Semester – VI

		L T P C
17MMU602A	**METRIC SPACES AND COMPLEX ANALYSIS**	**4 2 0 4**

Scope: On successful completion of course the learners gain about Metric spaces**,** Continuous mappings and Convergence of sequences and series.

Objectives: To enable the students to learn and gain knowledge about definite integrals of functions, Contour integrals and its geometrical applications.

UNIT I

METRIC SPACES

Definition and examples - Sequences in metric spaces - Cauchy sequences.

Complete Metric Spaces - Open and closed balls – neighbourhood - open set - interior of a set. Limit point of a set - closed set - Diameter of a set - Cantor's theorem – Subspaces - dense sets – separable spaces.

UNIT II

CONTINUOUS MAPPINGS

Continuous mappings - sequential criterion and other characterizations of continuity – Uniform Continuity – Homeomorphism - Contraction mappings - Banach Fixed point Theorem - Connectedness - connected subsets of R.

UNIT III

LIMITS

Limits - Limits involving the point at infinity - continuity. Properties of complex numbers – regions in the complex plane - functions of complex variable - mappings. Derivatives, differentiation formulas - Cauchy-Riemann equations, sufficient conditions for differentiability.

UNIT IV

ANALYTIC FUNCTIONS

Analytic functions - Examples of analytic functions - Exponential function - Logarithmic function - Trigonometric function - Derivatives of functions - Definite integrals of functions.

Contours: Contour integrals and its examples - Upper bounds for moduli of contour integrals - Cauchy-Goursat theorem - Cauchy integral formula.

UNIT V

CONVERGENCE

Liouville's theorem and the fundamental theorem of algebra. Convergence of sequences and series- Taylor series and its examples - Laurent series and its examples, absolute and uniform convergence of power series.

SUGGESTED READINGS

TEXT BOOK

1. Satish Shirali., and Harikishan L. Vasudeva., (2006). Metric Spaces, Springer Verlag, London.

REFERENCES

1. Kumaresan S., (2011). Topology of Metric Spaces, Second Edition., Narosa Publishing House, New Delhi.
2. Simmons G.F., (2004).Introduction to Topology and Modern Analysis, McGraw-Hill, New Delhi.
3. James Ward Brown., and Ruel V. Churchill., (2009). Complex Variables and Applications, Eighth Edition., McGraw – Hill International Edition, New Delhi.
4. Joseph Bak., and Donald J. Newman., (2010). Complex Analysis, Second Edition., Undergraduate Texts in Mathematics, Springer-Verlag New York.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: J. Jansi
Subject Name: Metric Spaces and Complex Analysis **Sub.Code:17MMU601**
Semester: VI **Class: III- B.Sc Mathematics**

S. No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1	1	Introduction on Metric and uniform metric spaces: Definition and Examples	T1: ch-1 P.No: 27-34 R2: ch-2 P.No : 51-53
2	1	Sequence in metric spaces: Convergences and Diverges, Theorm and Examples	T1: ch-1 P.No:37-44
3	1	Sequence in metric spaces: Convergences and Diverges, Theorm and Examples	T1: ch-1 P.No:37-44
4	1	Cauchy sequences and complete metric spaces	T1: ch-1 P.No: 44-57
5	1	Tuotorial 1	
6	1	Open and Closed balls: Definition and Examples	T1: ch-2 P.No:64-65 R1: ch-1 P.No : 15-17 R2: ch-2 P.No :64-65
7	1	Theorem on neighborhood and Interior Point of a set	T1: ch-2 P.No:66-69
8	1	Tutorial 2	
9	1	Limit Point of a set, closed set ,Examples Proposition and Theorems	T1: ch-2 P.No: 70-75
10	1	Diameter of a set, Cantor set, Cantor's theorem & Examples. Subspaces:	T1: ch-2 P.No:75-80

		Lemma & Theorem	
11	1	Tutorial 3	
12	1	Definition and examples of Dense & Separable sets and theorems	R1: ch-2 P.No:86-88 R2: ch-3 P.No: 96-97
13	1	Tutorial 4 Recapitulation & discussion of possible questions	
	Total No of Hours Planned For Unit 1=13		
		UNIT-II	
1	1	Introduction on Continuous Mapping: Definition and Theorem & Lemmas and Sequential criterion	T1: ch-3 P.No:103-108
2	1	Sequential criterion sequential criterion and other characterizations of continuity	T1: ch-3 P.No:109-113
3	1	Tutorial 5	
4	1	Uniform continuity and Homeomorphism	T1: ch-3 P.No:114-122
5	1	Tutorial 6	
6	1	Contraction mappings and Banach Fixed point Theorem	T1: ch-3 P.No:132-138
7	1	Linear Differential Equation and Picard's Theorem	T1: ch-3 P.No:135-140
8	1	Tutorial 7	
9	1	Connectedness: Intermediate Value Theorem & Continuous theorem, Theorem on Connected component of point	T1: ch-4 P.No: 156-160 R1: ch-5 P.No:106-107
10	1	connected subsets of R.	T1: ch-4 P.No:160-163
11		Tutorial 8 Recapitulation & discussion of possible questions	

	Total No of Hours Planned For Unit II=11 hrs		
		UNIT-III	
1	1	Introduction on Limits: Definition, Examples and Theorem on limit	R3: Ch-2 P.No: 45-55
2	1	Tutorial 9	R3: Ch-2 P.No:50-55
3	1	Properties of Complex number and complex plane	R3: Ch-1 P.No:24-26 R3: Ch-1 P.No:31-32
4	1	Functions of a complex variables	R3: Ch-2 P.No:35-37
5	1	Tutorial 10	
6	1	Mappings: Examples	R3: Ch-2 P.No:38-44
7	1	Derivatives, Differential formula examples	R3: Ch-2 P.No:56-62
8	1	Tutorial 11	R3: Ch-2 P.No:60-62
9	1	Cauchy-Riemann Equation: Theorems & Examples	R3: Ch-2 P.No:63-66 R4: Ch-3 P.No: 35-38
10	1	Sufficient condition for Differentiability theorem	R3: Ch-2 P.No: 66-67
11	1	Tutorial 12 Recapitulation & discussion of possible questions	
Total No of Hours Planned For Unit III=11 hrs			
		UNIT-IV	
1	1	Introduction on Analytic function and Examples	R3: Ch-2 P.No:73-77
2	1	Exponential function and Logarithmic function Examples	R3: Ch-3 P.No:89-94
3	1	Tutorial 13	
4	1	Banaches & Derivatives of Logarithmic	R3: Ch-3 P.No:95-99

		and Identities	
5	1	Trigonometric function	R3: Ch-3 P.No:104-107
6	1	Tutorial 14	
7	1	Derivatives of functions and Definite integrals	R3: Ch-4 P.No: 117-120
8	1	Contours: Definitions and Examples	R3: Ch-4 P.No:122-132
9	1	Tutorial 15	
10	1	Upper bounds for moduli of contour integrals	R3: Ch-4 P.No:137-139
11	1	Cauchy-Goursat theorem and Cauchy-Integral Formula	R3: Ch-4 P.No:150-167 R4: Ch-5 P.No: 59
12	1	Tutorial 16 Recapitulation & discussion of possible questions	
Total No of Hours Planned For Unit IV=12 hrs			
		UNIT-V	
1	1	Lioville’s theorem & The Fundamental theorem of algebra	R3: Ch-4 P.No:172-174 R4: Ch-5 P.No: 59-61
2	1	Maximum modulus principle	R3: Ch-4 P.No:175-178
3	1	Tutorial 17	
4	1	Convergence of sequence and series	R3: Ch-5 P.No:181-186
5	1	Corollary and Examples on convergence of series	R3: Ch-5 P.No:186-187

6		Tutorial 18-	
7	1	Taylor series: Examples	R3: Ch-5 P.No:189-195
8	1	Laurent series: Examples	R3: Ch-5 P.No:197-205
9		Absolute & Uniform convergent of power series	R3: Ch-5 P.No:208-211
10		Tutorial 19	
11	1	Tutorial 20-Recapitulation and discussion of possible questions	
12	1	Discussion of previous ESE question papers.	
13	1	Discussion of previous ESE question papers.	
14	1	Discussion of previous ESE question papers	
	Total No of Hours Planned for unit V=13hrs		
		Total Planned Hours	**60 hrs**

TEXT BOOK

1. Satish Shirali., and Harikishan L. Vasudeva., (2006). Metric Spaces, Springer Verlag, London.

REFERENCES

1. Kumaresan S., (2011). Topology of Metric Spaces, Second Edition., Narosa Publishing House, New Delhi.
2. Simmons G.F., (2004).Introduction to Topology and Modern Analysis, McGraw-Hill, New Delhi.
3. James Ward Brown., and Ruel V. Churchill., (2009). Complex Variables and Applications, Eighth Edition., McGraw – Hill International Edition, New Delhi.
4. Joseph Bak., and Donald J. Newman., (2010). Complex Analysis, Second Edition., UndergraduateTexts in Mathematics, Springer-Verlag New York.

Class Representative **Signature of the Faculty**

Tutor **Programme Co-ordinator** **HOD**

Name and Signature
of the Student Representative

Name and Signature
of Course Faculty

Name and Signature of Class Tutor

Name and Signature of Coordinator

Head of the Department

UNIT I

Syllabus

Metric spaces: definition and examples - Sequences in metric spaces – Cauchy sequences. Complete Metric Spaces - Open and closed balls – neighbourhood - open set - interior of a set. Limit point of a set - closed set - diameter of a set - Cantor's theorem – Subspaces - dense sets – separable spaces.

1. Metric Spaces

The notion of function, the concept of limit and the related concept of continuity play an important role in the study of mathematical analysis. The notion of limit can be formulated entirely in terms of distance. For example, a sequence $\{x_n\}_{n \geq 1}$ of real numbers converges to x if and only if for all $\epsilon > 0$ there exists a positive integer n_0 such that $|x_n - x| < \epsilon$ whenever $n > n_0$. A discerning reader will note that the above definition of convergence depends only on the properties of the distance $|a - b|$ between pairs a, b of real numbers, and that the algebraic properties of real numbers have no bearing on it, except insofar as they determine properties of the distance such as,

$|a - b| > 0$ when $a \neq b$, $|a - b| = |b - a|$ and $|a - g| \leq |a - b| + |b - g|$.

There are many other sets of elements for which 'distance between pairs of elements' can be defined, and doing so provides a general setting in which the notions of convergence and continuity can be studied. Such a setting is called a *metric space*. The approach through metric spaces illuminates many of the concepts of classical analysis and economises the intellectual effort involved in learning them.

We begin with the definition of a metric space.

Definition 1.2.1. A nonempty set X with a map $d : X \times X \to R$ is called a metric space if the map d has the following properties:

(MS1) $d(x, y) > 0 \quad x, y \in X$;

(MS2) $d(x, y) = 0$ if and only if $x = y$; (MS3) $d(x, y) = d(y, x)$ x, y 2 X;

(MS4) $d(x, y) \leq d(x, z)$ þ $d(z, y)$ x, y, z 2 X.

The map d is called the metric on X or sometimes the distance function on X. The phrase "(X, d) is a metric space" means that d is a metric on the set X. Property (MS4) is often called the triangle inequality.

The four properties (MS1)–(MS4) are abstracted from the familiar properties of distance between points in physical space. It is customary to refer to elements of any metric space as points and $d(x, y)$ as the distance between the points x and y.

We shall often omit all mention of the metric d and write "the metric space X" instead of "the metric space (X, d)". This abuse of language is unlikely to cause any confusion. Different choices of metrics on the same set X give rise to different metric spaces. In such a situation, careful distinction between them must be maintained.

Suppose that (X, d) is a metric space and Y is a nonempty subset of X. The restriction d_Y of d to $Y \times Y$ will serve as a metric for Y, as it clearly satisfies the metric space axioms (MS1)–(MS4); so (Y, d_Y) is a metric space. By abuse of language, we shall often write (Y, d) instead of (Y, d_Y). This metric space is called a subspace of X or of (X, d) and the restriction d_Y is called the metric induced by d on Y.

(ii) The space of bounded functions. Let S be any nonempty set and B(S) denote the set of all real- or complex-valued functions on *S*, each of which is bounded, i.e.,

Define If *f* and *g* belong to B(*S*), there exist $M > 0$ and $N > 0$ such that

$$\sup_{x \in S} |f(x)| \leq M \text{ and } \sup_{x \in S} |g(x)| \leq N:$$

It follows that $\sup_{x \in S} |f(x) - g(x)| < 1$. Indeed,

$$|f(x) - g(x)| \leq |f(x)| + |g(x)| \leq \sup |f(x)| + \sup |g(x)|,$$

$$0 \leq \sup |f(x) - g(x)| \leq M + N$$

$$d(f, g) = \sup |f(x) - g(x)|, \quad f, g \text{ belongs } B(S):$$

Evidently, $d(f, g) \geq 0$, $d(f, g) = 0$ if and only if $f(x) = g(x)$ for all $x \in S$ and $d(f, g) = d(g, f)$. It remains to verify the triangle inequality for B(*S*). By the triangle inequality for R, we have

$$d(f, g) \leq d(f, h) + d(h, g),$$

for all $f, g, h \in B(S)$. The metric *d* is called the uniform metric (or supremum metric).

(ii) The space of continuous functions. Let *X* be the set of all continuous functions defined on [*a*,*b*], an interval in R. For $f, g \in X$, define

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|:$$

The measure of distance between the functions *f* and *g* is the largest vertical distance between their graphs Since the difference of two continuous functions is continuous, the composition of two continuous functions is continu- ous, and a continuous function defined on the closed and bounded interval [*a*,*b*] is bounded, it follows that $d(f, g) \in R$ for all f, g . It may be verified as in Example.

(viii) that *d* is a metric on *X*. The space *X* with metric *d* defined as above is denoted by *C*[*a*,*b*]. All that we have said is valid whether all complex-valued continuous functions are taken into

consideration or only real-valued ones are. When it is necessary to specify which, we write $C_C[a, b]$ or $C_R[a, b]$. Note that $C[a, b]$ B[, b] and the metric described here is the one induced by the metric in Example
(viii) and is also called the uniform metric (or supremum metric).

(x) The set of all continuous functions on $[a,b]$ can also be equipped with the metric

$$d(f, g) = \int_a^b |f(x) - g(x)|\, dx.$$

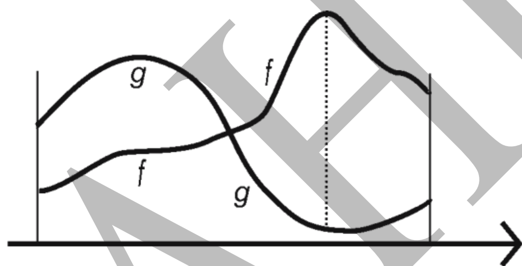

Figure 1.1

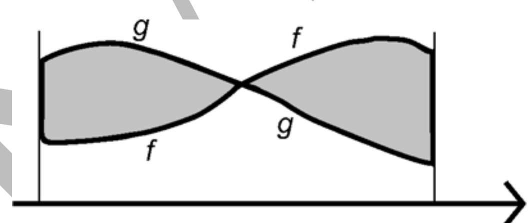

Figure 1.2

The measure of distance between the functions f and g represents the area between their graphs, indicated by shading[2] in Figure 1.2. If f, g $C[a, b]$, then f g $C[a, b]$, and the integral defining $d(f, g)$ is finite. It may be easily verified that d is a metric on $C[a, b]$. We note that the continuity of the functions enters into the verification of the 'only if' part of (MS2).

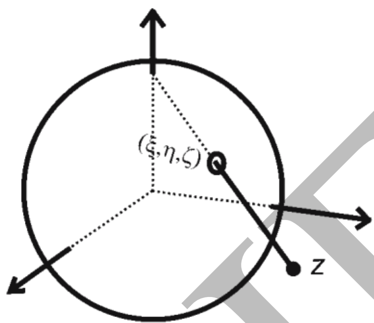

Figure 1.3

Analytically, the above representation is described by the formulae

Corresponding to the point at infinity, we have the point (0, 0, 1). Also,

We define the distance between the points of X by This is actually the chordal distance between those points on the sphere correspond- ing to the points Evidently, $d(z_1, z_2) > 0$ and $d(z_1,z_2)=0$

Definition: Let X be a nonempty set. A pseudometric on X is a mapping of X × X into R that satisfies the conditions:

(PMS1) $d(x, y) > 0$ x, y 2 X;
(PMS2) $d(x, y) = 0$ if x ¼ y;
(PMS3) $d(x, y) = d(y, x)$ x, y 2 X;
(PMS4) $d(x, y) \leq d(x, z)$ þ $d(z, y)$ x, y, z 2 X.

Another example of a pseudometric space is the following:

1.3. Sequences in Metric Spaces

As pointed out in Chapter 0, analysis is primarily concerned with matters involving limit processes. It is no wonder that mathematicians thinking about such matters studied and generalised the concept of convergence of sequences of real numbers and of continuous functions of a real variable. The reader will note that the basic facts about convergence are just as easily expressed in this setting.

Definition: Let (X,d) be a metric space. A sequence of points in X is a function f from N into X.

In other words, a sequence assigns to each n 2 N a uniquely determined element of X. If $f(n) = x_n$, it is customary to denote the sequence by the symbol $\{x_n\}_{n \, \$ \, 1}$ or $\{x_n\}$ or by $x_1, x_2, \ldots, x_n, \ldots$.

Definition 1.3.2. Let d be a metric on a set X and $\{x_n\}$ be a sequence in the set X. An element x 2 X is said to be a limit of $\{x_n\}$ if, for every e > 0, there exists a natural number n_0 such that

$$d(x_n, x) < e \text{ whenever } n > n_0:$$

In this case, we also say that $\{x_n\}$ converges to x, and write it in symbols as x_n x. If there is no such x, we say that the sequence diverges. A sequence is said to be convergent if it converges to some limit, and divergent otherwise.

Remark 1. By comparing the above with the definition of convergence in R (or C), we find that x_n ¼ x if and only if $\lim_{n!1}$ $d(x_n, x)$ 0, where d denotes the usual metric in R (or C).

Remark 2. In case there are two or more metrics on the set X, then it is necessary to specify which metric is intended to be used in applying the definition of convergence.

We next consider the notion of convergence in specific metric space for all

Cauchy Sequences

In real analysis (function theory), we have encountered Cauchy's principle of convergence. (Recall that a sequence $\{x_n\}_{n \geq 1}$ of numbers is said to be *Cauchy*, or to *satisfy the Cauchy criterion*, if and only if, for all e > 0, there exists an integer n_0(e) such that $|x_n - x_m| <$ e whenever $m \geq n_0$(e) and $n \geq n_0$(e). The Cauchy principle states that a sequence in R or C is convergent if and only if it is Cauchy.) The principle enables us to prove the convergence of a sequence without prior knowledge of its limit.

The real sequence

1 3 7 15

is such that for $m \geq n$ the distance between the terms is given by which tends to zero as *m*, *n* tend to infinity. In other words, the real sequence

$\{x_n\}_{n \geq 1}$, where $x_n = 1 - 1/2^n$, satisfies the Cauchy criterion and hence converges by Cauchy's principle of convergence.

A similar situation arises with sequences of functions; in fact, it comes up more often than with real or complex sequences. An extension of the idea of Cauchy sequences to metric spaces turns out to be useful.

Definition: Let *d* be a metric on a set *X*. A sequence $\{x_n\}_{n \geq 1}$ in the set *X* is said to be a Cauchy sequence if, for every e > 0, there exists a natural number n_0 such that

$$d(x_n, x_m) < e \text{ whenever } n > n_0 \text{ and } m > n_0:$$

Remark 1. A sequence $\{x_n\}$ in R or C is a Cauchy sequence in the sense familiar from elementary analysis if and only if it is a Cauchy sequence according to Definition
1.4.1 in the sense of the usual metric on R or C.

Remark 2. It is cumbersome to keep referring to a "sequence in a set *X* with metric *d*", especially if it is understood which metric is intended and no symbol such as *d* has been introduced to denote it. We shall therefore adopt the standard phrase "sequence in a metric space *X*".

(ii) In $C[0,1]$, the sequence f_1, f_2, f_3, ... given by

$$f_n(x) ¼ \frac{nx}{n þx}, \quad x\ 2\ [0, 1]$$

is Cauchy in the uniform metric. For m \$ n the function being continuous on [0, 1], assumes its maximum at some point x_0 2 [0,1]. So,

$d(f_m, f_n)$ ¼ sup {j$f_m(x)$ — $f_n(x)$j: x 2 [0, 1]} for large m and n. Moreover, the sequence $\{f_n\}_{n}$ \$ 1 converges to some limit. For

$f(x)$ ¼ x,

Therefore, $\{f_n\}_{n \$ 1}$ converges to the limit f, where $f(x)$ ¼ x for all x 2 [0, 1].

Proposition: A convergent sequence in a metric space is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a sequence in a set X with metric d, and let x be an element of X such ¼that $\lim_{n!1} x_n$ x. Given any e > 0, there exists some natural number n_0 such that $d(x_n, x)$ < e=2 whenever n \$ n_0. Consider any natural numbers n and m such that n \$ n_0 and m \$ n_0. Then $d(x_n, x)$ < e=2 and $d(x_m, x)$ < e=2. Therefore

$$d(x_n,x_m) \le d(x_n,x)+d(x_m,x)<\in \&$$

Does the converse of Proposition 1.4.3 hold? If a sequence $\{x_n\}_{n \$ 1}$ in a metric space (X, d) fulfills the Cauchy condition of Definition 1.4.1, does it follow that the sequence converges?

Examples 1.4.4. (i) Let X denote the set of all rational numbers with the usual metric, namely, $d(x, y)$ ¼ jx — yj for x, y 2 X. It is well known that the sequence

1:4, 1:41, 1:414, ...

converges to ffi 2. It is therefore Cauchy. However, it does not converge to a point of

X. So, a Cauchy sequence need not converge to a point of the space.

(ii) Another example of a Cauchy sequence that does not converge to a point of the space is the following: Let *X* ¼ *C*[0, 1] with metric *d* defined by

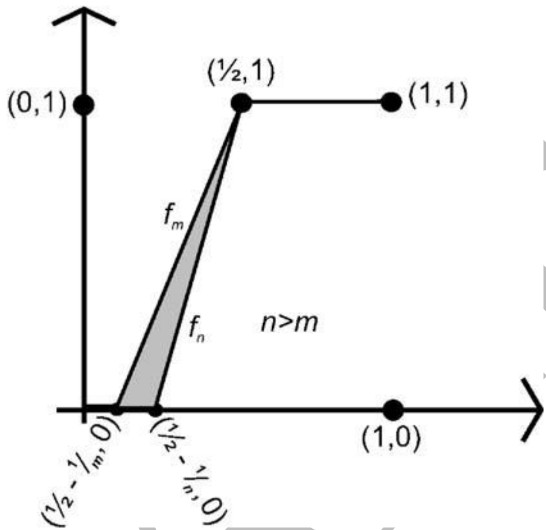

Figure 1.5

Suppose now that there is a continuous function *f* such that $d(f_n, f)$ ▪ 0. It will be shown that this leads to a contradiction. Since

j1 — *f* (*x*)j*dx* ¼ 0:

Since f is continuous, we see that $f(x)$ ¼ 0 for $0 \leq x \leq 1=2$ and $f(x)=1$ for $1=2 < x \leq 1$, which is impossible.

Thus, the metric spaces in which Cauchy sequences are guaranteed to converge are special and we need a name for them.

Definition: A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

It follows from Cauchy's principle of convergence that R, C and R^n equipped with their standard metrics$y$ $(y_1, y_2, \dots, y_n)$ in R^n) arecomplete metric spaces. The metric space (X, d), where X denotes the set of rationals and $d(x, y)$ x y for x, y X, has been observed to be an incomplete metric space (see Example 1.4.4(i)). That the metric space (X, d) of rationals is incomplete also follows on considering the sequence $\{x_n\}_{n \$ 1}$, where

$$x_n ¼ 1 þ \frac{1}{1!} þ \frac{1}{2!} þ \dots þ \frac{1}{n!},$$

as this is a Cauchy sequence but it converges to the irrational number e. In our next proposition, we need the following definition.

Definition: Let $\{x_n\}_{n \$ 1}$ be a given sequence in a metric space (X,d) and let $\{n_k\}_{k \$ 1}$ be a sequence of positive integers such that $n_1 < n_2 < n_3 < \dots$. Then the sequence $\{x_{n_k}\}_{k \$ 1}$ is called a subsequence of $\{x_n\}_{n \$ 1}$. If $\{x_{n_k}\}_{k \$ 1}$ converges, its limit is called a subsequential limit of $\{x_n\}_{n>1}$

It is clear that a sequence $\{x_n\}_{n \$ 1}$ in X converges to x if and only if every subsequence of it converges to x.

Proposition If a Cauchy sequence of points in a metric space (X,d) contains a convergent subsequence, then the sequence converges to the same limit as the subsequence.

Proof. Let $\{x_n\}_{n \,\$\, 1}$ be a Cauchy sequence in (X, d). Then for every positive number e there exists an integer n_0(e) such that

$$d(x_m, x_n) < \text{e whenever } m, n \,\$\, n_0(\text{e}):$$

Denote by $\{x_{n_k}\}$ a convergent subsequence of $\{x_n\}_{n \,\$\, 1}$ and its limit by x. It follows that

$$d(x_{n_m}, x_n) < \text{e whenever } m, n \,\$\, n_0(\text{e}),$$

since $\{n_k\}$ is a strictly increasing sequence of positive integers. Now,

$$d(x, x_n) \,\#\, d(x, x_{n_m}) \text{ þ } d(x_{n_m}, x_n) < d(x, x_{n_m}) \text{ þ e whenever } m, n \,\$\, n_0(\text{e}):$$

Letting $m \to 1$, we have

$$d(x, x_n) \leq \text{e whenever } n > n_0(\text{e}):$$

So, the sequence $\{x_n\}_{n \,\$\, 1}$ converges to x. &

We next show that the spaces $(\mathbb{R}^n, d_p)$, $B(S)$ and $C[a, b]$ are complete.

Proposition The space $B(S)$ of all real- or complex-valued functions f on S, each of which is bounded, with the uniform metric $d(f, g)$ ¼ $\sup\{\text{j}f(x) — g(x)\text{j}: x \text{ 2 } S\}$, is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $B(S)$. For each s 2 S, we have $\text{j}f_n(s) — f_m(s)\text{j} \leq d(f_n, f_m)$, so that the sequence $\{f_n(s)\}$ in C is a Cauchy sequence and therefore convergent. Define $f : S$! C by $f(s)$ ¼ $\lim_{n!1} f_n(s)$. We shall prove first that f 2 $B(S)$ and then prove that $\lim_{n!1} d(f_n, f)$ ¼ 0.

Since 1 > 0, therefore by the Cauchy property of $\{f_n\}$, there exists some n_0 such that

$$d(f_n, f_m) < 1 \text{ whenever } n \,\$\, n_0 \text{ and } m \,\$\, n_0:$$

In particular, $d(f_n, f_{n_0}) < 1$, and hence $\text{j}f_n(s) — f_{n_0}(s)\text{j} < 1$ for all s 2 S, whenever $n \,\$\, n_0$. Since f_{n_0} 2 $B(S)$, there exists some $M > 0$ such that $\text{j}f_{n_0}(s)\text{j} \,\#\, M$ for all s 2 S. Therefore, j$f_n(s)$j # j$f_n(s)$ — f_{n0} (s)jþ jf_{n0} (s)j # 1 þ M 8s 2 S whenever n \$ n_0:

Now consider any e > 0. By the Cauchy property of {f_n}, there exists some n_0

such that

$d(f_n, f_m)$ < e whenever n \$ n_0 and m \$ n_0:

Therefore, j$f_n(s)$ — $f_m(s)$j < e 8s 2 S whenever n \$ n_0 and m \$ n_0:

It follows upon letting $m \to \infty$ that

$$|f_n(s) - f(s)| \leq \varepsilon \quad \text{whenever } s \in S \text{ and } n \geq n_0.$$

This says that $\lim_{n\to\infty} d(f_n, f) = 0$. □

Proposition: Let $X = C[a, b]$ and $d(f, g) = \sup\{|f(x) - g(x)| : a \leq x \leq b\}$ be the associated metric. Then (X,d) is a complete metric space.

Proof. Let $\{f_n\}_{n \geq 1}$ be a Cauchy sequence in $C[a,b]$. Then for every $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that $m, n \geq n_0(\varepsilon)$ implies $d(f_m, f_n) = \sup\{|f_m(x) - f_n(x)| : a \leq x \leq b\} < \varepsilon$. In particular, for every $x \in [a, b]$, the sequence $\{f_n(x)\}_{n \geq 1}$ is a Cauchy sequence of numbers. By Cauchy's principle of convergence, $f_n(x) \to f(x)$, say, as $n \to \infty$. We have thus defined a function f with domain $[a,b]$. It remains to show that $f \in C[a, b]$ and that $\lim_{n\to\infty} d(f_n, f) = 0$.

Since for every $x \in [a, b]$,

$$|f_m(x) - f_n(x)| < \varepsilon$$

provided that $m, n \geq n_0(\varepsilon)$, it follows upon letting $m \to \infty$ that

$$|f_n(x) - f(x)| \leq \varepsilon \qquad (1.18)$$

for all $n \geq n_0(\varepsilon)$ and all $x \in [a, b]$.

To see why f is continuous, consider any $x_0 \in [a, b]$ and any $\eta > 0$. According to what has been noted in the preceding paragraph, there exists an integer $n_1(\eta)$ such that, for every $x \in [a, b]$, we have $|f_n(x) - f(x)| < \eta/3$ provided that $n \geq n_1(\eta)$. Select $m \geq n_1(\eta)$. Then

$$|f_m(x) - f_m(x)| < \frac{\eta}{3} \quad \text{for all } x \in [a, b]. \qquad (1.19)$$

Now use the continuity of f_m to obtain $\delta > 0$ such that

$$|f_m(x) - f_m(x_0)| < \frac{\eta}{3} \text{ for } |x - x_0| < \delta. \qquad (1.20)$$

Since $$|f(x) - f(x_0)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)|,$$

it follows from (1.19) and (1.20) that $|f(x) - f(x_0)| < \eta$ whenever $|x - x_0| < \delta$.

Therefore, $f \in C[a, b]$. Moreover, (1.18) says that $\lim_{n\to\infty} d(f_n, f) = 0$. As already noted, this completes the proof.

&

Examples: (i) Let *X* be any nonempty set and let *d* be defined by

$$d(x, y) \text{ - } \ ¼ \quad \begin{matrix} 0 & \text{if } x ¼ y, \\ 1 & \text{if } x \ 6¼ \ y. \end{matrix}$$

Then (*X*, *d*) is a complete metric space.

Indeed, if $\{x_n\}_{n \$ 1}$ is a Cauchy sequence, then for 0 < e < 1 there exists a positive integer n_0(e) such that $d(x_n, x_m)$ < e for all *m*, *n* $ n_0(e). So for *n* $ n_0(e), we have x_n ¼ x_{n_0} . Thus, any Cauchy sequence in (*X*,*d*) is of the form

$$(x_1, x_2, \ldots, x_{n_0}, x_{n_0}, \ldots),$$

which is clearly convergent to the limit x_{n_0} .

(ii) Let N denote the set of natural numbers. Define

$$d(m, n) \ ¼ \ j \frac{1}{m} - \frac{1}{n} j, \ m, n \ 2 \ \mathrm{N}:$$

Then (N, *d*) is an incomplete metric space.

That (N, *d*) is a metric space is clear. The sequence $\{n\}_{n \$ 1}$ can be shown to be Cauchy by arguing as follows. Let e > 0 and let n_0 be the least integer greater than 1=e. If *m*, *n* > n_0 then Cauchy Sequences suppose that it were to converge if possible to, say, *p* 2 N. Let n_1 be any integer greater than 2*p*. Then *n* $$n_1$ implies that

This shows that the sequence cannot converge to *p* and therefore does not converge at all.

Then (*X*, *d*) is a complete metric space.

Let $\{z_n\}_{n \$ 1}$ be a Cauchy sequence in (*X*,*d*). If the sequence $\{z_n\}_{n \$ 1}$ contains the point at infinity infinitely often, then it contains a convergent subsequence, namely the subsequence each of whose terms is 1. In this case, the sequence $\{z_n\}_{n \$ 1}$ converges to 1 by Proposition 1.4.7. On the other hand, if the point at infinity appears only finitely many times in the sequence, then we may assume without loss of generality that the sequence consists of points of C only, as the deletion (or insertion) of finitely many terms does not alter the convergence behaviour of a sequence.

Case I. If the sequence $\{|z_n|\}_{n \$ 1}$ is unbounded, then for every natural number *k* there exists a term z_{n_k} of the sequence such that $|z_{n_k}| > k$, where these terms can be chosen so that $n_{k þ 1} > n_k$, *k* ¼ 1, 2, ... Now,We thus have $\lim_{k ! 1} z_{n_k}$ ¼ 1 in (*X*, *d*). By Proposition 1.4.7, it follows that $\lim_{n ! 1} z_n$ ¼ 1.

Case II. The sequence $\{|z_n|\}_{n \$ 1}$ is bounded, say by *M* > 0. Let e > 0 be given. There exists n_0 2 N such that *m*, *n* $\$ n_0$ implies

Since $|z_n| < M$ for all *n*, it follows that $|z_n - z_m| < (1=2)e(1 þ M^2)$. This shows that

$\{z_n\}_{n \$ 1}$ is a Cauchy sequence in the usual metric in C, and hence $\lim_{n ! 1} |z_n - z|$ ¼ 0 for some *z* 2 C. Since $d(z_n, z_m) \# 2|z_n - z_m|$ always, it follows that $d(z_n, z)$<0 as *n*<1. Thus, (*X*, *d*) is a complete metric space.

Completion of a Metric Space

Let (*X*, *d*) be a metric space that is not complete. It is always possible to construct a larger space which is complete and contains *just* enough points so that every Cauchy sequence in *X* has a limit in the larger space. In fact, we need to adjoin new points to (*X*, *d*) and extend *d* to all these new points in such a way that the formerly nonconvergent Cauchy sequences find limits among these new points and the new points are limits of sequences in *X*.

Definition: Let (X, d) be an arbitrary metric space. A complete metric space (X^{m}, d^{m}) is said to be a completion of the metric space (X, d) if

(i) X is a subspace of X^{m};
(ii) every point of X^{m} is the limit of some sequence in X.

For example, the space of all real numbers is the completion of the space of rationals. Also, the closed interval [0,1] is the completion of (0,1), [0,1), (0,1] and itself. In fact, any complete metric space is its own completion. We note that the Weierstrass approximation theorem (Proposition 0.8.4) shows that the metric space $C_R[a, b]$ of Example 1.2.2(ix) is the completion of its subset consisting of polynomials.

Definition: Let (X,d) and (X^{0}, d^{0}) be two metric spaces. A mapping f of X into X^0 is an isometry if

$$d^{0}(f(x), f(y)) ¼ d(x, y)$$

for all x, y X. The mapping f is also called an isometric embedding of X into X^{0}. If, however, the mapping is onto, the spaces X and X^0 themselves, between which there exists an isometric mapping, are said to be isometric. It may be noted that an isometry is always one-to-one.

Theorem: Every metric space has a completion and any two completions are isometric to each other.

Proof. Let (X,d) be a metric space and let $\hat{X}$ denote the set of all Cauchy sequences in X. We define two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X to be equivalent if $\lim_{n\to\infty} d(x_n, y_n)$ ¼ 0 and write this in symbols as $\{x_n\} \sim \{y_n\}$. We shall now show that this is an equivalence relation in $\hat{X}$, i.e., the relation $\sim$ is reflexive, symmetric and transitive.

Reflexivity: $\{x_n\} \sim \{x_n\}$, since $d(x_n, x_n)$ ¼ 0 for every n and so $\lim_{n\to\infty} d(x_n, x_n)$ ¼ 0. Symmetry: If $\{x_n\} \sim \{y_n\}$, then $\lim_{n\to\infty} d(x_n, y_n)$ ¼ 0; but $d(x_n, y_n)$ ¼ $d(y_n, x_n)$ for every n, and, therefore, $\lim_{n\to\infty} d(y_n, x_n)$ ¼ 0, so that

$\{y_n\} \sim \{x_n\}$. Transitivity: If $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$, then $\lim_{n!1}$ $d(x_n, y_n)$ ¼ 0 and $\lim_{n!1}$ $d(y_n, z_n)$ ¼ 0. We shall show that $\lim_{n!1}$ $d(x_n, z_n)$ ¼ 0. Since

$$0 \leq d(x_n, z_n) \# d(x_n, y_n) þ d(y_n, z_n)$$

for all n, it follows that

$$0 \leq \lim_{n\to 1} d(x_n, z_n) \leq \lim_{n\to 1} d(x_n, y_n) þ \lim_{n\to 1} d(y_n, z_n) = 0,$$

i.e.,

$\lim_{n\to 1} d(x_n, z_n) = 0.$

∧

Thus, $\sim$ is an equivalence relation and X splits into equivalence classes. Any two members of the same equivalence class are equivalent, and no member of an equivalence class is equivalent to a member of any other equivalence class. Let $\tilde{X}$ denote the set of all equivalence classes; the elements of $\tilde{X}$ will be denoted by $\tilde{x}$, $\tilde{y}$, etc. Observe that if a Cauchy sequence $\{x_n\}$ has a limit $x \in X$, and if $\{y_n\}$ is equivalent to $\{x_n\}$, then $\lim_{n\to\infty} y_n = x$. This follows immediately from the follow- ing inequality:

$$d(y_n, x) \le d(y_n, x_n) + d(x_n, x).$$

Moreover, if $\{x_n\}$ and $\{y_n\}$ are two nonequivalent sequences, then $\lim_{n\to\infty} x_n \ne \lim_{n\to\infty} y_n$.

For, if $\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} y_n$, then the inequality

$$0 \le d(x_n, y_n) \le d(x_n, x) + d(x, y_n)$$

leads to $\lim_{n\to\infty} d(x_n, y_n) = 0$, contradicting the fact that $\{x_n\}$ and $\{y_n\}$ are two nonequivalent sequences. The constant sequence $(x, x, \dots, x, \dots)$ is evidently Cau- chy and has limit x.

Define a mapping $f: X \to \tilde{X}$ as follows: $f(x) = \tilde{x}$, where $\tilde{x}$ denotes the equiva- lence class each of whose members converges to x. Thus the constant sequence $(x, x, \dots, x, \dots)$ is a representative of $\tilde{x}$. In view of the observations made above, the mapping f is one-to-one. We next define a metric r in $\tilde{X}$. For $\tilde{x}, \tilde{y} \in \tilde{X}$, set

$$r(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n), \text{ where } \{x_n\} \in \tilde{x} \text{ and } \{y_n\} \in \tilde{y}.$$

Observe that

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_m, y_n),$$

where $\{x_n\} \in \tilde{x}$ and $\{y_n\} \in \tilde{y}$ and so $\{d(x_n, y_n)\}_{n\ge 1}$ is a Cauchy sequence of real numbers. Hence, $\lim_{n\to\infty} d(x_n, y_n)$ exists, for R is a complete metric space. We first show that r is well defined. Indeed, if $\{x_n'\} \sim \{x_n\}$ and $\{y_n'\} \sim \{y_n\}$, then

&

Remarks. (i) The proof explicitly assumes the completeness of R. Hence, the above method of completion cannot be employed for constructing the real number system from the rational number system.

(ii) There exist other methods of completion of an incomplete space. One such method will be provided in Example. 17 of Chapter 3 (Section 3.8).

Topology of a Metric Space

The real number system has two types of properties. The first type are algebraic properties, dealing with addition, multiplication and so on. The other type, called topological properties, have to do with the notion of distance between numbers and with the concept of limit. In this chapter, we study topological properties in the framework of metric spaces. We begin by looking at the notions of open and closed sets, limit points, closure and interior of a set and some elementary results involving them. The concept of base of a metric topology and related ideas are also discussed. In the final section, we deal with the important concept of category due to Baire and its usefulness in existence proofs. Also included are some theorems due to Baire.

Open and Closed Sets

There are special types of sets that play a distinguished role in analysis; these are the open and closed sets. To expedite the discussion, it is helpful to have the notion of a neighbourhood in metric spaces.

Definition: Let (X, d) be a metric space. The set

$S(x_0, r) = \{x\ 2\ X : d(x_0, x) < r\}$, where $r > 0$ and x_0 $2\ X$, is called the open ball of radius r and centre x_0. The set

$\bar{S}(x_0, r) = \{x\ 2\ X : d(x_0, x) \le r\}$, where $r > 0$ and $x_0\ 2\ X$,

is called the closed ball of radius r and centre x_0. A few concrete examples are in order.

Examples (i) The open ball $S(x_0, r)$ on the real line is the bounded open interval $(x_0\ r,\ \ x_0\ \ r)$ with midpoint x_0 and total length $2r$. Conversely, it is clear that any bounded open interval on the real line is an open ball. So the open balls on the real line are precisely the bounded open intervals. The closed balls $\bar{S}(x_0, r)$ on the real line are precisely the bounded closed intervals but containing more than one point.

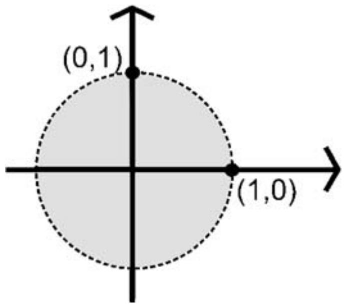

Figure 2.1

(ii) The open ball $S(x_0, r)$ in R^2 with metric d_2 (see Example 1.2.2(iii)) is the inside of the circle with centre x_0 and radius r as in Fig. 2.1. Open balls of radius 1 and centre (0,0), when the metric is d_1 or d_1 (see Example 1.2.2(iv) for the latter) are illustrated in Figs. 2.2 and 2.3.

(iii) If (X, d) denotes the discrete metric space (see Example 1.2.2(v)), then $S(x, r)$ ¼ $\{x\}$ for all x 2 X and any positive r # 1, whereas $S(x, r)$ ¼ X for all x 2 X and any $r > 1$.

(iv) Consider the metric space $C_R[a, b]$ of Example 1.2.2(ix). The open ball $S(x_0, r)$, where x_0 2 $C_R[a, b]$ and $r > 0$, consists of all continuous functions x 2 $C_R[a, b]$ whose graphs lie within a band of vertical width $2r$ and is centred around the graph of x_0. (See Fig. 2.4.)

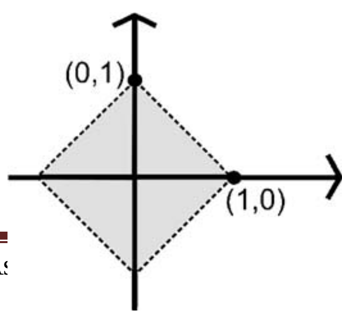

Figure 2.2

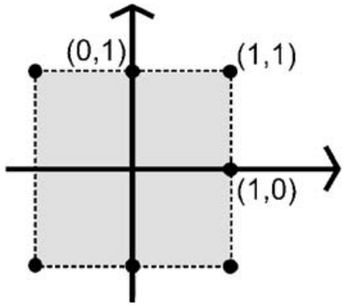

Figure 2.3

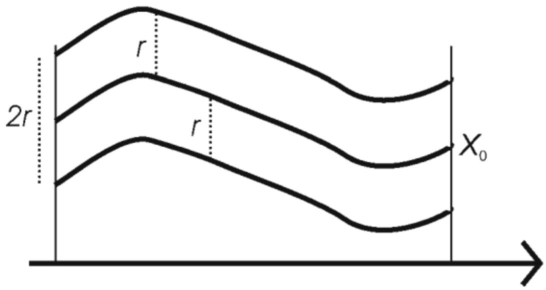

Figure 2.4

Definition: Let (X, d) be a metric space. A neighbourhood of the point x_0 2 X
is any open ball in (X, d) with centre x_0.

Definition: A subset G of a metric space (X, d) is said to be open if given any point x G, there exists $r > 0$ such that $S(x, r)$ G, i.e., each point of G is the centre of some open ball contained in G. Equivalently, every point of the set has a neighbourhood contained in the set.

Theorem: In any metric space (X, d), each open ball is an open set.

Proof. First observe that $S(x,r)$ is nonempty, since x 2 $S(x, r)$. Let y 2 $S(x, r)$, so that $d(y, x) < r$, and let $r' = r - d(y, x) > 0$. We shall show that $S(y, r') \leq S(x, r)$, as illustrated in Fig. 2.5. Consider any z 2 $S(y,$

r^{0}). Then we have

$$d(z,x) \leq d(z,y) \text{ þ } d(y,x) < r^{0} \text{ þ } d(y,x) \text{ ¼ } r,$$

which means z 2 $S(x, r)$. Thus, for each y 2 $S(x, r)$, there is an open ball
$S(y, r^{0}) \leq S(x, r)$. Therefore $S(x, r)$ is an open subset of X. &

Examples 2.1.6. (i) In R, any bounded open interval is an open subset because it is an open ball. It is easy to see that even an unbounded open interval is an open subset.

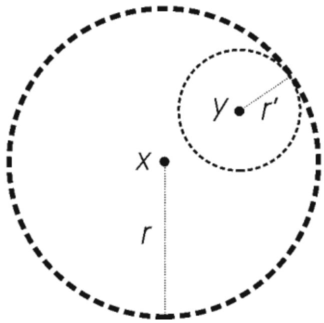

Figure 2.5

(ii) In a discrete metric space X, any subset G is open, because any x G is the centre of the open ball $S(x$, 1=2) which is nothing but $\{x\}$.

The following are fundamental properties of open sets.

Theorem: Let (X, d) be a metric space. Then

(i) **1** and X are open sets in (X, d);
(ii) the union of any finite, countable or uncountable family of open sets is open;
(iii) the intersection of any finite family of open sets is open.

Proof. (i) As the empty set contains no points, the requirement that each point in **1** is the centre of an open ball contained in it is automatically satisfied. The whole space X is open, since every open ball centred at any of its points is contained in X.

(ii) Let $\{G_a$:2a $L\}$ be an arbitrary family of open sets and H ¼ [$_{a2L}G_a$. If H is empty, then it is open by part (i). So assume H to be nonempty and consider any x 2 H. Then x 2 G_a for some a 2 L. Since G_a is open, there exists an $r > 0$ such that $S(x, r) \leq G_a \leq H$. Thus, for each x 2 H there exists an $r > 0$ such that $S(x, r) \leq H$.

Consequently, H is open.

(iii) Let $\{G_i : 1 \leq i \leq n\}$ be a finite family of open sets in X and let $G = \bigcap_{i=1}^{n} G_i$.

If G is empty, then it is open by part (i). Suppose G is nonempty and let $x \in G$. Then $x \in G_j$, $j = 1, \dots, n$. Since G_j is open, there exists $r_j > 0$ such that $S(x, r_j) \subseteq G_j$, $j = 1, \dots, n$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then $r > 0$ and $S(x, r) \subseteq S(x, r_j)$, $j = 1, \dots, n$. Therefore the ball $S(x,r)$ centred at x satisfies

$$S(x, r) \subseteq \bigcap_{j=1}^{n} S(x, r_j) \subseteq G:$$

$$S(x, r) \subseteq \bigcap_{j=1}^{n} S(x, r_j) \subseteq G:$$

This completes the proof. &

Remark: The intersection of an infinite number of open sets need not be open. To see why, let $S_n = S(0, \frac{1}{n}) \subseteq C$, $n = 1, 2, \dots$. Each S_n is an open ball in the complex plane and hence an open set in C. However,which is not open, since there exists no open ball in the complex plane with centre 0 that is contained in $\{0\}$.

The following theorem characterises open subsets in a metric space.

Theorem: A subset G in a metric space (X, d) is open if and only if it is the union of all open balls contained in G.

Proof. Suppose that G is open. If G is empty, then there are no open balls contained in it. Thus, the union of all open balls contained in G is a union of an empty class, which is empty and therefore equal to G. If G is nonempty, then since G is open, each of its points is the centre of an open ball contained entirely in G. So, G is the union of all open balls contained in it. The converse follows immediately from Theorem.

Remark: The above Theorem 2.1.9 describes the structure of open sets in a metric space. This information is the best possible in an arbitrary metric space. For open subsets of R, Theorem 2.1.9 can be improved.

Theorem: Each nonempty open subset of R is the union of a countable family of disjoint open intervals. Moreover, the endpoints of any open interval in the family lie in the complement of the set and are no less than the infimum and no greater than the supremum of the set.

Proof. Let G be a nonempty open subset of 2R and let x G. Since G is open, there exists a bounded open interval with centre x and contained in G. So there exists some $y > x$ such that (x, y) $\leq G$ and some $z < x$ such that $(z, x) \leq G$. Let

$$a = \inf \{z : (z, x) \leq G\} \text{ and } b \text{ ¼ } \sup \{y: (x, y) \leq G\}: \quad (1)$$

Then $a < x < b$ and I_x ¼ (a, b) is an open interval containing x. We shall show that The argument that b G is similar. Now suppose w I_x we shall show that w G. If w x, then of course w G. Let w x, so that either $a < w < x$ or $x < w < b$. We need consider only the former case: $a < w < x$. Since $a < w$, it follows from (1) that there exists some $z < w$ such that $(z, x) \leq G$. Since $w < x$, this implies that w 2 G.next show that any two intervals in the collection $\{I_x$: x $G\}$ are disjoint. Let (a, b) and (c, d) be two intervals in this collection with a point in common. Then we must have $c < b$ and $a < d$. Since c does not belong to G, it does not belong to (a, b) and so $c \leq a$. Since a does not belong to G, and hence also does not belong to (c, d), we also have $a \leq c$. Therefore, $c = a$. Similarly, $b = d$, which shows that (a,b) and (c, d) are actually the same in2terval. Thus, $\{I_x : x\ G\}$ is a collection of disjoint intervals.

Now we establish that the collection is countable. Each nonempty open interval contains a rational number. Since disjoint intervals cannot contain the same number and the rationals 2are countable, it follows that the collection $\{I_x : x \quad G\}$ is countable.

Finally, we note that it follows from (1) that a \$ inf G and b # sup G:

Definition: Let A be a subset of a metric space (X, d). A point 2 x X is called an interior point of A if there exists an open ball with centre x contained in A, i.e.,

$$x \; 2 \; S(x, r) \leq A \text{ for some } r > 0,$$

or equivalently, if x has a neighbourhood contained in A. The set of all interior points of A is called the interior of A and is denoted either by Int(A) or A$^\circ$. Thus

Int(A) ¼ A° ¼ $\{x \; 2 \; A$: $S(x, r) \leq A$ for some $r >$ 0}: Observe that Int(A) $\leq A$.

Example : The interior of the subset [0, 1] ≤ R can be shown to be 2 (0,1). Let x (0, 1). Since (0,1) is open, there exists $r > 0$ such that (x r, x r) [0, 1]. So, x is an interior point of [0,1]. Also, 0 is not an interior point of [0,1], because there exists no $r > 0$ such that (r, r) [0, 1]. Similarly, 1 is also not an interior point of [0, 1].

The next theorem relates interiors to open sets and provides a characterisation of open subsets in terms of interiors.

Theorem: Let A be a subset of a metric space (X, d). Then

(i) A° is an open subset of A that contains every open subset of A;
(ii) A is open if and only if $A = A^\circ$.

Proof. (i) Let 2 x A° be arbitrary. Then, by definition, there exists an open ball $S(x, r)$ A. But $S(x,r)$ being an open set (see Theorem 2.1.5), each point of it is the centre of some open ball contained in $S(x,r)$ and consequently also contained in A. Therefore each point of $S(x,r)$ is an interior point of A, i.e., $S(x, r)$ A°. Thus, x is the centre of an open ball contained in A°. Since x A° is arbitrary, it follows that each x A° has the property of being the centre of an open ball contained in A°. Hence, A° is open.

It remains to show that A° contains every open subset $G \leq A$. Let x 2 G. Since G is open, there exists an open ball $S(x, r) \leq G \leq A$. So x 2 A°. This shows that x 2 G ❯ x 2 A°. In other words, $G \leq A^\circ$.

(ii) is immediate from (i). &

The following are basic properties of interiors.

Theorem: Let (X, d) be a metric space and A, B be subsets of X. Then

(i) $A \leq B$ ❩ $A8 \leq B8$;
(ii) $(A \setminus B)8 = A8 \setminus B8$;
(iii) $(A$ ⁅ $B)8 \leq A8$ ⁅ $B8$.

Proof. (i) Let x 2 $A8$. Then there exists an $r > 0$ such that $S(x, r) \leq A$. Since $A \leq B$, we have $S(x, r) \leq B$, i.e., x 2 $B8$.

(ii) $A \setminus B \leq A$ as well as $A \setminus B \leq B$. It follows from (i) that $(A \setminus B)8 \leq A8$ as well as $(A \setminus B)8 \leq B8$, which implies that $(A \setminus B)8 \leq A8 \setminus B8$. On the other hand, let x 2 $A8 \setminus B8$. Then x 2 $A8$ and x 2 $B8$. Therefore, there exist $r_1 > 0$ and $r_2 > 0$ such that $S(x, r_1) \leq A$ and $S(x, r_2) \leq B$. Let $r = \min\{r_1, r_2\}$. Clearly, $r > 0$ and $S(x, r) \leq A \setminus B$, i.e., x 2 $(A \setminus B)8$.

(ii) $A \leq A$ ⁅ B as well as $B \leq A$ ⁅ B. Now apply (i). &

Remark: The following example shows that $(A$ ⁅ $B)8$ need not be the same as $A8$ ⁅ $B8$. Indeed, if A ¼ [0, 1] and B ¼ [1, 2], then A ⁅ B = [0, 2]. Since $A8$ ¼ (0, 1), $B8$ ¼ (1, 2) and $(A$ ⁅ $B)8$ ¼ (0, 2), we have $(A$ ⁅ $B)8$ 6¼ $A8$ ⁅ $B8$.

Definition: Let X be a metric space and F a subset of X. A point x 2 X is called a limit point of F if each open ball with centre x contains at least one point of F different from x, i.e.,

$$(S(x, r) — \{x\}) \setminus F \; 6= 1:$$

The set of all limit points of F is denoted by F ’ and is called the derived set of F.

Examples: (i) The subset F ¼ {1, 1=2, 1=3, .. . } of the real line has 0 as a limit point; in fact, 0 is its only limit point. Thus the derived set of F is {0}, i.e., F' ¼ {0}.

(ii) The subset Z of integers of the real line, consisting of all the integers, has no limit point. Its derived set Z’ is 1.

(iii) Each real number is a limit point of the subset of rationals: Q' ¼ R.

(iv) If (X, d) is a discrete metric space (see Example 1.2.2(v)) and $F \leq X$, then F has no limit points, since every open ball of radius 1 consists only of the centre.

(v) Consider the subset $F ¼ \{(x, y) 2 C : x > 0, y > 0\}$ of the complex plane. Each point of the subset $\{(x, y) C : x \$ 0, y \$ 0\}$ is a limit point of F. In fact, the latter set is precisely F'.

(vi) For an interval $I \leq$ R, the set I' consists of not only all the points of I but also any endpoints I may have, even if they do not belong to I. Thus $(0, 1)' ¼ (0, 1]' ¼ [0, 1)' ¼ [0, 1]' ¼ [0, 1]$.

Proposition: Let (X, d) be a metric space and $F \leq X$. If x_0 is a limit point of F, then every open ball $S(x_0, r)$, $r > 0$, contains an infinite number of points of F.

Proof. Suppose that the ball $S(x_0, r)$ contains only a finite number of points of F. Let $y_1, y_2, \ldots, y_n$ denote the points of $S(x_0, r) \setminus F$ that are distinct from x_0. Let

$$d=¼ \min \{d(y_1, x_0), d(y_2, x_0), \ldots, d(y_n, x_0)\}:$$

Then the ball $S(x_0, d)$ contains no point of F distinct from x_0, contradicting the assumption that x_0 is a limit point of F.

&

The following characterisation of the limit points of a set in a metric space is useful.

Proposition: Let (X, d) be a metric space and $F \leq X$. Then a point x_0 is a limit point of F if and only if it is possible to select from the set F a sequence of distinct points $x_1, x_2, \ldots, x_n, \ldots$ such that $\lim_n d(x_n, x_0)=¼ 0$.

Proof. If $\lim_n d(x_n, x_0) ¼ 0$, where $x_1, x_2, \ldots, x_n, \ldots$ is a sequence of distinct points of F, then every ball $S(x_0, r)$ with centre x_0 and radius r contains each of x_n, where $n \$ n_0$ for some suitably chosen n_0. As $x_1, x_2, \ldots, x_n, \ldots$ in F are distinct, it follows that $S(x_0, r)$ contains a point of F different from x_0. So, x_0 is a limit point of F.

On the other hand, assume that x_0 is a limit point of F. Choose a point x_1 2 F in the open ball $S(x_0, 1)$ such that x_1 is different from x_0. Next, choose a point x_2 2 F in the open ball $S(x_0, 1=2)$ different from x_0 as well as from x_1; this is possible by Proposition 2.1.19. Continuing this process in which, at the nth step of the process we choose a point x_n 2 F in $S(x_0, 1=n)$ different from $x_1, x_2, \ldots, x_{n-1}$, we have a sequence $\{x_n\}$ of distinct points of the set F

Such that $\lim_n d(x_n, x_0)$ ¼ 0.

Definition: A subset F of the metric space (X, d) is said to be closed if it contains each of its limit points, i.e., $F' \leq F$.

Examples. (i) The set Z of integers is a closed subset of the real line.

(ii) The set F ¼ $\{1, 1=2, 1=3, \ldots, 1=n, \ldots\}$ is not closed in R. In ¼ fact, F' $\{0\}$, which is not contained in F.

(iii) The set F ¼ $\{(x, y)$ 2 C: x \$0, y \$0$\}$ is a closed subset of the complex plane

C. In this case, the derived set ¼ is F' F.

(iv) Each subset of a discrete metric space is closed.

Proposition Let F be a subset of the metric space (X, d). The set of limit points of F, namely, F' is a closed subset of (X, d), i.e., $(F')' \leq F'$.

Proof. If ¼**1** or (F')¼ **1**, then there is nothing to prove. Let F' 6¼ **1** 2 and let x_0 $(F')'$. Choose an arbitrary open ball $S(x_0, r)$ with centre x_0 and radius r. By the definition of 2 limit point, there 2 exists ¼ a point y F' such that y $S(x_0, r)$. If r' r $d(y, x_0)$, then $S(y, r')$ contains infinitely many points of F by Proposition

But $S(y, r')$ $S(x_0, r)$ as in the proof of Theorem 2.1.5. So, infinitely many points of F lie in $S(x_0, r)$. Therefore, x_0 2 is a limit point of F, i.e., x_0 F'. Thus, F' contains all its limit points and hence F is closed.

Definition: Let F be a subset of a metric space (X, d). The set F [F' is called the closure of F and is denoted by $\bar{F}$.

Corollary: The closure $\bar{F}$ of $F \leq X$, where (X, d) is a metric space, is closed.

Proof. In fact, by Proposition 2.1.23 and Theorem 2.1.24(ii),

$$(\bar{F})' = (F \cup F')' = F' \cup (F')' \subseteq F' \cup F' = F' \subseteq \bar{F}:$$

Corollary: (i) Let F be a subset of a metric space(X, d). Then F is closed if and only if $F = \bar{F}$.

(i) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.

(ii) If $A \subseteq F$ and F is closed, then $\bar{A} \subseteq F$.

Proof. (i) If $F = \bar{F}$, then it follows from Corollary 2.1.26 that F is closed. On the other hand, suppose that F is closed; then

$$\bar{F} = F \cup F' = F \subseteq \bar{F}:$$

It follows from the above relations that $F = \bar{F}$.

(ii) This is an immediate consequence of Theorem 2.1.24(i).

(iii) This is an immediate consequence of (ii) above. &

Proposition : Let (X, d) be a metric space and $F \subseteq X$. Then the following statements are equivalent:

(i) $x \in \bar{F}$;

(ii) $S(x, \epsilon) \cap F \neq \emptyset$ for every open ball $S(x, \epsilon)$ centred at x;

(iii) there exists an infinite sequence $\{x_n\}$ of points (not necessarily distinct) of F such that $x_n \to x$.

Proof. (i)$\Rightarrow$(ii). Let $x \in \bar{F}$. If $x \in F$, then obviously $S(x, \epsilon) \cap F \neq \emptyset$. If $x \notin F$, then by the definition of closure, we have $x \in F'$. By definition of a limit point,

$$(S(x, \epsilon)\setminus\{x\}) \cap F \neq \emptyset$$

and, a fortiori, $$S(x, \epsilon) \cap F \neq \emptyset:$$

(ii)$\Rightarrow$(iii). For each positive integer n, choose $x_n \in S(x, 1/n) \cap F$. Then the sequence $\{x_n\}$ of points in F converges to x. In fact, upon choosing $n_0 > 1/\epsilon$, where e > 0 is arbitrary, we have $d(x_n, x) < 1=n < 1=n_0 <$ e, i.e., x_n 2 $S(x$, e) whenever $n \$ n_0$.

(iii)➢(i) If the sequence $\{x_n\}_{n\$ 1}$ of points in F consists of finitely many distinct points, then there exists a subsequence $\{x_{n_k}\}$ such that x_{n_k} ¼ x for all k. So, x 2 F. If however, $\{x_n\}_{n \$ 1}$ contains infinitely many distinct points, then there exists a subsequence $\{x_{n_k}\}$ consisting of distinct points and $\lim_k d(x_{n_k}, x)$ ¼ 0, for $\lim_n d(x_n, x)$ ¼ 0 by hypothesis. By Proposition 2.1.20, it follows that x 2 $F^0 \leq \bar{F}$: &

Condition (ii) of Definition 1.5.1 of a completion can be rephrased in view of condition (i) and Proposition 2.1.28 (iii) as saying that the closure of metric space X as a subset of its completion X^m must be the whole of X^m.

The following proposition is an easy consequence of Theorem 2.1.24.

Proposition. Let F_1, F_2 be subsets of a metric space (X, d).

(ii) $(F_1 \setminus F_2) \leq \bar{F}1 \setminus \bar{F}2$.

Proof. we have

$$(F_1 [F_2) ¼ (F_1 [F_2) [(F_1 [F_2)^0 ¼ (F_1 [F_2) [(F_1^0 [F_2^0)$$
$$¼ (F_1 [F_1^0) [(F_2 [F_2^0) ¼ \bar{F}1 [\bar{F}2,$$

which establishes (i). The proof of (ii) is equally simple.

Remarks 2.1.30. (i) It is not necessarily the case that the closure of an arbitrary union is the union of the closures of the subsets in the union. If $\{A_a\}_{a2L}$ is an infinite family of subsets of (X, d), it follows from Corollary 2.1.27 (ii) that

$$\underset{a2L}{[} \bar{A}a \leq \underset{a2A}{[} A_a:$$

Equality need not hold, as the following example shows: If A_n ¼ $\{r_n\}$, n

¼ 1, 2, ...

and $r_1, r_2, \ldots, r_n, \ldots$ is an enumeration of rationals, then A_n ¼ {r_n} ¼ S S

1¼ 1 A_n ¼ Q, whereas 1¼ 1 A_n ¼ $\bar{Q}$ ¼ R.

(ii) In Proposition 2.1.29 (ii), equality need not hold. For example, if F_1 denotes {r_n} and

the set of rationals in R and F_2 the set of irrationals in R, then ($F_1 \setminus F_2$) ¼ $\bar{1}$ ¼ 1

but $\bar{F}1$ ¼ $\bar{F}2$ ¼ R.

Proposition Let (X, d) be a metric space. The empty set 1 and the whole space X are closed sets.

Proof. Since the empty set has no limit points, the requirement that a closed set contain all its limit points is automatically satisfied by the empty set.

Since the whole space contains all points, it certainly contains all its limit points (if any), and is thus closed.

The following is a useful characterisation of closed sets in terms of open sets.

Theorem : Let (X, d) be a metric space and F be a subset of X. Then F is closed in X if and only if F^c is open in X.

Proof. Suppose F is closed in X. We show that F^c is open in X. If ¼ F 1(respectively, ¼ X), then F^c X (respectively, 1) and it is open by Theorem 6¼ 2.1.7(i); 6¼ so we may suppose that F 1 F^c. 62 ≤ Let x be a point in F^c. Since F is closed and x F, x cannot be a limit point of F. So there exists an $r > 0$ such that $S(x, r)$ F^c. Thus, each point of F^c is contained in an open ball contained in F^c. This means F^c is open.

For the converse, suppose F^c is open. We show that F is closed. Let 2 x X be a limit point of F. Suppose, 62 if possible, 2 ≤ that x F. Then x F^c, which is assumed to be open. Therefore, there exists $r > 0$ such that $S(x, r)$ F^c, i.e.,

$$S(x, r) \setminus F \text{ ¼ } 1:$$

Thus, x cannot be a limit point of F, which is a contradiction. Hence, x belongs to F.

Theorem: Let (*X*, *d*) be a metric space. Then

(i) 1 and *X* are closed;
(ii) any intersection of closed sets is closed;
(iii) a finite union of closed sets is closed.

Proof. (i) This is a restatement of Proposition 2.1.31.
Theorem : Let *F* be a nonempty bounded closed subset of R and let *a* ¼ inf *F*
and *b* ¼ sup*F*. Then *a* 2 *F* and *b* 2 *F*.

Proof. We need only show that if *a* 62 *F*, then *a* is a limit point of *F*. By the definition of infimum, for any e > 0, there exists at least one member *x* 2 *F* such that *a*# *x* < *a* þ e. But *a* 62 *F*, whereas *x* 2 *F*. So,

$$a < x < a \text{ þ } e:$$

Thus, every neighbourhood of *a* contains at least one 2 member *x F* which is different from *a*. Hence, *a* is a limit point of *F*.

&

Definition 2.1.37. Let *F* be a nonempty bounded subset of R ¼ and let *a* ¼inf *F* and *b* sup*F*. The closed interval [*a*, *b*] is called the smallest closed interval containing *F*.

Theorem: If [*a*, *b*] is the smallest closed interval containing *F*, where *F* is a nonempty bounded closed subset of R, then

$$[a, b]\backslash F \text{ ¼ } (a, b) \setminus F^c$$

and so is open in R.

Proof. Let x_0 2 [*a*, *b*]*F*; this means that x_0 2 [*a*, *b*], x_0 62 *F*. If x_0 62 *F*, then x_0 6¼ *a* and x_0 6¼ *b*, because *a* and *b* do belong to *F*, by Theorem 2.1.36. It follows that x_0 2 (*a*, *b*). Moreover, it is obvious that x_0 2 F^c, so that

$$[a, b]\backslash F \leq (a, b) \setminus F^c:$$

The reverse inclusion is obvious. &

The following characterisation of closed subsets of R is a direct consequence of Theorems 2.1.11 and 2.1.38.

Theorem 2.1.39. Let *F* be a nonempty bounded closed subset of R. Then *F* is either a closed interval or is obtained from some closed interval by removing a countable family of pairwise disjoint open intervals whose endpoints belong to *F*.

Proof. Let $[a, b]$ be the smallest closed interval containing *F*, where a ¼ inf F and
b ¼ sup$F$. By Theorem 2.1.38,

$$[a, b]\backslash F ¼ (a, b) \setminus F^c$$

is open and hence is a countable union of disjoint open intervals by Theorem 2.1.11. Moreover, the endpoints of the open intervals do not belong to $[a, b]\backslash F$ but do belong to $[a, b]$. So they belong to *F*. The set *F* thus has the desired property. &

This seemingly simple looking process of writing a nonempty bounded closed subset of R leads to some very interesting and useful examples. The following example, which is of particular importance, is due to Cantor.

Example 2.1.40. (Cantor) Divide the closed interval I ¼ [0, 1] into three equal parts by the points 1/3 and 2/3 and remove the open interval (1/3, 2/3) from *I*. Divide each of the remaining two closed intervals [0, 1/3] and [2/3, 1] into three equal parts by the points 1/9, 2/9 and by 7/9, 8/9, respectively, and remove the open intervals (1/9, 2/9) and (7/9, 8/9). Now divide each of the remaining four intervals [0, 1/9], [2/9, 1/3], [2/3, 7/9] and [8/9, 1] into three equal parts and remove the middle third open intervals. Continue this process indefinitely. The open set *G* removed in this way from *I* ¼ [0, 1] is the union of disjoint open intervals

$$G ¼ \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots :$$

The complement of *G* in [0,1], denoted by *P*, is called the Cantor set. Important properties of this set are listed in the Exercise 16 and Section 6.4.

The completeness of R can also be characterised in terms of nested sequences of bounded closed intervals. An analogue of this result for metric spaces is proved in Theorem 2.1.44. We begin with some relevant definitions.

Definition. Let (*X*, *d*) be a metric space and let *A* be a nonempty subset of *X*.
We say that *A* is bounded if there exists *M* > 0 such that

$$d(x, y) \# M \quad x, y\ 2\ A:$$

If *A* is bounded, we define the diameter of *A* as

$$\mathrm{diam}(A)\ ¼\ d(A)\ ¼\ \sup\{d(x, y): x, y\ 2\ A\}:$$

If *A* is not bounded, we write *d*(*A*) ¼ 1.
We define the distance between the point *x* 2 *X* and the subset *B* of *X* by

$$d(x, B)\ ¼\ \inf\{d(x, y) : y\ 2\ B\},$$

and, in an analogous manner, we define the distance between two nonempty subsets
B and *C* of *X* by

$$d(B, C)\ ¼\ \inf\{d(x, y) : x\ 2\ B, y\ 2\ C\}:$$

Examples. (i) Recall that a subset *A* of R (respectively, R^2) is bounded if and only if *A* is contained in an interval (respectively, square) of finite length (respect- ively, whose edge has finite length). Thus, our definition of bounded set in an arbitrary metric space is consistent with the definition of bounded set of real numbers (respectively, bounded set of pairs of real numbers).

(ii) The interval 1(0,) is not a bounded subset of R. However, if R is equipped with the discrete metric, then every subset *A* of this 1discrete space (in particular, the 2set (0, ¼)) is bounded, since *d*(*x*, *y*) # 1 for *x*, *y* *A*. Indeed, *d*(*A*) 1, provided *A* contains more than one point. Moreover, any subset of any discrete metric space has diameter 1 if it contains more than one point.

(iii) If R is equipped with the nondiscrete metric *d*(*x*, *y*) ¼ j*x* — *y*j=[1 þ j*x* — *y*j], then every subset is bounded and *d*(R) ¼ 1.

(iv) In the space (‘2, *d*) (see Example 1.2.2(vii)), consider the set

$$Y\ ¼\ \{e_1, e_2, \ldots, e_n, \ldots\},$$

—

Theorem : (Cantor) Let (X, d) be a metric space. Then (X, d) is complete if and only if, for every nested sequence $\{F_n\}_{n \geq 1}$ of nonempty closed subsets of X, that is,

(a) $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$ such that (b) $d(F_n) \to 0$ as $n \to \infty$, $\bigcap_{n=1}^{\infty} F_n$

Proof. First suppose that (X, d) is complete. For each positive integer n, let x_n be any point in F_n. Then by (a),

$$x_n, x_{n+1}, x_{n+2}, \ldots$$

all lie in F_n. Given $\epsilon > 0$, there exists by (b) some integer n_0 such that $d(F_{n_0}) < \epsilon$. Now, $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots$ all lie in F_{n_0}. For $m, n \geq n_0$, we then have $d(x_m, x_n) \leq d(F_{n_0}) < \epsilon$. This shows that the sequence $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in the complete metric space X. So, it is convergent. Let $x \in X$ be such that $\lim_{n \to \infty} x_n = x$. Now for any given n, we have the sequence $x_n, x_{n+1}, \ldots \subseteq F_n$. In view of this,

since F_n is closed. Hence,

If $\epsilon > 0$, then there exists a natural number n_0 such that $d(F_{n_0}) < \epsilon$. But $x \in F_{n_0}$

and thus $n \geq n_0$ implies $d(x_n, x) < \epsilon$.

&

Subspaces

Let (X, d) be a metric space and Y a nonempty subset of X. If d_Y denotes the restriction of the function d to the set $Y \times Y$, then d_Y is a metric for Y and (Y, d_Y) is

a metric space (see Section 1.2). If $Z \subseteq Y \subseteq X$, we may speak of Z being open (respectively, closed) relative to Y as well as open (respectively, closed) relative to X. It may happen that Z is an open (respectively, closed) subset of Y but not of X. For example, let X be $\mathbb{R}^2$ with metric d_2 and $Y = \{(x, 0): x \in \mathbb{R}\}$ with the induced metric. Then Y is a closed subset of X (for $Y^c = \{(x, y) \in \mathbb{R}^2: y \neq 0\}$ is open in X). If $Z = \{(x, 0): 0 < x < 1\}$, then Z considered as a subset of Y is open in Y. However, Z considered as a subset of X is not open in X. In fact, no point $(x, 0) \in Z$ is an interior point of Z (Z considered as a subset of X) because any neighbourhood of $(x,0)$

in X is the ball $S($ $(x, 0), r)$, $r > 0$, which is not contained in Z. Thus, Z $\{(x, 0) : 0 < x < 1\}$ is an open subset of Y $\{(x, 0) : x$ R$\}$ but not of X (R^2, d_2).

The above examples illustrate that the property of a set being open (respectively closed) depends on the metric space of which it is regarded a subset. The following theorem characterises open (respectively closed) sets in a subspace Y in terms of open (respectively closed) subsets in the space X. First we shall need a lemma.

Lemma 2.2.1. Let (X, d) be a metric space and Y a subspace of 2 X. Let z Y and $r > 0$. Then

$$S_Y(z, r) ¼ S_X(z, r) \setminus Y,$$

where $S_Y(z, r)$ (respectively $S_X(z, r)$) denotes the ball with centre z and radius r in Y (respectively X).

Proof. We have

$$S_X(z, r) \setminus Y ¼ \{x \; 2 \; X : d(x, z) < r\} \setminus Y$$
$$¼ \{x \; 2 \; Y : d(x, z) < r\}$$
$$¼ S_Y(z, r) \quad \text{since } Y \leq X: \qquad \&$$

Let X ¼ R^2 and Y ¼ {(x_1, x_2): 0 < x_1 # 1, 0 # x_2 < 1, x^2 þ x^2 $ 1}. Here, the

open ball in Y with centre (1, 0) and radius P ffi 1 2 2 is the entire space Y. (See Figure 2.6.)

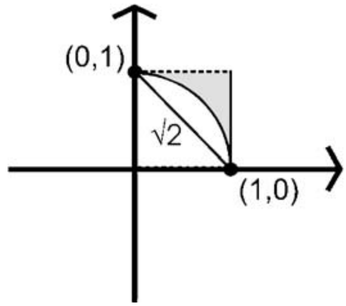

Figure 2.6

Theorem: Let (X, d) be a metric space and Y a subspace of X. Let Z be a subset of Y. Then

(i) Z is open in Y if and only if there exists an open set $G \leq X$ such that
Z ¼ $G \setminus Y$;

(ii) Z is closed in Y if and only if there exists a closed set $F \leq X$ such that
Z ¼ $F \setminus Y$.

Proof. (i) Let Z be open in Y. Then if z is any point of Z, there exists an open ball $S_Y(z, r)$ contained in Z. Observe that the radius r of the ball $S_Y(z, r)$ depends on the point z 2 Z. We then have

$_X(z, r)$ is open in X.

On the other hand, suppose that Z ¼ $G \setminus Y$, where G is open in X. If z 2 Z, then z is a point of G and so there exists an open ball $S_X(z, r)$ such that $S_X(z, r)$ G. Hence,

$S_Y(z, r)$ ¼ $S_X(z, r) \setminus Y$ by Lemma 2:2:1
$\leq G \setminus Y$ ¼ Z,

so that z is an interior point of the subset Z of Y. As z is an arbitrary point of Z, it follows that Z is open in Y.

(ii) Z is closed in Y if and only if $(X\setminus Z)\cap Y$ is open in Y. Hence, Z is closed in Y if and only if there exists an open set G in X such that

$$(X\setminus Z)\cap Y = G\cap Y$$

using (i) above. On taking complements in X on both sides, we have

$$Z\cup(X\setminus Y) = (X\setminus G)\cup(X\setminus Y).$$

Hence

$$\begin{aligned} Z &= Z\cap Y = (Z\cup(X\setminus Y))\cap Y \\ &= ((X\setminus G)\cup(X\setminus Y))\cap Y \\ &= (X\setminus G)\cap Y \end{aligned}$$

So, Z is the intersection of the closed set $X\setminus G$ and Y.

where $X \backslash F$ is open in X. Hence $(X \backslash Z) \setminus Y$ is open in Y, i.e., Z is closed in Y. &

Proposition : Let Y be a subspace of a metric space (X, d).

(i) Every subset of Y that is open in Y is also open in X if and only if Y is open in X.

(ii) Every subset of Y that is closed in Y is also closed in X if and only if Y is closed in X.

Proof. (i) Suppose every subset of Y open in Y is also open in X. We want to show that Y is open in X. Since Y is an open subset of Y, it must be open in X. Conversely, suppose Y is open in X. Let Z be an open subset of Y. By Theorem 2.2.2(i), there exists an open subset G of X such that Z ¼ $G \setminus Y$. Since G and Y are both open subsets of X, their intersection must be open in X, i.e., Z must be open in X.

(ii) The proof is equally easy and is, therefore, not included. &

Proposition : Let (X, d) be a metric space and $Z \leq Y \leq X$. If $\mathrm{cl}_X Z$ and $\mathrm{cl}_Y Z$ denote, respectively, the closures of Z in the metric spaces X and Y, then

$$\mathrm{cl}_Y Z \text{ ¼ } Y \setminus \mathrm{cl}_X Z :$$

Proof. Obviously, $Z \leq Y \setminus \mathrm{cl}_X Z$. Since $Y \setminus \mathrm{cl}_X Z$ is closed in Y (see Theorem 2.2.2(ii)), it follows that $\mathrm{cl}_Y Z \leq Y \setminus \mathrm{cl}_X Z$. On the other hand, by Theorem 2.2.2(ii), $\mathrm{cl}_Y Z$ ¼ $Y \setminus F$, where F is a closed subset of X. But then

$$Z \leq \mathrm{cl}_Y Z \leq F,$$

and hence, by Corollary 2.1.27(ii),

$$\mathrm{cl}_X Z \leq F:$$

Therefore,

$$\mathrm{cl}_Y Z \text{ ¼ } Y \setminus F \leq Y \setminus \mathrm{cl}_X Z :$$

This completes the proof. &

In contrast to the relative properties discussed above, there are some properties that are intrinsic. In fact, the property of *x* being a limit point of *F* holds in any subspace containing *x* and *F* as soon as it holds in the whole space, and conversely. Another such property is that of being complete. The following propositions describe relations between closed sets and complete sets.

Proposition : If *Y* is a nonempty subset of a metric space (*X*, *d*), and (Y , d_Y) is complete, then *Y* is closed in *X*.

Proof. Let *x* be any limit point of *Y*. Then *x* is the limit of a sequence $\{y_n\}n$ \$ 1 in *Y*. In view of Proposition 1.4.3, the sequence $\{y_n\}n$ \$ 1 is Cauchy, and hence, by assump- tion, converges to a point *y* of *Y*. But by Remark 3 following Definition 1.3.2, *y* ¼ *x*. Therefore, *x* 2 *Y*. This shows that *Y* is closed in *X*.

Proposition: Let (*X*, *d*) be a complete metric space and *Y* a closed subset of *X*. Then (Y, d_Y) is a complete space.

Proof. Let $\{y_n\}n$ \$ 1 be a Cauchy sequence in (Y, d_Y). Then $\{y_n\}n$ \$ 1 is also a Cauchy sequence in (*X*, *d*); so there exists an *x* 2 *X* such that $\lim_{n!1} y_n$ ¼ *x*. If follows (see Proposition 2.1.28) that *x* 2 $\bar{Y}$, which is the same set as *Y* by Corollary 2.1.27(i). &

Countability Axioms and Separability

Definition: Let (*X*, *d*) be a metric space and *x* 2 *X*. Let $\{G_l\}l2L$ be a family of open sets, each containing *x*. The family $\{G_l\}l2L$ is said to be a local base at *x* if, for every nonempty open set *G* containing *x*, there exists a set G_m in the family

$\{G_l\}l2L$ such that *x* 2 $G_m \leq G$.

Examples 2.3.2. (i) In the metric space R^2 with the Euclidean metric, let G_l ¼
S(*x*, *l*), where *x* ¼(x_1,x_2) 2 R^2 and 0 < *l*2 R. The family {G_l:0 < *l*2 R}¼

$\{S(x, \lambda): 0 < \lambda \in \mathbb{R}\}$ is a family of balls and is a local base at x. Note that $S(x, \lambda)$, where $x = (x_1, x_2)$, can also be described as $\{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 < \lambda^2\}$. (ii) Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $G_\lambda = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + 2(y_2 - x_2)^2 < \lambda\}$, where $0 < \lambda \in \mathbb{R}$. Then the family $\{G_\lambda : 0 < \lambda \in \mathbb{R}\}$ is a local base at x. To see why, consider any open set $G \subseteq \mathbb{R}^2$ such that $x \in G$. Since G is open, there exists $r > 0$ such that $S(x, r) \subseteq G$. Now $S(x, r) = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\}$. Let $\lambda = r^2$. Then $y \in G_\lambda \Rightarrow (y_1 - x_1)^2 + 2(y_2 - x_2)^2 < \lambda \Rightarrow (y_1 - x_1)^2 + (y_2 - x_2)^2 < \lambda \Rightarrow (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2 \Rightarrow y \in S(x, r)$, so that $G_\lambda \subseteq S(x, r) \subseteq G$. In this example, the sets G_λ are ellipses.

(iii) Let $x \in \mathbb{R}$. Consider the family of all open intervals (r, s) containing x and having rational endpoints r and s. This family is a local base at x. It consists of open balls, not necessarily centred at x. Moreover, the family is countable and thus constitutes what is called a countable base at x.

Proposition: In any metric space, there is a countable base at each point.

Proof. Let (X, d) be a metric space and $x \in X$. The family of open balls centred at x and having rational radii, i.e., $\{S(x, r): r$ rational and positive$\}$ is a countable base at x. In fact, if G is an open set and $x \in G$, then by the definition of an open set, there exists an $\varepsilon > 0$ (ε depending on x) such that $x \in S(x, \varepsilon) \subseteq G$. Let r be a positive rational number less than ε. Then

$$x \in S(x, r) \subseteq S(x, \varepsilon) \subseteq G.$$

Definition: A family $\{G_\lambda\}_{\lambda \in L}$ of nonempty open sets is called a base for the open sets of (X, d) if every open subset of X is a union of a subfamily of the family $\{G_\lambda\}_{\lambda \in L}$.

The condition of the above definition can be expressed in the following equiva- lent form: If G is an arbitrary nonempty open set and $x \in G$, then there exists a set G_μ in the family such that $x \in G_\mu \subseteq G$.

Proposition: The collection $\{S(x, \varepsilon): x \in X, \varepsilon > 0\}$ of all open balls in X is a base for the open sets of X.

Proof. Let G be a nonempty open subset of X and let $x G$. By the definition of an open subset, there exists a positive e(x) (depending upon x) such that

$$x \, 2 \, S(x, e(x)) \le G:$$

This completes the proof. &

Generally speaking, an open base is useful if its sets are simple in form. A space that has a countable base for the open sets has pleasant properties and goes by the name of "second countable".

Definition: A metric space is said to be second countable (or satisfy the second axiom of countability) if it has a countable base for its open sets.

The reason for the name *second* countable is that the property of having a countable base at each point, as in Proposition 2.3.3, is usually called *first* countability.

Examples: (i) Let (R,d) be the real line with the usual metric. The collection
$\{(x, y) : x, y \text{ rational}\}$ of all open intervals with rational endpoints form a countable base for the open sets of R.

(ii) The collection

$\{S(x, r): x = (x_1, x_2, \ldots, x_n), x_i$ rationals, $1 \le i \le n$, and r positive rational$\}$ of all r-balls with rational centres and rational radii is a countable base for the open sets of the metric space (R^n, d), where d may be any of the metrics on R^n described in Example 1.2.2(iii).

(iii) Let X have the discrete metric. Then any set $\{x\}$ containing a single point x is also the open ball $S(x, 1/2)$ and therefore must be a union of nonempty sets of any base. So any base has to contain each set $\{x\}$ as one of the sets in it. If X is nondenumerable, then the sets $\{x\}$ are also nondenumerable, forcing every base to be nondenumerable as well. Consequently, X does not satisfy the second axiom of countability when it is nondenumerable.

It is easy to see that any subspace of a second countable space is also a second countable space. In fact, the class of all intersections with the subspace of the sets of a base form a base for the open sets of the subspace.

Definition: Let (X, d) be a metric space and G be a collection of open sets 2in X. If for each x X2there is a m2mber G G such that x G, then G is called an open cover (or open covering) of X. A subcollection of G which is itself an open cover of X is called a subcover (or subcovering).

Examples 2.3.9. (i) The union of the family { … , (3, 1), (2, 0), (1, 1), (0, 2), … } of open intervals is R. The family is therefore an open covering of R. However, the family of open intervals { … , (2, 1), (1, 0), (0, 1), (1, 2), … } is not an open covering, because the intervals' union does not contain the integers. The aforementioned cover contains no subcovering besides itself, because, if we delete any interval from the family, the midpoint of the deleted interval will not belong to the union of the remaining intervals.

(ii) Let X be the discrete metric space consisting of the five elements a, b, c, d, e. The union of the family of subsets $\{\{a\}, \{b, c\}, \{c, d\}, \{a, d, e\}\}$ is X and all subsets are open. Therefore the family is an open cover. The family $\{\{b, c\}, \{c, d\}, \{a, d, e\}\}$ is a proper subcover.

(iii) Consider the set Z of all integers with the discrete metric. As in any discrete metric space, all subsets are open. Consider the family consisting of the three subsets

{3n : n 2 Z}, {3n þ 1 : n 2 Z} and {3n þ 2 : n 2 Z}:

Since every integer must be of the formþ 3n, 3nþ1 or 3n 2, the above three subsets form an open cover of Z. There is no proper subcover.

(iv) The family of intervals {(— n, n): n 2 N} is an open cover of R and the family consisting of the open balls {z 2 C : jz þ 17j < $n^{3=2}$, n 2 N} is an open cover of C. If we extract a subfamily by restricting n to be greater than some integer n_0, the subfamily is also an open cover. Indeed, if we delete a finite number of sets in the family, the remaining subfamily is an open cover. Thus, there are infinitely many open subcovers.

Definition: A metric space is said to be Lindelöf if each open covering of X contains a countable subcovering.

Proposition: Let (X, d) be a metric space. If X satisfies the second axiom of countability, then every open covering $\{U_a\}a2L$ of X contains a countable subcover- ing. In other words, a second countable metric space is Lindelöf.

Proof. Let $\{G_i : i$ ¼ 1, 2, ... } be a countable base of open sets for X. Since each U_a is a union of sets G_i, it follows that a subfamily $\{G_{ij} : j$ ¼ 1, 2, ... } of the base
$\{G_i : i$ ¼ 1, 2, ... } is a covering of X. Choose $U_{ij} \leq G_{ij}$ for each j. Then $\{U_{ij} : j$ ¼ 1, 2, ... } is the required countable subcovering. &

Definition: A subset X_0 of a metric space (X, d) is said to be everywhere dense or simply dense if X_0 ¼ X, i.e., if every point of X is either a point or a limit

point of X_0. This means that, given any point x of X, there exists a sequence of points of X_0 that converges to x.

It follows easily from this definition and the definition of interior that a subset of X_0 is dense if and only if X^c has empty interior.

It may be noted that X is always a dense subset of itself; interest centres around what *proper* subsets of a metric space are dense.

Examples: (i) The set of rationals is a dense subset of R (usual metric) and so is the set of irrationals. Note that the former is countable whereas the latter is not.

(ii) Consider the metric space (R^n, d) with any of the metrics described in Example 1.2.2(iii). Within any neighbourhood of any point in R^n, there is a point with rational coordinates. Thus,

$$Q^n \text{ ¼ } Q \times Q \times ... \times Q$$

is dense in R^n.

(iii) In the space C[0, 1] of Example 1.2.2(ix), we consider the set C_0 consisting of all polynomials with rational coefficients. We shall check that C_0 is dense in C[0, 1]. Let $x(t)$ C[0, 1]. By Weierstrass' theorem (Theorem 0.8.4), there exists a polyno- mial $P(t)$ such that

Definition: The metric space X is said to be separable if there exists a countable, everywhere dense set in X. In other words, X is said to be separable if there exists in X a sequence

$$\{x_1, x_2, \ldots\} \qquad (2{:}1)$$

such that for every x 2 X, some sequence in the range of (2.1) converges to x.

Examples: In Examples 2.3.13(i)–(iii) and (v), we saw dense sets that are countable. Therefore, the spaces concerned are separable. In (iv) however, the space is separable if and only if the set X is countable.

There are metric spaces other than the discrete metric space mentioned above which fail to satisfy the separability criterion. The next example is one such case. Let X denote the set of all bounded sequences of real numbers with metric

$$d(x, y) \text{ ¼ } \sup\{\text{j}x_i - y_i\text{j} : i \text{ ¼ } 1, 2, 3, \ldots\},$$

as in Example 1.2.2(vi). We shall show that X is inseparable.

First we consider the set A of elements ¼ x (x_1, x_2, …) of X for which each x_i is either 0 or 1 and show that it is uncountable. If E is any countable subset of A, then the elements of E can be arranged in a sequence s_1, s_2, … . We construct a sequence s as follows. If the m^{th} element of s_m is 1, then the m^{th} element of s is 0, and vice versa. Then the element s of X differs from each s_m in the m^{th} place and is therefore equal to none of them. So, s ∉ E although s ∈ A. This shows that any countable subset of A must be a proper subset of A. It follows that A is uncountable, for if it were to be countable, then it would have to be a proper subset of itself, which is absurd. We proceed to use the uncountability of the subset A to argue that X must be inseparable.

The distance between two distinct elements x ¼ (x_1, x_2, …) and y ¼ (y_1, y_2, …) of A is d(x, y) ¼ sup{ x_i y_i : i ¼ 1, 2, 3, … } ¼ 1. Suppose, if possible, that E_0 is a countable, everywhere dense subset of X. Consider the balls of radii 1/3 whose centres are the points of E_0. Their union is the entire space X, because E_0 is everywhere dense, and in particular contains A. Since the balls are countable in number while A is not, in at least one ball there must be two distinct elements x and y of A. Let x_0 denote the centre of such a ball. Then

which is, however, impossible. Consequently, (X, d) cannot be separable.

Proposition: Let (X, d) be a metric space and Y ≤ X. If X is separable, then Y with the induced metric is separable, too.

Proof. Let E ¼ {x_i : i ¼ 1, 2, … } be a countable dense subset of X. If E is contained in Y, then there is nothing to prove. Otherwise, we construct a countable dense subset of Y whose points are arbitrarily close to those of E. For positive integers n and m, let $S_{n,m}$ ¼ S(x_n, 1=m) and choose $y_{n,m}$ 2 $S_{n,m}$ ∖ Y whenever this set is nonempty. We show that the countable set {$y_{n,m}$: n and m positive integers} of Y is dense in Y.

For this purpose, let y 2 Y and e > 0. Let m be so large that 1=m < e=2 and find

x_n 2 S(y, 1=m). Then y 2 $S_{n,m}$ ∖ Y and n n

Thus, y_n, m 2 $S(y$, e). Since y 2 Y and e > 0 are arbitrary, the assertion is proved. &

The main result of this section is the following.

Theorem: Let (X, d) be a metric space. The following statements are equiva- lent:

(i) (X, d) is separable;
(ii) (X, d) satisfies the second axiom of countability;
(iii) (X, d) is Lindelöf.

Proof. (i)⟹(ii). Let E ¼ {x_i : i ¼ 1, 2, . . . } be a countable, dense subset of X and let {r_j: j ¼ 1, 2, ... } be an enumeration of positive rationals. Consider the countable collection of balls with centres at x_i, i ¼ 1, 2, ... and radii r_j, j ¼ 1, 2, ... ; i.e.,

{$S(x_i, r_j)$: x_i 2 E for i ¼ 1, 2, ... and r_j, is rational j ¼ 1, 2, .. . }:

Possible questions

2 MARK QUESTION:

1. Define Open set.
2. Define Pesudometric.
3. Define Cauchy Sequence.
4. Define Metric space
5. Define Closed set.

8 MARK QUESTION:

1. Prove that A convergent sequence in a metric space is a Cauchy sequence.

2. Let (X,d) be a metric space and A, B be subsets of X. Then
   i) A⊆B implies $A^o \subseteq B^{o;}$
   ii) $(A \cap B)^o = A^o \cap B^o$;
   iii) $A^o \cup B^o \subseteq (A \cup B)^o$.

3. Let (X,d) be a metric space.
   Define d': X x X → R by $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$.
   Then d' is a metric on X.

4. Let (X,d) be a metric space and F_1, F_2 be subsets of X. Then
   i) If $F_1 \subseteq F_2$, then $F_1' \subseteq F_2'$;
   ii) $(F_1 \cup F_2)' = F_1' \cup F_2'$;
   iii) $(F_1 \cap F_2)' \subseteq F_1' \cap F_2'$.

5. Let $\{x^{(n)}\}$ be a sequence in L_p space such that $\lim x_k^{(n)} = x_k$ as n tends to infinity for each k, where $x=\{x_k\}$ is an element of L_p. Suppose also that for every $\in>0$ there exists an integer $m_o(\in)$ such that

$$\left(\sum_{k=m+1}^{\infty}\left|x_k^{(n)}\right|\right)^{1/p} < \in \text{ for } m\geq m_o(\in) \text{ and for all}$$

Then $\lim d(x^{(n)},x)=0$ as n tends to ∞.

6. Let (X,d) be a metric space and $A \subseteq X$. If x_o is a limit point of A, then every open ball $S(x_o,r)$, $r>0$, contains an infinite number of points of A.

7. Let (X,d) be a metric space, Then prove that
   i) $\emptyset$ and X are open sets in (X,d):
   ii) the union of any finite family of open sets is open:
   iii) the intersection of any finite family of open sets is open.

8. Let (X,d) be a metric space and F be a subset of X. Then prove that F is closed in X if and only of F^c is open in X.

9. Prove that in any metric space (X,d), each open ball is an open set.

10. Let (X,d) be a metric space, Then prove that
    i) $\emptyset$ and X are closed sets in (X,d);
    ii) any intersection of closed sets is closed;
    iii) a finite union of closed sets is open.

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UNIT I

Question	OPTION 1	OPTION 2	OPTION 3	OPTION 4	Answer
The property d(x,y) is less then or equal to d(x,z) + d(z,y) is called_____.	Cauchy-Schwarz inequality	Minkowski's Inequality	Triangle inequality	Cauchy inequality	Triangle inequality
Let (X,d) be a metric space. If d(x,y) = 0 then_____	x<y	x>y	x=y	x=0	x=y
Let d(x,y) = 0 if x=y and d(x,y)= 1 if x not equal to y. This metric is called_____	Euclidean Metric	Discrete Metric	Standard Metric	Distance Metric	Discrete Metric
The second property of pseudometric is_____	d(x,y)=0 if x=y	x=y if d(x,y)=0	d(x,y)=0 iff x=y	d(x,y)=d(y,x)	d(x,y)=0 if x=y
A Convergent sequence in a metric space is called______	Bounded sequence	Cauchy sequence	Convergent sequence	Divergent sequence	Cauchy sequence
A metric space (X,d) is complete if every Cauchy sequence in X is called_____	Incomplete	Bounded	Divergent	Convergent	Convergent
A mapping f of X into X' is an isometry if d'(f(x),f(y))=____	d(x,y)	f(x)	f(y)	0	d(x,y)
Let (X,d) be a metric space.The set $s(x_0,r)$ is called the open ball if x belonges to X such that $d(x_0,x)$ is________.	less then r	less then or equal to r	grater then r	equal to r	less then r
Let (X,d) be a metric space. The intersection of any finite family of open sets is ____	closed	open	bounded	unbounded	open
Let A be a subset of a metric space (X,d).Then A is open iff	$A< A^0$	$A> A^0$	$A= A^0$	$A-A^0=1$	$A= A^0$
Let (X.d) be a metric space and A , B be subsets of X.Then $(AUB)^0$ is containing in	B^0 U A^0	A^0 U B^0	AUB	A=B	A^0 U B^0
Let (X,d) be a metric space. Then any intersection of closed sets is____________.	open set	empty set	singleton	closed set	closed set
Let (X,d) be a metric space. Then Space (X,d) is Lindelof it is equivalent to (X,d) is_________.	separable	inseparable	not countability	countability	separable
Let (X,d) be a metric space and F be a subset of X.Then F is Closed in X iff F^c is________ in X	subset	closed set	open set	cantor set	open set
A subset F of the metric space (X,d) is said to be ________ if it contains each of its limit points	subset	closed	open	limit	closed
Let A be the subset of metric space ,Int(A)={ x belongs to A such that contained in A for some r>0}.	s(x)	s(r)	s(r,x)	s(x,r)	s(x,r)
Which property is difference between metric spaces and pseudometric .	First	Second	Third	Fourth	Second
The spaces (R^n,d_p), B(s) are __________.	complete	incomplete	limit	completion	complete
The space of the real numbers is the _________of the space of the rational.	complete	isometry	completion	equivalent	completion
If A is bounded then sup{d(x,y): x,y belongs to A} is called____________.	supermom	distance	diameter	radious	diameter
If a subsequence is converges, its limit is called a ____________limit of {xn}.	subsequential	upper	lower	equal	subsequential
S(x,r) is denoted by__________.	open ball	closed ball	null ball	unit ball	open ball
S(x,r) is open ball where x is _______.	limit point	centre point	arbitrary point	interior point	centre point
Each open ball is an ___________set.	null	empty	closed	open	open
Each closed ball is an ___________set.	null	empty	closed	open	closed

The intersection of an infinite number of open sets is _____________.	not open	open	not closed	closed	not open
Let F be a subset of X. F is equal to closure of F if F is _____________.	null	empty	closed	open	closed
The union of a set F and derived set of F is called_____________.	open	closed	derived	closure	closure
$(F_1 \cup F_2)^1$ =___________.	$F_1^1 \cup F_2^1$	$F_1^1 F_2^1$	$F_1^1 + F_2^1$	$F_2^1 \cup F_1^1$	$F_1^1 \cup F_2^1$
S(x,r) intersect with F is not equal to____________.	null	empty	closed	open	empty
Let alpha = inf F and beta = sup F then the interval [alpha,beta] is called the __________interval containing F.	higest open	higest closed	smallest open	smallest closed	smallest closed
S(x,r) is open ball where r is _______.	diameter	radious	center	point	radious
Let (X,d) be a metric space. Then Space (X,d) is separable it is equivalent to (X,d) is_________.	countability	inseparable	not countable	Lindelof	Lindelof
Z is open in Y iff if there exists an open set G contained in X such that G intersection Y=________.	Z	G	Y	GUY	Z
Z is cloesd in Y iff if there exists an closed set F contained in X such that Z=______.	F intersection Y	F union Y	F interesction X	F union X	F intersection Y
The distance between the point and subset B of X is denoted by____________.	d(x,y)	d(x,B)	d(B,C)	d(B,X)	d(x,B)
d(A) is denoted by___________.	supermom	distance	diameter	radious	diameter
The distance between two nonempty subsets B and C by_________.	d(x,y)	d(x,B)	d(B,C)	d(B,X)	d(B,C)
The finite union of closed set is__________.	open	closed	not open	not closed	closed
If A is contained in F and F is closed then closure of A is____________-.	equal to F	not equal to F	contained in F	containing F	contained in F
If A is contained in B then closure of A is ________________.	equal to B	not equal to B	contained in B	containing B	contained in B
Every open ball contains an ____________ points.	one	two	finite	infinte	infinite
Two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X to be equivalent if the limit $d(x_n,y_n)$=______ as n tends to infinity.	0	1	2	3	0
The space of all real number is the ___________ of the space of rationals	complete	in complete	limite	completion	completion
The property of Triangle inequality d(x,y) is less then or equal to_________.	d(x,z) + d(z,y)	d(x,z) - d(z,y)	d(z,y)+d(x,z)	d(z,y)-d(x,z)	d(x,z) + d(z,y)
Let (X,d) be a metric space. If x=y then_____	d(x,y) < 0	d(x,y) > 0	d(x,y) = 0	d(x,y) = 1	d(x,y) = 0
A___________ sequence in a metric space is called Cauchy sequence	Bounded sequence	Cauchy sequence	Convergent sequence	Divergent sequence	Convergent sequence
A metric space (X,d) is________ if every Cauchy sequence in X is called Convergent.	complete	Bounded	Divergent	Convergent	complete
A subset F of the metric space (X,d) is said to be closed if it contains each of its ____.	subset	closed	open	limit	limit

UNIT II
SYLLABUS

Continuous mappings - sequential criterion and other characterizations of continuity – Uniform Continuity – Homeomorphism - Contraction mappings - Banach Fixed point Theorem - Connectedness - connected subsets of R.

Continuous Mappings

For a real-valued function f with domain A R, a rough and rather inaccurate description of continuity at a point a A is the statement '$f(x)$ is close to $f(a)$ when x is close to a'. The measure of 'closeness' of two numbers, or distance between them, is the absolute value of the difference of the numbers. In terms of the standard metric d on R, continuity involves a relationship between $d(x, a)$ and $d(f(x),f(a))$. This observation makes it possible to extend the concept of continuity to functions with domain and range in metric spaces.

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces and A belong X. A function $f : A$ into Y is said to be continuous at a belongs A, if for every e > 0, there exists some $d > 0$ such that

$$d_Y(f(x), f(a)) < e \text{ whenever } x\ 2\ A \text{ and } d_X(x, a) < d:$$

If f is continuous at every point of A, then it is said to be continuous on A.

Remark (i) If one positive number d satisfies this condition, then every positive number $d_1 < d$ also satisfies it. This is obvious because whenever x 2 A and $d_X(x, a) < d_1$, it is also true that x A and $d_X(x, a) < d$. Therefore, such a number d is far from being unique.

(ii) In the definition of continuity, we have placed no restriction whatever on the nature of the domain A of the function. It may happen that a is an isolated point of A, i.e., there is a neighbourhood of a that contains no point of A other than a. In this case, the function f is continuous at a irrespective of how it is defined at other points of the set A. However, if a is a limit point of A and $\{x_n\}$ is a sequence of points of A such that x_n a, it follows from the continuity of f at a that $f(x_n)$ $f(a)$. In fact, we have the following theorem:

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces and $A \leq X$. A function $f : A$ into Y is continuous at a 2 A if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

Proof. First suppose the function $f : A$ Y is continuous at a A and let $\{x_n\}$ be a sequence in A converging to a. We shall show that $\{f(x_n)\}$ converges to $f(a)$. Let e be any positive real number. By continuity of f at a, there exists some $d > 0$ such that x 2 A and $d_X(x, a) <$ *delta* $d_Y(f(x), f(a)) <$ e. Since $\lim_{n!1} x_n$ ¼ a, there exists some n_0 such that $n > n_0$ implies $d_X(x_n, a) < d$. Therefore $n > n_0$ $d_Y(f(x_n), f(a)) <$ e. Thus, $\lim_{n!1} f(x_n)$ ¼ $f(a)$.

Now suppose that every sequence $\{x_n\}$ in A converging to a has the property that $\lim_{n!1} f(x_n)=f(a)$. We shall show that f is continuous at a. Suppose, if possible, that f is not continuous at a. There must exist e > 0 for which no positive d can satisfy the requirement that x 2 A and $d_X(x, a) < d$ *implies* $d_Y(f(x), f(a)) <$ e. This means that for every $d > 0$, there exists x 2 A such that $d_X(x, a) < d$ but $d_Y(f(x), f(a))$ $ e. For every n 2 N, the number $1=n$ is positive and therefore there exists x_n 2 A such that $d_X(x_n, a) < 1=n$ but $d_Y(f(x_n), f(a))$ $ e. The sequence $\{x_n\}$ then converges to a but the sequence $\{f(x_n)\}$ does not converge to $f(a)$. This contradicts the assumption that every sequence $\{x_n\}$ in A converging to a has the property that $\lim_{n!1} f(x_n)$ ¼ $f(a)$. Therefore, the supposition that f is not con- tinuous at a must be false.

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces and $A \leq X$. Let f

: *A into Y* and *a* be a limit point of *A*. We write $lim_{x \to a} f(x)$ *b*, where *b* *Y* , if for every e > 0 there exists *d* > 0 such that

$$d_Y(f(x), b) < e \text{ whenever } x \ 2 \ A \text{ and } 0 < d_X(x, a) < d:$$

Remark. In the definition of limit, the point *a* in *X* need only be a limit point of *A* and does not have to belong to *A*. In addition, if *a* 2 *A*, we may have $\lim_{x \to a} f(x)$ 6¼ $f(a)$.

Proposition : Let (X, d_X), (Y, d_Y), *A*, *f* and *a* be as in the definition above. Then
if and only if

$$\lim_{x \to a} f(x) \quad b \qquad ¼$$

$$\lim_{n \to 1} f(x_n) \quad b \qquad ¼$$

for every sequence $\{x_n\}$ in *A* such that x_n 6¼ *a* and $\lim_{n \to 1} x_n$ ¼ *a*.

Proof. The argument is similar to that of Theorem 3.1.3 and is therefore not included.

Lemma : Let *f* : *X* *Y* be an arbitrary function and let *A contained in X* and *B contained in Y* .Then *f* (*A*) contained in *B* if and only if *A contained in* $f^{-1}(B)$.

The next characterisation of continuity follows immediately from Definitions

Proposition: Let (X, d_X) and (Y, d_Y) be metric spaces and *A* ≤ *X* . Let *f* : *A intoY* and *a* be a limit point of *A*. Then *f* is continuous ¼ at *a* if and only if $\lim_{x \to a} f(x)$ $f(a)$. If *a* is an isolated point of *A*, the function *f* is continuous at *a* irrespective of how it is defined at other points of *A*.

The following reformulation of the definition of continuity at a point *a* in terms of neighbourhoods is useful.

Proposition: A mapping *f* of a metric space (X, d_X) into a metric

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space (Y, d_Y) is continuous at a point a X if and only if for every e > 0, there exists $d > 0$ such that

$$S(a, d) \leq f^{-1}(S(f(a), e)),$$

where $S(x, r)$ denotes the open ball of radius r with centre x.

Proof. The mapping $f : X$ *into* Y is continuous at a *belongs* X if and only if for every e > 0 there exists $d > 0$ such that

$$d_Y(f(x), f(a)) < e \quad \text{for all } x \text{ satisfying } d_X(x, a) < d,$$

x belongs $S(a, d)$ implies $f(x)$ belongs $S(f(a), e)$

$$f(S(a, d)) \leq S(f(a), e):$$

This is equivalent to the condition

$$S(a, d) \leq f^{-1}(S(f(a), e)):$$

Theorem A mapping $f : X$ Y is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.

Proof. Suppose f is continuous on X and let G be an open subset of Y. We have to show $f^{-1}(G)$ is open in X. Since $\mathbf{1}$ and X are open, we may suppose that $f^{-1}(G)$ $\mathbf{1}$ and $f^{-1}(G)$ X. Let x $f^{-1}(G)$. Then $f(x)$ G. Since G is open, there exists e > 0 such that $S(f(x), e)$ contained in G. Since f is continuous at x, by Proposition 3.1.8, for this e there exists $d > 0$ such that

$$S(x, d) \leq f^{-1}(S(f(x), e)) \leq f^{-1}(G):$$

Thus, every point x of $f^{-1}(G)$ is an interior point, and so $f^{-1}(G)$ is open in X.

Suppose, conversely, that $f^{-1}(G)$ is open in X for all open subsets G of Y. Let x *belongs* X. For each e > 0, the set $S(f(x), e)$ is open and so $f^{-1}(S(f(x), e))$ is open in X. Since

$$x \; 2 \; f^{-1}(S(f(x), e)),$$

it follows that there exists $d > 0$ such that

$$S(x, d) \leq f^{-1}(S(f(x), e)):$$

it follows that f is continuous at x.

Theorem. A mapping $f : X$ Y is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y.

Proof. Let F be a closed subset of Y. Then $Y \setminus F$ is open in Y so that $f^{-1}(Y \setminus F)$ is open in X by Theorem 3.1.9. But

So $f^{-1}(F)$ is closed in X.Suppose, conversely, that $f^{-1}(F)$ is closed in X for all closed subsets F of Y. Then, by Theorem 2.1.31, $X \setminus f^{-1}(F)$ is open in X and so

$$f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$$

is open in X. Since every open subset of Y is a set of the type $Y \setminus F$, where F is a suitable closed set, it follows by using Theorem 3.1.9 that f is continuous.

The characterisation of continuity in terms of open sets leads to an elegant and brief proof of the fact that a composition of continuous maps is continuous.

Theorem Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and let $f : X$ Y and $g : Y$ Z be continuous. Then the composition $g \circ f$ is a continuous map of X into Z.

Proof. Let G be an open subset of Z. By Theorem 3.1.9, $g^{-1}(G)$ is an open subset of Y, and another application of the same theorem shows that $f^{-1}(g^{-1}(G))$ is an open subset of X. Since $(g \circ f)^{-1}(G)$ ¼ $f^{-1}(g^{-1}(G))$, it follows from the same theorem again that $g \circ f$ is continuous.

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X$ Y . Then the following statements are equivalent:

(i) f is continuous on X;
(ii) $\overline{f^{-1}(B)} \leq f^{-1}(\overline{B})$ for all subsets B of Y;
(iii) $f(\overline{A}) \leq \overline{f(A)}$ for all subsets A of X.

Proof. (i) implies(ii). Let B be a subset of Y. Since $\bar{B}$ is a closed subset of Y, $f^{-1}(\bar{B})$ is closed in X. Moreover, $f^{-1}(B)$ $f^{-1}(\bar{B})$, and so $f^{-1}(B)$ $f^{-1}(\bar{B})$. (Recall that $f^{-1}(B)$ is the smallest closed set containing $f^{-1}(B)$.)

(ii) Implies (iii). Let A be a subset of X. Then, if B ¼=$f(A)$, we have $A \leq f^{-1}(B)$ and

$\bar{A} \leq f^{-1}(B) \leq f^{-1}(\bar{B})$. Thus $f(\bar{A}) \leq f(f^{-1}(\bar{B})) = \bar{B}$ ¼ $f(A)$.

(ii) implies (i) Let F be a closed set in Y and set $f^{-1}(F) = F_1$ it is sufficient to show that F_1 is closed in X, that is, F_1 ¼ F_1. Now,

$$f(\bar{F}1) \leq f\overline{(f^{-1}(F))} \leq \bar{F} = F,$$

$$\bar{F}1 \leq f^{-1}(f(\bar{F}_1)) \leq f^{-1}(F) = F_1$$

$$|f_k(x) — f_k(y)| <|f(x) — f(y)|, k = 1, 2, .. . , n,$$

(ii) If (X, d) is a discrete metric space, then every function $f : X$! Y, where Y is any metric space, is continuous. Let a 2 X and $S(f(a)$, e) be an open ball centred at $f(a)$ with radius e. Choose d < 1. Then $S(a, d)$ ¼ $\{a\}$ and so $f(S(a, d))$ ¼ $\{f(a)\} \leq S(f(a)$, e).

SEQUENTIAL CRITERION AND OTHER CHARACTERIZATIONS OF CONTINUITY

Consider the function $f : (0, 1)$! R defined by $f(x)$ ¼ 1=x. There is no continu- ous function g defined on [0, 1) that agrees with f . In other words, f has no continuous 'extension' to [0, 1). The term 'extension' is formally defined below.

Definition Let X and Y be abstract sets and let A be a proper subset of X. If f is a mapping of A into Y, then a mapping $g : X$ Y is called an extension of f if $g(x)$ $f(x)$ for each x A; the function f is then called the restriction of g to A.

If X and Y are metric spaces, A X and $f : A$ Y is continous, then we might ask whether there exists a continuous extension g of f. Extension problems abound in analysis and have attracted the attention of many celebrated mathematicians. Below, we deal with some simple extension techniques.

Theorem Let (X, d_X) and (Y , d_Y) be metric spaces and let f : X , g : X ❢ Y be continuous maps. Then the set {x 2 X: f (x) ¼ g (x)} is a closed subset of X.

Proof. Let F ¼ {x 2 X: f (x) ¼ g (x)}. Then X\$F$ ¼ {x 2 X: f (x) 6¼ g (x)}. We shall show that X\$F$ is open. If X\$F$ ¼ **1**, then there is nothing to prove. So let X \$F$ 6¼ **1** and let a 2 X\$F$. Then f (a) 6¼ g (a). Let r > 0 be the distance d_Y (f (a), g (a)). For e ¼ r=3, there exists a d > 0 such that

d_X(x, a) < d implies d_Y (f (x), f (a)) < r=3 and d_Y (g (x), g (a)) < r=3:

By the triangle inequality, we have

d_Y(f (a), g (a)) # d_Y(f (a), f (x)) þ d_Y(f (x), g (x)) þ d_Y(g (x), g (a)), which implies

d_Y (f (x), g (x)) $ d_Y (f (a), g (a)) — d_Y (f (a), f (x)) — d_Y (g (x), g (a)) > r=3

for all x satisfying d_X(x, a) < d. Thus, for each x 2 S(a, d), d_Y (f (x), g (x)) > 0, i.e.,
f (x) 6¼ g (x). So,

$$S(a, d) \leq X \backslash F:$$

Hence, X\F is open and thus F is closed. &

Corollary Let (X, d_X) and (Y , d_Y) be metric spaces and let f : X ❢ Y , g : X ❢ Y be continuous maps. If F ¼ {x 2 X : f (x) ¼ g (x)} is dense in X, then f ¼ g .

Proof. By Theorem 3.2.2, F is closed. Since F is assumed dense in X, we have $X = \bar{F} = F$, i.e., $f(x) = g(x)$ for all $x \in X$.

Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces, A a dense subset of X and f a map from A to Y. Then f has a continuous extension $g : X \to Y$ if and only if for every $x \in X$ that is a limit point of A, the limit $\lim_{y \to x} f(y)$ not only exists in Y but also equals $f(x)$ in case $x \in A$. When the extension exists, it is unique. (Note that the stipulation $\lim_{y \to x} f(y) = f(x)$ when $x \in A$ says that f is continuous on A.)

Proof. Suppose that f has a continuous extension g, and consider any x X that is a limit point of X. Since A is dense, x must be a limit point of A as well, as we now argue. Any ball $S(x, \varepsilon)$ contains a point y X, y x. There exists $S(y, \varepsilon')$ $S(x, \varepsilon)$ such that x $S(y, \varepsilon')$. Since A is dense, $S(y, \varepsilon')$ contains a point a A. Thus, $S(x, \varepsilon)$ contains the point a A and a x.

Now

$$g(x) = \lim_{y \to x} g(y) \qquad (g \text{ is continuous})$$

$$= \lim_{y \to x} g(y) \text{ with } y \in A \qquad (x \text{ is a limit point of } A)$$

$$= \lim_{y \to x} f(y) \qquad (g \text{ is an extension of } f):$$

Thus, $\lim_{y \to x} f(y)$ exists and equals $g(x)$.

Conversely, suppose that for every limit point $x \in X$, $\lim_{y \to x} f(y)$ exists and that it equals $f(x)$ when $x \in A$. Define $g(x)$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \lim_{y \to x} f(y) & \text{if } x \notin A \text{ but } x \in A': \end{cases}$$

Since A is dense in X, the function g is defined on the whole of X. We need to show that g is continuous. By the definition of a limit, for every positive number e, there exists a positive number $d > 0$ such that

Consider any z 2 $S(x, d)$. In case z is an isolated point of X, then $g(z)$ 2 $S(g(x), e{=}2)$, in view of the observation above. If z is not an isolated point of X, then $g(z)$ is the limit of $f(y)$ as y ! z in $S(x, d) \setminus A$. Therefore,

$$g(z)\ 2\ \overline{f(A \setminus S(x, d))} \leq \overline{S(g(x), e{=}2)} \leq S(g(x), e),$$

so that g is continuous at x. Hence, g is continuous on X. By Corollary 3.2.3, it follows that g is the unique continuous extension of f.

Examples (i) Let $f(x)$ ¼ sin (1=x), x 2 R\{0}. We shall show that $\lim_{x\,!\,0}$ sin (1=x) does not exist. Hence, the function f cannot be extended to a continuous function on R.

Definition. Let X be a nonempty set. Given mappings f and g of X into C and a 2 C, we define the mappings f þ g , af , fg and jf j into C as follows:

$$(f \text{ þ } g)(t) \text{ ¼ } f(t) \text{ þ } g(t)$$
$$(af)(t) \text{ ¼ } af(t)$$
$$(fg)(t) \text{ ¼ } f(t)g(t)$$
$$\text{j}f\text{j}(t) \text{ ¼ j}f(t)\text{j}$$

for all t 2 X. Further, if $f(t)$ 6¼ 0 for all t 2 X, we define the mapping 1/f of X into C by

$$(1{=}f)(t) \text{ ¼ } \frac{1}{f(t)} \quad \text{for all } t \text{ 2 } X:$$

The proofs of the assertions in the following theorem are direct generalisations of the familiar proofs in the case where X is the real line.

Theorem Let *f* and *g* be continuous mappings of a metric space (X, d_X) into C and let *a* C. Then the mappings *f* *g* , *af* , *fg* and *f* are continuous on *X*, and so is the mapping 1/*f*, if it is defined.

Examples (i) Let *f* : R ! C be defined by

f (*x*) ¼ *x* þ *i*x^2:

We shall argue that *f* is continuous at 2 2 R. Consider any e > 0. Upon using the fact that the functions *g* : R ! R and *h*: R ! R defined by *g* (*x*) ¼ *x* and *h*(*x*) ¼ x^2 are continuous at 2, it follows that there exist d_1 > 0 and d_2 > 0 such that

(ii) Let *X* ¼ *C*[0,1] with the uniform metric. Define *f* : *X* ! C by *f* (*x*) ¼ *x*(0) whenever *x* 2 *X*. We shall show that *f* is continuous on *X*. Let {x_n}$_{n \$ 1}$ be a sequence in *X*, i.e., in *C*[0,1] such that $\lim_n$ x_n *x*. Since uniform convergence implies pointwise convergence, we have

$\lim_n$ *f* (x_n) ¼ $\lim_n$ x_n(0) ¼ *x*(0) ¼ *f* (*x*):

Thus, *f* is continuous on *X* ¼ *C*[0, 1].

(iii) Let *X* ¼ *C*[0,1] with the uniform metric. Define *f* : *X* ! C by

Uniform Continuity

Let (*X*, d_X) and (*Y* , d_Y) be two metric spaces and let *f* be a function continuous at each point x_0 of *X*. In the definition of continuity, when x_0 and e are specified, we make a definite choice of *d* so that

d_Y(*f* (*x*), *f* (x_0)) < e whenever d_X(*x*, x_0) < *d*:

This describes *d* as dependent upon x_0 and e, say *d* ¼ *d*(x_0, e). If *d*(x_0, e) can be chosen in such a way that its values have a lower positive bound when e is kept fixed and x_0 is allowed to vary over *X*, and if this happens for each positive e, then we have the notion of 'uniform continuity'. More precisely, we have the following definition:

Definition Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f: X \to Y$ is said to be uniformly continuous on X if, for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on ε alone) such that

$$d_Y(f(x_1), f(x_2)) < \varepsilon \text{ whenever } d_X(x_1, x_2) < \delta$$

for all $x_1, x_2 \in X$.

Every function $f: X \to Y$ which is uniformly continuous on X is necessarily continuous on X. However, the converse may not be true. We shall see later (see Theorem 5.4.10) that these two concepts agree on certain kinds of metric spaces called 'compact'.

(ii) Let A be a subset of the metric space (X, d). Define

$$f(x) = d(x, A) = \inf\{d(x, y): y \in A\}, \qquad x \in X.$$

We shall prove that f is uniformly continuous over X. For $y \in A$ and x, z *belongs* X, the triangle inequality gives

$$d(x, y) \leq d(x, z) + d(z, y).$$

On taking the infimum as y varies over A, we get

$$d(x, A) - d(z, A) \leq d(x, z), \qquad x, z \in X.$$

Interchanging x and z and observing that $d(x, z) = d(z, x)$, we get

$$d(z, A) - d(x, A) \leq d(x, z), \qquad x, z \in X.$$

Hence,

$$|f(x) - f(z)| = |d(x, A) - d(z, A)| \leq d(x, z), \qquad x, z \in X.$$

The uniform continuity of f results on choosing $\delta = \varepsilon$.

Proposition. Let (X, d) be a metric space and let $x \in X$ and $A \subseteq X$ be non- empty. Then $x \in \bar{A}$ if and only if $d(x, A) = 0$.

Proof. Suppose $d(x, A) = 0$. There are two possibilities: $x \in A$ or $x \notin A$. If $x \in A$, then $x \in \bar{A}$. We shall next show that if $x \notin A$, then x is a limit point of A. Let $\varepsilon > 0$ be given. By the definition of $d(x, A)$, there exists a *y* *A* such that *d*(*x*, *y*) < e, i.e., *y* *S*(*x*, e). Thus, every ball with centre *x* and radius e contains a point of *A* distinct from *x*; so *x* $\bar{A}$. Conversely, suppose *x* $\bar{A}$. If *x* *A*, then obviously *d*(*x*, *A*) 0. We shall next show that if *x* is a limit point of *A*, then *d*(*x*, *A*) 0. By the definition of limit point, every ball *S*(*x*, e) with centre *x* and radius e > 0 contains a point *y* 2 *A* distinct from *x*. Consequenly, *d*(*x*, *A*) < e, i.e., *d*(*x*, *A*) ¼ 0.

Theorem Let *A* and *B* be disjoint closed subsets of a metric space (*X*, *d*). Then there is a continuous real-valued function *f* on *X* such that *f* (*x*) ¼ 0 for all *x* 2 *A*, *f* (*x*) ¼ 1 for all *x* 2 *B* and 0 # *f* (*x*) # 1 for all *x* 2 *X*.

Proof. From Example (ii) above, it follows that the mappings *x* ❢ *d*(*x*, *A*) and *x* ❢ *d*(*x*, *B*) are continuous on *X*. Since *A* and *B* are closed and *A* \ *B* ¼ 1, Proposition 3.4.3 shows that *d*(*x*, *A*) þ *d*(*x*, *B*) > 0 for all *x* 2 *X*. Indeed, if *d*(*x*, *A*) þ *d*(*x*, *B*) ¼ 0 for some *x* 2 *X*, then *d*(*x*, *A*) ¼ *d*(*x*, *B*) ¼ 0; so *x* 2 $\bar{A}$ ¼ *A* and *x* 2 $\bar{B}$ ¼ *B*, and hence *x* 2 *A* \ *B*, a contradiction.

Now define a mapping *f* : *X* ❢ R by

$$f(x) \text{ ¼ } \frac{d(x, A)}{d(x, A) \text{ þ } d(x, B)}, \qquad x \text{ 2 } X:$$

Corollary Let (*X*, *d*) be a metric space and *A*, *B* be disjoint closed subsets of *X*. Then there exist open sets *G*, *H* such that *A* ≤ *G*, *B* ≤ *H* and *G* \ *H* ¼ 1.

Proof. Let *f* : *X* ❢ [0, 1] be any function guaranteed by Theorem 3.4.4, and let

Then *G* ¼ f^{-1}([0, 1=2)) and *H* ¼ f^{-1}(((1=2, 1]) are open subsets of *X*, being inverse images of open subsets of [0,1]. Moreover, *A* ≤ *G*, *B* ≤ *H* and *G* \ *H* ¼ 1. &

A composition of uniformly continuous mappings is again a uniformly continu- ous mapping. More precisely, we have the following theorem:

Theorem If f and g are two uniformly continuous mappings of metric spaces (X, d_X) to (Y, d_Y), and (Y, d_Y) to (Z, d_Z), respectively, then $g \circ f$ is a uniformly continuous mapping of (X, d_X) to (Z, d_Z).

Proof. Since g is uniformly continuous, for each e > 0, there exists a $d > 0$ such that

$$d_Y(f(x), f(y)) < d \text{ implies } d_Z((g \circ f)(x), (g \circ f)(y)) < e$$

for all $f(x), f(y)$ 2 Y.

As f is uniformly continuous, corresponding to $d > 0$, there exists an $h > 0$ such that

$$d_X(x, y) < h \text{ implies } d_Y(f(x), f(y)) < d$$

for all x, y 2 X.

Thus, for each e > 0, there exists an $h > 0$ such that

$$d_X(x, y) < h \text{ implies } d_Z((g \circ f)(x), (g \circ f)(y)) < e$$

for all x, y 2 X and so $g \circ f$ is uniformly continuous on X. &

A continuous function may not map a Cauchy sequence into a Cauchy sequence as the following example shows:

Example Let X ¼ (0, 1) with the induced usual metric of the reals and Y be the reals with the usual metric. The function $f : X$! Y defined by

$$f(x) \text{ ¼ } \frac{1}{x}, x \text{ 2 } X,$$

is continuous on X. Now $\{1=n\}_n$ $ 1 is a Cauchy sequence in X (because it is convergent in R). But $\{f(1=n)\}_n$ $ 1 $\{n\}_n$ $ 1 is not a Cauchy sequence in Y. Indeed, the absolute difference of any two distinct terms is at least as large as 1.

However, Cauchy sequences are mapped into Cauchy sequences by uniformly continuous functions.

Theorem . Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X$! Y be uniformly continuous. If $\{x_n\}n \$ 1$ is a Cauchy sequence in X, then so is $\{f(x_n)\}n \$ 1$ in Y.

Proof. Since f is uniformly continuous, for every e > 0, there exists a $d > 0$ such that

$$d_Y(f(x), f(y)) < e \quad \text{whenever } d_X(x, y) < d \qquad (3:4)$$

for all x, y 2 X.

Because the sequence $\{x_n\}n \$ 1$ is Cauchy, corresponding to $d > 0$, there exists n_0
such that

$$n, m \$ n_0 \text{ implies } d_X(x_n, x_m) < d: \qquad (3:5)$$

From (3.4) and (3.5), we conclude that

$$d_Y(f(x_n), f(x_m)) < e \quad \text{for } n, m \$ n_0,$$

and so $\{f(x_n)\}n \$ 1$ is a Cauchy sequence in Y.

Theorem Let f be a uniformly continuous mapping of a set A, dense in the metric space (X, d_X), into a complete metric space (Y, d_Y). Then there exists a unique continuous mapping $g : X \quad Y$ such that $g(x) \quad f(x)$ when $x \quad A$; more- over, g is uniformly continuous.

Proof. Since f is uniformly continuous, a fortiori, continuous, therefore, for every $x \quad A$ that is a limit point of X, the limit $\lim_{y ! x} f(y)$ not only exists in Y but also equals $f(x)$. Therefore, by Theorem 3.2.4, in order to prove the existence and uniqueness of such a continuous mapping $g : X \; Y$, it is sufficient to show for every $x \; X \backslash A$ that $f(y)$ tends to a limit as $y \; x$. (It is understood that $y \; A$, because the domain of f is A.)

Let x 2 X be arbitrary. Since A is dense in X, there exists a sequence $\{x_n\}_{n \$ 1}$ in A such that $\lim_{n!1} d_X(x_n, x)$ ¼ 0. Since $\{x_n\}n \$ 1$ is convergent, it is a fortiori Cauchy; so by Theorem 3.4.8, it follows that $\{f(x_n)\}n \$ 1$ is a Cauchy sequence in the complete metric space ¼ (Y, d_Y) and hence converges to a limit, which we shall denote by b. Now consider any sequence $\{x'_n\}_{n \$ 1}$ in A with x'_n 6¼ x for each n and $\lim_{n!1} x'_n$ x. It follows from uniform continuity of f that, for e > 0, there exists a $d > 0$ such that

$$d_Y(f(z), f(y)) < e \quad \text{whenever } d_X(z, y) < d: \qquad (3:6)$$

Since $\lim_{n!1} x_n$ ¼ x ¼ $\lim_{n!1} x'_n$, there exists an integer n_1 such that
$d_X(x_n, x'_n) < d$ whenever $n \$ n_1$. Therefore by (3.6)

$$d_Y(f(x_n), f(x'_n)) < e \quad \text{whenever } n \$ n_1:$$

Remark The condition that the metric space (Y, d_Y) is complete in Theorem 3.4.9 cannot be omitted. In fact, ¼ let X R with the usual ¼ metric and Y Q, the set of rationals with the metric ¼ induced from R. Let A Q. Observe that A is a dense subset of X. ¼ The function 2 $f : A$ Y defined by $f(x)$ x for every x A is uni- formly continuous but it possesses no continuous extension to X, as the only continuous *rational-valued* functions on X ¼ R are constant functions.

Homeomorphism

Definition 3.5.1. Let (X, d_X) and (Y, d_Y) be any two metric spaces. A function $f : X$ Y which is both one-to-one and onto is said to be a homeomorphism if and only if the mappings f and f^{-1} are continuous on X and Y, respectively. Two metric spaces X and Y are said to be homeomorphic if and only if there exists a homeomorphism of X onto Y, and in this case, Y is called a homeomorphic image of X.

If X and Y are homeomorphic, the homeomorphism puts their points in one-to- one correspondence in such a way that their open sets also correspond to one another.

For metric spaces X and Y, let $X \sim Y$ mean that X and Y are homeomorphic. It is easily verified that the relation is reflexive, symmetric and transitive.

Suppose that whenever a metric space (X, d) has the property 'P', every metric space homeomorphic to (X, d) also has the property; then we say that the property is 'preserved under homeomorphism'. There are a large number of properties that are not preserved under homeomorphism, as the following example shows:

Example Let X N and Y {1=n: n N}, each equipped with the usual absolute value metric. The function $f : X Y$ defined by $f(x)$ 1=x is a home- omorphism of X onto Y. Observe that X is a closed subset of R and since R is complete, it follows that X is complete. On the other hand, $\{1=n\}_{n \ \$ \ 1}$ is a Cauchy sequence in Y that does not converge; so Y is not complete. Besides, the space X is not bounded, whereas Y is bounded.

Recall from Definition 1.5.2 that a mapping f of X into Y is an isometry if

$$d_Y(f(x), f(y)) \ ¼ \ d_X(x, y)$$

for all x, y X . It is obvious that an isometry is one-to-one and uniformly continuous. Recall also that X and Y are said to be isometric if there exists an isometry between them that is onto. An isometry is necessarily a homeomorphism, but the converse is not true, as is evident from Examples 3.5.2 (i) and (ii) above.

By definition, it follows that isometric spaces possess the same metric properties. For metric spaces X and Y, let X Y mean that X and Y are isometric. It is easily verified that this relation between metric spaces is reflexive, symmetric and transitive.

Definition. Let d_1 and d_2 be metrics on a nonempty set X such that, for every sequence $\{x_n\}_{n}$ $ 1 in X and x 2 X,

$$\lim_{n ! 1} d_1(x_n, x) ¼ 0 \quad \text{if and only if} \quad \lim_{n ! 1} d_2(x_n, x) ¼ 0,$$

i.e., a sequence converges to x in (X, d_1) if and only if it converges to x in (X, d_2). We then say that d_1 and d_2 are equivalent metrics on X and that (X, d_1) and (X, d_2) are equivalent metric spaces.

Remark In view of Theorem 3.1.3, two metrics d_1 and d_2 on a nonempty set X are equivalent if and only if the identity maps id: (X, d_1) ! (X, d_2) and id: (X, d_2) ! (X, d_1) are both continuous, i.e., if and only if the identity mapping from (X, d_1) to (X, d_2) is a homeomorphism (as Definition in 3.5.1 above). Note that this amounts to saying that the families of open sets are the same in (X, d_1) and (X, d_2).

The following is a sufficient condition for two metrics on a set to be equivalent.

$$f_n(x) ¼ \tan^{-1}(nx),\ x \$ 0,$$

is uniformly convergent on $[a, 1)$ when $a > 0$, but is not uniformly convergent on $[0, 1)$. The pointwise limit function is

$$f(x) ¼ \lim_{n ! 1} f_n(x) ¼ \begin{cases} \frac{p}{2} & \text{if } x > 0, \\ 0 & \text{if } x ¼ 0. \end{cases}$$

We shall show that f_n ! f uniformly on $[a, 1)$ when $a > 0$. For $x > 0$,

$$|f_n(x) - f(x)| ¼ \cdot \tan^{-1}(nx) - \frac{p}{\cdot} ¼ \cot^{-1}(nx),$$

as we shall now prove. Since $0 < \tan^{-1} u < p=2$ for any $u > 0$, therefore when
$x > 0$, we have $0 < \tan^{-1}(nx) < p=2$ and hence

$$0 < \frac{p}{2} - \tan^{-1}(nx) < \frac{p}{2}: \qquad (3:10)$$

Also,

Now, it follows from (3.10) and (3.11) that $p=2$ ¼ $\tan^{-1}(nx)$ $\cot^{-1}(nx)$. It also follows from the first inequality in (3.10) that $|\tan^{-1}(nx)\ p=2|$ ¼ $p=2 \ \tan^{-1}(nx)$ for $x > 0$. Thus, $\tan^{-1}(nx)\ \ p=2\ \ \cot^{-1}(nx)$.

Let e > 0 be arbitrary. When $x \$ a$, the inequality $n > (\cot e)=a$ implies that $n > (\cot e)=x$, so that $nx > \cot e$ and hence $\cot^{-1} nx < e$ in view of the fact that $\cot^{-1}$ is a decreasing function. It follows that if n_0 is an integer greater than or equal to $(\cot e)=a$, then $f_n(x)\ \ f(x)\ \ \tan^{-1}(nx)\ \ p=2\ \ \cot^{-1} nx < e$ whenever $n \$ n_0$ and $x \$ a$. However, $(\cot e)=x$ as x 0, so that no integer n_0 exists for which $f_n(x)\ \ f(x) < e$ for all $n \$ n_0$ and all x [0,). Actually this proves that the convergence fails to be uniform even on the smaller set (0,).

The following basic result about transmission of the property of being continu- ous will be needed in the sequel.

Theorem Let (X, d_X) and (Y, d_Y) be metric spaces, $\{f_n\}_{n \$ 1}$ a sequence of functions, each defined on X with values in Y, and let $f : X \ ! \ Y$. Suppose that $f_n\ f$ uniformly over X and that each f_n is continuous over X. Then f is continuous over X. Briefly put, a uniform limit of continuous functions is continuous.

Proof. Let x_0 2 X be arbitrary and let e > 0 be given. Since $f_n\ !\ f$ uniformly over
X, there exists n_0 (depending on e only) such that for each x 2 X,

$$d_Y(f_n(x), f(x)) < \frac{e}{3} \qquad \text{for } n \$ n_0: \qquad (3:12)$$

Since f_n0 is continuous at x_0, we can choose d > 0 such that x 2 $S(x_0, d)$ ¼ {x 2 X: $d_X(x, x_0)$ < d} implies

Proposition (Cauchy Criterion) Let {f_n}n $ 1 a sequence of functions defined on a metric space (X, d_X) with values in a complete metric space (Y, d_Y). Then there exists a function f : X ! Y such that

f_n ! f uniformly on X

if and only if the following condition is satisfied: For every e > 0, there exists an integer n_0 such that

m, n $ n_0 implies $d_Y(f_m(x), f_n(x))$ < e

for every x 2 X.

Contraction Mappings and Applications

The concept of completeness of metric spaces has interesting and important applications in classical analysis. In this section, we show how various existence and unique- ness theorems in the theory of differential and integral equations follow from very simple facts about mappings in a complete metric space. The simple fact alluded to above is called the contraction mapping principle, which we now consider.

Definition Let (X, d) be a metric space. A mapping T of X into itself is said to be a contraction (or contraction mapping) if there exists a real number a,0 < a < 1, such that

$d(Tx, Ty)$ # $ad(x, y)$

for all x, y 2 X.

It is obvious that a contraction mapping is uniformly continuous (see Definition 3.4.1).

Theorem (Contraction Mapping Principle) Let $T : X \; X$ be a contraction of the complete metric space (X, d). Then T has a unique fixed point.

Proof. Let x_0 2 X and let $\{x_n\}_n$ \$ 1 be the sequence defined iteratively by $x_{n\text{þ}1}$ ¼ Tx_n for n ¼ 0, 1, 2, …. We shall prove that $\{x_n\}_n$ \$ 1 is a Cauchy sequence. For p ¼ 1, 2, … , we have

$$d(x_{p\text{þ}1}, x_p) \text{¼} d(Tx_p, Tx_{p—1}) \# ad(x_p, x_{p—1}),$$

(3:16) where $0 < a < 1$ is such that for all x, y 2 X.

$$d(Tx, Ty) = ad(x, y)$$

Repeated application of the inequality (3.16) gives

$$d(x_{p\text{þ}1}, x_p) = ad(x_p, x_{p—1})$$
$$= a^2 d(x_{p—1}, x_{p—2}) \# \ldots \# a^p d(x_1, x_0):$$

Now, let m, n be positive integers with $m > n$. By the triangle inequality,

$$d(x_m, x_n) = d(x_m, x_{m—1}) \text{ þ } d(x_{m—1}, x_{m—2}) \text{ þ } \ldots \text{ þ } d(x_{n\text{þ}1}, x_n)$$
$$= (a^{m—1} \text{ þ } a^{m—2} \text{ þ } \ldots \text{ þ } a^n) d(x_1, x_0)$$
$$= a^n (a^{m—n—1} \text{ þ } a^{m—n—2} \text{ þ } \ldots \text{ þ } 1) d(x_1, x_0)$$
$$= \frac{a^n}{1 — a} d(x_1, x_0):$$

But $\lim_{n!1} a^n$ ¼ 0. It follows that $\{x_n\}$ is a Cauchy sequence in (X, d), which is complete. Let y ¼ $\lim_{n!1} x_n$. Since T is a contraction, it is continuous. It follows that Ty ¼ $T(\lim_{n!1} x_n)$ ¼ $\lim_{n!1} Tx_n$ ¼ $\lim_{n!1} x_{n\text{þ}1}$ ¼ y. Thus, y is a fixed point of T. Moreover, it can be shown to be unique: If y 6¼ z are such that Ty ¼ y and Tz ¼ z, then $d(y, z)$ ¼ $d(Ty, Tz) \# ad(y, z) < d(y, z)$. This implies $d(y, z)$ ¼ 0, i.e., y ¼ z.

Connected Spaces

Definition 4.1.1. A metric space (X, d) is said to be disconnected if there exist two nonempty subsets A and B of X such that

(i) X ¼ $\underline{A}$ [B;
(ii) A \ $\overline{B}$ ¼ 1 and $\overline{A}$ \ B ¼ 1.

That is, the subsets must be nonempty, together they must constitute the whole space and neither may contain a point of the closure of the other. If no such subsets exist, then (X, d) is said to be connected; this means that if we do split X into two nonempty parts A and B having no points in common, then at least one of the subsets contains a limit point of the other.

A nonempty subset Y of a metric space (X, d) is said to be connected if the subspace (Y , dj$_Y$) with the metric induced from X is connected.

Theorem 4.1.3. Let (X, d) be a metric space. Then the following statements are equivalent:

(i) (X, d) is disconnected;
(ii) there exist two nonempty disjoint subsets A and B, both open in X, such that
X ¼ A [B;
(iii) there exist two nonempty disjoint subsets A and B, both closed in X, such that
X ¼ A [B;
(iv) there exists a proper subset of X that is both open and closed in X.

Proof. (i)) (ii). Let X ¼ A [B, where A and B are nonempty and
A \ $\overline{B}$ ¼ 1, $\overline{A}$ \ B ¼ 1. Then A ¼ X\$\overline{B}$. In fact, $\overline{A}$ ≤ X\$B$ ≤ X\$B$ ¼ A. So A is

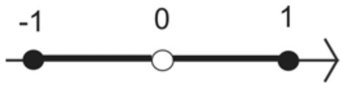

Figure 4.1

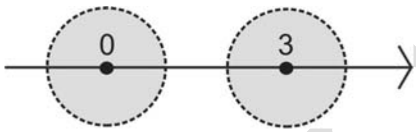

Figure 4.2

open in *X*. Similarly, *B* is open in *X*. Since $\bar{A}$ and *B* are disjoint, a fortiori, *A* and *B*
are disjoint, which proves (ii).

That (ii) and (iii) are equivalent is trivial.

(iii)⟹(iv) Since ¼*A X**B*, *A* is open. Thus *A* is both a closed and open proper subset of *X*, and so (iv) is proved.

(iv)⟹(i) Let *A* be a proper subset of *X* that is both open and closed in *X* and let *B* ¼ *X**A*. Then *X* ¼ *A* [*B*, *A* \ *B* ¼ 1. Since *A* ¼ *A* (*A* being closed), it follows that *A* \ *B* ¼ 1. Similarly, *A* \ *B* ¼ 1. This completes the proof.

Theorem Let (R, *d*) be the space of real numbers with the usual metric. A subset *I* R is connected if and only if *I* is an interval, i.e., *I* is of one of the following forms:

(*a*, *b*), [*a*, *b*), (*a*, *b*], [*a*, *b*], (— 1, *b*), (— 1, *b*], (*a*, 1), [*a*, 1), (— 1, 1):

Proof. Let *I* be a connected subset of real numbers and suppose, if possible, that *I* is not an interval. Then there exist real numbers *x*, *y*, *z* with *x* < *z* < *y* and *x*, *y* 2 *I* but *z* 62 *I*. Then *I* may expressed as *I* ¼ *A* [*B*, where

A ¼ (—1, *z*) \ *I* and *B* ¼ (*z*, 1) \ *I*:

Since $x \in A$ and $y \in B$, therefore, A and B are nonempty; also, they are clearly disjoint and open in I. Thus, I is disconnected.

To prove the converse, suppose I is an interval but is not connected. Then there are nonempty subsets A and B such that

$$I = A \cup B, \quad A \cap \bar{B} = \emptyset, \quad \bar{A} \cap B = \emptyset.$$

Pick $x \in A$ and $y \in B$ and assume (without loss of generality) that $x < y$. Observe that $[x, y] \subseteq I$, for I is an interval. Define

$$z = \sup (A \cap [x, y]).$$

The supremum exists since $A \cap [x, y]$ is bounded above by y and it is nonempty, as x is in it. Then $z \in \bar{A}$. (We shall show that if $z \notin A$, then z is a limit point of A. Let $\varepsilon > 0$ be arbitrary. By the definition of supremum, there exists some element $a \in A$ such that $z - \varepsilon < a \leq z$, i.e., every neighbourhood of z contains a point of A.) Hence, $z \notin B$, for $\bar{A} \cap B = \emptyset$; in particular, $x \leq z < y$.

If $z \notin A$, then $x < z < y$ and $z \notin I$. This contradicts the fact that $[x, y] \subseteq I$.

If $z \in A$, then $z \notin \bar{B}$, for $A \cap \bar{B} = \emptyset$. So there exists a $d > 0$ such that $(z - d, z + d) \cap B = \emptyset$. This implies that there exists $z_1 \notin B$ satisfying the inequal- ity $z < z_1 < y$. Then $x \leq z < z_1 < y$ and $z_1 \notin I$, for z_1 being greater than $\sup (A \cap [x, y])$ is not in A. This contradicts the fact that $[x, y] \subseteq I$.

Remark. It follows as a special case of Theorem 4.1.4 that the entire real line R is a connected set. It now follows from Theorem 4.1.3(iv) that the only subsets of R that are both open and closed are the empty set and R itself.

Let $X_0 = \{0, 1\}$ and let d_0 denote the discrete metric on X_0. We shall call (X_0, d_0) the discrete two point space. Definition 4.1.1 can be reformulated in the following handier fashion:

Theorem Let (X, d) be a metric space. Then the following statements are equivalent:

(i) (X, d) is disconnected;

(ii) there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0).

Proof. (i)❩(ii). Let X ¼ A [B, where A and B are disjoint nonempty open subsets (see Theorem 4.1.3(ii)). Define a mapping f : X ! X_0 by

$$f(x) ¼ \begin{cases} 0 & \text{if } x\ 2\ A, \\ 1 & \text{if } x\ 2\ B; \end{cases}$$

the mapping f is clearly onto. It remains to show that f is continuous from (X, d) to (X_0, d_0). The open subsets of the discrete metric space are precisely 1, {0}, {1} and {0,1}. Observe that f^{-1}(1) 1, f^{-1}({0, 1}) X and the subsets 1, X are open in (X, d). Moreover, f^{-1}({0}) A, f^{-1}({1}) B, which are open subsets of (X, d). Hence, f is continuous and thus (ii) is proved.

(ii) implies (i) Let $f : (X, d)$ (X_0, d_0) be continuous and onto. Let A ¼ f^{-1}({0}) and B f^{-1}({1}). Then A and B are nonempty disjoint subsets of X, both open and such that X A B. It follows upon using Theorem 4.1.3(ii) that X is disconnected.

Theorem Let (X, d_X) be a connected metric space and f : !(X, d_X) (Y, d_Y) be a continuous mapping. Then the space $f(X)$ with the metric induced from Y is connected.

Proof. The map !f : X $f(X)$ is continuous. If $f(X)$ were not connected, then there would be, by Theorem 4.1.6, a continuous mapping, g say, of $f(X)$ onto the discrete two element space (X_0, d_0). Then !g f : X X_0 would also be a continuous map- ping of X onto X_0, contradicting the connectedness of X.

The intermediate value theorem of real analysis (see Proposition 0.5.3) is a special case of Theorem 4.1.8.

Theorem (Intermediate Value Theorem) If *f* : [*a*, *b*] ❢ R is continuous over [*a*, *b*], then for every *y* such that $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$ there exists *x* 2 [*a*, *b*] for which *f* (*x*) ¼ *y*.

Proof. We need consider only the case when *f* (*a*) *y* *f* (*b*). Let *y* be any real number such that $f(a) < y < f(b)$. By Theorem 4.1.4, [*a*, *b*] is a connected subset of R. Hence, *f* ([*a*, *b*]) is an interval by Theorems 4.1.8 and 4.1.4. Therefore, there exists an *x* [*a*, *b*] such that *f* (*x*) *y*. The case where $f(b) < y < f(a)$ is dealt with in a similar way.

The following converse of the intermediate value theorem also holds.

Theorem Let (X, d_X) be a metric space. If every continuous function *f* : (X, d_X) ❢ (R, *d*) has the intermediate value property (i.e., if y_1, y_2 2 *f* (*X*) and *y* is a real number between y_1 and y_2, then there exists an *x* 2 *X* such that *f* (*x*) ¼ *y*), then (X, d_X) is a connected metric space.

Proof. Suppose, if possible, (X, d_X) is not connected. Then, by Theorem 4.1.6, there exists a continuous map $g:(X,d_X)$ ❢ (X_0, d_0) that is onto. Define a map $h:(X_0,d_0)$ ❢ (R,*d*) by *h*(0)¼0 and *h*(1)¼1. Then *h* is continuous. Consider the map $h \circ g$ $:(X,d_X)$ ❢ (R,*d*). Clearly, $h \circ g$ is continuous, being the composition of continuous maps *h* and *g*. Besides, $\{0,1\} \leq h \circ g(X)$. However, there exists no *x* 2*X* such that $h \circ g$ (*x*)¼1=2. In fact, $(h \circ g)^{-1}(\{1=2\})$¼$g^{-1}oh^{-1}(\{1=2\})$¼ g^{-1}(𝟏)¼𝟏.

An interesting application of the Weierstrass intermediate value theorem is the following "fixed point theorem":

Theorem. Let *I* ¼ [—1, 1] and let *f* : *I* ❢ *I* be continuous. Then there exists a point *c* 2 *I* such that *f* (*c*) ¼ *c*.

Proof. If *f* (— 1) ¼ —1 or *f* (1) ¼ 1, the required conclusion follows; hence, we can assume that *f* (— 1) > —1 and *f* (1) < 1. Define

$g(x)$ ¼ $f(x)$ — x, x 2 I :

Note that g is continuous, being the difference of continuous functions, and that it satisfies the inequalities g(—1) ¼ f(—1) þ 1 > 0 and g (1) ¼ f (1) — 1 < 0. Hence, by the Weierstrass intermediate value theorem, there exists c 2 (—1, 1) such that g (c) ¼ 0, that is, $f(c) = c$.

Maps(1,1) into itself and yet has no fixed point. Indeed, $f(t)$ t implies t 1. In the latter case, $f(t)$ t 1, 1 # t < , is continuous, maps [1,) into itself and yet has no fixed point, for $f(t)$ t implies 1 0.

(ii) The foregoing theorem is possibly the simplest case of the famous fixed point

theorem of L.E.J. Brouwer, according to which every continuous mapping of the closed unit ball in the Euclidean space R^n into itself has a fixed point. The proofs for the cases n \$ 2 are not easy and are beyond the scope of the present text.

Theorem If Y is a connected set in a metric space (X, d) then any set Z such that $Y \leq Z \leq \overline{Y}$ is connected.

Proof. Suppose A and B are two nonempty open sets in ½Z such that ¼ A B Z and A B 1; as Y is dense in Z , Y A and Y B are nonempty open sets in Y and we have

Y ¼ ($Y \setminus A$) [($Y \setminus B$), ($Y \setminus A$) \ ($Y \setminus B$) ¼ $Y \setminus$ ($A \setminus B$) ¼ 1,

a contradiction. &

Remark .Since Y $\overline{Y}$ $\overline{Y}$, it follows that $\overline{Y}$ is connected if Y is connected in (X, d).

Example Since Y {(x, y): y sin (1=x), 0 < x # 1} R^2 is a continuous image of (0,1], it follows that $\overline{Y}$ Y {(0, y): 1 # y # 1} is connected. Observe that with the omission of any subset of {(0, y): 1 # y #1}, the resulting set is still connected.

Definition The union $C(x)$ of all connected subsets containing the point x is called the connected component of x in (X, d).

Clearly, $C(x)$ is a maximal connected subset of X.

Examples (i) Let Q be the set of rationals in (R, d). The component of each x Q is the set consisting of x alone. In other words, any subset A of Q containing more than one point is disconnected. Indeed, if x, y A, $x < y$, then (, a) A and (a,) A provide a disconnection of A, when $x < a < y$ and a is irrational.

(ii) Let ≤Y R^2 be the subspace consisting of the segments joining the origin to 2 the points {(1, 1=n): n N} together with the segment (1/2, 1]. The line joining (0,0) and (1, 1/n) is the image of ¼ the connected set [0,1] under the continuous map y x=n and, therefore, connected. If Z denotes the union of these lines, then Z is connected since the origin is common to all the line segments. Finally, Y is such that

$$Z \subset Y \subset \bar{Z},$$

where $\bar{Z}$ ¼ Z [(0, 1], and so Y is connected, by Theorem 4.1.13 and Theorem

(See Figure 4.3.) However, $Y \setminus \{(0; 0)\}$ is not connected. In $Y \setminus \{(0; 0)\}$, the component of each point is the segment containing it.

Theorem . Let (X,d) be a metric space. Then

(i) each connected subset of (X,d) is contained in exactly one component;

(ii) each nonempty connected subset of (X,d) that is both open and closed in (X, d) is a component of (X,d);

(iii) each component of (X,d) is closed.

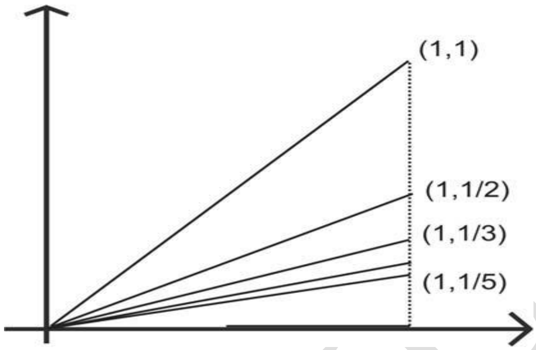

Figure 4.3

Proof. (i) Observe that if $C(x)$ 6¼ $C(x')$ 1, then $C(x)$ $C(x')$ is connected (see Theorem 4.1.16). This contradicts ¼ the maximality of $C(x)$ unless $C(x)$ $C(x')$. Thus, any two distinct connected components are disjoint. Now, let A be a connected subset of X containing x. By the maximality of $C(x)$, it follows that A $C(x)$. Since any two distinct components are disjoint, the statement

(i) follows.

(ii) Let A be a connected subset of (X, d) that is both open and closed in (X, d). Let x A, so that A $C(x)$. Then A is both open and closed in $(C(x), d$ $C(x))$ by Theorem 2.2.2 and consequently, A $C(x)$ (see Theorem 4.1.3(d)).

(iii) Since $C(x)$ is connected, so also is $C(x)$ (see Theorem 4.1.13); but the maximality of $C(x)$ implies $C(x) \leq C(x)$. Hence, $C(x)$ is closed.

Compact Spaces

One of the distinguishing properties of a bounded closed interval $[a, b]$ is that every sequence in it has a subsequence converging to a limit in the interval. This need not happen with an unbounded interval such as [0, 1) or a bounded nonclosed interval such as (0,1]; the former contains the sequence $\{n\}_{n}$ \$ 1, which has no

convergent subsequence, and the latter contains the sequence $\{1=n\}_{n \ \$ \ 1}$, which has no subse- quence converging to a limit belonging to the interval. In fact, it is true of any bounded closed subset of R that any sequence in it has a subsequence converging to a limit belonging to the subset. To see why, we first note that any sequence in a
bounded subset must, by the Bolzano-Weierstrass theorem (Proposition 0.4.2), have a convergent subsequence with limit in R; this limit must then be in the closed subset by the definition of a closed subset.

Definition A collection F of sets in X is said to have the finite intersection property if every finite subcollection of F has a nonempty intersection.

The following proposition now holds.

Proposition Let (X, d) be a metric space. The following statements are equivalent:

(i) (X, d) is compact;
(ii) every collection of closed sets in (X, d) with empty intersection has a finite subcollection with empty intersection;
(iii) every collection of closed sets in (X, d) with the finite intersection property has nonempty intersection.

Proof. That (i) is equivalent to (ii) has been proved in the paragraph preceding Example 5.1.2. The statements (ii) and (iii) are equivalent; in fact, each is the contrapositive of the other.

The reader will have noticed that the set considered in Example 5.1.2 (i) was not closedandtheoneconsideredin(ii) wasnotbounded. Thisisnotacoincidence. Infact, if a subset $Y$of a metric space (X, d) is compact, then it is both closed and bounded.

Definition The metric space (X, d) is said to be totally bounded if, for any e > 0, there exists a finite e-net for (X, d). A nonempty subset Y of X is said to be totally bounded if the subspace Y is totally bounded.

Proposition A totally bounded metric space is bounded.

Proof. Let (X, d) be totally bounded and suppose e > 0 has been given. Then there exists a finite e-net for X, say A. Since A is a finite set of points, $d(A)$ ¼ sup$\{d(y, z): y, z \quad A\} <$. Now, let x_1 and x_2 be any two points of X. There exist points y and z in A such that

$$d(x_1, y) < e \text{ and } d(x_2, z) < e:$$

It follows, using the triangle inequality, that

$$d(x_1, x_2) \# d(x_1, y) þ d(y, z) þ d(z, x_2)$$
$$\# d(A) þ 2e:$$
$$d(X) ¼ \sup\{d(x_1, x_2) : x_1, x_2\ 2\ X\} \# d(A) þ 2e$$

and, hence, X is bounded. &

Theorem Let Y be a subset of the metric space (X, d). Then Y is totally bounded if and only if every sequence in Y contains a Cauchy subsequence.

Proof. Suppose Y is totally bounded. Let $\{y_n\}_{n \$ 1}$ be a sequence in Y whose range may be assumed to be infinite. Choose a finite 1/2-net in Y. Then one of the balls of radius 1/2 with centre in the net contains infinitely many elements of the range of the sequence. We shall denote the subsequence formed by these elements by $\{y^{(1)}\}_{n \$ 1}$. Choose a finite 1/4-net in Y. Then one of the balls of radius 1/4 with centre in the finite 1/4-net contains infinitely many elements of the range of $\{y^{(1)}\}_{n \$ 1}$. We shall denote the subsequence formed as $\{y^{(2)}\}_{n \$ 1}$. Proceeding in this way, we obtain a sequence of sequences, each a subsequence of the preceding one, so that at the kth stage, the terms $\{y^{(k)}\}_{n \$ 1}$ lie in the ball of radius $1{=}2^k$ with centre in the $1{=}2^k$-net. Now $\{y^{(n)}\}_{n \$ 1}$ is a subsequence of $\{y_n\}_{n \$ 1}$. Let e > 0 be given. Choose n_0 so large that $1{=}2^{n_0 - 2} <$ e. Then, for $m > n > n_0$, we have

Conversely, suppose that every sequence in Y has a Cauchy subsequence. We shall show that Y is totally bounded. Let e be a positive real number and let y_1 2 Y . If Y \$S(y_1$, e) ¼ 1, we have found an e-net, namely, the set {y_1}. Otherwise choose y_2 2 Y \$S(y_1$, e). If Y \[$S(y_1$, e) [$S(y_2$, e)] ¼ 1, we have found an e-net, namely, the set {y_1, y_2}. It is enough to show that this process terminates after a finite number of steps. If it does not terminate, we shall obtain an infinite sequence $\{y_n\}_{n \$ 1}$ with the property that $d(y_n, y_m)$ \$ e, n m. Consequently, the sequence $\{y_n\}_{n \$ 1}$ would have no Cauchy subsequence, contrary to hypothesis.

Proposition Let (X, d) be a compact metric space.Then (X, d) is totally bounded.

Proof. For any given e > 0, the collection of all balls S(x, e) for x 2 X is an open cover of X. The compactness of X implies that this open cover contains a finite subcover. Hence, for e > 0, X is covered by a finite number of open balls of radius e, i.e., the centres of the balls in the finite subcover form a finite e-net for X. So, X is totally bounded.

Theorem Let (X, d) be a totally bounded and complete metric space. Then (X, d) is compact.

Proof. Suppose, if possible, that (X, d) is totally bounded and complete but is not compact. Then there exists an open covering $\{G_l\}_{l 2 L}$ of X that does not admit a finite subcovering.

Since (X, d) is totally bounded, it is bounded; hence, for some real number $r > 0$ and some x_0 X , we have X $S(x_0, r)$. Observe that X $S(x_0, r)$ implies X $S(x_0, r)$. Let e_n r=2^n.

We know that X, being totally bounded, can be covered by finitely many balls of radius e_1. By our hypothesis, at least one of these balls, say $S(x_1, e_1)$, cannot be covered by a finite number of sets G_l (for if each had a finite subcovering, the same would be true for X). Because $S(x_1, e_1)$ is itself totally bounded (any nonempty subset of a totally bounded set is totally bounded, as shown above), we can find an x_2 2 $S(x_1, e_1)$ such that $S(x_2, e_2)$ cannot be covered by a finite number of sets G_l.

In this way, a sequence $\{x_n\}_{n\ \$\ 1}$ may be defined with the property that for each n, $S(x_n$, e$_n)$ cannot be covered by a finite number of sets G_l (5.2) and $x_{nþ1}$ 2 $S(x_n$, e$_n)$.

We next show that the sequence $\{x_n\}_{n\ \$\ 1}$ is convergent. Since $x_{nþ1}$ 2 $S(x_n$, e$_n)$, it follows that $d(x_n, x_{nþ1})$ < e$_n$ and hence,

$$d(x_n, x_{nþp})\ \#\ d(x_n, x_{nþ1})\ þ\ d(x_{nþ1}, x_{nþ2})\ þ\ ...\ þ\ d(x_{nþp—1}, x_{nþp})$$
$$< e_n\ þ\ e_{nþ1}\ þ\ ...\ þ\ e_{nþp—1}$$
$$< \frac{r}{2^{n—1}}:$$

So $\{x_n\}_{n\ \$\ 1}$ is a Cauchy sequence in X, and since X is complete, it converges to y 2 X, say. Since y 2 X, there exists $\boldsymbol{l}_0$ 2 $\boldsymbol{L}$ such that y 2 G_{l0} . Because G_{l0} is open, it contains $S(y, d)$ for some d > 0. Choose n so large that $d(x_n, y) < d$=2 and e$_n$ < d=2. Then, for any x 2 X such that $d(x, x_n)$ < e$_n$, it follows that

$$d(x, y)\ \#\ d(x, x_n)\ þ\ d(x_n, y)$$
$$< \frac{1}{2} d\ þ\ \frac{1}{2} d\ ¼\ d,$$

so that $S(x_n$, e$_n)$ ≤ $S(y, d)$. Therefore, $S(x_n$, e$_n)$ admits a finite subcovering, namely by the set G_{l0} . Since this contradicts (5.2), the proof is complete.

POSSIBLE QUESTION

2 MARK QUESTION:

1. Define Homeomorphism.
2. Define fixed point.
3. Define Uniform Continuous
4. State Contraction Mapping Principle.
5. Define Continuous.

8 MARK QUESTION:

1. Let (X,d_x) and (Y,d_y) be two metric spaces and $f : X \to Y$ be uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in X, then so is $\{f(x_n)\}$ in Y.
2. Let (X,d_x) be a connected metric space and f : (X,dx) into (Y,dy) be a continuous mapping. Then the space f(X) with the metric induced from Y is connected.
3. Let (X,d) be a metric space and let $x \in X$ and $A \subseteq X$ be non empty. Then $x \in \bar{A}$ if and only if d(x,A) = 0.
4. State and Prove Contraction Mapping Principle.
5. Prove that a mapping f: X→Y is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y.
6. State and prove Intermediate value theorem.
7. If f and g are two uniform continuous mapping of (X,d_x) to (Y,d_y) and (Y,d_y) to (Z,d_z) ,respectively, then prove that $g_o f$ is uniform continuous mapping of (X,d_x) to (Z,d_z) .
8. Let (X,d) be a metric space. Then prove that the following statements are equivalent:
   i)(X,d) is Disconnected;
   ii) there exist two nonempty disjoint subsets A and B, both open in X, such that X= A U B;
   iii) there exist two nonempty disjoint subsets A and B, both closed in X, such that X= A U B;
   iv) there exists a proper subset of X that is both open and closed in X.
9. Prove that a mapping f: X→Y is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.
10. Let (X,d) be a metric space. Then prove that the following statements are equivalent:
    i)(X,d) is Disconnected;
    ii) there exist a continuous mapping of (X,d) onto the discrete two element space (X_o,d_o).

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UNIT II

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
If f from A into Y is continuous at every point of A then it is continuous on _______.	Y	X	A	N	A
Let f from X into Y be a function & A contained in X and B contained in Y then f(A)______.	Containing B	Contained in B	Containing A	Contained in A	Contained in B
Let f from X into Y and g from Y into Z be continuous then gof is_______.	Convergent	Divergent	Continuous	Discontinuous	Continuous
Let f from X into Y then f is ______ on X.	Continuous	Convergent	Divergent	Discontinuous	Continuous
Let f from X into Y then f(closure of A) contained in _______ for all subsets A of X.	f(X)	closure of f(X)	f(A)	closure of f(A)	closure of f(A)
dy (f(x1) , f(x2))< epsilon whenever dx(x1,x2) < delta is _________.	Continuous	Discontinuous	Uniform continuous	Not uniform continuous	Uniform continuous
d(x,A) - d(z,A)less then or equal to________.	d(A,A)	d(x.A)	d(z,a)	d(x,z)	d(x,z)
The function f from (0,1) into R defined by f(x) =1/x is ________.	Continuous	Discontinuous	Uniform continuous	Not uniform continuous	Not uniform continuous
Let (X,d) be a metric space then x belonges to closure of A if d(x,A)=___________.	1	infinity	0	finite	0
Let A and B be disjoint closed subsets of X then f(x)=_____ for all x belonges to A and 0<f(x)<1.	1	infinity	0	finite	0
Let A and B be disjoint closed subsets of X then f(x)=_____ for all x belonges to B and 0<f(x)<1	1	infinity	0	finite	1
Let A and B be disjoint closed subsets of X then there exists open sets G,H such that A contained in G, B contained in H and G interect with H=________	Empty	Non Empty	Finite	Infinite	Empty
If f and g are two uniform continuous mapping of (X,dx) into (Y,dy) and (Y,dy) into (Z,dz) then gof is _________ mapping of (X,dx) into (Z,dz) .	Continuous	Discontinuous	Uniform continuous	Not uniform continuous	Uniform continuous
A function f is homeomorphism if the mapping f and inverse of f are _________.	Continuous	Discontinuous	Uniform continuous	Not uniform continuous	Continuous
A continuous function f which is both one to one and onto is said to be ________.	Isomorphism	Homeomorphism	surjective	injective	Homeomorphism
The metric space X and Y are homeomorphism the Y is _______ image of X.	Homeomorphism	Non Homeomorphism	Homeomorphic	Non Homeomorphic	Homeomorphic
A sequence convergent to x in (X,d1) if it convergent to x in (x,d2) then d1 and d2 are_____ metric on X	Different	Equivalent	Not equivalent	sundet	Equivalent
The metrics d1 and d2 are equivalent if the identity maps id : (X,d1) into (X,d2) and id:(X,d2) into (X,d1) are both________.	Continuous	Discontinuous	Uniform continuous	Not uniform continuous	Continuous
The metrics d1 and d2 are equivalent if there exits a constant k such that d1(x,y) less then or equal to ___________.	1/k [d2(x,y)]	d(x,y)	k d2(x,y)	d2(y,x)	k d2(x,y)
A mapping T from X into X is a contraction mapping if there exists alpha , 0<aplpha <1 such that d(Tx,Ty) less then or equal to______.	d(x,y)	alpha d(x,y)	alpha (y,x)	alpha (x,y)	alpha d(x,y)
A point x is fixed point of the mapping T from X into X if Tx=__________.	t	x	T	X	x
A mapping T is a contraction of the complete metric space. Then T has a _______ fixed point.	finite	infinity	0	unique	unique
A metric space space (X,d) is disconnected if there exist two non empty subsets A and B of X such that A U B=______.	A	B	empty	X	X
A metric space space (X,d) is disconnected if there exist two non empty subsets A and B of X such that A interect with clouser of B=______.	A	B	empty	X	empty
A metric space space (X,d) is disconnected if there exist two non empty subsets A and B of X such that clouser of A interect with B=______.	A	B	empty	X	empty

A metric space space (X,d) is _________ if there exist two non empty subsets A and B of X such that A interect with clouser of B=empty	Continuous	Discontinuous	connected	disconnected	disconnected
If f from A into Y is _________ at every point of A then it is continuous on A	Continuous	Discontinuous	connected	disconnected	Continuous
Let f from X into Y be a function & A contained in _______ and B contained in_____ then f(A) contained in B	X,Y	Y,X	B,A	f(A),f(B)	X,Y
Let f from X into Y and g from Y into Z be______ then gof is continuous	Convergent	Divergent	Continuous	Discontinuous	Continuous
Let f from X into Y then f(closure of A) contained in closure of f(A) for all subsets ________ of X	X	B	A	Y	A
d(x,z) greater then or equal to	d(x,A) +d(z,A)	d(x,A) - d(z,A)	d(x,A) d(z,A)	d(z,A)-d(x,A)	d(x,A) - d(z,A)
The function f from (0,1) into R defined by f(x) =________ is not uniform	x	2/x	2x	1/x	1/x
Let (X,d) be a metric space then x belonges to __________ if d(x,A)=0	A	closure of A	interior of A	A'	closure of A
Let A and B be _________ subsets of X then f(x)= 0 for all x belonges to A and 0<f(x)<1	open	closed	disjoint open	disjoint closed	disjoint closed
Let A and B be _________ subsets of X then f(x)= 1 for all x belonges to B and 0<f(x)<1	open	closed	disjoint open	disjoint closed	disjoint closed
Let A and B be disjoint closed subsets of X then f(x)=0 for all x belonges to _____ and 0<f(x)<1	A	B	A union B	A intersection B	A
Let A and B be disjoint closed subsets of X then f(x)=1 for all x belonges to _____ and 0<f(x)<1	A	B	A union B	A intersection B	B
Let A and B be disjoint closed subsets of X then there exists ________ sets G,H such that A contained in G, B contained in H and G interect with H=empty	open	closed	disjoint open	disjoint closed	open
If f and g are two __________ mapping of (X,dx) into (Y,dy) and (Y,dy) into (Z,dz) then gof is uniform continuous mapping of (X,dx) into (Z,dz) .	Continuous	Discontinuous	Uniform continuous	Not uniform continuous	Uniform continuous
A function f is ________ if the mapping f and inverse of f are continuous.	Homeomorphism	Non Homeomorphism	Homeomorphic	Non Homeomorphic	homeomorphism
The metric space X and Y are _________ the Y is Homeomorphic image of X.	Homeomorphism	Non Homeomorphism	Homeomorphic	Non Homeomorphic	homeomorphism
A sequence ________ to x in (X,d1) if it _________ to x in (x,d2) then d1 and d2 are equaivalent metric on X	Convergent	Divergent	Continuous	Non continuous	Convergent
The metrics d1 and d2 are _________ if the identity maps id : (X,d1) into (X,d2) and id:(X,d2) into (X,d1) are both continuous.	Different	Equivalent	Not equivalent	Subset	Equivalent
A mapping T from X into X is a __________ mapping if there exists alpha , 0<aplpha <1 such that d(Tx,Ty) less then or equal to alpha d(x,y)	Onto	One to one	Bijective	Contraction	Contraction
A mapping T is a ________ of the complete metric space. Then T has a unique fixed point.	Onto	One to one	Bijective	Contraction	Contraction
A metric space space (X,d) is ____________ if there exist two non empty subsets A and B of X such that A U B=X.	Connected	Disconnected	Continuous	Discontinuous	Disconnected
A metric space space (X,d) is __________ if there exist two non empty subsets A and B of X such that clouser of A interect with B=empty	Connected	Disconnected	Continuous	Discontinuous	Disconnected
If f from A into Y is continuous at ______ point of A then it is continuous on A	One	Two	Limit	Every	Every
Let f from X into Y and g from Y into Z be continuous then gof is continuous from X into _____.	X	Y	Z	2X	Z

UNIT – I
SYLLABUS

Limits - Limits involving the point at infinity - continuity. Properties of complex numbers – regions in the complex plane - functions of complex variable - mappings. Derivatives, differentiation formulas - Cauchy-Riemann equations, sufficient conditions for differentiability.

PROPERTIES OF COMPLEX NUMBERS

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r (Fig. 1). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is true when θ is decreased by 2π. It is, therefore, evident from Fig. 10 that *two nonzero complex numbers*

$$z_1 = r_1e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2e^{i\theta_2}$$

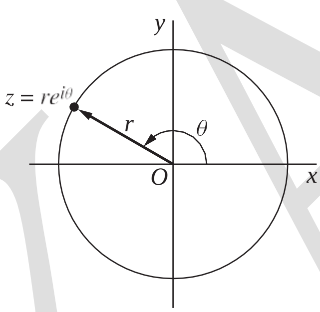


FIGURE 1

are equal if and only if $r_1 = r_2$ *and* $\theta_1 = \theta_2 + 2k\pi$, *where k is some integer ($k = 0, \pm 1, \pm 2,...$).*

This observation, together with the expression $z^n = r^n e^{in\theta}$ in Sec. 7 for integral powers of complex numbers $z = re^{i\theta}$, is useful in finding the nth roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$, where n has one of the values $n = 2, 3, \ldots$. The method starts with the fact that an nth root of z_0 is a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

According to the statement in italics just above, then,

$$r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi,$$

Consequently, the complex numbers are the nth roots of z_0. We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equal spaced every $2\pi/n$ radians, starting with argument θ_0/n. Evidently, then, all of the *distinct* roots are obtained when $k = 0, 1, 2, \ldots, n-1$, and no further roots arise with other values of k. We let c_k ($k = 0, 1, 2, \ldots, n-1$) denote these distinct roots.

The number is the length of each of the radius vectors representing the positive real number r_0, the symbol r_0 denotes the entire set of roots; and the symbol in expression (1) is reserved for one positive root. When the value of θ_0 that is used in expression (1) is the principal value of arg z_0 ($-\pi < \theta_0 \le \pi$),the number c_0 is referred to as the *principal root*. Thus when z_0 is a positive real number r_0, its principal root isObserve that if we write expression (1) for the roots of z_0. It follows from property of $e^{i\theta}$ that

$$\omega^k = \exp\left(i\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1)$$

and hence that

$$c_k = c_0\omega^k \quad (k = 0, 1, 2, \ldots, n-1).$$

The number c_0 here can, of course, be replaced by any particular nth root of z_0, since ω_n represents a counterclockwise rotation through $2\pi/n$ radians.

Finally, a convenient way to remember expression (1) is to write z_0 in its most general exponential form (compare with Example 2 in Sec. 6)

$$z_0 = r_0 e^{i(\theta_0 + 2k\pi)} \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

and to *formally* apply laws of fractional exponents involving real numbers, keeping in mind that there are precisely n roots.

The examples in the next section serve to illustrate this method for finding roots of complex numbers.

EXAMPLES

In each of the examples here, we start with expression (5), Sec. 9, and proceed in the manner described just after it.

EXAMPLE 1. Let us find all values of $(-8i)^{1/3}$, or the three cube roots of the number $-8i$. One need only write.

They lie at the vertices of an equilateral triangle, inscribed in the circle are equally spaced around that circle every $2\pi/3$ radians, starting with the equation.

Without any further calculations, it is then evident that $c_1 = 2i$; and, since c_2 is symmetric to c_0 with respect to the imaginary axis, we know that $c_2 = -\sqrt{3} - i$.

Note how it follows from expressions that these roots can be written

$$c_0,\ c_0\omega_3,\ c_0\omega_3^2 \quad \text{where } \omega_3 = \exp\left(i\frac{2\pi}{3}\right).$$

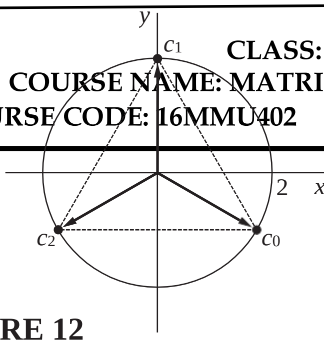


FIGURE 12

REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in the z plane, and their closeness to one another. Our basic tool is the concept of an ε *neighborhood*

(1) $$|z - z_0| < \varepsilon$$

of a given point z_0. It consists of all points z lying inside but not on a circle centered at z_0 and with a specified positive radius ε (Fig. 2). When the value of ε is understood or is immaterial in the discussion, the set (1) is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a *deleted neighborhood*, or punctured disk,

(2) $$0 < |z - z_0| < \varepsilon$$

consisting of all points z in an ε neighborhood of z_0 except for the point z_0 itself.

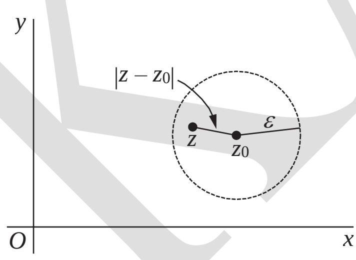


FIGURE 2

A point z_0 is said to be an *interior point* of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called an *exterior point* of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a *boundary point* of S. A boundary point is, therefore, a point all of whose neighborhoods contain at least one point in S and at least one point not in S. The totality of all boundary points is called the *boundary* of S. The circle $|z| = 1$, for instance, is the boundary of each of the sets

$$|z| < 1 \qquad \text{and} \qquad |z| \leq 1.$$

A set is *open* if it contains none of its boundary points. It is left as an exercise to show that a set is open if and only if each of its points is an interior point. A set is *closed* if it contains all of its boundary points, and the *closure* of a set S is the closed set consisting of all points in S together with the boundary of S. Note that the first of the sets (3) is open and that the second is its closure.

Some sets are, of course, neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Observe that the punctured disk $0 < |z| \leq 1$ is neither open nor closed. The set of all complex numbers is, on the other hand, both open and closed since it has no boundary points.

An open set S is *connected* if each pair of points z_1 and z_2 in it can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in S. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is, of course, open and it is also connected (see Fig. 3). A nonempty open set that is connected is called a *domain*. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*.

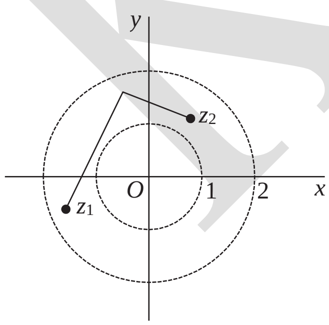


FIGURE 3

A set S is *bounded* if every point of S lies inside some circle $|z| = R$; otherwise, it is *unbounded*. Both of the sets (3) are bounded regions, and the half plane Re $z \geq 0$ is unbounded.

A point z_0 is said to be an *accumulation point* of a set S if each deleted neighborhood of z_0 contains at least one point of S. It follows that if a set S is closed, then it contains each of its accumulation points. For if an accumulation point z_0 were not in S, it would be a boundary point of S; but this contradicts the fact that a closed set contains all of its boundary points. It is left as an exercise to show that the converse is, in fact, true. Thus a set is closed if and only if it contains all of its accumulation points.

Evidently, a point z_0 is *not* an accumulation point of a set S whenever there exists some deleted neighborhood of z_0 that does not contain at least one point of S.

Note that the origin is the only accumulation point of the set $z_n = i/n$ $(n = 1, 2, \ldots)$.

FUNCTIONS OF A COMPLEX VARIABLE

Let S be a set of complex numbers. A *function* f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the *domain of definition* of f.*

It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Also, it is not always convenient to use notation that distinguishes between a given function and its values.

EXAMPLE 1. If f is defined on the set $z \neq 0$ by means of the equation $w = 1/z$, it may be referred to only as the function $w = 1/z$, or simply the function $1/z$.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy).$$

Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

(1) $$f(z) = u(x, y) + iv(x, y).$$

If the polar coordinates r and θ , instead of x and y, are used, then

$$u + iv = f(re^{i\theta})$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

(2) $$f(z) = u(r, \theta) + iv(r, \theta).$$

EXAMPLE 2. If $f(z) = z^2$, the

$$f(x + iy) = (x + iy) = x - y + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = 2xy.$$

When polar coordinates are used,

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Consequely,

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

If, in either of equations (1) and (2), the function v always has value zero, then the value of f is always real. That is, f is a *real-valued function* of a complex variable.

EXAMPLE 3. A real-valued function that is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0.$$

If n is zero or a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n /= 0$, the function

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$$

is a *polynomial* of degree n. Note that the sum here has a finite number of terms and that the domain of definition is the entire z

plane. Quotients *P (z)/Q(z)* of

polynomials are called *rational functions* and are defined at each point z where *Q(z)* ≠ 0. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

A generalization of the concept of function is a rule that assigns more than one value to a point *z* in the domain of definition. These *multiple-valued func- tions* occur in the theory of functions of a complex variable, just as they do in the case of a real variable. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function.

EXAMPLE 4. Let *z* denote any nonzero complex number. We know from Sec. 9 that $z^{1/2}$ has the two values

where $r = |z|$ and Θ $(-\pi < \Theta \leq \pi)$ is the *principal value* of arg *z*.

But, if we choose only the positive value of $\pm\sqrt{r}$ and write

(3) $$f(z) = \sqrt{r}\exp\frac{i\Theta}{2} \qquad (r > 0, -\pi < \Theta \leq \pi),$$

the (single-valued) function (3) is well defined on the set of nonzero numbers in the *z* plane. Since zero is the only square root of zero, we also write *f* (0) = 0. The function *f* is then well defined on the entire plane.

MAPPINGS

Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when *w* = *f* *(z)*, where *z* and *w* are complex, no such convenient graphical representation of the function *f* is available because each of the numbers *z* and *w* is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points *z* = *(x, y)* and *w(u, v)*. To do this, it is generally simpler to draw the z and w planes separately.

When a function f is thought of in this way, it is often referred to as a *mapping,* or transformation. The *image* of a point z in the domain of definition S is the point $w = f(z)$, and the set of images of all points in a set T that is contained in S is called the image of T. The image of the entire domain of definition S is called the *range* of f. The *inverse image* of a point w is the set of all points z in the domain of definition of f that have w as their image. The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when w is not in the range of f.

Terms such as *translation, rotation,* and *reflection* are used to convey domi- nant geometric characteristics of certain mappings. In such cases, it is sometimes convenient to consider the z and w planes to be the same. For example, the mapping

$$w = z + 1 = (x + 1) + iy,$$

where $z = x + iy$, can be thought of as a translation of each point z one unit to the right. Since $i = e^{i\pi/2}$, where $z = re^{i\theta}$, rotates the radius vector for each nonzero point z through a right angle about the origin in the counterclockwise direction; and the mapping

$$w = \bar{z} = x - iy$$

transforms each point $z = x + iy$ into its reflection in the real axis.

More information is usually exhibited by sketching images of curves and regions than by simply indicating images of individual points. In the following three examples, we illustrate this with the transformation $w = z^2$. We begin by finding the images of some *curves* in the z plane.

EXAMPLE 1. According to Example 2 in Sec. 12, the mapping $w = z^2$ can be thought of as the transformation

(1) $$u = x^2 - y^2,\ v = 2xy$$

from the xy plane into the uv plane. This form of the mapping is especially useful in finding the images of certain hyperbolas.

It is easy to show, for instance, that each branch of a hyperbola

(2) $$x^2 - y^2 = c_1 \quad (c_1 > 0)$$

is mapped in a one to one manner onto the vertical line $u = c_1$. We start by noting from the first of equations (1) that $u = c_1$ when (x, y) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the second of equations

(1) tells us that $v = 2y\sqrt{y^2 + c_1}$. Thus the image of the right-hand branch can be expressed parametrically as

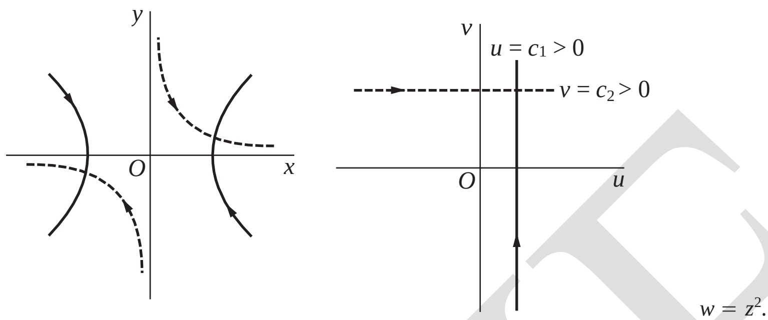
$$u = c_1, \quad v = 2y\sqrt{y^2 + c_1} \quad (-\infty < y < \infty);$$

and it is evident that the image of a point (x, y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction .

Likewise, since the pair of equations

$$u = c_1, \quad v = -2y\sqrt{y^2 + c_1} \quad (-\infty < y < \infty)$$

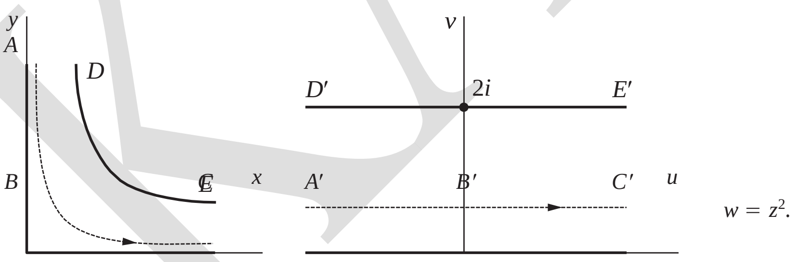
furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going *downward* along the entire left-hand branch is seen to move up the entire line $u = c_1$.



2
On the other hand, each branch of a hyperbola

$$2xy = c_2 \qquad (c_2 > 0)$$

EXAMPLE 2. The domain $x > 0, y > 0, xy < 1$ consists of all points lying on the upper branches of hyperbolas from the family $2xy\ \ c$, where $0 < c < 2$ (Fig. 18). We know from Example 1 that as a point travels downward along the entirety of such a branch, its image under the transformation $w\ z^2$ moves to the right along the entire line $v\ c$. Since, for all values of c between 0 and 2, these upper branches fill out the domain $x > 0, y > 0, xy < 1$, that domain is mapped onto the horizontal strip $0 < v < 2$.



In view of equations (1), the image of a point $(0, y)$ in the z plane is $(y^2, 0)$. Hence as $(0, y)$ travels downward to the origin along the y axis, its image moves to the right along the negative u axis and reaches the origin in the w plane. Then, since the image of a point $(x, 0)$ is $(x^2, 0)$, that image moves to the right from the origin along the u axis as $(x, 0)$ moves to the right from the origin along the x axis. The image of the upper branch of the hyperbola $xy = 1$ is, of course, the horizontal line $v = 2$. Evidently, then, the closed region $x \geq 0, y \geq 0, xy \leq 1$ is mapped onto the closed strip $0 \leq v \leq 2$, as indicated in Fig. 18.

Our last example here illustrates how polar coordinates can be useful in ana- lyzing certain mappings.

EXAMPLE 3. The mapping $w = z^2$ becomes

(4) $w = r^2 e^{i2\theta}$

when $z = re^{i\theta}$. Evidently, then, the image $w = \rho e^{i\varphi}$ of any nonzero point z is found by squaring the modulus $r = |z|$ and doubling the value θ of arg z that is used:

(5) $$\rho = r^2 \quad \text{and} \quad \varphi = 2\theta.$$

Observe that points $z = r_0 e^{i\theta}$ on a circle $r = r_0$ are transformed into points $w = r_0^2 e^{i2\theta}$ on the circle $\rho = r_0^2$. As a point on the first circle moves counterclockwise from the positive real axis to the positive imaginary axis, its image on the second circle moves counterclockwise from the positive real axis to the negative real axis (see Fig. 19). So, as all possible positive values of r_0 are chosen, the corresponding arcs in the z and w planes fill out the first quadrant and the upper half plane, respectively. The transformation $w = z^2$ is, then, a one to one map- ping of the first quadrant $r \geq 0, 0 \leq \theta \leq \pi/2$ in the z plane onto the upper half $\rho \geq 0, 0 \leq \varphi \leq \pi$ of the w plane, as indicated in Fig. 19. The point $z = 0$ is, of course, mapped onto the point $w = 0$.

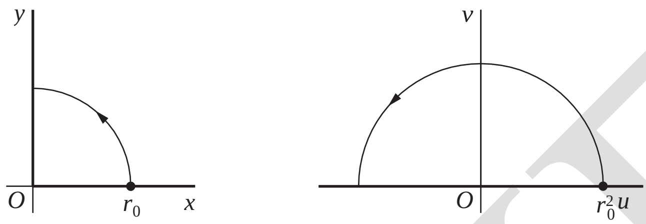


FIGURE 19

$w = z$.

The transformation $w = z^2$ also maps the upper half plane $r < 0$, $0 < \theta < \pi$ onto the entire w plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z plane are mapped onto the positive real axis in the w plane. When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $w = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \leq r_0$, $0 \leq \theta \leq 2\pi/n$ is mapped onto the disk $\rho \leq r_0^n$, but not in a one to one manner.

Other, but somewhat more involved, mappings by $w = z^2$ appear in Example 1, Sec. 97, and Exercises 1 through 4 of that section.

LIMITS

Let a function f be defined at all points z in some deleted neighborhood of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

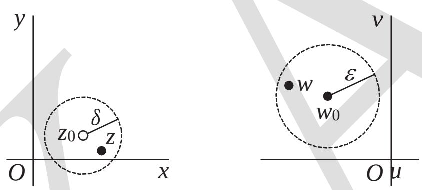
(1) $$\lim_{z \to z_0} f(z) = w_0,$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$(2) \qquad |f(z) - w_0| < \varepsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta.$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood. Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.



It is easy to show that *when a limit of a function $f(z)$ exists at a point z_0, it is unique*. To do this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for each positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta_0$$

$$|f(z) - w_1| < \varepsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta_1.$$

$$|w_1 - w_0| = |[f(z) - w_0] - [f(z) - w_1]| \le |f(z) - w_0| + |f(z) - w_1| < \varepsilon + \varepsilon = 2\varepsilon.$$

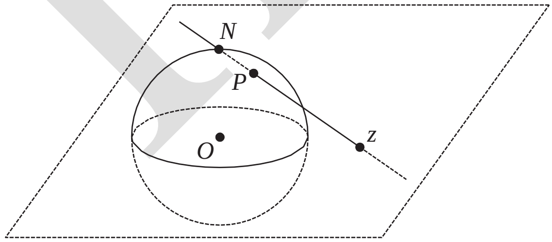
But $|w_1 - w_0|$ is a nonnegative constant, and ε can be chosen arbitrarily small. Hence

$$w_1 - w_0 = 0, \text{ or } \quad w_1 = w_0.$$

Definition (2) requires that f be defined at all points in some deleted neigh- borhood of z_0. Such a deleted neighborhood, of course, always exists when z_0 is an interior point of a region on which f is defined. We can extend the definition of limit to the case in which z_0 is a boundary point of the region by agreeing that the first of inequalities (2) need be satisfied by only those points z that lie in both the region *and* the deleted neighborhood.

LIMITS INVOLVING THE POINT AT INFINITY

It is sometimes convenient to include with the complex plane the *point at infinity,* denoted by ∞, and to use limits involving it. The complex plane together with this point is called the *extended* complex plane. To visualize the point at infinity, one can think of the complex plane as passing through the equator of a unit sphere centered at the origin To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is the point where the line through z and the north pole N intersects the sphere. In like manner, to each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one



point z in the plane. By letting the point N of the sphere correspond to the point at infinity, we obtain a one to one correspondence between the points of the sphere and the points of the extended complex plane. The sphere is known as the *Riemann sphere,* and the correspondence is called a *stereographic projection.*

Observe that the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and the point N deleted. Moreover, for each small positive number ε, those points in the complex plane exterior to the circle $|z| = 1/\varepsilon$ correspond to points on the sphere close to N. We thus call the set $|z| > 1/\varepsilon$ an *ε neighborhood,* or neighborhood, of ∞.

Let us agree that in referring to a point z, we mean a point in the *finite* plane. Hereafter, when the point at infinity is to be considered, it will be specifically mentioned.

A meaning is now readily given to the statement

$$\lim_{z\to z_0} f(z) = w_0$$

when either z_0 or w_0, or possibly each of these numbers, is replaced by the point ∞ at infinity. In the definition of limit in Sec. 15, we simply replace the appropriate neighborhoods of z_0 and w_0 by neighborhoods of ∞. The proof of the following theorem illustrates how this is done.

Theorem. If z_o and w_o are points in the z and w planes, respectively, then prove that

i) $\lim f(z)=\infty$ as $z\to z_o$ if and only if $\lim 1/f(z) = 0$ as $z\to z_o$ and

ii) $\lim f(z) = w_o$ as $z\to\infty$ if and only if $\lim f\left(\frac{1}{z}\right) = w_o$ as $z\to 0$

Moreover, $\lim f(z)=\infty$ as $z\to\infty$ if and only if $\lim \frac{1}{f(z)} = 0$ as $z\to 0$.

CONTINUITY

A function f is *continuous* at a point z_0 if all three of the following conditions are satisfied:

$\lim_{z\to z_0} f(z)$ exists, (1)

$f(z_0)$ exists, (2)

$\lim_{z\to z_0} f(z) = f(z_0)$. (3)

Observe that statement (3) actually contains statements (1) and (2), since the exis- tence of the quantity on each side of the equation there is needed. Statement (3) says, of course, that for each positive number ε, there is a positive number δ such that

(4) $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

A function of a complex variable is said to be continuous in a region R if it is continuous at each point in R.

If two functions are continuous at a point, their sum and product are also contin- uous at that point; their quotient is continuous at any such point if the denominator is not zero there. These observations are direct consequences of Theorem 2, Sec. 16. Note, too, that a polynomial is continuous in the entire plane because of limit (11) in Sec. 16.

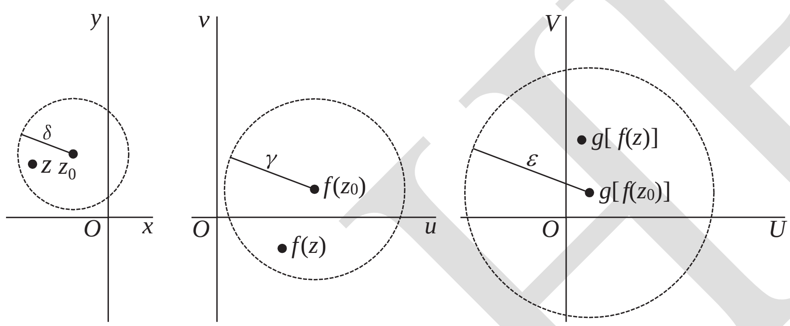
We turn now to two expected properties of continuous functions whose veri- fications are not so immediate. Our proofs depend on definition (4) of continuity, and we present the results as theorems.

Theorem . *A composition of continuous functions is itself continuous.*

A precise statement of this theorem is contained in the proof to follow. We let $w = f(z)$ be a function that is defined for all z in a neighborhood $|z - z_0| < \delta$ of a point z_0, and we let $W = g(w)$ be a function whose domain of definition contains the image of that neighborhood under f. The composition $W = g[f(z)]$ is, then, defined for all z in the neighborhood $|z - z_0| < \delta$. Suppose now that f is continuous at z_0 and that g is continuous at the point $f(z_0)$ in the w plane. In view of the continuity of g at $f(z_0)$, there is, for each positive number ε, a positive number γ such that

$$|g[f(z)] - g[f(z_0)]| < \varepsilon \quad \text{whenever} \quad |f(z) - f(z_0)| < \gamma.$$

But the continuity of f at z_0 ensures that the neighborhood $|z - z_0| < \delta$ can be made small enough that the second of these inequalities holds. The continuity of the composition $g[f(z)]$ is, therefore, established.



Theorem . *If a function $f(z)$ is continuous and nonzero at a point z_0 , then $f(z) \neq 0$ throughout some neighborhood of that point.*

Assuming that $f(z)$ is, in fact, continuous and nonzero at z_0, we can prove Theorem 2 by assigning the positive value $|f(z_0)|/2$ to the number ε in statement (4). This tells us that there is a positive number δ such that

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever } |z - z_0| < \delta.$$

So if there is a point z in the nighbourhood at which $f(z) = 0$, we have the contradiction

$$|f(z_0)| < \frac{|f(z_0)|}{2};$$

The continuity of a function

(5) $$f(z) = u(x, y) + iv(x, y)$$

is closely related to the continuity of its component functions *u(x, y)* and *v(x, y)*. We note, for instance, how it follows from Theorem 1 in Sec. 16 that *the function*

(5) *is continuous at a point* z_0 (x_0, y_0) *if and only if its component functions are continuous there.* Our proof of the next theorem illustrates the use of this state- ment. The theorem is extremely important and will be used often in later chapters, especially in applications. Before stating the theorem, we recall from Sec. 11 that a region *R* is *closed* if it contains all of its boundary points and that it is *bounded* if it lies inside some circle centered at the origin.

Theorem 3. *If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that*

(6) $$|f(z)| \le M \quad \text{for all points } z \text{ in } R,$$

where equality holds for at least one such z.

To prove this, we assume that the function *f* in equation (5) is continuous and note how it follows that the function

$$\sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is continuous throughout *R* and thus reaches a maximum value *M* somewhere in *R*.* Inequality (6) thus holds, and we say that *f is bounded on R*.

DERIVATIVES

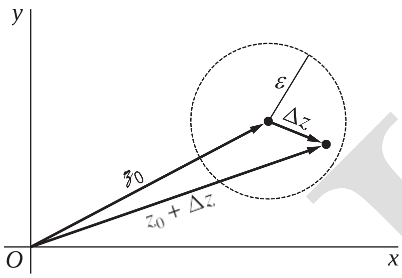
Let *f* be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The *derivative* of *f* at z_0 is the limit

(1) $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and the function f is said to be *differentiable* at z_0 when $f^j(z_0)$ exists.

By expressing the variable z in definition (1) in terms of the new complex variable

Because f is defined throughout a neighborhood of z_0, the number $f(z_0 + Oz)$ is always defined for $|Oz|$ sufficiently small.



When taking form (2) of the definition of derivative, we often drop the subscript on z_0 and introduce the number

$$Ow = f(z + Oz) - f(z),$$

which denotes the change in the value $w = f(z)$ of f corresponding to a change Oz in the point at which f is evaluated.

DIFFERENTIATION FORMULAS

The definition of derivative in Sec. 19 is identical in form to that of the derivative of a real-valued function of a real variable. In fact, the basic differentiation formulas given below can be derived from the definition in Sec. 19 by essentially the same steps as the ones used in calculus. In these formulas, the derivative of a function f at a point z is denoted by either depending on which notation is more convenient.

Let c be a complex constant, and let f be a function whose derivative exists at a point z. It is easy to show that

(1) —, $\frac{d}{dz}[cf(z)] \equiv cf^{j}(z)$.
dz

Also, if n is a positive —

This formula remains valid when n is a negative integer, provided that z
If the derivatives of two functions f and g exist at a point z, then

Let us derive formula (4). To do this, we write the following expression for the change in the product $w = f(z)g(z)$:

$$Ow = f(z + Oz)g(z + Oz) - f(z)g(z)$$
$$= f(z)[g(z + Oz) - g(z)] + [f(z + Oz) - f(z)]g(z + Oz).$$
$$\frac{Ow}{Oz} = f(z)\frac{g(z + Oz) - g(z)}{Oz} + \frac{f(z + Oz) - f(z)}{Oz}g(z + Oz);$$

and, letting Oz tend to zero, we arrive at the desired formula for the derivative of $f(z)g(z)$. Here we have used the fact that g is continuous at the point z.

CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions *u* and *v* of a function

$$(1) \qquad f(z) = u(x, y) + iv(x, y)$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also show how to express $f^{j}(z_0)$ in terms of those partial derivatives.

We start by writing

$$z_0 = x_0 + iy_0, \; Oz = Ox + iOy,$$

and

$$Ow = f(z_0 + Oz) - f(z_0)$$
$$= [u(x_0 + Ox, y_0 + Oy) - u(x_0, y_0)] + i[v(x_0 + Ox, y_0 + Oy) - v(x_0, y_0)].$$

Assuming that the derivative

Now it is important to keep in mind that expression (3) is valid as (Ox, Oy) tends to $(0, 0)$ in any manner that we may choose. In particular, we let (Ox, Oy) tend to $(0, 0)$ horizontally through the points $(Ox, 0)$, as indicated in Fig. 29 (Sec. 19). Inasmuch as $Oy = 0$, the quotient Ow/Oz becomes derivatives with respect to *x* of the functions *u* and *v*, respectively, at (x_0, y_0). Substitution of these limits into expression (3) tells us that

Equation not only give $f^{j}(z_0)$ in terms of partial derivatives of the component functions *u* and *v*, but they also provide necessary conditions for the existence of $f^{j}(z_0)$. To obtain those conditions, we need only equate the real parts and then the imaginary parts on the right-hand sides of equations (4) and (5) to see that the existence of $f^{j}(z_0)$ requires that

(6) $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Equations (6) are the *Cauchy–Riemann equations*, so named in honor of the French mathematician A. L. Cauchy (1789–1857), who discovered and used them, and in honor of the German mathematician G. F. B. Riemann (1826–1866), who made them fundamental in his development of the theory of functions of a complex variable.

We summarize the above results as follows.

Theorem. *Suppose that*

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy–Riemann equations

(7) $$u_x = v_y, \quad u_y = -v_x$$

there. Also, $f'(z_0)$ can be written

(8) $$f'(z_0) = u_x + iv_x,$$

where these partial derivatives are to be evaluated at (x_0, y_0).

EXAMPLE 1. we showed that the function

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

is differentiable everywhere and that $f'(z) = 2z$. To verify that the Cauchy–Riemann equations are satisfied everywhere, write

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Thus

$$u_x = 2x = v_y, \; u_y = -2y = -v_x.$$

Moreover, according to equation (8),

$$f'(z) = 2x + i2y = 2(x + iy) = 2z.$$

Since the Cauchy–Riemann equations are necessary conditions for the existence of the derivative of a function f at a point z_0, they can often be used to locate points at which f does *not* have a derivative.

SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

Satisfaction of the Cauchy–Riemann equations at a point z_0 (x_0, y_0) is not suffi- cient to ensure the existence of the derivative of a function $f(z)$ at that point. (See Exercise 6, Sec. 23.) But, with certain continuity conditions, we have the following useful theorem.

Theorem. *Let the function*

$$f(z) = u(x, y) + iv(x, y)$$

be defined throughout some ε *neighborhood of a point* $z_0 = x_0 + iy_0$, *and suppose that*

(a) *the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood;*

(b) *those partial derivatives are continuous at* (x_0, y_0) *and satisfy the Cauchy– Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0).

Then $f'(z_0)$ *exists, its value being*

$$f'(z_0) = u_x + iv_x$$

where the right-hand side is to be evaluated at (x_0, y_0).

To prove the theorem, we assume that conditions *(a)* and *(b)* in its hypothesis are satisfied and write $Oz = Ox + iOy$, where $0 < |Oz| < \varepsilon$, as well as

$$Ow = f(z_0 + Oz) - f(z_0).$$

The assumption that the first-order partial derivatives of u and v are continuous at the point (x_0, y_0) enables us to write*

(2) $$Ou = u_x(x_0, y_0)Ox + u_y(x_0, y_0)Oy + \varepsilon_1 Ox + \varepsilon_2 Oy$$

and

(3) $$Ov = v_x(x_0, y_0)Ox + v_y(x_0, y_0)Oy + \varepsilon_3 Ox + \varepsilon_4 Oy,$$

where ε_1, ε_2, ε_3, and ε_4 tend to zero as (Ox, Oy) approaches (0, 0) in the Oz plane. Substitution of expressions (2) and (3) into equation (1) now tells us that

(4) $$Ow = u_x(x_0, y_0)Ox + u_y(x_0, y_0)Oy + \varepsilon_1 Ox + \varepsilon_2 Oy + i[v_x(x_0, y_0)Ox + v_y(x_0, y_0)Oy + \varepsilon_3 Ox + \varepsilon_4 Oy].$$

Because the Cauchy–Riemann equations are assumed to be satisfied at (x_0, y_0), one can replace $u_y(x_0, y_0)$ by $-v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ by $u_x(x_0, y_0)$ in equation (4) and then divide through by the quantity $Oz = Ox + iOy$.

2 MARK QUESTIONS

1. Write the C-R Equation.
2. State sufficient condition for differentiability.
3. State the Cauchy Riemann Equation
4. Define Derivate.
5. Definition of Limit.

8 MARK QUESTIONS

1. State and Prove the sufficient conditions for differentiability.
2. Prove that a composition of continuous functions is itself continuous.
3. Let the functionf (z) = u(x, y) + iv(x, y)be defined throughout some ε eighborhood of a point $z_o = x_o + iy_o$, and supposethat
   i) the first-order partial derivatives of the functions u and v with respect to x andy exist everywhere in the neighborhood;
   ii) those partial derivatives are continuous at (x0, y0) and satisfy the C–R equations $u_x = v_y$, $u_y = -v_x$ at (x_o, y_o).
4. Prove that Cauchy Riemann equation.
5. If a function f (z) is continuous and nonzero at a point z_o , then prove thatf (z) ≠0 throughout some neighborhood of that point.
6. Let the functionf(z) = u(x, y) + iv(x, y)be defined throughout some ε neighborhood of apoint $z_o = x_o + iy_o$, and Then prove that f ‘(z_o) exists, its value being f ‘(z_o) = $u_x + iv_x$ where the right-hand side is to be evaluated at (x_o, y_o).
7. Prove that if a function f is continuous throughout a region R that is both closed and bounded, then there exists a nonnegative real number M such that |f (z)| ≤ M for all points z in R,where equality holds for at least one such z.
8. f f (z) = z, thenprove that $\frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$.
9. If z_o and w_o are points in the z and w planes, respectively, then prove that
   i)limf (z)=∞ as $z \to z_o$ if and only if lim 1/f (z)= 0 as $z \to z_o$ and
   ii)limf (z) = w_o as z→∞ if and only if lim $f\left(\frac{1}{z}\right) = w_o$ as z→0
   Moreover,limf (z)=∞ as z→∞ if and only if lim$\frac{1}{f(z)}$= 0 as z→0.
10. Suppose thatf (z) = u(x, y) + iv(x, y)and that f ‘(z) exists at a point $z_o = x_o + iy_o$. Then prove that the first-order partial derivativesof u and v must exist at (x0, y0), and they mustsatisfy the Cauchy–Riemann equations$u_x = v_y$, $u_y = -v_x$.

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UNIT III

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
For each positive integer epsilon, there exits a delta such that |f	Limit	Continuous	Convergent	Divergent	Limit
If zo and wo are points in the z and w planes, then lim f(z)= in	0	1	wo	Infinity	0
If zo and wo are points in the z and w planes, then lim f(z)= wo	0	1	wo	Infinity	wo
If zo and wo are points in the z and w planes, then lim f(z)= in	0	1	wo	Infinity	0
If zo and wo are points in the z and w planes, then lim f(z)= __	0	1	wo	Infinity	infinity
If zo and wo are points in the z and w planes, then lim f(z)= __	0	1	wo	Infinity	wo
If zo and wo are points in the z and w planes, then lim f(z)= __	0	1	wo	Infinity	infinity
lim (iz+3)/(z+1)=________ as z tends to -1.	0	1	z	Infinity	infinity
lim (z+1)/(iz+3)=_______ as z tends to -1.	0	1	z	Infinity	0
lim (2+iz)/(1+z)=________ as z tends to 0.	0	1	2	Infinity	2
lim (z+2z)/(2-z)=________ as z tends to 0.	0	1	2	Infinity	0
lim (z+2z)/(2-z)=________ as z tends to infinity.	0	1	z	Infinity	infinity
A function of a complex variable is _______ in a region R if it	Limit	Continuous	Convergent	Divergent	Continuous
A function of a complex variable is continuous in a region R if	Limit	Continuous	Convergent	Divergent	Continuous
A composition of continuous function is itself ____________.	Limit	Continuous	Convergent	Divergent	Continuous
If a function f(z) is continuous and non zero at a point zo, then	0	1	2	Infinity	0
If a function f(z) is ________ and non zero at a point zo, then f	Limit	Continuous	Convergent	Divergent	Continuous
If a function f is continuous throughout a region R that is Boun	Closed	Bounded	Continuous	Convergent	Closed
A set is _________ if it cointains none of its boundary points.	Open	Not open	Closed	Not cloed	Open
A set is open if each of its points is an________.	Arbitray	Interior	Closure	Limit	Interior
A set is closed if it cointains ______ of its boundary points.	One	Two	Finite	All	All
A set is ______ if every point of S lies inside some cirle |z| = R	Open	Bounded	Continuous	Closed	Bounded
A point zo is _________ point of a set S if each deleted nieghb	Arbitray	Interior	Closure	Accumulation	Accumulation
One to One function is called________	Injective	Surjective	Bijective	Into	Injective
Onto function is called______.	Injective	Surjective	Bijective	Into	Surjective

One to one and onto function is called ________.	Injective	Surjective	Bijective	Into	Bijective
The derivative of f at zo is denoted by__________.	f(z)	f '(z)	f "(z)	f '(zo)	f'(zo)
lim [f(z)-f(zo)]/[z-zo]=_______ as z tends to zo.	f(z)	f '(z)	f "(z)	f '(zo)	f'(zo)
lim [f(z)-f(zo)]/[z-zo]= f'(zo) as z tends to zo . Then the functio	Differentiable	Differentiable	Convergent at	Convergent at	Differentiable
d/dz[f(z)] is denoted by________	f(z)	f '(z)	f "(z)	f'(zo)	f'(z)
d/dz(c) = _____	0	1	2	Infinity	0
d/dz(z) = _____	0	1	2	Infinity	1
d/dz[cf(z)]	cf(z)	cf '(z)	f "(z)	f '(z)	cf '(z)
d/dz[f(z) + g(z)] = ___________.	f ' (z) + g ' (z)	f ' (z) - g ' (z)	f ' (z) g ' (z)	f ' (z) / g ' (z)	f ' (z) + g ' (z)
d/dz[f(z) g(z)] = ___________.	f(z)g ' (z) f '(z	f(z)g ' (z) - f '(	f(z)g ' (z) / f '(	f(z)g ' (z) + f '(	f(z)g ' (z) + f '(
d/dz[f(z) /g(z)] = ___________.	[f '(z)g(z)-f(z)	[f '(z)g(z)-f(z)	[f '(z)g(z)-f(z)	[f '(z)g(z)-f(z)	[f '(z)g(z)-f(z)g
d/dz[z^2]=________.	2z	3z	z	z^3	2z
u(x,y) + iv(x,y) is _________ in C-R equation	u	v	f(z)	f'(z)	f(z)
e^(x+iy)=_______	e^x	e^y	e^(x+y)	e^z	e^z
f(z)=e cos y + e sin y here u is _____	cos y	sin y	e cos y	e sin y	e cos y
f(z)=e cos y + e sin y here v is _____	cos y	sin y	e cos y	e sin y	e sin y
x+ iy is _________ in C-R equation	f(z)	z	x	y	z
u_x=v_y the another C-R equation is__________	u_y=v_x	u_y= -v_x	u_y+v_x	u_y-v_x	u_y= -v_x
u_y= -v_x the another C-R equation is__________	u_x=v_y	u_y= -v_x	u_y+v_x	u_y-v_x	u_x=v_y
u_x=v_y and u_y= -v_y is __________.	R equation	C equation	C-R equation	R-C equation	C-R equation
lim (3+iz)/(1+z)=________ as z tends to 0.	1	2	3	0	3
lim (5+iz)/(1+z)=________ as z tends to 0.	1	3	5	0	5

UNIT IV
SYLLABUS

Analytic functions, examples of analytic functions, exponential function, Logarithmic function, trigonometric function, derivatives of functions, definite integrals of functions. Contours: Contour integrals and its examples - upper bounds for moduli of contour integrals - Cauchy- Goursat theorem, Cauchy integral formula.

ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is *analytic at a point* z_0 if it has a derivative at each point in some neighborhood of z_0.* It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is *analytic in an open set* if it has a derivative everywhere in that set. If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

Note that the function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 3, Sec. 19.) An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a *singular point,* or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhere analytic.

A necessary, but by no means sufficient, condition for a function f to be analytic in a domain D is clearly the continuity of f

throughout *D*. Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in *D* are provided by the theorems in Secs. 22 and 23.

Other useful sufficient conditions are obtained from the differentiation formulas in Sec. 20. The derivatives of the sum and product of two functions exist wherever

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*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.

the functions themselves have derivatives. Thus, *if two functions are analytic in a domain D, their sum and their product are both analytic in D.* Similarly, *their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.* In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

From the chain rule for the derivative of a composite function, we find that *a composition of two analytic functions is analytic.* More precisely, suppose that a function $f(z)$ is analytic in a domain D and that the image (Sec. 13) of D under the transformation $w = f(z)$ is contained in the domain of definition of a function $g(w)$. Then the composition $g[f(z)]$ is analytic in D, with derivative

$$\frac{d}{dz} g[f(z)] = g'[f(z)]f'(z).$$

The following property of analytic functions is especially useful, in addition to being expected.

Theorem. *If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.*

We start the proof by writing $f(z) = u(x, y) + iv(x, y)$. Assuming that $f'(z) = 0$ in D, we note that $u_x + iv_x = 0$; and, in view of the Cauchy–Riemann equations, $v_y - iu_y = 0$. Consequently,

$$u_x = u_y = 0 \quad \text{and} \quad v_x = v_y = 0$$

at each point in D.

Next, we show that $u(x, y)$ is constant along any line segment L extending from a point P to a point P' and lying entirely in D. We let s denote the distance along L from the point P and let $\mathbf{U}$ denote the unit vector along L in the direction of increasing s (see Fig. 30). We know from calculus that the directional derivative du/ds can be written as the dot product

(1) $$\frac{du}{ds} = (\text{grad } u) \cdot \mathbf{U},$$

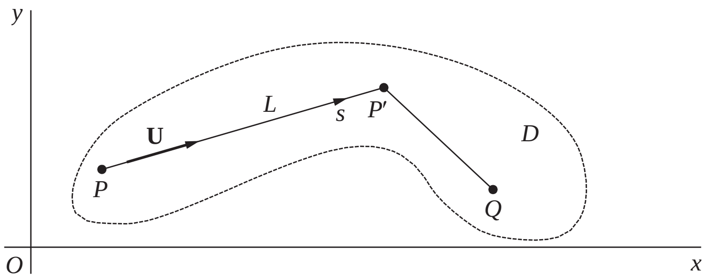

FIGURE 30

where grad u is the gradient vector

(2) $$\text{grad } u = u_x\mathbf{i} + u_y\mathbf{j}.$$

Because u_x and u_y are zero everywhere in D, grad u is evidently the zero vector at all points on L. Hence it follows from equation (1) that the derivative du/ds is zero along L; and this means that u is constant on L.

Finally, since there is always a finite number of such line segments, joined end to end, connecting any two points P and Q in D (Sec. 11), the values of u at P and Q must be the same. We may conclude, then, that there is a real constant a such that $u(x, y) = a$ throughout D. Similarly, $v(x, y) = b$; and we find that $f(z) = a + bi$ at each point in D.

EXAMPLES

As pointed out in Sec. 24, it is often possible to determine where a given function is analytic by simply recalling various differentiation formulas in Sec. 20.

EXAMPLE 1. The quotient

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

When a function is given in terms of its component functions

$u(x, y)$ and $v(x, y)$, its analyticity can be demonstrated by direct application of the Cauchy– Riemann equations.

EXAMPLE 2. If

$$f(z) = \cosh x \cos y + i \sinh x \sin y,$$

the component functions are

$$u(x, y) = \cosh x \cos y \quad \text{and} \quad v(x, y) = \sinh x \sin y.$$

Because

$$u_x = \sinh x \cos y = v_y \quad \text{and} \quad u_y = -\cosh x \sin y = -v_x$$

everywhere, it is clear from the theorem in Sec. 22 that f is entire.

Finally, we illustrate how the theorem in Sec. 24 can be used to obtain other properties of analytic functions.

EXAMPLE 3. Suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

and its conjugate

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

are *both* analytic in a given domain D. It is now easy to show that $f(z)$ must be constant throughout D.

To do this, we write $\overline{f(z)}$ as

$$\overline{f(z)} = U(x, y) + iV(x, y)$$

$$U(x, y) = u(x, y) \quad \text{and} \quad V(x, y) = -v(x, y).$$

Because of the analyticity of $f(z)$, the Cauchy–Riemann equations

(1) $$u_x = v_y, \quad u_y = -v_x$$

hold in D; and the analyticity of $\overline{f(z)}$ in D tells us that

(2) $$U_x = V_y, \quad U_y = -V_x.$$

In view of relations (1), equations (3) can also be written

(3) $$u_x = -v_y, \quad u_y = v_x.$$

By adding corresponding sides of the first of equations (2) and (4), we find that $u_x = 0$ in D. Similarly, subtraction involving corresponding sides of the second of equations (2) and (4) reveals that $v_x = 0$. According to expression (8) in Sec. 21, then,

$$f'(z) = u_x + iv_x = 0 + i0 = 0 ;$$

and it follows from the theorem in Sec. 24 that $f(z)$ is constant throughout D.

EXAMPLE 4. As in Example 3, we consider a function f that is analytic throughout a given domain D. Assuming further that the modulus $|f(z)|$ is constant throughout D, one can prove that $f(z)$ must be constant there too. This result is needed to obtain an important result later on in Chap. 4 (Sec. 54).

The proof is accomplished by writing

(4) $$|f(z)| = c \quad \text{for all } z \text{ in } D,$$

where c is a real constant. If $c = 0$, it follows that $f(z) = 0$ everywhere in D. If

$c \neq 0$, the fact that (see Sec. 5)

$$f(z)\overline{f(z)} = c^2$$

tells us that $f(z)$ is never zero in D. Hence

$$\overline{f(z)} = \frac{c^2}{f(z)} \text{ for all } z \text{ in } D,$$

and it follows from this that $\overline{f(z)}$ is analytic everywhere in D. The main result in Example 3 just above thus ensures that $f(z)$ is constant throughout D.

THE EXPONENTIAL FUNCTION

As anticipated earlier (Sec. 14), we define here the exponential function e^z by writing

(1) $$e^z = e^x e^{iy} \quad (z = x + iy),$$

where Euler's formula (see Sec. 6)

(2) $$e^{iy} = \cos y + i \sin y$$

is used and y is to be taken in radians. We see from this definition that e^z reduces to the usual exponential function in calculus when $y = 0$; and, following the con- vention used in calculus, we often write $\exp z$ for e^z.

Note that since the *positive* nth root $\sqrt[n]{e}$ of e is assigned to e^x when $x = 1/n$ $(n = 2, 3, \ldots)$, expression (1) tells us that the complex exponential function e^z is also $\sqrt[n]{e}$ when $z = 1/n$ $(n = 2, 3, \ldots)$. This is an exception to the convention (Sec. 9) that would ordinarily require us to interpret $e^{1/n}$ as the set of nth roots of e.

According to definition (1), $e^x e^{iy} = e^{x+iy}$; and, as already pointed out in Sec. 14, the definition is suggested by the additive property

$$e^{x_1} e^{x_2} = e^{x_1+x_2}$$

of e^x in calculus. That property's extension,

(3) $$e^{z_1} e^{z_2} = e^{z_1+z_2},$$

to complex analysis is easy to verify. To do this, we write

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Then

$$e^{z_1} e^{z_2} = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}) = (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2}).$$

But x_1 and x_2 are both real, and we know from Sec. 7 that

$$e^{iy_1} e^{iy_2} = e^{i(y_1+y_2)}.$$

and,

$$e^{z_1} e^{z_2} = e^{(x_1+x_2)} e^{i(y_1+y_2)};$$

since

$$(x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2,$$

the right-hand side of this last equation becomes $e^{z_1+z_2}$. Property (3) is now estab- lished.

Observe how property (3) enables us to write $e^{z_1-z_2} e^{z_2} = e^{z_1}$, or

(4) $$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

From this and the fact that $e^0 = 1$, it follows that $1/e^z = e^{-z}$.

There are a number of other important properties of e^z that are expected. According to Example 1 in

dz

everywhere in the z plane. Note that the differentiability of e^z for all z tells us that
e^z *is entire* (Sec. 24). It is also true that

(5) $$e^z \neq 0 \quad \text{for any complex number } z.$$

This is evident upon writing definition (1) in the form

$$e^z = \rho e^{i\varphi} \quad \text{where } \rho = e^x \text{ and } \varphi = y,$$

which tells us that

(7) $$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi \; (n = 0, \pm1, \pm2, \ldots).$$

Statement (6) then follows from the observation that $|e^z|$ is always positive.

Some properties of e^z are, however, *not* expected. For example, since

$$e^{z+2\pi i} = e^z e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1,$$

we find that e^z *is periodic*, with a pure imaginary period of $2\pi i$:

(8) $$e^{z+2\pi i} = e^z.$$

For another property of e^z that e^x does not have, we note that while e^x is always positive, e^z can be negative. We recall (Sec. 6), for instance, that $e^{i\pi}$ 1. In fact,

$$e^{i(2n+1)\pi} = e^{i2n\pi + i\pi} = e^{i2n\pi}e^{i\pi} = (1)(-1) = -1 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

There are, moreover, values of z such that e^z is any given nonzero complex number. This is shown in the next section, where the logarithmic function is developed, and is illustrated in the following example.

30. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z \tag{1}$$

for w, where z is any *nonzero* complex number. To do this, we note that when z and w are written $z = re^{i\copyright}$ $(-\pi < \copyright \le \pi)$ and $w = u + iv$, equation (1) becomes

$$e^u e^{iv} = re^{i\copyright}.$$

According to the statement in italics at the beginning of Sec. 9 about the equality of two complex numbers expressed in exponential form, this tells us that

$$e^u = r \quad \text{and} \quad v = \copyright + 2n\pi$$

where n is any integer. Since the equation $e^u = r$ is the same as u $\ln r$, it follows that equation (1) is satisfied if and only if w has one of the values

$$w = \ln r + i(\copyright + 2n\pi) \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus, if we write

$$\log z = \ln r + i(\copyright + 2n\pi) \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{2}$$

equation (1) tells us $e^{\log z} = z \quad (z \neq 0),$

that (3)

which serves to motivate expression (2) as the *definition* of the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$.

EXAMPLE 1. If $z = -1 - \sqrt{3}i$, then $r = 2$ and $\Theta = -2\pi/3$. Hence

$$\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

It should be emphasized that it is *not* true that the left-hand side of equation
(3) with the order of the exponential and logarithmic functions
reversed reduces to just z. More precisely, since expression (2) can
be written

$$\log z = \ln |z| + i \arg z$$

and since (Sec. 29)

$$|e^z| = e^x \text{ and } \arg(e^z) = y + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

when $z = x + iy$, we know that

$$\log(e^z) = \ln |e^z| + i \arg(e^z) = \ln(e^x) + i(y + 2n\pi) = (x + iy) + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

That

$$\log(e^z) = z + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

is, (4)

The *principal value* of $\log z$ is the value obtained from equation (2) when n

0 there and is denoted by Log z. Thus

(5) $$\text{Log}\, z = \ln r + i©.$$

Note that Log z is well defined and single-valued when z /= 0 and that

(6) $$\log z = \text{Log}\, z + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

It reduces to the usual logarithm in calculus when z is a positive real number $z = r$. To see this, one need only write $z = re^{i0}$, in which case equation (5) becomes Log $z = \ln r$. That is, Log $r = \ln r$.

EXAMPLE 2. From expression (2), we find that

$$\log 1 = \ln 1 + i(0 + 2n\pi) = 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

As anticipated, Log 1 = 0.

Our final example here reminds us that although we were unable to find loga- rithms of *negative* real numbers in calculus, we can now o so.

EXAMPLE 3. Observe that

$$\log(-1) = \ln 1 + i(\pi + 2n\pi) = (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and that $\text{Log}\,(-1) = \pi i$.

31. BRANCHES AND DERIVATIVES OF LOGARITHMS

If $z = re^{i\theta}$ is a nonzero complex number, the argument θ has any one of the values
θ = © + $2n\pi$ $(n = 0, \pm1, \pm2, \ldots)$, where © = Arg z. Hence the definition

$$\log z = \ln r + i(© + 2n\pi) \quad (n = 0, \pm1, \pm2, \ldots)$$

of the multiple-valued logarithmic function in Sec. 30 can be written

(1) $$\log z = \ln r + i\theta.$$

If we let α denote any real number and restrict the value of θ in expression (1)
so that $\alpha < \theta < \alpha + 2\pi$, the function

(2) $$\log z = \ln r + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi),$$

with

$$u(r, \theta) = \ln r \text{ and } v(r, \theta) = \theta,$$

components

(3)

is *single-valued* and continuous in the stated domain (Fig. 35). Note that if the function (2) were to be defined on the ray $\theta = \alpha$, it would not be continuous there. For if z is a point on that ray, there are points arbitrarily close to z at which the values of v are near α and also points such that the values of v are near $\alpha + 2\pi$.

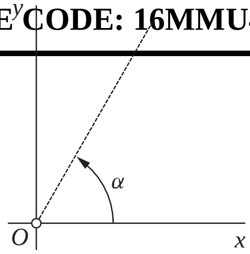

FIGURE 35

The function (2) is not only continuous but also analytic throughout the domain $r > 0$, $\alpha < \theta < \alpha + 2\pi$ since the first-order partial derivatives of u and v are continuous there and satisfy the polar form (Sec. 23)

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations. Furthermore, according to Sec. 23,

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}};$$

that is,

(4) $$\frac{d}{dz}\log z = \frac{1}{z} \quad (|z| > 0, \alpha < \arg z < \alpha + 2\pi).$$

In particular,

(5) $$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z} \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi).$$

A *branch* of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f. The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f. Observe that for each fixed α, the single-valued function (2) is a branch of the multiple-valued function (1). The function

(6) $$\text{Log}\, z = \ln r + i© \quad (r > 0, -\pi < © < \pi)$$

is called the *principal branch.*

A *branch cut* is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Points on the branch cut for F are singular points (Sec. 24) of F, and any point that is common to all branch cuts of f is called a *branch point.* The origin and the ray $\theta\ \alpha$ make up the branch cut for the branch (2) of the logarithmic function. The branch cut for the principal branch (6) consists of the origin and the ray © π. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

Special care must be taken in using branches of the logarithmic function, espe- cially since expected identities involving logarithms do not always carry over from calculus.

32. SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

(1) $$\log(z_1z_2) = \log z_1 + \log z_2.$$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

(2) $$\arg(z_1z_2) = \arg z_1 + \arg z_2$$

was in Sec. 8. That is, if values of two of the three logarithms are specified, then there is a value of the third such that equation (1) holds.

The verification of statement (1) can be based on statement (2) in the following way. Since $|z_1z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$$\ln|z_1z_2| = \ln|z_1| + \ln|z_2|.$$

So it follows from this and equation (2) that

$$\ln|z_1z_2| + i\arg(z_1z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2). \quad (3)$$

Finally, because of the way in which equations (1) and (2) are to be interpreted, equation (3) is the same as equation (1).

EXAMPLE. To illustrate statement (1), write $z_1 = z_2 = -1$ and recall from Examples 2 and 3 in Sec. 30 that

$$\log 1 = 2n\pi i \text{ and } \log(-1) = (2n+1)\pi i,$$

where $n = 0, \pm1, \pm2, \ldots$. Noting that $z_1z_2 = 1$ and using the values

$$\log(z_1z_2) = 0 \text{ and } \log z_1 = \pi i,$$

we find that equations (1) is satisfied when the value $\log z_2 = -\pi i$ is chosen.

If, on the other hand, the principal values

$$\text{Log } 1 = 0 \text{ and } \text{Log}(-1) = \pi i$$

are used,

$$\text{Log}(z_1z_2) = 0 \text{ and } \text{Log } z_1 + \log z_2 = 2\pi i$$

for the same numbers z_1 and z_2. Thus statement (1), which is sometimes true when *log* is replaced by *Log* (see Exercise 1), is not always true when principal values are used in all three of its terms.

Verification of the statement

(4) $$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2,$$

which is to be interpreted in the same way as statement (1), is left to the exercises. We include here two other properties of log z that will be of special interest in Sec. 33. If z is a nonzero complex number, then

(5) $$z^n = e^{n \log z} \quad (n = 0 \pm 1, \pm 2, \ldots)$$

for any value of log z that is taken. When $n = 1$, this reduces, of course, to relation (3), Sec. 30. Equation (5) is readily verified by writing $z = re^{i\theta}$ and noting that each side becomes $r^n e^{in\theta}$.

It is also true that when $z \neq 0$,

(6) $$z^{1/n} = \exp\left(\frac{1}{n} \log z\right) \quad (n = 1, 2, \ldots).$$

That is, the term on the right here has n distinct values, and those values are the nth roots of z. To prove this, we write $z = r \exp(i©)$, where © is the principal value of arg z. Then, in view of definition (2), Sec. 30, of log z,

$$\exp\left(\frac{1}{n} \log z\right) = \exp\left[\frac{1}{n} \ln r + \frac{i(© + 2k\pi)}{n}\right]$$

where $k = 0, \pm 1, \pm 2, \ldots$. Thus

(7) $$\exp\left(\frac{1}{n} \log z\right) = \sqrt[n]{r} \exp\left[i\left(\frac{©}{n} + \frac{2k\pi}{n}\right)\right] \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Because $\exp(i2k\pi/n)$ has distinct values only when $k = 0, 1, \ldots, n-1$, the right- hand side of equation (7) has only n values. That right-hand side is, in fact, an expression for the nth roots of z (Sec. 9), and so it can be written $z^{1/n}$. This establishes property (6), which is actually valid when n is a negative integer too

34. TRIGONOMETRIC FUNCTIONS

Euler's formula (Sec. 6) tells us that

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

for every real number *x*. Hence

$$e^{ix} - e^{-ix} = 2i \sin x \quad \text{and} \quad e^{ix} + e^{-ix} = 2 \cos x.$$

That is,

It is, therefore, natural to *define* the sine and cosine functions of a complex variable
z as follows:

(1) $$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

These functions are entire since they are linear combinations (Exercise 3, Sec. 25) of the entire functions e^{iz} and e^{-iz}. Knowing the derivatives

$$\frac{d}{dz} e^{iz} = ie^{iz} \quad \text{and} \quad \frac{d}{dz} e^{-iz} = -ie^{-iz}$$

of those exponential functions, we find from equations (1) that

(2) $$\frac{d}{dz} \sin z = \cos z \quad \text{and} \quad \frac{d}{dz} \cos z = -\sin z.$$

It is easy to see from definitions (1) that the sine and cosine functions remain odd and even, respectively:

(3) $$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z.$$

(4) $$e^{iz} = \cos z + i \sin z.$$

This is, of course, Euler's formula (Sec. 6) when z is real.

A variety of identities carry over from trigonometry. For instance (see Exercises 2 and 3),

(5) $$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

(5) $$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

From these, it follows readily that

(6) (7)

$$\sin 2z = 2\sin z\cos z, \quad \cos 2z = \cos^2 z - \sin^2 z,$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z,$$

and [Exercise 4(*a*)] (9)

$$\sin^2 z + \cos^2 z = 1.$$

The periodic character of sin *z* and cos *z* is also evident:

(10) $$\sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z,$$

(11) $$\cos(z + 2\pi) = \cos z, \quad \cos(z + \pi) = -\cos z.$$

When y is any real number, definitions (1) and the hyperbolic functions

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

from calculus can be used to write

(12) $$\sin(iy) = i \sinh y \quad \text{and} \quad \cos(iy) = \cosh y.$$

Also, the real and imaginary components of $\sin z$ and $\cos z$ can be displayed in terms of those hyperbolic functions:

(13) $$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

(14) $$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

where $z = x + iy$. To obtain expressions (13) and (14), we write

$$z_1 = x \quad \text{and} \quad z_2 = iy$$

in identities (5) and (6) and then refer to relations (12). Observe that once expres- sion (13) is obtained, relation (14) also follows from the fact (Sec. 21) that if the derivative of a function

$$f(z) = u(x, y) + iv(x, y)$$

exists at a point $z = (x, y)$, then

$$f'(z) = u_x(x, y) + iv_x(x, y).$$

Expressions (13) and (14) can be used (Exercise 7) to show that

(15) $$|\sin z|^2 = \sin^2 x + \sinh^2 y,$$

(16) $$|\cos z|^2 = \cos^2 x + \sinh^2 y.$$

Inasmuch as sinh y tends to infinity as y tends to infinity, it is clear from these two equations that sin z and cos z are *not bounded* on the complex plane, whereas the absolute values of sin x and cos x are less than or equal to unity for all values of x. (See the definition of a bounded function at the end of Sec. 18.)

A *zero* of a given function $f(z)$ is a number z_0 such that $f(z_0) = 0$. Since sin z becomes the usual sine function in calculus when z is real, we know that the real numbers $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ are all zeros of sin z. To show that *there are no other zeros*, we assume that sin $z = 0$ and note how it follows from equation (15) that

$$\sin^2 x + \sinh^2 y = 0.$$

This sum of two squares reveals that

$$\sin x = 0 \quad \text{and} \quad \sinh y = 0.$$

Evidently, then, $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ and $y = 0$; that is,

(17) $$\sin z = 0 \text{ if and only if } z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots).$$

Sinc
e

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

according to the second of identities (8),

(18) $\cos z = 0$ if and only if $z = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$

So, as was the case with sin *z*, the zeros of cos *z* are all real.

The other four trigonometric functions are defined in terms of the sine and cosine functions by the expected relations:

(19) $$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$

(20) $$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

Observe that the quotients tan *z* and sec *z* are analytic everywhere except at the singularities (Sec. 24)

$$z = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots),$$

which are the zeros of cos *z*. Likewise, cot *z* and csc *z* have singularities at the zeros of sin *z*, namely

$$z = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

By differentiating the right-hand sides of equations (19) and (20), we obtain the anticipated differentiation formulas

(21) $$\frac{d}{dz}\tan z = \sec^2 z, \quad \frac{d}{dz}\cot z = -\csc^2 z,$$

(22) $$\frac{d}{dz}\sec z = \sec z \tan z, \quad \frac{d}{dz}\csc z = -\csc z \cot z.$$

The periodicity of each of the trigonometric functions defined by equations (19) and
(20) follows readily from equations (10) and (11). For example,

(23) $$\tan(z + \pi) = \tan z.$$

Mapping properties of the transformation $w = \sin z$ are especially important
in the applications later on. A reader who wishes at this time to learn some of those properties is sufficiently prepared to read Sec. 96 (Chap. 8), where they are discussed.

36. DERIVATIVES OF FUNCTIONS *w* (*t*)

In order to introduce integrals of $f(z)$ in a fairly simple way, we need to first consider derivatives of complex-valued functions w of a *real* variable t . We write

(1) $$w(t) = u(t) + iv(t),$$

where the functions u and v are *real-valued* functions of t . The derivative

$$w^{j}(t), \text{ or } \frac{d}{dt} w(t),$$

of the function (1) at a point t is defined as

(2) $$w^{j}(t) = u^{j}(t) + iv^{j}(t),$$

provided each of the derivatives u^{j} and v^{j} exists at t .

From definition (2), it follows that for every complex constant $z_0 = x_0 + iy_0$,

$$\frac{d}{dt}[z_0 w(t)] = [(x_0 + iy_0)(u + iv)]^{j} = [(x_0u - y_0v) + i(y_0u + x_0v)]^{j}$$
$$= (x_0u - y_0v)^{j} + i(y_0u + x_0v)^{j} = (x_0u^{j} - y_0v^{j}) + i(y_0u^{j} + x_0v^{j})$$

But so (3)

and

$$(x_0u' - y_0v') + i(y_0u' + x_0v') = (x_0 + iy_0)(u' + iv') = z_0w'(t),$$

$$\frac{d}{dt}[z_0w(t)] = z_0w'(t)$$

Another expected rule that we shall often use is where $z_0 = x_0 + iy_0$. To verify this, we write

$$e^{z_0 t} = e^{x_0 t} e^{iy_0 t} = e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t$$

and refer to definition (2) to see that

$$\frac{d}{dt} e^{z_0 t} = (e^{x_0 t} \cos y_0 t)' + i(e^{x_0 t} \sin y_0 t)'.$$

Familiar rules from calculus and some simple algebra then lead us to the expression

$$\frac{d}{dt} e^{z_0 t} = (x_0 + iy_0)(e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t),$$

or

$$\frac{d}{dt} e^{z_0 t} = (x_0 + iy_0) e^{x_0 t} e^{iy_0 t}.$$

This is, of course, the same as equation (4).

Various other rules learned in calculus, such as the ones for differentiating sums and products, apply just as they do for real-valued functions of t. As was the case with property (3) and formula (4), verifications may be based on corresponding rules in calculus. It should be pointed out, however, that not every such rule carries over to functions of type (1). The following example illustrates this.

EXAMPLE. Suppose that $w(t)$ is continuous on an interval $a \le t \le b$; that is, its component functions $u(t)$ and $v(t)$ are continuous there. Even if $w'(t)$ exists when $a < t < b$, the mean value theorem for derivatives no longer applies. To be precise, it is not necessarily true that there is a number c in the interval $a < t < b$ such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}.$$

To see this, consider the function $w(t) = e^{it}$ on the interval $0 \le t \le 2\pi$. When that function is used, $|w'(t)| = |ie^{it}| = 1$; and this means that the derivative $w'(t)$ is never zero, while $w(2\pi) - w(0) = 0$.

39. CONTOURS

Integrals of complex-valued functions of a *complex* variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

A set of points $z = (x, y)$ in the complex plane is said to be an *arc* if

(1) $$x = x(t), \quad y = y(t) \qquad (a \le t \le b),$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. This definition establishes a continuous mapping of the interval $a \le t \le b$ into the xy, or z, plane; and the image points are ordered according to increasing values of t. It is convenient to describe the points of C by means of the equation

(2) $$z = z(t) \qquad (a \le t \le b),$$

wher $$z(t) = x(t) + iy(t).$$

The arc C is a *simple arc*, or a Jordan arc,* if it does not cross itself ; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a *simple closed curve*, or a Jordan curve. Such a curve is *positively oriented* when it is in the counterclockwise direction.

The geometric nature of a particular arc often suggests different notation for the parameter t in equation (2). This is, in fact, the case in the following examples.

EXAMPLE 1. The polygonal line (Sec. 11) defined by means of the equations

(4) $$z = \begin{cases} x + ix & \text{when } 0 \le x \le 1, \\ x + i & \text{when } 1 \le x \le 2 \end{cases}$$

and consisting of a line segment from 0 to $1 + i$ followed by one from $1 + i$ to $2 + i$ (Fig. 36) is a simple arc.

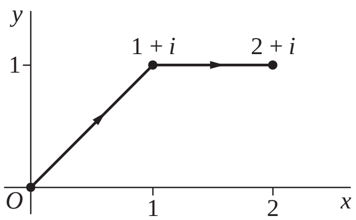

FIGURE 36

*Named for C. Jordan (1838–1922), pronounced *jor-don*.

EXAMPLE 2. The unit circle

(5) $$z = e^{i\theta} \quad (0 \le \theta \le 2\pi)$$

about the origin is a simple closed curve, oriented in the counterclockwise direction. So is the circle

(6) $$z = z_0 + Re^{i\theta} \quad (0 \le \theta \le 2\pi),$$

centered at the point z_0 and with radius R (see

Sec. 6). The same set of points can make

up different arcs.

EXAMPLE 3. The arc

(7) $$z = e^{-i\theta} \quad (0 \le \theta \le 2\pi)$$

is not the same as the arc described by equation (5). The set of points is the same, but now the circle is traversed in the *clockwise* direction.

EXAMPLE 4. The points on the arc

(8) $$z = e^{i2\theta} \quad (0 \le \theta \le 2\pi)$$

are the same as those making up the arcs (5) and (7). The arc here differs, however, from each of those arcs since the circle is traversed *twice* in the counterclockwise direction.

The parametric representation used for any given arc C is, of course, not unique. It is, in fact, possible to change the interval over which the parameter ranges to any other interval. To be specific, suppose that

(9) $$t = \varphi(\tau) \quad (\alpha \le \tau \le \beta),$$

where φ is a real-valued function mapping an interval $\alpha \le \tau \le \beta$ onto the interval $a \le t \le b$ in representation (2). (See Fig. 37.) We assume that φ is continuous with

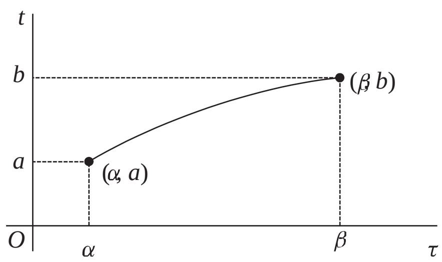

FIGURE 37
$t = \varphi(\tau)$

a continuous derivative. We also assume that $\varphi'(\tau) > 0$ for each τ; this ensures that t increases with τ. Representation (2) is then transformed by equation (9) into

(10) $$z = Z(\tau) \quad (\alpha \le \tau \le \beta),$$

where $$Z(\tau) = z[\varphi(\tau)].$$

(11)

This is illustrated in Exercise 3.

Suppose now that the components $x'(t)$ and $y'(t)$ of the derivative (Sec. 37)

(12) $$z'(t) = x'(t) + iy'(t)$$

of the function (3), used to represent C, are continuous on the entire interval $a \le t \le b$.

The arc is then called a *differentiable arc*, and the real-valued function

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over the interval $a \le t \le b$. In fact, according to the definition of arc length in calculus, the length of C is the number

(13) $$L = \int_a^b |z'(t)|\, dt.$$

The value of L is invariant under certain changes in the representation for C that is used, as one would expect. More precisely, with the change of variable indicated in equation (9), expression (13) takes the form [see Exercise 1*(b)*]

$$L = \int_\alpha^\beta |z'[\varphi(\tau)]|\varphi'(\tau)\, d\tau.$$

So, if representation (10) is used for C, the derivative (Exercise 4)

(14) $$Z'(\tau) = z'[\varphi(\tau)]\varphi'(\tau)$$

enables us to write expression (13) as

$$L = \int_\alpha^\beta |Z'(\tau)|\, d\tau.$$

Thus the same length of C would be obtained if representation (10) were to be used.

If equation (2) represents a differentiable arc and if $z'(t) \neq 0$ anywhere in the interval $a < t < b$, then the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all t in that open interval, with angle of inclination $\arg z'(t)$. Also, when **T** turns, it does so continuously as the parameter t varies over the entire interval

$a < t < b$. This expression for **T** is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be *smooth*. In referring to a smooth arc $z\ z(t)\ (a\ t\ b)$, then, we agree that the derivative $z^j(t)$ is continuous on the closed interval $a\ t\ b$ and nonzero throughout the open interval $a < t < b$.

A *contour*, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z^j(t)$ is piecewise continuous. The polygonal line
(4) is, for example, a contour. When only the initial and final values of $z(t)$ are
the same, a contour C is called a *simple closed contour*. Examples are the circles
(5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. It will be convenient to accept
this statement, known as the *Jordan curve theorem*, as geometrically evident; the proof is not easy.*

40. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z\ z_1$ to a point $z\ z_2$ in the complex plane. It is, therefore, a line integral ; and its value depends, in general, on the contour C as well as on the function f . It is written

$$\int_C f(z)\, dz \quad \text{or} \quad \int_{z1}^{z2} f(z)\, dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral may be defined directly as the limit of a sum, we choose to define it in terms of a definite integral of the type introduced in Sec. 38.

Suppose that the equation

$$(1) \qquad z = z(t) \quad (a \le t \le b)$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 38) on the interval $a \le t \le b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or *contour integral*, of f along C in terms of the parameter t :

$$(2) \qquad \int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt.$$

Note that since C is a contour, $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral (2) is ensured.

The value of a contour integral is invariant under a change in the representation of its contour when the change is of the type (11), Sec. 39. This can be seen by following the same general procedure that was used in Sec. 39 to show the invariance of arc length.

It follows immediately from definition (2) and properties of integrals of complex-valued functions *w(t)* mentioned in Sec. 38 that

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz, \tag{3}$$

for any complex constant z_0, and

$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz. \tag{4}$$

C C C

Associated with the contour C used in integral (2) is the contour $-C$, consisting of the same set of points but with the order reversed so that the new contour extends from the point z_2 to the point z_1 (Fig. 39). The contour $-C$ has parametric representation

$$z = z(-t) \qquad (-b \leq t \leq -a).$$

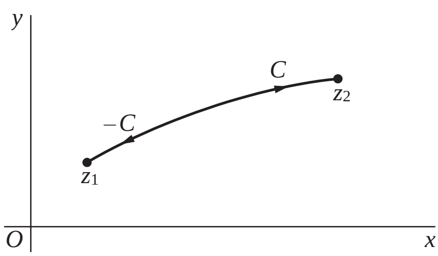

FIGURE 39

43. UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

We turn now to an inequality involving contour integrals that is extremely important in various applications. We present the result as a theorem but preface it with a needed lemma involving functions $w(t)$ of the type encountered in Secs. 37 and 38.

Lemma. *If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \leq t \leq b$, then*

This inequality clearly holds when the value of the integral on the left is zero. Thus, in the verification we may assume that its value is a *nonzero* complex number and write

Now the left-hand side of this equation is a real number, and so the right-hand side is too. Thus, using the fact that the real part of a real number is the number itself, we find that

$$r_0 = \text{Re} \int_a^b e^{-i\theta_0} w(t)\, dt,$$

or

$$\text{Re}[e^{-i\theta_0} w(t)] \le |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}|\,|w(t)| = |w(t)|,$$

and it follows from equation (3) that

Because r_0 is, in fact, the left-hand side of inequality (1), the verification of the lemma is complete.

46. CAUCHY–GOURSAT THEOREM

In Sec. 44, we saw that when a continuous function f has an antiderivative in a domain D, the integral of $f(z)$ around any given closed contour C lying entirely in D has value zero. In this section, we present a theorem giving other conditions on a function f which ensure that the value of the integral of $f(z)$ around a *simple* closed contour (Sec. 39) is zero. The theorem is central to the theory of functions of a complex variable; and some modifications of it, involving certain special types of domains, will be given in Secs. 48 and 49.

We let C denote a simple closed contour $z = z(t)$ $(a \le t \le b)$, described in the *positive sense* (counterclockwise), and we assume that f is analytic at each point interior to and on C. According to Sec. 40,

$$(1) \qquad \int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt;$$

and if

$$f(z) = u(x, y) + iv(x, y) \quad \text{and} \quad z(t) = x(t) + iy(t),$$

the integrand $f[z(t)]z'(t)$ in expression (1) is the product of the functions

$$u[x(t), y(t)] + iv[x(t), y(t)], \quad x'(t) + iy'(t)$$

of the real variable t. Thus

$$(2) \qquad \int_C f(z)\,dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt.$$

In terms of line integrals of real-valued functions of two real variables, then,

$$(3) \qquad \int_C f(z)\,dz = \int_C \qquad \int_C$$

u $v\,dx + u\,dy.$

dx

$-v$

dy

$+i$
$\int$

C C C

Observe that expression (3) can be obtained formally by replacing $f(z)$ and dz on the left with the binomials

$$u + iv \quad \text{and} \quad dx + i\,dy,$$

respectively, and expanding their product. Expression (3) is, of course, also valid when C is any contour, not necessarily a simple closed one, and when $f[z(t)]$ is only piecewise continuous on it.

We next recall a result from calculus that enables us to express the line inte- grals on the right in equation (3) as double integrals. Suppose that two real-valued functions $P(x, y)$ and $Q(x, y)$, together with their first-order partial derivatives, are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C. According to *Green's theorem*,

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.$$

Now f is continuous in R, since it is analytic there. Hence the functions u and v are also continuous in R. Likewise, if the derivative f' of f is continuous in R, so are the first-order partial derivatives of u and v. Green's theorem then enables us to rewrite equation (3) as

$$\int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dA + i \iint_R (u_x - v_y)\,dA. \tag{3}$$

But, in view of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

the integrands of these two double integrals are zero throughout R. So *when f is analytic in R and f' is continuous there,*

$$\int_C f(z)\,dz = 0. \tag{5}$$

This result was obtained by Cauchy in the early part of the nineteenth century.

Note that once it has been established that the value of this integral is zero, the orientation of C is immaterial. That is, statement (5) is also true if C is taken in the clockwise direction, since then

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz = 0.$$

EXAMPLE. If C is any simple closed contour, in either direction, then

$$\int_C \exp(z^3)\,dz = 0.$$

This is because the composite function $f(z) = \exp(z^3)$ is analytic everywhere and its derivative $f'(z) = 3z^2 \exp(z^3)$ is continuous everywhere.

Goursat* was the first to prove that *the condition of continuity on* f^{j} *can be omitted*. Its removal is important and will allow us to show, for example, that the derivative f^{j} of an analytic function f is analytic without having to assume the continuity of f^{j}, which follows as a consequence. We now state the revised form of Cauchy's result, known as the *Cauchy–Goursat theorem*.

Theorem. *If a function* f *is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\, dz = 0.$$

---

*E. Goursat (1858–1936), pronounced *gour-sah*j.

The proof is presented in the next section, where, to be specific, we assume that *C* is positively oriented. The reader who wishes to accept this theorem without proof may pass directly to Sec. 48.

47. PROOF OF THE THEOREM

We preface the proof of the Cauchy–Goursat theorem with a lemma. We start by forming subsets of the region *R* which consists of the points on a positively oriented simple closed contour *C* together with the points interior to *C*. To do this, we draw equally spaced lines parallel to the real and imaginary axes such that the distance between adjacent vertical lines is the same as that between adjacent horizontal lines. We thus form a finite number of closed square subregions, where each point of *R* lies in at least one such subregion and each subregion contains points of *R*. We refer to these square subregions simply as *squares*, always keeping in mind that by a square we mean a boundary together with the points interior to it. If a particular square contains points that are not in *R*, we remove those points and call what remains a *partial square*. We thus *cover* the region *R* with a finite number of squares and partial squares (Fig. 55), and our proof of the following lemma starts with this covering.

Lemma. *Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any positive number ε, the region R can be covered with a finite number of squares and partial squares, indexed by $j = 1, 2, \ldots, n$, such that in each one there is a fixed point z_j for which the inequality*

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \tag{1}$$

is satisfied by all points other than z_j in that square or partial square.

is satisfied by all points other than z_j in that square or partial square.

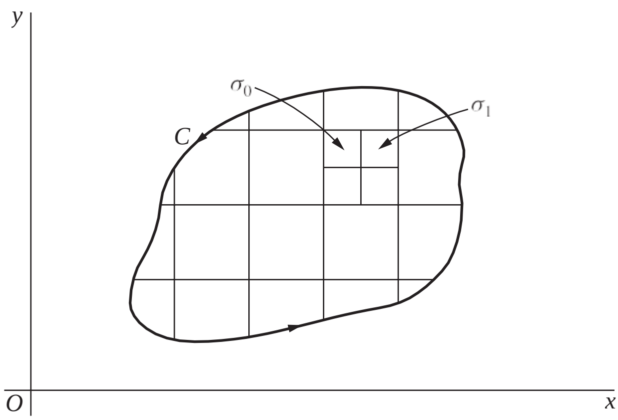

FIGURE 55

To start the proof, we consider the possibility that in the covering constructed just prior to the statement of the lemma, there is some square or partial square in which no point z_j exists such that inequality (1) holds for all other points z in it. If that subregion is a square, we construct four smaller squares by drawing line segments joining the midpoints of its opposite sides (Fig. 55). If the subregion is a partial square, we treat the whole square in the same manner and then let the portions that lie outside of R be discarded. If in any one of these smaller subre- gions, no point z_j exists such that inequality (1) holds for all other points z in it, we construct still smaller squares and partial squares, etc. When this is done to each of the original subregions that requires it, we find that *after a finite number of steps*, the region R can be covered with a finite number of squares and partial squares such that the lemma is true.

To verify this, we suppose that the needed points z_j do *not* exist after subdivid- ing one of the original subregions a finite number of times and reach a contradiction. We let σ_0 denote that subregion if it is a square; if it is a partial square, we let σ_0 denote the entire square of which it is a part. After we subdivide σ_0, at least one of the four smaller squares, denoted by σ_1, must contain points of R but no appropriate point z_j. We then subdivide σ_1 and continue in this manner. It may be that after a square σ_{k-1} $(k = 1, 2, \ldots)$ has been subdivided, more than one of the four smaller squares constructed from it can be chosen. To make a specific choice, we take σ_k to be the one lowest and then furthest to the left.

In view of the manner in which the nested infinite sequence

$$\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k, \ldots \tag{2}$$

of squares is constructed, it is easily shown (Exercise 9, Sec. 49) that there is a point z_0 common to each σ_k; also, each of these squares contains points of R other than possibly z_0. Recall how the sizes of the squares in the sequence are decreasing, and note that any δ

neighborhood $|z - z_0| < \delta$ of z_0 contains such squares when their diagonals have lengths less than δ. Every δ neighborhood $|z - z_0| < \delta$ therefore contains points of R distinct from z_0, and this means that z_0 is an accumulation point of R. Since the region R is a closed set, it follows that z_0 is a point in R. (See Sec. 11.) Now the function f is analytic throughout R and, in particular, at z_0. Consequently, $f'(z_0)$ exists, According to the definition of derivative (Sec. 19), there is, for each positive number ε, a δ neighborhood $|z - z_0| < \delta$ such that the inequality

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

is satisfied by all points distinct from z_0 in that neighborhood. But the neighborhood $|z - z_0| < \delta$ contains a square σ_K when the integer K is large enough that the length of a diagonal of that square is less than δ (Fig. 56). Consequently, z_0 serves as the point z_j in inequality (1) for the subregion consisting of the square σ_K or a part of σ_K. Contrary to the way in which the sequence (2) was formed, then, it is not necessary to subdivide σ_K. We thus arrive at a contradiction, and the proof of the lemma is complete.

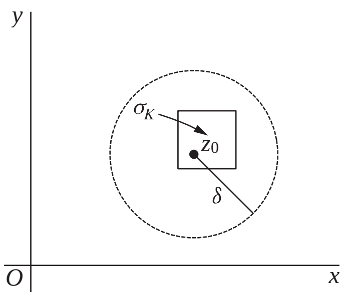

FIGURE 56

Continuing with a function f which is analytic throughout a region R consisting of a positively oriented simple closed contour C and points interior to it, we are now ready to prove the Cauchy–Goursat theorem, namely that

(3) $$\int_C f(z)\,dz = 0.$$

Given an arbitrary positive number ε, we consider the covering of R in the statement of the lemma. We then define on the j th square or partial square a function $\delta_j(z)$ whose values are $\delta_j(z_j) = 0$, where z_j is the fixed point in inequality (1), and

(4) $$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \quad \text{when } z \neq z_j.$$

According to inequality (1),

(5) $$|\delta_j(z)| < \varepsilon$$

at all points z in the subregion on which $\delta_j(z)$ is defined. Also, the function $\delta_j(z)$
is continuous throughout the subregion since $f(z)$ is continuous there and

50. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

(1) $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}.$$

Formula (1) is called the *Cauchy integral formula*. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

When the Cauchy integral formula is written as

(2) $$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0),$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, formula (2) tells us that

$$\int_C \frac{z/(9 - z^2)}{z - (-i)}\,dz = 2\pi i$$

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 66). Since the quotient $f(z)/(z - z_0)$ is analytic between and on the contours C_ρ and C, it follows from the principle of deformation of paths (Sec. 49) that

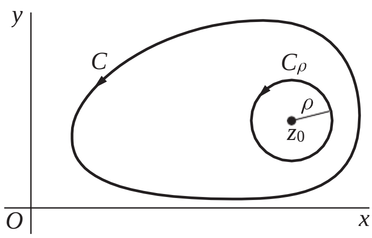

FIGURE 66

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, however small, there is a positive number δ such that

(5) $$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta.$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z - z_0| = \rho < \delta$ when z is on C_ρ, it follows that the *first* of inequalities (5) holds when z is such a point; and the theorem in Sec. 43, giving upper bounds for the moduli of contour integrals.

POSSIBLE QUESTION

2 MARK QUESTION

1. State the Cauchy Integral Formula.
2. Write the equation of logarithmic function.
3. Write the equation of Cauchy–Goursat theorem
4. Write the Cauchy integral formula.
5. State Cauchy–Goursat theorem.

8 MARK QUESTION

1. Give any two examples of analytic function.
2. Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_o is any point interior to C, then prove that f (z_o) = $1/2\pi i\left(\int \frac{f(z)dz}{z-z_o}\right)$.
3. Explain about Exponential Function.
4. Give an example of Contour Integrals.
5. Describe the logarithmic function.
6. State and Prove the Laurent series.
7. Explain the Trigonometric Function.
8. State and Prove Cauchy–Goursat theorem.
9. Discribe the Derivatives of Functions w(t).
10. If a function f is analytic at all points interior to and on a simple closed contour C,then prove that $\int$ f (z) dz = 0.

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UNIT IV

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
A function f is __________ in an open set if it has a derivative everywhere in that set.	Analytic	Derivative	Differentiation	Not analytic	Analytic
A function f(z)= 1/z is _________ at each non zero point in the finite plane	Derivative	Differentiation	Not analytic	Analytic	Analytic
A function f(z)= |z|^2 is _________ at each non zero point in the finite plane	Derivative	Differentiation	Not analytic	Analytic	Not analytic
If two function are analytic in a domain D then their sum is __________ in D	Derivative	Differentiation	Not analytic	Analytic	Analytic
If two function are analytic in a domain D then their product is __________ in D	Derivative	Differentiation	Not analytic	Analytic	Analytic
The quotient P(z)/Q(z) of two polynomials is analytic in any domain throughout which Q(z) is________.	Equal to 0	Not equal to 0	Equal to 1	Not equal to 1	Not equal to 0
d/dz{g[f(z)]}=_____	g '[f(z)] f '(z)	g '[f(z)]	f(z) f '(z)	g ' f '(z)	g '[f(z)] f '(z)
If f ' (z) = 0 everywhere in a domain D, then f(z) must to be __________ throughout D.	Analytic	Derivative	Differentiation	Constant	Constant
If f ' (z) = ___________ everywhere in a domain D, then f(z) must to be constant throughout D.	0	1	z	infiity	0
The conjugate of u+iv is_______	u	v	u-iv	v-iu	u-iv
The conjugate of u-iv is_______	u	v	u+iv	v+iu	u+iv
The conjugate of C-R equation is _________.	Ux=-Vy,Uy=Vx	Ux=Vy,Uy=-Vx	Ux=Vy,Uy=Vx	Ux=-Vy,Uy=-Vx	Ux=Vy,Uy=-Vx
The exponential function is writing by____________.	e	e^x	e^y	e^z	e^z
d/dz[e^z]=______	0	1	e^z	infinity	e^z
The value of e^z is ______ for any complex number z.	Equal to 0	Not equal to 0	Equal to 1	Not equal to 1	Not equal to 0
|e^z|=_____	e	e^x	e^y	e^z	e^x
e^ix=________.	cos x + isin x	cos x - isin x	cos x + sin x	cos x / isin x	cos x + isin x
2i sin x=______	(e^ix)- (e^-ix)	(e^ix)+(e^-ix)	(e^ix)-(e^ix)	(e^x)- (e^-x)	(e^ix)- (e^-ix)
2i cos x=______	(e^ix)- (e^-ix)	(e^ix)+(e^-ix)	(e^ix)-(e^ix)	(e^x)- (e^-x)	(e^ix)+ (e^-ix)
d/dz[e^iz]=______	e^iz	ie^z	ie^iz	e^z	ie^iz
d/dz[sin z]=_______.	sin z	cos z	-sin z	-cos z	cos z

d/dz[cos z]=_______.	sin z	cos z	-sin z	-cos z	-sin z
sin^2 [z] + cos ^2 [z] =_______.	0	1	sin z + cos z	sin z - cos z	1
sin (iy)=_______.	sinhy	i sinhy	coshy	i coshy	i sinhy
cos (iy)=______.	sinhy	i sinhy	coshy	i coshy	coshy
d/dz[sec z]=______.	sec z	tan z	sec z tan z	sec^2 z	sec z tan z
The derivative of complex valued function w of a real varriable t is written as w(t)=___________.	u(t)+v(t)	u(t)+iv(t)	u(t)-v(t)	u(t)-iv(t)	u(t)+iv(t)
The derivative of the function w '(t) is defined as__________.	u '(t)+v '(t)	u '(t)+iv '(t)	u '(t)-v '(t)	u '(t)-iv '(t)	u '(t)+iv '(t)
d/dz[tan z]=_____.	sec z	sec^2 z	cot z	cot ^2z	sec^2 z
cos z/sin z=________.	tan z	cot z	sec z	csc z	cot z
sin z/cos z=________.	tan z	cot z	sec z	csc z	tan z
1/cos z =_______.	tan z	cot z	sec z	csc z	sec z
1/sin z=________.	tan z	cot z	sec z	csc z	csc z
If z=n(22/7) where n=…-2,-1,0,1,2…. Then sin z=______.	0	1	-1	2	0
An arc consisting of a finite number of smooth arcs joined ene to end is called _________.	Smooth arc	Arcs	Curve	Contour	Contour
An arc consisting of a finite number of __________ joined ene to end is called contour	Smooth arc	Arcs	Curve	Contour	Smooth arc
If the equation z=z(t) is represent a contour, z(t) is ________.	Continuous	Discontinuous	Derivative	Arc	Continuous
If the equation z=z(t) is represent a __________, z(t) is continuous	Smooth arc	Arcs	Curve	Contour	Contour
When only the initial and final values of z(t) are the_______, a contour C is called a simple closed contour.	Same	Diffeent	0	1	Same
When only the initial and final values of z(t) are the same, a contour C is called a __________ contour.	Open	Simple open	Closed	Simple closed	Simple closed
(e^ix) - (e^-ix)=________.	2i sin x	sin x	2i cos x	cos x	2i sin x
(e^ix) + (e^-ix)=________.	2i sin x	sin x	2i cos x	cos x	cos x
sin(iy)=__________.	i sinhy	sinhy	i coshy	coshy	i sinhy
cos(iy)=_________.	i sinhy	sinhy	i coshy	coshy	coshy
A function f is analytic in an open set if it has a ________ everywhere in that set.	Analytic	Derivative	Differentiation	Not analytic	Derivative

The quotient P(z)/Q(z) of two polynomials is _______ in any domain throughout which Q(z) is not equal to 0.	Analytic	Derivative	Differentiation	Not analytic	Analytic

UNIT V
SYLLABUS

Liouville's theorem and the fundamental theorem of algebra. Convergence of sequences and series, Taylor series and its examples - Laurent series and its examples, absolute and uniform convergence of power series.

LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here,which is known as Liouville's theorem, states this result in a somewhat different way.

Theorem 1. If a function f is entire and bounded in the complex plane, then f(z) is constant througout the plane.

To start the proof, we assume that f is as stated and note that since f is entire, Theorem 3 in Sec. 52 can be applied with any choice of z_0 and R. In particular, Cauchy's inequality (2) in that theorem tells us that when n = 1,

(1) $$|f^{j}(z_0)| \le \frac{M_R}{R}.$$

Moreover, the boundedness condition on f tells us that a nonnegative constant M exists such that f(z) M for all z ; and, because the constant M_R in inequality (1) is always less than or equal to M, it follows that

(2) $$|f^{j}(z_0)| \le \frac{M}{R},$$

where R can be arbitrarily large. Now the number M in inequality (2) is independent of the value of R that is taken. Hence that inequality holds for arbitrarily large values of R only if f $^{j}(z_0)$ 0. Since the choice of z_0 was arbitrary, this means that

$f^j(z)$ 0 everywhere in the complex plane. Consequently, f is a constant function,
according to the theorem in Sec. 24.

The following theorem, called the fundamental theorem of algebra, follows readily from Liouville's theorem.

Theorem 2. Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n /= 0)$$

of degree n ($n \geq 1$) has at least one zero. That is, there exists at least one point z_0
such that $P(z_0) = 0$.

The proof here is by contradiction. Suppose that P (z) is not zero for any value of z. Then the reciprocal

is clearly entire, and it is also bounded in the complex plane.
To show that its is bounded, we first write
Next, we observe that a sufficiently large positive number R can be found such that the modulus of each of the quotients in expression (3) is less than the number a_n /(2n) when z > R. The generalized triangle inequality (10), Sec. 4, which applies to n complex numbers, thus shows that

$$|w| < \frac{|an|}{2} \quad \text{whenever} |z| > R.$$

Consequently,

$$|a_n + w| \geq ||a_n| - |w|| > \frac{|an|}{2} \quad \text{whenever} |z| > R.$$

This inequality and expression (4) enable us to write

(5) $|P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n$ whenever $|z| > R$.

Evidently, then,

So f is bounded in the region exterior to the disk $|z| \le R$. But f is continuous in that closed disk, and this means that f is bounded there too (Sec. 18). Hence f is bounded in the entire plane.

It now follows from Liouville's theorem that f (z), and consequently P (z), is constant. But P (z) is not constant, and we have reached a contradiction.

The fundamental theorem tells us that any polynomial P (z) of degree n ($n \ge 1$) can be expressed as a product of linear factors:

(6) $$P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n),$$

where c and z_k ($k = 1, 2, \ldots, n$) are complex constants. More precisely, the theorem ensures that P (z) has a zero z_1. Then, according to Exercise 9, Sec. 54,

$$P(z) = (z - z_1)Q_1(z),$$

where $Q_1(z)$ is a polynomial of degree $n - 1$. The same argument, applied to $Q_1(z)$, reveals that there is a number z_2 such that

$$P(z) = (z - z_1)(z - z_2)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n - 2$. Continuing in this way, we arrive at expression (6). Some of the constants z_k in expression (6) may, of course, appear more than once, and it is clear that P (z) can have no more than n distinct zeros.

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MAXIMUM MODULUS PRINCIPLE

In this section, we derive an important result involving maximum values of the moduli of analytic functions. We begin with a needed lemma.

Lemma. Suppose that $|f(z)| \le |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then f (z) has the constant value $f(z_0)$ through- out that neighborhood.

To prove this, we assume that f satisfies the stated conditions and let z_1 be any point other than z_0 in the given neighborhood. We then let ρ be the distance between z_1 and z_0. If C_ρ denotes the positively oriented circle $|z - z_0| = \rho$, centered at z_0 and passing through z_1 (Fig. 70), the Cauchy integral formula tells us that

the parametric representation

$$z = z_0 + \rho e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

for C_ρ enables us to write equation (1) as we note from expression (2) that when a function is analytic within and on a given circle, its value at the center is the arithmetic mean of its values on the circle. This result is called Gauss's mean value theorem.

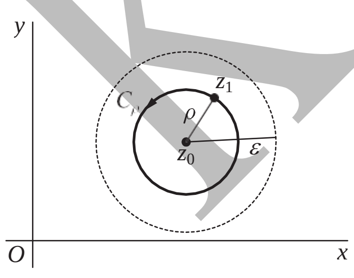

FIGURE 70

From equation we obtain the inequality

(4) $$|f(z_0 + \rho e^{i\theta})| \le |f(z_0)| \qquad (0 \le \theta \le 2\pi),$$

This lemma can be used to prove the following theorem, which is known as the maximum modulus principle.

Theorem. If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.

Given that f is analytic in D, we shall prove the theorem by assuming that f (z) does have a maximum value at some point z_0 in D and then showing that f (z) must be constant throughout D.

The general approach here is similar to that taken in the proof of the lemma in Sec. 27. We draw a polygonal line L lying in D and extending from z_0 to any other point P in D. Also, d represents the shortest distance from points on L to the boundary of D. When D is the entire plane, d may have any positive value. Next, we observe that there is a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L such that z_n coincides with the point P and

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$

In forming a finite sequence of neighborhoods (Fig. 71)

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n$$

where each N_k has center z_k and radius d, we see that f is analytic in each of these neighborhoods, which are all contained in D, and that the center of each neighborhood N_k $(k = 1, 2, \ldots, n)$ lies in the neighborhood N_{k-1}.

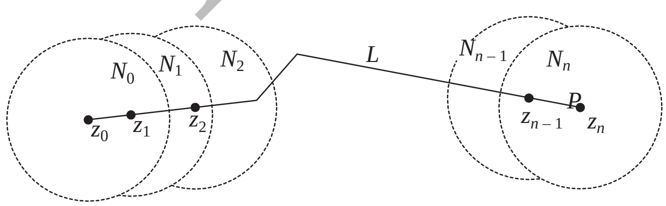

FIGURE 71

Since $|f(z)|$ was assumed to have a maximum value in D at z_0, it also has a maximum value in N_0 at that point. Hence, according to the preceding lemma, $f(z)$ has the constant value $f(z_0)$ throughout N_0. In particular, $f(z_1) = f(z_0)$. This means that $|f(z)| \le |f(z_1)|$ for each point z in N_1 ; and the lemma can be applied again, this time telling us that

$$f(z) = f(z_1) = f(z_0)$$

when z is in N_1. Since z_2 is in N_1, then, $f(z_2) = f(z_0)$. Hence $|f(z)| \le |f(z_2)|$

when z is in N_2 ; and the lemma is once again applicable, showing that

$$f(z) = f(z_2) = f(z_0)$$

when z is in N_2. Continuing in this manner, we eventually reach the neighborhood

N_n and arrive at the fact that $f(z_n) = f(z_0)$.

Recalling that z_n coincides with the point P , which is any point other than z_0 in D, we may conclude that $f(z) = f(z_0)$ for every point z in D. Inasmuch as $f(z)$ has now been shown to be constant throughout D, the theorem is proved.

If a function f that is analytic at each point in the interior of a closed bounded region R is also continuous throughout R, then the modulus $|f(z)|$ has a maximum value somewhere in R (Sec. 18). That is, there exists a nonnegative constant M such that $|f(z)| \le M$ for all points z in R, and equality holds for at least one such point.

If f is a constant function, then $|f(z)| = M$ for all z in R. If, however, $f(z)$ is not constant, then, according to the theorem just proved, $|f(z)| \ne M$ for any point z in the interior of R. We thus arrive at an important corollary.

Corollary. Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

EXAMPLE. Let R denote the rectangular region $0 \le x \le \pi$, $0 \le y \le 1$. The corollary tells us that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R that occurs somewhere on the boundary of R and not in its interior. This can be verified directly by writing

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$$

and noting that the term $\sin^2 x$ is greatest when $x = \pi/2$ and that the increasing function $\sinh^2 y$ is greatest when $y = 1$. Thus the maximum value of $|f(z)|$ in R occurs at the boundary point $z = (\pi/2, 1)$ and at no other point in R (Fig. 72).

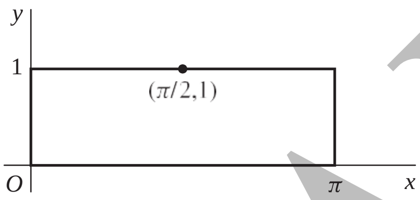

FIGURE 72

When the function f in the corollary is written $f(z) = u(x, y) + iv(x, y)$, the component function $u(x, y)$ also has a maximum value in R which is assumed on the boundary of R and never in the interior, where it is harmonic (Sec. 26). This is because the composite function $g(z) = \exp[f(z)]$ is continuous in R and analytic and not constant in the interior. Hence its modulus $|g(z)| = \exp[u(x, y)]$, which is continuous in R, must assume its maximum value in R on the boundary. In view of the increasing nature of the exponential function, it follows that the maximum value of $u(x, y)$ also occurs on the boundary.

Properties of minimum values of $|f(z)|$ and $u(x, y)$ are treated in the exercises.

CONVERGENCE OF SEQUENCES

An infinite sequence

(1) $$z_1, z_2, \ldots, z_n, \ldots$$

of complex numbers has a limit z if, for each positive number ε, there exists a positive integer n_0 such that

(2) $$|z_n - z| < \varepsilon \quad \text{whenever } n > n_0.$$

Geometrically, this means that for sufficiently large values of n, the points z_n lie in any given ε neighborhood of z (Fig. 73). Since we can choose ε as small as we please,

it follows that the points z_n become arbitrarily close to z as their subscripts increase. Note that the value of n_0 that is needed will, in general, depend on the value of ε.

The sequence (1) can have at most one limit. That is, a limit z is unique if it exists (Exercise 5, Sec. 56). When that limit exists, the sequence is said to converge to z ; and we write

$$\lim_{n\to\infty} z_n = z.$$

If the sequence has no limit, it diverges.

Theorem. Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $z = x + iy$. Then

(4) $$\lim_{n\to\infty} z_n = z$$

if and only if

(5) $$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

To prove this theorem, we first assume that conditions (5) hold and obtain condition (4) from it. According to conditions (5), there exist, for each positive number ε, positive integers n_1 and n_2 such that

Hence if n_0 is the larger of the two integers n_1 and n_2,

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever } n > n_0.$$

Since $|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|$,

$$|x_n - x| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

$$|y_n - y| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|;$$

and this means that

$$|x_n - x| < \varepsilon \quad \text{and} \quad |y_n - y| < \varepsilon \quad \text{whenever } n > n_0.$$

then,

Conversely, if we start with condition (4), we know that for each positive number ε, there exists a positive integer n_0 such that

$$|(x_n + iy_n) - (x + iy)| < \varepsilon \quad \text{whenever } n > n_0.$$

That is, conditions (5) are satisfied.

Note how the theorem enables us to write

$$\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

whenever we know that both limits on the right exist or that the one on the left exists.

CONVERGENCE OF SERIES

An infinite series

(1) $$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

of complex numbers converges to the sum S if the sequence

(2) $$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N \qquad (N = 1, 2, \ldots)$$

of partial sums converges to S; we then write

$$\sum_{n=1}^{\infty} z_n = S.$$

Note that since a sequence can have at most one limit, a series can have at most one sum. When a series does not converge, we say that it diverges.

Theorem. Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $S = X + iY$. Then

This theorem tells us, of course, that one can write

whenever it is known that the two series on the right converge or that the one on the left does.

To prove the theorem, we first write the partial sums (2) as

(5) $$S_N = X_N + iY_N,$$

(6) $$\lim_{N \to \infty} S_N = S;$$

and, in view of relation (5) and the theorem on sequences in Sec. 55, limit (6) holds if and only if

(7) $$\lim_{N \to \infty} X_N = X \quad \text{and} \quad \lim_{N \to \infty} Y_N = Y.$$

Limits (7) therefore imply statement (3), and conversely. Since X_N and Y_N are the partial sums of the series (4), the theorem here is proved.

This theorem can be useful in showing that a number of familiar properties of series in calculus carry over to series whose terms are complex numbers. To illustrate how this is done, we include here two such properties and present them as corollaries.

Corollary 1. If a series of complex numbers converges, the nth term converges to zero as n tends to infinity.

Assuming that series (1) converges, we know from the theorem that if

$$z_n = x_n + iy_n \ (n = 1, 2, \ldots),$$

then each of the series (8)

converges. We know, moreover, from calculus that the nth term of a convergent series of real numbers approaches zero as n tends to infinity.

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + 0 \cdot i = 0 \ ;$$

and the proof of Corollary 1 is complete.

It follows from this corollary that the terms of convergent series are bounded. That is, when series (1) converges, there exists a positive constant M such that $|z_n| \le M$ for each positive integer n. (See Exercise 9.)

For another important property of series of complex numbers that follows from a corresponding property in calculus, series (1) is said to be absolutely convergent if the series

Corollary 2. The absolute convergence of a series of complex numbers implies the convergence of that series.

To prove Corollary 2, we assume that series (1) converges absolutely. Since

$$|x_n| \le \sqrt{x_n^2 + y_n^2} \quad \text{and} \quad |y_n| \le \sqrt{x_n^2 + y_n^2},$$

we know from the comparison test in calculus that the two series

must converge. Moreover, since the absolute convergence of a series of real numbers implies the convergence of the series itself, it follows that the series (8) both converge. In view of the theorem in this section, then, series (1) converges. This finishes the proof of Corollary 2.

In establishing the fact that the sum of a series is a given number S, it is often convenient to define the remainder ρ_N after N terms, using the partial sums (2) :

$$\text{(9)} \qquad \rho_N = S - S_N .$$

Thus $S = S_N + \rho_N$; and, since $|S_N - S| = |\rho_N - 0|$, we see that a series converges to a number S if and only if the sequence of remainders tends to zero.

make considerable use of this observation in our treatment of power series. They are series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots,$$

where z_0 and the coefficients a_n are complex constants and z may be any point in a stated region containing z_0. In such series, involving a variable z, we shall denote sums, partial sums, and remainders by $S(z)$, $S_N(z)$, and $\rho_N(z)$, respectively.

TAYLOR SERIES

We turn now to Taylor's theorem, which is one of the most important results of the chapter.

Theorem. Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0 (Fig. 74). Then $f(z)$ has the power series represen- tation

(1) $$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad (|z-z_0| < R_0),$$

where

(2) $$\frac{f^{(n)}(z_0)}{}$$

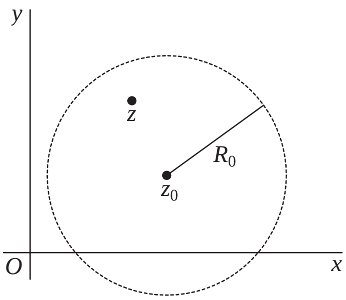

FIGURE 74

This is the expansion of f (z) into a Taylor series about the point z_0. It is the familiar Taylor series from calculus, adapted to functions of a complex variable. With the agreement that

$$f^{(0)}(z_0) = f(z_0) \quad \text{and} \quad 0! = 1,$$

series (1) can, of course, be written

When it is known that f is analytic everywhere inside a circle centered at z_0, convergence of its Taylor series about z_0 to f (z) for each point z within that circle is ensured; no test for the convergence of the series is even required. In fact, according to Taylor's theorem, the series converges to f (z) within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 at which f fails to be analytic. In Sec. 65, we shall find that this is actually the largest circle centered at z_0 such that the series converges to f (z) for all z interior to it.

In the following section, we shall first prove Taylor's theorem when z_0 0, in which case f is assumed to be analytic throughout a disk z < R_0 and series (1) becomes a Maclaurin series:

The proof when z_0 is arbitrary will follow as an immediate consequence. A reader who wishes to accept the proof of Taylor's theorem can easily skip to the examples in Sec. 59.

PROOF OF TAYLOR'S THEOREM

To begin the derivation of representation (4), Sec. 57, we write z r and let C_0 denote and positively oriented circle z r_0, where r < $r_0 < R_0$ (see Fig. 75). Since **f** is analytic inside and on the circle C_0 and since the point z is interior to

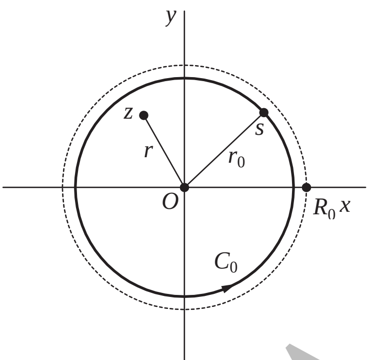

FIGURE 75

C_0, the Cauchy integral formula

To accomplish this, we recall that z r and that C_0 has radius r_0, where $r_0 > r$. Then, if s is a point on C_0, we can see that

$$|s - z| \geq ||s| - |z|| = r_0 - r.$$

Consequently, if M denotes the maximum value of $|\mathbf{f}(s)|$ on C_0,

Inasmuch as $(r/r_0) < 1$, limit (7) clearly holds.

To verify the theorem when the disk of radius R_0 is centered at an arbitrary point z_0, we suppose that **f** is analytic when $|z - z_0| < R_0$ and note that the composite function **f** $(z + z_0)$ must be analytic when $|(z + z_0) - z_0| < R_0$. This last inequality is, of course, just $|z| < R_0$; and, if we write $g(z) = \mathbf{f}(z + z_0)$, the analyticity of g in the disk $|z| < R_0$ ensures the existence of a Maclaurin series representation:

(z0) z^n $(|z| < R_0)$.

After replacing z by z z_0 in this equation and its condition of validity, we have the desired Taylor series expansion (1) in Sec. 57.

EXAMPLES

for all points z interior to some circle centered at z_0, then the power series here must be the Taylor series for f about z_0, regardless of how those constants arise. This observation often allows us to find the coefficients a_n in Taylor series in more efficient ways than by appealing directly to the formula $a_n f^{(n)}(z_0)/n!$ in Taylor's theorem. In the following examples, we use the formula in Taylor's theorem to find the Maclaurin series expansions of some fairly simple functions, and we emphasize the use of those expansions in finding other representations. In our examples, we shall freely use expected properties of convergent series, such as those verified in Exercises 7 and 8, Sec. 56.

EXAMPLE 1. Since the function $f(z) = e^z$ is entire, it has a Maclaurin series representation which is valid for all z. Here $f^{(n)}(z) = e^z$ $(n = 0, 1, 2, \ldots)$;

LAURENT SERIES

If a function f fails to be analytic at a point z_0, one cannot apply Taylor's theorem at that point. It is often possible, however, to find a series representation for f (z) involving both positive and negative powers of z z_0. (See Example 5, Sec. 59, and also Exercises 11, 12, and 13 for that section.) We now present the theory of uch representations, and we begin with Laurent's theorem.

Theorem. Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain (Fig. 76). Then, at each point in the domain, f (z) has the series representation

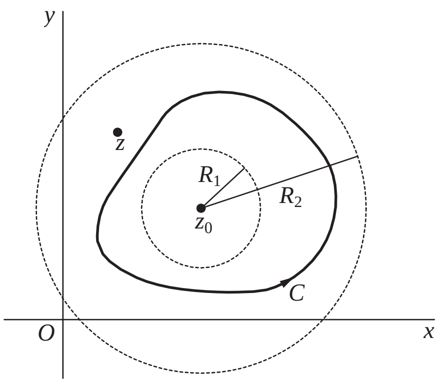

FIGURE 76

Note how replacing n by − n in the second series in representation (1) enables us to write that series as In either one of the forms (1) and (4), the representation of f (z) is called a Laurent series.

Observe that the integrand in expression (3) can be written f (z)(z−z_0)$^{n-1}$. Thus it is clear that when f is actually analytic throughout the disk |z-z_0| < R_2, this integrand is too. Hence all of the coefficients b_n are zero; and, because

If, however, f fails to be analytic at z_0 but is otherwise analytic in the disk |z − z_0| < R_2, the radius R_1 can be chosen arbitrarily small. Representation (1) is then valid in the punctured disk 0 < |z − z_0| < R_2. Similarly, if f is analytic at each point in the finite plane exterior to the circle |z − z_0| = R_1, the condition of validity is R_1 < |z − z_0| < ∞. Note that if f is analytic everywhere in the finite plane except at z_0, series (1) is valid at each point of analyticity, or when 0 < |z − z_0| < ∞.

We shall prove Laurent's theorem first when z_0 = 0, which means that the annulus is centered at the origin. The verification of the theorem when z_0 is arbitrary will follow readily; and, as was the case with Taylor's theorem, a reader can skip the entire proof without difficulty.

PROOF OF LAURENT'S THEOREM

We start the proof by forming a closed annular region $r_1 \le |z| \le r_2$ that is contained in the domain $R_1 < |z| < R_2$ and whose interior contains both the point z and the contour C (Fig. 77). We let C_1 and C_2 denote the circles $|z|= r_1$ and $|z|= r_2$,

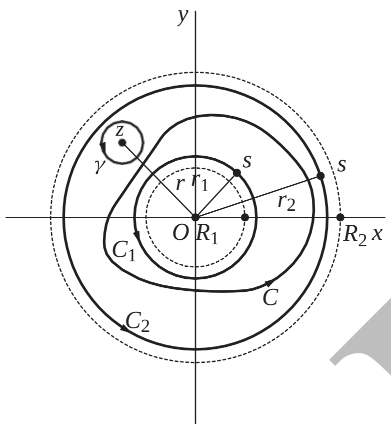

FIGURE 77

respectively, and we assign them a positive orientation. Observe that **f** is analytic on C_1 and C_2, as well as in the annular domain between them.

Next, we construct a positively oriented circle γ with center at z and small enough to be contained in the interior of the annular region $r_1 \le |z| \le r_2$, as shown in Fig. 77. It then follows from the adaptation of the Cauchy–Goursat theorem to integrals of analytic functions around oriented boundaries of multiply connected domains (Sec. 49) that

But, according to the Cauchy integral formula, the value of the third integral here is $2\pi i f(z)$. Hence

ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

This section and the three following it are devoted mainly to various properties of power series. A reader who wishes to simply accept the theorems and the corollary in these sections can easily skip the proofs in order to reach more quickly.We recall from that a series of complex numbers converges absolutely if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

POSSIBLE QUESTIONS

2 MARK QUESTIONS

1. State the Liouville's theorem.
2. State the absolute convergence of power series.
3. State the Laurent series.
4. Write the any two equations of Exponential Function.
5. Write the equation of Taylor series.

8 MARK QUESTIONS

1. If a function f is entire and bounded in the complex plane, then prove that f (z) is constant throughout the plane.
2. State and Prove the absolute convergence of power series.
3. If a series of complex numbers converges, then n th term converges to zero as n tends to infinity.
4. State and prove the Liouville's theorem.
5. Prove that any polynomial $P(z) = a_0 + a_1 z + a_2 z_2 + \cdots + a_n z_n$ ($a_n \neq 0$) of degree n ($n \geq 1$) has at least one zero. That is, there exists at least one point z_0 such that $P(z_o) = 0$.
6. State and Prove the Taylor series.
7. Suppose that $z_n = x_n + iy_n$ (n = 1, 2, . . .) and z = x + iy. Then $\lim z_n = z$ as $n\to\infty$ if and only if $\lim x_n = x$ as $n\to\infty$ and $\lim y_n = y$ as $n\to\infty$.
8. Suppose that a function f is analytic throughout a disk $|z - z_o| < R_o$, centered at z_o and with radius R_o. Then prove that f (z) has the power series representation
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0),$$
where $a_n = f(n)(z0)/n!$ (n = 0, 1, 2, . . .).That is, the series is converges to f (z) when z lies in the stated open disk.
9. Suppose that $z_n = x_n + iy_n$ (n = 1, 2, . . .) and S = X + iY. Then prove that $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = x$ and $\sum_{n=1}^{\infty} y_n = y$
10. Prove that the absolute convergence of a series of complex numbers implies the convergence of that series.

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UNIT V

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
If a function f is entire and bounded in the complex plan, then f(z) is ___________ throughtout the plan	Constant	Continuous	Derivative	Convergent	Constant
If a function f is entire and __________ in the complex plan, then f(z) is constant throughtout the plan	Closed	Open	Bounded	Convergent	Bounded
Any polynimial P(z) of degree n has at least _______ zero	1	2	Finite	Infinit	1
For each positive number epsilon , therre exists a positive integer m such that $\|z_n-z\|<$ epsilon whenever n>m. Here z is called _______ point	Limit	Arbitrary	Fixed	Interior	Limit
When that limit exists, the sequence is called_______.	Convergent	Diverrgent	Continuous	Discontinuous	Convergent
If the seqence has no limit, its called________.	Convergent	Diverrgent	Continuous	Discontinuous	Diverrgent
If the seqence has ______ limit, its called divergent	One	Two	Finite	No	No
If the seqence has ______ limit, its called Convergent	One	Two	Finite	No	One
The _______ of the seqence is called series	Sum	Product	Subration	Divition	Sum
If the n th trem of the serise is convergent to zero as n tends to infinity then the total series is ____________.	Convergent	Diverrgent	Continuous	Discontinuous	Convergent
The absolute convergent of a series of complex numbers implies the ____________ of the serries.	Convergent	Diverrgent	Continuous	Discontinuous	Convergent
Suppose that $z_n = x_n + I y_n$ and S=X + iY. Then z1+z2+z3+…..=_______ if x1+x2+x3+….=X and y1+y2+y3+….=Y	0	X	S	Y	S
A sequence of points in X is a function f from_____ into X.	R	N	X	Z	N
Suppose that $z_n = x_n + I y_n$ and S=X + iY. Then z1+z2+z3+…..=S if x1+x2+x3+….=_________ and y1+y2+y3+….=Y	0	X	S	Y	X
Suppose that $z_n = x_n + I y_n$ and S=X + iY. Then z1+z2+z3+…..=S if x1+x2+x3+….= X and y1+y2+y3+…. =	0	X	S	Y	Y
The ___________ convergent of a series of complex numbers implies the convergent of the serries.	Absolute	Uniform	Non Uniform	Finite	Absolute
If a function f is entire and bounded in the complex plan, then f(z) is constant throughtout the plan is called__________	Taylor	Liouville's	Laurent	Absolute	Liouville's
Replaceing z by 1/z in e^z we have ________ series.	Taylor	Liouville's	Laurent	Absolute	Laurent
z1,z2,z3,z4,z5,…… is called________	Sequence	Series	Elements	Points	Sequence
z1+z2+z3+z4+…. Is called_________.	Sequence	Series	Elements	Points	Series
lim (z+5)/(iz+3)=_______ as z tends to -5.	0	1	z	Infinity	0

If a function f is continuous throughout a region R that is closed and ________, there exists a non negative real number M such that |f(z)| less than or equal to M for all points z in R.	Open	Bounded	Continuous	Convergent	Bounded
lim f(z)=f(zo) as z tends to zo is called_______.	Discontinuou	Continuous	Limit	Function	Continuous
A set is _________ if it cointains all of its boundary points.	Open	Not open	Closed	Not cloed	Closed
A point x is ______ of the mapping T from X into X if Tx=x	Fixed point	Arbitary point	Interior point	Limit point	Fixed point
|conjugates of z|=_______.	z	Conjugate of z	|z|	|x|	|z|
(z+conjugate of z)/2 =_________.	Re z	Im z	z	2z	Re z
(z -conjugate of z)/2i =_________.	Re z	Im z	z	2z	Im z
The conjugate of x+iy is ________.	x	y	x+y	x-iy	x-iy
The conjugate of x-iy is ________.	x	y	x+y	x+iy	x+iy
The conjugate of 2+i5 is _______	2	5	i5	2-iy	2-iy
The conjugate of 2-i5 is _______	2	5	i5	2+iy	2+iy
The conjugate of -4-i5 is _______	-4	5	9	-4+i5	-4+i5
The conjugate of -4+i5 is _______	-4	5	9	-4-i5	-4-i5
The conjugate of iz is ________.	i	-i conjugate z	-iz	iz	-i conjugate z