KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: Dr. K. KALIDASS
SUBJECT NAME: REAL ANALYSIS SUB.CODE: 19MMU202
SEMESTER: II CLASS: I B.SC MATHEMATICS

S. No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1	1	Introduction to sets	S1, Ch 1,1-2
2	1	Theorems of finite sets	S1, Ch 1,3
3	1	Theorems of infinite sets	S1, Ch 1,4-5
4	1	Examples of countable sets	S1, Ch 1, 6
5	1	Examples of uncountable sets	S1, Ch 1, 7-8
6	1	Tutorial	
7	1	Theorems on bounded sets	S1, Ch 1, 9-10
8	1	Theorems on suprema and infima	S1, Ch 1,1-11-12
9	1	Completeness property of $\mathbb{R}$	S4, Ch 3, 25
10	1	Archimedean property of $\mathbb{R}$	S1, Ch 2,38
11	1	Intervals	S1, Ch 2, 46-47
12	1	Tutorial	
13	1	Recapitulation and discussion of possible questions	
Total No of Hours Planned for unit I -13			
		UNIT-II	
1	1	Introduction to sequences	S1, Ch 3, 55
2	1	bounded sequences	S1, Ch 3,56-57

3	1	Cacuchy's criterion	S1, Ch 3,58
4	1	Limit of a sequence	S3, Ch 3
5	1	Limit theorems	S1, Ch 3,63
6	1	Tutorial	
7	1	Cauchy's theorem	S1, Ch 3, 64
8	1	Order preservation	S1, Ch 3, 64-65
9	1	Squeeze theorem	S1, Ch 3, 70
10	1	Monotone sequences	S1, Ch 3, 70-71
11	1	Monotone sequences	S1, Ch 3, 72
12	1	Tutorial	
13	1	Recapitulation and discussion of possible questions	
Total No of Hours Planned for unit II -13			
		UNIT-III	
1	1	Infinite series	S1, Ch 3, 94
2	1	Cauchy convergence criterion for series	S1, Ch 3, 95
3	1	Positive term series	S1, Ch 3, 96-97
4	1	Geometric series	S1, Ch 3, 98
5	1	Comparison test	S1, Ch 3, 99
6	1	Tutorial	
7	1	Convergence of p series	S1, Ch 3, 100
8	1	Root test	S1, Ch 3, 101
9	1	Ratio test	S1, Ch 3, 102
10	1	Alternating series	S2, Ch 4,155
11	1	Leibnitz's test	S1, Ch 3, 102
12	1	Tutorial	
13	1	Absolute convergence	S1, Ch 3, 103
14	1	Conditional convergence	S1, Ch 3, 103
15	1	Recapitulation and discussion of possible questions	
Total No of Hours Planned For Unit III – 16			
		UNIT-IV	
1	1	Monotone sequences	S1, Ch 3, 70
2	1	Monotone convergence theorem	S1, Ch 3, 71-72
3	1	Subsequences	S1, Ch 3, 73

4	1	Divergence criteria	S1, Ch 3, 75
5	1	Monotone subsequence theorem	S1, Ch 3, 75-76
6	1	Tutorial	
7	1	Monotone subsequence theorem	S1, Ch 3, 77
8	1	Monotone subsequence theorem	S1, Ch 3, 77
9	1	Bolzano Weirstrass theorem	S1, Ch 3, 78
10	1	Bolzano Weirstrass theorem	S1, Ch 3, 79-80
11	1	Bolzano Weirstrass theorem	S1, Ch 3, 81-85
12		Tutorial	
13	1	Recapitulation and discussion of possible questions	
Total No of Hours Planned For Unit IV=15			
		UNIT-V	
1	1	Cauchy's sequence	S5, Ch 2, 52
2	1	Cauchy's convergence criteria	S1, Ch 9, 268
3	1	Cauchy's convergence criteria	S1, Ch 9, 269
4	1	Cauchy's convergence criteria	S1, Ch 9, 269-270
5	1	Cluster points	S1, Ch 9, 270
6	1	Tutorial	
7	1	Bolzano Weirstrass theorem	S1, Ch 9, 271-272
8	1	Properly divergence sequences	S1, Ch 9, 273
9	1	Infinite series	S1, Ch 9, 273-274
10	1	Infinite series	S1, Ch 9, 274
11	1	Infinite series	S1, Ch 9, 275
12	1	Tutorial	
13	1	Recapitulation and discussion of possible questions	
14	1	Discussion of previous ESE question papers.	
15	1	Discussion of previous ESE question papers.	
16	1	Discussion of previous ESE question papers.	
Total No of Hours Planned for unit V -16			
Total planned hours – **70**			

Chapter 1

Sets and functions

1.1 Introduction

[1] Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used in the definitions of nearly all mathematical objects.

The modern study of set theory was initiated by **Georg Cantor** and **Richard Dedekind** in the 1870s. After the discovery of paradoxes in naive set theory, such as the Russell's paradox, numerous axiom systems were proposed in the early twentieth century, of which the $Zermelo - Fraenkel$ axioms, with the axiom of choice, are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of $Zermelo - Fraenkel$ set theory with the axiom of choice. Beyond its foundational role, set

[1] source from wikipedia

theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

1.2 Basics of sets

Definition 1 *A collection of well defined objects is called a set.*

Definition 2 *Objects of a set are called elements or members.*

Remark 1

- *If x is an element of A, we say that $x \in A$.*
- *If x is not an element of A, we say that $x \notin A$.*

Example 1

- $A = \{x \colon x \text{ is an integer}\}$
- $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, *set of all natural numbers.*
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, *set of all integers.*[2]
- $\mathbb{Q} = \left\{\frac{p}{q} \colon p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$, *set of rational numbers*[3]

Definition 3 *A set that contains no elements is called the null set. It is denoted by $\emptyset$.*

Definition 4 *A set consisting of only one element is called a singleton set.*

[2] Z is for Zahlen - the German word for integers.
[3] Q is for quotient - which is how rational numbers are identified.

Definition 5 *If every element of a set A also belongs to a set B, we say that* $A \subseteq B$ *(or)* $B \supseteq A$.

Definition 6 *A set A is a proper subset of B if* $A \subseteq B$ *and there is atleast one element of B which is not in A.*

Definition 7 *Two sets A and B are said to be equal if* $A \subseteq B$ *and* $B \subseteq A$.

Definition 8 *The union of sets A and B is the set* $A \cup B = \{x\colon x \in A \text{ or } x \in B\}$.

Example 2 *Since* $\mathbb{N}$ *is the set of all natural numbers and* $\mathbb{Z}$ *is the set of all integers, we have* $\mathbb{N} = \{1, 2, 3, \cdots\}$ *and* $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
Then $\mathbb{N} \cup \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
and $\mathbb{N} \cup \mathbb{Z} = Z$

Remark 2 *(i) If* $A \subset B$, *then* $A \cup B = B$
(ii) Since $\emptyset \subset A$, *then* $\emptyset \cup A = A$.
(iii) Union of two sets is commutative.

Definition 9 *The intersection of the sets A and B is the set* $A \cap B = \{x\colon x \in A \text{ and } x \in B\}$.

Example 3 *Suppose* $A = \{1, 2, 3\}$ *and* $B = \{-2, -1, 0, 1\}$. *Then* $A \cap B = \{1\}$.

Definition 10 *The complement of B relative to A is the set $A - B = \{x\colon\ x \in A \text{ and } x \notin B\}$.*

Example 4 *Suppose $A = \{1, 2, 3, 4\}$. and $B = \{-2, -1, 0, 1\}$. Then $A - B = \{2, 3, 4\}$.*

Theorem 1 *For any three sets A, B and C, we have*

(i) $A \cup A = A$

(ii) $A \cup \emptyset = \emptyset \cup A = A$

(iii) $A \cup B = B \cup A$

(iv) $A \cup (B \cup C) = (A \cup B) \cup C$

(v) $A \cup B = B$ if and only if $A \subseteq B$

Proof

(iv) Let $x \in A \cup (B \cup C)$ be arbitrary

$$
\begin{aligned}
&\Rightarrow \quad x \in A \text{ (or) } x \in (B \cup C) \\
&\Rightarrow \quad x \in A \text{ (or) } x \in B \text{ (or) } x \in C \\
&\Rightarrow \quad (x \in A \text{ (or) } x \in B) \text{ (or) } x \in C \\
&\Rightarrow \quad x \in (A \cup B) \text{ (or) } x \in C \\
&\Rightarrow \quad x \in (A \cup B) \cup C \\
&\Rightarrow \quad A \in (B \cup C) \subseteq (A \cup B) \cup C
\end{aligned}
\tag{1.1}
$$

$$
\begin{aligned}
&\Rightarrow\ (x \in A \cup x \in B)\ \text{(or)}\ x \in C \\
&\Rightarrow\ x \in A\ \text{(or)}\ x \in B\ \text{(or)}\ x \in C \\
&\Rightarrow\ x \in A\ \text{(or)}\ (x \in B\ \text{(or)}\ x \in C) \\
&\Rightarrow\ x \in A\ \text{(or)}\ x \in (B \cup C) \\
&\Rightarrow\ x \in A \cup (B \cup C) \\
&\Rightarrow\ (A \cup B) \cup C \subseteq A \cup (B \cup C) \qquad (1.2)
\end{aligned}
$$

From (1.1) and (1.2), we have

$A \in (B \cup C) = (A \cup B) \cup C$

Theorem 2 *For any three sets* A, B *and* C*, we have*

(i) $A \cap A = A$.

(ii) $A \cap \emptyset = \emptyset \cap A = A$.

(iii) $A \cap B = B \cap A$.

(iv) $A \cap (B \cap C) = (A \cap B) \cap C$

(v) $A \cap B = B$ *if and only if* $A \subseteq B$

Definition 11 *Two sets* A *and* B *are said to be disjoint if* $A \cap B = \phi$

Example 5 *Let* $A = \{1, 3, 4\}$ *and* $B = \{5, 8, 9\}$ *then* $A \cap B = \phi$

Remark 3 *1.* $x \notin A \cup B \Leftrightarrow x \notin A$ *and* $x \notin B$

2. $x \notin A \cap B \Leftrightarrow x \notin A$ *(or)* $x \notin B$

Theorem 3 *If* A,B *and* C *are sets then*

(i) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

Proof

(i) Let $x \in A - (B \cup C)$ be arbitrary

$$\begin{aligned}
&\Rightarrow \quad x \in A \text{ and } x \notin (B \cup C) \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ and } x \notin C \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ and } x \in A \text{ and } x \notin C \\
&\Rightarrow \quad x \in (A - B) \text{ and } x \in (A - C) \\
&\Rightarrow \quad x \in (A - B) \cap (A - C)
\end{aligned}$$

Therefore, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$

similarly, we can prove

$(A - B) \cap (A - C) \subseteq A - (B \cup C)$

From the above, we have

$A - (B \cup C) = (A - B) \cap (A - C)$

(ii) Let $x \in A - (B \cap C)$ be arbitrary

$$\begin{aligned}
&\Rightarrow \quad x \in A \text{ and } x \notin (B \cap C) \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ or } x \notin C \\
&\Rightarrow \quad x \in A \text{ and } x \notin B \text{ or } x \in A and x \notin C \\
&\Rightarrow \quad x \in (A - B) \text{ or } x \in (A - C) \\
&\Rightarrow \quad x \in (A - B) \cup (A - C)
\end{aligned}$$

Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

similarly, we can prove

$(A - B) \cup (A - C) \subseteq A - (B \cap C)$

From the above, we have

$A - (B \cap C) = (A - B) \cup (A - C)$

Hence proved.

Definition 12 *If A and B are nonempty sets, then the cartesian product of A and B is denoted by AXB and is defined by* $AXB = \{(a, b) \colon a \in A and b \in B\}$

Definition 13 *A set S is said to be finite if it is either empty set (or) it has n elements for some* $n \in N$.

1.3 Functions

Definition 14 *Let A and B be nonempty sets. A function* $f \colon A \to B$ *which assigns to each element* $a \in A$*, a unique element* $b \in B$.

Remark 4 *The element b is called the image of a under f.*

Remark 5 *The element a is called preimage of b under f.*

Remark 6 *The set A is called domain of f and the set B is called co domain of f.*

Remark 7 *The set* $\{f(a) \colon a \in A\}$ *is called range of f, and is denoted by* $R(f)$.

Definition 15 *A function $f: A \to A$ is given by $f(x) = x\ \forall x$, is called identity function.*

Definition 16 *A function $f: A \to B$ is given by $f(x) = c$, a constant is called constant function.*

Remark 8

- *The range of constant function is always singleton set.*
- *Suppose $f: A \to B$ is an identity function, then $A = B$ or $A \subseteq B$.*

Definition 17 *A function $f: A \to B$ is one-one (injective) if distinct elements of A have distinct image in B.*

Remark 9 *f is one-one if $f(x) = f(y) \Rightarrow x = y$.*

Remark 10 *f is one-one if $x \neq y \Rightarrow f(x) \neq f(y)$.*

Definition 18 *A function $f: A \to B$ is onto(surjective) if range of f is equal to B.*

Definition 19 *A function $f: A \to B$ is called bijection if f is both one-one and onto function.*

Example 6 *Let $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = |x|\ \forall x \in \mathbb{Z}$.*
Here $f(-2) = f(2)$ but $-2 \neq 2$
Therefore, f is not one-one.

Example 7 *Consider $f\colon Z \to Z$ given by $f(x) = x + 3\ \forall\ x \in \mathbb{Z}$.*
Suppose

$$\begin{aligned} f(x) &= f(y) \\ x+3 &= y+3 \\ x &= y \end{aligned}$$

Therefore, f is one-one. Also $R_f = \mathbb{Z}$
Therefore, f is onto. Hence, f is bijection.

Definition 20 *Let $f\colon A \to B$ be a bijection. Then for each $b \in B$, there exists a unique element $a \in A$ such that $f(a) = b$.*
Define $f^{-1}\colon B \to A$ by $f^{-1}(b) = a$ Therefore, f^{-1} is called the inverse function of f.

Remark 11 *Suppose $f : A \to B$ is a bijection. Then A and B are said to be equivalent.*

1.4 Countable sets

Definition 21 *A set S is said to be countably infinite if there is a bijection between $\mathbb{N}$ and S,*

Example 8 *Let $E = \{2n : n \in \mathbb{N}\}$ is a even function.*
Let $f\colon \mathbb{N} \to E$ such that $f(x) = 2x$.

suppose

$$\begin{aligned} f(x) &= f(y) \\ 2x &= 2y \\ x &= y \end{aligned}$$

Therefore, f is one-one.

Also, $R_f = \{2, 4, 6, \cdots\} = E$

Therefore, f is onto.

$\therefore$ *f is bijection.*

$\therefore$ *E is countably finite.*

Example 9 *Let $A = \{\frac{1}{2}, \frac{2}{3}, \cdots\}$*

Solution

Let f be a function from $\mathbb{N} \to A$, such that

$f(n) = \frac{n}{n+1}$.

Suppose

$$\begin{aligned} f(n) &= f(m) \\ \frac{n}{n+1} &= \frac{m}{m+1} \\ n(m+1) &= m(n+1) \\ nm + n &= mn + m \end{aligned}$$

clearly f is one-one and onto function.

Therefore f is bijection.

Hence A is countably infinite.

Remark 12 *A subset of a countable set is countable.*

Theorem 4 $N \times N$ *is countable*

Proof

$N \times N = \{(a, b) \colon a, b \in N\}$

Take all orederd pairs (a, b) such that $a + b = 2$

There is only one element namely $(1, 1)$

Take all ordered pairs (a, b) such that $a + b = 3$

we have $(1, 2)$ and $(2, 1)$.

Next take all the ordered pairs (a, b) such that $a + b = 4$

we have $(1, 3), (2, 2)$ and $(3, 1)$

Proceeding like this and listing all the ordered pairs together from the begining, we get

$\{(1, 1), (1, 2), (2, 1), (1, 3), \cdots\}$

The set contains every ordered pair belonging to $\mathbb{N}\times\mathbb{N}$ exactly once

$\therefore \mathbb{N} \times \mathbb{N}$ is countable (or) countably infinite.

Remark 13 *If A and B are countable sets then* $A \times B$ *is also countable.*

Remark 14 *The set of all natural numbers is countable.*

Definition 22 *A set which is not countable is called uncountable.*

Theorem 5 $(0, 1]$ *is uncountable.*

Proof

Suppose $(0,1]$ is countable.

The elements of $(0,1]$ can be listed.

i.e., $(0,1] = \{x_1, x_2, \dots\}$, where

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\dots \\ x_2 &= 0.a_{21}a_{22}a_{23}\dots \\ &\vdots \end{aligned}$$

with $0 \leq a_{ij} \leq 9$

Let $y = 0.b_1b_2b_3\dots$, clearly $y \in (0,1]$

Now for each positive integer n select b_n such that $0 \leq b_n \leq 9$ and $b_n \neq a_{nn}$

Here y is different from each x_i atleast in the i^{th} place.

Which is contradiction to every elements of $(0,1] listed$.

Hence, $(0,1]$ is uncountable.

Remark 15 *The set of all real numbers $\mathbb{R}$ is uncountable.*

Remark 16 *The set of all irrational numbers is uncountable.*

1.5 The absolute value of a real number

Definition 23 *The absolute value of a real number a is denoted by $|a|$ is defined by*

$$|a| = \begin{cases} a & if \quad a > 0 \\ -a & if \quad a < 0 \end{cases}$$

Remark 17 *Suppose a is a real number* $|a| \geq 0$

Remark 18 $|a| = |-a|$

Theorem 6 *(a)* $|ab| = |a||b| forall a, b \in R$

(b) $|a|^2 = a^2 forall a \in R$

(c) If $c \geq 0$ *, then* $|a| \leq c \Leftrightarrow -c \leq a \leq c$

(d) $-|a| \leq a \leq |a|$ *for all* $a \in R$.

Proof (a) Case (i): Suppose

$$\begin{aligned} a &= 0 \\ |a| &= 0 \\ |a| \cdot |b| &= 0 \cdot |b| \\ &= 0 \end{aligned}$$

$$\begin{aligned} |a \cdot b| &= |0 \cdot b| \\ &= |0| \\ &= 0 \\ \text{Hence}|ab| &= |a||b| \end{aligned}$$

Case (ii): Suppose $b = 0$

$|b| = 0$

$|a|.|b| = |a|.0 = 0$

$|a.b| = |a.0| = |0| = 0$

$|ab| = 0 = |a|.|b|$

$|ab| = |a||b|$

Case (i): Suppose $a > 0$ and $b > 0$

$|a| = a$ and $|b| = b$

$|ab| = ab$, $(ab > 0)$

$= |a||b|$

$|ab| = |a||b|$

Case(iv): Suppose $a > 0$ and $b < 0$

Therefore, $|a| = a$ and $|b| = -b$

we have $ab < 0$

$|ab| = -(ab)$

$= a.(-b)$

$= |a||b|$

$|ab| = |a||b|$ case(v): Suppose $a < 0$ and $b < 0$

Therefore, $|a| = -a$ and $|b| = -b$

we have $ab > 0$

$|ab| = (ab)$

$= (-a).(-b)$

$= |a||b|$

$|ab| = |a||b|$ Hence $|ab| = |a.b|$ for all $a, b \in R$ (b) Let $a \in R$ be arbitrary

Then $a^2 \geq 0$

Now $|a^2| = a^2$

$= a.a$

$= |a||a|$

$= |a|^2|$

Hence, $|a|^2 = a^2$ for all $a \in R$

(c) Let us assume $c \geq 0$

Suppose $a \leq 0$

Then we have both $a \leq c$ and $-a \leq c$

since, $a \leq c$ and $-a \leq c$

$-c \leq a \leq -a \leq c$

$-c \leq a \leq c$

conversely, suppose $-c \leq a \leq c$

since $-c \leq a, c \geq -a$

$\therefore$ we have $a \leq c$ and $-a \leq c$, Then $|a| \leq c$

(d) Let $a \in R$ be arbitrary , Then $|a| \geq 0$

Let $c = |a|$ we know that, $|a| \leq |a|$

$\therefore -|a| \leq a \leq |a|$

1.6 Triangle inequality

Theorem 7 *If $a, b \in R$, then $|a + b| \leq |a| + |b|$*

Proof By (d) $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$

By adding above inequalities

$-|a| - |b| \leq a + b \leq |a| + |b|$

$-(|a| + |b|) \leq a + b \leq |a| + |b|$

$|a + b| \leq |a| + |b|$(by (c))

Remark 19 $|a + b| = |a| + |b|$ *iff* $ab > 0$

Theorem 8 *If* $a, b \in R$ *be arbitrary (a)* $||a| - |b|| \leq |a - b|$ *(b)* $|a - b| \leq |a| + |b|$.

Proof (a) Let $a, b \in R$ be arbitrary

Now

$$a = a - b + b$$

$$|a| = |a - b + b|$$

$$|a| = |(a - b) + b|$$

$$|a| \leq |a - b| + |b| (Bytriangleinequality)$$

$$|a| - |b| \leq |a - b| \tag{1.3}$$

Now

$$
\begin{aligned}
b &= b - a + a \\
|b| &= |b - a + a| \\
|b| &= |(b - a) + a| \\
|b| &\leq |b - a| + |a| (By triangle inequality) \\
|b| - |a| &\leq |b - a| \\
-|b| + |a| &\geq -|b - a| \qquad (1.4)
\end{aligned}
$$

From (1.3) and (1.4)

$-|a - b| \leq |a| - |b| \leq |a - b|$

$\therefore ||a| - |b|| \leq |a - b|$

Hence proved.

(b) Let a and b be any real numbers

since $b \in R$, $-b \in R$ (by triangle inequality)

$\therefore |a + (-b)| \leq |a| + |-b|$

$|a - b| \leq |a| + |b|$

Hence proved.

Let S be a non-empty subset of R.

1.7 Bounded sets

Definition 24 *Let S is said to be bounded above if there exists a number $u \in R$ such that $s \leq u \forall s \leq S$. Each such number u is called an upper bound of S.*

Definition 25 *The set S is said to be bounded below if there exists a number $u \in R$ such that $u \leq s \forall s \in S$. Each such number u is called as lower bound of S.*

Definition 26 *A set S is said to be bounded if it is both bounded above and bounded below.*

Definition 27 *A set S is said to be unbounded if it is not bounded.*

Example
Let $A = \{x \in R : 0 < x < 1\} = (0, 1)$
since all the elements of $A \geq 0$.
Therefore, A is bounded below.
since all the elements of $A \leq 1$
Therefore A is bounded above
Hence A is bounded.

Remark 20 *1. Every interval of the form (a, b),$[a, b)$,$(a, b]$ and $[a, b]$ are bounded subsets of R.*
2. Any finite subset of R is a bounded set.

Definition 28 *Let S be a nonempty subset of R. If S is bounded above, then a number u is said to be supremum (or) a least upper bound of S if (i) u is an upperbound of S. (ii) if v is an upperbound of S, then $u \leq v$*

Definition 29 *Let S be a nonempty subset of R. If S is bounded below, then a number w is said to be infimum (or) a greatest lower* *bound of S if (i) w is an lowerbound of S. (ii) if v is an lowerbound of S, then $v \leq w$*

Remark 21 *1. There can be only one supremum (infimum) of a given subset of R.*
2. If the supremum (or) the infimum of a set S exists, we will denote them by $supS$ or $infS$.

1.8 The completeness property of $\mathbb{R}$

(i) Every nonempty set of real numbers that has an upper bound and also has an supremum in *R*.
(ii) Every nonempty subset or real numbers that has a lower bound also has an infimum in *R*.

Example 10 *Let* $S = \{dfrac1n : n \in N\}$
$S = \{1, dfrac12, dfrac13, \ldots\}$
$infS = 0$ *and* $SupS = 1$.

1.9 Some properties of the supremum

Theorem 9 *Let S be a nonempty set of real numbers with a supremum, say bsup S. Then for every $a < b$ there is some $x \in S$ such that $a < x \leq b$.*

Proof

Let $b = sup\ S$.

Then we have $x \leq b$ for all $x \in S$.

To Prove: *$a < x \leq b$ for some $a \in S$ and $a < b$.*

Suppose $x \leq a$ for all $x \in S$ and $a < b$.

Therefore x is an upper bound of S and $a < b$.

Which is $\Rightarrow\Leftarrow$ b is a least upper upper bound.

$x > a$ for atleast one $x \in S$. $\square$

Definition 30 *Let S be a nonempty subset of R that is bounded above and let a be any number in R. Define $s = \{a + s : s \in S\}$.*

Theorem 10 *S be a nonempty subset of R. Suppose S is bounded above and $a \in R$. Then prove that $sup(a + S) = a + supS$.*

Proof

Let S be a nonempty bounded above subset of R. Therefore S has an upper bound. By completeness property of R, we have supremum of S exists.

Let $u \in supS$, Then $x \in u$ for all $x \in S$

Therefore, $a + x \leq u + a \forall x \in S$

$\therefore$ $u + a$ is an upperbound of $a + S$.

Let

$$m = sup(a + S)$$

$$\therefore m \leq u + a \tag{1.5}$$

suppose v is an upperbound of $a + S$

$\therefore$ $a + x \leq v$ for all $x \in S$

$\therefore x \leq v - a$ for all $x \in S$

Therefore $v - a$ is an upperbound of S

$u \leq v - a$

$a + u \leq v$

since v is an upperbound of $a + S$

$$a + u \leq m \tag{1.6}$$

From (1.4) and (1.5), we get

$a + u = m$

$a + supS = sup(a + S)$.

Theorem 11 *Suppose that A and B are nonempty subset of R, such that* $a \leq b \forall a \in A$ *and* $b \in B$

Then $supA \leq infB$.

Proof

Let B be arbitrary .

Then $a \leq b$ for all $a \in A$

b is an upper bound of A

$supA \leq b$

Therefore sup A is a lower bound of B

$\therefore supA \leq infB$

1.10 Archimedian property

If $x \in R$ then there exist $n_x \in N$ such that $x < n_x$

Proof

Let $x \in R$ be an arbitrary

To prove : There is atleast one $n_x \in N$ such that $x < n_x$

Suppose $n \leq x$ for all $n \in N$

$\therefore$ x is an upper bound of N.

By completeness property of R

$supN$ exists.

Let $u = supN$

Then $u - 1$ is not an upper bound of N

$\therefore$ $m \in N$ such that

$u - 1 < m$, $u < m + 1$

since $m + 1 \in N$, we must have $m + 1 \leq u$

$\therefore$ there exist $n_x \in N$ such that $x < n_x$

Example 11 *f(x) = 0, if x is even ,*

f(x) =1, if x is odd . $\therefore$ *Range of* $f = R_f$ $=\{0, 1\} \subseteq R$.

Example 12 $f(x) = |x|$

Range of $f = R_f = \{0, 1, 2, \dots\} \subseteq R$

Definition 31 *Given a function* $f : D \to R$*, we say that* f *is bounded if the set* $f(D)$ = *range of* $f = \{f(x) : x \in D\}$ *is bounded* *above in R. similarly, the function f is bounded below if $f(D)$ is bounded below in R. we say that, f is bounded if $f(D)$ is bounded below and bounded above (or) $|f(x)| \leq B, B \in RR$*

Example 13 *Let $f : N \to Q$ be a function defined by $f(n) = \frac{n}{n+1}$*
The range of $f = R_f = \{dfrac12, dfrac23, dfrac34, \dots\} \subseteq Q$
$SupR_f = supf(N) = 1$
$infR_f = inff(N) = \frac{1}{2}$
$\therefore$ *The given function is bounded.*

1.11 Multiple choice questions

1. Let $A = \{1, 2\}$. Then $A \times A$ is

   A. $\{(1, 1), (2, 2)\}$ B. $\{(1, 2), (2, 1)\}$
   C.$\{(1, 1), (2, 2), (1, 2)\}$ D. $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$

2. Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then number of elements in $A \times B$ is

   A. 2 B.3
   C. 2^3 D. 2×3

3. Suppose number of elements in A is n and number of elements in B is m. Then number of elements in $A \times B$ is

   A. $n + m$ B. $n \times m$
   C. n^m D. m^n

4. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then which of the following element does not belongs to $A \times B$

A. $(1, a)$ B. $(3, c)$

C. $(c, 2)$ D. $(1, c)$

5. Identify the domain of this relation
$\{(9, 10), (6, -1), (6, 10), (7, -2), (11, 5)\}$ is
A. $\{6, 7, 9, 11\}$ B. $\{6, 7, 9, 10\}$
C. $\{-1, -2, 5, 10\}$ D.$\{-1, -2, 5, 11\}$

6. Identify the range of this relation
$\{(9, 10), (6, -1), (6, 10), (7, -2), (11, 5)\}$ is

A. $\{6, 7, 9, 11\}$ B. $\{6, 7, 9, 10\}$
C. $\{-1, -2, 5, 10\}$ D. $\{-1, -2, 5, 11\}$

7. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(x) = x^2$ where $\mathbb{Z}$ is a set of all real numbers. Then the range of f is
A. $\mathbb{Z}$ B. $\mathbb{N}$
C. $\mathbb{W}$ D. $\{0, 1, 4, 9, \cdots\}$

8. The set of all positive integers $\{1, 2, \cdots\}$ is
A. finite B. infinite
C. countable D. uncountable

9. Greatest lower bound of set of all positive even integers is
A. 2 B. 0
C. 1 D. 4

10. Let S be a bounded above set of real numbers and $\sup S = u$. Then for $x \in S$, we have

A. $x > u$ B. $x < u$

C. $x \leq u$ D. $x \geq u$

11. Which equation does not represent a function?

A. $y = 2x$ B. $y = x^2 + 10$

C. $y = \frac{10}{x}$ D. $x^2 + y^2 = 95$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x$. Then f is

A.one-one B. onto

C. bijection D. neither onto nor one-one

13. Which of the following sets is countable?

A. $(0, \infty)$ B. $\mathbb{R}$

C. set of all irrational numbersD. set of all Fibonacci numbers

14. $B - (B - A) = A$ if

A. $B \subset A$ B. $A \subset B$

C. $A \cup B = A$ D. $A \cup B = A$

15. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then the number of distinct functions from A into B is

A. 8 B. 9

C. 6 D. 5

16. $\sup \{1 - \frac{1}{n} : n \in \mathbb{N}\}$=

A. -1 B. 1

C. 0 D. $\frac{1}{2}$

17. Let A be the set of letters in the word "trivial" and let B be the set of letters in the word difficult. Then $A - B =$

A. $\{a, r, v\}$ B. $\{d, f, c, u\}$

C.$\{i, l, t\}$ D. $\{a, i, l, r, t, v\}$

18. Let S be the set of all 26 letters in the alphabet and let A be the set of letters in the word "trivial". Then the number of elements in A^c is

A. 19 B. 20 C. 21 D. 22

19. Let $A = \{1, 2\}$. Then $A \times A$ is

A. $\{(1, 1), (2, 2)\}$ B. $\{(1, 2), (2, 1)\}$

C. $\{(1, 1), (2, 2), (1, 2)\}$ D. $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$

20. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then which of the following element does not belongs to $A \times B$

A. $(1, a)$ B. $(3, c)$

C. $(c, 2)$ D. $(1, c)$

21. Let F be a function and $(x, y) \in F$ and $(x, z) \in F$. Then we must have

A. $y \neq z$ B.$y < z$

C. $y > z$ D. $y = z$

22. Let $f : A \to B$ be a function and the range of f denoted by $\mathscr{R}(f)$. Which of the following is always is true?

A. $\mathscr{R}(f) \neq B$ B. $\mathscr{R}(f) \subseteq B$

C. $B \subset \mathscr{R}(f)$ D. $B \subseteq \mathscr{R}(f)$

23. If a function $f : A \to B$ is such that $\mathscr{R}(f) \neq B$ then f is a/an ?

A. into function B.onto function

C. surjective D. many to one

24. If a function $f : A \to B$ is such that $\mathscr{R}(f) = B$ then f is a/an ?

A. into function B. onto function

C. one to one function D. many to one

25. If $f : \{1, 2, \cdots\} \to \{0, \pm 1, \pm 2, \cdots\}$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \text{ is even} \\ -\left(\frac{x-1}{2}\right), & x \text{ is odd} \end{cases}$$

then $f^{-1}(100) =$

A. 100 B. 199

C. 200 D. 201

26. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is

A. one-to-one B. onto

C. bijection D. many to one

27. Let $f : X \to Y$ be a function. If f^{-1} is a function then f^{-1}

A. from $\mathscr{R}(f)$ to X B. from Y to X

C. from X to Y D. $\mathscr{R}(f)$ to Y

28. If f^{-1} is a function then

    A. f is one-to-one but not ontoB. f is onto but not one-to-one

    C. f is both one-to-one and onto D. f is neither onto nor one-to-one

29. Let $f : A \to B$ be a function. We call f as a sequence in B if

    A. $A = \{0, 1, 2, \cdots\}$ B. $A = \{1, 3, 5, \cdots\}$

    C.$A = \{1, 2, 3 \cdots\}$ D. $A = \{0, 2, 4, \cdots\}$

30. A set S is countable if it is

    A. both finite and countably infinite B. either finite or countably infinite

    C. neither finite nor countably infinite D. finite but not countably infinite

31. Let $\mathbb{R}$ be the set of all real numbers. Then number of elements in $\mathbb{R}$ is

    A. countably infinite B. uncountable

    C. finite D. zero

1.12 Two marks questions

1. Define an uncountable set.

2. Define countable set and give an example.

3. Give two examples for uncountable sets.

4. State the triangle inequality
5. Define bounded set.
6. Define supremum of a set
7. Define infimum of a set.
8. Define unbounded set.
9. Give two examples for unbounded set
10. Give two example for bounded set
11. Prove that $|a + b| = |a| + |b|$ iff $a = b = 0$
12. State archimedian property of $\mathbb{R}$
13. Define cluster point
14. Prove that $\mathbb{R}$ is uncountable
15. Let $S = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Find $\sup S$ and $\inf S$
16. Prove that the set of all rational number is countable.
17. If $a, b \in \mathbb{R}$, ptove that $|a + b| \leq |a| + |b|$

    OR State and prove triangle inequality
18. State and prove Archimedean property.
19. Let S be a subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Prove that $a + \sup S = \sup(a + S)$

20. Prove that the set of all real numbers is uncountable.

UNIT I

The set of all points between a and b is called ------------	integer	interval	elements	set	interval
The set {x: a < x < b} is --------------	(a, b)	[a, b]	(a, b]	[a, b)	(a, b)
A real number is called a positive integer if it belongs to -----------	interval	open interval	closed interval	inductive set	inductive set
Rational numbers is of the form -----------------	pq	p + q	p/q	p - q	p/q
e is ---------------	rational	irrational	prime	composite	irrational
An integer n is called ----------- if the only possible divisors of n are 1 and r	rational	irrational	prime	composite	prime
A set with no upper bound is called ------------	bounded above	bounded below	prime	function	bounded above
A set with no lower bound is called ------------	bounded above	bounded below	prime	function	bounded below
The least upper bound is called -----------	bounded above	bounded below	supremum	infimum	supremum
The greatest lower bound is called -----------	bounded above	bounded below	supremum	infimum	infimum
The supremum of {3, 4} is ----------	3	4	(3, 4)	[3, 4]	4
Every finite set of numbers is ----------	bounded	unbounded	prime	bounded above	bounded
A set S of real numbers which is bounded above and bounded below is ca	bounded set	inductive set	super set	subset	bounded set
The set N of natural numbers is ----------	bounded	not bounded	irrational	rational	not bounded
The infimum of {3, 4} is ------------	3	4	(3, 4)	[3, 4]	3

Sup C = Sup A + Sup B is called -------------- property	approximation	additive	archimedean	comparison	additive
For any real x, there is a positive integer n such that -----------	n > x	n < x	n = x	n = 0	n > x
If x > 0 and if y is an arbitrary real number, there is a positive number n su	approximation	additive	archimedean	comparison	archimedean
The set of positive integers is -------------	bounded above	bounded below	unbounded abo	unbounded belo	unbounded abo
The absolute value of x is denoted by ---------------	|x|	| |x| |	x < 0	x > 0	|x|
If x < 0 then ---------------	|x| = x	| |x| | = |x|	| |x| | = -x	|x| = -x	|x| = -x
If S = [0, 1) then sup S = ----------------	0	1	(0, 1)	[0,1]	1
Triangle inequality is -------------------	|a| + |b| greate	|a| > |a + b|	|b| > |a + b|	|a + b| less thar	|a + b| less thar
|x + y| greater than equal to -----------------	|x| + |y|	|x| |y|	|x| - |y|	| |x| - |y| |	| |x| - |y| |
If (x, y) belongs to F and (x, z) belongs to F, then -------------	x = z	x = y	xy = z	y = z	y = z
A mapping S into itself is called ------------	function	relation	domain	transformation	transformation
If F(x) = F(y) implies x =y is a --------------- function	one-one	onto	into	inverse	one-one
One-one function is also called -----------	injective	bijective	transformation	codomain	injective
S = {(a,b) : (b,a) is in S} is called ---------------	inverse	domain	codomain	converse	converse
If A and B are two sets andif there exists a one-one correspondence betw	denumerable	uncountable	finite	equinumerous	equinumerous
A set which is equinumerous with the set of all positive integers is called -	finite	infinite	countably infinit	countably finite	countably infinit

A set which is either finite or countably infinite is called ------------ set	countable	uncountable	similar	equal	countable
Uncountable sets are also called ------------- set	denumerable	non-denumerab	similar	equal	non-denumerab
Countable sets are also called --------------- set	denumerable	non-denumerab	similar	equal	denumerable
Every subset of a countable set is ------------	countable	uncountable	rational	irrational	countable
The set of all real numbers is ----------------	countable	uncountable	rational	irrational	uncountable
The cartesian product of the set of all positive integers is ----------	countable	uncountable	rational	irrational	countable
The set of those elements which belong either to A or to B or to both is ca	complement	intersection	union	disjoint	union
The set of those elements which belong to both A and B is called -----------	complement	intersection	union	disjoint	intersection
Union of sets is --------------	commutative	not commutativ	not associative	disjoint	commutative
The complement of A relative to B is denoted by --------------	B - A	B	A	A - B	B - A
If A intersection B is the empty set, then A and B are called --------	commutative	not commutativ	not associative	disjoint	disjoint
B - (union A) = -----------------	union (B -A)	B - (intersection	intersection (B -	{}	intersection (B -
B - (intersection A) = -----------------------	union (B -A)	B - (union A)	intersection (B -	{}	union (B -A)
Union of countable sets is -----------------	uncountable	infinite	countable	disjoint	countable
The set of all rational numbers is ---------------	uncountable	infinite	countable	disjoint	countable
The set S of intervals with rational end points is ---------- set	uncountable	infinite	countable	disjoint	countable

The product of two prime numbers will always be					
	even number	odd number	neither prime n	composite	composite
Let A be the set of all prime numbers. Then number of elements in A					
	countable	uncountable	finite	empty	countable

$A^\wedge c$

Chapter 2

Real sequences

2.1 Sequences and their limits

Definition 32 *A sequence in $\mathbb{R}$ is a function from $\mathbb{N}$ into $\mathbb{R}$.*

Remark 22 *(i) The sequence is denoted by the symbol $\{s_n\}$ or (s_n).*
(ii) The image of of n, s_n is called the n^{th} term of the sequence.

Example 14 *Let f be function from $\mathbb{N} \to \mathbb{R}$ such that $f(n) = 0$*
Range of f =$\{0\}$

Definition 33 *If $b \in R$, the sequence $B = \{b, b, b, \dots\}$ is called constant sequence.*

Definition 34 *The Fibnacci sequence $F = (f_n)$ is given by*

$$\begin{aligned} f_1 &= 1 \\ f_2 &= 2 \\ f_{n+1} &= f_n + f_{n-1}, \quad n \geq 2 \end{aligned}$$

Definition 35 *A sequence (x_n) in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ or x is said to be a limit of (x_n) if for every $\epsilon > 0$ there exists a positive integers N such that $|x_n - x| < \epsilon$ for all $n \geq N$.*
If a sequence has a limit, we say that the sequence is convergent, if it has no limit, we say that the sequence is divergent.

Remark 23 *Suppose a sequence (x_n) has limit x, Then we can write*
$\lim x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$

Theorem 12 *Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) be a sequence of positive real numbers with $\lim a_n = 0$ and if for some constant $c > 0$ and some $m \in N$ we have*
$|x_n - x| \leq ca_n \forall n \geq m$, *then* $\lim x_n = x$

Proof suppose let $\epsilon > 0$ be given, then $\frac{\epsilon}{c} > 0$
Given that $\lim a_n = 0$
Therefore for $\frac{\epsilon}{c} > 0$, there exist a positive integer N such that

$$\begin{aligned} |a_n - 0| &< \frac{\epsilon}{c} \\ |a_n| &< \frac{\epsilon}{c} \\ a_n &< \frac{\epsilon}{c} \quad \forall n \geq N \end{aligned}$$

Suppose for some $m \in N$ such that

$$\begin{aligned} |x_n - x| &\leq c.a_n \quad \forall n \geq N \\ &\leq c.\frac{\epsilon}{c} \\ &= \epsilon \end{aligned}$$

$\therefore x_n \to x$

Example 15 *If* $a > 0$*, then* $\lim(\frac{1}{1+na}) = 0$

Solution *Since* $a > 0$*,* $na > 0$*,* $0 < na < 1 + na$

$\therefore \frac{1}{na} > \frac{1}{1+na}$

Now

$$\begin{aligned} |\frac{1}{1+na} - 0| &= |\frac{1}{1+na}| \\ &= \frac{1}{1+na} < \frac{1}{na} \\ \therefore \ |\frac{1}{1+na} - 0| &< \frac{1}{a}(\frac{1}{n}) \end{aligned}$$

Since $\lim(\frac{1}{n}) = 0$*,* $\lim(\frac{1}{1+na}) = 0$

Remark 24 *Convergence of* $(|x_n|)$ *need not imply the convergence of* (x_n)*.*

Consider a sequence $((-1)^n)$

Then $(|(-1)^n|) = (1, 1, \dots)$

Clearly, $\lim |x_n| = 1$

Now $((-1)^n) = (-1, 1, -1, 1, \dots)$

This is not a convergent sequence.

2.2 Limit theorems

Theorem 13 *If* $0 < b < 1$*, then* $\lim(b^n) = 0$

Proof Suppose $0 < b < 1$

Then $b = \frac{1}{1+a}$ if $a > 0$

$$\begin{aligned} |b^n - 0| &= |b^n| \\ &= b^n \\ &= [\frac{1}{1+a}]^n] \\ &= \frac{1^n}{(1+a)^n} \\ &= \frac{1}{(1+a)^n} \\ &\leq \frac{1}{1+na} \\ &\leq 1na \\ &= c\frac{1}{n} \end{aligned} \tag{2.1}$$

Since $\lim x_n = 0$ and by previous theorem, $\lim(b^n) = 0$

Theorem 14 *If* $c > 0$*, then* $\lim(c^{\frac{1}{n})=1}$

Proof Case(i) suppose $c = 1$

Then $(c^{\frac{1}{n})}$ is a constant sequene and $\lim(c^{\frac{1}{n}}) = 1$

Case(ii) suppose $0 < c < 1$

Then $c^{\frac{1}{n}} = \frac{1}{1+h_n}$ where $h_n > 0$

$(c^{\frac{1}{n}})^n = (c^{\frac{1}{1+h_n}})^n$

$c = \frac{1}{(1+h_n)^n}$

$< \frac{1}{n.h_n}$

Now $|c^{\frac{1}{n}-1}| = |1 - c^{\frac{1}{n}}|$

$= |1 - \frac{1}{1+h_n}|$

$= |\frac{1+h_n-1}{1+h_n}$

$=|\frac{h_n}{1+h_n}|$

$< h_n$

since $c < \frac{1}{nh_n}$, $h_n < \frac{1}{nc}$

$\therefore |c^{\frac{1}{n}} - 1| < \frac{1}{nc}$

since $\frac{1}{c} > 0$ and

$lima_n = 0$ if $a_n = \frac{1}{n}$

Then

$lim(c^{\frac{1}{n}}) = 1$

case(iii)

suppose $c > 1$

Then $c^{\frac{1}{n}} = 1 + d_n$ where $d_n > 0$

Now $c = (1 + d_n)^n$

$= 1 + n.d_n + \cdots + d_n{}^n$

$\geq 1 + nd_n$

$\therefore c - 1 \geq nd_n$

$\frac{c-1}{n} \geq d_n$

Now $|c^{\frac{1}{n}-1}| = |d_n|$

$= d_n$

$\frac{c-1}{n}$

$= (c - 1).\frac{1}{n}$

Hence

$lim(c^{\frac{1}{n}}) = 1$

2.3 Bounded sequences

Definition 36 *A* (x_n) *of real numbers is said to bounded if there exists a real number* $M > 0$ *such that*
$|x_n| \leq M$ *for all* $n \in N$, $-M \leq x_n \leq M$

Theorem 15 *A convergent sequence of real numbers is bounded.*

Proof

suppose that

$lim(x_n) = x$

Let $\in= 1 > 0$

Then there exists a positive integer N such that $|x_n - x| < 1$ if $n \geq N$

Now $|x_n| = |x_n - x + x|$

$\leq |x_n - x| + |x|$

$\leq 1 + |x|$ if $n \geq N$

Then $|x_n| \leq M$ for all $n \geq 1$

Therefore (x_n) is bounded.

Definition 37 *If* $x = (x_n)$ *and* $y = (y_n)$ *are sequences of real number, we define their sum to be the sequene* $x + y = (x_n + y_n)$*, their difference to be the sequence* $x - y = (x_n - y_n)$ *and their product to be the sequence* $xy = (x_n y_n)$ *.*
If $c \in R$*, we define the sequence* $cx = (cx_n)$
If $z = (z_n)$ *is a sequence of non-zero real numbers, then we define*

the quotient of x and Z to be the sequence

$\frac{x}{Z} = \frac{(}{x_n} z_n)$.

Theorem 16 *Let $X = (x_n)$ and $Y = (y_n)$ converge to x and y respectively and $c \in R$. Then the sequence $x + y$ $x - y$, xy and cx converge to $x + y$, $x - y$, xy and cx respectively.*

Proof

Let $\in > 0$ be given. suppose $x_n \to x$ and $y_n \to y$

$\frac{\in}{2} > 0$ and $x_n \to x$

There exist a positive integer N_1 such that

$|x_n - x| < \frac{\in}{2}$

since $\frac{\in}{2} > 0$ and $y_n \to y$

Thereexist a positive integer N_2 such that

$|y_n - y| < \frac{\in}{2}\ \forall n \geq N_2$

Now $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$

$\leq |x_n - x| + |y_n - y|$

let $N = max\{N_1, N_2\}$

$|(x_n + y_n) - (x + y)| < \frac{\in}{2} + \frac{\in}{2}$

$= \in$

Therefore, $(x_n + y_n) \to x + y$

By using similar arguments, we have

The sequence $(x_n - y_n)$ converges to $x - y$

consider $|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$

$= |x_n(y_n - Y) + y(x_n - x)|$

$\leq |x_n(y_n - y)| + |y(x_n - x)|$

$= |x_n||y_n - y| + |y||x_n - x|$

since $(x_n) \to x$, Thereexist a positive real number M_1 such that

$|x_n| \leq M, \forall n \geq 1$

Hence $|x_n y_n - xy| \leq M$, $|y_n - y| + |y||x_n - x|$

let $M = \sup\{M_1, |y|\}$

$|x_n y_n - xy| \leq yM|y_n - y| + M|x_n - x|$

let $\in > 0$ be given

since $(x_n) \to x$, there exist a positive integer N_1 such that

$|x_n - x| < \frac{\in}{2M}\ \forall n \geq N_1$

since $(y_n) \to y$, there exist a positive integer N_2 such that

$|y_n - y| < \frac{\in}{2M}\ \forall n \geq N_2$

$N = sup\{N_1, N_2\}$

Therefore $|x_n y_n - xy| < M\frac{(}{\in}2M) + M\frac{(}{\in}2M)$ if $n \geq N$

Therefore $|x_n y_n - xy| < \in$ if $n \geq N$

i.e., $(x_n y_n) \to xy$

Let (y_n) be a constant sequence(c)

Then $(y_n) \to c$

By the above argument, $(x_n y_n) \to xc$

i.e.,$(x_n c) \to xc$

i.e.,$(cx_n) \to cx$

Theorem 17 *If $X = (x_n)$ converges to x and $z = (z_n)$ is a sequence of non-zero real numbers that converge to z and if $z \neq 0$, then the* *quotient sequence* $(\frac{x_n}{z_n}) \to \frac{x}{z}$

Proof

Let $\alpha = \frac{1}{z} > 0$

since $(z_n) \to z$, there exist a positive integer N_1 such that $|z_n - z| < \alpha$ if $n \geq N_1$

$-|z_n - z| > -\alpha$ if $n \geq N_1$ Therefore, $-\alpha < -|z_n - z| \leq |z_n| - |z|$ if $n \geq N_1$

$-\alpha < |z_n| - |z|$ if $n \geq N_1$

$\frac{1}{2}|z| = |z| - \frac{1}{2}|z|$

$= |z| - \alpha$

$< |z_n|$ if $n \geq N_1$

$\frac{1}{2}|z| \leq z_n$ if $n \geq N_1$

$\frac{2}{|z|} \geq \frac{1}{|z_n|}$ if $n \geq N_1$

Now $|\frac{1}{z_n} - \frac{1}{z}|$

$= \frac{|z - z_n|}{|z_n z|}$

$= \frac{|z - z_n|}{|z_n||z|} \leq \frac{|z_n - z}{|z|} . \frac{2}{|z|}$

$= \frac{2|z_n - 2|}{|z|^2}$

let $\in > 0$ be given

since $(z_n) \to z_1$ there exist a positive integer N_2 such that

$|z_n - z| < \frac{\in}{2}|z|^2$ if $n \geq N_2$

Hence $|\frac{1}{z_n} - \frac{1}{z}| \leq \frac{2}{|z|^2} \in |z|^2 2$ if $n \geq N = sup\{N_1, N_2\}$

Therefore, $(\frac{1}{z_n}) \to (\frac{1}{z})$

Theorem 18 *If (x_n) is a convergent sequence of real number and* *if* $x_n \geq 0$ *for all* $n \in N$*, then* $x = \lim(x_n) \geq 0$.

Proof

suppose $(x_n) \rightarrow x$

To prove

$x \geq 0$

suppose $x < 0$

Then $-x > 0$

Let $\in = -x > 0$

since $(x_n) \rightarrow x$, There esixt a positive integer N such that

$|x_n - x| < -x$ if $n \geq N$

Then $x < x_n - x < -x$ if $n \geq N$

Therefore, $x_n - x < -x$ if $n \geq N$

$x_n < -x + x$ if $n \geq N$

$x_n < 0$ if $n \geq N$

i.e., $x_N < 0, x_{N+1} < 0, \ldots$

$\Rightarrow x_n \geq 0 \ \forall n$

Hence $x_n \geq 0$

Note

(i) suppose sequence (x_n) is convergent to x and $x_n > 0$. Then $\lim(x_n) = x$ need not be greater than zero.

Theorem 19 *If* (x_n) *and* (y_n) *are convergent sequence of real numbers and if* $x_n \leq y_n$ *forall* $n \in N$*, then* $\lim(x_n) \leq \lim(y_n)$.

Proof Let $z_n = y_n - x_n$

Then (z_n) is a sequence of real numbers and $z_n \geq 0$.

By previous theorem,

$\lim(z_n) \geq 0$

$\lim(y_n - x_n) \geq 0$

$\lim(y_n) - \lim(x_n) \geq 0$

$\lim(y_n) \geq \lim(x_n)$

Theorem 20 *If (x_n) is a convergent sequence and if $a \leq x_n \leq b$ for all $n \in N$, then $a \leq \lim(x_n) \leq b$.*

Proof Let (y_n) be q sequence such that $y_n = b \forall n \in N$

since $a \leq x_n \leq b$, we have $n \leq y_n \ \forall n \in N$

By previous theorem, $\lim(a) \leq \lim(y_n) \leq \lim(b)$

$a \leq \lim(y_n) \leq b$

2.4 Squeeze theorem

Theorem 21 *suppose that (x_n), (y_n) and (z_n) are sequences of real numbers such that $x_n \leq y_n \leq z_n \ \forall n \in N$*

and $\lim(x_n) = \lim(z_n)$

$\lim(x_n) = \lim(y_n) = \lim(z_n)$

Proof Given that $\lim(x_n) = \lim(z_n)$

Then $\lim(x_n) = \lim(z_n) = w$

Let $\in > 0$ be given.

Then there exist positive integer N such that

$|x_n - w| <\in$ if $n \geq N$

and $|z_n - w| <\in$ if $n \geq N$

Also given that $x_n \leq y_n \leq z_n$, Then $x_n - w \leq y_n - w \leq z_n - w$

$\in< x_n - w < y_n - w < z_n - w <\in$

$- \in< y_n - w <\in$

$|y_n - w| <\in$ if $n \geq N$

Therefore, $\lim(y_n) = w$

Theorem 22 *Let the sequence (x_n) converges to x. Then the sequence $(|x_n|)$ of absolute values converges to $|x|$.*

Proof Let $\in> 0$ be given

There exist a positive integer N such that

$|x_n - x| <\in$ for all $n \geq N$

Now, $||x_n| - |x|| \leq |x_n - x| <\in$

$\therefore \lim(x_n) = |x|$

2.5 Monotone sequence

Definition 38 *Let (x_n) be a sequence of real numbers. we say that sequence (x_n) is increasing if $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots$*
we say that sequence (x_n) is decreasing if $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots$, we say that (x_n) is monotone if it is either increasing or decreasing.

Problem

Give an example of two divergent sequences X and Y such that (i) sum $x + y$ converges (ii) Product $X.Y$ converges.

Solution

Let $X = (-1)^n = (-1, 1, -1, 1, \dots)$ $Y = (-1)^{n+1} = (-1, 1, -1, 1, \dots)$

clearly X and Y are divergent

Now $X + Y = (0, 0, 0, \dots)$ converges

$X.Y = (-1, -1, -1, \dots)$ converges

Problem 1 *Show that if X and Y are sequences such that X and Y $X + Y$ are convergent then Y is convergent.*

Solution Given X and $X + Y$ are convergent. Then $X + Y - X$ is also convergent. i.e., Y is convergent.

2.6 Monotone convergence theorem

Theorem 23 *A monotone sequene of real numbers is convergent if and only if it is bounded. Moreover (i) If $X = (x_n)$ is a bounded increasing sequene, then $\lim(x_n) = sup\{x_n : n \in N\}$ (ii) If $Y = (y_n)$ is a bounded decreasing sequence, then $\lim(y_n) = inf\{y_n : n \in N\}$*

Proof Suppose a monotone sequence is convergent then the sequence is bounded. conversely, suppose a monotone sequence is bounded. since given sequence is monotone, we have either increasing or decreasing.

(i) Let X be a increasing sequence and bounded.

since, X is bounded, there is a real number M such that $x_n \leq M$ $\forall n \in N$

Therefore, $\{x_n : n \in N\}$ is bounded above.

By completeness property of $\mathbb{R}$, there exist the $sup\{x_n : n \in N\}$

$\in > 0$ be given

Then $x^* - \in$ is not an upper bound.

Therefore there exist a member of set x_n such that $x^* - \in$ ¡ x_k

Then $x^* - \in < x_n \; \forall n \geq k$

Hence $x^* - \in < x_k \leq x_n \leq x^* < x^* + \in$

$- \in < x_n - x^* < \in$ if $n \geq k$

$|x_n - x^*| < \in$ if $n \geq k$

$\lim(x_n) = x^*$

(ii) Let $Y = (y_n)$ be a bounded decreasing sequence

Then $X = -Y = (-y_n)$ is an increasing sequence

By (i) $\lim(-y_n) = sup\{-y_n : n \in N\}$

$= -inf\{y_n : n \in N\}$

$\lim X = -inf\{y_n : n \in N\}$

$\lim(-y) = -inf\{y_n : n \in N\}$

$-\lim(y) = -inf\{y_n : n \in N\}$

$\lim(y) = inf\{y_n : n \in N\}$

Problem 2 *show that* $\lim(\frac{1}{\sqrt{n}}) = 0$

Solution $\lim(\frac{1}{\sqrt{n}}) = x$ and $x = (\frac{1}{\sqrt{n}})$
Now $X.X = (\frac{1}{\sqrt{n}})(\frac{1}{\sqrt{n}})$
$= (\frac{1}{\sqrt{n}} \to 0)$
Therefore $x^2 = 0$ and $x = 0$

Problem 3 *consider a* (x_n) *with* $x_1 = 2$ *and* $x_{n+1} = 2 + \frac{1}{x_n}, n \in N$. *Find the limit of the sequence* (x_n).

Solution Let $\lim(x_n) = x$

since $x_n \geq 0\ \forall n$, we have $x \geq 0$

Moreover $x_n \geq 2$ and $x \neq 0$

Now $x = \lim(x_n)$

$= \lim(x_{n+1})$

$= \lim(2 + \frac{1}{x_n})$

Let $y_n = 2$ and $z_n = 1$

Then $\lim(y_n) = 2$ and $\lim(z_n) = 1$

$x = \lim(y_n + \frac{z_n}{x_n})$

$= \lim(y_n) + \lim(\frac{z_n}{x_n})$

$= \lim(y_n) + \frac{\lim(z_n)}{\lim(x_n)}$

$x = 2 + \frac{1}{x}$

$x^2 = 2x + 1$

$x^2 - 2x - 1 = 0$

Therefore, $x = 1 + \sqrt{2}$ (or) $x = 1 - \sqrt{2} < 0$

Problem 4 *Show that* $(-1)^n$ *is divergent*

Solution Suppose sequence $(-1)^n$ is convergent and $\lim(-1)^n = a$

Let $\in = 1 > 0$

There exists a positive integer N such that

$|(-1)^n - a| < 1$ if $n \geq N$

suppose n is even

$|1 - a|$ ¡ 1 if $n \geq N$

$-1 < 1 - a < 1$ if $n \geq N$

$-2 < -a < 0$ if $n \geq N$

$2 > a > 0$ if $n \geq N$ suppose n is odd

$| - 1 - a|$ ¡ 1 if $n \geq N$

$-1 < -1 - a < 1$ if $n \geq N$

$-1 + 1 < -a < 1 + 1$ if $n \geq N$

$0 > a > -2$ if $n \geq N$

Therefore we have $a > 0$ and $a < 0$

Hence $(-1)^n$ is diverges.

Theorem 24 *Let (x_n) be a sequence of positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) = L$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) =$ 0*

Proof since (x_n) is a sequence of positive real numbers. we have $(\frac{x_{n+1}}{x_n})$ is also a sequence of positive real numbers.

By previous theorem, $L \geq 0$

suppose $L < 1$, then $0 \leq L < 1$

let $r \in R$ such that $L < r < 1$

let $\in = r - L > 0$

since $(\frac{x_{n+1}}{x_n})$ converges, there exist a positive integer N, such that

$|\frac{x_{n+1}}{x_n} - L| < \in$ if $n \geq N$

Then $\frac{x_{n+1}}{x_n} < \in + L$ if $n \geq N$

$\frac{x_{n+1}}{x_n} < (r - L) + L$ if $n \geq N$

$\frac{x_{n+1}}{x_n} < r$ if $n \geq N$

Therefore $x_{n+1} < rx_n$ if $n \geq N$

$\therefore 0 \leq x_{n+1} < r.x_n < r^2 x_{n-1} < \cdots < r^{n-N+1} x_N$

Let $C = \frac{x_N}{r^N}$

$\therefore 0 \leq x_{n+1} < C.r^{n+1}$

since $0 < r < 1$, $\lim(x_n) = 0$

Problem 5 *Consider a sequence $\{x_n\}$ with $x_n = \frac{n}{2^n}$.Discuss about the convergent of (x_n) and find the limit of the sequence.*

Solution Given $x_n = \frac{n}{2^n}$. Then $x_{n+1} = \frac{n+1}{2^{n+1}}$

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{n+1}{2^{n+1}}.\frac{2^n}{n} \\
&= \frac{2^n.n+1}{2^n.2n} \\
&= \frac{n+1}{2n} \\
\lim\left(\frac{x_{n+1}}{x_n}\right) &= \frac{1}{2} < 1
\end{aligned}$$

By previous theorem, we have (x_n) converges and $\lim(x_n) = 0$

Problem 6 *Let $a > 0$ and construct a sequence (s_n) of real numbers such that* $\lim(s_n) = \sqrt{a}$

Solution Let $s_1 > 0$ be arbitrary and define $s_{n+1} = \frac{1}{2}(s_n + \frac{a}{s_n})$ for $n \in N$

Now

$$\begin{aligned} s_{n+1} &= \frac{1}{2}(\frac{s_n^2 + a}{s_n}) \\ 2s_{n+1} &= \frac{s_n^2 + a}{s_n} \\ 2s_{n+1}s_n &= s_n^2 + a \\ s_n^2 - 2s_{n+1}.s_n + a &= 0 \end{aligned}$$

since the quadratic has real roots, we must have ($n \in \mathbb{N}$)

$$\begin{aligned} 4.s_{n+1}^2 - 4a &\geq 0 \\ 4s_{n+1}^2 &\geq 4a \\ s_{n+1}^2 &\geq a \end{aligned}$$

Now

$$\begin{aligned} s_n - s_{n+1} &= s_n - \frac{1}{2}(s_n + \frac{a}{2n}) \\ &= \frac{1}{2}(\frac{s_n^2 - a}{s_n}) \\ s_n - s_{n+1} &\geq 0 \\ s_n &\geq s_{n+1},\ n \in \mathbb{N} \end{aligned}$$

clearly (s_n) is a monotone decreasing sequence.

$\therefore$ (s_n) is convergent

Let $\lim(s_n) = s$

$$\begin{aligned}
\lim(s_n) &= \lim(s_{n+1}) \\
&= \lim[\frac{1}{2}(s_n + \frac{a}{s_n})] \\
&= \lim[\frac{1}{2}(s_n + \frac{a}{2}\frac{1}{s_n})] \\
&= [\frac{1}{2}\lim(s_n) + \frac{a}{2}.\frac{1}{\lim(s_n)}] \\
&= \frac{1}{2}s + \frac{a}{2}.\frac{1}{s} \\
&= \frac{1}{2}(s + \frac{a}{s}) \\
2s^2 &= s^2 + a \\
s^2 &= a \\
s &= \sqrt{a} \text{ or } -\sqrt{a} \\
\therefore \lim(s_n) &= \sqrt{a} \text{ since } s > 0
\end{aligned}$$

Theorem 25 *Let* $e_n = (1 + \frac{1}{n})^n$, $n \in N$ *then,* $\lim(e_n) = e$

Proof Given $e_n = (1 + \frac{1}{n})^n$

since, the expression for e_n contains $n+1$ terms, and the expression for e_{n+1} contains $n + 2$ terms and each term appearing in $e_n \leq e_{n+1}$.

Therefore (e_n) is monotone increasing sequence

since $2^{p-1} \leq p!$, $(p = 1, 2, \ldots, n)$

$\frac{1}{2^{p-1}} \geq \frac{1}{p!}$

Hence $2 \leq e_n = 3$

$\therefore$ (e_n) is bounded.

Hence (e_n) is convergent and $\lim(e_n)$ lies between 2 and 3. We define the number e to be the limit of this sequence.

$\therefore \lim(e_n) = e$

2.7 Multiple choice questions

1. Suppose $\lim(x_n) = x$ and $\lim(-x_n) = x$. Then $x =$

   A. 1 B. $\frac{1}{2}$

   C. 0 D. -1

2. Suppose $\lim(x_n) = x$. For every $\epsilon > 0$, there is a +ve integer N such that we have

   A. $x - \epsilon < x_n$ B. $x + \epsilon > x_n$

   C. both A and B D. neither A nor B

3. The sequence $\left(\frac{1}{n}\right)$ is

   A. convergent B. bounded

   C. both A and B D. neither A nor B

4. The sequence $((-1)^n)$ is

   A. convergent B. bounded

   C. both A and B D. neither A nor B

5. Constant sequence is

   A. increasing B. decreasing

   C. both A and B D. neither A nor B

6. If $X = ((-1)^n))$ and $Y = \left((-1)^{n+1}\right)$ then $X + Y$

   A. coverges B. diverges

C. both A and B D. neither A nor B

7. If X and $X+Y$ are convergent, then Y

A. coverges B. diverges

C. both A and B D. neither A nor B

8. If $x_1 = 8$ and $x_{n+1} = \frac{x_n}{2} + 2$, (x_n) is

A.monotone B.bounded

C. both A and B D. neither A nor B

9. If $z_n = (a^n + b^n)^n$ and $0 < a < b$, then $\lim(z_n) =$

A. 0 B. 1

C. a D. b

10. If X converges to x and XY converges then Y converges if

A. $x \neq 0$ B. $x_n \neq 0$

C. both A and B D. neither A nor B

11. A sequence (x_n) in A is a function from —- to A

A. $\mathbb{R}$ B. $\mathbb{Z}$

C. $\mathbb{N}$ D. $\mathbb{W}$

12. The range of a real sequence is

A. $\mathbb{R}$ B. $\mathbb{Z}$

C. $\mathbb{N}$ D. $\mathbb{W}$

13. $\lim\left(\frac{3n+2}{n+1}\right) =$

A. 1 B. 2

C. 3 D. 4

14. $\lim(b^n) = 0$ if

A. $b > 1$ B. $0 \le b \le 1$

C. $b \ge 1$ D. $0 < b < 1$

15. $\lim\left(\frac{1}{1+na}\right) = 0$ if

A. $a > 0$ B. $0 \le a \le 1$

C. $a \ge 0$ D. $0 < a < 1$

16. $\lim\left(a^{\frac{1}{n}}\right) = 0$ if

A. $a > 0$ B. $0 \le a \le 1$

C. $a \ge 0$ D. $0 < a < 1$

17. The nth of the sequence $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8} \cdots$ is

A. $\frac{1}{2^n}$ B. $\frac{(-1)^n}{2^n}$

C. $\frac{(-1)^{n+1}}{2^n}$ D. $\frac{(-1)^{n+1}}{2^{n+1}}$

18. $\lim(b^n) =$

A. 0 B. 2

C. 3 D. 1

19. The sequence (a_n) where $a_n = \frac{n}{2^n}$ is

A. increasing B. decreasing

C. both A and B D. neither A nor B

20. $\lim\left(\frac{1}{\sqrt{n}}\right) =$

A. 0 B. 2

C. 3 D. 1

21. If $y_1 = 1$ and $y_{n+1} = \frac{1}{4}(2y_n + 3)$ for $n \geq 1$, $\lim(y_n) =$

    A. $\frac{1}{2}$ B. $\frac{3}{2}$

    C. $\frac{1}{3}$ D. $\frac{2}{3}$

22. If $s_1 > 0$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$ for $n \geq 1$, $\lim(s_n) =$

    A. a B. $\sqrt{a}$

    C. $\frac{1}{a}$ D. $\frac{1}{\sqrt{a}}$

23. If $s_n = \left(1 + \frac{1}{n}\right)^n$ for $n \geq 1$, $\lim(s_n) =$

    A. π B. $\sqrt{\pi}$

    C. $\frac{1}{e}$ D. e

24. Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \geq 1$. Then the sequence (x_n) is

    A. increasing B. decreasing

    C. both A and B D. neither A nor B

2.8 Two marks questions

1. Give an example for unbounded sequence.
2. Define a bounded sequence.
3. Define a convergent sequence.
4. Give an example for bounded sequence need not be a convergent sequence
5. Define a sequence.

6. Give an example monotonic sequence need not be a convergent sequence

7. Given an example for a monotonic sequence which is convergent

8. Prove that if $c > 0$, $\lim c^{\frac{1}{n}} = 1$

9. State and prove squeeze theorem

10. If $a > 0$, then prove that $\lim \frac{1}{1+na} = 0$

11. Prove that a convergent sequence of real numbers is bounded. Also prove that the converse is need not be true.

12. Prove that $\lim n^{\frac{1}{n}} = 0$

13. Let $X = (x_n)$ and $Y = (y_n)$ be sequence of real numbers that converges to x and y respectively. Prove that the sequences $X + Y$ and XY converge to $x + y$ and xy, respectively.

14. State and prove uniqueness theorem on limit.

15. Prove that a convergent sequence of real numbers is bounded

16. Prove that a sequence in R can have at most one limit.

UNIT II

If A is the set of even prime numbers and B is the	A is a subset of B	B is a subset of A	A and B are disjoin	A and B are not dis	A and B are disjoint
which relation is not a function?	{(2,5),(3,6).(4,7)}	{(2,1),(3,2).(4,7)}	{(2,1),(2,3).(3,4),(4	{(2,1),(3,3),(4,1)}	{(2,1),(2,3).(3,4),(4,1))}
Given the relation A={(5,2),(7,4),(9,10),(x,5)}. Which of	7	9	4	5	4
Let A be the set of letters in the word " trivial" and let	{a,r,v}	{d,f,c,u}	{I,l.t}	{a,I,l,r,t,v}	{a,r,v}
Let S be the set of of all 26 letters in the alphabet and	19 A^c	20	21	22	21
Let A={1,2}. Then A X A =	{(1,1),(2,2)}	{(1,2),(2,1)}	{(1,1)(1,2),(2,1),(2,	{(1,1),(2,2),(2,1)}	{(1,1)(1,2),(2,1),(2,2)}
Let A={1,2} and B={a,b,c}. Then number of elements in	2	3	2*2*2	2*3	2*3
Suppose n(A)=a and n(B)=b. Then number of elements	a	b	ab	a+b	ab
Let A={1,2} and B={a,b,c}. Then which of the following	(1,a)	(3,c)	(c,2)	(1,c)	(c,2)
Let F be a function and (x,y) in F and (x,z) in F. Then w	x=y	y=z	z=x	x=x	y=z
If the number of elements in a set S are %. Then the nu	5	6	16	32	32
If range of f is equal to codain set, then f is	into	onto	one-one	many to one	onto
Converse of function is a function only if f is	into	onto	one-one	bijection	bijection
Inverse function is always	into	onto	one-one	bijection	bijection
If A and B contains n elements then number bijection l	n!	n	n+1	n-1	n!

Let f be a function from A to B. Then we call f as a seq	set of positive inte	set of all real numl	set of all rationals	set of irrationals	set of positive integers
Two sets A and B are said to be similar iff there is a fur	into	one-one	onto	bijection	bijection
If two sets A={1,2,...,m} and B={1,2,..,n} are smilar ther	m<n	n<m	n=m	n>0	n=m
Which of the following is an example for countable?	set of real number	set of all irrational	set of all rationals	(0,1)	set of all rationals
Number of elements in the set of all real numbers is	finite	countably infinite	10000000000	uncountable	uncountable
The union of elements A and B is the set of elements b	either A or B	neither A not B	both A and B	A and not in B	either A or B
The set of elements belongs A and not in B is	B	A	B-A	A-B	A-B
The set of elements belongs B and not in A is	B	A	B-A	A-B	B-A
Countable union of countable set is	uncountable	countable	finite	countably infinite	countable
N X N is	uncountable	countable	finite	countably infinite	countable
Z X R is	uncountable	countable	finite	countably infinite	uncountable
R x R is	uncountable	countable	finite	countably infinite	uncountable
The set of sequences consists of only 1 and 0 is	uncountable	countbale	finite	countably infinite	uncountable
Every subset of a countable set is	uncountable	countable	finite	countably infinite	countable
Every subset of a finite set is	uncountable	countable	finite	countably infinite	finite
Fibonnaci numbers is an example for	uncountable set	countable set	finite set	infinte set	countable

Suppose A and B is countable then A X B is	uncountable	countable	finite	infinite	countable
A X B is similar to	A	B	A XA	A X B	A X B
The set of all even integers is	uncountable	countable	finite	infinite	countable
(0,1] is	uncountable	countable	finite	countably infinite	uncountable
{1,2,.....,100000}	uncountable	countable	infinite	countably infinite	countable
Suppose f is a one to one function. Then x not eqaul y	f(x) is not equal tc	f(x)=f(y)	f(x)<f(y)	f(x)>f(y)	f(x) is not equal to f(y)
Suppose f is a one to one function. Then f(x)=f(y) impl	x=-y	y=x+10	x=y	x is not eqaul y	x=y
Let f be a bijection between A and B and A is counatbl	uncountable	countable	finite	similar to R	countable
Let f be a function defined on A and itself such that f(x	onto	one to one	bijection	neither one to one	bijection
Constant function is an example for	onto	one to one	many to one	bijection	many to one
Stricly increasing function is	an onto function	one to one	many to one	bijection	one to one
Strictly decreasing function is	an onto function	one to one	many to one	bijection	one to one
If *g(x)* = 3x + x + 5, evaluate *g* (2)	8	9	13	17	13
A = {x: x ≠ x }represents	{1}	{}	{0}	{2}	{}
If a set A has n elements, then the total number of subsets of A	n!	2n	2^n	n	2^n

Chapter 3

Infinite series

3.1 Introduction

If $x = (x_n)$ is a sequence in R then the infinite series or series generated by x is the sequence $s = s_n$ defined by

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \end{aligned}$$

Remark 25 *1. clearly*

$$\begin{aligned} s_n &= x_1 + x_2 + \cdots + x_n \\ &= x_1 + x_2 + \cdots + x_{n-1} + x_n \\ &= s_{n-1} + x_n \end{aligned}$$

2. The numbers x_n are called the terms of the series and the numbers s_n is called the partial sum of this series.

3. If $\lim S$ *exists, we say that the series is convergent and this limit is the sum or the value of this series.*

4. If this limit does not exists, we say that the series is divergent.

5. It is convenient to use symbols such as $\sum(x_n)$ *to denote the infinite series.*

Example 16 *consider the series* $\sum \frac{1}{n(n+1)}$

Solution $\sum \frac{1}{n(n+1)}$

Now $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Then

$$\begin{aligned} s_n &= \frac{1}{1.2} + \frac{1}{2.3} + \dots \\ &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots \\ &= 1 - \frac{1}{n+1} \\ \lim s_n &= \lim\left(1 - \frac{1}{n+1}\right) \\ &= \lim(1) - \lim\left(\frac{1}{n+1}\right) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$\therefore \sum \frac{1}{n(n+1)}$ *is converges.*

3.2 Geometric series

Example 17 *Consider the series* $\sum r^n = 1 + r + r^2 + \dots$

Solution

$$
\begin{aligned}
s_n &= 1 + r + r^2 + \cdots + r^{n-1} \\
s_n(1-r) &= s_n - s_n r \\
&= 1 + r + r^2 + \cdots + r^{n-1} - (1 + r + r^2 + \cdots + r^{n-1}).r \\
&= 1 - r^n \\
s_n(1-r) &= 1 - r^n \\
s_n &= \frac{1}{1-r} - \frac{r^n}{1-r} \\
s_n - \frac{1}{1-r} &= -\frac{r^n}{1-r} \\
s_n - \frac{1}{1-r} &= -\frac{r^n}{1-r} \\
\lim\left(s_n - \frac{1}{1-r}\right) &= \lim\left(-\frac{r^n}{1-r}\right) \\
&= 0 \ \ if \ \ |r| < 1 \\
\lim s_n &= \frac{1}{1-r}
\end{aligned}
$$

$\therefore \sum r^n$ *converges if* $|r| < 1$

3.3 The nth term test

Theorem 26 *If the series* $\sum x_n$ *converges then* $\lim(x_n) = 0$

Proof Suppose $\sum x_n$ converges

Let s_n be the partial sum of $\sum x_n$

By definition of convergence of $\sum x_n$, we have $\lim(s_n) = x$

Now

$$\begin{aligned} s_n - s_{n-1} &= (x_1 + x_2 + \cdots + x_n) - (x_1 + x_2 + \ldots x_{n-1}) \\ &= x_n \\ \text{i.e. }, x_n &= s_n - s_{n-1} \\ \lim(x_n) &= \lim(s_n - s_{n-1}) \\ &= \lim(s_n) - \lim(s_{n-1)} \\ &= x - x = 0 \end{aligned}$$

Therefore $\lim(x_n) = 0$

Example 18 *Consider* $\sum \frac{1}{r(r+1)}$

Clearly the series converges.Also

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{1}{r(r+1)} &= \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{r(r+1)} + \cdots \\ \lim\left(\frac{1}{r(r+1)}\right) &= \lim(\frac{1}{r} - \frac{1}{r+1}) \\ &= \lim(\frac{1}{r}) - \lim(\frac{1}{r+1}) \\ &= 0 \end{aligned}$$

Example 19 *Consider the series* $\sum r^n$, $|r| < 1$

Clearly the series converges. Also

$$\begin{aligned} \sum_{n=1}^{\infty} r^n &= r^0 + r^1 + \cdots + r^n + \ldots \\ \lim(r^n) &= 0 \end{aligned}$$

Example 20 *Consider* $\sum(-1)^n$

$$\begin{aligned}\sum(-1)^n &= (1)^0 + (-1)^1 + \dots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \dots\end{aligned}$$

$\lim(s_n)$ *does not exist.*

There fore the sereis diverges.

Remark 26 *If* $\lim x_n \neq 0$*, then the series* $\sum\limits_{n=1}^{\infty} x_n$ *cannot converge.*

Theorem 27 *Let* (x_n) *be a sequence of nonnegative real numbers. Then the series* $\sum x_n$ *converges if and only if the sequence* $s = (s_k)$ *of partial sums is bounded. In this case* $\sum x_n = \lim(s_k) = sup\{s_k : k \in N\}$

Proof since $x_n > 0$, we have

$$\begin{aligned}s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ &= S_1 + x_2 \\ s_2 &> s_1 \\ s_3 &= x_1 + x_2 + x_3 \\ &= s_2 + x_3 \\ s_3 &> s_2\end{aligned}$$

$\therefore$, the sequence of partial sums satisfies $s_1 < s_2 < s_3 < \dots$

$\therefore (s_k)$ is monotone sequence.

Suppose $\sum x_n$ converges

By convergence definition, (s_k) converges.

$\therefore (s_k)$ is bounded.

Conversely (s_k) is bounded

i.e., (s_k) is monotone and bounded.

By monotone convergence theorem, (s_k) converges

Therefore $\sum x_k$ converges.

Moreover, $\lim(s_k) = sup\{s_k : k \in N\}$

$\therefore \sum x_k = sup\{S_k : k \in N\}$

Example 21 *Consider the series* $\sum(\frac{1}{n})$

Solution*Here*

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_3 &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\ &\vdots \end{aligned}$$

Therefore $s_1 < s_2 < \dots$

clearly (s_k) *is not bounded.*

$\therefore \sum(\frac{1}{n})$ *is divergent.*

Problem 7 *Show that* $\sum \frac{1}{(n+1)(n+2)} = 1$

Solution

$$\begin{aligned} s_n &= \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n+1)(n+2)} \\ &= 1 - \frac{1}{n+2} \quad since \quad \frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)} \end{aligned}$$

Therefore, sequence of partial sums (s_n) *bounded. Hence* $\sum \frac{1}{(n+1)(n+2)}$ *converges.*

Also

$$\begin{aligned} \lim(s_n) &= \lim(1 - \frac{1}{n+2}) \\ &= \lim(1) - \lim(\frac{1}{n+2}) \\ &= 1 - \lim(\frac{1}{n}) \\ &= 1 \end{aligned}$$

Hence $\sum \frac{1}{(n+1)(n+2)} = 1$

Theorem 28 *The p-series* $\sum \frac{1}{n^p}$ *diverges when* $0 < p \leq 1$

Proof We know that

$$\begin{aligned} n^p &\leq n \text{ if } \quad 0 < p \leq 1 \\ \frac{1}{n^p} &\geq \frac{1}{n} \\ i.e. \frac{1}{n} &\leq \frac{1}{n^p} \end{aligned}$$

Since the harmonic series, $\sum \frac{1}{n}$ diverges, we have $\sum \frac{1}{n^p}$ diverges.

3.4 Cauchy criterion

Theorem 29 *The series $\sum x_n$ converges if and only if for every $\in >$ 0 there exist $M(\in) \in N$ such that if $m > \geq M(\in)$ then $|S_m - S_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \in$*

3.5 Comparison test

Let $X = (x_n)$ and $Y = (y_n)$ be real sequences and suppose that for some $k \in N$ we have $0 \leq x_n \leq y_n$ for $n \geq k$

(a) Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.

(b) The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

Proof (a) suppose that $\sum y_n$ converges.
By cauchy criterion, for given $\epsilon > 0$ there exist $M(\epsilon) \in N$ such that

$$\begin{aligned}
|y_{n+1} + y_{n+2} + \cdots + y_m| &< \epsilon \text{ if } \quad m > n \geq M(\epsilon) \\
y_{n+1} + y_{n+2} + \cdots + y_m &< \epsilon \\
x_{n+1} + x_{n+2} + \cdots + x_m &< y_{n+1} + y_{n+2} + \cdots + y_m \\
x_{n+1} + x_{n+2} + \cdots + x_m &< \epsilon \text{ if } \quad m > n \geq M(\epsilon) \\
|x_{n+1} + x_{n+2} + \cdots + x_m| &< \epsilon
\end{aligned}$$

By cauchy criterion, $\sum x_n$ converges.
(b) Suppose $\sum x_n$ diverges
To prove $\sum y_n$ diverges
suppose $\sum y_n$ converges

by(a) $\sum x_n$ converges

$\Rightarrow\Leftarrow$ *to* $\sum x_n$ diverges.

$\therefore \sum y_n$ diverges.

3.6 Limit comparison test

Theorem 30 *suppose that $X = (x_n)$ and $Y = (y_n)$ are strictly positive sequences and suppose that the following limit exists in R.* $r = \lim(\frac{x_n}{y_n})$

(a) *If $r \neq 0$ then $\sum x_n$ is convergent if and only if $\sum y_n$ converges.*

(b) *If $r = 0$ and if $\sum y_n$ is convergent the $\sum x_n$ converges.*

Proof (a) Suppose $r = \lim(\frac{x_n}{y_n})$ and $r \neq 0$ then, clearly $r > 0$.
By convergence of sequence $(\frac{x_n}{y_n})$, for $\frac{r}{2} > 0$ there exist a N such

that

$$
\begin{aligned}
|\frac{x_n}{y_n} - r| < \frac{r}{2} \ \text{ if } \ n \geq N \\
\frac{-r}{2} < \frac{x_n}{y_n} - r < \frac{r}{2} \\
\frac{-r}{2} + r < \frac{x_n}{y_n} - r + r < \frac{r}{2} + r \\
\frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2} \\
\frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2} < 2r \\
\frac{r}{2} < \frac{x_n}{y_n} < 2r \\
\frac{r}{2} y_n \leq x_n < 2r.y_n
\end{aligned}
$$

Suppose $\sum y_n$ convergent.

$\sum (2r) y_n$ converges.

By comparison test, $\sum x_n$ converges.

By comparison test, $\sum (\frac{r}{2}) y_n$ converges.

Therefore $\sum y_n$ converges.

(b) Suppose $r = \lim(\frac{x_n}{y_n})$ and $r = 0$

For $\in = 1 > 0$, there exist N such that

$$
\begin{aligned}
|\frac{x_n}{y_n} - r| &< 1 \quad \text{if} n \geq N \\
|\frac{x_n}{y_n}| &< 1 \\
x_n &< y_n
\end{aligned}
$$

Suppose $\sum y_n$ converges, by comparison test, $\sum x_n$ converges.

Theorem 31 $\sum \frac{1}{n^2}$ *is convergent.*

Proof Let $k_1 = 2^1 - 1 = 2 - 1 = 1$

$$S_{k_1} = S_1 = 1(\text{sum of first term})$$

Let $k_2 = 2^2 - 1 = 4 - 1 = 3$

$$\begin{aligned}
S_{k_2} &= S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2} \\
&< Sk_2 = S_3 = 1 + \frac{1}{2^2} + \frac{1}{2^2} \\
&= 1 + \frac{2}{2^x} \\
&= 1 + \frac{1}{2}
\end{aligned}$$

Therefore $Sk_2 < 1 + (\frac{1}{2})^1$

$$\begin{aligned}
Sk_3 &= 7 \text{ sum of first 7 terms} \\
&= Sk_2 + (\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}) \\
&< 1 + \frac{1}{2} + (\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}) \\
&< 1 + \frac{1}{2} + \frac{1}{4} \\
&= 1 + \frac{1}{2} + \frac{1}{2^2}
\end{aligned}$$

Therefore $Sk_3 < 1 + (\frac{1}{2})^1 + (\frac{1}{2})^2$

By mathematical induction , $Sk_j < 1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^{j-1}$

Since the terms in the (R.H.S) is a partial sum of a geometric series $\sum r^n$ with $r = \frac{1}{2} < 1$

Also

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n &= \frac{1}{1-(\frac{1}{2})} \\ &= 2\end{aligned}$$

$\therefore$ The partial sum of $\sum \frac{1}{n^2}$ is bounded also $s_1 \leq s_2 \leq \dots$

$\therefore$ The sequence of partial sum is monotone.

By previous theorem, $\sum \frac{1}{n^2}$ converges.

Problem 8 *Prove that* $\sum \frac{1}{n^2+n}$ *converges.*

Solution clearly $0 < \frac{1}{n^2+n} < \frac{1}{n^2}, n \in N$

since the series $\sum \frac{1}{n^2}$ converges, by comparison test, $\sum \frac{1}{n^2+n}$ converges.

Problem 9 *Prove that the series* $\sum 1n^2 - n + 1$ *is convergent.*

Solution Let $x_n = \frac{1}{n^2-n+1}$ and $y_n = \frac{1}{n^2}$

Then

$$\begin{aligned}\frac{x_n}{y_n} &= \frac{\frac{1}{n^2-n+1}}{\frac{1}{n^2}} \\ &= \frac{n^2}{n^2-n+1} \\ \lim\left(\frac{x_n}{y_n}\right) &= \lim\left(\frac{1}{1-\frac{1}{n}+\frac{1}{n^2}}\right) \\ &= 1 \neq 0\end{aligned}$$

By limit comparison test, since $\sum \frac{1}{n^2}$ converges, we have

$\sum \frac{1}{n^2-n+1}$ converges.

Problem 10 *Prove that the series $\sum \frac{1}{\sqrt{n+1}}$ is divergent.*

Solution Let $x_n = \frac{1}{\sqrt{n+1}}$ and $y_n = \frac{1}{\sqrt{n}}$

$$\begin{aligned}\frac{x_n}{y_n} &= \sqrt{\frac{1}{\sqrt{n+1}}\frac{1}{\sqrt{n}}} \\ &= \frac{\sqrt{n}}{\sqrt{n+1}} \\ &= \sqrt{\frac{1}{1+\frac{1}{n}}} \\ &= 1 \neq 0\end{aligned}$$

By limit comparison test, since $\sum \frac{1}{\sqrt{n}}$ diverges then $\sum \frac{1}{\sqrt{n+1}}$ is also divergent.

3.7 Root Test

Theorem 32 *Given a series $\sum a_n$ of non-negative terms, Let $\rho =$* lim $\sqrt[n]{a_n}$

(a) *The series $\sum a_n$ converges if $\rho < 1$*

(b) *The series $\sum a_n$ diverges if $\rho > 1$*

(c) *The test is inconclusive if $\rho = 1$*

Proof (a) suppose $\rho < 1$

Let x be a real number such that $\rho < x < 1$ given that $\rho = \lim \sqrt[n]{a_n}$

Therefore there exist a positive integer N such that

$$\sqrt[n]{a_n} < \rho \quad \text{for all} \quad n \geq N$$
$$\sqrt[n]{a_n} < x < 1$$
$$a_n < x^n < 1$$

Since $\sum x^n$ converges, we have $\sum a_n$ converges.

(b) Suppose $\rho > 1$

Then

$$\begin{aligned} (a_n)^{\frac{1}{n}} &> 1 \text{ for infintely many} \\ (a_n) &> 1 \text{ for infintely many} \\ \lim(a_n) &> 1 \neq 0 \end{aligned}$$

$\therefore \sum a_n$ diverges.

(c) Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$

For both series $\rho = 1$

Clearly, $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges

Therefore, the test is inconclusive.

Problem 11 *Discuss about the convergence of* $\sum [\frac{n}{n+1}]^{n^2}$

Solution Let $a_n = \left[\frac{n}{n+1}\right]^{n^2}$

Therefore,

$$\begin{aligned}
\sqrt[n]{a_n} &= (a_n)^{\frac{1}{n}} \\
&= \left[\left[\frac{n}{n+1}\right]^{n^2}\right]^{\frac{1}{n}} \\
&= (\frac{n}{n+1})^n \\
\therefore \quad \lim \sqrt[n]{a_n} &= \lim(\frac{n}{n+1})^n \\
&= \lim\left[\frac{1}{(1+\frac{1}{n})^n}\right] \\
&= \frac{\lim(1)}{\lim(1+\frac{1}{n})^n} \\
&= \frac{1}{e} < 1
\end{aligned}$$

Therefore, $\rho < 1$

By root test, $\sum[\frac{n}{n+1}]^{n^2}$ converges.

Problem 12 *Discuss about the convergence of* $\sum(logn)^{-n}$

Solution Let

$$\begin{aligned} a_n &= (logn)^{-n} \\ \sqrt[n]{a_n} &= (a_n)^{\frac{1}{n}} \\ &= (logn)^{-1} \\ &= \frac{1}{logn} \\ \lim \sqrt[n]{(a_n)} &= \lim(\frac{1}{logn}) < 1 \\ \rho &< 1 \end{aligned}$$

By root test, $\sum(logn)^{-n}$ converges.

3.8 Ratio test

Theorem 33 *Let $\sum a_n$ be a series of positive terms such that* $\lim \frac{a_{n+1}}{a_n} = L$

(a) *The series $\sum a_n$ converges if $L < 1$.*

(b) *The series $\sum a_n$ diverges if $L > 1$.*

(c) *The test is inconclusive if $L = 1$*

Proof (a) suppose $L < 1$

Let x be a real number such that $L < x < 1$

Then there exist a positive integer N such that

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &< x \quad \text{for all } \ n \geq N \\
\frac{a_{n+1}}{a_n} &< \frac{x^{n+1}}{x^n} \\
a_{n+1} &< x^{n+1}\frac{a_n}{x^n} \\
&\leq x^{n+1}\frac{a_N}{x^N} \quad \text{for all } \ n \geq N \\
a_{n+1} &< c.x^{n+1} \text{ if } \ c = \frac{a_N}{x^N}
\end{aligned}$$

Since $x < 1$ and $\sum x^n$ converges for $|x| < 1$, we have $\sum a_n$ converges.

(b) suppose $L > 1$

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &> 1 \text{ for infinitely many} \\
a_{n+1} &> a_n \text{ for infinitely many}
\end{aligned}$$

$\therefore \sum a_n$ diverges.

(c) consider the series $\sum \frac{1}{n}$ and $\frac{1}{n^2}$.

For both series $L = 1$

Clearly, $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges.

$\therefore$ The test is inconclusive.

Remark 27 *Let* $\sum a_n$ *be a series of positive terms such that* $\lim \frac{a_n}{a_{n+1}} = L$

(a) *The series* $\sum a_n$ *converges if* $L > 1$

(b) *The series $\sum an$ diverges if $L < 1$*

(c) *The test is inconclusive if $L = 1$*

Problem 13 *Test the convergence of the series $\sum \frac{5^{n-1}}{n!}$*

Solution Here

$$\begin{aligned}
a_n &= n^{th} \text{ term} \\
&= \frac{5^{n-1}}{n!} \\
a_{n+1} &= n^{th} \text{ term} \\
&= \frac{5^n}{(n+1)!} \\
&= \frac{5^n}{n!(n+1)} \\
\frac{a_n}{a_{n+1}} &= \frac{5^{n-1}}{n!}\frac{n!(n+1)}{5^n} \\
&= \frac{n+1}{5} \\
\lim(\frac{a_n}{a_{n+1}}) &= \lim(\frac{n+1}{5}) > 1
\end{aligned}$$

Therefore, by ratio test, $\sum \frac{5^{n-1}}{n!}$ converges

Problem 14 *Test the convergence of the series $\sum \frac{2^n}{n^3+1}$*

Solution Here $a_n = n^{th} term = \frac{2^n}{n^3+1}$

$a_{n+1} = n^{th} term = \frac{2^{n+1}}{(n+1)^3+1}$

$\frac{a_n}{a_{n+1}} = (\frac{a^n}{n^3+1}).\frac{(n+1)^3+1}{2^n.2}$

$= \frac{1}{2} < 1$

$lim\frac{a_n}{a_{n+1}} = \frac{1}{2}$
By ratio test $\sum \frac{2^n}{n^3+1}$ is divergent.

Problem 15 *Test the convergence of the series* $\sum \frac{(n+1)^n}{n!}$

Solution $a_n = \sum \frac{(n+1)^n}{n!}$
$a_{n+1} = \sum \frac{(n+2)^{n+1}}{(n+1)!}$
$\frac{a_n}{a_{n+1}} = \frac{(n+1)^n}{n!} \frac{n!(n+1)}{(n+2)^{n+1}}$
$= \frac{(n+1)^{n+1}}{(n+2)^{n+1}}$
$= \frac{(n+1)^{n+1}}{[(n+1)+1]^{n+1}}$
$= \frac{1}{[1+\frac{1}{n+1}]^{n+1}}$
$\frac{a_n}{a_{n+1}} = \frac{1}{e}$ ¡ 1
$\therefore$ By ratio test $\sum \frac{(n+1)^n}{n!}$ is diverges.

Problem 16 *Test the convergence of the series* $\frac{2!}{3} + \frac{3!}{3^2} + \dots$

Solution Here $a_n = \frac{(n+1)!}{3^n}$
$a_{n+1} = \frac{(n+2)!}{3^{n+1}}$
$\frac{a_n}{a_{n+1}} = \frac{3}{n+2}$
$\lim(\frac{a_n}{a_{n+1}}) = \lim(\frac{3}{n+2}) = 0 < 1$
$\frac{(n+1)!}{3^n}$ is diverges.

Problem 17 *Test the convergene of the series* $\frac{1}{1+2} + \frac{2}{1+2^2} + \dots$

Solution Here $a_n = \frac{n}{1+2^n}$
$a_{n+1} = \frac{n+1}{1+2^{n+1}}$
$\frac{a_n}{a_{n+1}} = \frac{n(1+2^{n+1})}{1+2^n(n+1)}$

$lim\frac{a_n}{a_{n+1}} =$
$lim\frac{n(1+2^{n+1})}{1+2^n(n+1)}$
$= 2 > 1$

$\therefore$ The above series is convergent.

3.9 Alternating series

The series $\sum(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$ is alternating series where each $a_0 > 0$.

3.10 Leibniz's rule

Theorem 34 *If $\{a_n\}$ is an monotone decreasing sequence with limit 0, the alternating series $\sum(-1)^{n-1}a_n$ converges. If S denotes its sum and S_n its n^{th} partial sum, we also have $0 < (-1)^n(S - S_n) < a_{n+1}$ for all $n \geq 1$*

Proof

The partial sums S_{2n} form an increasing sequence.

$S_{2n+2} - S_{2n} = (a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} + a_{2n+1} - a_{2n+2}) - (a_1 - a_2 + a_3 - \cdots + a_{2n-1} - a_{2n}$

$= a_{2n+1} - a_{2n+2} > 0$

$= S_{2n+2} - S_{2n} > 0$

$\therefore S_{2n+2} > S_{2n}$

Also the partial sums S_{2n-1} form a decreasing sequence.

Both sequenes are bounded below by S_2 and bounded above by S_1.

$\therefore$ Each sequence (S_{2n}) and (S_{2n-1}) are monotone and bounded.

$\therefore$ By monotone convergence theorem (S_{2n}) and (S_{2n-1}) converges

$\therefore \lim S_{2n} = S^{'}$ and $\lim S_{2n-1} = S^{''}$

Now, $S^{'} - S^{''} = \lim S_{2n} - \lim S_{2n-1}$

$= \lim(S_{2n} - S_{2n-1})$

$= \lim(-a_{2n}) = -\lim a_{2n} = 0$

Therefore $S^{'} = S^{''} = S$ Therefore sequence of partial sums converges.

$\therefore \sum(-1)^{n-1}a_n$ converges.

since (S_{2n}) is a monotonicallly increasing sequence, we have

$S_{2n} < S_{2n+2} \leq S$

since (S_{2n-1}) is a monotonically decreasing sequence, we have

$S_{2n} < S_{2n+2} < S_{2n-1}$

$\therefore$ we have

$0 < S_{2n-1} - S \leq S_{2n-1} - S_{2n} = a_{2n+1}$

and $0 < S_{2n-1} - S \leq S_{2n-1} - S_{2n} = a_{2n}$

Hence we have. $0 < (-1)^n(S - S_n) < a_{n+1}$

3.11 Absolute convergence

Let $X = (x_n)$ be a sequence in R. we say that the series $\sum x_n$ is absolutely convergent if $|x_n|$ is convergent in R.

Conditional convergent

A series is said to be conditionally convergent but not absolutely convergent.

Example 22 *Consider a series* $\sum \frac{(-1)^n}{n}$
By Leibnitz's test, $\sum \frac{(-1)^n}{n}$ *converges.*
Now $\sum |\frac{(-1)^n}{n}| = \sum \frac{|(-1)^n|}{|n|} = \sum \frac{1}{n}$ *diverges is conditionally convergent.*

Remark 28 *A series of positive terms is absolutely convergent if and only if it is convergent.*

3.12 Two marks questions

1. Define a geometric series.
2. Define a geometric sequence.
3. Define a harmonic sequence.
4. State the nth term test.
5. Prove that the converse of the nth term test need not be true,
6. Define alternating harmonic series.
7. Give an example for alternating series
8. Define p-series
9. Estabilish the convergence or the divergence of the series whose nth term is $\frac{n}{(n+2)()n+3}$

10. Show that $\sum_{n=1}^{\infty} \frac{1}{n\cdot(n+1)\cdot(n+2)} = \frac{1}{4}$

11. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

12. State and prove limit comparison test.

13. Show that $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$

14. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ converges

15. Prove that if $\sum x_n$ converges then $\lim x_n = 0$

16. Prove that the 2-series converges.

17. State and prove the comparison test for the series

18. Discuss about the series (i) $\sum \frac{1}{n^2+n}$ (ii) $\sum \frac{1}{n!}$

19. State and prove the nth term test for series

20. State and prove Cauchy criterion for series.

21. Prove the p-series converges if $p > 1$.

UNIT III

If an increasing sequence is bounded above then	seqeunce converges to inf of its range	sequence converges to sup of its range	sequence converges to 1	sequence converges to 0	sequence converges to sup of its range
If an decreasing sequence is bounded below then	seqeunce converges to inf of its range	sequence converges to sup of its range	sequence converges to 2	sequence converges to 1	sequence converges to inf of its range
Fibonacci sequence is	an increasing sequence	a decresing sequence	constant sequence	bounded sequence	an incresing sequence
A sequence in a metric space (S,d) can converge	at least one point	more than two point	atmost one point	more than three point	atmost one point
Suppose a sequence in a metric space (S,d) converges to both a and b. Then we must have	a<b	a>b	a-b=1	a=b	a=b
In a metric space (S,d), a sequence converges to p. Then range of the sequence is	bounded	unbounded	finite	infinite	bounded
The range of a constant sequence is	infinite	countably infinite	uncountable	singlton set	singleton set
Suppose in a metric space (S,d), a sequence converges to p. Then the point p is	an adherent point of S	an accumulation point of S	an isolated point of S	not an adherent point of S	an adherent point of S
Suppose in a metric space (S,d) , a sequence converges to p and the rnage of the sequence is infinite. Then p is	an adherent point of S	an accumulation point of S	an isolated point of S	not an accumulation point of S	an accumulation point of S
Suppose in a metric space, a sequence converges. Then	every sequence in a metric space converges	subsequence of convergent sequence converges	subsequence of convergent sequence converges	some sequence in a metric space converges	subsequence of convergent sequence converges
A sequence is said to be bounded if if its range is	unbounded	bounded	countable	uncountable	bounded
The range of the sequence {1/n} is	finite	{1}	{}	infinite	infinite
The range of the sequence {1/n} is	unbounded	bounded	{}	{1,0}	bounded
The esequence {1/n}	converges	diverges	oscilates	converges to 1	converges

In Euclidean metric space every cauchy sequence is	convergent	divergent	oscilates	convergent to 0	converges
Every convergent sequence is a	constant seqeunce	cauchy sequence	increasing sequence	decreasing sequence	cauchy sequence
The sequence {n^2}	converges	diverges	oscilates	converges to 2	diverges
The range of the sequence {n^2} is	unbounded	bounded	{}	{0.1}	unbounded
The range of the sequence {n^2} is	finite	{1}	{}	infinite	infinite
The sequence {i^n}	converges	diverges	oscilates	converges to 0	diverges
The range of the sequence {i^n} is	unbounded	bounded	{}	{0,1}	bounded
The range of the sequence {i^n} is	finite	infinite	{}	{0,1}	finite
The sequence {1}	converges	diverges	oscilates	converges to 0	converges
The range of the sequence {1} is	{}	{1}	{1,0}	{1,2,3}	{1}
The range of the sequence {1} is	bounded	unbounded	{1,0}	{0}	bounded

Chapter 4

Subsequences

4.1 Subsequences

Definition 39 *Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural numbers. Then the sequence $X^{'} = (x_n k)$ given by $(x_{n_1}, x_{n_2}, \dots)$ is called a subsequence of X*

Example 23 *Consider a sequence $X = (1, \frac{1}{2}, \frac{1}{3}, \dots)$*
Let $X^{'} = (\frac{1}{2}, \frac{1}{4}, \dots)$
clearly, $x^{'}$ is a subsequence of X. note that $n_1 = 2, n_2 = 4, \dots$

Definition 40 *If $X(x_1, x_2, \dots)$ is a sequence of real numbers and if m is a given natural numbers, then the m-tail of X is the sequence.*
$X_m = (x_{m+1}, x_{m+2}, \dots)$

Remark 29 *A tail of a sequence is a special type of subsequence. (ii) Not every subsequence of a given sequence need be a tail of the sequence.*

Theorem 35 *If a sequence $X = (x_n)$, of real numbers converges to a real number x, then any subsequencece $x^{'} = (x_{n_k})$ of x, also converges to x.*

Proof

Given that,

$lim x_n = x$

$\therefore$ for given $\in > 0$, there exist a positive integer N such that $|x_n - x| < \in$ if $n \geq N$

Let $X^{'} = (x_{n_k})$ be a subsequence of X. The $n_1 < n_2 < n_3 < \dots$

clealy $n_k \geq k$

suppose $k \geq N$, then $n_k \geq N$

$|x_{n_k} - x| < \in$

Therefore (x_{n_k}) converges to x

Definition 41 *For a sequence (x_n), we say that the m^{th} term x_m of (x_n) if $x_m \geq x_n$ for all $n \geq M$.*

Remark 30 *In a decreasing sequence, every term is peak and in an increasing sequence no term is peak.*

4.2 The cauchy sequences

Definition 42 *A sequence $X = (x_n)$ of real number is said to be a cauchy sequence if for every $\in > 0$, there exist a natural number N*

such that $|x_n - x_m| \in$ *if* $n, m \geq N$

Theorem 36 *If* $X = (x_n)$ *is a convergent sequence of real numbers then X is a cauchy sequence.*

Proof

Let $X = (x_n)$ be a convergent sequence. Let $\lim x_n = x$

Let $\in > 0$ be arbitrary, then for $\frac{\in}{2} > 0$, there exist a positive integer N such that

$|x_n - x| < \frac{\in}{2}$ if $n \geq N$

Let $n, m \geq N$

Now $|x_n - x_m| = |x_n - x + x - x_m|$

$\leq |x_n - x| + |x - x_m|$ ¡ $\frac{\in}{2} + \frac{\in}{2} = \in$

$|x_n - x_m| < \in$ if $n, m \geq N$

Therefore (x_n) is a cauchy sequence.

Theorem 37 *A caushy sequence of real number is bounded*

Proof

Let $X = (x_n)$ be a cauchy sequence

Let $\in = 1$, then there exist a positive integer N such that

$|x_n - x_m| < 1$ if $n, m \geq N$

In particular, $|x_n - x_m| < 1$ if $n, m \geq N$

Now $|x_n| - |x_N| \leq |x_n - x_N| < 1$ if $n \geq N$

$\therefore |x_n| - |x_N| < 1$ if $n \geq N$

$|x_n| < 1 + |x_N|$ if $n \geq N$

Let $M = sup\{|x_1|, |x_2|, \ldots, |x_{N+1}|, 1 + |x_N|\}$

Then $|x_n| < M$ for all n

Therefore $-M < x_n < m$ for all n

Therefore (x_n) is bounded.

4.3 Cauchy convergence criterion

Theorem 38 *A sequence of real number is convergent if and only if it is cauchy sequence.*

Proof

Suppose $X = (x_n)$ is a convergent sequence

by previous theorem, X is a cauchy sequence.

Conversely suppose $X = (x_n)$ is a cauchy sequence. by previous theorem, X is bounded

By Bolzono theorem, X has a convergent subsequence.

Let $x_{n_k} \to x$

claim $x_n \to x$

since X is cauchy sequence, for given $\frac{\epsilon}{2} > 0$, there exist a positive integer N such that

$|x_n - x_m| < \frac{\epsilon}{2}$ if $n, m \geq N$

since (x_{n_k}) converges to x, for $\frac{\epsilon}{2} > 0$, there exist a positive integer $k \geq N$ such that

$|x_k - x| < \frac{\epsilon}{2}$ if $n \geq N$

Now $|x_n - x| = |x_n - x_k + x_k - x|$

$\leq |x_n - x_k| + |x_k - x| = \in$

i,e., $|x_n - x| < \in$ if $n \geq N$, therefore $x_n \to x$

i.e., X is a convergent sequence.

Problem 18 *Discuss the convergence of the series* $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \ldots$

Solution

Given series is an alternating series.

Let $a_n = \frac{1}{\sqrt{n}}$

$a_{n+1} = \frac{1}{\sqrt{n+1}}$

$a_{n+1} - a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}$

$= \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n}\sqrt{n+1}} < 0$

$a_{n+1} - a_n < 0$

$A_{n+1} < a_n$

$\therefore \{a_n\}$ is monotonically decreasing also $\lim a_n = \frac{1}{\sqrt{n}} = 0$

$\therefore$ The given **Solution** satisfies all the conditions of Leibnitz rule.

The given series converges.

Problem 19 *Discuss the convergence of* $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \ldots$

Solution

Given series is an alternating series

LEt $a_n = \frac{2n+3}{2n}$

$a_{n+1} = \frac{2(n+1)+3}{2(n+1)}$

$= \frac{2n+5}{2n+2}$

$a_{n+1} - a_n = \frac{-6}{2n(2n+2)} < 0$

$a_{n+1} < a_n$

$\therefore \{a_n\}$ is monotonically decreasing. Also

$lima_n = \frac{2n+3}{2n}$

$\frac{2+0}{2} = 1 \neq 0$

$\therefore$ the given series does not satisfies one of the condition of Leibnitz test.

$\therefore$ the given series diverges.

Problem 20 *Discuss the convergence of the series* $\frac{1}{log2} - \frac{1}{log3} + \frac{1}{log4} - \dots$

Solution

Given series is alternating series

Let $a_n = \frac{1}{log(n+1)}$

$a_{n+1} = \frac{1}{log(n+2)}$

$a_{n+1} - a_n = \frac{1}{log(n+2)} - \frac{1}{log(n+1)}$ ¡ 0

$a_{n+1} - a_n < 0$

$a_{n+1} < a_n$

$\therefore \{a_n\}$ is a monotonically decreasing.

$lima_n = \frac{1}{log(n+1)}$

$= \frac{1}{\infty} = 0$

Therefore the given series satisfies all the condition of leibnitz test.

The given series is convergent.

4.4 Two marks questions

1. Show that the sequence $\{x_n\}$ is monotone, where $x_n = \frac{3n+1}{2n-3}$ for all $n \geq 2$
2. Give an example of a bounded sequence that is not a Cauchy sequence.
3. Give an example for Cauchy sequence.
4. Define Cauchy sequence
5. State monotone subsequence theorem.
6. Define a subsequence.
7. Let $X = \{1, \frac{1}{2}, 3, \frac{1}{4}, \cdots\}$. Find any one susequence of X which is convergent
8. State monotone theorem.
9. State Cauchy criterion for a sequence
10. Show that the sequence $\{x_n\}$ is monotone, where $x_n = \frac{3n+1}{2n-3}$ for all $n \geq 2$
11. State and prove monotone subsequence theorem.
12. State and prove Cauchy convergence criterion for sequences
13. State and prove Bolzano- Weirstrass theorem.

14. Prove that every convergent sequence is Cauchy sequence. Also prove that the converse need not be true.

15. State and prove monotone subsequence theorem.

16. Prove that a bounded sequence converges to x if every subsequence converges to x.

17. Prove that any convergent sequence is a Cauchy sequence.

18. State and prove Cauchy convergence criterion

Unit IV

Constant sequence	converges	oscillates	diverges	converges to 1	Converges
The sequence {1,1,1,1,1,.....}	converges	oscillates	diverges	converges to 1	converges to 1
The sequence {1,0,1,0,1,0,...}	converges	oscillates	diverges	converges to 1	Oscillates
The harmonic series converges if	P=1	p>1	P<1	P=0	p>1
In limit comparison test both the series converges absolutely if	r=1	r=0	r is not equal to zero	R=2	r is not equal to zero
For the absolute convergence of the series, the ratio between n+1th term and nth term must be	Less than r	Greater than r	Less than or equal to r	Greater than equal to r	Less than or equal to r
For the absolute convergence of the series, the nth root of nth term must be	Less than r	Greater than r	Less than or equal to r	Greater than equal to r	Less than or equal to r
The alternating harmonic series	converges	oscillates	diverges	converges to 1	Converges
If a series converges absolutely, the series	converges	oscillates	diverges	converges to 1	Converges
A series converges iff converges absolutely if the series consists of ----terms	positive	negative	Non zero	Either a or b	Positive
The series 1-1+1-1+1-1+...	converges	oscillates	diverges	converges to 1	Diverges

Chapter 5

Sequences and series of functions

5.1 Sequences of functions

Definition 43 *Let $A \subseteq R$ be given and suppose that for each $n \in N$ there is a function $f_n\colon A \to R$, we say that (f_n) is a sequence of functions A to $B \to R$.*

Definition 44 *A sequence (f_n) of functions on $A \subseteq R$ to R, converges to a function $f\colon A \to B$ if for every $\in > 0$ there exist a positive integer $N(\in, x)$ such that $|f_n(x) - f(x)| < \in$ if $x \in A$ and $n \geq N$*

Remark 31 *(i) The positive integer N will depend on both $\in$ and $x \in A$.*
(ii) The sequence (f_n) converges on A to f, we have $f_n \to f$ (or) $f(x) = \lim f_n(x)$

Example 24 *Let* $f(x) = \frac{x}{n}$, $x \in R$

Now $f(x) = \lim f_n(x) =$

$lim(\frac{x}{n})$

$= \frac{\lim x}{\lim n} = \frac{x}{\infty} = 0$

Therefore, $f_n \to f$ *for all* $x \in R$.

Example 25 *Let* $f_n(x) = x^n, x \in R$

$f(x) = \lim f_n(x) = \lim x^n$

$f_n \to f(x) = 0, -1 < x < 1$ *(or)* $f_n \to f(x) = 1, x = 1$

Example 26 $f_n(x) = \frac{sin(nx+n)}{n}$, $x \in R$

$f(x) = \lim f_n(x) = \lim \frac{sin(nx+n)}{n} = 0$

5.2 Uniform convergence

A sequence (f_n) of functions on $A \subseteq R$ to R converges uniformly on A to a function $f : A \to R$ if for every $\in > 0$ there exist a positive integer N such that $|f_n(x) - f(x)| < \in$ if $n \geq N$

Uniform norm

If $A \subseteq R$ and $f : A \to B$ is a function an f is bounded we define the uniform norm of f on A by $||f||_A = sup\{|f(x)| : x \in A\}$

Example

Let $f(x) = frac1x$

Then $||f|| = 1$

Note

Suppose $\in > 0$, and $||f||_A \leq \in$

By definiton of norm of f,

$\|f\| = sup\{|f(x)|: x \in A\} \leq \in$

$|f(x)| \leq \in$

suppose $|f(x)| \leq \in$

for all $x \in A$

$\|f\| \leq \in$

Hence, $\|f\|_A \leq \in \Leftrightarrow |f(x)| \leq \in$ for all $x \in A$

Theorem 39 *A sequence (f_n) of bounded function on $A \subseteq R$ converges uniformly on A to $f \Leftrightarrow \|f_n - f\| \to 0$.*

Proof

Suppose $f_n \to f$ uniformly on A. Then for $\in > 0$, there exist a positive integer N such that $|f_n(x) - f(x)| < \in$ if $n \geq N$

by previous theorem, $\|f_n - f\| < \in$ if $n \geq N$

$\|f_n - f\| \to 0$

conversely suppose $\|f_n - f\| \to 0$

on A

Then for given $\in > 0, thereexistapositiveintegerN$ such that

$|(\|f_n - f\|) - 0| < \in$ if $n \geq N$

$\|f_n - f\| < \in$ if $n \geq N$

i.e., $\|f_n - f\| < \in$ if $n \geq N$

i.e., $|f_n(x) - f(x)| < \in$ if $n \geq N$

$\therefore f_n \to f$ uniformly on A.

5.3 Series of functions

Definition 45 *If (f_n) is a sequence of functions defined on a subset D of R with values in R, the sequence of partial sums (S_n) of the infinite series $\sum f_n$ is efined for x in D by*

$$\begin{aligned} S_1(x) &= f_1(x) \\ S_2(x) &= f_1(x) + f_2(x) \\ S_3(x) &= f_1(x) + f_2(x) + f_3(x) \\ &\vdots \end{aligned}$$

In case, the sequence (S_n) of functions converges on D to a function f, we say that the infinte series of functions $\sum f_n$ converges to f on D.

Definition 46 *If the series $\sum |f_n(x)|$ converges for each α in D, we say that $\sum f_n$ is absolutely convergent on D.*

Definition 47 *If the sequence (S_n) of partial sums is uniformly convergent on D to a function f, we say that $\sum f_n$ is uniformly convergent on D to f.*

5.4 Weierstross M - test

Theorem 40 *Let (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in D, n \in N$*
If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.

Proof

Suppose $m > n$

$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)|$

$\leq |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_m(x)|$

$\leq M_{n+1} + M_{n+2} + \cdots + M_m$

By cauchy criterion for series, The series $\sum x_n$ converges if and only if for every $\in> 0$ there exist a positive integer M that if $m > n \geq M(\in)$ then

$|S_m - S_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| <\in$

since $\in M_n$ converges, $|M_{n+1} + M_{n+2} + \cdots + M_m| <\in$

$M_{n+1} + M_{n+2} + \cdots + M_m <\in$

Therefore $|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| <\in$

By cauchy criterion for sequence of functions $|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| <\in$

$\therefore \sum f_n$ uniformly convergent on D.

5.5 Power series

Definition 48 *A series of real functions $\sum f_n$ is said to be a power series around $x = c$ if the function f_n is of the form $f_n(x) = a_n(x-c)^n$ where a_n and c belong to R and where $n = 0, 1, 2, 3, \ldots$*

Definition 49 *Let $\sum a_n X^n$ be a power series. If the sequence $(|a_n|^{\frac{1}{n}})$ is bounded, we get $\rho = \lim sup(|a_n|^{\frac{1}{n}}$*
If this sequence is not bounded, we get $\rho = +\infty$. we define the ra-

dius of convergence of $\sum a_n x^n$ *to be given by*

$R = 0$ *if* $\rho = +\infty$

$= \frac{1}{\rho} if 0 < \rho < \infty$

$= \infty$ *if* $\rho = 0$

Remark 32 *The radius of convergence of the series* $\sum a_n x^n$ *is also given by*

$\lim(|\frac{a_n}{a_{n+1}})$ *provided the limits exists.*

Problem 21 *Find the radius of convergence of the series* $\sum a_n x^n$ *there* $a_n = \frac{1}{n!}$

Solution

$a_n = \frac{1}{n!}$

$a_{n+1} = \frac{1}{(n+1!)}$

$|\frac{a_n}{a_{n+1}}| = |\frac{1}{n!} x \frac{(n+1)!}{1}|$

$= |n + 1| = n + 1$

$\lim \frac{|}{a_n} a_{n+1}| = lim(n + 1) = \infty$

Therefore, The radius of convergence is $+\infty$

5.6 Cauchy-Hadmard Theorem

Theorem 41 *If R is the radius of convergence of the power series* $\sum a_n x^n$, *then the series* $\sum a_n x^n$ *is absolutely convergent if* $|x| < R$ *an is divergent if* $|x| > R$

Proof

Suppose $0 < R < +\infty$ suppose $|x| < R$

i.e., $0 < |x| < R$, then there is a positive real number $c < 1$ such that $|x| < c.R$

Therefore $|x| < c.\frac{1}{\rho}$

$\Rightarrow \rho < \frac{c}{|x|}$

$\Rightarrow \lim sup \sqrt{|a_n|} < \frac{c}{|x|}$

Therefore $|a_n| < \frac{c^n}{|x|^n}$

$\Rightarrow |a_n||x|^n < c^n$

$\Rightarrow |a_n x^n| < c^n$

since $c < 1$, the geometric series $\sum c^n$ converges.

By comparison test, $\sum |a_n x^n|$ converges.

Therefore $\sum a_n x^n$ converges absolutely.

Suppose $|x| > R$

$|x| > \frac{1}{\rho}$

$\therefore \lim sup \sqrt{a_n} > \frac{1}{|x|}$

$\Rightarrow |a_n| \geq \frac{1}{|x|^n}$

$\Rightarrow |a_n x^n| \geq 1$ for infintely many n

By comparison test, $\sum a_n x^n$ diverges.

Problem 22 *Discuss the uniform convergence of* $\sum \frac{sinnx}{n^2}$

Solution

Given $f_n(x) = \frac{sinnx}{n^2}$

$|f_n(x)| = |\frac{sinnx}{n^2}|$

$= \frac{|sinnx|}{n^2}$

$\leq \frac{1}{n^2}$

since $\sum \frac{1}{n^2}$ converges, we have $\sum sinnxn^2$ converges uniformly.

5.7 Cluster Point

Definition 50 *Let $A \subseteq R$. A point $C \in R$ is a cluster point of A if for every $\in > 0$ there exist atleast one point $x \in A$, $x \neq C$, such that $|x - c| < \in$*

Example 27 *Let $A = \{1, 2\}$ 1 and 2 are not cluster point of A. Moreover A has no cluster points of A.*

Remark 33 *Finite set has no cluster points. Cluster point is also called limit point.*

5.8 Two marks questions

1. Define uniformly convergent of a series.

2. Define radius of convergence.

3. Define power series.

4. Define absolutely convergent of a series.

5. Find the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} a_n\ x^n$, where $a_n = \frac{n^n}{n!}$

6. Find the radius of convergence of the series $\sum n^2\ 3^n\ z^n$
7. State M-test
8. Define uniform norm of a function
9. State Cauchy criterion for sequence of functions
10. State and prove Weierstrass M test
11. State and prove Cauchy Hadamard theorem
12. If $\mathbb{R}$ is the radius of convergence of the power series $\sum a_n x^n$, prove that the series absolutely convergent if $|x| < R$ and divergent if $|x| > R$.
13. If $\sum a_n x^n$ and $\sum b_n x^n$ converges on some interval $(-r, r), r > 0$, to the same function f, then prove that $a_n = b_n$ for all $n? \geq N$.
14. State and prove Cauchy criterion for series of functions.
15. State and prove Weierstrass M test

UNIT V

If R is the radius of convergence of the series, the series converges absolutely if |x|	>R	=R	<R	Less than or equal to R	<R
If rho=infinity, the radius of convergence R is	0	1	2	3	0
If rho=0, the radius of convergence R is	0	1	2	infinity	Infinity
If rho is finite, the radius of convergence R is	0	Rho	Reciprocal of rho	infinity	Reciprocal of rho
If R is the radius of convergence of the series, the series diverges if |x|	>R	=R	<R	Less than or equal to R	>R
If R is the radius of convergence then the interval of convergence is	(-R,R]	[-R,R]	(-R,R)	[-R,R)	(-R,R)
The sequence of functions (x/n) converges to a function x=	0	1	2	3	0
The sequence of functions x power n converges to a function x=0 if x lies between	1 and 2	-1 and 1	0 and 1	-1 and 0	-1 and 1
A series of positive terms converges then the series	converges only	converges absolutely	both A and B	neither A nor B	both A and B
A convergent series contains only finite number of negative terms then it is	converges only	converges absolutely	both A and B	neither A nor B	converges absolutely
A convergent series contains only -------- number of negative terms then it is converges absolutely	infinite	10	finite	countable	finite
A convergent series contains only finite number of --------- terms then it is converges absolutely	negative	positive	zero	1	negative

Reg. No..................

19MMU202

Karpagam Academy of Higher Education
Coimbatore-21
Department of Mathematics
Second Semester- I Internal test
Real Analysis

Date: Time: 2 hours
Class: I B.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(x) = x^2$ where $\mathbb{Z}$ is a set of all real numbers. Then the range of f is
A. $\mathbb{Z}$ B. $\mathbb{N}$
C. $\mathbb{W}$ D. $\{0, 1, 4, 9, \cdots\}$

2. The set of all positive integers $\{1, 2, \cdots\}$ is
A. finite B. infinite
C. countable D. uncountable

3. If $f : \{1, 2, \cdots\} \to \{0, \pm 1, \pm 2, \cdots\}$ defined by
$$f(x) = \begin{cases} \frac{x}{2}, & x \text{ is even} \\ -\left(\frac{x-1}{2}\right), & x \text{ is odd} \end{cases}$$
then $f^{-1}(100) =$
A. 100 B. 99
C. 200 D. 201

4. Let S be a bounded above set of real numbers and $\sup S = u$. Then for $x \in S$, we have
A. $x > u$ B. $x < u$
C. $x \leq u$ D. $x \geq u$

5. Which equation does not represent a function?
A. $y = 2x$ B. $y = x^2 + 10$
C. $y = \frac{10}{x}$ D. $x^2 + y^2 = 95$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x$. Then f is
A.one-one B. onto
C. bijection D. neither onto nor one-one

7. Which of the following sets is countable?
A. $(0, \infty)$ B. $\mathbb{R}$
C. set of all irrational numbers D. set of all Fibonacci numbers

8. $B - (B - A) = A$ if
A. $B \subset A$ B. $A \subset B$
C. $A \cup B = A$ D. $A \cup B = A$

9. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then the number of distinct functions from A into B is
A. 8 B. 9
C. 6 D. 5

10. $\sup \{1 - \frac{1}{n} : n \in \mathbb{N}\}=$
A. -1 B. 1
C. 0 D. $\frac{1}{2}$

11. Which of the following is not true?
A. $\mathbb{Z} \sim \{0, 1, 2, \cdots\}$ B. $\{1, 2, 3, \cdots\} \sim \{2, 4, 6, \cdots\}$
C. $\{1, 2, 3, \cdots\} \sim (0, \infty)$ D. $\{1, 2, 3, \cdots\} \sim \{1, 3, 6, \cdots\}$

12. Two sets A and B are similar iff there exists a function f such that
A. f is one-to-one only B. f is bijection
C. f is onto only D. f is many one

13. If $\{1, 2, 3 \cdots, m\} \sim \{1, 2, 3, \cdots, n\}$ then
A. $m < n$ B. $m = n$
C. $m > n$ D. $m \neg n$

14. The cardinal number of $\emptyset$ is
A. 1 B. 0
C. both A and B D. neither A nor B

15. A set S is countable if it is
A. finite B. countably infinite
C. both A and B D. neither A nor B

16. Let $\mathbb{R}$ be the set of all real numbers. Then number of elements in $\mathbb{R}$ is
A. finite B. countably infinite
C. uncountable D. zero

17. The union $A_1 \cap A_2$ is the set of those elements belongs
A. A B. B
C. both A and B D. either A nor B

18. If a function $f : A \to B$ is such that $\mathscr{R}(f) = B$ then f is a/an
A. one to one B. onto
C. both A and B D. neither A nor B

19. Let $f : \mathbb{R} to \mathbb{R}$ be a function defined by $f(x) = [x]^2 + [x+1]3$ where [] denotes greatest integer function, then $f(x)$ is
A. manyone into B. manyone onto
C. oneone into D. oneone onto

20. The total number of points of undefined points of $f(x)$ in $x \in [2, 2]$ are
A. 1 B. 3
C. 4 D. infinite

Part B-($3 \times 2 = 6$ **marks)**

21. If $a, b \in \mathbb{R}$, prove that $|a+b| = |a| + |b|$ iff $ab \geq 0$

22. State the order properties of $\mathbb{R}$

23. Let $S = \{1, 2\}$ and $T = \{a, b, c\}$. Determine all different injections from S into T

Part C-($3 \times 8 = 24$ **marks)**

24. a) (i) State and prove triangle inequality. (4)
(ii) State and prove Archimedian property of $\mathbb{R}$

OR

b) Prove that $\mathbb{Z}$ is countable

25. a) State and prove Bernoulli's inequality

OR

b) Prove the following
(i) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.
(ii) $1 > 0$.
(iii) If $n \in \mathbb{N}$, then $n > 0$.

26. a) Prove that there does not exist a rational number r such that $r^2 = 2$.

OR

b) State and prove Cantor's theorem

Reg. No..................

16MMU103

Karpagam Academy of Higher Education
Coimbatore-21
Department of Mathematics
First Semester- II Internal test
Real Analysis

Date: Time: 2 hours
Class: I B.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. Suppose $\lim(x_n) = x$ and $\lim(-x_n) = x$. Then $x =$
   A. 1 B. $\frac{1}{2}$
   C. 0 D. -1

2. Suppose $\lim(x_n) = x$. For every $\epsilon > 0$, there is a +ve integer N such that we have
   A. $x - \epsilon < x_n$ B. $x + \epsilon > x_n$
   C. both A and B D. neither A nor B

3. The sequence $\left(\frac{1}{n}\right)$ is
   A. convergent B. bounded
   C. both A and B D. neither A nor B

4. The sequence $((-1)^n)$ is
   A. convergent B. bounded
   C. both A and B D. neither A nor B

5. Constant sequence is
   A. increasing B. decreasing
   C. both A and B D. neither A nor B

6. If $X = ((-1)^n))$ and $Y = \left((-1)^{n+1})\right)$ then $X + Y$
   A. coverges B. diverges
   C. both A and B D. neither A nor B

7. If X and $X + Y$ are convergent, then Y
   A. coverges B. diverges
   C. both A and B D. neither A nor B

8. A sequence in $\mathbb{R}$ has ——- one limit
   A. atmost B. atleast
   C. no D. all the above

9. 5. Fibonacci sequence is a ——- sequence
   A. Bounded B. decreasing
   C. increasing D. constant

10. $\{x_n\}$ is a constant sequence if $x_n = c$, —
    A. for some $n \in \mathbb{N}$ B. for all $n \in \mathbb{N}$
    C. for no $n \in \mathbb{N}$ D. for only one $n \in \mathbb{N}$

11. $\lim\left(\frac{2}{n}\right) = ---$
    A. 1 B. -1
    C. 0 D. ∞

12. $\lim((2n/(n+2)) = ---$
    A. 1 B. -1
    C. 0 D. ∞

13. If $\lim|x_n| = 0$ then $\lim x_n = ---$
    A. 1 B. -1
    C. 0 D. ∞

14. If $\lim|x_n| = 0$ then $\lim x_n = ---$
    A. 1 B. -1
    C. 0 D. ∞

15. $\lim\left(\frac{1}{n^2+1}\right) = ----$ A. 1 B. -1
    C. 0 D. ∞

16. (2^n) is —
A. convergent B. divergent
C. both A and B D. Either A or B

17. (1^n) is —
A. convergent B. divergent
C. both A and B D. Either A or B

18. $(\frac{1}{3^n})$ is —
A. convergent B. divergent
C. both A and B D. Either A or B

19. $((-2)^n n^2)$ is —
A. convergent B. bounded
C. both A and B D. Either A or B

20. $((-1)^n)$ is —
A. monotonic B. bounded
C. both A and B D. either A or B

Part B-($3 \times 2 = 6$ **marks)**

21. Give an example of an unbounded sequence that has a convergent subsequence

22. Prove that $\lim\left(\frac{3n+1}{n+1}\right) = 0$

23. State Fibonacci sequence

Part C-($3 \times 8 = 24$ **marks)**

24. a) Prove that convergent sequence of real numbers is bounded.

OR

b) Prove that if $c > 0$ then $\lim(c^1 n) = 0$

25. a) Find $\lim\left(\frac{n^2}{n!}\right)$

OR

b) Prove that $\lim(\sqrt{n+1} - \sqrt{n}) = 0$

26. a) Prove that if $0 < b < 1$ and $\lim(b^n) = 0$

OR

b) State and prove uniqueness of limit of a sequence

KARPAGAM UNIVERSITY

(Established Under Section 3 of UGC Act, 1956)

Real Analysis

1. Let $A = \{1, 2\}$. Then $A \times A$ is
   A. $\{(1, 1), (2, 2)\}$ B. $\{(1, 2), (2, 1)\}$
   C. $\{(1, 1), (2, 2), (1, 2)\}$ D. $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$

2. Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then number of elements in $A \times B$ is
   A. 2 B. 3
   C. 2^3 D. 2×3

3. Suppose number of elements in A is n and number of elements in B is m. Then number of elements in $A \times B$ is
   A. $n + m$ B. $n \times m$
   C. n^m D. m^n

4. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then which of the following element does not belongs to $A \times B$
   A. $(1, a)$ B. $(3, c)$
   C. $(c, 2)$ D. $(1, c)$

5. Identify the domain of this relation $\{(9, 10), (6, -1), (6, 10), (7, -2), (11, 5)\}$ is
   A. $\{6, 7, 9, 11\}$ B. $\{6, 7, 9, 10\}$
   C. $\{-1, -2, 5, 10\}$ D. $\{-1, -2, 5, 11\}$

6. Identify the range of this relation $\{(9, 10), (6, -1), (6, 10), (7, -2), (11, 5)\}$ is
   A. $\{6, 7, 9, 11\}$ B. $\{6, 7, 9, 10\}$
   C. $\{-1, -2, 5, 10\}$ D. $\{-1, -2, 5, 11\}$

7. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(x) = x^2$ where $\mathbb{Z}$ is a set of all real numbers. Then the range of f is
A. $\mathbb{Z}$ B. $\mathbb{N}$
C. $\mathbb{W}$ D. $\{0, 1, 4, 9, \cdots\}$

8. The set of all positive integers $\{1, 2, \cdots\}$ is
A. finite B. infinite
C. countable D. uncountable

9. Greatest lower bound of set of all positive even integers is
A. 2 B. 0
C. 1 D. 4

10. Let S be a bounded above set of real numbers and $\sup S = u$. Then for $x \in S$, we have
A. $x > u$ B. $x < u$
C. $x \leq u$ D. $x \geq u$

11. Which equation does not represent a function?
A. $y = 2x$ B. $y = x^2 + 10$
C. $y = \frac{10}{x}$ D. $x^2 + y^2 = 95$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x$. Then f is
A.one-one B. onto
C. bijection D. neither onto nor one-one

13. Which of the following sets is countable?
A. $(0, \infty)$ B. $\mathbb{R}$
C. set of all irrational numbers D. set of all Fibonacci numbers

14. $B - (B - A) = A$ if
A. $B \subset A$ B. $A \subset B$
C. $A \cup B = A$ D. $A \cup B = A$

15. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then the number of distinct functions from A into B is
A. 8 B. 9
C. 6 D. 5

16. sup $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$=
A. -1 B. 1
C. 0 D. $\frac{1}{2}$

17. Let A be the set of letters in the word "trivial" and let B be the set of letters in the word difficult. Then $A - B =$
A. $\{a, r, v\}$ B. $\{d, f, c, u\}$
C.$\{i, l, t\}$ D. $\{a, i, l, r, t, v\}$

18. Let S be the set of all 26 letters in the alphabet and let A be the set of letters in the word "trivial". Then the number of elements in A^c is

A. 19 B. 20 C. 21 D. 22

19. Let $A = \{1, 2\}$. Then $A \times A$ is

A. $\{(1, 1), (2, 2)\}$ B. $\{(1, 2), (2, 1)\}$
C. $\{(1, 1), (2, 2), (1, 2)\}$ D. $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$

20. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then which of the following element does not belongs to $A \times B$
A. $(1, a)$ B. $(3, c)$
C. $(c, 2)$ D. $(1, c)$

21. Let F be a function and $(x, y) \in F$ and $(x, z) \in F$. Then we must have
A. $y \neq z$ B.$y < z$
C. $y > z$ D. $y = z$

22. Let $f : A \to B$ be a function and the range of f denoted by $\mathscr{R}(f)$. Which of the following is always is true?
A. $\mathscr{R}(f) \neq B$ B. $\mathscr{R}(f) \subseteq B$
C. $B \subset \mathscr{R}(f)$ D. $B \subseteq \mathscr{R}(f)$

23. If a function $f : A \to B$ is such that $\mathscr{R}(f) \neq B$ then f is a/an ?
A. into function B.onto function
C. surjective D. many to one

24. If a function $f : A \to B$ is such that $\mathscr{R}(f) = B$ then f is a/an ?

A. into function B. onto function
C. one to one function D. many to one

25. If $f : \{1, 2, \cdots\} \to \{0, \pm 1, \pm 2, \cdots\}$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \text{ is even} \\ -\left(\frac{x-1}{2}\right), & x \text{ is odd} \end{cases}$$

then $f^{-1}(100) =$
A. 100 B. 199
C. 200 D. 201

26. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is
A. one-to-one B. onto
C. bijection D. many to one

27. Let $f : X \to Y$ be a function. If f^{-1} is a function then f^{-1}
A. from $\mathscr{R}(f)$ to X B. from Y to X
C. from X to Y D. $\mathscr{R}(f)$ to Y

28. If f^{-1} is a function then
A. f is one-to-one but not onto B. f is onto but not one-to-one
C. f is both one-to-one and onto D. f is neither onto nor one-to-one

29. Let $f : A \to B$ be a function. We call f as a sequence in B if
A. $A = \{0, 1, 2, \cdots\}$ B. $A = \{1, 3, 5, \cdots\}$
C.$A = \{1, 2, 3 \cdots\}$ D. $A = \{0, 2, 4, \cdots\}$

30. A set S is countable if it is
A. both finite and countably infinite B. either finite or countably infinite
C. neither finite nor countably infinite D. finite but not countably infinite

31. Let $\mathbb{R}$ be the set of all real numbers. Then number of elements in $\mathbb{R}$ is
A. countably infinite B. uncountable
C. finite D. zero

32. Suppose $\lim(x_n) = x$ and $\lim(-x_n) = x$. Then $x =$
A. 1 B. $\frac{1}{2}$
C. 0 D. -1

33. Suppose $\lim(x_n) = x$. For every $\epsilon > 0$, there is a +ve integer N such that we have
A. $x - \epsilon < x_n$ B. $x + \epsilon > x_n$
C. both A and B D. neither A nor B

34. The sequence $\left(\frac{1}{n}\right)$ is
A. convergent B. bounded
C. both A and B D. neither A nor B

35. The sequence $((-1)^n)$ is
A. convergent B. bounded
C. both A and B D. neither A nor B

36. Constant sequence is
A. increasing B. decreasing
C. both A and B D. neither A nor B

37. If $X = ((-1)^n))$ and $Y = \left((-1)^{n+1})\right)$ then $X + Y$
A. coverges B. diverges
C. both A and B D. neither A nor B

38. If X and $X + Y$ are convergent, then Y
A. coverges B. diverges
C. both A and B D. neither A nor B

39. If $x_1 = 8$ and $x_{n+1} = \frac{x_n}{2} + 2$, (x_n) is
A.monotone B.bounded
C. both A and B D. neither A nor B

40. If $z_n = (a^n + b^n)^n$ and $0 < a < b$, then $\lim(z_n) =$
A. 0 B. 1
C. a D. b

41. If X converges to x and XY converges then Y converges if
A. $x \neq 0$ B. $x_n \neq 0$
C. both A and B D. neither A nor B

42. A sequence (x_n) in A is a function from —- to A
A. $\mathbb{R}$ B. $\mathbb{Z}$
C. $\mathbb{N}$ D. $\mathbb{W}$

43. The range of a real sequence is
A. $\mathbb{R}$ B. $\mathbb{Z}$
C. $\mathbb{N}$ D. $\mathbb{W}$

44. $\lim\left(\frac{3n+2}{n+1}\right) =$
A. 1 B. 2
C. 3 D. 4

45. $\lim(b^n) = 0$ if
A. $b > 1$ B. $0 \leq b \leq 1$
C. $b \geq 1$ D. $0 < b < 1$

46. $\lim\left(\frac{1}{1+na}\right) = 0$ if
A. $a > 0$ B. $0 \leq a \leq 1$
C. $a \geq 0$ D. $0 < a < 1$

47. $\lim\left(a^{\frac{1}{n}}\right) = 0$ if
A. $a > 0$ B. $0 \leq a \leq 1$
C. $a \geq 0$ D. $0 < a < 1$

48. The nth of the sequence $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8} \cdots$ is
A. $\frac{1}{2^n}$ B. $\frac{(-1)^n}{2^n}$
C. $\frac{(-1)^{n+1}}{2^n}$ D. $\frac{(-1)^{n+1}}{2^{n+1}}$

49. $\lim(b^n) =$
A. 0 B. 2
C. 3 D. 1

50. The sequence (a_n) where $a_n = \frac{n}{2^n}$ is
A. increasing B. decreasing
C. both A and B D. neither A nor B

51. $\lim\left(\frac{1}{\sqrt{n}}\right) =$
A. 0 B. 2
C. 3 D. 1

52. If $y_1 = 1$ and $y_{n+1} = \frac{1}{4}(2y_n + 3)$ for $n \geq 1$, $\lim(y_n) =$
A. $\frac{1}{2}$ B. $\frac{3}{2}$
C. $\frac{1}{3}$ D. $\frac{2}{3}$

53. If $s_1 > 0$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$ for $n \geq 1$, $\lim(s_n) =$
A. a B. $\sqrt{a}$
C. $\frac{1}{a}$ D. $\frac{1}{\sqrt{a}}$

54. If $s_n = \left(1 + \frac{1}{n}\right)^n$ for $n \geq 1$, $\lim(s_n) =$
A. π B. $\sqrt{\pi}$
C. $\frac{1}{e}$ D. e

55. Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \geq 1$. Then the sequence (x_n) is
A. increasing B. decreasing
C. both A and B D. neither A nor B

Real Analysis

Unit I

1. Define an uncountable set.
2. Define countable set and give an example.
3. Give two examples for uncountable sets.
4. State the triangle inequality
5. Define bounded set.
6. Define supremum of a set
7. Define infimum of a set.
8. Define unbounded set.
9. Give two examples for unbounded set
10. Give two example for bounded set
11. Prove that $|a+b| = |a| + |b|$ iff $a = b = 0$
12. State archimedian property of $\mathbb{R}$
13. Define cluster point
14. Prove that $\mathbb{R}$ is uncountable
15. Let $S = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Find sup S and inf S

16. Prove that the set of all rational number is countable.
17. If $a, b \in \mathbb{R}$, ptove that $|a+b| \leq |a| + |b|$
    OR State and prove triangle inequality
18. State and prove Archimedean property.
19. Let S be a subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Prove that $a + \sup S = \sup(a + S)$
20. Prove that the set of all real numbers is uncountable.

Unit II

1. Give an example for unbounded sequence.
2. Define a bounded sequence.
3. Define a convergent sequence.
4. Give an example for bounded sequence need not be a convergent sequence
5. Define a sequence.
6. Give an example monotonic sequence need not be a convergent sequence
7. Given an example for a monotonic sequence which is convergent
8. Prove that if $c > 0$, $\lim c^{\frac{1}{n}} = 1$
9. State and prove squeeze theorem
10. If $a > 0$, then prove that $\lim \frac{1}{1+na} = 0$
11. Prove that a convergent sequence of real numbers is bounded. Also prove that the converse is need not be true.
12. Prove that $\lim n^{\frac{1}{n}} = 0$

13. Let $X = (x_n)$ and $Y = (y_n)$ be sequence of real numbers that converges to x and y respectively. Prove that the sequences $X + Y$ and XY converge to $x + y$ and xy, respectively.

14. State and prove uniqueness theorem on limit.

15. Prove that a convergent sequence of real numbers is bounded

16. Prove that a sequence in R can have at most one limit.

Unit III

1. Define a geometric series.

2. Define a geometric sequence.

3. Define a harmonic sequence.

4. State the nth term test.

5. Prove that the converse of the nth term test need not be true,

6. Define alternating harmonic series.

7. Give an example for alternating series

8. Define p-series

9. Estabilish the convergence or the divergence of the series whose nth term is $\frac{n}{(n+2)()n+3}$

10. Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n\cdot(n+1)\cdot(n+2)} = \frac{1}{4}$

11. Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+n}$

12. State and prove limit comparison test.

13. Show that $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$

14. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ converges

15. Prove that if $\sum x_n$ converges then $\lim x_n = 0$

16. Prove that the 2-series converges.

17. State and prove the comparison test for the series

18. Discuss about the series (i) $\sum \frac{1}{n^2+n}$ (ii) $\sum \frac{1}{n!}$

19. State and prove the nth term test for series

20. State and prove Cauchy criterion for series.

21. Prove the p-series converges if $p > 1$.

Unit IV

1. Show that the sequence $\{x_n\}$ is monotone, where $x_n = \frac{3n+1}{2n-3}$ for all $n \geq 2$

2. Give an example of a bounded sequence that is not a Cauchy sequence.

3. Give an example for Cauchy sequence.

4. Define Cauchy sequence

5. State monotone subsequence theorem.

6. Define a subsequence.

7. Let $X = \{1, \frac{1}{2}, 3, \frac{1}{4}, \cdots\}$. Find any one susequence of X which is convergent

8. State monotone theorem.

9. State Cauchy criterion for a sequence

10. Show that the sequence $\{x_n\}$ is monotone, where $x_n = \frac{3n+1}{2n-3}$ for all $n \geq 2$
11. State and prove monotone subsequence theorem.
12. State and prove Cauchy convergence criterion for sequences
13. State and prove Bolzano- Weirstrass theorem.
14. Prove that every convergent sequence is Cauchy sequence. Also prove that the converse need not be true.
15. State and prove monotone subsequence theorem.
16. Prove that a bounded sequence converges to x if every subsequence converges to x.
17. Prove that any convergent sequence is a Cauchy sequence.
18. State and prove Cauchy convergence criterion

Unit V

1. Define uniformly convergent of a series.
2. Define radius of convergence.
3. Define power series.
4. Define absolutely convergent of a series.
5. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = \frac{n^n}{n!}$
6. Find the radius of convergence of the series $\sum n^2\, 3^n\, z^n$
7. State M-test
8. Define uniform norm of a function

9. State Cauchy criterion for sequence of functions
10. State and prove Weierstrass M test
11. State and prove Cauchy Hadamard theorem
12. If $\mathbb{R}$ is the radius of convergence of the power series $\sum a_n x^n$, prove that the series absolutely convergent if $|x| < R$ and divergent if $|x| > R$.
13. If $\sum a_n x^n$ and $\sum b_n x^n$ converges on some interval $(-r, r)$, $r > 0$, to the same function f, then prove that $a_n = b_n$ for all $n? \geq N$.
14. State and prove Cauchy criterion for series of functions.
15. State and prove Weierstrass M test