theorems stated in Section 7.2.7.

8. Prove all the theorems stated in Section 7.2.8.

9. Write the matrix A associated with the graph of Figure 7.15. Find A^2, A^3, A^4, S, S^2, S^3, and verify the theorems of Sections 7.2.7 and 7.2.8.

10. Enumerate all possible four-cliques.

7.3 MATHEMATICAL MODELS IN TERMS OF SIGNED GRAPHS

7.3.1 Balance of Signed Graphs

A signed (or an algebraic) graph is one in which every edge has a positive or negative sign associated with it. Thus the four graphs of Figure 7.16 are signed graphs. Let positive sign denote friendship and negative sign denote enmity, then in graph (i) A is a friend of both B and C and B and C are

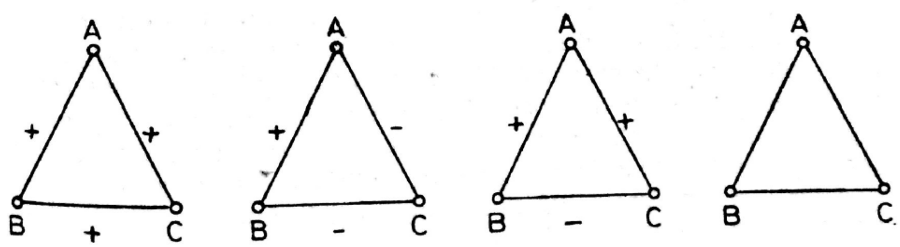


Figure 7.16

also friends. In graph (ii) A is friend of B and A and B are both jointly enemies of C. In graph (iii), A is a friend of both B and C, but B and C are enemies. In graph (iv) A is an enemy of both B and C, but B and C are not friends.

The first two graphs represents normal behaviour and are said to be balanced, while the last two graphs represent unbalanced situations since if A is a friend both B and C and B and C are enemies, this creates a tension in the system and there is a similar tension when B and C have a common enemy A, but are not friends of each other.

We define the sign of a cycle as the product of the signs of component edges. We find that in the two balanced cases, this sign is positive and in the two unbalanced cases, this is negative.

We say that a cycle of length three or a triangle is balanced if and only if its sign is positive. A complete algebraic graph is defined to be a complete graph such that between any two edges of it, there is a positive or negative sign. A complete algebraic graph is said to be balanced if all its triangles are balanced. An alternative definition states that a complete algebraic graph is balanced if all its cycles are positive. It can be shown that the two definitions are equivalent.

A graph is locally balanced at a point a if all the cycles passing through a are balanced. If a graph is locally balanced at all points of the graph, it will obviously be balanced. A graph is defined to be m-balanced if all its cycles of length m are positive. For an incomplete graph, it is preferable to define it to be balanced if all its cycles are positive. The definition in terms of triangle is not satisfactory, as there may be no triangles in the graph.

7.3.2 Structure Theorem and Its Implications

Theorem. The following four conditions are equivalent:

(i) The graph is balanced i.e. every cycle in it is positive.

(ii) All closed line-sequences in the graph are positive i.e. any sequence of edges starting from a given vertex and ending on it and possibly passing through the same vertex more than once is positive.

(iii) Any two line-sequences between two vertices have the same sign.

(iv) The set of all points of the graph can be partitioned into two disjoint sets such that every positive sign connects two points in the same set and every negative sign connects two points of different sets.

The last condition has an interesting interpretation with possibility of application. It states that if in a group of persons there are only two possible relationships viz. liking and disliking and if the algebraic graph representing these relationships is balanced, then the group will break up into two separate parties such that persons within a party like one another, but each person of one party dislikes every person of the other party. If a balanced situation is regarded as stable, this theorem can be interpreted to imply that a two-party political system is stable.

7.3.3 Antibalance and Duobalance of a Graph

An algebraic graph is said to be antibalanced if every cycles in it has an even number of positive edges. The concept can be obtained from that of a balanced graph by changing the signs of the edges. It will then be seen that an algebraic graph is antibalanced if and only if its vertices can be separated into two disjoint classes, such that each negative edge joins two vertices of the same class and each positive edge joins persons from different classes.

A signed graph is said to be duobalanced if it is both balanced and antibalanced.

7.3.4 The Degree of Unbalance of a Graph

For many purposes it is not enough to know that a situation is unbalanced. We may be interested in the degree of unbalance and the possibility of a balancing process which may enable one to pass from an unbalanced to a balanced graph. The possibility is interesting as it can give an approach to group dynamics and demonstrate that methods of graph theory can be applied to dynamic situations also.

Cartwright and Harary define the degree of balance of a group G to be the ratio of the positive cycles of G to the total number of cycles in G. This balance index obviously lies between 0 and 1. G_1 has six negative triangles viz (abc), (ade), (bcd), (bce), (bde), (cde) and has four positive triangles. G_2 has four negative triangles viz (abc), (abd), (bce) and (bde) and six positive triangle. The degree of balance of G_1 is therefore less than the degree of balance of G_2.

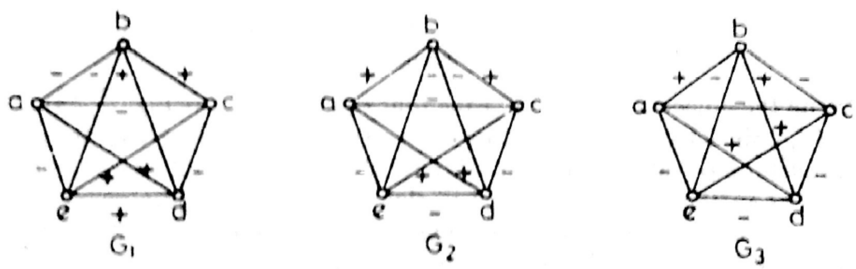


Figure 7.17

However in order to get a balanced graph from G_1, we have to change the sign of only two edges viz. bc and de and similarly to make G_2 balanced we have to change the signs of two edges viz bc and bd. From this point of view both G_1 and G_2 are equally unbalanced.

Abelson and Rosenberg therefore gave an alternative definition. They defined the degree of unbalance of an algebraic graph as the number of the smallest set of edges of G whose change of sign produces a balanced graph.

The degree of an antibalanced complete algebraic graph (i.e. of a graph all of whose triangles are negative) is given by $[n(n-2)+k]/4$ where $k=1$ if n is odd and $k=0$ if n is even. It has been conjectured that the degree of unbalancing of every other complete algebraic graph is less than or equal to this value.

EXERCISE 7.3

1. State which of the following graphs are balanced. If balanced, find the decomposition guaranteed by the structure theorem. If unbalanced, find the degree of unbalance.

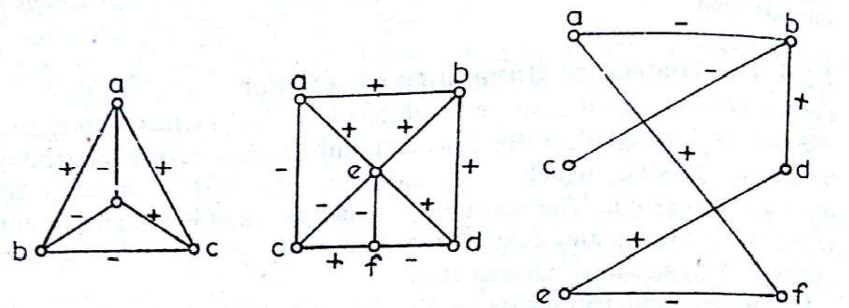


Figure 7.18

2. Draw some antibalanced graphs and verify the structure theorems for them.
3. The adjacency matrix of a signed graph is defined as follows:

$$\begin{aligned} a_{ij} &= 1 \text{ if there is } + \text{ sign associated with edge } i, j \\ &= -1 \text{ if there is } - \text{ sign associated with edge } i, j \\ &= 0 \text{ if there is no edge } i, j. \end{aligned}$$

Write the adjacency matrices of the four signed graphs is Figure 7.18.

4. A signed graph G is said to have an idealised party structure if the vertices of G can be partitioned into classes so that all edges joining the vertices in the same class have $+$ sign and all edges joining vertices in different sets have negative sign (a) Give an example of a signed graph which does not have an idealised party structure (b) Give an example of a graph which is not balanced but which has an idealised party structure.
5. Show that a signed graph has an idealised party structure if and only no circuit has exactly one $-$ sign.
6. Show that if all cycles of a signed graph are positive, then all its cycles are also positive. State and prove its converse also.

7.4 MATHEMATICAL MODELLING IN TERMS OF WEIGHTED DIGRAPHS

7.4.1 Communication Networks with Known Probabilities of Communication

In the communication graph of Figure 7.10, we know that a can communicate with both b and c only and in the absence of any other knowledge, we assigned equal probabilities to a's communicating with b or c. However we may have a priori knowledge that a's chances of communicating with b and c are in the ratio 3 : 2, then we assign probability .6 to a's communicating with b and .4 to a's communicating with c. Similarly we can associate a probability with every directed edge and we get the weighted digraph (Figure 7.19) with the associated matrix

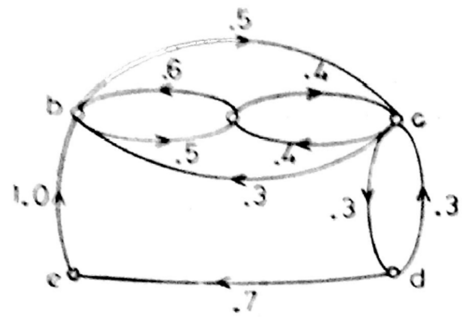


Figure 7.19

$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.4 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0 & .3 & 0 & 0.7 \\ 0 & 1.0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (6)$$

We note that the elements are all non-negative and the sum of the elements of every row is unity so that B is a stochastic matrix and unity is one of its eigenvalues. The eigenvector corresponding to this eigenvalues will be different from the eigenvector found in Section 7.2.6 and so the relative importance of the individuals depends both on the directed edges as well as on the weights associated with the edges.

7.4.2 Weighted Digraphs and Markov Chains

A Markovian system is characterised by a transition probability matrix. Thus if the states of a system are represented by 1, 2, . . ., n and p_{ij} gives the probability of transition from the ith state to jth state, the system is characterised by the transition probability matrix (t.p.m)

$$T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nj} & \cdots & p_{nn} \end{bmatrix} \qquad (7)$$

Since $\sum_{i=1}^{n} p_{ij}$ represents the probability of the system going from ith state to any other state or of remaining in the same state, this sum must be equal to unity. Thus the sum of elements of every row of a t.p.m. is unity.

Consider a set of N such Markov systems where N is large and suppose at any instant $NP_1, NP_2, \ldots, NP_n$ of these $(P_1 + P_2 + \ldots + P_n = 1)$ are in states $1, 2, 3, \ldots, n$ respectively. After one step, let the proportions in these states be denoted by $P_1', P_2', \ldots, P_n'$, then

$$\begin{aligned} P_1' &= P_1p_{11} + P_2p_{21} + P_3p_{31} + \ldots\ldots + P_np_{n1} \\ P_2' &= P_2p_{12} + P_2p_{22} + P_3p_{32} + \ldots\ldots + P_np_{n2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P_n' &= P_1p_{1n} + P_2p_{2n} + P_3p_{3n} + \ldots\ldots + P_np_{nn} \end{aligned} \tag{8}$$

or

$$P' = PT \tag{9}$$

where P and P' are row matrices representing the proportions of systems in various states before and after the step and T is the t.p.m.

We assume that the system has been in operation for a long time and the proportions $P_1, P_2, \ldots, P_n$ have reached equilibrium values. In this case

$$P = PT \quad \text{or} \quad P(I - T) = 0, \tag{10}$$

where I is the unit matrix. This represents a system of n equations for determining the equilibrium values of $P_1, P_2, \ldots, P_n$. If the equations are consistent, the determinant of the coefficient must vanish i.e. $|T - I| = 0$. This requires that unity must be an eigenvalue of T. However this, as we have seen already is true. This shows that an equilibrium state is always possible for a Markov chain.

A Markovian system can be represented by a weighted directed graph. Thus consider the Markovian system with the stochastic matrix

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Its weighted digraph is given in Figure 7.20.

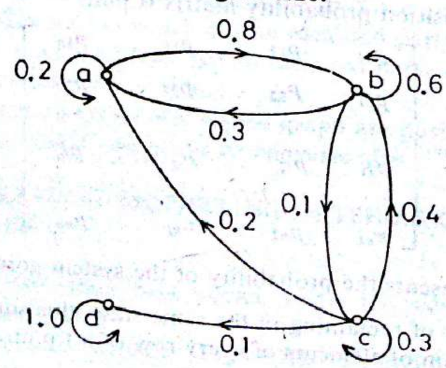


Figure 7.20

In this example d is an absorbing state or a state of equilibrium. Once a system reaches the state d, it stays there for ever.

It is clear from Figure 7.20, that in whichever state, the system may start, it will ultimately end in state d. However the number of steps that may be required to reach d depends on chance. Thus starting from c, the number of steps to reach d may be 1, 2, 3, 4, . . .; starting from b the number of steps to reach d may be 2, 3, 4, . . . and starting for a, the number of steps may be 3, 4, 5, . . . In each case, we can find the probability that the number of steps required is n and then we can find the expected number of steps to reach it.

Thus for the matrix

$$\begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \end{bmatrix} \end{array} \tag{12}$$

a is an absorbing state. Starting from b, we can reach a in 1, 2, 3, . . ., n steps with probabilities (1/3), (1/3) (2/3), (1/3) $(2/3)^2$, . . ., (1/3) $(2/3)^{n-1}$, . . ., so that the expected number of steps is

$$\sum_{n=1}^{\infty} n\frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 3 \tag{13}$$

7.4.3 General Communication Networks

So for we have considered communication networks in which the weight associated with a directed edge represents the probability of communication along that edge. We can however have more general networks e.g.

(a) for communication of messages where the directed edge represents the channel and the weight represents the capacity of the channel say in bits per second

(b) for communication of gas in pipelines where the weights are the capacities, say in gallons per hour

(c) communication roads where the weights are the capacities is cars per hour.

An interesting problem is to find the maximum flow rate, of whatever is being communicated, from any vertex of the communication network to any other. Useful graph-theoretic algorithms for this have been developed by Elias. Feinstein and Shannon as well as by Ford and Fulkerson.

7.4.4 More General Weighted Digraphs

In the most general case, the weight associated with a directed edge can be positive or negative. Thus Figure 7.21 means that a unit change at vertex 1 at time t causes changes of -2 units at vertex 2, of 2 units at vertex 4 and of 3 units at vertex 5 at time $t + 1$. Similarly a change of 1 unit

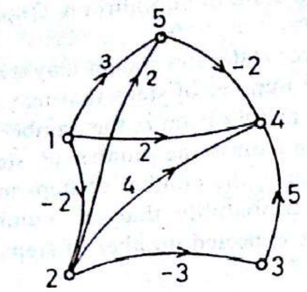


Figure 7.21

at vertex 2 causes a change of -3 units at 3 vertex, 4 units at vertex 4 and of 2 units at vertex 5 and so on. Given the values at all vertices at time t, we can find the values at times $t+1$, $t+2$, $t+3$, . . . The process of doing this systematically is known as the pulse rule.

These general weighted digraphs are useful for representing energy flows, monetary flows and changes in environmental conditions.

7.4.5. Signal Flow Graphs

The system of algebraic equations

$$\begin{aligned} x_1 &= 4y_0 + 6x_2 - 2x_3 \\ x_2 &= 2y_0 - 2x_1 + 2x_3 \\ x_3 &= 2x_1 - 2x_2 \end{aligned} \tag{14}$$

can be represented by the weighted digraph in Figure 7:22. For solving for x_1, we successively eliminate x_3 and x_2 to get the graphs in Figure 7.23 and finally we get

$$x_1 = 4y_0$$

We can similarly represent the solution of any number of linear equations graphically.

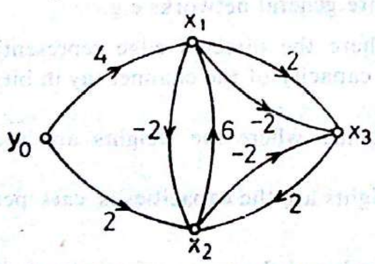


Figure 7.22

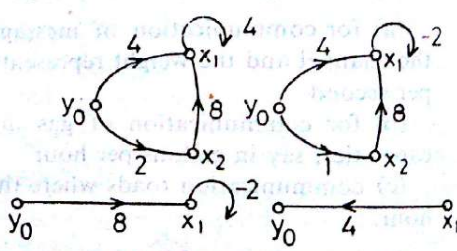


Figure 7.23

7.4.5 Weighted Bipartitic Digraphs and Difference Equations

Consider the system of difference equations

$$\begin{aligned} x_{t+1} &= a_{11}x_t + a_{12}y_t + a_{13}z_t \\ y_{t+1} &= a_{21}x_t + a_{22}y_t + a_{23}z_t \\ z_{t+1} &= a_{31}x_t + a_{32}y_t + a_{33}z_t \end{aligned} \tag{15}$$

This can be represented by a weighted bipartitic digraph (Figure 7.24). The weights can be positive or negative.

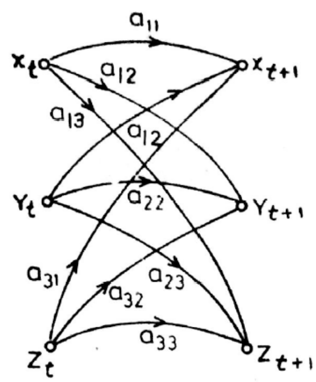


Figure 7.24

EXERCISE 17.4

1. A machine can be in any one of the states a, b, c. The transitions between states are governed by the transition probability matrix

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array} \tag{16}$$

Draw the weighted digraph and find the limiting probabilities for the machine to be found in each of the three states.

2. The entropy of a Markov machine is defined by

$$H = \sum_{i=1}^{n} P_i H_i = -\sum_{i=1}^{n}\sum_{j=1}^{n} P_i p_{ij} \ln p_{ij} \tag{17}$$

Show that

(a) When

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/4 & 3/4 & 0 \\ 3/4 & 0 & 1/4 \\ 1/8 & 3/4 & 1/8 \end{bmatrix} \end{array}$$

$$P_1 = 0.449, \quad P_2 = 0.429, \quad P_3 = 0.122$$
$$H_1 = 0.811, \quad H_2 = 0.811, \quad H_3 = 1.663$$

(b) When

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.6 & 0.4 & 0 \\ 0 & 0.6 & 0.4 & 0 \\ 0.3 & 0 & 0 & 0.7 \\ 0.3 & 0 & 0 & 0.7 \end{bmatrix} \end{array}$$

$$P_1 = 6/35, \quad P_2 = 9/35,$$
$$P_3 = 6/35, \quad P_4 = 14/35$$
$$H = 0.92$$

3. In a panel survey, a person is asked a question to which he can answer 'Yes' or 'No'. In the next survey, the probability of his being in state 1 (Yes) or state 2 (No) is given by

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \end{array} \tag{18}$$

Show that

(a) $p_1(t+1) = p_1(t)(1-\alpha) + p_2(t)\beta$

$$p_2(t+1) = p_1(t)\alpha + p_2(t)(1-\beta) \tag{19}$$

(b) $p_1(t) = \dfrac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^t\left[p_1(0) - \dfrac{\beta}{\alpha+\beta}\right]$

$$p_2(t) = \frac{\alpha}{\alpha+\beta} + (1-\alpha-\beta)^t\left[p_2(0) - \frac{\alpha}{\alpha+\beta}\right] \tag{20}$$

(c) $p_1(t)$, $p_2(t)$ approaches $\beta/(\alpha+\beta)$ and $\alpha/(\alpha+\beta)$ as $t \to \infty$ if $\alpha + \beta \leq 1$.

4. In Exercise 3, find the expected number of time units in which the system now in state 1(2) will change to state 2(1).

5. Interpret the models and results of Exercises 3 and 4 when states 1, 2 refer to

(a) a neuron being excited or not excited
(b) a machine being in working order or out of order
(c) a stimulus being or not being available in a learning situation
(d) a daily wage worker being employed or not employed.

6. Give the graphical solution of

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 4 \end{aligned} \tag{21}$$

7.5 MATHEMATICAL MODELLING IN TERMS OF UNORIENTED GRAPHS

7.5.1 Electrical Networks and Kirchoffs' Laws

For more than a hundred years after Euler solved the Königsberg problem in 1736, graph theory continued to deal with interesting puzzles only. It was in 1849 that Kirchoffs' formulation of his laws of electrical currents in graph-theoretic terms led to interest in serious applications of graph theory

An electrical circuit (Figures 7.25a, b) consists of resistors R_1, R_2, . . ., inductances L_1, L_2, . . ., capacitors C_1, C_2 and batteries B_1 B_2, etc.

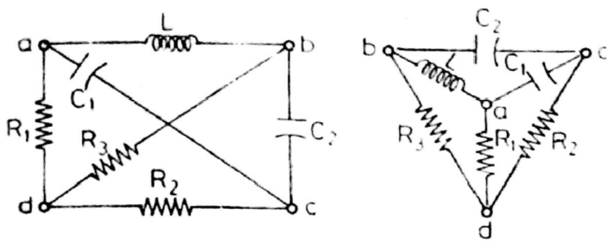


Figure 7.25

The network diagram represents two independent aspects of an electrical network. The first gives the interconnection between components and the second gives voltage-current relationship of each component. The first aspect is called network topology and can be modelled graphically. This aspect is independent of voltages and currents. The second aspects involves voltages and current and is modelled through differential equations.

For topological purposes, lengths and shapes of connections are not important and graphs of Figures 7.25(a), 7.25(b) and 7.25(c) are isomorphic.

For stating Kirchoff's laws, we need two incidence matrices accociated with the graph. If v and e denote the number of vertices and edges respectively, we define the *vertex* or *incidence* matrix $A = [a_{ij}]$ as follows:

$$a_{ij} = 1 \quad \text{if the edge } j \text{ is incident at vertex } i$$
$$a_{ij} = 0 \quad \text{if the edge } j \text{ is not incident at vertex } i$$

This consists of v rows and e columns. For graph 7.25, A is given by

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad (22)$$

We note that every column has two non-zero elements.

Similarly we define the circuit matrix $B = [b_{kj}]$ as follows

$$b_{kj} = 1 \quad \text{if element } j \text{ is in circuit } k$$
$$= 0 \quad \text{if element } j \text{ is not in circuit } k$$

The matrix B contains as many rows as there are circuits and it has e columns. In our case

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \tag{23}$$

Now Kirchoff's laws can be written in the matrix form as follows:

$$AI = 0 \text{ (Kirchoff's current law)} \tag{24}$$

$$BV = 0 \text{ (Kirchoff's voltage law)} \tag{25}$$

where I is an exl column matrix giving the e currents and V is exl column matrix giving e voltages.

Matrices A and B depend on the graph only, matrices I and V depend on currents and voltages only. A and B can be written independently of I and V. Now an important question is as to how many of the components of the current and voltage vectors are independent.

It can be proved that the rank of A is $v - 1$ and the rank of B is $e - v + 1$. Thus $v - 1$ and $e - v + 1$ are the numbers of linearly independent Kirchoffs current and voltage equations.

The graph-theoretic methods can now be used to (i) establish the validity of the circuit and vertex equations and find their generalisations (ii) conditions under which unique solutions of these equations exist (iii) justify the duality procedures used in network theory (iv) develop short-cut methods for writing equations (v) develop techniques for network synthesis.

7.5.2 Lumped Mechanical Systems

If the linear graph represents a lumped mechanical system with the vertices representing rigid bodies, matrices A and B arise for Newtons' force and displacements equations respectively and $v - 1$ and $e - v + 1$ represent the number of linearly independent force and displacement equations.

7.5.3 Map-Colouring Problems

The four colour problem that every plane map, however complex, can be coloured with four colours in such a way that two neighbouring regions get different colours, challenged and fascinated mathematicians for over one hundred years till it was finally solved by Appall and Haken in 1976 by using over 1000 hours of computer time. The problem is essentially graph-theoretic since the sizes and shapes of regions are not important. That four colours are necessary is easily seen by considering the simple graph in Figure 7.26. It was the proof of the sufficiency that took more than hundred years. However the efforts

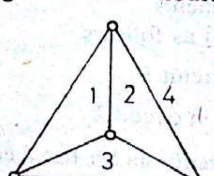


Figure 7.26

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to solve this problem led to the development of many other graph-theoretic models.

Similar map-colouring problems arise for colouring of maps on surface of a sphere, a torus or other surfaces. However many of these were solved even before the simpler-looking four-colour problem was disposed of.

7.5.4 Planar Graphs

In printing of T.V. and radio circuits; we want that the wires, all lying in a plane, should not intersect. In the graph of Figure 7.27a wires appear to intersect, but we can find an isomorphic graph in Figure 7.27(b) in which edges do not intersect. A graph which is such that we can draw a graph isomorphic to it in which edges do not intersect is called a planar graph.

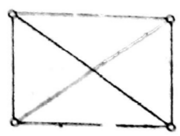

Figure 7.27 (a)

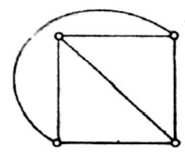

Figure 7.27 (b)

A complete graph with five vertices is not planar (Figure 7.28a). We can draw nine of the edges so that these do not intersect (Figure 7.28b) but however we may draw, we cannot draw all the ten edges without at least two of them intersecting. The proof of this depends on Jordan's theorem that every simple closed curve divides the plane into two regions, one inside the curve and one outside the curve. *ABCDE* in Figure 7.28(b) is a closed Jordan curve and we cannot draw three edges either inside it or outside it without intersecting.

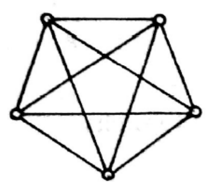

Figure 7.28 (a)

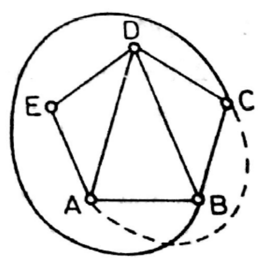

Figure 7.28 (b)

7.5.5 Euler's Formula for Polygonal Graphs

A polygonal graph with n vertices and n straight or curved edges has n vertices, n edges and two faces (one inside and one outside) so that for this graph

$$V - E + F = 2 \qquad (26)$$

If we add on one edge, another polygonal region of r vertices, we increase the number of vertices by $r - 2$, the number of edges by $r - 1$ and the number of faces by 1, so that the net increases in $V - E + F$ is zero and the formula (26) remains valid. It can be shown by using the principle of induction that (26) is valid for any polygonal graph with any number of regions.

To draw the dual graph G^* of G, we take a point inside each region and draw an edge through it intersecting one of the edges of the region. It is obvious that for this dual graph the number of vertices, edges and faces is given by

$$V^* = F, \quad E = E^*, \quad F^* = V, \tag{30}$$

so that

$$V^* - E^* + F^* = F - E + V = 2, \tag{31}$$

as expected.

7.5.6 Regular Solids

A polygonal graph G is said to be completely regular if both G and its dual G^* are regular i.e. if the degree of each vertex of G is the same (say ρ) and the degree of each vertex of G^* is the same (say ρ^*). From this definition, it follows

$$2E = \rho V = \rho^* F \tag{32}$$

or

$$E = \frac{1}{2}\rho V, \quad F = \frac{\rho}{\rho^*}V \tag{33}$$

Substituting (33) in (26)

$$V - \frac{1}{2}\rho V + \frac{\rho}{\rho^*}V = 2 \tag{34}$$

or

$$V(2\rho + 2\rho^* - \rho\rho^*) = 4\rho^* \tag{35}$$

Since V, ρ, ρ^* are positive integers

$$2\rho + 2\rho^* - \rho\rho^* > 0 \quad \text{or} \quad (\rho - 2)(\rho^* - 2) < 4 \tag{36}$$

If $\rho > 2$, $\rho^* > 2$, the only solutions of the inequality (36) are

$\rho = 3$, $\rho^* = 3$; $\rho = 3$; $\rho^* = 4$; $\rho = 3$, $\rho^* = 5$; $\rho = 4$, $\rho^* = 3$;
$\rho = 5$, $\rho^* = 3$.

Substituting in (35) and (33), we get the table and graphs

	ρ	V	E	F	ρ^*	V^*	E^*	F^*
(i)	3	4	6	4	3	4	6	4
(ii)	3	8	12	6	4	6	12	8
(iii)	3	20	30	12	5	12	30	20
(iv)	4	6	12	8	3	8	12	6
(v)	5	12	30	20	3	20	30	12

The corresponding graphs are given in Figure 7.29(a)-(e). It is obvious that tetrahedron graph is dual to itself, cube is dual of octahedron and Dodocahedron and Icosahedron are duals of each other.

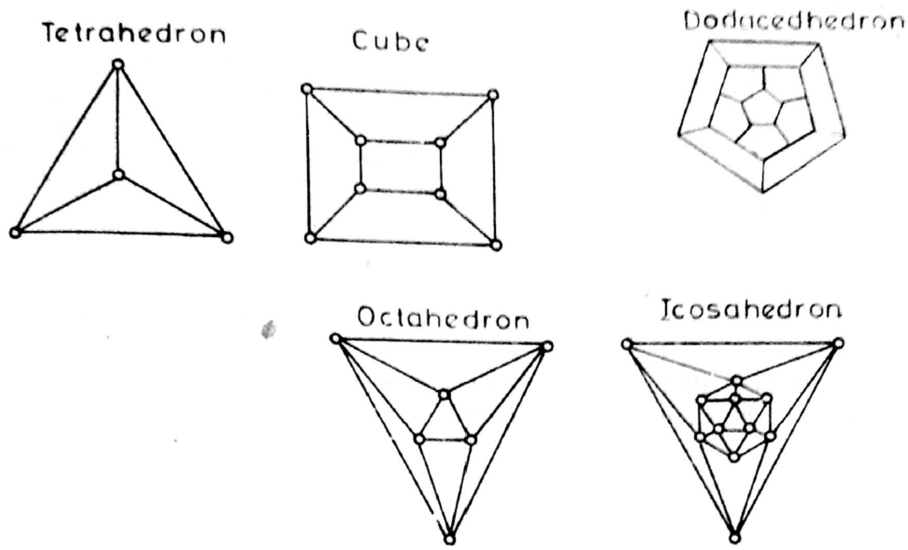

Figure 7.29

These five graphs corresponding to five Platonic regular solids (Figure 7.30).

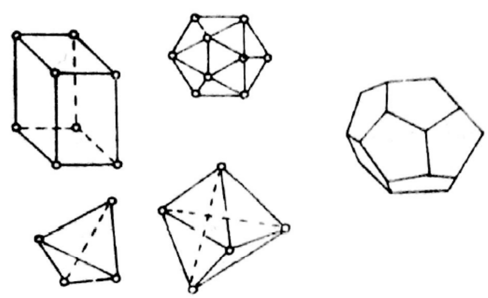

Figure 7.30

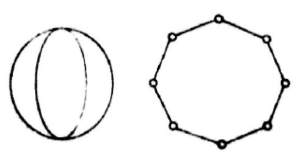

Figure 7.31

There is another solution of (36) viz. $\rho = 2$, $\rho^* = 2, 3, 4, \ldots$ The corresponding graphs G and G^* are shown in Figure 7.31.

EXERCISE 7.5

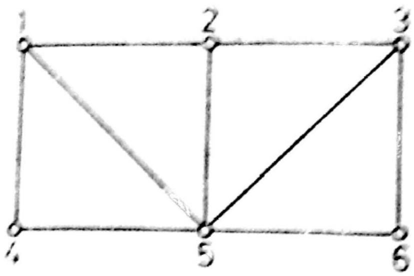

Figure 7.32

1. For the graph of Figure 7.32, write the adjacency matrix A and circuit matrix B and find their ranks. Find a set of independent circuits.

2. Prove that if the columns of matrices A and B are arranged in the same element order, then

$$AB^T = 0, \quad BA^T = 0 \tag{37}$$

3. Draw some polygonal graphs. Draw their duals and verify (26) and (31) for them.

4. Prove that all repetitive planar graph pattern or mosaics must be formed either by triangles or by quadrangles or by hexagon.

POSSIBLE QUESTIONS

Part B (6 Marks)

1.Explain in detail senior-subordinate relationship.

2.Write an explanatory note on planar graphs.

3.Discuss in detail weighted digraphs and markov chains.

4.Write a note on the following

i) Signal flow graphs

ii) Map-colouring problems

iii) Planar graphs

iv) Euler's formula for polygonal graphs

5.Explain about one-way traffic problems.

6.Discuss in detail about communication networks

7.Write a note on seven bridges problem.

8.Give a brief note on Genetic graphs.

Part C (10 Marks)

1.Discuss in detail about communication networks.

2.Give a detailed note on electrical networks and Kirchoff's laws.

3.Give a brief note on Genetic graphs.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Mathematical Modelling **Subject Code: 18MMP303**

Class : II - M.Sc. Mathematics **Semester : III**

Unit V

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Mathematics deals with both quantative and ___________ relationship	Qualitative	Numeric	Decimal	Integer	Qualitative
Apply mathematics deals with solution of ______________	Difference	Numeric	Decimal	Integer	Difference
Apply mathematics deals with solution of ______________	Integral	Numeric	Decimal	Integer	Integral
Apply mathematics deals with solution of ______________	Functional	Numeric	Decimal	Integer	Functional
Apply mathematics deals with solution of ______________	Algebraic	Numeric	Decimal	Integer	Algebraic
In graphical model the problem of 7 bridges is called ______________	Fick's	Routhwelt	Konigsberg	Gauss	Konigsberg
A graph is called ____________ if every pair of vertices is joined by an edge	Complete	Incomplete	Digraph	Continuous	Complete
A graph is called ___________ if every edge is directed with an arrow	Complete	Incomplete	Digraph	Continuous	Digraph
A graph is called __________ if every edge has either + or - sign associated with it	Complete	Incomplete	Digraph	signed graph	signed graph

A digraph is called ___________ if every directed edge has a weight associated wit it	Weighted digraph	signed graph	Digraph	Complete	Weighted digraph
A graph is called ________ if each of its vertices has same degree r.	regular	irregular	solid	unsolid	regular
If traffic is allowed from point a to b the edge can draw ______ from a to b.	undirected	directed	complete	incomplete	directed
If G is undirected connected graph then one can always direct ___________ edge of G	Non circuit	Vertex	Non vertex	Circuit	Circuit
In genetic graph, the local degree of incoming edges at eac vertex must be less then or equal to __________	1	2	3	4	2
The necessary condition for a directed graph is to be ___________	one way traffic	Two way traffic	Genetic	Nature	Genetic
The measure m(x) is always a __________ number	Real	complex	whole	natural	Whole
If x has no subordinates then measure m(x) equals	1	0	2	3	0
If witout oterwise changing the struture we move subordinate of a to a lower level relative to x then m(x)_________	Increases	decreases	stable	unstable	increases
If witout oterwise changing the struture we add a new individual subordinate to x then m(x)_________	Increases	decreases	stable	unstable	increases
In communication network a ________ graph can serve as a model	undirected	stable	directed	unstable	directed
An individual can send message direct to n individuals with propability	n	1/n	2n	3n	1/n
in a matrix representation an individual can send message to himself then _________ elements are 0	row	column	digonal	all	digonal
All the elements of matrix are non negative and the sum of elements of every row is unity, the matrix is _________	stochastic	propabilistic	direct	ergodic	stochastic

All the elements of matrix are non negative and the sum of elements of every row is _________, the matrix is stochastic	2	unity	3	4	unity
The markov chain is not __________	stochastic	propabilistic	direct	ergodic	ergodic
A subset is to form clique if every member of subset has a __________ relation with other member	symmetrical	non symmetrical	stable	unstable	symmetrical
A subset of persons in a socio - psychological group will set to form a _________	queue	clique	line	None	clique
The subset has atleast ___________ member	1	2	3	4	3
If the group consists of n persons then can represents the group by ___________ verties of graph	n+1	n-1	n	2n	n
For each communication netwrok can set up the corresponding ___________ propability matrix	Digonal	Unit	row	Transition	Transition
A ___________ graph Is one in which every edge has positive or negative sign	direct	undirect	signed	unsigned	signed
A ___________ graph Is one in which every edge has positive or negative sign	direct	undirect	algebraic	unsigned	algebraic
The graph is balanced, every cycle in it is __________	Positive	negative	both	none	positive
All closed line sequences in the graph is __________	Positive	negative	both	none	positive
Any two lines sequence between two verties have the same _________	sign	number	constant	variable	sign
Any two lines sequence between two verties have the ___________ sign	same	different	equal	none	same
The set of all points of graph can be partitioned into __________ disjoints sets	1	2	4	3	2
Every negative sign connects ___________ points of differents set	1	2	3	4	2
Every positive sign connects ___________ points of differents set	1	2	3	4	2

An algebraic grap is set to be antibalanced if every cycle in it has _________ no of positive edges	even	odd	real	distinct	even