KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: M. Santhi
Subject Name: Graph Theory **Sub.Code: 19MMP205A**
Semester: II **Class: I M.Sc Mathematics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1.	1	Isomorphism of graphs and sub graphs	R1:Chap:2.1:Pg.No:14- 16
2.	1	Walks, Paths, Circuits	R4:Chap:1.3:Pg.No:6-9
3.	1	Connected , connectedness of graphs and components of graphs	R1:Chap:2.5:Pg.No:19- 21
4.	1	Euler graphs and Euler graphs based on theorems	R1:Chap:2.6:Pg.No:21- 23
5.	1	Hamiltonian paths and circuits	R3:Chap:4.5:Pg.No:314- 316
6.	1	Theorems on some properties of trees and Distance and centers in tree	R6:Chap:3:Pg.No:39-41
7.	1	Rooted and binary trees and spanning trees, Fundamentals Circuits	R1:Chap:3.5:Pg.No:45- 57
8.	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit I=8			
		UNIT-II	
1.	1	Spanning trees in a Weights Grap	R8:Chap:3.10:Pg.No:58- 61
2.	1	Theorems on some properties of Cut Sets and all Cut Sets	R1:Chap:4.2:Pg.No:68- 71

3.	1	Fundamental Circuits and Cut Sets	R1:Chap:4.5:Pg.No:73- 75
4.	1	Connectivity and separability	R1:Chap:4.5:Pg.No:73- 75
5.	1	Network flows	R3:Chap:11:Pg.No:1377- 1380
6.	1	Theorems on some 1- Isomorphism	R1:Chap:4.7:Pg.No:80- 82
7.	1	Theorems on some 2- Isomorphism	R1:Chap:4.5:Pg.No:73- 75
8.	1	Combinational versus Geometric Graphs	R1:Chap:5.1:Pg.No:88- 89
9.	1	Different Representation of a Planar Graph	R1:Chap:5.4:Pg.No:90-99
10.	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit II=10			
UNIT-III			
1.	1	Introduction and definition of a incidence matrix	R1:Chap:7.1:Pg.No:137- 139
2.	1	Sub matrix and Circuits matrix based on problems	R1:Chap:7.3:Pg.No:142- 146
3.	1	Path matrix and adjacency matrix based on problems	R1:Chap:7.8:Pg.No:156- 161
4.	1	Chromatic Number theorems	R5:Chap:1.12:Pg.No:257 - 258
5.	1	Chromatic partitioning	R5:Chap:16.14:Pg.No:25 8- 259
6.	1	Chromatic polynomial, Matching, covering	R1:Chap:8.5:Pg.No:174- 190
7.	1	Four color problem	R5:Chap:2.1:Pg.No:31- 35

8.	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit III=8			
UNIT-IV			
1.	1	Introduction and definition of Directed Graphs	R9:Chap:3.1:Pg.No:163-165
2.	1	Some types of Directed Graphs	R1:Chap:9.2:Pg.No:197-198
3.	1	Types of enumeration	R1:Chap:10.1:Pg.No:238 - 240
4.	1	Counting labeled trees	R1:Chap:10.2:Pg.No:240 - 241
5.	1	Counting unlabeled trees	R1:Chap:10.3:Pg.No:241 - 250
6.	1	Polya's counting theorem	R1:Chap:10.4:Pg.No:250 - 260
7.	1	Graph enumeration with Polya's theorem	R1:Chap:10.5:Pg.No:260 - 264
8.	1	Recapitulation and Discussion of possible questions	
Total No of Hours Planned For Unit IV=8			
UNIT-V			
1.	1	Introduction Terminology and concepts	R1:Chap:1.1:Pg.No:15- 16
2.	1	Applications of Domination in graphs	R7:Chap:5.1:Pg.No:71-73
3.	1	Dominating set and Domination number	R2:Chap:1.2:Pg.No:16- 18
4.	1	Independent set and Independent number	R2:Chap:1.3:Pg.No:19- 20
5.	1	History of domination in graphs	R2:Chap:1.13:Pg.No:36- 37
6.	1	Recapitulation and Discussion of possible questions	
7.	1	Discuss on Previous ESE Question Papers	
8.	1	Discuss on Previous ESE Question Papers	

9.	1	Discuss on Previous ESE Question Papers	
Total No of Hours Planned for unit V=9			

S1: Deo N, (2007). Graph Theory with Applications to Engineering and Computer Science, Prentice Hall of India Pvt Ltd, New Delhi..
S2: Teresa W. Haynes, Stephen T. Hedetniemi and Peter J.Slater, (1998), Fundamentals of Domination in Graphs, Marcel Dekker, New York.
S3: Flouds C. R., (2009). Graph Theory Applications, Narosa Publishing House. New Delhi,India.

Total Planned Hours	40

UNIT-I

SYLLABUS

Graphs – Introduction – Isomorphism – Sub graphs – Walks, Paths, Circuits – Connectedness – Components – Euler Graphs – Hamiltonian Paths and Circuits – Trees – Properties of trees – Distance and Centers in Tree – Rooted and Binary Trees - Spanning trees – Fundamental Circuits.

INTRODUCTION

A *linear*† *graph* (or simply a *graph*) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called *vertices*, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the *end vertices* of e_k. The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices. Often this diagram itself is referred to as the graph. The object shown in Fig. 1-1, for instance, is a graph.

Observe that this definition permits an edge to be associated with a vertex pair (v_i, v_i). Such an edge having the same vertex as both its end vertices is called a *self-loop* (or simply a *loop*. The word loop, however, has a different meaning in electrical network theory; we shall therefore use the term self-loop to avoid confusion). Edge e_1 in Fig. 1-1 is a self-loop. Also note that

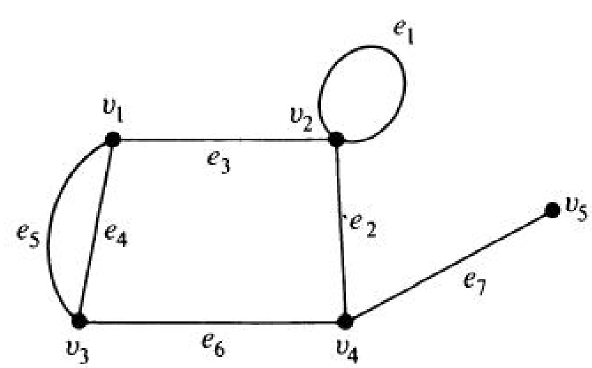


Fig. 1-1 Graph with five vertices and seven edges.

the definition allows more than one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in Fig. 1-1. Such edges are referred to as *parallel edges*.

A graph that has neither self-loops nor parallel edges is called a *simple graph*. In some graph-theory literature, a graph is defined to be only a simple graph, but in most engineering applications it is necessary that parallel edges and self-loops be allowed; this is why our definition includes graphs with self-loops and/or parallel edges. Some authors use the term *general graph* to emphasize that parallel edges and self-loops are allowed.

It should also be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short: what is important is the incidence between the edges and vertices. For example, the two graphs drawn incidence between the edges and vertices. For example, the two graphs drawn in Figs. 1-2(a) and (b) are the same, because incidence between edges and vertices is the same in both cases.

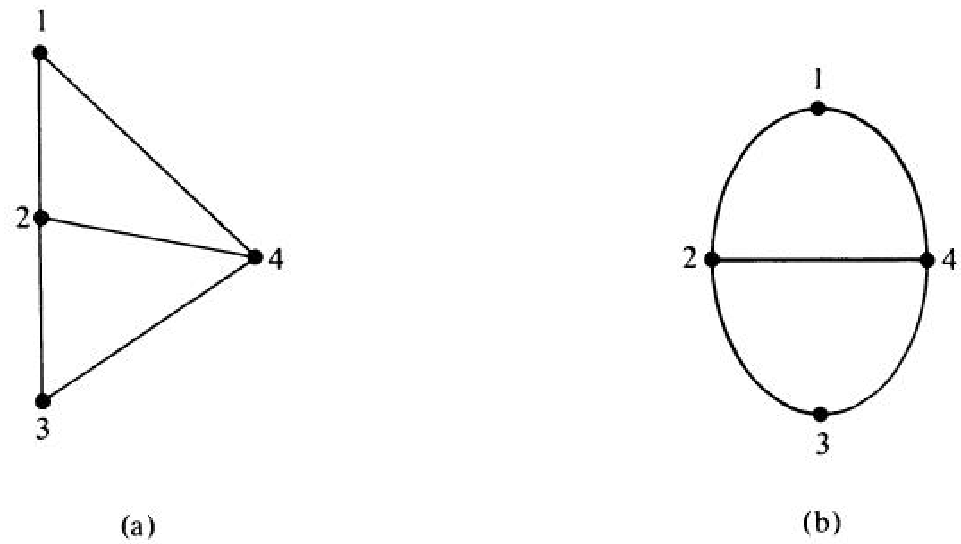


Fig. 1-2 Same graph drawn differently.

In a diagram of a graph, sometimes two edges may seem to intersect at a point that does not represent a vertex, for example, edges e and f in Fig. 1-3. Such edges should be thought of as being in different planes and thus having no common point. (Some authors break one of the two edges at such a crossing to emphasize this fact.)

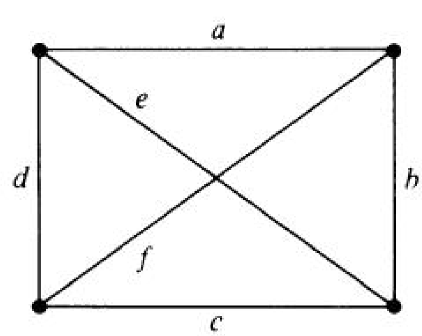
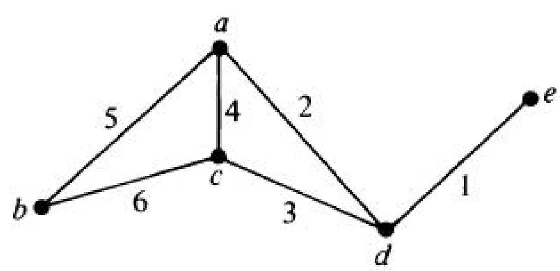


Fig. 1-3 Edges e and f have no common point.

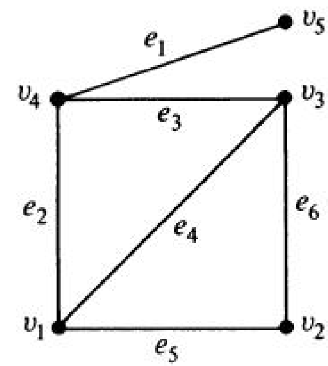
A graph is also called a *linear complex*, a *1-complex*, or a *one-dimensional complex*. A vertex is also referred to as a *node*, a *junction*, a *point*, *0-cell*, or an *0-simplex*. Other terms used for an edge are a *branch*, a *line*, an *element*, a *1-cell*, an *arc*, and a *1-simplex*. In this book we shall generally use the terms graph, vertex, and edge.

ISOMORPHISM

In geometry two figures are thought of as equivalent (and called congruent) if they have identical behavior in terms of geometric properties. Likewise, two graphs are thought of as equivalent (and called *isomorphic*) if they have identical behavior in terms of graph-theoretic properties. More precisely: Two graphs G and G' are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that edge e is incident on vertices v_1 and v_2 in G; then the corresponding edge e' in G' must be incident on the vertices v'_1 and v'_2 that correspond to



(a)



(b)

Fig. 2-1 Isomorphic graphs.

v_1 and v_2, respectively. For example, one can verify that the two graphs in Fig. 2-1 are isomorphic. The correspondence between the two graphs is as follows: The vertices a, b, c, d, and e correspond to v_1, v_2, v_3, v_4, and v_5, respectively. The edges 1, 2, 3, 4, 5, and 6 correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively.

Except for the labels (i.e., names) of their vertices and edges, isomorphic graphs are the same graph, perhaps drawn differently. As indicated in Chapter 1, a given graph can be drawn in many different ways. For example, Fig. 2-2 shows two different ways of drawing the same graph.

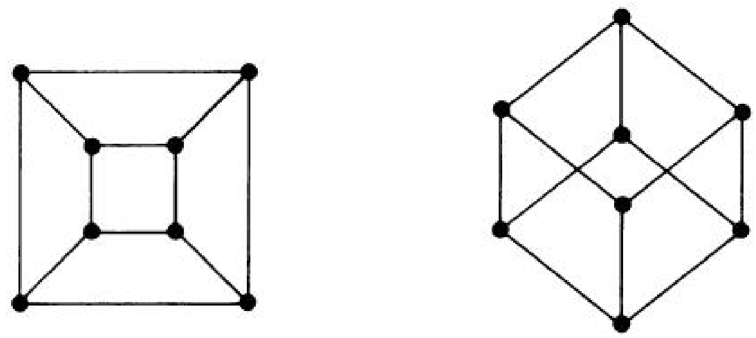
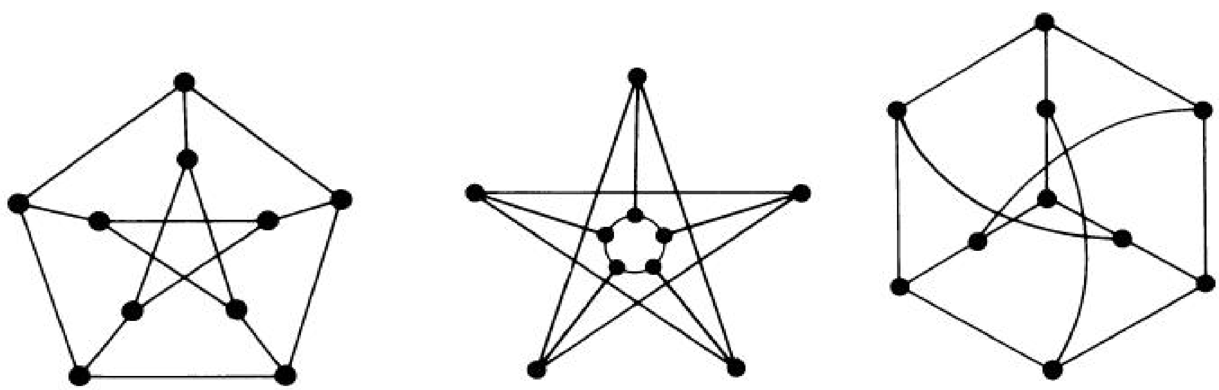


Fig. 2-2 Isomorphic graphs.

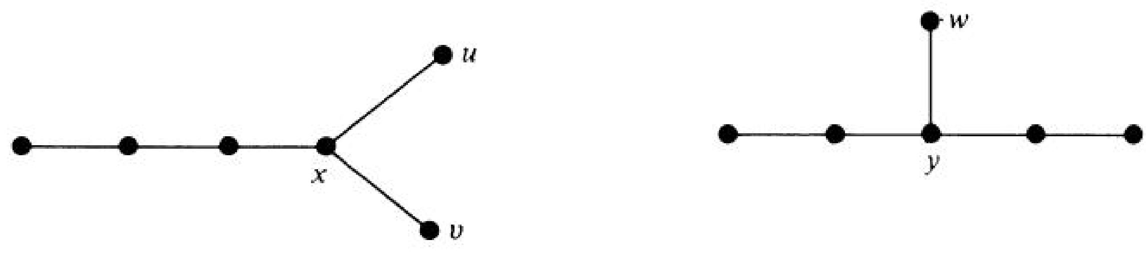
It is not always an easy task to determine whether or not two given graphs are isomorphic. For instance, the three graphs shown in Fig. 2-3 are all isomorphic, but just by looking at them you cannot tell that. It is left as an exercise for the reader to show, by redrawing and labeling the vertices and edges, that the three graphs in Fig. 2-3 are isomorphic (see Problem 2-3).

It is immediately apparent by the definition of isomorphism that two isomorphic graphs must have

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.



(a) (b) (c)



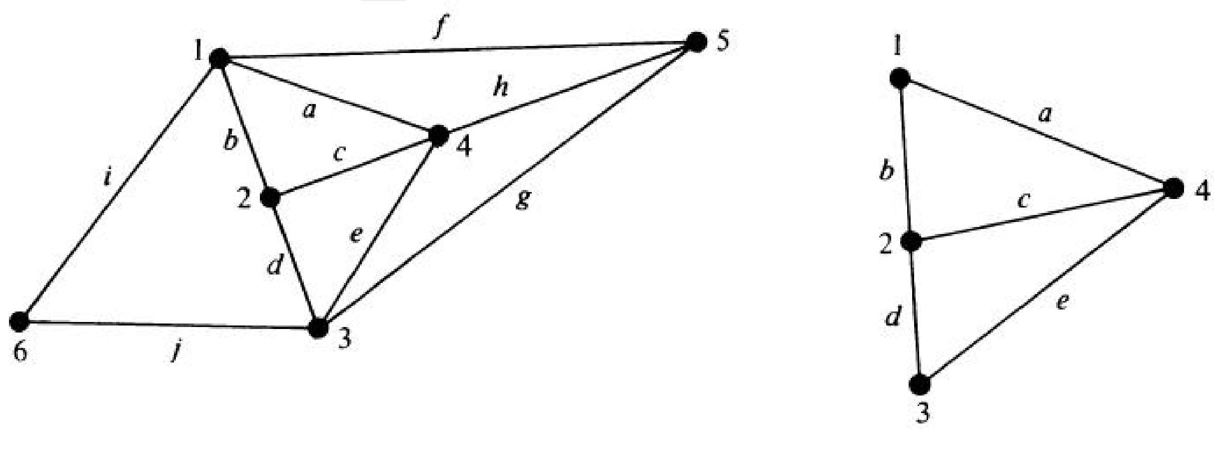
(a) (b)

However, these conditions are by no means sufficient. For instance, the two graphs shown in Fig. 2-4 satisfy all three conditions, yet they are not isomorphic. That the graphs in Figs. 2-4(a) and (b) are not isomorphic can be shown as follows: If the graph in Fig. 2-4(a) were to be isomorphic to the one in (b), vertex x must correspond to y, because there are no other vertices of degree three. Now in (b) there is only one pendant vertex, w, adjacent to y, while in (a) there are two pendant vertices, u and v, adjacent to x.

Finding a simple and efficient criterion for detection of isomorphism is still actively pursued and is an important unsolved problem in graph theory. In Chapter 11 we shall discuss various proposed algorithms and their programs for automatic detection of isomorphism by means of a computer. For now, we move to a different topic.

SUBGRAPHS

A graph g is said to be a *subgraph* of a graph G if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G. For instance, the graph in Fig. 2-5(b) is a subgraph of the one in Fig. 2-5(a). (Obviously, when considering a subgraph, the original graph must



not be altered by identifying two distinct vertices, or by adding new edges or vertices.) The concept of subgraph is akin to the concept of subset in set theory. A subgraph can be thought of as being contained in (or a part of) another graph. The symbol from set theory, $g \subset G$, is used in stating "g is a subgraph of G."

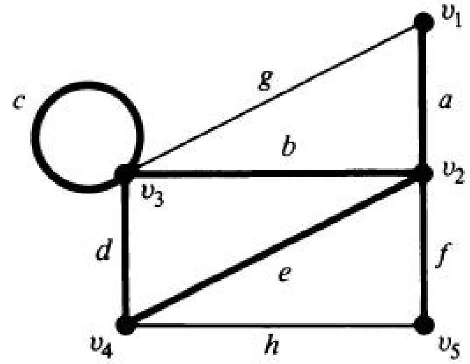
The following observations can be made immediately:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.

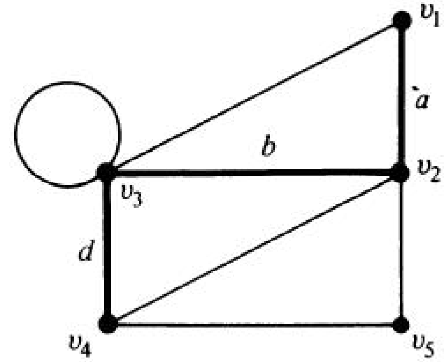
Edge-Disjoint Subgraphs: Two (or more) subgraphs g_1 and g_2 of a graph G are said to be *edge disjoint* if g_1 and g_2 do not have any edges in common. For example, the two graphs in Figs. 2-7(a) and (b) are edge-disjoint subgraphs of the graph in Fig. 2-6. Note that although edge-disjoint graphs do not have any edge in common, they may have vertices in common. Subgraphs that do not even have vertices in common are said to be *vertex disjoint*. (Obviously, graphs that have no vertices in common cannot possibly have edges in common.)

WALKS, PATHS, AND CIRCUITS

A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears (is covered or traversed) more than once in a walk. A vertex, however, may appear more than once. In Fig. 2-8(a), for instance, $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a walk shown with



(a) An Open Walk



(b) A Path of Length Three

Fig. 2-8 A walk and a path.

heavy lines. A walk is also referred to as an *edge train* or a *chain*. The set of vertices and edges constituting a given walk in a graph G is clearly a subgraph of G.

Vertices with which a walk begins and ends are called its *terminal vertices*. Vertices v_1 and v_5 are the terminal vertices of the walk shown in Fig. 2-8(a). It is possible for a walk to begin and end at the same vertex. Such a walk is called a *closed walk*. A walk that is not closed (i.e., the terminal vertices are distinct) is called an *open walk* [Fig. 2-8(a)].

An open walk in which no vertex appears more than once is called a *path* (or a *simple path* or an *elementary path*). In Fig. 2-8, $v_1\ a\ v_2\ b\ v_3\ d\ v_4$ is a path, whereas $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is not a path. In other words, a path does not intersect itself. The number of edges in a path is called the *length of a path*. It immediately follows, then, that an edge which is not a self-loop is a path of length one. It should also be noted that a self-loop can be included in a walk but not in a path (Fig. 2-8).

The terminal vertices of a path are of degree one, and the rest of the vertices (called *intermediate vertices*) are of degree two. This degree, of course, is counted only with respect to the edges included in the path and not the entire graph in which the path may be contained.

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a *circuit*. That is, a circuit is a closed, non-

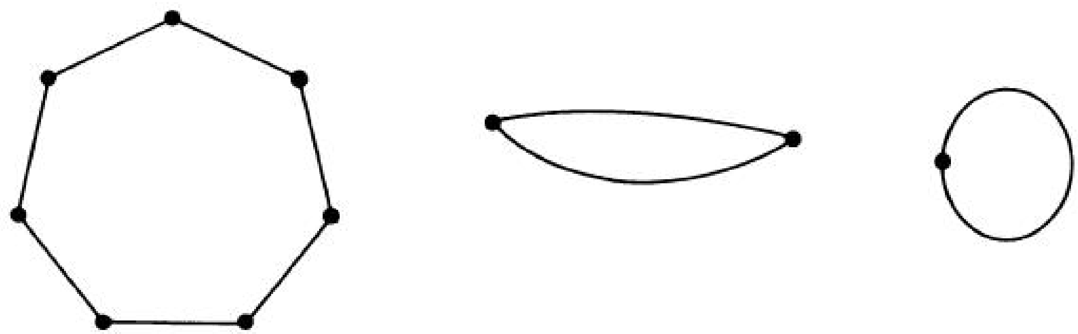


Fig. 2-9 Three different circuits.

intersecting walk. In Fig. 2-8(a), $v_2\ b\ v_3\ d\ v_4\ e\ v_2$ is, for example, a circuit. Three different circuits are shown in Fig. 2-9. Clearly, every vertex in a circuit is of degree two; again, if the circuit is a subgraph of another graph, one must count degrees contributed by the edges in the circuit only.

A circuit is also called a *cycle*, *elementary cycle*, *circular path*, and *polygon*. In electrical engineering a circuit is sometimes referred to as a *loop*—not to be confused with self-loop. (Every self-loop is a circuit, but not every circuit is a self-loop.)

The definitions in this section are summarized in Fig. 2-10. The arrows are in the direction of increasing restriction.

You may have observed that although the concepts of a path and a circuit are very simple, the formal definition becomes involved.

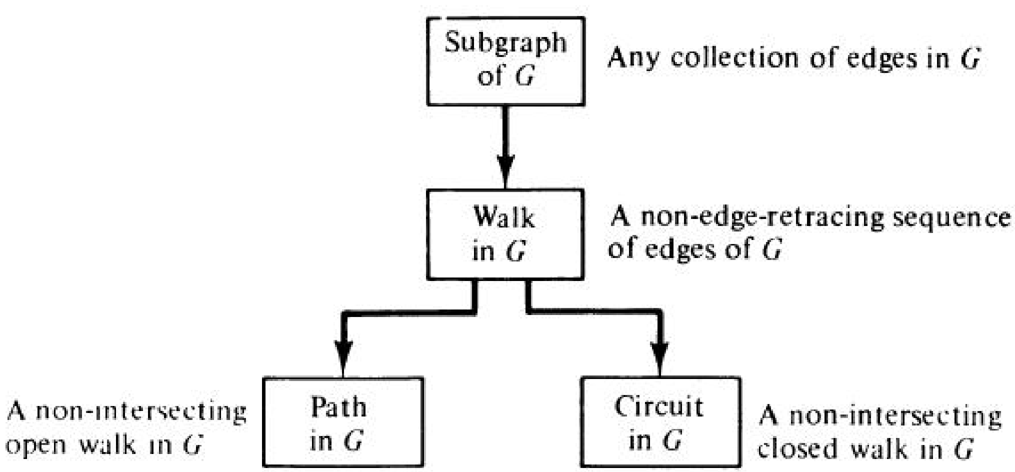


Fig. 2-10 Walks, paths, and circuits as subgraphs.

CONNECTED GRAPHS, DISCONNECTED GRAPHS, AND COMPONENTS

Intuitively, the concept of *connectedness* is obvious. A graph is connected if we can reach any vertex from any other vertex by traveling along the edges. More formally:

A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. Otherwise, G is *disconnected*. For instance, the graph in Fig. 2-8(a) is connected, but the one in Fig. 2-11 is disconnected. A null graph of more than one vertex is disconnected (Fig. 1-12).

It is easy to see that a disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a *component*. The graph in Fig. 2-11 consists of two components. Another way of looking at a component is as follows: Consider a vertex v_i in a disconnected graph G. By definition, not all vertices of G are joined by paths to v_i. Vertex v_i and all the vertices of G that have paths to v_i, together with all the edges incident on them, form a component. Obviously, a component itself is a graph.

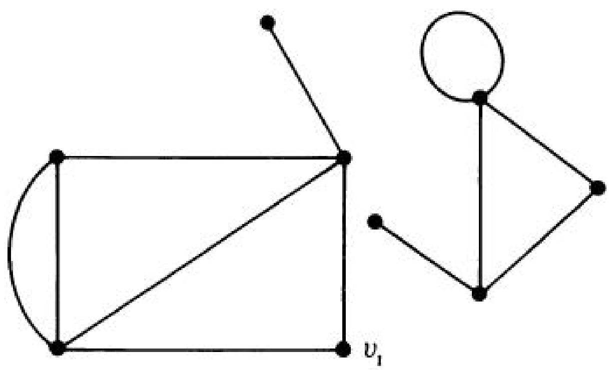


Fig. 2-11 A disconnected graph with two components.

THEOREM 2-1

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

Proof: Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b; otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition. ■

Two interesting and useful results involving connectedness are:

THEOREM 2-2

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof: Let G be a graph with all even vertices† except vertices v_1 and v_2, which are odd. From Theorem 1-1, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, v_1 and v_2 must belong to the same component, and hence must have a path between them. ■

THEOREM 2-3

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

Proof: Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \dots, n_k$. Thus we have

$$n_1 + n_2 + \cdots + n_k = n,$$
$$n_i \geq 1.$$

†For brevity, a vertex with odd degree is called an *odd vertex*, and a vertex with even degree an *even vertex*.

The proof of the theorem depends on an algebraic inequality†

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k). \tag{2-1}$$

Now the maximum number of edges in the ith component of G (which is a simple connected graph) is $\frac{1}{2}n_i(n_i - 1)$. (See Problem 1-12.) Therefore, the maximum number of edges in G is

$$\frac{1}{2}\sum_{i=1}^{k}(n_i - 1)n_i = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2} \tag{2-2}$$

$$\leq \frac{1}{2}[n^2 - (k-1)(2n-k)] - \frac{n}{2}, \qquad \text{from (2-1)}$$

$$= \frac{1}{2}\cdot(n-k)(n-k+1). \quad \blacksquare \tag{2-3}$$

It may be noted that Theorem 2-3 is a generalization of the result in Problem 1-12. The solution to Problem 1-12 is given by (2-3), where $k = 1$.

EULER GRAPHS

As mentioned in Chapter 1, graph theory was born in 1736 with Euler's famous paper in which he solved the Königsberg bridge problem. In the same paper, Euler posed (and then solved) a more general problem: In what type of graph G is it possible to find a closed walk running through every edge of G exactly once? Such a walk is now called an *Euler line*, and a graph that consists of an Euler line is called an *Euler graph*. More formally:

If some closed walk in a graph contains all the edges of the graph, then the walk is called an *Euler line* and the graph an *Euler graph*.

By its very definition a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an *Euler graph* is always connected, except for any isolated vertices the graph may have. Since isolated vertices do not contribute anything to the understanding of an Euler graph, it is hereafter assumed that Euler graphs do not have any isolated vertices and are therefore connected.

Now we shall state and prove an important theorem, which will enable us to tell immediately whether or not a given graph is an Euler graph.

†*Proof:* $\sum_{i=1}^{k}(n_i - 1) = n - k$. Squaring both sides,

$$\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$

or $\sum_{i=1}^{k}(n_i^2 - 2n_i) + k +$ nonnegative cross terms $= n^2 + k^2 - 2nk$ because $(n_i - 1) \geq 0$, for all i. Therefore, $\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n = n^2 - (k-1)(2n-k)$. $\blacksquare$

THEOREM 2-4

A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof: Suppose that G is an Euler graph. It therefore contains an Euler line (which is a closed walk). In tracing this walk we observe that every time the walk meets a vertex v it goes through two "new" edges incident on v—with one we "entered" v and with the other "exited." This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we "exited" and "entered" the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even.

To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but v. And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we just traced includes all the edges of G, G is an Euler graph. If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges. Since both G and h have all their vertices of even degree, the degrees of the vertices of h' are also even. Moreover, h' must touch h at least at one vertex a, because G is connected. Starting from a, we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph. ■

Königsberg Bridge Problem: Now looking at the graph of the Königsberg bridges (Fig. 1-5), we find that not all its vertices are of even degree. Hence, it is not an Euler graph. Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

One often encounters Euler lines in various puzzles. The problem common to these puzzles is to find how a given picture can be drawn in one continuous line without retracing and without lifting the pencil from the paper. Two such pictures are shown in Fig. 2-12. The drawing in Fig. 2-12(a) is called *Mohammed's scimitars* and is believed to have come from the Arabs. The one in Fig. 2-12(b) is, of course, the *star of David.* (Equal time!)

In defining an Euler line some authors drop the requirement that the walk be closed. For example, the walk a 1 c 2 d 3 a 4 b 5 d 6 e 7 b in Fig. 2-13, which includes all the edges of the graph and does not retrace any edge, is not closed. The initial vertex is a and the final vertex is b. We shall call such an open walk that includes (or traces or covers) all edges of a graph without retracing any edge a *unicursal line* or an *open Euler line*. A (connected) graph that has a unicursal line will be called a *unicursal graph*.

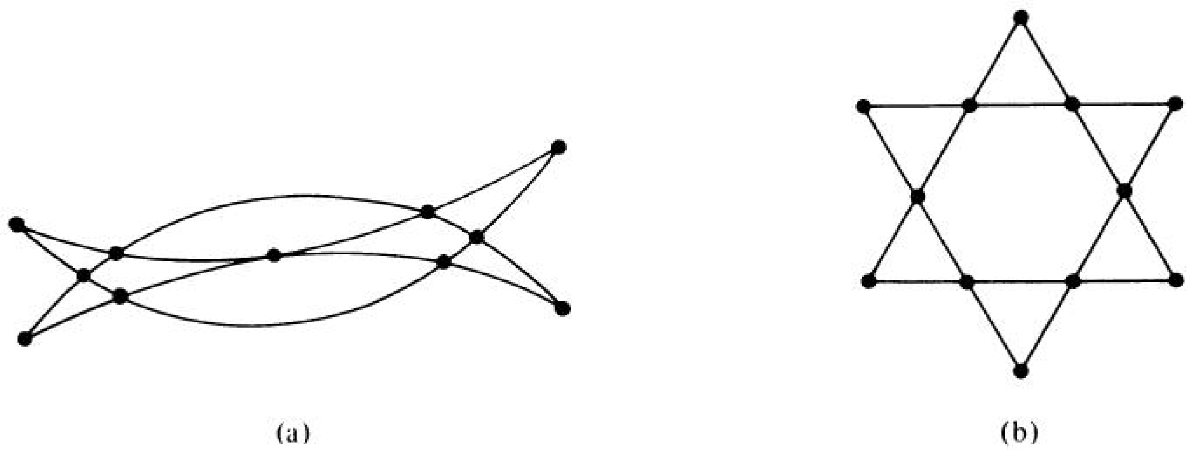


Fig. 2-12 Two Euler graphs.

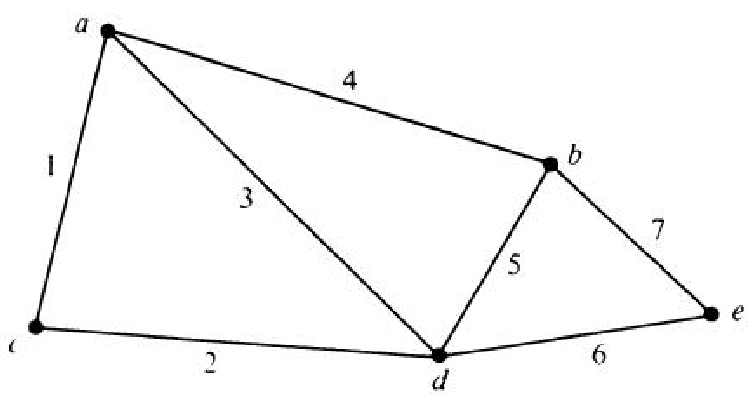


Fig. 2-13 Unicursal graph.

It is clear that by adding an edge between the initial and final vertices of a unicursal line we shall get an Euler line. Thus a connected graph is unicursal if and only if it has exactly two vertices of odd degree. This observation can be generalized as follows:

THEOREM 2-5

In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Proof: Let the odd vertices of the given graph G be named $v_1, v_2, \ldots, v_k$; $w_1, w_2, \ldots, w_k$ in any arbitrary order. Add k edges to G between the vertex pairs $(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)$ to form a new graph G'.

Since every vertex of G' is of even degree, G' consists of an Euler line ρ. Now if we remove from ρ the k edges we just added (no two of these edges are incident on the same vertex), ρ will be split into k walks, each of which is a unicursal line: The first removal will leave a single unicursal line; the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them. Thus the theorem. ■

THEOREM 2-6

A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

Proof: Suppose graph G can be decomposed into circuits; that is, G is a union of edge-disjoint circuits. Since the degree of every vertex in a circuit is two, the degree of every vertex in G is even. Hence G is an Euler graph.

Conversely, let G be an Euler graph. Consider a vertex v_1. There are at least two edges incident at v_1. Let one of these edges be between v_1 and v_2. Since vertex v_2 is also of even degree, it must have at least another edge, say between v_2 and v_3. Proceeding in this fashion, we eventually arrive at a vertex that has previously been traversed, thus forming a circuit Γ. Let us remove Γ from G. All vertices in the remaining graph (not necessarily connected) must also be of even degree. From the remaining graph remove another circuit in exactly the same way as we removed Γ from G. Continue this process until no edges are left. Hence the theorem. ■

Arbitrarily Traceable Graphs: Consider the graph in Fig. 2-17, which is an Euler graph. Suppose that we start from vertex a and trace the path $a\ b\ c$.

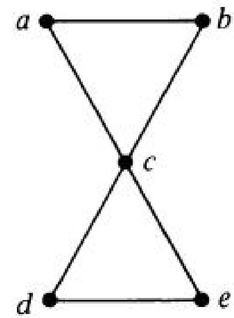


Fig. 2-17 Arbitrarily traceable graph from c.

Now at c we have the choice of going to a, d, or e. If we took the first choice, we would only trace the circuit $a\ b\ c\ a$, which is not an Euler line. Thus, starting from a, we cannot trace the entire Euler line simply by moving along any edge that has not already been traversed. This raises the following interesting question:

HAMILTONIAN PATHS AND CIRCUITS

An Euler line of a connected graph was characterized by the property of being a closed walk that traverses *every edge* of the graph (exactly once). A *Hamiltonian circuit* in a connected graph is defined as a closed walk that traverses *every vertex* of G exactly once, except of course the starting vertex, at which the walk also terminates. For example, in the graph of Fig. 2-20(a)

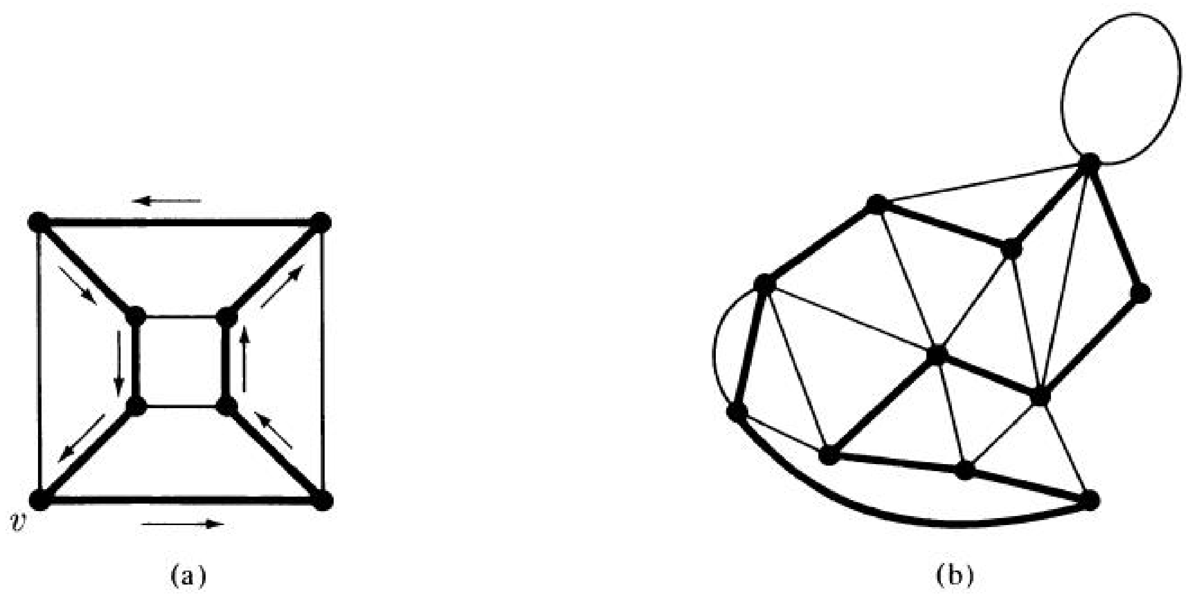


Fig. 2-20 Hamiltonian circuits.

starting at vertex v, if one traverses along the edges shown in heavy lines—passing through each vertex exactly once—one gets a Hamiltonian circuit. A Hamiltonian circuit for the graph in Fig. 2-20(b) is also shown by heavy lines. More formally:

A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices consists of exactly n edges.

Obviously, not every connected graph has a Hamiltonian circuit. For example, neither of the graphs shown in Figs. 2-17 and 2-18 has a Hamiltonian circuit. This raises the question: What is a necessary and sufficient condition for a connected graph G to have a Hamiltonian circuit?

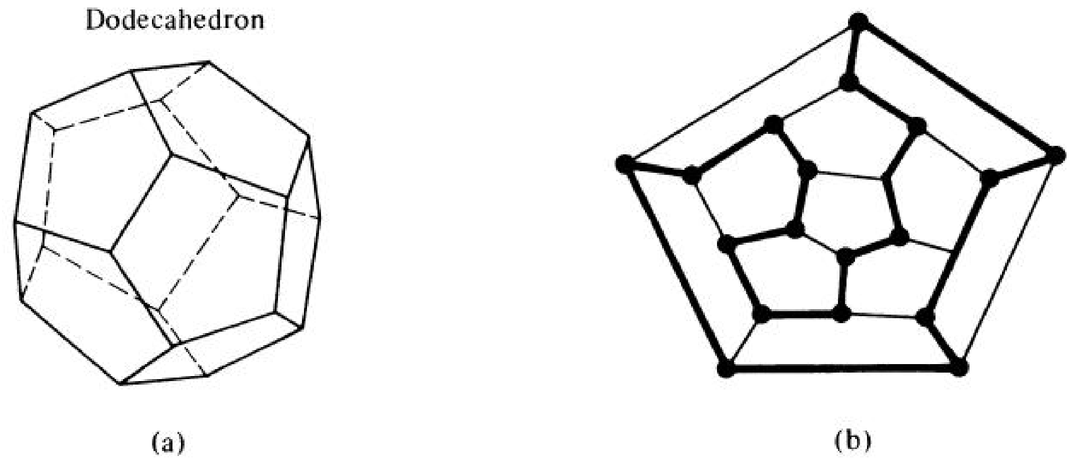


Fig. 2-21 Dodecahedron and its graph shown with a Hamiltonian circuit.

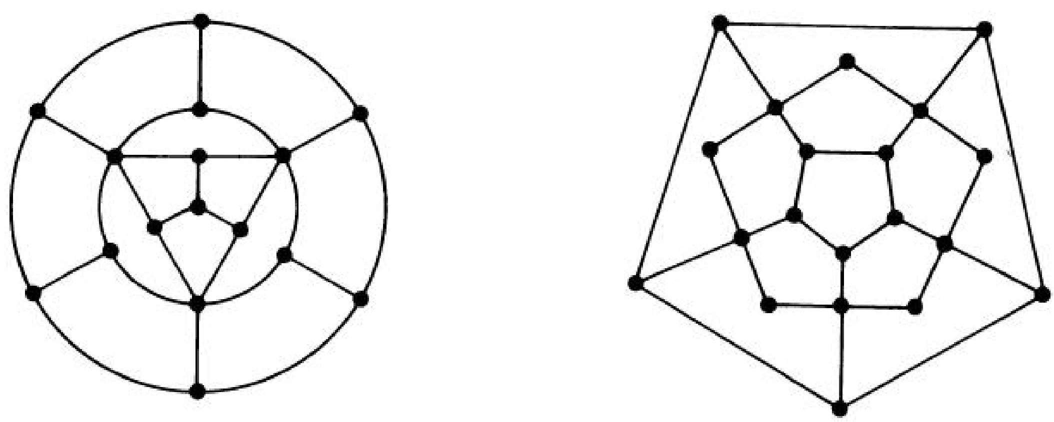
This problem, first posed by the famous Irish mathematician Sir William Rowan Hamilton in 1859, is still unsolved. As was mentioned in Chapter 1, Hamilton made a regular dodecahedron of wood [see Fig. 2-21(a)], each of whose 20 corners was marked with the name of a city. The puzzle was to start from any city and find a route along the edge of the dodecahedron that passes through every city exactly once and returns to the city of origin. The graph of the dodecahedron is given in Fig. 2-21(b), and one of many such routes (a Hamiltonian circuit) is shown by heavy lines.

The resemblance between the problem of an Euler line and that of a Hamiltonian circuit is deceptive. The latter is infinitely more complex. Although one can find Hamiltonian circuits in many specific graphs, such as those shown in Figs. 2-20 and 2-21, there is no known criterion we can apply to determine the existence of a Hamiltonian circuit in general. There are, however, certain types of graphs that always contain Hamiltonian circuits, as will be presently shown.

Hamiltonian Path: If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called a *Hamiltonian path*. Clearly, a Hamiltonian path in a graph G traverses every vertex of G. Since a Hamiltonian path is a subgraph of a Hamiltonian circuit (which in turn is a subgraph of another graph), every graph that has a Hamiltonian circuit also has a Hamiltonian path. There are, however, many graphs with Hamiltonian paths that have no Hamiltonian circuits (Problem 2-23). The length of a Hamiltonian path (if it exists) in a connected graph of n vertices is $n - 1$.

In considering the existence of a Hamiltonian circuit (or path), we need only consider simple graphs. This is because a Hamiltonian circuit (or path) traverses every vertex exactly once. Hence it cannot include a self-loop or a set of parallel edges. Thus a general graph may be made simple by removing parallel edges and self-loops before looking for a Hamiltonian circuit in it.

It is left as an exercise for the reader to show that neither of the two graphs



(a) (b)

Fig. 2-22 Graphs without Hamiltonian circuits.

shown in Fig. 2-22 has a Hamiltonian circuit (or Hamiltonian path). See Problem 2-24.

shown in Fig. 2-22 has a Hamiltonian circuit (or Hamiltonian path). See Problem 2-24.

What general class of graphs is guaranteed to have a Hamiltonian circuit? Complete graphs of three or more vertices constitute one such class.

Complete Graph: A simple graph in which there exists an edge between every pair of vertices is called a *complete graph*. Complete graphs of two, three, four, and five vertices are shown in Fig. 2-23. A complete graph is

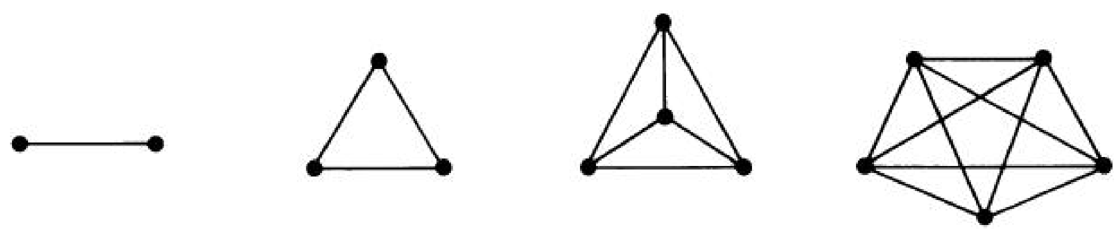


Fig. 2-23 Complete graphs of two, three, four, and five vertices.

sometimes also referred to as a *universal graph* or a *clique*. Since every vertex is joined with every other vertex through one edge, the degree of every vertex is $n - 1$ in a complete graph G of n vertices. Also the total number of edges in G is $n(n - 1)/2$. See Problem 1-12.

It is easy to construct a Hamiltonian circuit in a complete graph of n vertices. Let the vertices be numbered $v_1, v_2, \ldots, v_n$. Since an edge exists between any two vertices, we can start from v_1 and traverse to v_2, and v_3, and so on to v_n, and finally from v_n to v_1. This is a Hamiltonian circuit.

Number of Hamiltonian Circuits in a Graph: A given graph may contain more than one Hamiltonian circuit. Of interest are all the edge-disjoint Hamiltonian circuits in a graph. The determination of the exact number of *edge-disjoint* Hamiltonian circuits (or paths) in a graph in general is also an unsolved problem. However, the number of edge-disjoint Hamiltonian circuits in a complete graph with odd number of vertices is given by Theorem 2-8.

THEOREM 2-8

In a complete graph with n vertices there are $(n - 1)/2$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

Proof: A complete graph G of n vertices has $n(n-1)/2$ edges, and a Hamiltonian circuit in G consists of n edges. Therefore, the number of edge-disjoint Hamiltonian circuits in G cannot exceed $(n-1)/2$. That there are $(n-1)/2$ edge-disjoint Hamiltonian circuits, when n is odd, can be shown as follows:

The subgraph (of a complete graph of n vertices) in Fig. 2-24 is a Hamiltonian circuit. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise

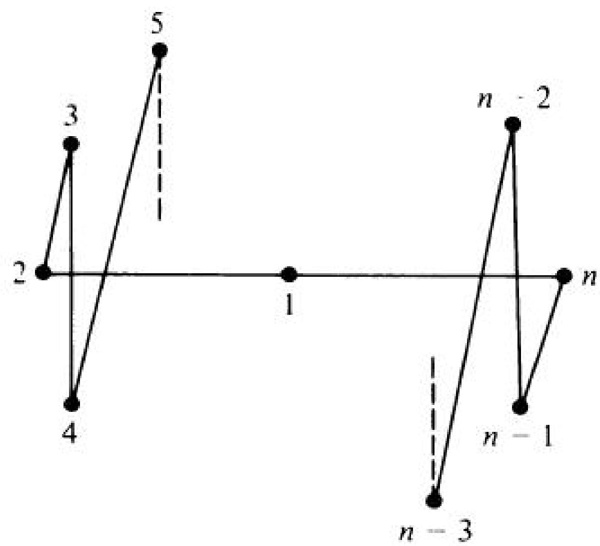


Fig. 2-24 Hamiltonian circuit; n is odd.

by $360/(n-1), 2\cdot 360/(n-1), 3\cdot 360/(n-1), \ldots, (n-3)/2\cdot 360/(n-1)$ degrees. Observe that each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones. Thus we have $(n-3)/2$ new Hamiltonian circuits, all edge disjoint from the one in Fig. 2-24 and also edge disjoint among themselves. Hence the theorem. ■

This theorem enables us to solve the problem of the seating arrangement at a round table, introduced in Chapter 1, as follows:

Representing a member x by a vertex and the possibility of his sitting next to another member y by an edge between x and y, we construct a graph G. Since every member is allowed to sit next to any other member, G is a complete graph of nine vertices—nine being the number of people to be seated around the table. Every seating arrangement around the table is clearly a Hamiltonian circuit.

The first day of their meeting they can sit in any order, and it will be a Hamiltonian circuit H_1. The second day, if they are to sit such that every member must have different neighbors, we have to find another Hamiltonian circuit H_2 in G, with an entirely different set of edges from those in H_1; that is,

H_1 and H_2 are edge-disjoint Hamiltonian circuits. From Theorem 2-8 the number of edge-disjoint Hamiltonian circuits in G is four; therefore, only four such arrangements exist among nine people.

Another interesting result on the question of existence of Hamiltonian circuits in a graph, obtained by G. A. Dirac, is:

THEOREM 2-9

A sufficient (but by no means necessary) condition for a simple graph G to have a Hamiltonian circuit is that the degree of every vertex in G be at least $n/2$, where n is the number of vertices in G.

Proof: For proof the reader is referred to the original paper by Dirac [2-3].

TREES

A *tree* is a connected graph without any circuits. The graph in Fig. 3-1, for instance, is a tree. Trees with one, two, three, and four vertices are shown in Fig. 3-2. As pointed out in Chapter 1, a graph must have at least one vertex, and therefore so must a tree. Some authors allow the *null tree*, a tree without any vertices. We have excluded such an entity from being a tree. Similarly, as we are considering only finite graphs, our trees are also finite.

It follows immediately from the definition that a tree has to be a simple graph, that is, having neither a self-loop nor parallel edges (because they both form circuits).

Trees appear in numerous instances. The genealogy of a family is often

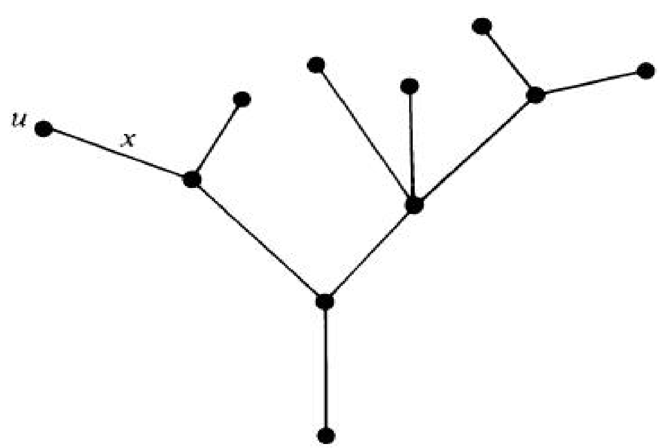


Fig. 3-1 Tree.

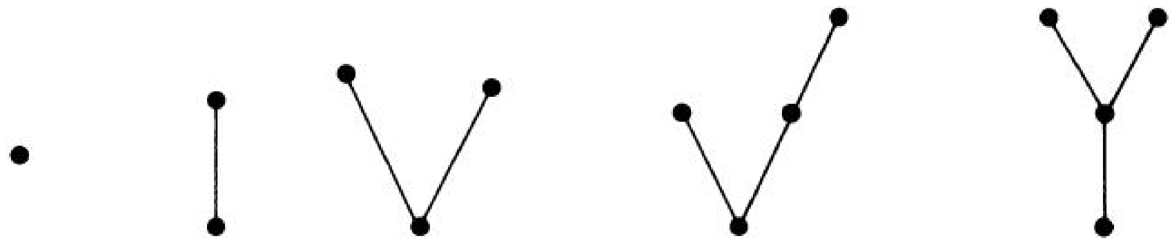


Fig. 3-2 Trees with one, two, three, and four vertices.

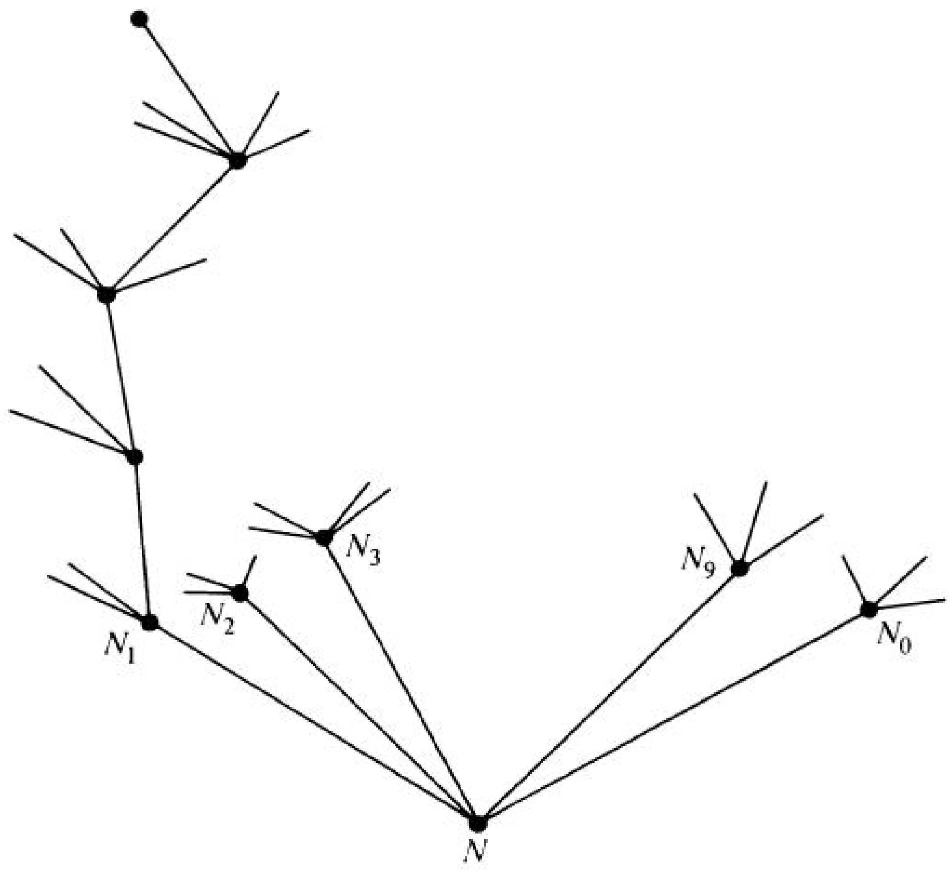


Fig. 3-3 Decision tree.

represented by means of a tree (in fact the term *tree* comes from *family tree*). A river with its tributaries and subtributaries can be represented by a tree. The sorting of mail according to zip code and the sorting of punched cards are done according to a tree (called *decision tree* or *sorting tree*).

Figure 3-3 might represent the flow of mail. All the mail arrives at some local office, vertex N. The most significant digit in the zip code is read at N, and the mail is divided into 10 piles $N_1, N_2, \ldots, N_9$, and N_0, depending on the most significant digit. Each pile is further divided into 10 piles according to the second most significant digit, and so on, till the mail is subdivided into 10^5 possible piles, each representing a unique five-digit zip code.

In many sorting problems we have only two alternatives (instead of 10 as in the preceding example) at each intermediate vertex, representing a dichotomy, such as large or small, good or bad, 0 or 1. Such a decision tree with two choices at each vertex occurs frequently in computer programming and switching theory. We shall deal with such trees and their applications in Section 3-5. Let us first obtain a few simple but important theorems on the general properties of trees.

SOME PROPERTIES OF TREES

THEOREM 3-1

There is one and only one path between every pair of vertices in a tree, T.

Proof: Since T is a connected graph, there must exist at least one path between every pair of vertices in T. Now suppose that between two vertices a and b of T there are two distinct paths. The union of these two paths will contain a circuit and T cannot be a tree. ■

Conversely:

THEOREM 3-2

If in a graph G there is one and only one path between every pair of vertices, G is a tree.

Proof: Existence of a path between every pair of vertices assures that G is connected. A circuit in a graph (with two or more vertices) implies that there is at least one pair of vertices a, b such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, G can have no circuit. Therefore, G is a tree. ■

THEOREM 3-3

A tree with n vertices has $n - 1$ edges.

Proof: The theorem will be proved by induction on the number of vertices.

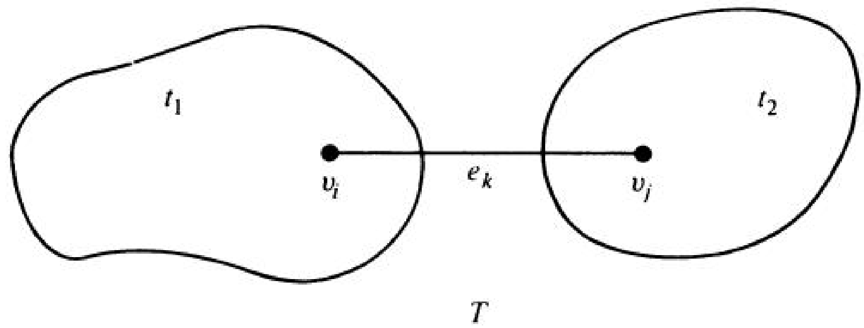


Fig. 3-4 Tree T with n vertices.

It is easy to see that the theorem is true for $n = 1, 2$, and 3 (see Fig. 3-2). Assume that the theorem holds for all trees with fewer than n vertices.

Let us now consider a tree T with n vertices. In T let e_k be an edge with end vertices v_i and v_j. According to Theorem 3-1, there is no other path between v_i and v_j except e_k. Therefore, deletion of e_k from T will disconnect the graph, as shown in Fig. 3-4. Furthermore, $T - e_k$ consists of exactly two components, and since there were no circuits in T to begin with, each of these components is a tree. Both these trees, t_1 and t_2, have fewer than n vertices each, and therefore, by the induction hypothesis, each contains one less edge than the number of vertices in it. Thus $T - e_k$ consists of $n - 2$ edges (and n vertices). Hence T has exactly $n - 1$ edges. ■

THEOREM 3-4

Any connected graph with n vertices and $n - 1$ edges is a tree.

Proof: The proof of the theorem is left to the reader as an exercise (Problem 3-5).

You may have noticed another important feature of a tree: its vertices are connected together with the minimum number of edges. A connected graph is said to be *minimally connected* if removal of any one edge from it disconnects the graph. A minimally connected graph cannot have a circuit; otherwise, we could remove one of the edges in the circuit and still leave the graph connected. Thus a minimally connected graph is a tree. Conversely, if a connected graph G is not minimally connected, there must exist an edge e_i in G such that $G - e_i$ is connected. Therefore, e_i is in some circuit, which implies that G is not a tree. Hence the following theorem:

THEOREM 3-5

A graph is a tree if and only if it is minimally connected.

The significance of Theorem 3-5 is obvious. Intuitively, one can see that to interconnect n distinct points, the minimum number of line segments needed is $n - 1$. It requires no background in electrical engineering to realize

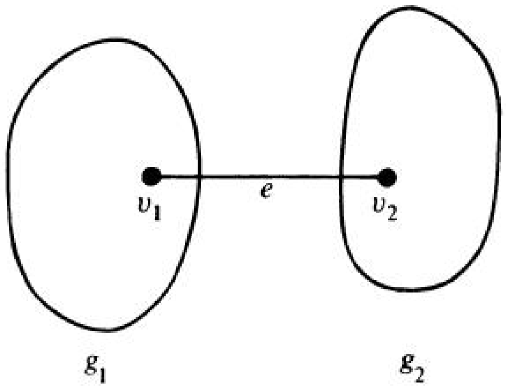


Fig. 3-5 Edge e added to $G = g_1 \cup g_2$.

that to short (electrically) n pins together, one needs at least $n - 1$ pieces of wire. The resulting structure, according to Theorem 3-5, is a tree.

We showed that a connected graph with n vertices and without any circuits has $n - 1$ edges. We can also show that a graph with n vertices which has no circuit and has $n - 1$ edges is always connected (i.e., it is a tree), in the following theorem.

THEOREM 3-6

A graph G with n vertices, $n - 1$ edges, and no circuits is connected.

Proof: Suppose there exists a circuitless graph G with n vertices and $n - 1$ edges which is disconnected. In that case G will consist of two or more circuitless components. Without loss of generality, let G consist of two components, g_1 and g_2. Add an edge e between a vertex v_1 in g_1 and v_2 in g_2 (Fig. 3-5). Since there was no path between v_1 and v_2 in G, adding e did not create a circuit. Thus $G \cup e$ is a circuitless, connected graph (i.e., a tree) of n vertices and n edges, which is not possible, because of Theorem 3-3. ■

The results of the preceding six theorems can be summarized by saying that the following are five different but equivalent definitions of a tree. That is, a graph G with n vertices is called a tree if

1. G is *connected* and is *circuitless*, or
2. G is *connected* and has $n - 1$ *edges*, or
3. G is *circuitless* and has $n - 1$ *edges*, or
4. There is *exactly one path* between every pair of vertices in G, or
5. G is a *minimally connected* graph.

PENDANT VERTICES IN A TREE

You must have observed that each of the trees shown in the figures has several pendant vertices (a pendant vertex was defined as a vertex of degree

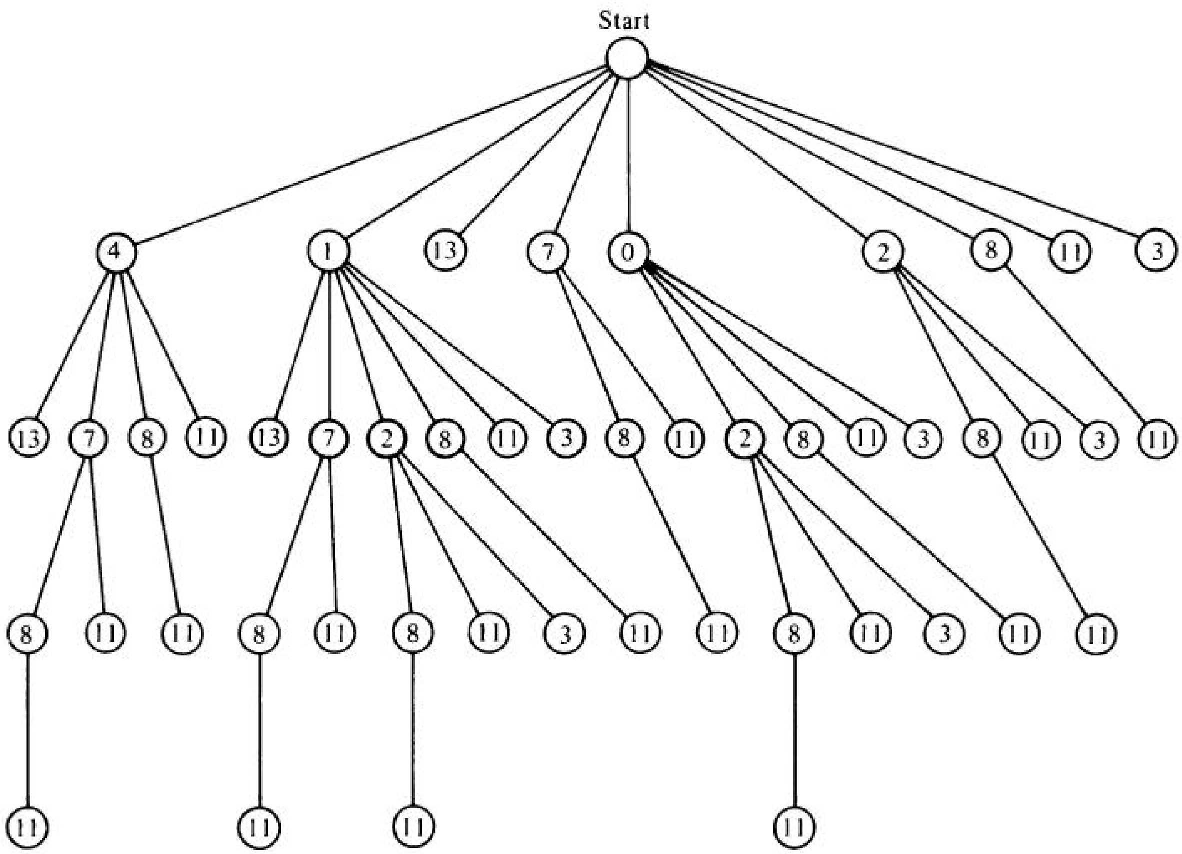


Fig. 3-6 Tree of the monotonically increasing sequences in 4, 1, 13, 7, 0, 2, 8, 11, 3.

one). The reason is that in a tree of n vertices we have $n - 1$ edges, and hence $2(n - 1)$ degrees to be divided among n vertices. Since no vertex can be of zero degree, we must have at least two vertices of degree one in a tree. This of course makes sense only if $n \geq 2$. More formally:

THEOREM 3-7

In any tree (with two or more vertices), there are at least two pendant vertices.

An Application: The following problem is used in teaching computer programming. Given a sequence of integers, no two of which are the same, find the largest monotonically increasing subsequence in it. Suppose that the sequence given to us is 4, 1, 13, 7, 0, 2, 8, 11, 3; it can be represented by a tree in which the vertices (except the start vertex) represent individual numbers in the sequence, and the path from the start vertex to a particular vertex v describes the monotonically increasing subsequence terminating in v. As shown in Fig. 3-6, this sequence contains four longest monotonically increasing subsequences, that is, (4, 7, 8, 11), (1, 7, 8, 11), (1, 2, 8, 11), and (0, 2, 8, 11). Each is of length four. Such a tree used in representing data is referred to as a data tree by computer programmers.

DISTANCE AND CENTERS IN A TREE

The tree in Fig. 3-7 has four vertices. Intuitively, it seems that vertex b is located more "centrally" than any of the other three vertices. We shall ex-

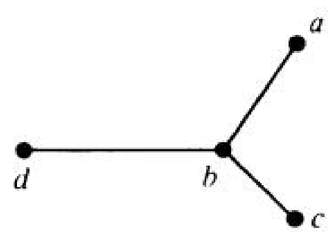


Fig. 3-7 Tree.

plore this idea further and see if in a tree there exists a "center" (or centers). Inherent in the concept of a center is the idea of "distance," so we must define distance before we can talk of a center.

In a connected graph G, the *distance* $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path (i.e., the number of edges in the shortest path) between them.

The definition of distance between any two vertices is valid for any connected graph (not necessarily a tree). In a graph that is not a tree, there are generally several paths between a pair of vertices. We have to enumerate all these paths and find the length of the shortest one. (There may be several shortest paths.)

For instance, some of the paths between vertices v_1 and v_2 in Fig. 3-8 are (a, e), (a, c, f), (b, c, e), (b, f), (b, g, h), and (b, g, i, k). There are two shortest paths, (a, e) and (b, f), each of length two. Hence $d(v_1, v_2) = 2$.

In a tree, since there is exactly one path between any two vertices (Theorem 3-1), the determination of distance is much easier. For instance, in the tree of Fig. 3-7, $d(a, b) = 1$, $d(a, c) = 2$, $d(c, b) = 1$, and so on.

A Metric: Before we can legitimately call a function $f(x, y)$ of two variables a "distance" between them, this function must satisfy certain requirements. These are

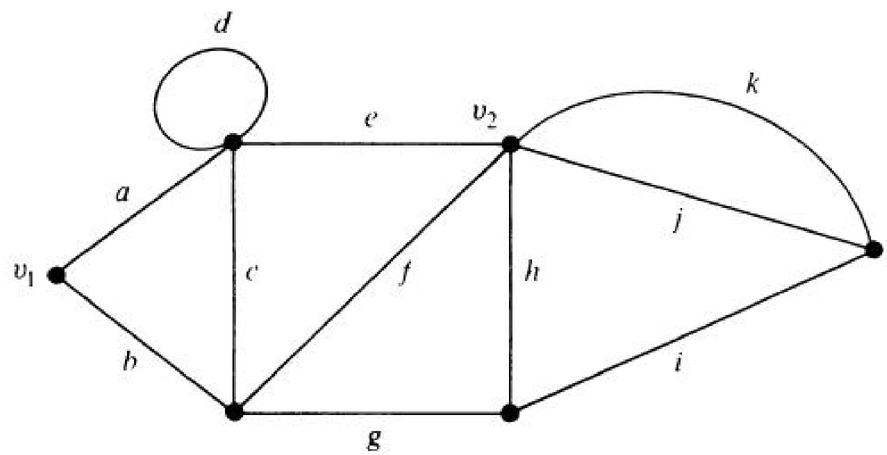


Fig. 3-8 Distance between v_1 and v_2 is two.

1. Nonnegativity: $f(x, y) \geq 0$, and $f(x, y) = 0$ if and only if $x = y$.
2. Symmetry: $f(x, y) = f(y, x)$.
3. Triangle inequality: $f(x, y) \leq f(x, z) + f(z, y)$ for any z.

A function that satisfies these three conditions is called a *metric*. That the distance $d(v_i, v_j)$ between two vertices of a connected graph satisfies conditions 1 and 2 is immediately evident. Since $d(v_i, v_j)$ is the length of the shortest path between vertices v_i and v_j, this path cannot be longer than another path between v_i and v_j, which goes through a specified vertex v_k. Hence $d(v_i, v_j) \leq d(v_i, v_k) + d(v_k, v_j)$. Therefore,

THEOREM 3-8

The distance between vertices of a connected graph is a metric.

Coming back to our original topic of relative location of different vertices in a tree, let us define another term called *eccentricity* (also referred to as *associated number* or *separation*) of a vertex in a graph.

The eccentricity $E(v)$ of a vertex v in a graph G is the distance from v to the vertex farthest from v in G; that is,

$$E(v) = \max_{v_i \in G} d(v, v_i).$$

A vertex with minimum eccentricity in graph G is called a *center* of G. The eccentricities of the four vertices in Fig. 3-7 are $E(a) = 2$, $E(b) = 1$, $E(c) = 2$, and $E(d) = 2$. Hence vertex b is the center of that tree. On the other hand, consider the tree in Fig. 3-9. The eccentricity of each of its six vertices is shown next to the vertex. This tree has two vertices having the same minimum eccentricity. Hence this tree has two centers. Some authors refer to such centers as *bicenters;* we shall call them just centers, because there will be no occasion for confusion.

The reader can easily verify that a graph, in general, has many centers. For example, in a graph that consists of just a circuit (a polygon), every vertex is a center. In the case of a tree, however, König [1-7] proved the following theorem:

THEOREM 3-9

Every tree has either one or two centers.

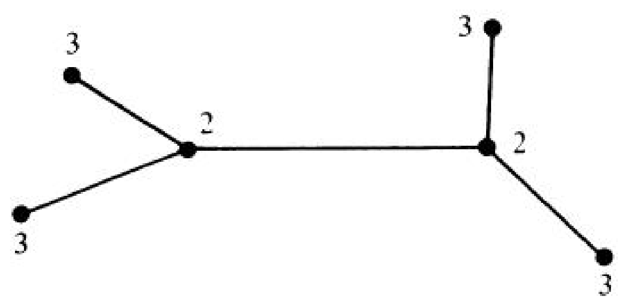


Fig 3-9 Eccentricities of the vertices of a tree.

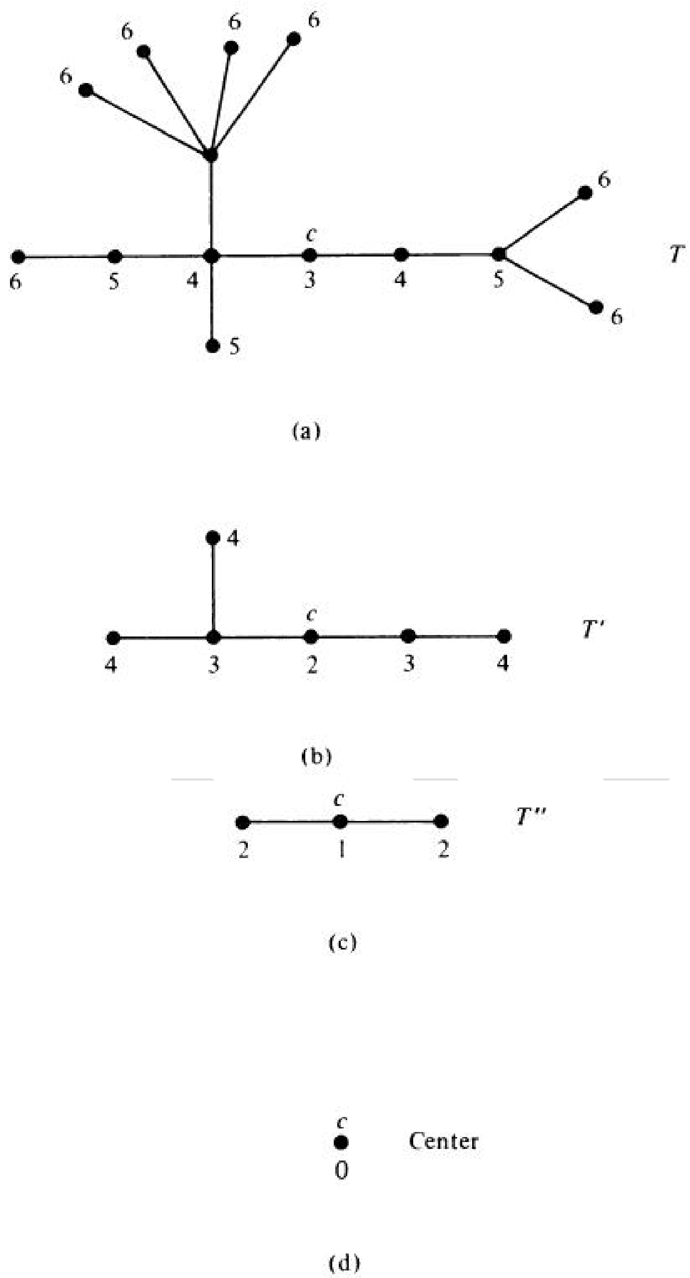


Fig. 3-10 Finding a center of a tree.

Proof: The maximum distance, max $d(v, v_i)$, from a given vertex v to any other vertex v_i occurs only when v_i is a pendant vertex. With this observation, let us start with a tree T having more than two vertices. Tree T must have two or more pendant vertices (Theorem 3-7). Delete all the pendant vertices from T. The resulting graph T' is still a tree. What about the eccentricities of the vertices in T'? A little deliberation will reveal that removal of all pendant vertices from T uniformly reduced the eccentricities of the remaining vertices (i.e., vertices in T') by one. Therefore, all vertices that T had as centers will still remain centers in T'. From T' we can again remove all pendant vertices and get another tree T''. We continue this process (which is illustrated in Fig. 3-10) until there is left either a vertex (which is the center of T) or an edge (whose end vertices are the two centers of T). Thus the theorem. ■

COROLLARY

From the argument used in proving Theorem 3-9, we see that if a tree T has two centers, the two centers must be adjacent.

A Sociological Application: Suppose that the communication among a group of 14 persons in a society is represented by the graph in Fig. 3-10(a), where the vertices represent the persons and an edge represents the communication link between its two end vertices. Since the graph is connected, we know that all the members can be reached by any member, either directly or through some other members. But it is also important to note that the graph is a tree—minimally connected. The group cannot afford to lose any of the communication links.

The eccentricity of each vertex, $E(v)$, represents how close v is to the farthest member of the group. In Fig. 3-10(a), vertex c should be the leader of the group, if closeness of communication were the criterion for leadership.

Radius and Diameter: If a tree has a center (or two centers), does it have a radius also? Yes. The eccentricity of a center (which is the distance from the center of the tree to the farthest vertex) in a tree is defined as the *radius* of the tree. For instance, the radius of the tree in Fig. 3-10(a) is three. The *diameter* of a tree T, on the other hand, is defined as the length of the longest path in T. It is left as an exercise for the reader (Problem 3-6) to show that a radius in a tree is not necessarily half its diameter.

ROOTED AND BINARY TREES

A tree in which one vertex (called the *root*) is distinguished from all the others is called a *rooted tree*. For instance, in Fig. 3-3 vertex N, from where all the mail goes out, is distinguished from the rest of the vertices. Hence N can be considered the root of the tree, and so the tree is rooted. Similarly, in Fig. 3-6 the start vertex may be considered as the root of the tree shown. In a diagram of a rooted tree, the root is generally marked distinctly. We will show the root enclosed in a small triangle. All rooted trees with four vertices are shown in Fig. 3-11. Generally, the term *tree* means trees without any root. However, for emphasis they are sometimes called *free trees* (or *nonrooted trees*) to differentiate them from the rooted kind.

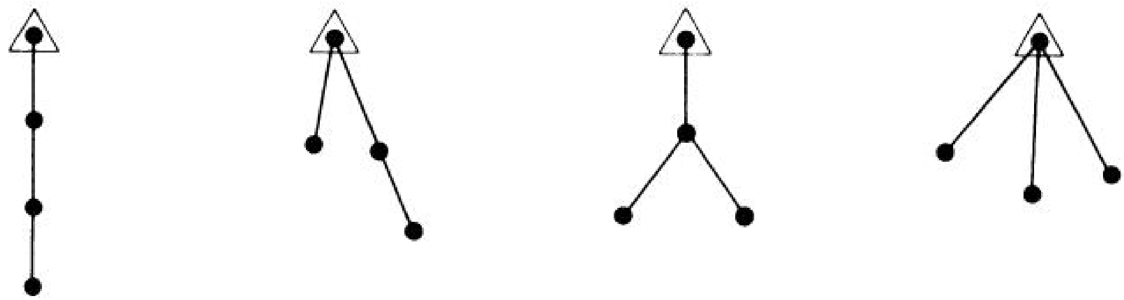


Fig. 3-11 Rooted trees with four vertices.

Binary Trees: A special class of rooted trees, called binary rooted trees, is of particular interest, since they are extensively used in the study of computer search methods, binary identification problems, and variable-length binary codes. A *binary tree* is defined as a tree in which there is exactly one vertex of degree two, and each of the remaining vertices is of degree one or three (Fig. 3-12). (Obviously, we are talking about trees with three or more vertices.) Since the vertex of degree two is distinct from all other vertices, this vertex serves as a root. Thus every binary tree is a rooted tree. Two properties of binary trees follow directly from the definition:

1. The number of vertices n in a binary tree is always odd. This is because there is exactly one vertex of even degree, and the remaining $n - 1$ vertices are of odd degrees. Since from Theorem 1-1 the number of vertices of odd degrees is even, $n - 1$ is even. Hence n is odd.

2. Let p be the number of pendant vertices in a binary tree T. Then $n - p - 1$ is the number of vertices of degree three. Therefore, the number of edges in T equals

$$\frac{1}{2}[p + 3(n - p - 1) + 2] = n - 1;$$

hence

$$p = \frac{n+1}{2}. \qquad (3\text{-}1)$$

A nonpendant vertex in a tree is called an *internal vertex*. It follows from Eq. (3-1) that the number of internal vertices in a binary tree is one less than the number of pendant vertices. In a binary tree a vertex v_i is said to be at *level* l_i if v_i is at a distance of l_i from the root. Thus the root is at level 0. A 13-vertex, four-level binary tree is shown in Fig. 3-12. The number of vertices at levels 1, 2, 3, and 4 are 2, 2, 4, and 4, respectively.

One of the most straightforward applications of binary trees is in search procedures. Each vertex of a binary tree represents a test with two possible

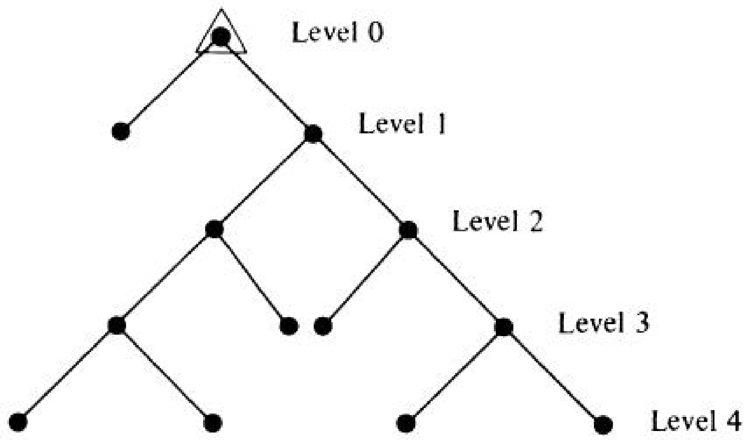


Fig. 3-12 A 13-vertex, 4-level binary tree.

outcomes. We start at the root, and the outcome of the test at the root sends us to one of the two vertices at the next level, where further tests are made, and so on. Reaching a specified pendant vertex (the goal of the search) terminates the search. For such a search procedure it is often important to construct a binary tree in which, for a given number of vertices n, the vertex farthest from the root is as close to the root as possible. Clearly, there can be only one vertex (the root) at level 0, at most two vertices at level 1, at most four vertices at level 2, and so on. Therefore, the maximum number of vertices possible in a k-level binary tree is

$$2^0 + 2^1 + 2^2 + \cdots + 2^k \geq n.$$

The maximum level, $l_{\max}$, of any vertex in a binary tree is called the *height* of the tree. It is easy to see that the minimum possible height of an n-vertex binary tree is

$$2^0 + 2^1 + 2^2 + \cdots + 2^k \geq n.$$

The maximum level, l_{max}, of any vertex in a binary tree is called the *height* of the tree. It is easy to see that the minimum possible height of an n-vertex binary tree is

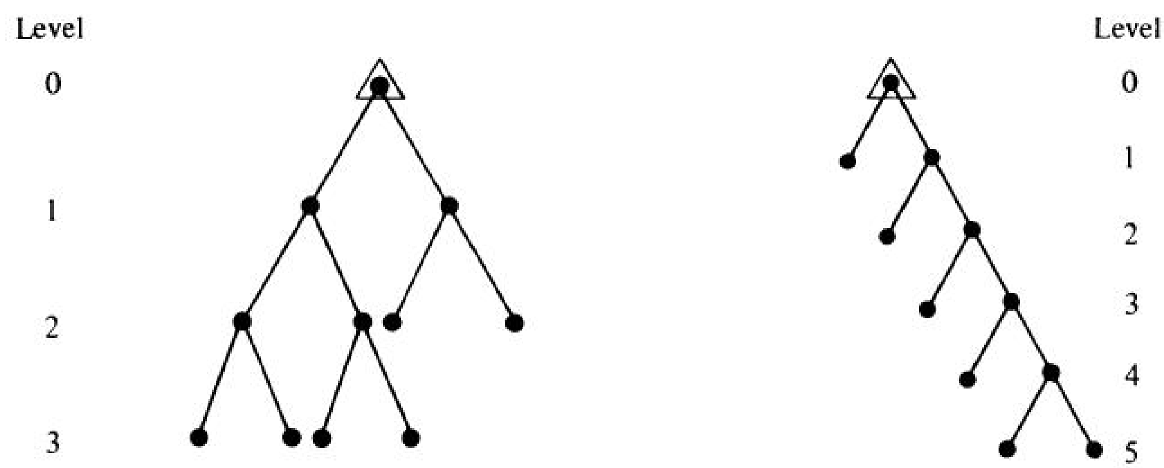
$$\min l_{max} = \lceil \log_2 (n+1) - 1 \rceil, \tag{3-2}$$

where $\lceil n \rceil$ denotes the smallest integer greater than or equal to n.

On the other hand, to construct a binary tree for a given n such that the farthest vertex is as far as possible from the root, we must have exactly two vertices at each level, except at the 0 level. Therefore,

$$\max l_{max} = \frac{n-1}{2}. \tag{3-3}$$

For $n = 11$, binary trees realizing both these extremes are shown in Fig. 3-13.



$\min l_{max} = \lceil (\log_2 12) - 1 \rceil$

(a)

$\max l_{max} = \frac{11-1}{2} = 5$

(b)

Fig. 3-13 Two 11-vertex binary trees.

In analysis of algorithms we are generally interested in computing the sum of the levels of all pendant vertices. This quantity, known as *the path length* (or external path length) *of a tree*, can be defined as the sum of the path lengths from the root to all pendant vertices. The path length of the binary tree in Fig. 3-12, for example, is

$$1 + 3 + 3 + 4 + 4 + 4 + 4 = 23.$$

The path lengths of trees in Figs. 3-13(a) and (b) are 16 and 20, respectively. The importance of the path length of a tree lies in the fact that this quantity is often directly related to the execution time of an algorithm.

It can be shown that the type of binary tree in Fig. 3-13(a) (i.e., a tree with $2^{l_{\max}-1}$ vertices at level $l_{\max} - 1$) yields the minimum path length for a given n.

Weighted Path Length: In some applications, every pendant vertex v_j of a binary tree has associated with it a positive real number w_j. Given w_1, $w_2, \ldots, w_m$ the problem is to construct a binary tree (with m pendant vertices) that minimizes

$$\sum w_j l_j,$$

where l_j is the level of pendant vertex v_j, and the sum is taken over all pendant vertices. Let us illustrate the significance of this problem with a simple example.

A Coke machine is to identify, by a sequence of tests, the coin that is put into the machine. Only pennies, nickels, dimes, and quarters can go through the slot. Let us assume that the probabilities of a coin being a penny, a nickel, a dime, and a quarter are .05, .15, .5, and .30, respectively. Each test has the effect of partitioning the four types of coins into two complementary sets and asserting the unknown coin to be in one of the two sets. Thus for four coins we have $2^3 - 1$ such tests. If the time taken for each test is the same, what sequence of tests will minimize the expected time taken by the Coke machine to identify the coin?

The solution requires the construction of a binary tree with four pendant vertices (and therefore three internal vertices) v_1, v_2, v_3, and v_4 and corresponding weights $w_1 = .05$, $w_2 = .15$, $w_3 = .5$, and $w_4 = .3$, such that the quantity $\sum l_i w_i$ is minimized. The solution is given in Fig. 3-14(a), for which the expected time is $1.7t$, where t is the time taken for each test. Contrast this with Fig. 3-14(b), for which the expected time is $2t$. An algorithm for constructing a binary tree with minimum weighted path length can be found in [3-6].

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In this problem of a Coke machine, many interesting variations are possible. For example, not all possible tests may be available, or they may not all consume the same time.

Binary trees with minimum weighted path length have also been used in

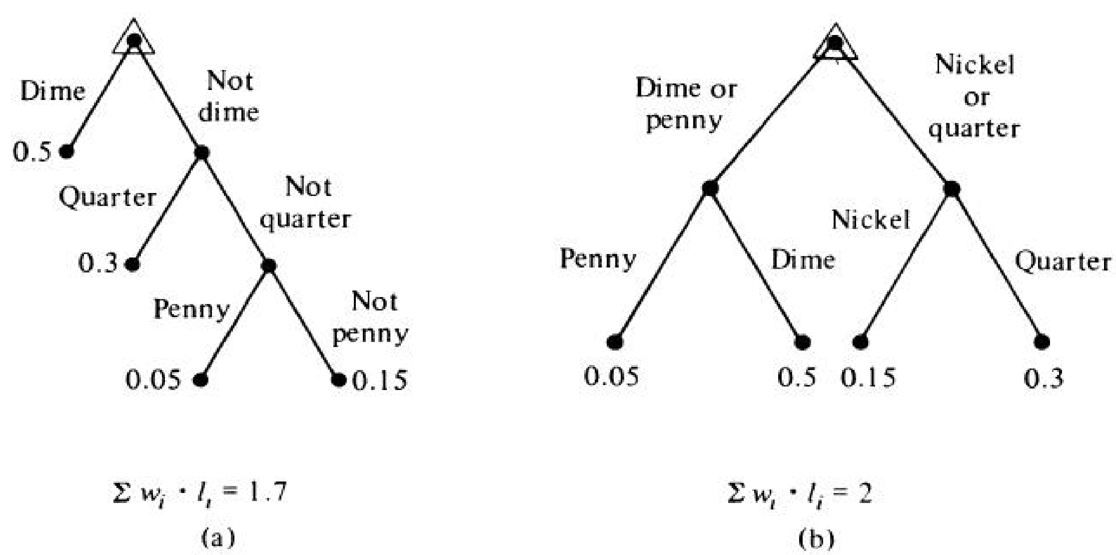


Fig. 3-14 Two binary trees with weighted pendant vertices.

constructing variable-length binary codes, where the letters of the alphabet (A, B, C, . . . , Z) are represented by binary digits. Since different letters have different frequencies of occurrence (frequencies are interpreted as weights $w_1, w_2, \ldots, w_{26}$), a binary tree with minimum weighted path length corresponds to a binary code of minimum cost; see [3-6]. For more on minimum-path binary trees and their applications the reader is referred to [3-5] and [3-7].

ON COUNTING TREES

In 1857, Arthur Cayley discovered trees while he was trying to count the number of structural isomers of the saturated hydrocarbons (or paraffin series) C_kH_{2k+2}. He used a connected graph to represent the C_kH_{2k+2} molecule. Corresponding to their chemical valencies, a carbon atom was represented by a vertex of degree four and a hydrogen atom by a vertex of degree one (pendant vertices). The total number of vertices in such a graph is

$$n = 3k + 2,$$

and the total number of edges is

$$e = \frac{1}{2}(\text{sum of degrees}) = \frac{1}{2}(4k + 2k + 2)$$
$$= 3k + 1.$$

Since the graph is connected and the number of edges is one less than the number of vertices, it is a tree. Thus the problem of counting structural isomers of a given hydrocarbon becomes the problem of counting trees (with certain qualifying properties, to be sure).

The first question Cayley asked was: what is the number of different trees that one can construct with n distinct (or labeled) vertices? If $n = 4$, for instance, we have 16 trees, as shown in Fig. 3-15. The reader can satisfy himself that there are no more trees of four vertices. (Of course, some of these trees are isomorphic—to be discussed later.)

A graph in which each vertex is assigned a unique name or label (i.e., no two vertices have the same label), as in Fig. 3-15, is called a *labeled graph*. The distinction between a labeled and an unlabeled graph is very important when we are counting the number of different graphs. For instance, the four graphs in the first row in Fig. 3-15 are counted as four different trees (even though they are isomorphic) only because the vertices are labeled. If there

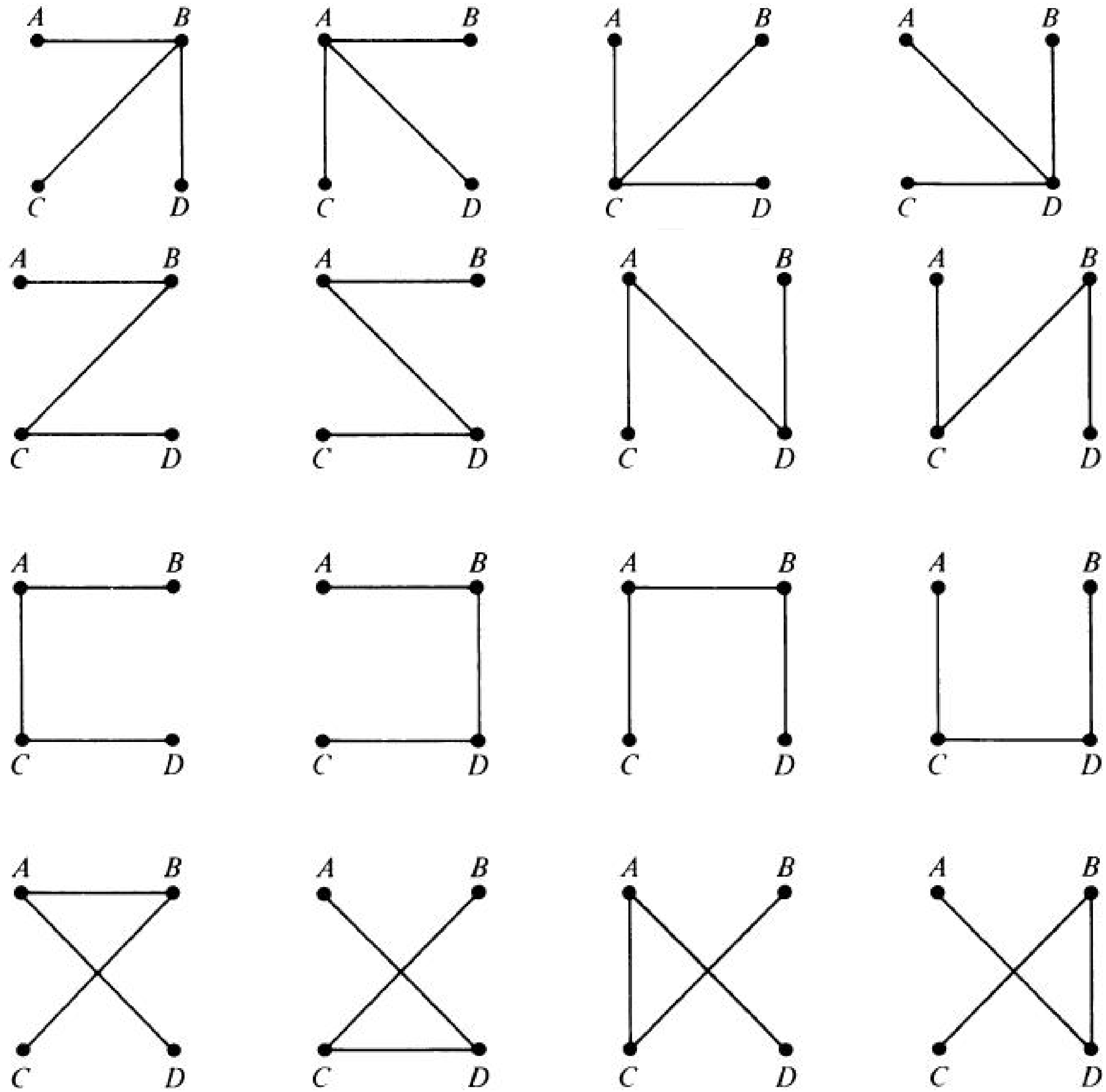


Fig. 3-15 All 16 trees of four labeled vertices.

were no distinction made between A, B, C, or D, these four trees would be counted as one. A careful inspection of the graphs in Fig. 3-15 will reveal that the number of unlabeled trees with four vertices (no distinction made between A, B, C, and D) is only two. But first we shall continue with counting labeled trees.

The following well-known theorem for counting trees was first stated and proved by Cayley, and is therefore called Cayley's theorem.

THEOREM 3-10

The number of labeled trees with n vertices ($n \geq 2$) is n^{n-2}.

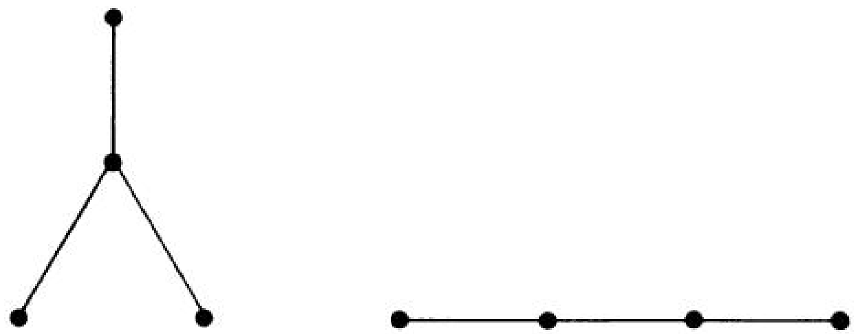
Proof: The result was first stated and proved by Cayley. Many different proofs with various approaches (all somewhat involved) have been published since. An excellent summary of 10 such proofs is given by Moon [3-9]. We will give one proof in Chapter 10.

Unlabeled Trees: In the actual counting of isomers of C_kH_{2k+2}, Theorem 3-10 is not enough. In addition to the constraints on the degree of the vertices, two observations should be made:

1. Since the vertices representing hydrogen are pendant, they go with carbon atoms only one way, and hence make no contribution to isomerism. Therefore, we need not show any hydrogen vertices.

2. Thus the tree representing C_kH_{2k+2} reduces to one with k vertices, each representing a carbon atom. In this tree no distinction can be made between vertices, and therefore it is unlabeled.

Thus for butane, C_4H_{10}, there are only two distinct trees (Fig. 3-16). As every organic chemist knows, there are indeed exactly two different types of butanes: *n*-butane and isobutane. It may be noted in passing that the four trees in the first row of Fig. 3-15 are isomorphic to the one in Fig. 3-16(a); and the other 12 are isomorphic to Fig. 3-16(b).



(a) (b)

Fig. 3-16 All trees of four unlabeled vertices.

The problem of counting trees of different types will be taken up again and discussed more thoroughly in Chapter 10.

SPANNING TREES

So far we have discussed the tree and its properties when it occurs as a graph by itself. Now we shall study the tree as a subgraph of another graph. A given graph has numerous subgraphs—from e edges, 2^e distinct combinations are possible. Obviously, some of these subgraphs will be trees. Out of these trees we are particularly interested in certain types of trees, called *spanning trees*—as defined next.

A tree T is said to be a *spanning tree* of a connected graph G if T is a subgraph of G and T contains all vertices of G. For instance, the subgraph in heavy lines in Fig. 3-17 is a spanning tree of the graph shown.

Since the vertices of G are barely hanging together in a spanning tree, it is a sort of skeleton of the original graph G. This is why a spanning tree is sometimes referred to as a *skeleton* or *scaffolding* of G. Since spanning trees are the largest (with maximum number of edges) trees among all trees in G, it is also quite appropriate to call a spanning tree a *maximal tree subgraph* or *maximal tree* of G.

It is to be noted that a spanning tree is defined only for a connected graph, because a tree is always connected, and in a disconnected graph of n vertices we cannot find a connected subgraph with n vertices. Each component (which by definition is connected) of a disconnected graph, however, does have a spanning tree. Thus a disconnected graph with k components has a *spanning forest* consisting of k spanning trees. (A collection of trees is called a *forest*.)

Finding a spanning tree of a connected graph G is simple. If G has no circuit, it is its own spanning tree. If G has a circuit, delete an edge from the circuit. This will still leave the graph connected (Problem 2-10). If there are more circuits, repeat the operation till an edge from the last circuit is deleted—leaving a connected, circuit-free graph that contains all the vertices of G. Thus we have

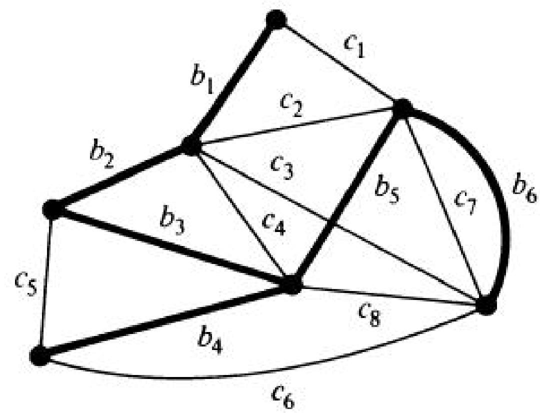


Fig. 3-17 Spanning tree.

THEOREM 3-11

Every connected graph has at least one spanning tree.

An edge in a spanning tree T is called a *branch* of T. An edge of G that is not in a given spanning tree T is called a *chord*. In electrical engineering a chord is sometimes referred to as a *tie* or a *link*. For instance, edges b_1, b_2, b_3, b_4, b_5, and b_6 are branches of the spanning tree shown in Fig. 3-17, while edges c_1, c_2, c_3, c_4, c_5, c_6, c_7, and c_8 are chords. It must be kept in mind that branches and chords are defined only with respect to a given spanning tree. An edge that is a branch of one spanning tree T_1 (in a graph G) may be a chord with respect to another spanning tree T_2.

It is sometimes convenient to consider a connected graph G as a union of two subgraphs, T and $\bar{T}$; that is,

$$T \cup \bar{T} = G,$$

where T is a spanning tree, and $\bar{T}$ is the complement of T in G. Since the subgraph $\bar{T}$ is the collection of chords, it is quite appropriately referred to as the *chord set* (or *tie set* or *cotree*) of T. From the definition, and from Theorem 3-3, the following theorem is evident.

THEOREM 3-12

With respect to any of its spanning trees, a connected graph of n vertices and e edges has $n - 1$ tree branches and $e - n + 1$ chords.

For example, the graph in Fig. 3-17 (with $n = 7$, $e = 14$), has six tree branches and eight chords with respect to the spanning tree $\{b_1, b_2, b_3, b_4, b_5, b_6\}$. Any other spanning tree will yield the same numbers.

If we have an electric network with e elements (edges) and n nodes (vertices), what is the minimum number of elements we must remove to eliminate all circuits in the network? The answer is $e - n + 1$. Similarly, if we have a farm consisting of six walled plots of land, as shown in Fig. 3-18, and these plots are full of water, how many walls will have to be broken so that all the water can be drained out? Here $n = 10$ and $e = 15$. We shall have to select

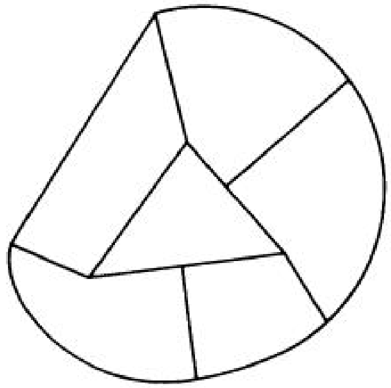


Fig. 3-18 Farm with walled plots of land.

a set of six $(15 - 10 + 1 = 6)$ walls such that the remaining nine constitute a spanning tree. Breaking these six walls will drain the water out.

Rank and Nullity: When someone specifies a graph G, the first thing he is most likely to mention is n, the number of vertices in G. Immediately following comes e, the number of edges in G. Then k, the number of components G has. If $k = 1$, G is connected. How are these three numbers of a graph related? Since every component of a graph must have at least one vertex, $n \geq k$. Moreoever, the number of edges in a component can be no less than the number of vertices in that component minus one. Therefore, $e \geq n - k$. Apart from the constraints $n - k \geq 0$ and $e - n + k \geq 0$, these three numbers n, e, and k are independent, and they are fundamental numbers in graphs. (Needless to mention, these numbers alone are not enough to specify a graph, except for trivial cases.)

From these three numbers are derived two other important numbers called *rank* and *nullity*, defined as

$$\text{rank} \qquad r = n - k,$$

$$\text{nullity} \qquad \mu = e - n + k.$$

The rank of a connected graph is $n - 1$, and the nullity, $e - n + 1$. Although the real significance of these numbers will be clear in Chapter 7, it may be observed here that

$$\text{rank of } G = \text{number of branches in any spanning tree (or forest) of } G,$$

$$\text{nullity of } G = \text{number of chords in } G,$$

$$\text{rank} + \text{nullity} = \text{number of edges in } G.$$

The nullity of a graph is also referred to as its *cyclomatic number*, or *first Betti number*.

FUNDAMENTAL CIRCUITS

THEOREM 3-13

A connected graph G is a tree if and only if adding an edge between any two vertices in G creates exactly one circuit.

Let us now consider a spanning tree T in a connected graph G. Adding any one chord to T will create exactly one circuit. Such a circuit, formed by adding a chord to a spanning tree, is called a *fundamental circuit*.

How many fundamental circuits does a graph have? Exactly as many as the number of chords, μ $(= e - n + k)$. How many circuits does a graph have in all? We know that one circuit is created by adding any one chord to a tree. Suppose that we add one more chord. Will it create exactly one more circuit? What happens if we add all the chords simultaneously to the tree?

Let us look at the tree $\{b_1, b_2, b_3, b_4, b_5, b_6\}$ in Fig. 3-17. Adding c_1 to it, we get a subgraph $\{b_1, b_2, b_3, b_4, b_5, b_6, c_1\}$, which has one circuit (fundamental circuit), $\{b_1, b_2, b_3, b_5, c_1\}$. Had we added the chord c_2 (instead of c_1) to the tree, we would have obtained a different fundamental circuit, $\{b_2, b_3, b_5, c_2\}$. Now suppose that we add both chords c_1 and c_2 to the tree. The subgraph $\{b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2\}$ has not only the fundamental circuits we just mentioned, but it has also a third circuit, $\{b_1, c_1, c_2\}$, which is not a fundamental circuit. Although there are 75 circuits in Fig. 3-17 (enumerated by computer), only eight are fundamental circuits, each formed by one chord (together with the tree branches).

Two comments may be appropriate here. First, a circuit is a fundamental circuit only with respect to a given spanning tree. A given circuit may be fundamental with respect to one spanning tree, but not with respect to a different spanning tree of the same graph. Although the number of fundamental circuits (as well as the total number of circuits) in a graph is fixed, the circuits that become fundamental change with the spanning trees.

Second, in most applications we are not interested in all the circuits of a graph, but only in a set of fundamental circuits, which fortuitously are a lot easier to track. The concept of a fundamental circuit, introduced by Kirchhoff, is of enormous significance in electrical network analysis. What Kirchhoff showed, which now every sophomore in electrical engineering knows, is that no matter how many circuits a network contains we need consider only fundamental circuits with respect to any spanning tree. The rest of the circuits (as we shall prove rigorously in Chapter 7) are combinations of some fundamental circuits.

UNIT-II

SYLLABUS

SPANNING TREES

Spanning Trees in a Weighted Graph – Cut Sets – Properties of Cut Set – All Cut Sets – Fundamental Circuits and Cut Sets – Connectivity and separability – Network flows – 1-Isomorphism – 2-Isomorphism – Combinational versus Geometric Graphs – Planer Graphs – Different Representation of a Planer Graph.

FINDING ALL SPANNING TREES OF A GRAPH

Usually, in a given connected graph there are a large number of spanning trees. In many applications we require all spanning trees. One reasonable way to generate spanning trees of a graph is to start with a given spanning tree, say tree T_1 ($a\ b\ c\ d$ in Fig. 3-19). Add a chord, say h, to the tree T_1. This forms a fundamental circuit ($b\ c\ h\ d$ in Fig. 3-19). Removal of any branch, say c, from the fundamental circuit $b\ c\ h\ d$ just formed will create a new

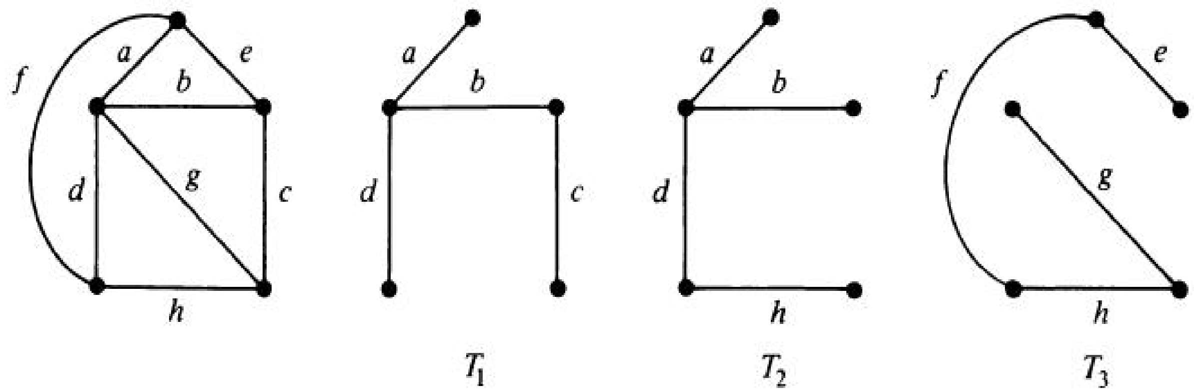


Fig. 3-19 Graph and three of its spanning trees.

spanning tree T_2. This generation of one spanning tree from another, through addition of a chord and deletion of an appropriate branch, is called a *cyclic interchange* or *elementary tree transformation*. (Such a transformation is a standard operation in the iteration sequence for solving certain transportation problems.)

In the above procedure, instead of deleting branch c, we could have deleted d or b and thus would have obtained two additional spanning trees $a\ b\ c\ h$ and $a\ c\ h\ d$. Moreover, after generating these three trees, each with chord h in it, we can restart with T_1 and add a different chord ($e, f,$ or g) and repeat the process of obtaining a different spanning tree each time a branch is deleted from the fundamental circuit formed. Thus we have a procedure for generating spanning trees for any given graph.

As we shall see in Chapter 13, the topological analysis of a linear electrical network essentially reduces to the generation of trees in the corresponding graph. Therefore, finding an efficient procedure for generating all trees of a graph is a very important practical problem.

The procedure outlined above raises many questions. Can we start from any spanning tree and get a desired spanning tree by a number of *cyclic exchanges*? Can we get all spanning trees of a given graph in this fashion? How long will we have to continue exchanging edges? Out of all possible spanning trees that we can start with, is there a preferred one for starting? Let us try to answer some of these questions; others will have to wait until Chapters 7, 10, and 11.

The distance between two spanning trees T_i and T_j of a graph G is defined as the number of edges of G present in one tree but not in the other. This distance may be written as $d(T_i, T_j)$. For instance, in Fig. 3-19 $d(T_2, T_3) = 3$.

Let $T_i \oplus T_j$ be the ring sum of two spanning trees T_i and T_j of G (as defined in Chapter 2, $T_i \oplus T_j$ is the subgraph of G containing all edges of G that are either in T_i or in T_j but not in both). Let $N(g)$ denote the number of edges in a graph g. Then, from definition,

$$d(T_i, T_j) = \frac{1}{2} N(T_i \oplus T_j).$$

It is not difficult to see that the number $d(T_i, T_j)$ is the minimum number of cyclic interchanges involved in going from T_i to T_j. The reader is encouraged to prove the following two theorems.

THEOREM 3-14

The distance between the spanning trees of a graph is a *metric*. That is, it satisfies

$$d(T_i, T_j) \geq 0 \quad \text{and} \quad d(T_i, T_j) = 0 \text{ if and only if } T_i = T_j,$$
$$d(T_i, T_j) = d(T_j, T_i),$$
$$d(T_i, T_j) \leq d(T_i, T_k) + d(T_k, T_j).$$

THEOREM 3-15

Starting from any spanning tree of a graph G, we can obtain every spanning tree of G by successive cyclic exchanges.

Since in a connected graph G of rank r (i.e., of $r + 1$ vertices) a spanning tree has r edges, we have the following results:

The maximum distance between any two spanning trees in G is

$$\max d(T_i, T_j) = \frac{1}{2} \max N(T_i \oplus T_j)$$
$$\leq r, \text{ the rank of } G.$$

Also, if μ is the nullity of G, we know that no more than μ edges of a spanning tree T_i can be replaced to get another tree T_j.

Hence $$\max d(T_i, T_j) \leq \mu;$$

combining the two,

$$\max d(T_i, T_j) \leq \min(\mu, r),$$

where $\min(\mu, r)$ is the smaller of the two numbers μ and r of the graph G.

Central Tree: For a spanning tree T_0 of a graph G, let $\max_i d(T_0, T_i)$ denote the maximal distance between T_0 and any other spanning tree of G. Then T_0 is called a *central tree* of G if

$$\max_i d(T_0, T_i) \leq \max_j d(T, T_j) \qquad \text{for every tree } T \text{ of } G.$$

The concept of a central tree is useful in enumerating all trees of a given graph. A central tree in a graph is, in general, not unique. For more on central trees the reader should see [3-1] and [3-4].

Tree Graph: The *tree graph* of a given graph G is defined as a graph in which each vertex corresponds to a spanning tree of G, and each edge corresponds to a *cyclic interchange* between the spanning trees of G represented by the two end vertices of the edge. From Theorem 3-15 we know that starting from any spanning tree we can obtain all other spanning trees through cyclic interchanges (or elementary tree transformations). Therefore, the tree graph of any given (finite, connected) graph is connected. For additional properties of tree graphs, the reader should see [3-3].

SPANNING TREES IN A WEIGHTED GRAPH

As discussed earlier in this chapter, a spanning tree in a graph G is a minimal subgraph connecting all the vertices of G. If graph G is a weighted graph (i.e., if there is a real number associated with each edge of G), then the *weight of a spanning tree T* of G is defined as the sum of the weights of all the branches in T. In general, different spanning trees of G will have different weights. Among all the spanning trees of G, one with the *smallest* weight is of practical significance. (There may be several spanning trees with the smallest weight; for instance, in a graph of n vertices in which every edge has unit weight, all spanning trees have a weight of $n - 1$ units.) A spanning tree with the smallest weight in a weighted graph is called a *shortest spanning tree* or *shortest-distance spanning tree* or *minimal spanning tree*.

One possible application of the shortest spanning tree is as follows: Suppose that we are to connect n cities $v_1, v_2, \ldots, v_n$ through a network of roads. The cost c_{ij} of building a direct road between v_i and v_j is given for all pairs of cities where roads can be built. (There may be pairs of cities between which no direct road can be built.) The problem is then to find the least expensive network that connects all n cities together. It is immediately evident that this connected network must be a tree: otherwise, we can always remove some edges and get a connected graph with smaller weight. Thus the problem of connecting n cities with a least expensive network is the problem of finding a shortest spanning tree in a connected weighted graph of n vertices. A necessary and sufficient condition for a spanning tree to be shortest is given by

THEOREM 3-16

A spanning tree T (of a given weighted connected graph G) is a shortest spanning tree (of G) if and only if there exists no other spanning tree (of G) at a distance of one from T whose weight is smaller than that of T.

Proof: The necessary or the "only if" condition is obvious; otherwise, we shall get another tree shorter than T by a cyclic interchange. The fact that this condition is also sufficient is remarkable and is not obvious. It can be proved as follows:

Let T_1 be a spanning tree in G satisfying the hypothesis of the theorem (i.e., there is no spanning tree at a distance of one from T_1 which is shorter than T_1).

The proof will be completed by showing that if T_2 is a shortest spanning tree (different from T_1) in G, the weight of T_1 will also be equal to that of T_2. Let T_2 be a shortest spanning tree in G. Clearly, T_2 must also satisfy the hypothesis of the theorem (otherwise there will be a spanning tree shorter than T_2 at a distance of one from T_2, violating the assumption that T_2 is shortest).

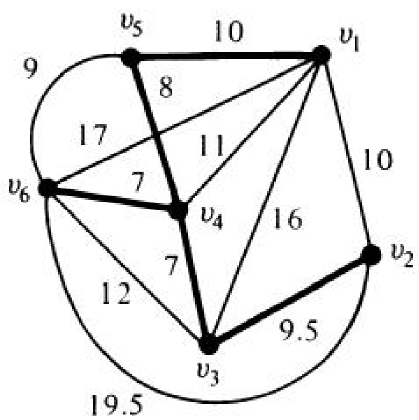
Consider an edge e in T_2 which is not in T_1. Adding e to T_1 forms a fundamental circuit with branches in T_1. Some, but not all, of the branches in T_1 that form the fundamental circuit with e may also be in T_2; each of these branches in T_1 has a weight smaller than or equal to that of e, because of the assumption on T_1. Amongst all those edges in this circuit which are not in T_2 at least one, say b_j, must form some fundamental circuit (with respect to T_2) containing e. Because of the minimality assumption on T_2 weight of b_j cannot be less than that of e. Therefore b_j must have the same weight as e. Hence the spanning tree $T_1' = (T_1 \cup e - b_j)$, obtained from T_1 through one cycle exchange, has the same weight as T_1. But T_1 has one edge more in common with T_2, and it satisfies the condition of Theorem 3-16. This argument can be repeated, producing a series of trees of equal weight, $T_1, T_1, T_1, \ldots$, each a unit distance closer to T_2, until we get T_2 itself.

This proves that if none of the spanning trees at a unit distance from T is shorter than T, no spanning tree shorter than T exists in the graph. ■

Algorithm for Shortest Spanning Tree: There are several methods available for actually finding a shortest spanning tree in a given graph, both by hand and by computer. One algorithm due to Kruskal [3-8] is as follows: List all edges of the graph G in order of nondecreasing weight. Next, select a smallest edge of G. Then for each successive step select (from all remaining edges of G) another smallest edge that makes no circuit with the previously selected edges. Continue until $n - 1$ edges have been selected, and these edges will constitute the desired shortest spanning tree. The validity of the method follows from Theorem 3-16.

Another algorithm, which does not require listing all edges in order of nondecreasing weight or checking at each step if a newly selected edge forms a circuit, is due to Prim [3-10]. For Prim's algorithm, draw n isolated vertices and label them $v_1, v_2, \ldots, v_n$. Tabulate the given weights of the edges of G in an n by n table. (Note that the entries in the table are symmetric with respect to the diagonal, and the diagonal is empty.) Set the weights of nonexistent edges (corresponding to those pairs of cities between which no direct road can be built) as very large.

Start from vertex v_1 and connect it to its nearest neighbor (i.e., to the vertex which has the smallest entry in row 1 of the table), say v_k. Now consider v_1 and v_k as one subgraph, and connect this subgraph to its closest neighbor (i.e., to a vertex other than v_1 and v_k that has the smallest entry among all entries in rows 1 and k). Let this new vertex be v_i. Next regard the tree with vertices v_1, v_k, and v_i as one subgraph, and continue the process until all n vertices have been connected by $n - 1$ edges. Let us now illustrate this method of finding a shortest spanning tree.



(a)

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\left[\begin{array}{cccccc}
- & 10 & 16 & 11 & 10 & 17 \\
10 & - & 9.5 & \infty & \infty & 19.5 \\
16 & 9\ 5 & - & 7 & \infty & 12 \\
11 & \infty & 7 & - & 8 & 7 \\
10 & \infty & \infty & 8 & - & 9 \\
17 & 19.5 & 12 & 7 & 9 & -
\end{array}\right]
\end{array}$$

(b)

Fig. 3-20 Shortest spanning tree in a weighted graph.

A connected weighted graph with 6 vertices and 12 edges is shown in Fig. 3-20(a). The weight of its edges is tabulated in Fig. 3-20(b). We start with v_1 and pick the smallest entry in row 1, which is either (v_1, v_2) or (v_1, v_5). Let us pick (v_1, v_5). [Had we picked (v_1, v_2) we would have obtained a different shortest tree with the same weight.] The closest neighbor of subgraph (v_1, v_5) is v_4, as can be seen by examining all the entries in rows 1 and 5. The three remaining edges selected following the above procedure turn out to be (v_4, v_6), (v_4, v_3), and (v_3, v_2) in that sequence. The resulting tree—a shortest spanning tree—is shown in Fig. 3-20(a) in heavy lines. The weight of this tree is 41.5 units.

Degree-Constrained Shortest Spanning Tree: In a shortest spanning tree resulting from the preceding construction, a vertex v_i can end up with any degree; that is, $1 \leq d(v_i) \leq n - 1$. In some practical cases an upper limit on the degree of every vertex (of the resulting spanning tree) has to be imposed.

For instance, in an electrical wiring problem, one may be required to wire together n pins (using as little wire as possible) with no more than three wires wrapped around any individual pin. Thus, in this particular case,

$$d(v_i) \leq 3 \qquad \text{for every } v_i.$$

Such a spanning tree is called a *degree-constrained shortest spanning tree*.

In general, the problem may be stated as follows: Given a weighted connected graph G, find a shortest spanning tree T in G such that

$$d(v_i) \leq k \qquad \text{for every vertex } v_i \text{ in } T.$$

If $k = 2$, this problem, in fact, reduces to the problem of finding the shortest Hamiltonian path, as well as the traveling-salesman problem (without the salesman returning to his home base), discussed at the end of Chapter 2. So far, no efficient method of finding an arbitrarily degree-constrained shortest spanning tree has been found.

CUT-SETS

In a connected graph G, a *cut-set* is a set of edges† whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G. For instance, in Fig. 4-1 the set of edges $\{a, c, d, f\}$ is a cut-set. There are many other cut-sets, such as $\{a, b, g\}$, $\{a, b, e, f\}$, and $\{d, h, f\}$. Edge $\{k\}$ alone is also a cut-set. The set of edges $\{a, c, h, d\}$, on the other hand, is *not* a cut-set, because one of its proper subsets, $\{a, c, h\}$, is a cut-set.

To emphasize the fact that no proper subset of a cut-set can be a cut-set, some authors refer to a cut-set as a *minimal cut-set*, a *proper cut-set*, or a *simple cut-set*. Sometimes a cut-set is also called a *cocycle*. We shall just use the term *cut-set*.

A cut-set always "cuts" a graph into two. Therefore, a cut-set can also be defined as a minimal set of edges in a connected graph whose removal reduces the rank of the graph by one. The rank of the graph in Fig. 4.1(b), for in-

†Since a set of edges (together with their end vertices) constitutes a subgraph, a cut-set in G is a subgraph of G.

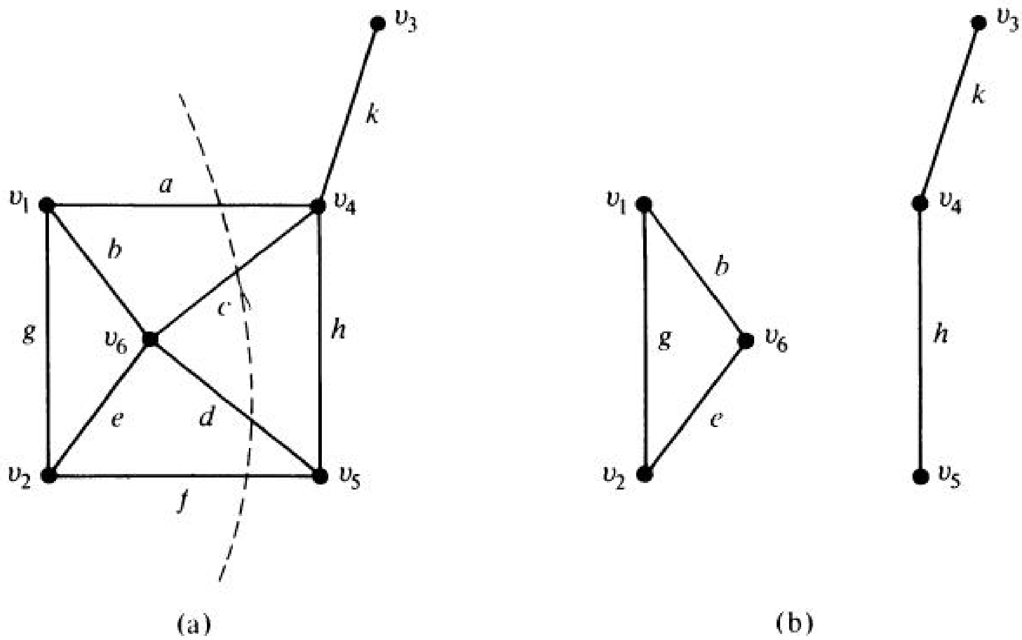


Fig. 4-1 Removal of a cut-set $\{a, c, d, f\}$ from a graph "cuts" it into two.

stance, is four, one less than that of the graph in Fig. 4.1(a). Another way of looking at a cut-set is this: if we partition all the vertices of a connected graph G into two mutually exclusive subsets, a cut-set is a minimal number of edges whose removal from G destroys all paths between these two sets of vertices. For example, in Fig. 4-1(a) cut-set $\{a, c, d, f\}$ connects vertex set $\{v_1, v_2, v_6\}$ with $\{v_3, v_4, v_5\}$. (Note that one or both of these two subsets of vertices may consist of just one vertex.) Since removal of any edge from a tree breaks the tree into two parts, *every edge of a tree is a cut-set*.

Cut-sets are of great importance in studying properties of communication and transportation networks. Suppose, for example, that the six vertices in Fig. 4-1(a) represent six cities connected by telephone lines (edges). We wish to find out if there are any weak spots in the network that need strengthening by means of additional telephone lines. We look at all cut-sets of the graph, and the one with the smallest number of edges is the most vulnerable. In Fig. 4-1(a), the city represented by vertex v_3 can be severed from the rest of the network by the destruction of just one edge. After some additional study of the properties of cut-sets, we shall return to their applications.

SOME PROPERTIES OF A CUT-SET

Consider a spanning tree T in a connected graph G and an arbitrary cut-set S in G. Is it possible for S not to have any edge in common with T? The answer is *no*. Otherwise, removal of the cut-set S from G would not disconnect the graph. Therefore,

THEOREM 4-1

Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.

Will the converse also be true? In other words, will any minimal set of edges containing at least one branch of every spanning tree be a cut-set? The answer is *yes*, by the following reasoning:

In a given connected graph G, let Q be a minimal set of edges containing at least one branch of every spanning tree of G. Consider $G - Q$, the subgraph that remains after removing the edges in Q from G. Since the subgraph $G - Q$ contains no spanning tree of G, $G - Q$ is disconnected (one component of which may just consist of an isolated vertex). Also, since Q is a minimal set of edges with this property, any edge e from Q returned to $G - Q$ will create at least one spanning tree. Thus the subgraph $G - Q + e$ will be a connected graph. Therefore, Q is a minimal set of edges whose removal from G disconnects G. This, by definition, is a cut-set. Hence

THEOREM 4-2

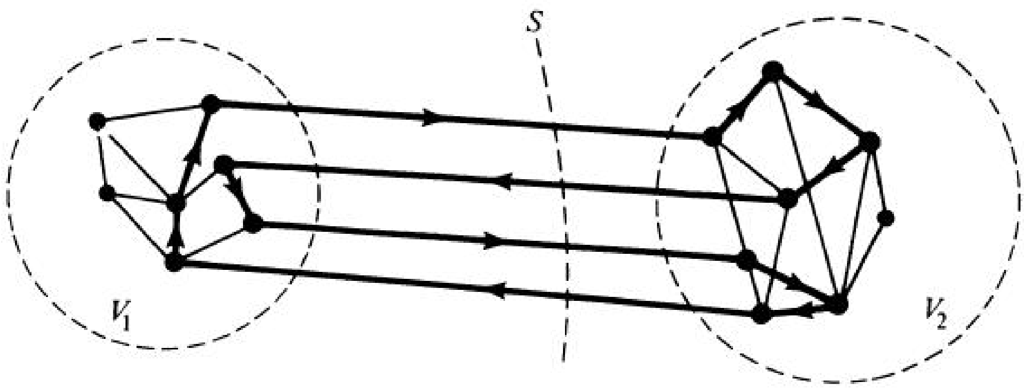
In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.

THEOREM 4-3

Every circuit has an even number of edges in common with any cut-set.

Proof: Consider a cut-set S in graph G (Fig. 4-2). Let the removal of S partition the vertices of G into two (mutually exclusive or disjoint) subsets V_1 and V_2. Consider a circuit Γ in G. If all the vertices in Γ are entirely within vertex set V_1 (or V_2), the number of edges common to S and Γ is zero; that is, $N(S \cap \Gamma) = 0$, an even number.†

If, on the other hand, some vertices in Γ are in V_1 and some in V_2, we traverse



Circuit Γ shown in heavy lines, and is traversed along the direction of the arrows

Fig. 4-2 Circuit and a cut-set in G.

back and forth between the sets V_1 and V_2 as we traverse the circuit (see Fig. 4-2). Because of the closed nature of a circuit, the number of edges we traverse between V_1 and V_2 must be even. And since very edge in S has one end in V_1 and the other in V_2, and no other edge in G has this property (of separating sets V_1 and V_2), the number of edges common to S and Γ is even. ■

ALL CUT-SETS IN A GRAPH

Fundamental Cut-Sets: Consider a spanning tree T of a connected graph G. Take any branch b in T. Since $\{b\}$ is a cut-set in T, $\{b\}$ partitions all vertices of T into two disjoint sets—one at each end of b. Consider the same partition of vertices in G, and the cut set S in G that corresponds to this partition. Cut-set S will contain only one branch b of T, and the rest (if any) of the edges in S are chords with respect to T. Such a cut-set S containing exactly one branch of a tree T is called a *fundamental cut-set* with respect to T. Sometimes a fundamental cut-set is also called a *basic cut-set*. In Fig. 4-3, a spanning tree

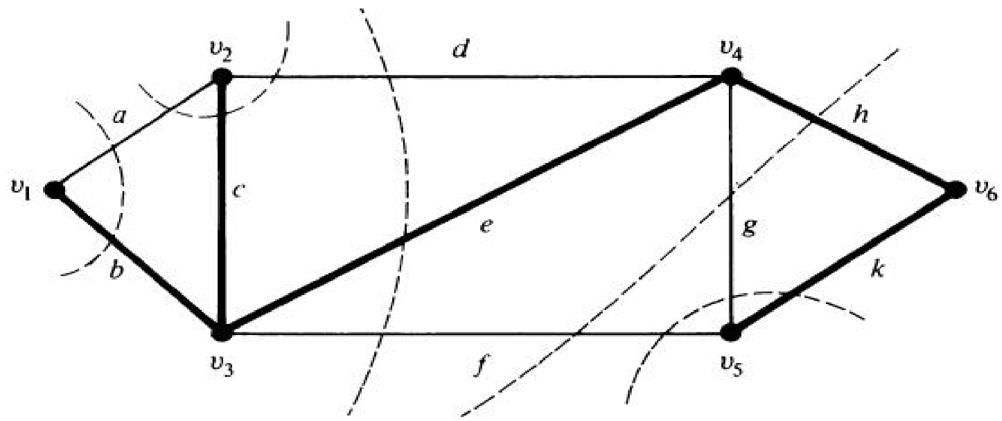


Fig. 4-3 Fundamental cut-sets of a graph.

T (in heavy lines) and all five of the fundamental cut-sets with respect to T are shown (broken lines "cutting" through each cut-set).

Just as every chord of a spanning tree defines a *unique* fundamental circuit, every branch of a spanning tree defines a *unique* fundamental cut-set. It must also be kept in mind that the term fundamental cut-set (like the term fundamental circuit) has meaning only with respect to a *given* spanning tree.

Now we shall show how other cut-sets of a graph can be obtained from a given set of cut-sets.

THEOREM 4-4

The ring sum of any two cut-sets in a graph is either a third cut-set or an edge-disjoint union of cut-sets.

Outline of Proof: Let S_1 and S_2 be two cut-sets in a given connected graph G. Let V_1 and V_2 be the (unique and disjoint) partitioning of the vertex set V of G corresponding to S_1. Let V_3 and V_4 be the partitioning corresponding to S_2. Clearly [see Figs. 4-4(a) and (b)],

$$V_1 \cup V_2 = V \quad \text{and} \quad V_1 \cap V_2 = \varnothing,$$
$$V_3 \cup V_4 = V \quad \text{and} \quad V_3 \cap V_4 = \varnothing.$$

Now let the subset $(V_1 \cap V_4) \cup (V_2 \cap V_3)$ be called V_5, and this by definition is the same as the ring sum $V_1 \oplus V_3$. Similarly, let the subset $(V_1 \cap V_3) \cup (V_2 \cap V_4)$ be called V_6, which is the same as $V_2 \oplus V_3$. See Fig. 4-4(c).

The ring sum of the two cut-sets $S_1 \oplus S_2$ can be seen to consist only of edges that join vertices in V_5 to those in V_6. Also, there are no edges outside $S_1 \oplus S_2$ that join vertices in V_5 to those in V_6.

Thus the set of edges $S_1 \oplus S_2$ produces a partitioning of V into V_5 and V_6 such that

$$V_5 \cup V_6 = V \quad \text{and} \quad V_5 \cap V_6 = \varnothing .$$

Hence $S_1 \oplus S_2$ is a cut-set if the subgraphs containing V_5 and V_6 each remain connected after $S_1 \oplus S_2$ is removed from G. Otherwise, $S_1 \oplus S_2$ is an edge-disjoint union of cut-sets.

Example: In Fig. 4-3 let us consider ring sums of the following three pairs of cut-sets.

Example: In Fig. 4-3 let us consider ring sums of the following three pairs of cut-sets.

$$\{d, e, f\} \oplus \{f, g, h\} = \{d, e, g, h\}, \qquad \text{another cut-set,}$$

$$\{a, b\} \oplus \{b, c, e, f\} = \{a, c, e, f\}, \qquad \text{another cut-set,}$$

$$\{d, e, g, h\} \oplus \{f, g, k\} = \{d, e, f, h, k\}$$

$$= \{d, e, f\} \cup \{h, k\}, \text{ an edge-disjoint union of cut-sets.} \quad \blacksquare$$

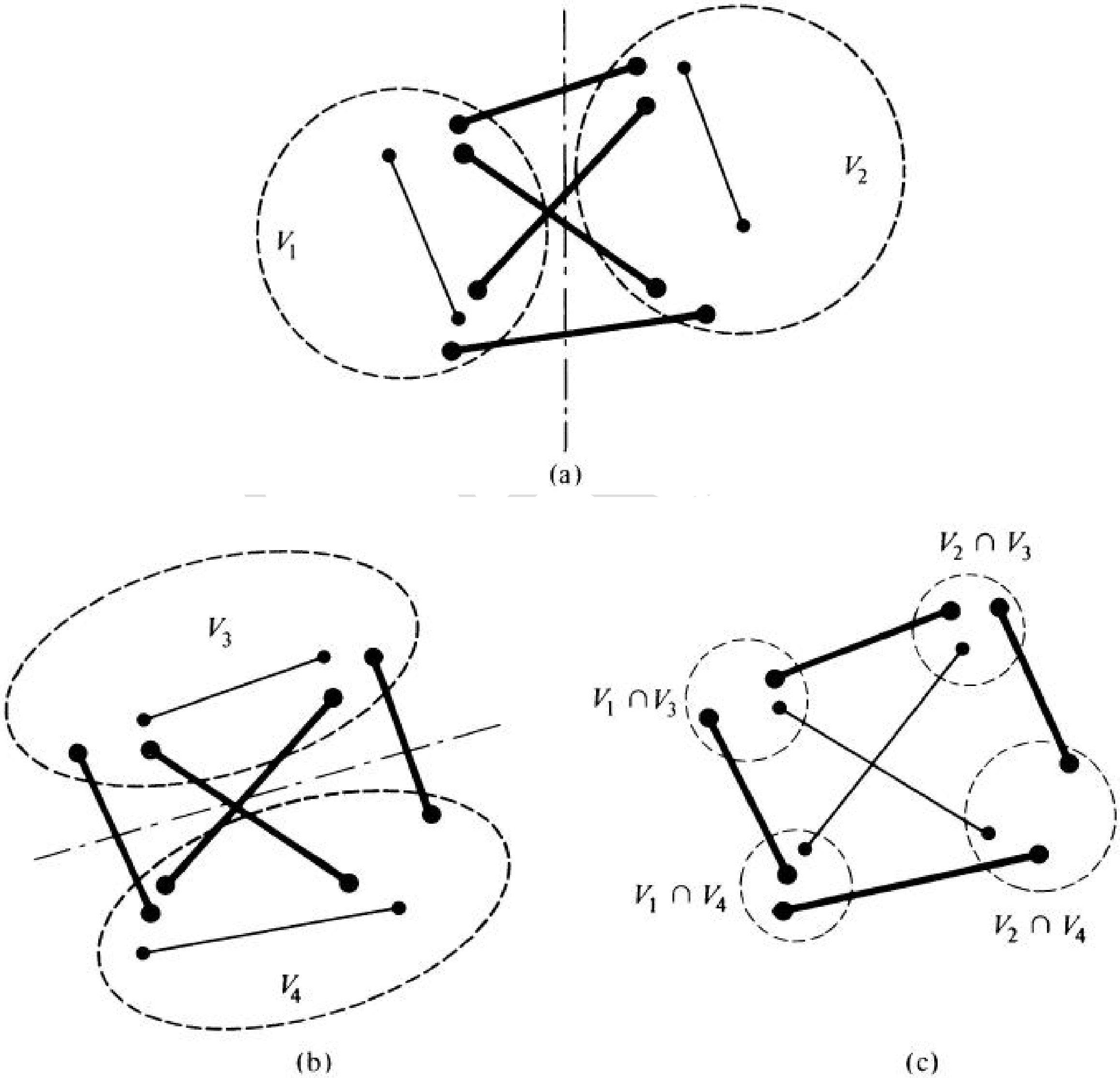


Fig. 4-4 Two cut-sets and their partitionings.

So we have a method of generating additional cut-sets from a number of given cut-sets. Obviously, we cannot start with any two cut-sets in a given graph and hope to obtain all its cut-sets by this method. What then is a minimal set of cut-sets from which we can obtain every cut-set of G by taking ring sums? The answer (to be proved in Chapter 6) is the set of all fundamental cut-sets with respect to a given spanning tree.

FUNDAMENTAL CIRCUITS AND CUT-SETS

Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T, and let the fundamental circuit made by c_i be called Γ, consisting of k branches $b_1, b_2, \ldots, b_k$ in addition to the chord c_i; that is,

$\Gamma = \{c_i, b_1, b_2, \ldots, b_k\}$ is a fundamental circuit with respect to T.

Every branch of any spanning tree has a fundamental cut-set associated with it. Let S_1 be the fundamental cut-set associated with b_1, consisting of q chords in addition to the branch b_1; that is,

$S_1 = \{b_1, c_1, c_2, \ldots, c_q\}$ is a fundamental cut-set with respect to T.

Because of Theorem 4-3, there must be an even number of edges common to Γ and S_1. Edge b_1 is in both Γ and S_1, and there is only one other edge in Γ (which is c_i) that can possibly also be in S_1. Therefore, we must have two edges b_1 and c_i common to S_1 and Γ. Thus the chord c_i is one of the chords $c_1, c_2, \ldots, c_q$.

Exactly the same argument holds for fundamental cut-sets associated with $b_2, b_3, \ldots$, and b_k. Therefore, the chord c_i is contained in every fundamental cut-set associated with branches in Γ.

Is it possible for the chord c_i to be in any other fundamental cut-set S' (with respect to T, of course) besides those associated with $b_1, b_2, \ldots$ and b_k? The answer is *no*. Otherwise (since none of the branches in Γ are in S'), there would be only one edge c_i common to S' and Γ, a contradiction to Theorem 4-3. Thus we have an important result.

THEOREM 4-5

With respect to a given spanning tree T, a chord c_i that determines a fundamental circuit Γ occurs in every fundamental cut-set associated with the branches in Γ and in no other.

As an example, consider the spanning tree $\{b, c, e, h, k\}$, shown in heavy lines, in Fig. 4-3. The fundamental circuit made by chord f is

$$\{f, e, h, k\}.$$

The three fundamental cut-sets determined by the three branches e, h, and k are

determined by branch e: $\{d, e, f\}$,

determined by branch h: $\{f, g, h\}$,

determined by branch k: $\{f, g, k\}$.

Chord f occurs in each of these three fundamental cut-sets, and there is no other fundamental cut-set that contains f. The converse of Theorem 4-5 is also true.

THEOREM 4-6

With respect to a given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S, and in no others.

As an example, consider the spanning tree $\{b, c, e, h, k\}$, shown in heavy lines, in Fig. 4-3. The fundamental circuit made by chord f is

$$\{f, e, h, k\}.$$

The three fundamental cut-sets determined by the three branches e, h, and k are

determined by branch e: $\{d, e, f\}$,

determined by branch h: $\{f, g, h\}$,

determined by branch k: $\{f, g, k\}$.

Chord f occurs in each of these three fundamental cut-sets, and there is no other fundamental cut-set that contains f. The converse of Theorem 4-5 is also true.

THEOREM 4-6

With respect to a given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S, and in no others.

Proof: The proof consists of arguments similar to those that led to Theorem 4-5. Let the fundamental cut-set S determined by a branch b_i be

$$S = \{b_i, c_1, c_2, \ldots, c_p\},$$

and let Γ_1 be the fundamental circuit determined by chord c_1:

$$\Gamma_1 = \{c_1, b_1, b_2, \ldots, b_q\}.$$

Since the number of edges common to S and Γ_1 must be even, b_i must be in Γ_1. The same is true for the fundamental circuits made by chords $c_2, c_3, \ldots, c_p$.

On the other hand, suppose that b_i occurs in a fundamental circuit Γ_{p+1} made by a chord other than $c_1, c_2, \ldots, c_p$. Since none of the chords $c_1, c_2, \ldots, c_p$ is in Γ_{p+1}, there is only one edge b_i common to a circuit Γ_{p+1} and the cut-set S, which is not possible. Hence the theorem. ■

Turning again for illustration to the graph in Fig. 4-3, consider branch e of spanning tree $\{b, c, e, h, k\}$. The fundamental cut-set determined by e is

$$\{e, d, f\}.$$

The two fundamental circuits determined by chords d and f are

determined by chord d: $\{d, c, e\}$,

determined by chord f: $\{f, e, h, k\}$.

Branch e is contained in both these fundamental circuits, and none of the remaining three fundamental circuits contains branch e.

CONNECTIVITY AND SEPARABILITY

Edge Connectivity: Each cut-set of a connected graph G consists of a certain number of edges. The number of edges in the smallest cut-set (i.e., cut-set with fewest number of edges) is defined as the *edge connectivity* of G. Equivalently, the edge connectivity of a connected graph† can be defined as the minimum number of edges whose removal (i.e., deletion) reduces the rank of the graph by one. The edge connectivity of a tree, for instance, is one. The edge connectivities of the graphs in Figs. 4-1(a), 4-3, 4-5 are one, two, and three, respectively.

Vertex Connectivity: On examining the graph in Fig. 4-5, we find that although removal of no single edge (or even a pair of edges) disconnects the

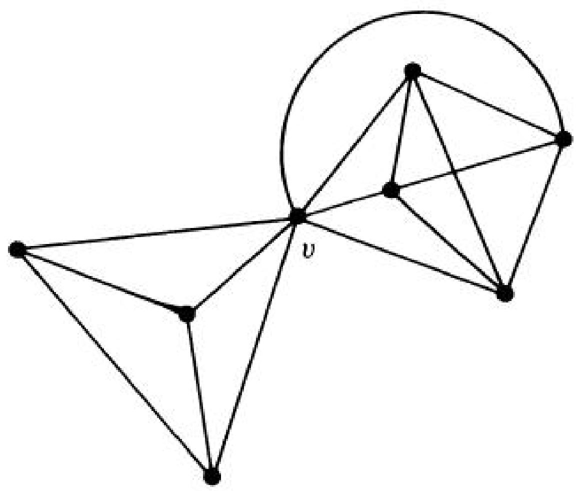


Fig. 4-5 Separable graph.

graph, the removal of the single vertex v does.† Therefore, we define another analogous term called *vertex connectivity*. The *vertex connectivity* (or simply *connectivity*) of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.‡ Again, the vertex connectivity of a tree is one. The vertex connectivities of the graphs in Figs. 4-1(a), 4-3, and 4-5 are one, two, and one, respectively. Note that from the way we have defined it vertex connectivity is meaningful only for graphs that have three or more vertices and are not complete.

Separable Graph: A connected graph is said to be *separable* if its vertex connectivity is one. All other connected graphs are called *nonseparable*. An equivalent definition is that a connected graph G is said to be separable if there exists a subgraph g in G such that $\bar{g}$ (the complement of g in G) and g have only one vertex in common. That these two definitions are equivalent can be easily seen (Problem 4-7). In a separable graph a vertex whose removal disconnects the graph is called a *cut-vertex*, a *cut-node*, or an *articulation point*.

For example, in Fig. 4-5 the vertex v is a cut-vertex, and in Fig. 4-1(a) vertex v_4 is a cut-vertex. It can be shown (Problem 4-18) that in a tree every vertex with degree greater than one is a cut-vertex. Moreover:

THEOREM 4-7

A vertex v in a connected graph G is a cut-vertex if and only if there exist two vertices x and y in G such that every path between x and y passes through v.

The proof of the theorem is quite easy and is left as an exercise (Problem 4-17). The implication of the theorem is very significant. It states that v is a crucial vertex in the sense that any communication between x and y (if G represented a communication network) must "pass through" v.

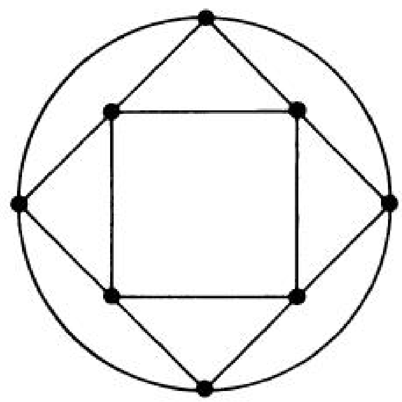


Fig. 4-6 Graph with 8 vertices and 16 edges.

An Application: Suppose we are given n stations that are to be connected by means of e lines (telephone lines, bridges, railroads, tunnels, or highways) where $e \geq n - 1$. What is the best way of connecting? By "best" we mean that the network should be as invulnerable to destruction of individual stations and individual lines as possible. In other words, construct a graph with n vertices and e edges that has the maximum possible edge connectivity and vertex connectivity.

For example, the graph in Fig. 4-5 has $n = 8$, $e = 16$, and has vertex connectivity of one and edge connectivity of three. Another graph with the same number of vertices and edges (8 and 16, respectively) can be drawn as shown in Fig. 4-6.

It can easily be seen that the edge connectivity as well as the vertex connectivity of this graph is four. Consequently, even after any three stations are bombed, or any three lines destroyed, the remaining stations can still continue to "communicate" with each other. Thus the network of Fig. 4-6 is better connected than that of Fig. 4-5 (although both consist of the same number of lines—16).

THEOREM 4-8

The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Proof: Let vertex v_i be the vertex with the smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from G by removing the $d(v_i)$ edges incident on vertex v_i. Hence the theorem. ■

THEOREM 4-9

The vertex connectivity of any graph G can never exceed the edge connectivity of G.

Proof: Let α denote the edge connectivity of G. Therefore, there exists a cut-set S in G with α edges. Let S partition the vertices of G into subsets V_1 and V_2. By removing at most α vertices from V_1 (or V_2) on which the edges in S are incident, we can effect the removal of S (together with all other edges incident on these vertices) from G. Hence the theorem. ■

COROLLARY

Every cut-set in a nonseparable graph with more than two vertices contains at least two edges.

THEOREM 4-10

The maximum vertex connectivity one can achieve with a graph G of n vertices and e edges ($e \geq n - 1$) is the integral part of the number $2e/n$; that is, $\lfloor 2e/n \rfloor$.

Proof: Every edge in G contributes two degrees. The total ($2e$ degrees) is divided among n vertices. Therefore, there must be at least one vertex in G whose degree is equal to or less than the number $2e/n$. The vertex connectivity of G cannot exceed this number, in light of Theorems 4-8 and 4-9.

To show that this value can actually be achieved, one can first construct an n-vertex regular graph of degree $\lfloor 2e/n \rfloor$ and then add the remaining $e - (n/2) \cdot \lfloor 2e/n \rfloor$ edges arbitrarily. The completion of the proof is left as an exercise. ■

The results of Theorems 4-8, 4-9, and 4-10 can be summarized as follows:

$$\text{vertex connectivity} \leq \text{edge connectivity} \leq \frac{2e}{n},$$

and

$$\text{maximum vertex connectivity possible} = \left\lfloor \frac{2e}{n} \right\rfloor.$$

Thus, for a graph with 8 vertices and 16 edges (Figs. 4-5 and 4-6), for example, we can achieve a vertex connectivity (and therefore edge connectivity) as high as four ($= 2 \cdot 16/8$).

A graph G is said to be *k-connected* if the vertex connectivity of G is k; therefore, a *1-connected* graph is the same as a separable graph.

THEOREM 4-11

A connected graph G is k-connected if and only if every pair of vertices in G is joined by k or more paths that do not intersect,† and at least one pair of vertices is joined by exactly k nonintersecting paths.

THEOREM 4-12

The edge connectivity of a graph G is k if and only if every pair of vertices in G is joined by k or more edge-disjoint paths (i.e., paths that may intersect, but have no edges in common), and at least one pair of vertices is joined by exactly k edge-disjoint paths.

The reader is referred to Chapter 5 of [1-5] for the proofs of Theorems 4-11 and 4-12. Note that our definition of k-connectedness is slightly different from the one given in [1-5]. A special result of Theorem 4-11 is that a graph G is nonseparable if and only if any pair of vertices in G can be placed in a circuit (Problem 4-13).

The reader is encouraged to verify these theorems by enumerating all edge-disjoint and vertex-disjoint paths between each of the 15 pairs of vertices in Fig. 4-3.

NETWORK FLOWS

In a network of telephone lines, highways, railroads, pipelines of oil (or gas or water), and so on, it is important to know the maximum rate of flow that is possible from one station to another in the network. This type of network is represented by a weighted connected graph in which the vertices are the stations and the edges are lines through which the given commodity (oil, gas, water, number of messages, number of cars, etc.) flows. The weight, a real positive number, associated with each edge represents the capacity of the line, that is, the maximum amount of flow possible per unit of time. The graph in Fig. 4-7, for example, represents a flow network consisting of 12 stations and 31 lines. The capacity of each of these lines is also indicated in the figure.

It is assumed that at each intermediate vertex the total rate of commodity entering is equal to the rate leaving. In other words, there is no accumulation or generation of the commodity at any vertex along the way. Furthermore, the flow through a vertex is limited only by the capacities of the edges incident on it. In other words, the vertex itself can handle as much flow as allowed through the edges. Finally, the lines are lossless.

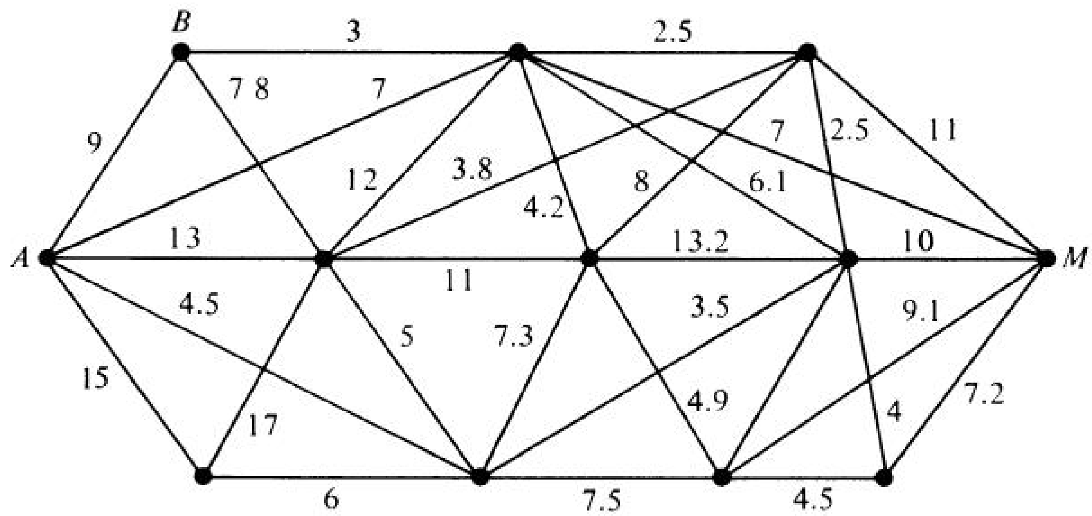


Fig. 4-7 Graph of a flow network.

In such a flow problem the questions to be answered are

1. What is the maximum flow possible through the network between a specified pair of vertices—say, from B to M in Fig. 4-7?

2. How do we achieve this flow (i.e., determine the actual flow through each edge when the maximum flow exists)?

Theorem 4-13, perhaps the most important result in the theory of transport networks, answers the first question. The second question is answered implicitly by a constructive proof of the theorem. To facilitate the statement and proof of the theorem, let us define a few terms.

A *cut-set with respect to a pair of vertices* a and b in a connected graph G puts a and b into two different components (i.e., separates vertices a and b). For instance, in Fig. 4-3 cut-set $\{d, e, f\}$ is a cut-set with respect to v_1 and v_6. The set $\{f, g, h\}$ is also a cut-set with respect to v_1 and v_6. But the cut-set $\{f, g, h\}$ is *not* a cut-set with respect to v_1 and v_6. The *capacity of cut-set S* in a weighted connected graph G (in which the weight of each edge represents its flow capacity) is defined as the sum of the weights of all the edges in S.

THEOREM 4-13

The maximum flow possible between two vertices a and b in a network is equal to the minimum of the capacities of all cut-sets with respect to a and b.

Proof: Consider any cut-set S with respect to vertices a and b in G. In the subgraph $G - S$ (the subgraph left after removing S from G) there is no path between a and b. Therefore, every path in G between a and b must contain at least one edge of S. Thus every flow from a to b (or from b to a) must pass through one or more edges of S. Hence the total flow rate between these two vertices cannot exceed the capacity of S. Since this holds for all cut-sets with respect to a and b, the flow rate cannot exceed the minimum of their capacities. ■

1-ISOMORPHISM

A separable graph consists of two or more nonseparable subgraphs. Each of the largest nonseparable subgraphs is called a *block*. (Some authors use the term *component*, but to avoid confusion with components of a disconnected graph, we shall use the term block.) The graph in Fig. 4-5 has two blocks. The graph in Fig. 4-8 has five blocks (and three cut-vertices a, b, and c); each block

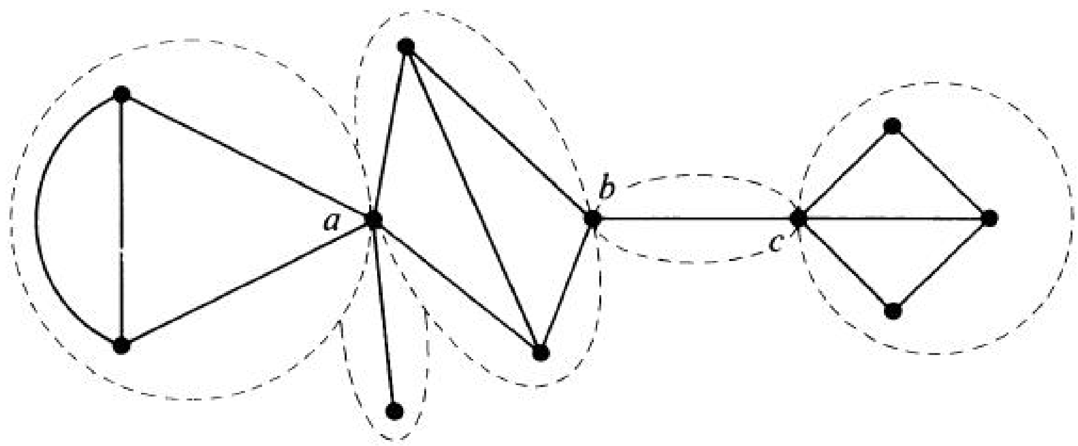


Fig. 4-8 Separable graph with three cut-vertices and five blocks.

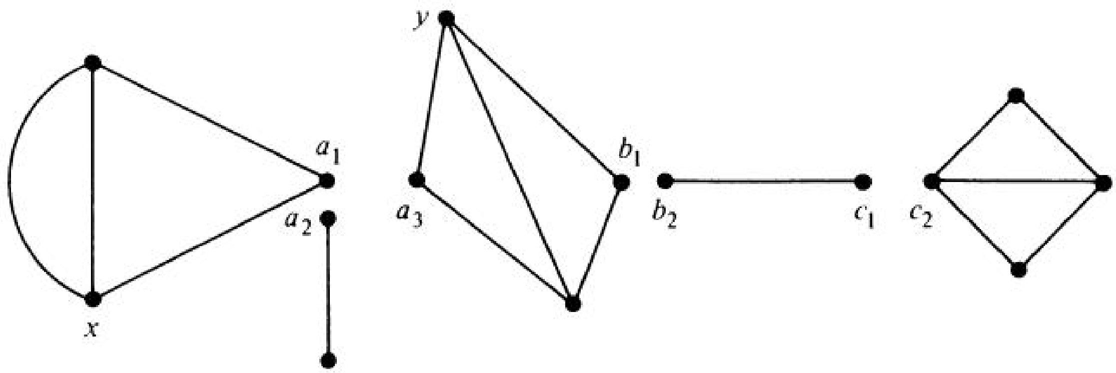


Fig. 4-9 Disconnected graph 1-isomorphic to Fig. 4-8.

is shown enclosed by a broken line. Note that a nonseparable connected graph consists of just one block.

Visually compare the disconnected graph in Fig. 4-9 with the one in Fig. 4-8. These two graphs are certainly not isomorphic (they do not have the same number of vertices), but they are related by the fact that the blocks of the graph in Fig. 4-8 are isomorphic to the components of the graph in Fig. 4-9. Such graphs are said to be *1-isomorphic*. More formally:

Two graphs G_1 and G_2 are said to be *1-isomorphic* if they become isomorphic to each other under repeated application of the following operation.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs.

From this definition it is apparent that two nonseparable graphs are 1-isomorphic if and only if they are isomorphic.

THEOREM 4-14

If G_1 and G_2 are two 1-isomorphic graphs, the rank of G_1 equals the rank of G_2 and the nullity of G_1 equals the nullity of G_2.

Proof: Under operation 1, whenever a cut-vertex in a graph G is "split" into two vertices, the number of components in G increases by one. Therefore, the rank of G which is

$$\text{number of vertices in } G - \text{number of components in } G$$

remains invariant under operation 1.

Also, since no edges are destroyed or new edges created by operation 1, two 1-isomorphic graphs have the same number of edges. Two graphs with equal rank and with equal numbers of edges must have the same nullity, because

$$\text{nullity} = \text{number of edges} - \text{rank}. \quad \blacksquare$$

What if we join two components of Fig. 4-9 by "gluing" together two vertices (say vertex x to y)? We obtain the graph shown in Fig. 4-10.

Clearly, the graph in Fig. 4-10 is 1-isomorphic to the graph in Fig. 4-9. Since the blocks of the graph in Fig. 4-10 are isomorphic to the blocks of the graph in Fig. 4-8, these two graphs are also 1-isomorphic. Thus the three graphs in Figs. 4-8, 4-9, and 4-10 are 1-isomorphic to one another.

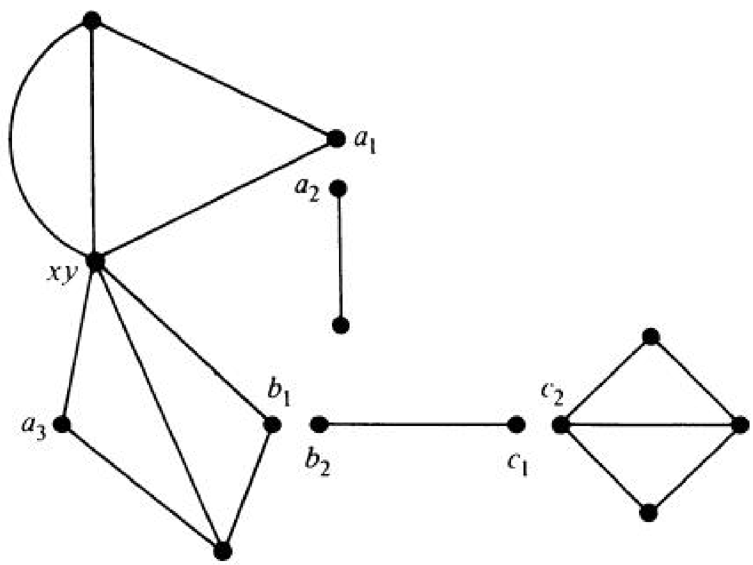


Fig. 4-10 Graph 1-isomorphic to Figs. 4-8 and 4-9.

2-ISOMORPHISM

In Section 4-7 we generalized the concept of isomorphism by introducing 1-isomorphism. A graph G_1 was 1-isomorphic to graph G_2 if the blocks of G_1 were isomorphic to the blocks of G_2. Since a nonseparable graph is just one block, 1-isomorphism for nonseparable graphs is the same as isomorphism. However, for separable graphs (i.e., graphs with vertex connectivity of one), 1-isomorphism is different from isomorphism. Graphs that are isomorphic are also 1-isomorphic, but 1-isomorphic graphs may not be isomorphic. This generalized isomorphism is very useful in the study of separable graphs.

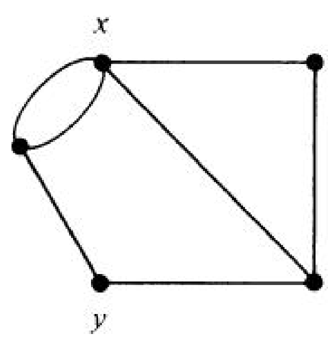
We can generalize this concept further to broaden its scope for 2-connected graphs (i.e., graphs with vertex connectivity of two), as follows:

In a 2-connected graph G let vertices x and y be a pair of vertices whose removal from G will leave the remaining graph disconnected. In other words, G consists of a subgraph g_1 and its complement $\bar{g}_1$ such that g_1 and $\bar{g}_1$ have exactly two vertices, x and y, in common. Suppose that we perform the following *operation 2* on G (after which, of course, G no longer remains the original graph).

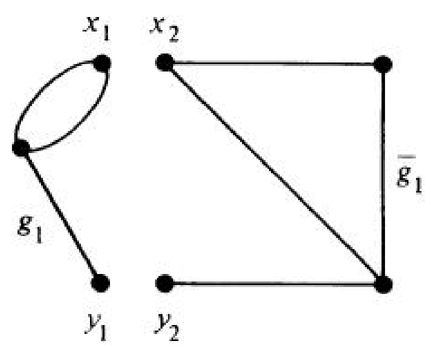
Operation 2: "Split" the vertex x into x_1 and x_2 and the vertex y into y_1 and y_2 such that G is split into g_1 and $\bar{g}_1$. Let vertices x_1 and y_1 go with g_1 and x_2 and y_2 with $\bar{g}_1$. Now rejoin the graphs g_1 and $\bar{g}_1$ by merging x_1 with y_2 and x_2 with y_1. (Clearly, edges whose end vertices were x and y in G could have gone with g_1 or $\bar{g}_1$, without affecting the final graph.)

Two graphs are said to be *2-isomorphic* if they become isomorphic after undergoing operation 1 (in Section 4-7) or operation 2, or both operations any number of times. For example, Fig. 4-11 shows how the two graphs in Figs. 4-11(a) and (d) are 2-isomorphic. Note that in (a) the degree of vertex x is four, but in (d) no vertex is of degree four.

From the definition it follows immediately that isomorphic graphs are always 1-isomorphic, and 1-isomorphic graphs are always 2-isomorphic. But 2-isomorphic graphs are not necessarily 1-isomorphic, and 1-isomorphic



(a)



(b)

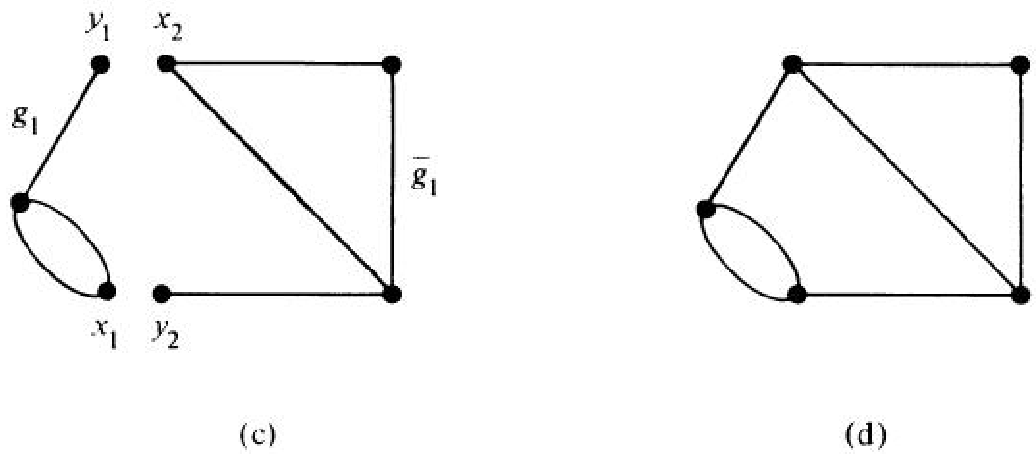


Fig. 4-11 2-isomorphic graphs (a) and (d).

graphs are not necessarily isomorphic. However, for graphs with connectivity three or more, isomorphism, 1-isomorphism, and 2-isomorphism are synonymous.

It is clear that no edges or vertices are created or destroyed under operation 2. Therefore, the rank and nullity of a graph remain unchanged under operation 2. And as shown in Section 4-7, the rank or nullity of a graph does not change under operation 1. Therefore, 2-isomorphic graphs are equal in rank and equal in nullity. The fact that the rank r and nullity μ are not enough to specify a graph within 2-isomorphism can easily be shown by constructing a counterexample (Problem 4-23).

Circuit Correspondence: Two graphs G_1 and G_2 are said to have a *circuit correspondence* if they meet the following condition: There is a one-to-one correspondence between the edges of G_1 and G_2 and a one-to-one correspondence between the circuits of G_1 and G_2, such that a circuit in G_1 formed by certain edges of G_1 has a corresponding circuit in G_2 formed by the corresponding edges of G_2, and vice versa. Isomorphic graphs, obviously, have circuit correspondence.

Since in a separable graph G every circuit is confined to a particular block (Problem 4-15), every circuit in G retains its edges as G undergoes *operation 1* (in Section 4-7). Hence 1-isomorphic graphs have circuit correpondence.

Similarly, let us consider what happens to a circuit in a graph G when it undergoes operation 2, as defined in this section. A circuit Γ in G will fall in one of three categories:

1. Γ is made of edges all in g_1, or
2. Γ is made of edges all in $\bar{g}_1$, or
3. Γ is made of edges from both g_1 and $\bar{g}_1$, and in that case Γ must include both vertices x and y.

In cases 1 and 2, Γ is unaffected by operation 2. In case 3, Γ still has the original edges, except that the path between vertices x and y in g_1, which constituted a part of Γ, is "flipped around." Thus every circuit in a graph undergoing operation 2 retains its original edges. Therefore, 2-isomorphic graphs also have circuit correspondence.

Theorem 4-15, which is considered the most important result for 2-isomorphic graphs, is due to H. Whitney.

THEOREM 4-15

Two graphs are 2-isomorphic if and only if they have circuit correspondence.

Proof: The "only if" part has already been shown in the argument preceding the theorem. The "if" part is more involved, and the reader is referred to Whitney's original paper [4-7].

As we shall observe in subsequent chapters, the ideas of 2-isomorphism and circuit correspondence play important roles in the theory of contact networks, electrical networks, and in duality of graphs.

PLANAR GRAPHS

A graph G is said to be *planar* if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.† A graph that cannot be drawn on a plane without a crossover between its edges is called *nonplanar*.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called *embedding*. Thus, to declare that a graph G is nonplanar, we have to show that of all possible geometric representations of G none can be embedded in a plane. Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane.

Otherwise, G is nonplanar. An embedding of a planar graph G on a plane is called a *plane representation* of G.

For instance, consider the graph represented by Fig. 1-3. The geometric representation shown in Fig. 1-3 clearly is not embedded in a plane, because the edges e and f are intersecting. But if we redraw edge f outside the quadrilateral, leaving the other edges unchanged, we have embedded the new geometric graph in the plane, thus showing that the graph which is being represented by Fig. 1-3 is planar. As another example, the two isomorphic diagrams in Fig. 2-2 are different geometric representations of one and the same graph. One of the diagrams is a plane representation; the other one is not. The graph, of course, is planar. On the other hand, you will not be able to draw any of the three configurations in Fig. 2-3 on a plane without edges intersecting. The reason is that the graph which these three different diagrams in Fig. 2-3 represent is nonplanar.

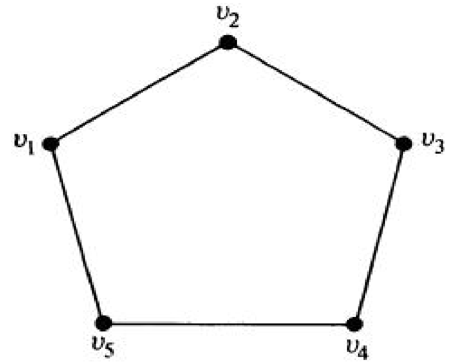
A natural question now is: How can we tell if a graph G [which may be given by an abstract notation $G = (V, E, \Psi)$ or by one of its geometric representations] is planar or nonplanar? To answer this question, let us first discuss two specific nonplanar graphs which are of fundamental importance. These are called Kuratowski's graphs, after the Polish mathematician Kasimir Kuratowski, who discovered their unique property.

KURATOWSKI'S TWO GRAPHS

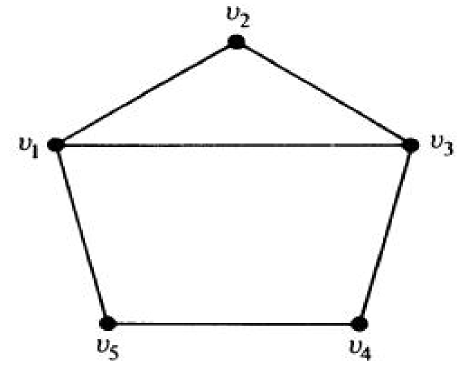
THEOREM 5-1

The complete graph of five vertices is nonplanar.

Proof: Let the five vertices in the complete graph be named v_1, v_2, v_3, v_4, and v_5. A complete graph, as you may recall, is a simple graph in which every vertex is joined to every other vertex by means of an edge. This being the case, we must



(a)



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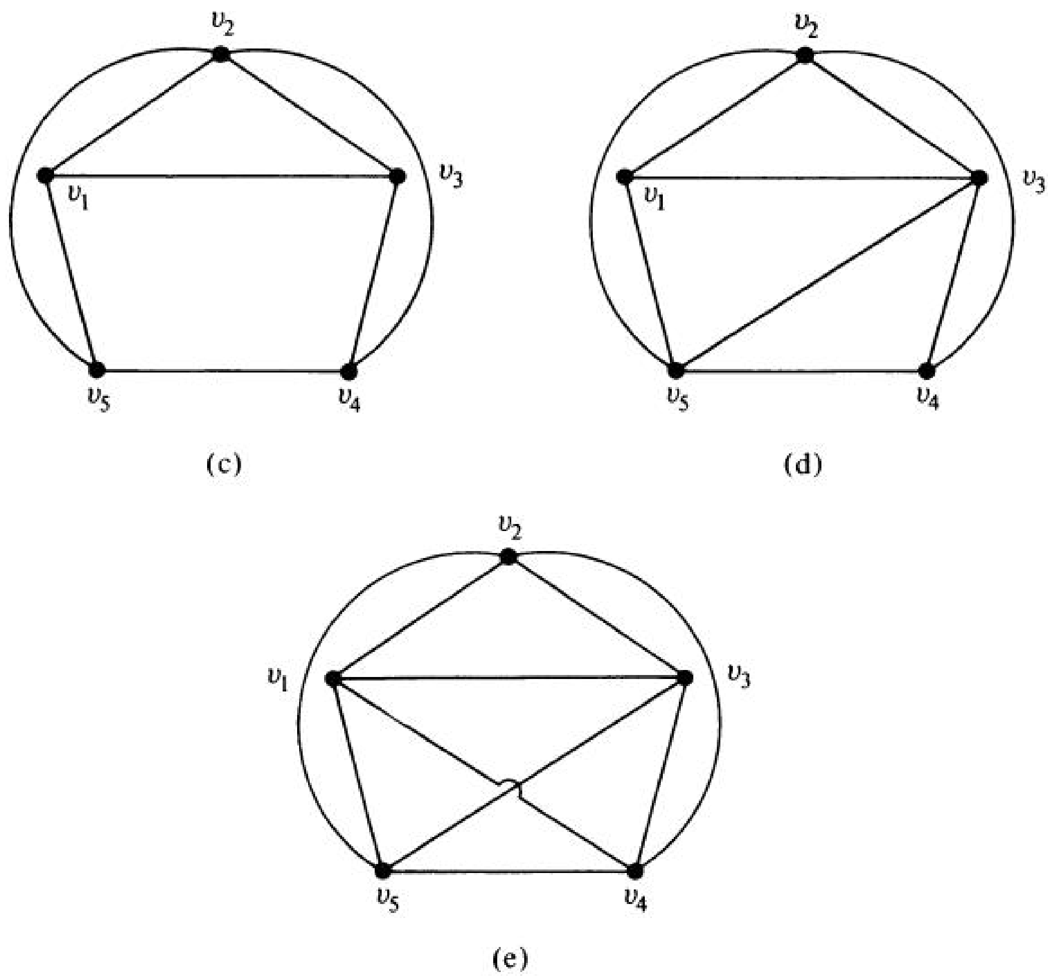


Fig. 5-1 Building up of the five-vertex complete graph.

have a circuit going from v_1 to v_2 to v_3 to v_4 to v_5 to v_1—that is, a pentagon. See Fig. 5-1(a). This pentagon must divide the plane of the paper into two regions, one *inside* and the other *outside* (Jordan curve theorem).

Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Suppose that we choose to draw a line from v_1 to v_3 inside the pentagon. See Fig. 5-1(b). (If we choose outside, we end up with the same argument.) Now we have to draw an edge from v_2 to v_4 and another one from v_2 to v_5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon. See Fig. 5-1(c). The edge connecting v_3 and v_5 cannot be drawn outside the pentagon without crossing the edge between v_2 and v_4. Therefore, v_3 and v_5 have to be connected with an edge inside the pentagon. See Fig. 5-1(d).

Now we have yet to draw an edge between v_1 and v_4. This edge cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane. See Fig. 5-1(e). ■

Some readers may find this proof somewhat unsatisfactory because it depends so heavily on visual intuition. Do not despair; we shall provide you with an algebraic nonvisual proof in the next section.

A complete graph with five vertices is the first of the two graphs of Kuratowski. The second graph of Kuratowski is a regular† connected graph with six vertices and nine edges, shown in its two common geometric representations in Figs. 5-2(a) and (b), where it is fairly easy to see that the graphs are isomorphic.

Employing visual geometric arguments similar to those used in proving Theorem 5-1, it can be shown that the second graph of Kuratowski is also nonplanar. The proof of Theorem 5-2 is, therefore, left as an exercise (Problem 5-1).

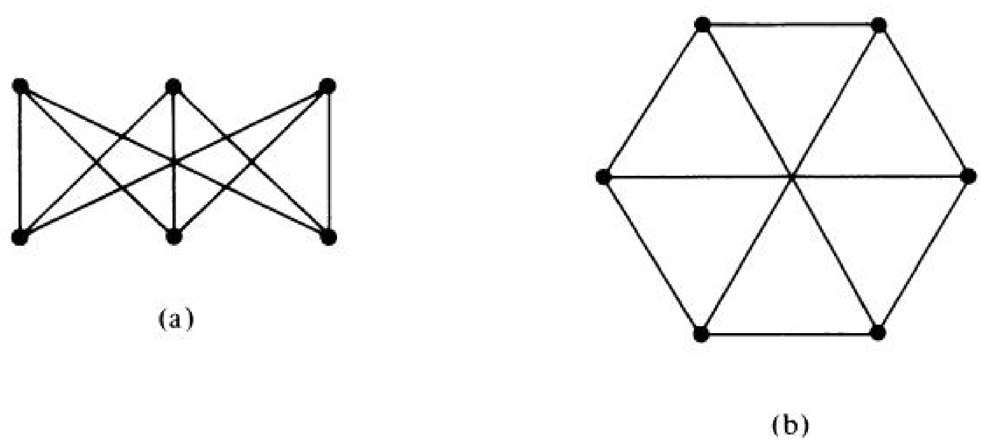


Fig. 5-2 Kuratowski's second graph.

THEOREM 5-2

Kuratowski's second graph is also nonplanar.

You may have noticed several properties common to the two graphs of Kuratowski. These are

1. Both are regular graphs.

2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the smallest number of edges. Thus both are the simplest nonplanar graphs.

In the literature, Kuratowski's first graph is usually denoted by K_5 and the second graph by $K_{3,3}$—letter K being for Kuratowski.

DIFFERENT REPRESENTATIONS OF A PLANAR GRAPH

THEOREM 5-3

Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

Proof: The proof is involved and does not contribute much to the understanding of planarity. The interested reader is, therefore, referred to pages 74–77 in [1-2] or to the original paper of Fary [5-4]. As an illustration, the graph in Fig. 5-1(d) can be redrawn using straight line segments to look like Fig. 5-3. In this theorem, it is necessary for the graph to be simple because a self-loop or one of two parallel edges cannot be drawn by a straight line segment. ■

Region: A plane representation of a graph divides the plane into *regions* (also called *windows*, *faces*, or *meshes*), as shown in Fig. 5-4. A region is

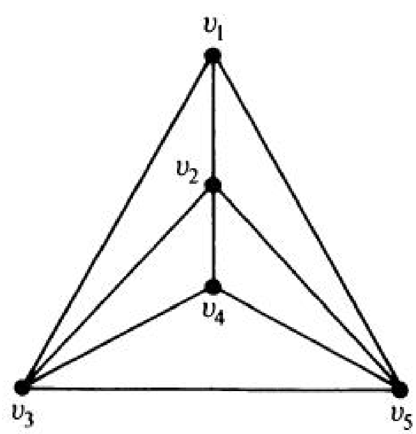


Fig. 5-3 Straight-line representation of the graph in Fig. 5-1(d).

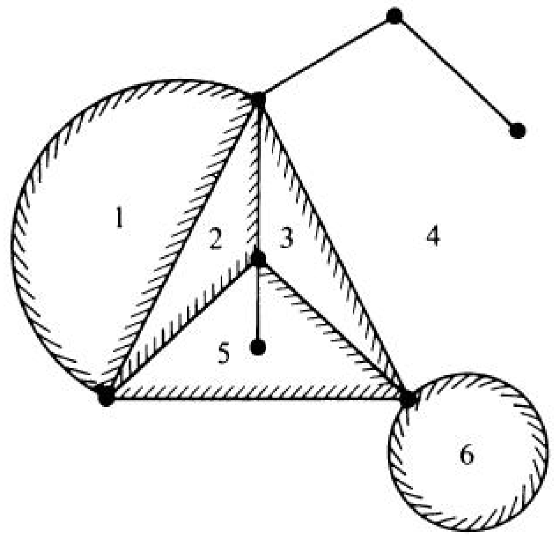


Fig. 5-4 Plane representation (the numbers stand for regions).

characterized by the set of edges (or the set of vertices) forming its *boundary*. Note that a region is not defined in a nonplanar graph or even in a planar graph not embedded in a plane. For example, the geometric graph in Fig. 1-3 does not have regions. Thus a region is a property of the specific plane representation of a graph and not of an abstract graph per se.

Infinite Region: The portion of the plane lying outside a graph embedded in a plane, such as region 4 in Fig. 5-4, is infinite in its extent. Such a region is called the *infinite*, *unbounded*, *outer*, or *exterior* region for that particular plane representation. Like other regions, the infinite region is also characterized by a set of edges (or vertices). Clearly, by changing the embedding of a given planar graph, we can change the infinite region. For instance, Figs. 5-1(d) and 5-3 are two different embeddings of the same graph. The finite region $v_1 v_3 v_5$ in Fig. 5-1(d) becomes the infinite region in Fig. 5-3. In fact, we shall shortly show that any region can be made the infinite region by proper embedding.

Embedding on a Sphere: To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere. It is accomplished by stereographic projection of a sphere on a plane. Put the sphere on the plane and call the point of contact SP (south pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north pole). See Fig. 5-5.

Now, corresponding to any point p on the plane, there exists a unique point p' on the sphere and vice versa, where p' is the point at which the straight line from point p to point NP intersects the surface of the sphere. Thus there is a one-to-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane correspond to the point NP on the sphere.

From this construction, it is clear that any graph that can be embedded in

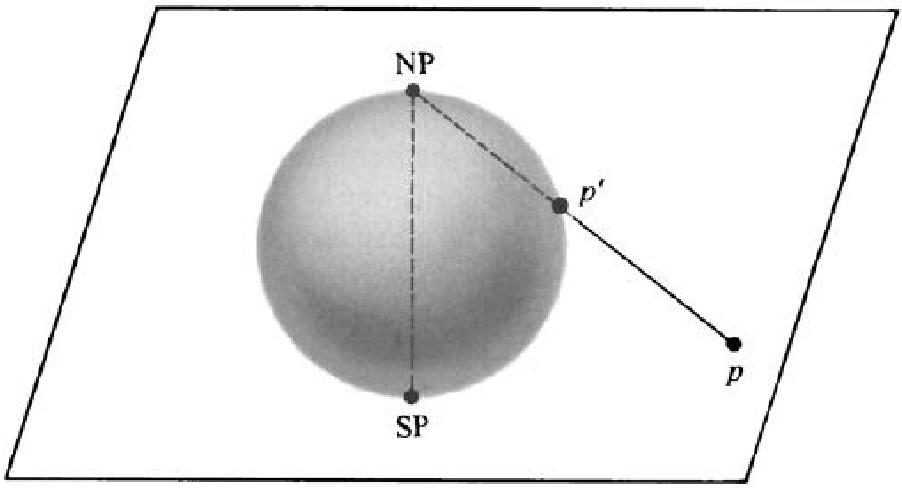


Fig. 5-5 Stereographic projection.

a plane (i.e., drawn on a plane such that its edges do not intersect) can also be embedded in the surface of the sphere, and vice versa. Hence

THEOREM 5-4

A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

A planar graph embedded in the surface of a sphere divides the surface into different regions. Each region on the sphere is finite, the infinite region on the plane having been mapped onto the region containing the point NP. Now it is clear that by suitably rotating the sphere we can make any specified region map onto the infinite region on the plane. From this we obtain

THEOREM 5-5

A planar graph may be embedded in a plane such that any specified region (i.e., specified by the edges forming it) can be made the infinite region.

Thinking in terms of the regions on the sphere, we see that there is no real difference between the infinite region and the finite regions on the plane. Therefore, when we talk of the regions in a plane regresentation of a graph, we include the infinite region. Also, since there is no essential difference between an embedding of a planar graph on a plane or on a sphere (a plane may be regarded as the surface of a sphere of infinitely large radius), the term "plane representation" of a graph is often used to include spherical as well as planar embedding.

Euler's Formula: Since a planar graph may have different plane representations, we may ask if the number of regions resulting from each embedding is the same. The answer is *yes*. Theorem 5-6, known as Euler's formula, gives the number of regions in any planar graph.

THEOREM 5-6

A connected planar graph with n vertices and e edges has $e - n + 2$ regions.

Proof: It will suffice to prove the theorem for a simple graph, because adding a self-loop or a parallel edge simply adds one region to the graph and simultaneously increases the value of e by one. We can also disregard (i.e., remove) all edges that do not form boundaries of any region. Three such edges are shown in Fig. 5-4. Addition (or removal) of any such edge increases (or decreases) e by one and increases (or decreases) n by one, keeping the quantity $e - n$ unaltered.

Since any simple planar graph can have a plane representation such that each edge is a straight line (Theorem 5-3), any planar graph can be drawn such that each region is a polygon (a polygonal net). Let the polygonal net representing the given graph consist of f regions or faces, and let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3 \cdot k_3 + 4 \cdot k_4 + 5 \cdot k_5 + \cdots + r \cdot k_r = 2 \cdot e, \qquad (5\text{-}1)$$

where k_r is the number of polygons, with maximum edges.
Also,

$$k_3 + k_4 + k_5 + \cdots + k_r = f. \qquad (5\text{-}2)$$

The sum of all angles subtended at each vertex in the polygonal net is

$$2\pi n. \qquad (5\text{-}3)$$

Recalling that the sum of all interior angles of a p-sided polygon is $\pi(p-2)$, and the sum of the exterior angles is $\pi(p+2)$, let us compute the expression in (5-3) as the grand sum of all interior angles of $f-1$ finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$$\begin{aligned}&\pi(3-2)\cdot k_3 + \pi(4-2)\cdot k_4 + \cdots + \pi(r-2)\cdot k_r + 4\pi \\ &= \pi(2e-2f) + 4\pi. \end{aligned} \tag{5-4}$$

Equating (5-4) to (5-3), we get

$$2\pi(e-f) + 4\pi = 2\pi n,$$

or

$$e - f + 2 = n.$$

Therefore, the number of regions is

$$f = e - n + 2. \quad \blacksquare$$

COROLLARY

In any simple, connected planar graph with f regions, n vertices, and e edges ($e > 2$), the following inequalities must hold:

$$e \geq \frac{3}{2}f, \tag{5-5}$$

$$e \leq 3n - 6. \tag{5-6}$$

Proof: Since each region is bounded by at least three edges and each edge belongs to exactly two regions,

$$2e \geq 3f$$

or

$$e \geq \frac{3}{2}f.$$

Substituting for f from Euler's formula in inequality (5-5),

$$e \geq \frac{3}{2}(e - n + 2)$$

or

$$e \leq 3n - 6. \quad \blacksquare$$

Inequality (5-6) is often useful in finding out if a graph is nonplanar. For example, in the case of K_5, the complete graph of five vertices [Fig. 5-1(e)],

$$n = 5, \qquad e = 10, \qquad 3n - 6 = 9 < e.$$

Thus the graph violates inequality (5-6), and hence it is not planar.

Incidentally, this is an alternative and independent proof of the nonplanarity of Kuratowski's first graph, as promised in Section 5-3.

The reader must be warned that inequality (5-6) is only a necessary, but *not* a sufficient, condition for the planarity of a graph. In other words, although every simple planar graph must satisfy (5-6), the mere satisfaction of this inequality does not guarantee the planarity of a graph. For example, Kuratowski's second graph, $K_{3,3}$, satisfies (5-6), because

$$e = 9,$$
$$3n - 6 = 3\cdot 6 - 6 = 12.$$

Yet the graph is nonplanar.

To prove the nonplanarity of Kuratowski's second graph, we make use of the additional fact that no region in this graph can be bounded with fewer than four edges. Hence, if this graph were planar, we would have

$$2e \geq 4f,$$

and, substituting for f from Euler's formula,

$$2e \geq 4(e - n + 2),$$

or
$$2\cdot 9 \geq 4(9 - 6 + 2),$$

or
$$18 \geq 20, \quad \text{a contradiction.}$$

Hence the graph cannot be planar.

Plane Representation and Connectivity: In a disconnected graph the embedding of each component can be considered independently. Therefore, it is clear that a disconnected graph is planar if and only if each of its components is planar. Similarly, in a separable (or 1-connected) graph the embedding of each block (i.e., maximal nonseparable subgraph) can be considered independently. Hence a separable graph is planar if and only if each of its blocks is planar.

Therefore, in questions of embedding or planarity, one need consider only nonseparable graphs.

THEOREM 5-7

The spherical embedding of every planar 3-connected graph is unique.

This theorem plays a very important role in determining if a graph is

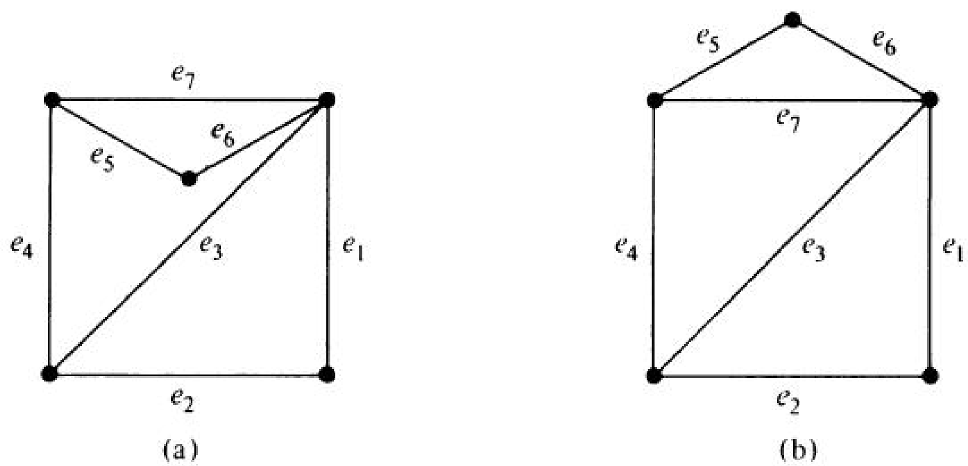


Fig. 5-6 Two distinct plane representations of the same graph.

planar or not. The theorem states that a 3-connected graph, if it can be embedded at all, can be embedded in only one way.

DETECTION OF PLANARITY

How to tell if a given graph G is planar or nonplanar is an important problem, and "find out by drawing it" is obviously not a good answer. We must have some simple and efficient criterion. Toward that goal, we take the following simplifying steps:

Elementary Reduction

Step 1: Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set

$$G = \{G_1, G_2, \ldots, G_k\},$$

where each G_i is a nonseparable block of G. Then we have to test each G_i for planarity.

Step 2: Since addition or removal of self-loops does not affect planarity, remove all self-loops.

Step 3: Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.

Step 4: Elimination of a vertex of degree two by merging two edges in series† does not affect planarity. Therefore, eliminate all edges in series.

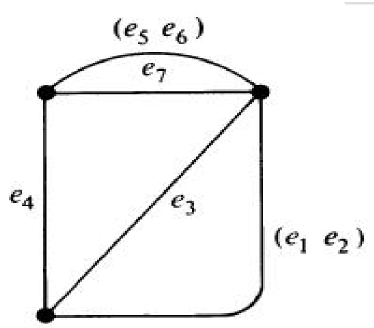
Repeated application of steps 3 and 4 will usually reduce a graph drastically. For example, Fig. 5-7 illustrates the series-parallel reduction of the graph of Fig. 5-6(b).

Let the nonseparable connected graph G_i be reduced to a new graph H_i after the repeated application of steps 3 and 4. What will graph H_i look like? Theorem 5-8 has the answer.

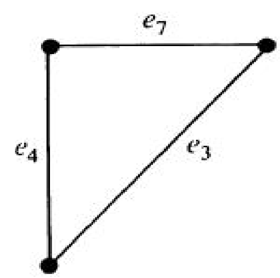
THEOREM 5-8

Graph H_i is

1. A single edge, or
2. A complete graph of four vertices, or
3. A nonseparable, simple graph with $n \geq 5$ and $e \geq 7$.



(a) Series Reduced



(b) Parallel Reduced

(c) Series Reduced (d) Parallel Reduced

Fig. 5-7 Series-parallel reduction of the graph in Fig. 5-6(b).

Proof: The theorem can be proved by considering all connected nonseparable graphs of six edges or less. The proof is left as an exercise (Problem 5-9).

In Theorem 5-8, all H_i falling in categories 1 or 2 are planar and need not be checked further.

From now on, therefore, we need to investigate only *simple*, *connected*, *nonseparable graphs of at least five vertices and with every vertex of degree three or more*. Next, we can check to see if $e \leq 3n - 6$. If this inequality is not satisfied, the graph H_i is nonplanar. If the inequality is satisfied, we have to test the graph further and, with this, we come to Kuratowski's theorem (Theorem 5-9), perhaps the most important result of this chapter.

Homeomorphic Graphs: Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by the creation of edges in series (i.e., by insertion of vertices of degree two) or by the merger of edges in series. The three graphs in Fig. 5-8 are homeomorphic to each other, for instance. A graph G is planar if and only if every graph that is homeomorphic to G is planar. (This is a restatement of series reduction, step 4 in this section.)

THEOREM 5-9

A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them.

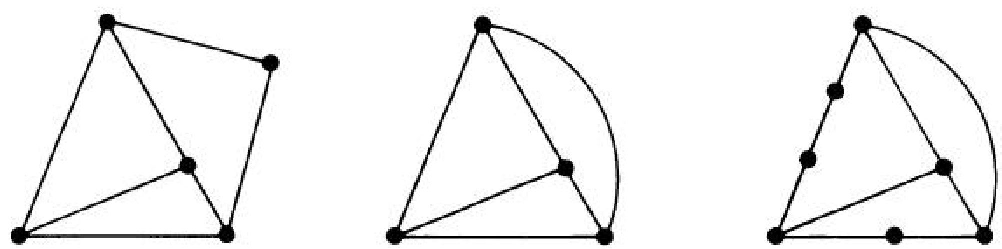


Fig. 5-8 Three graphs homeomorphic to each other.

Proof: The necessary condition is clear, because a graph G cannot be embedded in a plane if G has a subgraph that cannot be embedded. That this condition is also sufficient is surprising, and its proof is involved. Several different proofs of the theorem have appeared since Kuratowski stated and proved it in 1930. For a complete proof of the theorem, the reader is referred to Harary [1-5], pages 108–112, Berge [1-1], pages 211–213, or Busacker and Saaty [1-2], pages 70–73.

Note that it is *not* necessary for a nonplanar graph to have either of the Kuratowski graphs as a subgraph, as this theorem is sometimes misstated. The nonplanar graph may have a subgraph homeomorphic to a Kuratowski graph. For example, the graph in Fig. 5-9(a) is nonplanar, and yet it does not have either of the Kuratowski graphs as a subgraph. However, if we remove

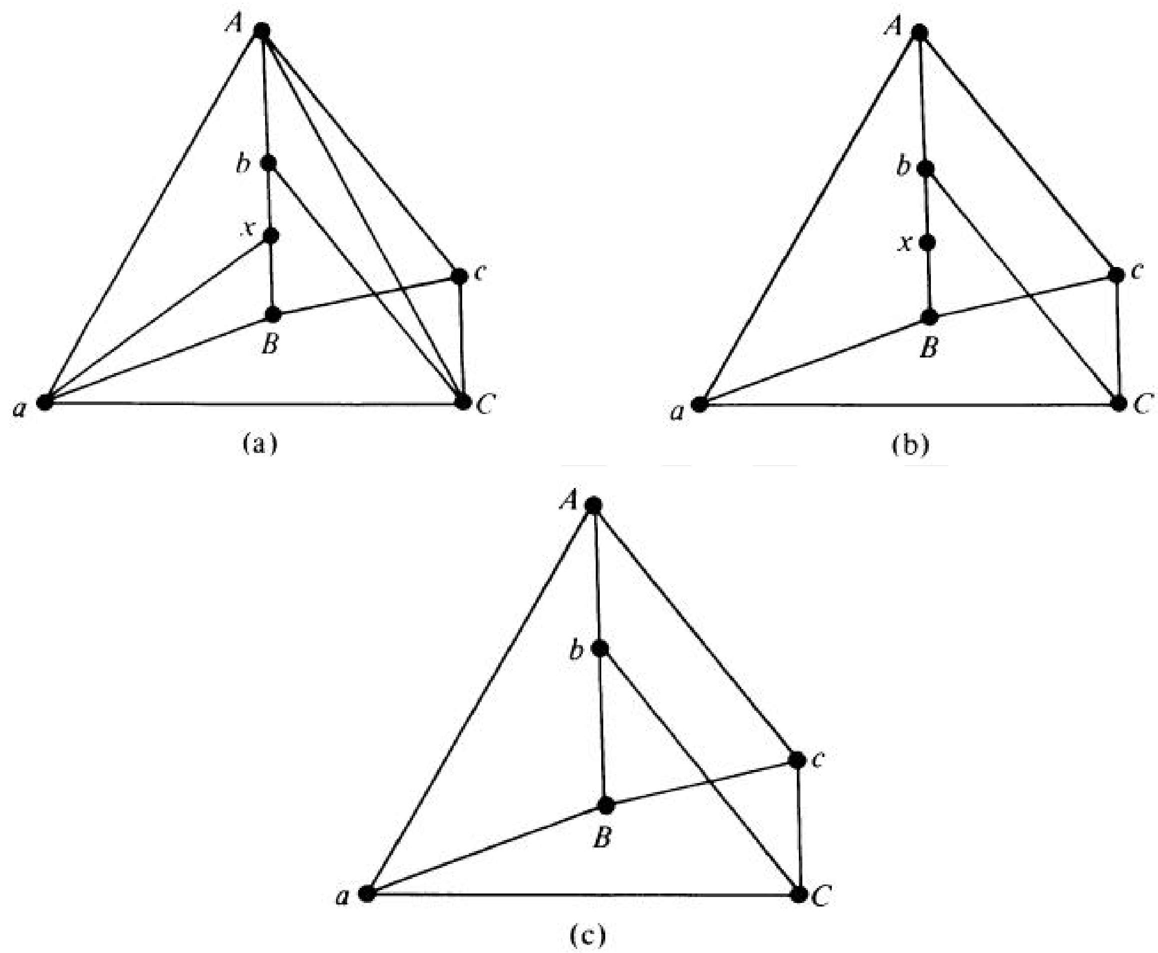


Fig. 5-9 Nonplanar graph with a subgraph homeomorphic to $K_{3,3}$.

edges (a, x) and (A, C) from this graph, we get a subgraph, as shown in Fig. 5-9(b). This subgraph is homeomorphic (merge two series edges at vertex x) to the one shown in Fig. 5-9(c). The graph of Fig. 5-9(c) clearly is isomorphic to $K_{3,3}$, Kuratowski's second graph, and this demonstrates the nonplanarity of the graph in Fig. 5-9(a).

UNIT-III

SYLLABUS

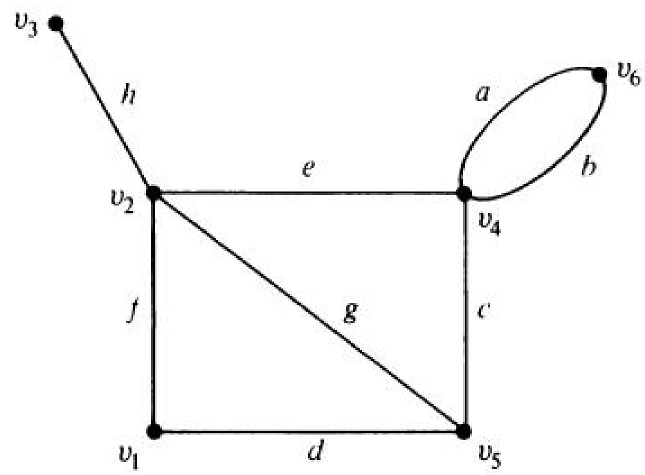
Incidence matrix – Sub matrices – Circuit Matrix – Path Matrix – Adjacency Matrix – Chromatic Numb
Chromatic partitioning – Chromatic polynomial - Matching - Covering – Four Color Problem.

INCIDENCE MATRIX

Let G be a graph with n vertices, e edges, and no self-loops. Define an n by e matrix $A = [a_{ij}]$, whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows:

The matrix element

$$a_{ij} = 1, \quad \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i, \text{ and}$$
$$= 0, \quad \text{otherwise.}$$



(a)

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

(b)

Fig. 7-1 Graph and its incidence matrix.

Such a matrix A is called the *vertex-edge incidence matrix*, or simply *incidence matrix*. Matrix A for a graph G is sometimes also written as A(G). A graph and its incidence matrix are shown in Fig. 7-1.

The incidence matrix contains only two elements, 0 and 1. Such a matrix is called a *binary matrix* or a *(0, 1)-matrix*. Let us stipulate that these two elements are from Galois field modulo 2.† Given any geometric representation of a graph without self-loops, we can readily write its incidence matrix.

On the other hand, if we are given an incidence matrix A(G), we can construct its geometric graph G without ambiguity. The incidence matrix and the geometric graph contain the same information†—they are simply two alternative ways of representing the same (abstract) graph.

The following observations about the incidence matrix A can readily be made:

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix, for example, columns 1 and 2 in Fig. 7-1.
5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix A(G) of graph G can be written in a block-diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right], \tag{7-1}$$

where A(g_1) and A(g_2) are the incidence matrices of components g_1 and g_2. This observation results from the fact that no edge in g_1 is incident on vertices of g_2, and vice versa. Obviously, this remark is also true for a disconnected graph with any number of components.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph. This observation leads us to Theorem 7-1.

THEOREM 7-1

Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $\mathsf{A}(G_1)$ and $\mathsf{A}(G_2)$ differ only by permutations of rows and columns.

Rank of the Incidence Matrix: Each row in an incidence matrix $\mathsf{A}(G)$ may be regarded as a vector over GF(2) in the vector space of graph G. Let the vector in the first row be called A_1, in the second row A_2, and so on. Thus

$$\mathsf{A}(G) = \begin{bmatrix} \mathsf{A}_1 \\ \mathsf{A}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathsf{A}_n \end{bmatrix}, \tag{7-2}$$

Since there are exactly two 1's in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries). Thus vectors $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ are not linearly independent. Therefore, the rank of A is less than n; that is, rank $\mathsf{A} \leq n - 1$.

Now consider the sum of any m of these n vectors ($m \leq n - 1$). If the graph is connected, $\mathsf{A}(G)$ cannot be partitioned, as in Eq. (7-1), such that $\mathsf{A}(g_1)$ is with m rows and $\mathsf{A}(g_2)$ with $n - m$ rows. In other words, no m by m submatrix of $\mathsf{A}(G)$ can be found, for $m \leq n - 1$, such that the modulo 2 sum of those m rows is equal to zero.

Since there are only two constants 0 and 1 in this field, the additions of all vectors taken m at a time for $m = 1, 2, \ldots, n - 1$ exhausts all possible linear combinations of $n - 1$ row vectors. Thus we have just shown that no linear combination of m row vectors of A (for $m \leq n - 1$) can be equal to zero. Therefore, the rank of $\mathsf{A}(G)$ must be at least $n - 1$.

Since the rank of $\mathsf{A}(G)$ is no more than $n - 1$ and is no less than $n - 1$, it must be exactly equal to $n - 1$. Hence Theorem 7-2.

THEOREM 7-2

If $A(G)$ is an incidence matrix of a connected graph G with n vertices, the rank of $A(G)$ is $n - 1$.

The argument leading to Theorem 7-2 can be extended to prove that the rank of $A(G)$ is $n - k$, if G is a disconnected graph with n vertices and k components (Problem 7-3). This is the reason why the number $n - k$ has been called the rank of a graph with k components.

If we remove any one row from the incidence matrix of a connected graph, the remaining $(n - 1)$ by e submatrix is of rank $n - 1$ (Theorem 7-2). In other words, the remaining $n - 1$ row vectors are linearly independent. Thus we need only $n - 1$ rows of an incidence matrix to specify the corresponding graph completely, for $n - 1$ rows contain the same amount of information as the entire matrix. (This is obvious, since given $n - 1$ rows we can easily reconstitute the missing row, because each column in the matrix has exactly two 1's.)

Such an $(n - 1)$ by e submatrix A_f of A is called a *reduced incidence matrix*. The vertex corresponding to the deleted row in A_f is called the *reference vertex*. Clearly, any vertex of a connected graph can be made the reference vertex.

Since a tree is a connected graph with n vertices and $n - 1$ edges, its reduced incidence matrix is a square matrix of order and rank $n - 1$. In other words,

COROLLARY

The reduced incidence matrix of a tree is nonsingular.

A graph with n vertices and $n - 1$ edges that is not a tree is disconnected. The rank of the incidence matrix of such a graph will be less than $n - 1$. Therefore, the $(n - 1)$ by $(n - 1)$ reduced incidence matrix of such a graph will not be nonsingular. In other words, the reduced incidence matrix of a graph is nonsingular if and only if the graph is a tree.

SUBMATRICES OF A(G)

Let g be a subgraph of a graph G, and let $A(g)$ and $A(G)$ be the incidence matrices of g and G, respectively. Clearly, $A(g)$ is a submatrix of $A(G)$ (possibly with rows or columns permuted). In fact, there is a one-to-one correspondence between each n by k submatrix of $A(G)$ and a subgraph of G with k edges, k being any positive integer less than e and n being the number of vertices in G.

Submatrices of $A(G)$ corresponding to special types of subgraphs, such as circuits, spanning trees, or cut-sets in G, will undoubtedly exhibit special properties. Theorem 7-3 gives one such property.

THEOREM 7-3

Let $A(G)$ be an incidence matrix of a connected graph G with n vertices. An $(n-1)$ by $(n-1)$ submatrix of $A(G)$ is nonsingular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitute a spanning tree in G.

Proof: Every square submatrix of order $n-1$ in $A(G)$ is the reduced incidence matrix of the same subgraph in G with $n-1$ edges, and vice versa. From the remarks following Theorem 7-2, it is clear that a square submatrix of $A(G)$ is nonsingular if and only if the corresponding subgraph is a tree. The tree in this case is a spanning tree, because it contains $n-1$ edges of the n-vertex graph. Thus the theorem. ■

CIRCUIT MATRIX

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a *circuit matrix* $B = [b_{ij}]$ of G is a q by e, (0, 1)-matrix defined as follows:

$$\begin{aligned} b_{ij} &= 1, \quad \text{if } i\text{th circuit includes } j\text{th edge, and} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

To emphasize the fact that B is a circuit matrix of graph G, the circuit matrix may also be written as $B(G)$.

The graph in Fig. 7-1(a) has four different circuits, $\{a, b\}$, $\{c, e, g\}$, $\{d, f, g\}$, and $\{c, d, f, e\}$. Therefore, its circuit matrix is a 4 by 8, (0, 1)-matrix as shown:

$$\mathsf{B}(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}. \qquad (7\text{-}3)$$

The following observations can be made about a circuit matrix $\mathsf{B}(G)$ of a graph G:

1. A column of all zeros corresponds to a noncircuit edge (i.e., an edge that does not belong to any circuit).
2. Each row of $\mathsf{B}(G)$ is a circuit vector.
3. Unlike the incidence matrix, a circuit matrix is capable of representing a self-loop—the corresponding row will have a single 1.
4. The number of 1's in a row is equal to the number of edges in the corresponding circuit.
5. If graph G is separable (or disconnected) and consists of two blocks (or components) g_1 and g_2, the circuit matrix $\mathsf{B}(G)$ can be written in a block-diagonal form as

$$\mathsf{B}(G) = \left[\begin{array}{c|c} \mathsf{B}(g_1) & 0 \\ \hline 0 & \mathsf{B}(g_2) \end{array}\right],$$

where $\mathsf{B}(g_1)$ and $\mathsf{B}(g_2)$ are the circuit matrices of g_1 and g_2. This observation results from the fact that circuits in g_1 have no edges belonging to g_2, and vice versa (Problem 4-14).

6. Permutation of any two rows or columns in a circuit matrix simply corresponds to relabeling the circuits and edges.
7. Two graphs G_1 and G_2 will have the same circuit matrix if and only if G_1 and G_2 are 2-isomorphic (Theorem 4-15). In other words, (unlike

an incidence matrix) the circuit matrix does not specify a graph completely. It only specifies the graph within 2-isomorphism. For instance, it can be easily verified that the two graphs in Figs. 4-11(a) and (d) have the same circuit matrix, yet the graphs are not isomorphic.

An important theorem relating the incidence matrix and the circuit matrix of a self-loop-free graph G is

THEOREM 7-4

Let B and A be, respectively, the circuit matrix and the incidence matrix (of a self-loop-free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A; that is,

$$A \cdot B^T = B \cdot A^T = 0 \qquad (\text{mod } 2), \tag{7-4}$$

where superscript T denotes the transposed matrix.

Proof: Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the number of those edges in the circuit Γ that are incident on v is exactly two.

With this remark in mind, consider the ith row in A and the jth row in B. Since the edges are arranged in the same order, the nonzero entries in the corresponding positions occur only if the particular edge is incident on the ith vertex and is also in the jth circuit.

If the ith vertex is not in the jth circuit, there is no such nonzero entry, and the dot product of the two rows is zero. If the ith vertex is in the jth circuit, there will be exactly two 1's in the sum of the products of individual entries. Since $1 + 1 = 0$ (mod 2), the dot product of the two arbitrary rows—one from A and the other from B—is zero. Hence the theorem. ■

As an example, let us multiply the incidence matrix and transposed circuit of the graph in Fig. 7-1(a), after making sure that the edges are in the same order in both.

$$A \cdot B^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

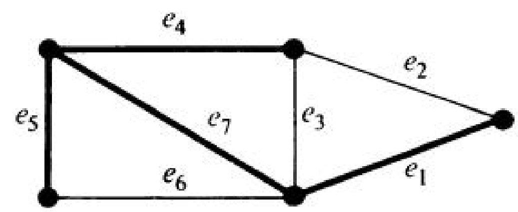
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(mod 2)}.$$

FUNDAMENTAL CIRCUIT MATRIX AND RANK OF *B*

A submatrix (of a circuit matrix) in which all rows correspond to a set of fundamental circuits is called a *fundamental circuit matrix* B_f. A graph and its fundamental circuit matrix with respect to a spanning tree (indicated by heavy lines) are shown in Fig. 7-2.

As in matrices A and B, permutations of rows (and/or of columns) do not affect B_f. If n is the number of vertices and e the number of edges in a connected graph, then B_f is an $(e - n + 1)$ by e matrix, because the number of fundamental circuits is $e - n + 1$, each fundamental circuit being produced by one chord.

Let us arrange the columns in B_f such that all the $e - n + 1$ chords correspond to the first $e - n + 1$ columns. Furthermore, let us rearrange the rows such that the first row corresponds to the fundamental circuit made



(a)

$$\begin{array}{c}
\begin{array}{ccccccc} e_2 & e_3 & e_6 & e_1 & e_4 & e_5 & e_7 \end{array} \\
\left[\begin{array}{ccc|cccc}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}\right]
\end{array}$$

(b)

Fig. 7-2 Graph and its fundamental circuit matrix (with respect to the spanning tree shown in heavy lines).

by the chord in the first column, the second row to the fundamental circuit made by the second, and so on. This indeed is how the fundamental circuit matrix is arranged in Fig. 7-2(b).

A matrix B_f thus arranged can be written as

$$B_f = [I_\mu \mid B_t], \tag{7-5}$$

where I_μ is an identity matrix of order $\mu = e - n + 1$, and B_t is the remaining μ by $(n - 1)$ submatrix, corresponding to the branches of the spanning tree.

From Eq. (7-5) it is clear that the

$$\text{rank of } B_f = \mu = e - n + 1.$$

Since B_f is a submatrix of the circuit matrix B, the

$$\text{rank of } B \geq e - n + 1.$$

In fact, we can prove Theorem 7-5.

THEOREM 7-5

If B is a circuit matrix of a connected graph G with e edges and n vertices,

$$\text{rank of B} = e - n + 1.$$

Proof: If A is an incidence matrix of G, from Eq. (7-4) we have

$$\mathsf{A} \cdot \mathsf{B}^T = 0 \pmod 2.$$

Therefore, according to Sylvester's theorem (Appendix B),

$$\text{rank of A} + \text{rank of B} \leq e;$$

that is,

$$\text{rank of B} \leq e - \text{rank of A}.$$

Since $\qquad \text{rank of A} = n - 1$

we have $\qquad \text{rank of B} \leq e - n + 1.$

But $\qquad \text{rank of B} \geq e - n + 1.$

Therefore, we must have

$$\text{rank of B} = e - n + 1. \quad \blacksquare$$

An Alternative Proof: Theorem 7-5 can also be proved by considering the circuit subspace W_Γ in the vector space W_G of a graph, as discussed in Chapter 6.

Every row in circuit matrix B is a vector in W_Γ, and since the rank of any matrix is equal to the number of linearly independent rows (or columns) in the matrix, we have.

$$\text{rank of matrix B} = \text{number of linearly independent rows in B};$$

but the number of linearly independent rows in B $\leq$ number of linearly independent vectors in W_Γ, and the number of linearly independent vectors in W_Γ = dimension of $W_\Gamma = \mu$. Therefore, rank of B $\leq e - n + 1$. Since we already showed that rank of B $\geq e - n + 1$, Theorem 7-5 follows. $\blacksquare$

Note that in talking of spanning trees of a graph G it is necessary to assume that G is connected. In the case of a disconnected graph, we would have to consider a spanning forest and fundamental circuits with respect to this forest. It is not difficult to show (considering component by component) that if G is a disconnected graph with k components, e edges, and n vertices,

$$\text{rank of B} = \mu = e - n + k.$$

PATH MATRIX

Another (0, 1)-matrix often convenient to use in communication and transportation networks is the *path matrix*. A path matrix is defined for a specific pair of vertices in a graph, say (x, y), and is written as $P(x, y)$. The rows in $P(x, y)$ correspond to different paths between vertices x and y, and the columns correspond to the edges in G. That is, the path matrix for (x, y) vertices is $P(x, y) = [p_{ij}]$, where

$$\begin{aligned} p_{ij} &= 1, \quad \text{if } j\text{th edge lies in } i\text{th path, and} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

As an illustration, consider all paths between vertices v_3 and v_4 in Fig. 7-1(a). There are three different paths; $\{h, e\}$, $\{h, g, c\}$, and $\{h, f, d, c\}$. Let us number them 1, 2, and 3, respectively. Then we get the 3 by 8 path matrix $P(v_3, v_4)$:

$$P(v_3, v_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Some of the observations one can make at once about a path matrix $P(x, y)$ of a graph G are

1. A column of all 0's corresponds to an edge that does not lie in any path between x and y.
2. A column of all 1's corresponds to an edge that lies in every path between x and y.
3. There is no row with all 0's.
4. The ring sum of any two rows in $P(x, y)$ corresponds to a circuit or an edge-disjoint union of circuits.

THEOREM 7-7

If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix $P(x, y)$, then the product (mod 2)

$$A \cdot P^T(x, y) = M,$$

where the matrix M has 1's in two rows x and y, and the rest of the $n-2$ rows are all 0's.

Proof: The proof is left as an exercise for the reader (Problem 7-14).

As an example, multiply the incidence matrix in Fig. 7-1 to the transposed $P(v_3, v_4)$, just discussed.

$$A \cdot P^T(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

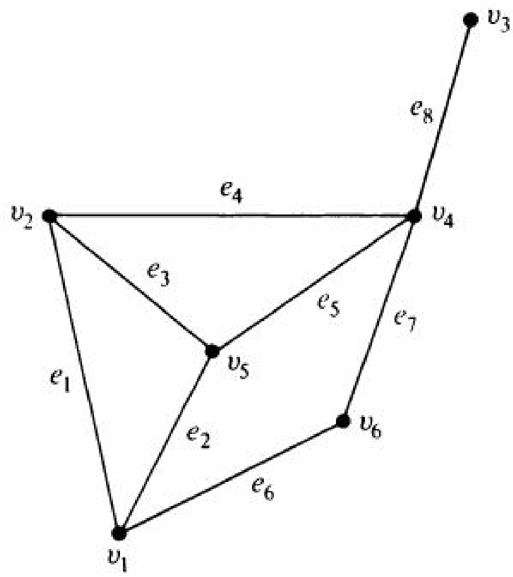
$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \pmod 2.$$

Other properties of the path matrix, such as the rank, are left for the reader to investigate on his own. It should be noted that a path matrix contains less information about the graph in general than any of the matrices A, B, or C does.

ADJACENCY MATRIX

As an alternative to the incidence matrix, it is sometimes more convenient to represent a graph by its *adjacency matrix* or *connection matrix*. The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = 1, \quad \text{if there is an edge between } i\text{th and } j\text{th vertices, and}$$
$$= 0, \quad \text{if there is no edge between them.}$$



$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 7-7 Simple graph and its adjacency matrix.

A simple graph and its adjacency matrix are shown in Fig. 7-7.

Observations that can be made immediately about the adjacency matrix X of a graph G are

1. The entries along the principal diagonal of X are all 0's if and only if the graph has no self-loops. A self-loop at the ith vertex corresponds to $x_{ii} = 1$.
2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix X was defined for graphs without parallel edges.†

3. If the graph has no self-loops (and no parallel edges, of course), the degree of a vertex equals the number of 1's in the corresponding row or column of X.
4. Permutations of rows and of the corresponding columns imply reordering the vertices. It must be noted, however, that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged. Hence two graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related:

$$X(G_2) = R^{-1} \cdot X(G_1) \cdot R,$$

where R is a permutation matrix.
5. A graph G is disconnected and is in two components g_1 and g_2 if and only if its adjacency matrix $X(G)$ can be partitioned as

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right],$$

where $X(g_1)$ is the adjacency matrix of the component g_1 and $X(g_2)$ is that of the component g_2.

This partitioning clearly implies that there exists no edge joining any vertex in subgraph g_1 to any vertex in subgraph g_2.
6. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

Powers of X: Let us multiply by itself the 6 by 6 adjacency matrix of the simple graph in Fig. 7-7. The result, another 6 by 6 symmetric matrix X^2, is shown below (note that this is ordinary matrix multiplication in the ring of integers and *not* mod 2 multiplication):

$$X^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}.$$

The value of an off-diagonal entry in X^2, that is, ijth entry ($i \neq j$) in X^2,
= number of 1's in the dot product of ith row and jth column (or jth row) of X.
= number of positions in which both ith and jth rows of X have 1's.
= number of vertices that are adjacent to both ith and jth vertices.
= number of different paths of length two between ith and jth vertices.

Similarly, the ith diagonal entry in X^2 is the number of 1's in the ith row (or column) of matrix X. Thus the value of each diagonal entry in X^2 equals the degree of the corresponding vertex, if the graph has no self-loops.

Since a matrix commutes with matrices that are its own power,

$$X \cdot X^2 = X^2 \cdot X = X^3.$$

And since the product of two square symmetric matrices that commute is also a symmetric matrix, X^3 is a symmetric matrix. (Again note that this is an ordinary product and not mod 2.)

The matrix X^3 for the graph of Fig. 7-7 is

$$X^3 = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}.$$

Let us now consider the ijth entry of X^3.

ijth entry of X^3 = dot product of ith row X^2 and jth column (or row) of X.

$= \sum_{k=1}^{n} ik$th entry of $X^2 \cdot kj$th entry of X.

$= \sum_{k=1}^{n}$ number of all different edge sequences† of three edges from ith to jth vertex via kth vertex.

= number of different edge sequences of three edges between ith and jth vertices.

For example, consider how the 1,5th entry on X^3 for the graph of Fig. 7-7 is formed. It is given by the dot product

$$\text{row 1 of } X^2 \cdot \text{row 5 of } X = (3, 1, 0, 3, 1, 0) \cdot (1, 1, 0, 1, 0, 0)$$
$$= 3 + 1 + 0 + 3 + 0 + 0 = 7.$$

These seven different edge sequences of three edges between v_1 and v_5 are

$$\{e_1, e_1, e_2\}, \quad \{e_2, e_2, e_2\}, \quad \{e_6, e_6, e_2\}, \quad \{e_2, e_3, e_3\},$$
$$\{e_6, e_7, e_5\}, \quad \{e_2, e_5, e_5\}, \quad \{e_1, e_4, e_5\}.$$

Clearly this list includes all the paths of length three between v_1 and v_5, that is, $\{e_6, e_7, e_5\}$ and $\{e_1, e_4, e_5\}$.

It is left as an exercise for the reader to show (Problem 7-19) that the iith entry in X^3 equals twice the number of different circuits of length three (i.e., triangles) in the graph passing through the corresponding vertex v_i.

The general result that includes the properties of X, X^2, and X^3 discussed so far is expressed in Theorem 7-8.

THEOREM 7-8

Let X be the adjacency matrix of a simple graph G. Then the ijth entry in X^r is the number of different edge sequences of r edges between vertices v_i and v_j.

Proof: The theorem holds for $r = 1$, and it has been proved for $r = 2$ and 3 also. It can be proved for any positive integer r, by induction.

In other words, assume that it holds for $r - 1$, and then evaluate the ijth entry in X, with the help of the relation

$$X^r = X^{r-1} \cdot X,$$

as was done for X^3.

COROLLARY A

In a connected graph, the distance between two vertices v_i and v_j (for $i \neq j$) is k, if and only if k is the smallest integer for which the i, jth entry in x^k is nonzero.

This is a useful result in determining the distances between different pairs of vertices.

COROLLARY B

If X is the adjacency matrix of a graph G with n vertices, and

$$Y = X + X^2 + X^3 + \cdots + X^{n-1}, \qquad \text{(in the ring of integers)},$$

then G is disconnected if and only if there exists at least one entry in matrix Y that is zero.

Relationship Between A(G) *and* X(G)*:* Recall that if a graph G has no self-loops, its incidence matrix A(G) contains all the information about G. Likewise, if G has no parallel edges, its adjacency matrix X(G) contains all the information about G. Therefore, if a graph G has neither self-loops nor parallel edges (i.e., G is a simple graph), both A(G) and X(G) contain the entire information. Thus it is natural to expect that either matrix can be obtained directly from the other, in the case of a simple graph. This relationship is given in Problem 7-23.

CHROMATIC NUMBER

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the *proper coloring* (or sometimes simply *coloring*) of a graph. A graph in which every vertex has been assigned a color

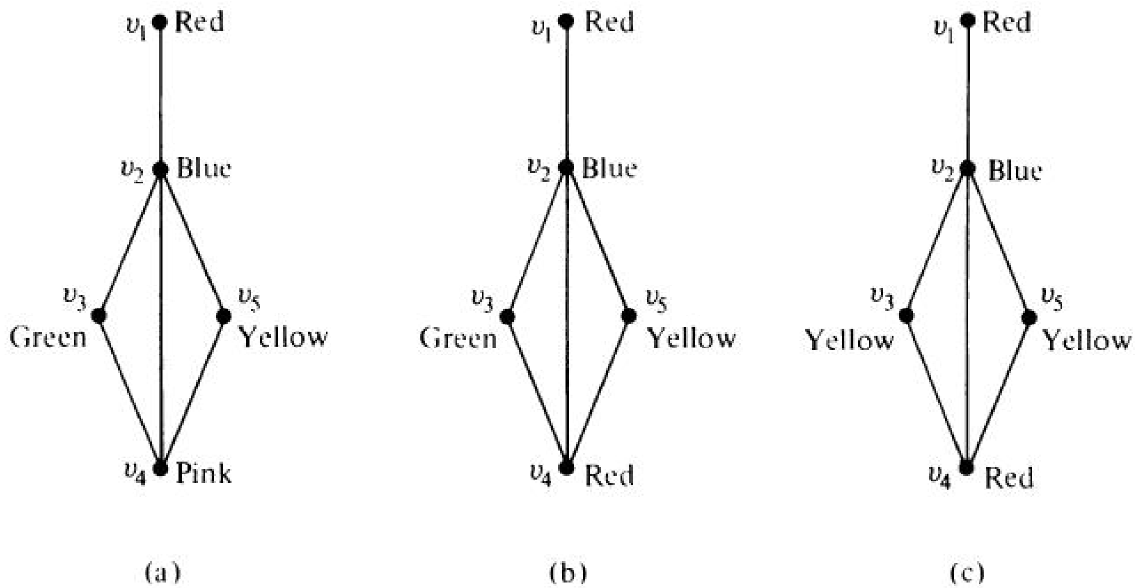


Fig. 8-1 Proper colorings of a graph.

according to a proper coloring is called a *properly colored* graph. Usually a given graph can be properly colored in many different ways. Figure 8-1 shows three different proper colorings of a graph.

The proper coloring which is of interest to us is one that requires the minimum number of colors. A graph G that requires κ different colors for its proper coloring, and no less, is called a *κ-chromatic* graph, and the number κ is called the *chromatic number* of G. You can verify that the graph in Fig. 8-1 is 3-chromatic.

In coloring graphs there is no point in considering disconnected graphs. How we color vertices in one component of a disconnected graph has no effect on the coloring of the other components. Therefore, it is usual to investigate coloring of connected graphs only. All parallel edges between two vertices can be replaced by a single edge without affecting adjacency of vertices. Self-loops must be disregarded. Thus for coloring problems we need to consider only simple, connected graphs.

Some observations that follow directly from the definitions just introduced are

1. A graph consisting of only isolated vertices is 1-chromatic.
2. A graph with one or more edges (not a self-loop, of course) is at least *2-chromatic* (also called *bichromatic*).

3. A complete graph of n vertices is n-chromatic, as all its vertices are adjacent. Hence a graph containing a complete graph of r vertices is at least r-chromatic. For instance, every graph having a triangle is at least 3-chromatic.

4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd. (This can be seen by numbering vertices 1, 2, . . . , n in sequence and assigning one color to odd vertices and another to even. If n is even, no adjacent vertices will have the same color. If n is odd, the nth and first vertex will be adjacent and will have the same color, thus requiring a third color for proper coloring.)

Proper coloring of a given graph is simple enough, but a proper coloring with the minimum number of colors is, in general, a difficult task. In fact, there has not yet been found a simple way of characterizing a κ-chromatic graph. (The brute-force method of using all possible combinations can, of course, always be applied, as in any combinatorial problem. But brute force is highly unsatisfactory, because it gets out of hand as soon as the size of the graph increases beyond a few vertices.) Chromatic numbers of some specific types of graphs will be discussed in the rest of this section.

THEOREM 8-1

Every tree with two or more vertices is 2-chromatic.

Proof: Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with color 1. Paint all vertices adjacent to v with color 2. Next, paint the vertices adjacent to these (those that just have been colored with 2) using color 1. Continue this process till every vertex in T has been painted. (See Fig. 8-2). Now in T we find that all vertices at odd distances from v have color 2, while v and vertices at even distances from v have color 1.

Now along any path in T the vertices are of alternating colors. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same color. Thus T has been properly colored with two colors. One color would not have been enough (observation 2 in this section). ■

Though a tree is 2-chromatic, not every 2-chromatic graph is a tree. (The utilities graph, for instance, is not a tree.) What then is the characterization

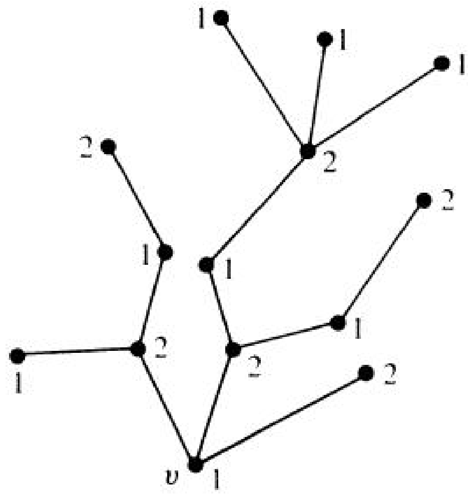


Fig. 8-2 Proper coloring of a tree.

of a 2-chromatic graph? Theorem 8-2 (due to König) characterizes all 2-chromatic graphs.

THEOREM 8-2

A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof: Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. Using the coloring procedure and the result of Theorem 8-1, let us properly color T with two colors. Now add the chords to T one by one. Since G had no circuits of odd length, the end vertices of every chord being replaced are differently colored in T. Thus G is colored with two colors, with no adjacent vertices having the same color. That is, G is 2-chromatic.

Conversely, if G has a circuit of odd length, we would need at least three colors just for that circuit (observation 4 in this section). Thus the theorem. ■

An upper limit on the chromatic number of a graph is given by Theorem 8-3, whose proof is left as an exercise (Problem 8-1).

THEOREM 8-3

If $d_{\max}$ is the maximum degree of the vertices in a graph G,

$$\text{chromatic number of } G \leq 1 + d_{\max}.$$

Brooks [8-1] showed that this upper bound can be improved by 1 if G has no complete graph of $d_{\max} + 1$ vertices. In that case

$$\text{chromatic number of } G \leq d_{\max}.$$

A graph G is called *bipartite* if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2. Thus every tree is a bipartite graph. So are the graphs in Figs. 8-6 and 8-8. Obviously, a bipartite graph can have no self-loop. A set of parallel edges between a pair of vertices can all be replaced with one edge without affecting bipartiteness of a graph.

Clearly, every 2-chromatic graph is bipartite because the coloring partitions the vertex set into two subsets V_1 and V_2 such that no two vertices in V_1 (or V_2) are adjacent. Similarly, every bipartite graph is 2-chromatic, with one trivial exception; a graph of two or more isolated vertices and with no edges is bipartite but is 1-chromatic.

In generalizing this concept, a graph G is called p-partite if its vertex set can be decomposed into p disjoint subsets $V_1, V_2, \ldots, V_p$, such that no edge in G joins the vertices in the same subset. Clearly, a κ-chromatic graph is p-partite if and only if

$$\kappa \leq p.$$

With this qualification, the results of this section on κ-chromatic graphs are applicable to κ-partite graphs also.

8-2. CHROMATIC PARTITIONING

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the coloring in Fig. 8-1(c) produces the partitioning

$$\{v_1, v_4\}, \quad \{v_2\}, \quad \text{and} \quad \{v_3, v_5\}.$$

No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set; more formally:

A set of vertices in a graph is said to be an *independent set* of vertices or simply an *independent set* (or an *internally stable set*) if no two vertices in the set are adjacent. For example, in Fig. 8-3, $\{a, c, d\}$ is an independent set. A single vertex in any graph constitutes an independent set.

A *maximal independent set* (or *maximal internally stable set*) is an independent set to which no other vertex can be added without destroying its independence property. The set $\{a, c, d, f\}$ in Fig. 8-3 is a maximal independent set. The set $\{b, f\}$ is another maximal independent set. The set $\{b, g\}$ is a third one. From the preceding example, it is clear that a graph, in general, has many maximal independent sets; and they may be of different sizes. Among all maximal independent sets, one with the largest number of vertices is often of particular interest.

Suppose that the graph in Fig. 8-3 describes the following problem. Each of the seven vertices of the graph is a possible code word to be used in some communication. Some words are so close (say, in sound) to others that they might be confused for each other. Pairs of such words that may be mistaken for one another are joined by edges. Find a largest set of code words for a reliable communication. This is a problem of finding a maximal independent set with largest number of vertices. In this simple example, $\{a, c, d, f\}$ is an answer.

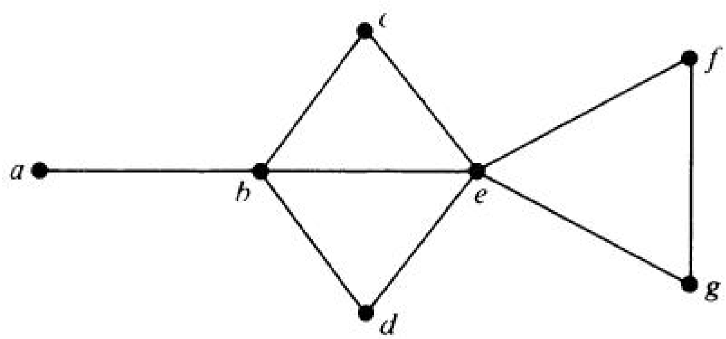


Fig. 8-3

The number of vertices in the largest independent set of a graph G is called the *independence number* (or *coefficient of internal stability*), $\beta(G)$.

Consider a κ-chromatic graph G of n vertices properly colored with κ different colors. Since the largest number of vertices in G with the same color cannot exceed the independence number $\beta(G)$, we have the inequality

$$\beta(G) \geq \frac{n}{\kappa} \cdot$$

Finding a Maximal Independent Set: A reasonable method of finding a maximal independent set in a graph G will be to start with any vertex v of G in the set. Add more vertices to the set, selecting at each stage a vertex that is not adjacent to any of those already selected. This procedure will ultimately produce a maximal independent set. This set, however, is not necessarily a maximal independent set with a largest number of vertices.

Finding All Maximal Independent Sets: A reasonable (but not very efficient for large graphs) method for obtaining all maximal independent sets in any graph can be developed using Boolean arithmetic on the vertices. Let each vertex in the graph be treated as a Boolean variable. Let the logical (or Boolean) sum $a + b$ denote the operation of including vertex a or b or both; let the logical multiplication ab denote the operation of including both vertices a and b, and let the Boolean complement a' denote that vertex a is not included.

For a given graph G we must find a maximal subset of vertices that does not include the two end vertices of any edge in G. Let us express an edge (x, y) as a Boolean product, xy, of its end vertices x and y, and let us sum all such products in G to get a Boolean expression

$$\varphi = \Sigma\, xy \qquad \text{for all } (x, y) \text{ in } G.$$

Let us further take the Boolean complement φ' of this expression, and express it as a sum of Boolean products:

$$\varphi' = f_1 + f_2 + \cdots + f_k.$$

A vertex set is a maximal independent set if and only if $\varphi = 0$ (logically false), which is possible if and only if $\varphi' = 1$ (true), which is possible if and only if at least one $f_i = 1$, which is possible if and only if each vertex appearing in f_i (in complemented form) is excluded from the vertex set of G. Thus each f_i will yield a maximal independent set, and every maximal independent set will be produced by this method. This procedure can be best explained by an example. For the graph G in Fig. 8-3,

$$\varphi = ab + bc + bd + be + ce + de + ef + eg + fg,$$
$$\varphi' = (a' + b')(b' + c')(b' + d')(b' + e')(c' + e')(d' + e')$$
$$(e' + f')(e' + g')(f' + g').$$

Multiplying these out and employing the usual identities of Boolean arithmetic, such as

$$aa = a,$$
$$a + a = a,$$
$$a + ab = a,$$

we get

$$\varphi' = b'e'f' + b'e'g' + a'c'd'e'f' + a'c'd'e'g' + b'c'd'f'g'.$$

Now if we exclude from the vertex set of G vertices appearing in any one of these five terms, we get a maximal independent set. The five maximal independent sets are

$$acdf, \quad acdg, \quad bg, \quad bf, \quad \text{and} \quad ae.$$

These are all the maximal independent sets of the graph.

Finding Independence and Chromatic Numbers: Once all the maximal independent sets of G have been obtained, we find the size of the one with the largest number of vertices to get the independence number $\beta(G)$. The independence number of the graph in Fig. 8-3 is four.

To find the chromatic number of G, we must find the minimum number of these (maximal independent) sets, which collectively include all the vertices of G. For the graph in Fig. 8-3, sets $\{a, c, d, f\}$, $\{b, g\}$, and $\{a, e\}$, for example, satisfy this condition. Thus the graph is 3-chromatic.

Chromatic Partitioning: Given a simple, connected graph G, partition all vertices of G into the smallest possible number of disjoint, independent sets. This problem, known as the *chromatic partitioning* of graphs, is perhaps the most important problem in partitioning of graphs.

By enumerating all maximal independent sets and then selecting the smallest number of sets that include all vertices of the graph, we just solved this problem. The following four are some chromatic partitions of the graph in Fig. 8-3, for example.

$$\{(a, c, d, f), (b, g), (e)\},$$
$$\{(a, c, d, g), (b, f), (e)\},$$
$$\{(c, d, f), (b, g), (a, e)\},$$
$$\{(c, d, g), (b, f), (a, e)\}.$$

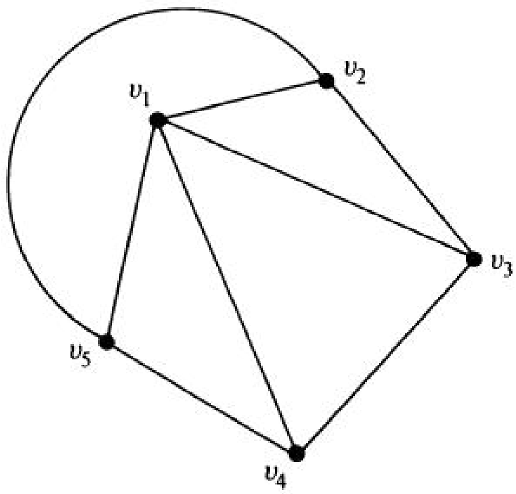


Fig. 8-4 A 3-chromatic graph.

This method of chromatic partitioning (requiring enumeration of all maximal independent sets) is inefficient and needs prohibitively large amounts of computer memory. A more efficient method for computer implementation is proposed in [8-6].

Uniquely Colorable Graphs: A graph that has only one chromatic partition is called a *uniquely colorable* graph. The graph in Fig. 8-3 is *not* a uniquely colorable graph, but the one in Fig. 8-4 is (Problem 8-2). For some interesting properties of uniquely colorable graphs, the reader is referred to Chapter 12 of [1-5].

A concept related to that of the independent set and chromatic partitioning is the dominating set, to be discussed next.

Dominating Sets: A *dominating set* (or an *externally stable set*) in a graph G is a set of vertices that dominates every vertex v in G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set. For instance, the vertex set $\{b, g\}$ is a dominating set in Fig. 8-3. So is the set $\{a, b, c, d, f\}$ a dominating set. A dominating set need not be independent. For example, the set of all its vertices is trivially a dominating set in every graph.

In many applications one is interested in finding minimal dominating sets defined as follows:

A *minimal dominating set* is a dominating set from which no vertex can be removed without destroying its dominance property. For example, in Fig. 8-3, $\{b, e\}$ is a minimal dominating set. And so is $\{a, c, d, f\}$. Observations that follow from these definitions are

1. Any one vertex in a complete graph constitutes a minimal dominating set.
2. Every dominating set contains at least one minimal dominating set.
3. A graph may have many minimal dominating sets, and of different sizes. [The number of vertices in the smallest minimal dominating set of a graph G is called the *domination number*, $\alpha(G)$.]
4. A minimal dominating set may or may not be independent.
5. Every maximal independent set is a dominating set. For if an independent set does not dominate the graph, there is at least one vertex that is neither in the set nor adjacent to any vertex in the set. Such a vertex can be added to the independent set without destroying its independence. But then the independent set could not have been maximal.
6. An independent set has the dominance property only if it is a maximal independent set. Thus an *independent dominating set* is the same as a maximal independent set.
7. In any graph G,

$$\alpha(G) \leq \beta(G).$$

Finding Minimal Dominating Sets: A method for obtaining all minimal dominating sets in a graph will now be developed. The method, like the one for finding all maximal independent sets, also uses Boolean arithmetic.

To dominate a vertex v_i we must either include v_i or any of the vertices adjacent to v_i. A minimum set satisfying this condition for every vertex v_i is a desired set. Therefore, for every vertex v_i in G let us form a Boolean product of sums $(v_i + v_{i_1} + v_{i_2} + \cdots + v_{i_d})$, where $v_{i_1}, v_{i_2}, \ldots, v_{i_d}$ are the vertices adjacent to v_i, and d is the degree of v_i:

$$\theta = \prod (v_i + v_{i_1} + v_{i_2} + \cdots + v_{i_d}) \qquad \text{for all } v_i \text{ in } G.$$

When θ is expressed as a sum of products, each term in it will represent a minimal dominating set. Let us illustrate this algorithm using the graph of Fig. 8-3:

Consider the following expression θ for Fig. 8-3:

$$\theta = (a + b)(b + c + d + e + a)(c + b + e)(d + b + e)$$
$$(e + b + c + d + f + g)(f + e + g)(g + e + f).$$

Since in Boolean arithmetic $(x + y)x = x$,

$$\theta = (a + b)(b + c + e)(b + d + e)(e + f + g)$$
$$= ae + be + bf + bg + acdf + acdg.$$

Each of the six terms in the preceding expression represents a minimal dominating set. Clearly, $\alpha(G) = 2$, for this example.

CHROMATIC POLYNOMIAL

In general, a given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the *chromatic polynomial* of G and is defined as follows:

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.

Let c_i be the different ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in

$$\binom{\lambda}{i} \qquad \text{different ways,}$$

there are $c_i\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.

Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is,

$$c_n = n!.$$

As an illustration, let us find the chromatic polynomial of the graph given in Fig. 8-4.

$$P_5(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}.$$

Since the graph in Fig. 8-4 has a triangle, it will require at least three different colors for proper coloring. Therefore,

$$c_1 = c_2 = 0 \quad \text{and} \quad c_5 = 5!.$$

Moreover, to evaluate c_3, suppose that we have three colors x, y, and z. These three colors can be assigned properly to vertices v_1, v_2, and v_3 in $3! = 6$ different ways. Having done that, we have no more choices left, because vertex v_5 must have the same color as v_3, and v_4 must have the same color as v_2. Therefore,

$$c_3 = 6.$$

Similarly, with four colors, v_1, v_2, and v_3 can be properly colored in $4 \cdot 6 = 24$ different ways. The fourth color can be assigned to v_4 or v_5, thus providing two choices. The fifth vertex provides no additional choice. Therefore,

$$c_4 = 24 \cdot 2 = 48.$$

Substituting these coefficients in $P_5(\lambda)$, we get, for the graph in Fig. 8-4,

$$\begin{aligned} P_5(\lambda) &= \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) \\ &\quad + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7). \end{aligned}$$

The presence of factors $\lambda - 1$ and $\lambda - 2$ indicates that G is at least 3-chromatic.

Chromatic polynomials have been studied in great detail in the literature. The interested reader is referred to [8-5] for a more thorough discussion of their properties. Theorems 8-4, 8-5, and 8-6 should provide a glimpse into the colorful world of chromatic polynomials.

THEOREM 8-4

A graph of n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1).$$

Proof: With λ colors, there are λ different ways of coloring any selected vertex of a graph. A second vertex can be colored properly in exactly $\lambda - 1$ ways, the third in $\lambda - 2$ ways, the fourth in $\lambda - 3$ ways, . . . , and the nth in $\lambda - n + 1$ ways if and only if every vertex is adjacent to every other. That is, if and only if the graph is complete. ■

THEOREM 8-5

An n-vertex graph is a tree if and only if its chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-1}.$$

Proof: That the theorem holds for $n = 1, 2$ is immediately evident. It is left as an exercise to prove the theorem by induction (Problem 8-9).

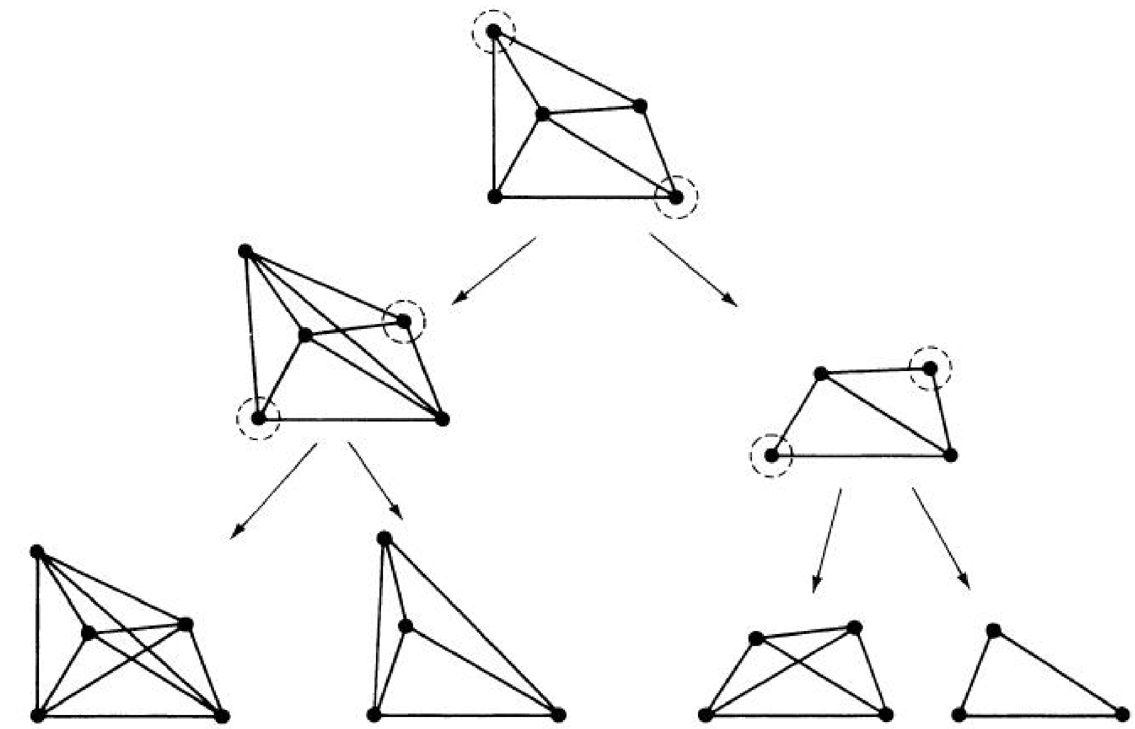
THEOREM 8-6

Let a and b be two nonadjacent vertices in a graph G. Let G' be a graph obtained by adding an edge between a and b. Let G'' be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

Proof: The number of ways of properly coloring G can be grouped into two cases, one such that vertices a and b are of the same color and the other such that a and b are of different colors. Since the number of ways of properly coloring G such that a and b have different colors = number of ways of properly coloring G', and

number of ways of properly coloring G such that a and b have the same color



$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$ $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$ $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$ $\lambda(\lambda-1)(\lambda-2)$

$$\begin{aligned} P_5(\lambda) \text{ of } G &= \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) \\ &\quad + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7) \end{aligned}$$

Fig. 8-5 Evaluation of a chromatic polynomial.

= number of ways of properly coloring G'',

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''. \quad \blacksquare$$

Theorem 8-6 is often used in evaluating the chromatic polynomial of a graph. For example, Fig. 8-5 illustrates how the chromatic polynomial of a graph G is expressed as a sum of the chromatic polynomials of four complete graphs. The pair of nonadjacent vertices shown enclosed in circles is the one used for reduction at that stage.

In the last three sections we have been concerned with proper coloring of the vertices in a graph. Suppose that we are interested in coloring the edges rather than the vertices. It is reasonable to call two edges *adjacent* if they have one end vertex in common (but are not parallel). A proper coloring of edges then requires that adjacent edges should be of different colors. Some results on proper coloring of edges, similar to the results given in Sections 8-1 and 8-2, can be derived (Problem 8-19).

Moreover, a set of edges in which no two are adjacent is similar to an independent set of vertices. Such a set of edges is called a *matching*, the subject of the next section.

MATCHINGS

Suppose that four applicants a_1, a_2, a_3, and a_4 are available to fill six vacant positions p_1, p_2, p_3, p_4, p_5, and p_6. Applicant a_1 is qualified to fill position p_2 or p_5. Applicant a_2 can fill p_2 or p_5. Applicant a_3 is qualified for p_1, p_2, p_3, p_4, or p_6. Applicant a_4 can fill jobs p_2 or p_5. This situation is represented by the graph in Fig. 8-6. The vacant positions and applicants are represented by vertices. The edges represent the qualifications of each applicant for filling

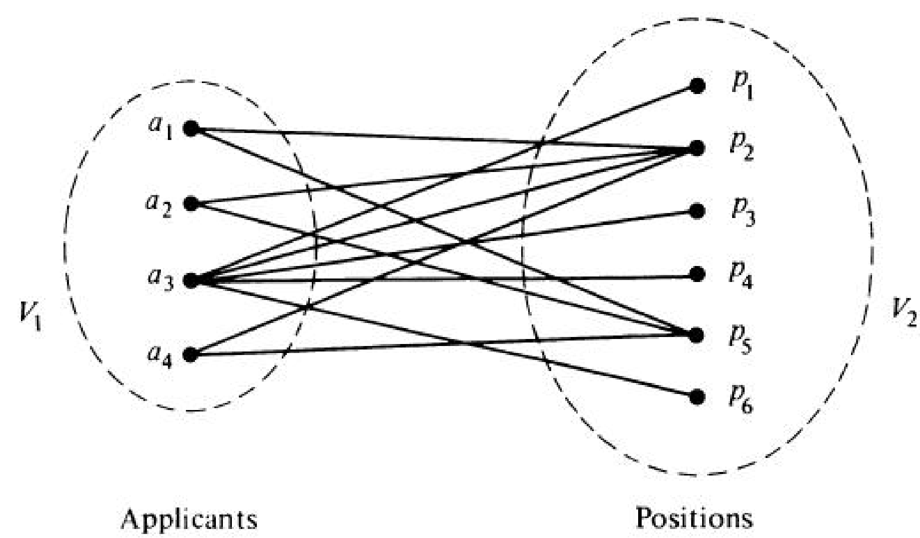


Fig. 8-6 Bipartite graph.

(a) (b)

Fig. 8-7 Graph and two of its maximal matchings.

different positions. The graph clearly is bipartite, the vertices falling into two sets $V_1 = \{a_1, a_2, a_3, a_4\}$ and $V_2 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$.

The questions one is most likely to ask in this situation are: Is it possible to hire all the applicants and assign each a position for which he is suitable? If the answer is no, what is the maximum number of positions that can be filled from the given set of applicants?

This is a problem of *matching* (or *assignment*) of one set of vertices into another. More formally, a *matching* in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is obviously a matching.

A *maximal matching* is a matching to which no edge in the graph can be added. For example, in a complete graph of three vertices (i.e., a triangle) any single edge is a maximal matching. The edges shown by heavy lines in Fig. 8-7 are two maximal matchings. Clearly, a graph may have many different maximal matchings, and of different sizes. Among these, the maximal *matchings*. In Fig. 8-7(b), a largest maximal matching is shown in heavy lines. The number of edges in a largest maximal matching is called the *matching number* of the graph.

Although matching is defined for any graph, it is mostly studied in the context of bipartite graphs, as suggested by the introduction to this section. In a bipartite graph having a vertex partition V_1 and V_2, a *complete matching* of vertices in set V_1 into those in V_2 is a matching in which there is one edge incident with every vertex in V_1. In other words, every vertex in V_1 is matched against some vertex in V_2. Clearly, a complete matching (if it exists) is a largest maximal matching, whereas the converse is not necessarily true.

For the existence of a complete matching of set V_1 into set V_2, first we must have at least as many vertices in V_2 as there are in V_1. In other words, there must be at least as many vacant positions as the number of applicants if all the applicants are to be hired. This condition, however, is not sufficient. For example, in Fig. 8-6, although there are six positions and four applicants, a complete matching does not exist. Of the three applicants a_1, a_2, and a_4, each qualifies for the same two positions p_2 and p_5, and therefore one of the three applicants cannot be matched.

This leads us to another necessary condition for a complete matching: Every subset of r vertices in V_1 must collectively be adjacent to at least r vertices in V_2, for all values of $r = 1, 2, \ldots, |V_1|$. This condition is not satisfied in Fig. 8-6. The subset $\{a_1, a_2, a_4\}$ of three vertices has only two vertices p_2 and p_5 adjacent to them. That this condition is also sufficient for existence of a complete matching is indeed surprising. Theorem 8-7 is a formal statement and proof of this result.

THEOREM 8-7

A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r.

Proof: The "only if" part (i.e., the necessity of a subset of r applicants collectively qualifying for at least r jobs) is immediate and has already been pointed out. The sufficiency (i.e., the "if" part) can be proved by induction on r, as the theorem trivially holds for $r = 1$. For a complete proof, the student is referred to Theorem 11-1 in [8-3], Theorem 5-19 in [4-5], or Chapter 4 in [1-9].

Problem of Distinct Representatives: Five senators s_1, s_2, s_3, s_4, and s_5 are members of three committees, c_1, c_2, and c_3. The membership is shown in Fig. 8-8. One member from each committee is to be represented in a super-committee. Is it possible to send one distinct representative from each of the committees†?

This problem is one of finding a complete matching of a set V_1 into set V_2 in a bipartite graph. Let us use Theorem 8-7 and check if r vertices from V_1 are collectively adjacent to at least r vertices from V_2, for all values of r. The result is shown in Table 8-1 (ignore the last column for the time being).

Thus for this example the condition for the existence of a complete matching is satisfied as stated in Theorem 8-7. Hence it is possible to form the super-committee with one distinct representative from each committee.

The problem of distinct representatives just solved was a small one. A

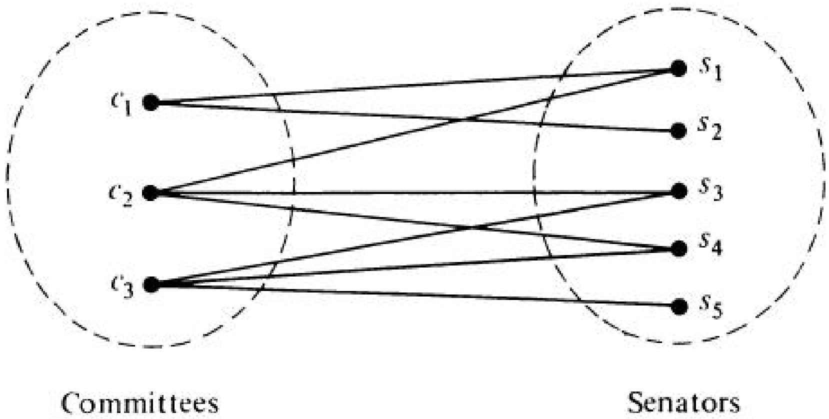


Fig. 8-8 Membership of committees.

	V_1	V_2	$r-q$
$r=1$	$\{c_1\}$	$\{s_1, s_2\}$	-1
	$\{c_2\}$	$\{s_1, s_3, s_4\}$	-2
	$\{c_3\}$	$\{s_3, s_4, s_5\}$	-2
$r=2$	$\{c_1, c_2\}$	$\{s_1, s_2, s_3, s_4\}$	-2
	$\{c_2, c_3\}$	$\{s_1, s_3, s_4, s_5\}$	-2
	$\{c_3, c_1\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-3
$r=3$	$\{c_1, c_2, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-2

Table 8-1

larger problem would have become unwieldy. If there are M vertices in V_1, Theorem 8-7 requires that we take all $2^M - 1$ nonempty subsets of V_1 and find the number of vertices of V_2 adjacent collectively to each of these. In most cases, however, the following simplified version of Theorem 8-7 will suffice for detection of a complete matching in any large graph.

THEOREM 8-8

In a bipartite graph a complete matching of V_1 into V_2 exists if (but not only if) there is a positive integer m for which the following condition is satisfied:

$$\text{degree of every vertex in } V_1 \geq m \geq \text{degree of every vertex in } V_2.$$

Proof: Consider a subset of r vertices in V_1. These r vertices have at least $m \cdot r$ edges incident on them. Each $m \cdot r$ edge is incident to some vertex in V_2. Since the degree of every vertex in set V_2 is no greater than m, these $m \cdot r$ edges are incident on at least $(m \cdot r)/m = r$ vertices in V_2.

Thus any subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2. Therefore, according to Theorem 8-7, there exists a complete matching of V_1 into V_2. ■

In the bipartite graph of Fig. 8-8,

$$\text{degree of every vertex in } V_1 \geq 2 \geq \text{degree of every vertex in } V_2.$$

Therefore, there exists a complete matching.

In the bipartite graph of Fig. 8-6 no such number is found, because the degree of $p_2 = 4 >$ degree of a_1.

It must be emphasized that the condition of Theorem 8-8 is a sufficient condition and not necessary for the existence of a complete matching. It will be instructive for the reader to sketch a bipartite graph that does not satisfy Theorem 8-8 and yet has a complete matching (Problem 8-15).

The matching problem or the problem of distinct representatives is also called the *marriage problem* (whose solution, unfortunately, is of little use to those with real marital problems!) See Problem 8-16.

If one fails to find a complete matching, he is most likely to be interested in finding a maximal matching, that is, to pair off as many vertices of V_1 with those in V_2 as possible. For this purpose, let us define a new term called *deficiency*, $\delta(G)$, of a bipartite graph G.

A set of r vertices in V_1 is collectively incident on, say, q vertices of V_2. Then the maximum value of the number $r - q$ taken over all values of $r = 1, 2, \ldots$ and all subsets of V_1 is called the deficiency $\delta(G)$ of the bipartite graph G.

Theorem 8-7, expressed in terms of the deficiency, states that a complete matching in a bipartite graph G exists if and only if

$$\delta(G) \leq 0.$$

For example, the deficiency of the bipartite graph in Fig. 8-7 is -1 (the largest number in the last column of Table 8-1). It is suggested that you prepare a table for the graph of Fig. 8-6, similar to Table 8-1, and verify that the deficiency is $+1$ for this graph (Problem 8-17).

Theorem 8-9 gives the size of the maximal matching for a bipartite graph with a positive deficiency.

THEOREM 8-9

The maximal number of vertices in set V_1 that can be matched into V_2 is equal to

$$\text{number of vertices in } V_1 - \delta(G).$$

The proof of Theorem 8-9 can be found in [8-3], page 288. The size of a maximal matching in Fig. 8-6, using Theorem 8-9, is obtained as follows:

$$\text{number of vertices in } V_1 - \delta(G) = 4 - 1 = 3.$$

Matching and Adjacency Matrix: Consider a bipartite graph G with non-adjacent sets of vertices V_1 and V_2, having number of vertices n_1 and n_2, respectively, and let $n_1 \leq n_2$, $n_1 + n_2 = n$, the number of vertices in G. The adjacency matrix $\mathsf{X}(G)$ of G can be written in the form

$$\mathsf{X}(G) = \left[\begin{array}{c|c} 0 & \mathsf{X}_{12} \\ \hline \mathsf{X}_{12}^T & 0 \end{array}\right],$$

where the submatrix X_{12} is the n_1 by n_2, (0, 1)-matrix containing the information as to which of the n_1 vertices of V_1 are connected to which of the n_2 vertices of V_2. Matrix X_{12}^T is the transpose of X_{12}.

Clearly, all the information about the bipartite graph G is contained in its X_{12} matrix.

A matching V_1 into V_2 corresponds to a selection of the 1's in the matrix X_{12} such that no line (i.e., a row or a column) has more than one 1.

The matching is complete if the n_1 by n_2 matrix made of selected 1's has exactly one 1 in every row. For example, the X_{12} matrix for Fig. 8-8 is

$$\mathsf{X}_{12} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array},$$

$$n_1 = 3, \quad n_2 = 5, \quad n = 8, \quad \text{and} \quad n_1 \leq n_2,$$

$$V_1 = \{c_1, c_2, c_3\}$$

$$V_2 = \{s_1, s_2, s_3, s_4, s_5\}.$$

A complete matching of V_1 into V_2 is given by

$$\mathsf{M} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

A maximal matching corresponds to the selection of a largest possible number of 1's from X_{12} such that no row in it has more than one 1. Therefore, according to Theorem 8-9, in matrix X_{12} the largest number of 1's, no two of which are in one row, is equal to

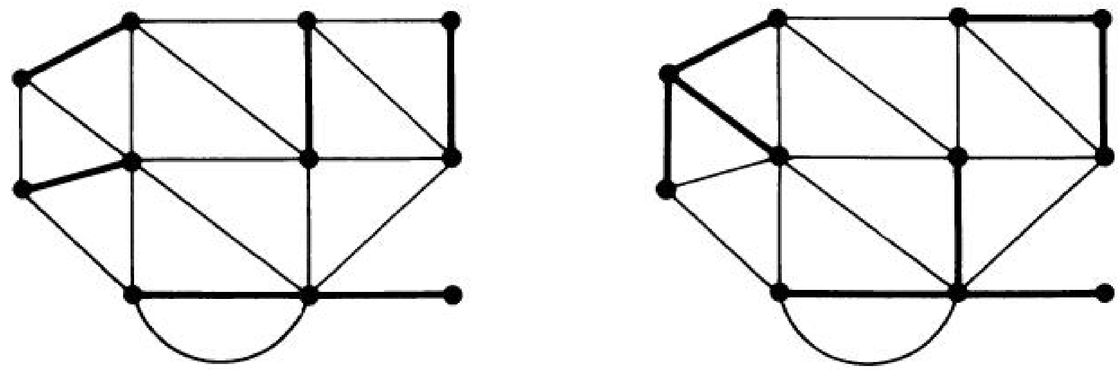
$$\text{number of vertices in } V_1 - \delta(G).$$

Matching problems in bipartite graphs can also be formulated in terms of the flow problem (see Section 14-5). All edges are assumed to be of unit capacity, and the problem of finding a maximal matching is reduced to the problem of maximizing flow from the source to the sink (also see [8-3]).

COVERINGS

In a graph G, a set g of edges is said to *cover* G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be an *edge covering*, a *covering subgraph*, or simply a *covering* of G. For example, a graph G is trivially its own covering. A spanning tree in a connected graph (or a spanning forest in an unconnected graph) is another covering. A Hamiltonian circuit (if it exists) in a graph is also a covering.

Just any covering is too general to be of much interest. We have already dealt with some coverings with specific properties, such as spanning trees and



(a) (b)

Fig. 8-9 Graph and two of its minimal coverings.

Hamiltonian circuits. In this section we shall investigate the *minimal covering*—a covering from which no edge can be removed without destroying its ability to cover the graph. In Fig. 8-9 a graph and two of its minimal coverings are shown in heavy lines.

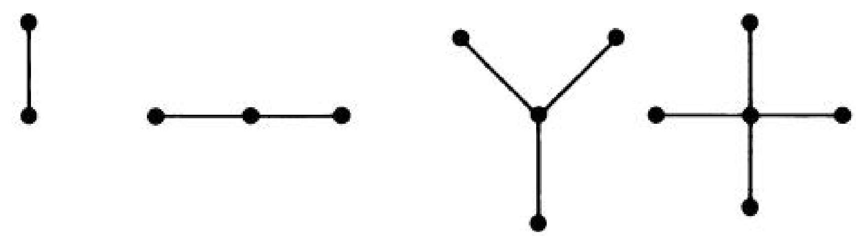
The following observations should be made:

1. A covering exists for a graph if and only if the graph has no isolated vertex.
2. A covering of an n-vertex graph will have at least $[n/2]$ edges. ($[x]$ denotes the smallest integer not less than x.)
3. Every pendant edge in a graph is included in every covering of the graph.
4. Every covering contains a minimal covering.
5. If we denote the remaining edges of a graph by $(G - g)$, the set of edges g is a covering if and only if, for every vertex v, the degree of vertex in $(G - g) \leq$ (degree of vertex v in G) $- 1$.
6. No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an n-vertex graph can contain no more than $n - 1$ edges.

7. A graph, in general, has many minimal coverings, and they may be of different sizes (i.e., consisting of different numbers of edges). The number of edges in a minimal covering of the smallest size is called the *covering number* of the graph.

THEOREM 8-10

A covering g of a graph is minimal if and only if g contains no paths of length three or more.



(a) (b) (c) (d)

Fig. 8-10 Star graphs of one, two, three, and four edges.

Proof: Suppose that a covering g contains a path of length three, and it is

$$v_1 e_1 v_2 e_2 v_3 e_3 v_4.$$

Edge e_2 can be removed without leaving its end vertices v_2 and v_3 uncovered. Therefore, g is not a minimal covering.

Conversely, if a covering g contains no path of length three or more, all its components must be *star graphs* (i.e., graphs in the shape of stars; see Fig. 8-10). From a star graph no edge can be removed without leaving a vertex uncovered. That is, g must be a minimal covering. ■

Suppose that the graph in Fig. 8-9 represents the street map of a part of a city. Each of the vertices is a potential trouble spot and must be kept under the surveillance of a patrol car. How will you assign a minimum number of patrol cars to keep every vertex covered?

The answer is a smallest minimal covering. The covering shown in Fig. 8-9(a) is an answer, and it requires six patrol cars. Clearly, since there are 11 vertices and no edge can cover more than two, less than six edges cannot cover the graph.

Minimization of Switching Functions†: An important step in the logical design of a digital machine is to minimize Boolean functions before implementing them. Suppose we are interested in building a logical circuit that gives the following function F of four Boolean variables w, x, y, and z.

$$F = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}yz + \bar{w}xy\bar{z} + \bar{w}xyz + wxyz,$$

where $+$ denotes logical OR, xy denotes x AND y, and $\bar{x}$ denotes NOT x.

Let us represent each of the seven terms in F by a vertex, and join every pair of vertices that differ only in one variable. Such a graph is shown in Fig. 8-11.

An edge between two vertices represents a term with three variables.

A minimal cover of this graph will represent a simplified form of F, performing the same function as F, but with less logic hardware.

The pendant edges 1 and 7 must be included in every covering of the

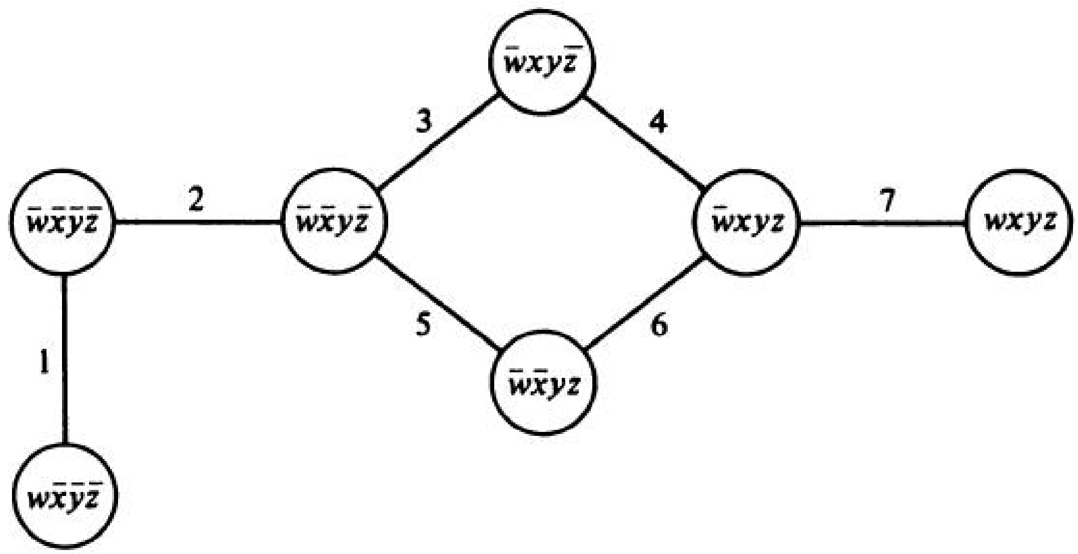


Fig. 8-11 Graph representation of a Boolean function.

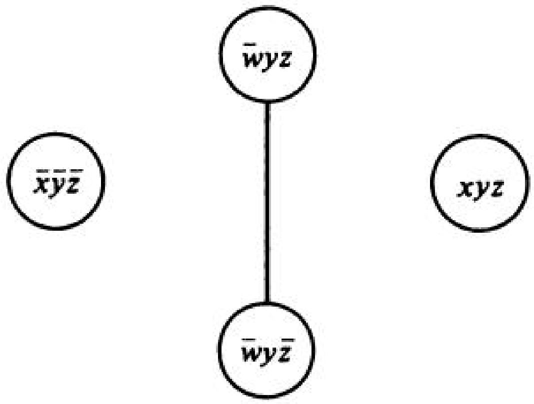


Fig. 8-12

graph. Therefore, the terms

$$\bar{x}\bar{y}\bar{z} \quad \text{and} \quad xyz \quad \text{are essential.}$$

Two additional edges 3 and 6 (or 4 and 5 or 3 and 5) will cover the remainder. Thus a simplified version of F is

$$F = \bar{x}\bar{y}\bar{z} + xyz + \bar{w}y\bar{z} + \bar{w}yz.$$

This expression can again be represented by a graph of four vertices, as shown in Fig. 8-12.

The essential terms $\bar{x}\bar{y}\bar{z}$ and xyz cannot be covered by any edge, and hence cannot be minimized further. One edge will cover the remaining two vertices in Fig. 8-12. Thus the minimized Boolean expression is

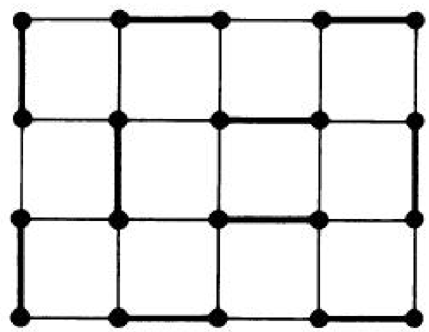
$$F = \bar{x}\bar{y}\bar{z} + xyz + \bar{w}y.$$

Dimer Problem: In crystal physics, a crystal is represented by a three-dimensional lattice. Each vertex in the lattice represents an atom, and an edge between vertices represents the bond between the two atoms. In the study of the surface properties of crystals, one is interested in two-dimensional lattices, such as the two shown in Fig. 1-10.

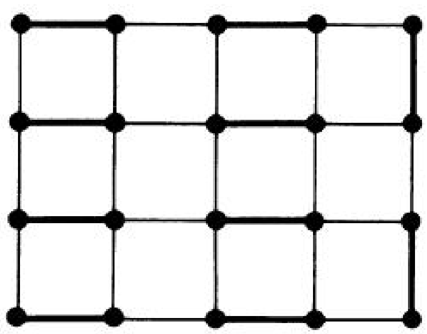
To obtain an analytic expression for certain surface properties of crystals consisting of diatomic molecules (also called *dimers*), one is required to find the number of ways in which all atoms on a two-dimensional lattice can be paired off as molecules (each consisting of two atoms). The problem is equivalent to finding all different coverings of a given graph such that every vertex in the covering is of degree one. Such a covering in which every vertex is of degree one is called a *dimer covering* or a *1-factor*. A dimer covering is obviously a matching because no two edges in it are adjacent. Moreover, a dimer covering is a maximal matching. This is why a dimer covering is often referred to as a *perfect matching*.

Two different dimer coverings are shown in heavy lines in the graph in Fig. 8-13.

Clearly, a graph must have an even number of vertices to have a dimer covering. This condition, however, is not enough (Problem 8-21).



(a)



(b)

Fig. 8-13 Two dimer coverings of a graph.

FOUR-COLOR PROBLEM

So far we have considered proper coloring of vertices and proper coloring of edges. Let us briefly consider the *proper coloring of regions* in a planar graph (embedded on a plane or sphere). Just as in coloring of vertices and edges, the regions of a planar graph are said to be properly colored if no two *contiguous* or *adjacent regions* have the same color. (Two regions are said to be adjacent if they have a common edge between them. Note that one or more vertices in common does not make two regions adjacent.) The proper coloring of regions is also called *map coloring*, referring to the fact that in an atlas different countries are colored such that countries with common boundaries are shown in different colors.

Once again we are not interested in just properly coloring the regions of a given graph. We are interested in a coloring that uses the minimum number of colors. This leads us to the most famous conjecture in graph theory. The conjecture is that every map (i.e., a planar graph) can be properly colored with four colors. The *four-color conjecture*, already referred to in Chapter 1, has been worked on by many famous mathematicians for the past 100 years. No one has yet been able to either prove the theorem or come up with a map (in a plane) that requires more than four colors.

That at least four colors are necessary to properly color a graph is immediate from Fig. 8-14, and that five colors will suffice for any planar graph will be shown shortly.

Two remarks may be made here in passing. Paradoxically, for surfaces more complicated than the plane (or sphere) corresponding theorems have been proved. For example, it has been proved that seven colors are necessary and sufficient for properly coloring maps on the surface of a torus.† Second, it has been proved that all maps containing less than 40 regions can be properly colored with four colors. Therefore, if in general the four-color conjecture is false, the counterexample has to be a very complicated and large one.

Vertex Coloring Versus Region Coloring: From Chapter 5 we know that a graph has a dual if and only if it is planar. Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G^*, and

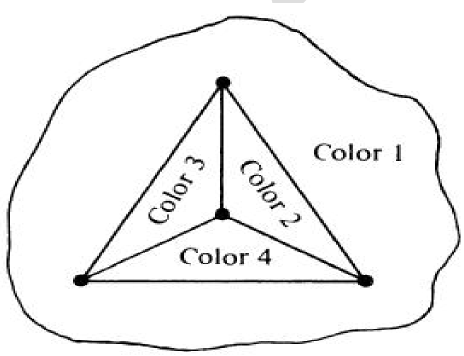


Fig. 8-14 Necessity of four colors.

vice versa. Thus the four-color conjecture can be restated as follows: *Every planar graph has a chromatic number of four or less.*

Five-Color Theorem: We shall now show that every planar map can be properly colored with five colors:

THEOREM 8-11

The vertices of every planar graph can be properly colored with five colors.

Proof: The theorem will be proved by induction. Since the vertices of all graphs (self-loop-free, of course†) with 1, 2, 3, 4, or 5 vertices can be properly colored with five colors, let us assume that vertices of every planar graph with $n - 1$ vertices can be properly colored with five colors. Then, if we prove that any planar graph G with n vertices will require no more than five colors, we shall have proved the theorem.

Consider the planar graph G with n vertices. Since G is planar, it must have at least one vertex with degree five or less (Problem 5-4). Let this vertex be v.

Let G' be a graph (of $n - 1$ vertices) obtained from G by deleting vertex v (i.e., v and all edges incident on v). Graph G' requires no more than five colors, according to the induction hypothesis. Suppose that the vertices in G' have been properly colored, and now we add to it v and all edges incident on v. If the degree of v is 1, 2, 3, or 4, we have no difficulty in assigning a proper color to v.

1, 2, 3, or 4, we have no difficulty in assigning a proper color to v.

This leaves only the case in which the degree of v is five, and all the five colors have been used in coloring the vertices adjacent to v, as shown in Fig. 8-15(a). (Note that Fig. 8-15 is part of a planar representation of graph G'.)

Suppose that there is a path in G' between vertices a and c colored alternately with colors 1 and 3, as shown in Fig. 8-15(b). Then a similar path between b and d, colored alternately with colors 2 and 4, cannot exist; otherwise, these two paths

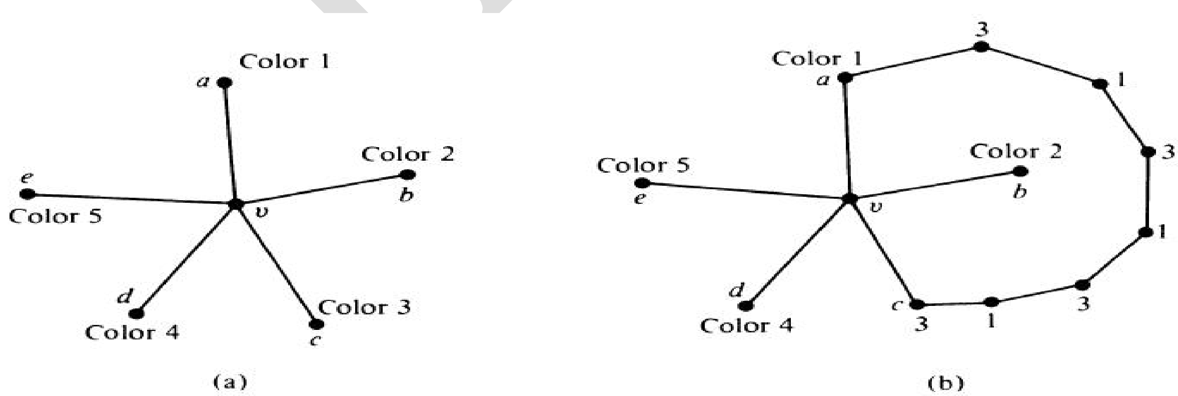


Fig. 8-15 Reassigning of colors.

will intersect and cause G to be nonplanar. (This is a consequence of the Jordan curve theorem, used in Section 5-3, also.)

If there is no path between b and d colored alternately with colors 2 and 4, starting from vertex b we can interchange colors 2 and 4 of all vertices connected to b through vertices of alternating colors 2 and 4. This interchange will paint vertex b with color 4 and yet keep G' properly colored. Since vertex d is still with color 4, we have color 2 left over with which to paint vertex v.

Had we assumed that there was no path between a and c of vertices painted alternately with colors 1 and 3, we would have released color 1 or 3 instead of color 2. And thus the theorem. ■

Regularization of a Planar Graph: Removing every vertex of degree one (together with the pendant edge) from the graph G does not affect the regions of a planar graph. Nor does the elimination of every vertex of degree two, by merging the two edges in series (Fig. 5-6), have any effect on the regions of a planar graph.

Now consider a typical vertex v of degree four or more in a planar graph. Let us replace vertex v by a small circle with as many vertices as there were incidences on v. This results in a number of vertices each of degree three (see Fig. 8-16).

Performing this transformation on every vertex of degree four or more in a planar graph G will produce another planar graph H in which every vertex is of degree three. When the regions of H have been properly colored, a proper coloring of the regions of G can be obtained simply by shrinking each of the new regions back to the original vertex.

Such a transformation may be called *regularization* of a planar graph, because it converts a planar graph G into a regular planar graph H of degree three. Clearly, if H can be colored with four colors, so can G. Thus, for map-coloring problems, it is sufficient to confine oneself to (connected) planar, regular graphs of degree three. And the four-color conjecture may be restated as follows:

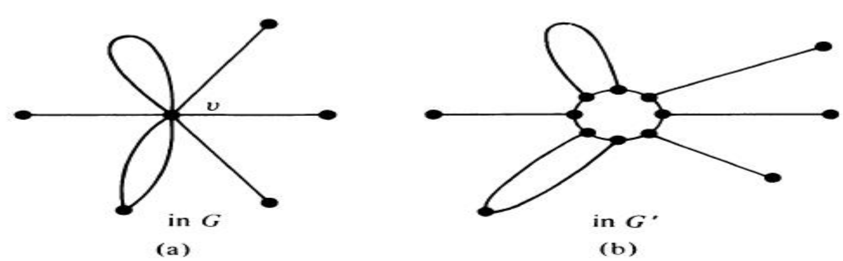


Fig. 8-16 Regularization of a graph.

The regions of every planar, regular graph of degree three can be colored properly with four colors.

If, in a planar graph G, every vertex is of degree three, its dual G^* is a planar graph in which every region is bounded by three edges; that is, G^* is a triangular graph. Thus the four-color conjecture may again be restated as follows: The chromatic number of every triangular, planar graph is four or less.

UNIT-IV

SYLLABUS

Directed Graphs – Types of Directed Graphs - Types of enumeration, counting labeled trees, counting unlabelled trees, Polya's counting theorem, graph enumeration with Polya's theorem.

DIRECTED GRAPH

A *directed graph* (or a *digraph* for short) G consists of a set of vertices $V = \{v_1, v_2, \ldots\}$, a set of edges $E = \{e_1, e_2, \ldots\}$, and a mapping Ψ that maps

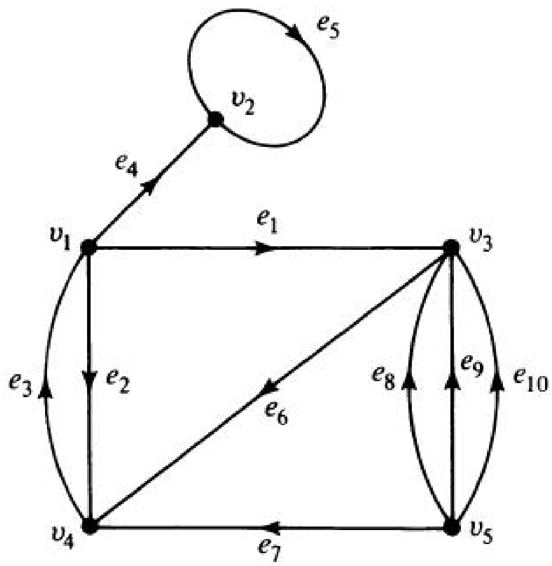


Fig. 9-1 Directed graph with 5 vertices and 10 edges.

every edge onto some *ordered* pair of vertices (v_i, v_j). As in the case of undirected graphs, a vertex is represented by a point and an edge by a line segment between v_i and v_j with an arrow directed from v_i to v_j. For example, Fig. 9-1 shows a digraph with five vertices and ten edges. A digraph is also referred to as an *oriented graph*.†

In a digraph an edge is not only incident on a vertex, but is also *incident out of* a vertex and *incident into* a vertex. The vertex v_i, which edge e_k is incident out of, is called the *initial vertex* of e_k. The vertex v_j, which e_k is incident into, is called the *terminal vertex* of e_k. In Fig. 9-1, v_5 is the initial vertex and v_4 is the terminal vertex of edge e_7. An edge for which the initial and terminal vertices are the same forms a *self-loop*, such as e_5. (Some authors reserve the term *arc* for an oriented or directed edge. We use the term edge to mean either an undirected or a directed edge. Whenever there is a possibility of confusion, we shall explicitly state directed or undirected edge.)

The number of edges incident out of a vertex v_i is called the *out-degree* (or *out-valence* or *outward demidegree*) of v_i and is written $d^+(v_i)$. The number of edges incident into v_i is called the *in-degree* (or *in-valence* or *inward demi-degree*) of v_i and is written as $d^-(v_i)$. In Fig. 9-1, for example,

$$d^+(v_1) = 3, \qquad d^-(v_1) = 1,$$
$$d^+(v_2) = 1, \qquad d^-(v_2) = 2,$$
$$d^+(v_5) = 4, \qquad d^-(v_5) = 0.$$

It is not difficult to prove (Problem 9-1) that in any digraph G the sum of all in-degrees is equal to the sum of all out-degrees, each sum being equal to the number of edges in G; that is,

$$\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i).$$

An *isolated vertex* is a vertex in which the in-degree and the out-degree are both equal to zero. A vertex v in a digraph is called *pendant* if it is of degree one, that is, if

$$d^+(v) + d^-(v) = 1.$$

Two directed edges are said to be *parallel* if they are mapped onto the same ordered pair of vertices. That is, in addition to being parallel in the sense of undirected edges, parallel directed edges must also agree in the direction of their arrows. In Fig. 9-1, edges e_8, e_9, and e_{10} are parallel, whereas edges e_2 and e_3 are not.

Since many properties of directed graphs are the same as those of undirected ones, it is often convenient to disregard the orientations of edges in a digraph. Such an undirected graph obtained from a directed graph G will be called the *undirected graph corresponding to G*.

On the other hand, given an undirected graph H, we can assign each edge of H some arbitrary direction. The resulting digraph, designated by $\vec{H}$ is called an *orientation of H* (or a *digraph associated with H*). Note that while a given digraph has a unique (within isomorphism) undirected graph corresponding to it, a given undirected graph may have "different" orientations possible. This is why we say *the* undirected graph corresponding to a digraph, but *an* orientation of a graph.

Isomorphic Digraphs: Isomorphic graphs were defined such that they have identical behavior in terms of graph properties. In other words, if their labels are removed, two isomorphic graphs are indistinguishable. For two digraphs

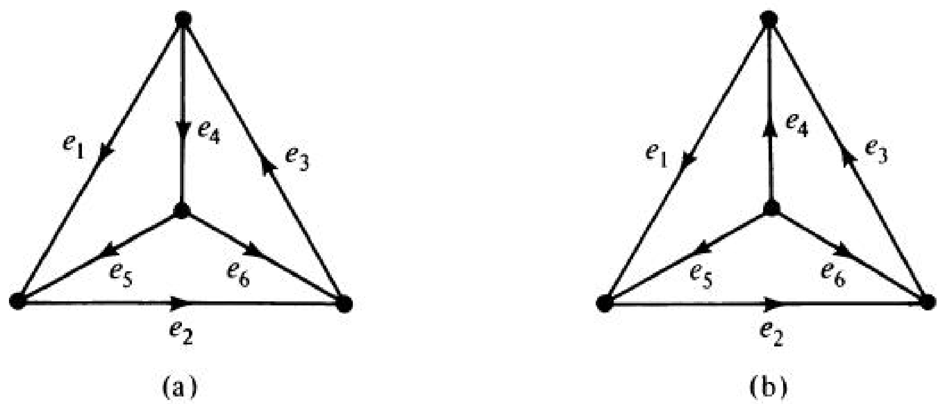


Fig. 9-2 Two nonisomorphic digraphs.

to be isomorphic not only must their corresponding undirected graphs be isomorphic, but the directions of the corresponding edges must also agree. For example, Fig. 9-2 shows two digraphs that are not isomorphic, although they are orientations of the same undirected graph.

Figure 9-2 immediately suggests a problem. What is the number of distinct (i.e., nonisomorphic) orientations of a given undirected graph? The problem was solved by F. Harary and E. M. Palmer in 1966. Some specific cases are left as an exercise (Problem 9-3).

SOME TYPES OF DIGRAPHS

Like their undirected sisters, digraphs come in many varieties. In fact, due to the choice of assigning a direction to each edge, directed graphs have more varieties than undirected ones.

Simple Digraphs: A digraph that has no self-loop or parallel edges is called a simple digraph (Figs. 9-2 and 9-3, for example).

Asymmetric Digraphs: Digraphs that have at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called *asymmetric* or *antisymmetric*.

Symmetric Digraphs: Digraphs in which for every edge (a, b) (i.e., from vertex a to b) there is also an edge (b, a).

A digraph that is both simple and symmetric is called a *simple symmetric digraph*. Similarly, a digraph that is both simple and asymmetric is *simple asymmetric*. The reason for the terms symmetric and asymmetric will be apparent in the context of binary relations in Section 9-3.

Complete Digraphs: A complete undirected graph was defined as a simple graph in which every vertex is joined to every other vertex exactly by one edge. For digraphs we have two types of complete graphs. A *complete symmetric digraph* is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex (Fig. 9-3), and a *complete asymmetric digraph* is an asymmetric digraph in which there is exactly one edge between every pair of vertices (Fig. 9-2).

A complete asymmetric digraph of n vertices contains $n(n-1)/2$ edges, but a complete symmetric digraph of n vertices contains $n(n-1)$ edges. A complete asymmetric digraph is also called a *tournament* or a *complete tournament* (the reason for this term will be made clear in Section 9-10).

A digraph is said to be *balanced* if for every vertex v_i the in-degree equals the out-degree; that is, $d^+(v_i) = d^-(v_i)$. (A balanced digraph is also referred to as a *pseudosymmetric* digraph, or an *isograph*.) A balanced digraph is said to be *regular* if every vertex has the same in-degree and out-degree as every other vertex.

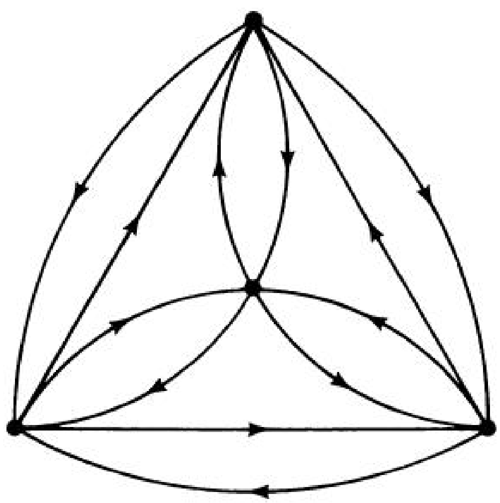


Fig. 9-3 Complete symmetric digraph of four vertices.

DIGRAPHS AND BINARY RELATIONS

The theory of graphs and the calculus of binary relations are closely related (so much so that some people often mistakenly come to regard graph theory as a branch of the theory of relations).

In a set of objects, X, where

$$X = \{x_1, x_2, \ldots\},$$

a *binary relation R* between pairs (x_i, x_j) may exist. In which case, we write

$$x_i R x_j$$

and say that x_i *has relation R to* x_j.

Relation R may for instance be "is parallel to," "is orthogonal to," or "is congruent to" in geometry. It may be "is greater than," "is a factor of," "is equal to," and so on, in the case when X consists of numbers. On the other hand, if the set X is composed of people, the relation R may be "is son of," "is spouse of," "is friend of," and so forth. Each of these relations is defined only on pairs of numbers of the set, and this is why the name *binary relation*. Although there are relations other than binary (x_i "is a product of" x_j and x_k, for example, will be a tertiary relation), binary relations are the most important in mathematics, and the word "relation" implies a binary relation.

A digraph is the most natural way of representing a binary relation on a set X. Each $x_i \in X$ is represented by a vertex x_i. If x_i has the specified relation R to x_j, a directed edge is drawn from vertex x_i to x_j, for every pair (x_i, x_j). For example, the digraph in Fig. 9-4 represents the relation "is greater than" on a set consisting of five numbers {3, 4, 7, 5, 8}.

Clearly, every binary relation on a finite set can be represented by a digraph without parallel edges. Conversely, every digraph without parallel edges defines a binary relation on the set of its vertices.

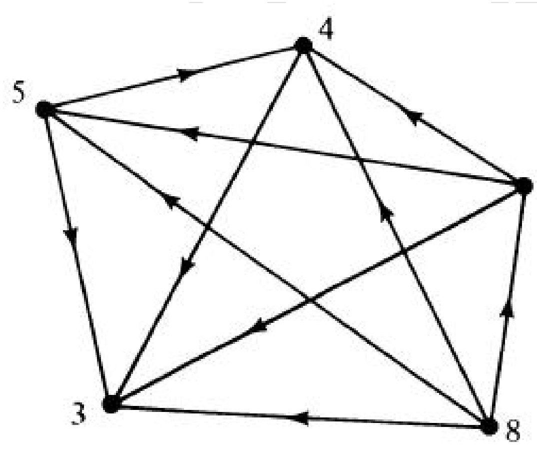


Fig. 9-4 Digraph of a binary relation.

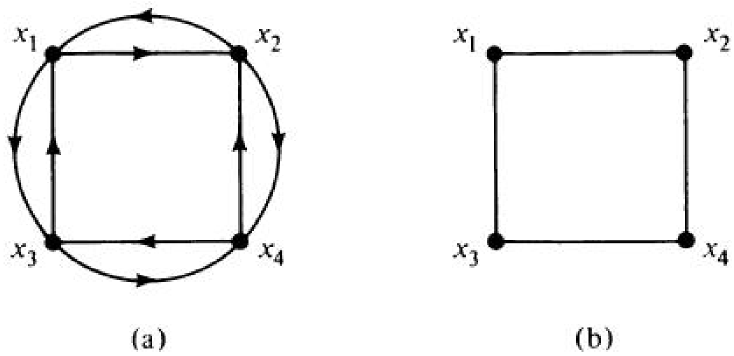


Fig. 9-5 Graphs of symmetric binary relation.

Reflexive Relation: For some relation R it may happen that every element is in relation R to itself. For example, a number is always equal to itself, or a line is always parallel to itself. Such a relation R on set X that satisfies

$$x_i R x_i$$

for every $x_i \in X$ is called a *reflexive* relation. The digraph of a reflexive relation will have a self-loop at every vertex. Such a digraph representing a reflexive binary relation on its vertex set may be called a *reflexive digraph*. A digraph in which no vertex has a self-loop is called an *irreflexive digraph*.

Symmetric Relation: For some relation R it may happen that for all x_i and x_j, if

$$x_i R x_j \text{ holds, then } x_j R x_i \text{ also holds.}$$

Such a relation is called a *symmetric relation*. "Is spouse of" is a symmetric but irreflexive relation. "Is equal to" is both symmetric and reflexive.

The digraph of a symmetric relation is a *symmetric digraph* because for every directed edge from vertex x_i to x_j there is a directed edge from x_j to x_i. Figure 9-5(a) shows the graph of an irreflexive, symmetric binary relation on a set of four elements. The same relation can also be represented by drawing just one undirected edge between every pair of vertices that are related, as in Fig. 9-5(b). Thus every undirected graph is a representation of some symmetric binary relation (on the set of its vertices). Furthermore, every undirected graph with e edges can be thought of as a symmetric digraph with $2e$ directed edges. (A two-way street is equivalent to two one-way streets pointed in opposite directions.)

Transitive Relation: A relation R is said to be *transitive* if for any three elements x_i, x_j, and x_k in the set,

$$x_i R x_j \quad \text{and} \quad x_j R x_k$$

always imply

$$x_i R x_k.$$

The binary relation "is greater than," for example, is a transitive relation. If $x_i > x_j$ and $x_j > x_k$, clearly $x_i > x_k$. "Is descendent of" is another example of a transitive relation.

The digraph of a transitive (but irreflexive and asymmetric) binary relation is shown in Fig. 9-4. Note the triangular subgraphs. A digraph representing a transitive relation (on its vertex set) is called a *transitive directed graph.*

Equivalence Relation: A binary relation is called an *equivalence relation* if it is reflexive, symmetric, and transitive. Some examples of equivalence relations are "is parallel to," "is equal to," "is congruent to," "is equal to modulo m," and "is isomorphic to."

The graph representing an equivalence relation may be called an *equivalence graph*. What does an equivalence graph look like? Let us look at an example, consisting of the equivalence relation "is congruent to modulo 3" defined on the set of 11 integers, 10 through 20. The graph is shown in Fig. 9-6. (Recall that each undirected edge in Fig. 9-6 represents two parallel but oppositely directed edges.)

In Fig. 9-6 we see that the vertex set of the graph is divided into three disjoint classes, each in a separate component. Each component is an undirected subgraph (due to symmetry) with a self-loop at each vertex (due to reflexivity). Furthermore, in each component every vertex is related to (i.e., joined by an edge to) every other vertex.

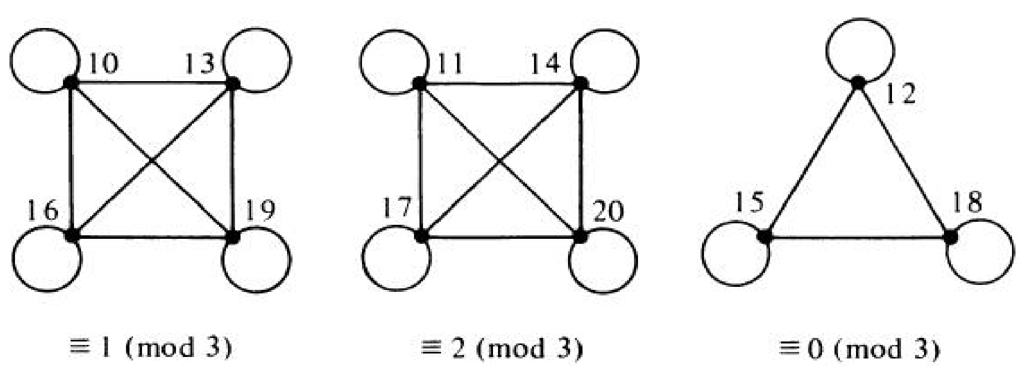


Fig. 9-6 Equivalence graph.

In general, an equivalence relation on a set partitions the elements of the set into classes (called *equivalence classes*) such that two elements are in the same class if and only if they are related. Symmetry ensures that there is no ambiguity regarding membership in the equivalence class; otherwise, x_i may have been related to x_j but not vice versa. Transitivity ensures that in each component every vertex is joined to every other vertex, because if a is related to b and b is related to c, a is also related to c. Transitivity also guarantees that no element can be in more than one class. Reflexivity allows an element to be in a class by itself, if it is not related to any other element in the set.

Relation Matrices: A binary relation R on a set can also be represented by a matrix, called a *relation matrix*. It is a (0, 1), n by n matrix, where n is the number of elements in the set. The i, jth entry in the matrix is 1 if $x_i R x_j$ is true, and is 0, otherwise. For example, the relation matrix of the relation "is greater than" on the set of integers {3, 4, 7, 5, 8} is

$$\begin{array}{c c} & \begin{array}{ccccc} 3 & 4 & 7 & 5 & 8 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 7 \\ 5 \\ 8 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

We shall see in Section 9-8 that this is precisely the adjacency matrix of the digraph representing the binary relation.

DIRECTED PATHS AND CONNECTEDNESS

Walks, paths, and circuits in a directed graph, in addition to being what they are in the corresponding undirected graph, have the added consideration of orientation. For example, in Fig. 9-1, the sequence of vertices and edges $v_5\, e_8\, v_3\, e_6\, v_4\, e_3\, v_1$ is a path "directed" from v_5 to v_1, whereas $v_5\, e_7\, v_4\, e_6\, v_3\, e_1\, v_1$ (although a path in the corresponding undirected graph) has no such consistent direction from v_5 to v_1. A distinction must be made between these two types of paths. It is natural to call the first one a *directed path* from v_5 to v_1, and the second one a *semipath*. The word "path" in a digraph could mean either a directed path or a semipath, and similarly for walks, circuits, and cutsets. More precisely:

A *directed walk* from a vertex v_i to v_j is an alternating sequence of vertices and edges, beginning with v_i and ending with v_j, such that each edge is oriented from the vertex preceding it to the vertex following it. Of course, no edge in a directed walk appears more than once, but a vertex may appear more than once, just as in the case of undirected graphs. A *semiwalk* in a directed graph is a walk in the corresponding undirected graph, but is *not* a *directed walk*. A *walk* in a digraph can mean either a directed walk or a semiwalk.

The definitions of *circuit*, *semicircuit*, and *directed circuit* can be written similarly. Let us turn to Fig. 9-1 once more. The set of edges $\{e_1, e_6, e_3\}$ is a directed circuit. But $\{e_1, e_6, e_2\}$ is a semicircuit. Both of them are circuits.

Connected Digraphs: In Chapter 2 a graph (i.e., undirected graph) was defined as connected if there was at least one path between every pair of vertices. In a digraph there are two different types of paths. Consequently, we have two different types of connectedness in digraphs. A digraph G is said to be *strongly connected* if there is at least one directed path from every vertex to every other vertex. A digraph G is said to be *weakly connected* if its corresponding undirected graph is connected but G is not strongly connected. In Fig. 9-2 one of the digraphs is strongly connected, and the other one is weakly connected. The statement that a digraph G is connected simply means that its corresponding undirected graph is connected; and thus G may be strongly or weakly connected. A directed graph that is not connected is dubbed as disconnected.

Since there are two types of connectedness in a digraph, we can define two types of components also. Each maximal connected (weakly or strongly) subgraph of a digraph G will still be called a *component* of G. But within each component of G the maximal strongly connected subgraphs will be called the *fragments* (or *strongly connected fragments*) of G.

For example, the digraph in Fig. 9-7 consists of two components. The component g_1 contains three fragments $\{e_1, e_2\}$, $\{e_5, e_6, e_7, e_8\}$, and $\{e_{10}\}$. Observe that e_3, e_4, and e_9 do not appear in any fragment of g_1.

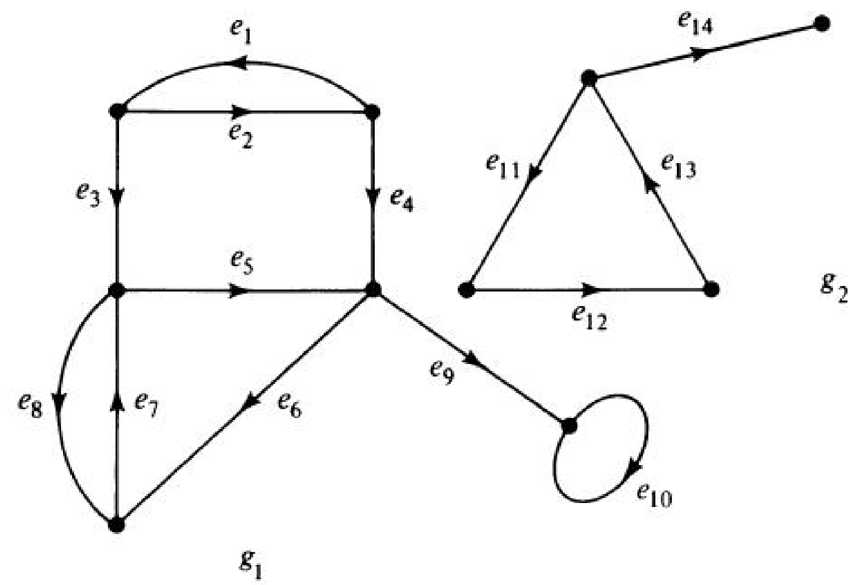


Fig. 9-7 Disconnected digraph with two components.

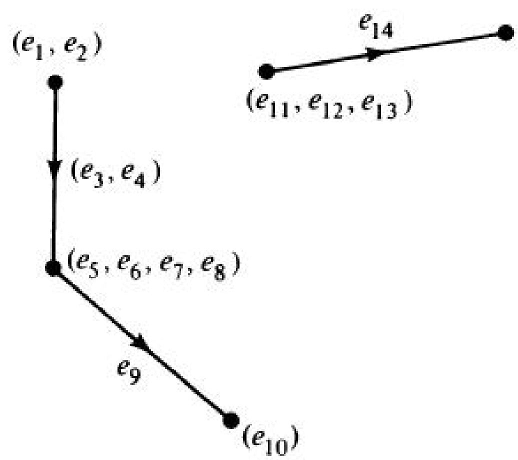


Fig. 9-8 Condensation of Fig. 9-7.

Condensation: The *condensation* G_c of a digraph G is a digraph in which each strongly connected fragment is replaced by a vertex, and all directed edges from one strongly connected component to another are replaced by a single directed edge. The condensation of the digraph G in Fig. 9-7 is shown in Fig. 9-8.

Two observations can be made from the definition:

1. The condensation of a strongly connected digraph is simply a vertex.
2. The condensation of a digraph has no directed circuit.

Accessibility: In a digraph a vertex b is said to be *accessible* (or *reachable*) from vertex a if there is a directed path from a to b. Clearly, a digraph G is strongly connected if and only if every vertex in G is accessible from every other vertex.

TYPES OF ENUMERATION

All graph-enumeration problems fall into two categories:

1. Counting the number of different graphs (or digraphs) of a particular kind, for example, all connected, simple graphs with eight vertices and two circuits.
2. Counting the number of subgraphs of a particular type in a given graph G, such as the number of edge-disjoint paths of length k between vertices a and b in G.

The second type of problem usually involves a matrix representation of graph G and manipulations of this matrix. Such problems, although often encountered in practical applications, are not as varied and interesting as those in the first category. We shall not consider such problems in this chapter.

In problems of type 1 the word "different" is of utmost importance and must be clearly understood. If the graphs are labeled (i.e., each vertex is assigned a name distinct from all others), all graphs are counted. On the other hand, in the case of unlabeled graphs the word "different" means non-isomorphic, and each set of isomorphic graphs is counted as one.

As an example, let us consider the problem of constructing all simple graphs with n vertices and e edges. There are $n(n-1)/2$ unordered pairs of vertices. If we regard the vertices as distinguishable from one another (i.e., labeled graphs), there are

$$\binom{\frac{n(n-1)}{2}}{e} \qquad (10\text{-}1)$$

ways of selecting e edges to form the graph. Thus expression (10-1) gives the number of simple *labeled* graphs with n vertices and e edges.

Many of these graphs, however, are isomorphic (that is, they are the same except for the labels of their vertices). Hence the number of simple, *unlabeled* graphs of n vertices and e edges is much smaller than that given by (10-1).

Among a collection of graphs, isomorphism is an equivalence relation (Problem 10-1). The number of different unlabeled graphs (of a certain type) equals the number of equivalence classes, under isomorphism, of the labeled graphs. For example, we have 16 different labeled trees of four vertices (Fig. 3-15), and these trees fall into two equivalence classes, under isomorphism. In Fig. 3-15 the 4 trees in the top row fall into one equivalence class, and the remaining 12 into another. Thus we have only two different unlabeled trees of four vertices (Fig. 3-16).

Let us now proceed with counting certain specific types of graphs.

THEOREM 10-1

The number of simple, labeled graphs of n vertices is

$$2^{n(n-1)/2}. \tag{10-2}$$

Proof: The numbers of simple graphs of n vertices and $0, 1, 2, \ldots, n(n-1)/2$ edges are obtained by substituting $0, 1, 2, \ldots, n(n-1)/2$ for e in expression (10-1). The sum of all such numbers is the number of all simple graphs with n vertices. Then the use of the following identity proves the theorem:

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k-1} + \binom{k}{k} = 2^k. \quad \blacksquare$$

COUNTING LABELED TREES

THEOREM 3-10

There are n^{n-2} labeled trees with n vertices ($n \geq 2$).

Proof of Theorem 3-10: Let the n vertices of a tree T be labeled $1, 2, 3, \ldots, n$. Remove the pendant vertex (and the edge incident on it) having the smallest label, which is, say, a_1. Suppose that b_1 was the vertex adjacent to a_1. Among the remaining $n - 1$ vertices let a_2 be the pendant vertex with the smallest label, and b_2 be the vertex adjacent to a_2. Remove the edge (a_2, b_2). This operation is repeated on the remaining $n - 2$ vertices, and then on $n - 3$ vertices, and so on. The process is terminated after $n - 2$ steps, when only two vertices are left. The tree T defines the sequence

$$(b_1, b_2, \ldots, b_{n-2}) \tag{10-3}$$

uniquely. For example, for the tree in Fig. 10-1 the sequence is (1, 1, 3, 5, 5, 5, 9). Note that a vertex i appears in sequence (10-3) if and only if it is not pendant (see Problem 10-2).

Conversely, given a sequence (10-3) of $n - 2$ labels, an n-vertex tree can be

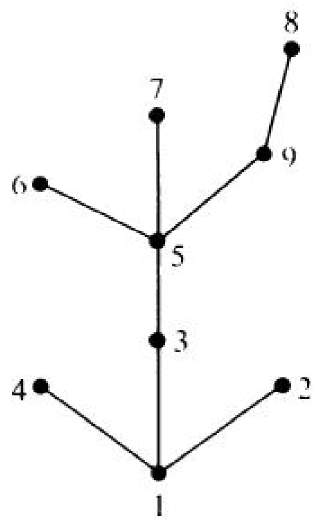


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1, 3, 5, 5, 5, 9).

constructed uniquely, as follows: Determine the first number in the sequence

$$1, 2, 3, \ldots, n \tag{10-4}$$

that does not appear in sequence (10-3). This number clearly is a_1. And thus the edge (a_1, b_1) is defined. Remove b_1 from sequence (10-3) and a_1 from (10-4). In the remaining sequence of (10-4) find the first number that does not appear in the remainder of (10-3). This would be a_2, and thus the edge (a_2, b_2) is defined. The construction is continued till the sequence (10-3) has no element left. Finally, the last two vertices remaining in (10-4) are joined. For example, given a sequence

$$(4, 4, 3, 1, 1),$$

we can construct a seven-vertex tree as follows: (2, 4) is the first edge. The second is (5, 4). Next, (4, 3). Then (3, 1), (6, 1), and finally (7, 1), as shown in Fig. 10-2.

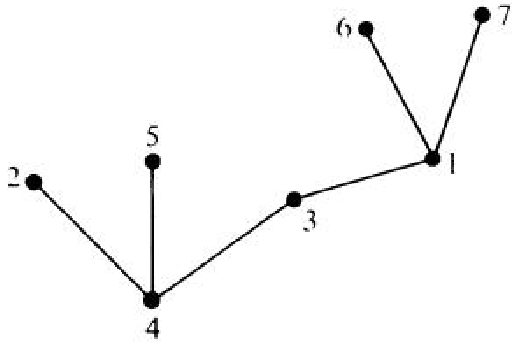


Fig. 10-2 Tree constructed from sequence (4, 4, 3, 1, 1).

For each of the $n-2$ elements in sequence (10-3) we can choose any one of n numbers, thus forming

$$n^{n-2} \tag{10-5}$$

$(n-2)$-tuples, each defining a distinct labeled tree of n vertices. And since each tree defines one of these sequences uniquely, there is a one-to-one correspondence between the trees and the n^{n-2} sequences. Hence the theorem. ■

Rooted Labeled Trees: In a rooted graph one vertex is marked as the root. For each of the n^{n-2} labeled trees we have n rooted labeled trees, because any of the n vertices can be made a root. Therefore,

THEOREM 10-2

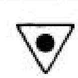
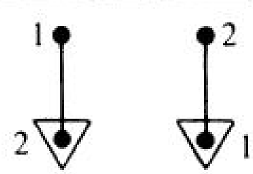
The number of different rooted, labeled trees with n vertices is

$$n^{n-1}. \tag{10-6}$$

All rooted trees for $n = 1, 2,$ and 3 are given in Fig. 10-3.

COUNTING UNLABFLED TREES

The problem of enumeration of unlabeled trees is more involved and requires familiarity with the concepts of *generating functions* and *partitions*.

n	Labeled free trees	Labeled rooted trees
1		
2		

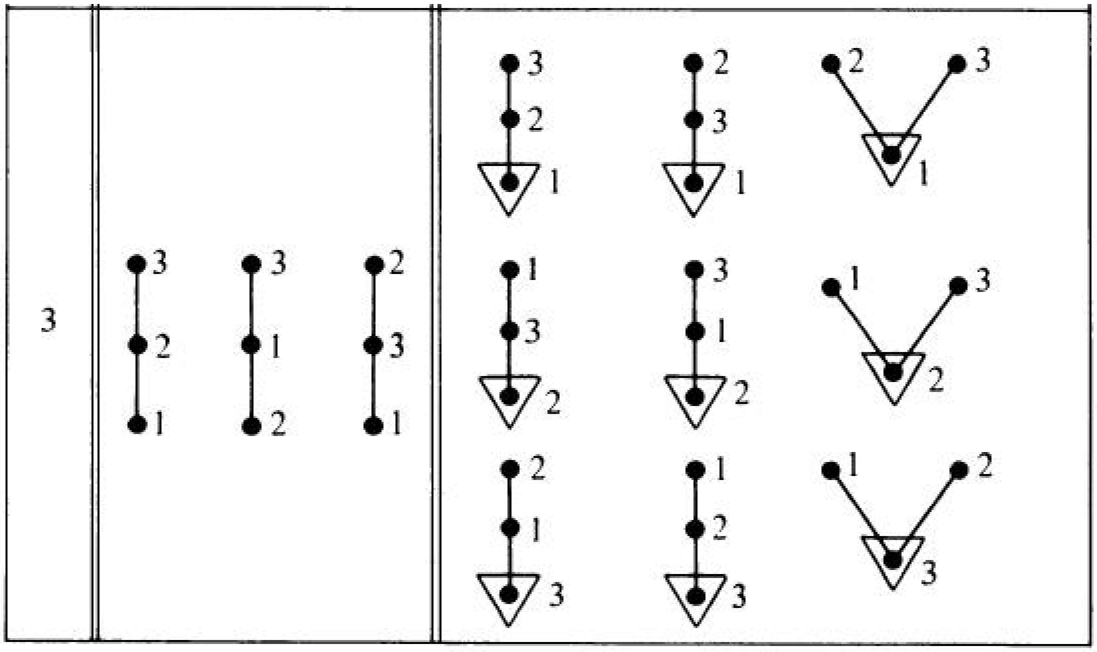


Fig. 10-3 Rooted labeled trees of one, two, and three vertices.

Centroid

In a tree T, at any vertex v of degree d, there are d subtrees with only vertex v in common. The *weight of each subtree* at v is defined as the number of branches in the subtree. Then the *weight of the vertex* v is defined as the weight of the heaviest of the subtrees at v. A vertex with the smallest weight in the entire tree T is called a *centroid* of T.

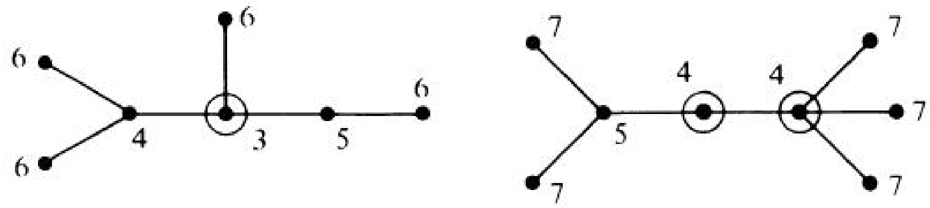
Just as in the case of centers of a tree (Section 3-4), it can be shown that every tree has either one centroid or two centroids. It can also be shown that if a tree has two centroids, the centroids are adjacent. In Fig. 10-6 a tree with a centroid (called a *centroidal tree*) and a tree with two centroids (called a *bicentroidal tree*) are shown. The centroids are shown enclosed in circles, and the numbers next to the vertices are the weights.

Free Unlabeled Trees

Let $t'(x)$ be the counting series for centroidal trees, and $t''(x)$ be the counting series for bicentroidal trees. Then $t(x)$, the counting series for all (unlabeled, free) trees, is the sum of the two. That is,

$$t(x) = t'(x) + t''(x). \tag{10-22}$$

To obtain $t''(x)$, observe that an n-vertex bicentroidal tree can be regarded as consisting of two rooted trees each with $n/2 = m$ vertices, and joined at their roots by an edge. (A bicentroidal tree will always have an even number of vertices; why?) Thus the number of bicentroidal trees with $n = 2m$ vertices is



(a) Centroidal Tree (b) Bicentroidal Tree

given by

$$t''_n = \binom{u_m + 1}{2} = \frac{u_m(u_m + 1)}{2},$$

and therefore

$$\begin{aligned} t''(x) &= \sum_{m=1}^{\infty} \frac{u_m(u_m + 1)}{2} x^{2m} \\ &= \frac{1}{2} \sum_{m=1}^{\infty} u_m x^{2m} + \frac{1}{2} \sum_{m=1}^{\infty} (u_m x^m)^2 \qquad (10\text{-}23) \\ &= \frac{1}{2} u(x^2) + \frac{1}{2} \sum_{m=1}^{\infty} (u_m x^m)^2. \end{aligned}$$

The number of vertices, n, in a centroidal tree can be odd or even. If n is odd, the maximum weight the centroid could have is $\frac{1}{2}(n - 1)$. This maximum is achieved only when the tree consists of a path of $n - 1$ edges. On the other hand, if n is even and the tree is centroidal, the maximum weight the centroid could possibly have is $\frac{1}{2}(n - 2)$. This maximum is achieved when the degree of the centroid is three, and one of the subtrees consists of just one edge.

Thus, regardless whether n is odd or even, it is clear that an n-vertex (free) centroidal tree can be regarded as composed of several rooted trees, rooted at the centroid, and none of these rooted trees can have more than $\lfloor (n-1)/2 \rfloor$ edges, where $\lfloor x \rfloor$ denotes the largest integer no greater than x. In view of this observation, an involved manipulation of Eq. (10-21) leads to the following (for missing steps see [10-3]):

$$t'(x) = u(x) - \frac{1}{2}u^2(x) - \frac{1}{2}\sum_{m=1}^{\infty}(u_m x^m)^2. \tag{10-24}$$

Adding (10-23) and (10-24), we get the desired counting series:

$$t(x) = u(x) - \frac{1}{2}\Big(u^2(x) - u(x^2)\Big)\cdot \tag{10-25}$$

This relation, which gives the tree-counting series in terms of the rooted-tree counting series, was first obtained by Richard Otter in 1948 and is known as Otter's formula. The first 10 terms of (10-25) are

$$t(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 11x^7 + 23x^8 + 47x^9 + 106x^{10} + \cdots.$$

The reader is encouraged to extend it by another 10 terms. The first 26 terms of both $u(x)$ and $t(x)$ are given in Riordan's book [3-11], page 138.

By now you must have the impression that enumeration of graphs is an involved subject. And indeed it is. So far we have enumerated only four types of graphs—rooted and free trees, both labeled and unlabeled varieties. It is difficult to proceed further without some additional enumerative tool. This is provided by a general counting theorem due to Pólya. We shall first state and discuss Pólya's theorem and then show how it can be applied for counting graphs.

PÓLYA'S COUNTING THEOREM

Permutation

On a finite set A of some objects, a permutation π is a one-to-one mapping from A onto itself. For example, consider a set $\{a, b, c, d\}$. A permutation

$$\pi_1 = \begin{pmatrix} a\ b\ c\ d \\ b\ d\ c\ a \end{pmatrix}$$

takes a into b, b into d, c into c, and d into a. Alternatively, we could write

$$\pi_1(a) = b,$$
$$\pi_1(b) = d,$$

$$\pi_1(c) = c,$$
$$\pi_1(d) = a.$$

The number of elements in the object set on which a permutation acts is called the *degree* of the permutation. The degree of π_1 in the above example is four.

A permutation can also be represented by a digraph, in which each vertex represents an element of the object set and the directed edges represent the mapping. For example, the permutation $\pi_1 = \begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}$ is represented diagrammatically by Fig. 10-7.

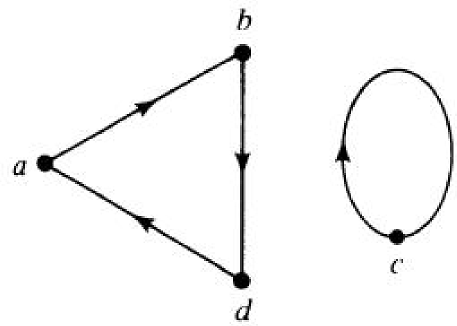


Fig. 10-7 Digraph of a permutation.

Observe that the in-degree and the out-degree of every vertex in the digraph of a permutation is one. Such a digraph must decompose into one or more vertex-disjoint directed circuits (why?). This suggests yet another way of representing a permutation—as a collection of the vertex-disjoint, directed circuits (called the *cycles of the permutation*). Permutation $\begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}$ can thus be written as $(a\ b\ d)(c)$. This compact and popular representation is called the *cyclic representation* of a permutation. The number of edges in a permutation cycle is called the *length of the cycle in the permutation*.

Often the only information of interest about a permutation is the number of cycles of various lengths. A permutation π of degree k is said to be of *type* $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ if π has σ_i cycles of length i for $i = 1, 2, \ldots, k$. For example, permutation $(a\ b\ d)(c)$ is of type $(1, 0, 1, 0)$ and permutation $(a\ b\ f)(c)(d\ e\ h)(g)$ is of type $(2, 0, 2, 0, 0, 0, 0, 0)$. Clearly,

$$1\sigma_1 + 2\sigma_2 + 3\sigma_3 + \cdots + k\sigma_k = k. \tag{10-26}$$

Another useful method for indicating the type of a permutation is to introduce k dummy variables, say, $y_1, y_2, \ldots, y_k$, and then show the type of permutation by the expression

$$y_1^{\sigma_1} y_2^{\sigma_2} \ldots y_k^{\sigma_k}. \tag{10-27}$$

Expression (10-27) is called the *cycle structure* of π. For example, the cycle structure of the eight-degree permutation $(a\,b\,f)(c)(d\,e\,h)(g)$ is

$$y_1^2\, y_2^0\, y_3^2\, y_4^0\, y_5^0\, y_6^0\, y_7^0\, y_8^0 = y_1^2\, y_3^2.$$

Note that the dummy variable y_i has no significance except as a symbol to which subscripts (indicating the lengths) and exponents (indicating the number of cycles) are attached. Two distinct permutations (acting on the same object set) may have the same cycle structure (page 149 in [10-1]).

So far we have discussed only the representation and properties of a permutation individually. Let us now examine a set of permutations collectively.

On a set A with k objects, we have a total of $k!$ possible permutations—including the *identity permutation*, which takes every element into itself. For example, the following are the six permutations on a set of three elements $\{a, b, c\}$:

$$(a)(b)(c), \quad (a\,b)(c), \quad (a\,c)(b), \quad (a)(b\,c), \quad (a\,b\,c), \quad (a\,c\,b).$$

Their cycle structures, respectively, are

$$y_1^3, \quad y_1y_2, \quad y_1y_2, \quad y_1y_2, \quad y_3, \quad y_3. \tag{10-28}$$

Composition of Permutations

Consider the two permutations π_1 and π_2 on an object set $\{1, 2, 3, 4, 5\}$:

$$\pi_1 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,1\,4\,5\,3 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,4\,1\,2\,5 \end{pmatrix}.$$

A *composition* of these two permutations $\pi_2\pi_1$ is another permutation obtained by first applying π_1 and then applying π_2 on the resultant. That is,

$$\pi_2\pi_1(1) = \pi_2(2) = 4,$$
$$\pi_2\pi_1(2) = \pi_2(1) = 3,$$
$$\pi_2\pi_1(3) = \pi_2(4) = 2,$$
$$\pi_2\pi_1(4) = \pi_2(5) = 5,$$
$$\pi_2\pi_1(5) = \pi_2(3) = 1.$$

Thus
$$\pi_2\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}.$$

Thus among a collection of permutations on the same object set, composition is a binary operation.

Permutation Group

A collection of m permutations $P = \{\pi_1, \pi_2, \ldots, \pi_m\}$ acting on a set

$$A = \{a_1, a_2, \ldots, a_k\}$$

forms a group under composition, if the four postulates† of a group, that is, closure, associativity, identity, and inverse (see Section 6-1), are satisfied. Such a group is called a *permutation group*. For example, it can be easily verified that the set of four permutations

$$\{(a)(b)(c)(d), \quad (a\ c)(b\ d), \quad (a\ b\ c\ d), \quad (a\ d\ c\ b)\} \tag{10-29}$$

acting on the object set $\{a, b, c, d\}$ forms a permutation group.

The number of permutations m in a permutation group is called its *order*, and the number of elements in the object set on which the permutations are acting is called the *degree of the permutation group*. In the example just cited, both the degree and order of the permutation group is four. It can be shown that the set of all $k!$ permutations on a set A of k elements forms a permutation group. Such a group, of order $k!$ and degree k, is called the *full symmetric group*, S_k.

Pólya's Counting Theorem

Let us consider two finite sets, domain D and range R, together with a permutation group P on D. To each element $\rho \in R$ let us assign a quantity $w[\rho]$ and call it the *content* (or *weight*) of the element ρ. The weight $w[\rho]$ can be a symbol or a real number. A mapping f from D to R can be described by a sequence of $|D|$ elements of set R such that the ith element in the sequence is the image of the ith element of set D under f. Therefore the content $W(f)$ of a mapping f can be defined as the product of the contents of all its images. That is,

$$W(f) = \prod_{d \in D} w[f(d)].$$

Clearly, all functions belonging to the same equivalence class defined by (10-33) have identical weights. Therefore, we define the weight of an entire equivalence class (of functions from domain D to range R) to be the (common) weight of the functions in this class. Our problem is to count the number of equivalence classes with various weights, given D, R, permutation group P on D, and weights $w[\rho]$ for each $\rho \in R$. This is exactly what Pólya's counting theorem gives.

In Pólya's terminology, elements ρ of set R are called *figures*, and functions f from D to R are called *configurations*. Often the weights of the elements of R can be expressed as powers of some common quantity x. In that case the weight assignment to elements of set R can be neatly described by means of a counting series $A(x)$

$$A(x) = \sum_{q=0}^{\infty} a_q x^q, \tag{10-34}$$

where a_q is the number of elements in set R with weight x^q.† Likewise, the number of configurations can be expressed in terms of another series, called *configuration counting series* $B(x)$, such that

$$B(x) = \sum_{m=0}^{\infty} b_m x^m, \tag{10-35}$$

where b_m is the number of different configurations having weight x^m. Now we can state the following powerful result known as Pólya's counting theorem.

THEOREM 10-3

The configuration-counting series $B(x)$ is obtained by substituting the figure-counting series $A(x^i)$ for each y_i in the cycle index $Z(P; y_1, y_2, \ldots, y_k)$ of the permutation group P. That is,

$$B(x) = Z(P; \sum a_q x^q, \sum a_q x^{2q}, \sum a_q x^{3q}, \ldots, \sum a_q x^{kq}). \tag{10-36}$$

The proof of Pólya's theorem, although not complicated, is not particularly illuminating and is therefore left out. The reader can find it in [10-1], page 157. Our interest is mainly in the application of the theorem; let us illustrate it with some examples.

Example 1: Suppose that we are given a cube and four (identical) balls. In how many ways can the balls be arranged on the corners of the cube? Two arrangements are considered the same if by any rotation of the cube they can be transformed into each other.

The answer is seven, as can be seen by inspection in Fig. 10-9. In Pólya's terms the domain D is the set of the eight corners of the cube, and the range

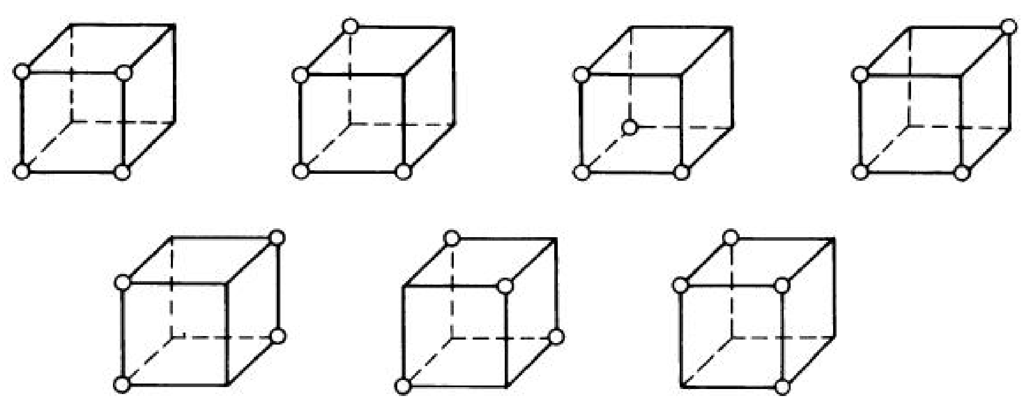


Fig. 10-9 Attaching four balls to corners of a cube.

R consists of two elements (i.e., figures), "presence of a ball" or "absence of a ball," with contents x^1 and x^0, respectively. The figure-counting series is

$$A(x) = \sum_{q=0}^{\infty} a_q x^q = a_0 x^0 + a_1 x^1 = 1 + x, \tag{10-37}$$

since a_0, the number of figures with content 0, is one, and a_1, the number of figures with content 1, is also one. The configurations are $2^8 = 256$ different mappings that assign balls to the corners of the cube. The permutation group P on D is the set of all those permutations that can be produced by rotations of the cube. These permutations with their cycle structures are

1. One identity permutation. Its cycle structure is y_1^8.
2. Three 180° rotations around lines connecting the centers of opposite faces. Its cycle structure is y_2^4.
3. Six 90° rotations (clockwise and counterclockwise) around lines connecting the centers of opposite faces. The cycle structure is y_4^2.
4. Six 180° rotations around lines connecting the midpoints of opposite edges. The corresponding cycle structure is y_2^4.
5. Eight 120° rotations around lines connecting opposite corners in the cube. The cycle structure of the corresponding permutation is $y_1^2 y_3^3$.

The cycle index of this group consisting of these 24 permutations is, therefore,

$$Z(P) = \frac{1}{24}(y_1^8 + 9y_2^4 + 6y_4^2 + 8y_1^2 y_3^2). \qquad (10\text{-}38)$$

Using Pólya's theorem, we now substitute the figure-counting series, that is $1 + x$ for y_1, $1 + x^2$ for y_2, $1 + x^3$ for y_3, and $1 + x^4$ for y_4. This yields the configuration-counting series.

$$B(x) = 1 + x + 3x^2 + 3x^3 + 7x^4 + 3x^5 + 3x^6 + x^7 + x^8. \qquad (10\text{-}39)$$

The coefficient of x^4 in $B(x)$ gives the number of P-inequivalent configurations of content x^4 (i.e., with four balls). This verifies the answer obtained by exhaustive inspection in Fig. 10-9.

The total number of P-inequivalent configurations (with contents $x^0, x^1, x^2, \ldots, x^8$) is obtained by adding all coefficients in (10-39), which is 23. It may be observed that this is the number of distinct ways of painting the eight vertices of a cube with two colors (one color corresponds to the "presence of a ball" and the other with the "absence of a ball").

Example 2: In example 1 we were given four identical balls. Now suppose that we are given two red balls and two blue balls, and are again asked to find the number of distinct arrangements on the corners of the cube. Clearly, D, P, and $Z(P)$ will remain the same as they were in example 1. Only the range R and the figure-counting series $A(x)$ will change. The range will contain three elements: (1) presence of no ball, (2) presence of a red ball, and (3) presence of a blue ball. Choosing x to indicate the presence of a red ball and x' to indicate the presence of a blue ball, the three elements in the range mentioned above will have the contents $x^0x'^0$, $x^1x'^0$, and $x^0x'^1$, respectively. Therefore the figure-counting series is

$$A(x, x') = x^0x'^0 + x^1x'^0 + x^0x'^1 = 1 + x + x'.$$

Substituting this figure-counting series in (10-38), we get the configuration-counting series

$$\begin{aligned} B(x, x') = \frac{1}{24}[&(1 + x + x')^8 + 9(1 + x^2 + x'^2)^4 + 6(1 + x^4 + x'^4)^2 \\ &+ 8(1 + x + x')^2(1 + x^3 + x'^3)^2] \\ = 1 &+ x + x' + 3x^2 + 3x'^2 + 3xx' + 3x^3 + 3x'^3 \\ &+ 7x^2x' + 7xx'^2 + 7x^4 + 7x'^4 + 13x^3x' + 13xx'^3 \\ &+ 22x^2x'^2 + 3x^5 + 3x'^5 + 13x^4x' + 13xx'^4 \\ &+ 24x^3x'^2 + 24x^2x'^3 + 3x^6 + 3x'^6 + 7x^5x' \\ &+ 7xx'^5 + 22x^4x'^2 + 22x^2x'^4 + 24x^3x'^3 + x^7 + x'^7 \\ &+ 3x^6x' + 3xx'^6 + 7x^5x'^2 + 7x^2x'^5 + 13x^3x'^4 + 13x^4x'^3 \\ &+ x^8 + x'^8 + x^7x' + xx'^7 + 3x^6x'^2 + 3x^2x'^6 + 3x^5x'^3 \\ &+ 3x^3x'^5 + 7x^4x'^4. \end{aligned} \tag{10-40}$$

The coefficient of $x^rx'^b$ in (10-40) is the number of distinct arrangements with r red balls, b blue balls and $8 - r - b$ corners with no balls. The number of arrangements with two red and two blue balls is, therefore, 22.

For some other non-graph-theoretic examples of the applications of Pólya's theorem, the reader should work out Problems 10-10, 10-11, 10-14, and 10-15. Let us now return to the counting of graphs.

GRAPH ENUMERATION WITH PÓLYA'S THEOREM

Enumeration of Simple Graphs: Let us consider the problem of counting all unlabeled, simple graphs of n vertices. Any such graph G can be regarded as a mapping (i.e., configuration) of the set D of all $\frac{1}{2}n(n-1)$ unordered pairs of vertices (for digraphs $n(n-1)$ pairs of vertices). Range R consists of two elements s and t, with contents x^1 and x^0, respectively. If a vertex pair is joined by an edge in G, the vertex pair maps into s, an element with content x^1; otherwise, into t, an element with content $x^0 = 1$. Thus the figure-counting series is

$$A(x) = \sum a_q x^q = 1 + x.$$

The relevant permutation group in this case is R_n, the group of permutations on the pairs of vertices induced by S_n (the full symmetric group on the n vertices of the graph).† Therefore, the configuration-counting series is obtained by substituting $1 + x$ for y_1, $1 + x^2$ for y_2, $1 + x^3$ for y_3, and so on in $Z(R_n)$. Some specific cases are

(1) For $n = 3$,

$$Z(R_3) = \frac{1}{6}(y_1^3 + 3y_1y_2 + 2y_3).$$

Therefore, the configuration-counting series is

$$B(x) = \frac{1}{6}[(1+x)^3 + 3(1+x)(1+x^2) + 2(1+x^3)]$$
$$= 1 + x + x^2 + x^3.$$

The coefficient of x^i in $B(x)$ is the number of configurations with content x^i. The content of a configuration here is the number of edges in the corresponding graph. Thus the number of nonisomorphic simple graphs of three vertices with 0, 1, 2, and 3 edges is each one. This is how it should be, as shown in Fig. 10-10.

(2) For $n = 4$, the cycle index $Z(R_4)$ is given in (10-32). Substituting $1 + x^i$ for y_i in (10-32), we get

$$B(x) = \frac{1}{24}[(1 + x)^6 + 9(1 + x)^2(1 + x^2)^2 + 8(1 + x^3)^2 + 6(1 + x^2)(1 + x^4)] \tag{10-41}$$
$$= 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6.$$

In (10-41) the coefficient of x^r gives the number of simple graphs with four vertices and r edges. The validity of series (10-41) is verified in Fig. 10-11.

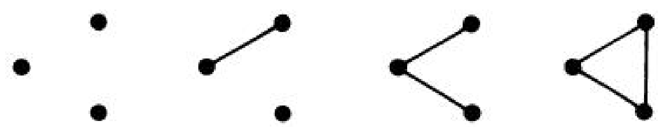


Fig. 10-10 Simple unlabeled graphs of three vertices.

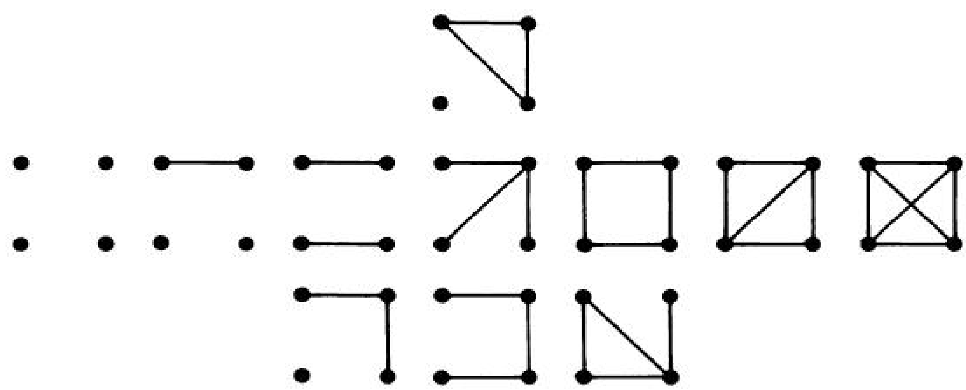


Fig. 10-11 Simple unlabeled graphs of four vertices.

(3) For $n = 5$, the cycle index $Z(R_5)$ is given in Problem 10-9. Substituting $1 + x^i$ for y_i in $Z(R_5)$, we get the counting series $B(x)$ for simple graphs of five vertices, as follows:

$$B(x) = \frac{1}{120}[(1 + x)^{10} + 10(1 + x)^4(1 + x^2)^3 + 20(1 + x)(1 + x^3)^3 + 15(1 + x)^2(1 + x^2)^4 + 30(1 + x^2)(1 + x^4)^2 + 20(1 + x)(1 + x^3)(1 + x^6) + 24(1 + x^5)^2] \tag{10-42}$$
$$= 1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 + 4x^7 + 2x^8 + x^9 + x^{10}.$$

Again, for each r the coefficient of x^r in (10-42) gives the number of simple graphs of five vertices and r edges.

The number of simple, unlabeled graphs with n vertices for any n can be counted similarly.

Enumeration of Multigraphs: Suppose that we are interested in counting multigraphs of n vertices, in which at most two edges are allowed between a pair of vertices.

In this case the domain and the permutation group are the same as they were for simple graphs. The range, however, is different. A pair of vertices may be joined by (1) no edge, (2) one edge, or (3) two edges. Thus range R contains three elements, say, s, t, u, with contents x^0, x^1, and x^2, respectively; that is, x^i indicates the presence of i edges between a vertex pair, for $i = 0, 1, 2$. Threfore, the figure-counting series becomes

$$1 + x + x^2. \tag{10-43}$$

Substitution of $1 + x^r + x^{2r}$ for y_r in $Z(R_n)$ will yield the desired configuration-counting series. For $n = 4$, using the cycle index from (10-32), we get

$$\begin{aligned} &\frac{1}{24}[(1 + x + x^2)^6 + 9(1 + x + x^2)^2(1 + x^2 + x^4)^2 + 8(1 + x^3 + x^6)^2 \\ &\quad + 6(1 + x^2 + x^4)(1 + x^4 + x^8)] \\ &= 1 + x + 3x^2 + 5x^3 + 8x^4 + 9x^5 + 12x^6 + 9x^7 + 8x^8 \\ &\quad + 5x^9 + 3x^{10} + x^{11} + x^{12}. \end{aligned} \tag{10-44}$$

The coefficient of x^i in (10-44) is the number of distinct, unlabeled, multigraphs of four vertices and i edges (such that there are at most two parallel edges between any vertex pair). For example. the coefficient of x^3 is 5, and these five multigraphs are shown in Fig. 10-12.

Instead of allowing at most two parallel edges between a pair of vertices,

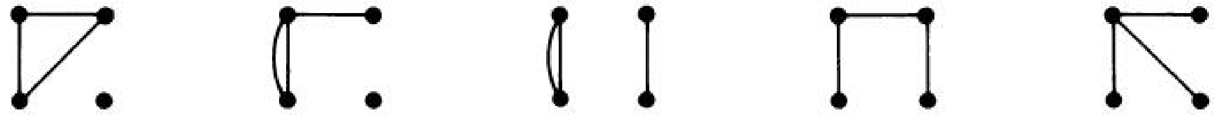


Fig. 10-12 Unlabeled multigraphs of four vertices, three edges, and at most two parallel edges.

had we allowed any number of parallel edges the figure-counting series would be the infinite series

$$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}. \tag{10-45}$$

Enumeration of Digraphs: For enumerating digraphs we have to consider all $n(n-1)$ *ordered* pairs of vertices as constituting the domain. The relevant permutation group will consist of permutations induced on all ordered pairs of vertices by S_n. The cycle index of this permutation group, M_n, can be obtained in the same fashion as was done in the case of R_n. For example, for $n = 4$, Table 10-4 gives the terms in $Z(M_n)$ induced by each term in $Z(S_n)$.

Term in $Z(S_4)$	Induced Term in $Z(M_4)$
y_1^4	y_1^{12}
$y_1^2 y_2$	$y_1^2 y_2^5$
$y_1 y_3$	y_3^4
y_2^2	y_2^6
y_4	y_4^3

Table 10-4

Therefore, the cycle index is

$$Z(M_4) = \frac{1}{24}(y_1^{12} + 6y_1^2 y_2^5 + 8y_3^4 + 3y_2^6 + 6y_4^3). \tag{10-46}$$

For a simple digraph the figure-counting series $A(x) = 1 + x$ is applicable, because a given ordered pair of vertices (a, b) either does or does not have an edge (directed) from a to b. On substituting $1 + x^i$ for every y_i in (10-46), we get the following configuration-counting series for four-vertex, simple digraphs.

$$\begin{aligned} B(x) &= \frac{1}{24}[(1+x)^{12} + 6(1+x)^2(1+x^2)^5 + 8(1+x^3)^4 \\ &\quad + 3(1+x^2)^6 + 6(1+x^4)^3] \\ &= 1 + x + 5x^2 + 13x^3 + 27x^4 + 38x^5 + 48x^6 \\ &\quad + 38x^7 + 27x^8 + 13x^9 + 5x^{10} + x^{11} + x^{12}. \end{aligned} \tag{10-47}$$

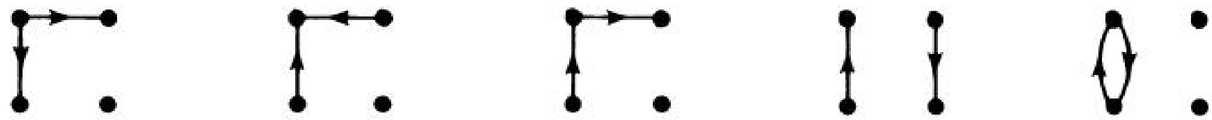


Fig. 10-13 Simple unlabeled digraphs of four vertices and two edges.

The coefficient of x^j in (10-47) is the number of simple digraphs with four vertices and j edges. For example, the five digraphs of two edges are shown in Fig. 10-13.

The general expression for the cycle index, $Z(M_n)$, of the permutation group on $n(n-1)$ ordered pairs induced by S_n is given in [1-5], page 180. Digraphs with parallel edges can be enumerated by substituting the appropriate figure-counting series, say (10-43), in $Z(M_n)$.

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Domination in graphs

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Domination in Graphs

by

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Table of Contents

List of Tables

List of Figures

Domination in Graphs

Jennifer M. Tarr

ABSTRACT

Vizing conjectured in 1963 that the domination number of the Cartesian product of two graphs is at least the product of their domination numbers; this remains one of the biggest open problems in the study of domination in graphs. Several partial results have been proven, but the conjecture has yet to be proven in general. The purpose of this thesis was to study Vizing's conjecture, related results, and open problems related to the conjecture. We give a survey of classes of graphs that are known to satisfy the conjecture, and of Vizing-like inequalities and conjectures for different types of domination and graph products. We also give an improvement of the Clark-Suen inequality [17]. Some partial results about fair domination are presented, and we summarize some open problems related to Vizing's conjecture.

Chapter 1
Introduction

Mathematical study of domination in graphs began around 1960. The following is a brief history of domination in graphs; in particular we discuss results related to Vizing's conjecture. We then provide some basic definitions about graph theory in general, followed by a discussion of domination in graphs.

1.1 History

Although mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [21] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [42] reported three basic types of problems that chess players studied during this time. These include the following:

1. **Covering:** Determine the minimum number of chess pieces of a given type that are necessary to cover (attack) every square of an $n \times n$ chess board.

2. **Independent Covering:** Determine the smallest number of mutually nonattacking chess pieces of a given type that are necessary to dominate every square of an $n \times n$ board.

3. **Independence:** Determine the maximum number of chess pieces of a given type that can be placed on an $n \times n$ chess board such that no two pieces attack each other. Note that if the chess piece being considered is the queen, this type of problem is commonly known as the N-queens Problem.

The study of domination in graphs was further developed in the late 1950's and 1960's, beginning with Claude Berge [5] in 1958. Berge wrote a book on graph theory, in which he introduced the "coefficient of external stability," which is now known as the domination number of a graph. Oystein Ore [39] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [48]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [16] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

Vizing's conjecture is perhaps the biggest open problem in the field of domination theory in graphs. Vizing [45] in 1963 first posed a question about the domination number of the Cartesian product of two graphs, defined in section 1.2. Vizing stated his conjecture that for any graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ in 1968 [46].

This problem did not receive much immediate attention after being conjectured; however, since the late 1970s, several results have been published. These results establish the truth of Vizing's conjecture for certain classes of graphs, and for graphs that meet certain criteria. Note that we say a graph G satisfies Vizing's conjecture if, for any graph H, the conjectured inequality holds. The first major result related to Vizing's conjecture was a theorem from Barcalkin and German [4] in 1979. They studied what is referred to as decomposable graphs and established a class of graphs known as BG-graphs for which Vizing's conjecture holds. A corollary of this result is that Vizing's conjecture holds for all graphs with domination number equal to 2, graphs with domination number equal to 2-packing number, and trees. The result that Vizing's conjecture is true for trees was also proved separately by Faudree, Schelp and Shreve [22], and Chen, Piotrowski and Shreve [13].

Hartnell and Rall [27] in 1995 established Vizing's conjecture for a larger class of graphs. They found a new way of partitioning the vertices of a graph that is slightly different from the way Barcalkin and German partitioned the vertices in decomposable graphs. The Type $\mathcal{X}$ class of graphs that resulted from Hartnell and Rall's work is an extension of the class of BG-graphs.

Another approach to Vizing's conjecture is to find a constant $c > 0$ such that $\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$. In 2000, Clark and Suen [17] were able to prove this inequality for $c = 1/2$. They used what is commonly referred to as the double projection method in their proof. As will be proven, this result can be improved to $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$.

One of the most recent results related to Vizing's conjecture deals with the new concept of fair reception, which was first defined by Brešar and Rall [11] in 2009. They defined the fair domination number of a graph G, denoted $\gamma_F(G)$, and proved that $\gamma(G\Box H) \geq \max\{\gamma(G)\gamma_F(H), \gamma_F(G)\gamma(H)\}$. Thus, for any graph G having $\gamma(G) = \gamma_F(G)$, Vizing's conjecture holds. Brešar and Rall showed that the class of such graphs is an extension of the BG-graphs distinct from Type $\mathcal{X}$ graphs.

1.2 Graph-Theoretic Definitions

The study of domination in graphs came about partially as a result of the study of games and recreational mathematics. In particular, mathematicians studied how chess pieces of a particular type could be placed on a chessboard in such a way that they would attack, or dominate, every square on the board. With this in mind, graph theoretical definitions will be related to the game of chess where applicable.

A *graph* $G = (V, E)$ consists of a set V of vertices and a set E of edges. We shall only consider simple graphs, which contain no loops and no repeated edges. That is, E is a set of unordered pairs $\{u, v\}$ of distinct elements from V. The *order* of G is $|V(G)| = n$, and the $size$ of G is $|E(G)| = m$. If $e = \{v_i, v_j\} \in E(G)$, then v_i and v_j are *adjacent*. Vertex v_i and edge e are said to be *incident*.

Envision a standard 8×8 chessboard, as can be seen in Figure 1. Each square can be represented by a vertex in a graph G. Consider placing several queens on the board. A queen may move any number of spaces vertically, horizontally, or diagonally. Any square (or vertex) to which a queen is able to move is adjacent to the square containing the queen. Therefore, there is an edge between those two squares, or vertices of the graph G. Since the chessboard is 8×8, with each square reprented by a vertex of the graph G, the order of G is 64. The size of G depends on the number, type, and placement of chess pieces on the board.

We call the set of vertices adjacent to a vertex v in a graph G the *open neighborhood* $N(v)$ of v. The open neighborhood of a set of vertices $S \subset V(G)$ is $N(S) = \bigcup\limits_{v \in S} N(v)$. The *closed neighborhood* $N[v]$ of v is $N(v) \cup \{v\}$, and the closed neighborhood of a set of vertices $S \subset V(G)$ is $N[S] = N(S) \cup S$.

The *degree* of a vertex v, denoted $\deg(v)$ is the number of edges incident with v. Alternatively, we can define $\deg(v) = |N(v)|$. The minimum and maximum degrees of vertices in $V(G)$ are

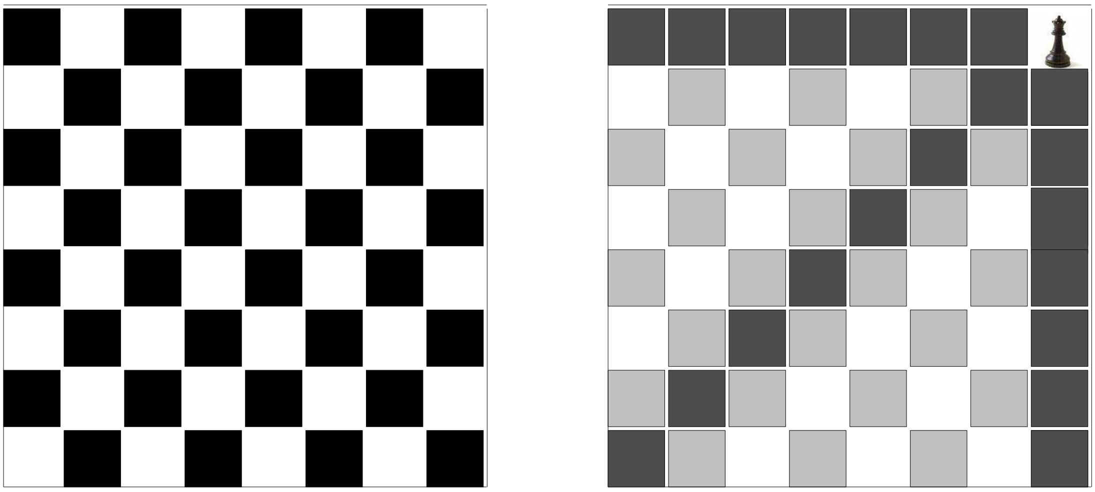

Figure 1.: The first image depicts a standard 8×8 chessboard. The second image has a queen placed in the upper right corner. If we represent every square on the board by a vertex in a graph, then we would draw an edge from the queen to every vertex representing one of the shaded squares.

denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $\delta(G) = \Delta(G) = r$, then the graph G is *regular* of degree r, or *r-regular*.

Consider, once again, placing several queens on a chessboard. Assume the space occupied by one of the queens is denoted by vertex v. Then the number of possible moves for the queen occupying that space, including those occupied by other queens, is equal to $\deg(v)$. If we count the number of possible spaces to which the queen in Figure 1 can move, we see that it has 21 possible moves. Thus, if we represent that chessboard by a graph and denote the space containing the queen as vertex v, we have $\deg(v) = 21$.

A *walk* of length k is a sequence $w = v_0, v_1, v_2, \ldots, v_k$ of vertices where v_i is adjacent to v_{i+1} for $i = 0, 1, \ldots, k-1$. A walk consisting of $k+1$ distinct vertices $v_0, v_1, ..., v_k$ is a *path*, and if $v_o = v_k$ then these vertices form a *cycle*. A graph G is *connected* if for every pair of vertices v and x in $V(G)$, there is a v-x path. Otherwise, G is *disconnected*. A *component* of G is a connected subgraph of G which is not properly contained in any other connected subgraph.

If there is at least one v-x walk in the graph G then the *distance* $d(v, x)$ is the minumum length of a v-x walk. If no v-x walk exists, we say that $d(v, x) = \infty$.

We now consider a few different types of graphs. The *cycle* C_n of order $n \geq 3$ has size $m = n$, is connected and 2-regular. See Figure 2 for the graphs C_4 and C_5. A $tree\ T$ is a connected graph

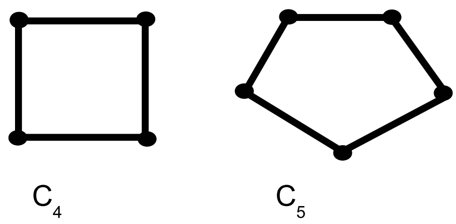

Figure 2.: Cycles C_4 and C_5

with no cycles. Every tree T with n vertices has $m = n - 1$ edges. The *star* $K_{1,n-1}$ has one vertex of degree $n-1$ and $n-1$ vertices of degree 1. Observe that a star is a type of tree. Refer to Figure 3 for examples of a tree and a star.

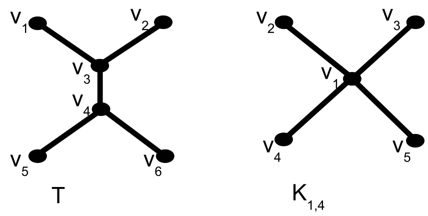

Figure 3.: A tree T and the star $K_{1,4}$

In any graph a vertex of degree one is an *endvertex*. An edge incident with an endvertex is a *pendant edge*. We can see that the graphs T and $K_{1,4}$ in Figure 3 each have four pendant edges and four endvertices. Specifically, in T, the endvertices are v_1, v_2, v_5, and v_6, and pendant edges are $\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_5\}$, and $\{v_4, v_6\}$.

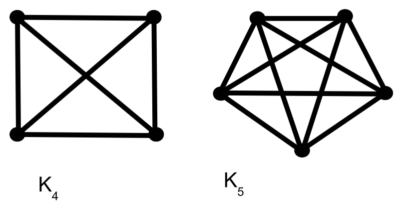

Figure 4.: Complete graphs K_4 and K_5

The *complete graph* K_n has the maximum possible edges $n(n-1)/2$. See Figure 4 for the graphs of K_4 and K_5. The *complement* $\bar{G}$ of a graph G has $V(\bar{G}) = V(G)$ and $\{u, v\} \in E(G)$ if and only if $\{u, v\} \notin E(\bar{G})$. Thus, the complement of a complete graph is the empty graph.

A *bipartite graph* is one that can be partitioned as $V = V_1 \cup V_2$ with no two adjacent vertices in the same V_i. We define the *chromatic number* of a graph G to be the minimum k such that $V(G)$ can be partitioned into sets $S_1, S_2, \ldots, S_k$ and each S_i is independent. That is, for each i, no two vertices in S_i are adjacent. Denote the chromatic number of G by $\chi(G)$. If $\chi(G) = k$, then G is k-colorable which means we can color the vertices of G with k colors in such a way that no two adjacent vertices are the same color. Observe that a graph is 2-colorable if and only if it is a bipartite graph.

The graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H satisfies the property that for every pair of vertices u and v in $V(H)$, the edge $\{u, v\}$ is in $E(H)$ if and only if $\{u, v\} \in E(G)$ then H is an *induced subgraph* of G. The induced subgraph H with $S = V(H)$ is called the *subgraph induced by S*. This is denoted by $G[S]$.

There are several different products of graphs G and H; we shall define the Cartesian product, strong direct product, and categorical product. All three of these products have vertex set $V(G) \times V(H)$. The *Cartesian product* of G and H, denoted by $G \Box H$, has edge set

$$E(G \Box H) = \{\{(u_1, v_1), (u_2, v_2)\} \mid u_1 = u_2 \text{ and } \{v_1, v_2\} \in E(H);$$

$$\text{or } \{u_1, u_2\} \in E(G) \text{ and } v_1 = v_2\}.$$

The *strong direct product* of G and H has edge set

$$E(G \Box H) \cup \{\{(u_1, v_1), (u_2, v_2)\} \mid \{u_1, u_2\} \in E(G) \text{ and } \{v_1, v_2\} \in E(H)\}$$

and is denoted by $G \boxtimes H$. The *categorical product*, denoted by $G \times H$, has edge set

$$E(G \times H) = \{\{(u_1, v_1), (u_2, v_2)\} \mid \{u_1, u_2\} \in E(G) \text{ and } \{v_1, v_2\} \in E(H)\}.$$

1.3 Domination in Graphs

We now introduce the concept of dominating sets in graphs. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a *dominating set* if every vertex $v \in V$ is an element of S or adjacent to an element of S. Alternatively, we can say that $S \subseteq V$ is a dominating set of G if $N[S] = V(G)$. A dominating set S is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. We call such a set a γ-set of G.

For a graph $G = (V, E)$ and $S \subseteq V$ a vertex $v \in S$ is an *enclave* of S if $N[v] \subseteq S$. For $S \subseteq V$ a vertex $v \in S$ is an *isolate* of S if $N(v) \subseteq V - S$. We say that a set is *enclaveless* if it does not contain any enclaves. Note that S is a dominating set of a graph $G = (V, E)$ if and only if $V - S$ is enclaveless.

Theorem 1.1 *[39] A dominating set S of a graph G is a minimal dominating set if and only if for any $u \in S$,*

1. *u is an isolate of S, or*

2. *There is $v \in V - S$ for which $N[v] \cap S = \{u\}$.*

Proof. [39] Let S be a γ-set of G. Then for every vertex $u \in S$, $S - \{u\}$ is not a dominating set of G. Thus, there is a vertex $v \in (V - S) \cup \{u\}$ that is not dominated by any vertex in $S - \{u\}$. Now, either $v = u$, which implies u is an isolate of S; or $v \in V - S$, in which case v is not dominated by $S - \{u\}$, and is dominated by S. This shows that $N[v] \cap S = \{u\}$.

In order to prove the converse, we assume S is a dominating set and for all $u \in S$, either u is an isolate of S or there is $v \in V - S$ for which $N[v] \cap S = \{u\}$. We assume to the contrary that S is not a γ-set of G. Thus, there is a vertex $u \in S$ such that $S - \{u\}$ is a dominating set of G. Hence, u is adjacent to at least one vertex in $S - \{u\}$, so condition (1) does not hold. Also, if $S - \{u\}$ is a dominating set, then every vertex in $V - S$ is adjacent to at least one vertex in $S - \{u\}$, so condition (2) does not hold for u. Therefore, neither (1) nor (2) holds, contradicting our assumption. $\square$

Theorem 1.2 *[39] Let G be a graph with no isolated vertices. If D is a γ-set of G, then $V(G) - D$ is also a dominating set.*

Proof. [39] Let D be a γ-set of the graph G and assume $V(G) - D$ is not a dominating set of G. This means that for some vertex $v \in D$, there is no edge from v to any vertex in $V(G) - D$. But then the set $D - v$ would be a dominating set, contradicting the minimality of D. We conclude that $V(G) - D$ is a dominating set of G. $\square$

Theorem 1.3 *[39] If a graph G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Proof. Let G be a graph with no isolated vertices and let D be a γ-set of G. Assume to the contrary that $\gamma(G) > \frac{n}{2}$. By Theorem 1.2, $V(G) - D$ is a dominating set of G. But $|V(G) - D| < n - \frac{n}{2}$, contradicting the minimality of $\gamma(G)$. We conclude that $\gamma(G) \leq \frac{n}{2}$. $\square$

Theorem 1.4 *[36] For any graph G,*

$$\gamma(G) + \gamma(\bar{G}) \leq n + 1 \tag{1.1}$$

$$\gamma(G)\gamma(\bar{G}) \leq n \tag{1.2}$$

Proof. [36] We show (1.1) first. If the graphs G and $\bar{G}$ have no isolated vertices, then Theorem 1.3 implies $\gamma(G)+\gamma(\bar{G}) \leq n$. If G has an isolated vertex, then $\gamma(G) \leq n$ and $\gamma(\bar{G}) = 1$. Then we have $\gamma(G) + \gamma(\bar{G}) \leq n + 1$. Similarly, if $\bar{G}$ has an isolated vertex, we have $\gamma(\bar{G}) \leq n$ and $\gamma(G) = 1$, which implies $\gamma(G) + \gamma(\bar{G}) \leq n + 1$.

Now we prove (1.2). Define for $X \subseteq V(G)$ the following sets:

$$D_0(X) = \{u \in V(G) - X \mid \{u, v\} \in E(G) \text{ for all } v \in X\},$$

and

$$D_1(X) = \{u \in X \mid \{u, v\} \in E(G) \text{ for all } v \in X\}.$$

Now, let $D = \{v_1, v_2, ..., v_{\gamma(G)}\}$ be a γ-set of G and partition the vertices of $V(G)$ into sets Π_i such that $v_i \in \Pi_i$ for each $i = 1, 2, ..., \gamma(G)$ and if $v \in \Pi_i$ then $v = v_i$ or $\{v, v_i\} \in E(G)$. Choose this partition in such a way that $\sum_{i=1}^{\gamma(G)} |D_1(\Pi_i)|$ is a maximum.

Suppose $|D_0(\Pi_j)| \geq 1$ for some j. Then there is a vertex $v \in \Pi_k$, for $k \neq j$, such that $\{u, v\} \in E(G)$ for all $u \in \Pi_j$.

If $v \in D_0(\Pi_k)$ then $(D - \{v_j, v_k\}) \cup \{v\}$ is a dominating set of G with cardinality smaller than $\gamma(G)$, a contradiction. Thus, $v \notin D_0(\Pi_k)$.

Now we can re-partition the vertices of G so that $\Pi'_l = \Pi_l$ for $l \neq j$ and $l \neq k$, $\Pi'_j = \Pi_j \cup \{v\}$ and $\Pi'_k = \Pi_k - \{v\}$. But then $|D_1(\Pi'_l)| = |D_1(\Pi_l)|$, $|D_1(\Pi'_j)| = |D_1(\Pi_j)| + 1$, and $|D_1(\Pi'_k)| \geq |D_1(\Pi_k)|$. This contradicts the choice of our original partition of G.

We conclude that $|D_0(\Pi_i)| = 0$ for all $i = 1, 2, \ldots, \gamma(G)$. As any set X with $|D_0(X)| = 0$ dominates $\bar{G}$, each set Π_i dominates $\bar{G}$ and so $\gamma(\bar{G}) \leq |\Pi_i|$. Therefore, we have

$$n = \sum_{i=1}^{\gamma(G)} |\Pi_i| \geq \gamma(G)\gamma(\bar{G}).$$

□

We define the corona G of graphs G_1 and G_2 as follows. The *corona* $G = G_1 \circ G_2$ is the graph formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. Refer to Figure 5 for an example of a corona of two graphs. We take the original graph G and, as $|V(G)| = 4$, we have four copies of H. Both vertices in the ith copy of H are adjacent to the ith vertex in G for each $i = 1, ..., 4$.

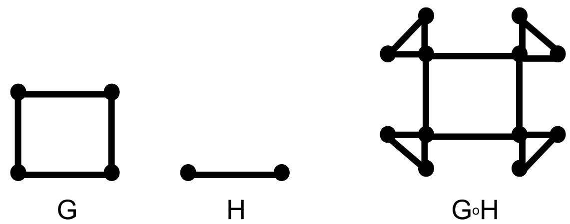

Figure 5.: Graphs G and H, and the corona $G \circ H$

The following theorem, which was proved independently by Payan and Xuong and by Fink, Jacobson, Kinch and Roberts, tells us which graphs have domination number equal to $\frac{n}{2}$. Thus, we can use this result to find extremal examples of graphs which achieve the upper bound in Theorem 1.3.

Theorem 1.5 *[23] [40] For a graph G with even order n and no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Proof. [40] It can easily be verified that if the components of a graph G are C_4 or the corona $H \circ K_1$ for a connected graph H, then $\gamma(G) = \frac{n}{2}$.

Now we assume that $\gamma(G) = \frac{n}{2}$. We may assume that G is connected. Let $C = \{S_1, S_2, \ldots, S_p\}$ be a minimal set of stars which cover all vertices of G. Since $\gamma(G) = \frac{n}{2}$, C must be a maximal matching of $p = \frac{n}{2}$ edges. For each $S_i \in C$, let $S_i = \{x_i, y_i\}$. We consider two cases.

If $p \geq 3$ then for every i, either x_i or y_i has degree 1. If not, there is i such that $\deg(x_i) \geq 2$ and $\deg(y_i) \geq 2$. But then we can find a dominating set of G with cardinality less than $\frac{n}{2}$. This implies G is a corona $H \circ K_1$ for some connected graph H.

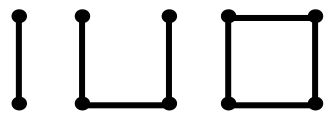

Figure 6.: Coronas $K_1 \circ K_1$ and $K_2 \circ K_1$ and cycle C_4.

If $p \leq 2$ then G is isomorphic to one of the graphs in Figure 6. Note that the first two graphs are coronas and the third is the cycle C_4.

We conclude that $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ where H is a connected graph. $\square$

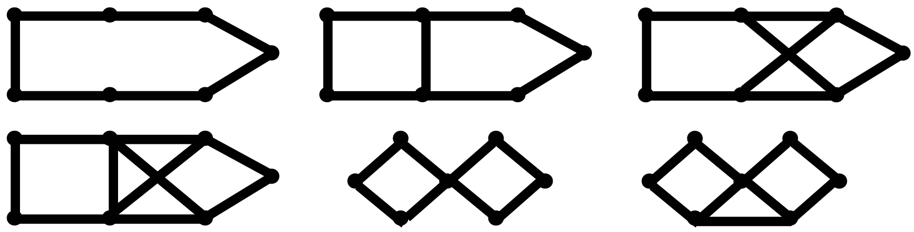

Figure 7.: Family $\mathcal{A}$

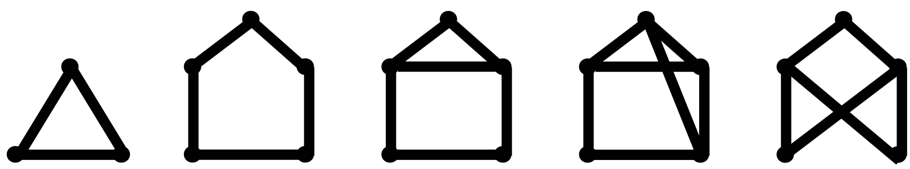

Figure 8.: Family $\mathcal{B}$

We now characterize connected graphs with $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ by defining the following six classes of graphs. These results were proved independently by Cockayne, Haynes and Hedetniemi [15] and by Randerath and Volkmann [41].

1. $G_1 = \{C_4\} \cup \{G \mid G = H \circ K_1 \text{ where } H \text{ is connected}\}$.

2. $G_2 = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are the families of graphs depicted in Figure 7 and Figure 8.

3. $G_3 = \bigcup\limits_{H} S(H)$ where $S(H)$ denotes the set of connected graphs, each of which can be formed from $H \circ K_1$ by adding a new vertex x and edges joining x to at least one vertex in H.

4. $G_4 = \{\Theta(G) \mid G \in G_3\}$ where $y \in V(C_4)$ and for $G \in G_3$, $\Theta(G)$ is obtained by joining G to C_4 with the single edge $\{x, y\}$, where x is the new vertex added in forming G.

5. $G_5 = \bigcup_H \mathcal{P}(H)$ where u, v, w is a vertex sequence of a path P_3. For any graph H, $\mathcal{P}(H)$ is the set of connected graphs which may be formed from $H \circ K_1$ by joining each of u and w to one or more vertices of H.

6. $G_6 = \bigcup_{H,X} \mathcal{R}(H, X)$ where H is a graph, $X \in \mathcal{B}$, and $\mathcal{R}(H, X)$ is the set of connected graphs obtained from $H \circ K_1$ by joining each vertex of $U \subseteq V(X)$ to one or more vertices of H such that no set with fewer than $\gamma(X)$ vertices of X dominates $V(X) - U$.

Theorem 1.6 *[15] [41] A connected graph G satisfies $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G \in \mathcal{G} = \bigcup_{i=1}^{6} G_i$.*

As a result of Theorem 1.5 and Theorem 1.6, we can completely classify graphs with domination number $\gamma(G) = \lfloor \frac{n}{2} \rfloor$.

We now define several additional types of domination in graphs. We shall show Vizing-like inequalities and conjectures for these types of domination in Section 2.2.

Let $f : V(G) \to [0, 1]$ be a function defined on the vertices of a graph G; this is a *fractional-dominating function* if the sum of the values of f over any closed neighborhood in G is at least 1. The *fractional domination number* of a graph G is denoted $\gamma_f(G)$ and is the minimum weight of a fractional-dominating function, where the weight of the function is the sum over all vertices of its values. A similar type of domination is integer domination. Let $k \geq 1$ and let $f : V(G) \to \{0, 1, \ldots, k\}$ be a function defined on the vertices of a graph G. This is a $\{k\}$*-dominating function* if the sum of the function values over any closed neighborhood of G is at least k. As with fractional domination, the weight of a $\{k\}$-dominating function is the sum of its function values over all vertices. We define the $\{k\}$*-domination number* of G to be the minimum weight of a $\{k\}$-dominating function of G. This is denoted by $\gamma_{\{k\}}(G)$.

The maximum cardinality of a minimal dominating set of a graph G is called the *upper domination number* and is denoted by $\Gamma(G)$. We say that a set $S \subset V(G)$ is *independent* if for all u and v in S, $\{u, v\} \notin E(G)$. The maximum cardinality of a maximal independent set in G is the *independence number* $\alpha(G)$, and the minimum cardinality of a maximal independent set is the *lower independence number* $i(G)$. Note that the lower independence number is also often referred to as the *independent domination number*.

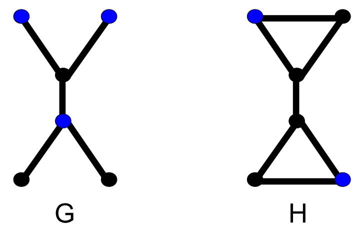

Figure 9.: Independent domination in non-claw-free and claw-free graphs

Observe that claw-free graphs, or graphs that do not contain a copy of $K_{1,3}$ as an induced subgraph, have $\gamma(G) = i(G)$. This result was proved by Allan and Laskar in 1978 [3]. Refer to Figure 9. It can easily be verified that the graphs G and H both have domination number equal to 2. The graph G is not claw-free and $i(G) = 3$; an example of a minimal independent dominating set of G is indicated by the blue vertices. The graph H, on the other hand, is claw-free and has $i(H) = 2$. We can see that the blue vertices in H form an independent dominating set.

A set $S \subseteq V(G)$ is a *total dominating set* of G if $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set. Note that a dominating set S is a total dominating set if $G[S]$, the subgraph induced by S has no isolated vertices. The *upper total domination number* of G, denoted by $\Gamma_t(G)$, is the maximum cardinality of a minimal total dominating set of a graph G. The function $f : V(G) \rightarrow \{0, 1, \ldots, k\}$ is a *total* $\{k\}$*-dominating function* if the sum of its function values over any open neighborhood is at least k. The *total* $\{k\}$*-domination number* $\gamma_t^{\{k\}}$ of a graph G is the minimum weight of a total {k}-dominating function of G.

The above defined parameters of a graph G are related by the following lemma.

Lemma 1.1 *[38] For any graph G, $\gamma_f(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$. If G has no isolated vertices, then $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.*

For any graph G, a matching is a set of independent edges in G, and a perfect matching of G is one which matches every vertex in G. The set $D \subseteq V(G)$ is a *paired dominating set* of G if D dominates G and the induced subgraph $G[D]$ has a perfect matching. We denote the *paired domination number*, or the minimum cardinality of a paired dominating set, by $\gamma_{pr}(G)$.

The *independence domination number* of a graph G, denoted by $\gamma^i(G)$, is the maximum, over all independent sets I in G, of the minimum number of vertices required to dominate I. Note that this is different from the independent domination number.

There are several other types of domination, defined below, for which we will not present further Vizing-like results.

Let $G = (V, E)$ be a bipartite graph, with partite sets V_1 and V_2. If a set of vertices $S \subseteq V_1$ dominates V_2, we say that S is a *bipartite dominating set* of G.

A *connected dominating set* is a dominating set that induces a connected subgraph of the graph G. We denote by $\gamma_c(G)$ the *connected domination number*, or the minimum cardinality of a dominating set S such that $G[S]$ is connected. Clearly, $\gamma(G) \leq \gamma_c(G)$.

Observe that when $\gamma(G) = 1$, $\gamma(G) = \gamma_c(G) = i(G) = 1$. This implies that if G is a complete graph or a star, the domination number, connected domination number, and independent domination number all equal 1. Also, since a connected dominating set of G is also a total dominating set of G, we have $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$. An example of the sharpness of this bound can be seen in the complete bipartite graph $K_{r,s}$, in which $\gamma(K_{r,s}) = \gamma_t(K_{r,s}) = \gamma_c(K_{r,s}) = 2$. See Figure 10, which depicts the graph $K_{2,3}$. The blue vertices form both a minimal dominating set and a total dominating set.

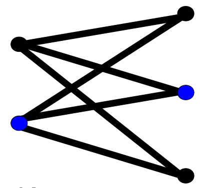

Figure 10.: An example of equality in domination and total domination

If D is a dominating set of G and $G[D]$ is complete, then we call D a *dominating clique*. The minimum cardinality of a dominating clique is the *clique domination number,* denoted $\gamma_{cl}(G)$. Not every graph has a dominating clique; for example, any cycle C_n where $n \geq 5$ does not contain a dominating clique. Clearly, if $\gamma(G) = 1$, then $\gamma(G) = \gamma_c(G) = \gamma_{cl}(G) = 1$. If G has a dominating clique and $\gamma(G) \geq 2$ then $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq \gamma_{cl}(G)$. An example of the sharpness of these bounds can be seen in the corona $K_p \circ K_1$, which has $\gamma(K_p \circ K_1) = \gamma_t(K_p \circ K_1) = \gamma_c(K_p \circ K_1) = \gamma_{cl}(K_p \circ K_1) = p$. The blue vertices in the graph of the corona $K_3 \circ K_1$ in Figure 11 form a minimal dominating set which is also a total dominating set, connected dominating set, and a dominating clique.

A *cycle dominating set* is a dominating set of G whose vertices form a cycle.

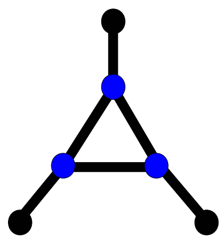

Figure 11.: An example of equality in domination, total domination, connected domination, and clique domination

Chapter 2
Vizing's Conjecture

Since Vizing's conjecture was first stated in the 1960s, several results have been published which establish the truth of the conjecture for classes of graphs satisfying certain criteria. As the problem has not yet been solved in general, researchers have also studied similar problems for different types of graph products and for other types of domination. Some of these similar problems also remain conjectures, while others have been proven. Here, we describe the classes of graphs which are known to satisfy Vizing's conjecture and provide a brief discussion of the similar Vizing-like conjectures which have also been studied. Another common approach to solving the conjecture is to find a constant c such that for any graphs G and H, $\gamma(G \square H) \geq c\gamma(G)\gamma(H)$. As Clark and Suen [17] proved in 2000, this is true for $c = \frac{1}{2}$. We provide a slight improvement of this lower bound by tightening their arguments.

2.1 Classes of Graphs Satisfying Vizing's Conjecture

Vizing's conjecture is that for any two graphs, the domination number of the Cartesian product graph of G and H is greater than or equal to the product of the domination numbers of G and H. The conjecture is stated as follows:

Conjecture 2.1 *[46] For any graphs G and H, $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.*

Recall that the Cartesian product of graphs G and H has vertex set

$$V(G \square H) = V(G) \times V(H) = \{(x, y) \mid x \in V(G) \text{ and } y \in V(H)\}$$

and it has edge set

$$E(G \square H) = \{\{(x_1, y_1), (x_2, y_2)\} \mid x_1 = x_2 \text{ and } \{y_1, y_2\} \in E(H);$$

or $\{x_1, x_2\} \in E(G)$ and $y_1 = y_2\}$.

Define a *2-packing* of G as a set $X \subset V(G)$ of vertices such that $N[x] \cap N[y] = \emptyset$ for each pair of distinct vertices $x, y \in X$. Alternatively, we can define a 2-packing as a set X of vertices in G such that for any pair of vertices x and y in X, $d(x, y) > 2$. The maximum cardinality of a 2-packing of G is called the *2-packing number of* G and is denoted by $\rho_2(G)$.

Observe that for any graph G, $\rho_2(G) \leq \gamma(G)$. Let S be a maximal 2-packing of G. Then, as $d(u, v) > 2$ for every pair of vertices u and v in S, we need at least one vertex in $V(G)$ to dominate each vertex in S. Hence, the cardinality of a minimal dominating set is greater than or equal to the cardinality of a maximal 2-packing.

Note that we say a graph G satisfies Vizing's conjecture if, for any graph H, the conjectured inequality holds. Several results establish the truth of Vizing's conjecture for graphs satisfying certain criteria. The case where $\gamma(G) = 1$ is trivial. A corollary of Barcalkin and German's [4] proof that Vizing's conjecture holds for decomposable graphs is that Vizing's conjecture is true for any graph G with $\gamma(G) \leq 2$. In 2004, Sun [44] verified Vizing's conjecture holds for any graph G with $\gamma(G) \leq 3$.

We now consider classes of graphs that are proven to satisfy Vizing's conjecture.

Lemma 2.1 *[26] If G satisfies Vizing's conjecture and K is a spanning subgraph of G such that $\gamma(G) = \gamma(K)$, then K satisfies Vizing's conjecture.*

Proof. Let K be a spanning subgraph of G obtained by a finite sequence of edge removals which does not change the domination number. Since K is a subgraph of G, $K \square H$ is a subgraph of $G \square H$. Thus we have $\gamma(K \square H) \geq \gamma(G \square H) \geq \gamma(G)\gamma(H)$ by assumption on G. By assumption on K, we have $\gamma(G)\gamma(H) = \gamma(K)\gamma(H)$. We conclude that K satisfies Vizing's conjecture. $\square$

Theorem 2.1 *[28] Let G be a graph and let $x \in V(G)$ such that $\gamma(G - x) < \gamma(G)$. Then if G satisfies Vizing's conjecture, the graph $G - x$ satisfies Vizing's conjecture.*

Proof. [28] Let G be a graph which satisfies Vizing's conjecture, and assume $\gamma(G - x) < \gamma(G)$ for some $x \in V(G)$. Then $\gamma(G - x) = \gamma(G) - 1$. Now assume there is a graph H such that $\gamma((G - x)\square H) < \gamma(G - x)\gamma(H)$. Let A be a γ-set of $(G - x)\square H$ and let B be a γ-set of H. Define $D = A \cup \{(x, b) \mid b \in B\}$. Clearly D is a dominating set of $G \square H$ of cardinality

$|A| + |B| < \gamma(G-x)\gamma(H) + \gamma(H) = (\gamma(G-x)+1)\gamma(H) = \gamma(G)\gamma(H)$. This contradicts our assumption that G satisfies Vizing's conjecture, and so we conclude that $G - x$ satisfies Vizing's conjecture. $\square$

Note that, if the converse of this theorem does not hold, we would have a counterexample to Vizing's conjecture. Consider a graph K that satisfies Vizing's conjecture, and let $S \subseteq V(K)$ be a set of vertices such that no vertex of S belongs to any γ-set of K and such that $\gamma(K - S) = \gamma(K)$. We can form a graph G from K by adding a new vertex v and all edges $\{u, v\}$ where u is in S. If the resulting graph G does not satisfy Vizing's conjecture then obviously we have a counterexample. If, on the other hand, we can prove that the graph G satisfies Vizing's conjecture, then this result would contribute to an attempt to prove Vizing's conjecture by using a finite sequence of constructive operations. The idea is to begin with a class C of graphs for which we know Vizing's conjecture is true and find a collection of operations to apply to graphs from C, each of which results in a graph which satisfies Vizing's conjecture. At this point, the goal would be to show that any graph can be obtained from a seed graph in C by applying a finite set of these operations. This type of approach has obviously not yet been successful, but Hartnell and Rall [28] define several operations which could potentially lead to a proof of Vizing's conjecture using a constructive method.

Lemma 2.2 *[20] For any graphs G and H, $\gamma(G\Box H) \geq \min\{|V(G)|, |V(H)|\}$.*

Proof. [20] Let D be a γ-set of the product graph $G\Box H$, and assume to the contrary that $|D| < \min\{|V(G)|, |V(H)|\}$. Then there is a column of vertices $H_u = \{u\} \times V(H)$ and a row of vertices $G_v = V(G) \times \{v\}$ such that $D \cap H_u = D \cap G_v = \emptyset$. But then $(u, v) \notin N[D]$, a contradiction. Therefore, $\gamma(G\Box H) \geq \min\{|V(G)|, |V(H)|\}$. $\square$

The following result providing a lower bound for $\gamma(G\Box H)$ was proved by Jacobson and Kinch [34]. Their proof considers a dominating set for the product graph $G\Box H$ and counts the way the dominating set intersects each set of vertices $V(G) \times \{v\}$, where $v \in V(H)$.

Theorem 2.2 *[34] For any graphs G and H, $\gamma(G\Box H) \geq \frac{|H|}{\Delta(H)+1}\gamma(G)$.*

Observe that this theorem implies Vizing's conjecture holds for cycles of length $3k$. Consider the cycle C_{3k}, for $k \geq 1$ an integer. We have $\Delta(C_{3k}) = 2$ and $\gamma(C_{3k}) = k$, so therefore $\frac{|C_{3k}|}{\Delta(C_{3k})+1} = \frac{3k}{3} = k = \gamma(C_{3k})$.

Theorem 2.3 *[45] For any graphs G and H, $\gamma(G\square H) \leq \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}$.*

Proof. Let A be a γ-set of G. Now let $D = \{A \times \{v\} \mid v \in V(H)\}$. Then D is a dominating set of $G\square H$ of cardinality $\gamma(G)|V(H)|$. Similarly, we can let B be a γ-set of H and define $D = \{\{u\} \times B \mid u \in V(G)\}$. Thus, we have $\gamma(G\square H) \leq \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}$. $\square$

Theorem 2.4 *[35] For any graphs G and H,*

$$\gamma(G\square H) \geq \max\{\gamma(G)\rho_2(H), \rho_2(G)\gamma(H)\}.$$

Notice that this result from Jacobson and Kinch can be improved by the following theorem from Chen, Piotrowski and Shreve.

Theorem 2.5 *[13] For any graphs G and H,*

$$\gamma(G\square H) \geq \gamma(G)\rho_2(H) + \rho_2(G)(\gamma(H) - \rho_2(H)).$$

The earliest significant result related to the domination number of a Cartesian product was produced by Barcalkin and German [4] in 1979. Barcalkin and German studied graphs G which have domination number equal to the chromatic number of $\bar{G}$. Recall that the chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G in such a way that no two adjacent vertices are the same color. Observe that any proper coloring of $\bar{G}$ is a partition of the vertices of G into cliques, or complete subgraphs of G. A single vertex may be chosen from each clique to form a dominating set of G and, therefore, it is always true that $\gamma(G) \leq \chi(\bar{G})$.

Barcalkin and German defined *decomposable graphs* as follows. Let G be a graph with $\gamma(G) = k$, and assume $V(G)$ can be partitioned into k sets $C_1, C_2, ..., C_k$ such that each induced subgraph $G[C_i]$ is a complete subgraph of G. If G satisfies these conditions, then it is a decomposable graph. They also define the *A-class*, which consists of all graphs G' that are spanning subgraphs of a decomposable graph G, where $\gamma(G') = \gamma(G)$. The result of Barcalkin and German's 1979 paper established Vizing's conjecture for any graph which belongs to the A-class. Note that we now commonly refer to this class of graphs as BG-graphs.

Theorem 2.6 *[4] Let G be a decomposable graph and let K be a spanning subgraph of G with $\gamma(G) = \gamma(K)$. Then K satisfies Vizing's conjecture.*

Proof. [28] We assume that G is a decomposable graph with $\gamma(G) = k$. Let $\{C_i \mid G[C_i]$ is a complete subgraph of $G, 1 \leq i \leq k\}$ be a partition of $V(G)$. We now consider the partition $\{C_i \times V(H) \mid i = 1, \ldots, k\}$ of $V(G \Box H)$ for H an arbitrary graph. Let D be a γ-set of $G \Box H$.

Denote by D_j the set of vertices in D that are also in $C_j \times V(H)$. That is,

$$D_j = D \cap (C_j \times V(H)) \text{ for } j = 1, \ldots, k.$$

Let $u_j \in C_j$ and denote by P_j the projection of vertices in $C_j \times V(H)$ onto $\{u_j\} \times V(H)$.

Let L_j be the set of all vertices v such that (u_j, v) is not dominated by $P_j(D_j)$. That is,

$$L_j = \{v \mid (u_j, v) \notin N[P_j(D_j)]\}.$$

We observe that if $v \in L_i$, then the vertices $C_j \times \{v\}$ are dominated "horizontally". Obviously, if $P_j(D_j)$ dominates $u_j \times V(H)$, $|L_j| = 0$. However, if $|D_j| = \gamma(H) - m$ then we have

$$|D_j| + |L_j| \geq |P_j(D_j)| + |L_j| \geq \gamma(H).$$

This implies that $|L_j| \geq m$.

We now consider $v \in V(H)$ such that $v \in L_i$ for at least one $i = 1, \ldots, k$. Define the sets D_v, S_v, and A_v as follows. We let $S_v = \{C_i \mid v \in L_i$ and $i = 1, \ldots, k\}$. Define A_v to be the set of cliques C_j such that there is at least one edge from a vertex in C_j to a member of S_v and $D \cap (C_j \times \{v\}) \neq \emptyset$. Finally, we let $D_v = \{u \in V(G) \mid (u, v) \in D$ and $u \in C_j \in A_v\}$.

We observe that $|D_v| \geq |S_v| + |A_v|$, for otherwise we would have

$$\hat{D_v} = D_v \cup \{(u_j, v) | C_j \notin S_v \cup A_v\}$$

is a dominating set of $V(G) \times \{v\}$ of cardinality less than k.

Also observe that for each $i = 1, \ldots, k$ either $|D_i| \geq \gamma(H)$, in which case summing over i gives the desired inequality; or $|D_i| = \gamma(H) - m$. In the latter case, we have shown that $|D_v| \geq |S_v| + |A_v|$. From this, we have

$$|S_v| \leq \sum_{u \in D_v} (|D \cap (C_j \times \{u\})| - 1). \tag{2.1}$$

Thus, we have sufficient extra vertices in D in neighboring cliques so that we still have an average of $\gamma(H)$ for each $|D_j|$. We conclude that $\gamma(G \Box H) = |D| \geq \gamma(G)\gamma(H)$.

If K is a spanning subgraph of a decomposable graph G satisfying $\gamma(G) = \gamma(K)$, then we apply Lemma 2.1 to prove that K also satisfies Vizing's conjecture. $\square$

Corollary 2.1 *[4] Let G be a graph satisfying $\gamma(G) = 2$ or $\rho_2(G) = \gamma(G)$. Then G satisfies Vizing's conjecture.*

This corollary follows from the previous theorem. Any graph G with $\gamma(G) = 2$ is a subgraph of a decomposable graph. To establish the second part of the corollary, we assume G is a graph satisfying $\gamma(G) = \rho_2(G)$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a 2-packing of G. Then we can add edges to G to make $N[v_1], N[v_2], \ldots, N[v_{k-1}]$ and $V(G) - (N[v_1] \cup N[v_2] \cup \ldots \cup N[v_{k-1}])$ into cliques. The resulting graph is decomposable and still has k pairwise disjoint closed neighborhoods. Hence, it follows from Theorem 2.6 that any graph with $\gamma(G) = \rho_2(G)$ satisfies Vizing's conjecture. An example of this can be seen in Figure 12. The labeled vertices v_1, v_2, and v_3 in G form a 2-packing of the graph. We can add edges as described above to get the decomposable graph H.

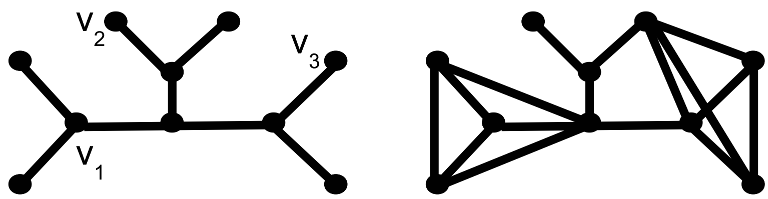

Figure 12.: A graph G with $\gamma(G) = \rho_2(G)$ and a decomposable graph H formed by adding edges to G.

Observe that this corollary implies Vizing's conjecture is true for any tree. We also have the following result from Hartnell and Rall as a corollary of Theorem 2.6 and Corollary 2.1.

Corollary 2.2 *[28] Let G be a graph such that $\bar{G}$ is 3-colorable. Then G satisfies Vizing's conjecture.*

Proof. We consider three cases based on the chromatic number of $\bar{G}$.

- Case 1: $\chi(\bar{G}) = 1$. Then G is a complete graph and the result holds.
- Case 2: $\chi(\bar{G}) = 2$. Then G belongs to the A-class and Vizing's conjecture holds.

- Case 3: $\chi(\bar{G}) = 3$. If $\gamma(G) = 3$ then G is decomposable and result holds by Theorem 2.6. Otherwise $\gamma(G) \leq 2$ and result holds by Corollary 2.1.

□

We now define Type $\mathcal{X}$ graphs, as introduced by Hartnell and Rall [27] in 1995. This class of graphs contains the BG-graphs as a proper subset and, hence, is an improvement of Barcalkin and German's [4] 1979 result. Hartnell and Rall, in defining Type $\mathcal{X}$ graphs, took an approach similar to that of Barcalkin and German in that they considered a particular way of partitioning a graph G. The difference is that not every set in the partition of a Type $\mathcal{X}$ graph induces a complete subgraph.

Type $\mathcal{X}$ graphs are defined as follows. Let k, t, r be nonnegative integers, not all zero. Let G be a graph with $\gamma(G) = k + t + r + 1$ whose vertices can be partitioned as $S \cup SC \cup BC \cup C$, where S, SC, BC, and C satisfy the following.

- Let $BC = B_1 \cup B_2 \cup \ldots \cup B_t$. Each B_i for $i = 1, \ldots, t$ is referred to as a *buffer clique*.

- Let $C = C_1 \cup C_2 \cup \ldots \cup C_r$.

- Each of $SC, B_1, \ldots, B_k, C_1, \ldots, C_r$ induces a clique.

- Every $v \in SC$ has at least one neighbor outside of SC. The set SC is called a *special clique*.

- Each B_i, for $i = 1 \ldots, k$ has at least one vertex which has no neighbors outside of B_i.

- Let $S = S_1 \cup S_2 \cup \ldots \cup S_k$ where each S_i is star-like. That is, each S_i has a vertex v_i which is adjacent to all $v \in S_i - v_i$. The vertex v_i has no neighbors other than those in S_i. Note that S_i does not induce a clique, and no edges may be added to S_i without decreasing the domination number of G.

- There are no edges between vertices in S and vertices in C.

Observe that not every graph that is Type $\mathcal{X}$ has a special clique. We can also have t, r, or k equal to zero. The example in Figure 13, is a Type $\mathcal{X}$ graph with a special clique. In this graph, the blue vertices represent the set S, the red vertices represent the buffer clique B, and the green vertices represent the special clique SC. One can easily verify that this graph satisfies the definition of Type $\mathcal{X}$ graphs above.

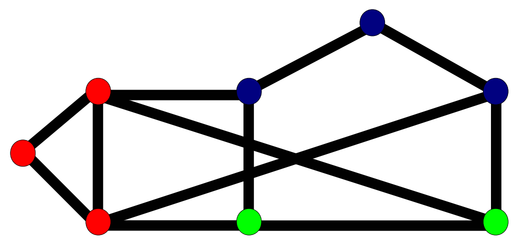

Figure 13.: Example of a Type $\mathcal{X}$ graph with a special clique

Theorem 2.7 *[27] Let G be a Type $\mathcal{X}$ graph. Then for any graph H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.*

The proof of Hartnell and Rall's theorem is similar to the proof that Vizing's conjecture is true for BG-graphs. We partition the vertices of G as indicated by the definition of a Type $\mathcal{X}$ graph and consider any dominating set D of $G\Box H$. Hartnell and Rall used the idea that some vertices in the product graph must be dominated "horizontally" and found $\gamma(G)$ disjoint sets in D, each of which have cardinality at least $\gamma(H)$, thus implying that Vizing's conjecture holds for any Type $\mathcal{X}$ graph.

Theorem 2.8 *[27] Let G be a Type $\mathcal{X}$ graph and let K be a spanning subgraph of G such that $\gamma(G) = \gamma(K)$. Then Vizing's conjecture is true for K.*

This theorem can be proved in the same way we showed that any spanning subgraph K of a decomposable graph G with $\gamma(G) = \gamma(K)$ satisfies Vizing's conjecture.

Hartnell and Rall were also able to show that any graph with domination number one more than its 2-packing number is a Type $\mathcal{X}$ graph and, hence, we have the following result.

Corollary 2.3 *[27] Let G be a graph satisfying $\gamma(G) = \rho_2(G) + 1$. Then Vizing's conjecture is true for G.*

Brešar and Rall [11] recently discovered a new class of graphs which satisfy Vizing's conjecture. They defined fair domination and proved that any graph with fair domination number equal to its domination number satisfies the conjecture. Furthermore, they proved that this class of graphs is an extension of the BG-graphs distinct from Type $\mathcal{X}$ graphs. Their results are presented in Chapter 3.

2.2 Vizing-Like Conjectures for Other Domination Types

As Vizing's conjecture has not yet been proven in general, researchers such as Fisher, Ryan, Domke and Majumdar [25]; Nowakowski and Rall [38]; Brešar [7]; and Dorbec, Henning and Rall [19] have studied variations of the original problem. These similar problems deal with other types of graph products and different graph parameters. As we will see, several of these variations remain open conjectures, while others have been proven.

Fractional Domination

One of the first Vizing-like results was proved for the fractional domination number. Recall that the fractional domination number of a graph G is the minimum weight of a fractional-dominating function, where the weight of the function is the sum over all vertices of its values. We note that for any graph G, $\gamma_f(G) \leq \gamma(G)$. Fisher, Ryan, Domke, and Majumdar proved the following result in their 1994 paper.

Theorem 2.9 *[25] For any pair of graphs G and H, $\gamma_f(G \Box H) \geq \gamma_f(G)\gamma_f(H)$.*

This theorem can be proved by first showing that $\gamma_f(G \boxtimes H) = \gamma_f(G)\gamma_f(H)$. Recall that $G \boxtimes H$ denotes the strong direct product of G and H, which has vertex set $V(G) \times V(H)$ and edge set $E(G \Box H) \cup \{\{(u_1, v_1), (u_2, v_2)\} \mid \{u_1, u_2\} \in E(G)$ and $\{v_1, v_2\} \in E(H)\}$. Since $G \Box H$ is a subgraph of $G \boxtimes H$, we have $\gamma_f(G \Box H) \geq \gamma_f(G \boxtimes H)$.

Fisher [24] also proved the following similar theorem in 1994; an improved proof was given by Brešar [6] in 2001.

Theorem 2.10 *[24] For any pair of graphs G and H, $\gamma(G \Box H) \geq \gamma_f(G)\gamma_((H)$.*

An obvious corollary of this theorem is that Vizing's conjecture is true for any graph with fractional domination number equal to domination number.

Integer Domination

A related concept to fractional domination is integer domination, which was studied first by Domke, Hedetniemi, Laskar, and Fricke [18]. We recall that the weight of a $\{k\}$-dominating function is the sum of its function values over all vertices, and the $\{k\}$-domination number of G, $\gamma_{\{k\}}(G)$ is the minimum weight of a $\{k\}$-dominating function of G. Domke, et. al. proved the following theorem relating fractional domination to integer domination.

Theorem 2.11 *[18] For any graph G, $\gamma_f(G) = \min_{k \in \mathbb{N}} \frac{\gamma_{\{k\}}(G)}{k}$.*

The following Vizing-like conjecture for integer domination is from Hou and Lu [33].

Conjecture 2.2 *[33] For any pair of graphs G and H and any integer $k \geq 1$, $\gamma_{\{k\}}(G\Box H) \geq \frac{1}{k}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H)$.*

This conjecture remains open, but Brešar, Henning and Klavžar [9] prove several related results in their 2006 paper. Note that if this conjecture is true for all k, in particular $k = 1$, then Vizing's conjecture is true.

Upper Domination

Nowakowski and Rall's [38] 1996 paper gives results and conjectures on several associative graph products, two of which are the Cartesian product and the categorical product, as previously defined in Section 1.2.

Recall that the upper domination number $\Gamma(G)$ of a graph G is the maximum cardinality of a minimal dominating set of G. Also recall that the minimum cardinality of a maximal independent set is the independent domination number $i(G)$.

Nowakowski and Rall [38] made the following conjectures in their 1996 paper.

- $i(G \times H) \geq i(G)i(H)$
- $\Gamma(G \times H) \geq \Gamma(G)\Gamma(H)$
- $\Gamma(G\Box H) \geq \Gamma(G)\Gamma(H)$

The last of these conjectures was proved by Brešar [7] in 2005. In fact, he provided a slight improvement of the conjectured lower bound.

Theorem 2.12 *[7] For any nontrivial graphs G and H,*

$$\Gamma(G\Box H) \geq \Gamma(G)\Gamma(H) + 1.$$

The proof Brešar provided for this theorem is constructive in nature. He begins with arbitrary graphs G and H and creates a minimal dominating set D of the product graph $G\Box H$ which contains at least $\Gamma(G)\Gamma(H) + 1$ vertices.

Total Domination

Henning and Rall's [30] 2005 paper was the first to introduce results on total domination in Cartesian products of graphs. Recall that a set $D \subset V(G)$ is a total dominating set if $N(D) = V(G)$. The total domination number is the minimum cardinality of a total dominating set of G and is denoted by $\gamma_t(G)$. Henning and Rall conjectured that $2\gamma_t(G \square H) \geq \gamma_t(G)\gamma_t(H)$ and they proved this inequality holds for certain classes of graphs G with no isolated vertices and any graph H without isolated vertices. This conjecture was proved for graphs without isolated vertices by Ho.

Theorem 2.13 *[32] Let G and H be graphs without isolated vertices. Then*

$$2\gamma_t(G \square H) \geq \gamma_t(G)\gamma_t(H).$$

Recall that the total $\{k\}$-domination number $\gamma_t^{\{k\}}(G)$ is defined as the minimum cardinality of a total k-dominating set D of a graph. In 2008, Li and Hou [37] proved that for any graphs G and H without isolated vertices, $\gamma_t^{\{k\}}(G)\gamma_t^{\{k\}}(H) \leq k(k+1)\gamma_t^{\{k\}}(G \square H)$. Note that Theorem 2.13 is easily proved using this inequality.

Upper Total Domination

Recall that we define the upper total domination number of G, denoted by $\Gamma_t(G)$, to be the maximum cardinality of a minimal total dominating set of a graph G. Dorbec, Henning and Rall [19] published results in 2008 on a Vizing-like inequality for the upper total domination number. They achieved the following two results.

Theorem 2.14 *[19] If G and H are connected graphs of order at least 3 and $\Gamma_t(G) \geq \Gamma_t(H)$, then*

$$2\Gamma_t(G \square H) \geq \Gamma_t(G)(\Gamma_t(H) + 1)$$

and this bound is sharp.

In order to prove this theorem we must first define the sets $epn(S, v)$, $ipn(v, S)$, and $pn(v, S)$. Let $S \subset V(G)$ and let $v \in S$. The set $epn(v, S)$ of external private neighbors of v is $epn(v, S) = \{u \in V(G) - S \mid N(u) \cap S = \{v\}\}$. The set of internal private neighbors of $v \in S$ is $ipn(v, S) = \{u \in S \mid N(u) \cap S = \{v\}\}$. We denote the set of all private neighbors of $v \in S$ by $pn(v, S)$. This is the union of all external and internal private neighbors of v. That is, $pn(v, S) = epn(v, S) \cup ipn(v, S)$. Cockayne, et. al. make the following observation.

Observation 2.1 *[14] Let S be a total dominating set in a graph G with no isolated vertices. Then S is a minimal total dominating set if and only if for all $v \in S$,*

1. *$epn(v, S) \neq \emptyset$, or*

2. *$pn(v, S) = ipn(v, S) \neq \emptyset$.*

We will also need the following lemma.

Lemma 2.3 *[19] Let G be a graph. Every $\Gamma_t(G)$-set contains as a subset a γ-set D such that $|D| \geq \frac{1}{2}\Gamma_t(G)$ and for all $v \in D$, $|epn(v, D)| \geq 1$.*

We will now prove Theorem 2.14.

Proof. [19] We assume G and H are connected graphs with order at least 3, where $\Gamma_t(G) \geq \Gamma_t(H)$. By the above lemma, there is a γ-set S of G with $|S| \geq \frac{1}{2}\Gamma_t(G)$ and for each $v \in S$, $|epn(v, S)| \geq 1$. For each $u \in V(G)$, denote $H_u = \{u\} \times V(H)$. Similarly, for $w \in V(H)$, let $G_w = V(G) \times \{w\}$.

Now, let $D = S \times V(H)$, and observe that D dominates $G\Box H$ since S dominates V(G). Also, for each $u \in S$, the vertices $V(H_u)$ are totally dominated "vertically"; thus, D is a total dominating set of $G\Box H$. We claim that D is a minimal total dominating set of $G\Box H$.

Let $(u, w) \in D$ and consider (u', w), where $u' \in epn(u, S)$ in G. Then $(u', w) \in epn((u, w), D)$ in $G\Box H$. Thus, for all $(u, w) \in D$, $|epn((u, w), D)| \geq 1$. Then, by Observation 2.1, D is a minimal total dominating set of $G\Box H$ and so $\Gamma_t(G\Box H) \geq |D|$. Note that since H is a connected graph with order at least 3, $|V(H)| \geq \Gamma_t(H) + 1$. Therefore,

$$\Gamma_t(G\Box H) \geq |D| = |S| \times |V(H)| \geq \frac{1}{2}\Gamma_t(G)(\Gamma_t(H) + 1).$$

Equality holds when both G and H are *daisies* with $k \geq 2$ *petals*. That is, we begin with k copies of K_3 and identify one vertex from each copy to form a single vertex. The resulting graph is a daisy. Figure 14 shows the daisy with 3 petals. $\square$

The following theorem is easily proved using Theorem 2.14 and the fact that for a graph G with no isolated vertices, $\Gamma_t(G)\Gamma_t(K_2) \leq 2\Gamma_t(G\Box K_2)$. Equality holds if and only if G is a disjoint union of copies of K_2. Let $u \in V(K_2)$. Then $V(G) \times \{u\}$ is a minimal total dominating set of $G\Box K_2$, giving that

$$\Gamma_t(G\Box K_2) \geq |V(G)| \geq \Gamma_t(G) = \frac{1}{2}\Gamma_t(G)\Gamma_t(K_2).$$

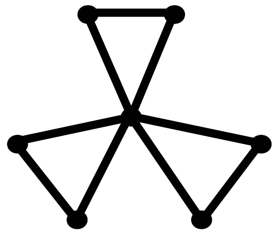

Figure 14.: The daisy with 3 petals

In order for equality to hold, we must have $\Gamma_t(G) = |V(G)|$, and so G must be a disjoint union of copies of K_2.

Theorem 2.15 *[19] If G and H have no isolated vertices, then*

$$2\Gamma_t(G\Box H) \geq \Gamma_t(G)\Gamma_t(H)$$

with equality if and only if both G and H are disjoint unions of copies of K_2.

Paired Domination

Brešar, Henning and Rall [10] published results in 2007 about Vizing-like inequalities for paired domination. Recall that a set $D \subseteq V(G)$ is a paired dominating set of G if D dominates G and the induced subgraph $G[D]$ has a perfect matching. Note that in every graph without isolated vertices, a maximal matching forms a paired dominating set. The paired domination number $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set.

The inequalities established by Brešar, Henning and Rall relate the paired domination number of the Cartesian product of G and H to the 3-packing number of G. Recall that a 2-packing of a graph G is a set of vertices $S \subset V(G)$ such that for any vertices u and v in S, $d(u, v) > 2$. We define a 3-packing similarly. That is, a 3-packing of the graph G is a set S of vertices such that the distance between any pair of vertices in S is greater than 3. The 3-packing number of G, denoted $\rho_3(G)$, is the maximum cardinality of a 3-packing in G.

Theorem 2.16 *[10] If G and H are graphs without isolated vertices, then*

$$\gamma_{pr}(G\Box H) \geq \max\{\gamma_{pr}(G)\rho_3(H), \gamma_{pr}(H)\rho_3(G)\}.$$

Brešar, Henning and Rall were also able to show that $\gamma_{pr}(T) = 2\rho_3(T)$ in any nontrivial tree T. Thus, the following result follows from Theorem 2.16.

Theorem 2.17 *[10] Let T be a nontrivial tree. Then for any graph H without isolated vertices, $\gamma_{pr}(T\Box H) \geq \frac{1}{2}\gamma_{pr}(T)\gamma_{pr}(H)$, and this bound is sharp.*

The final major result from Brešar, Henning and Rall in 2007 is the following theorem relating paired domination in the Cartesian product of G and H to the 3-packing numbers of G and H.

Theorem 2.18 *[10] If G and H have no isolated vertices, then $\gamma_{pr}(G\Box H) \geq 2\rho_3(G)\rho_3(H)$.*

Independence Domination

Aharoni and Szabó [2] in 2009 provided a Vizing-like result for the independence domination number. Recall that this is different from the independent domination number; we let the independence domination number $\gamma^i(G)$ denote the maximum, over all independent sets I in G, of the minimum number of vertices required to dominate I. It was proven by Aharoni, Berger and Ziv [1] that $\gamma(G) = \gamma^i(G)$ for any chordal graph G, where a graph is chordal if any cycle of more than four vertices contains at least one chord, or edge connecting vertices that are not adjacent in the cycle. Aharoni and Szabó proved the following theorem.

Theorem 2.19 *[2] For arbitrary graphs G and H, $\gamma(G\Box H) \geq \gamma^i(G)\gamma(H)$.*

Proof. [2] Let G and H be graphs. We may assume that G has no isolated vertices, for if it did have an isolated vertex v then the validity of the theorem for $G - v$ implies the validity for G.

Assume $I \subset V(G)$ is an independent set which requires at least $\gamma^i(G)$ vertices to dominate it. We will show that $\gamma(I\Box H) \geq \gamma^i(G)\gamma(H)$ by showing that $|D| \geq \gamma^i(G)\gamma(H)$, where D is a set that dominates $I \times V(H)$.

Let $\{v_1, v_2, \ldots, v_{\gamma(H)}\}$ be a γ-set of H. Use these vertices to partition $V(H)$ into sets $\{\Pi_i \mid v_i \in \Pi_i$ and $v \in \Pi_i$ if and only if $v = v_i$ or $\{v, v_i\} \in E(H)\}$. Note that, for every $J \subseteq \{1, 2, \ldots, \gamma(H)\}$, we have

$$\gamma(\bigcup_{j\in J} \Pi_j) \geq |J| \tag{2.2}$$

Let $S_u = \{i \mid \{u\} \times \Pi_i$ is dominated vertically by some vertices $(u, v) \in D\}$, and let $S_i = \{u \in I \mid \{u\} \times \Pi_i$ is dominated vertically by some vertices $(u, v) \in D\}$. Summing S_u and S_i, we have

$$\mathcal{S} = \sum_{u\in I} S_u = \sum_{i=1}^{\gamma(H)} S_i$$

By (2.2), for each $u \in I$ we have

$$|D \cap (\{u\} \times V(H))| \geq |S_u|.$$

Sum over $v \in I$ to get

$$|D \cap (I \times V(H))| \geq |\mathcal{S}|. \tag{2.3}$$

Now consider $k \leq \gamma(H)$; each set of vertices $\{u\} \times \Pi_k$ which is not in $\mathcal{S}$ contains at least one vertex (u, v) which is not dominated by any vertex in $\{u\} \times V(H)$. Thus, (u, v) is dominated "horizontally" by some vertex (w, v) where $w = w(v)$. Note that $w \notin I$ since I is independent and so the set $\{w(v) \mid \{v\} \times \Pi_k \notin \mathcal{S}\}$ dominates $|I| - |S_j|$ vertices in I and has cardinality at least $\gamma^i(G) - |S_j|$. Sum over k to get

$$|D \cap ((V(G) - I) \times V(H))| \geq \gamma^i(G)\gamma(H) - |\mathcal{S}|. \tag{2.4}$$

Combine equations (2.3) and (2.4) to get

$$\gamma(G \Box H) \geq \gamma^i(G)\gamma(H).$$

$\square$

Combining this result with that of Aharoni, Berger and Ziv [1], an obvious corollary is that Vizing's conjecture holds for chordal graphs.

Independent Domination

Brešar, et. al. [8] provide a few open conjectures in their survey paper, including the following.

Conjecture 2.3 *[8] For any graphs G and H, $\gamma(G \Box H) \geq \min\{i(G)\gamma(H), \gamma(G)i(H)\}$.*

The truth of this conjecture would immediately imply Vizing's conjecture holds for any pair of graphs G and H, as $\gamma(G) \leq i(G)$ by Lemma 1.1. We also have the following conjecture, which is implied by Vizing's conjecture. Brešar, et. al. suggest that perhaps this could be established without first proving Vizing's conjecture.

Conjecture 2.4 *[8] For any graphs G and H, $i(G \Box H) \geq \gamma(G)\gamma(H)$.*

In addition, the survey paper makes the following partition conjecture, which would also imply the truth of Vizing's conjecture.

Conjecture 2.5 *[8] Let G and H be arbitrary graphs. There is a partition of $V(G)$ into $\gamma(G)$ sets $\Pi_1, \ldots, \Pi_{\gamma(G)}$ such that there is a minimal dominating set D of $G\Box H$ such that the projection of $D \cap (\Pi_i \times V(H))$ onto H dominates H for all $i = 1, \ldots, \gamma(G)$.*

2.3 Clark-Suen Inequality and Improvement

We have given several results establishing the truth of Vizing's conjecture for classes of graphs satisfying certain properties. Another approach to proving Vizing's conjecture is to find a constant c such that for any graphs G and H, $\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$. Clark and Suen [17] in 2000 proved that this inequality is true for $c = \frac{1}{2}$. Here, we present an improvement of this result.

Theorem 2.20 *For any graphs G and H, $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$.*

Proof. Let G and H be arbitrary graphs, and let D be a γ-set of the Cartesian product $G\Box H$. Let $\{u_1, u_2, ..., u_{\gamma(G)}\}$ be a γ-set of G. Partition $V(G)$ into $\gamma(G)$ sets $\Pi_{1\bullet}, \Pi_{2\bullet}, ..., \Pi_{\gamma(G)\bullet}$, where $u_i \in \Pi_{i\bullet}$ for all $i = 1, 2, ..., \gamma(G)$ and if $u \in \Pi_{i\bullet}$ then $u = u_i$ or $\{u, u_i\} \in E(G)$.

Let $P_{i\bullet}$ denote the projection of $(\Pi_{i\bullet} \times V(H)) \cap D$ onto H. That is,

$$P_{i\bullet} = \{v \in V(H) \mid (u, v) \in D \text{ for some } u \in \Pi_{i\bullet}\}.$$

Define $C_{i\bullet} = V(H) - N_H[P_{i\bullet}]$ as the complement of $N_H[P_{i\bullet}]$, where $N_H[X]$ is the set of closed neighbors of X in graph H. As $P_i \cup C_i$ is a dominating set of H, we have

$$|P_{i\bullet}| + |C_{i\bullet}| \geq \gamma(H), \qquad i = 1, 2, \ldots, \gamma(G). \tag{2.5}$$

For $v \in V(H)$, let

$$D_{\bullet v} = \{u \mid (u, v) \in D\} \quad \text{and} \quad S_{\bullet v} = \{i \mid v \in C_{i\bullet}\}.$$

Observe that if $i \in S_{\bullet v}$ then the vertices in $\Pi_{i\bullet} \times \{v\}$ are dominated "horizontally" by vertices in $D_{\bullet v} \times \{v\}$. Let S_H be the number of pairs (i, v) where $i = 1, 2, \ldots, \gamma(G)$ and $v \in C_{i\bullet}$. Then obviously

$$S_H = \sum_{v \in V(H)} |S_{\bullet v}| = \sum_{i=1}^{\gamma(G)} |C_{i\bullet}|.$$

Since $D_{\bullet v} \cup \{u_i \mid i \notin S_{\bullet v}\}$ is a dominating set of G, we have

$$|D_{\bullet v}| + (\gamma(G) - |S_{\bullet v}|) \geq \gamma(G),$$

giving that

$$|S_{\cdot v}| \leq |D_{\cdot v}|. \tag{2.6}$$

Summing over $v \in V(H)$, we have

$$S_H \leq |D|. \tag{2.7}$$

We now consider two cases based on (2.5).

Case 1 *Assume $|P_{i\cdot}| + |C_{i\cdot}| > \gamma(H)$ for all $i = 1, ..., \gamma(G)$. Then as $|(\Pi_{i\cdot} \times V(H)) \cap D| \geq |P_{i\cdot}|$, we have*

$$\sum_{i=1}^{\gamma(G)} (|C_{i\cdot}| + |(\Pi_{i\cdot} \times V(H)) \cap D|) \geq \sum_{i=1}^{\gamma(G)} (\gamma(H) + 1),$$

which implies that

$$S_H + |D| \geq \gamma(G)\gamma(H) + \gamma(G). \tag{2.8}$$

Combining (2.7) *and* (2.8) *gives that*

$$\gamma(G\Box H) = |D| \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(G). \tag{2.9}$$

Case 2 *Assume $|P_{i\cdot}| + |C_{i\cdot}| = \gamma(H)$ for some $i = 1, ..., \gamma(G)$. Note that $P_{i\cdot} \cup C_{i\cdot}$ is a γ-set of H. We now use this γ-set of H to partition $V(H)$ in the same way as $V(G)$ is partitioned above. That is, label the vertices in $P_{i\cdot} \cup C_{i\cdot}$ as $v_1, v_2, ..., v_{\gamma(H)}$, and let $\{\Pi_{\cdot j} \mid 1 \leq j \leq \gamma(H)\}$ be a partition of H such that for all $j = 1, ..., \gamma(H)$, $v_j \in \Pi_{\cdot j}$ and if $v \in \Pi_{\cdot j}$, either $v = v_j$ or $\{v, v_j\} \in E(H)$. We next define the sets $P_{\cdot j}, C_{\cdot j}, S_{u\cdot}$ and $D_{u\cdot}$ in the same way $P_{i\cdot}, C_{i\cdot}, S_{\cdot v}$ and $D_{\cdot v}$ are defined above. To be specific, for $1 \leq j \leq \gamma(H)$, let*

$$P_{\cdot j} = \{u \in V(G) \mid (u, v) \in D \text{ for some } v \in \Pi_{\cdot j}\}, \quad \text{and} \quad C_{\cdot j} = V(G) - N_G[P_{\cdot j}],$$

and for $u \in V(G)$, let

$$D_{u\cdot} = \{v \mid (u, v) \in D\} \quad \text{and} \quad S_{u\cdot} = \{j \mid u \in C_{\cdot j}\}.$$

Similarly, we have

$$S_G = \sum_{u \in V(G)} |S_{u\cdot}| = \sum_{j=1}^{\gamma(H)} C_{\cdot j}.$$

For $u \in V(G)$, let $\hat{D}_{u\cdot} = \{v_j \mid (u, v_j) \in D_{u\cdot}, 1 \leq j \leq \gamma(H)\}$. We claim that

$$|S_{u\cdot}| \leq |D_{u\cdot}| - |\hat{D}_{u\cdot}|. \tag{2.10}$$

This is because $D_{u\boldsymbol{\cdot}} \cup \{v_j \mid j \notin S_{u\boldsymbol{\cdot}}\}$ is a dominating set of H, with

$$D_{u\boldsymbol{\cdot}} \cap \{v_j \mid j \notin S_{u\boldsymbol{\cdot}}\} = \hat{D}_{u\boldsymbol{\cdot}},$$

and the argument for proving (2.10) *follows in the same way as* (2.6) *is proved. To make use of the claim, we note that when we partition the vertices of H, we have at least $\gamma(H)$ vertices in D that are of the form (u, v_k). Indeed, for each $k = 1, 2, \ldots, \gamma(H)$, either $v_k \in P_{i\boldsymbol{\cdot}}$, which implies $(u, v_k) \in D$ for some $u \in \Pi_{i\boldsymbol{\cdot}}$, or $v_k \in C_{i\boldsymbol{\cdot}}$, which implies that the vertices in $\Pi_{i\boldsymbol{\cdot}} \times \{v_k\}$ are dominated "horizontally" by some vertices $(u', v_k) \in D$. It therefore follows that*

$$\sum_{u \in V(G)} |\hat{D}_{u\boldsymbol{\cdot}}| \geq \gamma(H),$$

and hence summming both sides of (2.10)

$$\sum_{u \in V(G)} |S_{u\boldsymbol{\cdot}}| \leq \sum_{u \in V(G)} (|D_{u\boldsymbol{\cdot}}| - |\hat{D}_{u\boldsymbol{\cdot}}|)$$

gives that

$$S_G \leq |D| - \gamma(H). \tag{2.11}$$

To complete the proof, we note that similar to (2.5)*, we have*

$$|P_{\boldsymbol{\cdot}j}| + |C_{\boldsymbol{\cdot}j}| \geq \gamma(G), \qquad j = 1, 2, \ldots, \gamma(H),$$

and summing over j gives that

$$|D| + S_G \geq \gamma(G)\gamma(H). \tag{2.12}$$

Combining (2.11) *and* (2.12)*, we obtain*

$$\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(H). \tag{2.13}$$

As either (2.9) or (2.13) holds, it follows that

$$\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

□

This approach may also be used to prove a similar inequality involving the independence number of a graph, where G is a claw-free graph. Recall that the independence number of a graph G is the maximum cardinality of a maximal independent set in G, and is denoted by $\alpha(G)$. Also recall that a graph is claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. Brešar, et. al. [8] proved the following.

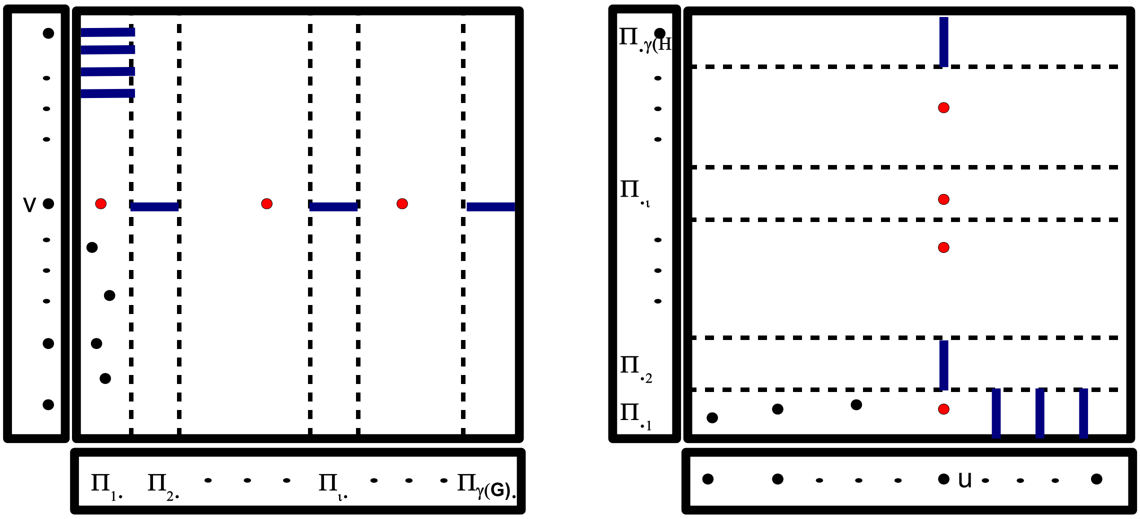

Figure 15.: Partitions $\Pi_{i\bullet}$ and the sets $D_{\bullet v}, S_{\bullet v}$, and $C_{i\bullet}$; and partitions $\Pi_{\bullet j}$ and the sets $D_{u\bullet}, S_{u\bullet}$, and $C_{\bullet j}$

Theorem 2.21 *[8] Let G be a claw-free graph and let H be a graph without isolated vertices. Then*

$$\gamma(G\Box H) \geq \frac{1}{2}\alpha(G)(\gamma(H)+1).$$

Observe that $\gamma(G) \leq \alpha(G)$ for every graph G, so we have the following corollary.

Corollary 2.4 *[8] Let G be a claw-free graph and let H be a graph without isolated vertices. Then*

$$\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)(\gamma(H)+1).$$

From this corollary we can conclude that any claw-free graph satisfying $\alpha(G) = 2\gamma(G)$ satisfies Vizing's conjecture.

Chapter 3
Fair Domination

A recent development in attempts to prove Vizing's conjecture is Brešar and Rall's [11] idea of fair domination. Their 2009 paper defines this concept and establishes the truth of Vizing's conjecture for graphs with fair domination number equal to domination number. Furthermore, they verify that the class of such graphs contains the BG-graphs and is distinct from the Type $\mathcal{X}$ graphs defined by Hartnell and Rall. We will define fair reception and fair domination, provide a proof that Vizing's conjecture holds for the class of graphs with fair domination number equal to domination number, examine fair domination in edge-critical graphs, and summarize some open questions related to fair domination.

3.1 Definition and General Results

A recent paper by Brešar and Rall [11] published in 2009 introduces the concept of fair domination of a graph. Brešar and Rall were able to verify that Vizing's conjecture holds for any graph G with a fair reception of size $\gamma(G)$.

In order to define fair domination, we must first define external domination. We say that a set $X \subset V(G)$ *externally dominates* set $U \subset V(G)$ if $U \cap X = \emptyset$ and for each $u \in U$ there is $x \in X$ such that $\{u, x\} \in E(G)$.

Let G be a graph and let $S_1, ..., S_k$ be pair-wise disjoint sets of vertices of G. Let $\mathcal{S} = S_1 \cup S_2 \cup ... \cup S_k$ and let $Z = V(G) - \mathcal{S}$. The sets $S_1, ..., S_k$ form a *fair reception of size k* if for each $l \in \mathbb{Z}$, $1 \leq l \leq k$, and any choice of l sets $S_{i_1}, ..., S_{i_l}$ the following holds: if D externally dominates $S_{i_1} \cup ... \cup S_{i_l}$ then

$$|D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) \geq l.$$

Notice that on the left-hand side of the above inequality, we count all the vertices of D that are not in $\mathcal{S}$. For vertices of D that are in some S_j, we count all but one from $D \cap S_j$. The largest k such that there exists a fair reception of size k in graph G is called the *fair domination number* of G and is denoted by $\gamma_F(G)$.

Proposition 3.1 *[11] For any graph G, $\rho_2(G) \leq \gamma_F(G) \leq \gamma(G)$.*

Proof. Let T be a 2-packing of G. Let each S_i consist of exactly one vertex $v \in T$. This gives us a fair reception of size $|T|$. Thus, $\rho_2(G) \leq \gamma_F(G)$. Now assume there exists a graph G with $r = \gamma(G) < \gamma_F(G) = k$. Let D be a γ-set of G and let $S_1, ..., S_k$ form a fair reception of size k in G. Since $r < k$, D must be disjoint from at least one S_i. We assume $D \cap S_i = \emptyset$ for $1 \leq i \leq t$ and $D \cap S_j \neq \emptyset$ for $t+1 \leq j \leq k$. Then D externally dominates $S_1 \cup S_2 \cup ... \cup S_t$, and so by the definition of fair reception, we have

$$t \leq |D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) = |D \cap Z| + \sum_{j=t+1}^{k} |S_j \cap D| - (k-t) = |D| - k + t.$$

Then $k \leq |D|$ and we have a contradiction. Therefore, $\rho_2(G) \leq \gamma_F(G) \leq \gamma(G)$. $\square$

Theorem 3.1 *[11] For any graphs G and H,*

$$\gamma(G \Box H) \geq \max\{\gamma_F(G)\gamma(H), \gamma(G)\gamma_F(H)\} \tag{3.1}$$

Proof. Let G and H be arbitrary graphs. Let D be a γ-set of $G \Box H$ and let the sets $S_1, S_2, ..., S_k$ form a fair reception of H, where $\gamma_F(H) = k$. As in the definition of fair reception, we let $S = \bigcup_{i=1}^{k} S_i$ and $Z = V(H) - S$.

Let $D_{u\cdot}$ be the set of vertices in $\{u\} \times V(H)$ that are also in D and let $P_{u\cdot}$ denote the projection of $D_{u\cdot}$ onto H. That is, $D_{u\cdot} = (\{u\} \times V(H)) \cap D$ and $P_{u\cdot} = \{v \in V(H) \mid \{u, v\} \in \{u\} \times V(H)\}$.

Let $D_{\cdot i}$ be the set of vertices in $V(G) \times S_i$ that are also in D and let $P_{\cdot i}$ denote the projection of $D_{\cdot i}$ onto G. That is, $D_{\cdot i} = (V(G) \times S_i) \cap D$ and $P_{\cdot i} = \{u \in V(G) \mid \{u, v\} \in V(G) \times S_i\}$.

Let $D_{ui} = (\{u\} \times S_i) \cap D$.

Let $D_{\cdot Z} = (V(G) \times Z) \cap D$ and let $D_{uZ} = (\{u\} \times Z) \cap D$.

Now define d_{ui} as follows.

$$d_{ui} = \begin{cases} |D_{ui}| - 1 & \text{if } D_{ui} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Observe that d_{ui} counts the vertices in D_{ui} that are not uniquely projected onto G.

Now we define $T_{\cdot i} = \{u \in V(G) \mid u \notin N[P_{\cdot i}]\}$. Observe that $T_{\cdot i} \cup P_{\cdot i}$ is a dominating set of G and, thus,

$$|T_{\cdot i}| + |P_{\cdot i}| \geq \gamma(G) \tag{3.2}$$

Let $T_{u\cdot} = \{i \mid u \in T_{\cdot i}, i = 1, 2, \ldots, k\}$. By definition of $T_{\cdot i}$ and $T_{u\cdot}$, the following holds

$$\sum_{i=1}^{k} |T_{\cdot i}| = \sum_{u \in V(G)} |T_{u\cdot}| \tag{3.3}$$

Observe that if $i \in T_{u\cdot}$ the vertices $\{u\} \times S_i$ are not dominated by $D_{\cdot i}$, and so $P_{u\cdot}$ externally dominates S_i for all $i \in T_{u\cdot}$. Therefore, by definition of fair reception, we have

$$|D_{uZ}| + \sum_{i=1}^{k} d_{ui} \geq |T_{u\cdot}| \tag{3.4}$$

Now, we have:

$$\begin{aligned}
|D| = \sum_{i=1}^{k} |D_{\cdot i}| + |D_{\cdot Z}| &= \sum_{i=1}^{k} (|P_{\cdot i}| + (|D_{\cdot i}| - |P_{\cdot i}|)) + |D_{\cdot Z}| \\
&= \sum_{i=1}^{k} |P_{\cdot i}| + \sum_{i=1}^{k} \sum_{u \in V(G)} d_{ui} + |D_{\cdot Z}| = \sum_{i=1}^{k} |P_{\cdot i}| + \sum_{u \in V(G)} (\sum_{i=1}^{k} d_{ui} + |D_{uZ}|) \\
&\geq \sum_{i=1}^{k} |P_{\cdot i}| + \sum_{u \in V(G)} |T_{u\cdot}| && (3.5) \\
&= \sum_{i=1}^{k} (|P_{\cdot i}| + |T_{\cdot i}|) && (3.6) \\
&\geq k\gamma(G) = \gamma(G)\gamma_F(H). && (3.7)
\end{aligned}$$

Note, (3.5) holds by (3.4), (3.6) holds by (3.3), and (3.7) holds by (3.2).

Similarly, we define a fair reception of G and repeat the proof with the roles of G and H reversed to conclude that $\gamma(G \Box H) \geq \gamma_F(G)\gamma(H)$. Therefore, we conclude that $\gamma(G \Box H) \geq \max\{\gamma_F(G)\gamma(H), \gamma(G)\gamma_F(H)\}$. $\Box$

Corollary 3.1 *Let G be a graph with $\gamma(G) = \gamma_F(G)$. Then G satisfies Vizing's conjecture.*

There are some known examples of graphs G for which $\gamma_F(G) \neq \gamma(G)$. One such example can be seen in Figure 16. It can be easily verified that this graph G has $\gamma(G) = 3$. Brešar, et. al. [8] verified by computer that $\gamma_F(G) = 2$.

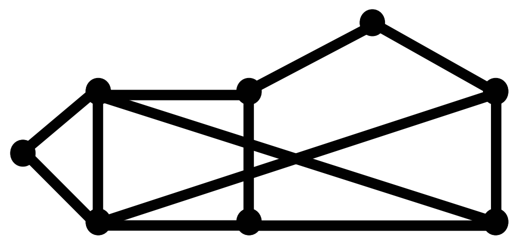

Figure 16.: Example of a graph with $\gamma(G) = \gamma_F(G) + 1$

Brešar and Rall observe that the class of graphs satisfying $\gamma(G) = \gamma_F(G)$ is an extension of the class of BG-graphs which is distinct from Type $\mathcal{X}$ graphs. An open question regarding fair domination is whether a lower bound may be found for $\gamma_F(G)$ in terms of $\gamma(G)$. If, for example, one could find a constant $c > \frac{1}{2}$ such that $\gamma_F(G) \geq c\gamma(G)$, that would improve the Clark-Suen inequality.

3.2 Edge Critical Graphs

There are two classes of graphs that are critical with respect to the domination number: *edge-critical graphs* and *vertex-critical graphs*. In an edge-critical graph, the domination number decreases if an edge is added; in vertex-critical graphs, the domination number decreases if a vertex is deleted. Here, we concentrate on the class of edge-critical graphs.

A graph G is *k-edge-domination-critical*, or simply k-edge-critical if $\gamma(G) = k$ and for every pair of nonadjacent vertices $u, v \in V(G)$, $\gamma(G + \{u, v\}) = k - 1$. In other words, the domination number decreases if any missing edge is added to the graph G.

Note that a graph G is 1-edge-critical if and only if G is a complete graph. It is also straightforward to characterize 2-edge-critical graphs, using the following theorem.

Theorem 3.2 *[43] A graph G is 2-edge-critical if and only if $\bar{G} = \bigcup_{i=1}^{t} K_{1,p_i}$ for some $t \geq 1$.*

In other words, the only 2-edge-critical graphs are complements of unions of stars. Although Vizing's Conjecture has already been established for graphs G with $\gamma(G) = 2$, we can provide a different method of proof for 2-edge-critical graphs. We will show that the domination number equals the fair domination number in a 2-edge-critical graph and, therefore, we can apply Brešar and Rall's [11] result to show that Vizing's conjecture holds.

Theorem 3.3 *For any 2-edge-critical graph G, $\gamma(G) = \gamma_F(G)$.*

Proof. Let G be a 2-edge-critical graph. By Theorem 3.2, every 2-edge-critical graph is the complement of a union of stars. Consider $H = K_{1,n_1-1} \cup K_{1,n_2-1} \cup \ldots \cup K_{1,n_t-1}$, where $|V(K_{1,n_i-1})| = n_i$ and $t \geq 1$. Let v_i be the vertex of maximum degree in K_{1,n_i-1} for $i = 1, 2, \ldots, t$. Now let $G = \bar{H}$. Let $S_1 = \{v_i \mid i = 1, 2, \ldots, t\}$ and let $S_2 = V(G) - S_1$.

We need to show that S_1 and S_2 form a fair reception of G. Consider the set S_1. In order to externally dominate this set, we need at least 2 vertices from S_2. Take $v \in K_{1,n_i-1}$ where $v \neq v_i$. Then v externally dominates v_j for all $j \neq i$. Thus we must choose at least one more vertex from S_2 to externally dominate v_i. This implies $|D \cap S_2| - 1 \geq 1$ for all sets of vertices D that externally dominate S_1.

Now consider S_2. Choose $v_i \in S_1$. Then v_i externally dominates all vertices of S_2 except those that were in the star K_{1,n_i-1} in H. Thus, we must choose an additional vertex $v_j \neq v_i$ to externally dominate those vertices. We have $|D \cap S_1| - 1 \geq 1$.

Therefore, for any 2-edge-critical graph G, $\gamma_F(G) \geq 2$. But $\gamma_F(G) \leq \gamma(G)$ by Proposition 3.1 and since $\gamma(G) = 2$, we have $\gamma_F(G) = \gamma(G) = 2$. □

Since we know that Vizing's conjecture holds for any graph G that has $\gamma(G) = \gamma_F(G)$, this result implies Vizing's conjecture holds for all 2-edge-critical graphs.

Unfortunately, 3-edge-critical graphs are not easily characterized as 1- and 2-edge-critical graphs are. We provide a few examples of 3-edge-critical graphs.

Figure 17 provides seven examples of 3-edge-critical graphs. Observe that we can find a fair reception of size 3 in five of these graphs, as shown in Figure 18; however, it is difficult to tell if there is a fair reception of size 3 in the remaining two graphs in Figure 17. We do have the following result which may help in finding fair reception of size $\gamma(G)$ in an edge-critical graph G.

Theorem 3.4 *Let G be k-edge-critical with $\gamma(G) = \gamma_F(G) = k$. Then if $S_1, S_2, \ldots, S_k$ form a fair reception of G, each S_i for $i = 1, 2, \ldots, k$ is a complete subgraph of G.*

Proof. Assume G is k-edge-critical and that $\gamma(G) = \gamma_F(G) = k$. Let $S_1, S_2, ..., S_k$ form a fair reception of G. Without loss of generality, assume S_1 does not form a complete subgraph of G. For $u, v \in S_1$ such that $\{u, v\} \notin E(G)$, draw the edge $\{u, v\}$. Then we still have a fair reception of size

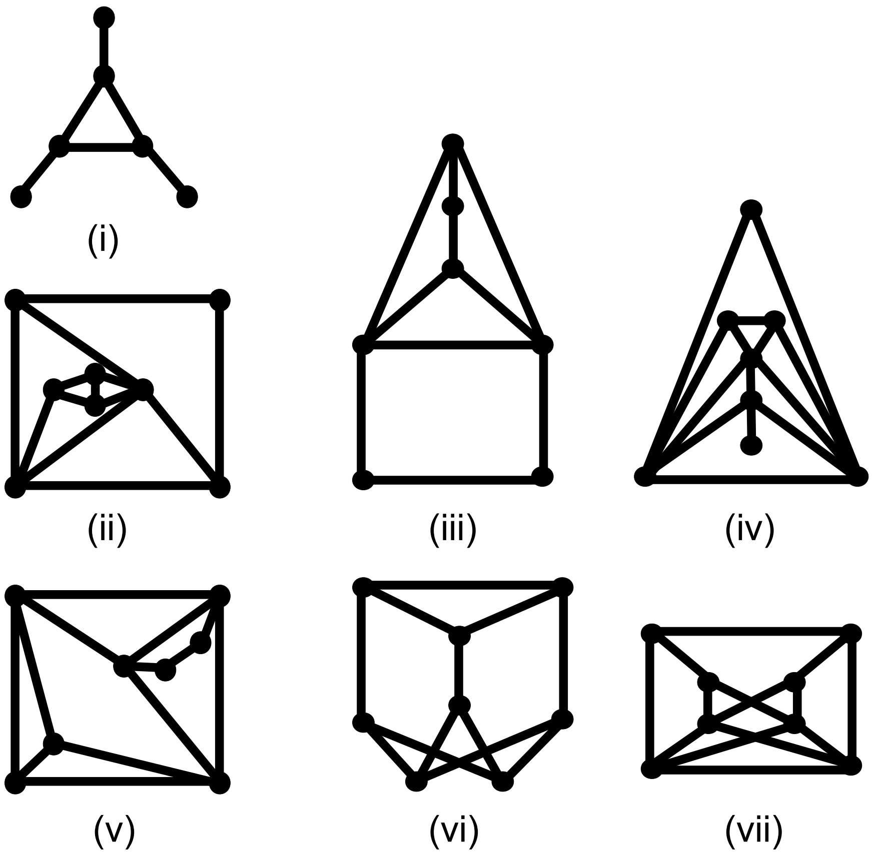

Figure 17.: Examples of 3-edge-critical graphs

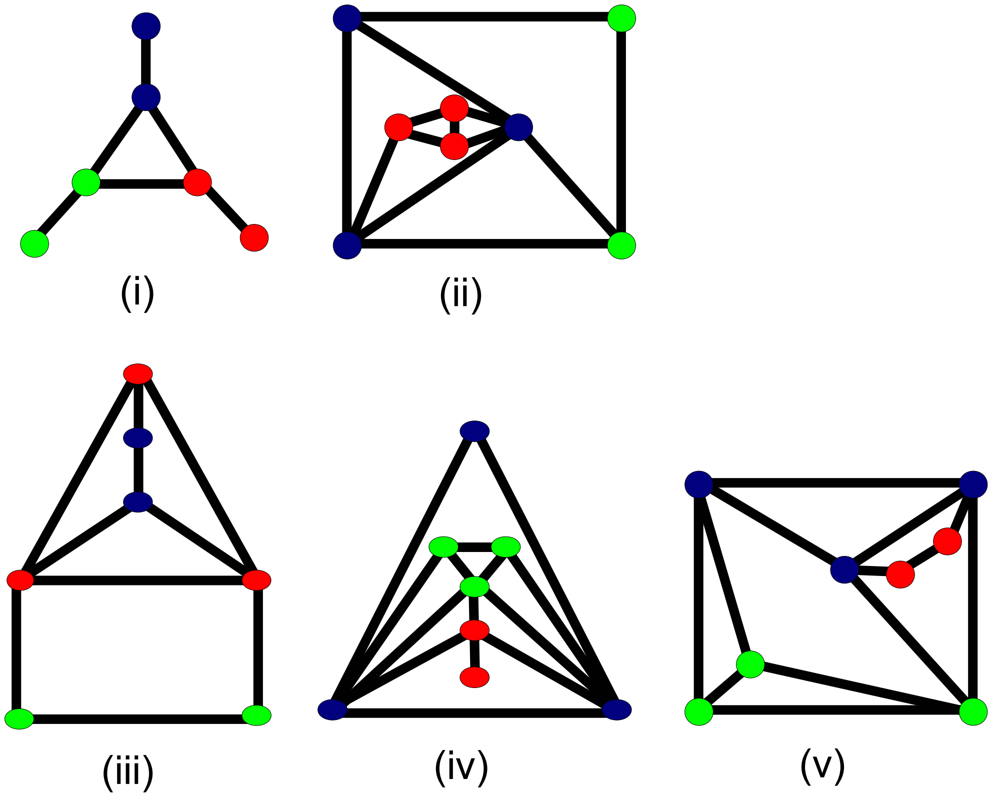

Figure 18.: Fair domination in 3-edge-critical graphs: In each graph, let S_1 = the vertices that are blue, S_2 = set of green vertices, and S_3 = red vertices. These sets form a fair reception of each graph of size 3.

k. But adding $\{u, v\}$ decreases $\gamma(G)$, so now we have $\gamma_F(G) > \gamma(G)$, a contradiction. Therefore, for each $i = 1, 2, ..., k$, S_i forms a complete subgraph of G. $\square$

We define the *line graph* of the complete graph on $[k]$ as follows: let $[k]$ denote the k-set $\{1, 2, \ldots, k\}$ and consider the set of 2-subsets of $[k]$. Let these $\binom{n}{2}$ 2-subsets be the vertices $v_1, v_2, \ldots, v_{\binom{k}{2}}$ of the line graph G_k. There is an edge $\{v_1, v_2\}$ between vertices $v_1, v_2 \in V(G)$ if and only if $v_1 \cap v_2 \neq \emptyset$. For the line graph G_k, $\gamma(G_k) = \lceil \frac{k-1}{2} \rceil$. If k is even then a γ-set of G_k is $\{1, 2\}, \{3, 4\}, \ldots, \{k-1, k\}$, and $\gamma(G) = \lceil \frac{k-1}{2} \rceil$. If k is odd then a γ-set of G_k is $\{1, 2\}, \{3, 4\}, \ldots, \{k-2, k-1\}$, and $\gamma(G) = \frac{k-1}{2}$.

Lemma 3.1 *If k is even, then the line graph G_k is edge-critical.*

Proof. Let D be a dominating set for G_k, where k is even. Without loss of generality, let $D = \{\{1, 2\}, \{3, 4\}, \ldots, \{k-1, k\}\}$. Now add an edge between two vertices in D, say $\{\{1, 2\}, \{3, 4\}\}$ to form the graph G'_k. Then $D' = \{\{1, 2\}, \{5, 6\}, \ldots, \{k-1, k\}\}$ is a dominating set of G'_k and $|D'| = |D| - 1$. Hence, G_k is edge-critical when k is even. $\square$

Consequently, if there is a fair reception of G_k of size $\lceil \frac{k-1}{2} \rceil$, then each set S_i, $i = 1, 2, \ldots, \frac{k-1}{2}$ is a complete subgraph of G_k.

Note that for any k, we can find a fair reception of G_k of size $\lfloor \frac{k}{3} \rfloor$. Consider partitioning the set $[k]$ into 3-subsets; without loss of generality, say we have $\{1, 2, 3\}, \{4, 5, 6\}$, and so on. Then the vertices generated by each set form the sets $S_1, S_2, \ldots, S_{\lfloor \frac{k}{3} \rfloor}$. So we have, for example, $S_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. By forming our sets S_i in this way, we ensure that no vertex in S_j dominates a vertex in S_i for $i \neq j$. We also require at least two vertices from $V(G_k) - S$ to dominate each S_i and so these sets satisfy the criteria to be a fair reception. As $\gamma(G_k) = \lceil \frac{k-1}{2} \rceil$ we have $\gamma_F(G_k) \geq \lfloor \frac{k}{3} \rfloor \geq \frac{2}{3}\gamma(G_k)$. Now, observe that we have a lower bound on $\gamma_F(G_{k+6})$ in terms of $\gamma_F(G_k)$.

Lemma 3.2 *For any k, $\gamma_F(G_{k+6}) \geq \gamma_F(G_k) + 2$.*

Proof. Let $S_1, S_2, \ldots, S_{\gamma_F(G_k)}$ form a fair reception of $\gamma(G_k)$. Now add the 6 points $\{1, 2, 3, 4, 5, 6\}$ to $[k]$ and consider G_{k+6}. We can form a fair reception of this graph by adding $S_{\gamma_F(G_k)+1} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $S_{\gamma_F(G_k)+2} = \{\{4, 5\}, \{4, 6\}, \{5, 6\}\}$ to $S_1, S_2, \ldots, S_{\gamma_F(G_k)}$. Thus, $\gamma_F(G_{k+6}) \geq \gamma_F(G_k) + 2$. $\square$

Note that finding a good upper bound for $\gamma_F(G_k)$ is much more difficult. We know that $\gamma_F(G_k) \leq \gamma(G_k)$. It remains an open problem whether we can improve this upper bound.

We observe that the line graph G_k is claw-free, and so we can apply Corollary 2.4, which states that for a claw-free graph and any graph H without isolated vertices,

$$\gamma(G_k \Box H) \geq \frac{1}{2}\gamma(G_k)(\gamma(H) + 1).$$

Note, also, that we can apply Theorem 3.1 to get $\gamma(G_k \Box H) \geq \gamma_F(G_k)\gamma(H) \geq \frac{2}{3}\gamma(G_k)\gamma(H)$.

Chapter 4

Conclusion

Vizing's conjecture, as stated in 1963, is that the domination number of the Cartesian product of two graphs is at least the product of their domination numbers. The first major result related to the conjecture was from Barcalkin and German [4] in 1979 when they defined decomposable graphs and proved Vizing's conjecture holds for the so-called A-class, now commonly called BG-graphs. Hartnell and Rall's [27] 1995 breakthrough established the truth of Vizing's conjecture for what they called Type $\mathcal{X}$ graphs; this class of graphs is an extension of the BG-graphs. Brešar and Rall [11] in 2009 defined fair reception and fair domination. They proved that Vizing's conjecture holds for graphs with domination number equal to fair domination number. The class of such graphs is an extension of the BG-graphs which is distinct from Type $\mathcal{X}$ graphs. We also know that Vizing's conjecture is true for any graph with domination number less than 4; this was proved in 2004 by Sun [44].

Another approach to proving Vizing's conjecture is to find a constant $c > 0$ so that $\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$, with the hope that eventually this constant will improve to 1. Clark and Suen [17] were able to do this in 2000 for $c = \frac{1}{2}$, and we were able to tighten their arguments to prove a slightly improved inequality.

As Vizing's conjecture is not yet proved for all graphs, several researchers have studied Vizing-like conjectures for other graph products and other types of domination. We provided a summary of some Vizing-like results for fractional domination, integer domination, upper domination, upper total domination, paired domination, and independence domination. In addition, we stated a few conjectures which remain open problems and would contribute to efforts to prove Vizing's conjecture. Two of these conjectures involve independent domination, and one is known as the projection conjecture (Conjecture 2.5). A proof of any of these three conjectures would imply the truth of Vizing's conjecture.

We also defined fair reception and fair domination, as introduced by Brešar and Rall, and included a proof of their Vizing-like inequality relating the domination number of the Cartesian product of graphs G and H to the fair domination numbers of G and H. It remains an open question whether we can find a constant $c > \frac{1}{2}$ so that $\gamma_F(G) \geq c\gamma(G)$ for any graph G. We do know that there are graphs for which $\gamma_F(G) = \gamma(G) - 1$, and we believe the line graph G_k could have fair domination number much smaller than domination number; however it remains difficult to find a lower bound on the fair domination number of a graph in terms of the domination number.

Finally, we considered fair domination in edge critical graphs. We found that a fair reception of an edge-critical graph G of size $\gamma(G)$ must have each set S_i induce a complete subgraph of G. We also provided a proof that Vizing's conjecture is true for 2-edge-critical graphs. This result, of course, was already known since we know Vizing's conjecture holds for any graph with domination number less than 4; however, it is an example of how we might use the idea of fair domination to prove that Vizing's conjecture is true for certain graphs.

Note that a common method of proof in most of the Vizing-like results is to partition a dominating set D of $G\Box H$ and project the vertices of D onto G or H. It is unclear whether this particular method will be useful to prove Vizing's conjecture. As long as Vizing's conjecture remains unresolved, possible next steps in attempt to prove it are to continue studying Vizing-like conjectures, particularly those relating domination and independent domination. One might also study fair domination further, with hopes of finding a lower bound on the fair domination number of a graph. We also note that the BG-graphs, Type $\mathcal{X}$ graphs, and graphs with fair domination number equal to domination number are all defined by a partition of the vertex set of a graph. It could be useful to find a new way of partitioning the vertices of a graph in such a way that we can establish the truth of Vizing's conjecture for an even larger class of graphs.

Table 1: Symbols

Symbol	Description
$\gamma(G)$	Domination number
$\gamma_F(G)$	Fair domination number
$\delta(G)$	Minimum vertex degree
$\Delta(G)$	Maximum vertex degree
$\chi(G)$	Chromatic number
$i(G)$	Independent domination number
$\gamma_t(G)$	Total domination number
$\gamma_c(G)$	Connected domination number
$\gamma_{cl}(G)$	Clique domination number
$\rho_2(G)$	2-packing number
$\gamma_f(G)$	Fractional domination number
$\gamma_{\{k\}}(G)$	$\{k\}$-domination number
$\alpha(G)$	Independence number
$\Gamma(G)$	Upper domination number
$\gamma_t^{\{k\}}(G)$	Total $\{k\}$-domination number
$\Gamma_t(G)$	Upper total domination number
$\gamma_{pr}(G)$	Paired domination number
$\gamma^i(G)$	Independence domination number

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