KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act1956)

Coimbatore – 641 021.

DEPARTMENT OF MATHEMATICS

SEMESTER-II

19MMP204 PARTIAL DIFFERENTIAL EQUATIONS **Semester – II 4H – 4C**

Instruction Hours / week: L: 4 T: 0 P: 0 **Marks:** Internal: **40** External: **60** Total: **100**
End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- The basic concepts of solution of PDE and its applications.
- About initial and boundary value problems for PDEs of first and second order which includes Laplace Equation, Diffusion Equation and Wave Equation.

Course Outcomes (COs)

On successful completion of this course the students will be able to

1. Classify linear and Nonlinear first order differential equations with constant coefficients.
2. Describe the method of separable variables and integral transforms.
3. Solve the elementary Laplace equation with symmetry.
4. Acquire the knowledge of wave equation and vibrating membranes.
5. Enrich their knowledge about diffusion equations with sources.

UNIT I

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Non linear partial differential equation of first order –Compatible systems of first order equations – Special type of first order equations- Partial differential equations of second order – The origin of second order equations – Linear partial differential equations with constant coefficient equations with variable coefficients.

UNIT II

SEPARATION OF VARIABLES

Method of separation of variables –The method of integral transforms.

UNIT III

LAPLACE EQUATION

Elementary solutions of Laplace equations- Families of Equi-potential surfaces - Boundary Value problems-separation of variables-problems with axial symmetry.

UNIT IV

WAVE EQUATION

Elementary solutions of one dimensional wave equation-Vibrating membrane - Applications of calculus of variations-Green's functions for the wave equation.

UNIT V

DIFFUSION EQUATION

The resolution of Boundary value problems for the Diffusion equation- Elementary solutions of diffusion equation - Separation of variables- use of Green's functions- Diffusion with Sources.

SUGGESTED READINGS

1. Sharma, J. N, Keharsingh, (2009). Partial Differential Equations for Engineering and Scientists, Narosa Publishing House, New Delhi.
2. Ian. N. Sneedon, (2006). Elementary Partial differential equations, Tata Mcgraw Hill Ltd.
3. Geraold. B. Folland, (2001). Introduction to Partial Differential Equations, Prentice Hall of India Private limited, New Delhi.
4. SankaraRao. K, (2010). Introduction to Partial Differential Equations, Third edition, Prentice Hall of India Private limited, New Delhi.
5. Veerarajan, T, (2004). Partial Differential Equations and Integral Transforms, Tata McGraw-Hill Publishing Company limited, New Delhi.
6. John, F, (1991). Partial Differential equations, Third edition, Narosa publication co, New Delhi.
7. Tyn-Myint-U andLokenathDebnath(2008). Linear Partial Differential Equations for Scientists and Engineers, Fourth Edition, Birkhauser, Berlin.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: M.Sangeetha
Subject Name: Partial Differential Equation **Sub.Code:19MMP204**
Semester: II **Class: I- M.Sc Mathematics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1.	1	Problems on Non linear partial differential equation of first order	S3:ch2:pg 56-58
2.	1	Theorems on Compatible systems of first order	S3:ch2:pg 58-60
3.	1	Special type of first order equations-problems	S4: ch0 :pg 42
4.	1	Problems on Partial differential equations of second order	S5:ch2:pg 57-59
5.		The origin of second order equations	S6:ch2:pg 59-60
6.	1	The origin of second order equations-Theorems	S5:ch2:pg 60-61
7.	1	Linear partial differential equations with constant .Coefficient Theorems	S4:ch10:pg 195
8.	1	Linear partial differential equations coefficient equations with variable coefficients	S4:ch10:pg 196
9.	1	Recapitulation and discussion of possible questions	
	Total No of Hours =9		
		UNIT-II	
1.	1	Introduction to method of separation of variables	S2:ch14:pg 154-155
2.	1	Continuation of method of separation of variables	S1:ch4:pg 137-139

3.	1	The method of integral transforms.	S3:ch2:pg 68-69
4.	1	Continuation of the method of integral transforms.	S4:ch6:pg 124-125
5.	1	Continuation of the method of integral transforms.	S4:ch6:pg 125-126
6.	1	Recapitulation and discussion of possible questions	
	Total No of Hours =6 hrs		
		UNIT-III	
1.	1	Elementary solutions of Laplace equations-	S3:ch2:pg 104-106
2.	1	Continuation of Elementary solutions of Laplace equations-	S3:ch2:pg 149-151
3.	1	Problems on Families of Equi potential surfaces	S4:ch8:pg 151-152
4.	1	Continuation Problems on Families of Equi potential surfaces	S4:ch8:pg 152-154
5.	1	Boundary Value problems	S4:ch8:pg 154-155
6.	1	separation of variables and problems with axial symmetry	S3:ch2:pg 109-111
7.	1	Recapitulation and discussion of possible questions	
Total No of Hours =7 hrs			
		UNIT-IV	
1.	1	Derivative on elementary solutions of one dimensional wave equation	S2:ch3:pg 170-171
2.	1	Problems on elementary solutions of one dimensional wave equation	S2:ch3:pg 172-173
3.	1	Vibrating membranes	S1:ch2:pg 52-53
4.	1	Problems on vibrating membranes	S1:ch2:pg 53-54

5.	1	Applications of calculus of variations	S4:ch5:pg 95-96
6.	1	Green's functions for the wave equation	S4:ch5:pg 96-97
7.	1	Applications of Green's functions for the wave equation	S4:ch5:pg 98-99
8.	1	Recapitulation and discussion of possible questions	
Total No of Hours =8 hrs			
		UNIT-V	
1.	1	The resolution of boundary value problems for the diffusion equation	S1:ch4:pg 126-127
2.	1	Problems on the resolution of boundary value problems for the diffusion equation	S1:ch4:pg 127-129
3.	1	Elementary solutions of diffusion equation	s4:ch24:pg 399
4.	1	Continuation of Elementary solutions of diffusion equation	s4:ch24:pg 399-400
5.	1	Separation of variables	s4:ch24:pg 403-405
6.	1	Problems on Green's functions	s7:ch4:pg 135-140
7.	1	Recapitulation and discussion of possible questions	
8.	1	Discussion of previous ESE question papers.	
9.	1	Discussion of previous ESE question papers.	
10.	1	Discussion of previous ESE question papers	
	Total No of Hours =10 hrs		
		Total Planned Hours	**40 hrs**

SUGGESTED READINGS

1. Sharma, J. N, Keharsingh, (2009). Partial Differential Equations for Engineering and Scientists, Narosa Publishing House, New Delhi.
2. Ian. N. Sneedon, (2006). Elementary Partial differential equations, Tata Mcgraw Hill Ltd.
3. Geraold. B. Folland, (2001). Introduction to Partial Differential Equations, Prentice Hall of India Private limited, New Delhi.
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7. Tyn-Myint-U andLokenathDebnath(2008). Linear Partial Differential Equations for Scientists and Engineers, Fourth Edition, Birkhauser, Berlin

Class Representative **Signature of the Faculty**

Tutor **Programme Co-ordinator** **HOD**

Name and Signature of the Student Representative

Name and Signature of Course Faculty

Name and Signature of Class Tutor

Name and Signature of Coordinator

Head of the Department

UNIT-I
SYLLABUS

First Order Partial Differential Equations:

Non linear partial differential equation of first order –Compatible systems of first order equations – Special type of first order equations- Partial differential equations of second order – The origin of second order equations – Linear partial differential equations with constant coefficient equations with variable coefficients.

UNIT I

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS:

METHODS TO SOLVE THE FIRST ORDER PDE:

TYPE I:

Given f(p,q)=0.

Let z = ax+by+c which is the solution of the given equation f(p,q)=0.

Then p = $\frac{\partial z}{\partial x}$= a,

and q = $\frac{\partial z}{\partial y}$= b.

This implies p = a, q = b.

Hence the complete solution is z = ax+by+c, where f(a,b)=0.

EXAMPLE 1:

Solve $\sqrt{p}+\sqrt{q}$ =1.

Solution:

Given:

F(p,q)=0.

Therefore the complete integral is,

z = ax+by+c.

Therefore the given equation becomes

$\sqrt{a}+\sqrt{b}$ =1.

Therefore $\sqrt{b}$ = 1- $\sqrt{a}$

Taking square on both sides we get,

$$b = (1-\sqrt{a})^2$$

Substitute b = ± $(1-\sqrt{a})^2$ in z = ax+by+c.

$$z = ax+(1-\sqrt{a})^2y+c.$$

Differentiating partially with respect to a we get,

$$0 = x+\frac{(1-\sqrt{a})}{\sqrt{a}}y.$$

Therefore there is no singular integral.

To find the general integral:

Let c = f(a) in z = ax+ (± $(1-\sqrt{a})^2$) y+c.

z = ax+ (± $(1-\sqrt{a})^2$) y+f(a).

Differentiating partially with respect to a we get,

$0 = x+\frac{(1-\sqrt{a})}{\sqrt{a}}y + f'(a).$

By eliminating a in between above two equation we get the general solution.

EXAMPLE 2:

Solve: $p^2 + q^2 = n^2$

Solution:

Given: $p^2 + q^2 = n^2$

It is of the form F(p,q) = 0

Therefore the complete integral is,

$$a^2 + b^2 = n^2$$

The value of b is given by,

b = $\sqrt{n^2 - a^2}$

Substituting the value of b in the equation $z = ax + by + c$ we get,

$$z = ax + \sqrt{n^2 - a^2}y + c$$

Differentiating partially with respect to a we get,

$0 = x + \frac{1}{2\sqrt{n^2-a^2}}(-2ay) + \varphi'(a)$

$0 = x - \frac{ay}{\sqrt{n^2-a^2}} + \varphi'(a)$

Eliminating a in between above two equation we get the general integral.

TYPE II:

It is of the form,

z = px+qy+f(p,q).

Now z = ax+by+f(a,b) …..(1)

Which is the complete integral.

Differentiate (1) partially with respect to a and b, we get

$0 = x + \frac{\partial f}{\partial a}$ …..(2)

$0 = y + \frac{\partial f}{\partial b}$ …..(3)

Eliminating a and b from equations (1),(2) and (3) we get the singular integral,

Let $b = \varphi(a)$.

Therefore equation (1) becomes,

$z = ax + \varphi(a)\, y + f(a, \varphi(a))$ ….(4)

Differentiate partially with respect to a,

$0 = x + \varphi'(a)y + f'(a, \varphi(a))$ ….(5)

Eliminating a between (4) and (5) we will get the general integral.

EXAMPLE 1:

Solve $px + qy + \sqrt{1+p^2+q^2}$

Solution:

Given:

$z = px + qy + \sqrt{1+p^2+q^2}$

Therefore the complete integral,

$z = ax + by + \sqrt{1+a^2+b^2};$

Differentiate partially with respect to a and b we get,

$0 = x + \frac{a}{\sqrt{1+a^2+b^2}}$

This implies,

$x = -\frac{a}{\sqrt{1+a^2+b^2}}$

$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$

This implies,

$y = -\frac{b}{\sqrt{1+a^2+b^2}}$

Eliminating a and b from the above equation we get the singular integral,

$x^2 = \frac{a^2}{1+a^2+b^2}$,

$y^2 = \frac{b^2}{1+a^2+b^2}$,

Therefore,

$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$,

$1-(x^2+ y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$

This implies,

$1-(x^2+ y^2) = \frac{1}{1+a^2+b^2}$

$1 + a^2 + b^2 = \frac{1}{1-x^2-y^2}$

$\sqrt{1 + a^2 + b^2} = \frac{1}{1-x^2-y^2}$

Substituting we get,

$a = \frac{-x}{\sqrt{1-x^2-y^2}}$

$b = \frac{-y}{\sqrt{1-x^2-y^2}}$

By substituting we get,

$z = \sqrt{1 - x^2 - y^2}$

taking square on both sides, we get

$z^2 = 1 - x^2 - y^2$

$x^2 + y^2 + z^2 = 1$

Which is a required solution.

EXAMPLE 2:

Solve: $z = px + qy + pq$

Solution:

Given: $z = px + qy + pq$

Now $z = ax + by + ab$

Which is the complete integral.

Differentiate partially with respect to a and b we get,

$$0 = x + b \text{ ....(1)}$$

$$0 = y + a \text{ ....(2)}$$

Therefore the values of a and b are as follows,

$$a = -x \; and \; b = -y$$

Substituting the values of a and b in the given equation we get,

$$z = -xy - xy + xy$$

$$z = -xy$$

Eliminating a and b from the equation (1) & (2) we get the singular integral,

Let $b = \varphi(a)$

Therefore the equation becomes,

$$z = ax + \varphi(a)y + a\,\varphi(a)$$

Differentiating partially with respect to a we get,

$$z = ax + \varphi(a)y + \varphi'(a)a + \varphi(a)$$

Eliminating a and b we get the general integral.

TYPE III

This is of the form F(z,p,q)=0 does not contain x and y explicitly.

Let us take z=f(U)

Where U=x+ay

$p=\frac{\partial z}{\partial x}$

$q=\frac{\partial z}{\partial y}$

therefore, p and q becomes,

$p=\frac{\partial z}{\partial U}\frac{\partial U}{\partial x}$ and

$q=\frac{\partial z}{\partial U}\frac{\partial U}{\partial y}$

Therefore, $p=\frac{dz}{dU}$ and

$q= a.\frac{dz}{dU}$

Therefore,

$F(z,\frac{dz}{dU}, a\,\frac{dz}{dU}) = 0,$

Which is of the form ordinary differential equations,

$\frac{dz}{dU}= \varphi\,(z,a)$

This implies,

$\frac{dz}{\varphi\,(z,a)} = dU$

Integrating on both sides we get,

$\int \frac{dz}{\varphi\,(z,a)} = \int dU$

This implies,

f(z,a) = U+ c

Case 1:

If the equation is of the type,

F(x,p,q) = 0

Let q = a and

p = f(x,a)

By applying total derivative we get,

$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$

Therefore, z = pdx+qdy

Integrating we get,

$z = \int pdx + \int pdy$

$= \int f(x,a)dx + \int ady$

z = F(x,a) + ay + c

Case 2:

If the equation is of the type,

F(y,p,q) = 0

Let p = a and

q = f(y,a)

By applying total derivative we get,

$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$

Therefore, z = pdx+qdy

Integrating we get,

$z = \int pdx + \int pdy$

$= \int adx + \int f(y, a)dy$ z

= ax + F(y,a) + c

EXAMPLE 1:

Find the complete integral of z = pq

Solution:

This is of the form f(z,p,q) = 0.

Let z = f(X) = f(x+ay)

where X = (x+ay)

put $p = \frac{dz}{dX}$ and

$q = a\frac{dz}{dX}$

By substituting the values of p and q we get the given equation as,

$z = (\frac{dz}{dX}) a (\frac{dz}{dX})$

$z = a (\frac{dz}{dX})^2$

Taking square roots on both sides we get,

$$\sqrt{z} = \pm\sqrt{a}\,(\frac{dz}{dX})$$

$dX = \pm\sqrt{\frac{a}{z}}\,dz$

Integrating we get,

X + c = $\pm 2\sqrt{a}\sqrt{z}$

X + c = $\pm 2\sqrt{az}$

Substituting the value of X in the above equation we get,

(x+ay) + c = $\pm 2\sqrt{az}$

Taking square on both sides we get,

$(x + ay + c)^2 = 4\,az$

EXAMPLE 2:

Find the complete integral of $p^3 + q^3 = 3pqz$

Solution:

This is of the form f(z,p,q) = 0.

Let z = f(X) = f(x+ay)

where X = (x+ay)

put p = $\frac{dz}{dX}$ and

q = a $\frac{dz}{dX}$

Substituting the values of p and q we get,

$$\frac{dz^3}{dX} + a\frac{dz^3}{dX} = 3az\frac{dz}{dX}\frac{dz}{dX}$$

$$\frac{dz}{dX}^3(1 + a^3) = 3za\frac{dz}{dX}^2$$

$$\frac{dz}{dX}(1 + a^3) = 3za$$

Therefore the equations becomes,

$\frac{1}{(1+a^3)}\ dz\ = 3z\frac{1}{dz}$

$(1+a^3)\frac{1}{z}dz = 3a\ dz$

Then by integrating on both sides we get,

$$\int \frac{1}{z}dz = 3\ \frac{a}{(1+a^3)}\int dX$$

$$\log z = 3\ \frac{a}{(1+a^3)}X + c$$

Substituting the value of X we get,

$$\log z = 3\ \frac{a}{(1+a^3)}(x+ay) + c$$

Therefore we get,

$(1+a^3)\log z = 3a\ (x+ay) + c.$

NON LINEAR PARTIAL DIFFERENTIAL EQUATION OF FIRST ORDER:

CASE 1:

Equations of the form F $(x^m p\ ,\ y^n\ q)$ = 0.

Let us X = x^{1-m} and Y = y^{1-n}, where m,n$\neq$ 1.

Therefore,

$p = P(1-m)x^{1-m-1}$

$p = P(1-m)x^{-m}$

$q = Q(1-n)y^{1-n-1}$

$q = Q(1-n)y^{-n}$

Therefore,

F$[P(1-m)]Q[(1-n)]=0$

This is of the form $F[P,Q]=0$.

CASE 2:

Equations of the form F $(x^m p\ ,\ y^n\ q)$ = 0.

Where, m=n=1

Therefore $px = P$; $qy = Q$

Therefore F $(x^{m-1}P, y^{n-1}\ Q)$ = 0.

Therefore $F[px\ ,\ qy] = 0$.

CASE 3:

Equations of the form F $(z^k p\ ,\ z^k\ q)$ = 0.

Where k= constant

CASE 3.1:

If k$\neq -1\ put\ Z = z^{k+1}$, differentiate partially with respect to x and y we get,

Hence the equation reduces to the form,

$$F[P,Q] = 0.$$

Hence the equation reduces to the form,

$$F[P,Q] = 0.$$

Where $= \frac{\partial z}{\partial x}\ ,\ Q\ =\ \frac{\partial z}{\partial y}$.

CASE 3.2:

If k= 1 put z = log z, differentiate partially with respect to x and y,

Therefore $P\ = \frac{p}{z}\ and\ Q\ = \frac{q}{z}$.

This also reduces to the form of $F[P, Q] = 0$.

CASE 4:

Equations of the form F $(x^m z^k p\ , y^n z^k q\ ) = 0$.

If m $\neq 1$ and k $\neq -1$, $n \neq 1$.

We can take it as X = x^{1-m} ; Y = y^{1-n} .

Z = z^{k+1}.

Then the equation reduces to $F[P, Q] = 0$.

EXAMPLE 1:

Solve $x^2p^2 + y^2p^2 = z^2$.

Solution:

The given equation,

$x^2p^2 + y^2p^2 = z^2$.

By dividing z^2 *on both sides of given equation we get,*

$$\left(\frac{xp}{z}\right)^2 \left(\frac{yp}{z}\right)^2 = 1.$$

Here m = n = 1.

Let X = $\log x$, $Y = \log y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}\frac{dX}{dx} = \frac{P}{x}.$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}\frac{dY}{dy} = \frac{Q}{y}.$$

Therefore $p = \frac{P}{x}$; $q = \frac{Q}{y}$

This implies, $px = P$; $qy = Q$.

Substituting in given equation,

$$\left(\frac{xP}{x}\right)^2 \left(\frac{yQ}{y}\right)^2 = z^2.$$

Therefore ,

$P^2 + Q^2 = Z^2$

$P^2 + Q^2 - Z^2 = 0.$

COMPATIBLE SYSTEM:

Definition:

If every solution of the 1st order partial differential equation is of the form f$(x, y, z, p, q) =$ 0, is a solution of the partial differential equation g$(x, y, z, p, q) = 0$.

Then the equation is said to be compatible.

Statement:

The necessary condition that the two equations are compatible is

$$[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0.$$

EXAMPLE 1:

Show that the system of equations $xp = yq, z(xp + yq) = 2xy$ are compatible and solve them.

Solution:

Given:

$f = xp - yq \; and \; g = z(xp + yq) - 2xy.$

$$[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0.$$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp - 2y & 2x \end{vmatrix} = 2xy.$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ px+qy & zx \end{vmatrix} = -px^2 - qxy.$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -2xy.$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ px+qy & zy \end{vmatrix} = (pxy + qy^2).$$

$[f,g] = 2xy + p(-px^2 - qxy) - 2xy + q\,(pxy + qy^2).$

$[f,g] = 0.$

Therefore the given equations are compatible.

Now, $xp = yq$,

$z(xp + xp) = 2xy$

$2xpz = 2xy$

$pz = y$

$p = \frac{y}{z}$

Then, $z(yq + yq) = 2xy$

$2zyq = 2xy$

$zq = x$

$q = \frac{x}{z}$

Therefore, $p = \frac{y}{z}$ and $q = \frac{x}{z}$

We know that,

$\varphi\, dx + \Psi dy - dz = 0$

$dz = pdx + qdy$

$dz = \frac{y}{z} dx + \frac{x}{z} dy$

By integrating,

$\int dz = \int \frac{y}{z}\, dx + \int \frac{x}{z}\, dy$

$\int z\, dz = \int y\, dx + \int x\, dy$

Therefore,

$\frac{z^2}{2} = xy + xy$

$z^2 = 2\,(2xy)$

Therefore, $z^2 = 4\,xy + c$.

EXAMPLE 2:

Solve the system of equations $\frac{\partial z}{\partial x} = 6x + 3y$, $\frac{\partial z}{\partial y} = 3x - 4y$ are compatible and solve them.

Solution:

Given: $p = 6x + 3y$, $q = 3x - 4y$

Therefore, $f = p - 6x - 3y \; and \; f = q - 3x + 4y$

Since,

$[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\,\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\,\frac{\partial(f,g)}{\partial(z,q)} = 0.$

$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} -6 & 1 \\ -3 & 0 \end{vmatrix} = 3.$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0.$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} = -3.$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0.$$

$[f, g] = 3 + 0 - 3 + 0 = 0.$

$[f, g] = 0$

Since,

$p = 6x + 3y\,, q = 3x - 4y.$

By integrating,

$\int dz = \int(6x + 3y)dx \int(3x - 4y)dy$

Therefore,

$z = 3x^2 - 2y^2 + 6xy.$

SPECIAL TYPES OF FIRST ORDER EQUATIONS:

Equations involving ‘p’ and ‘q’ only:

Let $(f, p) = 0$ be the partial differential equation, then the charpit’s equations are,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} = \frac{dg}{0}$$

Since, $f_x = f_y = f_z = 0$, *we have*

$$\frac{dp}{0} = \frac{dq}{0}$$

This implies, $p = 0 \ and \ q = 0$

Therefore, $f(p, q) = 0$

$\Rightarrow f(a, b) = 0$

$\Rightarrow$b $= \varphi(a)$

Therefore, $p = a \ and \ q = \varphi(a)$

Thus $dz = pdx + qdy$

$\Rightarrow dz = adx + bdy$

$\Rightarrow z = ax + \varphi(a)y + b$

EXAMPLE 1:

Consider the equation, p+q = pq.

Solution:

The given equation can be written as $f = p + q - pq = 0$

This equation contains p and q only,

From the charpit'sequationwe get,

$p = a \ and \ q = b.$

Substitute these values in above equation we get,

$a + b - ab = 0$

$a + b(1 - a) = 0$

$b(1 - a) = -a$

$$b = \frac{-a}{(1 - a)}$$

$therefore,$

$b = \frac{a}{(a-1)}.$

From the charpit's method we have,

$dz = pdx + qdy$

Therefore, $z = ax + \varphi(a)y + b$

$$\Rightarrow ax + \frac{a}{(a-1)}y + b$$

$\Rightarrow z = (a(a-1)x + ay + b)(\frac{1}{(a-1)})$

Therefore,

$(ax - z)(a - c) + ay = b$, which is the required solution.

Equations not involving independent variable:

Let $(z, f, p) = 0 \ldots .(1)$, be the partial differential equation, then the charpit's equations are,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} = \frac{dg}{0}$$

This implies,

$$\frac{dp}{p} = \frac{dq}{q}$$

Where, $f_x = f_y = 0.$

That is,

$\frac{dp}{p} = \frac{dq}{q} = a.$

$\Rightarrow p = aq \ldots .(2) where\ 'a' is\ constant.$

Solving equations (1) and (2) for substituting the values of p and q in the charpit's equation,

$$dz = pdx + qdy$$

We get the solution of the equation (1).

EXAMPLE 1:

Consider the partial differential equation $z = p^2 - q^2$(not involving independent variable x and y).

Solution:

Given equations $(z, p, q) = p^2 - q^2 - z \ldots.(1)$

From the charpit's equation we get,

$\frac{dp}{p} = \frac{dq}{q} = a.$

$\Rightarrow p = aq \ldots..(2)$

Where 'a' is constant.

Substitute $p = aq \; in \; (1)$ we get,

$p^2 - q^2 - z = 0$

$a^2q^2 - q^2 - z = 0$

$q^2(a^2 - 1) = z$

$$q^2 = \frac{z}{a^2 - 1}$$

Taking square roots we get,

$$q = \frac{\sqrt{z}}{\sqrt{a^2 - 1}}$$

$\Rightarrow p = aq$

$\Rightarrow p = \frac{\sqrt{z}}{\sqrt{a^2-1}}$

Therefore from the charpit's equation,

$dz = pdx + qdy$

$dz = \frac{a\sqrt{z}}{\sqrt{a^2-1}}dx + \frac{\sqrt{z}}{\sqrt{a^2-1}}dy$

$\sqrt{a^2-1}\,dz = a\sqrt{z}\,dx + \sqrt{z}\,dy$

$\frac{\sqrt{a^2-1}}{\sqrt{z}}\,dz = a\,dx + dy$

Integrating we get,

$2\sqrt{a^2-1}\sqrt{z} = ax + y + c$

Which is the required solution.

EXAMPLE 2:

Solve the following partial differential equation $(p+q)(px+qy) = 1$.

Solution:

Let the equation is of the form $f(x, y, z, p, q) = 0$

Therefore $f = (p+q)(px+qy) - 1 \ldots (1)$

The charpit's equation,

$\frac{dp}{p} = \frac{dq}{q} = a$

$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} = \frac{dg}{0}$

That is,

$\frac{dp}{p} = \frac{dq}{q} = c_1$

$\Rightarrow p = c_1 q$

Substituting in given equation,

$(c_1q + q)(c_1qx + qy) = 0.$

$q^2(c_1 + 1)(c_1x + y) = 0$

Taking square roots on both sides we get,

$$q = \frac{1}{\sqrt{(c_1 + 1)(c_1x + y)}}$$

Therefore the charpit's equation,

$dz = pdx + qdy$

$dz = q(c_1dx + dy)$

$$dz = \frac{1}{\sqrt{(c_1 + 1)(c_1x + y)}}(c_1dx + dy)$$

Integrating we get,

$$z + b = \frac{(c_1x + y)}{\sqrt{(c_1 + 1)(c_1x + y)}}$$

$$z + b = \frac{\sqrt{(c_1x+y)}}{\sqrt{(c_1+1)}}.$$

ORIGIN OF SECOND ORDER EQUATION:

Equations that can be integrated by inspection.

Since, $p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$; $s = \frac{\partial^2 z}{\partial x \partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$; $r = \frac{\partial^2 z}{\partial x^2}$

EXAMPLE 1:

Solve: $s = 2x + 2y$.

Solution:

Given: $s = 2x + 2y$.

Substituting the values of 's' we get,

$$\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$$

Integrating with respect to 'x' and keeping 'y' as constant we get,

$$\frac{\partial z}{\partial y} = x^2 + 2xy + \varphi(y)$$

Integrating with respect 'y' and keeping 'x' as constant we get,

$$z = x^2 y + 2x\frac{y^2}{2} + \int \varphi(y) dy + f(x)$$

$$z = x^2 y + xy^2 + \int \varphi(y) dy + f(x)$$

$$z = x^2 y + xy^2 + F(x) + f(x).$$

EXAMPLE 2:

Solve: $r = 6x$.

Solution:

Given: $r = 6x$

Substituting the values of 'r' we get,

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

Integrating with respect to 'x' and keeping 'y' as constant,

$$\frac{\partial z}{\partial x} = 3x^2 + \varphi(y)$$

Integrating with respect to 'x' and keeping 'y' as constant,

$$z = 3\frac{x^3}{3} + x\,\varphi(y) + \Psi(y).$$

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answers
A partial differential equation is one which involves __________ derivatives	single	ordinary	partial	linear	partial
The three variables involves in Pdx+Qdy+Rdz=0 is called_______________	pfaffian	lagrange	recursive	quadratic	pfaffian
The general solution of PDE is of the form	C.F+P.I	C.F-P.I	C.F*P.I	C.F/P.I	C.F+P.I
The Equation is of the form Z=px+qy+f(p,q) is called_________	clairaut	charpit	crout	separable	clairaut
f(x,p)=g(y,q) is called _______equation	clairaut	charpit	crout	separable	separable
Reducible equation is defined as te product of _______factors.	linear	nonlinear	polynomial	recursive	linear
If the operator F(D,D`) is reducible te order in which the linear factors occur is ___	important	unimportant	considerabl e	reluctable	unimportant
If u is the C.F and z_1 is particular P.I then the general solution is	u+z1	u-z1	u*z1	u/z1	u+z1
L(z)+f(x,y,z,p,q)=0 where L is the ________ operator	laplace	differential	lagrange	longdivisio	differential
If S^2-4RT>0 then it is	elliptic	parabolic	hyperbolic	diffusion	hyperbolic
If S^2-4RT<0 then it is	elliptic	parabolic	hyperbolic	diffusion	elliptic
If S^2-4RT=0 then it is	elliptic	parabolic	hyperbolic	diffusion	parabolic
The order of PDE to be the order of the derivative of ______ order occurring in it.	lowest	highest	first	second	highest
In Rr+Ss+Tt+Pp+Qq=W, W is the function of ________	x	y	x and y	z	x and y
In F(D,D')=0the term D' denotes about the variable	x	y	z	p	y
The solution of the PDE consists ______ main parts	2	3	4	5	2
The Fourier transform is defined in the interval	(-∞,∞)	(0, ∞)	(∞,0)	(0,pi)	(-∞,∞)
The Integral transform reduce the PDE to	ODE	DE	Integral	homogeneo us	DE
TheLaplace transform is defined in the interval	(-∞,∞)	(0, ∞)	(∞,0)	(0,pi)	(0, ∞)
If f and g are said to be compatible then it have _______ solution	unique	different	linear	non linear	unique

UNIT-II
SYLLABUS

Method of separation of variables-The method of Integral transforms

Second Order Linear Partial Differential Equations

One-dimensional undamped wave equation; D'Alembert solution of the wave equation; damped wave equation and the general wave equation; two-dimensional Laplace equation

The second type of second order linear partial differential equations in 2 independent variables is the one-dimensional wave equation. Together with the heat conduction equation, they are sometimes referred to as the "*evolution equations*" because their solutions "evolve", or change, with passing time. The simplest instance of the one-dimensional wave equation problem can be illustrated by the equation that describes the standing wave exhibited by the motion of a piece of undamped vibrating elastic string.

Undamped One-Dimensional Wave Equation: Vibrations of an Elastic String

Consider a piece of thin flexible string of length *L*, of negligible weight. Suppose the two ends of the string are firmly secured ("clamped") at some supports so they will not move. Assume the set-up has no damping. Then, the vertical displacement of the string, $0 < x < L$, and at any time $t > 0$, is given by the displacement function $u(x, t)$. It satisfies the *homogeneous one-dimensional undamped wave equation*:

$$a^2 u_{xx} = u_{tt}$$

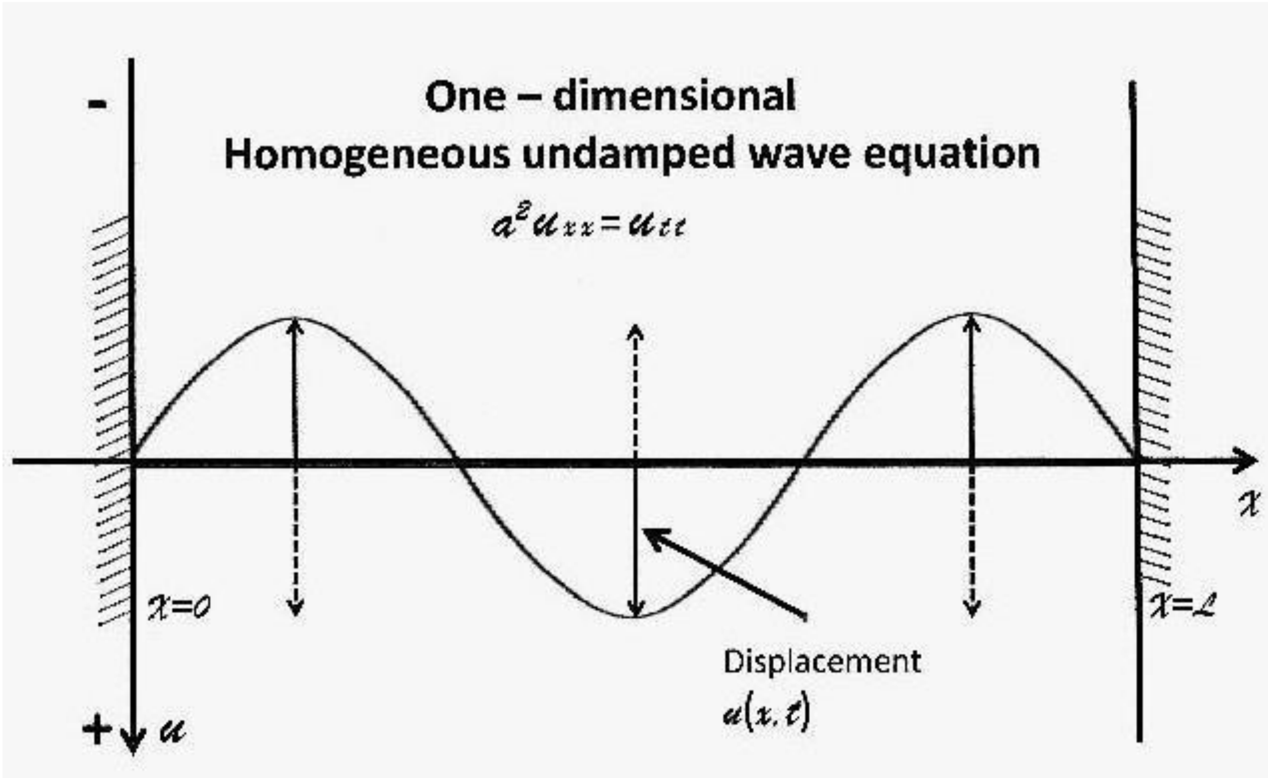
Where the constant coefficient a^2 is given by the formula $a^2 = T/\rho$, such that a = horizontal propagation speed (also known as *phase velocity*) of the wave motion, T = force of tension exerted on the string, ρ = mass density (mass per unit length). It is subjected to the homogeneous boundary conditions

$$u(0, t) = 0, \quad \text{and} \qquad u(L, t) = 0, \quad t > 0.$$

The two boundary conditions reflect that the two ends of the string are clamped in fixed positions. Therefore, they are held motionless at alltime.

The equation comes with 2 initial conditions, due to the fact that it contains the second partial derivative of time, u_{tt}. The two initial conditions are the initial (vertical) displacement $u(x, 0)$, and the initial (vertical) velocity $u_t(x, 0)$*, both are arbitrary functions of x alone. (Note that the string is merely the medium for the wave, it does not itself move horizontally, it only vibrates, vertically, in place. The resulting undulation, or the wave-like "shape" of the string, is what moves horizontally.)

* Velocity = rate of change of displacement with respect to time. The other first partial derivative u_x represents the slope of the string at a point x and time t.



Hence, what we have is the following initial-boundary value problem:

(Wave equation)	$a^2 u_{xx} = u_{tt}$,	$0 < x < L$, $t > 0$,
(Boundary conditions)	$u(0, t) = 0$,and	$u(L, t) = 0$,
(Initial conditions)	$u(x, 0) = f(x)$,and	$u_t(x, 0) = g(x)$.

We first let $u(x, t) = X(x)T(t)$ and separate the wave equation into two ordinary differential equations. Substituting $u_{xx} = X'' T$ and $u_{tt} = X T''$ into the wave equation, it becomes

$$a^2 X'' T = X T''.$$

Dividing both sides by $a^2 X T$:

$$\frac{X''}{X} = \frac{T''}{a^2 T}$$

As for the heat conduction equation, it is customary to consider the constant a^2 as a function of t and group it with the rest of t-terms. Insert the constant of separation and break apart the equation:

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$$\frac{X''}{X} = -\lambda \quad \rightarrow \quad X'' = -\lambda X \quad \rightarrow \quad X'' + \lambda X = 0,$$

$$\frac{T''}{a^2 T} = -\lambda \quad \rightarrow \quad T'' = -a^2 \lambda T \quad \rightarrow \quad T'' + a^2 \lambda T = 0.$$

The boundary conditions also separate:

$u(0, t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$ or $T(t) = 0$
$u(L, t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$ or $T(t) = 0$

As usual, in order to obtain nontrivial solutions, we need to choose $X(0) = 0$ and $X(L) = 0$ as the new boundary conditions. The result, after separation of variables, is the following simultaneous system of ordinary differential equations, with a set of boundary conditions:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

$$T'' + a^2 \lambda T = 0.$$

The next step is to solve the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X(L) = 0.$$

We have already solved this eigenvalue problem, recall. The solutions are

Eigenvalues: $\lambda = \dfrac{n^2\pi^2}{L^2}$, $n = 1, 2, 3, \ldots$

Eigenfunctions: $X_n = \sin\dfrac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$

Next, substitute the eigenvalues found above into the second equation to find $T(t)$. After putting eigenvalues λ into it, the equation of T becomes

$$T'' + a^2\frac{n^2\pi^2}{L^2}T = 0.$$

It is a second order homogeneous linear equation with constant coefficients. It's characteristic have a pair of purely imaginary complex conjugate roots:

$$r = \pm\frac{an\pi}{L}i.$$

Thus, the solutions are simple harmonic:

$$T_n(t) = A_n\cos\frac{an\pi t}{L} + B_n\sin\frac{an\pi t}{L}, \qquad n = 1, 2, 3, \ldots$$

Multiplying each pair of X_n and T_n together and sum them up, we find the general solution of the one-dimensional wave equation, with both ends fixed, to be

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}.$$

There are two sets of (infinitely many) arbitrary coefficients. We can solve for them using the two initial conditions.

Set $t = 0$ and apply the first initial condition, the initial (vertical) displacement of the string $u(x, 0) = f(x)$, we have

$$u(x,0) = \sum_{n=1}^{\infty}\left(A_n \cos(0) + B_n \sin(0)\right)\sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

Therefore, we see that the initial displacement $f(x)$ needs to be a Fourier sine series. Since $f(x)$ can be an arbitrary function, this usually means that we need to expand it into its odd periodic extension (of period $2L$). The coefficients A_n are then found by the relation $A_n = b_n$, where b_n are the corresponding Fourier sine coefficients of $f(x)$. That is

$$A_n = b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Notice that the entire sequence of the coefficients A_n are determined exactly by the initial displacement. They are completely independent of the other sequence of coefficients B_n, which are determined solely by the second initial condition, the initial (vertical) velocity of the string. To find B_n, we differentiate $u(x, t)$ with respect to t and apply the initial velocity, $u_t(x, 0) = g(x)$.

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-A_n \frac{an\pi}{L} \sin \frac{an\pi t}{L} + B_n \frac{an\pi}{L} \cos \frac{an\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

Set $t = 0$ and equate it with $g(x)$:

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{an\pi}{L} \sin \frac{n\pi x}{L} = g(x) \, .$$

We see that $g(x)$ needs also be a Fourier sine series. Expand it into its odd periodic extension (period $2L$), if necessary. Once $g(x)$ is written into a sine series, the previous equation becomes

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{an\pi}{L} \sin \frac{n\pi x}{L} = g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Compare the coefficients of the like sine terms, we see

$$B_n \frac{an\pi}{L} = b_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \, .$$

Therefore,

$$B_n = \frac{L}{an\pi} b_n = \frac{2}{an\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \, .$$

As we have seen, half of the particular solution is determined by the initial displacement, the other half by the initial velocity. The two halves are determined independent of each other. Hence, if the initial displacement $f(x) = 0$, then all $A_n = 0$ and $u(x, t)$ contains no sine-terms of t. If the initial velocity $g(x) = 0$, then all $B_n = 0$ and $u(x, t)$ contains no cosine-terms of t.

Let us take a closer look and summarize the result for these 2 easy special cases, when either $f(x)$ or $g(x)$ is zero.

Special case I: Nonzero initial displacement, zero initial velocity: $f(x) \neq 0$, $g(x) = 0$.

Since $g(x) = 0$, then $B_n = 0$ for all n.

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \ldots$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos\frac{an\pi t}{L}\sin\frac{n\pi x}{L}.$$

The D'Alembert Solution

In 1746, Jean D'Alembert[†] produced an alternate form of solution to the wave equation. His solution takes on an especially simple form in the above case of zero initial velocity.

Use the product formula $\sin(A)\cos(B) = [\sin(A-B) + \sin(A+B)]/2$, the solution above can be rewritten as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n\left(\sin\frac{n\pi(x-at)}{L} + \sin\frac{n\pi(x+at)}{L}\right)$$

Therefore, the solution of the undamped one-dimensional wave equation with zero initial velocity can be alternatively expressed as

$$u(x,t) = [F(x-at) + F(x+at)]\,/\,2.$$

In which $F(x)$ is the odd periodic extension (period $2L$) of the initial displacement $f(x)$.

An interesting aspect of the D'Alembert solution is that it readily shows that the starting waveform given by the initial displacement would keep its general shape, but it would also split exactly into two halves. The two halves of the wave form travel in the opposite directions at the same finite speed of propagation a. This can be seen by the fact that the two halves of the wave form, in terms of x, are being translated/moved in the opposite direction, to the right and left, in the form of phase shifts, at the rate of distance a units per unit time. Hence the value a is also known as the wave's phase velocity.

[†] Jean le Rond d"Alembert (1717 – 1783) was a French mathematician and physicist. He is perhaps best known to calculus students as the inventor of the *Ratio Test* for convergence.

Furthermore, once the "wave front" has passed over a point on the string, the displacement at that point will be restored to its previous state before the arrival of the wave. In physics, this aspect of a clearly-defined, echo-less, wave motion of a one-dimensional wave is called the *Huygens' Principle*. (The principle also holds for solutions of a three-dimensional wave equation. But it is not true for two-dimensional waves.)

Special case II: Zero initial displacement, nonzero initial velocity: $f(x) = 0$, $g(x) \neq 0$.

Since $f(x) = 0$, then $A_n = 0$ for all n.

$$B_n = \frac{2}{an\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, 3, \ldots$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{an\pi t}{L} \sin \frac{n\pi x}{L}.$$

Example: Solve the one-dimensional wave problem

$$9\,u_{xx} = u_{tt}\ , \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \text{ and } \quad u(5, t) = 0,$$
$$u(x, 0) = 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x),$$
$$u_t(x, 0) = 0.$$

First note that $a^2 = 9$ (so $a = 3$), and $L = 5$.

The general solution is, therefore,

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5} \right) \sin\frac{n\pi x}{5}.$$

Since $g(x) = 0$, it must be that all $B_n = 0$. We just need to find A_n. We also see that $u(x, 0) = f(x)$ is already in the form of a Fourier sine series. Therefore, we just need to extract the corresponding Fourier sine coefficients:

$$A_5 = b_5 = 4,$$
$$A_{10} = b_{10} = -1,$$
$$A_{25} = b_{25} = -3,$$
$$A_n = b_n = 0, \text{ for all other } n,\ n \neq 5, 10, \text{ or } 25.$$

Hence, the particular solution is

$$u(x, t) = 4\cos(3\pi t)\sin(\pi x) - \cos(6\pi t)\sin(2\pi x) - 3\cos(15\pi t)\sin(5\pi x).$$

We can also solve the previous example using D'Alembert's solution. The problem has zero initial velocity and its initial displacement has already been expanded into the required Fourier sine series, $u(x,0) = 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x) = F(x)$. Therefore, the solution can also be found by using the formula $u(x, t) = [F(x - at) + F(x + at)] / 2$, where $a = 3$. Thus

$$u(x, t) = [\ [\ 4\sin(\pi(x + 3t)) + 4\sin(\pi(x - 3t))\ ] - [\sin(2\pi(x+ 3t)) + \sin(2\pi(x + 3t))\ ] - [3\sin(5\pi(x + 3t)) + 3\sin(5\pi(x + 3t))\ ]\ ] / 2$$

Indeed, you could easily verify (do this as an exercise) that the solution obtained this way is identical to our previous answer. Just apply the addition formula of sine function ($\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$) to each term in the above solution and simplify.

Example: Solve the one-dimensional wave problem

$$9\,u_{xx} = u_{tt}, \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \text{ and } \quad u(5, t) = 0,$$
$$u(x, 0) = 0,$$
$$u_t(x, 0) = 4.$$

As in the previous example, $a^2 = 9$ (so $a = 3$), and $L = 5$. Therefore, the general solution remains

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5}\right)\sin\frac{n\pi x}{5}.$$

Now, $f(x) = 0$, consequently all $A_n = 0$. We just need to find B_n. The initial velocity $g(x) = 4$ is a constant function. It is not an odd periodic function. Therefore, we need to expand it into its odd periodic extension (period $T = 10$), then equate it with $u_t(x, 0)$. In short:

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{3n\pi}\int_0^5 4\sin\frac{n\pi x}{5}\,dx$$

$$= \begin{cases} \dfrac{80}{3n^2\pi^2}, & n = odd \\ 0, & n = even \end{cases}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{80}{3(2n-1)^2\pi^2}\sin\frac{3(2n-1)\pi t}{5}\sin\frac{(2n-1)\pi x}{5}.$$

The Structure of the Solutions of the Wave Equation

In addition to the fact that the constant a is the standing wave's propagation speed, several other observations can be readily made from the solution of the wave equation that give insights to the nature of the solution.

To reduce the clutter, let us look at the form of the solution when there is no initial velocity (when $g(x) = 0$). The solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{an\pi t}{L} \sin \frac{n\pi x}{L}.$$

The sine terms are functions of x. They described the spatial wave patterns (the wavy "shape" of the string that we could visually observe), called the *normal modes*, or *natural modes*. The frequencies of those sine waves that we could see, $n\pi / L$, are called the *spatial frequencies* of the wave. They are also known as the *wave numbers*. It measures the angular motion, in radians, per unit distance that the wave travels. The "period" of each spatial (sine) function, $2/(n\pi/L) = 2L/n$, is the *wave length* of each term. Meanwhile, the cosine terms are functions of t, they give the vertical displacement of the string relative to its equilibrium position (which is just the horizontal, or the x-axis). They describe the up-and-down vibrating motion of the string at each point of the string. These *temporal frequencies* (the frequencies of functions of t; in this case, the cosines') are the actual frequencies of oscillating motion of vertical displacement. Since this is the undamped wave equation, the motion of the string is simple harmonic. The frequencies of the cosine terms, $an\pi / L$ (measured in radians per second), are called the *natural frequencies* of the string. In a string instrument, they are the frequencies of the sound that we could hear. The corresponding *natural periods* (= 2π/natural frequency) are, therefore, $T = 2L/an$.

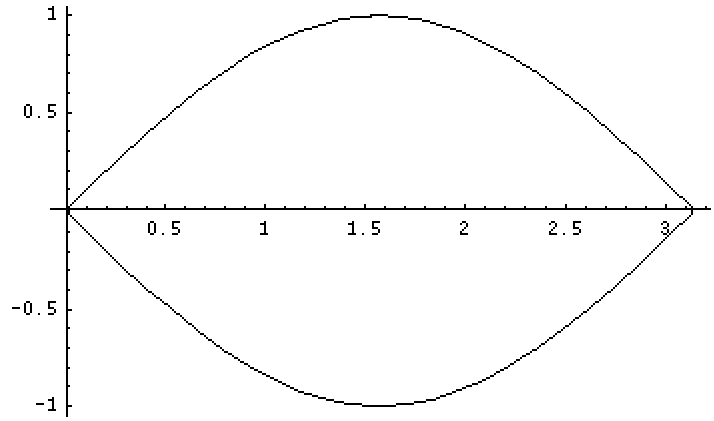
For $n = 1$, the observable spatial wave pattern is that of $\sin(\pi x / L)$. The wave length is $2L$, meaning the length L string carries only a half period of the sinusoidal motion. It is the string's first natural mode. The first natural frequency of oscillation, $a\pi / L$, is called the *fundamental frequency* of the string. It is, given the set-up, the lowest frequency note the vibrating string can produce. It is also called, in acoustics, as the *first harmonic* of the string.

For $n = 2$, the spatial wave pattern is $\sin(2\pi x / L)$ is the second natural mode. Its wavelength is L, which is the length of the string itself. The second natural frequency of oscillation, $2a\pi / L$, is also called the second harmonic, or the *first overtone*, of the string. It is exactly twice of the string's fundamental frequency; hence its wavelength (= L) is only half as long. Acoustically, it produces a tone that is exactly one *octave* higher than the first harmonic. For $n = 3$, the third natural frequency, $3a\pi / L$, is also called the third harmonic, or the second overtone. It is 3 times larger than the fundamental frequency and, at a 3:2 ratio over the second harmonic, is situated exactly halfway between the adjacent octaves (at the second and the fourth harmonics). The fourth natural frequency (fourth harmonic/ third overtone), $4a\pi / L$, is four times larger than the fundamental frequency and twice of that the second natural frequency. The tone it produces is, therefore, exactly 2 octaves and 1 octave higher than those generated by the first and second harmonics, respectively. Together, the sequence of all positive integer multiples of the fundamental frequency is called a *harmonic series* (not to be confused with that other harmonic series that you have studied in calculus).

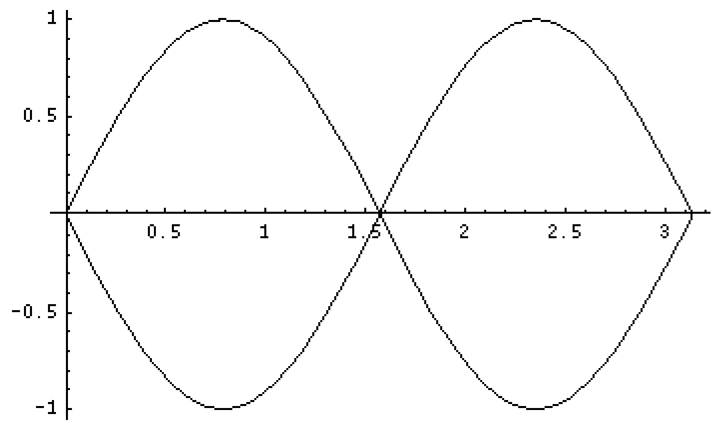
The motion of the string is the combination of all its natural modes, as indicated by the terms of the infinite series of the general solution. The presence, and magnitude, of the nature modes are solely determined by the (Fourier sine series expansion of) initial conditions.

Lastly, notice that the "wavelike" behavior of the solution of the undamped wave equation, quite unlike the solution of the heat conduction equation discussed earlier, does not decrease in amplitude/intensity with time. It never reaches a steady state (unless the solution is trivial, $u(x, t) = 0$, which occurs when $f(x) = g(x) = 0$). This is a consequence of the fact that the undamped wave motion is a thermodynamically reversible process that needs not obey the second law of Thermodynamics.

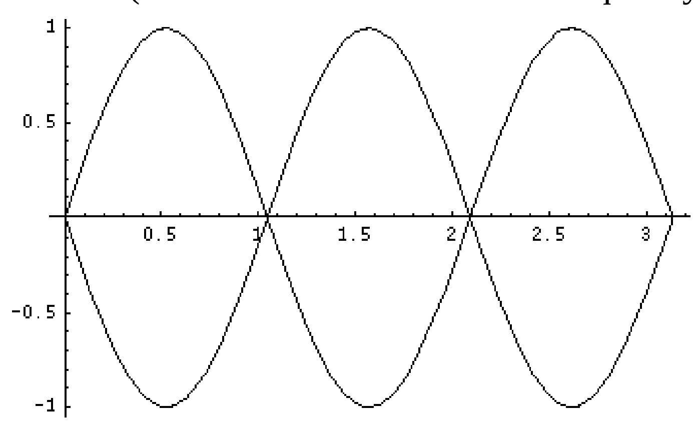
First natural mode (oscillates at the fundamental frequency / 1st harmonic):



Second natural mode (oscillates at the 2nd natural frequency / 2nd harmonic):



Third natural mode (oscillates at the 3rd natural frequency / 3rd harmonic):



Summary of Wave Equation: Vibrating String Problems

The vertical displacement of a vibrating string of length L, securely clamped at both ends, of negligible weight and without damping, is described by the homogeneous undamped wave equation initial-boundary value problem:

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0,$$
$$u(0, t) = 0, \text{ and } \qquad u(L, t) = 0,$$
$$u(x, 0) = f(x), \text{ and } u_t(x,0) = g(x).$$

The general solution is

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{an\pi t}{L} + B_n \sin \frac{an\pi t}{L} \right) \sin \frac{n\pi x}{L}.$$

The particular solution can be found by the formulas:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad \text{and}$$

$$B_n = \frac{2}{an\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

The solution waveform has a constant (horizontal) propagation speed, in both directions of the x-axis, of a. The vibrating motion has a (vertical) velocity given by $u_t(x, t)$ at any location $0 < x < L$ along the string.

Exercises E-4.1:

1. Solve the vibrating string problem of the given initial conditions.

$$4\,u_{xx} = u_{tt}\quad , \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(0, t) = 0, \qquad u(\pi, t) = 0,$$

(a) $u(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x)$,
$u_t(x, 0) = 0$.

(b) $u(x, 0) = 0$,
$u_t(x, 0) = 6$.

(c) $u(x, 0) = 0$,
$u_t(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x)$.

2. Solve the vibrating string problem.

$$100\,u_{xx} = u_{tt}\quad , \qquad 0 < x < 2, \qquad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(2, t) = 0,$$
$$u(x, 0) = 32\sin(\pi x) + e^2\sin(3\pi x) + 25\sin(6\pi x),$$
$$u_t(x, 0) = 6\sin(2\pi x) - 16\sin(5\pi x / 2).$$

3. Solve the vibrating string problem.

$$25\,u_{xx} = u_{tt}\quad , \qquad 0 < x < 1, \qquad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(2, t) = 0,$$
$$u(x, 0) = x - x^2,$$
$$u_t(x, 0) = \pi.$$

4. Verify that the D'Alembert solution, $u(x, t) = [F(x - at) + F(x + at)] / 2$, where $F(x)$ is an odd periodic function of period $2L$ such that $F(x) = f(x)$ on the interval $0 < x < L$, indeed satisfies the given initial-boundary value problem by checking that it satisfies the wave equation, boundary conditions, and initial conditions.

$$
\begin{aligned}
&a^2 u_{xx} = u_{tt}\ , && 0 < x < L, \quad t > 0,\\
&u(0, t) = 0, && u(L, t) = 0,\\
&u(x, 0) = f(x), && u_t(x, 0) = 0.
\end{aligned}
$$

5. Use the method of separation of variables to solve the following wave equation problem where the string is rigid, but not fixed in place, at both ends (i.e., it is inflexible at the endpoints such that the slope of displacement curve is always zero at both ends, but the two ends of the string are allowed to freely slide in the vertical direction).

$$
\begin{aligned}
&a^2 u_{xx} = u_{tt}\ , && 0 < x < L, \quad t > 0,\\
&u_x(0, t) = 0, && u_x(L, t) = 0,\\
&u(x, 0) = f(x), && u_t(x, 0) = g(x).
\end{aligned}
$$

6. What is the steady-state displacement of the string in #5? What is $\lim_{t \to \infty} u(x, t)$? Are they the same?

Answers E-4.1:

1. (a) $u(x, t) = 12\cos(4t)\sin(2x) - 16\cos(10t)\sin(5x) + 24\cos(12t)\sin(6x)$.
   (c) $u(x, t) = 3\sin(4t)\sin(2x) - 1.6\sin(10t)\sin(5x) + 2\sin(12t)\sin(6x)$.

5. The general solution is

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L}\right)\cos\frac{n\pi x}{L}.$$

The particular solution can be found by the formulas:

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad B_0 = \frac{1}{L}\int_0^L g(x)\,dx, \text{ and}$$

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx.$$

6. The steady-state displacement is the constant term of the solution, A_0. The limit does not exist unless $u(x, t) = C$ is a constant function, which happens when $f(x) = C$ and $g(x) = 0$, in which case the limit is C. They are not the same otherwise.

The General Wave Equation

The most general form of the one-dimensional wave equation is:

$$a^2 u_{xx} + F(x, t) = u_{tt} + \gamma\, u_t + k\, u.$$

Where a = the propagation speed of the wave,
γ = the damping constant
k = (external) restoration factor, such as when vibrations occur in an elastic medium.
$F(x, t)$ = arbitrary external forcing function (If $F = 0$ then the equation is homogeneous, else it is nonhomogeneous.)

The Telegraph Equation

The most well-known example of (a homogeneous version of) the general wave equation is the *telegraph equation*. It describes the voltage $u(x, t)$ inside a piece of telegraph / transmission wire, whose electrical properties per unit length are: resistance R, inductance L, capacitance C, and conductance of leakage current G:

$$a^2 u_{xx} = u_{tt} + \gamma u_t + k u.$$

Where $a^2 = 1 / LC$, $\gamma = G / C + R / L$, and $k = GR / CL$.

Example: The One-Dimensional Damped Wave Equation

$$a^2 u_{xx} = u_{tt} + \gamma u_t , \qquad \gamma \neq 0.$$

Suppose boundary conditions remain as the same (both ends fixed): $(0, t) = 0$, and $u(L, t) = 0$.

The equation can be separated as follow. First rewrite it as:

$$a^2 X'' T = X T'' + \gamma X T',$$

Divide both sides by $a^2 X T$, and insert a constant of separation:

$$\frac{X''}{X} = \frac{T'' + \gamma T'}{a^2 T} = -\lambda .$$

Rewrite it into 2 equations:

$$X'' = -\lambda X \qquad \rightarrow \qquad X'' + \lambda X = 0,$$

$$T'' + \gamma T' = -a^2 \lambda T \qquad \rightarrow \qquad T'' + \gamma T' + a^2 \lambda T = 0.$$

The boundary conditions also are separated, as usual:

$u(0, t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$ or $T(t) = 0$
$u(L, t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$ or $T(t) = 0$

As before, setting $T(t) = 0$ would result in the constant zero solution only. Therefore, we must choose the two (nontrivial) conditions in terms of x: $X(0) = 0$, and $X(L) = 0$.

After separation of variables, we have the system

$$X'' + \lambda X = 0, \qquad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

$$T'' + \gamma T' + \alpha^2 \lambda T = 0.$$

The next step is to find the eigenvalues and their corresponding eigenfunctions of the boundary value problem

$$X'' + \lambda X = 0, \qquad X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

This is a familiar problem that we have encountered more than once previously. The eigenvalues and eigenfunctions are, recall,

Eigenvalues: $\lambda = \dfrac{n^2\pi^2}{L^2}$, $n = 1, 2, 3, \ldots$

Eigenfunctions: $X_n = \sin\dfrac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$

The equation of *t*, however, has different kind of solutions depending on the roots of its characteristic equation.

(Optional topic) Nonhomogeneous Undamped Wave Equation

Problems of partial differential equation that contains a nonzero forcing function (which would make the equation itself a *nonhomogeneous partial differential equation*) can sometimes be solved using the same idea that we have used to handle nonhomogeneous boundary conditions – by considering the solution in 2 parts, a steady-state part and a transient part. This is possible when the forcing function is independent of time t, which then could be used to determine the steady-state solution. The transient solution would then satisfy a certain homogeneous equation. The 2 parts are thus solved separately and their solutions are added together to give the final result. Let us illustrate this idea with a simple example: when the string's weight is no longer "negligible".

Example: A flexible string of length L has its two ends firmly secured. Assume there is no damping. Suppose the string has a weight density of 1 Newton per meter. That is, it is subject to, uniformly across its length, a constant force of $F(x, t) = 1$ unit per unit length due to its own weight. Let $u(x,t)$ be the vertical displacement of the string, $0 < x < L$, and at any time $t > 0$. It satisfies the nonhomogeneous one-dimensional undamped wave equation:

$$a^2 u_{xx} + 1 = u_{tt}.$$

The usual boundary conditions $u(0, t) = 0$, and $u(L, t) = 0$, apply. Plus the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$.

Since the forcing function is independent of time t, its effect is to impart, permanently, a displacement on the string that depends only on the location (the effect is subject to the boundary conditions, thus might change with x). That is, the effect is to introduce a nonzero steady-state displacement, $v(x)$. Hence, we can rewrite the solution $u(x, t)$ as:

$$u(x, t) = v(x) + w(x, t).$$

By setting t to be a constant and rewrite the equation and the boundary conditions to be dependent of x only, the steady-state solution $v(x)$ must satisfy:

$$a^2 v'' + 1 = 0,$$
$$v(0) = 0, \quad v(L) = 0.$$

Rewrite the equation as $v'' = -1 / a^2$, and integrate twice, we get

$$v(x) = \frac{-1}{2a^2} x^2 + C_1 x + C_2.$$

Apply the boundary conditions to find $C_1 = L / 2a^2$ and $C_2 = 0$:

$$v(x) = \frac{-1}{2a^2} x^2 + \frac{L}{2a^2} x.$$

Comment: Thus, the sag of a wire or cable due to its own weight can be seen as a manifestation of the steady-solution of the wave equation. The sag is also parabolic, rather than sinusoidal, as one might have reasonably assumed, in nature.

We can then subtract out $v(x)$ from the equation, boundary conditions, and the initial conditions (try this as an exercise), the transient solution $w(x, t)$ must satisfy:

$$a^2 w_{xx} = w_{tt}, \qquad 0 < x < L, \qquad t > 0,$$
$$w(0, t) = 0, \qquad w(L, t) = 0,$$
$$w(x, 0) = f(x) - v(x), \quad w_t(x, 0) = g(x).$$

The problem is now transformed to the homogeneous problem we have already solved. The solution is just

$$w(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{an\pi t}{L}+B_n\sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}.$$

Combining the steady-state and transient solutions, the general solution is found to be

$$u(x,t)=v(x)+w(x,t)=\frac{-1}{2a^2}x^2+\frac{L}{2a^2}x+\sum_{n=1}^{\infty}\left(A_n\cos\frac{an\pi t}{L}+B_n\sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

The coefficients can be calculated and the particular solution determined by using the formulas:

$$A_n=\frac{2}{L}\int_0^L\big(f(x)-v(x)\big)\sin\frac{n\pi x}{L}\,dx,$$ and

$$B_n=\frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Note: Since the velocity $u_t(x,t)=v_t(x)+w_t(x,t)=0+w_t(x,t)=w_t(x,t)$. The initial velocity does not need any adjustment, as $u_t(x,0)=w_t(x,0)=g(x)$.

Comment: We can clearly see that, even though a nonzero steady-state solution exists, the displacement of the string still will not converge to it as $t\to\infty$.

The Laplace Equation / Potential Equation

The last type of the second order linear partial differential equation in 2 independent variables is the two-dimensional *Laplace equation*, also called the *potential equation*. Unlike the other equations we have seen, a solution of the Laplace equation is always a steady-state (i.e. time-independent) solution. Indeed, the variable t is not even present in the Laplace equation. The Laplace equation describes systems that are in a state of equilibrium whose behavior does not change with time. Some applications of the Laplace equation are finding the potential function of an object acted upon by a gravitational / electric / magnetic field, finding the steady-state temperature distribution of the (2- or 3-dimensional) heat conduction equation, and the steady-state flow of an ideal fluid (where the flow velocity forms a vector field that has zero curl and zero divergence).

Since the time variable is not present in the Laplace equation, any problem of the Laplace equation will not, therefore, have any initial condition. A Laplace equation problem has only boundary conditions.

Let $u(x,y)$ be the potential function at a point (x, y), then it is governed by the two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Any real-valued function having continuous first and second partial derivatives that satisfies the two-dimensional Laplace equation is called a *harmonic function*.

Similarly, suppose $u(x, y, z)$ is the potential function at a point (x, y, z), then it is governed by the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Comment: The one-dimensional Laplace equation is rather dull. It is merely $u_{xx} = 0$, where u is a function of x alone. It is not a partial differential equation, but rather a simple integration problem of $u'' = 0$. (What is its solution? Where have we seen it just very recently?)

The boundary conditions that accompany a 2-dimensional Laplace equation describe the conditions on the boundary curve that encloses the 2-dimensional region in question. While those accompany a 3-dimensional Laplace equation describe the conditions on the boundary surface that encloses the 3-dimensional spatial region in question.

The Relationships among Laplace, Heat, and Wave Equations (Optional topic)

Now let us take a step back and see the bigger picture: how the homogeneous heat conduction and wave equations are structured, and how they are related to the Laplace equation of the same spatial dimension.

Suppose $u(x, y)$ is a function of two variables, the expression $u_{xx} + u_{yy}$ is called the *Laplacian* of u. It is often denoted by

$$\nabla^2 u = u_{xx} + u_{yy}.$$

Similarly, for a three-variable function $u(x, y, z)$, the 3-dimensional Laplacian is then

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}.$$

(As we have just noted, in the one-variable case, the Laplaian of $u(x)$, degenerates into $\nabla^2 u = u''$.)

The homogeneous heat conduction equations of 1-, 2-, and 3- spatial dimension can then be expressed in terms of the Laplacians as:

$$\boxed{\alpha^2 \nabla^2 u = u_t,}$$

where α^2 is the thermo diffusivity constant of the conducting material. Thus, the homogeneous heat conduction equations of 1-, 2-, and 3- dimension are, respectively,

$$\alpha^2 u_{xx} = u_t$$

$$\alpha^2 (u_{xx} + u_{yy}) = u_t$$

$$\alpha^2 (u_{xx} + u_{yy} + u_{zz}) = u_t$$

As well, the homogeneous wave equations of 1-, 2-, and 3- spatial dimension can then be similarly expressed in terms of the Laplacians as:

$$a^2 \nabla^2 u = u_{tt},$$

where the constant a is the propagation velocity of the wave motion. Thus, the homogeneous wave equations of 1-, 2-, and 3-dimension are, respectively,

$$a^2 u_{xx} = u_{tt}$$

$$a^2 (u_{xx} + u_{yy}) = u_{tt}$$

$$a^2 (u_{xx} + u_{yy} + u_{zz}) = u_{tt}$$ [‡]

Now let us consider the steady-state solutions of these heat conduction and wave equations. In each case, the steady-state solution, being independent of time, must have all zero as its partial derivatives with respect to t. Therefore, in every instance, the steady-state solution can be found by setting, respectively, u_t or u_{tt} to zero in the heat conduction or the wave equations and solve the resulting equation. That is, the steady-state solution of a heat conduction equation satisfies

$$\alpha^2 \nabla^2 u = 0,$$

and the steady-state solution of a wave equation satisfies

$$a^2 \nabla^2 u = 0.$$

[‡] Even the electromagnetic waves are described by this equation. It can be easily shown by vector calculus that any electric field ***E*** and magnetic field ***B*** satisfying the Maxwell's Equations will also satisfy the 3-dimensional wave equation, with propagation speed $a = c \approx 299792$ km/s, the speed of light in vacuum.

In all cases, we can divide out the (always positive) coefficient α^2 or a^2 from the equations, and obtain a "universal" equation:

$$\nabla^2 u = 0.$$

This universal equation that all the steady-state solutions of heat conduction and wave equations have to satisfy is the Laplace / potential equation!

Consequently, the 1-, 2-, and 3-dimensional Laplace equations are, respectively,

$$u_{xx} = 0,$$

$$u_{xx} + u_{yy} = 0,$$

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Therefore, the Laplace equation, among other applications, is used to solve the steady-state solution of the other two types of equations. And all solutions of a Laplace equation are steady-state solutions. To answer the earlier question, we have had seen and used the one-dimensional Laplace equation (which, with only one independent variable, x, is a very simple ordinary differential equation, $u'' = 0$, and is not a PDE) when we were trying to find the steady-state solution of the one-dimensional homogeneous heat conduction equation earlier.

Laplace Equation for a rectangular region

Consider a rectangular region of length a and width b. Suppose the top, bottom, and left sides border free-space; while beyond the right side there lies a source of heat/gravity/magnetic flux, whose strength is given by $f(y)$. The potential function at any point (x, y) within this rectangular region, $u(x, y)$, is then described by the boundary value problem:

(2-dim. Laplace eq.)	$u_{xx} + u_{yy} = 0,$	$0 < x < a,$	$0 < y < b,$
(Boundary conditions)	$u(x, 0) = 0$, and	$u(x, b) = 0,$	
	$u(0, y) = 0$, and	$u(a, y) = f(y).$	

The separation of variables proceeds similarly. A slight difference here is that $Y(y)$ is used in the place of $T(t)$. Let $u(x,y) = X(x)Y(y)$ and substituting $u_{xx} = X''\,Y$ and $u_{yy} = X\,Y''$ into the wave equation, it becomes

$$X''\,Y + X\,Y'' = 0,$$

$$X''\,Y = -X\,Y''.$$

Dividing both sides by $X\,Y$:

$$\frac{X'}{X} = -\frac{Y'}{Y}$$

Now that the independent variables are separated to the two sides, we can insert the constant of separation. Unlike the previous instances, it is more convenient to denote the constant as positive λ instead.

$$\frac{X'}{X} = -\frac{Y'}{Y} = \lambda$$

$$\frac{X'}{X} = \lambda \quad \rightarrow \quad X'' = \lambda X \quad \rightarrow \quad X'' - \lambda X = 0,$$

$$-\frac{Y'}{Y} = \lambda \quad \rightarrow \quad Y'' = -\lambda Y \quad \rightarrow \quad Y'' + \lambda Y = 0.$$

The boundary conditions also separate:

$u(x, 0) = 0 \rightarrow X(x)Y(0) = 0 \rightarrow \quad X(x) = 0$ or $Y(0) = 0$
$u(x, b) = 0 \rightarrow X(x)Y(b) = 0 \rightarrow \quad X(x) = 0$ or $Y(b) = 0$
$u(0, y) = 0 \rightarrow X(0)Y(y) = 0 \rightarrow \quad X(0) = 0$ or $Y(y) = 0$
$u(a, y) = f(y) \rightarrow X(a)Y(y) = f(y)$ [cannot be simplified further]

As usual, in order to obtain nontrivial solutions, we need to ignore the constant zero function in the solution sets above, and instead choose $Y(0) = 0$, $Y(b) = 0$, and $X(0) = 0$ as the new boundary conditions. The fourth boundary condition, however, cannot be simplified this way. So we shall leave it as-is. (Don't worry. It will play a useful role later.) The result, after separation of variables, is the following simultaneous system of ordinary differential equations, with a set of boundary conditions:

$X'' - \lambda X = 0, \qquad X(0) = 0,$

$Y'' + \lambda Y = 0, \qquad Y(0) = 0$ and $Y(b) = 0.$

Plus the fourth boundary condition, $u(a, y) = f(y)$.

The next step is to solve the eigenvalue problem. Notice that there is another slight difference. Namely that this time it is the equation of Y that gives rise to the two-point boundary value problem which we need to solve.

$$Y'' + \lambda Y = 0, \qquad Y(0) = 0, \qquad Y(b) = 0.$$

However, except for the fact that the variable is y and the function is Y, rather than x and X, respectively, we have already seen this problem before (more than once, as a matter of fact; here the constant $L = b$). The eigenvalues of this problem are

$$\lambda = \sigma^2 = \frac{n^2\pi^2}{b^2}, \qquad n = 1, 2, 3, \ldots$$

Their corresponding eigenfunctions are

$$Y_n = \sin\frac{n\pi y}{b}, \qquad n = 1, 2, 3, \ldots$$

Once we have found the eigenvalues, substitute λ into the equation of x. We have the equation, together with one boundary condition:

$$X' - \frac{n^2\pi^2}{b^2}X = 0, \qquad X(0) = 0.$$

Its characteristic equation, $r^2 - \frac{n^2\pi^2}{b^2} = 0$, has real roots $r = \pm\frac{n\pi}{b}$.

Hence, the general solution for the equation of x is

$$X = C_1 e^{\frac{n\pi}{b}x} + C_2 e^{\frac{-n\pi}{b}x}.$$

The single boundary condition gives

$$X(0) = 0 = C_1 + C_2 \quad \rightarrow \quad C_2 = -C_1.$$

Therefore, for $n = 1, 2, 3, \ldots$,

$$X_n = C_n\left(e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi}{b}x}\right).$$

Because of the identity for the hyperbolic sine function

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2},$$

the previous expression is often rewritten in terms of hyperbolic sine:

$$X_n = K_n \sinh\frac{n\pi x}{b}, \qquad n = 1, 2, 3, \ldots$$

The coefficients satisfy the relation: $K_n = 2C_n$.

Combining the solutions of the two equations, we get the set of solutions that satisfies the two-dimensional Laplace equation, given the specified boundary conditions:

$$u_n(x, y) = X_n(x)\,Y_n(y) = K_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b},$$

$n = 1, 2, 3, \ldots$

The general solution, as usual, is just the linear combination of all the above, linearly independent, functions $u_n(x, y)$. That is,

$$u(x, y) = \sum_{n=1}^{\infty} K_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}.$$

This solution, of course, is specific to the set of boundary conditions

$$u(x, 0) = 0, \text{ and} \qquad u(x, b) = 0,$$
$$u(0, y) = 0, \text{ and} \qquad u(a, y) = f(y).$$

To find the particular solution, we will use the fourth boundary condition, namely, $u(a, y) = f(y)$.

$$u(a, y) = \sum_{n=1}^{\infty} K_n \sinh \frac{an\pi}{b} \sin \frac{n\pi y}{b} = f(y)$$

We have seen this story before. There is nothing really new here. The summation above is a sine series whose Fourier sine coefficients are $b_n = K_n \sinh(an\pi / b)$. Therefore, the above relation says that the last boundary condition, $f(y)$, must either be an odd periodic function (period = $2b$), or it needs to be expanded into one. Once we have $f(y)$ as a Fourier sine series, the coefficients K_n of the particular solution can then be computed:

$$K_n \sinh \frac{an\pi}{b} = b_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy$$

Therefore,

$$K_n = \frac{b_n}{\sinh \frac{an\pi}{b}} = \frac{2}{b \sinh \frac{an\pi}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy .$$

(Optional topic) Laplace Equation in Polar Coordinates

The steady-state solution of the two-dimensional heat conduction or wave equation within a circular region (the interior of a circular disc of radius k, that is, on the region $r < k$) in polar coordinates, $u(r, \theta)$, is described by the polar version of the two-dimensional Laplace equation

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 .$$

The boundary condition, in this set-up, specifying the condition on the circular boundary of the disc, i.e., on the curve $r = k$, is given in the form $u(k, \theta) = f(\theta)$, where f is a function defined on the interval $[0, 2\pi)$. Note that there is only one set of boundary condition, prescribed on a circle. This will cause a slight complication. Furthermore, the nature of the coordinate system implies that u and f must be periodic functions of θ, of period 2π. Namely, $u(r, \theta) = u(r, \theta + 2\pi)$, and $f(\theta) = f(\theta + 2\pi)$.

By letting $u(r, \theta) = R(r)\Theta(\theta)$, the equation becomes

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0 .$$

Which can then be separated to obtain

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda .$$

This equation above can be rewritten into two ordinary differential equations:

$$r^2 R'' + rR' - \lambda R = 0,$$

$$\Theta'' + \lambda\Theta = 0.$$

The eigenvalues are not found by straight forward computation. Rather, they are found by a little deductive reasoning. Based solely on the fact that Θ must be a periodic function of period 2π, we can conclude that $\lambda = 0$ and $\lambda = n^2$, $n = 1, 2, 3\ldots$, are the eigenvalues. The corresponding eigenfunctions are $\Theta_0 = 1$ and $\Theta_n = A_n \cos n\theta + B_n \sin n\theta$. The equation of r is an *Euler equation* (the solution of which is outside of the scope of this course).

The general solution of the Laplace equation in polar coordinates is

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos n\theta + B_n \sin n\theta\right) r^n .$$

Applying the boundary condition $u(k, \theta) = f(\theta)$, we see that

$$u(k,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n k^n \cos n\theta + B_n k^n \sin n\theta\right) = f(\theta) .$$

Since $f(\theta)$ is a periodic function of period 2π, it would already have a suitable Fourier series representation. Namely,

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) .$$

Hence, $A_0 = a_0$, $A_n = a_n / k^n$, and $B_n = b_n / k^n$, $n = 1, 2, 3\ldots$

For a problem on the unit circle, whose radius $k = 1$, the coefficients A_n and B_n are exactly identical to, respectively, the Fourier coefficients a_n and b_n of the boundary condition $f(\theta)$.

(Optional topic) Undamped Wave Equation in Polar Coordinates

The vibrating motion of an elastic membrane that is circular in shape can be described by the two-dimensional wave equation in polar coordinates:

$$u_{rr} + (1 / r)\, u_r + (1 / r^2)\, u_{\theta\theta} = a^{-2}\, u_{tt}.$$

The solution is $u(r, \theta, t)$, a function of 3 independent variables that describes the vertical displacement of each point (r, θ) of the membrane at any time t.

POSSIBLE QUESTIONS
UNIT II
PART-B(5X6=30)

1. Find the solution of equation $Rr + Ss + Tt + Pp + Qq + Zz = F$ using separation variables.
2. Solve $\frac{\partial^2 z}{\partial x^2} = \frac{1}{k} \cdot \frac{\partial z}{\partial t}$
3. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{k} \cdot \frac{\partial z}{\partial t}$
4. Determine the solution of the equation $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0$ $(-\infty < x < \infty, y > 0)$ satisfying the condition.
   (i) z and its partial derivatives tend to zero as $x \to \pm\infty$
   (ii) $z = f(x)$, $\frac{\partial x}{\partial y} = 0$ on $y = 0$

PART-C (1X10=10)

1.. Discuss about the equation with variable coefficients.

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answers
A powerful method of finding solutions of second order linear _____differential equations is applicable in certain circumstances.	partial	ordinary	Hankel	Kennal	partial
Z=X(x)Y(y) is called ________ of variables	integratione	separation	differentiation	induction	separation
The separation principle can readily be extended to ___ number of variables.	smaller	unique	larger	contrary	larger
The separation principle can readily be extended to larger number of ________.	Constants	variables	coefficients	sequences	variables
In Separation of variables cn denotes______	variables	coefficients	sequences	Constants	Constants
The solution of PDE satisfies for all values of ______	n	1	2	3	n
The _______ of PDE satisfies for all values of n	unity	existance	solution	formal	solution
If Z tends to zero then t tends to infinity is the property of the solution of ___________	Ode	Pde	C.I	P.I	Pde
If Z tends to ______ then t tends to infinity is the property of the solution of PDE	1	0	3	4	0
If Z tends to zero then t tends to ______ is the property of the solution of PDE	infinity	1	0	2	infinity
If Z tends to zero then t tends to infinity is the __________ of the solution of PDE	definition	result	property	note	property
Z=X(x)Y(y) is separable in the variables_________	x&y	x&z	y&z	x+y	x&y
Z=X(x)Y(y)T(z) is the extension of ______________ variables	integratione	separation	differentiation	induction	separation
Z=X(x)Y(y)T(z) is separable in the variables_________	x&y	x&z	y&z	x,y&z	y&z
The use of the theory of integral transforms is the solution of ______________	Ode	Pde	C.I	P.I	Pde
The use of the theory of integral transforms is the _______ of PDE	unity	existance	solution	formal	solution
The use of the theory of _________ transforms is the solution of PDE	Constants	variables	coefficients	integral	integral
In the method of integral transforms L denotes ________operator.	non linear	Constants	linear	variables	linear
In the method of integral transforms _____denotes linear operator.	A	B	L	U	L
In the method of integral transforms lamda denotes________	Constants	variables	coefficients	sequences	Constants
In the method of integral transforms the variable x lies between alpha and ________	Lamda	Gamma	epsilon	Beta	Beta
In the method of integral transforms the variable x lies between ______ and beta	epsilon	alpha	Gamma	Lamda	alpha
In the method of integral transforms the variable _______ lies between alpha and beta	x	y	z	mew	x
In the method of integral transforms K is the ______ of the function	kennal	k(x,y)	kernal	k(z)	kernal

KARPAGAM ACADEMY OF HIGHER EDUCATION

Subject :PARTIAL DIFFERENTIAL EQUATIONS SEMESTER: II L T P C
SUBJECT CODE: 19MMP204 CLASS : I M.Sc MATHEMATICS 4 0 0 4

UNIT III
SYLLABUS

Laplace Equation:
Elementary solutions of Laplace equations- Families of Equi-potential surfaces - Boundary Value problems-separation of variables-problems with axial symmetry.

Definition :

Laplace's Equation

Laplace's equation

$$\Delta u = 0$$

and its inhomogeneous version, **Poisson's equation**,

$$-\Delta u = f.$$

We say a function u satisfying Laplace's equation is a **harmonic function**.

3.1 The Fundamental Solution

Consider Laplace's equation in $\mathbb{R}^n$,

$$\Delta u = 0 \qquad x \in \mathbb{R}^n.$$

Clearly, there are a lot of functions u which satisfy this equation. In particular, any constant function is harmonic. In addition, any function of the form $u(x) = a_1x_1 + \ldots + a_nx_n$ for constants a_i is also a solution. Of course, we can list a number of others. Here, however, we are interested in finding a particular solution of Laplace's equation which will allow us to solve Poisson's equation.

Given the symmetric nature of Laplace's equation, we look for a *radial* solution. That is, we look for a harmonic function u on $\mathbb{R}^n$ such that $u(x) = v(|x|)$. In addition, to being a natural choice due to the symmetry of Laplace's equation, radial solutions are natural to look for because they reduce a PDE to an ODE, which is generally easier to solve. Therefore, we look for a radial solution.

If $u(x) = v(|x|)$, then

$$u_{x_i} = \frac{x_i}{|x|} v'(|x|) \qquad |x| \neq 0,$$

which implies

$$u_{x_i x_i} = \frac{1}{|x|} v'(|x|) - \frac{x_i^2}{|x|^3} v'(|x|) + \frac{x_i^2}{|x|^2} v''(|x|) \qquad |x| \neq 0.$$

$|x| \qquad |x|^3 \qquad |x|^2$

Therefore,

$$\Delta u = \frac{n-1}{|x|} v'(|x|) + v''(|x|).$$

Letting $r = |x|$, we see that $u(x) = v(|x|)$ is a radial solution of Laplace's equation implies v satisfies

$$\frac{n-1}{r} v'(r) + v''(r) = 0.$$

Therefore,

$$v'' = \frac{1-n}{r} v'$$

$$\Longrightarrow \frac{v''}{v'} = \frac{1-n}{r}$$

$$\Longrightarrow \ln v' = (1-n) \ln r + C$$

$$\Longrightarrow v'(r) = \frac{C}{r^{n-1}}$$

which implies

$$v(r) = \begin{cases} c_1 \ln r + c_2 & n = 2 \\ \dfrac{c_1}{(2-n) r^{n-2}} + c_2 & n \geq 3. \end{cases}$$

From these calculations, we see that for any constants c_1, c_2, the function

$$u(x) \equiv \begin{cases} c_1 \ln|x| + c_2 & n = 2 \\ \dfrac{c_1}{(2-n)|x|^{n-2}} + c_2 & n \geq 3. \end{cases} \tag{3.1}$$

for $x \in \mathbb{R}^n$, $|x| \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^n - \{0\}$. We notice that the function u defined in (3.1) satisfies $\Delta u(x) = 0$ for $x \neq 0$, but at $x = 0$, $\Delta u(0)$ is undefined. We claim that we can choose constants c_1 and c_2 appropriately so that

$$-\Delta_x u = \delta_0$$

in the sense of distributions. Recall that δ_0 is the distribution which is defined as follows. For all $\varphi \in \mathcal{D}$,

$$(\delta_0, \varphi) = \varphi(0).$$

Below, we will prove this claim. For now, though, assume we can prove this. That is, assume we can find constants c_1, c_2 such that u defined in (3.1) satisfies

$$-\Delta_x u = \delta_0. \tag{3.2}$$

Let Φ denote the solution of (3.2). Then, define

$$v(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy.$$

Formally, we compute the Laplacian of v as follows,

$$\begin{aligned} -\Delta_x v &= -\int_{\mathbb{R}^n} \Delta_x \Phi(x - y) f(y)\, dy \\ &= \int_{\mathbb{R}^n} \Delta_y \Phi(x - y) f(y)\, dy \\ &= \int_{\mathbb{R}^n} \delta_x f(y)\, dy = f(x). \end{aligned}$$

That is, v is a solution of Poisson's equation! Of course, this set of equalities above is entirely formal. We have not proven anything yet. However, we have motivated a solution formula for Poisson's equation from a solution to (3.2). We now return to using the radial solution (3.1) to find a solution of (3.2).

Define the function Φ as follows. For $|x| \neq 0$, let

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \ln|x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}} & n \geq 3, \end{cases} \tag{3.3}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. We see that Φ satisfies Laplace's equation on $\mathbb{R}^n - \{0\}$. As we will show in the following claim, Φ satisfies $-\Delta_x \Phi = \delta_0$. For this reason, we call Φ the **fundamental solution** of Laplace's equation.

Claim 1. *For Φ defined in* (3.3)*, Φ satisfies*

$$-\Delta_x \Phi = \delta_0$$

in the sense of distributions. That is, for all $g \in D$,

$$-\int_{\mathbb{R}^n} \Phi(x)\Delta_x g(x)\, dx = g(0).$$

Proof. Let F_Φ be the distribution associated with the fundamental solution Φ. That is, let $F_\Phi : D \to \mathbb{R}$ be defined such that

$$(F_\Phi, g) = \int_{\mathbb{R}^n} \Phi(x)g(x)\, dx$$

for all $g \in D$. Recall that the derivative of a distribution F is defined as the distribution G such that

$$(G, g) = -(F, g')$$

for all $g \in D$. Therefore, the *distributional Laplacian* of Φ is defined as the distribution $F_{\Delta\Phi}$ such that

$$(F_{\Delta\Phi}, g) = (F_\Phi, \Delta g)$$

for all $g \in D$. We will show that

$$(F_\Phi, \Delta g) = -(\delta_0, g) = -g(0),$$

and, therefore,

$$(F_{\Delta\Phi}, g) = -g(0),$$

which means $-\Delta_x \Phi = \delta_0$ in the sense of distributions.

By definition,

$$(F_\Phi, \Delta g) = \int_{\mathbb{R}^n} \Phi(x)\Delta g(x)\, dx.$$

Now, we would like to apply the divergence theorem, but Φ has a singularity at $x = 0$. We get around this, by breaking up the integral into two pieces: one piece consisting of the ball of radius δ about the origin, $B(0, \delta)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^n$. Therefore, we have

$$\begin{aligned}(F_\Phi, \Delta g) &= \int_{\mathbb{R}^n} \Phi(x)\Delta g(x)\, dx \\ &= \int_{B(0,\delta)} \Phi(x)\Delta g(x)\, dx + \int_{\mathbb{R}^n - B(0,\delta)} \Phi(x)\Delta g(x)\, dx \\ &= I + J.\end{aligned}$$

We look first at term I. For $n = 2$, term I is bounded as follows,

$$\left|\int_{B(0,\delta)} \frac{1}{2\pi}\ln|x|\Delta g(x)\,dx\right| \le C|\Delta g|_{L^\infty}\left|\int_{B(0,\delta)} \ln|x|\,dx\right|$$
$$\le C\left|\int_0^{2\pi}\int_0^\delta \ln|r|\,r\,dr\,d\theta\right|$$
$$\le C\left|\int_0^\delta \ln|r|\,r\,dr\right|$$
$$\le C\ln|\delta|\delta^2.$$

For $n \ge 3$, term I is bounded as follows,

$$\left|\int_{B(0,\delta)} \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}\Delta g(x)\,dx\right| \le C|\Delta g|_{L^\infty}\int_{B(0,\delta)} \frac{1}{|x|^{n-2}}\,dx$$
$$\le C\int_0^\delta\left(\int_{\partial B(0,r)} \frac{1}{|y|^{n-2}}\,dS(y)\right)dr$$
$$= \int_0^\delta \frac{1}{r^{n-2}}\left(\int_{\partial B(0,r)} dS(y)\right)dr$$
$$= \int_0^\delta \frac{1}{r^{n-2}}\,n\alpha(n)r^{n-1}\,dr$$
$$= n\alpha(n)\int_0^\delta r\,dr = \frac{n\alpha(n)}{2}\delta^2.$$

Therefore, as $\delta \to 0^+$, $|I| \to 0$.

Next, we look at term J. Applying the divergence theorem, we have

$$\int_{\mathbb{R}^n-B(0,\delta)} \Phi(x)\Delta_x g(x)\,dx = \int_{\mathbb{R}^n-B(0,\delta)} \Delta_x\Phi(x)g(x)\,dx - \int_{\partial(\mathbb{R}^n-B(0,\delta))} \frac{\partial\Phi}{\partial\nu}g(x)\,dS(x)$$
$$+ \int_{\partial(\mathbb{R}^n-B(0,\delta))} \Phi(x)\frac{\partial g}{\partial\nu}\,dS(x)$$
$$= -\int_{\partial(\mathbb{R}^n-B(0,\delta))} \frac{\partial\Phi}{\partial\nu}g(x)\,dS(x) + \int_{\partial(\mathbb{R}^n-B(0,\delta))} \Phi(x)\frac{\partial g}{\partial\nu}\,dS(x)$$
$$\equiv J_1 + J_2.$$

using the fact that $\Delta_x\Phi(x) = 0$ for $x \in \mathbb{R}^n - B(0,\delta)$.

We first look at term J_1. Now, by assumption, $g \in \mathcal{D}$, and, therefore, g vanishes at ∞. Consequently, we only need to calculate the integral over $\partial B(0,s)$ where the normal derivative ν is the outer normal to $\mathbb{R}^n - B(0,\delta)$. By a straightforward calculation, we see that

$$\nabla_x\Phi(x) = -\frac{x}{n\alpha(n)|x|^n}.$$

We look first at term I. For $n = 2$, term I is bounded as follows,

The outer unit normal to $\mathbb{R}^n - B(0, \delta)$ on $\partial B(0, \delta)$ is given by

$$\nu = -\frac{x}{|x|}.$$

Therefore, the normal derivative of Φ on $B(0, \delta)$ is given by

$$\frac{\partial \Phi}{\partial \nu} = \left(-\frac{x}{n\alpha(n)|x|^n} \right) \cdot \left(-\frac{x}{|x|} \right) = \frac{1}{n\alpha(n)|x|^{n-1}}.$$

Therefore, $J1$ can be written as

$$-\int_{\partial B(0,\delta)} \frac{1}{n\alpha(n)|x|^{n-1}} g(x)\, dS(x) = -\frac{1}{n\alpha(n)\delta^{n-1}} \int_{\partial B(0,\delta)} g(x)\, dS(x) = -⨍_{\partial B(0,\delta)} g(x)\, dS(x).$$

Now if g is a continuous function, then

$$-⨍ g(x)\, dS(x) \to -g(0) \qquad \text{as } \delta \to 0.$$

Lastly, we look at term $J2$. Now using the fact that g vanishes as $|x| \to +\infty$, we only need to integrate over $\partial B(0, \delta)$. Using the fact that $g \in D$ and, therefore, infinitely differentiable, we have

$$\left| \int_{\partial B(0,\delta)} \Phi(x) \frac{\partial g}{\partial \nu}\, dS(x) \right| \leq \left\| \frac{\partial g}{\partial \nu} \right\|_{L^\infty(\partial B(0,\delta))} \int_{\partial B(0,\delta)} |\Phi(x)|\, dS(x)$$

$$\leq C \int_{\partial B(0,\delta)} |\Phi(x)|\, dS(x).$$

Now first, for $n = 2$,

$$\begin{aligned}\int_{\partial B(0,\delta)} |\Phi(x)|\, dS(x) &= C \int_{\partial B(0,\delta)} |\ln|x||\, dS(x) \\ &\leq C|\ln \delta| \int_{\partial B(0,\delta)} dS(x) \\ &= C|\ln|\delta||(2\pi\delta) \leq C\delta|\ln|\delta||.\end{aligned}$$

Next, for $n \geq 3$,

$$\begin{aligned}\int_{\partial B(0,\delta)} |\Phi(x)|\, dS(x) &= C \int_{\partial B(0,\delta)} \frac{1}{|x|^{n-2}}\, dS(x) \\ &\leq \frac{C}{\delta^{n-2}} \int_{\partial B(0,\delta)} dS(x) \\ &= \frac{C}{\delta^{n-2}} n\alpha(n)\delta^{n-1} \leq C\delta.\end{aligned}$$

Therefore, we conclude that term $J2$ is bounded in absolute value by

$$\begin{cases} C\delta|\ln \delta| & n = 2 \\ C\delta & n \geq 3. \end{cases}$$

Therefore, $|J2| \to 0$ as $\delta \to 0^+$.

Combining these estimates, we see that

$$\int_{\mathbb{R}^n} \Phi(x)\Delta_x g(x)\, dx = \lim_{\delta\to 0^+} I + J1 + J2 = -g(0).$$

Therefore, our claim is proved.

□

Solving Poisson's Equation. We now return to solving Poisson's equation

$$-\Delta u = f \qquad x \in \mathbb{R}^n.$$

From our discussion before the above claim, we *expect* the function

$$v(x) \equiv \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

to give us a solution of Poisson's equation. We now prove that this is in fact true. First, we make a remark.

Remark. If we hope that the function v defined above solves Poisson's equation, we must first verify that this integral actually converges. If we assume f has compact support on some bounded set K in $\mathbb{R}^n$, then we see that

$$\int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \le |f|_{L^\infty} \int_K |\Phi(x-y)|\, dy.$$

If we additionally assume that f is bounded, then $|f|_{L^\infty} \le C$. It is left as an exercise to verify that

$$\int_K |\Phi(x-y)|\, dy < +\infty$$

on any compact set K.

Theorem 2. *Assume $f \in C^2(\mathbb{R}^n)$ and has compact support. Let*

$$u(x) \equiv \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

where Φ is the fundamental solution of Laplace's equation (3.3). *Then*

1. $u \in C^2(\mathbb{R}^n)$
2. $-\Delta u = f$ *in* $\mathbb{R}^n$.

Ref: Evans, p. 23.

Proof. 1. By a change of variables, we write

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy.$$

Let

$$e_i = (\ldots, 0, 1, 0, \ldots)$$

be the unit vector in $\mathbb{R}^n$ with a 1 in the i^{th} slot. Then

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x+he_i-y)-f(x-y)}{h}\right] dy.$$

Now $f \in C^2$ implies

$$\frac{f(x+he_i-y)-f(x-y)}{h} \to \frac{\partial f}{\partial x_i}(x-y) \text{ as } h \to 0$$

uniformly on $\mathbb{R}^n$. Therefore,

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y)\, dy.$$

Similarly,

$$\frac{\partial^2 u}{\partial x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i x_j}(x-y)\, dy.$$

This function is continuous because the right-hand side is continuous.

2. By the above calculations and Claim 1, we see that

$$\begin{aligned}
\Delta_x u(x) &= \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y)\, dy \\
&= \int_{\mathbb{R}^n} \Phi(y) \Delta_y f(x-y)\, dy \\
&= -f(x).
\end{aligned}$$

□

Properties of Harmonic Functions

Mean Value Property

In this section, we prove a mean value property which all harmonic functions satisfy. First, we give some definitions. Let

$B(x, r)$ = ball of radius r about x in $\mathbb{R}^n$

$\partial B(x, r)$ = boundary of ball of radius r about x in $\mathbb{R}^n$

$\alpha(n)$ = volume of unit ball in $\mathbb{R}^n$

$n\alpha(n)$ = surface area of unit ball in $\mathbb{R}^n$.

For a function u defined on $B(x, r)$, the **average of** u **on** $B(x, r)$ is given by

$$⨍_{B(x,r)} u(y)\, dy = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u(y)\, dy.$$

$B(x,r)$ $B(x,r)$

Let

For a function u defined on $\partial B(x, r)$, the **average of u on $\partial B(x, r)$** is given by

$$\fint_{\partial B(x,r)} u(y)\, dS(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u(y)\, dS(y).$$

Theorem 3. (Mean-Value Formulas) *Let $\Omega \subset \mathbb{R}^n$. If $u \in C^2(\Omega)$ is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y) = \fint_{B(x,r)} u(y)\, dy$$

for every ball $B(x, r) \subset \Omega$.

Proof. Assume $u \in C^2(\Omega)$ is harmonic. For $r > 0$, define

$$\varphi(r) = \fint_{\partial B(x,r)} u(y)\, dS(y).$$

For $r = 0$, define $\varphi(r) = u(x)$. Notice that if u is a smooth function, then $\lim_{r\to 0^+} \varphi(r) = u(x)$, and, therefore, φ is a continuous function. Therefore, if we can show that $\varphi'(r) = 0$, then we can conclude that φ is a constant function, and, therefore,

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y).$$

We prove $\varphi'(r) = 0$ as follows. First, making a change of variables, we have

$$\begin{aligned}\varphi(r) &= \fint_{\partial B(x,r)} u(y)\, dS(y)\\ &= \fint_{\partial B(0,1)} u(x + rz)\, dS(z).\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi'(r) &= \fint_{\partial B(0,1)} \nabla u(x + rz) \cdot z\, dS(z)\\ &= \fint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r}\, dS(y)\\ &= -_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\, dS(y)\\ &= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\, dS(y)\\ &= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \nabla \cdot (\nabla u)\, dy \text{ (by the Divergence Theorem)}\\ &= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy = 0,\end{aligned}$$

using the fact that u is harmonic. Therefore, we have proven the first part of the theorem. It remains to prove that

$$u(x) = \fint_{B(x,r)} u(y)\, dy.$$

We do so as follows, using the first result,

$$\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_0^r \left(\int_{\partial B(x,s)} u(y)\, dS(y) \right) ds \\
&= \int_0^r n\alpha(n) s^{n-1} \left(\fint_{\partial B(x,s)} u(y)\, dS(y) \right) ds \\
&= \int_0^r n\alpha(n) s^{n-1} u(x)\, ds \\
&= n\alpha(n) u(x) \int_0^r s^{n-1}\, ds \\
&= n\alpha(n) u(x) \int_0^r s^{n-1}\, ds \\
&= \alpha(n) u(x)\, s^n \big|_{s=0}^{s=r} \\
&= \alpha(n) u(x) r^n.
\end{aligned}$$

Therefore,

$$\int_{B(x,r)} u(y)\, dy = \alpha(n) r^n u(x),$$

which implies

$$u(x) = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u(y)\, dy = \fint_{B(x,r)} u(y)\, dy,$$

as claimed.

□

Converse to Mean Value Property

In this section, we prove that *if* a smooth function u satisfies the mean value property described above, then u must be harmonic.

Theorem 4. *If $u \in C^2(\Omega)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y)$$

for all $B(x, r) \subset \Omega$, then u is harmonic.

Proof. Let

$$\varphi(r) = \fint_{\partial B(x,r)}$$

If $u(y)\,dS(y)$.
using the fact that u is harmonic. Therefore, we have proven the first part of the theorem.

$$u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y)$$

for all $B(x, r) \subset \Omega$, then $\varphi'(r) = 0$. As described in the previous theorem,

$$\varphi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy.$$

Suppose u is not harmonic. Then there exists some ball $B(x, r) \subset \Omega$ such that $\Delta u > 0$ or $\Delta u < 0$. Without loss of generality, we assume there is some ball $B(x, r)$ such that $\Delta u > 0$. Therefore,

$$\varphi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy > 0,$$

which contradicts the fact that $\varphi'(r) = 0$. Therefore, u must be harmonic.

□

Maximum Principle

In this section, we prove that if u is a harmonic function on a bounded domain Ω in $\mathbb{R}^n$, then u attains its maximum value on the boundary of Ω.

Theorem 5. *Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic. Then*

1. (Maximum principle)

$$\max_{\overline{\Omega}} u(x) = \max_{\partial\Omega} u(x).$$

2. (Strong maximum principle) *If Ω is connected and there exists a point $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{\overline{\Omega}} u(x),$$

then u is constant within Ω.

Proof. We prove the second assertion. The first follows from the second. Suppose there exists a point x_0 in Ω such that

$$u(x_0) = M = \max_{\overline{\Omega}} u(x).$$

Then for $0 < r < \text{dist}(x_0, \partial\Omega)$, the mean value property says

$$M = u(x_0) = \fint_{B(x_0,r)} u(y)\, dy \leq M.$$

But, therefore,

$$\fint_{B(x_0,r)} u(y)\, dy = M,$$

and $M = \max_{\overline{\Omega}} u(x)$. Therefore, $u(y) \equiv M$ for $y \in B(x_0, r)$. To prove $u \equiv M$ throughout Ω, you continue with this argument, filling Ω with balls.

□

Remark. By replacing u by $-u$ above, we can prove the Minimum Principle.

Next, we use the maximum principle to prove uniqueness of solutions to Poisson's equation on bounded domains Ω in $\mathbb{R}^n$.

Theorem 6. (Uniqueness) *There exists at most one solution* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *of the boundary-value problem,*

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

Proof. Suppose there are two solutions u and v. Let $w = u - v$ and let $\tilde{w} = v - u$. Then w and $\tilde{w}$ satisfy

$$\begin{cases} \Delta w = 0 & x \in \Omega \\ w = 0 & x \in \partial\Omega. \end{cases}$$

Therefore, using the maximum principle, we conclude

$$\max_{\Omega} |u - v| = \max_{\partial\Omega} |u - v| = 0.$$

□

Smoothness of Harmonic Functions

In this section, we prove that harmonic functions are C^∞.

Theorem 7. *Let* Ω *be an open, bounded subset of* $\mathbb{R}^n$. *If* $u \in C(\Omega)$ *and* u *satisfies the mean value property,*

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y)$$

for every ball $B(x, r) \subset \Omega$, *then* $u \in C^\infty(\Omega)$.

Remarks.

1. As proven earlier, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and u is harmonic, then u satisfies the mean value property, and, therefore, $u \in C^\infty(\Omega)$.
2. In fact, if u satisfies the hypothesis of the above theorem, then u is analytic, but we will not prove that here. (See Evans.)

Proof. First, we introduce the function η such that

$$\eta(x) \equiv \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where the constant C is chosen such that $\int_{\mathbb{R}^n} \eta(x)\, dx = 1$. Notice that $\eta \in C^\infty(\mathbb{R}^n)$ and η has compact support. Now define the function $\eta_s(x)$ such that

$$\eta_s(x) \equiv \frac{1}{s^n}\eta\left(\frac{x}{s}\right).$$

Therefore, $\eta_s \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\eta_s) \subset \{x : |x| < s\}$. Further,

$$\int \eta_s(x)\, dx = 1.$$

Now choose s such that $s <$ dist$(x, \partial\Omega)$. Define

$$u_s(x) = \int_\Omega \eta_s(x - y)u(y)\, dy.$$

Now we claim

1. $u_s \in C^\infty$
2. $u_s(x) = u(x)$.

First, for (1), $u_s \in C^\infty$ because $\eta_s \in C^\infty$. We prove (2) as follows. Using the fact that supp$\eta_s(x - y) \subset \{y : |x - y| < s\}$. Therefore,

$$\begin{aligned}
u_s(x) &= \int_{B(x,s)} \eta_s(x - y)u(y)\, dy \\
&= \frac{1}{s^n}\int_{B(x,s)} \eta\left(\frac{|x-y|}{s}\right) u(y)\, dy \\
&= \frac{1}{s^n}\int_0^s \left(\int_{\partial B(x,r)} \eta\left(\frac{|x-y|}{s}\right) u(y)\, dS(y)\right) dr \\
&= \frac{1}{s^n}\int_0^s \eta\left(\frac{r}{s}\right) \left(\int_{\partial B(x,r)} u(y)\, dS(y)\right) dr \\
&= \frac{1}{s^n}\int_0^s \eta\left(\frac{r}{s}\right) \int_{\partial B(x,r)} u(y)\, dS(y)\, dr \\
&= \frac{1}{s^n}\int_0^s \eta\left(\frac{r}{s}\right) na(n)r^{n-1} ⨍_{\partial B(x,r)} u(y)\, dS(y)\, dr \\
&= \frac{1}{s^n}\int_0^s \eta\left(\frac{r}{s}\right) na(n)r^{n-1}u(x)\, dr \\
&= u(x)\frac{1}{s^n}\int_0^s \eta\left(\frac{r}{s}\right)\int_{\partial B(0,r)} dS(y)\, dr \\
&= u(x)\frac{1}{s^n}\int_{B(0,s)} \eta\left(\frac{|y|}{s}\right) dy \\
&= u(x)\int_{B(0,s)} \eta_s(y)\, dy \\
&= u(x).
\end{aligned}$$

□

Liouville's Theorem

In this section, we show that the only functions which are bounded and harmonic on $\mathbb{R}^n$ are constant functions.

Theorem 8. *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.*

Proof. Let $x_0 \in \mathbb{R}^n$. By the mean value property,

$$u(x_0) = \fint_{B(x_0,r)} u(y)\, dy$$

for all $B(x_0, r)$. Now by the previous theorem, we know that if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and u is harmonic, then u is C^∞. Therefore,

$$\Delta u = 0 \implies \Delta u_{x_i} = 0$$

for $i = 1, \ldots, n$. Therefore, u_{x_i} is harmonic and satisfies the mean value property. Therefore,

$$\begin{aligned} u_{x_i}(x_0) &= \fint_{B(x_0,r)} u_{x_i}(y)\, dy \\ &= \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u_{x_i}(y)\, dy \\ &= \frac{1}{\alpha(n)r^n} \int_{\partial B(x_0,r)} u\nu_i\, dS(y), \end{aligned}$$

by the Divergence theorem, where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal to $B(x_0, r)$. Therefore,

$$\begin{aligned} |u_{x_i}(x_0)| &\le \left| \frac{1}{\alpha(n)r^n} \int_{\partial B(x_0,r)} u\nu_i\, dS(y) \right| \\ &\le |u|_{L^\infty(\partial B(x_0,r))} |\nu_i|_{L^\infty(\partial B(x_0,r))} \left| \frac{1}{\alpha(n)r^n} \int_{\partial B(x_0,r)} dS(y) \right| \\ &\le |u|_{L^\infty(\mathbb{R}^n)} \cdot \left| \frac{n\alpha(n)r^{n-1}}{\alpha(n)r^n} \right| \\ &\le \frac{n}{r} |u|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

Therefore,

$$\begin{aligned} |u_{x_i}(x_0)| &\le \frac{n}{r} |u|_{L^\infty(\mathbb{R}^n)} \\ &\le C\frac{n}{r}, \end{aligned}$$

by the assumption that u is bounded. Now this is true for all r. Taking the limit as $r \to +\infty$, we see that $|u_{x_i}(x_0)| = 0$. Therefore, $u_{x_i}(x_0) = 0$. This is true for $i = 1, \ldots, n$ and for all $x_0 \in \mathbb{R}^n$. Therefore, we conclude that $u \equiv$constant.

□

As a corollary of Liouville's Theorem, we have the following representation formula for all bounded solutions of Poisson's equation on $\mathbb{R}^n$, $n \geq 3$.

Theorem 9. (Representation Formula) *Let $f \in C^2(\mathbb{R}^n)$ with compact support. Let $n \geq 3$. Then every bounded solution of*

$$-\Delta u = f \qquad x \in \mathbb{R}^n \tag{3.4}$$

has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy + C$$

for some constant C, where $\Phi(x)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^n$.

Proof. Recall that the fundamental solution of Laplace's equation in $\mathbb{R}^n$, $n \geq 3$ is given by

$$\Phi(x) = \frac{K}{|x|^{n-2}}$$

where $K = 1/n(n-2)\alpha(n)$. As shown earlier,

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a solution of (3.4). Here we show this is a bounded solution for $n \geq 3$. Fix $s > 0$. Then, we have

$$\begin{aligned}
|u(x)| &= \left| \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy \right| \\
&= \left| K \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y)\, dy \right| \\
&= \left| K \int_{B(x,s)} \frac{1}{|x-y|^{n-2}} f(y)\, dy \right| + \left| K \int_{\mathbb{R}^n - B(x,s)} \frac{1}{|x-y|^{n-2}} f(y)\, dy \right| \\
&\leq \|f(y)\|_{L^\infty} \int_{B(x,s)} \frac{1}{|x-y|^{n-2}}\, dy + C \int_{\mathbb{R}^n - B(x,s)} |f(y)|\, dy.
\end{aligned}$$

It is easy to see that the first term on the right-hand side is bounded. The second term on the right-hand side is bounded, using the assumption that $f \in C^2(\mathbb{R}^n)$ with compact support. Therefore, we conclude that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a bounded solution of (3.4). Now suppose there is another bounded solution of (3.4). Let $\tilde{u}$ be such a solution. Let

$$w(x) = u(x) - \tilde{u}(x).$$

Then w is a bounded, harmonic function on $\mathbb{R}^n$. Then, by Liouville's Theorem, w must be constant. Therefore, we conclude that

$$\begin{aligned}\tilde{u}(x) &= u(x) + C \\ &= \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy + C,\end{aligned}$$

as claimed.

□

Solving Laplace's Equation on Bounded Domains

Laplace's Equation on a Rectangle

In this section, we will solve Laplace's equation on a rectangle in $\mathbb{R}^2$. First, we consider the case of Dirichlet boundary conditions. That is, we consider the following boundary value problem. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < a,\ 0 < y < b\}$. We want to look for a solution of the following,

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u(0, y) = g_1(y),\ u(a, y) = g_2(y) & 0 < y < b \\ u(x, 0) = g_3(x),\ u(x, b) = g_4(y) & 0 < x < a. \end{cases} \tag{3.5}$$

In order to do so, we consider the following simpler example. From this, we will show how to solve the more general problem above.

Example 10. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}$. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x, y) \in \Omega \\ u(0, y) = g_1(y),\ u(a, y) = 0 & 0 < y < b \\ u(x, 0) = 0,\ u(x, b) = 0 & 0 < x < a. \end{cases} \tag{3.6}$$

We use separation of variables. We look for a solution of the form

$$u(x, y) = X(x)\,Y(y).$$

Plugging this into our equation, we get

$$X''Y + XY'' = 0.$$

Now dividing by XY, we arrive at

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

which implies

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

for some constant λ. By our boundary conditions, we want $Y(0) = 0 = Y(b)$. Therefore, we begin by solving the eigenvalue problem,

$$\begin{cases} Y'' = -\lambda Y & 0 < y < b \\ Y(0) = 0 = Y(b). \end{cases}$$

As we know, the solutions of this eigenvalue problem are given by

$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right), \quad \lambda_n = \left(\frac{n\pi}{b}\right)^2.$$

We now turn to solving

$$X'' = \left(\frac{n\pi}{b}\right)^2 X$$

with the boundary condition $X(a) = 0$. The solutions of this ODE are given by

$$X_n(x) = A_n\cosh\left(\frac{n\pi}{b}x\right) + B_n\sinh\left(\frac{n\pi}{b}x\right).$$

Now the boundary condition $X(a) = 0$ implies

$$A_n\cosh\left(\frac{n\pi}{b}a\right) + B_n\sinh\left(\frac{n\pi}{b}a\right) = 0.$$

Therefore,

$$u_n(x, y) = X_n(x)Y_n(y) = \left(A_n\cosh\left(\frac{n\pi}{b}x\right) + B_n\sinh\left(\frac{n\pi}{b}x\right)\right)\sin\left(\frac{n\pi}{b}y\right)$$

where A_n, B_n satisfy the condition

$$A_n\cosh\left(\frac{n\pi}{b}a\right) + B_n\sinh\left(\frac{n\pi}{b}a\right) = 0.$$

is a solution of Laplace's equation on Ω which satisfies the boundary conditions $u(x, 0) = 0$, $u(x, b) = 0$, and $u(a, y) = 0$. As we know, Laplace's equation is linear. Therefore, we can take any combination of solutions $\{u_n\}$ and get a solution of Laplace's equation which satisfies these three boundary conditions. Therefore, we look for a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty}\left(A_n\cosh\left(\frac{n\pi}{b}x\right) + B_n\sinh\left(\frac{n\pi}{b}x\right)\right)\sin\left(\frac{n\pi}{b}y\right)$$

where A_n, B_n satisfy

$$A_n\cosh\left(\frac{n\pi}{b}a\right) + B_n\sinh\left(\frac{n\pi}{b}a\right) = 0. \tag{3.7}$$

To solve our boundary-value problem (3.6), it remains to find coefficients A_n, B_n which not only satisfy (3.7), but also satisfy the condition $u(0, y) = g_1(y)$. That is, we need

$$u(0, y) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{b}y\right) = g_1(y).$$

That is, we want to be able to express g_1 in terms of its Fourier sine series on the interval $[0, b]$. Assuming g_1 is a "nice" function, we can do this. From our earlier discussion of Fourier series, we know that the Fourier sine series of a function g_1 is given by

$$g_1(y) \sim \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{b} y\right)$$

where the coefficients A_n are given by

$$A_n = \frac{\left(g_1, \sin\left(\frac{n\pi}{b} y\right)\right)_b}{\left(\sin\left(\frac{n\pi}{b} y\right), \sin\left(\frac{n\pi}{b} y\right)\right)_b}$$

where the L^2-inner product is taken over the interval $[0, b]$.

Therefore, to summarize, we have found a solution of (3.6) given by

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} \left[A_n \cosh\left(\frac{n\pi}{b} x\right) + B_n \sinh\left(\frac{n\pi}{b} x\right)\right] \sin\left(\frac{n\pi}{b} y\right)}$$

where

$$\boxed{A_n = \frac{\left(g_1, \sin\left(\frac{n\pi}{b} y\right)\right)}{\left(\sin\left(\frac{n\pi}{b} y\right), \sin\left(\frac{n\pi}{b} y\right)\right)}}$$

and

$$\boxed{B_n = -\coth\left(\frac{n\pi}{b} a\right) A_n.}$$

¤

Now we return to considering (3.5). For the general boundary value problem on a rectangle with Dirichlet boundary conditions, we can find a solution by finding four separate solutions u_i for $i = 1, \ldots, 4$ such that each u_i is identically zero on three of the sides and satisfies the boundary condition on the fourth side. For example, for the boundary value problem (3.5), we use the procedure in the above example to find a function $u_1(x, y)$ which is harmonic on Ω and such that $u_1(0, y) = g_1(y)$ and $u_1(a, y) = 0$ for $0 < y < b$, and $u_1(x, 0) = 0 = u_1(x, b)$ for $0 < x < a$. Similar we find functions u_2, u_3 and u_4 which vanish on three of the sides but satisfy the fourth boundary condition.

We now consider an example where we have a mixed boundary condition on one side.

Example 11. Let $\Omega = \{(x, y) \in \mathbb{R}^2, 0 < x < L, 0 < y < H\}$. Consider the following boundary value problem,

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x, y) \in \Omega \\ u(0, y) = 0, \; u(L, y) = 0 & 0 < y < H \\ u(x, 0) - u_y(x, 0) = 0, \; u(x, H) = f(x) & 0 < x < L. \end{cases} \tag{3.8}$$

Using separation of variables, we have

That is, we want to be able to express g_1 in terms of its Fourier sine series on the interval

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

We first look to solve

$$\begin{cases} X'' = \lambda X & 0 < x < L \\ X(0) = 0 = X(L). \end{cases}$$

As we know, the solutions of this eigenvalue problem are given by

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Now we need to solve

$$-Y'' = -\left(\frac{n\pi}{L}\right)^2 Y$$

with the boundary condition $Y(0) - Y'(0) = 0$. The solutions of this ODE are given by

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{L}y\right) + B_n \sinh\left(\frac{n\pi}{L}y\right).$$

The boundary condition $Y(0) - Y'(0) = 0$ implies

$$A_n - B_n\frac{n\pi}{L} = 0.$$

Therefore,

$$Y_n(y) = B_n\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + B_n \sinh\left(\frac{n\pi}{L}y\right).$$

Therefore, we look for a solution of (3.8) of the form

$$u(x, y) = \sum_{n=1} B_n \sin\left(\frac{n\pi}{L}x\right)\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right).$$

Substituting in the condition $u(x, H) = f(x)$, we have

$$u(x, H) = \sum_{n=1} B_n \sin\left(\frac{n\pi}{L}x\right)\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right)\right) = f(x).$$

Recall the Fourier sine series of f on $[0, L]$ is given by

$$f \sim \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

where

$$A_n = \frac{\left(f, \sin\left(\frac{n\pi}{L}x\right)\right)}{\left(\sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right)}$$

where the L^2-inner product is taken over $(0, L)$. Therefore, in order for our boundary condition $u(x, H) = f(x)$ to be satisfied, we need B_n to satisfy

$$B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right)\right) = \frac{\left(f, \sin\left(\frac{n\pi}{L}x\right)\right)}{\left(\sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right)}$$

Using the fact that

$$\left(\sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right) = \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx = \frac{L}{2},$$

the solution of (3.8) is given by

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right)$$

where

$$B_n = \frac{2}{L}\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right)\right)^{-1}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) dx.$$

¤

Laplace's Equation on a Disk

In this section, we consider Laplace's Equation on a disk in $\mathbb{R}^2$. That is, let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x, y) \in \Omega \\ u = h(\theta) & (x, y) \in \partial\Omega. \end{cases} \qquad (3.9)$$

To solve, we write this equation in polar coordinates as follows. To transform our equation in to polar coordinates, we will write the operators ∂_x and ∂_y in polar coordinates. We will use the fact that

$$x^2 + y^2 = r^2$$
$$\frac{y}{x} = \tan\theta.$$

Consider a function u such that $u = u(r, \theta)$, where $r = r(x, y)$ and $\theta = \theta(x, y)$. That is,

$$u = u(r(x, y), \theta(x, y)).$$

Then

$$\begin{aligned} \frac{\partial}{\partial x} u(r(x, y), \theta(x, y)) &= u_r r_x + u_\theta \theta_x \\ &= u_r \frac{x}{(x^2 + y^2)^{1/2}} - u_\theta \frac{y}{x^2 \sec^2\theta} \\ &= u_r \cos\theta - \frac{\sin\theta}{r} u_\theta. \end{aligned}$$

Therefore, the operator $\frac{\partial}{\partial x}$ can be written in polar coordinates as

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}.$$

Similarly, the operator $\frac{\partial}{\partial y}$ can be written in polar coordinates as

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}.$$

Now squaring these operators we have

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Similarly,

$$\frac{\partial^2}{\partial y^2} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Combining the above terms, we can write the operator $\partial_x^2 + \partial_y^2$ in polar coordinates as follows,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Therefore, in polar coordinates, Laplace's equation is written as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \qquad (3.10)$$

Now we will solve it using separation of variables. In particular, we look for a solution of the form $u(r, \theta) = R(r)\Theta(\theta)$. Then letting $\tilde{u}(x, y) = u(r(x, y), \theta(x, y))$, we will arrive at a solution of Laplace's equation on the disk.

Substitutingafunction ofthe form $u(r, \theta) = R(r)\Theta(\theta)$, into(3.10), ourequation is written as

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing by $R\Theta$,

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = 0.$$

Multiplying by r^2, we are led to the equations

$$\frac{\Theta''}{\Theta} = -\frac{r^2R''}{R} - \frac{rR'}{R} = -\lambda$$

for some scalar λ. The boundary condition for this problem is $u = h(\theta)$ for $(x, y) \in \partial\Omega$. Therefore, we are led to the following eigenvalue problem with periodic boundary conditions,

$$\Theta^{jj} = \quad \lambda\Theta \qquad\qquad 0 < \theta < 2\pi$$
$$\Theta(0) = \Theta(2\pi),\ \Theta^{j}(0) = \Theta^{j}(2\pi).$$

Recall from our earlier work that periodic boundary conditions imply our eigenfunctions and eigenvalues are

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \qquad \lambda_n = n^2 \qquad n = 0, 1, 2, \ldots$$

For each λ_n, we need to solve

$$r^2 R_n'' + r R_n' = \lambda_n R_n.$$

That is, we need to solve the second-order ODE,

$$r^2 R_n'' + r R_n' - n^2 R_n = 0$$

for $n = 0, 1, 2, \ldots$. Recall that a second-order ODE will have two linearly independent solutions. We look for a solution of the form $R(r) = r^a$ for some a. Doing so, our ODE becomes

$$(a^2 - n^2) r^a = 0.$$

Therefore, for $n \geq 1$, we have found two linearly independent solutions, $R_n(r) = r^n$ and $R_n(r) = r^{-n}$. Now for $n = 0$, we have only found one linearly independent solution so far, $R_0(r) = 1$. We look for another linearly independent solution. If $n = 0$, our equation can be written as

$$r^2 R'' + r R' = 0.$$

Dividing by r, our equation becomes

$$r R'' + R' = 0.$$

A linearly independent solution of this equation is $R_0(r) = \ln r$. Therefore, for each $n \geq 0$, we have found a solution of (3.10) of the form

$$u_n(r, \theta) = R_n(r)\Theta_n(\theta) = \begin{cases} \left[C_n r^n + \dfrac{D_n}{r^n}\right]\left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] \\ A_0 \left[C_0 + D_0 \ln r\right]. \end{cases}$$

But, we don't want a solution which blows up as $r \to 0^+$. Therefore, we reject the solutions $\frac{1}{r^n}$ and $\ln r$. Therefore, we consider a solution of (3.10) of the form

$$u(r, \theta) = \sum_{n=0} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right].$$

Now in order to solve (3.9), we need $u(a, \theta) = h(\theta)$. That is, we need

$$\sum_{n=0} a^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] = h(\theta).$$

Using the fact that our eigenfunctions are orthogonal on $[0, 2\pi]$, we can solve for our coefficients A_n and B_n as follows. Multiplying the above equation by $\cos(n\theta)$ and integrating over $[0, 2\pi]$, we have

$$A_n = \frac{1}{a^n} \frac{(h(\theta), \cos(n\theta))}{(\cos(n\theta), \cos(n\theta))} = \frac{1}{\pi a^n} \int_0^{2\pi} h(\theta) \cos(n\theta)\, d\theta \qquad \text{for } n = 1, 2, \ldots$$

0

$$A_0 = \frac{(h(\theta), 1)}{(1, 1)} = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, d\theta.$$

Similarly, multiplying by $\sin(n\theta)$ and integrating over $[0, 2\pi]$, we have

$$B_n = \frac{1}{a^n}\frac{(h(\theta), \sin(n\theta))}{(\sin(n\theta), \sin(n\theta))} = \frac{1}{\pi a^n}\int_0^{2\pi} h(\theta)\sin(n\theta)\, d\theta.$$

To summarize, we have found a solution of Laplace's equation on the disk in polar coordinates, given by

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

where

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\, d\theta$$

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\theta)\cos(n\theta)\, d\theta$$

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\theta)\sin(n\theta)\, d\theta.$$

Now we will rewrite this solution in terms of a single integral by substituting A_n and B_n into the series solution above. Doing so, we have

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\, d\varphi \\ &\quad + \sum_{n=1}^{\infty} r^n \left[\left(\frac{1}{\pi a^n}\int_0^{2\pi} h(\varphi)\cos(n\varphi)\, d\varphi\right)\cos(n\theta) + \left(\frac{1}{\pi a^n}\int_0^{2\pi} h(\varphi)\sin(n\varphi)\, d\varphi\right)\sin(n\theta)\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\left[1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}[\cos(n\varphi)\cos(n\theta) + \sin(n\varphi)\sin(n\theta)]\right] d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\left[1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos(n(\theta - \varphi))\right] d\varphi. \end{aligned}$$

Now

$$\begin{aligned} \left[1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos(n(\theta - \varphi))\right] &= \left[1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\left(\frac{e^{in(\theta-\varphi)} + e^{-in(\theta-\varphi)}}{2}\right)\right] \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-\varphi)}}{a}\right)^n + \sum_{n=1}^{\infty}\left(\frac{re^{-i(\theta-\varphi)}}{a}\right)^n \end{aligned}$$

$n=1$

•

$$= 1 + \frac{re^{i(\theta-\varphi)}}{a - re^{i(\theta-\varphi)}} + \frac{re^{-i(\theta-\varphi)}}{a - re^{-i(\theta-\varphi)}}$$

$$= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \varphi) + r^2}.$$

Therefore,

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \varphi) + r^2}\,d\varphi.$$

We can write this in rectangular coordinates as follows. Let x be a point in the disk Ω with polar coordinates (r, θ). Let x' be a point on the boundary of the disk Ω with polar coordinates (a, φ). Therefore, $|x - x'|^2 = a^2 + r^2 - 2ar\cos(\theta - \varphi)$ by the law of cosines. Therefore,

$$u(x) = \frac{1}{2\pi}\int_{|x'|=a} \frac{u(x')(a^2 - |x|^2)}{|x - x'|^2}\frac{ds}{a},$$

using the fact that $ds = a\,d\varphi$ is the arc length of the curve. Rewriting this, we have

$$\boxed{u(x) = \frac{a^2 - |x|^2}{2\pi a}\int_{|x'|=a} \frac{u(x')}{|x - x'|^2}\,ds.}$$

This is known as **Poisson's formula** for the solution of Laplace's equation on the disk.

POSSIBLE QUESTIONS
UNIT III
PART B
(5X6=30 Marks)

1. Find the elementary solutions of the Laplace's equation.

2 Prove that $\lim_{r\to\infty} r\psi(r) = M$ where M=$\int \rho(r')dr'$, $\rho > 0$ and $\psi(r) = \int \frac{\rho(r')d\tau'}{|r-r'|}$.

3. Find the elementary solutions of the Laplace's equation.

4. Describe about the families of equipotential surfaces.

5. Prove the uniqueness of interior Dirichlet problem.

PART-C (1X10=10)

1. Write down the form of Ψ for points on the axial symmetry.
2. Show that the surfaces $x^2 + y^2 + z^2 = cx^{\frac{2}{3}}$ can form a family of equipotential surfaces and find the general form of the corresponding potential function.

Question	Opt 1	Opt 2
Both inside and outside the attracting matter the force of attraction expressed in _______ potential	actual	gravitation
In Physics the field equations reduced to ___________ equation	Fourier	Kennal
At any point at which the density of gravitation matter satisfies ________equation	Fourier	Kennal
In _______ the field equations reduced to Laplace equation	Maths	Physics
At any point at which the ________ of gravitation matter satisfies Poisson equation	velocity	acceleration
In occurrence of laplace equation there is no ____________________	singularity	Constants
The velocity of a perfect fluid in irrotational motionexpressed in terms of ________ potential.	velocity	acceleration
The velocity of a ________fluid in irrotational motionexpressed in terms of velocitypotential.	proportional	passive
The velocity of a perfect fluid in ________ motion expressed in terms of velocitypotential.	rotational	moving
The function Shi has no singularities except_____________	proportional	passive
The function Shi has no singularities except_____________	Source	velocity
The function shi is ________________	Constants	variables
The function shi is ________________	variables	conductor
At each point of the conductor n is the outward drawn ______ to the conductor	proportional	normal
At the surface of an ______ at which a battery is providing charge at a definite potential of the function.	node	electrode
In the presence of dilectrics the_________ potential is defined	grad	potential
The magnetic vector is defined in terms of ________ potential	grad	magnetostatic
Steady currents are defined through ________ current	induction	conduction
The determination of the potential due to uniform circular wire _______________	diameter	radius
The boundary S of a simply connected region satisfies _______ equation	Fourier	Kennal

Opt 3	Opt 4	Answers
increasing	decreasing	gravitation
Kernal	Laplace	Laplace
Kernal	Poisson	Poisson
Chemistry	Biology	Physics
density	potential	density
variable	sequences	singularity
density	kennal	velocity
perfect	sink	perfect
non moving	irrotational	irrotational
perfect	sink	irrotational
acceleration	density	Source
coefficients	sequences	Constants
coefficients	sequences	conductor
perpendicular	series	normal
grad	potential	electrode
electrostatic	node	electrostatic
electrostatic	node	magnetostatic
node	potential	conduction
node	potential	radius
Kernal	Laplace	Laplace

KARPAGAM ACADEMY OF HIGHER EDUCATION

Subject :PARTIAL DIFFERENTIAL EQUATIONS SEMESTER: II L T P C
SUBJECT CODE: 19MMP204 CLASS : I M.Sc MATHEMATICS 4 0 0 4

UNIT IV
SYLLABUS

Wave Equation:
Elementary solutions of one dimensional wave equation-Vibrating membranes - Applications of calculus of variations- Green's functions for the wave equation.

The Wave Equation

Another classical example of a hyperbolic PDE is a wave equation. The wave equa- tion is a second-order linear hyperbolic PDE that describes the propagation of a variety of waves, such as sound or water waves. It arises in different fields such as acoustics, electromagnetics, or fluid dynamics. In its simplest form, the wave equa-
tion refers to a scalar function $u = u(\mathbf{r},t)$, $\mathbf{r} \in \mathbb{R}^n$ that satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{4.1}$$

Here ∇^2 denotes the Laplacian in $\mathbb{R}^n$ and c is a constant speed of the wave propaga- tion. An even more compact form of Eq. (4.1) is given by

$$Q^2 u = 0,$$

where $Q^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is the d'Alembertian.

The Wave Equation in 1D

The wave equation for the scalar u in the one dimensional case reads

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{4.2}$$

The one-dimensional wave equation (4.2) can be solved exactly by d'Alembert's method, using a Fourier transform method, or via separation of variables. To illus- trate the idea of the d'Alembert

method, let us introduce new coordinates (ξ, η) by use of the transformation

$$\xi = x - ct, \qquad \eta = x + ct. \tag{4.3}$$

In the new coordinate system one can write

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \qquad \frac{1}{c^2}u_{tt} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta},$$

and Eq. (4.2) becomes

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0. \tag{4.4}$$

That is, the function u remains constant along the curves (4.3), i.e., Eq. (4.3) de- scribes characteristic curves of the wave equation (4.2)

(see App. B). Moreover, one can see that the derivative $\partial u/\partial\xi$ does not depends on η, i.e.,

$$\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}\right) = 0 \Leftrightarrow \frac{\partial u}{\partial\xi} = f(\xi).$$

After integration with respect to ξ one obtains

$$u(\xi, \eta) = F(\xi) + G(\eta),$$

where F is the primitive function of f and G is the "constant" of integration, in general the function of η. Turning back to the coordinates (x, t) one obtains the general solution of Eq. (4.2)

$$\boxed{u(x,t) = F(x-ct) + G(x+ct).} \tag{4.5}$$

Solution of the IVP

Now let us consider an initial value problem for Eq. (4.2):

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad t \geq 0, \\ u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x). \end{aligned} \tag{4.6}$$

To write down the general solution of the IVP for Eq. (4.2), one needs to exspress the arbitrary function F and G in terms of initial data f and g. Using the relation

$$\frac{\partial}{\partial t}F(x-ct) = -cF'(x-ct), \quad \text{where } F'(x-ct) := \frac{\partial}{\partial\xi}F(\xi)$$

one becomes: $\partial \xi$

$$u(x, 0) = F(x) + G(x) = f(x);$$
$$u_t(x, 0) = c(-F'(x) + G'(x)) = g(x).$$

After differentiation of the first equation with respect to x one can solve the system in terms of $F'(x)$ and $G'(x)$, i.e.,

$$F'(x) = \frac{1}{2}\left[f'(x) - \frac{1}{c}g(x)\right], \qquad G'(x) = \frac{1}{2}\left[f'(x) + \frac{1}{c}g(x)\right].$$

Hence

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y)\,dy + C, \quad G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x$$

where the integration constant C is chosen in such a way that the initial condition $F(x) + G(x) = f(x)$ is fullfield. Alltogether one obtains:

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy}. \qquad (4.7)$$

Numerical Treatment

A Simple Explicit Method

The first idea is just to use central differences for both time and space derivatives, i.e.,

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\triangle t^2} = c^2\,\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\triangle x^2}, \qquad (4.8)$$

or, with $\alpha = c\triangle t/\triangle x$

$$\boxed{u_i^{j+1} = -u_i^{j-1} + 2(1-\alpha^2)u_i^j + \alpha^2(u_{i+1}^j + u_{i-1}^j)}. \qquad (4.9)$$

Schematical representation of the scheme (4.9) is shown on Fig. 4.1.

Note that one should also implement initial conditions (4.6). In order to imple- ment the second initial condition one needs the virtual point u_i^{-1},

$$u_t(x_i, 0) = g(x_i) = \frac{u_i^1 - u_i^{-1}}{2\triangle t} + O(\triangle t^2).$$

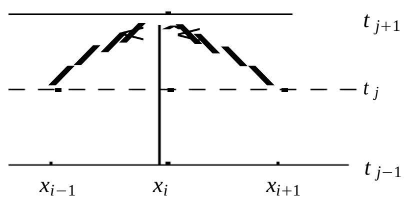


Fig. 4.1 Schematical visu- alization of the numerical scheme (4.9) for (4.2).

With $g_i := g(x_i)$ one can rewrite the last expression as

$$u_i^{-1} = u_i^1 - 2\triangle t g_i + O(\triangle t^2),$$

and the second time row can be calculated as

$$\boxed{u_i^1 = \triangle t g_i + (1-\alpha^2) f_i + \frac{1}{2}\alpha^2 (f_{i-1} + f_{i+1}),} \qquad (4.10)$$

where $u(x_i, 0) = u_i^0 = f(x_i) = f_i$.

von Neumann Stability Analysis

In order to investigate the stability of the explicit scheme (4.9) we start with the usual ansatz (1.21)

$$\varepsilon_i^{j+1} = g^j e^{ikx_i},$$

which leads to the following expression for the amplification factor $g(k)$

$$g^2 = 2(1-\alpha^2)g - 1 + 2\alpha^2 g\cos(k\triangle x).$$

After several transformations the last expression becomes just a quadratic equation for g, namely

$$g^2 - 2\beta g + 1 = 0, \qquad (4.11)$$

where

$$\beta = 1 - 2\alpha^2 \sin^2\left(\frac{k\triangle x}{2}\right).$$

Solutions of the equation for $g(k)$ read

$$g_{1,2} = \beta \pm \sqrt{\beta^2 - 1}.$$

Notice that if $\beta > 1$ then at least one of absolute value of $g_{1,2}$ is bigger that one. Therefor one should desire for $\beta < 1$, i.e.,

$$g_{1,2} = \beta \pm i\sqrt{\beta^2 - 1}$$

and

$$|g|^2 = \beta^2 + 1 - \beta^2 = 1.$$

That is, the scheme (4.9) is conditional stable. The stability condition reads

$$-1 \le 1 - 2\alpha^2 \sin^2\left(\frac{k\triangle x}{2}\right) \le 1,$$

what is equivalent to the standart CFL condition (2.7)

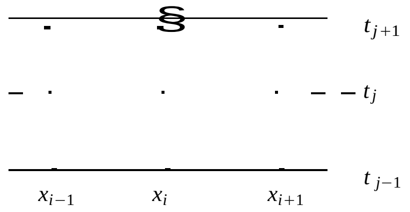


Fig. 4.2 Schematical visualization of the implicit numerical scheme (4.12) for(4.2).

$$\alpha = \frac{c\triangle t}{\triangle x} \leq 1 \,.$$

An Implicit Method

One can try to overcome the problems with conditional stability by introducing *an implicit scheme*. The simplest way to do it is just to replace all terms on the right hand side of (4.8) by an average from

the values to the time steps $j + 1$ and $j - 1$, i.e,

$$\frac{u_i^{j+1} - 2u_i^{j} + u_i^{j-1}}{\triangle t^2} = \frac{c^2}{2\triangle x^2}\left(u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1} + u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} \right). \quad (4.12)$$

Schematical diagramm of the numerical scheme (4.12) is shown on Fig. (4.2).

Let us check the stability of the implicit scheme (4.12). To this aim we use the standart ansatz

$$\varepsilon_i^{j+1} = g^{\,j} e^{ikx_i}$$

leading to the equation for $g(k)$

$$\beta\, g^2 - 2g + \beta = 0$$

with

$$\beta = 1 + \alpha^2 \sin^2\left(\frac{k\triangle x}{2}\right).$$

One can see that $\beta \geq 1$ for all k. Hence the solutions $g_{1,2}$ take the form

$$g_{1,2} = \frac{1 \pm i\sqrt{1 - \beta^2}}{\beta}$$

and

$$|g|^2 = \frac{1 - (1 - \beta^2)}{\beta^2} = 1 \,.$$

That is, the implicit scheme (4.12) is *absolute stable*.

Now, the question is, whether the implicit scheme (4.12) is better than the explicit scheme (4.9) form numerical point of view. To answer this question, let us analyse dispersion relation for the wave equation (4.2) as well as for both schemes (4.9) and

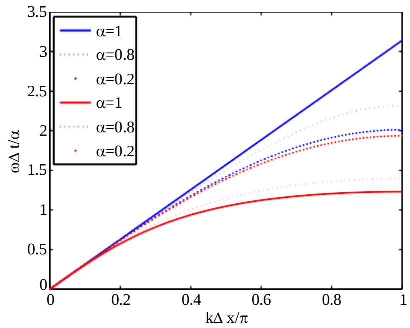


Fig. 4.3 Dispersion relation for the one-dimensional wave equation (4.2), calculated using the explicit (blue curves) and implicit (red curves) methods (4.9) and (4.12).

(4.12). The exact dispersion relation is

$$\omega = \pm ck\,,$$

i. e, all Fourier modes propagate without dispersion with the same phase velocity $\omega/k = \pm c$. Using the ansatz $u_i^j \sim e^{ikx_i - i\omega t_j}$ for the explicit method (4.9) one obtains:

$$\cos(\omega\triangle t) = 1 - \alpha^2(1 - \cos(k\triangle x)),$$

(4.1

3) while for the implicit method (4.12)

$$\cos(\omega\triangle t) = \frac{1}{1 + \alpha^2(1 - \cos(k\triangle x))}\,. \qquad (4.14)$$

One can see that for $\alpha \to 0$ both methods provide the same result, otherwise the explicit scheme (4.9) always exceeds the implicit one (see Fig. (4.3)). For $\alpha = 1$ the scheme (4.9) becomes exact, while (4.12) deviates more and more from the exact value of ω for increasing α. Hence, for Eq. (4.2) there are no motivation to use implicit scheme instead of the explicit one.

Examples

Example 1.

Use the explicit method (4.9) to solve the one-dimansional wave equation (4.2):

$$u_{tt} = 4\,u_{xx} \text{ for } x \in [0, L] \text{ and } t \in [0, T]$$

(4.15) with boundary conditions

$$u(0, t) = 0 \quad u(L, t) = 0\,.$$

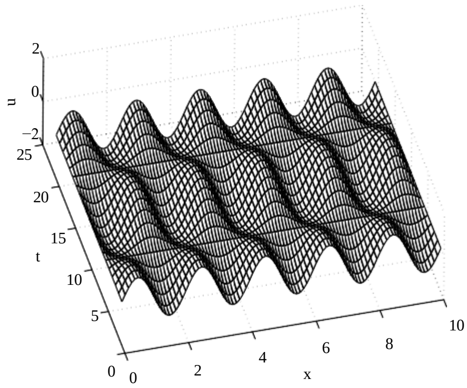


Fig. 4.4 Space-time evolution of Eq. (4.15) with the initial distribution $u(x,0) = \sin(\pi x)$, $u_t(x, 0) = 0$.

Assume that the initial position and velocity are

$$u(x,0) = f(x) = \sin(\pi x), \quad \text{and} \quad u_t(x,0) = g(x) = 0.$$

Other parameters are:

Space interval $L=10$
Space discretization step $\Delta x = 0.1$
Time discretization step $\Delta t = 0.05$
Amount of time steps $T = 20$

First one can find the d'Alambert solution. In the case of zero initial velocity Eq. (4.7) becomes

$$u(x,t) = \frac{f(x-2t) + f(x+2t)}{2} = \frac{\sin\pi(x-2t) + \sin\pi(x+2t)}{2} = \sin(\pi x)\cos(2\pi t),$$

i.e., the solution is just a sum of a travelling waves with initial form, given by $\frac{f(x)}{2}$. Numerical solution of (4.15) is shown on Fig. (4.4).

Example 2.

Solve Eq. (4.15) with the same boundary conditions. Assume now, that initial dis- tributions of position and velocity are

$$u(x,0) = f(x) = 0 \quad \text{and} \quad u_t(x,0) = g(x) = \begin{cases} 0, & x \in [0, x_1]; \\ g_0, & x \in [x_1, x_2]; \\ 0, & x \in [x_2, L]. \end{cases}$$

Other parameters are:

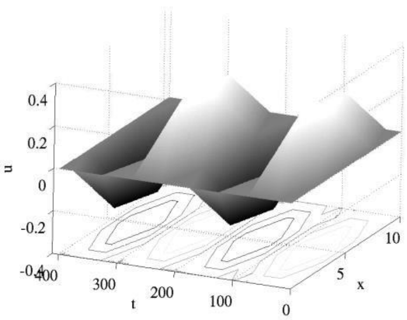


Fig. 4.5 Space-time evolution of Eq. (4.15) with the initial distribution $u(x,0) = 0$, $u_t(x, 0) = g(x)$.

Initial nonzero velocity $g_0=0.5$
Initial space intervals $x_1 = L/4$, $x_2 = 3L/4$
Space interval $L=10$
Space discretization step $\Delta x = 0.1$
Time discretization step $\Delta t = 0.05$
Amount of time steps $T = 400$

Numerical solution of the problem is shown on Fig. (4.5).

Example 3. Vibrating String

Use the explicit method (4.9) to solve the wave equation for a vibrating string:

$$u_{tt} = c^2 u_{xx} \text{ for } x \in [0, L] \text{ and } t \in [0, T], \quad (4.16)$$

where $c = 1$ with the boundary conditions

$$u(0,t) = 0 \qquad u(L,t) = 0.$$

Assume that the initial position and velocity are

$$u(x, 0) = f(x) = \sin(n\pi x/L), \quad \text{and} \quad u_t(x, 0) = g(x) = 0, \quad n = 1, 2, 3, \ldots.$$

Other parameters are:

Space interval $L=1$
Space discretization step $\Delta x = 0.01$
Time discretization step $\Delta t = 0.0025$
Amount of time steps $T = 2000$

Usually a vibrating string produces a sound whose frequency is constant. There- fore, since frequency characterizes the pitch, the sound produced is a constant note. Vibrating strings are the basis of any string instrument like guitar or cello. If the speed of propagation c is known, one can calculate the frequency of the sound produced by the string. The speed of propagation of a wave c is equal to the wavelength multiplied by the frequency f :

$$c = \lambda f$$

If the length of the string is L, the fundamental harmonic is the one produced by the vibration whose nodes are the two ends of the string, so L is half of thewavelength of the fundamental harmonic, so

$$f = \frac{c}{2L}$$

Solutions of the equation in question are given in form of standing waves. The stand- ing wave is a wave that remains in a constant position. This phenomenon can occur because the medium is moving in the opposite direction to the wave, or it can arise in a stationary medium as a result of interference between two waves traveling in opposite directions (see Fig. (4.6))

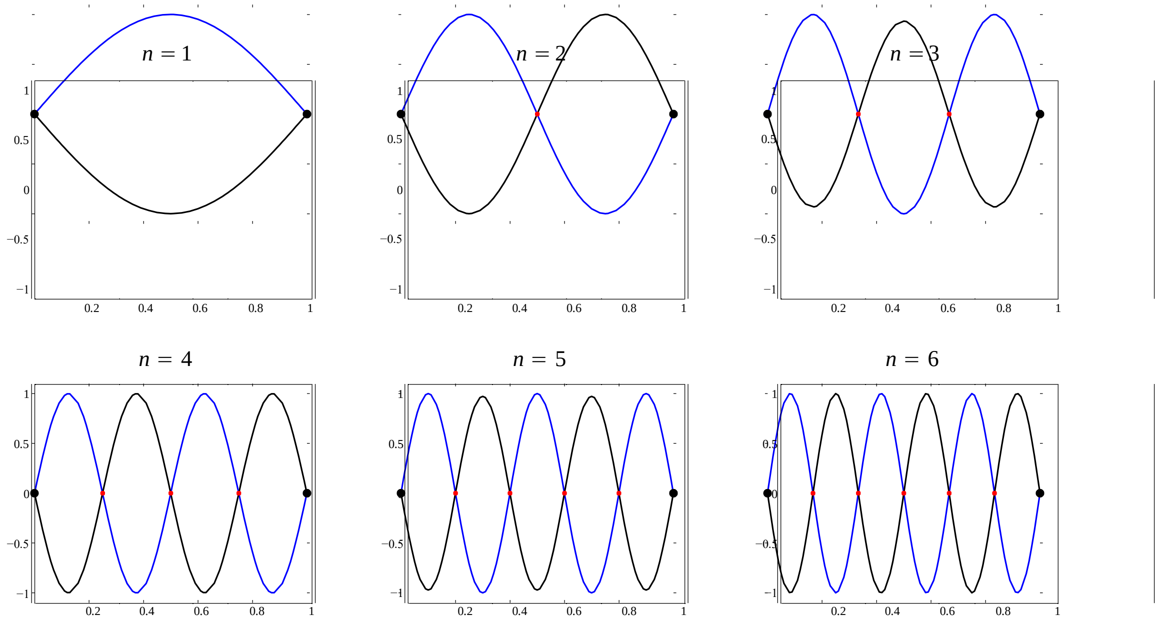


Fig. 4.6 Standing waves in a string. The fundamental mode and the first five overtones are shown. The red dots represent the wave nodes.

The Wave Equation in 2D

Examples

Example 1.

Use the standard five-point explicit method (4.9) to solve a two-dimansional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad u = u(x,y,t)$$

on the rectangular domain $[0, L] \times [0, L]$ with Dirichlet boundary conditions. Other parameters are:

Space interval L=1
Space discretization step $\Delta x = \Delta y = 0.01$ Time discretization step Δt = 0.0025 Amount of time steps T = 2000
Initial condition $u(x, y, 0) = 4x^2y(1-x)(1-y)$

Numerical solution of the problem for two different time moments $t = 0$ and $t = 500$ can be seen on Fig. (4.7)

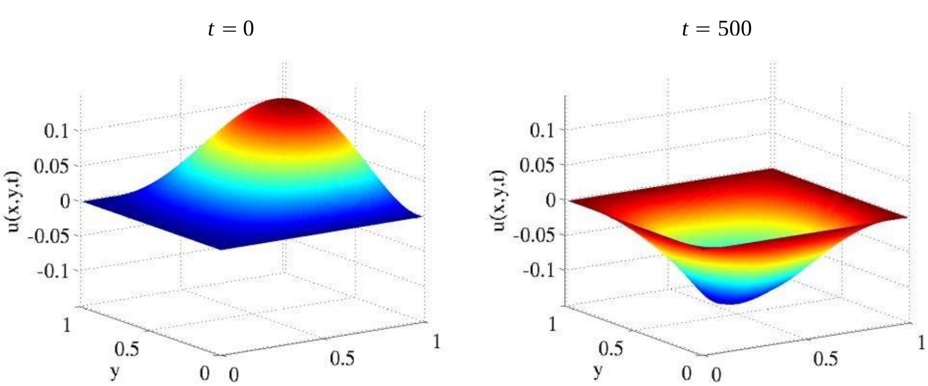


Fig. 4.7 Numerical solution of the two-dimensional wave equation, shown fo

Wave equation examples

The wave equation is discussed in detail in the Dawkins online text,.

The function *u(x, t)* is a solution to the classical one-dimensional wave equation if it satisfies the PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}$$

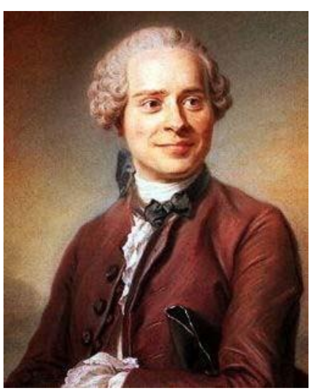
$\partial^2 u$ ∂t^2

The **wave function** *u* is the amplitude of the wave as a function of time and position. The constant *v* is the wave's velocity in the x direction.

For a derivation of the wave equation, see

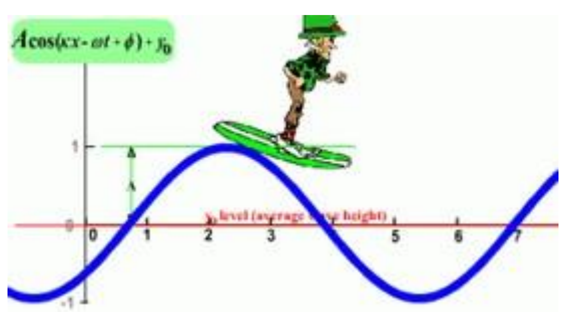
Since the wave equation is a linear second order PDE, given **any** two twice-differentiable functions of a single variable (call them f_1 and f_2), the most general solution is

$\boldsymbol{u(x, t) = f_1(x + vt) + f_2(x - vt)}$**.** That's almost all there is to it! (except for the details – ah, the details).



This was first noted by Jean D'Alembert, 18th century French mathematician and *bon vivant*. The plus/minus signs in $x + vt$ and $x - vt$ indicate the direction of wave travel: $f_2(x - vt)$ is traveling to the right and $f_1(x + vt)$ is traveling to the left. How can you remember that? Think of surfing a wave: you want to stay in the same relative position, riding the wave crest. As time goes on (t increases, you and the wave both move to the right (your x position increases). In order to keep the same relative point on the wave function, you'd better be surfing $f(x - vt)$. Was D'Alembert a surfer? With that hair? Not likely.

A leprechaun caught surfin' the cosine wave off Malibu. As t and x increase, he rides x – vt, staying at the same wave height.



Example Let $f_1(x + vt) = \cos(x + vt)$ and $f_2(x - vt) = 1 - (x - vt)^2$.

Then $u(x, t) = 1 - (x - vt)^2 + \cos(x + vt)$ is shown to be a wave function if it satisfies the wave equation. Show that it fits the PDE. Graph the function in *x* and *t* (especially using **Animate** to plot the function of x and animate it in time).

The form of the solution to the wave equation is determined by both the initial conditions (what is the value of u when t = 0?) and the boundary conditions (what must the wave function do at end points of the domain?).

No boundaries: traveling waves on a very long string

D'Alembert's analysis of wave functions leads to several important results.

First, we analyze the wave equation with ICs only. Let's write the wave equation as

$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$ or $u_{tt} = v^2 u_{xx}$ and the wave function $u(x, t) = f_1(x + vt) + f_2(x - vt)$.

a. We are given an initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = 0$. This is a guitar string plucked with finger or pick (although D'Alembert would have studied the harpsichord).

Applying the second (velocity) IC first, take the required derivatives of u:

$$u_t(x,0) = 0 = vf_1'(x) - vf_2'(x) \quad \text{or} \quad f_1' = f_2'$$

We integrate this directly to obtain $f_1 = f_2 + C$ and therefore

$u(x, t) = f_1(x + vt) + f_1(x - vt) - C$

Applying the first IC $u(x,0) = f(x)$,

$$f(x) = f_1(x) + f_2(x) = 2f_1 - C \text{ or } f_1 = \frac{f(x) + C}{2}$$

Combine:

$$u(x, t) = \frac{1}{2} f(x + vt) + \frac{C}{2} + \frac{1}{2} f(x - vt) + \frac{C}{2} - C$$

$$= \frac{1}{2} f(x + vt) + \frac{1}{2} f(x - vt)$$

The solution function is therefore always a sum (superposition) of ½ of the function that describes the shape of the string pluck; the constant cancels.

b. An initial velocity $\boldsymbol{u_t(x,0) = g(x)}$ is given and the initial displacement $\boldsymbol{u(0,t) = 0}$. This is a piano string struck by pressing a key or the very cool instrument known as a hammered dulcimer.

Use the method above to eliminate f_2:

$u(x, t) = f_1(x - vt) + f_1(x + vt)$ and then show that $f_1(x) = -\frac{1}{2v}\int_0^x g(s)ds$, where s is a dummy variable that disappears upon integration.

Combine to obtain $u(x,t) = \frac{1}{2v}\int_{x-vt}^{x+vt} g(s)ds$

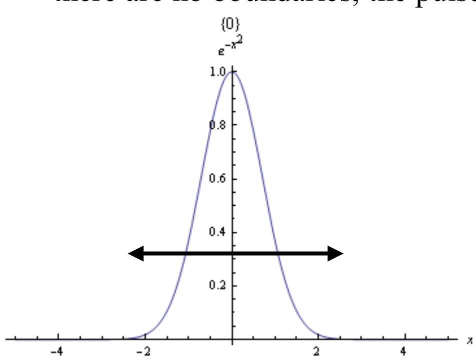
c. Combination of conditions: $\boldsymbol{u(0,t) = f(x)}$ and $\boldsymbol{u_t(x,0) = g(x)}$

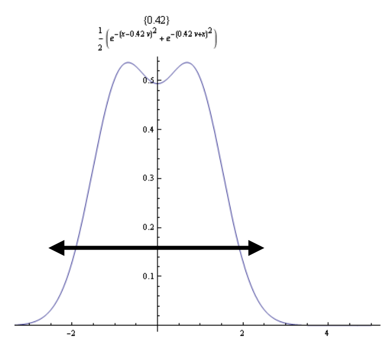
Combined ICs yield combined solutions known as **D'Alembert's Formula**:

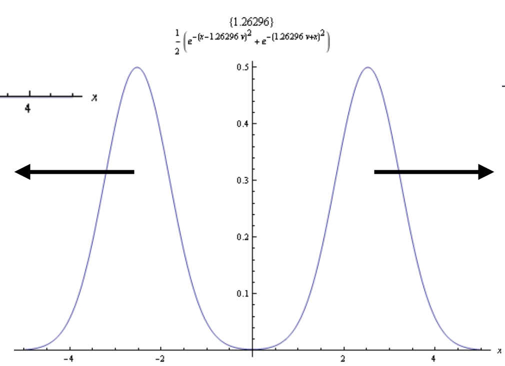
$$u(x,t) = \frac{1}{2} f(x - vt) + \frac{1}{2} f(x + vt) + \frac{1}{2v}\int_{x-vt}^{x+vt} g(s)ds$$

Example: Apply D'Alembert's Formula to form the wave function given by the initial condition (pluck) $u(x, 0) = \frac{1}{2} e^{-x^2}$ with $v = 4$.

We see immediately that $u(x, t) = \frac{1}{2} e^{-(x+2t)^2} + \frac{1}{2} e^{-(x-2t)^2}$, as illustrated below. The initial pulse starts at x = 0 and splits in two, one traveling left, the other traveling right. Since there are no boundaries, the pulses continue moving away from each other … forever.







Example

Suppose the ICs are $u(x, 0) = \frac{1}{2} e^{-x^2}$ and $u_t(x, 0) = -e^{-x^2}$ for $v = 4$.

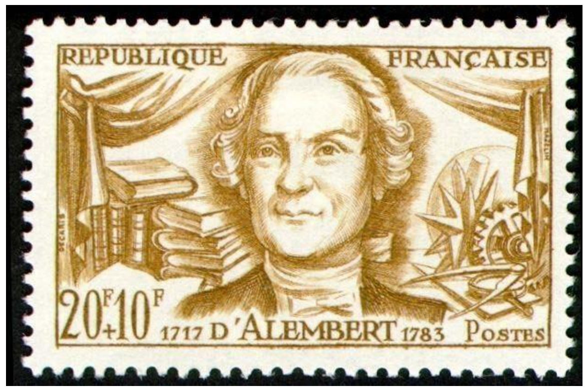
Use the D'Alembert Formula to find the wave function *u(x,t)*.

Once the functions f and g are defined, this statement will find values of u[x,t] using the D'Alembert Formula:

```
u[c_,f_,g_,t_,x_]:= .5(f[x + c t] + f[x - c t])+ (1/(2 c)) Integrate[g[x1], {x1,x - c t, x + c t}]
```

Woddya know? A stamp!

http://jeff560.tripod.com/images/dalemb.jpg



Now that we know the form of the solutions, we can look at some BVPs

We'll start with a one-sided boundary: Suppose a horizontal string is tied at one end (say x = 0), where it cannot move and thus $u(0,t) = 0$ and $u_t(0,t) = 0$

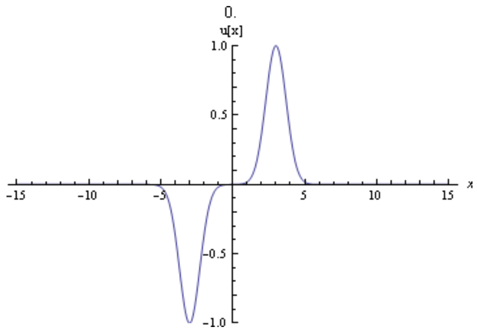
In order to prevent any displacement at the bound end, a "reflection" will be generated – a wave of opposite polarity will originate at the boundary. When the incoming wave and the reflected wave are superimposed, they cancel.

Example: The pulse begins at x = 3 so that $u(x, 0) = \frac{1}{2} e^{-(x-3)^2}$.

We must form a function that extends a negative of our wave function into x < 0 so that the sum of the wave displacement is 0.

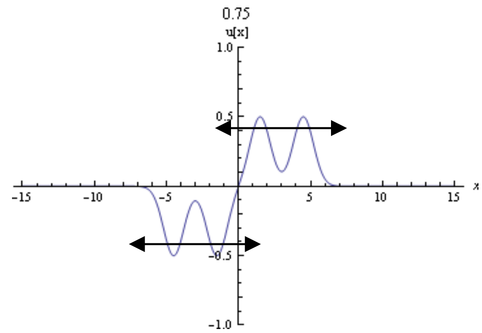
In general, this can be done by turning the wave function into an odd functions by an 'odd flip:' *y(x)* is redefined as *–y(-x)* for *x<0*. The most compact way to do this (but certainly not the only way) is as follows:

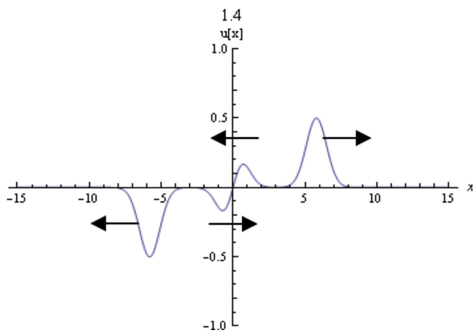
$$u(x, t) = \frac{1}{2}\left[\text{Sign}[x + vt]\, f(\text{Abs}[x + vt]) + \text{Sign}[x - vt]\, f(\text{Abs}[x - vt])\right]$$



This really creates two wave functions – the one we see from the starting point of the pulse and a mirror image (in reversed polarity) starting from x < 0.

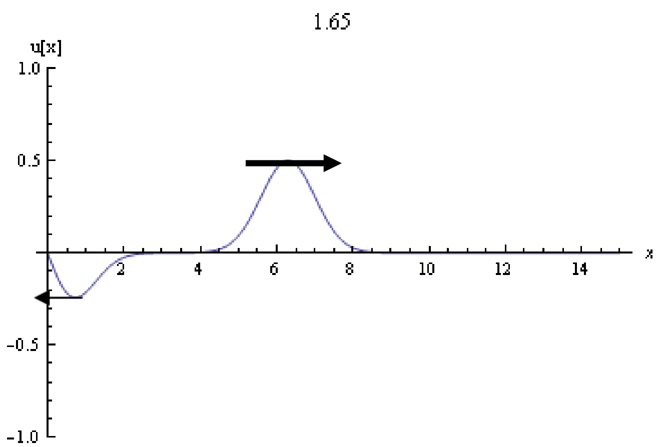
Each pulse splits, with one half moving left and the other right.





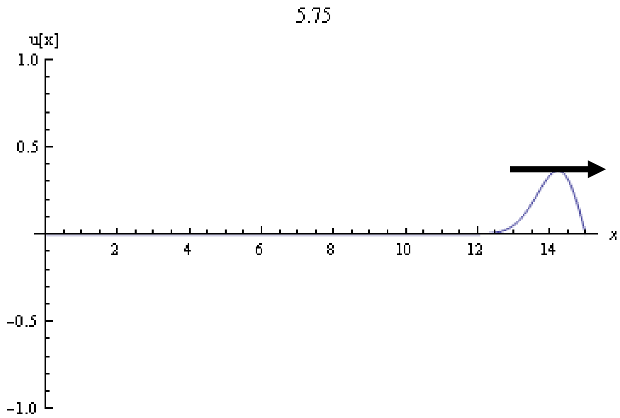
When the ‘real’ wave and the mirror image pass through each other at the boundary, they cancel out.

But we are only interested in what happens with x > 0, so it looks like the original pulse is reflected at x = 0; then both pulses move to the right.



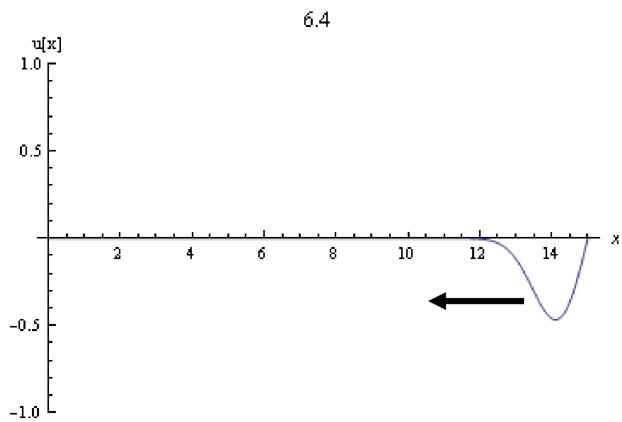
Verify that the wave function *u(x,t)* = 0 at a reflecting boundary for all values of t.

It is also possible to reflect at the right hand boundary.

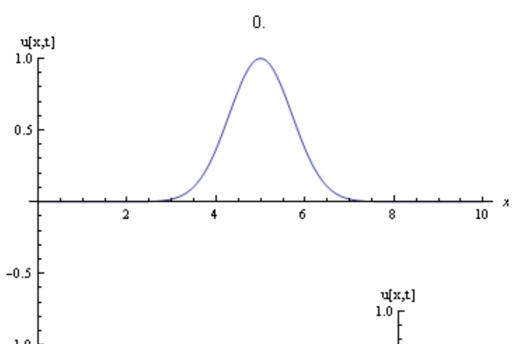


Pulse moving right, striking boundary at x = 15.

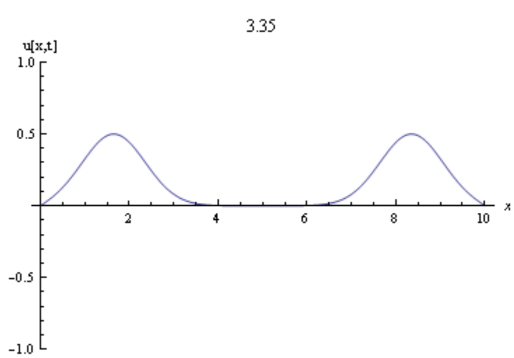
Reflected pulse (reverse polarity) now moving to the left.



Reflections at both boundaries are also possible – but require additional trickery

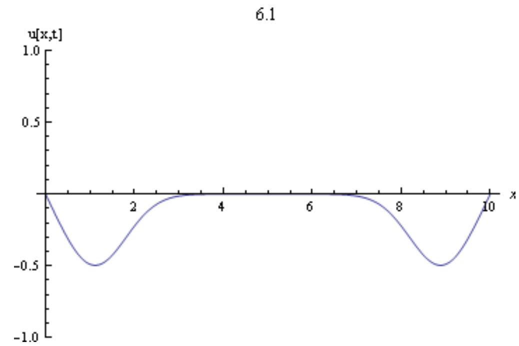


Pulse begins at x = 5, t = 0



Pulse splits, parts move left and right, about to strike boundaries at x = 0 and x = 10.

Reflected pulses now moving back towards x = 5



This required *conditional function definition* using **/;** *Mathematica's* conditional definition operator.

```
pulse2[x_, x0_, left_, right_] := e^-(x-x0)^2 /; left ≤ x < right
pulse2[x_, x0_, left_, right_] := -e^-(-x-x0)^2 /; x < left (* left is presumed to be zero *)
pulse2[x_, x0_, left_, right_] := -e^-(right+x0-x)^2 /; x ≥ right

Plot[.5 pulse2[x + t, 5, left, right] + .5 pulse2[x - t, 5, left, right], {x, left, right},
 PlotRange → {-1, 1}, AxesLabel → {x, "u[x,t]"}, PlotLabel → t]
```

Values for **left** and **right** (the x position of the boundaries) can be explicitly assigned prior to the **Plot[ ]** or set with a list replacement within the **Plot[ ]**.

We have defined the velocity of the wave as the value v in the wave equation. What is the derivative $u_t(x,t)$ represent in physical terms?

What happens if we use a continuous cosine $u(x, t) = ½ \cos(x - vt) + ½ \cos(x + vt)$ instead of a discrete pulse? Try it!

A more general means of finding a wave function when there are boundary conditions involves the technique of Separation of Variables. **Work that lab before continuing below.**

See http://www.math.duke.edu/education/ccp/materials/engin/wave/index.html. Work through all parts of this webpage and answer the questions in the summary. We will get to Fourier Series solutions after a while; for the moment, just think of them as an approximation to the given function formed by adding sines and cosines.

We will use Separation of Variables to consider each of the following cases, each specified by a different set of boundary conditions.

See notes in **wave equationBVP.pdf**

1. String of length L tied at both ends (standing waves)

   Boundary conditions **u(0,t) = u(L,t) = 0.** An initial amplitude **u(x,0)** or particle velocity $\mathbf{u_t(x,0)}$ or some combination of these ICs may be specified.

For an excellent animation of a standing wave on a string, see
http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/forces%20on%20wave.swf

2. Tube of length L open at one end (standing waves)

   Boundary conditions **u(0,t) = 0 u(L,t) = A,** for an amplitude value A.

3. String of length L tied at one end and shaken with amplitude A from the other end (traveling waves)

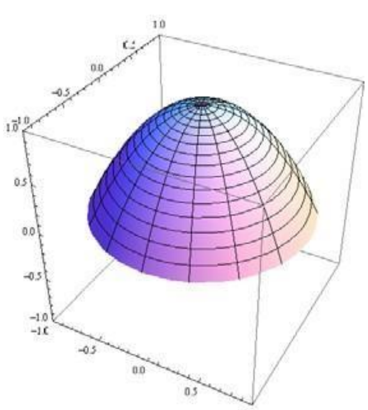
   Boundary conditions **u(0,t) = 0 u(L,t) = A.** An initial position or velocity must be specified.

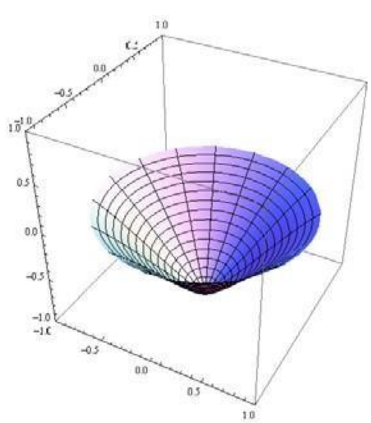
The vibrating drumhead (circular case)

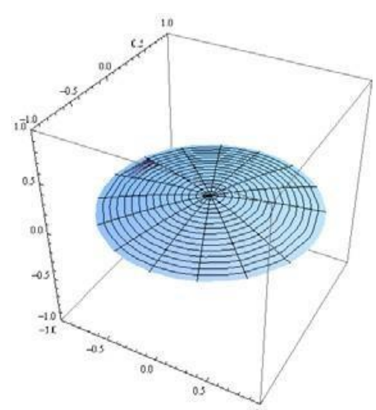
The two-dimensional wave equation can be expressed in polar coordinates.

$\frac{\partial^2 u}{\partial t^2} = \frac{1}{c2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ becomes $\frac{\partial^2 U}{\partial t^2} = \frac{1}{c^2}\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right)$. Here, *u(x, y, t)* is a amplitude displacement function in rectangular coordinates and *U(r, θ, t)* is the displacement function transposed into polar coordinates. Good news: It is still variables separable!

Suitable boundary conditions might be fixed edges at r = 1 and an initial displacement or velocity at the center.







UNIT V

PART B

(5X6=30 Marks)

1. Describe a method of boundary value problems for the generalised diffusion equation.
2. State and prove Duhamel's Theorem.
3. Describe a method of boundary value problems for the generalised diffusion equation.
4. State and prove Duhamel's Theorem.
5. Use Green's function to find the solution of the boundary value problem of diffusion equation.

PART – C

(1X10=10)

1) Find the solution of the Helmholtz equation by using the method separation of variables.
2. Discuss in detail about the diffusion equation with sources

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answers
The solution elementary solution of one dimentional wave equation is _________	D Alembert's	Kennal	Kernal	Laplace	D Alemberts
The motion of semi infinite string is _______	diameter	radius	positive	potential	positive
The one dimensional wave equation _____ solution follows	Riemann	conduction	node	potential	Riemann
Vibrating membrane is the application of _______ of variation.	grad	calculus	electrostatic	potential	calculus
In vibrating membrane ________ roots are positive	n	1	2	3	n
In vibrating membrane n roots are _____________	negative	positive	linear	non linear	positive
The variational approach to ____ value problem is useful in the derivation of approximating solution	Euclid	Kernal	eigen	node	eigen
The Bessel function of the first kind with argument is _______	x	y	z	none	z
In wave equation all singularities lie ________ the boundary	inside	outside	in	on	outside
In Greens theorem the concept of _________ theorem is applied	weber	route	null	kennel	weber
In the Riemann-Volterra solution of the one dimensional wave equation the variable x transformed to____	epsilon	eta	geta	gamma	geta
In the Riemann-Volterra solution of the one dimensional wave equation the variable y transformed to____	epsilon	eta	geta	gamma	eta
The curve gamma is the projection of c with equation___________	u(x,y)=0	u(x,y)=1	u(x,y)=2	u(x,y)=n	u(x,y)=0
The Greens function w must satisfy the condition_________	Lw=1	Lw=0	L/w=0	L-w=0	Lw=0
In the Riemann-Volterra solution the sector is called the ______ domain of influence(x0,y0)	cononical	unique	initial	long	initial

KARPAGAM ACADEMY OF HIGHER EDUCATION

SUB :PARTIAL DIFFERENTIAL EQUATIONS SEMESTER: II **L T P C**

SUBJECT CODE: 19MMP204 Class : I M.Sc MATHEMATICS **4 0 0 4**

UNIT V
SYLLABUS

Diffusion Equation:
The resolution of Boundary value problems for the Diffusion equation- Elementary solutions of diffusion equation - Separation of variables- use of Green's functions- Diffusion with Sources.

1. Concepts, Definitions, and the Diffusion Equation

Environmental fluid mechanics is the study of fluid mechanical processes that affect the fate and transport of substances through the hydrosphere and atmosphere at the local or regional scale[1] (up to 100 km). In general, the substances of interest are mass, momentum and heat. More specifically, mass can represent any of a wide variety of passive and reactive tracers, such as dissolved oxygen, salinity, heavy metals, nutrients, and many others. Part I of this textbook, "Mass Transfer and Diffusion," discusses the passive process affecting the fate and transport of species in a homogeneous natural environment. Part II, "Stratified Flow and Buoyant Mixing," incorporates the effects of buoyancy and stratification to deal with active mixing problems.

This chapter introduces the concept of mass transfer (transport) and focuses on the physics of diffusion. Because the concept of diffusion is fundamental to this part of the course, we single it out here and derive its mathematical representation from first principles to the solution of the governing partial differential equation. The mathematical rigor of this section is deemed appropriate so that the student gains a fundamental and complete understanding of diffusion and the diffusion equation. This foundation will make the complicated processes discussed in the remaining chapters tractable and will start to build the engineering intuition needed to solve problems in environmental fluid mechanics.

Concepts and definitions

Stated simply, Environmental Fluid Mechanics is the study of natural processes that change concentrations.

These processes can be categorized into two broad groups: transport and transformation. Transport refers to those processes which move substances through the hydrosphere

the fact that mass fraction and concentration are often used interchangeably in dilute and atmosphere by physical means. As an analogy to the postal service, transport is the process by which a letter goes from one location to another. The postal truck is the analogy for our fluid, and the letter itself is the analogy for our chemical species. The two primary modes of transport in environmental fluid mechanics are advection (transport associated with the flow of a fluid) and diffusion (transport associated with random motions within a fluid). Transformation refers to those processes that change asubstance

of interest into another substance. Keeping with our analogy, transformation is the paper recycling factory that turns our letter into a shoe box. The two primary modes of transformation are physical (transformations caused by physical laws, such as radioactive decay) and chemical(transformations caused bychemicalor biological reactions, such as dissolution).

The glossary at the end of this text provides a list of important terms and their definitions in environmental fluid mechanics (with the associated German term).

Expressing Concentration

The fundamental quantity of interest in environmental fluid mechanics is concentration. In common usage, thetermconcentration expresses ameasureoftheamountofasubstance within a mixture.

Mathematically, the concentration C is the ratio of the mass of a substance M_i to the total volume of a mixture V expressed

$$C = \frac{M_i}{V}. \tag{1.1}$$

The units of concentration are [M/L^3], commonly reported in mg/l, kg/m^3, lb/gal, etc. Forone- andtwo-dimensionalproblems, concentration can also be expressed as themass per unit segment length [M/L] or per unit area, [M/L^2].

A related quantity, the mass fraction x is the ratio of the mass of a substance M_i to the total mass of a mixture M , written

$$x = \frac{M_i}{M}. \tag{1.2}$$

Mass fraction is unitless, but is often expressed using mixed units, such as mg/kg, parts per million (ppm), or parts per billion (ppb).

A popular concentration measure used by chemists is the molar concentration θ. Molar concentration is defined as the ratio of the number of moles of a substance N_i to the total volume of the mixture

$$\theta = \frac{N_i}{V}. \tag{1.3}$$

The units of molar concentration are [number of molecules/L^3]; typical examples are mol/l and μmol/l. To work with molar concentration, recall that the atomic weight of an atom is reported in the Periodic Table in units of g/mol and that a mole is $6.022 \cdot 10^{23}$ molecules.

The measure chosen to express concentration is essentially a matter of taste. Always use caution and confirm that the units chosen for concentration are consistent with the equations used to predict fate and transport. A common source of confusion arises from aqueous systems. This comes about because the density of pure water at 4°C is 1 g/cm^3, making values for concentration in mg/l and mass fraction in ppm identical. Extreme caution should be used in other solutions, as in seawater or the atmosphere, where ppm and mg/l are *not* identical. The conclusion to be drawn is: always check your units!

the fact that mass fraction and concentration are often used interchangeably in dilute

A very powerful analytical technique that we will use throughout this course is dimensional analysis. The concept behind dimensional analysis is that if we can define the parameters that a process depends on, then we should be able to use these parameters, usually in the form of dimensionless variables, to describe that process at all scales (not just the scales we measure in the laboratory or the field).

Dimensional analysis as a method is based on the Buckingham π-theorem (see e.g. Fischer et al. 1979). Consider a process that can be described by m dimensional variables. This full set of variables contains n different physical dimensions (length, time, mass, temperature, etc.). The Buckingham π-theorem states that there are, then, $m-n$ independent non-dimensional groups that can be formed from these governing variables (Fischer et al. 1979). When forming the dimensionless groups, we try to keep the dependent variable (the one we want to predict) in only one of the dimensionless groups (i.e. try not to repeat the use of the dependent variable).

Once we have the $m - n$ dimensionless variables, the Buckingham π-theorem further tells us that the variables can be related according to

$$\pi_1 = f(\pi_2, \pi_i, \ldots, \pi_{m-n}) \qquad (1.4)$$

where π_i is the ith dimensionless variable. As we will see, this method is a powerful way to find engineering solutions to very complex physical problems.

As an example, consider how we might predict when a fluid flow becomes turbulent. Here, our dependent variable is a quality (turbulent or laminar) and does not have a dimension. The variables it depends on are the velocity u, the flow disturbances, characterized by a typical length scale L, and the fluid properties, as described by its density ρ, temperature T, and viscosity μ. First, we must recognize that ρ and μ are functions of T; thus, all three of these variables cannot be treated as independent. The most compact and traditional approach is to retain ρ and μ in the form of the kinematic viscosity $\nu = \mu/\rho$. Thus, we have $m = 3$ dimensional variables (u, L, and ν) in $n = 2$ physical dimensions (length and time).

The next step is to form the dimensionless group $\pi_1 = f(u, L, \nu)$. This can be done by assuming each variable has a different exponent and writing separate equations for each dimension. That is

$$\pi_1 = u^a L^b \nu^c, \qquad (1.5)$$

and we want each dimension to cancel out, giving us two equations

$$\text{T gives: } 0 = -a - c$$
$$\text{L gives: } 0 = a + b + 2c.$$

From the T-equation, we have $a = -c$, and from the L-equation we get $b = -c$. Since the system is under-defined, we are free to choose the value of c. To get the most simplified form, choose $c = 1$, leaving us with $a = b = -1$. Thus, we have

$$\pi_1 = \frac{\nu}{uL}. \quad (1.6)$$

This non-dimensional combination is just the inverse of the well-known Reynolds number *Re*; thus, we have shown through dimensional analysis, that the turbulent state of the fluid should depend on the Reynolds number

$$Re = \frac{uL}{\nu}, \quad (1.7)$$

which is a classical result in fluid mechanics.

Diffusion

A fundamental transport process in environmental fluid mechanics is diffusion. Diffusion differs from advection in that it is random in nature (does not necessarily follow a fluid particle). A well-known example is the diffusion of perfume in an empty room. If a bottle of perfume is opened and allowed to evaporate into the air, soon the whole room will be scented. We know also from experience that the scent will be stronger near the source and weaker as we moveaway,but fragrance molecules will have wondered throughout the room due to random molecular and turbulent motions. Thus, diffusion has twoprimary properties: it is random in nature, and transport is from regions of high concentration to low concentration, with an equilibrium state of uniform concentration.

Fickian diffusion

We just observed in our perfume example that regions of high concentration tend to spread into regions of low concentration under the action of diffusion. Here, we want to derive a mathematical expression that predicts this spreading-out process, and we will follow an argument presented in Fischer et al. (1979).

To derive a diffusive flux equation, consider two rows of molecules side-by-side and centered at $x = 0$, as shown in Figure 1.1(a.). Each of these molecules moves about randomly in response to the temperature (in a random process called Brownian motion). Here, for didactic purposes, we will consider only one component of their three-dimensional motion: motion right or left along the x-axis. We further define the mass of particles on the left as M_l, the mass of particles on the right as M_r, and the probability (transfer rate per time) that a particles moves across $x = 0$ as k, with units [T^{-1}].

After some time δt an *average* of half of the particles have taken steps to the right and half have taken steps to the left, as depicted through Figure 1.1(b.) and (c.). Looking at the particle histograms also in Figure 1.1, we see that in this random process, maximum concentrations decrease, while the total region containing particles increases (the cloud spreads out).

Mathematically, the average flux of particles from the left-hand column to the right is kM_l, and the average flux of particles from the right-hand column to the left is $-kM_r$, where the minus sign is used to distinguish direction. Thus, the net flux of particles q_x is

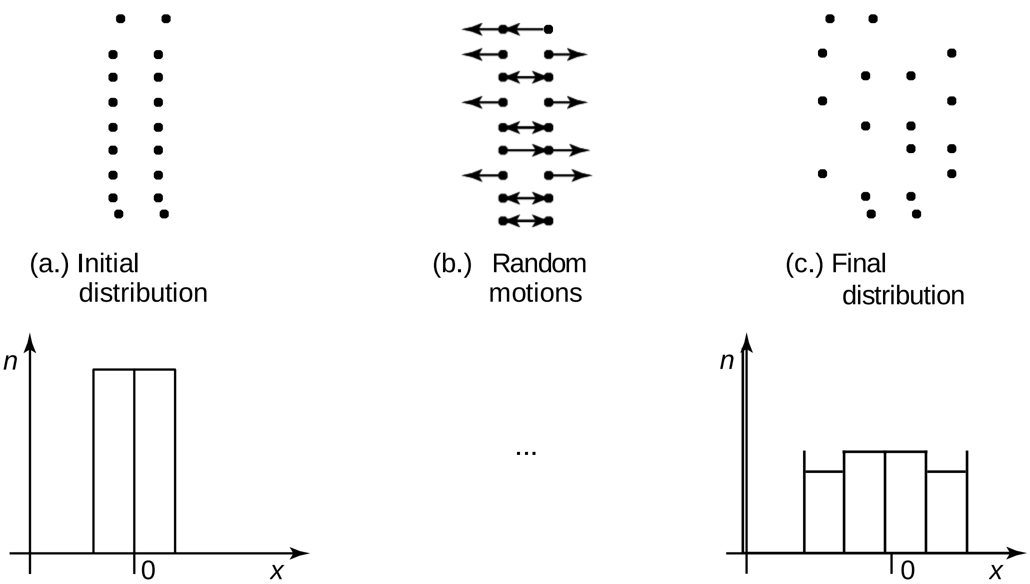


Fig. 1.1. Schematic of the one-dimensional molecular (Brownian) motion of a group of molecules illustrating the Fickian diffusion model. The upper part of the figure shows the particles themselves; the lower part of the figure gives the corresponding histogram of particle location, which is analogous to concentration.

$$q_x = k(M_l - M_r). \tag{1.8}$$

For the one-dimensional case, concentration is mass per unit line segment, and we can write (1.8) in terms of concentrations using

$$C_l = M_l/(\delta x \delta y \delta z) \tag{1.9}$$

$$C_r = M_r/(\delta x \delta y \delta z) \tag{1.10}$$

where δx is the width, δy is the breadth, and δz is the height of each column. Physically, δx is the average step along the x-axis taken by a molecule in the time δt. For the one-dimensional case, we want q_x to represent the flux in the x-direction per unit area perpendicular to x; hence, we will take $\delta y \delta z = 1$. Next, we note that a finite difference approximation for dC/dx is

$$\frac{dC}{dx} = \frac{C_r - C_l}{x_r - x_l} = \frac{M_r - M_l}{\delta x(x_r - x_l)}, \tag{1.11}$$

which gives us a second expression for $(M_l - M_r)$, namely,

$$(M_l - M_r) = -\delta x(x_r - x_l)\frac{dC}{dx}. \tag{1.12}$$

Substituting (1.12) into (1.8) yields

$$q_x = -k(\delta x)^2 \frac{dC}{dx}. \tag{1.13}$$

(1.13) contains two unknowns, k and δx. Fischer et al. (1979) argue that since q cannot depend on an arbitrary δx, we must assume that $k(\delta x)^2$ is a constant, which we will

Example Box 1.1:
Diffusive flux at the air-water interface.

The time-average oxygen profile $C(z)$ in the laminar sub-layer at the surface of a lake is

$$C(z) = C_{sat} - (C_{sat} - C_l)\,\mathrm{erf}\left(\frac{z}{\delta\sqrt{2}}\right)$$

where C_{sat} is the saturation oxygen concentration in the water, C_l is the oxygen concentration in the body of the lake, δ is the concentration boundary layer thickness, and z is defined positive downward. Turbulence in the body of the lake is responsible for keeping δ constant. Find an expression for the total rate of mass flux of oxygen into the lake.

Fick's law tells us that the concentration gradient in the oxygen profile will result in a diffusive flux of oxygen into the lake. Since the concentration is uniform in x and y, we have from (1.14) the diffusive flux

$$q_z = -D\frac{dC}{dz}.$$

The derivative of the concentration gradient is

$$\frac{dC}{dz} = -(C_{sat} - C_l)\frac{d}{dz}\,\mathrm{erf}\left(\frac{z}{\delta\sqrt{2}}\right)$$
$$= -\frac{2(C_{sat} - C_l)}{\sqrt{\pi}\,\delta\sqrt{2}}e^{-\left(\frac{z}{\delta\sqrt{2}}\right)^2}$$

At the surface of the lake, z is zero and the diffusive flux is

$$q_z = (C_{sat} - C_l)\frac{D\sqrt{2}}{\delta\sqrt{\pi}}.$$

The units of q_z are in $[M/(L^2 \cdot T)]$. To get the total mass flux rate, we must multiply by a surface area, in this case the surface of the lake A_l. Thus, the total rate of mass flux of oxygen into the lake is

$$\dot{m} = A_l(C_{sat} - C_l)\frac{D\sqrt{2}}{\delta\sqrt{\pi}}.$$

For $C_l < C_{sat}$ the mass flux is positive, indicating flux down, into the lake. More sophisticated models for gas transfer that develop predictive expressions for δ are discussed later in Chapter 5.

callthe diffusion coefficient, D. Substituting, we obtain the one-dimensional diffusive flux equation

$$q_x = -D\frac{dC}{dx}. \qquad (1.14)$$

It is important to note that diffusive flux is a vector quantity and, since concentration is expressed in units of $[M/L^3]$, it has units of $[M/(L^2T)]$. To compute the total mass flux rate $\dot{m}$, in units $[M/T]$, the diffusive flux must be integrated over a surface area. For the one-dimensional case we would have $\dot{m} = Aq_x$.

Generalizing to three dimensions, we can write the diffusive flux vector at a point by adding the other two dimensions, yielding (in various types of notation)

$$\begin{aligned} q &= -D\left(\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial z}\right) \\ &= -D\nabla C \\ &= -D\frac{\partial C}{\partial x_i}. \end{aligned} \qquad (1.15)$$

Diffusion processes that obey this relationship are called Fickian diffusion, and (1.15) is called Fick's law. To obtain the total mass flux rate we must integrate the normal component of q over a surface area, as in

$$\dot{m} = \iint_A q \cdot n\,dA \qquad (1.16)$$

where n is the unit vector normal to the surface A.

Table 1.1. Molecular diffusion coefficients for typical solutes in water at standard pressure and at two temperatures (20°C and 10°C).[a]

Solute name	Chemical symbol	Diffusion coefficient[b] (10^{-4} cm^2/s)	Diffusion coefficient[c] (10^{-4} cm^2/s)
hydrogen ion	H^+	0.85	0.70
hydroxide ion	OH^-	0.48	0.37
oxygen	O_2	0.20	0.15
carbon dioxide	CO_2	0.17	0.12
bicarbonate	HCO_3^-	0.11	0.08
carbonate	CO_3^{2-}	0.08	0.06
methane	CH_4	0.16	0.12
ammonium	NH_4^+	0.18	0.14
ammonia	NH_3	0.20	0.15
nitrate	NO_3^-	0.17	0.13
phosphoric acid	H_3PO_4	0.08	0.06
dihydrogen phosphate	$H_2PO_4^-$	0.08	0.06
hydrogen phosphate	HPO_4^{2-}	0.07	0.05
phosphate	PO_4^{3-}	0.05	0.04
hydrogen sulfide	H_2S	0.17	0.13
hydrogen sulfide ion	HS^-	0.16	0.13
sulfate	SO_4^{2-}	0.10	0.07
silica	H_4SiO_4	0.10	0.07
calcium ion	Ca^{2+}	0.07	0.05
magnesium ion	Mg^{2+}	0.06	0.05
iron ion	Fe^{2+}	0.06	0.05
manganese ion	Mn^{2+}	0.06	0.05

[a] Taken from http://www.talknet.de/~alke.spreckelsen/roger/thermo/difcoef.html
[b] for water at 20°C with salinity of 0.5 ppt.
[c] for water at 10°C with salinity of 0.5 ppt.

Diffusion coefficients

From the definition $D = k(\delta x)^2$, we see that D has units L^2/T. Since we derived Fick's law for molecules moving in Brownian motion, D is a molecular diffusion coefficient, which we will sometimes call D_m to be specific. The intensity (energy and freedom of motion) of these Brownian motions controls the value of D. Thus, D depends on the phase (solid, liquid or gas), temperature, and molecule size. For dilute solutes in water, D is generally of order $2 \cdot 10^{-9}$ m^2/s; whereas, for dispersed gases in air, D is of order $2 \cdot 10^{-5}$ m^2/s, a difference of 10^4.

Table 1.1 gives a detailed accounting of D for a range of solutes in water with low salinity (0.5 ppt). We see from the table that for a given temperature, D can range over about $\pm 10^1$ in response to molecular size (large molecules have smaller D). The table also shows the sensitivity of D to temperature; for a 10°C change in water temperature, D

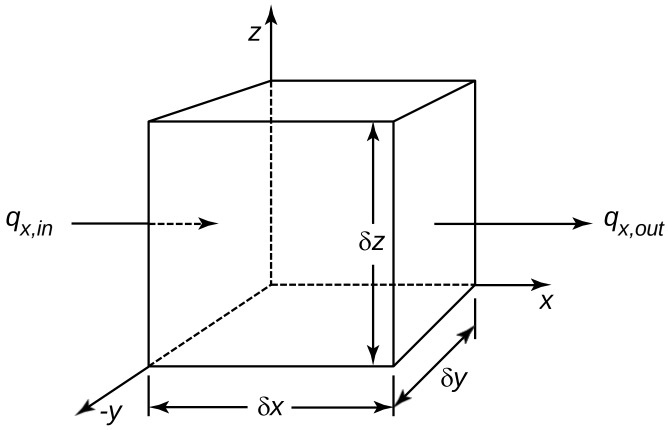


Fig. 1.2. Differential control volume for derivation of the diffusion equation.

can change by a factor of ±2. These observations can be summarized by the insight that faster and less confined motions result in higher diffusion coefficients.

Diffusion equation

Although Fick's law gives us an expression for the flux of mass due to the process of diffusion, we still require an equation that predicts the change in concentration of the diffusing mass over time at a point. In this section we will see that such an equation can be derived using the law of conservation of mass.

To derive the diffusion equation, consider the control volume (CV) depicted in Figure 1.2. The change in mass M of dissolved tracer in this CV over time is given by the mass conservation law

$$\frac{\partial M}{\partial t} = \sum \dot{m}_{in} - \sum \dot{m}_{out}. \quad (1.17)$$

To compute the diffusive mass fluxes in and out of the CV, we use Fick's law, which for the x-direction gives

$$q_{x,in} = -D \left.\frac{\partial C}{\partial x}\right|_1 \quad (1.18)$$

$$q_{x,out} = -D \left.\frac{\partial C}{\partial x}\right|_2 \quad (1.19)$$

where the locations 1 and 2 are the inflow and outflow faces in the figure. To obtain total mass flux $\dot{m}$ we multiply q_x by the CV surface area $A = \delta y \delta z$. Thus, we can write the net flux in the x-direction as

$$\delta \dot{m}|_x = -D\delta y \delta z \left(\left.\frac{\partial C}{\partial x}\right|_1 - \left.\frac{\partial C}{\partial x}\right|_2 \right) \quad (1.20)$$

which is the x-direction contribution to the right-hand-side of (1.17).

To continue we must find a method to evaluate $\partial C/\partial x$ at point 2. For this, we use linear Taylor series expansion, an important tool for linearly approximating functions. The general form of Taylor series expansion is

$$f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} \delta x + \text{HOTs}, \tag{1.21}$$

where HOTs stands for "higher order terms." Substituting $\partial C/\partial x$ for $f(x)$ in the Taylor series expansion yields

$$\left.\frac{\partial C}{\partial x}\right|_2 = \left.\frac{\partial C}{\partial x}\right|_1 + \left.\frac{\partial}{\partial x}\left(\frac{\partial C}{\partial x}\right)\right|_1 \delta x + HOTs. \tag{1.22}$$

For linear Taylor series expansion, we ignore the HOTs. Substituting this expression into the net flux equation (1.20) and dropping the subscript 1, gives

$$\delta \dot{m}|_x = D\delta y\delta z \frac{\partial^2 C}{\partial x^2}\delta x. \tag{1.23}$$

Similarly, in the y- and z-directions, the net fluxes through the control volume are

$$\delta \dot{m}|_y = D\delta x\delta z \frac{\partial^2 C}{\partial y^2}\delta y \tag{1.24}$$

$$\delta \dot{m}|_z = D\delta x\delta y \frac{\partial^2 C}{\partial z^2}\delta z. \tag{1.25}$$

Before substituting these results into (1.17), we also convert M to concentration by recognizing $M = C\delta x\delta y\delta z$. After substitution of the concentration C and net fluxes $\delta \dot{m}$ into (1.17), we obtain the three-dimensional diffusion equation (in various types of notation)

$$\begin{aligned}\frac{\partial C}{\partial t} &= D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right)\\ &= D\nabla^2 C\\ &= D\frac{\partial^2 C}{\partial x_i^2},\end{aligned} \tag{1.26}$$

which is a fundamental equation in environmental fluid mechanics. For the last line in (1.26), we have used the Einsteinian notation of repeated indices as a short-hand for the ∇^2 operator.

One-dimensional diffusion equation

In the one-dimensional case, concentration gradients in the y- and z-direction are zero, and we have the one-dimensional diffusion equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}. \tag{1.27}$$

We pause here to consider (1.27) and to point out a few key observations. First, (1.27) is first-order in time. Thus, we must supply and impose one initial condition for its solution, and its solutions will be unsteady, or transient, meaning they will vary with time. To

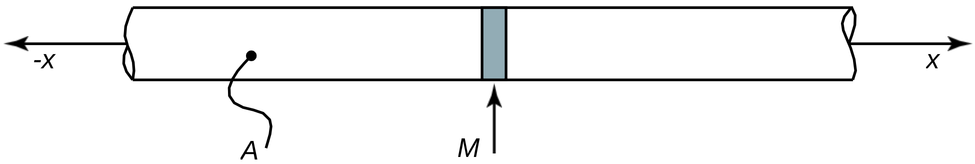


Fig. 1.3. Definitions sketch for one-dimensional pure diffusion in an infinite pipe.

solve for the steady, invariant solution of (1.27), we must set $\partial C/\partial t = 0$ and we no longer require an initial condition; the steady form of (1.27) is the well-known Laplace equation. Second, (1.27) is second-order in space. Thus, we can impose two boundary conditions, and its solution will vary in space. Third, the form of (1.27) is exactly the same as the heat equation, where D is replaced by the heat transfer coefficient κ. This observation agrees well with our intuition since we know that heat conducts (diffuses) away from hot sources toward cold regions (just as concentration diffuses from high concentration toward low concentration). This observation is also useful since many solutions to the heat equation are already known.

Similarity solution to the one-dimensional diffusion equation

Because (1.26) is of such fundamental importance in environmental fluid mechanics, we demonstrate here one of its solutions for the one-dimensional case in detail. There are multiple methods that can be used to solve (1.26), but we will follow the methodology of Fischer et al. (1979) and choose the so-called similarity method in order to demonstrate the usefulness of dimensional analysis as presented in Section 1.1.2.

Consider the one-dimensional problem of a narrow, infinite pipe (radius a) as depicted in Figure 1.3. A mass of tracer M is injected uniformly across the cross-section of area $A = \pi a^2$ at the point $x = 0$ at time $t = 0$. The initial width of the tracer is infinitesimally small. We seek a solution for the spread of tracer in time due to molecular diffusion alone.

As this is a one-dimensional ($\partial C/\partial y = 0$ and $\partial C/\partial z = 0$) unsteady diffusion problem, (1.27) is the governing equation, and we require two boundary conditions and an initial condition. As boundary conditions, we impose that the concentration at $\pm\infty$ remain zero

$$C(\pm\infty,\ t) = 0. \tag{1.28}$$

The initial condition is that the dye tracer is injected uniformly across the cross-section over an infinitesimally small width in the x-direction. To specify such an initial condition, we use the Dirac delta function

$$C(x, 0) = (M/A)\delta(x) \tag{1.29}$$

where $\delta(x)$ is zero everywhere accept at $x = 0$, where it is infinite, but the integral of the delta function from $-\infty$ to ∞ is 1. Thus, the total injected mass is given by

Table 1.2. Dimensional variables for one-dimensional pipe diffusion.

	Variable	Dimensions
dependent variable	C	M/L^3
independent variables	M/A	M/L^2
	D	L^2/T
	x	L
	t	T

$$M = \int_V C(x,t)\,dV \tag{1.30}$$

$$= \int_{-\infty}^{\infty}\int_0^a (M/A)\,\delta(x)\,2\pi r\,dr\,dx. \tag{1.31}$$

To use dimensional analysis, we must consider all the parameters that control the solution. Table 1.2 summarizes the dependent and independent variables for our problem. There are $m = 5$ parameters and $n = 3$ dimensions; thus, we can form two dimensionless groups

$$\pi_1 = \frac{C}{M/(A\sqrt{Dt})} \tag{1.32}$$

$$\pi_2 = \frac{x}{\sqrt{Dt}} \tag{1.33}$$

From dimensional analysis we have that $\pi_1 = f(\pi_2)$, which implies for the solution of C

$$C = \frac{M}{A\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right) \tag{1.34}$$

where f is a yet-unknown function with argument π_2. (1.34) is called a similarity solution because C has the same shape in x at all times t (see also Example Box 1.3). Now we need to find f in order to know what that shape is. Before we find the solution formally, compare (1.34) with the actual solution given by (1.53). Through this comparison, we see that dimensional analysis can go a long way toward finding solutions to physical problems.

The function f can be found in two primary ways. First, experiments can be conducted and then a smooth curve can be fit to the data using the coordinates π_1 and π_2. Second, (1.34) can be used as the solution to a differential equation and f solved for analytically. This is what we will do here. The power of a similarity solution is that it turns a partial differential equation (PDE) into an ordinary differential equation (ODE), which is the goal of any solution method for PDEs.

The similarity solution (1.34) is really just a coordinate transformation. We will call our new similarity variable $\eta = x/\sqrt{Dt}$. To substitute (1.34) into the diffusion equation, we will need the two derivatives

$$\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t} \tag{1.35}$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{Dt}}. \qquad (1.36)$$

We first use the chain rule to compute $\partial C/\partial t$ as follows

$$\begin{aligned}
\frac{\partial C}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{M}{A\sqrt{Dt}} f(\eta)\right) \\
&= \frac{\partial}{\partial t}\left(\frac{M}{A\sqrt{Dt}}\right) f(\eta) + \frac{M}{A\sqrt{Dt}} \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial t} \\
&= \frac{M}{A\sqrt{Dt}}\left(-\frac{1}{2t}\right) f(\eta) + \frac{M}{A\sqrt{Dt}}\frac{\partial f}{\partial \eta}\left(-\frac{\eta}{2t}\right) \\
&= -\frac{M}{2At\sqrt{Dt}}\left(f + \eta\frac{\partial f}{\partial \eta}\right).
\end{aligned} \qquad (1.37)$$

Similarly, we use the chain rule to compute $\partial^2 C/\partial x^2$ as follows

$$\begin{aligned}
\frac{\partial^2 C}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{M}{A\sqrt{Dt}} f(\eta)\right)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{M}{A\sqrt{Dt}}\frac{\partial \eta}{\partial x}\frac{\partial f}{\partial \eta}\right) \\
&= \frac{M}{ADt\sqrt{Dt}}\frac{\partial^2 f}{\partial \eta^2}.
\end{aligned} \qquad (1.38)$$

Upon substituting these two results into the diffusion equation, we obtain the ordinary differential equation in η

$$\frac{d^2 f}{d\eta^2} + \frac{1}{2}\left(f + \eta\frac{df}{d\eta}\right) = 0. \qquad (1.39)$$

To solve (1.39), we should also convert the boundary and initial conditions to two new constraints on f. As we will see shortly, both boundary conditions and the initial condition can be satisfied through a single condition on f. The other constraint (remember that second order equations require two constrains) is taken from the conservation of mass, given by (1.30). Substituting $dx = d\eta\sqrt{Dt}$ into (1.30) and simplifying, we obtain

$$\int_{-\infty}^{\infty} f(\eta)\, d\eta = 1. \qquad (1.40)$$

Solving (1.39) requires a couple of integrations. First, we rearrange the equation using the identity

$$\frac{d(f\eta)}{d\eta} = f + \eta\frac{df}{d\eta}, \qquad (1.41)$$

which gives us

$$\frac{d}{d\eta}\left(\frac{df}{d\eta} + \frac{1}{2}f\eta\right) = 0. \qquad (1.42)$$

Integrating once leaves us with

$$df \quad 1$$

$$d\eta + {}_2 f\eta = C_0. \qquad (1.43)$$

1. Concepts, Definitions, and the Diffusion Equation

It can be shown that choosing $C_0 = 0$ satisfies both boundary conditions and the initial condition (see Appendix A for more details).

With $C_0 = 0$ we have a homogeneous ordinary differential equation whose solution can readily be found. Moving the second term to the right hand side we have

$$\frac{df}{d\eta} = -\frac{1}{2} f\eta. \tag{1.44}$$

The solution is found by collecting the f - and η-terms on separate sides of the equation

$$\frac{df}{f} = -\frac{1}{2}\eta d\eta. \tag{1.45}$$

Integrating both sides gives

$$\ln(f) = -\frac{1}{2}\frac{\eta^2}{2} + C_1 \tag{1.46}$$

which after taking the exponential of both sides gives

$$f = C_1 \exp\left(-\frac{\eta^2}{4}\right). \tag{1.47}$$

To find C_1 we must use the remaining constraint given in (1.40)

$$\int_{-\infty}^{\infty} C_1 \exp\left(-\frac{\eta^2}{4}\right) d\eta = 1. \tag{1.48}$$

To solve this integral, we should use integral tables; therefore, we have to make one more change of variables to remove the 1/4 from the exponential. Thus, we introduce ζ such that

$$\zeta^2 = \frac{1}{4}\eta^2 \tag{1.49}$$

$$2d\zeta = d\eta. \tag{1.50}$$

Substituting this coordinate transformation and solving for C_1 leaves

$$C_1 = \frac{1}{2\int_{-\infty}^{\infty} \exp(-\zeta^2) d\zeta}. \tag{1.51}$$

After looking up the integral in a table, we obtain $C_1 = 1/(2\sqrt{\pi})$. Thus,

$$f(\eta) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{\eta^2}{4}\right). \tag{1.52}$$

Replacing f in our similarity solution (1.34) gives

$$C(x,t) = \frac{M}{A\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) \tag{1.53}$$

which is a classic result in environmental fluid mechanics, and an equation that will be used thoroughly throughout this text. Generalizing to three dimensions, Fischer et al. (1979) give the the solution

$$C(x, y, z, t) = \frac{M}{4\pi t\sqrt{4\pi D_x D_y D_z t}} \exp\left(-\frac{x^2}{4D_x t} - \frac{y^2}{4D_y t} - \frac{z^2}{4D_z t}\right) \tag{1.54}$$

(1.54)

which they derive using the separation of variables method.

Example Box 1.2:
Maximum concentrations.

For the three-dimensional instantaneous point-source solution given in (1.54), find an expression for the maximum concentration. Where is the maximum concentration located?

The classical approach for finding maximaof functions is to look for zero-points in the derivative of the function. For many concentration distributions, it is easier to take a qualitative look at the functional form of the equation. The instantaneous point-source solution has the form

$$C(\mathbf{x}, t) = C_1(t)\exp(-|f(\mathbf{x}, t)|).$$

$C_1(t)$ is an amplification factor independent of space. The exponential function has a negative argument, which means it is maximum when the argument is zero. Hence, the maximum concentration is

$$C_{max}(t) = C_1(t).$$

Applying this result to (1.54) gives

$$C_{max}(t) = \frac{M}{4\pi t\sqrt{4\pi D_x D_y D_z t}}.$$

The maximum concentration occurs at the point where the exponential is zero. In this case $\mathbf{x}(C_{max}) = (0, 0, 0)$.

We can apply this same analysis to other concentration distributions as well. For example, consider the error function concentration distribution

$$C(x, t) = \frac{C_0}{2}\left(1 - \mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right).$$

The error function ranges over $[-1, 1]$ as its argument ranges from $[-\infty, \infty]$. The maximum concentration occurs when $\mathrm{erf}(\cdot) = -1$, and gives,

$$C_{max}(t) = C_0.$$

C_{max} occurs when the argument of the error function is $-\infty$. At $t = 0$, the maximum concentration occurs for all points $x < 0$, and for $t > 0$, the maximum concentration occurs only at $x = -\infty$.

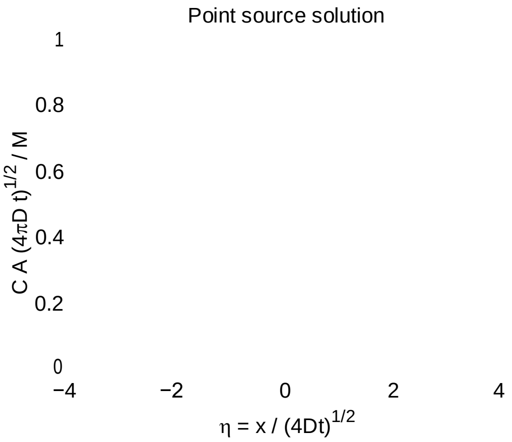


Fig. 1.4. Self-similarity solution for one-dimensional diffusion of an instantaneous point source in an infinite domain.

1.3.1 Interpretation of the similarity solution

Figure 1.4 shows the one-dimensional solution (1.53) in non-dimensional space. Comparing (1.53) with the Gaussian probability distribution reveals that (1.53) is the normal bell-shaped curve with a standard deviation σ, of width

$$\sigma^2 = 2Dt. \tag{1.55}$$

Theconceptofselfsimilarity is nowalso evident: theconcentration profile shape is always Gaussian. By plotting in non-dimensional space, the profiles also collapse into a single profile; thus, profiles for all times $t > 0$ are given by the result in the figure.

The Gaussian distribution can also be used to predict how much tracer is within a certain region. Looking at Figure 1.4 it appears that most of the tracer is between -2 and 2. Gaussian probability tables, available in any statistics book, can help make this observation more quantitative. Within $\pm\sigma$, 64.2% of the tracer is found and between $\pm 2\sigma$, 95.4% of the tracer is found. As an engineering rule-of-thumb, we will say that a diffusing tracer is distributed over a region of width 4σ, that is, $\pm 2\sigma$.

> **Example Box 1.3:**
> **Profile shape and self similarity.**
>
> For the one-dimensional, instantaneous point-source solution, show that the ratio C/C_{max} can be written as a function of the single parameter α defined such that $x = \alpha\sigma$. How might this be used to estimate the diffusion coefficient from concentration profile data?
>
> From the previous example, we know that $C_{max} = M/\sqrt{4\pi Dt}$, and we can re-write (1.53) as
>
> $$\frac{C(x,t)}{C_{max}(t)} = \exp\left(-\frac{x^2}{4Dt}\right).$$
>
> We now substitute $\sigma = \sqrt{2Dt}$ and $x = \alpha\sigma$ to obtain
>
> $$\frac{C}{C_{max}} = \exp\left(-\alpha^2/2\right).$$
>
> Here, α is a parameter that specifies the point to calculate C based on the number of standard deviations the point is away from the center of mass. This illustrates very clearly the notion of self similarity: regardless of the time t, the amount of mass M, or the value of D, the ratio C/C_{max} is always the same value at the same position αx.
>
> This relationship is very helpful for calculating diffusion coefficients. Often, we do not know the value of M. We can, however, always normalize a concentration profile measured at a given time t by $C_{max}(t)$. Then we pick a value of α, say 1.0. We know from the relationship above that $C/C_{max} = 0.61$ at $x = \sigma$. Next, find the locations where $C/C_{max} = 0.61$ in the experimental profile and use them to measure σ. We then use the relationship $\sigma = \sqrt{2Dt}$ and the value of t to estimate D.

Application: Diffusion in a lake

With a solid background now in diffusion, consider the following example adapted from Nepf (1995).

As shown in Figures 1.5 and 1.6, a small alpine lake is mildly stratified, with a thermocline (region of steepest density gradient) at 3 m depth, and is contaminated by arsenic. Determine the magnitude and direction of the diffusive flux of arsenic through the thermocline (cross-sectional area at the thermocline is $A = 2 \cdot 10^4$ m^2) and discuss the nature of the arsenic source. The molecular diffusion coefficient is $D_m = 1 \cdot 10^{-10}$ m^2/s.

Molecular diffusion. To compute the molecular diffusive flux through the thermocline, we use the one-dimensional version of Fick's law, given above in (1.14)

$$q_z = -D_m \frac{\partial C}{\partial z}. \quad (1.56)$$

We calculate the concentration gradient at $z = 3$ from the concentration profile using a finite difference approximation. Substituting the appropriate values, we have

$$q_z = -D_m \frac{\partial C}{\partial z}$$

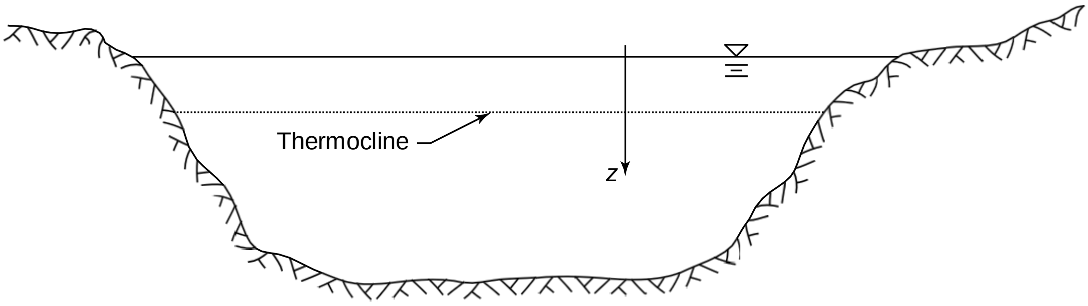


Fig. 1.5. Schematic of a stratified alpine lake.

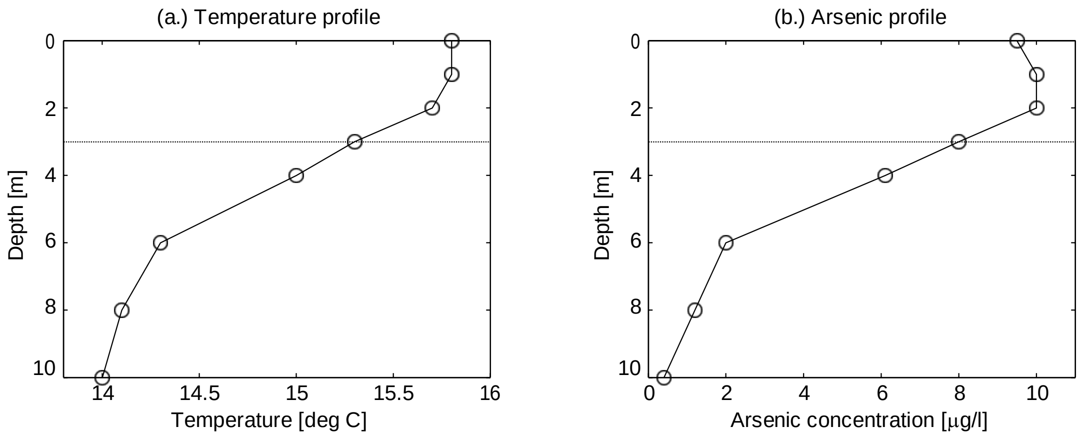


Fig. 1.6. Profiles of temperature and arsenic concentration in an alpine lake. The dotted line at 3 m indicates the location of the thermocline (region of highest density gradient).

$$= -(1 \cdot 10^{-10}) \frac{(10 - 6.1)}{(2 -} \cdot \frac{1000\,\mathrm{l}}{}$$

$$= +1.95 \cdot 10^{-7}\,\mu\mathrm{g}/(\mathrm{m}^2 \cdot \mathrm{s}) \tag{1.57}$$

where the plus sign indicates that the flux is downward. The total mass flux is obtained by multiplying over the area: $\dot{m} = Aq_z = 0.0039\ \mu\mathrm{g/s}$.

Turbulent diffusion. As wepointed out in the discussion on diffusion coefficients, faster random motions lead to larger diffusion coefficients. As we will see in Chapter 3, turbulence also causes a kind of random motion that behaves asymptotically like Fickian diffusion. Because theturbulent motions aremuchlargerthan molecularmotions, turbulentdiffusion coefficients are much larger than molecular diffusion coefficients.

Sources of turbulence at the thermocline of a small lake can include surface inflows, wind stirring, boundary mixing, convection currents, and others. Based on studies in

this lake, a turbulent diffusion coefficient can be taken as $D_t = 1.5 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$. Since turbulent diffusion obeys the same Fickian flux law, then the turbulent diffusive flux $q_{z,t}$ can be related to the molecular diffusive flux $q_{z,t} = q_z$ by the equation

$$q_{z,t} = q_{z,m} \frac{D_t}{D_m} \tag{1.58}$$

$$= +2.93 \cdot 10^{-3}\ \mu g/(m^2 \cdot s). \quad (1.59)$$

Hence, weseethatturbulent diffusive transport is muchgreater than moleculardiffusion. As a warning, however, if the concentration gradients are very high and the turbulence is low, molecular diffusion can become surprisingly significant!

Implications. Here,wehaveshown thattheconcentration gradientresults in anetdiffusive flux of arsenic intothe hypolimnion (region below the thermocline). Assuming no other transport processes are at work, wecan conclude that the arsenic source is at the surface. If the diffusive transport continues, the hypolimnion concentrations will increase. The next chapterconsiders howthe situation mightchangeif weinclude another typeof transport: advection.

Summary

This chapterintroduced thesubject ofenvironmental fluid mechanics and focused on the important transport process of diffusion. Fick's law was derived to represent the mass flux (transport) due to diffusion, and Fick's lawwas used to derive the diffusion equation, which isusedtopredictthetime-evolution ofaconcentration field in spaceduetodiffusive transport. A similarity method wasused through the aid of dimensional analysis to find a one-dimensional solution to thediffusion equation for an instantaneous pointsource. As illustrated through an example, diffusive transport results when concentration gradients existand plays an importantrole in predictingthe concentrations ofcontaminants as they move through the environment.

Exercises

Definitions. Write a short, qualitative definition of the following terms:

Concentration.
Mass fraction.
Density.
Diffusion.
Brownian motion.
Instantaneouspointsource.
Similarity method.
Partial differential equation.
Standard deviation.
Chemical fate.
Chemical transport.
Transport equation.
Fick's law.

Concentrations in water. A student adds 1.00 mg of pure Rhodamine WT (a common fluorescent tracer used in field experiments) to 1.000 l of water at 20°C. Assuming the solution is dilute so that we can neglect the equation of state of the solution, compute the concentration of the Rhodamine WT mixture in the units of mg/l, mg/kg, ppm, and ppb.

Concentration in air. Air consists of 21% oxygen. For air with a density of 1.4 kg/m^3, compute the concentration of oxygen in the units of mg/l, mg/kg, mol/l, and ppm.

Instantaneous point source. Consider the pipe section depicted in Figure 1.3. A student injects 5 ml of 20% Rhodamine-WT solution (specific gravity 1.15) instantaneously and uniformly over the pipe cross-section (A = 0.8 cm^3) at the point x = 0 and the time t = 0. The pipe is filled with stagnant water. Assume the molecular diffusion coefficient is $D_m = 0.13 \cdot 10^{-4}\ cm^2/s$.

- What is the concentration at x = 0 at the time t = 0?
- What is the standard deviation of the concentration distribution 1 s after injection?
- Plot the maximum concentration in the pipe, $C_{max}(t)$, as a function of time over the interval $t = [0, 24\,h]$.
- How long does it take until the concentration over the region $x = \pm 1$ m can be treated as uniform? Define a uniform concentration distribution as one where the minimum concentration within a region is no less than 95% of the maximum concentration within that same region.

Advection versus diffusion. Rivers can often be approximated as advection dominated (downstream transport due to currents is much faster than diffusive transport) or diffusion dominated (diffusive transport is much faster than downstream transport due to currents). This property is described by a non-dimensional parameter (called the Peclet number) $Pe = f(u, D, x)$, where u is the stream velocity, D is the diffusion coefficient, and x is the distance downstream to the point of interest. Using dimensional analysis, find the form of Pe such that Pe 1 is advection dominated and Pe 1 is diffusion dominated. For a stream with u = 0.3 m/s and D = 0.05 m^2/s, where are diffusion and advection equally important?

Maximum concentrations. Referring to Figure 1.4, we note that the maximum concentration in space is always found at the center of the distribution (x = 0). For a point at $x = r$, however, the maximum concentration over time occurs at one specific time t_{max}. Using (1.53) find an equation for the time t_{max} at which the maximum concentration occurs at the point $x = r$.

Diffusion in a river. The Rhein river can be approximated as having a uniform depth (h = 5 m), width (B = 300 m) and mean flow velocity (u = 0.7 m/s). Under these conditions, 100 kg of tracer is injected as a point source (the injection is evenly distributed transversely over the cross-section). The cloud is expected to diffuse laterally as a one-dimensional point source in a moving coordinate system, moving at the mean stream velocity. The river has an enhanced mixing coefficient of D = 10 m^2/s. How long does it take the cloud to reach a point x = 15000 m downstream? What is the maximum concentration that passes the point x? How wide is the cloud (take the cloud width as 4σ) when it passes this point?

Table 1.3. Measured concentration and time for a point source diffusing in three-dimensions for problem num- ber 18.

Time (days)	Concentration (μg/cm^3 ±0.03)
0.5	0.02
1.0	0.50
1.5	2.08
2.0	3.66
2.5	4.81
3.0	5.50
3.5	5.80
4.0	5.91
4.5	5.81
5.0	5.70
5.5	5.54
6.0	5.28
6.5	5.05
7.0	4.87
7.5	4.65
8.0	4.40
8.5	4.24
9.0	4.00
9.5	3.84
10.0	3.66

Measuring diffusion coefficients 1. A chemist is trying to calculate the diffusion coefficient for a new chemical. In his experiments, he measured the concentration as a function of time at a point 5 cm away from a virtual point source diffusing in three dimensions. Select a set of coordinates such that, when plotting the data in Table 1.3, D is the slope of a best-fit line through the data. Based on this coordinate transformation, what is more important to measure precisely, concentration or time? What recommendation would you give to this scientist to improve the accuracy of his estimate for the diffusion coefficient?
Measuring diffusion coefficients 2.[1] As part of a water quality study, you have been asked to assess the diffusion of a new fluorescent dye. To accomplish this, you do a dye study in a laboratory tank (depth h = 40 cm). You release the dye at a depth of 20 cm (spread evenly over the area of the tank) and monitor its development over time. Vertical profiles of dye concentration in the tank are shown in Figure 1.7; the x-axis represents the reading on your fluorometer and the y-axis represents the depth.

- Estimate the molecular diffusion coefficient of the dye, D_m, based on the evolution of the dyecloud.

[1] This problem is adapted from Nepf (1995).

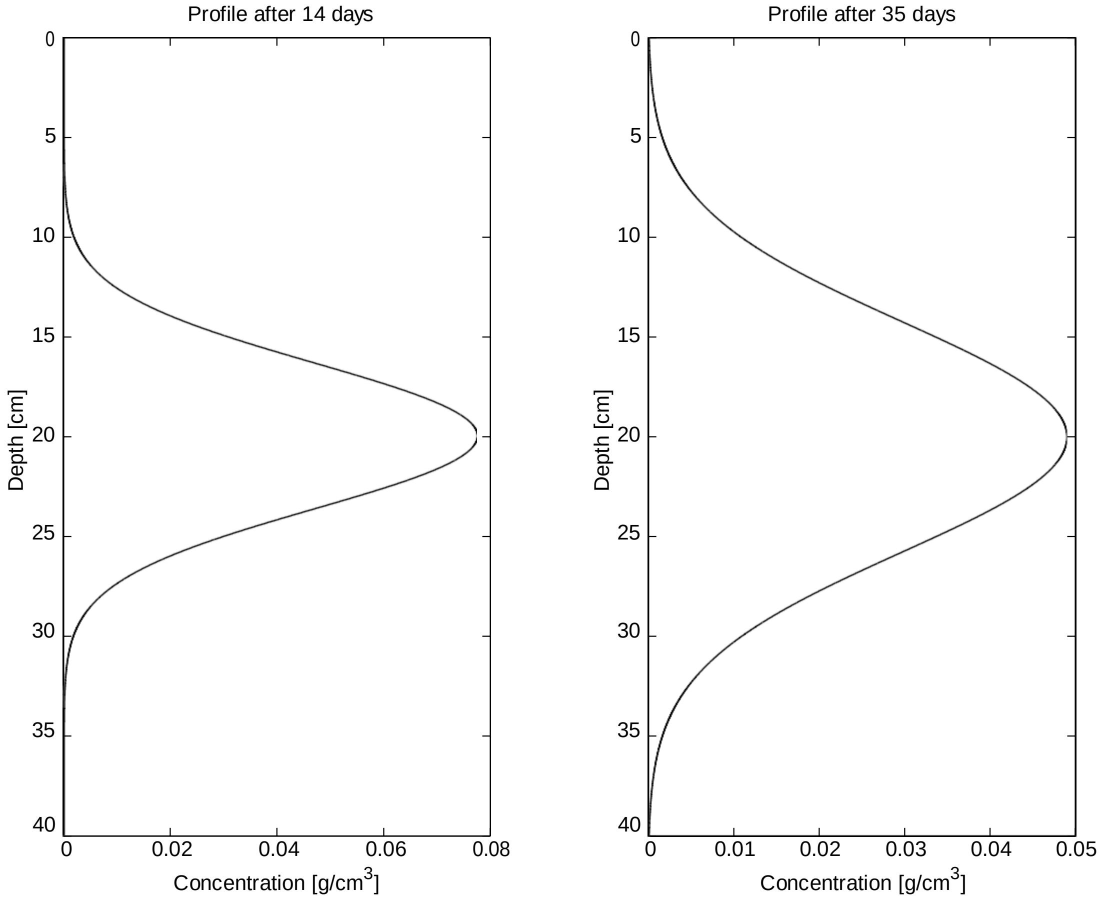


Fig. 1.7. Concentration profiles of fluorescent dye for two different measurement times. Refer to problem number 1.9.

- Predict at what time the vertical distribution of the dye will be affected by the boundaries of the tank.

Radiative heaters. A student heats his apartment (surface area $A_r = 32\ m^2$ and ceiling height $h = 3$ m) with a radiative heater. The heater has a total surface area of $A_h = 0.8\ m^2$; the thickness of the heater wall separating the heater fluid from the outside air is $\delta x = 3$ mm (refer to Figure 1.8). The conduction of heat through the heater wall is given by the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{1.60}$$

where T is the temperature in °C and $\kappa = 1.1 \cdot 10^{-2}$ kcal/(s°Cm) is the thermal conductivity of the metal for the heater wall. The heat flux q through the heater wall is given by

$$q = -\kappa \nabla T. \tag{1.61}$$

Recall that 1 kcal = 4184 J and 1 Watt = 1 J/s.

- The conduction of heat normal to the heater wall can be treated as one-dimensional. Write (1.60) and (1.61) for the steady-state, one-dimensional case.

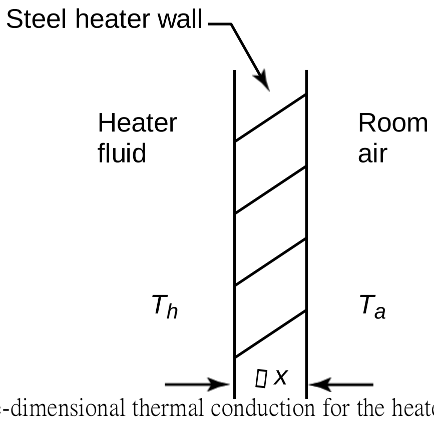


Fig. 1.8. Definitions sketch for one-dimensional thermal conduction for the heater wall in problem number 1.10.

- Solve (1.60) for the steady-state, one-dimensional temperature profile through the heater wall with boundary conditions $T(0) = T_h$ and $T(\delta x) = T_r$ (refer to Figure 1.8).
- The water in the heater and the air in the room move past the heater wall such that $T_h = 85^\circ$C and $T_r = 35^\circ$C. Compute the heat flux from (1.61) using the steady-state, one-dimensional solution just obtained.
- How many 300 Watt lamps are required to equal the heat output of the heater assuming 100% efficiency?
- Assume the specific heat capacity of the air is $c_v = 0.172$ kcal/(kg·K) and the density is $\rho_a = 1.4$ kg/m^3. How much heat is required to raise the temperature of the apartment by 5°C?
- Given the heat output of the heater and the heat needed to heat the room, how might you explain that the student is able to keep the heater turned on all the time?

POSSIBLE QUESTIONS

UNIT V

PART B

(5X6=30 Marks)

1. Describe a method of boundary value problems for the generalised diffusion equation.
2. State and prove Duhamel's Theorem.
3. Describe a method of boundary value problems for the generalised diffusion equation.
4. State and prove Duhamel's Theorem.
5. Use Green's function to find the solution of the boundary value problem ofdiffusion equation.

PART – C

(1X10=10)

1)Find the solution of the Helmholtz equation by using the method separation of variables.

2. Discuss in detail about the diffusion equation with sources

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answers
The generalisation of the typical parabolic equation is ___________equation	wave	laplace	fourier	diffusion	diffusion
The generalisation of the typical __________ equation is diffusion equation	hyperbolic	parabolic	elliptic	cubic	parabolic
The conduction of heat in solids the temperature is denoted as_________	alpha	beta	theta	gamma	theta
The flow of heat through a _____ element of volume shows the variation of theta	large	small	unique	linear	small
The conduction of heat in solids the thermal conductivity is denoted as_________	k	h	a	b	h
The conduction of heat in solids the density is denoted as_________	row	h	a	b	row
The conduction of heat in solids the specific heat of the solid is denoted as_________	k	h	c	b	c
The conduction of heat in solids the temperature of every point is denoted as_________	r	h	c	b	r
Diffusion in isotropic substances the current vector is denoted as________	r	h	c	J	J
Solvation of Diffusion in isotropic substances ________ concept is used	grad	integration	addition	subtraction	grad
_________ concept is used to solve the diffusion in isotropic substances.	div	integration	addition	subtraction	div
_________ law is used to solve the diffusion in isotropic substances.	Finks	Kirchoffs	Ficks	Newton	Finks
________ equation is used in conducting media	Finks	Maxwell	Ficks	Newton	Maxwell
____ concept is used to solve in conducting media	curl	div	grad	addition	curl
The method of separation of variablesapplied to diffusion equation is similar to________theory	potential	grad	calculus	electrostatic	potential
The method of separation of variablesapplied to diffusion equation is similar to________motion	wave	laplace	fourier	kennel	wave
_______ equation is used to solve in separation of variables	Finks	Kirchoffs	Ficks	helmholtz	helmholtz
The first region bounded in the use of integral transform is ________	S1	A1	B1	C1	S1
The second region bounded in the use of integral transform is ________	S1	S2	B1	C1	S2
In the use of Integral transforms _____ number of regions are bounded	one	two	three	four	two