Semester – II

19MMU201 **DIFFERENTIAL EQUATIONS** **5H – 5C**

Instruction Hours / week: L: 5 T: 0 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- First order exact differential equations, linear homogeneous and non homogeneous equations of higher order with constant coefficients.
- The complete solution of a non-homogeneous differential equation with constant coefficients by the method of undetermined coefficients.
- The transform of a periodic function.
- The applications of the inverse Laplace transform.

Course Outcomes (COs)

On successful completion of this course, the student will be able to

1. Understand the concepts of explicit, implicit and singular solutions of a differential equation.
2. Acquire knowledge on linear and bernoulli's equaitons.
3. Know the concepts of population model.
4. Understand the method of solving differential equation using variation of parameters.
5. Identify the applications of differential equations.

UNIT I

DIFFERENTIAL EQUATIONS

Differential equations and mathematical models. General,particular, explicit, implicit and singular solutions of a differential equation.

UNIT II

TYPES OF DIFFERENTIAL EQUATIONS

Exact differential equations and integrating factors - Separable equations and equations reducible to this form - Linear equation and Bernoulli equations - Special integrating factors and transformations.

UNIT III

SECOND ORDER LINEAR EQUATIONS

General solution of homogeneous equation of second order - Principle of super position for homogeneous equation –Wronskian: its properties and applications - Linear homogeneous andnon-homogeneous equations of higher order with constant coefficients - Euler's equation - Method of undetermined coefficients - Method of variation of parameters.

UNIT IV

LAPLACE TRANSFORMS

Definition-Sufficient conditions for the existence of the Laplace Transform - Laplace Transform of periodic functions- Some general theorems-Evaluation of integrals using Laplace Transform.

UNIT V

INVERSE LAPLACE TRANSFORMS

Solving ordinary differential equations with constant coefficients using Laplace Transforms-Solving a system of differential equations using Laplace Transforms.

SUGGESTED READINGS

1. Ross S.L., (2016). Differential Equations, Third Edition, John Wiley and Sons, India.
2. Martha L Abell., and James P Braselton., (2004). Differential Equations with MATHEMATICA, Third Edition, Elsevier Academic Press.
3. Sneddon I.,(2006). Elements of Partial Differential Equations, McGraw-Hill, International Edition, New Delhi.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University) (Established Under Section 3 of UGC Act 1956) Coimbatore – 641 021.

LESSON PLAN

DEPARTMENT OF MATHEMATICS

NAME OF THE FACULTY : M.JANNATH BEGAM

SUBJECT : DIFFERENTIAL EQUATIONS

SUBJECT CODE : 19MMU201

CLASS : I B.Sc MATHEMATICS

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
UNIT I			
1	1	Introduction of Differential equations	S1:Chap1:Pg.No:3-4
2	1	Mathematical models related examples	S1:Chap1:Pg.No:5-6
3	1	General solutions of a differential equation Problems	S1:Chap1:Pg.No:7-8
4	1	Particular solutions of a differential equation Problems	S1:Chap1:Pg.No:9-10
5	1	Explicit solutions of a differential equation Problems	S1:Chap1:Pg.No:11-12
6	1	Implicit solutions of a differential equation Problems	S2:Chap1:Pg.No:6-7
7	1	Continuation of problems on implicit solutions of a differential equation	S2:Chap1:Pg.No:7-9
8	1	Singular solutions of a differential equation -Problems	S2:Chap1:Pg.No:10-11
9	1	Continuation of problems on Singular solutions of a differential equation	S2:Chap1:Pg.No:11-13
10	1	Recapitulation and discussion of important questions.	
Total	10 hrs		

UNIT II			
1	1	Introduction on concept of Exact differential equations	S1:Chap2:Pg.No:36-40
2	1	Integrating factors Problems	S1:Chap2:Pg.No:42-44
3	1	Separable equations Problems	S2:Ch2.1:Pg.No:46-47
4		Continuation of problems on Separable equations	S2:Ch2.1:Pg.No:47-49
5	1	Equations reducible to this form linear equation Problems	S1:Chap2:Pg.No:50-53
6	1	Bernoulli equations related Problems	S1:Chap2:Pg.No:56-59
7	1	Continuation of problems on Bernoulli equations	S1:Chap2:Pg.No:60-62
8	1	Special integrating factors and transformations related Problems	S1:Chap2:Pg.No:68-70
9	1	Continuation of problems on Special integrating factors and transformations	S1:Chap2:Pg.No:70-74
10	1	Recapitulation and discussion of important questions	
Total	10 hrs		
UNIT III			
1	1	Introduction on general solution of homogeneous equation of second order related Problems	S2:Chap:4:Pg.No:196-199
2	1	Principle of super position for homogeneous equation	RS2:Chap:4:Pg.No:200-202
3	1	Wronskian: its properties and applications	S2:Chap:4:Pg.No:239-242
4	1	Linear homogeneous equations of higher order with constant coefficients related Problems	S2:Chap:4:Pg.No:200-202
5	1	Non- homogeneous of higher order with constant coefficients related Problems	S2:Chap:4:Pg.No:202-205
6	1	Euler's equation related Problems	S2:Chap:4:Pg.No:255-258
7	1	Method of undetermined coefficients related Problems	S2:Chap:4:Pg.No:222-223

8	1	Method of variation of parameters related Problems	S2:Chap:4:Pg.No:248-249
9	1	Continuation of problems on Method of variation of parameters	S2:Chap:4:Pg.No:249-251
10	1	Recapitulation and discussion of important questions	
Total 10 hrs			
Unit- IV			
1	1	Definition of Laplace Transform	S3:Chap:4:Pg.No:141-142
2	1	Sufficient conditions for the existence of the Laplace Transform	S3:Chap:4:Pg.No:143-145
3	1	Laplace Transform of periodic functions	S1:Chap:9:Pg.No:428-429
4	1	Continuation of problems on Laplace Transform of periodic functions	S1:Chap 9:Pg.No:429-430
5	1	Some general theorems	S3:Chap 4:Pg.No:164-167
6	1	Continuation on some general theorems	S1:Chap 9:Pg.No:437-438
7	1	Evaluation of integrals using Laplace Transform	S1:Chap 9:Pg.No:439-440
8	1	Continuation of problems on integrals using Laplace Transform	S1:Chap 9:Pg.No:441-442
9	1	Continuation of problems on integrals using Laplace Transform	S1:Chap 9:Pg.No:442-443
10	1	Recapitulation and discussion of important questions	
Total 10 hrs			
Unit- V			
1	1	Solving ordinary differential equations with constant coefficients using inverse Laplace Transforms	S1:Chap:9:Pg.No:441-447
2	1	Continuation on Solving ODE with constant with constant coefficients using inverse Laplace Transforms	S1:Chap:9:Pg.No:447-452

3	1	Solving a system of differential equations using Laplace Transforms.	S1:Chap:9:Pg.No:453-455
4	1	Continuation on Solving a system of differential equations using Laplace Transforms	S1:Chap:9:Pg.No:456-458
5	1	Continuation on Solving a system of differential equations using Laplace Transforms	S1:Chap:9:Pg.No:459-460
7	1	Recapitulation and discussion of important questions	
8	1	Discuss on Previous ESE question papers	
9	1	Discuss on Previous ESE question papers	
10	1	Discuss on Previous ESE question papers	
Total	10 hrs		

SUGGESTED READINGS

R1. Ross S.L., (2004). Differential Equations, Third Edition, John Wiley and Sons, India.

R2. Martha L Abell., and James P Braselton., (2004). Differential Equations with MATHEMATICA, Third Edition, Elsevier Academic Press.

R3. Sneddon I.,(2006). Elements of Partial Differential Equations, McGraw-Hill, International Edition, New Delhi.

Signature of the class Representative **Signature of the Faculty**

Tutor **Programme co-ordinator** **HOD**

UNIT-I

SYLLABUS

Differential equations and mathematical models. General, particular, explicit, implicit and singular solutions of a differential equation.

Introduction

Differential equations finds its application in a variety of real world problems such as growth and decay problems. Newton's law of cooling can be used to determine the time of death of a person. Torricelli's law can be used to determine the time when the tank gets drained off completely and many other problems in science and engineering can be solved by using differential equations. In this chapter, we will first discuss the concept of differential equations and the method of solving a first order differential equation. In the next section, we will discuss various applications of differential equations.

Basic Terminology

Variable: Variable is that quantity which takes on different quantitative values. Example: memory test scores, height of individuals, yield of rice etc.

Dependent Variable: A variable that depends on the other variable is called a dependent variable. For instance, if the demand of gold depends on its price, then demand of gold is a dependent variable.

Independent Variable: Variables which takes on values independently are called independent variables. In the above example, price is an independent variable.

Derivative: Let y = f(x) be a function. Then the derivative $\frac{dy}{dx} = f'(x)$ of the function f is the rate at which the function y = f(x) is changing with respect to the independent variable.

Differential Equation: An equation which relates an independent variable, dependent variable and one or more of its derivatives with respect to independent variable is called a differential equation.

Ordinary differential equation: A differential equation in which the dependent variable (unknown function) depends only on a single independent variable is called an ordinary differential equation.

Partial Differential equation: A differential equation in which the dependent variable is a function of two or more independent variables is called a partial differential equation.

Order of a differential equation: The order of a differential equation is defined as the order of the highest order derivative appearing in the differential equation. The order of a differential equation is a positive integer.

First order differential equation

A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is called a differential equation of first order. If initial condition $y(x_0) = y_0$ is also specified , then it is called an initial value problem.

Degree of a differential equation: The exponent of the highest order derivative appearing in the differential equation, when all derivatives are made free from radicals and fractions, is called degree of the differential equation. In other words, it is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in derivatives.

Differential Equations and Mathematical Models

In this section, we illustrate the use of differential equations in science and engineering and in coordinate geometry through the following examples.

Application in coordinate geometry

Example: In the following problems, a function y = h(x) is described by some geometric property of its graph. Write a differential equation of the form $\frac{dy}{dx} = f(x, y)$ having the function h as its solution.

(a) Every straight line normal to the graph of h passes through the point (0,1).

(b) The line tangent to the graph of h at (x,y) passes through the point (-y , x)

(c) The graph of h is normal to every curve of the form $y = x^2 + k$, k is a constant ,where they meet.

Solution: (a) Slope of tangent at the point (x,y) = $\frac{dy}{dx}$. Then slope of the normal = $\frac{-1}{dy/dx}$

Equation of straight line passing through the point (0,1) and slope $\frac{-1}{dy/dx}$ is

$$(y-1) = \frac{-1}{dy/dx}(x-0)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y-1}.$$

Thus, the equation of the normal passes through the point (0,1) is $\frac{dy}{dx} = \frac{-x}{y-1}$.

(b) Slope of tangent to the graph at (x, y) = dy/dx. Equation of tangent line with slope $\frac{dy}{dx}$ and passing through the point (-y , x) is

$$y - x = \frac{dy}{dx}(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x}$$

(c) Slope of the tangent = $m = \frac{dy}{dx}$

Slope of the normal to the curve $y = x^2 + k$ is $m' = \frac{d(x^2+k)}{dx} = 2x$

By condition of orthogonality, $mm' = -1 \quad \Rightarrow \frac{dy}{dx}.2x = -1 \quad \Rightarrow \frac{dy}{dx} = \frac{-1}{2x}$

Therefore, the required differential equation is $\frac{dy}{dx} = \frac{-1}{2x}$

Applications of Differential Equation in science and Engineering

Velocity: The rate of change of displacement with time is called velocity. It is given by dx/dt where x =x(t) gives the position of a moving particle at any time t .

Acceleration: The rate of change of velocity with time is called acceleration. It is given by dv/dt where v =v(t) gives the velocity of a moving particle at any time t .

Let the motion of a particle is given by the position function x = f(t)

Then velocity = $v(t)=\frac{dx}{dt}=f'(t)$ and acceleration = $a(t)=\frac{d^2x}{dt^2}=\frac{dv}{dt}$

By Newton's second law of motion,

F = ma

where F is the force, m is the mass of the particle, a is the acceleration.

Then, $F=m\frac{dv}{dt}$ or $\frac{dv}{dt}=\frac{\mathrm{F}}{\mathrm{m}}$ ....(1)

For instance, suppose that the force F, and therefore acceleration a = F/m are constant.

Then (1) gives $\frac{dv}{dt}=\mathrm{a}$.

Integrating both sides we get

$v=at+c$, where c is constant of integration. ....(2)

Let $v=v_0$ at $\mathrm{t}=0$. Then (2) gives $c=v_0$.

Put this value of c in (2) we get

$$v=at+v_0 .$$

This is the velocity function.

Now, put $v=\frac{dx}{dt}$ in it we get

$\frac{dx}{dt}=at+v_0 \quad \Rightarrow \mathrm{dx}=(at+v_0)dt$ ......(3)

Integrating (3) on both sides we get

$x(t)=\frac{1}{2}at^2+v_0t+k,$ *where* k is a constant of integration.

Put $x=x_0$ at $\mathrm{t}=0$ in the above equation we get k = x_0

Then, $x(t)=\frac{1}{2}at^2+v_0t+x_0$ is the position of the particle at any time t.

Example A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

Solution: We are given $x_0 = 400, v_0 = 0, a = -32\,ft/s^2$ (acceleration is negative because height is decreasing).When the ball strikes the ground, x = 0

We know that $x = \frac{1}{2}at^2 + v_0 t + x_0$

$$0 = \frac{1}{2}(-32)t^2 + 0 \times t + 400$$

$\Rightarrow t = \frac{400}{16} = 5$ sec. Therefore, it will take 5 seconds to reach the ground.

We have $v = v_0 + at$

$\Rightarrow v = 0 - 32 \times 5 = -160$ ft/s. Therefore, the ball will strike the ground with a velocity of 160 ft/s.

Example Find the velocity function v(t) and position function x(t) of a moving particle with the given acceleration a(t), initial position x_0 =x(0), and initial velocity v_0 = v(0) where $a(t) = 50, v_0 = 10, x_0 = 20$

Solution: We know that $a(t) = \frac{dv}{dt}$ ....(1)

Put a(t)=50 in (1) we get $\frac{dv}{dt} = 50$ ....(2)

We rewrite (2) as dv=50dt ....(3)

Integrating both sides of (3) we get

$$\int dv = 50\int dt$$

$v = 50t + c$ where c is a constant

Put $v_0 = 10$ i.e., v =10 at t = 0 we get

10 = 50(0)+ c or c = 10

Then v = 50 t +10 is the velocity function.

Also, $v = \frac{dx}{dt}$.

Put v = 50 t +10 in it we get

$$\frac{dx}{dt} = 50t + 10$$

$$\Rightarrow dx = (50t + 10)dt \qquad \ldots\ldots\ldots\ldots(4)$$

Integrating (4) on both sides, we get

$$\int dx = \int (50t + 10)dt$$

$$\Rightarrow x = 25t^2 + 10t + c \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Put $x_0 = 20$ i.e., x = 20 at t =0 in (5) we get c = 20.

Then (5) gives $x = 25t^2 + 10t + 20$ as the required position function.

Example Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $\frac{dv}{dt} = kv^2$. The initial speed of the motorboat is v(0)=10 m/s and v is decreasing at the rate of 1 m/s^2 when v = 5 m/s. How long does it take for the velocity of the boat to decrease to 1 m/s ? To 1/10 m/s? When does the boat come to a stop?

Solution: We are given that $\frac{dv}{dt} = kv^2 \ldots\ldots(1)$

$$\Rightarrow \frac{dv}{v^2} = kdt \qquad \ldots\ldots.(2)$$

Integrating both sides of (2) we get

$$\int \frac{dv}{v^2} = k\int dt$$

$\Rightarrow -\frac{1}{v} = kt + c$, where c is the constant of integration. ....(3)

Put v(0) = 10 i.e., v = 10 at t = 0 in (3), we get

$$\Rightarrow -\frac{1}{10} = k(0) + c \qquad \Rightarrow c = -\frac{1}{10}$$

Put this value of c in (3) , we get

$$-\frac{1}{v} = kt - \frac{1}{10} \qquad \ldots\ldots..(4)$$

Since v is decreasing at the rate of 1 m/s^2 when v = 5, it means

$$\frac{dv}{dt} = -1 \text{ when } v = 5.$$

Put these values in (1) we get $1 = k(5)^2 \quad \Rightarrow k = \frac{-1}{25}$

Put this value of k in (4) we get

$$-\frac{1}{v} = \frac{-1}{25}t - \frac{1}{10} \qquad \text{........(5)}$$

Now we find t when v = 1. For this put v=1 in (5) we get

$$-\frac{1}{1} = \frac{-1}{25}t - \frac{1}{10} \qquad \Rightarrow t = 22.5$$

Therefore, the motorboat will take 22.5 seconds for the velocity of the boat to decrease to 1 m/s.

Now put v = 1/10 in (5), we get

$$-\frac{1}{1/10} = \frac{-1}{25}t - \frac{1}{10} \qquad \Rightarrow t = 247.5$$

Therefore, the motorboat will take 247.5 seconds for the velocity of the boat to decrease to 1/10 m/s.

The boat comes to stop when v $\rightarrow$0.

It is clear from (5) that when v $\rightarrow$0 then $t \to \infty$. It means that v(t) approaches zero as t increases without bound.

Example Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

Solution: When the car skids 15m while moving at 50 km/h and the brakes are applied ,then $x(t) = \frac{15}{1000} km, x_0 = 0, v_0 = 50, v = 0, a = ?$

Now, $v = v_0 + at \quad \Rightarrow 0 = 50 + at \quad \Rightarrow at = -50$ ....(1)

Also, $x = \frac{1}{2}at^2 + v_0 t + x_0$

$$\Rightarrow \frac{15}{1000} = \frac{1}{2}(-50)t + 50t + 0$$

$$\Rightarrow t = 6\times 10^{-4}$$

Put the value of t in (1) we get $a = \frac{-50}{6\times10^{-4}} = -83333.3$

Now if the car is moving with a speed of 100 km/h when the brakes are applied then,

$$v = 0, v_0 = 100, a = -83333.3, x_0 = 0$$

Then, $v = v_0 + at \;\Rightarrow 0=100\text{-}83333.3t \quad \Rightarrow t=1.2\times10^{-3}$

Now, $x = \frac{1}{2}at^2 + v_0t + x_0$

$$\Rightarrow x = \frac{1}{2}\times(-83333.3)\times\left(1.2\times10^{-3}\right)^2 + 100\times\left(1.2\times10^{-3}\right)+0$$

$$\Rightarrow x = 0.061km = 61m.$$

Example A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

Solution: When a stone is dropped from rest at an initial height h above the surface of the earth, then $v_0 = 0, x_0 = 0, a = g, x = h, v = ?$

Now, $v = v_0 + at \;\Rightarrow v = 0+gt$ ....(1)

Also, $x = \frac{1}{2}at^2 + v_0t + x_0$

$$\Rightarrow h = \frac{1}{2}\times g\times t^2 + 0\times t + 0$$

$$\Rightarrow t^2 = \frac{2h}{g} \qquad \Rightarrow t = \sqrt{\frac{2h}{g}}$$

Put this value in (1) we get $v = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}$

Hence proved.

Growth and Decay

Natural Growth Equation: The differential equation $\frac{dx}{dt} = kx,\ x(t) > 0,\ k > 0$ is called a natural growth equation or exponential equation.

Natural Decay equation: The differential equation $\frac{dx}{dt} = kx,\ x(t) > 0,\ k < 0$ is called a natural decay equation.

Population growth: Let P(t) be the population having constant birth and death rates. Then the time rate of change of population P(t) is proportional to the size of the population. Then, we have

$$\frac{dP}{dt} = kP, \text{ where } k \text{ is a constant of proportionality.}$$

$$\Rightarrow \frac{dP}{P} = kdt \qquad \text{..........(1)}$$

Integrating (1) on both sides , we get

$$\int \frac{dP}{P} = k\int dt$$

$$\Rightarrow \log P = kt + c, \text{ where c is the constant of integration.} \qquad \text{.........(2)}$$

Let the population be P_0 initially. It means $P(0)=P_0$ i.e., $P=P_0$ at t = 0.

Put this value in (2) we get $\log P_0 = c$.

Then (2) gives $\log P = kt + \log P_0 \Rightarrow P = P_0 e^{kt}$. This is the population at any time t if the initial population is P_0.

Solution of a differential equation:

It is a relation between the variables involved in the differential equation which satisfies the differential equation. Such a relation when substituted in the differential equation with its derivatives, makes left hand side and right hand side identically equal.

Example1: $\frac{dy}{dx} = 2y$ is a differential equation which involves an independent variable x, dependent variable y, first derivative of y with respect to x. This equation involves the unknown function y of the independent variable x and first derivative $\frac{dy}{dx}$ of y w.r.t. x

Example2: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ is a differential equation which consists of an unknown function y of the independent variable x and the first two derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of y w.r.t. x.

Example3: In the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx} + 3\right)^4 = 0$, the order of the highest order derivative is 3, so it is a differential equation of order 3.

Example 4: In the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 - 6\left(\frac{d^2y}{dx^2}\right)^4 - 4y = 0$, the highest order derivative is $\frac{d^3y}{dx^3}$ and its exponent or power is 2. So, it is a differential equation of order 3 and degree 2.

Example 5: Consider the differential equation $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(c\frac{d^2y}{dx^2}\right)^{1/3}$. To express the differential equation as a polynomial in derivatives, we proceed as follows:

Squaring both sides, we get

$$1+\left(\frac{dy}{dx}\right)^2 = \left(c\frac{d^2y}{dx^2}\right)^{2/3}$$

Cubing both sides , we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(c\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^6 + 3\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right)^4 = c^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow c^2\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^6 - 3\left(\frac{dy}{dx}\right)^4 - 3\left(\frac{dy}{dx}\right)^2 - 1 = 0$$

Now, the highest order derivative appearing in the polynomial form of the given differential equation is $\frac{d^2y}{dx^2}$. Its exponent is 2. Therefore, degree of the given differential equation is 2. Infact, its order is also 2.

Example6: Consider a differential equation $y' + 2y = 0$ where $y' = \frac{dy}{dx}$. Then it can be easily verified that $y = 3e^{-2x}$ is the solution of the given differential equation by proceeding as follows.

Differentiating $y = 3e^{-2x}$ w.r.t. x, we get

$$y' = -6e^{-2x}$$

Substituting the values of $y \text{ and } y'$ in the L.H.S of the given differential equation, we get

$$L.H.S = y' + 2y = -6e^{-2x} + 2\left(3e^{-2x}\right) = -6e^{-2x} + 6e^{-2x} = 0 = R.H.S$$

$\therefore y = 3e^{-2x}$ satisfy the given differential equation and thus is a solution of it.

Example7: Consider a differential equation $y'' + y = 3\cos 2x$. Then $y = \cos x - \cos 2x$ is the solution of this differential equation. It can be seen as follows.

We have

$$y = \cos x - \cos 2x \qquad ....(1)$$

Differentiating (1) w.r.t. x on both sides , we get

$$y' = -\sin x + 2\sin 2x \qquad ....(2)$$

Differentiating (2) w.r.t. x we get

$$y'' = -\cos x + 4\cos 2x$$

Substituting the values of y and y'' in the L.H.S of the given differential equation $y'' + y = 3\cos 2x$, we get

L.H.S = -cosx +4 cos2x+cosx –cos2x

= 3 cos2x = R.H.S

Therefore, $y = \cos x - \cos 2x$ is the solution of the given differential equation.

One more thing to be noted here is that y = sinx - cos2x is also a solution of the given differential equation.

It can be seen as follows. We have

$y = \sin x - \cos 2x$ ....(3)

Differentiating (3) w.r.t. x , we get

$y' = \cos x + 2\sin 2x$ ....(4)

Differentiating (4) w.r.t x , we get

$y'' = -\sin x + 4\cos 2x$

Substituting the values of y and y'' in the L.H.S of the given differential equation $y'' + y = 3\cos 2x$, we get

L.H.S = -sinx + 4 cos2x +sinx – cos2x

= 3cos2x = R.H.S

Therefore, y = sinx - cos2x is also a solution of the given differential equation.

Example8: Substitute $y = e^{rx}$ in to the following differential equation to determine all values of the constant *r* for which $y = e^{rx}$ is the solution of the equation $3y'' + 3y' - 4y = 0$.

Solution: Consider $3y'' + 3y' - 4y = 0$ ....(1)

We have

$y = e^{rx}$ ....(2)

Differentiating (2) w.r.t. x, we get

$y' = re^{rx}$

Again differentiating w.r.t. x, we get

$$y'' = r^2 e^{rx}$$

Substituting the values of y, y' and y'' in the given differential equation, we get

$$3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0$$

$$\Rightarrow e^{rx}\left(3r^2 + 3r - 4\right) = 0$$

$$\Rightarrow 3r^2 + 3r - 4 = 0 \quad \because e^{rx} \neq 0 \text{ forany real value of } r$$

$$\Rightarrow r = \frac{-3 \pm \sqrt{57}}{6}$$

Therefore, e^{rx} is the solution of (1) for $r = \frac{-3 \pm \sqrt{57}}{6}$.

Example 9: If k is a constant , show that a general (1–parameter)solution of the differential equation $\frac{dx}{dt} = kx^2$ is given by $x(t) = \frac{1}{C - kt}$ where C is an arbitrary constant.

Solution: We have $x(t) = \frac{1}{C - kt}$ ....(1)

Differentiating both sides of (1) we get

$$\frac{dx}{dt} = \frac{k}{(C - kt)^2}$$

Put this value in the L.H.S of $\frac{dx}{dt} = kx^2$, we get

$$L.H.S = \frac{dx}{dt} = \frac{k}{(C - kt)^2} = kx^2 = R.H.S.$$

Therefore, $x(t) = \frac{1}{C - kt}$ is the solution of the differential equation.

Actually, $x(t) = \frac{1}{C - kt}$ defines a one parameter family of solution of $\frac{dx}{dt} = kx^2$, one for each value of the arbitrary constant or parameter C.

Integrals as General, Particular and Singular Solutions

General solution: A solution which contains as many arbitrary constants as the order of the differential equation is called a general solution of the differential equation.

Particular solution: A solution obtained by giving particular values to the arbitrary constants in the general solution of the differential equation is called a particular solution.

Singular Solution: A solution which cannot be obtained from the general solution by any choice of the arbitrary constants is called a singular solution.

Example 10: Consider a differential equation $\frac{dy}{dx} = 2\sqrt{y} \quad .....(1)$.

We can rewrite (1) as $\frac{dy}{\sqrt{y}} = 2dx \quad .....(2)$

Integrating both sides of (2), we get

$y = (x+c)^2$, c is a constant of integration. ....(3)

This solution contains one arbitrary constant c .This is the general solution as it contains only one arbitrary constant which is same as the order of the given differential equation.

If we put the initial condition y(0)=0 , i.e, y = 0 at x=0 in (3) then we get c = 0. In such a case $y = x^2$ is a particular solution.

Evidently, y = 0 is also a solution of (1) but it cannot be obtained from (3) by any choice of c. Thus the function y = 0 is a singular solution of (1).

Example 11: Solve the initial value problem $\frac{dy}{dx} = x\sqrt{x^2+9}, y(-4) = 0$.

Solution: We have $\frac{dy}{dx} = x\sqrt{x^2+9} \quad ....(1)$

$\Rightarrow dy = x\sqrt{x^2+9}dx \quad ....(2)$

Integrating both sides of (2) we get

$$\int dy = \int x\sqrt{x^2+9}dx \quad ....(3)$$

Let $I = \int x\sqrt{x^2+9}dx$

Put $x^2+9=t \Rightarrow$ 2xdx=dt

Then $I=\frac{1}{2}\int\sqrt{t}dt=\frac{1}{2}t^{3/2}.\frac{2}{3}+c=\frac{1}{3}t^{3/2}+c=\frac{1}{3}\left(x^2+9\right)^{3/2}+c$, where c is a constant.

Now, from (3) we get $y=\frac{1}{3}\left(x^2+9\right)^{3/2}+c$ ....(4)

Put y(-4)=0 i.e. x= -4 and y = 0 in (4) we get

$$0=\frac{1}{3}(16+9)^{3/2}+c \quad \Rightarrow c=\frac{-125}{3}.$$

Put this value of c in (4) we get

$$y=\frac{1}{3}\left(x^2+9\right)^{3/2}-\frac{125}{3}.$$

It is the required solution.

Example 12: Find the general solutions of the following differential equations.

(a) $(1-x^2)\frac{dy}{dx}=2y$

(b) $\frac{dy}{dx}=\frac{(x-1)y^5}{x^2\left(2y^3-y\right)}$

(c) $x^2\frac{dy}{dx}=1-x^2+y^2-x^2y^2$

Solution: (a) We have $(1-x^2)\frac{dy}{dx}=2y$

$$\Rightarrow\frac{dy}{2y}=\frac{dx}{(1-x^2)} \quad ....(1)$$

Integrating (1) on both sides, we get

$$\int\frac{dy}{2y}=\int\frac{dx}{(1-x^2)}$$

$$\Rightarrow\frac{1}{2}\log y=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)+\log c$$

$\Rightarrow y=c\left(\frac{1+x}{1-x}\right)$ is the general solution.

(b)We have $\frac{dy}{dx}=\frac{(x-1)y^5}{x^2(2y^3-y)}$

$$\Rightarrow \frac{(2y^3-y)dy}{y^5}=\frac{(x-1)dx}{x^2}.$$

$$\Rightarrow \left(\frac{2}{y^2}-\frac{1}{y^4}\right)dy=\left(\frac{1}{x}-\frac{1}{x^2}\right)dx \quad ....(1)$$

Integrating (1) on both sides we get

$$\int\left(\frac{2}{y^2}-\frac{1}{y^4}\right)dy=\int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$\Rightarrow \frac{-2}{y}+\frac{1}{3y^3}=\log|x|+\frac{1}{x}+c$, where c is constant of integration.

(c) We have $x^2\frac{dy}{dx}=1-x^2+y^2-x^2y^2$

$$\Rightarrow x^2\frac{dy}{dx}=(1-x^2)+y^2(1-x^2)$$

$$\Rightarrow x^2\frac{dy}{dx}=(1-x^2)(1+y^2)$$

$$\Rightarrow \frac{dy}{1+y^2}=\frac{(1-x^2)}{x^2}dx$$

$$\Rightarrow \frac{dy}{1+y^2}=\left(\frac{1}{x^2}-1\right)dx$$

Integrating both sides, we get

$$\tan^{-1}y=-\frac{1}{x}-x+c$$

$\Rightarrow y=\tan\left(-\frac{1}{x}-x+c\right)$, where c is a constant of integration.

Example 13: Find the explicit particular solution of the initial value problem $2y\frac{dy}{dx}=\frac{x}{\sqrt{x^2-16}}, y(5)=2$.

Solution: We have $2y\frac{dy}{dx}=\frac{x}{\sqrt{x^2-16}}$

$$\Rightarrow 2ydy=\frac{x}{\sqrt{x^2-16}}dx$$

Integrating on both sides we get

$$\int 2ydy=\int\frac{x}{\sqrt{x^2-16}}dx \qquad \text{...........(1)}$$

Let I = $\int\frac{x}{\sqrt{x^2-16}}dx$

Put $x^2-16=t$ $\Rightarrow$ 2xdx=dt

$$I=\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\sqrt{t}+c=\sqrt{x^2-16}+c$$

From (1) we get,

$$y^2=\sqrt{x^2-16}+c \qquad \text{.............(2)}$$

Put y(5)=2 in (2) i.e., y = 2 at x =5 .

$$4=\sqrt{(5)^2-16}+c \quad \Rightarrow c=1$$

$$y^2=\sqrt{x^2-16}+1$$

POSSIBLE QUESTIONS

PART - B (5 x 2 = 10 Marks)

1. Define Differential equation with example.
2. Define Partial Differential equation with example.
3. Expalin linear differential equation.
4. Explain singular solutions of the differential equation.
5. Explain the order of the differential equation with example

PART – C (5 x 6 = 30 Marks)

1. Show that $5x^2y^2 - 2x^3y^2 = 1$ is an implicit solution of the differential equation $x\frac{dy}{dx} + y = x^3y^3$ on the interval
0< x < 5/2.
2. Write the definition of general, particular, explicit, implicit and singular solutions of Differential equations.
3. Show that every function f defined by $f(x) = (x^3 + c)e^{-3x}$ where c is arbitrary equation is a solution of the
Differential equation $\frac{dy}{dx} + 3y = 3\,x^2e^{-3x}$.
4. Show that the function f defined by f(x)=$3e^{2x} - 2xe^{2x} - cos2x$ satisfies the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + \quad 4y = -8\sin 2x$ and also the condition that f(0)=2 and f'(0)=4
5. Write a note on solution of differential equations.
6. Show that the function for all x by f(x)= 2 sin x +3cos x is an explicit solution of the
7. Differential equation $\frac{d^2y}{dx^2} + y = 0$ for all real x.
8. Show that the function defined by f(x) = x + 3e^{-x} is a solution of differential equation $\frac{dy}{dx}$ + y = x + 1 on every interval a < x < b of the x-axis.
9. Briefly explain linear and nonlinear differential equations with examples.
10. Find the general solutions of the differential equations (1- x^2)$\frac{dy}{dx} = 2y$.
11. Show that $x^3 + 3xy^2$ is an implicit solution of the differential equation 2xy ($\frac{dy}{dx}$) + x^2 +y^2=0 on the interval 0 < x < 1.

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Subject: Differential Equations **Subject Code: 17MMU201**

Class : I B.Sc Mathematics **Semester : II**

UNIT -I

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
An equation involving one or more dependent variables with respect to one or more independent variables is called……………………...	differential equations	intergral equation	constant equation	Eulers equation	differential equations
An equation involving one or more ……….. variables with respect to one or more independent variables is called differential equations	single	dependent	independent	constant	dependent
An equation involving one or more dependent variables with respect to one or more………..variables is called differential equations	dependent	independent	single	different	independent
A differential equation involving ordinary derivatives of one or moredependentvariables with respect to single independent variables is called ……………	differential equations	partial differential equations	ordinary differential equations	total differential equations	ordinary differential equations
A differential equation involving ordinary derivatives of one or more dependentvariables with respect to………. independent variables is called ordinary differential equations	zero	single	different	one or more	single
A differential equation involving ………… derivatives of one or more dependentvariables with respect to single independent variables is called ordinary differential equations	partial	different	total	ordinary	ordinary

A differential equation involving partial derivatives of one or more dependent variables with respect to oneor more independent variables is called ……………	differential equations	partial differential equations	ordinary differential equations	total differential equations	partial differential equations
A differential equation involving partial derivatives of one or more dependentvariables with respect to………. independent variables is called partial differential equations	zero	single	different	one or more	oneormore
A differential equation involving ………… derivatives of one or more dependentvariables with respect to one or moreindependent variables is called partial differential equations	partial	different	total	ordinary	partial
The order of ……...derivatives involvedin the differential equations is called order of the differential equation	zero	lowest	highest	infinite	highest
The order of highest derivatives involvedin the differential equations is called ……………... of the differential equation	order	power	value	root	order
The order of highest ………………... involvedin the differential equations is called order of the differential equation	derivatives	intergral	power	value	derivatives
The order of the differential equations is (d^2 y)/ dx ^2 +xy(dy/dx)^2=1	0	1	2	4	2
A non linear ordinary differential equation is an ordinary differential equation that is not ………..	linear	non linear	differential	intergral	linear
A ……………..ordinary differential equation is an ordinary differential equation that is not linear	linear	non linear	differential	intergral	non linear
A non linear ordinary differential equation is an ……………. differential equation that is not linear	ordinary	partial	single	constant	ordinary
………….. ordinary differential equations are further classified according to the nature of the coefficients of the dependent variables and its derivatives	linear	non linear	differential	intergral	linear
Linear ………………. differential equations are further classified according to the nature of the coefficients of the dependent variables and its derivatives	ordinary	partial	single	constant	ordinary

Linear ordinary differential equations are further classified according to the nature of the coefficients of the ……………….variables and its derivatives	single	dependent	independent	constant	dependent
Linear ordinary differential equations are further classified according to the nature of the coefficients of the dependent variables and its ……………………..	integrals	constant	derivatives	roots	derivatives
Both explicit and implicit solutions will usually be called simply ……………	solutions	constant	equations	values	solutions
Both ……………….. solutions will usually be called simply solutions.	general and particular	singular and non singular	ordinary and partial	explicit and implicit	explicit and implicit
Let f be a real function defined for all x in a real interval I and having nth order derivatives then the function f is called …………solution of the differential equations	constant	implicit	explicit	general	explicit
Let f be a real function defined for all x in a real interval I and having …………..order derivatives then the function f is called explicit solution of the differential equations	1st	2nd	nth	(n+1)th	nth
The relation g(x,y)=0 is called the ………solution of F[x,y,(dy/dx)………..(dy/dx)^n]=0	constant	implicit	explicit	general	implicit

UNIT – II

SYLLABUS

Exact differential equations and integrating factors, separable equations and equations reducible to this form, linear equation and Bernoulli equations, special integrating factors and transformations.

2.1 Separable Variables

Definition 2.1: A first order differential equation of the form

$$\frac{dx}{dy} = g(x)h(y), \text{ where } g(x) \text{ and } h(y) \text{ are functions of } x \,\&\, y \text{ only, respectively.}$$

is called **separable** or to have **separable variables**.

Method or Procedure for solving separable differential equations

(i) If h(y) = 1, then

$$\frac{dy}{dx} = g(x)$$

or dy = g(x) dx

Integrating both sides we get

$$\int dy = \int g(x)d(x) + c$$

or

$$y = \int g(x)d(x) + c$$

where c is the constant of integral

We can write

$$y = G(x) + c$$

where *G(x)* is an anti-derivative (indefinite integral) of *g(x)*

(ii) Let $\frac{dy}{dx} = f(x, y)$

where $f(x, y) = g(x)h(y)$,

that is f(x,y) can be written as the product of two functions, one function of variable x and other of y. Equation

$$\frac{dy}{dx} = g(x)h(y)$$

can be written as

$$\frac{1}{h(y)}dy = g(x)dx$$

By integrating both sides we get

$$\int p(y)dy = \int g(x)dx + C$$

where $p(y) = \frac{1}{h(y)}$

or $H(y) = G(x) + C$

where *H(y)* and *G(x)* are anti-derivatives of $p(y) = \frac{1}{h(y)}$ and $g(x)$, respectively.

Example 2.1: Solve the differential equation

$$y' = y/x$$

Solution: Here $g(x) = \frac{1}{x}$, $h(y) = y$ and $p(y) = \frac{1}{y}$

$H(y) = \ln y$, $G(x) = \ln x$

Hence

H(y) = G(x) + C

or lny = lnx + lnc **(See Appendix)**

lny – lnx = lnc

$$\ln\frac{y}{x} = \ln c$$

$$\frac{y}{x} = c$$

$y = cx$

Example 2.2: Solve the initial-value problem

$$\frac{dy}{dx} = -\frac{x}{y}, y(4) = 3$$

Solution: $g(x) = x$, $h(y) = -1/y$, $p(y) = -y$

$H(y) = G(x) + c$

$$-\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

or $y^2 = -x^2 - 2c$

or $x^2 + y^2 = c_1^2$

where $c_1^2 = -2c$

By given initial-value condition

$16+9 = c_1^2$

or $c_1 = \pm 5$

or $x^2 + y^2 = 25$

Thus the initial value problem determines

$$x^2 + y^2 = 25$$

Example 2.3: Solve the following differential equation

$$\frac{dy}{dx} = \cos 5x$$

Solution: $dy = \cos 5x dx$

Integrating both sides we get

$$\int dy = \int \cos 5x dx + c$$

$$y = \frac{\sin 5x}{5} + c$$

2.2 Exact Differential Equations

We consider here a special kind of non-separable differential equation called an **exact differential equation.** We recall that the **total differential** of a function of two variables U(x,y) is given by

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \qquad (2.1)$$

Definition 2.2.1 : The first order differential equation

M(x,y)dx + N(x,y)dy=0 *(2.2)*

is called an **exact differential equation** if left hand side of (2.2) is the total differential of some function U(x,y).

Remark 2.2.1: (a) It is clear that a differential equation of the form (2.2) is exact if there is a function of two variables U(x,y) such that

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = M(x,y)dx + N(x,y)dy$$

or $$\frac{\partial U}{\partial x} = M(x,y), \qquad \frac{\partial U}{\partial y} = N(x,y)$$

(b) Let M(x,y) and N(x,y) be continuous and have continuous first derivatives in a rectangular region R defined by a<x<b, c<y<d. Then a necessary and sufficient condition that M(x,y)dx + N(x,y)dy be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (2.3)$$

For proof of Remark 2.2.1(a) see solution of Exercise 22 of this chapter.

Procedure of Solution 2.2:

Step 1: Check whether differential equation written in the form (2.2) satisfies (2.3) or not.

Step 2: If for given equation (2.3) is satisfied then there exists a function *f* for which

$$\frac{\partial f}{\partial x} = M(x, y) \qquad (2.4)$$

Integrating (2.4) with respect to x, while holding y constant, we get

$$f(x, y) = \int M(x, y)dx + g(y) \qquad (2.5)$$

where the arbitrary function g(y) is constant of integration.

Step 3: Differentiate (2.5) with respect to y and assume $\frac{\partial f}{\partial y}$ =N(x,y), we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)dx + g'(y) = N(x, y)$$

or

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx \qquad (2.6)$$

Step 4: Integrate (2.6) with respect to y and substitute this value in (2.5) to obtain f(x,y)=c, the solution of the given equation.

Remark 2.2.2: (a) Right hand side of (2.6) is independent of variable x, because

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x, y)dx\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

(b) We could just start the above mentioned procedure with the assumption that

$$\frac{\partial f}{\partial y} = N(x, y)$$

By integrating N(x,y) with respect to y and differentiating the resultant expression, we would find the analogues of (2.5) and (2.6) to be, respectively,

$$f(x,y)=\int N(x,y)dy+h(x) \quad \text{and}$$

$$h'(x)=M(x,y)-\frac{\partial}{\partial x}\int N(x,y)dy$$

Example 2.4: Check whether $x^2y^3dx + x^3y^2dy = 0$ is exact or not?

Solution: In view of Remark 2.2.1(b) we must check whether $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, where M(x,y)= x^2y^3, N(x,y)=x^3y^2

$$\frac{\partial M}{\partial y}=3x^2y^2, \quad \frac{\partial N}{\partial x}=3x^2y^2$$

This shows that $\frac{\partial M}{\partial y}=3x^2y^2=\frac{\partial N}{\partial x}$

Hence the given equation is exact.

Example 2.5: Determine whether the following differential equations are exact. If they are exact solve them by the procedure given in this section.

(a) $(2x-1)dx + (3y+7)dy=0$

(b) $(2x+y)dx - (x+6y)dy=0$

(c) $(3x^2y+e^y)dx + (x^3+xe^y-2y)dy=0$

Solution of (a) M(x,y) = 2x-1, N(x,y)=3y+7

$$\frac{\partial M}{\partial y}=0, \qquad \frac{\partial N}{\partial x}=0. \quad \text{Thus}$$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ and so the given equation is exact.

Apply procedure of solution 2.2 for finding the solution.

Put $\frac{\partial f}{\partial x}=2x-1.$ Integrating and choosing h(y) as the constant of integration we get

$$\int\frac{\partial f}{\partial x}=f(x,y)=x^2-x+h(y)$$

$h'(y) = N(x, y) = 3y + 7,$ and by integrating with respect to y we obtain

$$h(y) = \frac{3}{2}y^2 + 7y$$

The solution is

$$f(x, y) = x^2 - x + \frac{3}{2}y^2 + 7y = c$$

Solution of (b): It is not exact as

$$M(x, y) = 2x + y, N(x, y) = -x - 6y$$

and $\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1$

Solution of (c): $M(x,y) = 3x^2y + e^y$

$$N(x,y) = x^3 + xe^y - 2y$$

$$\frac{\partial M}{\partial y} = 3x^2 + e^y$$

$$\frac{\partial N}{\partial x} = 3x^2 + e^y$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, that is

the equation is exact.

Apply procedure of solution 2.2

Let $\frac{\partial f}{\partial x} = 3x^2y + e^y$

Integrating with respect to x, we obtain

$$f(x, y) = x^3y + xe^y + g(y)$$

where g(y) is a constant of integration

Differentiating with respect to y we obtain

$$\frac{\partial f}{\partial y} = x^3 + xe^y + g'(y)$$

This gives

$$N(x, y) = \frac{\partial f}{\partial y} = x^3 + xe^y + g'(y)$$

or $g'(y) = -2y$

or $g(y) = -y^2$

Substituting this value of g(y) we get

$f(x,y) = x^3y + xe^y - y^2 = c$. Thus

$x^3y + xe^y - y^2 = c$ is the solution of the given differential equation.

2.2.1 Equations Reducible to Exact Form

There are non-exact differential equations of first-order which can be made into exact differential equations by multiplication with an expression called an integrating factor. Finding an integrating factor for a non-exact equation is equivalent to solving it since we can find the solution by the method described in Section 2.2. There is no general rule for finding integrating factors for non-exact equations. We mention here two important cases for finding integrating factors. It may be seen from examples given below that integrating factors are not unique in general.

Computation of Integrating Factor

Let $M(x.y)dx+N(x,y)dy=0$

be a non-exact equation.

Then

(i) $$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

is an integrating factor, where M_y, N_x are partial derivatives of M and N with respect to y and x and $\frac{M_y - N_x}{N}$ is a function of x alone.

(ii) $$\mu(x) = e^{\int \frac{N_x - M_y}{M} dy}$$

is an integrating factor, where M_y and N_x are as in the case (i) and $\frac{N_x - M_y}{M}$ is a function of y alone.

Example 2.6: (a) Let us consider non-exact differential equation.

$(x^2/y)\, dy+2xdx=0$

$\frac{1}{x^2}$ and y are integrating factors of this equation.

(b) e^x is an integrating factor of the equation

$$\frac{dy}{dx}+y=x$$

Example 2.7: Solve the differential equation of the first-order:

$xydx+(2x^2+3y^2-20)dy=0$

Solution: $M(x,y)=xy$, $N(x,y)=2x^2+3y^2-20$

$M_y=x$ and $N_x=4x$. This shows that the differential equation is not exact.

$$\frac{M_y - N_x}{N} = \frac{-3x}{2x^2+3y^2-20}$$

leads us nowhere, as $\frac{M_y - N_x}{N}$ is a function of both x and y. However,

$\frac{N_x - M_y}{M} = \frac{4x-x}{xy} = \frac{3}{y}$ is a function of y only. Hence

$e^{\int 3\frac{dy}{y}} = e^{3\ln y} = e^{\ln y^3} = y^3$ is an integrating factor.

After multiplying the given differential equation by y^3 we obtain

$xy^4dx+(2x^2y^3+3y^5-20y^3)dy=0$

This is an exact differentiation equation. Applying the method of the previous section we get

$$\frac{1}{2}x^2y^4 + \frac{3}{6}y^6 - 5y^4 = C$$

Example 2.8: Solve the following differential equation:

$(2y^2+3x)dx+2xydy=0$

Solution: The given differential equation is not exact, that is

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, where

$M(x,y)=2y^2+3x$

$N(x,y)=2xy$

$(M_y-N_x)/N = 1/x$ is a function of x only.

Hence $e^{\int dx/x} = x$ is an integrating factor.

By multiplying the given equation by x we get $(2y^2x+3x^2)dx+2x^2ydy=0$

This is an exact equation as

$$\frac{\partial}{\partial y}(2y^2x+3x^2) = \frac{\partial}{\partial x}(2x^2y)$$

Applying the method for solving exact differential equation, we get $f=x^2y^2+x^3+h(y)$, $h'(y)=0$, and $h(y)=c$ if we put $f_x=2xy^2+3x^2$. The solution of the differential equation is $x^2y^2+x^3=c$.

2.3 Linear Equations

Definition 2.3.1: A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is called a **linear equation.**

if $a_1(x) \neq 0$, we can write this differential equation in the form

$$\frac{dy}{dx} + P(x)y = f(x) \qquad (2.7),$$

where $P(x) = \dfrac{a_0(x)}{a_1(x)}$, $f(x) = \dfrac{g(x)}{a_1(x)}$

(2.7) is called the standard form of a linear differential equation of the first order

Definition 2.3.2: $e^{\int P(x)dx}$ is called the integrating factor of the standard form of a linear differential equation (2.7).

Remark 2.3.1: (a) A linear differential equation of first order can be made exact by multiplying with the integrating factor. Finding the integrating factor is equivalent to solving the equation.

(b) Variation of parameters method is a procedure for finding a particular solution of 2.7. For details of **variation of parameters method** see the solution of Exercise 39 of this chapter.

Procedure of Solution 2.3:

Step 1: Put the equation in the standard form (2.7) if it is not given in this form.

Step 2: Identify P(x) and compute the integrating factor I(x) = $e^{\int P(x)dx}$

Step 3: Multiply the standard form by I(x).

Step 4: The solution is

$$y.I(x) = \int f(x).I(x)dx + c$$

Example 2.9: Find the general solution of the following differential equations:

(a) $\dfrac{dy}{dx} = 8y$ (b) $x\dfrac{dy}{dx} + 2y = 3$

(c) $x\dfrac{dy}{dx} + (3x+1)y = e^{-3x}$

Solution: (a) $\dfrac{dy}{dx} - 8y = 0$

P(x)= – 8

Integrating function = $I(x) = e^{\int -8dx} = e^{-8x}$

$$y.e^{-8x} = \int 0.e^{-8x}dx + c$$

or $y = ce^{8x}$, $-\infty<x<\infty$

(b) $$\frac{dy}{dx} + \frac{2}{x}y = \frac{3}{x}$$

Integrating factor = $I(x) = e^{\int P(x)dx}$, where

$$P(x) = \frac{2}{x}$$

$$I(x) = e^{\int \frac{2}{x}dx} = x^2$$

Solution is given by

$$y.I(x) = \int f(x).I(x)dx + c$$

where $I(x) = x^2$, $f(x) = \frac{3}{x}$

Thus

$$yx^2 = \int \frac{3}{x}.x^2dx + c$$

$$= \int 3xdx + c = \frac{3}{2}x^2 + c$$

or $y = \frac{3}{2} + \frac{c}{x^2}$, $0<x<\infty$

(c) Standard form is

$$\frac{dy}{dx} = \left(3 + \frac{1}{x}\right)y = \frac{e^{-3x}}{x}$$

$$P(x) = 3 + \frac{1}{x}, \quad f(x) = \frac{e^{-3x}}{x}$$

Integrating factor = $I(x) = e^{\int P(x)dx}$

$= e^{\int\left(3+\frac{1}{x}\right)dx} = xe^{3x}$

$$y.xe^{3x} = \int \frac{e^{-3x}}{x}.xe^{3x}dx + c$$

$$= \int e^{0}dx + c = x + c$$

or $y = e^{-3x} + \frac{c}{x}e^{-3x}$ for $0<x<\infty$.

2.4 Solutions by Substitutions

A first-order differential equation can be changed into a separable differential equation (Definition 2.1) or into a linear differential equation of standard form (Equation (2.7)) by appropriate substitution. We discuss here two classes of differential equations, one class comprises homogeneous equations and the other class consists of Bernoulli's equation.

2.4.1 Homogenous Equations

A function f(.,.) of two variables is called homogeneous function of degree α if

$f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α.

A first order differential equation, M(x,y)dx + N(x,y)dy = 0 is called **homogenous** if both coefficients M(x,y) and N(x,y) are homogenous functions of the same degree.

Method of Solution for Homogenous Equations: A homogeneous differential equation can be solved by either substituting y=ux or x=vy, where u and v are new dependent variables. This substitution will reduce the equation to a separable first-order differential equation.

Example 2.10: Solve the following homogenous equations:

(a) (x-y)dx + xdy = 0

(b) $(y^2+yx)dx + x^2dy = 0$

Solution: (a) Let y=ux, then the given equation takes the form

(x-ux)dx + x(udx + xdu) = 0

or $dx + xdu = 0$

or $\frac{dx}{x} + du = 0$

or $\ln|x| + u = c$

or $x\ln|x| + y = cx$

(b) Let $y=ux$, then the given equation takes the form

$(u^2x^2 + ux^2)dx + x^2(udx + xdu) = 0$

or $(u^2 + 2u)dx + xdu = 0$

or $\frac{dx}{x} + \frac{du}{u(u+2)} = 0$

Solving this separable differential equation, we get

$$\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| = c$$

or $\frac{x^2u}{u+2} = c_1$ where $c_1=2c$

or $x^2\frac{y}{x} = c_1\left(\frac{y}{x} + 2\right)$

or $x^2y = c_1(y + 2x)$

2.4.2 Bernoulli's Equation

An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{2.8}$$

is called a **Bernoulli's differential equation**. If $n\neq 0$ or 1, then the Bernoulli's equation (2.8) can be reduced to a linear equation of first-order by the substitution.

$$v = y^{1-n}$$

The linear equation can be solved by the method described in the previous section.

Example 2.11: Solve the following differential equations:

(a) $\frac{dy}{dx}+\frac{1}{x}y=3y^3$

(b) $\frac{dy}{dx}-y=e^x y^2$

Solution: (a) Let $v=y^{1-n}=y^{-2}$ (n=3)

$$\frac{dv}{dx}=-2y^{-3}\frac{dy}{dx}$$

or $$\frac{dy}{dx}.\frac{1}{y^3}=-\frac{1}{2}\frac{dv}{dx}$$

Substituting these values into the given differential equation, we get

$$-\frac{1}{2}\frac{dv}{dx}+\frac{1}{x}v=3$$

or $$\frac{dv}{dx}-\frac{2}{x}v=-6$$

This equation is of the standard form, (2.7) and so the method of Section 2.3 is applicable.

Integrating factor $I(x)=e^{\int P(x)dx}$

where $P(x)=-\frac{2}{x}$. Therefore I (x) $=x^{-2}$

Solution is given by

$$v.x^{-2}=\int -6x^{-2}\,dx+c$$

or

$v.x^{-2}=6x^{-1}+c$

or

$v = 6x + cx^2$

Since

$v = y^{-2}$ we get

$y^{-2} = 6x + cx^2$

or $$y = \pm \frac{1}{\sqrt{6x + cx^2}}$$

(b) Let $w = y^{-1}$, then the equation

$$\frac{dy}{dx} - y = e^x y^2$$

takes the form

$$\frac{dw}{dx} + w = -e^x$$

integrating factor $I(x) = = e^{\int P(x)dx}$, where $P(x) = 1$

or $I(x) = e^{\int P(x)dx} = e^x$

Solution is given by

$e^x.w = - \int e^{2x}dx + c$

$$= -\frac{1}{2}e^{2x} + c$$

$$or \qquad e^x \frac{1}{y} = -\frac{1}{2}e^{2x} + c$$

$$or \qquad y^{-1} = -\frac{1}{2}e^x + ce^{-x}$$

Special Integrating Factors

Given the O.D.E. $M(x,y)\,dx + N(x,y)\,dy = 0$, assume it is non-exact. Suppose that $n(x,y)$ is an integrating factor of the equation, then

$$n(x,y)\,M(x,y)\,dx + n(x,y)\,N(x,y)\,dy = 0$$

is an exact equation.
Therefore,

$$\frac{\partial}{\partial y}[n(x,y)M(x,y)] = \frac{\partial}{\partial x}[n(x,y)N(x,y)]$$

or

$$n(x,y)\frac{\partial M(x,y)}{\partial y} + \frac{\partial n(x,y)}{\partial y}M(x,y) = n(x,y)\frac{\partial N(x,y)}{\partial x} + \frac{\partial n(x,y)}{\partial x}N(x,y)$$

or

$$n(x,y)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = N\frac{\partial n}{\partial x} - M\frac{\partial n}{\partial y} \quad (1)$$

$n(x,y)$ is an unknown function that satisfies equation (1), but equation (1) is a partial differential equation. So, in order to find $n(x,y)$ we have to solve a P.D.E. and we do not know how to do it.
Therefore, we have to impose some restriction on $n(x,y)$.
Assume that n is function of only one variable, let's say of the variable x,

then $n(x)$ and $\frac{\partial n}{\partial y} = 0,\ \frac{\partial n}{\partial x} = \frac{dn}{dx}$

So, equation (1) reduces to

$$n(x)\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = N(x,y)\frac{dn}{dx}$$

or

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]dx = \frac{dn}{n}$$

If the left hand side of the above equation is only function of x, then the equation is separable and $n(x) = \exp\left(\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx\right)$.

Conclusion: The equation $M(x,y)\,dx + N(x,y)\,dy = 0$ has an integrating factor $n(x)$ that depends only on x if the expression $\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$ depends only on x.

Now, let's assume the n depends only on the variable y,

then n(y) and $\frac{\partial n}{\partial x} = 0$, $\frac{\partial n}{\partial y} = \frac{dn}{dy}$

So, equation (1) reduces to

$$n(y)\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = -M(x,y)\frac{dn}{dy}$$

or

$$-\frac{1}{M(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]dy = \frac{dn}{n}$$

or

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]dy = \frac{dn}{n}$$

If the left hand side of the above equation is only function of y, then the equation is separable and $n(y) = \exp\left(\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\right)$.

<u>Conclusion</u>: The equation M(x,y) dx + N(x,y) dy = 0 has an integrating factor n(y) that depends only on y if the expression $\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$ depends only on y.

Examples: Find the integrating factor

1) $(4xy + 3y^2 - x)\,dx + x(x + 2y)\,dy = 0$

$M(x,y) = 4xy + 3y^2 - x$ and $N(x,y) = x(x + 2y)$

$$\frac{\partial M(x,y)}{\partial y} = 4x + 6y \text{ and } \frac{\partial N(x,y)}{\partial x} = 2x + 2y$$

$$\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = 4x + 6y - 2x - 2y = 2x + 4y$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{1}{x(x+2y)}(2x + 4y) = \frac{2(x+2y)}{x(x+2y))} = \frac{2}{x}$$

Since it depends on x, only, there is an integrating factor n(x), given by

$$n(x) = \exp\left(\int 2\frac{dx}{x}\right) = \exp(2\ln|x|) = x^2$$

Multiply the original equation by n(x), we get the exact equation

$(4x^3y + 3x^2y^2 - x^3)\,dx + (x^4 + 2x^3y)\,dy = 0$

by grouping we get

$(4x^3y\,dx + x^4\,dy) + (3x^2y^2\,dx + 2x^3y\,dy) - x^3\,dx = 0$

$d(x^4y) + d(x^3y^2) - d(¼\,x^4) = d(c)$

$x^4y + x^3y^2 - ¼\,x^4 = c$

2) $y(x + y)\,dx + (x + 2y - 1)\,dy = 0$

$M(x,y) = y(x + y)$ and $N(x,y) = x + 2y - 1$

$$\frac{\partial M(x,y)}{\partial y} = x + 2y \text{ and } \frac{\partial N(x,y)}{\partial x} = 1$$

$$\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = x + 2y - 1$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{1}{x+2y-1}(x+2y-1) = 1$$

Since, the expression is constant, there is an integrating factor n(x)

$$n(x) = e^{\int dx} = e^x$$

Multiplying the original equation by n(x), we obtain the exact equation

$$ye^x(x + y)\,dx + e^x(x + 2y - 1)\,dy = 0$$

$$F(x,y) = \int M(x,y)dx = \int\left(xye^x + y^2e^x\right)dx = y\left(xe^x - e^x\right) + y^2e^x + B(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\left[y\left(xe^x - e^x\right) + y^2e^x + B(y)\right] = xe^x - e^x + 2ye^x + B'(y)$$

then

$$B'(y) = 0 \text{ and } B(y) = c$$

The solution is: $xe^x - e^x + 2ye^x = k$

3) $y(x + y + 1)\,dx + x(x + 3y + 2)\,dy = 0$

$M(x,y) = y(x + y + 1)$ and $N(x,y) = x(x + 3y + 2)$

$$\frac{\partial M(x,y)}{\partial y} = x + 2y + 1 \text{ and } \frac{\partial N(x,y)}{\partial x} = 2x + 3y + 2$$

$$\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = x + 2y + 1 - 2x - 3y - 2 = -(x + y + 1)$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{-(x+y+1)}{2x+3y+2} \text{ depends on } x \text{ and } y$$

consider

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x}-\frac{\partial M(x,y)}{\partial y}\right]=\frac{1}{y(x+y+1)}(x+y+1)=\frac{1}{y}$$

Since, it depends only on y, there is an integrating factor n(y)

$$n(y) = e^{\int \frac{dy}{y}} = e^{\ln y} = y$$

Multiplying the original equation by n(y), we obtain the exact equation

$$y^2(x + y + 1)\,dx + yx(x + 3y + 2)\,dy = 0$$

$$F(x,y)=\int M(x,y)dx=\int\left(xy^2+y^3+y^2\right)dx=\frac{x^2}{2}y^2+xy^3+xy^2+B(y)$$

$$\frac{\partial F(x,y)}{\partial y}=N(x,y)=\frac{\partial}{\partial y}\left[\frac{x^2}{2}y^2+xy^3+xy^2+B(y)\right]=x^2y+3xy^2+2xy+B'(y)$$

then B'(y) = 0 and therefore B(y) = c

The solution is: $\frac{1}{2}x^2y + xy^3 + xy^2 = k$.

Special Transformation

There are certain equations that can be transformed into a more basic type using a suitable transformation.

The equations have the form:

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants.

There are two different kind of transformations according to relationships among the constants.

Case 1: $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$

Solve the system

$$a_1h + b_1k + c_1 = 0$$
$$a_2h + b_2k + c_2 = 0$$

because of the imposed condition the system has a unique solution (h,k).

Then, the transformation:

$$x = X + h$$
$$y = Y + k$$

will change the original equation into a homogeneous equation in the variable X and Y,

$$(a_1X + b_1Y)\,dX + (a_2X + b_2Y)\,dY = 0$$

Case 2: $\frac{a_2}{a_1} = \frac{b_1}{b_2} = k$

The transformation $z = a_1x + b_1y$ changes the original equation into a separable equation in the variables z and x.

Examples: Solve the equations

1) $(2x - 5x + 3)\, dx - (2x + 4y - 6)\, dy = 0$

Since $2/2 \neq 4/-5$, let's solve the system

$$2h - 5k + 3 = 0$$
$$2h + 4k - 6 = 0$$

Subtract the second equation from the first one, to get

$$2h - 5h = -3$$
$$\underline{2h + 4k = 6}$$
$$0 - 9k = -9$$

then $k = 1$ and $2h = -3 + 5$ or $h = 1$.

So, the transformation:

$$x = X + 1, \quad dx = dX$$
$$y = Y + 1, \quad dy = dY$$

reduces the given equation to

$$(2X + 2 - 5Y - 5 + 3)\, dX - (2X + 2 + 4Y + 4 - 6)\, dY = 0$$
$$(2X - 5Y)\, dX - (2X + 4Y)\, dY = 0$$

which is homogeneous.

Using the transformation $Y = VX$, and $dY = VdX + XdV$,

We get the equation

$$(2 - 5V)\, dX - (2 + 4V)(VdX + XdV) = 0$$
$$(2 - 7V - 4V^2)\, dX - X(2 + 4V)\, dV = 0$$

$$\frac{dX}{X} - \frac{2+4V}{2-7V-4V^2}dV = 0$$

$$\frac{dX}{X} + \frac{4V+2}{4V^2+7V-2}dV = 0$$

$$\frac{4V+2}{4V^2+7V-2} = \frac{A}{4V-1} + \frac{B}{V+2}$$

$$A = \frac{4}{3},\ B = \frac{2}{3}$$

$$\frac{dX}{X} + \frac{4}{3}\frac{dV}{4V-1} + \frac{2}{3}\frac{dV}{V+2} = 0$$

Integrating

$$\ln|X| + \frac{1}{3}\ln|4V-1| + \frac{2}{3}\ln|V+2| = \ln|c|$$

$$X^3(4V-1)(V+2)^2 = k$$

replacing V by $\dfrac{Y}{X}$

$$X^3\left(4\frac{Y}{X}-1\right)\left(\frac{Y}{X}+2\right)^2 = (4Y-X)(Y+2X)^2 = K$$

replacing X by $\mathbf{x}-1$ and Y by $y-1$,

$$(4y-\mathbf{x}-3)(y+2\mathbf{x}-3)^2 = K$$

2) $(x + y)\, dx + (3x + 3y - 4)\, dy = 0$

Since $3/1 = 3/1$, we must take the transformation $z = x + y$.

We use $y = z - x$, $dy = dz - dx$ to obtain

$$z\, dx + (3z - 4)(dz - dx) = 0$$

or $\quad z\, dx - 3z\, dx + 4\, dx + (3z - 4)\, dz = 0$

or $\quad (4 - 2z)\, dx + (3z - 4)\, dz = 0$

and this is a separable equation,

POSSIBLE QUESTIONS

PART - B (5 x 2 = 10 Marks)

1. Write the standard forms of the Second order differential equations.
2. Explain integrating factor of the differential equation.
3. Define separable equations with examples.
4. Write the general form of Bernoulli's equation.
5. Define integrating factor of the differential equation.

PART – C (5 x 6 = 30 Marks)

1. Explain about exact differential equations with examples.
2. Solve the equation ($3x^2$+ 4xy) dx + ($2x^2$ +2 y) dy =0.
3. Write a note on integration factor of differential equations.
4. Determine whether the given equations are exact or not and also solve that there is exact.

   (2 xy+1) dx+ ($x^2 + 4y)dy$ =0.
5. Determine the most general function N(x,y) such that the equation is exact

   $(x^3 + xy^2)$ dx + N(x,y) dy=0.
6. Explain Separable equations with examples.
7. Solve the equation (x-4) y^4 dx - $x^3(y^2 - 3)$ dy = 0.
8. Determine whether the differential equation is homogeneous or not

   $(x^2 - 3y^2)dx + 2xy\,dy = 0$.
9. Define Bernoulli's equation with example.
10. Solve the differential equation $\frac{dy}{dx} + y = xy^3$.

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Subject: Differential Equations **Subject Code: 17MMU201**

Class : I B.Sc Mathematics **Semester : II**

UNIT -II

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The standard form of first order differential equations derivative form is…………	(dy/dx)=f(x,y)	(dx/dy)=f(x,y)	(dy/dx)=-f(x,y)	(dy/dx)=0	(dy/dx)=f(x,y)
The standard form of first order differential equations differential form is…………	M(x,y)dx-N(x,y)dy=0	M(x,y)dx*N(x,y)dy=0	M(x,y)dx/N(x,y)dy=0	M(x,y)dx+N(x,y)dy=0	M(x,y)dx+N(x,y)dy=0
The expression M(x,y)dx+N(x,y)dy=0 is called an ……………….. differential equations in a domain D.	ordinary	partial	exact	different	exact
The expression ………………….. is called an exact differential equations in a domain D.	M(x,y)dx+N(x,y)dy=0	M(x,y)dx*N(x,y)dy=0	M(x,y)dx/N(x,y)dy=0	M(x,y)dx-N(x,y)dy=0	M(x,y)dx+N(x,y)dy=0
The expression M(x,y)dx+N(x,y)dy=0 is called an exact differential equations in a domain D if there exists a function of……… variable such that the expression equals the total differential for all (x,y)in D	zero	one	two	three	two
The expression M(x,y)dx+N(x,y)dy=0 is called an exact differential equations in a domain D if there exists a function of two variable such that the expression equals the …………for all (x,y)in D	differential	ordinary differential	partial differential	total differential	total differential
If M(x,y)dx+N(x,y)dy is an exact differential then the differential equation M(x,y)dx+N(x,y)dy=…………is called exact differential equation	0	1	2	3	0

If M(x,y)dx+N(x,y)dy is an …………..differential then the differential equation M(x,y)dx+N(x,y)dy=0 is called exact differential equation	ordinary	partial	exact	different	exact
If M(x,y)dx+N(x,y)dy is not an exact differential in D then the differential equation …………………. in D the μ(x,y) is called integrating factor of the differentialequation	μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx-μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx*μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx/μ(x,y)N(x,y)dy=0	μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0
If M(x,y)dx+N(x,y)dy is not an ……………. differential in D then the differential equation μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0 in D the μ(x,y) is called integrating factor of the differentialequation	ordinary	partial	exact	different	exact
If M(x,y)dx+N(x,y)dy is not an exact differential in D then the differential equation μ(x,y)M(x,y)dx+μ(x,y)N(x,y)dy=0 in D the μ(x,y) is called …………factor of the differentialequation	differential	integrating	common	exact	integrating
An equation of the form ……………...is called an equation with variables separable orsimply a separable equations.	F(x)G(y) dx+f(x)g(y) dy=0	F(x)G(y) dx/f(x)g(y) dy=0	F(x) dx+g(y) dy=0	G(y) dx+f(x) dy=0	F(x)G(y) dx+f(x)g(y) dy=0
An equation of the form F(x)G(y) dx+f(x)g(y) dy=0 is called an ……………...or simply a separable equations.	equation with function separable	equation with constant separable	equation with roots separable	equation with variables separable	equation with variables separable
An equation of the form F(x)G(y) dx+f(x)g(y) dy=0 is called an equation with variables separable or simply a……………equations.	differential	integral	separable	variable	separable
The first order differential equation M(x,y)dx+N(x,y)dy=0 is said to be………………... if the derivativeof the form (dy/dx)=f(x,y) there exists a function g suchthat f(x,y) can be expressed in the form g(y/x)	homogeneous	non homogeneous	singular	non singular	homogeneous
The first order differential equation M(x,y)dx+N(x,y)dy=0 is said to be homogeneous if the derivativeof the form ………….. there exists a function g suchthat f(x,y) can be expressed in the form g(y/x)	(dy/dx)=0	(dy/dx)=f(x,y)	(dy/dx)=1/f(x,y)	(dy/dx)= -f(x,y)	(dy/dx)=f(x,y)

The first order differential equation M(x,y)dx+N(x,y)dy=0 is said to be……………….. if the derivativeof the form (dy/dx)=f(x,y) there exists a function g suchthat f(x,y) can be expressed in the form…………	g(x/y)	g(1/x)	g(1/y)	g(y/x)	g(y/x)
A first order differential equation is linear in the dependent variable y and the independent variable x if it is can be written in the form ……………………	(dy/dx)=P(x)y+Q(x).	(dy/dx)+P(x)y/Q(x)=0.	(dy/dx)+P(x)y=Q(x).	(dy/dx)+P(x)y=0	(dy/dx)+P(x)y=Q(x).
A first order differential equation is ………………. in the dependent variable y and the independent variable x if it is can be written in the form (dy/dx)+P(x)y=Q(x).	linear	nonlinear	zero	separable	linear
A …………..order differential equation is linear in the dependent variable y and the independent variable x if it is can be written in the form (dy/dx)+P(x)y=Q(x).	first	second	third	n th	first
An equation of the form ……………………….is called a Bernoulli differential equation .	(dy/dx=P(x) y^n	(dy/dx)+P(x)y/Q(x)=0.	(dy/dx)+P(x)y=Q(x) y^n	(dy/dx)+P(x)y=0	(dy/dx)+P(x)y=Q(x) y^n
An equation of the form (dy/dx)+P(x)y=Q(x) y^n is called ……………….. differential equation .					
In bernoulli equation when n=……….. then the equation is called linear equation.	0 or 1	1 or 2	0 or 2	0 or -1	0 or 1
In bernoulli equation when n=0 or 1 then the equation is called ……………. equation.	ordinary	partial	nonlinear	linear	linear
In ………………. equation when n=0 or 1 then the equation is called linear equation.	ordinary	Bernoulli	Euler	partial	Bernoulli

UNIT – III

SYLLABUS

Introduction to compartmental model, exponential decay model, lake pollution model (case study of Lake Burley Griffin), drug assimilation into the blood (case of a single cold pill, case of a course of cold pills), exponential growth of population, limited growth of population, limited growth with harvesting.

Introduction:

Compartmental models provides a means to formulate models for processes which have inputs and/or outputs over time. In this chapter, we will study modelling of radioactive decay processes, pollution levels in lake systems and the absorption of drugs into the bloodstream, exponential growth model, density dependent growth, limited growth harvesting using compartmental techniques.

Compartmental Model:

Definition: Compartmental Model is a model in which there is a place called compartment which has amount of substance in and amount of substance out over time. One example of compartmental model is the pollution into and out of a lake where lake is the compartment. Another example is the amount of carbon-di-oxide in the Earth's atmosphere. The compartment is the atmosphere where the input of CO_2 occurs through many processes such as burning and output of CO_2 occurs through plant respiration. It can be shown in the form of a diagram called compartmental diagram which is shown below.

Fig. 1: Input – output compartmental diagram for CO_2

Balance Law:

Statement: *The rate of change of quantity of substance is equal to 'Rate in'minus 'Rate out' of the compartment.*

Symbolically, if X(t) is the amount of quantity in the compartment, then

$$\frac{dX}{dt} = Rate\ In - Rate\ Out$$

Compartmental Diagram:

Word Equation:

In words, balance law can be written as :

$$\left\{\begin{array}{l}\textit{Net Rate} \text{ of change} \\ \text{of a substance}\end{array}\right\} = \{\text{Rate } in\} - \{\text{Rate out}\}$$

This Equation is known as **WORD EQUATION** of the model.

Exponential Decay Model and Radioactivity:

Radioactive elements are those elements which are not stable and emit α-particles, β- particles or photons while decaying into isotopes of other elements. Exponential decay model for radioactive decay can be considered as a compartmental model with compartment being the radioactive material with no input but output as decay of radioactive sample over time.

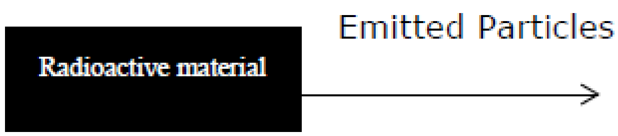

Fig. 2: Input – output compartmental diagram for radioactive nuclei

Word equation: By Balance Law, word equation can be written as :

$$\left\{\begin{array}{l}\textit{Rate} \text{ of change} \\ \text{of radioactive material} \\ \text{at time t}\end{array}\right\} = \left\{\begin{array}{l}\textit{Rate} \text{ amount of} \\ \text{radioactive material} \\ \text{decayed}\end{array}\right\}$$

Assumptions for the radioactive Decay Model:

1. Amount of an element present is large enough so that we are justified in ignoring random fluctuations.
2. The process is continuous in time.
3. We assume a fixed rate of decay for an element.
4. There is no increase in mass of the body of material.

Formulating the differential equation:

Let N(t) be the number of radioactive nuclei at time t

n_0 = initial radioactive nuclei present at time t_0

Since the rate of change of nuclei is directly proportional to the number of nuclei at the start of time period therefore, $C(t) = C_{in}$

$\Rightarrow \quad \frac{dN}{dt} = -KN$, where K is the constant of proportionality indicating rate of decay per nucleus in unit time.

At initial condition, number of radioactive nuclei is n_0 therefore, N(0) = n_0
Hence initial value problem corresponding to exponential decay model is given by:

$$\boxed{\frac{dN}{dt} = -KN \quad ; N(0) = n_0 \; ; K > 0}$$

Solution of the differential equation of Exponential Decay Model:

We have $\frac{dN}{dt} = -KN$

$\Rightarrow \quad \frac{dN}{N} = -Kdt$

Integrating both sides, we get

$$\int \frac{dN}{N} = -K\int dt$$

$\Rightarrow \quad \ln N = -Kt + \ln C$, where C is the constant of integration.

$\Rightarrow \quad \ln\left(\frac{N}{C}\right) = -Kt$

$\Rightarrow \quad \left(\frac{N}{C}\right) = e^{-Kt}$

$\Rightarrow \quad N = Ce^{-Kt}$ ....(1)

Put initial condition, N(0) = n_0 i.e., at t = 0, N = n_0 we get

$n_0 = Ce^{-K(0)}$

$\Rightarrow \quad n_0 = C \quad \because e^0 = 1$

Put C = n_0 in equation (1) we get $N = n_0 e^{-Kt}$ where K is the constant of proportionality. Moreover the value of K depends on the particular radioactive material.

Lake Pollution Model:

Lake pollution model is concerned with the pollution in lakes and rivers that has become a major problem particularly over the last 50 years. This model can be considered as a compartmental model with a single compartment, the lake. It can be represented in the form of the following compartmental diagram.

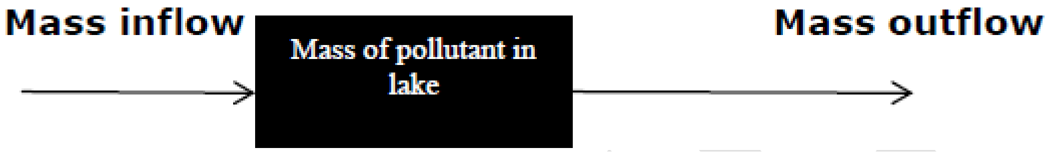

Fig 3 : Input – output compartmental diagram for pollution in a lake.

Balance Law in Lake Pollution Model: There is an input of polluted water from the river flowing into the lake, or due to a pollution dump into the lake, and an output as water flows from lake carrying some pollution with it. This gives the word equation as follows:

Word Equation:

$$\begin{Bmatrix} \textit{Rate} \text{ of change} \\ \text{of pollution in lake} \end{Bmatrix} = \{\textit{Pollution in}\} - \{\textit{Pollution} \text{ out}\}$$

Compartmental Model of Lake Pollution:

Assumption of lake pollution model: The lake has a constant volume V and that it is continuously well mixed so that the pollution is uniform throughout.

Let $x(t)$ = Amount of pollutant in the lake at time t.

V = Volume of the lake.

V_1 = Volume of water flowing in and out of the lake.

C_{in} = Concentration of pollution of incoming water.

Since volume of lake is constant,

$$V_1 = \begin{Bmatrix} \textit{Flow} \text{ of mixture} \\ \text{into lake} \end{Bmatrix} = \begin{Bmatrix} \textit{Flow} \text{ of mixture} \\ \text{out of lake} \end{Bmatrix}$$

By Balance Law, rate of change of pollution = Pollution in – Pollution out

$$\Rightarrow \quad \frac{dx}{dt} = Amount \text{ in - Amount out} \qquad \text{............(i)}$$

Also, Amount in = Concentration in × Volume = $C_{in} \times V_1$

Amount out = Concentration out × Volume = $C_{out} \times V_1 = \frac{x(t)}{V} . V_1$ because $C_{out} = \frac{Total\ Amount}{Volume} = \frac{x(t)}{V}$

Substituting the values of Amount in and Amount out in equation (i), we get

$$\frac{dx}{dt} = V_1.C_{in} - \frac{V_1\, x}{V}$$

At initial condition, $x(0) = x_0$

Therefore, the initial value problem is $\frac{dx}{dt} = V_1.C_{in} - \frac{V_1\, x}{V}$; $x(0) = x_0$ ....(ii)

Let C = Concentration of pollutant in lake at any time t, then $x = CV \Rightarrow \frac{dx}{dt} = V\frac{dC}{dt}$

Then equation (ii) implies that $V\frac{dC}{dt} = V_1 C_{in} - \frac{V_1 CV}{V}$

$$\Rightarrow \quad V\frac{dC}{dt} = V_1\left(C_{in} - C\right)$$

$$\Rightarrow \quad \frac{dC}{dt} = \frac{V_1}{V}\left(C_{in} - C\right); \mathrm{C}(0) = \mathrm{C}_0$$

This is the differential equation of the lake pollution model.

Solution of the Differential Equation:

Separating the variables, we get

$$\frac{dC}{\left(C_{in} - C\right)} = \frac{V_1}{V}\mathrm{dt}$$

Integrating both sides, we get

$$\int \frac{dC}{\left(C_{in} - C\right)} = \int \frac{V_1}{V}\mathrm{dt}$$

$\Rightarrow \quad -\ln\left|C_{in} - C\right| = \frac{V_1}{V}t + K$, K is constant of integration.

$$\Rightarrow \quad \left|C_{in} - C\right| = e^{-\frac{V_1}{V}t} e^{-k}$$

$$\Rightarrow \quad C(t) = C_{in} - e^{-\frac{V_1}{V}t} e^{-k} \qquad \text{.........(iii)}$$

Put the initial condition C(0) = C_0, we get

$$\Rightarrow \quad C_0 = C_{in} - e^{-\frac{V_1}{V}(0)} e^{-k}$$

$$\Rightarrow \quad C_0 = C_{in} - e^{-k} \text{ or } e^{-k} = C_{in} - C_0$$

Put it in (iii) we get

$$C(t) = C_{in} - e^{-\frac{V_1}{V}t}\left(C_{in} - C_0\right)$$

$$\Rightarrow \quad C(t) = \left(C_{in} - C_{in} e^{-\frac{V_1}{V}t}\right) + \left(C_0 e^{-\frac{V_1}{V}t}\right)$$

This is the solution of lake pollution model in which the expression in the first bracket is the contribution from the pollution inflow into the system and the expression in the second bracket is the contribution from the initial data.

Consider the case when t is very large i.e., $t \to \infty$

When $t \to \infty$ then $e^{-\frac{V_1}{V}t} = 0$. This gives $\lim_{t\to\infty} C(t) = C_{in}$

Example1: Let in a lake, the pollution level is 5%. If the fresh water at the rate of 10000litres per day is allowed to enter and same amount of water leaves the lake. Find the time when pollution level is 2.5% if volume of lake is 500000litres. Further, if safety level is 0.1%, then after how much time, water is suitable for drinking.

Solution: We are given $V_1 = 10000, V = 500000, C_{in} = 0, C(t) = 0.025, C_0 = 0.05$. We find t.

We know that

$$C(t) = \left(C_{in} - C_{in} e^{-\frac{V_1}{V}t}\right) + \left(C_0 e^{-\frac{V_1}{V}t}\right)$$

$$\Rightarrow \quad C(t) = 0.05 e^{\frac{-1}{50}t} \quad \because C_{in} = 0 \qquad \text{.........(i)}$$

$$\Rightarrow \quad 0.025 = 0.05 e^{\frac{-1}{50}t}$$

$$\Rightarrow \quad \frac{0.025}{0.05} = e^{\frac{-1}{50}t}$$

Take log on both sides, we get

$$\log\left(\frac{0.025}{0.05}\right) = -\frac{1}{50}t\log e$$

$$\Rightarrow \quad \log(0.5) = -\frac{1}{50}t\log e$$

$$\Rightarrow \quad -0.3010 = -\frac{1}{50}t(0.4343)$$

$$\Rightarrow \quad t = \frac{0.3010\times 50}{0.4343} = 34.65\ days$$

Therefore, at t = 34.65 days, pollution level is 2.5%.
Now, we find t at C = 0.001. For this, put C = 0.001 in (i), we get

$$0.001 = 0.05e^{\frac{-1}{50}t}$$

$$\Rightarrow \quad \frac{0.001}{0.05} = e^{\frac{-1}{50}t}$$

Take log on both sides, we get

$$\log\left(\frac{0.001}{0.05}\right) = -\frac{1}{50}t\log e$$

$$\Rightarrow \quad \log(0.02) = -\frac{1}{50}t\log e$$

$$\Rightarrow \quad -1.6990 = -\frac{1}{50}t(0.4343)$$

$$\Rightarrow \quad t = \frac{1.699\times 50}{0.4343} = 195.6\ days$$

Hence, water is suitable for drinking after t = 195.6 days.

Example2: Let in a lake, the pollution level is 7%. If the concentration of incoming water is 2% and 10000 litres per day water is allowed to enter the lake, find time when pollution level is 5%. Volume of the lake is 200000 litres. Also, find pollution after 32 days.

Solution: We are given $V_1 = 10000, V = 200000, C_{in} = 0.02, C(t) = 0.05, C_0 = 0.07$. We find t.
We know that

$$C(t) = \left(C_{in} - C_{in} e^{-\frac{V_1}{V}t} \right) + \left(C_0 e^{-\frac{V_1}{V}t} \right)$$

$$\Rightarrow \quad C(t) = 0.02 - (0.02 - 0.07) e^{\frac{-1}{20}t} \quad \text{.........(i)}$$

$$\Rightarrow \quad 0.05 = 0.02 + 0.05 e^{\frac{-1}{20}t}$$

$$\Rightarrow \quad \frac{0.03}{0.05} = e^{\frac{-1}{20}t}$$

Take log on both sides, we get

$$\log\left(\frac{0.03}{0.05}\right) = -\frac{1}{20} t \log e$$

$$\Rightarrow \quad \log(0.6) = -\frac{1}{20} t \log e$$

$$\Rightarrow \quad -0.2218 = -\frac{1}{20} t (0.4343)$$

$$\Rightarrow \quad t = \frac{0.2218 \times 20}{0.4343} = 10.214 days$$

Therefore, at t = 10.214 days, pollution level is 5%.

Now, we take t = 32 and find C(t). Put t = 32 in (i)

$$C(t) = 0.02 - (0.02 - 0.07) e^{\frac{-1}{20}\times 32} \quad \text{.........(i)}$$

$$= 0.02 + 0.05 e^{-1.6}$$

$$= 0.02 + 0.05 \times 0.2019 = 0.030095 = 3\% \text{ approx}$$

Case study : Lake Burley Griffin:

The information from this case study is adapted from the research paper Burges and Olive (1975).

Lake Burley Griffin in Canberra, the capital city of Australia, was created artificially in 1962 for both recreational and aesthetic purposes. In 1974, the public health authorities indicated that pollution standards set down for safe recreational use were being violated and that this was attributable to the sewage works in Queanbeyan upstream.

After extensive measurements of pollution levels taken in the 1970s it was established that, while the sewage plants (of which there are three above the lake) certainly exacerbated the problem, there were significant contributions from rural and urban runoff as well, particularly during summer rainstorms. These contributed to dramatic increases in pollution levels and at times were totally responsible for lifting the concentration levels above the safety limits. As a point of interest, Queanbeyan (where the sewage plants are situated) is in the state of New South Wales (NSW), while the lake is in the Australian Capital Territory, and although they are a ten-minute drive apart the safety levels/standards for those who swim in NSW are different from the standards for those who swim in the Capital Territory.

Example3: In 1974, the mean concentration of the bacteria faecal coliform count was approximately 10^7 bacteria per m^3 at the point where the river feeds into the lake. The safety threshold for this faecal coliform count in the water is such that for contact recreational sports no more than 10% of total samples over a 30- day period should exceed 4×10^6 bacteria per m^3.

Given that the lake was polluted it is of interest to examine how, if sewage management were improved, the lake would flush out and if and when the pollution levels would drop below the safety threshold.

Solution: Let the assume the following for the given system.

- Flow V_1 into the lake is assumed equal to flow out of the lake
- Volume V of the lake is constant and is approximately 28×10^6 m^3.
- The lake is well mixed in the sense that the pollution concentration throughout will be taken as constant.

Under the following assumptions, the differential equation for the pollutant concentration is:

$$\frac{dC}{dt} = \frac{V_1}{V}\left(C_{in} - C\right); \mathrm{C}(0) = \mathrm{C}_0$$

The solution of the differential equation is:

$$C(t) = \left(C_{in} - C_{in}e^{-\frac{V_1}{V}t}\right) + \left(C_0 e^{-\frac{V_1}{V}t}\right) \qquad \text{.........(i)}$$

Since only the fresh water is entering into the lake, $C_{in} = 0$.

$$\frac{dy}{dt} = K_1 x - K_2 \mathrm{y} \quad ; \mathrm{y}(0) = 0$$

Put these values in equation (i) and find t.

$$4 \times 10^6 = 10^7 e^{-\frac{4\times 10^6}{28\times 10^6}t}$$

$$\Rightarrow \qquad 0.4 = e^{-\frac{1}{7}t}$$

Take log on both sides, we get

$$\log 0.4 = -\frac{1}{7} t \log e$$

$$\Rightarrow \qquad -0.39794 = -\frac{1}{7} t(0.4343)$$

$$\Rightarrow \qquad t = \frac{0.39794 \times 7}{0.4343} \square\ 6.4 months$$

Therefore, the lake will take approximately 6.4 months for the pollution level to drop below the safety threshold.

Drug Assimilation Model:

Compartmental Model: Drug Assimilation Model can be considered as a compartmental model with two compartments, corresponding to GI tract (gastrointestinal tract) and bloodstream. The GI tract compartment has a single input and output and the bloodstream compartment has a single input and output.

Compartmental diagram:

Fig 4 : Input – output compartmental diagram for drug assimilation.

Word Equation: The application of balance law gives the following two word equations, one for each compartment.

$$\begin{Bmatrix} \text{Rate of change of} \\ \text{drug in GI tract} \end{Bmatrix} = \begin{Bmatrix} \textit{Rate of} \\ \textit{drug} \text{ intake} \end{Bmatrix} - \begin{Bmatrix} \textit{Rate} \text{ drug leaves} \\ \textit{GI tract} \end{Bmatrix} \quad \text{and}$$

$$\begin{Bmatrix} \textit{Rate} \text{ of change of} \\ \text{drug in blood} \end{Bmatrix} = \begin{Bmatrix} \textit{Rate} \text{ drug} \\ \textit{enters} \text{ blood} \end{Bmatrix} - \begin{Bmatrix} \textit{Rate} \text{ drug} \\ \text{leaves blood} \end{Bmatrix}$$

Model I : Case of Single Cold Pill:

In case of single cold pill, there is no ingestion of the drug except that it occurs initially.

Let $x(t)$ = amount of drug in GI tract at time t.

$y(t)$ = amount of drug in blood at time t.

K_1 = transfusion rate of GI tract.

K_2 = transfusion rate of blood.

x_0 = amount of drug in the pill initially.

In GI tract, we consider the pill to have been swallowed, so after this event, we have nothing more entering the GI tract. The pill dissolves and diffuses into the bloodstream from the GI tract. So, there is only one output term for the GI tract.

Rate of drug intake in GI tract = 0

Assuming output rate is proportional to GI tract drug concentration, Rate of drug leaving GI tract = K_1x where K_1 is the constant of proportionality.

So, by Balance Law, $\dfrac{dx}{dt} = 0 - K_1 x \quad ; x(0) = x_0$

The pill dissolves and diffuses into the bloodstream from the GI tract. In the bloodstream, the initial amount of the drug is zero, so y(0) = 0. The level increases as drug diffuses from GI tract and decreases as Kidneys and liver remove it.

Rate of drug entering blood = K_1 x

Rate of drug leaving blood = K_2 y

By Balance Law for bloodstream, we get

$$\frac{dy}{dt} = K_1 x - K_2 y \quad ; y(0) = 0 \text{ where } K_1 \neq K_2$$

The coefficients of proportionality, K_1 and K_2 are different for different component drugs in pill. K_1 and K_2 depends on age and health of the person involved and the concentration of drug may also depend on person's body mass which means for some person, the dose may peak faster than for an average person.

The differential equation for Drug Assimilation Model in case of single pill is:

For GI tract:

$$\frac{dx}{dt} = -K_1 x \quad ; x(0) = x_0$$

For Bloodstream:

$$\frac{dy}{dt} = K_1 x - K_2 y \quad ; y(0) = 0 \text{ where } K_1 \neq K_2$$

Solution of the GI tract Differential Equation:

$$\frac{dx}{dt} = -K_1 x \quad ; x(0) = x_0 \qquad \text{........(i)}$$

Separating the variables in (i), we get

$$t \to \infty \quad \frac{dx}{x} = -K_1 dt$$

Integrating both sides, we get

$$\int \frac{dx}{x} = -K_1 \int dt$$

$\Rightarrow \quad \ln x = -K_1 t + \ln C$, where C is constant of integration.

$$\Rightarrow \quad x = Ce^{-K_1 t} \qquad \text{..........(ii)}$$

At initial condition, $x(0) = x_0$ i.e., Put $x = x_0$ at $t = 0$ in (ii), we get

$$x_0 = Ce^{-K_1(0)}$$

$$\Rightarrow \quad C = x_0$$

Substituting this value of C in (ii), we get

$$x(t) = x_0 e^{-K_1 t} \qquad \text{..........(iii)}$$

Solution of the Blood Stream Differential Equation:

$$\frac{dy}{dt} = K_1 x - K_2 y \quad ; y(0) = 0 \text{ where } K_1 \neq K_2 \qquad \text{.......... (iv)}$$

Using (iii) in (iv), we get

$$\frac{dy}{dt} = K_1 x_0 e^{-K_1 t} - K_2 y$$

$$\frac{dy}{dt} + K_2 y = K_1 x_0 e^{-K_1 t}$$

It is a linear equation of the form

$$\frac{dy}{dt} + Py = Q(t) \text{ where } P = K_2 \text{ and } Q(t) = K_1 x_0 e^{-K_1 t}$$

Integrating factor = I.F = $e^{\int P(t)dt} = e^{\int K_2 dt} = e^{K_2 t}$

Solution is :

$$y(I.F) = \int Q(t) I.F dt$$

$$\Rightarrow \quad y(e^{K_2 t}) = \int e^{K_2 t} K_1 x_0 e^{-K_1 t} dt$$

$$\Rightarrow \quad y(e^{K_2 t}) = K_1 x_0 \frac{e^{(K_2 - K_1)t}}{K_2 - K_1} + C \text{ where C is constant of integration. ....(v)}$$

Put y(0) = 0 i.e., y = 0 at t = 0 in equation (v), we get

$$0 = K_1 x_0 \frac{e^{(K_2 - K_1)(0)}}{K_2 - K_1} + C$$

$$\Rightarrow \quad 0 = \frac{K_1 x_0}{K_2 - K_1} + C$$

$$\Rightarrow \qquad C = \frac{K_1 x_0}{K_1 - K_2}$$

Substitute the value of C in (v), we get

$$y(e^{K_2 t}) = \frac{K_1 x_0 e^{(K_2 - K_1)t}}{K_2 - K_1} - \frac{K_1 x_0}{K_1 - K_2}$$

$$\Rightarrow \qquad y(e^{K_2 t}) = \frac{K_1 x_0}{K_1 - K_2}\left(1 - e^{(K_2 - K_1)t}\right)$$

$$\Rightarrow \qquad y = \frac{K_1 x_0}{K_1 - K_2}\left(e^{-K_2 t} - e^{-K_1 t}\right) \quad \text{where } K_1 \neq K_2$$

Special case: When time is very large i.e., $t \to \infty$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x_0 e^{-K_1 t} = x_0(0) = 0 \qquad \text{and}$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{K_1 x_0\left(e^{-K_2 t} - e^{-K_1 t}\right)}{K_1 - K_2} = \frac{K_1 x_0 (0-0)}{K_1 - K_2} = 0$$

Drug Assimilation Model (A course of Pills): In reality, we take a course of pills rather than just one, particularly in case of cough and cold. In such cases, there is a continuous flow of drugs into the GI- tract.

Let $x(t)$ = amount of drug in GI tract at time t.

$y(t)$ = amount of drug in blood at time t.

K_1 = diffusion constant for GI tract.

K_2 = diffusion constant for blood.

x_0 = amount of drug consumed after a fixed time period regularly.

Compartmental diagram for Drug Assimilation in case of Course of Pills is :

Amount in = x_0 **Amount in = K_1 x**

Amount out = K_1 x **Amount Out = K_2y**

Fig 5 : Input – output compartmental diagram for drug assimilation in case of course of pills.

Since there is continuous flow of drugs into GI tract, therefore,

Rate of Drug Intake into GI tract = x_0

Assuming output rate is proportional to GI tract drug concentration,

Rate of drug leaving GI tract = K_1x where K_1 is the constant of proportionality.

So, by Balance Law, $\frac{dx}{dt} = x_0 - K_1x \quad ; x(0) = 0$

The pill dissolves and diffuses into the bloodstream from the GI tract. In the bloodstream, the initial amount of the drug is zero, so y(0) = 0. The level increases as drug diffuses from GI tract and decreases as Kidneys and liver remove it.

Rate of drug entering blood = K_1 x

Rate of drug leaving blood = K_2 y

By Balance Law for bloodstream, we get

$$\frac{dy}{dt} = K_1x - K_2y \quad ; y(0) = 0 \text{ where } K_1 \neq K_2$$

Solution of the Drug Assimilation Model (Course of pills):

GI tract: $x(t) = \frac{x_0}{K_1}\left(1 - e^{-K_1 t}\right)$

Blood Stream: $y(\mathrm{t}) = \frac{x_0\left(1 - e^{-K_2 t}\right)}{K_2} + \frac{x_0}{K_2 - K_1}\left(e^{-K_2 t} - e^{-K_1 t}\right)$ where $K_1 \neq K_2$

Special case: When time is very large i.e., $t \to \infty$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x_0}{K_1}\left(1 - e^{-K_1 t}\right) = \frac{x_0}{K_1}$$ and

$$\lim_{t\to\infty} y(\mathrm{t}) = \lim_{t\to\infty} \frac{x_0\left(1 - e^{-K_2 t}\right)}{K_2} + \lim_{t\to\infty} \frac{x_0}{K_2 - K_1}\left(e^{-K_2 t} - e^{-K_1 t}\right)$$

$$\Rightarrow \quad \lim_{t\to\infty} y(\mathrm{t}) = \frac{x_0}{K_2}$$

Exponential Growth Model (Population Growth):

Population growth model can be considered as a compartmental model with the compartment being world, town, ocean etc. There is an input of population in the compartment through birth and an output of population from the compartment through death.

Input – Output Compartmental Diagram for Population

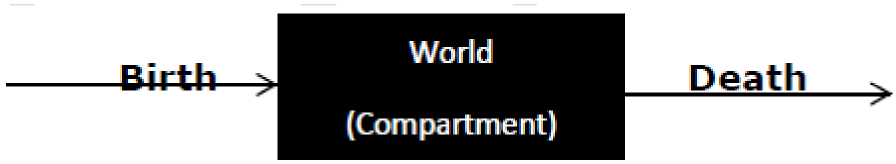

Fig 6 : Input – output compartmental diagram for Population Growth

Word equation: By Balance Law, word equation can be written as

$$\left\{\begin{matrix} \text{Rate of change} \\ \text{of population} \\ \text{size} \end{matrix}\right\} = \left\{\begin{matrix} \text{Rate of} \\ \text{births} \end{matrix}\right\} - \left\{\begin{matrix} \text{Rate of} \\ \text{Deaths} \end{matrix}\right\}$$

Formulating the differential equation

Let x(t) be the population at time *t*

x_0 = initial population

α = constant per capita death rate per individual per unit of time.

β = constant per capita birth rate per individual per unit of time.

Rate of change of population at any time t is directly proportional to the size of population at that time therefore, $\frac{dx}{dt} \alpha\, x$

Rate of deaths = α x(t)

Rate of births = β x(t)

By Balance Law,

$$\frac{dx}{dt} = \beta x - \alpha x = (\beta - \alpha)x$$

Let r = β - α , r is the growth rate for the population.

Clearly, r > 0 in exponential growth.

At initial condition, population is $x(0) = x_0$

Hence initial value problem corresponding to exponential growth model is given by:

Exponential or Natural Growth Equation

$$\frac{dx}{dt} = r\mathrm{x} \quad ; \mathrm{x}(0) = \mathrm{x}_0 \quad , \mathrm{r} > 0$$

Because of the presence of exponential function in its solution, the differential equation given above is called exponential or natural growth equation.

Solution of Exponential Growth Model :

$$x = x_0 e^{rt}$$

Limited Population Growth Model with Harvesting:

The effect of harvesting a population on a constant basis is of paramount importance in many industries such as fishing industry. This model is useful in answering the questions like "Will a high harvesting rate destroy the population? Or Will a low harvesting rate destroy the viability of the industry. Limited population growth model with harvesting is a compartmental model with compartment being the world, there is an input of population through births and an output through deaths.

Word equation: By Balance Law, word equation can be written as

$$\left\{\begin{matrix}\textit{Rate}\text{ of change}\\ \text{in population}\end{matrix}\right\}=\left\{\begin{matrix}\textit{Rate}\text{ of}\\ \textit{births}\end{matrix}\right\}-\left\{\begin{matrix}\textit{Normal}\text{ rate}\\ \text{of deaths}\end{matrix}\right\}-\left\{\begin{matrix}\textit{Rate}\text{ of deaths by}\\ \text{crowding or density}\end{matrix}\right\}-\left\{\begin{matrix}\textit{Rate}\text{ of deaths}\\ \text{by harvesting}\end{matrix}\right\}$$

Limited Population Growth Model with Harvesting:

$$\frac{dx}{dt}=\mathrm{r}x\left(1-\frac{x}{K}\right)-\mathrm{h},\ \mathrm{x}(0)=\mathrm{x}_0$$

Example : In a fish farm, fish are harvested at a constant rate of 2100 fish per week. The per capita death rate for the fish is 0.2 fish per day per fish and per capita birth rate is 0.7 fish per day per fish.

(a) Write down a word equation and differential equation for rate of change of fish population.
(b) Find when the fish population is in equilibrium.
(c) If initial fish population is 700, find fish population after a week.

Solution: We are given that h = 2100fish per week = 2100/7 = 300 fish per day. Also $\alpha = 0.2$, $\beta = 0.7$

Word Equation:

$$\left\{\begin{matrix}\textit{Rate}\text{ of change}\\ \text{in fish population}\end{matrix}\right\}=\left\{\begin{matrix}\textit{Rate}\text{ of}\\ \textit{births}\end{matrix}\right\}-\left\{\begin{matrix}\text{Rate of}\\ \text{normal deaths}\end{matrix}\right\}-\left\{\begin{matrix}\textit{Rate}\text{ of deaths}\\ \text{by harvesting}\end{matrix}\right\}$$

Differential equation:

Let $x(t)$ be the fish population at any time t.
α = constant per capita death rate.

β = constant per capita birth rate

h= constant rate of harvesting i.e., deaths due to harvesting per unit time.

Then differential equation is :

$$\frac{dx}{dt} = \beta x - \alpha x - \mathrm{h}$$

$$\Rightarrow \quad \frac{dx}{dt} = 0.7x - 0.2x - 300 = 0.5x - 300$$

$$\Rightarrow \quad \frac{dx}{dt} = 0.5x - 300$$

(b) At equilibrium,

$$\frac{dx}{dt} = 0$$

$$\Rightarrow \quad 0.5x - 300 = 0$$

$$\Rightarrow \quad x = 600$$

Therefore, at x = 600 fishes, fish population is in equilibrium.

(c) We have

$$\frac{dx}{dt} = 0.5x - 300; x(0) = 700$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}x - 300 = \frac{x-600}{2}$$

$$\Rightarrow \quad \frac{dx}{x-600} = \frac{1}{2}dt$$

Integrating both sides, we get

$$\int \frac{dx}{x-600} = \frac{1}{2}\int dt$$

$\Rightarrow \quad \log(x-600) = \frac{1}{2}t + C$, where C is the constant of integration. .....(i)

Put x(0) = 700, we get

$$\log(700-600) = \frac{1}{2}(0) + C$$

$$\Rightarrow \quad \log 100 = C$$

Put the value of C in (i), we get

$$\log(x-600) = \frac{1}{2}t + \log 100$$

$$\Rightarrow \quad \log\left(\frac{x-600}{100}\right) = \frac{1}{2}t$$

$$\Rightarrow \quad \left(\frac{x-600}{100}\right) = e^{t/2}$$

When t = 7, we have

$$\left(\frac{x-600}{100}\right) = e^{7/2}$$

$$\Rightarrow \quad \left(\frac{x-600}{100}\right) = 33.115$$

$$\Rightarrow \quad x = 3911.5\, fishes$$

After a week, fish population is 3912 fishes approximately.

POSSIBLE QUESTIONS

PART – B (5 x 2 = 10)

1. Define Compartmental Model.
2. Write the basic assumption for Exponential Decay Model.
3. Define Lake Pollution Model.
4. Explain the balance Law.
5. Explain Limited growth of population

PART – C (5 x 6 = 30 Marks)

1. Briefly explain Compartmental model with example.
2. Let in a lake, the pollution level is 5%. If the fresh water at the rate of 10000litres per day is allowed to enter and same amount of water leaves the lake. Find the time when pollution level is 2.5% if volume of lake is 500000litres. Further, if safety level is 0.1%, then after how much time, water is suitable for drinking.
3. Let in a lake, the pollution level is 7%. If the concentration of incoming water is 2% and 10000 litres per day water is allowed to enter the lake, find time when pollution level is 5%. Volume of the lake is 200000 litres. Also, find pollution after 32 days.
4. Explain Exponential Decay model.
5. How long will it take for the lake's pollution level to reach 5% of its initial level, if only fresh water flows into the lake. Consider two American Lakes where Lake Erie flows into the Lake Ontario. Volume of Lake Erie is 458×109 m^3 and flow of water into the lake is 480×106 m^3/day. Volume of Lake Ontario is 1636×109 m^3 and flow of water into the lake is 572×106 m^3/day. How long will it take for the lake's pollution level to reach 5% of its initial level, if only fresh water flows into the lake.
6. Explain Lake Pollution Model with example.
7. In a population, the initial population is 100. Suppose the population can be modeled using the differential equation with a time step of one month. Find predicted population after 2 months
8. Explain Limited growth with harvesting model with examples.
9. Explain Exponential growth of population model
10. In a fish farm, fish are harvested at a constant rate of 2100 fish per week. The per capita death rate for the fish is 0.2 fish per day per fish and per capita birth rate is 0.7 fish per day per fish.
    (i)Write down a word equation and differential equation for rate of change of fish population.
    (ii) Find when the fish population is in equilibrium& if initial fish population is 700, find fish population after a week.

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Coimbatore –641 021

Subject: Differential Equations **Subject Code: 17MMU201**

Class : I B.Sc Mathematics **Semester : II**

UNIT -III

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Compartmental Model is a model in which there is a place called …………….. which has amount of substance in and amount of substance out over time.	predator-prey	compartment	substance	epidemic	compartment
Compartmental Model is a model in which there is a place called compartment which has …………....	amount of substance in and amount of substance out over time	amount of substance out and amount of substance in over time	amount of substance in over time	amount of substance out over time	amount of substance in and amount of substance out over time
Net Rate of change of a subtance =………………………	Rate in + Rate out	Rate in- Rate out	Rate in * Rate out	Rate in / Rate out	Rate in - Rate out
……………….. model for radioactive decay can be considered as a compartmental model	Exponential decay	Epidemic	Exponential growth	Exponential pollution	Exponential decay
Exponential decay model for radioactive decay can be considered as a ……………………..	compartmental model	Exponential growth model	Exponential pollution model	Epidemic model	compartmental model
Exponential decay model for ………..decay can be considered as a compartmental model	positive	negative	radioactive	pollution	radioactive
In Exponential decay model with compartment being the radioactive material with …………….as decay of radioactive sample over time.	both input and output	no input but output	input but no output	no input but no output	no input but output

In ……………. assumption the amount of an element present is large enough so that we are justified in ignoring random fluctuations	radioactive model	Exponential growth model	Exponential pollution model	Epidemic model	radioactive model
In radioactive model assumption the amount of an element present is ………….. enough so that we are justified in ignoring random fluctuations	zero	large	small	infinite	large
In radioactive model assumption the amount of an element present is large enough so that we are justified in ignoring random ………………..	effects	fluctuations	values	forces	fluctuations
In radioactive model assumption the process is …………….. in time	continuous	dis continuous	finite	infinite	continuous
In radioactive model,we assume a ………………….. of decay for an element	constant rate	variable rate	unfixed rate	fixed rate	fixed rate
There is no ……….. in mass of the body of material in the basic assumption in radioactive model	increase	decrease	change	value	increase
There is no increase in………….. of the body of material in the basic assumption in radioactive model	force	mass	velocity	distance	mass
The initial value problem corresponding to exponential decay model is ………	dN/dt=-K, N(0)=n0,	dN/dt=N, N(0)=n0,	dN/dt=-KN, N(0)=n0,	dN/dt=KN, N(0)=0,	dN/dt=-KN, N(0)=n0,
The initial value problem corresponding to exponential decay model is dN/dt=-KN, N(0)=n0,where K=………..	K=0	K<0	K>0	K≥0	K>0
………………………can be considered as a compartmental model with two compartments	Exponential growth model	Drug Assimilation Model	Exponential pollution model	Epidemic model	Drug Assimilation Model
Drug Assimilation Model can be considered as a compartmental model with ………. compartments	zero	one	two	three	two
Exponential or Natural Growth Equation dx/dt=rx ; x(0) = x 0, r > 0	dx/dt=r-x ; x(0) = x 0, r > 0	dx/dt=r+x ; x(0) = x 0, r > 0	dx/dt=rx ; x(0) = x 0, r > 0	dx/dt=r/x ; x(0) = x 0, r > 0	dx/dt=rx ; x(0) = x 0, r > 0
………………... solution is the point at which there is no change in population	finite	infinite	Equilibrium	separable	Equilibrium
Equilibrium solution is the ………………. at which there is no change in population	solution	point	constant	variable	point

Equilibrium solution is the point at which there is ……... change in population	no	finite	infinite	limited	no
At ……………….. solution, rate of birth balance rate of death dX/dt=0.	equilibrium	finite	infinite	differential	equilibrium
At equilibrium solution, rate of ……….. balance rate of death dX/dt=0.	birth	death	decay	population	birth
At equilibrium solution, rate of birth balance rate of death ……….....	$dX/dt<0$	$dX/dt>0$	$dX/dt\neq0$	$dX/dt=0$	$dX/dt=0$
Rate of change of population=---------	Rate of births+Rate of deaths	Rate of births-Rate of deaths	Rate of births*Rate of deaths	Rate of births/Rate of deaths	Rate of births-Rate of deaths
Rate of change of drug in GI tract=………………………	Rate of drug intake/Rate of drug leaves GI tract	Rate of drug intake*Rate of drug leaves GI tract	Rate of drug intake+Rate of drug leaves GI tract	Rate of drug intake-Rate of drug leaves GI tract	Rate of drug intake-Rate of drug leaves GI tract
The differential equation for Drug Assimilation Model in case of single pill is for GI tract………	$dX/dt=1, x(0)=x0$	$dX/dt=K1(X), x(0)=x0$	$dX/dt=0, x(0)=x0$	$dX/dt=K1(X), x(0)=0$	$dX/dt=-K1(X), x(0)=x0$
The differential equation for Drug Assimilation Model in case of single pill is for bloodstream………	$dY/dt=K1(x)K2(y), y(0)=0, K1\neq K2$	$dY/dt=K1(x)+K2(y), y(0)=0, K1\neq K2$	$dY/dt=K1(x)-K2(y), y(0)=0, K1\neq K2$	$dY/dt=K1(x)/K2(y), y(0)=0, K1\neq K2$	$dY/dt=K1(x)-K2(y), y(0)=0, K1\neq K2$

Unit-IV

Syllabus

LAPLACE TRANSFORMS
Definition-Sufficient conditions for the existence of the Laplace Transform, Laplace Transform of periodic functions- Some general theorems-Evaluation of integrals using Laplace Transform.

DEFINITION, EXISTENCE, AND BASIC PROPERTIES OF THE LAPLACE TRANSFORM

DEFINITION

Let f be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

for all values of s for which this integral exists. The function F defined by the integral (9.1) is called the Laplace transform *of the function f. We shall denote the Laplace transform F of f by $\mathscr{L}\{f\}$ and shall denote $F(s)$ by $\mathscr{L}\{f(t)\}$.*

Example

Consider the function f defined by

$$f(t) = 1, \qquad \text{for } t > 0.$$

Then

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot 1\, dt = \lim_{R\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{1}{s} - \frac{e^{-sR}}{s}\right] = \frac{1}{s}$$

for all $s > 0$. Thus we have

$$\mathscr{L}\{1\} = \frac{1}{s} \qquad (s > 0).$$

Example

Consider the function f defined by

$$f(t) = t, \qquad \text{for } t > 0.$$

Then

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\,dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot t\,dt = \lim_{R\to\infty} \left[-\frac{e^{-st}}{s^2}(st+1) \right]_0^R$$
$$= \lim_{R\to\infty} \left[\frac{1}{s^2} - \frac{e^{-sR}}{s^2}(sR+1) \right] = \frac{1}{s^2}$$

for all $s > 0$. Thus

$$\mathscr{L}\{t\} = \frac{1}{s^2} \qquad (s > 0).$$

Example

Consider the function f defined by

$$f(t) = e^{at}, \qquad \text{for } t > 0.$$

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \lim_{R\to\infty} \int_0^R e^{(a-s)t}\,dt = \lim_{R\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^R$$
$$= \lim_{R\to\infty} \left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s} \right] = -\frac{1}{a-s} = \frac{1}{s-a} \qquad \text{for all } s > a.$$

Thus

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \qquad (s > a).$$

Example

Consider the function f defined by

$$f(t) = \sin bt \qquad \text{for } t > 0.$$

$$\begin{aligned}
\mathscr{L}\{\sin bt\} &= \int_0^\infty e^{-st} \cdot \sin bt\, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot \sin bt\, dt \\
&= \lim_{R\to\infty} \left[-\frac{e^{-st}}{s^2+b^2} (s \sin bt + b \cos bt) \right]_0^R \\
&= \lim_{R\to\infty} \left[\frac{b}{s^2+b^2} - \frac{e^{-sR}}{s^2+b^2} (s \sin bR + b \cos bR) \right] \\
&= \frac{b}{s^2+b^2} \qquad \text{for all } s > 0.
\end{aligned}$$

Thus

$$\mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \qquad (s > 0).$$

Example

Consider the function f defined by

$$f(t) = \cos bt \qquad \text{for } t > 0.$$

$$\begin{aligned}
\mathscr{L}\{\cos bt\} &= \int_0^\infty e^{-st} \cdot \cos bt\, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cos bt\, dt \\
&= \lim_{R\to\infty} \left[\frac{e^{-st}}{s^2+b^2} (-s \cos bt + b \sin bt) \right]_0^R \\
&= \lim_{R\to\infty} \left[\frac{e^{-sR}}{s^2+b^2} (-s \cos bR + b \sin bR) + \frac{s}{s^2+b^2} \right] \\
&= \frac{s}{s^2+b^2} \qquad \text{for all } s > 0.
\end{aligned}$$

Thus

$$\mathscr{L}\{\cos bt\} = \frac{s}{s^2+b^2} \qquad (s > 0).$$

THEOREM **Comparison Test for Improper Integrals**

Hypothesis

1. Let g and G be real functions such that

$$0 \leq g(t) \leq G(t) \qquad on \quad a \leq t < \infty.$$

2. Suppose $\int_a^\infty G(t)\,dt$ *exists.*
3. Suppose g is integrable on every finite closed subinterval of $a \leq t < \infty$.

Conclusion. *Then* $\int_a^\infty g(t)\,dt$ *exists.*

THEOREM

Hypothesis

1. Suppose the real function g is integrable on every finite closed subinterval of $a \leq t \leq \infty$.
2. Suppose $\int_a^\infty |g(t)|\,dt$ *exists.*

Conclusion. *Then* $\int_a^\infty g(t)\,dt$ *exists.*

We now state and prove an existence theorem for Laplace transforms.

THEOREM

Hypothesis. *Let f be a real function that has the following properties:*
1. f is piecewise continuous in every finite closed interval $0 \le t \le b$ $(b > 0)$.
2. f is of exponential order; that is, there exists α, $M > 0$, and $t_0 > 0$ such that

$$e^{-\alpha t}|f(t)| < M \qquad \textit{for} \quad t > t_0.$$

Conclusion. *The Laplace transform*

$$\int_0^\infty e^{-st} f(t)\, dt$$

of f exists for $s > \alpha$.

Proof. We have

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^{t_0} e^{-st} f(t)\, dt + \int_{t_0}^\infty e^{-st} f(t)\, dt.$$

By Hypothesis 1, the first integral of the right member exists. By Hypothesis 2,

$$e^{-st}|f(t)| < e^{-st} M e^{\alpha t} = M e^{-(s-\alpha)t}$$

for $t > t_0$. Also

$$\int_{t_0}^\infty M e^{-(s-\alpha)t}\, dt = \lim_{R \to \infty} \int_{t_0}^R M e^{-(s-\alpha)t}\, dt = \lim_{R \to \infty} \left[-\frac{M e^{-(s-\alpha)t}}{s-\alpha} \right]_{t_0}^R$$

$$= \lim_{R \to \infty} \left[\frac{M}{s-\alpha} \right] [e^{-(s-\alpha)t_0} - e^{-(s-\alpha)R}]$$

$$= \left[\frac{M}{s-\alpha} \right] e^{-(s-\alpha)t_0} \qquad \text{if} \quad s > \alpha.$$

Thus

$$\int_{t_0}^\infty M e^{-(s-\alpha)t}\, dt \qquad \text{exists for} \quad s > \alpha.$$

Finally, by Hypothesis 1, $e^{-st}|f(t)|$ is integrable on every finite closed subinterval of $t_0 \le t < \infty$. Thus, applying Theorem A with $g(t) = e^{-st}|f(t)|$ and $G(t) = Me^{-(s-\alpha)t}$, we see that

$$\int_{t_0}^{\infty} e^{-st}|f(t)|\,dt \qquad \text{exists if} \quad s > \alpha.$$

In other words,

$$\int_{t_0}^{\infty} |e^{-st}f(t)|\,dt \qquad \text{exists if} \quad s > \alpha,$$

and so by Theorem B

$$\int_{t_0}^{\infty} e^{-st}f(t)\,dt$$

also exists if $s > \alpha$. Thus the Laplace transform of f exists for $s > \alpha$.

Let us look back at this proof for a moment. Actually we showed that if f satisfies the hypotheses stated, then

$$\int_{t_0}^{\infty} e^{-st}|f(t)|\,dt \qquad \text{exists if} \quad s > \alpha.$$

Further, Hypothesis 1 shows that

$$\int_{0}^{t_0} e^{-st}|f(t)|\,dt \qquad \text{exists.}$$

Thus

$$\int_{0}^{\infty} e^{-st}|f(t)|\,dt \qquad \text{exists if} \quad s > \alpha.$$

In other words, if f satisfies the hypotheses of Theorem 9.1, then not only does $\mathscr{L}\{f\}$ exist for $s > \alpha$, but also $\mathscr{L}\{|f|\}$ exists for $s > \alpha$. That is,

$$\int_{0}^{\infty} e^{-st}f(t)\,dt \qquad \text{is absolutely convergent for} \quad s > \alpha.$$

Basic Properties of the Laplace Transform

Let f_1 and f_2 be functions whose Laplace transforms exist, and let c_1 and c_2 be constants. Then

$$\mathscr{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathscr{L}\{f_1(t)\} + c_2\mathscr{L}\{f_2(t)\}.$$

Translation Property

Hypothesis. *Suppose f is such that $\mathscr{L}\{f\}$ exists for $s > \alpha$.*

Conclusion. *For any constant a,*

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

for $s > \alpha + a$, where $F(s)$ denotes $\mathscr{L}\{f(t)\}$.

Example

Find $\mathscr{L}\{e^{at}\sin bt\}$. We let $f(t) = \sin bt$. Then $\mathscr{L}\{e^{at}\sin bt\} = F(s-a)$, where

$$F(s) = \mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \qquad (s > 0).$$

Thus

$$F(s-a) = \frac{b}{(s-a)^2+b^2}$$

and so

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \qquad (s > a).$$

Example

Find the Laplace transform of

$$g(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \sin t, & t > \dfrac{\pi}{2}. \end{cases}$$

$$g(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2}, \\ \cos\left(t - \frac{\pi}{2}\right), & t > \frac{\pi}{2}. \end{cases}$$

$$u_{\pi/2}(t)f(t - \pi/2) = \begin{cases} 0, & 0 < t < \frac{\pi}{2}, \\ \cos\left(t - \frac{\pi}{2}\right), & t > \frac{\pi}{2}, \end{cases}$$

$$F(s) = \mathscr{L}\{\cos t\} = \frac{s}{s^2 + 1}.$$

$a = \pi/2$, we obtain

$$\mathscr{L}\{g(t)\} = \mathscr{L}\{u_{\pi/2}(t)f(t - \pi/2)\} = \frac{se^{-(\pi/2)s}}{s^2 + 1}.$$

THEOREM

Hypothesis. *Suppose f is a periodic function of period P which satisfies the hypotheses of Theorem :*

Then

$$\mathscr{L}\{f(t)\} = \frac{\int_0^P e^{-st}f(t)\,dt}{1 - e^{-Ps}}.$$

By definition of the Laplace transform,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

The integral on the right can be broken up into the infinite series of integrals

$$\int_0^P e^{-st} f(t)\, dt + \int_P^{2P} e^{-st} f(t)\, dt + \int_{2P}^{3P} e^{-st} f(t)\, dt + \cdots + \int_{nP}^{(n+1)P} e^{-st} f(t)\, dt \cdots. \quad (9.28)$$

We now transform each integral in this series. For each $n = 0, 1, 2, \ldots$, let $t = u + nP$ in the corresponding integral

$$\int_{nP}^{(n+1)P} e^{-st} f(t)\, dt.$$

Then for each $n = 0, 1, 2, \ldots$, this becomes

$$\int_0^P e^{-s(u+nP)} f(u + nP)\, du.$$

$$e^{-nPs} \int_0^P e^{-su} f(u)\, du.$$

Hence the infinite series takes the form

$$\int_0^P e^{-su} f(u)\, du + e^{-Ps} \int_0^P e^{-su} f(u)\, du + e^{-2Ps} \int_0^P e^{-su} f(u)\, du + \cdots + e^{-nPs} \int_0^P e^{-su} f(u)\, du + \cdots$$

$$= [1 + e^{-Ps} + e^{-2Ps} + \cdots + e^{-nPs} + \cdots] \int_0^P e^{-su} f(u)\, du.$$

Now observe that the infinite series in brackets is a geometric series of first term 1 and common ratio $r = e^{-Ps} < 1$. Such a series converges to $1/(1-r)$, and hence the series in brackets converges to $1/(1-e^{-Ps})$. Therefore the right member of (9.30), and hence that of reduces to

$$\frac{\int_0^P e^{-su} f(u)\, du}{1 - e^{-Ps}}.$$

we have

$$\mathscr{L}\{f(t)\} = \frac{\int_0^P e^{-st} f(t)\, dt}{1 - e^{-Ps}}$$

Find the Laplace transform of f defined on $0 \le t < 4$ by

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ -1, & 2 \le t < 4, \end{cases}$$

and for all other positive t by the periodicity condition

$$f(t+4) = f(t).$$

The graph of f is shown in Figure 9.5. Clearly this function f is periodic of period $P = 4$. Applying formula (9.26) of Theorem 9.8, we find

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{\int_0^4 e^{-st} f(t)\,dt}{1 - e^{-4s}} \\
&= \frac{1}{1 - e^{-4s}} \left[\int_0^2 e^{-st}(1)\,dt + \int_2^4 e^{-st}(-1)\,dt \right] \\
&= \frac{1}{1 - e^{-4s}} \left[\frac{-e^{-st}}{s}\bigg|_0^2 + \frac{e^{-st}}{s}\bigg|_2^4 \right] \\
&= \frac{1}{1 - e^{-4s}} \left(\frac{1}{s}\right)[-e^{-2s} + 1 + e^{-4s} - e^{-2s}] \\
&= \frac{1 - 2e^{-2s} + e^{-4s}}{s(1 - e^{-4s})} = \frac{(1 - e^{-2s})^2}{s(1 - e^{-2s})(1 + e^{-2s})} \\
&= \frac{1 - e^{-2s}}{s(1 + e^{-2s})}.
\end{aligned}$$

POSSIBLE QUESTIONS

PART - B (5 x 2 = 10 Marks)

1. Define Laplace Transform
2. Give sufficient condition for the existence of Laplace Transform
3. Find L(1),L(t) and L(t^2) .
4. Find L(t^2 +2t+3)
5. Define piecewise continuity.

PART – C (5 x 6 = 30 Marks)

1. Find $L\{f(t)\}$ where $f(t) = 0$ when $0 < t \leq 2$

$$= 3 \; when \; t > 2.$$

2. Find $L\{f(t)\}$ where $f(t) = 1$ when $0 < t < b$

$$= -1 \; when \; b < t < 2b.$$

3. Find $L\{f(t)\}$ where $f(t) = t$ when $0 < t < b$

$$= 2b - t \; when \; b < t < 2b.$$

4. Find $L(te^{-t} sin\ t)$.

5. Find $L(\sin^3 2t)$.

6. Prove that $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\, dt = \log 2$.

7. Evaluate $\int_0^\infty t\, e^{-3t} \cos t\, dt$.

8. If $L\{f(t)\} = F(s)$ then prove that $L\{tf(t)\} = -\frac{d}{ds} F(S)$.

9. Find the Laplace transform of $\frac{\sin at}{t}$.

10. Find the Laplace transform of $\frac{1-e^t}{t}$.

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Class : I B.Sc Mathematics **Semester : II**

UNIT -IV

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If f1,f2… … ….fm are m given functions and c1,c2… … … …cm are m constants then the expression ………………...is called a linear combination of f1,f2… … …fm.	c1 f1+c2 f2+…………+cm fm	c1 f1*c2 f2*…………*cm fm	c1 f1/c2 f2/…………/cm fm	c1 f1-c2 f2-…………-cm fm	c1 f1+c2 f2+…………+cm fm
If f1,f2… … ….fm are m given functions and c1,c2… … … …cm are m constants then the expression c1 f1+c2 f2+ … … … ….+cm fm is called a …………….. of f1,f2… … …fm.	non linear combination	homogeneous equation	non homogeneous equation	linear combination	linear combination
Any ………….. combination of solutions of the homogeneous linear differential equation is also a solution of homogeneous equation.	linear	nonlinear	zero	separable	linear
Any lienar combination of solutions of the ……………… linear differential equation is also a solution of homogeneous equation.	homogeneous	non homogeneous	singular	non singular	homogeneous
Any lienar combination of solutions of the homogeneous linear differential equation is also a …………………...of homogeneous equation.	value	separable	solution	exact	solution
The n functions f1,f2… … ….fn are called …………….. on $a \leq x \leq b$ ifthere existsa constants c1,c2… … … …cn not all zero,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x.	linearly dependent	linearly independent	finite	infinite	linearly dependent

The n functions f1,f2… … ….fn are called linearly dependent on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not ………………..,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x.	all zero	one zero	two zero	n zero	all zero
The n functions f1,f2… … ….fn are called linearly dependent on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not all zero,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=…….. for all x.	1	2	3	0	0
The functions f1,f2… … ….fn are called ………………….. on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x implies that c1=c2=… … … …=cn=0.	linearly dependent	linearly independent	finite	infinite	linearly independent
The functions f1,f2… … ….fn are called linearly independent on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x implies that c1=c2=… … … …=cn=…………..	0	1	2	3	0
The functions f1,f2… … ….fn are called linearly independent on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)…………………….. for all x implies that c1=c2=… … … …=cn=0	equal to 0	< 0	> 0	not equal to 0	equal to 0
The nth order ………………….linear differential equations always possess n solutions that are linealy independent.	homogeneous	non homogeneous	singular	non singular	homogeneous
The nth order homogeneous linear …………………. equations always possess n solutions that are linealy independent.	differential	integral	bernoulli	euler	differential
The nth order homogeneous linear differential equations always possess ……………………solutions that are linealy independent.	zero	finite	inifinite	n	n
The nth order homogeneous linear differential equations always possess n solutions that are …………………	linearly dependent	linearly independent	finite	infinite	linearly independent
Let f1, f2,.. .. ..fn be n…………….functions each of which has an (n-1)st derivative on real interval a ≤ x ≤ b	real	complex	finite	infinite	real

Let f1, f2,.. .. ..fn be n real functions each of which has an ------------derivative on real interval $a \le x \le b$	n	n-1	n+1	n+2	n-1
Let f1, f2,.. .. ..fn be n real functions each of which has an (n-1)st derivative on ------------ interval $a \le x \le b$	real	complex	finite	infinite	real
The ……………...solution of homogeneous equation is called the complementary function of equation.	explicit	implicit	general	particular	general
The general solution of -------------- equation is called the complementary function of equation.	homogeneous	non homogeneous	singular	non singular	homogeneous
The general solution of homogeneous equation is called the ------------------ function of equation.	real	complex	complementary	particular	complementary
Any ………………….solution of linear differential equation involving no arbitrary constants is called particular integralof this equation.	explicit	implicit	general	particular	particular
Any particular solution of linear differential equation involving -------------- arbitrary constants is called particular integralof this equation.	finite	infinite	no	one	no
Any particular solution of linear differential equation involving no arbitrary constants is called ……………. integralof this equation.	general	particular	finite	infinite	particular
The soluation--------------- is called the general solutionsof linear differential equations.	yc-yp	yc+yp	yc*yp	yc/yp	yc+yp
The soluation yc+yp is called the --------------- solutionsof linear differential equations.	explicit	implicit	general	particular	general
In general solution yc+yp where yc is ……………..function	real	complex	complementary	particular	complementary
In general solution yc+yp where yp is ……………..function	explicit	implicit	general	particular	particular

UNIT – V

SYLLABUS

Equilibrium points, Interpretation of the phase plane, predatory-prey model and its analysis, epidemic model of influenza and its analysis, battle model and its analysis.

Equilibrium points

Let us consider the linear differential equations (coupled) of first order

$$\frac{dX}{dt} = Y, \qquad \frac{dY}{dt} = -X.$$

A point where the solutions of a coupled system of differential equations are constant is known as *equilibrium points*, i.e. where $dX/dt = 0$ and $dY/dt = 0$, simultaneously.

Therefore, from the given differential equations, we get

$$Y = 0, \qquad X = 0$$

So $(X,Y) = (0,0)$ is the equilibrium solution.

Trajectories and phase-plane diagram

Let us consider the (X,Y)-plane: known as the phase-plane. Dividing the plane into four quadrants as shown in the figure below, in the first quadrant we have $X > 0$ and $Y > 0$, i.e. $dX/dt = Y > 0$ *and* $dY/dt = -X < 0$. Hence $X(t)$ is increasing and $Y(t)$ is decreasing and for any solution in that quadrant, we get a direction vector, given by the arrow in figure below. Similarly we can consider each quadrant. Hence we can conclude that the phase-plane trajectories moving in a clockwise direction are the solutions.

Using the chain rule

Since on the right hand side, none of the differential equation has time variable t i.e. time variable t does not explicitly used in any differential equation. But the derivatives on the left hand side are with respect to time so the solutions will be time dependent. Hence we find an relation between X and Y, independent of t. In other word, we may express Y as a function of X. That is, we are making X the independent variable while previously it was a time t dependent variable.

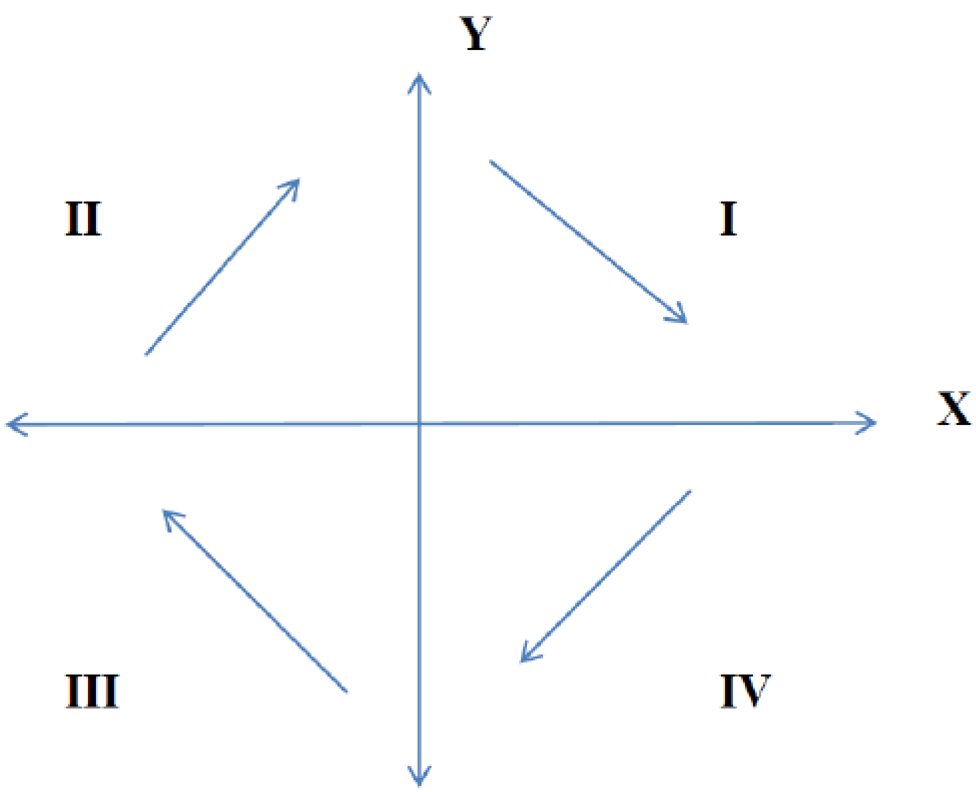


Fig: Direction vector for the trajectories in phase plane of above equations

An expression for the chain rule is

$$\frac{dY}{dt} = \frac{dY}{dX}\frac{dX}{dt}$$

which gives the derivative of Y with respect to t in terms of the derivative of Y with respect to X and the derivative of X with respect to t. Dividing by $\frac{dX}{dt}$

both side, we get

$$\frac{dY}{dX} = \frac{dY/dt}{dX/dt} \qquad 2$$

We substitute the value from the equations (1) into (2) which give

$$\frac{dY}{dX} = -\frac{Y}{X}. \qquad 3$$

Hence we get a first-order differential equation with Y a function of X.

Now we can easily solve the differential equation (3) by variable separable mothod, we can write

$$Y\frac{dY}{dX} = -X. \qquad 4$$

Solution: Integrate both sides with respect to the independent variable X we get

$$\int Y\frac{dY}{dX}\,dX = \int -X\,dX.$$

$$\Rightarrow \quad \int Y\,dY = \int -X\,dX.$$

Which gives

$$\frac{1}{2}Y^2 = -\frac{1}{2}X^2 + A$$

Where A is the integration constant. If we multiplying both side by 2, we have t

$$X^2 + Y^2 = B$$

where $B = 2A$. With the help of initial condition we can obtain the value of B This solution is the equation of a circle. It defines the paths drawn out by the (X,Y) pair over time, depending on starting conditions or the initial values. This will be the exact solutions to the phase-plane trajectories.

Interpretation of the phase-plane

After the analysis or interpretation of these trajectories, we found that the system start at the point (x_0, y_0) in the phase-plane if a system has its initial values x_0 and y_0, as time changes, it gives the trajectories curve (circle in above discussed case) in clockwise direction as sketched in above figure. The value of $X(t)$ and $Y(t)$ will be the coordinates of this trajectory at any consequent time. In the case of closed trajectory, the motion will be repeated continuously as in above discussed case of circle.

Generally we need an exact solution of the original coupled equations to check how the system varies with time. Sometimes it is not possible then we use the chain rule to gather valuable information about the system.

Phase-plane analysis is an easy technique to understand some common feature of the system which can be done by drawing a phase-plane diagram together with the phase-plane trajectories. If the differential equations are adequately simple, we may get an exact expression which relates the two dependent variables and describes the trajectory path by eliminating the time variable with chain rule.
The behavior of solutions for a variety of initial conditions can be easily understood with the phase-plane diagram. In the above example, we saw that all solutions of the differential equations have phase-trajectories as circles. We see here, as we move along the trajectory, both the variable X and Y shall return to their original values so the plot for both variables as functions of time should be oscillations. Also, the amplitude of the oscillation is reduced as the initial point approaches the equilibrium point, with the equilibrium point itself corresponding to a solution which is constant in time.

Note: To get a single first-order equation by reducing the coupled differential equations, we have to pay some price and we lost information about time in this procedure

Skill developed:

- Understand the theory of equilibrium solution.
- Create the direction of trajectories.
- Able to draw phase-plane based on the information on equilibrium points and trajectory direction.
- Use of chain rule to eliminate time variable and to reduce a coupled pair of differential equation into a single differential equation.

result of the model we obtain a pair of coupled differential equations, where the numbers of soldiers in the green and yellow army is denoted by $G(t)$ and $Y(t)$ denote, respectively. We supposed, both armies used only aimed fire. And we obtain a pair of the differential equations.

$$\frac{dG}{dt} = -\lambda_1 Y, \qquad \frac{dY}{dt} = -\lambda_2 G, \qquad (15)$$

Where λ_1 and λ_2 are attrition coefficients (positive constant).

Applying the chain rule to eliminate the time variable t

We can write

$$\frac{dY}{dG} = \frac{dY/dt}{dG/dt} = \frac{\lambda_2}{\lambda_1}\frac{G}{Y}.$$

$$\Rightarrow \quad \frac{dY}{dG} = \lambda \frac{G}{Y} \qquad (16)$$

Thus we get a single first order differential equation independent of time variable t, which relates Y and G.

Example: Find the solution of the differential equation

$$\frac{dY}{dG} = \lambda \frac{G}{Y}$$

Solution: Using separation of variable and then integrate both side with respect to independent variable G. It gives

$$\int Y \frac{dY}{dG} dG = \int \frac{\lambda_2}{\lambda_1} G dG.$$

$$\Rightarrow \quad \int Y\, dY = \int \frac{\lambda_2}{\lambda_1} G\, dG.$$

Integrate both side to get the equation

$$\frac{1}{2} Y^2 = \frac{\lambda_2}{2\lambda_1} G^2 + A,$$

Where A is constant of integration. Multiplying both sides of the equation by 2 we get

$$Y^2 = \frac{\lambda_2}{\lambda_1} G^2 + B$$

Where $B = 2A$ is also an arbitrary constant.

Relating the initial condition $G(0) = g_0 \text{ and } Y(0) = y_0$ we get

$$y_0^{\,2} = \frac{\lambda_2}{\lambda_1} g_0^2 + B$$

So that

$$B = y_0^{\,2} - \frac{\lambda_2}{\lambda_1} g_0^2$$

Example : Find the equilibrium [points of the differential equations $\frac{dX}{dt} = \lambda_1 X - c_1 XY \text{ and } \frac{dY}{dt} = c_2 XY - \lambda_2 Y$

Solution: we set $dX/dt = 0 \text{ and } dY/dt = 0$ to get the equations

$$\lambda_1 X - c_1 XY = 0, \qquad -\lambda_2 Y + c_2 XY = 0$$

Or in factor form we can write

$$X(\lambda_1 - c_1 Y) = 0, \qquad (17)$$

$$Y(-\lambda_2 + c_2 X) = 0. \qquad (18)$$

Two possible solutions arise from (17) which are: $X = 0$ *or* $\lambda_1 - c_1 Y = 0$. Each case is necessary to consider. All the parameters $\lambda_1, \lambda_2, c_1$ *and* c_2 are positive (non-zero) constants.

If $X = 0$, then putting this into (18) gives $-\lambda_2 Y = 0$ hence $Y = 0$. so $(X, Y) = (0,0)$ is one possible solutions of both equations.

If , $\lambda_1 - c_1 Y = 0$, then $Y = \lambda_1 / c_1$. Put this into (18) gives $-\lambda_2 + c_2 X = 0$ or $X = \lambda_2 / c_2$. Hence the second solution of both equations is $(X, Y) = (\lambda_2 / c_2, \lambda_1 / c_1)$.

Predators and prey

In this section, we develop a simple predator-prey model for omnivores using the evolution of population of small insect pests which interact with another population of beetle predators.

Background of model

There are several types of predator-prey interactions: that of carnivores which eat animal species, that of herbivores which eat plant species, that of cannibals which eat their own species and that of leeches which lives on or in another species (the host).

Model assumptions

Initially, few preliminary assumptions are made to build the model, which are as follows:

- To neglect random differences between individuals we assume the populations are sufficiently large.
- Initially, DDT effect is discounted, but later the model is modified to incorporate its influence on the system.
- We assume that the predator and the prey are only two populations, which affect the environment.
- In the absence of a predator, the prey population can grows exponentially.

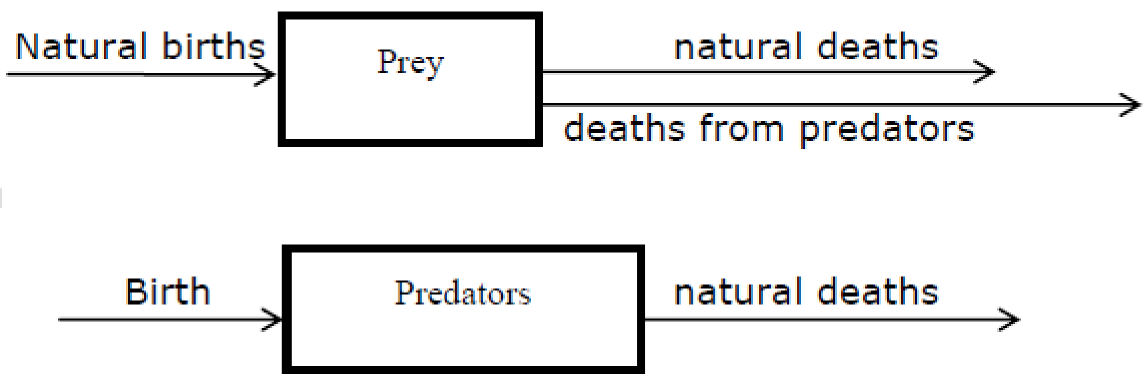
Compartmental model

The number of prey and the number of predators are two separates quantities which vary with time. It is better to consider the population density i.e. number per unit area, rather than population size, for population of animal as we do here. The system can be defined in two word equations, one for the rate of change of predator density and one for the rate of change of prey density.

Example

Determine a word equation and appropriate compartment diagram for the prey and predator both.

Solution: Births is the only inputs and Deaths is the only outputs for each population. Though, capturing and eating by the predator is the cause for the prey deaths. This is shown in the input-output compartmental diagram of figure 2. Here we consider two reasons for prey; one is natural prey deaths and the other prey deaths due to predators. Similarly we consider and differentiate between natural predator births, taking place in absence of prey, and additional births and that would occur due to the prey being eaten by predators. The input–output diagram for the predator-prey model is



input-output diagram for 2- species predator and prey model.

The applicable word equations for the model are

$$\left\{\begin{array}{l}\text{rate of change of}\\ \text{prey}\end{array}\right\}=\left\{\begin{array}{l}\text{rate of natural prey}\\ \text{births}\end{array}\right\}-\left\{\begin{array}{l}\text{rate of natural prey}\\ \text{deaths}\end{array}\right\}-\left\{\begin{array}{l}\text{rate of prey killed}\\ \text{by perdator}\end{array}\right\},$$

$$\left\{\begin{array}{l}\text{rate of change of}\\ \text{predator}\end{array}\right\}=\left\{\begin{array}{l}\text{rate of predator}\\ \text{births}\end{array}\right\}-\left\{\begin{array}{l}\text{rate of natural predator}\\ \text{deaths}\end{array}\right\} \qquad (5)$$

The rate of births from an *individual* prey is defined as per-capita birth rate and it does not depend on the density of predators. Let us assume that a constant b_1 is the per-capita birth rate for the prey (the scale insect). Similarly, the natural pre-capita death rate of the scale insect is a constant a_1. On the other hand the per-capita death rate of prey due to being killed by the predators will depend on the predator density and it will be directly proportional to density of predators. For simplicity let us consider this per-capita rate is proportional to the predator density. If the density of predators is more, the probability of an individual prey will be eaten is more. We assume that prey density does not affect a constant per-capita death rate for the predators. It is difficult to get the per-capita birth rate of predator. We assume that the important necessity for the births of the predator are prey, so the per-capita birth-rate for the predators will be the sum of a natural birth rate (which is constant) and a supplementary birth rate which is proportional to the rate of prey killed by predator. It is obvious that if the amount of prey available (food) is more at any time, the per-capita birth rate of predator will increase at that time.

Example

Formulate differential equations for the predator and prey density using the above assumptions and word equations 5.

Solution: Let the number of prey per unit area is denoted by $X(t)$ and the number of predators per unit area $Y(t)$. Let us assume that a constant b_1 is the per-capita birth rate for the prey (the scale insect). Similarly, the natural pre-capita death rate of the prey is a constant a_1 and per capita death rate of the predator is given by a_2.

The overall rate can be obtained by multiplying the per-capita rates by the corresponding population densities, we can write,

$$\begin{aligned} \{rate\ of\ \ prey\ births\} &= b_1 X(t), \\ \{rate\ of\ \ prey\ natural\ deaths\} &= a_1 X(t), \\ \{rate\ of\ predator\ deaths\} &= a_2 Y(t) \end{aligned} \tag{6}$$

Since deaths of prey (killed) is proportional to the predator density, for the prey deaths, the per-capita death rate is defined as $c_1Y(t)$, with c_1 as the positive constant of proportionality. Thus the rate at which prey are killed or eaten by predators is given by $c_1Y(t)X(t)$. The birth rate of predator has a factor which is proportional to this rate of prey killed or eaten by predators, so we write

$$\{rate\ of\ prey\ killed\ by\ predators\} = c_1Y(t)X(t),$$
$$\{rate\ of\ predator\ births\} = b_2Y + kc_1Y(t)X(t) \qquad (7)$$

Where b_2 is per-capita birth is rate of predator and k is positive constant of proportionality.

Now we change the word equation (5) into the pair of differential equations with the help of the equations (6-7).

$$\frac{dX}{dt} = b_1X - a_1X - c_1XY \Rightarrow \frac{dX}{dt} = (b_1 - a_1)X - c_1XY,$$
$$\frac{dY}{dt} = b_2Y + fc_1XY - a_2Y \Rightarrow \frac{dY}{dt} = (b_2 - a_2)Y + fc_1XY.$$

Let $\lambda_1 = b_1 - a_1,\ -\lambda_2 = b_2 - a_2$ and $c_2 = fc_1$, then

$$\frac{dX}{dt} = \lambda_1X - c_1XY, \qquad \frac{dY}{dt} = c_2XY - \lambda_2Y \qquad (8)$$

Where, $\lambda_1, \lambda_2, c_1$ and c_2 are all positive constants.

This system of equation is called the *Lotka-Volterra predator-prey system*. The constraints c_1 and c_2 are known as interaction parameters. Since on the right hand side of each differential equation we have positive and negative terms, we can expect the growth or decline in population. Further, the differential equations in (8) are coupled as solution of one equation will be used to solve other differential equation. These differential equations are nonlinear as they have the product XY. The product XY can be interpreted as it is proportional to the rate of contacts (encounters) between the two species i.e. predator and prey.

Example

Check the Predator-Prey model in the restrictive cases of prey without predator, and predator without prey.

Solution: Suppose there are no predator i.e number of prey is zero so that $Y = 0$. The equations then reduce to

$$\frac{dX}{dt} = \lambda_1 X$$

$$\Rightarrow \quad \frac{dX}{X} = \lambda_1 dt$$

$$\Rightarrow \quad X(t) = e^{\lambda_1 t}$$

Hence the prey grows exponentially in the absence of predators.

Similarly, If there are no prey then $X = 0$ and the equation reduce to

$$\frac{dY}{dt} = -\lambda_2 Y \Rightarrow Y(t) = e^{-\lambda_2 t}$$

That is, the predator population will decay exponential and die out in the absence of prey.

An epidemic model for influenza

Here a model is developed to describe the spread of disease in population and apply it to describe the influenza in city. To do so the population is divided into three groups: those susceptible to catching the disease, those infected with disease and capable of spreading it and those who have recovered and are immune from the disease. A system of two coupled differential equations is obtained by modelling these interacting groups.

Model assumptions

In the case of considering a disease, the population can be categorized into different classes; susceptible $S(t)$ and infectious infectives $I(t)$, where t denotes the time. The population liable to catch the disease is called the susceptibles, while the infectious infectives are those infected with the diseases that are capable to transfer it to a susceptible. The last category is of those who have recovered from the disease and who are now safe from further infection of the disease.

Initially, some assumptions are made to build the model, which are as follows:

- To ignore the random differences between individuals, we assume the populations of susceptibles and infectious infectives are large.
- We assume that the disease is spread by contact only and ignore the births and deaths in this model.
- We set the latent period for the disease equal to zero.
- We assume all those who recover from the disease are then safe (at least within the time period considered).
- At any time, the population is mixed homogenously, i.e. we assume that the susceptibles and infectious infectives are always randomly distributed over the area in which the population lives.

Formulating the differential equations

The rate of change in the number of susceptibles and infectious infectives describe in word equations with the help of an input-output compartment diagram. This process is illustrated in the following example.

Example

Create a compartmental diagram for the model and develop appropriate word equation for the rate of change of susceptibles and infectious infectives.

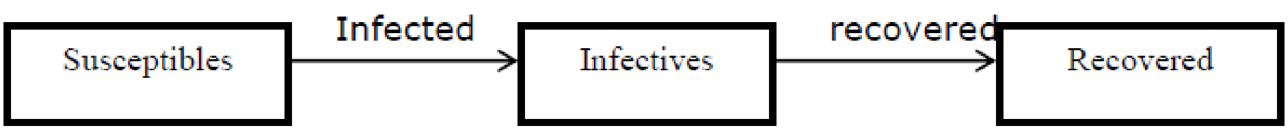
Solution: Since births are ignored in the model and infectious infectives cannot become susceptibles again i.e. the loss of those who become infected is the only way to change the number of susceptibles. The number of infectives decreases due to those infectives who die, become safe or are isolates and changes due to the susceptibles becoming infected.

The appropriate word equations are

$$\{\text{rate of change in no. of susceptibles}\} = -\{\textit{rate of susceptibles become infected}\}$$

$$\{\text{rate of change in no. of infectives}\} = \left\{\begin{matrix}\textit{rate of susceptibles become} \\ \textit{infected}\end{matrix}\right\} - \left\{\begin{matrix}\textit{rate of infectives} \\ \textit{have recovered}\end{matrix}\right\} \quad (1)$$

$$\{\textit{rate of change in no. of recovered}\} = \{\textit{rate of infectives have recovered}\}$$



Compartmental diagram for the epidemic model of influenza in a city, where there is no reinfection.

Let us first consider to model the total rate of susceptibles infected that only a single infective spread the infection in susceptibles. It is clear that the growth in the number of infectives will be greater due to greater the number of susceptibles. Thus, the rate of susceptibles diseased by a single infective will be an increasing function of the number of susceptibles. For ease, let us assume that the rate of susceptibles infected by a single infective is directly proportional to the number of susceptibles. Let $S(t)$ be the number of susceptibles at time t and $I(t)$ be the number of infectives at time t, then

$$\{\textit{rate of susceptible infected}\} \propto S(t)$$

$$\Rightarrow \{\textit{rate of susceptible infected}\} = \lambda S(t)$$

Where, constant λ is called the *transmission coefficient* or infection rate (Proportionality constant).

Hence, $\lambda S(t)$ will be the rate of susceptibles infected by a single infective and if we multiply $\lambda S(t)$ to the number of infectives, we will get total rate of susceptibles infected by infectives. Hence

$$\{rate\ of\ susceptible\ \inf ected\} = \lambda S(t)I(t) \qquad (2).$$

We must also account for those who have recovered from disease. In general, those infectives who died due to disease, those who become protected to the disease and those who become isolated will be counted as removed. The number of infectives removed in the time interval should depend only on the number of infectives, but not upon the number of susceptibles. Let the rate of infectives recovered from the disease is directly proportional to the number of infectives. We write

$$\{rate\ of\ \inf ectives\ recoverd\ from\ the\ disease\} \propto I(t)$$
$$\Rightarrow \{rate\ of\ \inf ectives\ recoverd\ from\ the\ disease\} = \delta I(t) \qquad (3)$$

Where constant δ is called *recovery rate* or the removal rate (constant of proportionality). This rate is a per-capita rate. The residence time in the infective compartment, i.e. the mean that an individual is infectious can be recognized as the reciprocal of the recovery rate i.e. δ^{-1}. Normally the infectious period for influenza is 1-3 days.

dS/dt is the rate of change in the number of susceptibles with respect to time and the rate of change in the number of infectives with respect to time is given by dI/dt. The rate of change in the number of recovered from the disease i.e. recovered is given by dR/dt. Finally the population word equations 1can be written in mathematical form with the use of equations 2 and 3 .

$$\begin{aligned} \frac{dS}{dt} &= -\lambda SI, \\ \frac{dI}{dt} &= \lambda SI - \delta I, \qquad (4) \\ \frac{dR}{dt} &= \delta I, \end{aligned}$$

with initial condition $S(0)=s_0$, $I(0)=i_0$ and $R(0)=0$.
Equation 4, a coupled system of nonlinear differential equations, were originally derived by Kermack and McKendrick in 1927 (Kermack and McKendrick, 1927). Since the R variable does appear only in the third differential equation. So the coupled system in (4) without third differential equation can be studied as a system on its own.

Model of a Battle

Now we study an original type of population interaction: battle between two contrasting groups or a destructive struggle. These may be fights between two aggressive insect groups, human armies or athletic teams. We will develop the model for the battle of two human armies. Many other example can modeled after generalizing of the model.

Background

Since the ancient times we have seen/heard about the battles between armies. In ancient times battles were mostly fought hand-to-hand. After the development of many disasters weapons, the battle has been fought with weapons like gun machine etc. Although there are many reasons to influence the battle outcome but numerical superiority and superior military training are crucial. F. W. Lanchester who was famous for his contributions to the theory of fight first developed this model in 1920s.

We want to develop a simple model to predict the soldier's strength in each army at any given time, provided we know the initial number of soldiers in each army.

Model assumptions

Initially few basic assumptions are made and then develop the model based on these.

- To neglect the random differences between armies, we assume the number of soldiers is sufficiently large.
- There are no backups and no functioning loses (i.e. due to desertion or disease).

Few assumptions can be easily relaxed at a later stage in case of inadequate model. We take an example of army to develop this model.

Example

Let us suppose that green army and yellow army are two opposite groups or populations. Draw the suitable compartment diagram and linked word equations for the number of soldiers in both the green and yellow armies.

Solution: Since there are backups or operational losses, the number of soldiers who are injured by the other army can change each population size. So we can prepare an input-output diagram of figure 3.

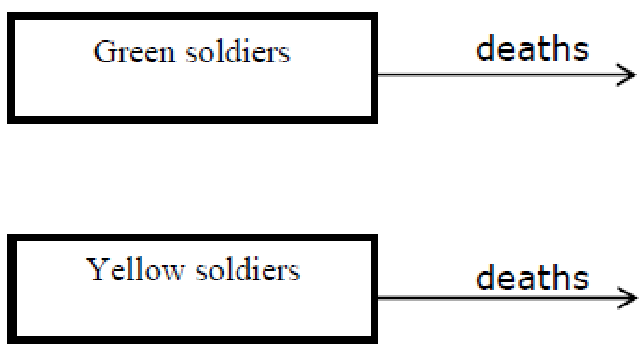


Figure 3: Compartment diagram for the simple battle model.

Thus, the word equations for the battle model at any time,

$$\{\text{rate of change of green soldiers}\} = -\{\text{rate red soldiers wounded by yellow army}\}$$
$$\{\text{rate of change of yellow soldiers}\} = -\{\text{rate blue soldiers wounded by green army}\} \quad (9)$$

In a real battle situation there is a combination of shots; (a) one fired into an area where the chances that the enemy will be hidden are more and (b) one fired directly at a soldier of the opposite army. The method of firing can dominate some battles. We consider these two ideologies of shots as *target fire* and *arbitrary fire*. For both the armies we assume only targeted fire in the model.

In the targeted fire ideology, we consider all targets are visible to army persons firing at them. If the yellow army used targeted fire on the green army, then each time an individual green soldier is targeted by a yellow soldier. The rate of injured green army soldiers gets affected only by the number of yellow soldiers firing at them but not on the number of green soldiers. For arbitrary fire a soldier firing a gun on hidden target, into a region where opponent soldiers are known to be hidden. So in arbitrary fire we consider the rate of enemy soldiers wounded will depend on both the army strength i.e. number of soldiers firing and the soldiers being fired at.

Formulating the differential equations

Let the number of soldiers of the green army is denoted by $G(t)$ and the number of soldiers of the yellow army is denoted by $Y(t)$. We assume that both ate armies fired on visible target.

After the above discussion we can make the following assumptions:

- The rate at which the soldiers are wounded is directly proportional to the number of enemy/opponent soldiers only for targeted fire.
- The rate of soldiers wounded is directly proportional to both number of soldiers in arbitrary fire.

These assumptions can be expressed mathematically by writing

$$\{rate\ green\ soldiers\ wounded\ by\ yellow\ army\} = \lambda_1 Y(t),$$

$$\{rate\ yellow\ soldiers\ wounded\ by\ green\ army\} = \lambda_2 G(t) \qquad (10)$$

Where λ_1 and λ_2 are positive constant of proportionality, and are called *attrition coefficients*. They measure the effectiveness of yellow army and green army respectively.

We also assume that attrition rates depend only on the firing rates and are a measure of the success of each firing.

Now we put equation (10) into the word equation (9), where dG/dt denotes the rate of change in the number of green soldiers and it is dY/dt f represent the change in the number of yellow soldiers. So the two simultaneous differential equations are

$$\frac{dG}{dt} = -\lambda_1 Y, \qquad \frac{dY}{dt} = -\lambda_2 G \qquad (11)$$

Interpretation of parameters

Now to refine the model we try to express the parameters λ_1 and λ_2 in terms of possible quantities which could be measured. The soldiers are wounded at a rate which depends on both the firing rate and probability of a shot hitting a target.

Now again from equations (10). Consider a single yellow soldier firing at the green army. Let f_y be a constant rate at which each yellow soldier fires (rate of bullet fired) . Then

$$\begin{Bmatrix} \text{rate of green soldiers} \\ \text{wounded by single} \\ \text{yellow soldier} \end{Bmatrix} = \begin{Bmatrix} \text{rate of bullets} \\ \text{fired in time} \\ \text{interval} \end{Bmatrix} \times \begin{Bmatrix} \text{probability of} \\ \text{a single bullet} \\ \text{hitting target} \end{Bmatrix} = f_y p_y$$

Where p_y is the probability (constant) that a green soldier is wounded by a single bullet from the yellow soldiers. Hence the green soldiers wounded by the entire yellow army (per unit time) can be counted by multiply by the number of yellow soldiers. This gives

$$\{\text{rate of green soldiers wounded by yellow army}\} = f_y p_y Y(t) \qquad (12)$$

Similarly,

$$\{\text{rate of yellow soldiers wounded by green army}\} = f_g p_g G(t)$$

Equating equation (10) and equation (12) ,we get the attrition rates, or coefficients, λ_1 and λ_2 as

$$\lambda_1 == f_y p_y, \qquad \lambda_2 = f_g p_g \qquad (13)$$

Where f_g denotes the firing rate by the single green soldier and the probability that a single green bullet hits its target is denoted by p_g.

The probability of a single bullet wounding a soldier cannot be constant for arbitrary fire. It will fluctuate and depend on the number of target soldiers actually occurs within a targeted area. Thus, this probability will get affected by the number of target soldiers and the area into which the opposite army fired both.

POSSIBLE QUESTIONS

PART – B (5 x 2 = 10)

1. Define equilibrium points.
2. Write short note on predator-prey model.
3. Write the basic assumptions for predator-prey model
4. Write short note on Interpretation of the phase plane.
5. Write short note on predator-prey model.

PART – C (5 x 6 = 30 Marks)

1. Find the equilibrium points of the differential equations $\frac{dX}{dt} = \lambda_1 X - c_1 XY$ and $\frac{dY}{dt} = c_2 XY - \lambda_2 Y$.
2. Explain the Epidemic Model of Influenza.
3. Determine a word equation and appropriate compartment diagram for the prey and predator both
4. Apply chain rule to find relation between X and Y for the differential equations $\frac{dX}{dt} = -2xy$ and $\frac{dY}{dt} = -3y$.
5. Formulate differential equations for the predator and prey density model.
6. Find equilibrium points for the following equations $x' = 2x - 3xy$, $y' = xy - 2y$.
7. Write an note on Interpretation of the phase plane.
8. Find equilibrium points for the following equations $x' = 3x - xy$, $y' = y - 2xy$.
9. Explain the battle model with examples.
10. Formulating the differential equations for the predator and prey density .

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Subject: Differential Equations **Subject Code: 17MMU201**

Class : I B.Sc Mathematics **Semester : II**

UNIT - V

PART A (20x1=20 Marks)

(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
An …………………….. for influenza is developed to describe the spread of disease in population	radioactive model	Exponential growth model	Exponential pollution model	Epidemic model	Epidemic model
An epidemic model for influenza is developed to describe the ……………………. in population	spread of virus	spread of medicine	spread of disease	spread of blood	spread of disease
An epidemic model for influenza is developed to describe the spread of disease in population.It is divided into ……….. groups	one	two	three	four	three
A system of …….. coupled differential equations is obtained by modelling these interacting groups	one	two	three	four	two
A system of two coupled ……………...equations is obtained by modelling these interacting groups	differential	integral	linear	non linear	differential
The population liable to catch the disease is called the………………	susceptibles	in susceptibles	infectious	disinfectious	susceptibles
To ignore the random differences between individuals, we assume the populations of susceptibles and infectious infectives are ………..	zero	small	large	infinite	large
The latent period for the disease equal to …………	zero	one	two	three	zero
The rate of change in the number of ………………………describe in word equations with the help of an input-output compartment diagram.	susceptibles and fectious infectives	unsusceptibles and infectious infectives	unsusceptibles and fectious infectives	susceptibles and infectious infectives	susceptibles and infectious infectives

The rate of change in the number of susceptibles and infectious infectives describe in word equations with the help of an ………………diagram.	birth-death	growth-deccay	input-output compartment	radioactive	input-output compartment
The formulation differential equation for the Predator-prey epidemic model the constant ʎ is called the ……………. or infection rate	transmission coefficient	partial coefficient	differential coefficient	integral coefficient	transmission coefficient
The formulation differential equation for the Predator-prey epidemic model the constant ʎ is called the transmissio coefficient or …………………..	susceptibles rate	birth rate	death rate	infection rate	infection rate
Rate of susceptible infected=………………	ʎ S(t) I(t)	ʎ S(t)/ I(t)	ʎ/ S(t) I(t)	ʎ S(t)	ʎ S(t) I(t)
The number of …………... removed in the time interval should depend only on the number of infectives	susceptibles	in susceptibles	infectives	disinfectives	infectives
The number of infectives removed in the time interval should depend only on the number of infectives	infectives	disinfectives	susceptibles	unsusceptibles	infectives
The rate at which susceptible converted into ……………. is proportionate to the number of susceptibles and infectives both	susceptibles	in susceptibles	infectived	disinfectived	infectived
The rate at which susceptible converted into infected is proportionate to the number of ………………	only susceptibles and not infectives	only infectives but not susceptibles	only susceptibles	susceptibles and infectives both	susceptibles and infectives both
The rate at which infectives recover and are removed is proportionate to the number of ………………..	only susceptibles and not infectives	only infectives	only susceptibles	only infectives	only infectives
The number of prey and the number of predators are two separates quantities which vary with ……	time	variable	constant	value	time
A simple ………………model for omnivores using the evolution of population of small insect pests	predator-prey	Exponential growth model	Exponential pollution model	Epidemic model	predator-prey
A simple predator-prey model for ……………..using the evolution of population of small insect pests	herbivores	omnivores	carnivores	both omnivores & herbivores	omnivores
In the absence of a …………..., the prey population can grows exponentially	predator	susceptibles	infective	Epidemic	predator

In the absence of a predator, the prey population can grows ……………….	graduately	geometrically	differentially	exponentially	exponentially
The …………………. are only two populations, which affect the environment.	predator-prey	Exponential growth model	Exponential pollution model	Epidemic model	predator-prey
The predator and the prey are only ………. populations, which affect the environment.	one	two	three	four	two
The predator and the prey are only two populations, which affect the ……………..	environment	animals	insects	birth	environment
capturing and eating by the predator is the cause for the prey……………….	birth	death	population	growth	death
……………….. by the predator is the cause for the prey deaths	capturing	eating	circulating	capturing and eating	capturing and eating
The rate of …………... from an individual prey is defined as per-capita birth rate and it does not depend on the density of predators.	births	deaths	growths	decays	births
The rate of births from an individual prey is defined as per-capita birth rate and it does not depend on the …………of predators.	velocity	force	dencity	mass	dencity
The phase-plane ………. moving in a clockwise direction are the solutions	angle	trajectories	density	force	trajectories
The phase-plane trajectories moving in a …………... direction are the solutions	clockwise	anticlockwise	positive	negative	clockwise
The phase-plane trajectories moving in a clockwise direction are the …………..	solutions	equations	constant	vatiables	solutions
……….. variable t does not explicitly used in any differential equation	constant	time	mass	density	time
Time variable t does not ……………….used in any differential equation	implicitly	explicitly	finitely	infinitely	explicitly
Time variable t does not explicitly used in any ……………...equation	constant	differential	integral	exponential	differential