# KARPAGAM ACADEMY OF HIGHER EDUCATION

**(Deemed to be University Established Under Section 3 of UGC Act 1956)**

**Coimbatore – 641 021.**

**LECTURE PLAN**

**DEPARTMENT OF MATHEMATICS**

**STAFF NAME: Dr.S.SOWMIYA**

**SUBJECT NAME: QUANTITATIVE TECHNIQUES** **SUB.CODE: 19MBAP204**

**SEMESTER: II** **CLASS: I M.B.A**

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|---|---|---|---|
| | | **UNIT I** | |
| 1 | 1 | Concept and Scope of operation research and Phase of OR study and models in OR | S5:Chap-1 Pg.No:3-7&11-18 |
| 2 | 1 | Advantages, limitations and rules of computers in OR | S5:Chap-1 Pg.No:30-33 |
| 3 | 1 | Formulation linear programming models-graphical solution of linear programming model- problems | S1:Chap-3 Pg.No:25-28 |
| 4 | 1 | **Tutorial-I** | |
| 5 | 1 | The simplex method –outline and computing procedure-problems | S2:Chap-4 Pg.No:106-115 |
| 6 | 1 | Use of artificial variables and Big-M method-problems | S5:Chap-2 Pg.No:158-165 |
| 7 | 1 | Problems on Two phase method | S5:Chap-2 Pg.No:166-176 |
| 8 | 1 | **Tutorial-II** | |
| 9 | 1 | Recapitulation & discussion of possible questions | |
| | **Total No of Hours Planned For Unit I=9** | | |
| | | **UNIT II** | |
| 1 | 1 | Introduction to Transportation problem and initial basic feasible solution to transportation cost-Northwest corner rule | S5:Chap-9 Pg.No:217-219 |
| 2 | 1 | Problems on Least cost method – Vogel's approximation method | S5:Chap-9 Pg.No:219-224 |
| 3 | 1 | Find optimal solution by using Modified Distribution method, Degeneracy in TP and unbalanced TP-problems | S2:Chap-9 Pg.No:286-290 |
| 4 | 1 | **Tutorial-I** | |
| 5 | 1 | Find alternative optimal solutions and maximization in transformation problems | S2:Chap-9 Pg.No:297-299 |

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|---|---|---|---|
| 6 | 1 | Assignment problem- Hungarian method of solving assignment problem | S2:Chap-10 Pg.No:317-320 |
| 7 | 1 | Problems on multiple optimum solution on maximization in AP, unbalanced AP and restrictions in AP | S2:Chap-10 Pg.No:326-329 & 337-339 |
| 8 | 1 | **Tutorial-III** | |
| 9 | 1 | Recapitulation & discussion of possible questions | |
| | **Total No of Hours Planned For Unit II=9** | | |
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| | | **UNIT III** | |
| 1 | 1 | Network analysis and construction of networks, Components and Precedence relationship | S6:Chap-25 Pg.No:763-765 |
| 2 | 1 | Events-Activates-rules of network constructions- problems | S6:Chap-25 Pg.No:765-768 |
| 3 | 1 | Concept on errors and dummies in network. | S4:Chap-6 Pg.No:277-280 |
| 4 | 1 | PERT/CPM networks- Project scheduling with uncertain activity times | S7:Chap-15 Pg.No:15.4-15.8 |
| 5 | 1 | **Tutorial-I** | |
| 6 | 1 | Critical Path Analysis-forward and backward pass method based problems | S7:Chap-15 Pg.No:15.17-15.21 |
| 7 | 1 | Concept on Float(or slack) of an activity and event- time | S6:Chap-25 Pg.No:795-805 |
| 8 | 1 | Cost trade-offs- crashing activity times based problems | S6:Chap-25 Pg.No:795-805 |
| 9 | 1 | **Tutorial-II** | |
| 10 | 1 | Recapitulation & discussion of possible questions | |
| | **Total No of Hours Planned For Unit III=10** | | |
| | | **UNIT IV** | |
| 1 | 1 | Introduction to inventory model and Economic order quality models | S5:Chap-12 Pg.No:880-893 |
| 2 | 1 | Quantity discount model and stochastic inventory model problems | S5:Chap-12 Pg.No:898-904 |
| 3 | 1 | Multi product models and inventory control models in practices | S5:Chap-12 Pg.No:924-930 |
| 4 | 1 | Introduction on Queueing models, queueing systems and structure | S5:Chap-10 Pg.No:785-789 |
| 5 | 1 | **Tutorial-I** | |

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| 6 | 1 | Notation parameter, single server and multi server models problems | S6:Chap-25 Pg.No:591-596 |
| 7 | 1 | Poisson input- exponential service, Constant rate service and infinite populations problems | S6:Chap-25 Pg.No:588-590 |
| 8 | 1 | **Tutorial-III** | |
| 9 | 1 | Recapitulation & discussion of possible questions | |
| | **Total No of Hours Planned For Unit IV=9** | | |
| | | **UNIT V** | |
| 1 | 1 | Introduction to decision models, anatomy of decision theory, decision models | S6:Chap-16 Pg.No:415-417 |
| 2 | 1 | Probabilistic decision models , Maximum likelihood Rule, Expected payoff creation, Competitive decision models problems | S6:Chap-16 Pg.No:417-419 |
| 3 | 1 | Maximin, Minimax, Savage, Hurwicz, Laplace decision Models- problems | S6:Chap-16 Pg.No:419-423 |
| 4 | 1 | **Tutorial-I** | |
| 5 | 1 | Introduction on Game theory and Two person zero sum games- graphical solution, Algebraic solutions, linear programming solution | S6:Chap-17 Pg.No:443-455 |
| 6 | 1 | Replacement models- models based on service life- economic life | S7:Chap-11 Pg.No:11.2-11.9 |
| 7 | 1 | Single/ Multi variable search technique- dynamic programming | S7:Chap-10 Pg.No:10.1-10.22 |
| 8 | 1 | Simulation techniques- introductions and types of simulation- Monte Carlo simulation | S7:Chap-17 Pg.No:17.1-17.5 |
| 9 | 1 | **Tutorial-II** | |
| 10 | 1 | Recapitulation & discussion of possible questions | |
| 11 | 1 | Discussion of previous ESE question papers | |
| 12 | 1 | Discussion of previous ESE question papers | |
| 13 | 1 | Discussion of previous ESE question papers | |
| | **Total No of Hours Planned For Unit I=13** | | |

**Suggested Reading**

1. Frederick S.Hillier, Gerald J. Lieberman, (2017). Introduction to Operations Research, $10^{th}$ Edition, McGraw Hill Education, New Delhi.

2. Sharma J.K., (2017). Operations Research -Theory Applications, Macmillan India Ltd, 6th Edition, Lakshmi Publications, New Delhi.
3. Srinivasan G.,(2017). Operations Research -Principles and Applications, PHI, New Delhi.
4. Hamdy A.Taha., (2014).Operations Research-An Introduction, 9th Edition ,Pearson Education, New Delhi.
5. Gupta P K., D.S.Hira(1976). Operations Research , Sultan Chand and Sons, New Delhi.
6. Kanthi Swarup, Gupta P.K.,and Man Mohan.,(2016). Operations Research, Sultan Chand and Sons, New Delhi.
7. Sundaresan V., Ganapathy Subramanian K.S., and Ganesan K.,(2014). Resource Management Techniques, A. R. Publications, Nagapatinam.

Signature student Representative | Signature of the Course Faculty

Signature of the Class Tutor | Signature of Coordinator

Head of the Department

# UNIT I

Concepts and Scope of Operations Research (OR) – Phases of OR study – Models in OR – Advantages and limitations of OR – Role of computers in OR- Formulating Linear programming models, graphical solution of linear programming models, the simplex method-outline, and computing procedure, use of artificial variables, Big M- method and Two phase method.

## LINEAR PROGRAMMING PROBLEM

### Introduction:

LPP deals with determining optimal allocations of limited resources to meet given objectives. The resources may be in the form of men, raw materials, market demand, money and machines, etc. The objective is usually maximizing utility etc.

LPP deals with the optimization of a function of decision variables known as objective function. Subject to a set of simultaneous linear equation or inequality known as constraints.

The term linear means that all the variables occurring in the objective function and the constraints are of the $1^{st}$ degree in the problem under consideration and the term programming means the process of determining the particular course of action.

### Mathematical formulation of LPP:

If $x_j$ (j = 1,2,…..n) are n decision variables of the problem and if the system is subject to m constraints.

∴ The general model can be redundant the form,

Optimize z = f($x_1, x_2, x_3,\ldots\ldots\ldots x_n$)

Subject to the constraints are,

$g_j(x_1, x_2, x_3,\ldots\ldots\ldots x_n) \leq, =, \geq, b_i$(i = 1, 2, …..,m) and

$x_1, x_2, x_3,\ldots\ldots\ldots x_n$ >= 0 (non negativity constraints)

### Procedure for forming a LPP model:

**Step 1:** Identify the unknown decision variables to be determined and assign symbols to them.

**Step 2:** Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.

**Step 3:** Identify the objective or aim and represent it also as a linear function of decision variables.

**Step 4:** Express the complete formulating of LPP as a general mathematical model.

## Problems:

1. A firm manufactures two types of product A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines $M_1$and $M_2$. Type A requires 1 minute of processing time on $M_1$ and 2 minutes on $M_2$. Type B requires 1 minute on $M_1$ and 1 minute on $M_2$. Machine $M_1$ is available for not more than 6 hours 40 minutes, while machines $M_2$ is a available for 10 hours during any work hours. Formulate the problem as LPP so as to maximize the profit.

## Solution:

Let us consider $x_1$ be the no. of units in Type A and $x_2$ be the no. of units in Type B.

To produce these units of Type A and Type B product it requires,

$x_1 + x_2$ processing minutes on $M_1$

$2x_1 + x_2$ processing minutes on $M_2$

Since $M_1$ is available for not more than 400 minutes and $M_2$ is available for not more then 600 minutes.

Therefore the constraints are:

$$x_1 + x_2 \leq 400$$

$$2x_1 + x_2 \leq 600$$

and $x_1, x_2 \geq 0$

since the profit from Type A is Rs. 2 and profit from Type B is Rs. 3.

∴ The total Profit is $2x_1 + 3x_2$

∴ Here the objective is to maximize the profit

∴ The objective function is,

Maximize z = $2x_1 + 3x_2$

The complete formulation of the LPP is

Maximize z = $2x_1 + 3x_2$

Subject to the constraints,

$x_1 + x_2 \leq 400$ ……….. (i)

$2x_1 + x_2 \leq 600$ ……….. (ii)

and $x_1, x_2 \geq 0$ ……….. (iii)

2.A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from 4 different types of foods. The yields per unit of these foods are given in the following table.

| Food Type | Yield/unit | | | Cost / unit (Rs) |
|---|---|---|---|---|
| | Proteins | Fats | Carbohydrates | |
| 1 | 3 | 2 | 6 | 45 |
| 2 | 4 | 2 | 4 | 40 |
| 3 | 8 | 7 | 7 | 85 |
| 4 | 6 | 5 | 4 | 65 |
| Minimum Requirement | 800 | 200 | 700 | |

Formulate the LPP model for this problem.

## Solution:

Let $x_1, x_2, x_3, x_4$ be the no. of units in the food type 1,2,3 and 4 respectively.

In this problem the main objective is to minimize the cost. ∴ The objective function is,

Minimize Z = $45x_1 + 40x_2 + 85x_3 + 65x_4$

The minimize requirement for proteins, fats and carbohydrates are 800, 200 and 700 respectively.

∴ The subject to the constraints are:

$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$

$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$

$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$

And the complete formation of LPP is

Minimize Z = $45x_1 + 40x_2 + 85x_3 + 65x_4$

∴ The subject to the constraints:

$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$

$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$

$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$

and $x_1, x_2, x_3, x_4 \geq 0$.

## Graphical Method of the solution of the LPP:

Linear programming problems involving only 2 variables can be effective solved by a graphical method which provides a pictorial representation of the problems and its solutions. And which gives the basic concept used in solving general LPP which may involve any finite no. of variables.

## Working procedure for graphical method:

Given a LPP optimize Z = f($x_i$),

Subject to the constraints,

$g_j(x_i) \leq, =, \ or \geq b_j$, (i=1,2,…..,n), (j=1,2,…..,m)

and $x_i \geq 0$ (non-negativity restrictions)

**Step 1:** Consider the inequality constraints as equalities. Draw the straight lines in the XOY plane corresponding to each equality and non-negativity restrictions.

**Step 2:** Find the permissible region (feasible region or solution space) for the values of the variable which is the region bound by the lines drawn in step1.

**Step 3:** Find the points of intersection of the bound lines by solving the equations of the corresponding lines.

**Step 4:** Find the values of Z at all vertices of the permissible region.

**Step 5:** (i) For minimization problem choose the vertex for which Z is maximum.

(ii) For minimization problem choose the vertex for which Z is minimum.

## Problems:

1. Solve the following LPP by graphical method.

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to the constraints are,

$$-2x_1 + x_2 \leq 1$$

$$x_1 \leq 2$$

$$x_1 + x_2 \leq 3 \quad \text{and } x_1, x_2 \geq 0$$

## Solution:

Consider the inequality constraints as equality,

$$-2x_1 + x_2 = 1 \quad \ldots\ldots\ldots\ldots (1)$$

$$x_1 = 2 \quad \ldots\ldots\ldots\ldots (2)$$

$$x_1 + x_2 = 3 \quad \ldots\ldots\ldots\ldots (3)$$

$$x_1 = 0 \quad \ldots\ldots\ldots\ldots (4)$$

$$x_2 = 0 \quad \ldots\ldots\ldots\ldots (5)$$

From equation (1), putting $x_1 = 0$.

We get $-2x_1 + x_2 = 1$

$x_2 = 1$

The point $(x_1, x_2) = (0,1)$

Similarly, putting $x_2 = 0$, we get

$$-2x_1 + 0 = 1$$

$$x_1 = {-1}/{2} = -0.5$$

∴ The point $(x_1, x_2) = (-0.5, 0)$

∴ The point (0, 1) and (-0.5, 0) lies on the line $-2x_1 + x_2 = 1$

From equation (2) we get, the points (2,1) and (2,2) lies on the line $x_1 = 2$.

From equation (3) putting $x_1 = 0$,

we get $x_1 + x_2 = 3$

$$x_2 = 3$$

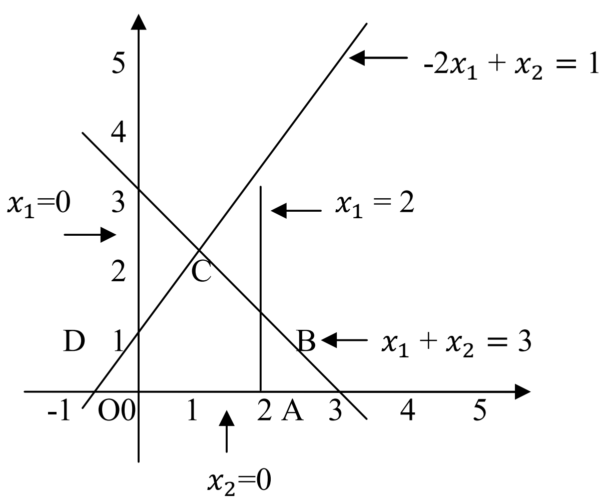
The point $(x_1, x_2) = (0, 3)$

Similarly putting $x_2 = 0$, we get $x_1 + 0 = 3$

$$x_1 = 3.$$

The point $(x_1, x_2) = (3, 0)$

∴ The point (0, 3) and (3, 0) lies on the line $x_1 + x_2 = 3$




∴ From the graph the vertices of the solution space are, O(0,0), A(2,0), B(2,1), C(0.7,2.3), D(0,1)

The values of the Z at these vertices are given by,

| Vertex | Z = $3x_1 + 2x_2$ |
|---|---|
| O(0,0) | 0 |
| A(2,0) | 6 |
| B(2,1) | 8 |
| C(0.7,2.3) | 6.7 |
| D(0,1) | 2 |

Since the problem is of maximization type.

∴ The optimum solution to the LPP is,

Max Z = 8

∴ $x_1 = 2$ and $x_2 = 1$

**2**.Maximize Z = $5x_1 + 8x_2$

Subject to constraints, $15x_1 + 10x_2 \leq 180$

$10x_1 + 20x_2 \leq 200$

$15x_1 + 20x_2 \leq 210$

and $x_1, x_2 \geq 0$

## Solution:

Consider the inequality constraints as equality,

$15x_1 + 10x_2 = 180$ ……….. (1)

$10x_1 + 20x_2 = 200$ ……….. (2)

$15x_1 + 20x_2 = 210$ ……….. (3)

$x_1 = 0$ ……….. (4)

$x_2 = 0$ ……….. (5)

From the (1) equation, putting $x_1 = 0$ , we get

$$15(0) + 10x_2 = 180$$

$$10x_2 = 180$$

$$x_2 = 18$$

∴ The point $(x_1, x_2) = (0, 18)$

Similarly putting $x_2 = 0$, we get

$$15x_1 + 10(0) = 180$$

$$15x_1 = 180$$

$$x_1 = 12$$

∴ The point $(x_1, x_2) = (12, 0)$

From the (2) equation putting $x_1 = 0$, we get

$$10(0) + 20x_2 = 200$$

$$20x_2 = 200$$

$$x_2 = 10$$

∴ The point $(x_1, x_2) = (0, 10)$

Similarly putting $x_2 = 0$, we get

$$10x_1 + 20(0) = 200$$

$$10x_1 = 200$$

$$x_1 = 20$$

∴ The point $(x_1, x_2) = (20, 0)$

From the 3rd equation putting $x_1 = 0$, we get

$$15(0) + 20x_2 = 210$$

$$20x_2 = 210$$

$$x_2 = 10.5$$

∴ The point $(x_1, x_2)$ = (0, 10.5)

Similarly putting $x_2 = 0$, we get

$$15x_1 = 210$$

$$x_1 = 14$$

∴ The point $(x_1, x_2)$ = (14, 0)

| Vertex | Z = $5x_1 + 8x_2$ |
|---|---|
| O(0,0) | 0 |
| A(12,0) | 60 |
| B(10,3) | 74 |
| C(2,9) | 82 |
| D(0,10) | 80 |

∴ The solution Max Z = 82

∴ $x_1 = 2$, $x_2 = 9$

## Some more cases in a LPP:

In general a LPP may have

1. A unique optimal solution.
2. An Infinite no. of optimal solutions.
3. An Unbounded solution.
4. No solution.

**Problems**:

1. Solve the following LPP graphically,

   Maximize Z = $100x_1 + 40x_2$

   Subject to the constraints are:

$$5x_1 + 2x_2 \leq 1000$$

$$3x_1 + 2x_2 \leq 900$$

$x_1 + 2x_2 \leq 500$

and $x_1, x_2 \geq 0$

**Solution:**

$5x_1 + 2x_2 = 1000$ …………… (1)

$3x_1 + 2x_2 = 900$ …………… (2)

$x_1 + 2x_2 = 500$ …………… (3)

From (1) equation putting $x_2 = 0$, we get

$5(0) + 2x_2 = 1000$

$2x_2 = 1000$

$x_2 = 500$

∴ The point $(x_1, x_2)$ = (0, 500)

Similarly, $x_2 = 0$, we get $5x_1 + 0 = 1000$

$x_1 = 200$

∴ The point $(x_1, x_2)$ = (200, 0)

From equation (2) putting $x_1 = 0$, we get

$3(0) + 2x_2 = 900$

$x_2 = 450$

∴ The point $(x_1, x_2)$ = (0, 450)

Similarly $x_2 = 0$, we get, $3x_1 + 2(0) = 900$

$3x_1 = 900$

$x_1 = 300$

∴ The point $(x_1, x_2)$ = (300, 0)

From equation (3) putting $x_1 = 0$, we get

$0 + 2x_2 = 500$

$x_2 = 250$

∴ The point $(x_1, x_2)$ = (0, 250)

Similarly $x_2 = 0$, we get, $x_1 + 0 = 500$

$x_1 = 500$

∴ The point $(x_1, x_2)$ = (500, 0)

| vertex | Z = $100x_1 + 40x_2$ |
|---|---|
| O(0,0) | 0 |
| A(200,0) | 20,000 |
| B(125,187.5) | 20,000 |
| C(0,250) | 10,000 |

Since the problem is of maximization type in the above table the two vertices have the same maximization Z value.

Hence the given LPP has an infinite no. of solution. For this problem the optimum solution is Maximize Z = 20,000

$x_1$ = 200 (or) 125

$x_2$ = 187.5 (or) 0

2. Max Z = $2x_1 + 3x_2$
   Subject to the constraints:

   $x_1 - x_2 \leq 2$

   $x_1 + x_2 \leq 4$ and $x_1, x_2 \geq 0$

## Solution:

$x_1 - x_2 = 2$ ……………. (1)

$x_1 + x_2 = 4$ ……………. (2)

From (1) equation $x_1 = 0$ we get,

$0 - x_2 = 2$

$x_2 = -2$

∴ The point $(x_1, x_2) = (0, -2)$

Similarly putting $x_2 = 0$ , we get

$x_1 - 0 = 2$

$x_1 = 2$

∴ The point $(x_1, x_2) = (2, 0)$

From (2) equation $x_1 = 0$ we get

$0 + x_2 = 4$

$x_2 = 4$

∴ The point $(x_1, x_2) = (0, 4)$

Similarly putting $x_2 = 0$ , we get

$x_1 + 0 = 4$
$x_1 = 4$

∴ The point $(x_1, x_2) = (4, 0)$

| vertex | Max Z = $2x_1 + 3x_2$ |
|---|---|
| A(0,4) | 12 |
| B(3,1) | 9 |

From the above graph, the maximization type with two vertex and have the unbounded solution.

**General Linear Programming Problem:**

**Simplex Methods:**

**General Linear Programming Problem:**

The Linear Programming Problem involving more than 2 variables will be expressed as follows:

Maximize (or) Minimize Z = $C_1x_1+C_2x_2+\ldots\ldots+C_nx_n$

Subject to the constraints:

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n \leq, =, \text{(or)} \geq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n \leq, =, \text{(or)} \geq b_2$

…………………………………………………..

…………………………………………………..

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n \leq, =, \text{(or)} \geq b_m$

and the non-negativity restrictions is, $x_1, x_2, \ldots\ldots\ldots.., x_n \geq 0$

**Definition – 1:**

A set of values $x_1, x_2, \ldots\ldots\ldots.., x_n$ which satisfies the constraints of the LPP is called its ***solution***.

**Definition – 2:**

Any solution to a LPP which satisfies the non-negativity restrictions of LPP is called its ***Feasible Solution***.

**Definition – 3:**

Any feasible solution which optimizes the objective function of the LPP is called its ***optimum or optimal solution.***

**Definition – 4:**

If a constraints of a general LPP, $\sum_{j=1}^{n} a_{ij}\, x_j \leq b_i$ (i=1,2,3,…..,k)

Then the non – negative variables $s_i$ which are introduced to convert inequalities to the equalities (i.e.) $\sum_{j=1}^{n} a_{ij}\, x_j + s_i = b_i$ are called ***slack variables***.

**Definition – 5:**

If a constraints of a general LPP, $\sum_{j=1}^{n} a_{ij}\, x_j \geq b_i$ (i=1,2,3,…..,k)

Then the non – negative variables $s_i$ which are introduced to convert inequalities to the equalities (i.e.) $\sum_{j=1}^{n} a_{ij}\, x_j - s_i = b_i$ are called ***surplus variables***.

## Canonical and standard forms of LPP:

After the formulation of LPP the next step is to obtain its solution. But before any method is used to find its solution the problem must be presented in a suitable form.

There are 2 forms: (1) Canonical form

(2) Standard form

### 1. The Canonical Form:

The general LPP can always be expressed in the following form,

Maximize Z = $C_1x_1 + C_2x_2 + \ldots\ldots + C_nx_n$

Subject to the constraints:

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n \leq b_2$

…………………………………………………..

…………………………………………………..

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n \leq b_m$

and the non-negativity restrictions is, $x_1, x_2, \ldots\ldots\ldots.., x_n \geq 0$

This form of LPP is called the ***Canonical form of LPP.***

### Characteristics of the Canonical Form:

1. The objective function is of Maximization type.
2. All constraints are of less or equal to type.
3. All variables $x_i$ are non-negative.

### 1. The Standard Form:

The general LPP in the form,

Maximize Z = $C_1x_1+C_2x_2+\ldots\ldots+C_nx_n$

Subject to the constraints:

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$

…………………………………………

…………………………………………

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = b_m$

and the non-negativity restrictions is, $x_1, x_2, \ldots\ldots\ldots.., x_n \geq 0$

is known as ***Standard from.***

## Characteristics of the Standard Form:

1. The objective function is of Maximization type.
2. All constraints are expressed as equation type.
3. Right hand sides of each constraint are non-negative.
4. All variables $x_i$ are non-negative.

## Problems:

1. Express the following LPP in Standard form.

Minimize Z = $5x_1 + 7x_2$

Subject to the constraints:

$x_1 + x_2 \leq 8$

$3x_1 + 4x_2 \geq 3$

$6x_1 + 7x_2 \geq 5$ and $x_1, x_2 \geq 0$

## Solution:

Since Minimize Z = - Max(-Z)

= -Max($Z^*$)
= - ($5x_1 + 7x_2$)

= - $5x_1$ - $7x_2$

The given LPP becomes,

Maximize $Z^*$ = - $5x_1$ - $7x_2$

Subject to the constraints:

$x_1 + x_2 \leq 8$

$3x_1 + 4x_2 \geq 3$

$6x_1 + 7x_2 \geq 5$ and $x_1, x_2 \geq 0$

By introducing slack variables $s_1$ surplus variables $s_2$, $s_3$ then the standard form of the LPP is given by,

Maximize $Z^*$ = - $5x_1$ - $7x_2$+ $0s_1 + 0s_2 + 0s_3$

Subject to the constraints:

$x_1 + x_2 + s_1 = 8$

$3x_1 + 4x_2 - s_2 = 3$

$6x_1 + 7x_2 - s_3 = 5$ and $x_1,\ x_2, s_1, s_2\ , s_3 \geq 0$

2. Express the following LPP in Standard form.

Maximize Z = $4x_1 + 2x_2$+ $6x_3$

Subject to the constraints:

$2x_1 + 3x_2 + 2x_3 \geq 6$

$3x_1 + 4x_2 = 8$

$6x_1$ - $7x_2 + x_3 \leq 10$ and $x_1, x_2, x_3 \geq 0$

**Solution:**

The given LPP becomes,

Max Z = $4x_1 + 2x_2 + 6x_3$

By introducing slack variables $s_2$ surplus variables $s_1$, then the standard form of the LPP is given by,

Max $Z^*$ = $4x_1 + 2x_2 + 6x_3 + 0s_1 + 0s_2$

Subject to the constraints:

$2x_1 + 3x_2 + 2x_3 - s_1 = 6$

$3x_1 + 4x_2 + 0x_3 = 8$

$6x_1 - 7x_2 + x_3 + s_2 = 10$

and $x_1, x_2, x_3, s_1, s_2 \geq 0$

3. Express the following LPP in Canonical form.

Minimize Z = $x_1 + 4x_2$

Subject to the constraints:

$3x_1 + x_2 \leq 5$

$-2x_1 + 4x_2 \geq -7$

and $x_1, x_2 \geq 0$

**Solution:**

The canonical form of the given LPP becomes,

Maximize $Z^*$ = $-x_1 - 4x_2$

Subject to the constraints:

$3x_1 + x_2 \leq 5$

$2x_1 - 4x_2 \leq 7$

and $x_1, x_2 \geq 0$

4. Express the following LPP in the canonical form:

Maximize Z = $2x_1 + 3x_2 + x_3$

Subject to the constraints:

$4x_1 - 3x_2 + x_3 \leq 6$

$x_1 + 5x_2 - 7x_3 \geq -4$

and $x_1, x_3 \geq 0$ and $x_2$ is unrestricted.

**Solution:**

Here $x_2$ is unrestricted, ∴ $x_2$ can be written as the difference of two non-negative variables, (i.e.) $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$

∴ The given LPP becomes,

Max Z = $2x_1 + 3(x_2' - x_2'') + x_3$

Subject to the constraints:

$4x_1 - 3(x_2' - x_2'') + x_3 \leq 6$

$-x_1 - 5(x_2' - x_2'') + 7x_3 \leq 4$

and $x_1, x_2', x_2'', x_3 \geq 0$

**The Simplex Method:**

**Definition – 1:**

Given a system of m linear equations with n variables (m<n). The solution obtained by setting (n-m) variables = 0 and solving for the remaining m variables is called a ***Basic Solution.***

The m variables are called Basic variables and they form Basic Solution. The (n-m) variables which are put to 0 are called as ***Non- basic Variables***.

**Definition – 2:**

A basic solution is said to be a non-degenerate Basic Solution if none of the Basic variables is zero.

**Definition – 3:**

A basic solution is said to be a degenerate basic solution if one or more of the basic variables are zero.

**Definition – 4:**

A feasible solution which is also basic is called a ***Basic feasible solution***.

**The Simplex Algorithm:**

Assuming the existence of an initial basic feasible solution, an optimum solution to any LPP by simplex method is found as follows:

***Step 1:*** Check whether the objective function is to be maximized or minimized. If it is to be minimized, then convert it into a problem of maximization, by Minimize Z = -Maximize(-Z)

***Step 2:*** Check whether all $b_i$'s are positive. If any of the $b_i$'s is negative, multiply both sides of that constraint by -1 so as to make its right hand positive.

***Step 3:*** By introducing slack / surplus variables, convert the inequality constraints into equations and express the given LPP into its standard form.

***Step 4:*** Find an initial basic feasible solution and express the above information conveniently in the following simplex table.

| | | $C_j$ | ($C_1$ | $C_2$ | $C_3$ | ……. | 0 | 0 | 0 | …….) |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | ……. | $s_1$ | $s_2$ | $s_3$ | ……. |
| $C_{B1}$ | $S_1$ | $b_1$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | ……. | 1 | 0 | 0 | ……. |
| $C_{B2}$ | $S_2$ | $b_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ | ……. | 0 | 1 | 0 | ……. |
| $C_{B3}$ | $S_3$ | $b_3$ | $a_{31}$ | $a_{32}$ | $a_{33}$ | ……. | 0 | 0 | 1 | ……. |
| . | . | . | . | . | . | ……. | | | | ……. |
| . | . | . | . | . | . | | | | | |
| . | . | . | . | . | . | ……. | | | | ……. |
| . | . | . | . | . | . | | | | | |
| . | . | . | . | . | . | ……. | | | | ……. |
| . | . | . | . | . | . | | | | | |

| | | | | |
|---|---|---|---|---|
| . . <br> . . | . <br> . | Body matrix | ……. | Unit matrix |
| $(Z_j - C_j)$ | $Z_0$ | $(Z_1 - C_1)$ $(Z_2 - C_2)$ | ……. | …………………… |

(Where $C_j$ – row denotes the coefficients of the variables in the objective function. $C_B$ – column denotes the coefficients of the basic variables in the objective function. $Y_B$ – column denotes the basic variables. $X_B$ – column denotes the values of the basic variables. The coefficients of the non-basic variables in the constraint equations constitute the body matrix while the coefficients of the basic variables constitute the unit matrix. The row $(Z_j - C_j)$ denotes the evaluations (or) index for each column).

***Step 5:*** Compute the net evaluations $(Z_j - C_j)$ (j = 1,2,…..,n) by using the relation $Z_j - C_j = C_B\, a_j - C_j$.

Examine the sign of $Z_j - C_j$

(a) If all $(Z_j - C_j) \geq 0$ then the current basic feasible solution $X_B$ is optimal.

(b) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal, go to the next step.

***Step 6: (To find the entering variable)***

The entering variable is the non-basic variable corresponding to the most negative value of $(Z_j - C_j)$. Let it be $x_r$ for some j = r. The entering variable column is known as the key column (or) pivot column which is shown marked with an arrow at the bottom. If more than one variable has the same most negative $(Z_j - C_j)$, any of these variables may be selected arbitrarily as the entering variable.

***Step 7: (To find the leaving variable)***

Compute the ratio $\theta = \text{Min}\left\{\frac{X_{Bi}}{a_{ir}}, a_{ir} > 0\right\}$ (i.e., the ration between the solution column and the entering variable column by considering only the positive denominators)

(a) If all $a_{ir} \leq 0$, then there is an unbounded solution to the given LPP.

(b) If atleast one $a_{ir} > 0$, then the leaving variable is the basic variable corresponding to the minimum ratio $\theta$. If $\theta = \frac{X_{Bk}}{a_{kr}}$, then the basic variable $x_k$ leaves the basis. The leaving variable row is called the key row or pivot equation, and the element at the intersection of the pivot column and pivot row is called the pivot element (or) leading element.

***Step 8:*** Drop the leaving variable and introduce the entering variable along with its associated value under $C_B$ column. Convert the pivot element to unity by dividing the pivot equation by the pivot element and all other elements in its column to zero by making use of

(i) New pivot equation = old pivot equation ÷ pivot element

(ii) New equation (all other rows including $(Z_j - C_j)$ row)

$$= \text{Old equation} - \begin{pmatrix}\text{Corresponding}\\ \text{Column}\\ \text{Coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New pivot}\\ \text{element}\end{pmatrix}$$

***Step 9:*** Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

***Note(1): For maximization problems:***

(i) If all $(Z_j - C_j) \geq 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non-basic variable corresponding to the most negative value of $(Z_j - C_j)$.

***Note(2): For minimization problems:***

(i) If all $(Z_j - C_j) \leq 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) > 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non-basic variable corresponding to the most positive value of $(Z_j - C_j)$.

***Note(3):*** For both maximization and minimization problems, the leaving variable is the basic variable corresponding to the minimum ratio $\theta$.

## Problems:

1. Use simplex method to solve following LPP.

Maximize $Z = 4x_1 + 10x_2$

Subject to the constraints:

$2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$

and $x_1, x_2 \geq 0$

**Solution:**

By introducing the slack variables $s_1$, $s_2$, $s_3$. The standard form of the given LPP becomes,

Maximize Z = $4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraints:

$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 \leq 50$

$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 \leq 100$

$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 \leq 90$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$

since there are 3 equations with 5 variables. ∴ The initial basic Feasible Solution is obtained by equality, (5-3) = 2 to 0.

∴ The initial basic Feasible Solution (IFBS) , $s_1$=50, $s_2$=100, $s_3$=90

**Initial iteration:**

| | | | $c_j$ | 4 | 10 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| | $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | Ratio |
| | 0 | $s_1$ | 50 | 2 | 1 | 1 | 0 | 0 | $50/1$ = 50 |
| ← | 0 | $s_2$ | 100 | 2 | [5] | 0 | 1 | 0 | $100/5$ = 20 |
| | 0 | $s_3$ | 90 | 2 | 3 | 0 | 0 | 1 | $90/3$ = 30 |
| | | $Z_j - C_j$ | 0 | -4 | -10 ↑ | 0 | 0 | 0 | |

$x_2$ is entering variable,

$s_2$ is leaving variable,

5 is pivot element

**I iteration:**

| | | $c_j$ | 4 | 10 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ |
| 0 | $s_1$ | 30 | $^8/_5$ | 0 | 1 | $^{-1}/_5$ | 0 |
| 0 | $x_2$ | 20 | $^2/_5$ | 1 | 0 | $^1/_5$ | 0 |
| 0 | $s_3$ | 30 | $^4/_5$ | 0 | 0 | $^{-3}/_5$ | 1 |
| | $Z_j - C_j$ | 200 | 0 | 0 | 0 | 2 | 0 |

Since all $Z_j - C_j \geq 0$. ∴ The current Basic Feasible solution is optimal.

∴ The optimal solution is Max Z = 200, $x_1 = 0$, $x_2 = 20$.

2. Find the non-negative values of $x_1, x_2, x_3$ which

Max Z = $3x_1 + 2x_2 + 5x_3$

Subject to the constraints,

$x_1 + 4x_2 \leq 420$
$3x_1 + 2x_3 \leq 460$
$x_1 + 2x_2 + x_3 \leq 430$

and $x_1, x_2, x_3 \geq 0$

**Solution:**

By introducing slack variables $s_1, s_2, s_3$. The standard form of the given

Max Z = $3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraints,

$x_1 + 4x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 420$

$3x_1 + 0x_2 + 2x_3 + 0s_1 + s_2 + 0s_3 = 460$
$x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 430$

and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

The IFBS is given by, $s_1 = 420, s_2 = 460, s_3 = 430$

**Initial iteration:**

| | | $c_j$ | 3 | 2 | 5 | 0 | 0 | 0 | Ratio $\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | |
| 0 | $s_1$ | 420 | 1 | 4 | 0 | 1 | 0 | 0 | $420/0=\infty$ |
| 0 ← | $s_2$ | 460 | 3 | 0 | [2] | 0 | 1 | 0 | $460/2=230$ |
| 0 | $s_3$ | 430 | 1 | 2 | 1 | 0 | 0 | 1 | $430/1=430$ |
| | $Z_j - C_j$ | 0 | -3 | -2 | -5 ↑ | 0 | 0 | 0 | |

$s_2$ is leaving variable, $x_3$ is entering variable, 2 is pivot element

**I iteration:**

| | | $c_j$ | 3 | 2 | 5 | 0 | 0 | 0 | Ratio $\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | |
| 0 | $s_1$ | 420 | 1 | 4 | 0 | 1 | 0 | 0 | $420/4=105$ |
| 5 | $x_3$ | 230 | $3/2$ | 0 | 1 | 0 | $1/2$ | 0 | $230/0=\infty$ |
| 0 ← | $s_3$ | 200 | $-1/2$ | [2] | 0 | 0 | $-1/2$ | 1 | $200/2=100$ |
| | $Z_j - C_j$ | 1150 | $9/2$ | -2 ↑ | 0 | 0 | $5/2$ | 0 | |

$s_3$ is leaving variable, $x_2$ is entering variable, 2 is pivot element

**II iteration:**

| | | $c_j$ | 3 | 2 | 5 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ |
| 0 | $s_1$ | 20 | 2 | 0 | 0 | 1 | 0 | -2 |
| 5 | $x_3$ | 230 | $3/2$ | 0 | 1 | 0 | $1/2$ | 0 |
| 2 | $x_2$ | 100 | $-1/4$ | 1 | 0 | 0 | $-1/4$ | $1/2$ |
| | $Z_j - C_j$ | 1350 | 4 | 0 | 0 | 0 | 2 | 1 |

$\therefore$ Max Z = 1350, $x_1 = 0$, $x_2 = 100$, $x_3 = 230$

## Artificial Variable Techniques:

To solve a LPP by a simplex method. We have to start with initial Basic feasible solution and construct the initial simplex table. In the previous problems the slack variable provided the

IFBS. However in some problems the slack variables cannot provide the IFBS. In these problems atleast one of the constraints is of equal to or greater than or equal to type. To solve such a LPP there are 2 methods available:

(1) Big-M method (or) M-Technique (or) The Method of penalties
(2) Two Phase method.

**Problems:**

1. Use Big- M \method to solve,

Min Z = $4x_1 + 3x_2$

Subject to the constraints,

$2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \leq 6$ and $x_1, x_2 \geq 0$

**Solution:**

The standard form of the given LPP is

Max $Z^* = -4x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 - MR_1 - MR_2$

Subject to the constraints,

$2x_1 + x_2 - s_1 + R_1 = 10$

$-3x_1 + 2x_2 + s_2 = 6$

$x_1 + x_2 - s_3 + R_2 = 6$

and $x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0$

∴ The IFBS is given by, $R_1 = 10, s_2 = 6, R_2 = 6$

**Initial Iteration:**

| | $c_j$ | -4 | -3 | 0 | 0 | 0 | -M | -M | Ratio |
|---|---|---|---|---|---|---|---|---|---|

| | $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $R_1$ | $R_2$ | $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ← | -M | $R_1$ | 10 | [2] | 1 | -1 | 0 | 0 | 1 | 0 | $10/2 = 5$ |
| | 0 | $s_2$ | 6 | -3 | 2 | 0 | 1 | 0 | 0 | 0 | $-6/3 = -2$ |
| | -M | $R_2$ | 6 | 1 | 1 | 0 | 0 | -1 | 0 | 1 | $6/1 = 6$ |
| | | $Z_j - C_j$ | -16M | -3M+4 ↑ | -2M+3 | M | 0 | M | 0 | 0 | |

$R_1$ is leaving variable, $x_1$ is entering variable, 2 is pivot element

**I iteration:**

| | | | $c_j$ | -4 | -3 | 0 | 0 | 0 | -M | Ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| | $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $R_2$ | $\theta$ |
| | -4 | $x_1$ | 5 | 1 | $1/2$ | $-1/2$ | 0 | 0 | 0 | 10 |
| | 0 | $s_2$ | 21 | 0 | $7/2$ | $-3/2$ | 1 | 0 | 0 | 6 |
| ← | -M | $R_2$ | 1 | 0 | [$1/2$] | $1/2$ | 0 | -1 | 1 | 2 |
| | | $Z_j - C_j$ | -20-M | 0 | $2 - M/2$ ↑ | $4 - M/2$ | 0 | M | 0 | |

$R_2$ is leaving variable, $x_2$ is entering variable, $1/2$ is pivot element

**II iteration:**

| | | $c_j$ | -4 | -3 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ |
| -4 | $x_1$ | 4 | 1 | 0 | -1 | 0 | 1 |
| 0 | $s_2$ | 14 | 0 | 0 | -5 | 1 | 7 |
| -3 | $x_2$ | 2 | 0 | 1 | 1 | 0 | -2 |
| | $Z_j - C_j$ | -22 | 0 | 0 | 1 | 0 | 2 |

Since all $Z_j - C_j \geq 0$, ∴ the current solution is optimal.

The optimal solution is, Max $Z^* = -22$, $x_1 = 4$, $x_2 = 2$

But, Min Z = - (Max (-Z))

= - (- 22)

Min Z = 22, $x_1$ = 4, $x_2$ = 2

2. Solve the following LPP by Simplex Method:
   Max Z = $3x_1 + 2x_2$
   Subject to the constraints,
   $2x_1 + x_2 \leq 2$
   $3x_1 + 4x_2 \geq 12$ and $x_1, x_2 \geq 0$

**Solution:**

By introducing the slack variable $s_1$ and surplus variable $s_2$,

artificial variable $R_1$. The standard form of the LPP becomes,

Max Z = $3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 - MR_1$

Subject to the constraints,

$2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 + R_1 = 12$

and $x_1, x_2, s_1, s_2, s_3, R_1 \geq 0$

∴ The IBFS, $s_1$ = 2, $R_1$= 12

**Initial Iteration:**

| | | $c_j$ | 3 | 2 | 0 | 0 | -M | Ratio $\theta$ |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $R_1$ | |
| ← 0 | $s_1$ | 2 | 2 | [1] | 1 | 0 | 0 | 2 |
| -M | $R_1$ | 12 | 3 | 4 | 0 | -1 | 1 | 3 |
| | $Z_j - C_j$ | -12M | -3M-3 | -4M-2 ↑ | 0 | M | 0 | |

$s_1$ is leaving variable, $x_2$ is entering variable, 1 is pivot element

**I Iteration:**

| | | $c_j$ | 3 | 2 | 0 | 0 | -M |
|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $R_1$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | $x_2$ | 2 | 2 | 1 | 1 | 0 | 0 |
| -M | $R_1$ | 4 | -5 | 0 | -4 | -1 | 1 |
| | $Z_j - C_j$ | 4-4M | 1+5M | 0 | 4M+2 | M | 0 |

∴ Since all $Z_j - C_j \geq 0$ and artificial variable $R_1$ appears in the basis at the non – zero level.
∴ The given LPP does not possess any feasible solution. But the given LPP Possess a pseudo optimal solution.

3. Use penalty method to solve,

Max Z = $2x_1 + x_2 + x_3$

Subject to the constraints,

$4x_1 + 6x_2 + 3x_3 \leq 8$
$3x_1 - 6x_2 - 4x_3 \leq 1$
$2x_1 + 3x_2 - 5x_3 \geq 4$

and $x_1, x_2, x_3 \geq 0$

**Solution:**

By introducing the slack variable $s_1$,$s_2$ and surplus variable $s_3$. Artificial variable $R_1$. The standard form of the LPP becomes,

Max Z = $2x_1 + x_2 + x_3 + 0s_1 + 0s_2 + 0s_3 - MR_1$

Subject to the constraints,

$4x_1 + 6x_2 + 3x_3 + s_1 = 8$
$3x_1 - 6x_2 - 4x_3 + s_2 = 1$
$2x_1 + 3x_2 - 5x_3 - s_3 + R_1 = 4$

and $x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$

∴ The IBFS is $s_1 = 8, s_2 = 1, R_1 = 4$

**Initial Iteration:**

| | | $c_j$ | 2 | 1 | 1 | 0 | 0 | 0 | -M | Ratio $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $R_1$ | |
| 0 | $s_1$ | 8 | 4 | 6 | 3 | 1 | 0 | 0 | 0 | $8/6 = 4/3$ |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | $s_2$ | 1 | 3 | -6 | -4 | 0 | 1 | 0 | 0 | --- |
| ← | -M | $R_1$ | 4 | 2 | **3** | -5 | 0 | 0 | -1 | 1 | $^4/_3$ |
| | | $Z_j - C_j$ | -4M | -2M-2 | -3M-1 | 5M-1 | 0 | 0 | M | 0 | |
| | | | | | ↑ | | | | | | |

$R_1$ is leaving variable, $x_2$ is entering variable, 3 is pivot element

**I Iteration:**

| | | | $c_j$ | 2 | 1 | 1 | 0 | 0 | 0 | Ratio $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | |
| ← | 0 | $s_1$ | 0 | 0 | 0 | **13** | 1 | 0 | 2 | 0 |
| | 0 | $s_2$ | 9 | 7 | 0 | -14 | 0 | 1 | -2 | ---- |
| | 1 | $x_2$ | $^4/_3$ | $^2/_3$ | 1 | $-^5/_3$ | 0 | 0 | $-^1/_3$ | ---- |
| | | $Z_j - C_j$ | $^4/_3$ | $-^4/_3$ | 0 | $-^8/_3$ | 0 | 0 | $-^1/_3$ | |
| | | | | | | ↑ | | | | |

$s_1$ is leaving variable, $x_3$ is entering variable, 13 is pivot element

**II iteration:**

| | | | $c_j$ | 2 | 1 | 1 | 0 | 0 | 0 | Ratio $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | |
| | 1 | $x_3$ | 0 | 0 | 0 | 1 | $^1/_{13}$ | 0 | $^2/_{13}$ | ---- |
| ← | 0 | $s_2$ | 9 | **7** | 0 | 0 | $^{14}/_{13}$ | 1 | $^2/_{13}$ | $^9/_7$ |
| | 1 | $x_2$ | $^4/_3$ | $^2/_3$ | 1 | 0 | $^5/_{39}$ | 0 | $-^1/_{13}$ | 2 |
| | | $Z_j - C_j$ | $^4/_3$ | $-^4/_3$ | 0 | 0 | $^8/_{39}$ | 0 | $^1/_{13}$ | |
| | | | | ↑ | | | | | | |

$s_2$ is leaving variable, $x_1$ is entering variable, 7 is pivot element

**III Iteration:**

| | | $c_j$ | 2 | 1 | 1 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|
| $c_B$ | $Y_B$ | $X_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | $x_3$ | 0 | 0 | 0 | 0 | $1/13$ | 0 | $2/13$ |
| 2 | $x_1$ | $9/7$ | 1 | 0 | 0 | $2/13$ | $1/7$ | $2/91$ |
| 1 | $x_2$ | $10/21$ | 0 | 1 | 0 | $1/39$ | $2/21$ | $-17/273$ |
| | $Z_j - C_j$ | $64/21$ | 0 | 0 | 0 | $16/39$ | $8/21$ | $37/273$ |

∴ The optimal solution is Max Z = $64/21$ , $x_1 = 9/7$, $x_2 = 10/21$, $x_3 = 0$

## Possible Questions Part B (2 Marks)

1. Give a definition for operation Research.
2. Define initial basic feasible solution.
3. Define artificial variables.
4. Define surplus variables.
5. Define slack variable.
6. Write any two applications of OR.

## Possible Questions Part C (5 Marks)

1. A pine apple firm produces two products canned pineapple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below:

| | Canned Juice | Canned Pineapple | Available resources |
|---|---|---|---|
| Labour (Man hours) | 3 | 2 | 12 |
| Equipment (M/c hours) | 1 | 2.3 | 6.9 |
| Material (Unit) | 1 | 1.4 | 4.9 |

Assuming one unit of canned juice and canned Pineapple has profit margins Rs.2 and Rs. 1 respectively. Formulate this as a L.P.P and solve it graphically also.

2. Use two phase simplex method tosolve the following LPP

Max $Z=5x_1 + 3x_2$

Subject to

$2x_1+ x_2 \leq 1$

$x_1 + 4x_2 \geq 6$

$x_1 \geq 0$ , $x_2 \geq 0$

3. A manufacturer produces 3 models (I, II and III) of a certain product. He uses 2 raw materials A and B of which 4000 and 6000 units respectively are available. The raw materials per unit of 3 models are given below.

| Raw materials | I | II | III |
|---|---|---|---|
| A | 2 | 3 | 5 |

| | | | |
|---|---|---|---|
| B | 4 | 2 | 7 |

4. Thelabour time for each unit of model I is twice that of model II and thrice that of model III. The entire labour force of factory can produce an equivalent of 2500 units of model I. A model survey indicates that the minimum demand of 3 models is 500, 500 and 375 units respectively. However the ratio of number of units produced must be equal to 3:2:5. Assume that profits per unit of model are 60, 40 and 100 respectively. Formulate a LPP.
5. Use graphical method to solve the following LPP

   Max $Z = 80x_1 + 55x_2$

   Subject to

   $4x_1 + 2x_2 \leq 40$

   $2x_1 + 4x_2 \leq 32$

   $x_1 \geq 0$ , $x_2 \geq 0$

6. A company produces 2 types of hats. Each hat of the I type requires twice as much as labour time as the II type. The company can produce a total of 500 hats a day. The market limits daily sales of I and II types to 150 and 250 hats. Assuming that the profit per hat are Rs.8 for type A and Rs. 5 for type B. Formulate a LPP models in order to determine the number of hats to be produced of each type so as to maximize the profit.
7. Use graphical method to solve the following LPP

   Maximize $Z = 5x_1 + 4x_2$

   Subject to

   $6x_1 + 4x_2 \leq 24$

   $x_1 + 2x_2 \leq 6$

   $x_2 - x_1 \leq 1$

   $x_2 \leq 2$

   $x_1 \geq 0, x_2 \geq 0$
8. A company owns 2 oil mills A and B which have different production capacities for low, high and medium grade oil. The company enters into a contract to supply oil to a firm every week with 12, 8, 24 barrels of each grade respectively. It costs the company Rs 1000 and Rs 800 per day to run the mills A and B. On a day A produces 6, 2, 4 barrels of each grade and B produces 2, 2, 12 barrels of each grade. Formulate an LPP.

## Possible Questions Part D (10 Marks)

1. Solve the following LPP by Big-M method $Maximize\ Z = 3x_1 + 2x_2$

   $subject\ \ to \quad 2x_1 + x_2 \leq 2$

   $3x_1 + 4x_2 \geq 12$

   $with\ \ x_1, x_2 \geq 0.$
2. Use simplex method to solve the following LPP

Maximize $Z = 2x_1 + 4x_2 + 3x_3$

Subject to

$3x_1 + 4x_2 + 3x_3 \leq 60$

$2x_1 + 1x_2 + 3x_3 \leq 40$

$x_1 + 3x_2 + 3x_3 \leq 80$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

3. Maximize $Z = 5x_1 + 8x_2$

Subject to constraints, $15x_1 + 10x_2 \leq 180$

$10x_1 + 20x_2 \leq 200$

$15x_1 + 20x_2 \leq 210$

and $x_1, x_2 \geq 0$

4. A firm manufactures two types of product A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines $M_1$and $M_2$. Type A requires 1 minute of processing time on $M_1$ and 2 minutes on $M_2$. Type B requires 1 minute on $M_1$ and 1 minute on $M_2$. Machine $M_1$ is available for not more than 6 hours 40 minutes, while machines $M_2$ is a available for 10 hours during any work hours. Formulate the problem as LPP so as to maximize the profit.

**UNIT II**

Transportation Problems (TP) – Initial basic feasible solution to Transportation Cost – Northwest corner rule, Least cost method – Vogel's approximation method, Optimal solution using Modified Distribution (MODI) method, Degeneracy in TP, Unbalanced TP, Alternative optimal solutions, Maximization in TP – Assignment Problems – Hungarian method of solving assignment problem, Multiple optimum solutions, Maximisation in Assignment Problems, Unbalanced Assignment Problems, Restrictions in Assignment Problems.

# TRANSPORTATION MODEL

## Introduction

Transportation deals with the transportation of a commodity (single product) from 'm' sources (origins or supply or capacity centers) to 'n' destinations (sinks or demand or requirement centers). It is assumed that

(i) Level of supply at each source and the amount of demand at each destination and
(ii) The unit transportation cost of transportation is linear.

It is also assumed that the cost of transportation is linear.

The objective is to determine the amount to be shifted from each sources to each destination such that the total transportation cost is minimum.

**Note:** The transportation model also can be modified to Account for multiple commodities.

1. **Mathematical Formulation of a Transportation problem:**
Let us assume that there are m sources and n destinations.
Let $a_i$ be the supply (capacity) at source i, $b_j$ be the demand at destination j, $c_{ij}$ be the unit transportation cost from source *i* to destination j and $x_{ij}$ be the number of units shifted from sources *i* to destination *j*.
Then the transportation problems can be expressed mathematically as

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to the constraints

$\sum_{j=1}^{n} x_{ij} = a_i,$ $i = 1, 2, 3, \ldots\ldots m$

$\sum_{i=1}^{m} x_{ij} = b_j,$ $j = 1, 2, 3, \ldots\ldots N.$

And $x_{ij} \geq 0$, for all *i* and *j*.

***Note 1:*** The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

*(total supply)* *(total demand)*

**Which is the necessary and sufficient condition for a transportation problems to have a feasible solution**. Problems satisfying this condition are **balanced transportation problems.**

**Note 2:** If $\sum a_i \neq \sum b_j$

**Note 3:** For any transportation problems, the coefficient of all $x_{ij}$ in the constraints are unity.

**Note 4:** The objective function and the constraints being all linear, the transportation problems is a special class of linear programming problem. Therefore it can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we can look for some other technique which would be simpler that the usual simplex method.

## Standard transportation table:

Transportation problem is explicitly represented by the following transportation table.

**Destination**

| | | $D_1$ | $D_1$ | $D_1$ | ..... | $D_1$ | ..... | $D_1$ | supply |
|---|---|---|---|---|---|---|---|---|---|
| | $S_1$ | $C_{11}$ | $C_{12}$ | $C_{13}$ | | $C_{1j}$ | | $C_{1n}$ | $a_1$ |
| Source | $S_1$ | $C_{21}$ | $C_{22}$ | $C_{23}$ | | $C_{2j}$ | | $C_{2n}$ | $a_2$ |
| | | | | | | | | | . . |
| | $S_1$ | $C_{i1}$ | $C_{i2}$ | | | $C_{ij}$ | | $C_{in}$ | . |
| | $S_1$ | $C_{m1}$ | $C_{m2}$ | | | $C_{mj}$ | | $C_{mn}$ | $a_n$ |
| Demand | | $b_1$ | $b_2$ | $b_3$ | ..... | ..... | ..... | $b_n$ | $\sum a_i = \sum b_j$ |

The *mn* squares are called **cells.** The unit transportation cost $c_{ij}$ from the $i^{th}$ source to the $j^{th}$ destination is displayed in the **upper left side of the (i,j)$^{th}$ cell.** Any feasible solution us shown in the table by entering the value of $x_{ij}$ **in the center of the** $(i,j)^{th}$ cell. The various a's and b's are called **rim requirements.** The feasibility of a solution can be verified by summing the values if $x_{ij}$ along the rows and down the columns.

**Definition 1:** A set of non-negative values $x_{ij}$, i=1,2……………m; j=1,2………..n. that satisfies the constrains (rim conditions and also the non-negativity restrictions) is called a **feasible solution** to the transportation problems.

**Note:** A balanced transportation problems will always have a feasible solution.

**Definition 2:** A feasible solution to a (m x n) transportation problems that contains no more than m + n-1 non-negative allocations is called a **basic feasible solution** (BFS) to the transportation problem.

**Definition 3:** A basic feasible solution to a (m x n) transportation problem is said to be a **non-degenerate basic feasible solution** if it contains exactly m + n-1 non-negative allocations in independent positions.

**Definition 4:** A basic feasible solution that contains less than m + n -1 non-negative allocations is said to be a degenerate basic feasible solution.

**Definition 5:** A feasible solution (not necessarily basic) is said to be an **optimal solution** if it minimize is atmost m + n -1.

**Note:** The number of non-basic variables in an m x n balanced transportation problem is almost m + n -1.

**Note:** The number of non-basic variables in an m x n balanced transportation problem is atleast mn – (m + n -1).

## II. Methods for finding initial basic feasible solution

**The transportation problems has a solution is and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given transportation problem is balanced.** If not once has to balance the transportation problems first. The way to doing this is discussed in section 7.4 page 7.40. In this section all the given transportation problems are balanced.

**Method I: North west corner rule:**

**Step I:** The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is $x_{11}$ = min {$a_1$, b1}.

**Case (i):** If min {$a_1$, b1} =$a_1$, then put $x_{11}$=$a_1$, decrease $b_1$ by $a_1$ and move vertically to the 2$^{nd}$ row (i.e.,) to the cell (2, 1) cross out the first row.

**Case (ii):** If min {$a_1$, b1} =$b_1$, then put $x_{11}$=$b_1$, decrease $a_1$ by $b_1$ and move horizontally right (i.e.,) to the cell (2, 1) cross out the first column.

**Case (iii):** If min {$a_1$, b1} = $a_1$ =$b_1$, the put $x_{11}$ = $a_1$=$b_1$ and move diagonally to the cell (2, 2) cross out the first row and the first column.

**Step 2:** Repeat the procedure until all the rim requirements are satisfied.

**Method 2: Lest cost method (or) Matrix minima method (or) Lowest cost entry method:**

**Step 1:** Identify the cell with smallest cost and allocate $x_{ij}$ = Min $\{a_i,b_j\}$

**Case (i):** If min $\{a_i,b_j\}= a_i$, then put $x_{ij}$ = a, cross out the ith row and decrease $b_j$ by $a_i$, go to step(2).

**Case (ii):** If min $\{a_i,b_j\}= b_j$, then put $x_{ij} = b_j$, cross out the jth column and decrease $a_j$ by $b_j$, go to step(2).

**Case (iii):** If min $\{a_i,b_j\}=a_i= b_j$, then put $x_{ij}=a_i = b_j$, cross out either ith row and jth column but not both, go to step(2).

**Step 2:** Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

**Method 3: Vogel's approximation method (VAM) (or) Unit cost penalty method:**

**Step 1:** Find the difference (penalty) between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

**Step 2:** Identify the row (or) column with large penalty. If a tie occurs, break the tie arbitrarily. Choose the sell with smallest cost in that selected row or column and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3).

**Step 3:** Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

**Example 1: Determine basic feasible solution to the following transportation problems using North West Corner Rule:**

| | | Sink | | | | | |
|---|---|---|---|---|---|---|---|
| | | **A** | **B** | **C** | **D** | **E** | **Supply** |
| **Origin** | **P** | **2** | **11** | **10** | **3** | **7** | **4** |
| | **Q** | **1** | **4** | **7** | **2** | **1** | **8** |
| | **R** | **3** | **9** | **4** | **8** | **12** | **9** |
| **Demand** | | **3** | **3** | **4** | **5** | **6** | |

**Solution:** Since $a_i = b_j$ = 21, the given problem is balanced. ∴ There exists a feasible solution to the transportation problem.

| | | | | | |
|---|---|---|---|---|---|
| **2** **3** | 11 | 10 | 3 | 7 | 4 |

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 7 | 2 | 1 | 8 |
| 3 | 9 | 4 | 8 | 12 | 9 |
| 3 | 3 | 4 | 5 | 6 | |

Following North West Corner rule, the first allocation is made in the cell(1,1)

Here $x_{11}$ = min $\{a_1, b_1\}$ = min {4,3} = 3

Allocate 3 to the cell(1,1) and decrease 4 by 3 i.e., 4 – 3 = 1

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

| | | | | |
|---|---|---|---|---|
| **11** **1** | **10** | 3 | 7 | 1 |
| 4 | 7 | 2 | 1 | 8 |
| 9 | 4 | 8 | 12 | 9 |
| 3 | 4 | 5 | 6 | |

Here the North West Corner cell is (1,2).

So allocate $x_{11}$= min {1, 3} = 1 to the cell (1,2) and move vertically to cell (2, 2). The resulting transportation table is

| | | | | |
|---|---|---|---|---|
| **4** **2** | 7 | 2 | 1 | 8 |
| **9** | 4 | 8 | 12 | 9 |
| 2 | 4 | 5 | 6 | |

Allocate $x_{22}$= min {8, 2} = 2 to the cell (2, 2) and move horizontally to cell (2, 3). The resulting transportation table is

| | | | |
|---|---|---|---|
| **7** **4** | 2 | 1 | 6 |
| **4** | 8 | 12 | 9 |

4 5 6

Allocate $x_{23}$= min {6, 4} = 4 and move horizontally to cell (2, 4). The resulting reduced transportation table is

| | | |
|---|---|---|
| **2** **2** | **1** | 2 |
| 8 | 12 | 9 |
| 5 | 6 | |

Allocate $x_{24}$= min {2, 5} = 2 and move vertically to cell (3, 4). The resulting reduced transportation table is

| | | |
|---|---|---|
| **8** **3** | **12** | 9 |
| 3 | 6 | |

Allocate $x_{34}$= min {9, 3} = 3 and move horizontally to cell (3, 5).which is

| | |
|---|---|
| 12 **6** | 6 |
| 6 | |

Allocate $x_{35}$= min {6, 6} = 6

Finally the initial basic feasible solution is as shown in the following table.

| | | | | |
|---|---|---|---|---|
| 2 **3** | 11 **1** | 10 | 3 | 7 |
| 1 | 4 **2** | 7 **4** | 2 **2** | 1 |
| 3 | 9 | 4 | 8 **3** | 12 **6** |

From this table we see that the number of positive independent allocations is equal to

$m + n - 1 = 3 + 5 - 1 = 7$. This ensures that the solution is non degenerate basic feasible.

∴The initial transportation cost = Rs. 2 x 3 + 11 x 1 + 4 x 2 + 7 x 4 + 2 x 2 + 8 x 3 + 12 x 6

= Rs. 153/-

**Example 2:**

Find the initial basic feasible solution for the following transportation problem by Least Cost Method.

| | To | | | | Supply |
|---|---|---|---|---|---|
| From | 1 | 2 | 1 | 4 | 30 |
| | 3 | 3 | 2 | 1 | 50 |
| | 4 | 2 | 5 | 9 | 20 |
| Demand | 20 | 40 | 30 | 10 | |

**Solution:**

Since $\sum a_i = \sum b_j$ = 100, the given TPP is balanced. There exists a feasible solution to the transportation problem.

| | | | | |
|---|---|---|---|---|
| **1** **20** | 2 | 1 | 4 | 30 |
| **3** | 3 | 2 | 1 | 50 |
| **4** | 2 | 5 | 9 | 20 |
| 20 | 40 | 30 | 10 | |

By least cost method, min $c_{ij} = c_{11} = c_{13} = c_{24} = 1$

Since more than one cell having the same minimum $c_{ij}$, break the tie.

Let us choose the cell (1,1) and allocate $x_{11}$= min $\{a_1, b_1\}$ = min {30,20} = 20 and cross out the satisfied column and decrease 30 by 20.

The resulting reduced transportation table is

| | | | |
|---|---|---|---|
| 2 | 1 **10** | 4 | 10 |

| | | | |
|---|---|---|---|
| 3 | 2 | 1 | 50 |
| 2 | 5 | 9 | 20 |
| 40 | 30 | 10 | |

Here min $c_{ij} = c_{13} = c_{24} = 1$.Choose the cell (1,3) and allocate $x_{13}=$ min $\{a_1, b_3\}$ = min {10,30} = 10 and cross out the satisfied row.

The resulting reduced transportation table is

| | | | |
|---|---|---|---|
| 3 | 2 | 1 **10** | 50 |
| 2 | 5 | 9 | 20 |
| 40 | 20 | 10 | |

Here min $c_{ij} = c_{24} = 1$

∴ Allocate $x_{24}=$ min $\{a_2, b_4\}$ = min (50,10) = 10 and cross out the satisfied column.

The resulting transportation is

| | | |
|---|---|---|
| 3 | **2** **20** | 40 |
| 2 | **5** | 20 |
| 40 | 20 | |

Here $c_{ij} = c_{23} = c_{32} = 2$. Choose the cell (2,3) and allocate $x_{23}=$ min $\{a_2, b_3\}$=

min (40,20) = 10 and cross out the satisfied column.

The resulting reduced transportation table is

| | |
|---|---|
| 3 | 20 |
| 2 20 | 20 |
| 40 | |

Here min = $c_{ij}= c_{32} = 2$. Choose the cell (3,2) and allocate $x_{32}=$ min $\{a_3, b_2\}$=

min (20,40) = 20 and cross out the satisfied row.

The resulting reduced transportation table is

| | |
|---|---|
| 3<br>**20** | 20 |
| 20 | |

Finally the initial basic feasible solution is as shown in the following table.

| | | | |
|---|---|---|---|
| 1<br>**20** | 2 | 1<br>**10** | 4 |
| 3 | 3<br>**20** | 2<br>**20** | 1<br>**10** |
| 4 | 2<br>**20** | 5 | 9 |

From this table we see that the number of positive independent allocations is equal to

$m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

∴ The initial transportation Cost = Rs. 1 x 20 + 1 x 10 + 3 x 20 + 2 x 20 +1 x 10 + 2 x 20

= Rs. 20 + 10 + 60 + 40 + 10 + 40

= Rs. 180/-

## Example 3:

Find the starting solution of the following transportation model

| | | | |
|---|---|---|---|
| 1 | 2 | 6 | 7 |
| 0 | 4 | 2 | 12 |
| 3 | 1 | 5 | 11 |
| 10 | 10 | 10 | |

Using ***(i). North West Corner rule***

***(ii). Least Cost method***

***(iii). Vogel's approximation method.***

## Solution:

Since $\sum a_i = \sum b_j = 100$, the given Transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

**(i). North West Corner rule:** Using this method, the allocation are shown in the tables below:

| | | | |
|---|---|---|---|
| **1** **7** | **2** | **6** | 7 |
| 0 | 4 | 2 | 12 |
| 3 | 1 | 5 | 11 |
| 10 | 10 | 10 | |

***(i)***

| | | | |
|---|---|---|---|
| **0** **3** | 4 | 2 | 12 |
| **3** | 1 | 5 | 11 |
| 3 | 10 | 10 | |

***(ii)***

| | | |
|---|---|---|
| **4** **8** | **2** | 9 |
| 1 | 5 | 11 |
| 10 | 10 | |

***(iii)***

| | | |
|---|---|---|
| 1 **1** | 5 | 11 |
| 1 | 10 | |

***(iv)***

| | |
|---|---|
| 5 10 | 10 |
| 10 | |

***(v)***

The starting solution is as shown in the following table

| | | |
|---|---|---|
| 1 **7** | 2 | 6 |

| | | |
|---|---|---|
| 0 **3** | 4 **9** | 2 |
| 3 | 1 **1** | 5 **10** |

∴ The initial transportation cost = Rs. 1 x 7 + 0 x 3 + 4 x 9 + 1 x 1 + 5 x 10

= Rs. 94/-

**(ii).** **Least Cost method:** Using this method, the allocation are as shown in the table below:

| | | | |
|---|---|---|---|
| 1 | 2 | 6 7 | 7 |
| 0 10 | 4 | 2 | 12 |
| 3 | 1 | 5 | 11 |
| 10 | 10 | 10 | |

***(i)***

| | | |
|---|---|---|
| 2 | 6 | 7 |
| 4 | 2 | 2 |
| 1 10 | 5 | 11 |
| 10 | 10 | |

***(ii)***

| | |
|---|---|
| 6 | 7 |
| 2 2 | 2 |
| 5 | 1 |
| 10 | |

***(iii)***

| | |
|---|---|
| 6 | 7 |
| 5 1 | 1 |
| 8 | |

| | |
|---|---|
| 6<br>7 | 7 |
| 7 | |

*(iv)* *(v)*

The starting solution is as shown in the following table:

| | | |
|---|---|---|
| 1 | 2 | 6 **7** |
| 0 **10** | 4 | 2 **2** |
| 3 | 1 **10** | 5 **1** |

∴ The initial transportation cost = Rs. 6 x 7 + 0 x 10 + 2 x 2 + 1 x 10 + 5 x 1

= Rs. 61/-

**(iii). Vogel's approximation Method:** Using this method, the allocations are shown in the table below:

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 6 | 7 | (1) |
| 0 | 4 | 2 10 | 12 | (2) |
| 3 | 1 | 5 | 11 | (2) |
| 10 | 10 | 10 | | |
| (1) | (1) | **(3)** | | |

*(i)*

| | | | |
|---|---|---|---|
| 1 | 2 | 7 | (1) |
| 0 2 | 4 | 2 | **(4)** |
| 3 | 1 | 11 | (2) |
| 10 | 10 | | |
| (1) | (1) | | |

*(ii)*

| | | | |
|---|---|---|---|
| 1 | 2 | 7 | (1) |
| 3 | 1 10 | 11 | **(2)** |

8 10
(2) (1)

***(iii)***

| | |
|---|---|
| 1 7 | 7 |
| 3 | 1 |
| 8 | |

***(iv)***

| | |
|---|---|
| 3 1 | 1 |
| 1 | |

***(v)***

The starting solution is as shown in the following table:

| | | |
|---|---|---|
| 1 **7** | 2 | 6 |
| 0 **2** | 4 | 2 **10** |
| 3 **1** | 1 **10** | 5 |

∴ The initial transportation cost = Rs.1 x 7 + 0 x 2 + 2 x 10 + 3 x 1 + 1 x 10

= Rs. 40/-

**Note:** For the above problem, the number of positive allocation in independent positions is (m + n -1) (i.e., m + n – 1 = 3 + 3 – 1 = 5). This ensures that the given problem has a non-degenerate basic feasible solution by using all the three methods. This need not be the case in all the problems.

## Transportation Algorithm (or) MODI Method (modified distribution method) (Test for optimal solution).

**Step 1:** Find the initial basic feasible solution of the given problems by Northwest Corner rule (or) Least Cost method or VAM.

**Step 2:** Check the number of occupied cells. If these are less than m + n – 1, there exists degeneracy and we introduce a very small positive assignment of $\in$ ($\approx$ 0) in suitable independent positions, so that the number of occupied cells is exactly equal to m + n – 1.

**Step 3:** Find the set of values $u_i$, $v_j$ (i= 1,2,3,….m; j=1,2,3…n) from the relation $c_{ij} = u_i + v_j$ for each occupied cell (i,j), by starting initially with $u_i$ =0 or $v_j$ =0 preferably for which the corresponding row or column has maximum number of individual allocations.

**Step 4:** Find $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding cell (i,j).

**Step 5:** Find the cell evaluations $d_{ij} = c_{ij} – (u_i + v_j)$ ($d_{ij}$ = upper left – upper right) for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding cell (i,j).

**Step 6:** Examine the cell evolutions $d_{ij}$ for all unoccupied cells (i,j) and conclude that

(i) If all $d_{ij}$>0, then the solution under the test is optimal and unique.
(ii) If all $d_{ij}$>0, with atleast one $d_{ij}$=0, then the solution under the test is optimal and an alternative optimal solution exists.
(iii) If atleast one $d_{ij}$<0, then the solution is not optimal. Go to the next step.

**Step 7:** Form a new B>F>S by giving maximum allocation to the cell for which $d_{ij}$is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal and vertical lines beginning and ending at the cell for which $d_{ij}$ is most negative and having its **other corners at some allocated cells.** Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. Choose minimum of the allocations from the cells having $-\theta$. Add this minimum allocation to the cells with $+\theta$ and subtract this minimum allocation from the allocation to the cells with $-\theta$.

**Step 8:** Repeat steps (2) to (6) to test the optimality of this new basic feasible solution.

**Step 9:** Continue the above procedure till an optimum solution is attained.

**Note:** The Vogels approximation method (VAM) takes into account not only the least cost $c_{ij}$ but also the costs that just exceed the least cost $c_{ij}$ and therefore yields better initial solution than obtained from other methods in general. This can be justified by the above example (4). So to find the initial solution, give preference to VAM unless otherwise specified.

**Example 1: Solve the transportation problem:**

| | **1** | **2** | **3** | **4** | **Supply** |
|---|---|---|---|---|---|
| **I** | **21** | **16** | **25** | **13** | **11** |
| **II** | **17** | **18** | **14** | **23** | **13** |
| **III** | **32** | **27** | **18** | **41** | **19** |
| **Demand** | **6** | **10** | **12** | **15** | |

**Solution:** Since $\sum a_i = \sum b_j$ =43, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is an shown in the following table.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 21 | 16 | 25 | 13 **11** | (3) | - | - | - |
| 17 **6** | 18 **3** | 14 | 23 **4** | (3) | (3) | (3) | (3) |
| 32 | 27 **7** | 18 **12** | 41 | (9) | (9) | (9) | (9) |
| (4) | (2) | (4) | **(10)** | | | | |
| (15) | (9) | (4) | **(18)** | | | | |
| **(15)** | (9) | (4) | | | | | |
| | **(9)** | (4) | | | | | |

That is

| | | | |
|---|---|---|---|
| 21 | 16 | 25 | 13 **11** |
| 17 **6** | 18 **3** | 14 | 23 **4** |
| 32 | 27 **7** | 18 **12** | 41 |

From this table, we see that the number of non-negative independent allocations is (m + n – 1) = (3+4-1) =6. Hence the solution is non-degenerate basic feasible.

∴ The initial transportation cost.

= Rs. 13 x 11 + 17 x 6 +18 x 3 +23 x 4 + 27 x 7 + 18 x 12

= Rs. 796/-

**To find the optimal solution**

Consider the above transportation table. Since m+n-1=6, we apply MODI method,

Now we determine a set of values $u_i$ and $v_j$ for each occupied cell (i,j) by using the relation $c_{ij} = u_i + v_j$. As the 2nd row contains maximum number of allocations, we choose $u_2=0$.

Therefore

$C_{21} = u_2 + v_1 \Rightarrow 17 = 0 + v_1 \Rightarrow v_1 = 17$

$C_{22} = u_2 + v_2 \Rightarrow 18 = 0 + v_2 \Rightarrow v_2 = 18$

$C_{24} = u_2 + v_4 \Rightarrow 23 = 0 + v_4 \Rightarrow v_4 = 23$

$C_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = -10$

$C_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 9$

$C_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$

Thus we have the following transportation table:

| | | | | |
|---|---|---|---|---|
| 21 | 16 | 25 | 13 **11** | $u_1 = -10$ |
| 17 **6** | 18 **3** | 14 | 23 **4** | $u_2 = 0$ |
| 32 | 27 **7** | 18 **12** | 41 | $u_3 = 9$ |
| $v_1 = 17$ | $v_2 = 18$ | $v_3 = 9$ | $v_4 = 23$ | |

Now we find $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding unoccupied cell(i,j).

Then we find the cell evalutions $d_{ij} = c_{ij} - (u_i + v_j)$ (ie., upper left corner – upper right corner) for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding unoccupied cell (i,j).

| | | | | |
|---|---|---|---|---|
| 21 7 / 14 | 16 8 / 8 | 25 -1 / 26 | 13 **11** | $u_1 = -10$ |
| 17 **6** | 18 **3** | 14 9 / 5 | 23 **4** | $u_2 = 0$ |
| 32 26 / 6 | 27 **7** | 18 **12** | 41 32 / 9 | $u_3 = 9$ |
| $v_1 = 17$ | $v_2 = 18$ | $v_3 = 9$ | $v_4 = 23$ | |

Since all $d_{ij}>0$, with $d_{32} = 0$, the current solution is optimal and unique.

∴ The optimum allocation schedule is given by $x_{14} = 11$, $x_{21} = 6$, $x_{22} =3$, $x_{24} =4$, $x_{32} =7$, $x_{33} =12$, and the optimum (minimum) transportation cost

= Rs. 13x 11 + 17 x 6 + 18 x 3 +23 x 4 + 27 x 7 + 18 x 12

= Rs. 796/-

**Example 2:**

Obtain on optimum has feasible solution to the following transportation problem:

| | To | | | Available |
|---|---|---|---|---|
| From | 7 | 3 | 2 | 2 |
| | 2 | 1 | 3 | 3 |
| | 3 | 4 | 6 | 5 |
| Demand | 4 | 1 | 5 | 10 |

**Solution:**

Since $\sum a_i = \sum b_j$ =43, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

| | | | |
|---|---|---|---|
| 7 | 3 | 2 **2** | **(1)** **(5)** |
| 2 | 1 **1** | 3 **2** | (1) (1) (1) |
| 3 **4** | 4 | 6 **1** | (1) (3) **(3)** |
| (1) | **(2)** | (1) | |
| (1) | | (1) | |
| (1) | | (3) | |

That is

| | | |
|---|---|---|
| 7 | 3 | 2 **2** |
| 2 | 1 **1** | 3 **2** |
| 3 **4** | 4 | 6 **1** |

From this table we see that the number of non-negative allocation is $m + n - 1 = (3 + 3 - 1) = 5$.

Hence the solution is non-degenerate basic feasible

∴ The initial transportation cost = Rs. 2 x 2 + 1 x 1 + 3 x 2 + 3 x 4 + 6 x 1 = Rs. 29/-

***For optimality:*** since the number of non – negative independent allocation is m + n -1, we apply MODI method.

Since the third column contains maximum number of allocations, we choose $v_3 = 0$.

Now we determine a set of values $u_i$ and $v_j$ by using the occupied cells and the relation $c_{ij} = u_i + v_j$.

That is

| | | | |
|---|---|---|---|
| 7 -1 | 3 0 | 2 **2** | $u_1 = 2$ |
| 2 | 1 **1** | 3 **2** | $u_2 = 3$ |

| | | | |
|---|---|---|---|
| 3 **4** | 4 | 6 **1** | $u_3 = 6$ |
| $v_1 = -3$ | $v_2 = -2$ | $v_3 = 0$ | |

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the corresponding unoccupied cell (I,j).

Then we find the cell evaluations $d_{ij} = c_{ij} – (u_i + v_j)$ for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j).

Thus we get the following table

| | | | |
|---|---|---|---|
| 7 -1 8 | 3 0 3 | 2 **2** | $u_1 = 2$ |
| 2 0 2 | 1 **1** | 3 **2** | $u_2 = 3$ |
| 3 **4** | 4 4 0 | 6 **1** | $u_3 = 6$ |
| $v_1 = -3$ | $v_2 = -2$ | $v_3 = 0$ | |

Since all $d_{ij}>0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

∴ The optimum allocation schedule is given by $x_{13} = 2$, $x_{32} = 1$, $x_{23} =2$, $x_{31} =4$, $x_{33} =1$, and the optimum (minimum) transportation cost

= Rs.2x2+1x1+3x2+3x4+6x1 = Rs.29/-

**Example 3:** Find the optimal transportation cost of the following matrix using least cost method for finding the critical solution.

| | | A | B | C | D | E | Available |
|---|---|---|---|---|---|---|---|
| | P | 4 | 1 | 2 | 6 | 9 | 100 |
| Factory | Q | 6 | 4 | 3 | 5 | 7 | 120 |

| | | | | | | |
|---|---|---|---|---|---|---|
| R | 5 | 2 | 6 | 4 | 8 | 120 |
| Demand | 40 | 50 | 70 | 90 | 90 | |

**Solution:**

Since $\sum a_i = \sum b_j$= 340, the given transportation problem is balanced.∴ There exists a basic feasible solution to this problem.

By using Least cost method, the initial solution is an shown in the following table:

| | | | | |
|---|---|---|---|---|
| 4 | 1 **50** | 2 **50** | 6 | 9 |
| 6 **10** | 4 | 3 **20** | 5 | 7 **90** |
| 5 **30** | 2 | 6 | 4 **90** | 8 |

∴ The initial transportation cost = Rs. 1 x 50 + 2 x 50 + 6 x 10 + 3 x 20 + 7 x 90 + 5 x 30 + 4 x 90

= Rs. 1410/-

***For optimality:*** Since the number of non – negative independent allocations is ( m + n -1), we apply MODI method:

| | | | | | |
|---|---|---|---|---|---|
| 4 5 -1 | 1 **50** | 2 **50** | 6 4 2 | 9 6 3 | $u_1$ = -1 |
| 6 **10** | 4 2 2 | 3 **20** | 5 5 0 | 7 **90** | $u_2$ = 0 |

| | | | | | |
|---|---|---|---|---|---|
| 5 <br> **30** | 2 1 <br> 1 | 6 2 <br> 4 | 4 <br> **90** | 8 6 <br> 2 | $u_3 = -1$ |
| $v_1 = 6$ | $v_2 = 2$ | $v_3 = 3$ | $v_4 = 5$ | $v_5 = 7$ | |

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (i,j) for which $d_{ij}$ is most negative by making an occupied cell empty. Here the cell (1,1) having the negative value $d_{11}$=-1. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,1) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $+\theta$ alternatively at the corners. We have

| | | | | |
|---|---|---|---|---|
| 4 <br> $+\theta$ | 1 <br> **50** | 2 <br> **50** $-\theta$ | 6 | 9 |
| 6 <br> **10** <br> $-\theta$ | 4 | 3 <br> **20** <br> $+\theta$ | 5 | 7 <br> **90** |
| 5 <br> **30** | 2 | 6 | 4 <br> **90** | 8 |

From the two cells (1,3), (2,1) having $+\theta$, we find that the minimum of the allocations 50,10 is 10. Add this cells with $+\theta$ and subtract this 10 to the cells with$+\theta$.

Hence the new basic feasible solution is displayed in the following table:

| | | | | |
|---|---|---|---|---|
| 4 <br> **10** | 1 <br> **50** | 2 <br> **40** | 6 | 9 |

| | | | | |
|---|---|---|---|---|
| 6 | 4 | 3<br>**30** | 5 | 7<br>**90** |
| 5<br>**30** | 2 | 6 | 4<br>**90** | 8 |

We see that the above table satisfies the rim conditions with (m + n – 1) non-negative allocations at independent position. So we apply MODI method.

| | | | | | |
|---|---|---|---|---|---|
| 4<br>**10** | 1<br>**50** | 2<br>**40** | 6 3<br>3 | 9 6<br>3 | $u_1 = 0$ |
| 6 5<br>1 | 4 2<br>2 | 3<br>**30** | 5 4<br>1 | 7<br>**90** | $u_2 = 1$ |
| 5<br>**30** | 2 2<br>0 | 6 3<br>3 | 4<br>**90** | 8 7<br>1 | $u_3 = 1$ |
| $v_1 = 4$ | $v_2 = 1$ | $v_3 = 2$ | $v_4 = 3$ | $v_5 = 6$ | |

Since all $d_{ij}>0$, with $d_{32}=0$, the current solution is optimal and there exists an alternative optimal solution.

The optimum allocation schedule is given by $x_{11}=10$, $x_{12}=50$, $x_{13}=40$, $x_{23}=30$, $x_{25}=90$, $x_{31}=30$, $x_{34}=90$ and the optimum (minimum) transportation cost.

= Rs. 4 x 10 + 1 x 50 + 2 x 40 + 3 x 30 + 7 x 90 + 5 x 30 + 4 x 90.

= Rs. 1400/-

**Degeneracy in Transportation Problems**

In transportation problems, whenever the number of non-negative independent allocations in less than m + n – 1, the transportation problems is said to be **degenerate** one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy, we allocate an extremely small amount (close to zero)to one or more empty cells of the transportation table(generally minimum cost cells if possible), so that the total number of occupied cells becomes (m + n – 1) at independent positions. WE denote this small amount by ∈ (epsilon) satisfying the following conditions:

(i) $0 < \in < x_{ij}$, for all $x_{ij} > 0$
(ii) $X_{ij} \pm \in = x_{ij,}$ for all $x_{ij} > 0$

The cells containing ∈ are then treated like other occupied cells and the problems is solved in the usual way. The∈'s are kept till the optimum solution is attained. Then we let each $\in \rightarrow 0$.

**Example 1: find the non-degenerate basic feasible solution for the following transportation problems using**

**(i) North west corner rule**
**(ii) Least cost method**
**(iii) Vogel's approximation method.**

| | To | | | | supply |
|---|---|---|---|---|---|
| | **10** | **20** | **5** | **7** | **10** |
| | **13** | **9** | **12** | **8** | **20** |
| **From** | **4** | **5** | **7** | **9** | **30** |
| | **14** | **7** | **1** | **0** | **40** |
| | **3** | **12** | **5** | **19** | **50** |
| **Demand** | **60** | **60** | **20** | **10** | |

**Solution:** Since $\sum a_i = \sum b_j$= 150, the given transportation problems is balanced.

∴ There exists a basic feasible solution to this problem.

**(i) The starting solution by NWC rule is an shown in the following table.**

| | | | |
|---|---|---|---|
| 10 **10** | 20 | 5 | 7 |

| | | | |
|---|---|---|---|
| 13 **20** | 9 | 12 | 8 |
| 4 **30** | 5 | 7 | 9 |
| 14 | 7 **40** | 1 | 0 |
| 3 | 12 **20** | 5 **20** | 19 **10** |

Since the number of non-negative allocations at independent positions is 7 which is less than (m + n – 1) = (5 + 4 – 1) = 8, this basic feasible solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity ∈ to the unoccupied cell (5,1) so that the number of occupied cells becomes (m+n-1) (m + n – 1). Hence the non-degenerate basic feasible solution is an shown in the following table.

| | | | |
|---|---|---|---|
| 10 **10** | 20 | 5 | 7 |
| 13 **20** | 9 | 12 | 8 |
| 4 **30** | 5 | 7 | 9 |
| 14 | 7 **40** | 1 | 0 |
| 3 ∈ | 12 **20** | 5 **20** | 19 **10** |

The initial transportation cost = Rs.

10x10+13x20+4x30+7x40+3x∈+20x20+5x20+19x10

= Rs.(1290 = 3∈) = Rs. 1290/- as ∈→ 0.

**(ii) Least cost method:** Using this method the starting solution is an shown in the following table:

| | | | |
|---|---|---|---|
| 10 | 20<br>**10** | 5 | 7 |
| 13 | 9<br>**20** | 12 | 8 |
| 4<br>**10** | 5<br>**20** | 7 | 9 |
| 14 | 7<br>**10** | 1<br>**20** | 0<br>**10** |
| 3<br>**50** | 12 | 5 | 19 |

Since the number of non-negative allocations at independent positions is (m + n – 1) =8, the solution is non-degenerate basic feasible.

The initial transportation cost = Rs.20x10+9x20+4x10+5x20+7x10+1x20+0x10+3x50

= Rs. 760/-

**(iii) Vogel's approximation method:** The starting solution by this method is an shown in the following table:

| | | | |
|---|---|---|---|
| 10<br>**10** | 20 | 5 | 7 |

| | | | |
|---|---|---|---|
| 13 | 9<br>**20** | 12 | 8 |
| 4 | 5<br>**30** | 7 | 9 |
| 14 | 7<br>**10** | 1<br>**20** | 0<br>**10** |
| 3<br>**50** | 12 | 5 | 19 |

Since the number of non-negative allocations is 7 which is less than (m + n – 1) = (5+4-1)=8., this basic solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity ∈ to the unoccupied cell(5,2) so that the number of occupied cells becomes (m + n – 1). Hence the non-degenerate basic feasible solution is an shown in the following table.

| | | | |
|---|---|---|---|
| 10<br>**10** | 20 | 5 | 7 |
| 13 | 9<br>**20** | 12 | 8 |
| 4 | 5<br>**30** | 7 | 9 |
| 14 | 7<br>**10** | 1<br>**20** | 0<br>**10** |
| 3<br>**50** | 12<br>∈ | 5 | 19 |

∴ The initial transportation cost

= Rs. 10 x 10 + 9 x 20 + 5 x 30 + 7 x 10 + 1 x 20 + 0 x 10 + 3 x 50 + 12 x ∈

= Rs. (670 +12∈)

= Rs.670/- = as ∈→0.

**Example 2: Solve the following transportation problems using vogel's method.**

| | | A | B | C | D | E | F | Available |
|---|---|---|---|---|---|---|---|---|
| | **1** | **9** | **12** | **9** | **6** | **9** | **10** | **5** |
| **Factory** | **2** | **7** | **3** | **7** | **7** | **5** | **5** | **6** |
| | **3** | **6** | **5** | **9** | **11** | **3** | **11** | **2** |
| | **4** | **6** | **8** | **11** | **2** | **2** | **10** | **9** |
| **Requirement** | | **4** | **4** | **6** | **2** | **4** | **2** | |

**Solution:** Since $\sum a_i = \sum b_j$= 22, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem. By Vogel's approximation method, the initial solution is as shown in the following table:

| | | | | | |
|---|---|---|---|---|---|
| 9 | 12 | 9<br>**5** | 6 | 9 | 10 |
| 7 | 3<br>**4** | 7 | 7 | 5 | 5<br>**2** |
| 6<br>**1** | 5<br>∈ | 9<br>**1** | 11 | 3 | 11 |
| 6<br>**3** | 8 | 11 | 2<br>**2** | 2<br>**4** | 10 |

Since the number of non-negative allocations is 8 which is less than (m + n – 1)=(4+6-1)=9, this basic solution is degenerate one.

To resolve degeneracy, we allocate a very small quantity ∈ to the cell (3,2), so that the number of occupied cells becomes (m + n – 1). Hence the non-degenerate basic feasible solution is as shown in the following table.

| | | | | | |
|---|---|---|---|---|---|
| 9 | 12 | 9 **5** | 6 | 9 | 10 |
| 7 | 3 **4** | 7 | 7 | 5 | 5 **2** |
| 6 **1** | 5 ∈ | 9 **1** | 11 | 3 | 11 |
| 6 **3** | 8 | 11 | 2 **2** | 2 **4** | 10 |

The initial transportation cost = Rs.9 x 5 + 3 x 4 + 5 x 2 + 6 x 1 + 5 x ∈ + 9 x 1

+ 6 x 3 + 2 x 2 + 2 x 4

=Rs.(112+5∈) = Rs.112/-, ∈→0.

**To find the optimal solution**

Now the number of non-negative allocations at independent positions is (m + n – 1). We apply the MODI method.

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 6 3 | 12 5 7 | 9 **5** | 6 2 4 | 9 2 7 | 10 7 3 | $u_1$ = 0 |
| 7 4 3 | 3 **4** | 7 7 0 | 7 0 7 | 5 0 5 | 5 **2** | $u_2$ = -2 |
| 6 **1** | 5 ∈ | 9 **1** | 11 2 9 | 3 2 1 | 11 7 4 | $u_3$ = 0 |
| 6 **3** | 8 5 3 | 11 9 2 | 2 **2** | 2 **4** | 10 7 3 | $u_4$ = 0 |

| | | | | | |
|---|---|---|---|---|---|
| $v_1 = 6$ | $v_2 = 5$ | $v_3 = 9$ | $v_4 = 2$ | $v_5 = 2$ | $v_6 = 7$ |

Since all $d_{ij}>0$ with $d_{23}=0$, the solution under the test is optimal and an alternative optimal solution is also exists.

∴ The optimum allocation schedule is given by $x_{14}=5$, $x_{22}=4, x_{26}=2, x_{31}=1, x_{32}=\epsilon, x_{33}=1, x_{41}=3, x_{44}=2, x_{45}=4$ and the optimum(minimum) transportation cost is

=Rs.9x5+3x4+5x2+6x1+5x∈+9x1+6x3+2x2+2x4

=Rs. (112+5∈)

=Rs. 112 as ∈→0.

**Example 3: Solve the following transportation problem to minimize the total cost of transportation.**

| | To | | | | Supply |
|---|---|---|---|---|---|
| | **1** | **2** | **3** | **4** | **6** |
| **From** | **4** | **3** | **2** | **0** | **8** |
| | **0** | **2** | **2** | **1** | **10** |
| **Demand** | **4** | **6** | **8** | **6** | |

**Solution:** Since $\sum a_i = \sum b_j = 24$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table:

| | | | |
|---|---|---|---|
| 1 | 2 **6** | 3 | 4 |
| 4 | 3 | 2 **2** | 0 **6** |
| 0 **4** | 2 | 2 **6** | 1 |
| | 4 | | |

Since the number of non-negative allocations is 5, which is less than $(m + n - 1)=(3+4-1)=6$, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (1,4), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given in the following table

| | | | |
|---|---|---|---|
| 1 | 2 **6** | 3 | 4 |
| 4 | 3 | 2 **2** | 0 **6** |
| 0 **4** | 2 $\in$ | 2 **6** | 1 |

$\therefore$ The initial transportation cost $=$ Rs. $2 \times 6 + 2 \times 2 + 0 \times 6 + 0 \times 4 + 2 \times \in + 2 \times 6$

$= \text{Rs. } (28 + 2\in)$

$= \text{Rs. } 28/\text{-, as } \in \longrightarrow 0.$

**To find the optimum solution:**

Now the number of non-negative allocations at independent positions is $(m + n - 1)$. We apply MODI method.

| | | | | |
|---|---|---|---|---|
| 1 0 1 | 2 **6** | 3 2 1 | 4 0 4 | $u_1 = 0$ |
| 4 0 4 | 3 2 1 | 2 **2** | 0 **6** | $u_2 = 0$ |
| 0 **4** | 2 $\in$ | 2 **6** | 1 0 1 | $u_3 = 0$ |
| $v_1 = 0$ | $v_2 = 2$ | $v_3 = 2$ | $v_4 = 0$ | |

Since all $d_{ij} > 0$ the solution under the test is optimal and unique.

∴ The optimal allocation schedule is given by $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = \in$, $x_{33} = 6$ and the optimum (minimum) transportation cost

= Rs. 2 x 6 + 2 x 2 + 0 x 6 + 0 x 4 + 2 x ∈ + 2 x 6

= Rs. (28 + 2∈) = Rs. 28, as $\in \longrightarrow 0$.

## Unbalanced Transportation Problems

If the given transportation problems is unbalanced one, i.e., if $\sum a_i \neq \sum b_j$, then convert this into a balanced one by introducing a dummy source or dummy destination with zero cost vector (zero unit transportation costs) as the case may be and then solve by usual method.

When the total supply if greater than the total demand, a dummy destination is included in the matrix with zero cost vectors. The excess supply is entered as a rim requirement for the dummy destination.

When the total demand is greater than the total supply, a dummy source is included in the matrix with zero cost vectors. The excess demand is entered as rim requirements for the dummy source.

**Example 1:** Solve the transportation problem

Destination

| | A | B | C | D | supply |
|---|---|---|---|---|---|
| 1 | 11 | 20 | 7 | 8 | 50 |
| Source 2 | 21 | 16 | 20 | 12 | 40 |
| 3 | 8 | 12 | 18 | 9 | 70 |
| Demand | 30 | 25 | 35 | 40 | |

**Solution:** Since the total supply ($\sum a_i$ =160) is greater than the total demand ($\sum b_j$ =130), the given problem is an unbalanced transportation problem. To convert this i9nto a balanced one, we introduce a dummy destination E with zero unit transportation costs and having demand equal to 160-130=30 units.

∴ The given problem becomes

Destination

| | A | B | C | D | E | supply |
|---|---|---|---|---|---|---|
| 1 | 11 | 20 | 7 | 8 | 0 | 50 |
| Source 2 | 21 | 16 | 20 | 12 | 0 | 40 |
| 3 | 8 | 12 | 18 | 9 | 0 | 70 |
| Demand | 30 | 25 | 35 | 40 | 30 | 160 |

By using VAM the initial solution is as shown in the following table

| | | | | |
|---|---|---|---|---|
| 11 | 20 | 7<br>**35** | 8<br>**15** | 0 |
| 21 | 16 | 20 | 12<br>**10** | 0<br>**30** |
| 8<br>**30** | 12<br>**25** | 18 | 9<br>**15** | 0 |

∴ The initial transportation cost

=Rs. 7 x 35 + 8 x 15 + 12 x 10 + 0 x 30 + 8 x 30 + 12 x 25 + 9 x 15

=Rs. 1160/-

**For Optimality:** Since the number non-negative allocations at independent position is (m + n – 1), we apply the MODI method.

| | | | | | |
|---|---|---|---|---|---|
| 11 7<br>4 | 20 11<br>9 | 7<br>**35** | 8<br>**15** | 0 -4<br>4 | $u_1$ = 8 |
| 21 11<br>10 | 16 15<br>1 | 20 11<br>9 | 12<br>**10** | 0<br>**30** | $u_2$ = 12 |
| 8<br>**30** | 12<br>**25** | 18 8<br>10 | 9<br>**15** | 0 -3<br>3 | $u_3$ = 9 |
| $v_1$ = -1 | $v_2$ = 3 | $v_3$ = -1 | $v_4$ = 0 | $v_5$ = -12 | |

Since all $d_{ij}>0$, the solution under the test is optimum and unique.

∴ The optimum allocation schedule is $x_{13}$ = 35, $x_{14}$ = 15, $x_{24}$ = 10, $x_{25}$ = 30, $x_{31}$ = 30, $x_{32}$ = 25, $x_{34}$ = 15

It can be noted that $x_{25}$=30 means that 30 units are dispatched from source 2 to the dummy destination E. In other words, 30 units are left undespatched from source 2.

The optimum (minimum) transportation cost

= Rs. 7 x 35 + 8 x 15 + 12 x 10 + 0 x 30 + 8 x 30 + 12 x 25 + 9 x 15

=Rs. 1160/-

**Example 2: Solve the transportation problem with unit transportation costs, demands and supplies as given below:**

Destination

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | 6 | 1 | 9 | 3 | 70 |
| Source $S_2$ | 11 | 5 | 2 | 8 | 55 |
| $S_3$ | 10 | 12 | 4 | 7 | 70 |
| Demand | 85 | 35 | 50 | 45 | |

**Solution:** Since the total demand ($\sum b_j$ =215) is greater than the total supply ($\sum a_i$ =195), the given problem is unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy source $S_4$ with zero unit transportation costs and having supply equal to 215-195=20 units. ∴ The given problems becomes

Destination

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | 6 | 1 | 9 | 3 | 70 |
| Source $S_2$ | 11 | 5 | 2 | 8 | 55 |
| $S_3$ | 10 | 12 | 4 | 7 | 70 |
| $S_4$ | 0 | 0 | 0 | 0 | 20 |
| Demand | 85 | 35 | 50 | 45 | 215 |

As this problem is balanced, there exists a basic feasible solution to this problem. By using Vogel's approximation method, the initial solution is as shown in the following table.

| | | | |
|---|---|---|---|
| 6 **65** | 1 **5** | 9 | 3 |
| 11 | 5 **30** | 2 **25** | 8 |
| 10 | 12 | 4 **25** | 7 **45** |
| 0 **20** | 0 | 0 | 0 |

∴ The initial transportation cost

= Rs. 6x65+1x5+5x30+2x25+4x25+7x45+0x20

=Rs.1010/-

**For optimality:** Since number of non-negative allocations at independent positions is (m + n – 1), we apply the MODI method.

| | | | | |
|---|---|---|---|---|
| 6 **65** | 1 **5** | 9 -2 11 | 3 1 2 | $u_1$ = 6 |
| 11 10 1 | 5 **30** | 2 **25** | 8 5 3 | $u_2$ = 10 |
| 10 12 -2 | 12 7 5 | 4 **25** | 7 **45** | $u_3$ = 12 |
| 0 **20** | 0 -5 5 | 0 -8 8 | 0 -5 5 | $u_4$ = 0 |
| $v_1$ = 0 | $v_2$ = -5 | $v_3$ = -8 | $v_4$ = -5 | |

Since $d_{31}$=-2<0, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum empty. For this, we draw a closed path consisting of horizontal and vertical lines beginning and ending at this cell (3,1) and having its other corners at some occupied cells. Along this closed loop, indicate +$\theta$ and -$\theta$ alternatively at the corners.

We have,

| | | | |
|---|---|---|---|
| 6 **65** -$\theta$ | 1 **5** +$\theta$ | 9 | 3 |
| 11 | 5 **30** | 2 **25** | 8 |

| | | | |
|---|---|---|---|
| | $-\theta$ | $+\theta$ | |
| 10 $+\theta$ | 12 | 4 **25** $-\theta$ | 7 **45** |
| 0 **20** | 0 | 0 | 0 |

From the three cells (1,1), (2,2), (3,3) having $-\theta$ , we find that the minimum of the allocations 65,30,25 is 25. Add this 25 to the cells with $+\theta$ and subtract this 25 to this cells with $-\theta$. Finally, the new feasible solution is displayed in the following table.

| | | | |
|---|---|---|---|
| 6 **40** | 1 **30** | 9 | 3 |
| 11 | 5 **5** | 2 **50** | 8 |
| 10 **25** | 12 | 4 | 7 **45** |
| 0 **20** | 0 | 0 | 0 |

We see that the above table satisfies the rim conditions with (m + n – 1) non-negative allocations at independent positions. Now we check for optimality.

| | | | |
|---|---|---|---|
| 6 **40** | 1 **30** | 9 -2 11 | 3 3 0 |
| 11 10 1 | 5 **5** | 2 **50** | 8 7 1 |
| 10 **25** | 12 5 7 | 4 2 2 | 7 **45** |

| | | | |
|---|---|---|---|
| 0 **20** | 0 -5 5 | 0 -8 8 | 0 -3 3 |

Since all $d_{ij}$>0 with $d_{14}$=0, the solution under the test is optimal and an alternative optimal solution exists.

∴ The optimum allocation schedule is given by $x_{13} = 35$, $x_{14} = 15$, $x_{24} = 10$, $x_{25} = 30$, $x_{31} = 30$, $x_{32} = 25$, $x_{34} = 15$, $x_{41} = 20$.

It can be noted that $x_{41}$=20 means that 20 units are dispatched from the dummy source $S_4$ to the destination$D_1$. In other words, 20 units are not fulfilled for the destination $D_1$.

The optimum (minimum) transportation cost

= Rs. 6 x 40 + 1 x 30 + 5 x 5 + 2 x 50 + 10 x 25 + 7 x 45 + 0 x 20

=Rs.960/-

## Example 3:

Solve the transportation problem with unit transportation costs in rupees, demand and supplies as given below:

Destination

| | $D_1$ | $D_2$ | $D_3$ | Supply(units) |
|---|---|---|---|---|
| A | 5 | 6 | 9 | 100 |
| Origin B | 3 | 5 | 10 | 75 |
| C | 6 | 7 | 6 | 50 |
| D | 6 | 4 | 10 | 75 |
| Demand (units) | 70 | 80 | 120 | |

**Solution:** Since the total supply ($\sum a_i$ =270), the given transportation problem is unbalanced.

To convert this into a balanced one, we introduce a dummy source $D_4$ with zero unit transportation costs and having demand equal to 300-270=30 units. ∴ The given problem becomes

**Destination**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply(units) |
|---|---|---|---|---|---|
| A | 5 | 6 | 9 | 0 | 100 |
| Origin B | 3 | 5 | 10 | 0 | 75 |
| C | 6 | 7 | 6 | 0 | 50 |
| D | 6 | 4 | 10 | 0 | 75 |
| Demand (units) | 70 | 80 | 120 | 30 | 300 |

By using VAM the initial solution is given by

| | | | |
|---|---|---|---|
| 5 | 6 | 9<br>**100** | 0 |
| 3<br>**70** | 5<br>**5** | 10 | 0 |
| 6 | 7 | 6<br>**20** | 0<br>**30** |
| 6 | 4<br>**75** | 10 | 0 |

Since the number of non-negative allocations is 6, which is less than (m + n – 1) = 4+4-1 =7, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity ∈ to the cell (2,4), so that the number of occupied cells becomes (m + n – 1). Hence the non-degenerate basic feasible solution is given in the following table.

| | | | |
|---|---|---|---|
| 5 | 6 | 9<br>**100** | 0 |
| 3<br>**70** | 5<br>**5** | 10 | 0<br>∈ |
| 6 | 7 | 6<br>**20** | 0<br>**30** |
| 6 | 4<br>**75** | 10 | 0 |

Now the number of non-negative allocations at independent positions is (m + n – 1). We apply MODI method.

| | | | | |
|---|---|---|---|---|
| 5 6<br>-1 | 6 8<br>-2 | 9<br>**100** | 0 3<br>-3 | $u_1 = 3$ |
| 3<br>**70** | 5<br>**5** | 10 6<br>4 | 0<br>∈ | $u_2 = 0$ |
| 6 3<br>3 | 7 5<br>2 | 6<br>**20** | 0<br>**30** | $u_3 = 0$ |
| 6 2<br>4 | 4<br>**75** | 10 5<br>5 | 0 -1<br>1 | $u_4 = -1$ |
| $v_1 = 3$ | $v_2 = 5$ | $v_3 = 6$ | $v_4 = 0$ | |

Since there are some $d_{ij} < 0$, the current solution is not optimal.

Since $d_{14}$= -3 is the most negative, let is form a new basic feasible solution is giving maximum allocations to the corresponding cell(1,4) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,4) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ Alternatively at the corners.

| | | | |
|---|---|---|---|
| 5 | 6 | 9<br>**100**<br>$-\theta$ | 0<br>$+\theta$ |
| 3<br>**70** | 5<br>**5** | 10 | 0<br>∈ |
| 6 | 7 | 6<br>**20**<br>$+\theta$ | 0<br>**30**<br>$-\theta$ |
| 6 | 4<br>**75** | 10 | 0 |

From the two cells (1, 3), (3, 4) having $-\theta$, we find that the minimum of the allocations 100, 30 is 30. Add this 30 the cells with $+\theta$ and subtract this 30 to the cells with $-\theta$. Hence the new basic feasible solution is given in the following table.

| | | | |
|---|---|---|---|
| 5 | 6 | 9<br>**70** | 0<br>**30** |

| | | | |
|---|---|---|---|
| 3<br>**70** | 5<br>**5** | 10 | 0<br>∈ |
| 6 | 7 | 6<br>**50** | 0 |
| 6 | 4<br>**75** | 10 | 0 |

We see that the above table satisfies the rim conditions with $(m + n - 1)$ non-negative allocations at independent positions. We apply MODI method.

| | | | | |
|---|---|---|---|---|
| 5 3<br>2 | 6 5<br>1 | 9<br>**70** | 0<br>**30** | $u_1 = 0$ |
| 3<br>**70** | 5<br>**5** | 10 9<br>1 | 0<br>∈ | $u_2 = 0$ |
| 6 0<br>6 | 7 2<br>5 | 6<br>**50** | 0 | $u_3 = -3$ |
| 6 2<br>4 | 4<br>**75** | 10 8<br>2 | 0 -1<br>1 | $u_4 = -1$ |
| $v_1 = 3$ | $v_2 = 5$ | $v_3 = 9$ | $v_4 = 0$ | |

Since all $d_{ij} > 0$, the current solution is optimal and unique.

The optimum allocation schedule is given by $x_{13} = 70$, $x_{14} = 30$, $x_{21} = 70$, $x_{22} = 5$, $x_{24} = \in$, $x_{33} = 50$, $x_{42} = 75$ and the optimum transportation cost (minimum)

= Rs. 9 x 70 + 0 x 30 + 3 x 70 + 5 x 5 + 0 x ∈ + 6 x 50 + 4 x 75

=Rs. 1465/-

## Maximization case in Transportation Problems

So far we have discussed the transportation problems in which the objectives has been to minimize the total transportation cost and algorithms have been designed accordingly.

If we have a transportation problems where the objective is to maximize the total profit, first we have to convert the maximization problem into a minimization problem by multiplying all the entries by -1 (or) by subtracting all the entries from the highest entry in the given transportation table. The modified minimization problem can be solved in the usual manner.

# ASSIGNMENT PROBLEM

## Introduction

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (Jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit).

Suppose that we have '*n*' jobs to be performed on '*m*' machines (one Job to one machine) and our objective is to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of *m* x *n* matrix ($c_{ij}$) called a cost matrix (or) Effectiveness matrix where $c_{ij}$ is the cost of assigning $i^{th}$ machine to the $j^{th}$ job.

| | 1 | 2 | 3 | …………… | n |
|---|---|---|---|---|---|
| 1 | $c_{11}$ | $c_{12}$ | $c_{13}$ | …………… | $c_{1n}$ |
| 2 | $c_{21}$ | $c_{22}$ | $c_{23}$ | …………… | $c_{2n}$ |
| Machines 3 | $c_{31}$ | $c_{32}$ | $c_{33}$ | …………… | $c_{3n}$ |
| . | ….. | ….. | ….. | …………… | ….. |
| . | | | | | |
| . | ….. | ….. | ….. | …………… | ….. |
| . | | | | | |
| . | ….. | ….. | ….. | …………… | ….. |
| . | | | | | |
| m | $c_{m1}$ | $c_{m2}$ | $c_{m3}$ | …………… | $c_{mn}$ |

## Mathematical formulation of an assignment problem.

Consider an assignment problem of assigning *n* jobs to *n* machines (one job to one machine). Let $c_{ij}$ be the unit cost of assigning $i^{th}$ nachine to the $j^{th}$ job and

$$\text{Let } x_{ij} = \begin{cases} 1, \text{ if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, \text{ if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is then given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,2,\ldots,n$$

$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,\ldots,n$

and $x_{ij} = 0$ (or) 1.

**Difference between the transportation problem and the assignment problem.**

| *Transportation problem* | *Assignment problem* |
|---|---|
| (a) Supply at any source may be any positive quantity $a_i$ | Supply at any source (machine) will be 1 i.e., $a_i = 1$. |
| (b) Demand at any destination may be any positive $b_j$ | Demand at any destination (job) will be 1 i.e., $b_j = 1$. |
| (c) One or more source to any Number of destinations | One source (machine) to only one destination (job). |

**Assignment Algorithm (or) Hungarian Method.**

First check whether the number of rows is equal to the number of columns. If it is so, the assignment problem is said to be **balanced**. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm.

**Step 1:** Subtract the smallest cost element of each row from all the elements in the row of the row of the given cost matrix. See that each row contains atleast one zero.

**Step 2:** Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1.

**Step 3: (Assigning the zeros)**

(a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeros in the column of this enriched zero, as these will not be considered for any future assignment. Continues in this way until all the rows have been examined.
(b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

**Step 4: (Apply optimal Test)**

(a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

(b) It atleast one row/column is without an assignment (i.e., if there is atleast one row/column is without one encircled zero), then the current assignment is not optimal. Go to step 5.

**Step 5:** Cover all the zeros by drawing a minimum number of straight lines as follows.

(a) Mark (✓) the rows that do not have assignment.
(b) Mark (✓) the columns (not already marked) that have zeros in marked columns.
(c) Mark (✓) the rows (not already marked) that have assignments in marked columns.
(d) Repeat (b) and (c) until no more marking is required.
(e) Draw lines through all unmarked rows and columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution otherwise not.

**Step 6:** Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

**Step 7:** Repeat steps (1) to (6). Until an optimum assignment is attained.

**Note 1:** In case some rows or columns contain more than one zero, encircle any unmarked zero, encircle any unmarked zero arbitrarily and cross all other zeros in its column or row. Proceed in this way until no zero is left unmarked or encircled.

**Note 2:** The above assignment algorithm is only for minimization problems.

**Note 3:** If the given assignment problem is of maximization type, convert it to a minimization assignment problem by max Z = - min (-Z) and multiply all the given cost elements by -1 in the cost matrix and then solve by assignment algorithm.

**Note 4:** Sometimes a final cost matrix contains more than required number of zeros at independent positions. This implies that there is more than one optimal solution (multiple optimal solutions) with the same optimum assignment cost.

**Example 1:**

**Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:**

**Job**

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **A** | **8** | **4** | **2** | **6** | **1** |

| | | | | | | |
|---|---|---|---|---|---|---|
| | **B** | **0** | **9** | **5** | **5** | **4** |
| **From** | **C** | **3** | **8** | **9** | **2** | **6** |
| | **D** | **4** | **3** | **1** | **0** | **3** |
| | **E** | **9** | **5** | **8** | **9** | **5** |

**Determine the optimum assignment schedule.**

**Solution:** The cost matrix of the given assignment problem is

$$\begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

**Step 1:** Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix

$$\begin{pmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{pmatrix}$$

**Step 2:** select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix.

$$\begin{pmatrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{pmatrix}$$

Since each row and each column at least one zero, we shall make assignments in the reduced matrix.

**Step 3:** Examine the rows successively until a row with exactly one unmarked zero is found. Since the 2nd row contains a single zero, encircle this zero and cross all other zeros of its column. The 3rd row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 4th row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 1st row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Finally the last row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Likewise examine the columns successively. The assignments in rows and columns in the reduced matrix is given by

$$\begin{pmatrix} 7 & 3 & 0 & 5 & (0) \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & 0 & 3 \\ 4 & (0) & 2 & 4 & 0 \end{pmatrix}$$

**Step 4:** Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

∴ The optimal assignment schedule is given by A ⟶ 5, B ⟶ 1, C ⟶ 4, D ⟶ 3, E ⟶ 2.

The optimum (minimum) assignment cost = (1 + 0 + 2 + 1 + 5) cost units = 9 units of cost.

**Example 2:**

The processing time in hours for the when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum.

| | Machines | | | | |
|---|---|---|---|---|---|
| | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ |
| $J_1$ | 9 | 22 | 58 | 11 | 19 |
| $J_2$ | 43 | 78 | 72 | 50 | 63 |
| Jobs $J_3$ | 41 | 28 | 91 | 37 | 45 |
| $J_4$ | 74 | 42 | 27 | 49 | 39 |
| $J_5$ | 36 | 11 | 57 | 22 | 25 |

**Solution:**

The cost matrix of the given problem is

| | | | | |
|---|---|---|---|---|
| 9 | 22 | 58 | 11 | 19 |
| 43 | 78 | 72 | 50 | 63 |
| 41 | 28 | 91 | 37 | 45 |
| 74 | 42 | 27 | 49 | 39 |
| 36 | 11 | 57 | 22 | 25 |

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

**Step 1:** select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{pmatrix}$$

**Step 2:** Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the following reduced matrix.

$$\begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{pmatrix}$$

**Step 3:** Now we shall examine the rows successively. Second row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Third row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Fourth row contains a single unmarked zero, encircle this zero and cross all other zero in its column. After this no row is with exactly one unmarked zero. So go for columns.

Examine the columns successively. Fourth column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. After examining all the rows and columns. We get

$$\begin{pmatrix} 0 & 13 & 49 & (0) & 0 \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{pmatrix}$$

**Step 4:** Since the $5^{th}$ column do not have any assignment, the current assignment is not optimal.

**Step 5:** Cover all the zeros by drawing a minimum number of straight lines as follows:

(a) Mark (✓) the rows that do not have assignment. The row 5 is marked.
(b) Mark (✓) the columns (not already marked) that have zeros in marked rows. Thus column 2 is marked.

(c) Mark the rows (not already marked) that have assignment in, marked columns. Thus row 3 is marked.
(d) Repeat (b) and (c) until no more marking is required. In the present case this repetition is not necessary.
(e) Draw lines through all unmarked rows (rows 1, 2 and 4). And marked columns (column 2). We get

$$\begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & (4) \end{pmatrix}$$

**Step 6:** Here 4 is the smallest element not covered by these straight lines. Subtract this 4 from all the uncovered element and add this 4 to all those elements which are lying in the intersections of these straight lines and do not change the remaining elements which lie on these straight lines. We get the following matrix.

$$\begin{pmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{pmatrix}$$

Since each row and each column contains at least one zero, we examine the rows and columns successively, i.e., repeat step 3 above, we get

$$\begin{pmatrix} 0 & 17 & 49 & (0) & 0 \\ (0) & 39 & 29 & 5 & 10 \\ 9 & (0) & 59 & 3 & 3 \\ 47 & 19 & (0) & 20 & 2 \\ 21 & 0 & 42 & 5 & (0) \end{pmatrix}$$

In the above matrix, each row and each column contains exactly one assignment (i.e., exactly one encircled zero), therefore the current assignment is optimal.

∴ The optimum assignment schedule is $J_1 \longrightarrow M_4$, $J_2 \longrightarrow M_1$, $J_3 \longrightarrow M_2$, $j_4 \longrightarrow M_3$,

$J_5 \longrightarrow M_5$ and the optimum (minimum) processing time

= 11+43+28+27+25 hours = 134 hours.

## Unbalanced Assignment Models

If the number of rows is not equal to the number columns in the cost matrix of the given assignment problems, then the given assignment problems is said to be unbalanced.

First convert the unbalanced assignment problems in to a balanced one by adding dummy rows or dummy columns with zero cost element in the cost matrix depending upon whether m<n or m>n and then solve by the usual method.

**Example 1: A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.**

**Machines**

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **A** | **18** | **24** | **28** | **32** |
| **B** | **8** | **13** | **17** | **19** |
| **Jobs C** | **10** | **15** | **19** | **22** |

**What are job assignments which will minimize the cost?**

**Solution:**

The cost matrix of the given assignment problems is

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problems is unbalanced.

To make it a balanced one, add a dummy job D (row) with zero cost elements. The balanced cost matrix is given by

| | | | |
|---|---|---|---|
| 18 | 24 | 28 | 32 |
| 8 | 13 | 17 | 19 |
| 10 | 15 | 19 | 22 |

$$\begin{matrix} 0 & 0 & 0 & 0 \end{matrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (columns), we get the reduced matrix

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this reduced matrix, we shall make the assignment in rows and columns having single zero. We have

$$\begin{pmatrix} (0) & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & (0) & 0 & 0 \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover the all zeros by drawing a minimum number of straight lines. Choose the smallest cost element not covered by these straight lines.

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & (5) & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here 5 is the smallest cost element not covered by these straight lines. Subtract this 5 from all the uncovered element, add this 5 to those elements which lie in the intersections of these straight lines and do not change the remaining element which lie on the straight lines. We get

$$\begin{matrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{matrix}$$

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 1 & 5 & 9 \\ 0 & (0) & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & (0) & 0 \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & (4) & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Choose the smallest cost element not covered by these straight line, subtract this from all the uncovered elements, add this to those elements which are in the intersection of the lines and do not change the remaining elements which lie on these straight lines. Thus we get

$$\begin{pmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 1 & 1 & 5 \\ 0 & (0) & 0 & 2 \\ 0 & 0 & (0) & 3 \\ 9 & 4 & 0 & (0) \end{pmatrix}$$

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

∴ The optimum assignment schedule is given by A ⟶ 1, B ⟶ 2, C ⟶ 3, D ⟶ 4 and the optimum (minimum) assignment cost

= (18+13+19+0) cost unit = 50/- units of cost

**Note 1:** For this problem, the alternative optimum schedule is A⟶1, B ⟶ 2, C⟶ 3, D⟶4, with the same optimum assignment cost= Rs. (18+17+15+0) = 50/- units of cost.

**Note 2:** Here the assignment D⟶ 4 means that the dummy Job D is assigned to the 4th Machine. It means that machine 4 is left without any assignment.

## Maximization case in Assignment Problems

In an assignment problem, we may have to deal with maximization of an objective function. For example, we may have to assign persons to jobs in such a way that the total profit is maximized. The maximization problems has to be converted into an equivalent minimization problem and then solved by the usual Hungarian Method.

The conversion of the maximization problem into an equivalent minimization problems can be done by any of the following methods:

(i) Since max Z = - min (-Z), multiply all the cost element $c_{ij}$ of the cost matrix by -1.
(ii) Subtract all the cost elements $c_{ij}$ of the cost matrix from the highest cost element in that cost matrix.

**Example:**

Solve the assignment problem for maximization given the profit matrix (profit in rupees).

Machines

| P | Q | R | S |
|---|---|---|---|

| | | | | | |
|---|---|---|---|---|---|
| | A | 51 | 53 | 54 | 50 |
| | B | 47 | 50 | 48 | 50 |
| Jobs | C | 49 | 50 | 60 | 61 |
| | D | 63 | 64 | 60 | 60 |

**Solution:**

The profit matrix of the given assignment problem is

$$\begin{pmatrix} 51 & 53 & 54 & 50 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the profit elements in the profit from the highest profit element 64 of this profit matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the cost elements of the corresponding column. We get

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 2 & 1 & (0) & 4 \\ 2 & (0) & 2 & 0 \\ 11 & 11 & 1 & (0) \\ (0) & 0 & 4 & 4 \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

∴ The optimum assignment schedule is given by A ⟶ R, B ⟶ Q, C ⟶ S, D ⟶ P and the optimum (maximum) profit = Rs. (54 + 50 + 61 + 63) = Rs. 228/-

## Possible Questions Part B (2 Marks)

1. Define unbalanced TP
2. Give an example for unbalanced assignment problem.
3. Give an example for balanced TP.
4. Define optimal solution.
5. Define initial basic feasible solution.
6. Solve non-degenerate basic feasible solution.

## Possible Questions Part C (5 Marks)

1. Solve the transportation problem.

| | To | | | | Supply |
|---|---|---|---|---|---|
| From | 1 | 2 | 3 | 4 | 6 |
| | 4 | 3 | 2 | 0 | 8 |
| | 0 | 2 | 2 | 1 | 10 |
| Demand | 4 | 6 | 8 | 6 | |

2. Solve the assignment problem.

| | A | B | C | D |
|---|---|---|---|---|
| I | 18 | 24 | 28 | 32 |
| II | 8 | 13 | 17 | 19 |
| III | 10 | 15 | 19 | 22 |

3. Find the initial basic feasible solution by using North-West Corner Rule

| W→ F ↓ | $W_1$ | $W_2$ | $W_3$ | $W_4$ | Factory Capacity |
|---|---|---|---|---|---|
| $F_1$ | 19 | 30 | 50 | 10 | 7 |
| $F_2$ | 70 | 30 | 40 | 60 | 9 |

| | | | | | |
|---|---|---|---|---|---|
| $F_3$ | 40 | 8 | 70 | 20 | 18 |
| Warehouse Requirement | 5 | 8 | 7 | 14 | 34 |

4. Solve the assignment problem.

| | A | B | C | D |
|---|---|---|---|---|
| I | 18 | 24 | 28 | 32 |
| II | 8 | 13 | 17 | 19 |
| III | 10 | 15 | 19 | 22 |

5. Find the starting solution of the following transportation model

| | | | |
|---|---|---|---|
| 1 | 2 | 6 | 7 |
| 0 | 4 | 2 | 12 |
| 3 | 1 | 5 | 11 |
| 10 | 10 | 10 | |

Using (i) North West Corner rule
(ii) Vogel's approximation method.

6. Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:

| | Job 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | 8 | 4 | 2 | 6 | 1 |
| B | 0 | 9 | 5 | 5 | 4 |
| From C | 3 | 8 | 9 | 2 | 6 |
| D | 4 | 3 | 1 | 0 | 3 |
| E | 9 | 5 | 8 | 9 | 5 |

7. Determine an initial basic feasible solution to the following transportation problem using least

cost method.

| | | I | II | III | IV | Supply |
|---|---|---|---|---|---|---|
| | A | 13 | 11 | 15 | 20 | 2000 |
| From | B | 17 | 14 | 12 | 13 | 6000 |
| | C | 18 | 18 | 15 | 12 | 7000 |
| | Demand | 3000 | 3000 | 4000 | 5000 | |

8. Write algorithm for assignment problem (Hungarian Method)

## Possible Questions Part D (10 Marks)

1. Solve the assignment problem whose effectiveness matrix is given in the table

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 49 | 60 | 45 | 61 |
| B | 55 | 63 | 45 | 69 |
| C | 52 | 62 | 49 | 68 |
| D | 55 | 64 | 48 | 66 |

2. Obtain on optimum has feasible solution to the following transportation problem:

| | To | | | Available |
|---|---|---|---|---|
| | 7 | 3 | 2 | 2 |
| From | 2 | 1 | 3 | 3 |
| | 3 | 4 | 6 | 5 |
| Demand | 4 | 1 | 5 | 10 |

3. **Solve the following transportation problems using vogel's method.**

| | | A | B | C | D | E | F | Available |
|---|---|---|---|---|---|---|---|---|
| | **1** | **9** | **12** | **9** | **6** | **9** | **10** | **5** |
| **Factory** | **2** | **7** | **3** | **7** | **7** | **5** | **5** | **6** |
| | **3** | **6** | **5** | **9** | **11** | **3** | **11** | **2** |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **4** | **6** | **8** | **11** | **2** | **2** | **10** | **9** |
| **Requirement** | **4** | **4** | **6** | **2** | **4** | **2** | |

4. Solve the assignment problem for maximization given the profit matrix (profit in rupees).

Machines

| | P | Q | R | S |
|---|---|---|---|---|
| A | 51 | 53 | 54 | 50 |
| B | 47 | 50 | 48 | 50 |
| Jobs C | 49 | 50 | 60 | 61 |
| D | 63 | 64 | 60 | 60 |

**UNIT-III**

**SYLLABUS**

**PERT and CPM:** Arrow networks - time estimates- earliest expected time, latest allowable occurrence time and slack - critical path - probability of meeting scheduled date of completion of project calculations on CPM network - various floats for activities - critical path - updating project - operation time cost trade off curve - project time cost trade off curve - selection of schedule based on cost analysis.

# PERT and CPM

## Introduction

A **project**is defined as a combination of interrelated activities all of which must be executed in a certain order to achieve a set goal. A large and complex project involves usually a number of interrelated activities requiring men, machines and materials. It is impossible for the management to make and execute an optimum schedule for such a project just by intuition, based on the organizational capabilities and work experience. A systematic scientific approach has become a necessity for such project. So a number of methods applying networks scheduling techniques has been developed: **Programme Evaluation Review Technique** (PERT) and **Critical Path** method (CPM) are two of the many network techniques which are widely used for planning, scheduling and controlling large complex projects.

## The main managerial functions for any project:

The main managerial functions for any project are

1. Planning
2. Scheduling
3. Control

**Planning**

This phase involves a listing of tasks or jobs that must be performed to complete a project under consideration. In this phase, men, machines and materials required for the project in addition to the estimates of costs and durations of various activities of the project are also determined

## Scheduling

This phase involves the laying out of the actual activities of the project in a **logical sequence** of time in which they have to be performed.

Men and material requirements as well as the **expected completion time** of each activity at each stage of the project are also determined.

## Control

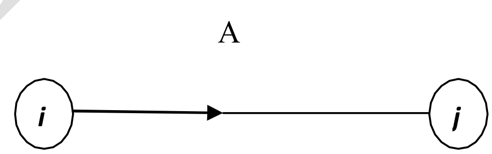
This phase consists of reviewing the progress of the project whether the actual performance is according to the planned schedule and finding the reasons for difference, if any, between the schedule and performance. The basic aspect of control is to analyse and correct this difference by taking remedial action whether possible.

PERT and CPM are especially useful for scheduling and controlling

## Basic Terminologies

**Activity** is a task or an item of work to be done in a project. An activity consumer resource like time, money, labour etc.

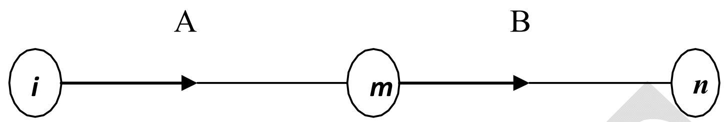
An activity is represented by an arrow with a node (event) at the beginning and a node (event) at the end indicating the start and termination (finish) of the activity. Nodes are denoted by circles. Since this is a logical diagram length or shape of the arrow has no meaning. The direction indicates the progress of the activity. Initial node and the terminal node are numbered as i-j(j<i) respectively. For example If A is the activity whose initial node is I and the terminal node is j then it is denoted diagrammatically by



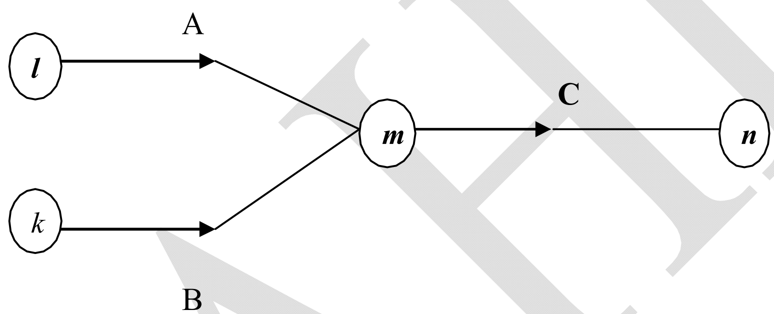

The name of the activity is written over the arrow, **not inside the circle.** The diagram

In which arrow represents an activity is called **arrow diagram.** The Initial and terminal nodes of activities are also called tail and head events.

If an activity B can start immediately after an activity A then it is denoted by



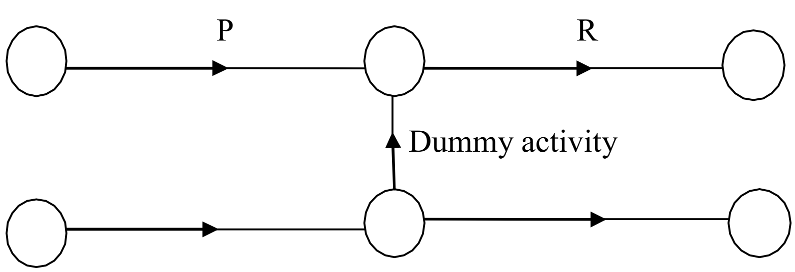

A is called the **immediate predecessor** of B and B is called the **immediate successor** of A. If C can start only after completing activites A and B then it is diagrammatically represented as follows:




**Notation:** “A is a predecessor of B” is denoted as “A<B‟,”B is a successor of A” is denoted by “B>A”.

If the project contains two or more activities which have some of their immediate predecessors in common then there is a need for introducing what is called **dummy activity.** Dummy activity is an imaginary activity which does not consume any resources and which serves the purpose of indicating the predecessor or successor relationship clearly in any activity on arrow diagram. The need for a dummy activity is illustrated by the following usual example.

Let P, Q be the predecessors of R and Q be the only predecessors of S.



Q S

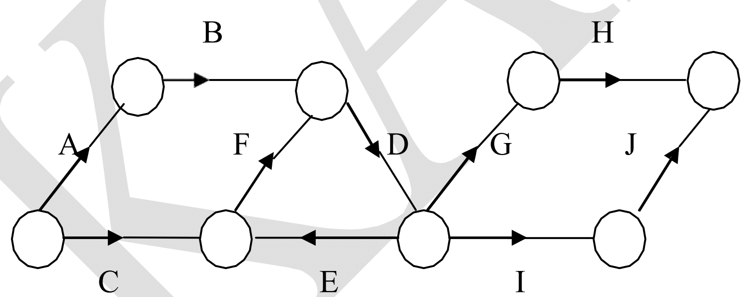
Activities which have no **predecessors** are called **start activities**of the project. All the **start activities** can be made to have the **same initial node**. Activities which have **no successors** are called **terminal activities** of the project. These can be made to have the **same** terminal node (end node) of the project.

A project consists of a number of activities to be performed in some technological sequence. For example while constructing a building the activity of laying the foundation should be done before the activity of erecting the walls for the building. The diagram denoting all the activities of a project by arrows taking into account the technological sequence of the activities is called the project network represented by **activity on arrow diagram** or simply **arrow diagram**.

**Note:** There is another representation of a project network representing activites on nodes called AON diagram. To avoid confusion we use only activity on arrow diagram throughout the text.

## Rules for constructing a project network

1. There must be no loops. For example, the activities F,D,E.




Obviously form a loop which is obviously not possible is any real project network.

2. Only one activity should connect any two nodes.

3. No dangling should appear in a project network i.e., no node of any activity except the terminal node of the project should be left without any activity emanating from it such a node can be joined to the terminal node of the project to avoid.

## The Rules for numbering the Nodes:

Nodes may be numbered using the rule given below:

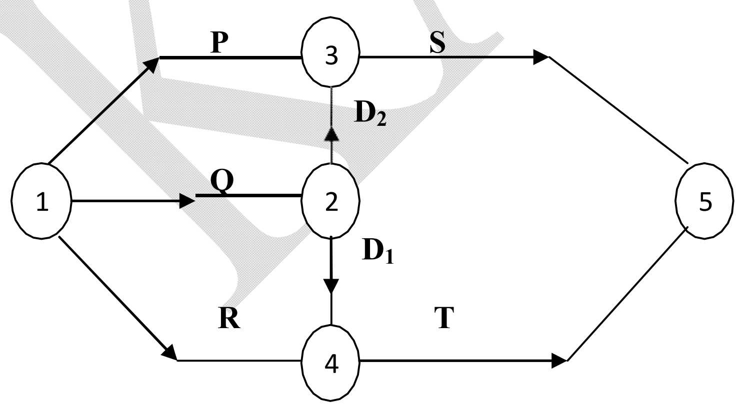
***(Ford and Fulkerson's Rule)***

1. Number the start node which has no predecessor activity, as 1.
2. Delete all the activites emanating from this node 1.
3. Number all the resulting start nodes without any predecessor as 2,3………
4. Delete all the activities originating from the start nodes 2,3,….in step 3.
5. Number all the resulting new start nodes without any predecessor next to the last number used in step(3).
6. Repeat the process until the terminal node without any successor activity is reached and number this terminal node suitably.

**Immediate predecessor (successor) will be simply called as predecessor (successor) unless otherwise stated.**

**Example 1:** If there are five activities P, Q, R, S and T such that P, Q, R have no immediate predecessors but S and T have immediate predecessors P, Q, R respectively. Represent this situation by a network.

**Solution:**



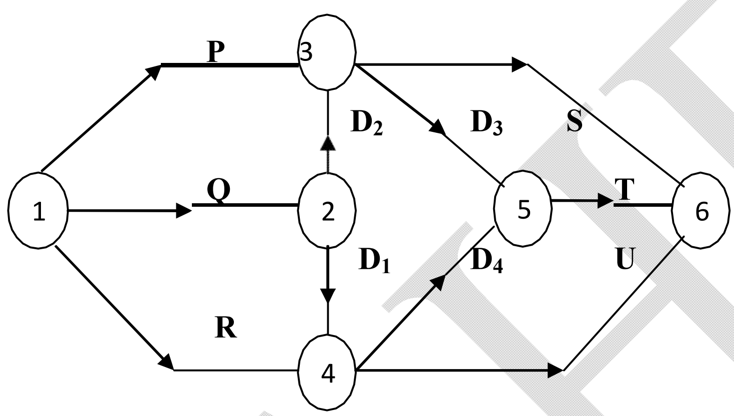

$D_1$ and $D_2$ are dummy activities.

**Example 2:**

Draw the network for the project whose activities and their precedence relationship are given below:

| Activity : | P | Q | R | S | T | U |
|---|---|---|---|---|---|---|
| Predecessor: | - | - | - | P, Q | P, R | Q, R |

**Solution:**




$D_1$, $D_2$, $D_3$, $D_4$ are dummy activities.

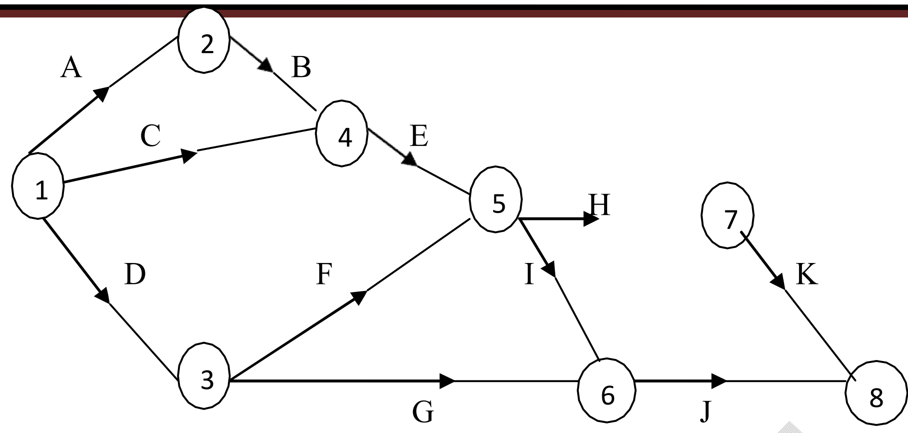
**Example 3:**

Draw the network for the project whose activities with their predecessor relationship are given below:

A, C, D can start simultaneously : E > B, C ; F, G > D ; H, I > E,F ; J > I,G ; K > H; B> A.

**Solution:**

Identify the start activities i.e., activities which have no predecessors. They are A,C and D as given. These three activities should start with the same start node. Also identify the terminal activities which have no successors. They are J and K. These two activities should end with the same end node, the last terminal node indicating the completion of the project. Taking into account the predecessors relationship given, the required network is as follows:




**Example 4:**

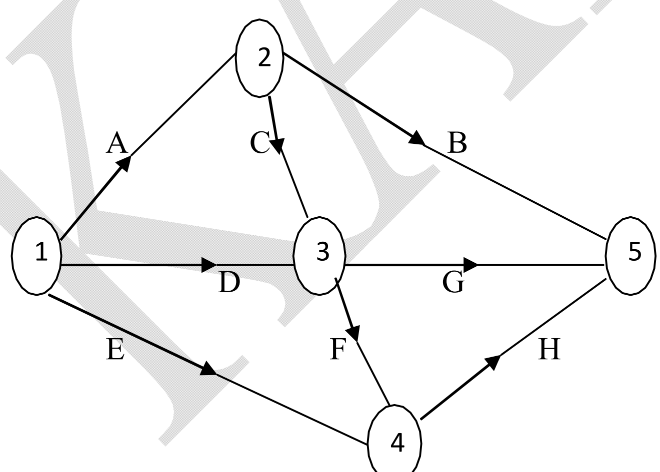
Construct the network for the project whose activities and their relationships are as given below:

Activities : A, D, E can start simultaneously.

Activities : B, C > A; G, F > D, C ; H > E, F.

**Solution:**

Start activities are A, D, E.
End activities are H, G, B.
The required network is



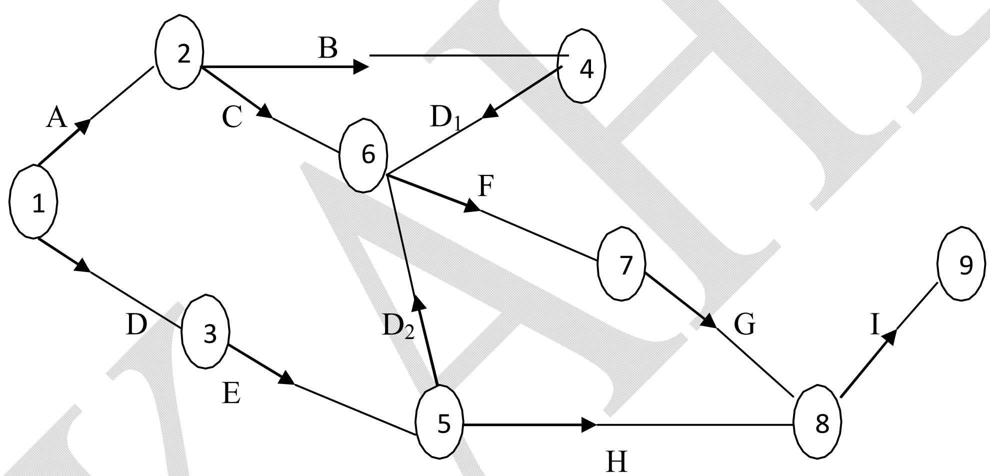

**Note :** see how the nodes of the activity F are numbered. Can we number C as 2 – 4 and F as 4 – 3?

**Example 5:** Draw the network for the project whose activities and their precedence relationships are as given below:

| Activities : | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| Immediate Predecessor: | - | A | A | - | D | B,C,E | F | E | G,H |

**Solution:**

Start activities : A,D, Terminal activities : I only. Activities B and C starting with the same node are both the predecessors of the activity F. Also the activity E has to be the predecessor of both F and H. Therefore dummy activities are necessary.Thus the required network is




$D_1$ and $D_2$ are dummy activities.

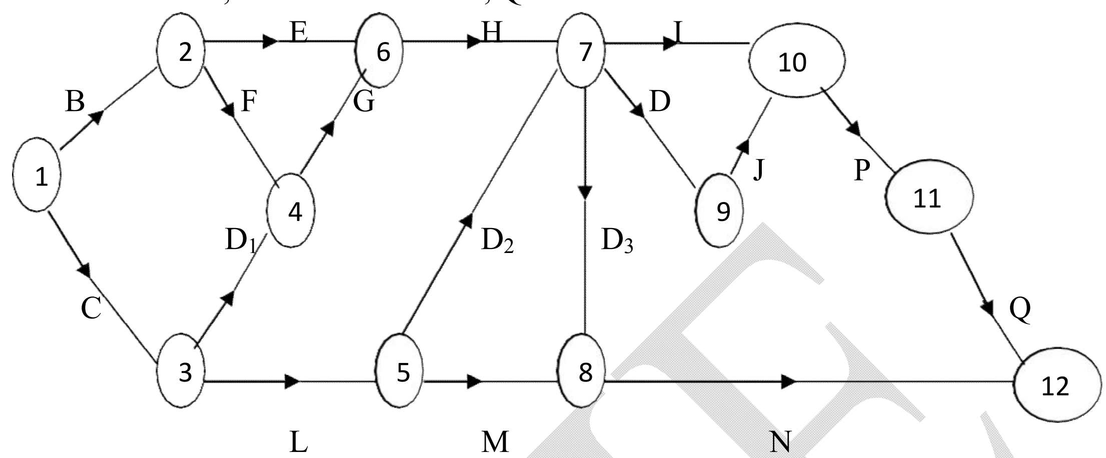
**Note: S**ometimes while constructing a network you may introduce more dummy activities than necessary. Redundant dummy activities can always be found out when one checks whether all the given precedence relationships given in the problem are satisfied exactly. (Nothing more, nothing less).

**Example 6:**

Construct the network for the project whose precedence relationships are as given below:
B < E, F ; C < G, L ; E, G < H ; L, H < I ; L < M ; H,M < N ; A < J ; I, J < P ; P < Q.

**Solution:**

Start activities: B,C End activities : N, Q



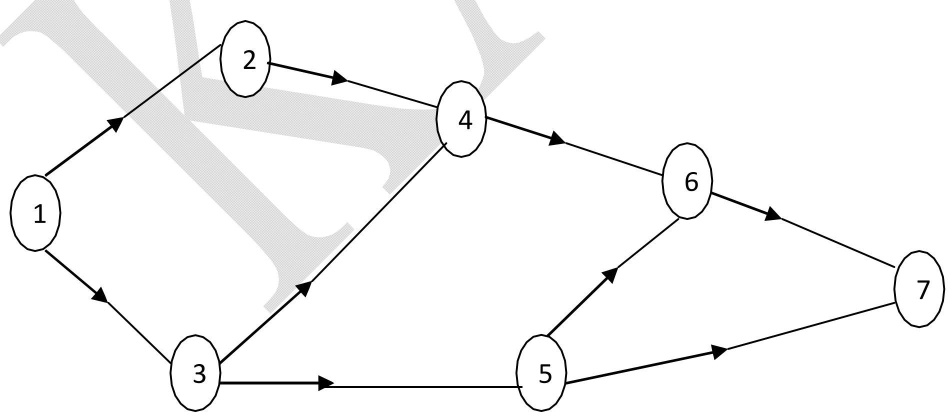

$D_1$ , $D_2$, $D_3$ and $D_4$ are dummy activities.

## Example 7:

Draw the event network for the following data:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Event No : | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Immediate Predecessors: | - | 1 | 1 | 2,3 | 3 | 4,5 | 5,6 |

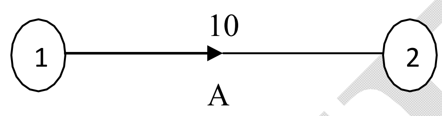
**Solution:**




**Network Computations and Critical Path**

(Earliest Completion time of a Project and Critical path)

It is obvious that the completion time of the project is one of the very important things to be calculated knowing the durations of each activity. In real world situation the duration of any activity has an element of uncertainty because of sudden unexpected shortage of labour, machines, materials etc. Hence the completion time of the project also has an element of uncertainty. We first consider the situation where the duration of each activity is deterministic without taking the uncertainty into account.




The above diagram represents an activity whose direction is 10 time unit(hour or days or weeks or month etc)

The first network calculation one does is the computation of earliest start and earliest finish (completion) time of each activity given the duration of each activity. The method used is called forward pass calculation and it is best illustrated by means of the following example.

**Example 1:**

Compute the earliest start, earliest finish latest start and latest finish of each activity of the project given below:

| Activity | 1-2 | 1-3 | 2-4 | 2-5 | 3-4 | 4-5 |
|---|---|---|---|---|---|---|
| Duration (in days) | 8 | 4 | 10 | 2 | 5 | 3 |

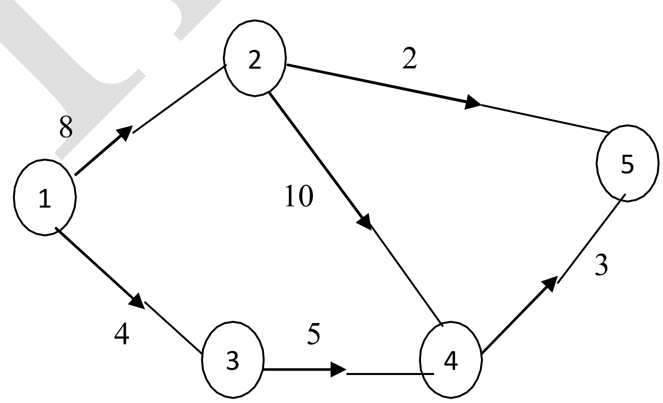


**Figure 1**

**To compute the Earliest start andEarliest finish of each activity:**

We take the earliest time of all the start activities as zero.

So earliest starts of 1-2 and 1-3 are zero.

To find earliest start of 2-4.

The activity 2-4 can start only after finishing the only preceding activity 1-2 i.e., after 8 days.

∴ Earliest start of 2-4 is 8 days. Similarly earliest start of 2-5 is also 8 days.

Similarly earliest start of 3-4 is 4 days.

To find the earliest start of 4-5 we first notice that the activity 4-5 has more than one predecessor and also the activity 4-5 can start only after finishing all its preceding activities.

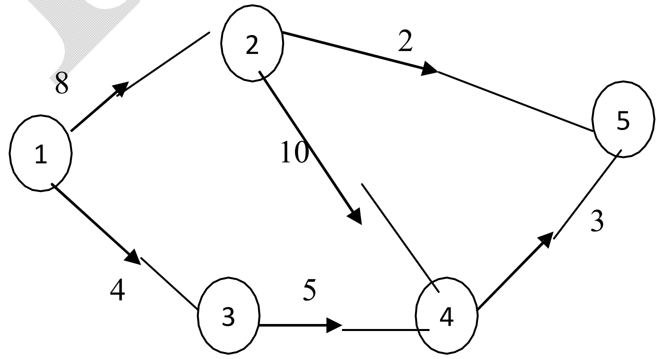
There are two paths leading to the activity 4-5: namely 1-2-4 which takes 18 days and 1-3-4 which takes 9 days. Obviously after 18 days all the activities 1-2, 1-3, 2-4, 3-4 can be finished but not earlier than that.

∴Earliest start of 4-5 is 18 days.

**Note:** Earliest start of an activity i-j can be denoted as $ES_i$ or $ES_{ij}$. It can also be called the earliest occurrence of the event i.

Earliest finish of any activity i-j is got by adding the duration of the activity denoted by $t_{ij}$ to the earliest start of i-j.

Hence the earliest finish of 1-2,1-3,2-4,2-5,3-4,4-5 are 8,4,18,10,9,21 respectively.



Obviously earliest completion time of the project is 21 days, the greater number among these since all the activities can be finished only after 21 days.

Formula for Earliest Start of an activity i-j in a project network is given by

$$ES_j = \text{Max } [ES_i+t_{ij}] \text{ where}$$

$ES_j$ denoted the earliest start time of all the activities emanating from node i and $t_{ij}$ is the estimated during of the activity i-j.

**To compute the latest finish and latest start of each activity:**

The method used here is called backward pass calculation since we start with the terminal activity and go back to the very first node.

We first calculate the latest finish of each activity as follows:

Latest finish of all the terminating (end) activities is taken as the earliest completion time of the project. Similarly latest finish of all the start activities is obviously taken as the same as the earliest start of these start activities.

Thus the latest finish of the terminal activities 2-5 and 4-5 are 21 days which is the earliest completion time of the project.

Latest finish of the activity 2-4 and 3-4 is 21-3 = 18 days.

Latest finish of 1-3 is 18-5 = 13 days

To find the latest finish of the activity 1-2, we observe that the activity 1-2 has more than one successor activity. Therefore the latest finish of the activity 1-2 is the smaller of the two numbers 21 - 2 = 19 and 18 – 10 = 8. i.e. 8 days.

**Note :** Latest finish of an activity can be denoted by $LF_j$ or $LF_{ij}$. It can also be called the latest occurrence of the event j.Latest start of each activity is the latest finish of that activity minus the duration of that activity. The latest start of the activities 4 -5, 2 – 5,2 – 4, 3 – 4,1 – 3, 1 – 2 are 21, 21, 18,18,13,8 respectively.

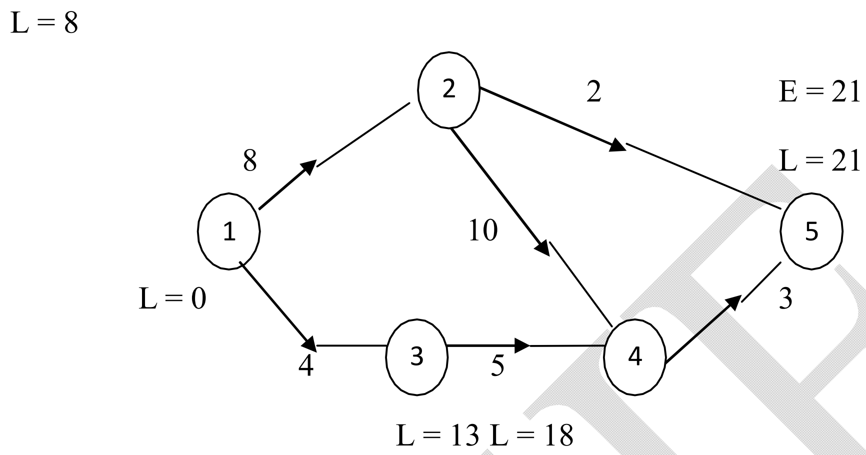



**Figure 3**

Formula for the latest start time of all the activities emanating from, the event i of the activity i – j, $LS_i$ = Min [$LS_j$ – $t_{ij}$] for all defined i-j activities where $t_{ij}$ is the estimated duration of the activity i – j.

We can tabulate the results and represents these earliest and latest occurrences of the events in the network diagram as follows:

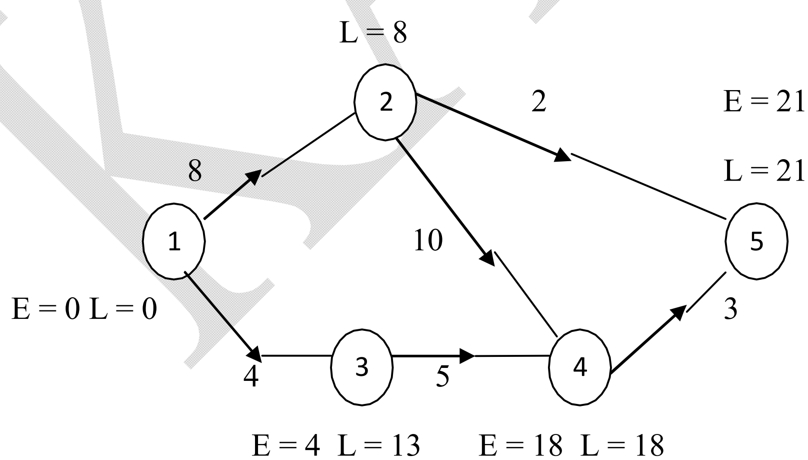


**Figure 4**

| | | Earliest | | Latest | |
|---|---|---|---|---|---|
| Activity | Duration days | Start ES | Finish EF ES = ES+$t_{ij}$ | Start LS LF – $t_{ij}$ | Finish LF |
| 1 – 2 | 8 | 0 | 8 | 0 | 8 |
| 1 – 3 | 4 | 0 | 4 | 9 | 13 |
| 2 – 4 | 10 | 8 | 18 | 8 | 18 |
| 2 – 5 | 2 | 8 | 10 | 19 | 21 |
| 3 – 4 | 5 | 4 | 9 | 13 | 18 |
| 4 – 5 | 3 | 18 | 21 | 18 | 21 |

**Note:** For small networks, it is not difficult to draw the network with E and L values calculated directly by looking at the diagram itself and constructing the table given above.

**Critical path:**

Path, connecting the first initial node to the very last terminal node, of longest duration in any project network is called the critical path.

All the activities in any critical path are called critical activities. Critical path is 1 – 2 – 4 – 5, usually denoted by doubt lines. (Ref fig.4)

Critical path plays a very important role in project scheduling problems.

**Floats**

Total float of an activity (T.F) is defined as the difference between the latest finish and the earliest finish of the activity or the difference between the latest start and the earliest start of the activity.

Total float of an activity i - j = $(LF)_{ij} – (EF)_{ij}$ Or = $(LS)_{ij} – (ES)_{ij}$.

Total float of an activity is the amount of time by which that particular activity may be delayed without affecting the duration of the project. If the total float is positive then it may indicate that the resources for the activity are more than adequate. If the total float of an activity is zero it may indicate that the resources are just adequate for that activity. If the total float is negative, it may indicate that the resources for that activity are inadequate.

Note: (L – E) of an event of I – j is called the slack of the event j.

There are three other types of floats for an activity, namely, Free float, Independent float and interference (interfering) float.

**Free Float:**

Free Float of an activity (F.F) is that portion of the total float which can be used for rescheduling that activity without affecting the succeeding activity. It can be calculated as follows:

Free float of an activity i – j = Total float of i – j – (L – E) of the event j

= Total float of i – j – slack of the head event j

= Total float of I – J – slack of the head event j

Where L = Latest occurrence , E= Earliest occurrence

Obviously Free Float ≤ Total float for any activity.

**Independent float (I.F):**

Independent float (I.F) of an activity is the amount of time by which the activity can be rescheduled without affecting the preceding or succeeding activities of that activity.

Independent float of an activity i – j = Free float of i- j – (L – E) of event i.

= Free float of i – j – Slack of the tail event j.

Clearly, Independent float ≤ Free float for any activity. Thus I.F ≤ F.F ≤ T.F.

Interfering Float or Interference Float of an activity i – j is nothing but the slack of the head event j. Obviously, Interfering Float of i – j = Total Float of i – j – Free Float of i – j.

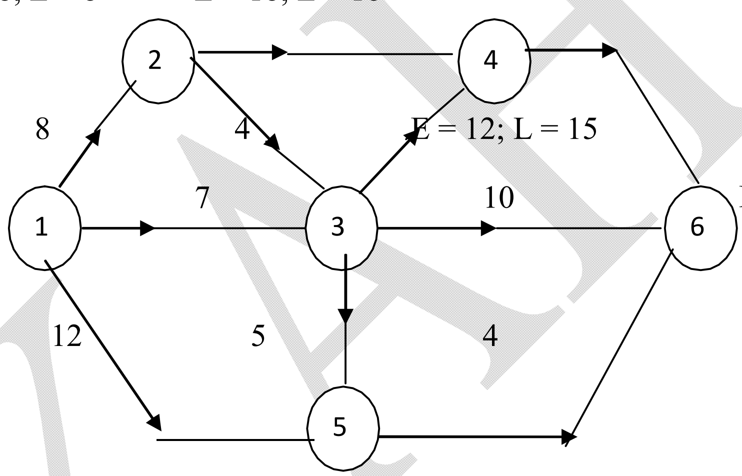
**Example 2:**

Calculate the total float, free float and independent float for the project whose activities are given below:

| Activity | 1-2 | 1-3 | 1-5 | 2-3 | 2-4 | 3-4 | 3-5 | 3-6 | 4-6 | 5-6 |
|---|---|---|---|---|---|---|---|---|---|---|
| Duration (in days) | 8 | 7 | 12 | 4 | 10 | 3 | 5 | 10 | 7 | 4 |

The data is the same as given in example 2 above. The network with L and E of every event is given by

**Solution:** E = 8; L = 8 E = 18; L = 18




E = 17; L = 21

| | | Earliest | | Latest | | Floats | | |
|---|---|---|---|---|---|---|---|---|
| Activity | Duration (in weeks) | Start | Finish | Start | Finish | TF | FF | IF |
| 1 – 2 | 8 | 0 | 8 | 0 | 8 | 0 | 0 | 0 |
| 1 – 3 | 7 | 0 | 7 | 8 | 15 | 8 | 5 | 5 |
| 1 – 5 | 12 | 0 | 12 | 9 | 21 | 9 | 5 | 5 |
| 2 – 3 | 4 | 8 | 12 | 11 | 15 | 3 | 0 | 0 |

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 – 4 | 10 | 8 | 18 | 8 | 18 | 0 | 0 | 0 |
| 3 – 4 | 3 | 12 | 15 | 15 | 18 | 3 | 3 | 0 |
| 3 – 5 | 5 | 12 | 17 | 16 | 21 | 4 | 0 | -3 |
| 3 – 6 | 10 | 12 | 22 | 15 | 25 | 3 | 3 | 0 |
| 4 – 5 | 7 | 18 | 25 | 18 | 25 | 0 | 0 | 0 |
| 5 – 6 | 4 | 17 | 21 | 21 | 25 | 4 | 4 | 0 |

**Explanation:**

To find the total float of 2 – 3.

Total float of (2 – 3) = (LF – EF) of (2 – 3) = 15 – 12 = 3 from the table against the activity 2 – 3.

Free Float of (2- 3) = Total float of (2 – 3) – (L – E) of event 3

= 3 – (15 12) from the figure for event 3 = 0

Free Float of (1 – 5) = Total float of (1 – 5) – (L – F) of event 5

= (21 – 12) – (21 – 17) from the figure for event 5

= 9 – 4 = 5

Independent float of (1 – 5) = Free Float of (1 – 5) – (L – E) of event 1

= 5 – (0 – 0) = 5

**Important Note:**

Note that all the critical activities have their total float as zero. In fact the critical path can also be defined as the path of least (zero) total float. As we have noticed total float is 3 for the activity 2 – 3. This means that the activity 2 – 3 can be delayed by 3 weeks without delaying the duration (completion date) of the project.

Free float of 3 – 4 is 3. This means that the activity 3 – 4 can be delayed by 3 weeks without affecting its succeeding activity 4 – 6.

Independent float of 1 – 5 is 5 means that the activity 1 – 5 can be delayed by 5 weeks without affecting its preceding or succeeding activity. Of course 1 – 5 has no preceding activity.

**Uses of floats:**

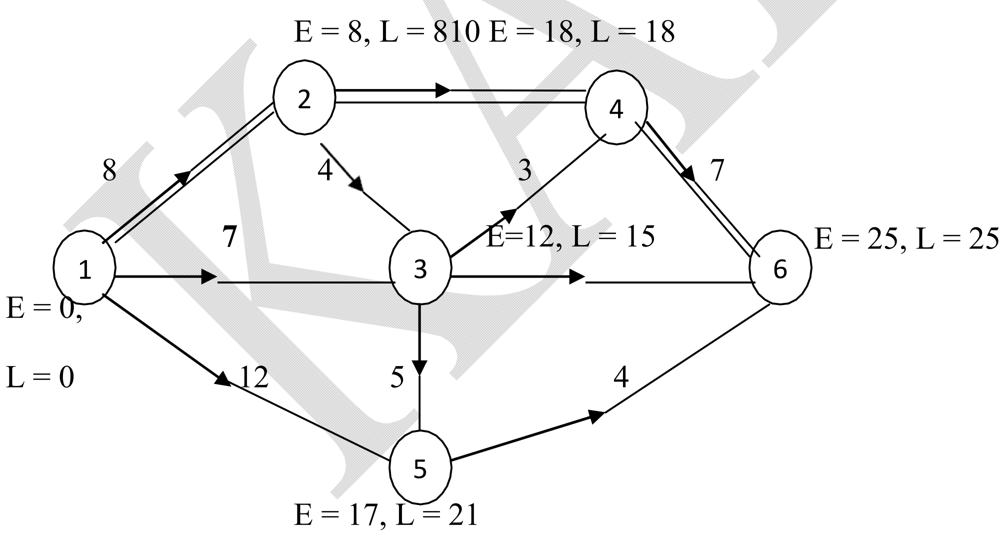
Floats are useful in resources leveling and recourse allocation problems which will be discussed in the last section of this chapter. Floats give some flexibility in rescheduling some activities so as to smoothen the level of resources or allocate the limited resources as best as possible.

**Example 3:**

**Calculate the earliest start, earliest finish, latest start and latest finish of each activity of the project given below and determine the critical path of the project.**

| | | | | | |
|---|---|---|---|---|---|
| Activity | 1 – 2 | 1 – 3 | 1 – 5 | 2 – 3 | 2 – 4 |
| Duration (in weeks) | 8 | 7 | 12 | 4 | 10 |
| Activity | 3 – 4 | 3 – 5 | 3 – 6 | 4 – 6 | 5 – 6 |
| Duration (in weeks) | 3 | 5 | 10 | 7 | 4 |

**Solution:**



## Program Evaluation Review Techniques: (PERT)

This technique, unlike CPM, take into account the uncertainly of project durations into account.

***Optimistic (least) time estimate: ($t_0$ or a)*** is the duration of any activity when everything goes on very well during the project. i.e., labourers are available and come in time, machines are working properly, money is available whenever needed, there is no scarcity of raw material needed etc.

***Pessimistic (greatest) time estimate: ($t_p$ or b)*** is the duration of any activity when almost everything goes against our will and a lot of difficulties is faced while doing a project.

***Most likely time estimate: ($t_m$ or m)*** is the duration of any activity when sometimes things go on very well, sometimes things go on very bed while doing the project.

Two main assumption make in PERT calculations are

(i) The activity durations are independent. Ie., the time required to complete an activity will have no bearing on the completion times of any other activity of the project.

(ii) The activity durations follow $\beta$ – distribution.

$\beta$ Distribution is a probability distribution with density function $k(t-a)^{\alpha}(b-t)^{\beta}$ with mean $t_e = \frac{1}{3}[2t_m + \frac{1}{2}(t_0 - t_p)]$ and the standard deviation $\sigma_t = \frac{t_p - t_0}{6}$

**PERT procedure**

(1) Draw the project network

(2) Compute the expected duration of each activity $t_e = \frac{t_0 + 4t_m + t_p}{6}$

(3) Compute the expected variance $\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$ of each activity.

(4) Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.
(5) Determine the critical path and identity critical activites.
(6) Compute the expected variance of the project length (also called the variance of the critical path) $\sigma_c^2$ which is the sum of the variance of all the critical activities.
(7) Compute the expected standard deviation of the project length $\sigma_c$ and calculate the standard normal deviate $\frac{T_S - T_E}{\sigma_C}$ where

$T_s$ = Specified or Schedule time to complete the project

$T_E$ = Normal expected project duration

$\sigma_c$ = Expected standard deviation of the project length.

(8) Using (7) one can estimate the probability of completing the project within a specified time, using the normal curve (Area) tables.

**Example 1:**

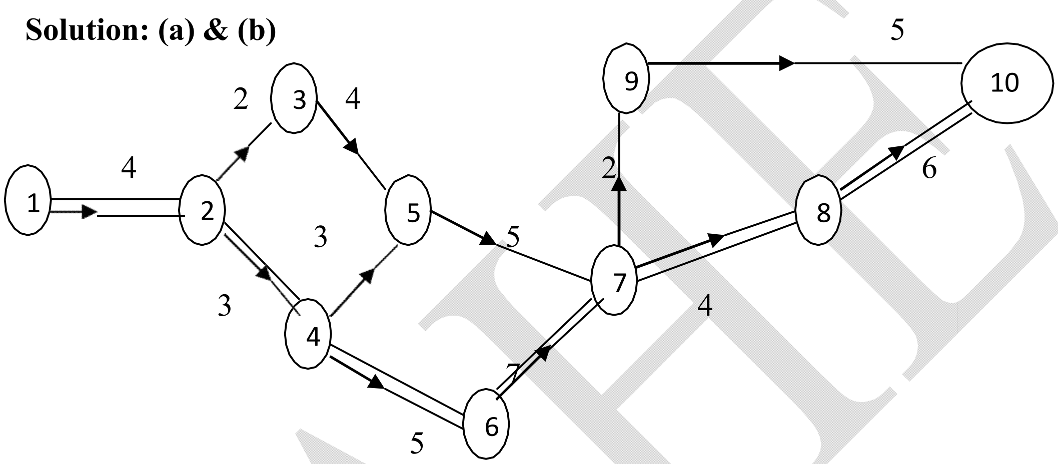
**Construct the network for the project whose activities and the three time estimates of these activities (in weeks) are given below. Compute**

(a) Excepted duration of each activity.
(b) Excepted variance of each activity.
(c) Excepted variance of the project length.

| Activity | $t_0$ | $t_m$ | $t_p$ |
|---|---|---|---|
| 1 – 2 | 3 | 4 | 5 |
| 2 – 3 | 1 | 2 | 3 |
| 2 – 4 | 2 | 3 | 4 |
| 3 – 5 | 3 | 4 | 5 |
| 4 – 5 | 1 | 3 | 5 |
| 4 – 6 | 3 | 5 | 7 |
| 5 – 7 | 4 | 5 | 6 |

| | | | |
|---|---|---|---|
| 6 – 7 | 6 | 7 | 8 |
| 7 – 8 | 2 | 4 | 6 |
| 7 – 9 | 1 | 2 | 3 |
| 8 – 10 | 4 | 6 | 8 |
| 9 – 10 | 3 | 5 | 7 |

**Solution: (a) & (b)**




| Activity | $t_0$ | $t_m$ | $t_p$ | Excepted duration $t_e = \frac{t_0 + 4t_m + t_p}{6}$ | Excepted variance $\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$ |
|---|---|---|---|---|---|
| 1 – 2 | 3 | 4 | 5 | 4 | **$^1/_9$ = 0.11** nearly |
| 2 – 3 | 1 | 2 | 3 | 2 | $^1/_9$ = 0.11 |
| 2 – 4 | 2 | 3 | 4 | 3 | **$^1/_9$ = 0.11** |
| 3 – 5 | 3 | 4 | 5 | 4 | $^1/_9$ = 0.11 |
| 4 – 5 | 1 | 3 | 5 | 3 | $^4/_9$ = 0.44 |
| 4 – 6 | 3 | 5 | 7 | 5 | **$^4/_9$ = 0.44** |
| 5 – 7 | 4 | 5 | 6 | 5 | $^1/_9$ = 0.11 |
| 6 – 7 | 6 | 7 | 8 | 7 | **$^1/_9$ = 0.11** |
| 7 – 8 | 2 | 4 | 6 | 4 | **$^4/_9$ = 0.44** |
| 7 – 9 | 1 | 2 | 3 | 2 | $^1/_9$ = 0.11 |
| 8 – 10 | 4 | 6 | 8 | 6 | **$^4/_9$ = 0.44** |
| 9 – 10 | 3 | 5 | 7 | 5 | $^4/_9$ = 0.44 |

Critical path 1 – 2 – 4 – 6 – 7 – 8 – 10. Excepted project duration = 29 weeks.

( c) Excepted variance of the project length = Sum of the expected variances of all the critical activities

$= \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{15}{9} = \frac{15}{9} = \frac{5}{3} = 1.67$

or (0.11 + 0.11 + 0.44 + 0.11 + 0.44 + 0.44 = 1.32 + 0.33 = 1.65)

**Example 2:**

The following table indicates the details of a project. The duration are in days. „a" refers to optimistic time, „m" refers to most likely time and „b" refers to pessimistic time duration.

| Activity | 1 – 2 | 1 – 3 | 1 – 4 | 2 – 4 | 2 – 5 | 3 – 5 | 4 – 5 |
|---|---|---|---|---|---|---|---|
| *a* | 2 | 3 | 4 | 8 | 6 | 2 | 2 |
| *m* | 4 | 4 | 5 | 9 | 8 | 3 | 5 |
| *b* | 5 | 6 | 6 | 11 | 12 | 4 | 7 |

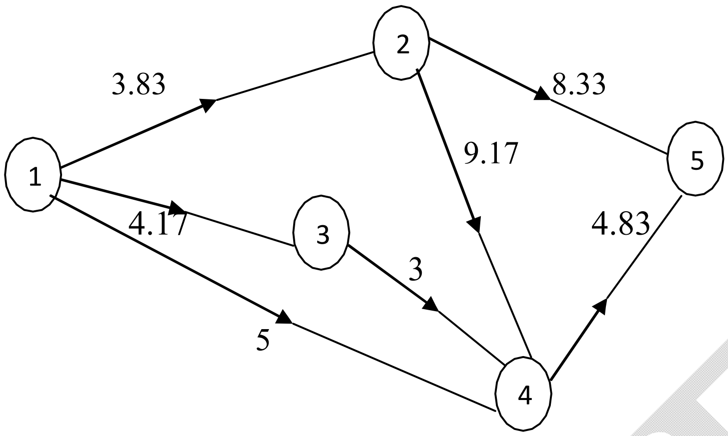
(a) Draw the network

(b) Find the critical path

(c) Determine the excepted standard deviation of the completion time.

**Solution:**

| Activity | *a* | *m* | *b* | Excepted duration $t_e$ | Excepted variance $\sigma^2$ |
|---|---|---|---|---|---|
| **1 – 2** | 2 | 4 | 5 | 3.83 | $\frac{1}{4}$ |
| 1 – 3 | 3 | 4 | 6 | 4.17 | $\frac{1}{4}$ |
| 1 – 4 | 4 | 5 | 6 | 5 | $\frac{1}{9}$ |
| **2 – 4** | 8 | 9 | 11 | 9.17 | $\frac{1}{4}$ |
| 2 – 5 | 6 | 8 | 12 | 8.33 | 1 |
| 3 – 4 | 2 | 3 | 4 | 3 | $\frac{1}{9}$ |
| **4 – 5** | 2 | 5 | 7 | 4.83 | $\frac{25}{36}$ |




Critical path 1 – 2 – 4 – 5

Excepted project duration = 17.83 days

Excepted variance of the completion time = $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{43}{36}$

Excepted standard deviation of the completion time = $\sqrt{\frac{43}{36}}$ = 1.09 nearly

### Example 3:

**A project consists of the following activities and time estimates:**

| *Activity* | *Least time (days)* | *Greatest time (days)* | *Most likely time (days)* |
|---|---|---|---|
| **1 – 2** | **3** | **15** | **6** |
| **2 – 3** | **2** | **14** | **5** |
| **1 – 4** | **6** | **13** | **12** |
| **2 – 5** | **2** | **8** | **5** |
| **2 – 6** | **5** | **17** | **11** |
| **3 – 6** | **3** | **15** | **6** |
| **4 – 7** | **3** | **27** | **9** |
| **5 – 7** | **1** | **7** | **4** |
| **6 – 7** | **2** | **8** | **5** |

**(a) Draw the network**
**(b) What is the probability that the project will be completed in 27 days?**

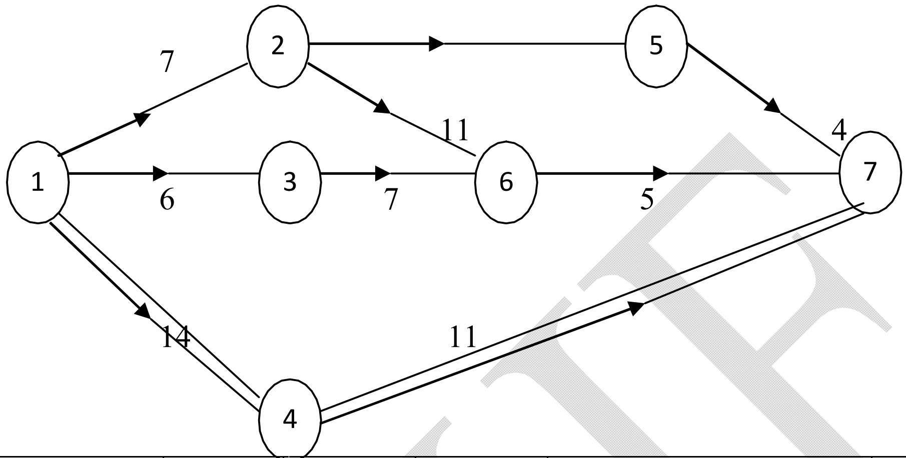
**Solution:**

Obviously Greatest time = Pessimistic time = $t_p$

Least time = Optimistic time = $t_0$

Most Likely time = $t_m$

(a) 5




| *Activity* | $t_0$ | $t_p$ | $t_m$ | $t_e = \frac{t_0 + 4t_m + t_p}{6}$ | $\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$ |
|---|---|---|---|---|---|
| 1 – 2 | 3 | 15 | 6 | 7 | 4 |
| 2 – 3 | 2 | 14 | 5 | 6 | 4 |
| 1 – 4 | 6 | 13 | 12 | 14 | 16 |
| 2 – 5 | 2 | 8 | 5 | 5 | 1 |
| 2 – 6 | 5 | 17 | 11 | 11 | 4 |
| 3 – 6 | 3 | 15 | 6 | 7 | 4 |
| 4 – 7 | 3 | 27 | 9 | 11 | 16 |
| 5 – 7 | 1 | 7 | 4 | 4 | 1 |
| 6 – 7 | 2 | 8 | 5 | 5 | 1 |

Critical path 1 – 4 – 7

Excepted project duration = 25 days

Excepted variance of the project length = Sum of the excepted variance of all the critical activities

= 16 + 16 = 32.

$\sigma_c$ = Standard deviation of the project length = $\sqrt{32} = 4\sqrt{2}$ = 5.656

$Z = \frac{T_S - T_E}{\sigma_C} = \frac{27-25}{5.656} = 0.35$

Probability that the project will be completed in 27 days

$= P(T_s \leq 27) = P(Z \leq 0.35)$

$= 0.6368 = 63.7\%$

**Basic difference between PERT and CPM**

**PERT**

1. PERT was developed in a brand new R and D project it had to consider and deal with the uncertainties associated with such projects. Thus the project duration is regarded as a random variables and therefore probabilities are calculated so as to characteristics it.
2. Emphasis is given to important stages of completion of task rather than the activities required to be performed to reach a particular event or task in the analysis of network. i.e., PERT network is essentially an event – oriented network.
3. PERT is usually used for projects in which time estimates are uncertain. Example: R & D activities which are usually non-repetitive.
4. PERT helps in identifying critical areas in a project so that suitable necessary adjustments may be made to meet the scheduled completion date of the project.

**CPM**

1. CPM was developed fir conventional projects like construction project which consists of well known routine tasks whose resources requirement and duration were known with certainty.
2. CPM is suited to establish a trade off for optimum balancing between schedule time and cost of the project.
3. CPM is used for projects involving well known activities of repetitive in nature, However the distinction between PERT and CPM is mostly historical.

## POSSIBLE QUESTIONS:

### PART-B ( 2 MARKS)

1. Define event.
2. What are the errors of network construction?
3. What are the dummies in network construction?
4. What are the rules of network construction.
5. Define most likely time.

### PART-C ( 5 MARKS)

1. A Project has the following characteristics

| Activity : | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Duration: | 6 | 8 | 4 | 9 | 2 | 7 |
| Preceding activity : | _ | A | A | B | C | D |

Draw the network diagram and find the critical path.

2. Calculate the total float, free float and independent float for theproject whose activities are given below.

| Activity: | 1-2 | 1-3 | 2-4 | 3-4 | 3-5 | 4-5 | 4-6 | 5-6 |
|---|---|---|---|---|---|---|---|---|
| Duration: | 6 | 5 | 10 | 3 | 4 | 6 | 2 | 9 |

3. Draw the network and find the critical path of the project given below.

| Activity | 1-2 | 1-3 | 2-4 | 2-5 | 3-4 | 4-5 |
|---|---|---|---|---|---|---|
| Duration | 8 | 4 | 10 | 2 | 5 | 3 |

4. Calculate the total float, free float and independent float for the project whose activities are given below:

| Activity | 1-2 | 1-3 | 1-5 | 2-3 | 2-4 | 3-4 | 3-5 | 3-6 | 4-6 | 5-6 |
|---|---|---|---|---|---|---|---|---|---|---|
| Duration (in days) | 8 | 7 | 12 | 4 | 10 | 3 | 5 | 10 | 7 | 4 |

5. Draw the network diagram and find the critical path.

| Job | Predecessors | Duration(days) |
|---|---|---|

| | | |
|---|---|---|
| A | - | 10 |
| B | - | 5 |
| C | B | 3 |
| D | A,C | 4 |
| E | A,C | 6 |
| F | D | 6 |
| G | E | 5 |
| H | F,G | 5 |

6. Write the difference between CPM and PERT.

7. A Project consists of the following activities and time estimates:

| Activity | Least time(days) | Greatest time(days) | Most likely Time (days) |
|---|---|---|---|
| 1-2 | 2 | 4 | 5 |
| 1-3 | 3 | 4 | 6 |
| 1-4 | 4 | 5 | 6 |
| 2-4 | 8 | 9 | 11 |
| 2-5 | 6 | 8 | 12 |
| 3-4 | 2 | 3 | 4 |
| 4-5 | 2 | 5 | 7 |

i) Draw the network and find the critical path.

ii) Find the expected standard deviation of completion of time.

8. A Project consists of the following activities and time estimates:

| Activity | $t_o$ | $t_m$ | $t_p$ |
|---|---|---|---|
| 1-2 | 0.8 | 1.0 | 1.2 |
| 2-3 | 3.7 | 5.6 | 9.9 |
| 2-4 | 6.2 | 6.6 | 15.4 |
| 3-4 | 2.1 | 2.7 | 6.1 |
| 4-5 | 0.8 | 3.4 | 3.0 |
| 5-6 | 0.9 | 3.4 | 2.7 |

i) Find the critical path.

ii) Determine the expected duration and standard deviation of each activity

iii) Find the probability that the project will be completed in 2 months earlier than expected.

8. A project consists of the following activities and time estimates:

| Activity | Least time (days) | Greatest time (days) | Most likely time (days) |
|---|---|---|---|
| 1 – 2 | 3 | 15 | 6 |
| 2 – 3 | 2 | 14 | 5 |
| 1 – 4 | 6 | 13 | 12 |
| 2 – 5 | 2 | 8 | 5 |
| 2 – 6 | 5 | 17 | 11 |
| 3 – 6 | 3 | 15 | 6 |
| 4 – 7 | 3 | 27 | 9 |
| 5 – 7 | 1 | 7 | 4 |
| 6 – 7 | 2 | 8 | 5 |

(c) Draw the network

(d) What is the probability that the project will be completed in 27 days?

9. The following table indicates the details of a project. The duration are in days. „a" refers to optimistic time, „m" refers to most likely time and „b" refers to pessimistic time duration.

| Activity | 1 – 2 | 1 – 3 | 1 – 4 | 2 – 4 | 2 – 5 | 3 – 5 | 4 – 5 |
|---|---|---|---|---|---|---|---|
| *a* | 2 | 3 | 4 | 8 | 6 | 2 | 2 |
| *m* | 4 | 4 | 5 | 9 | 8 | 3 | 5 |
| *b* | 5 | 6 | 6 | 11 | 12 | 4 | 7 |

(a) Draw the network

(b) Find the critical path

(c) Determine the excepted standard deviation of the completion time.

## PART-D (10 MARKS)

**COMPULSORY:**

1. The three estimates for the activities of a project are given below:

| Activity | Estimated duration (days) | | |
|---|---|---|---|
| | a | m | b |
| 1 – 2 | 5 | 6 | 7 |
| 1 – 3 | 1 | 1 | 7 |
| 1 – 4 | 2 | 4 | 12 |
| 2 – 5 | 3 | 6 | 15 |
| 3- 5 | 1 | 1 | 1 |
| 4 – 6 | 2 | 2 | 8 |
| 5 – 6 | 1 | 4 | 7 |

i) Draw the project network.
ii) What is the probability that the project will be completed on 22 days ?

2. The following table lists the jobs of a network with their time estimate

| Jobs | Optimistic | Duration days<br>Most likely | Pessimistic |
|---|---|---|---|
| 1-2 | 3 | 6 | 15 |
| 1-6 | 2 | 5 | 14 |
| 2-3 | 6 | 12 | 30 |
| 2-4 | 2 | 5 | 8 |
| 3-5 | 5 | 11 | 17 |
| 4-5 | 3 | 6 | 15 |

| | | | |
|---|---|---|---|
| 6-7 | 3 | 9 | 27 |
| 5-8 | 1 | 4 | 7 |
| 7-8 | 4 | 19 | 28 |

Draw the project network and calculate the length and variance of the critical path.

3. A Project consists of the following activities and time estimates:

| Activity | $t_o$ | $t_m$ | $t_p$ |
|---|---|---|---|
| 1-2 | 3 | 4 | 5 |
| 2-3 | 2 | 3 | 10 |
| 2-4 | 4 | 4 | 10 |
| 3-5 | 3 | 5 | 7 |
| 4-5 | 1 | 7 | 7 |
| 4-6 | 2 | 9 | 10 |
| 5-6 | 2 | 3 | 4 |

i) Find the critical path.
ii) Determine the expected project completion time and its variance

**UNIT IV**

Inventory models – Economic order quantity models – Quantity discount models – Stochastic inventory models – Multi product models – Inventory control models in practice - Queueing models – Queueing systems and structures – Notation parameter – Single server and multi server models – Poisson input – Exponential service – Constant rate service – Infinite population.

# INVENTORY MODELS

## Introduction

Inventory may be defined as the stock of goods, commodities or other economic resources that are stored or reserved for smooth and efficient running of business affairs. The Inventory may be kept in any one of the following forms:

i. Raw material Inventory.
   Raw materials which are kept in stock for using in production of goods.
ii. Work – in process Inventory.
   Semi finished goods which are stored during production process.
iii. Finished goods Inventory.
   Finished goods awaiting shipments from the factory.

## Type of Inventory

***i. Fluctuation Inventorie***
In real – life problems, there are fluctuations in the demand and lead time that affect the production of the items. Such types of safety stock are called ***Fluctuation Inventories.***

***ii. Anticipated Inventories***
These are built up in advance for the season of large sales, a promotion programme or a plant shut down period. Anticipated Inventories keep men and machine hours for future participation.

***iii. Lot – size Inventories***

Generally rate of consumption is different from rate of production or purchasing. Therefore the items are produced in larger quantities, which result in ***Lot – size Inventories***

**Reasons for maintaining Inventory**

1. Inventory helps in smooth and efficient running of business.
2. It provides service to the customers at short notice.
3. Because of long – uninterrupted runs, production cost is less.
4. It acts as a buffer stock if shop rejections are too many.
5. It takes care of economic fluctuations.

**Costs involved in Inventory Problems**

1. **Holding Cost ($C_1$)**

The cost associated with carrying or holding the goods in stock is known as ***holding cost (or) carrying cost*** per unit of time. Holding cost is assumed to directly vary with the size of inventory as well as the time the item is held in stock. The following components constitute holding cost.

(a) ***Interested capital cost***: This is the interest charge over the capital invested.
(b) Record keeping and Administrative costs.
(c) ***Handling cost:*** These include costs associated with movement of stock, such as cost of labour etc.
(d) Storage costs.
(e) Depreciation costs.
(f) Taxes and Insurance costs.
(g) Purchase price or production costs.

Purchase price per unit item is affected by the quantity purchased due to quantity discounts or price breaks. If P is the purchase price of an item and 1 is the stock holding cost per unit time expressed as a fraction of stock value (in rupees), then the holding cost $C_1$ = IP.

2. **Shortage Cost ($C_2$)**

The penalty costs that are incurred as a result of running out of stock (i.e., shortage) are known as ***shortage or stock – out costs***. These are denoted by $C_2$. In case where the unfilled demand for the goods may be satisfied at a latter date, these costs are assumed to vary directly with both the shortage quantity and the delaying time on the other hand if the unfilled demand is lost (no backlog case) shortage costs become proportional to shortage quantity only.

3. **Set – up cost($C_3$)**

These costs are associated with obtaining goods through placing an order or purchasing or manufacturing or setting up a machinery before starting production. So they include costs of purchase, requisition, follow up receiving the goods, quality control etc. These are called ***Ordering costs or replenishment costs***, or set-up cost usually denoted by $C_3$ per production run (cycle). They are assumed to be independent of the quantity ordered or produced.

**Variables in Inventory Problem:**The variables in inventory model are to two types.

(a) Controlled Variables
(b) Uncontrolled Variables

***(a) Controlled Variables***

1. How much quantity acquired.
2. The frequency or timing of acquisition.
3. The completion stage of stocked items.

***(b) Uncontrolled Variables***

These include holding costs, shortage costs, set-up cost and demand.

**Lead time, Reorder Level (R.O.L)**

***Lead time:*** Elapsed time between the placement of the order and its receipts in inventory is known as Lead time.

***Reorder Level:*** This is the time when we should place an order by taking into consideration the interval between placing the order and receiving the supply. For e.g., we would like to place a new order precisely at the time when Inventory Level reaches zero.

***Definition: Economic Order Quantity (E.O.Q) or Economic lot size formula***

*Economic order quantity (EOQ)* is that size or order which minimizes total annual cost of carrying inventory and the cost of ordering under the assumed conditions of certainty and that annual demands are known.

**Deterministic Inventory Models**

There are 4 types under this category which we shall study as follows:

***Model I :*** Purchasing model with no shortages.

***Model II :*** Manufacturing model with no shortages.

***Model III :*** Purchasing model with shortages.

***Model IV :*** Manufacturing model with shortages.

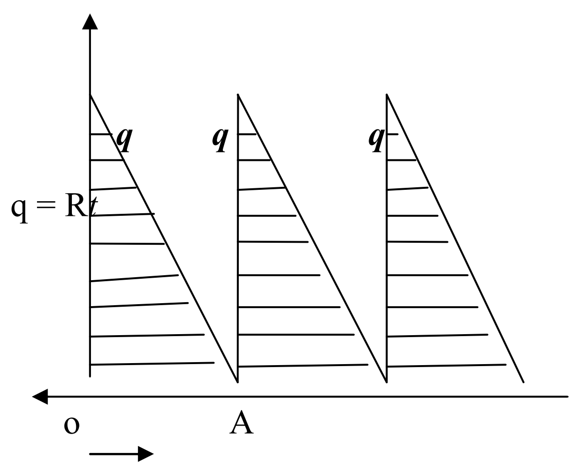
***Model I : Purchasing model with no shortages.***

*(Demand rate uniform, Production rate infinite)*

A manufacturer has order to supply goods at a uniform rate of R per unit time. No shortages are allowed, consequently the shortage cost is Infinity. He starts a production run every *t* time units, where *t* is fixed and the set up cost per production run is $C_3$. Production time is negligible. (Replacement Instantaneous) $C_1$ is the cost of holding one unit in inventory for a unit of time. The manufacturer's problem is to determine

1. How frequently he should make a production run
2. How many units should be made per production run.

Diagrammatic representation of this *model*.




If a production run is made at intervals *t*, a quantity $q = Rt$ must be produced in each run. Since the stock in small time *dt* is R*tdt*, the stock if period *t* will be

$$\int_0^t Rt\,dt = \frac{1}{2}Rt^2$$

$$= \frac{1}{2}qt = \text{Area of Inventory triangle OAP.}$$

Cost of holding inventory per production run $= \frac{1}{2}C_1Rt^2$

Set of cost per production run $= C_3$

Total cost per production run $= \frac{1}{2}C_1Rt^2 + C_3$.

Average total cost per unit time $C(t) = \frac{1}{2}C_1Rt + \frac{C_3}{t}$ .........(1)

C will be minimum if $\frac{dC(t)}{dt} = 0$ and $\frac{d^2C(t)}{dt^2}$ is positive.

Differentiating (1) w.r.t *t* and equating to zero,

$$\frac{dC(t)}{dt} = \frac{1}{2}C_1Rt - \frac{C_3}{t^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Which gives $t = \sqrt{\frac{2C_3}{C_1R}}$

Differentiating (2) w.r.t. *t* $\frac{d^2C(t)}{dt^2} = \frac{2C_3}{t_3}$ which is positive for value of *t* given by the above equation.

Thus C(*t*) is minimum for optimum time interval $t_0 = \sqrt{\frac{2C_3}{C_1R}}$.

Optimum quantity $q_0$ to be produced during each production run,

$$q_0 = Rt_0 = \sqrt{\frac{2C_3R}{C_1}}$$

which is known as the **optimal lot – size formula due to R.H. Wilson.**

The resulting minimum average cost per unit time,

$$C_0(q) = \frac{1}{2}C_1R\sqrt{\frac{2C_3}{C_1R}} + C_3\sqrt{\frac{C_1R}{2C_3}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{c_1c_3R} + \frac{1}{\sqrt{2}}\sqrt{c_1c_3R}$$

$$= \sqrt{2c_1c_3R}$$

***Remarks:***

1. If the demand rate is not uniform, and if D is the total demand to be satisfied during the period T then R = $\frac{D}{T}$ in the above formula.
2. $q_0, t_0, C_0$ are sometimes referred as $q^*, t^*, c^*$.

**Example 1:**

The annual demand for an item is 3200 units. The unit cost is Rs. 6/- and inventory carrying charges 25% per annum. If the cost of one procurement is Rs. 150/- determine

i. Economic order quantity.
ii. Time between two consecutive orders
iii. Number of orders per year
iv. The optimal total cost.

**Solution:**

R = 3200 units, $C_1 = \frac{25}{100} X\ 6 = \frac{3}{2}$

$C_3$ = 150 Rs, $\therefore q^* = \sqrt{\frac{2C_3R}{C_1}}$

$= \sqrt{\frac{2X150X3200}{\frac{3}{2}}}$ = 800 units.

(ii) $t_0 = \{= t^*\} = \frac{q^*}{R}$

$= \frac{800}{3200} = \frac{1}{4}$ th of a year

(iii) Number of orders = $\frac{1}{t_0} = \frac{1}{\frac{1}{4}} = 4$

Total = (R x price per unit) + $C_0$

(iv) Optimal cost = (6 x 3200) + $\sqrt{2C_1C_3R}$

= Rs. 20,400 ***[Ans]***

Otherwise optimal total cost = $\frac{R}{q}C_3 + \left(\frac{q}{2}\ XC_1\right)$ + RP

**Example 2:**

A manufacturing company purchases 9000 parts of a machine for its annual requirements, ordering one month usage at a time. Each part costs Rs 20. The ordering cost per order is Rs. 15 and the carrying charges are 15% of the average inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer, and how much would it save the company per year?

**Solution:**

Hear R = 9000 parts per year

$C_1$ = 15% unit cost

(Here 15% of average Inventory per year means that the carrying cost per unit per year is 15% of the unit cost)

$= 20 \text{ x } \frac{15}{100}$ = Rs. 3 each part per year

$C_3$ = Rs. 15 per order

$$\therefore q^* = \sqrt{\frac{2C_3R}{C_1}}$$

$$= \sqrt{\frac{2X150X3200}{\frac{3}{2}}} = 300 \text{ units}$$

$$t^* = \frac{q^*}{R} = \frac{300}{9000} = \frac{1}{30} \text{ year}$$

$$= \frac{365}{30} = 12 \text{ days}$$

$$C_{min} = \sqrt{2C_1C_3R}$$

$$= \sqrt{2 \; X \; 3 \; X \; 15 \; X \; 9000}$$

If the company follows the policy of ordering every month, then the annual ordering cost becomes = 12 x 15 = Rs. 180

and lot – size of Inventory at any time $q = \frac{9000}{12}$ = 750 parts.

Average Inventory at any time = $\frac{1}{2}q$ = 375 parts.

Shortage cost at any time = 375 $C_1$

= 375 x 3 = Rs. 1125.

Total annual cost = 1125 + 180 = Rs. 1305

∴ The company purchases 300 parts at time intervals of 12 days instead of ordering 750 parts each month. So there will be a net saving of

Rs. 1305 – Rs. 900 = Rs. 405 per year

**Example 3:**

A certain item costs Rs. 235 per ton. The monthly requirements are 5 tons, and each item the stock is replenished, there is a setup cost of Rs. 1000. The cost of carrying inventory has been estimated at 10% of the average inventory per year. What is the optimum order quantity.

**Solution:**

R = 5 tons/month

= 60 tons/year

$C_3$ = Rs. 1000

$C_1$ = 10% of unit cost per year

= Rs. 235 x $\frac{10}{100}$

= Rs. 23.5 per item per year

$\therefore q^* = \sqrt{\frac{2C_3R}{C_1}}$

$= \sqrt{\frac{2X1000X60}{23.5}}$ = 71.458 tons

**Example 4:**

A manufacturer has to supply his customer with 600 units of his products per year. Shortage are not allowed and storage cost amounts to 60 paise per unit per year. The set up cost is Rs. 80.00 find

i. The economic order quantity
ii. The minimum average yearly cost
iii. The optimum number of orders per year
iv. The optimum period of supply per optimum order

**Solution:**

R = 600 units/year

$C_1$ = Rs. 80

$C_3$ = 0.60 per unit/year

i. $\therefore q^* = \sqrt{\frac{2C_3R}{C_1}}$

$= \sqrt{\frac{2X600X80}{0.60}}$ = 400 units/year

ii. $C^* = \sqrt{2C_1C_3R}$ $= \sqrt{2\ X\ 0.60\ X\ 80\ X\ 600}$

= Rs. 240

iii. $N^* = \frac{demand}{EOQ} = \frac{600}{400} = \frac{3}{2}$

iv. $t^* = \frac{1}{N^*} = \frac{2}{3}$ of a year

***Model II : Manufacturing model with no shortages.****(Demand Rate uniform, production rate finite)*

It is assumed that run sizes are constant and that a new run will be started whenever Inventory is zero. Let

R = number of items required per unit time

K = number of items produced per unit time

$C_1$ = cost of holding per item per unit time

$C_3$ = cost of setting up a production run

$q$ = number of items produced per run, $q$ = R$t$

$t$ = time interval between runs.

Here each production run of length $t$ consists of two parts $t_1$ and $t_2$,

where (i) $t_1$ is the time during which the stock is building up at constant rate of K – R units per unit time. (ii) $t_2$ is the time during which there is no production (or supply) and inventory is decreasing at a constant rate R per unit time.

Let $I_m$ be the maximum Inventory available at the end of time $t_1$ which is expected to be consumed during the remaining period $t_2$ at the demand rate R.

Then $I_m$ = (K – R) $t_1$ (or)

$$t_1 = \frac{I_m}{K - R} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Now the total quantity produced during time $t_1$ id $q$ and quantity consumed during the same period is R$t$, therefore the remaining quantity available at the end of time $t_1$ is

$$t_1 = q - Rt_1$$

$$= q - \frac{R.I_m}{K - R} \qquad \text{from(1)}$$

$$\therefore I_m\left(1 + \frac{R}{k-R}\right) = q \text{ (or) } I_m = \frac{k-R}{k}q \ \ldots\ldots(2)$$

Now holding cost per production run for time period $t = \frac{1}{2}I_m t C_1$

And set up cost per production run $= C_3$

∴ Total average cost per unit time C($I_m$, $t$) $= \frac{1}{2}I_m C_1 + \frac{C_3}{t}$

$$C(q, t) = \frac{1}{2}\left(\frac{k-R}{k}q\right)C_1 + \frac{C_3}{t}$$

$$C(q) = \frac{1}{2}\left(\frac{k-R}{k}q\right)C_1 + \frac{C_3}{\frac{q}{R}}$$

$$= \frac{1}{2}\frac{k-R}{k}C_1 q + \frac{C_3 R}{q}$$

For minimum value of C($q$)

$$\frac{d}{dq}[c(q)] = \frac{1}{2}\frac{k-R}{k}C_1 - \frac{C_3 R}{q^2} = 0$$

Which gives $q = \sqrt{\frac{2C_3}{c_1}\frac{RK}{K-R}}$

∴ Optimum lot size $q_0 = \sqrt{\frac{K}{K-R}}\sqrt{\frac{2C_3 R}{C_1}}$

∴ Optimum time interval $t_0 = \frac{q_0}{R}$

$$= \sqrt{\frac{K}{K-R}}\sqrt{\frac{2C_3}{C_1 R}}$$

Optimum average cost/unit time

$$C_0 = \frac{1}{2}\frac{k-R}{k}\ C_1\sqrt{\frac{2C_3}{c_1}\frac{RK}{K-R}} + C_3R\sqrt{\frac{C_1(K-R)}{2C_3RK}}$$

$$= \sqrt{2C_1C_3R\frac{K-R}{K}}$$

$$= \sqrt{\frac{K-R}{K}}\sqrt{2C_1C_3R}$$

***Note:*** *(i)* If K = R then $C_0$= 0, (i.e.,) there will be no holding cost and set up cost

*(ii)* If K = ∞, (i.e.,) production rate is Infinite, this model reduces to model I.

**Example 1:**

A contractor has to supply 10,000 bearings per month to an automobile manufacturer. He finds that when he starts a production run he can produce 25,000 bearings per month. The cost of holding a bearing in stock for one year is Rs. 2 and the set up cost of a production run is Rs. 180. How frequently should the production run be made?

**Solution:**

R = 10,000/month x 12 = 1,20,000 per year

$C_1$ = Rs. 2 per year

$C_3$ = Rs. 180

K = 25,000 x 12 = 3,00,000 per year

$$\therefore q^* = \sqrt{\frac{2C_3R}{C_1}}\sqrt{\frac{K}{K-R}}$$

$$= \sqrt{\frac{3,00,000}{30,000-120000}}\ X\ \sqrt{\frac{2\ X\ 180\ X\ 120000}{2}}$$

$$= 1.29 \text{ x } \sqrt{21600000} \ = \ 6000 \text{ units}$$

$$t^* = \frac{q^*}{R} = \frac{6000}{120000} = 0.05 \text{ years (i.e.,) 18 days.}$$

**Example 2:**

The annual demand for a product is 1,00,000 units. The rate or production is 2,00,000 units per year. The set – up cost per production run is Rs. 5000, and the variable production cost of each item is Rs. 10. The annual holding cost per unit is 20% of the value of the unit. Find the optimum production lot – size, and the length of production run.

**Solution:**

R = 1,00,000 per year

$C_1 = \frac{20}{100}$ x 10 Rs. Per year

$C_3$ = Rs. 5000

K = 2,00,000

$\therefore$ EOQ = $\sqrt{\frac{K}{K-R}}\sqrt{\frac{2C_3R}{C_1}}$

$= \sqrt{\frac{2,00,000}{1,00,000}} \text{ x } \sqrt{\frac{2 X 1,00,000 X 5000}{\frac{20}{100} X 10}}$

= 1.4142 x 22360.6

= 31622 units (=$q^*$)

$t^* = \frac{q^*}{R} = \frac{31622}{1,00,000}$ = 0.31622 years = 115 days

**Example 3:**

An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 items per day. If the set up cost is Rs. 100 per set up and holding cost is Rs. 0.01 per unit of item per day, find the economic lot size for one run, assuming that shortages are not permitted. Also find the time of cycle and minimum total cost for one run.

**Solution:**

R = 25 items per day

$C_1$ = Rs. 0.01 per unit per day

$C_3$ = Rs. 100 per set up

K = 50 items per day

$\therefore$ EOQ = $\sqrt{\frac{K}{K-R}}\sqrt{\frac{2C_3R}{C_1}}$

$= \sqrt{\frac{2 \, X \, 100 \, X \, 25}{0.01}}$ x $\sqrt{\frac{50}{25}}$

= 1000 items

$t_0 = \frac{q_0}{R} = \frac{1000}{25}$ = 40 days

Minimum daily cost = $\sqrt{2C_1C_3R}\sqrt{\frac{K-R}{K}}$

= Rs. $\sqrt{2 \, X \, 0.01 \, X \, 100 \, X \, 25 \, X \, \frac{25}{50}}$

= Rs. 5

Minimum total cost per run = 5 x 40

= Rs. 200

**Example 4:**

A company has a demand of 12,000 units/year for an item and it can produce 2000 such items per month. The cost of one setup is Rs. 400 and the holding cost/unit/month is Rs. 0.15. Find the optimum lot size, max inventory, manufacturing time, total time.

**Solution:**

R = 12,000 units/year

$C_1$ = Rs. 400/ set up

$C_3$ = Rs. 0.15 x 12 = Rs. 1.80/unit/year.

K = 2000 x 12 = 24,000 units/year

$\therefore \; q_0 = \sqrt{\frac{K}{K-R}}\sqrt{\frac{2C_3R}{C_1}}$

$= \sqrt{\frac{2 \, X \, 400 \, X \, 12{,}000}{1.80}}$ x $\sqrt{\frac{24{,}000}{12{,}000}}$

= 3266 units/set up

Max inventory $I_{m_o} = \frac{K-R}{K} q_0$

$$= \frac{24{,}000-12{,}000}{24{,}000} \text{ x } 3266 \qquad = 1632 \text{ units .}$$

$$\text{Manufacturing time } t_1 = \frac{I_{m_0}}{K-R} = \frac{1632}{12{,}000} \qquad = 0.136 \text{ years.}$$

$$\text{Total time} \quad t_0 = \frac{q_0}{R} = \frac{3264}{12{,}000} = 0.272 \text{ years.}$$

**Example 5:**

A certain item costs Rs. 250 per ton. The monthly requirements are 10 tons and each time the stock is replenished there is a setup cost of Rs. 1000. The cost of carrying inventory has been estimated as 12% of the value of the stock per year. What is the optimal order quantity and how frequently should orders be placed? **Solution:**

$$C_1 = \frac{12}{100} \text{ x } 250$$

$$C_3 = \text{Rs. } 1000$$

$$R = 10 \text{ x } 12 = 120 \text{ tons/year}$$

$$\therefore \text{EOQ} = \sqrt{\frac{2C_3R}{C_1}}$$

$$= \sqrt{\frac{2 X 1000 X 120}{\frac{12}{100} X 250}}$$

$$= \sqrt{\frac{24{,}000}{30}} = \sqrt{8000}$$

$$= 89.44 \text{ units}$$

$$t_0 = \frac{q_0}{R} = \frac{89.44}{120} = 0.745 \text{ year} \qquad \approx 9 \text{ months.}$$

***Model III : Purchasing model with shortages.***

*(Demand rate uniform, Production rate infinite, shortages allowed)*

Assumptions are the same as model I, but shortages are allowed, consequently, a cost of shortage is incurred.

$C_1$ – Holding cost or carrying cost.

$C_3$ – Setup cost or Ordering cost.

$C_2$ – Shortage cost

R – Demand Rate.

The optimum quantities of this model are

a) The Economic order quantity $q^* = \sqrt{\frac{2C_3R}{C_1}}\sqrt{\frac{C_1+C_2}{C_2}}$

b) Time between two consecutive orders $t^* = \frac{q^*}{R}$

c) Number of orders per year $N^* = \frac{R}{q^*}$

**Example 1:**

The demand for an item is 18,000 units per year. The holding cost per unit time is Rs. 1.20 and the cost of shortage is Rs. 5.00, the production cost is Rs. 400. Assuming that replenishment rate is Instantaneous, determine the optimal order quantity.

**Solution:**

R = 18,000

$C_1$ = Rs. 1.20

$C_2$ = Rs. 5.00

$C_3$ = Rs. 400

$$\therefore q^* = \sqrt{\frac{2C_3R}{C_1}}\sqrt{\frac{C_1+C_2}{C_2}}$$

$$= \sqrt{\frac{2 \, X \, 400 \, X \, 18{,}000}{1.20}}\sqrt{\frac{1.20+5}{5}}$$

= 1.113 x 3,464.10

= 3856 units (app)

$$t^* = \frac{q^*}{R} = \frac{3856}{18{,}000} = 0.214 \text{ year}$$

$$N^* = \frac{R}{q^*} = 4.67 \text{ orders per year}$$

**Example 2:**

A certain product has a demand of 25 units per month and the items are withdrawn uniformly. Each time a production run is made the setup cost is Rs. 15. The production cost is Rs. 1 per item and inventory holding cost is Rs. 0.30 per item per month. If shortage cost is Rs. 1.50 per item per month, determine how often to make a production run and what size it should be ?

**Solution:**

Though the production cost is given, the cost equation remain the same.

$$\Longrightarrow \quad q^* = \sqrt{\frac{2C_3R}{C_1}}\sqrt{\frac{C_1 + C_2}{C_2}}$$

Given

R = 25 units/month

$C_1$ = 0.30 per item per month

$C_2$ = Rs. 1.50 per item per month

$C_3$ = Rs. 15

$$\therefore q^* = \sqrt{\frac{(1.50+0.30)2\ X\ 15\ X\ 25}{0.30\ X\ 1.50}}$$

$$= \sqrt{\frac{54.77}{25}} \text{ units}$$

$$t^* = \frac{q^*}{R} = \frac{54.77}{25} = 2.19 \text{ month}$$

***Model IV : Manufacturing model with shortages.***

*(Demand rate uniform, Production rate finite, shortages allowed)*

Assumptions are the same as model II, but shortages are allowed.

The optimum quantities of this model are

$C_1$ – Holding cost or carrying cost.

$C_3$ – Setup cost or Ordering cost.

$C_2$ – Shortage cost

R – Demand Rate.

K – Production Rate.

a) The Economic order quantity $q^* = \sqrt{\frac{2C_3R}{C_1}}\sqrt{\frac{C_1+C_2}{C_2}}\sqrt{\frac{K}{K-R}}$

b) Number of shortages $S = \frac{C_1}{C_1+C_2} q^* \left(1 - \frac{R}{K}\right)$

c) Time between two consecutive orders $t^* = \frac{q^*}{R}$

d) Number of orders per year $N^* = \frac{R}{q^*}$

e) Manufacturing time $= \frac{q^*}{K}$

**Example 1:**

The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3000 per month. The cost of one setup cost is Rs. 500.00 and the holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs. 20 per month. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and time between set – ups.

**Solution:**

R = 18,000 units per year

= 1500 units per month

K = 3000 units per month

$C_1$ = Rs. 0.15 per month

$C_2$ = Rs. 20.00

$C_3$ = Rs. 500

$$q^* = \sqrt{\frac{2C_3R}{C_1}}\sqrt{\frac{C_1+C_2}{C_2}}\sqrt{\frac{K}{K-R}}$$

$$= \sqrt{\frac{2\,X\,500\,X\,1500}{0.15}}\sqrt{\frac{0.15+20}{20}}\sqrt{\frac{3000}{3000-1500}}$$

$$= \frac{1224{,}744}{0.3872} \text{ x } 1.0037 \text{ x } 1.4142$$

$$= \frac{1738.458}{0.3872} = 4490 \text{ units (app)}$$

Number of shortages $S = \frac{C_1}{C_1 + C_2}\, q^* \left(1 - \frac{R}{K}\right)$

$$= \frac{0.15}{0.15+20} \text{ x } 4490 \left(1 - \frac{1500}{3000}\right)$$

$$= \frac{336.75}{20.15} = 16.71 \text{ units.}$$

Manufacturing time $= \frac{q^*}{K} = \frac{4490}{3000\,X\,12} = 0.1247$ years

Time between setup's $= \frac{q^*}{R} = \frac{4490}{18{,}000} = 0.2494$ years

**Inventory Models with Price breaks:**

In this section we shall consider a class of inventory problems in which the production (or) purchase cost per unit is a variable. This depends on the quality manufacture or purchased. This usually happen when discount are offered for the purchase of large quantities. These discounts take the form of Price-Breaks.

Consider the following three cases

Where $c_0(q) = \sqrt{2c_3\boldsymbol{k_1}PR} + k_1R + \frac{1}{2}c_3p$ ……(1)

and $q = \sqrt{\frac{2c_3R}{\boldsymbol{k_1P}}}$,

Total expected cost per unit time

$C\,(q) = \frac{C_3R}{q} + \frac{1}{2}\,q\,PI + PR$ ……(2)

$K_1$ = purchasing cost of care unit

p = holding cost/month expressed as a fraction of the values of the unit

Case (i): Purchase Inventory model with single price-break

| Given: | Unit purchasing cost | Range of quality |
|---|---|---|
| | $K_{11}$ | $0<q_1<b_1$ |
| | $K_{12}$ | $q_2 \geq b_1$ |

(i) If $b>q_2$ and $c_2(b) > c_0(q_1)$, the optimal lot size is $q_1$ and minimum values of $c(q) = c_0(q_1)$
(ii) If $b>q_2$ and $c_2(b) < c_0(q)$, the optimal lot size is b and min $c(q) = c_2(b)$
(iii) If $b<q_2$, the optimal lot size is $q_2$ and min $c(q) = c_0(q_2)$

Case (ii): Purchase Inventory Model with 2 prices – breaks

| Unit purchasing cost | Range of quality |
|---|---|
| $K_{11}$ | $0<q_1<b_1$ |
| $K_{12}$ | $b_1 \leq q_2 < b_2$ |
| $K_{13}$ | $b_2 \leq q_3$ |

The optimal purchase quality is determined in the following way

(i) Calculate $q_3$, If $q_3>b_2$, optimal purchase quality is $q_3$
(ii) If $q_3 \leq b_2$, calculate $q_2$ since $q_3<b_2$, the necessarily $q_2<b_2$. As a consequence we have $q_2<b_1$ or $q_2>b_1$.
(iii) If $q_3<b_2$and $b_1<q_2<b_2$, compare $c_0\ (q_2)$ with $c_3\ (b_2)$. The smaller of these qualities will be optimal purchase quantity.
(iv) If $q_3<b_2$ and $q_2<b_1$. Calculated $c_3\ (q_1)$ which will necessarily satisfy the inequality $q_1<b_1$. In this case compared $c_0\ (q_1)$, $c_2\ (b_1)$ and $c_3\ (b_2)$ to determine optimum purchase quantity.

Case (iii): Purchase inventory model with 'n' price breaks

When there are n price breaks, the situation can be represented as follows:

| Unit purchasing cost | Range of quality |
|---|---|
| $K_{11}$ | $0<q<b_1$ |
| $K_{12}$ | $b_1 \leq q<b_2$ |
| ………… | ……….. |
| $K_{1n}$ | $b_{n-1} \leq q$ |

(i) Calculate $q_n$. If $q_n > b_{n-1}$, optimal purchase quality is $q_n$

(ii) If $q_n < b_{n-1}$, calculate $q_{n-1}$. If $q_{n-1} \geq b_{n-2}$ proceed as in the case of one price break; (i.e.,) compare $c_0\,(q_{n-1})$ with $c(b_{n-1})$ to determine optimum purchase quality.

(iii) If $q_{n-1} < b_{n-2}$, compute $q_{n-2}$. If $q_{n-2} \geq b_{n-3}$, proceed as in the case of 2 price breaks: (i.e.,) compare $c_0(q_{n-2})$ with $c(b_{n-1})$ and $c(b_{n-2})$ to determine optimal purchase quality.

(iv) If $q_{n-2} < b_{n-3}$ compute $q_{n-3}$
If $q_{n-3} \geq b_{n-4}$ compare $c_0(q_{n-3})$ with $C(b_{n-3})$, $C(b_{n-2})$ and $C(b_{n-1})$.

(v) Continue in this number until $q_{n-j} \geq b_n - (j+1)(0 \leq j \leq n-1)$ and then compare $C_0(q_{n-j})$ with $C(b_{n-j})$, $c(b_{n-j+1})$, $C(b_{n-j+2})$……….$C(b_{n-1})$. This procedure involves only a finite number of steps.

**Example 1:**

Find the optimal order quantity for a product for which the price – break is as follows:

| Quantity | unit cost |
|---|---|
| $0 \leq Q_1 < 50$ | Rs. 10 |
| $50 \leq Q_1 < 100$ | Rs. 9 |
| $100 \leq Q_3$ | Rs. 8 |

The monthly demand for the product is 200 units, the cost of the storage is 25% of the unit cost and ordering cost is Rs. 20.00 per order.

Here R=200 units, P=0.25,$C_3$= Rs.20.00

$$Q_3^0 = \sqrt{\frac{2\ X\ 20\ X\ 200}{8\ X\ 0.25}}$$

= 63 units

Clearly 63 < 100(i.e.,) $Q_{3^o} < b_2$

∴ We compute $Q_2^0 = \sqrt{\frac{2\ X\ 20\ X\ 200}{9\ X\ 0.25}}$

= 60 units

Now since $Q_2^0 > b_1 (= 50)$ the optimum purchase quantity is determined by comparing $C_A(Q_{2^o})$ with $C_A\ (b_2\ )$

Now $C_A(Q_2^0) = 20 \text{ x } \frac{200}{60} + 200 \text{ x } 9 + 9 \text{ x } 0.25 \text{ x } \frac{60}{2}$

= Rs. 1934.16

$C_A\ (b_2\ ) = 20 \text{ x } \frac{200}{100}\ + 200 \text{ x } 8 + 8 \text{ x } 0.25 \text{ x } \frac{100}{2}$

= Rs.1740.00

Since $C_A(Q_2^0) > C_A\ (b_2\ )$, the optimum purchase quantity is $Q^0 = b_2 = 100$ units.

**Example 2:**

Find the optimal order quantity for which the price breaks are as follows:

| Quantity | Unit cost |
|---|---|
| $0 \leq q_1 < 500$ | Rs.10 |
| $500 \leq q_2 < 750$ | Rs.9.25 |
| $750 \leq q_3$ | Rs.8.75 |

The monthly demand for the product is 200 units, shortage cost is 2% of the unit cost and the cost of ordering is Rs. 100.

**Solution:**

$$q_3 = \sqrt{\frac{2c_3\ R}{k_{13}P}} = \sqrt{\frac{2\ X\ 100\ X\ 200}{8.75\ X\ 0.2}} = 478 \text{ Units}$$

$b_2 = 750$

$q_3 < b_2$, we calculate $q_2$

$$q_2 = \sqrt{\frac{2c_3\ R}{k_{12}P}}$$

$$= \sqrt{\frac{2\ X\ 100\ X\ 200}{9.25\ X\ 0.02}} = 465 \text{ Units}$$

$b_1 = 500$ Units , $q_2 < b_1$, we compute $q_1$

$$q_2 = \sqrt{\frac{2c_3\ R}{k_{11}P}} = \sqrt{\frac{2\ X\ 100\ X\ 200}{10\ X\ 0.02}} = 447 \text{ Units}$$

Next we compute

*Now $C_0\,(q_1) = \sqrt{2c_3\,k_{11}PR} + k_{11}R + \frac{1}{2}c_3P$

$= Rs.\ [\sqrt{2\ X\ 100\ X\ 10\ X\ 0.02\ X\ 200} + 10 \times 200 + \frac{1}{2} \times 100 \times 0.02]$

= Rs. 2090.42

$C_2\,(b_1) = c_3\frac{R}{q} + k_{12}\,R + \frac{1}{2}c_3P + \frac{1}{2}k_{12}\,pq$

$= Rs.\ [100 \times \frac{200}{500} + 9.25 \times 200 + \frac{1}{2} \times 100 \times 0.02 \times \frac{1}{2} \times 9.25 \times 0.02 \times 500]$

= 1937.25

$C_3\,(b_2) = c_3\frac{R}{q} + k_{13}\,R + \frac{1}{2}c_3P + \frac{1}{2}k_{13}pq$

$= Rs.\ [100 \times \frac{200}{750} + 8.75 \times 200 + \frac{1}{2} \times 100 \times 0.02 \times \frac{1}{2} \times 8.75 \times 0.02 \times 750]$

= Rs. 1843.29

Since $C_3\,(b_2) < C_2\,(b_1) < C_0\,(q_1)$, the optimal order quantity is $b_2$=750 units. *we can use formula (2) under 12.9 also.

**Example 3:**

Find the optimum order quantity for a quantity for which the price breaks are as follows:

| Quantity | Unit cost |
|---|---|
| $0 \le Q_1 < 100$ | Rs.10 |
| $500 < Q_2$ | Rs.9.25 |

The monthly demand for the product is 200 units, the cost of storage is 2% of the unit cost and the cost of ordering Rs.350.00

**Solution:**

$$Q_2^0 = \sqrt{\frac{2\ X\ 350\ X\ 200}{9.25\ X\ 0.2}}$$

= 870 units

as $Q_2^0 > b_1$,(870 > 500)

Optimum purchase quantity = 870 units.

**Example 4:**

Find the optimum order quantity for a quantity for which the price breaks are as follows:

| Quantity | Unit cost |
|---|---|
| $0 \le Q_1 < 500$ | Rs.20 per unit |
| $500 \le Q_2 < 750$ | Rs.18 per unit |
| $750 \le Q_3$ | Rs.16 per unit |

The monthly demand for the product is 400 units. The shortage cost is 20% of the unit cost of the product and the cost of ordering is 25.00 per month.

**Solution:**

R = 100 units, I = Re. 0.20, $C_3$ = Rs.25.00

$$Q_3^0 = \sqrt{\frac{2c_3\,R}{k_1 P}}$$

$$= \sqrt{\frac{2\ X\ 25\ X\ 400}{16\ X\ 0.20}} \qquad = 79 \text{ units}$$

Since $Q_{3}o < b_2$, we compute $Q_2^0$,

∴ We have $Q_2^0 = \sqrt{\frac{2\ X\ 25\ X\ 400}{20\ X\ 0.20}}$ = 75 units

Now since $Q_2^0 < b_1$(= 100) we next compute $Q_1^0$

Next we compute

Now $C_A(Q_1^0)$ = 25 x $\frac{400}{70}$ + 400 x 20 + 20 x 0.20 x $\frac{70}{2}$ = Rs. 8283.00

$C_A\ (b_1\ )$ = 25 x $\frac{400}{100}$ + 400 x 18 + 18 x 0.20 x $\frac{100}{2}$ = Rs.7480.00

$C_A\ (b_2\ )$ = 25 x $\frac{400}{200}$ + 400 x 16 + 16 x 0.20 x $\frac{200}{2}$ = Rs.6770.00

Since $C_A\ (b_2\ )\ < C_A\ (b_1\ ) < C_A(Q_1^0)$, the optimal purchase quantity is $Q^0$ = $b_2$ = 200 units.

**Example 5:**

Find the optimal quantity for a product where the annual demand for the product is 500 units. The cost of storage per unit per year is 10% of the unit cost and the ordering cost per order is Rs.180.00. The unit costs are given below.

| Quantity | Unit cost |
|---|---|
| $0 \le Q_1 < 500$ | Rs. 25 |
| $500 \le Q_2 < 1500$ | Rs. 24.80 |
| $1500 < Q_3 < 3000$ | Rs. 24.60 |
| $3000 < Q_4$ | Rs. 24.40 |

**Solution:**

R = 500 units, P = Rs. 0.10, $C_3$ = Rs.180.00

$$Q_4^0 = \sqrt{\frac{2c_3\,R}{k_4P}}$$

$$= \sqrt{\frac{2\ X\ 180\ X\ 500}{(24.40)\ X\ 0.10}} \qquad = 272$$

Since $Q_4^0 < b_3$, we compute $Q_3^0$,

$$Q_3^0 = \sqrt{\frac{2c_3\,R}{k_3P}} = \sqrt{\frac{2\ X\ 180\ X\ 500}{(24.60)\ X\ 0.10}} \qquad = 270$$

Since $Q_3^0 < b_2$ (= 1500) ∴ we calculate

$$Q_2^0 = \sqrt{\frac{2c_3\,R}{k_2P}} = \sqrt{\frac{2\ X\ 180\ X\ 500}{(24.80)\ X\ 0.10}} \qquad = 269$$

Since $Q_2^0 < b_1$ (= 500) ∴ we calculate

$$Q_1^0 = \sqrt{\frac{2c_3\,R}{k_1P}} = \sqrt{\frac{2\ X\ 180\ X\ 500}{25\ X\ 0.10}} \qquad = 268$$

Now $C_A(Q_1^0) = 180 \text{ x } \frac{500}{268} + 500 \text{ x } 25 + 25 \text{ x } 0.10 \text{ x } \frac{268}{2}$ = Rs. 13,170.82

$C_A\ (b_1\ ) = 180 \text{ x } \frac{500}{500} + 500 \text{ x } 24.80 + (24.80) \text{ x } 0.10 \text{ x } \frac{500}{2}$

= Rs.13,200.00

$C_A\ (b_2\ ) = 180 \text{ x } \frac{500}{1500} + 500 \text{ x } 24.60 + (24.60) \text{ x } 0.10 \text{ x } \frac{1500}{2}$

= Rs.14,205.00

$C_A\ (b_3\ ) = 180 \text{ x } \frac{500}{3000} + 500 \text{ x } 24.40 + (24.40) \text{ x } 0.10 \text{ x } \frac{3000}{2}$

= Rs.15,890.00

Since $C_A\ (b_3) > C_A\ (b_2\ ) > C_A\ (b_1\ ) > C_A(Q_1^0)$, the optimum purchase quantity is

$Q^0 = Q_1^0$ = 268 units.

**Example 6:**

Find the optimal order quantity for the following annual demand = 3600 units, order cost = Rs. 50, cost of storage = 20% of the unit cost

| | | |
|---|---|---|
| Price break | $0 < Q_1 < 100$ | Rs. 20 |
| | $750 \leq Q_2$ | Rs. 18 |

**Solution:**

Given R = 3600 units per year,

I = Rs.$\frac{20}{100}$,

$k_1$ = Rs.200.00

$k_2$ = Rs.18.00

$Q_2^0 = \sqrt{\frac{2\ X\ 50\ X\ 3600}{18\ X\ 0.20}}$ = 316.20

Now $b$ = 100 as $Q_2^0 > b$ , Optimum purchase quantity = 316.20

# QUEUING THEORY

## Introduction

In everyday life it is seen that a number of people arrive at a cinema ticket window. If the people arrive too frequently they will have to wait for getting tickets or sometimes do without it. Such problems arise in Railways, Airline etc. Under such circumstances the only alternative is to form a Queue called the Waiting Line in order to get the service more effectively. If we have too many counters for service then expenditure may be high. On the other hand if we have only few countries then Queue may become longer resulting in the dissatisfaction or loss of customers. Queuing models are aids to determine the optimal number of counters so as to satisfy the customers keeping the total cost minimum. Here the arriving people are called customers and the person issuing the tickets is called a server.

Servers may be in parallel or in series. When in parallel, the arriving customers may form a single Queue or several Queues as is common in biog post offices. Service time may be constant or variable and customers may be served singly or in batches.

### Queuing System

A queuing system can be completely described by

(a) The input (or arrival pattern).
(b) The service mechanism (or service pattern).
(c) The Queue discipline.
(d) Customer's behavior.

### (a)The input (or arrival pattern)

The input describes the way in which the customers arrive and join the system. Generally the customers arrive in more or less random fashion which is not worth making the prediction. Thus the arrival pattern can best be described in term of probabilities and consequently the probability distribution for inter arrival times (the time between two successive arrivals) must be defined. We deal with those Queuing system in which the customers arrive in 'Poisson' fashion. Mean arrival rate is denoted by

### (b) The service mechanism (or service pattern)

The service pattern is specified when it is known how many customers can be served at a time, what the statistical distribution of the service time is, and when the service is available. Service time may be constant or a random variable. Distribution of service time which is important in practice is the **negative exponential distribution.** The mean service rate is denoted by$\mu$.

**(c) The Queue discipline**

The queue discipline is the rule determining the formation of the Queue, the manner of the customer's behavior while waiting, and the manner in which they are chosen for service. The simplest discipline id first come, First Served" according to which the customers are served in the order of their arrival. Such type of Queue discipline is observed at a ration shop. If the order is reversed, we have 'Last come, first served' discipline, as in the case of a big godown the items which come last are taken out first.

Some of the queue service disciplines are:

FIFO – First in, First out or (FCFS)

LIFO – Last in, First out, (LCFS)

SIRO – Service in Random order.

**(d)Customer's behavior:**

The customer generally behaves in 4 ways:

(i) Balking: A customer may leave the Queue, if there is no waiting space.

(ii) Reneging: This occurs when the waiting customer leaves the Queue due to Impatience.

(iii) Priorities: In certain applications some customers are served before others regardless of their order of arrival.

(iv) Jockeying: Customers may jump from one, waiting line to another.

## Transient and Steady States

A system is said to be in **Transient State** when its operating characteristics are dependent on time.

**Steady State:** A system is said to be in **Steady State** when the behavior of the system is independent of time. Let $P_n(t)$ denote the probability that there are 'n' units in the system at time t. Then in steady state

$$=> \lim_{t\to\infty} P_n'(t) = 0$$

## Kendal'a Notation for representing Queuing models

Generally Queuing Model may be completely specified in the following symbol form :( a|b|c) :( d|e):

Where a=Probability law for the arrival

b=Probability law according to which customers are served.

c=Number of channels (or Service stations).

d=Capacity of the system.

e=Queue discipline.

**Distribution of Arrivals "The Poisson Process" Arrival Distribution Theorem. (Pure Birth Process)**

**If the arrivals are completely random, then the probability distribution of a number of arrivals in a fixed – time interval follows a Poisson distribution.**

## Model 1: (M|M|I): (∞/FCFS) – Birth and Death Model

**With usual notation, show that probability distribution of Queue length is given by $\rho_n (1-\rho)$ where $\rho = \frac{\lambda}{\mu} < 1$ and n≥0.**

## Measure of Model I

1. **To find the average (expected) number of units in the system, L.**
   **Solution:**
   By definition of excepted value

$$L_s = \sum_{n=1}^{\infty} nP_n = \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left[1 + 2\frac{\lambda}{\mu} + 3\left(\frac{\lambda}{\mu}\right)^2 + \dots\dots\dots + \right]$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right)^{-2} \text{ using Bionomial series}$$

$$= \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}$$

$$L_s = \frac{\rho}{1-\rho} \text{ where } \rho = \frac{\lambda}{\mu} < 1$$

2. **To find the average length of Queue, $L_q$**

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= \frac{\rho^2}{1-\rho}$$

3. **Excepted waiting time in the system**

$$W_s = \frac{L_s}{\lambda}$$

$$= \frac{1}{\mu-\lambda}$$

4. **Waiting time in the Queue,**

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)}$$

5. **Excepted waiting time of a customer who has to wait (W | W > 0)**

$$= \frac{1}{\mu-\lambda}$$

6. **Excepted length of the non – empty Queue, (L | L > 0)**

$$= \frac{\mu}{\mu-\lambda}$$

7. **Probability of Queue size ≥ N is $\rho^N$**
8. **Probability [Waiting time in the system ≥ *t*]**

$$= \int_t^{\infty} (\mu - \lambda) e^{-(\mu-\lambda)w} dw$$

9. **Probability [Waiting time in the queue ≥ *t*]**

$$= \int_t^{\infty} \rho(\mu - \lambda) e^{-(\mu-\lambda)w} dw$$

10. **Traffic Intensity = $\frac{\lambda}{\mu}$**

**Example 1:**

In a railway Marshalling yard, goods train arrive at a rate of 30 Trains per day. Assuming that inter arrival time follows an exponential distribution and the service time distribution is also exponential, with an average of 36 minutes. Calculate the following:

(i) The mean Queue size (line length)

(ii) The probability that Queue size exceeds 10

(iii) If the input of the train increases to an average 33 per dya, what will be the changes in (i), (ii)?

**Solution:**

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48}, \quad \mu = \frac{1}{36} \text{ trains per minute}$$

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

(i) $L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3$ trains

(ii) $P(\geq 10) = (0.75)^{10} = 0.056$

(iii) When the input increases to 33 trains per day,

We have $\lambda = \frac{30}{60 \times 24} = \frac{1}{480}$ and $\mu = \frac{1}{36}$ trains per minute

Now, $L_s = \frac{\rho}{1-\rho}$ where $\rho = \frac{\lambda}{\mu}$; $\rho = 0.825$

$\therefore L_s = \frac{0.825}{1-0.825} = 5$ trains (app)

Also $P(\geq 10) = \rho^{10} = (0.825)^{10}$

$= 0.1460$

**Example 2:**

In a super market, the average arrival rate of customer is 10 in every 30 minutes following Poisson process. The average time taken by the cashier to list and calculate the customer's purchases is 2.5 minutes, following exponential distribution. What is the probability that the Queue length exceeds 6? What is the excepted time spent by a customer in the system?

**Solution:**

Here the mean arrival rate

$$\lambda = \frac{10}{30} \text{ per minute}$$

and mean service rate $= \frac{1}{2.5}$ per minute

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{3}}{\frac{1}{2.5}} = 0.8333$$

(i) (The probability of Queue size > n) = $\rho^n$

When $n = 6 \Longrightarrow (0.8333)^6 = 0.3348$

(ii) $W_s = \frac{L_s}{\lambda} = \frac{\frac{\rho}{1-\rho}}{\lambda}, \rho = \frac{\lambda}{\mu}$

$$= \frac{0.8333}{1-0.8333} \text{ x } 3 = \frac{2.499}{0.167}$$

= 14.96 minutes

**Example 3:**

In a public telephone booth the arrivals are on the average 15 per hour. A call on the average takes 3 minutes. If there is just one phone, find (i) Excepted number of callers in the booth at any time (ii) The proportion of the time the booth is excepted to be Idle?

**Solution:**

Mean arrival rate λ = 15 per hour

Mean service rate $\mu = \frac{1}{3}$ x 60 = 20 per hr.

∴ (i). Excepted length of the non-empty

$$\text{Queue} = \frac{\mu}{\mu-\lambda} = \frac{20}{20-15} = 4$$

(ii). The service is busy means $= \frac{\lambda}{\mu} = \frac{15}{20} = \frac{3}{4}$

∴ The booth excepted to Idle for $1 - \frac{3}{4} = \frac{1}{4}$ hrs

= 15 minutes

**Example 4:**

A T.V repairman finds that the time spent on his job has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they came in and if the arrival of sets is Poisson with an average rate of 10 per 8 hour day, what is his excepted Idle time day? How many hobs are ahead of the average set just brought in?

**Solution:**

Mean service rate $\mu = \frac{1}{30}$ per minute

$= \frac{1}{30} \text{ x } 60 = 2$ sets per hour

Mean arrival rate $= \frac{10}{8}$ per hr

$\rho = \frac{\lambda}{\mu}$ where $\mu = 2$ per hr.

$\rho = \frac{5}{4}$ per hr

The utilization factor $\frac{\lambda}{\mu}$ is $\frac{5}{4 \text{ x } 2} = \frac{5}{8}$

For 8 hr day, Repairman's busy time = 8 x $\frac{5}{8}$ = 5 hrs

∴ Idle time of repairman = 8 – 5 hrs = 3 hrs

The number of jobs ahead = No. of units in the system

$$= \frac{\rho}{1-\rho} = \frac{\frac{5}{8}}{1-\frac{5}{8}} = \frac{\frac{5}{8}}{\frac{3}{8}} = \frac{5}{3}$$

= 2 app, TV sets

**Example 5:**

At a public Telephone booth in a Post Office arrivals are considered to be Poisson with an average inter-arrival time of 12 minutes. The length of the phone call may be assumed to be distributed exponentially with an average of 4 minutes. Calculate the following:

(a) What is the probability that a fresh arrival will not have to wait for the phone?
(b) What is the probability than an arrival will have to wait more than 10 minutes before the phone is free?
(c) What is the average length of Queues formed from time to time?

**Solution:**

Mean arrival rate, $\lambda = \frac{1}{12}$, $\mu = \frac{1}{4}$

Mean service rate, $\frac{\lambda}{\mu} = \frac{4}{12} = \frac{1}{3} = 0.33$

(a) Probability that a fresh arrival will not have to wait

$= 1 - \frac{\lambda}{\mu} = 1 - 0.33$

$= 0.67$

(b) Probability that an arrival will have to wait for atleast 10 minutes

$= \int_t^\infty \left(\frac{\lambda}{\mu}\right)(\mu - \lambda)e^{-(\mu-\lambda)t}dt$

$= \int_t^\infty (0.33)(0.25 - 0.083)e^{-0.167t}dt$

$= 0.05511\left[\frac{e^{-0.167t}}{-0.167}\right]_{10}^{\infty}$

$= 0.0621$

(c) The average length of Queues from time to time

$(L > 0) = \frac{\mu}{\mu-\lambda}$

$= \frac{0.25}{0.25-0.085}$

$= 1.5$

**Example 6:**

**People arrive at a Theatre ticket booth in Poisson distributed arrival rate of 25 per hour. Service time is constant at 2 minutes. Calculate**

**(a) The mean number in the waiting line**
**(b) The mean waiting time**
**(c) The utilization factor.**

**Solution:**

$\lambda =$ per hr;

$\mu = \frac{1}{2}$ x 60 = 30 per hr.

$\therefore \rho = \frac{\lambda}{\mu} = \frac{25}{30} = \frac{5}{6} = 0.833$

(i) Length of the Queue

$$L_q = \frac{\rho^2}{1-\rho}$$

$$= \frac{(0.833)^2}{1-0.833}$$

$$= \frac{0.693889}{0.167} = 4 \text{ (app)}$$

(ii) Mean waiting time $= \frac{L_q}{\lambda}$

$$= \frac{4}{25}$$

$$= 9.6 \text{ minutes}$$

(iii) Utilization factor $\rho = \frac{\lambda}{\mu} = 0.833.$

**Model II:**

**(M | M | I) : (N | FCFS)**

**Here the capacity of the system is limited, say N. Infact arrivals will not exceed N in any case. The various measures of this Model are**

**1.** $P_0 = \frac{1-\rho}{1-\rho^{N+1}}$ where $\rho = \frac{\lambda}{\mu}, \left\{\frac{\lambda}{\mu} > 1 \text{ is allowed}\right\}$

**2.** $P_n = \frac{1-\rho}{1-\rho^{N+1}} \rho^n$ for $n = 0, 1, 2,\ldots N$

**3.** $L_s = P_0 \sum_{n=0}^{N} n\rho^n$

**4.** $L_q = L_s - \frac{\lambda}{\mu}$

**5.** $W_s = \frac{L_s}{\lambda}$

**6.** $W_q = \frac{L_q}{\lambda}$

**Example 1:**

**If for period of 2 hours in a day (8 – 10 AM) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period**

**(a) The probability that the yard is empty**
**(b) Average Queue length, assuming that capacity of the yard is 4 trains only.**

**Solution:**

Here $\rho = \frac{36}{20} = 1.8$, N = 4

(a) $P_0 = \frac{\rho - 1}{\rho^5 - 1} = 0.04$

(b) Average Queue size

$= P_0 \sum_{n=0}^{N} n\rho^n$

$= 0.04\ (\rho + 2\rho^2 - 3\rho^3 + 4\rho^4)$

$= 2.9 \approx 3$ trains

**Example 2:**

**In a railway marshalling yard, goods training trains arrive at a rate of 30 trains per day. Assume that the inter arrival – time follows an exponential distribution and the service time distribution is also exponential with an average of 36 minutes, calculate**

**(a) The probability that the yard is empty**
**(b) Average queue length assuming that the line capacity of the yard is 9 trains.**

**Solution:**

Here $\rho = \frac{\lambda}{\mu} \Longrightarrow \rho = 0.75$

(a) The probability that the queue size is zero is given by

$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$ where N = 9

$P_0 = \frac{1-0.75}{1-(0.75)^{10}} = \frac{0.25}{0.90} = 0.2649$

(b) Average Queue length is given by the formula,

$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n$

$L_s = \frac{1-0.75}{1-(0.75)^{10}} \sum_{n=0}^{9} n(0.75)^n$

$= 0.28 \times 9.58 \approx 3$ trains.

**Example 3:**

**A barbershop has space to accommodate only 10 customers. He can service only one person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customers randomly, arrive at an average rate λ = 10 per hours and the barbers service time is negative exponential with an average of $\frac{1}{\mu}$ = 5 minutes per customer. Find $P_0$, $P_n$.**

**Solution:**

Here N = 10, $\lambda = \frac{10}{60}$, $\mu = \frac{1}{5}$

$\rho = \frac{\lambda}{\mu} = \frac{5}{6}$

$P_0 = \frac{1-\rho}{1-\rho^{11}} = \frac{1-\frac{5}{6}}{1-\left(\frac{5}{6}\right)^{11}}$

$= \frac{0.1667}{0.8655} = 0.1926$

$P_n = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$

$= (0.1926) \times \left(\frac{5}{6}\right)^n$, n = 0, 1, 2, …..,10

**Example 4:**

**A car park contains 5 cars. The arrival of cars is poisson at a mean rate of 10 per hour. The length of time each car spends in the car park is negative exponential distribution with mean of 2 hours. How many cars are in the car park on average?**

**Solution:**

N = 5, $\lambda = \frac{10}{60}$, $\mu = \frac{1}{2 \times 60}$, $\rho = \frac{\lambda}{\mu} = 20$

$P_0 = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)$

$= \frac{1-20}{1-20^6} = \frac{-19}{-6399} = 2.962 \times 10^{-7}$

$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n$

$= (2.9692 \times 10^{-3}) \times \sum_{n=0}^{5} n(2.9692 \times 10^{-3})^{n}$

$= (2.9692 \times 10^{-3}) \times [0 + (2.9692 \times 10^{-3}) + 2 \times (2.9692 \times 10^{-3})^{2} + 3 \times (2.9692 \times 10^{-3})^{3} + 4 \times (2.9692 \times 10^{-3})^{4} + 5 \times (2.9692 \times 10^{-3})^{5}]$

$= (2.9384 \times 10^{-3}) \times [0 + (2.9692 \times 10^{-3}) + 2 \times (2.9384 \times 10^{-3}) + 3 \times (2.9692 \times 10^{-3})^{2} + 4 \times (2.9692 \times 10^{-3})^{3} + 5 \times (2.9692 \times 10^{-3})^{4}]$

= 5 (app)

**Example 5:**

**At a one-man barber shop, the customers arrive following poisson process at an average rate of 5 per hour and they are served according to exponential distribution with an average service rate of 10 minutes. Assuming that only 5 seats are available for waiting customers, find the average time a customers, find the average time a customer spends in the system.**

**Solution:** $W_s = \frac{P_0}{\lambda} \sum_{n=0}^{N} n\rho^{n}$

Here $\lambda = 5$ per hr

$\mu = \frac{1}{10} \times 60$

= 6 per hr and N = 5

$\therefore \frac{1}{\mu} = \frac{5}{6} = \rho$

$P_0 = \frac{1-\rho}{1-\rho^{6}} = \frac{1-\frac{5}{6}}{1-\left(\frac{5}{6}\right)^{6}}$

$= \frac{1-\frac{1}{6}}{1-\left(\frac{1}{6}\right)^{6}} = \frac{\frac{1}{6}}{1-1.07 \times 10^{-4}}$

$= \frac{0.1666}{1-0.0001} = \frac{0.1666}{1}$

= 0.1666

$\frac{L_s}{\lambda} = W_s$

Where $L_s = 0.166 \times \sum_{n=0}^{N} n\rho^{n}$

$$= 0.166 \text{ x } \sum_{n=0}^{N} n\left(\frac{5}{6}\right)^n$$

$$= 0.166 \text{ x } [\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5]$$

$$= 0.166\left(\frac{5}{6} + 2\left(\frac{5}{6}\right)^2 + 3\left(\frac{5}{6}\right)^3 + 4\left(\frac{5}{6}\right)^4 + 5\left(\frac{5}{6}\right)^5\right)$$

$$= 0.166\ [0.833 + (2 \text{ x } 0.694) + (3 \text{ x } 0.5782) + (4 \text{ x } 0.4816) + (5 \text{ x } 0.4012)]$$

$$W_s = \frac{0.1666}{5}\left[0.833 + 1.388 + 1.7346 + 1.9264 + 2.006\right]$$

$$= \frac{0.1666 \text{x } 7.88}{5} = \frac{1.3094}{5} = 0.26 \text{ hrs} \approx 16 \text{ minutes}$$

## Possible Questions Part B (2 Marks)

1. Write the Kendal notation for queuing models.
2. Define Poisson random variables
3. Define exponential random variable.
4. Find the mean of Poisson random variable.
5. Fine the mean of exponential random variable.

## Possible Questions Part C (5 Marks)

1. The annual demand for an item is 3200 units. The unit cost is Rs.6/- and inventory carrying charges 25%per annum. If the cost of one procurement is Rs.150/- determine (i) economic order quantity (ii) time between two consecutive orders (iii) number of orders per year (iv) the optimal total cost.
2. If for period of 2 hours in a day ( 8 – 10AM) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period i) the Probability that the yard is empty ii) average Queue length, assuming that capacity of the yard is 4 trains only.
3. At a one-man barber shop, the customers arrive following Poisson process at an average rate of 5 per hour and they are served according to exponential distribution with an average service rate of 10 minutes. Assuming that only 5 seats are availablefor waiting customers, find the average time a customers, find the average time a customer spends in the system.

**(OR)**

4. Find the optimal order quantity for a product for which the price- break is as follows:

Quantity Unit cost

| | |
|---|---|
| $0 \le Q_1 < 50$ | Rs. 10 |
| $50 \le Q_2 < 100$ | Rs. 9 |
| $100 \le Q_3$ | Rs. 8 |

The monthly demand for the product is 200 units, the cost of the storage is 25% of the unit cost and ordering cost is Rs.20.00 per order.

5. In a super market, the average arrival rate of customer is 10 in every 30 minutes following Poisson process. The average time taken by the cashier to list and calculate the customer's purchases is 2.5 minutes, following exponential distribution. What is the probability that the Queue length exceeds 6? What is the excepted time spent by a customer in the system?
6. Find the optimal order quantity for a product where the annual demand for the product is 500 units. The cost of storage per unit per year is 10% of the unit cost and the ordering per order is Rs.180.00. The unit costs are given below.

| Quantity | Unit cost |
|---|---|
| $0 \le Q_1 < 500$ | Rs. 25 |
| $500 \le Q_2 < 1500$ | Rs. 24.80 |
| $1500 \le Q_3 < 3000$ | Rs.24.60 |
| $3000 < Q_4$ | Rs.24. |

## Possible Questions Part D (10 Marks)

1. In a railway marshaling yard, goods trains arrive at a rate of 30 trains per day. Assume that the inter arrival- time follows an exponential distribution and the Service time distribution is also exponential with an average of 36 minutes. Calculate the mean queue size and the probability that queue size exceeds 10
2. A company has a demand of 12,000 units per year for an item and it can produce2000 such items per month. The cost of one set up is Rs 400 and the holding costunit/month is Rs.0.15. Find
   1) The optimum lot size
   2) Maximum inventory
   3) Manufacturing time
   4) Total time
3. At a one-man barber shop, the customers arrive following poisson process at an average rate of 5 per hour and they are served according to exponential distribution with an average service rate of 10 minutes. Assuming that only 5 seats are available for waiting customers, find the average time a customers, find the average time a customer spends in the system.
4. A company has a demand of 18,000 units per year for an item and it can produce 3000such items per month. The cost of one set up is Rs500. and the holding cost per unit per month is Rs.0.15. The shortage cost of one unit is Rs.20 per month. Determine the optimum