19BEAE201/19BEME201 MATHEMATICS II 3 1 0 4

(Calculus, Ordinary DifferentialEquations and Complex Variable for Mechanical and Automobile Engineering)

COURSE OBJECTIVES

The objective of this course is to familiarize the prospective engineers with techniques in Multivariate integration, ordinary and partial differential equations and complex variables. It aims to equip the students to deal with advanced level of mathematics and applications thatwould be essential for their disciplines.

INTENDED OUTCOME

The students will learn:

- The mathematical tools needed in evaluating multiple integrals and their usage.
- The effective mathematical tools for the solutions of differential equations that model physical processes.
- The tools of differentiation and integration of functions of a complex variable that are used in various techniques dealing engineering Problems.

UNIT I: Multivariable Calculus (Integration) 12

Multiple Integration: double and triple integrals (Cartesian and polar), change of order of integration in double integrals, Applications: areas and volumes, Center of mass and Gravity (constant and variable densities). Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds.

UNIT II: First order ordinary differential equations 12

Exact, linear and Bernoulli's equations, Euler's equations, Equations not of first degree :equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.

UNIT III: Ordinary differential equations of higher orders 12

Second order linear differential equations with variable coefficients, method of variation of parameters, Cauchy-Euler equation; Power series solutions; Legendre polynomials, Bessel functions of the first kind and their properties.

UNIT IV: Analytic Functions 12

Cauchy-Riemann equations, analytic functions, harmonic functions, finding harmonic conjugate; elementary analytic functions (exponential, trigonometric, logarithm)and their properties; Conformal mappings, Mobius transformations.

UNIT V: Complex Integration 12

Contour integrals, Cauchy- Goursat theorem (without proof), Cauchy Integral formula(without proof), zeros of analytic functions, singularities, Taylor's series, Laurent's series, Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral involving sine and cosine. **Total: 60**

TEXT/REFERENCE BOOKS

S. NO.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAROF PUBLICATION
1	Hemamalini. P.T	Engineering Mathematics	McGraw Hill Education (India) Private Limited, New Delhi.	2014
2	G.B. Thomas and R.L. Finney	Calculus and Analytic geometry, 9th Edition	Pearson	2002
3	Erwin kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley & Sons	2006
4	W. E. Boyce and R. C. DiPrima	Elementary Differential Equations and Boundary Value Problems9th Edn.	Wiley India	2009
5	S. L. Ross	Differential Equations, 3rd Ed.	Wiley India	1984
6	E. A. Coddington	An Introduction to Ordinary Differential Equations	Prentice Hall, India	1995
7	E. L. Ince	Ordinary Differential Equations	Dover Publications	1958
8	J. W. Brown and R. V. Churchill	Complex Variables and Applications, 7th Ed.	Mc-Graw Hill	2004
9	N.P. Bali and Manish Goyal	A text book of Engineering Mathematics	Laxmi Publications	2008
10	B.S. Grewal	Higher Engineering Mathematics, 36th Edition	Khanna Publishers	2010

Faculty Incharge **HoD**

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

COIMBATORE-641 021

DEPARTMENT OF SCIENCE AND HUMANITIES
FACULTY OF ENGINEERING

I B.E MECHANICAL / AUTOMOBILE ENGINEERING

LECTURE PLAN

Subject : MATHEMATICS – II
(Calculus, Ordinary Differential Equations and Complex Variable)

Code : 19BEME201/19BEAE201

S.NO	Topics covered	No. of hours
	UNIT I First order ordinary differential equations	
1	Introduction of Multiple Integration: double and triple integrals	1
2	Multiple Integration: double integral	1
3	Multiple Integration: double and triple integrals (Cartesian and polar),	1
4	Multiple Integration: Triple integrals	1
5	change of order of integration in double integrals	1
6	change of order of integration in double integrals	1
7	Tutorial 1 - Problems based on change of order of integration in double integrals	1
8	Applications: areas and volumes	1
9	Applications: areas and volumes	1
10	Center of mass and Gravity (constant and variable densities).	1
11	Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds.	1
12	Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds.	1
13	Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds.	1
14	Tutorial 2 - Problems based on Theorems of Green, Gauss and Stokes	1
	TOTAL	**14**
	UNIT II First order ordinary differential equations	
15	Introduction of first order differential equations	1
16	Exact, linear and Bernoulli's equations	1
17	Exact, linear and Bernoulli's equations	**1**
18	Euler's equations	1
19	Tutorial 3 - Problems based on Exact, linear and Bernoulli's equations	1
20	Equations not of first degree:Equations solvable for p	1
21	Equations not of first degree:Equations solvable for p	1
22	Equations solvable for y	1
23	Equations solvable for y	1
24	Equations solvable for x	1
25	Equations solvable for x	1
26	Clairaut's type	1
27	Clairaut's type	1
28	Tutorial 4 - Problems based on Clairaut's type, Equations solving for x and y, p	1

	TOTAL	**14**
	UNIT III Ordinary differential equations of higher orders	
29	Introduction of ordinary differential equations	1
30	Second order linear differential equations with variable coefficients	1
31	Second order linear differential equations with variable coefficients	1
32	Second order linear differential equations with variable coefficients	1
33	Second order linear differential equations with variable coefficients	1
34	Second order linear differential equations with variable coefficients	1
35	Tutorial 5 - Problems based on second order differential equations with variable coefficients	1
36	Method of variation of parameters	1
37	Cauchy-Euler equation	1
38	Power series solutions; Legendre polynomials	1
39	Power series solutions; Legendre polynomials	1
40	Bessel functions of the first kind and their properties	1
41	Bessel functions of the first kind and their properties	1
42	Tutorial 6 - Problems based on Bessel functions and Legendre polynomials	1
	TOTAL	**14**
	UNIT IV: Analytic Functions	
43	Introduction – Analytic Function	1
44	Necessary and Sufficient conditions for an analytic function- Cauchy-Riemann equations –Cartesian form	1
45	Necessary and Sufficient conditions for an analytic function- Cauchy-Riemann equations –Cartesian form	1
46	Cauchy-Riemann equations – Polar form	1
47	Harmonic functions and its conjugate	1
48	Tutorial 7-Cauchy-Riemann equations Harmonic functions	1
49	Properties of analytic functions	1
50	Construction of an Analytic Function Milne-Thomson method	1
51	Construction of an Analytic Function Milne-Thomson method	1
52	Conformal mapping: The transformations w = z+a, az	1
53	Conformal mapping: The transformations w = 1/z, Z^2	1
54	Bilinear transformation	1
55	Mobius transformations	1
56	Tutorial 8 - Conformal mapping, Mobius transformations	1
	TOTAL	**14**
	UNIT V Complex Integration	
57	Introduction - Complex Integration, Line integral	1
58	Problems solving using Cauchy’s integral theorem	1
59	Problems solving using Cauchy’s integral formula	1
60	Taylor’s Series Problems	1
61	Taylor’s Series Problems	1
62	Laurent series problems	1
63	Laurent series problems	1
64	Tutorial 9 - Taylor’s and Laurent’s series problems	1
65	Theory of Residues	1
66	Cauchy Residue theorem (without proof)	1
67	Cauchy Residue theorem- Problems	1
68	Evaluation of definite integral involving sine and cosine.	1

69	Evaluation of definite integral involving sine and cosine.	1
70	Tutorial 10 - Cauchy’s residue theorem, Applications	1
	TOTAL	**14**
	GRAND TOTAL	**70**

Staff- Incharge **HoD**

Unit VIII

Vector Integration

Chapter 20: Line Integral, Surface Integral and Integral Theorems

20 Line Integral, Surface Integral and Integral Theorems

Chapter Outline

- Introduction
- Integration of Vectors
- Line Integral
- Circulation
- Application of Line Integrals
- Surfaces
- Surface Integrals
- Volume Integrals
- Integral Theorems

20.1 ❑ INTRODUCTION

In multiple integrals, we generalized integration from one variable to several variables. Our goal in this chapter is to generalize integration still further to include integration over curves or paths and surfaces. We will define integration not just of functions but also of vector fields. Integrals of vector fields are particularly important in applications involving the "field theories" of physics, such as the theory of electromagnetism, heat transfer, fluid dynamics and aerodynamics.

In this chapter, we shall define line integrals and surface integrals. We shall see that a line integral is a natural generalization of a define integral and a surface integral is a generalization of a double integral. Line integrals can be transformed into double integrals or into surface integrals and conversely. Triple integrals can be transformed into surface integrals and vice versa. These transformations are of great practical importance. Theorems of Green, Gauss and Stokes serve as powerful tools in many applications as well as in theoretical problems.

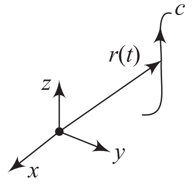

Fig. 20.1

In this chapter, we study the three main theorems of Vector Analysis: Green's Theorem, Stokes' Theorem and the Divergence Theorem. This is a fitting conclusion to the text because each of these theorems is a vector generalization of the Fundamental Theorem of calculus. This chapter is thus the culmination of efforts to extend the concepts and methods of single-variable calculus to the multivariable setting. However, far from being a terminal point, vector analysis the gateway to the field theories of mathematics physics and engineering. This includes, first and foremost, the theory of electricity and magnetism as expressed by the famous *Maxwell's equations*. It also includes fluid dynamics, aerodynamics, analysis of continuous matter, and at a more advanced level, fundamental physical theories such as general relativity and the theory of elementary particles.

Curves

Curves in space are important in calculus and in physics (for instance, as paths of moving bodies).

A curve C in space can be represented by a vector function

$$\begin{aligned}\vec{r}(t) &= [x(t), y(t), z(t)] \\ &= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\end{aligned} \tag{20.1}$$

where x, y, z are Cartesian coordinates. This is called a **parametric representation** of the curve (Fig. 20.1), t is called the **parameter** of the representation. To each value t_0 of t, there corresponds a point of C with position vector $\vec{r}(t_0)$, that is with coordinates $x(t_0)$, $y(t_0)$ and $z(t_0)$.

The parameter t may be time or something else. Equation (20.1) gives the **orientation** of C, a direction of travelling along C, so that t increasing is called the **positive sense** on C given by (20.1) and that of decreasing t is the **negative sense**.

• Examples

Straight line, ellipse, circle, etc.

The concept of a line integral is a simple and natural generalization of a definite

integral $\displaystyle\int_a^b f(x)dx$ (20.2)

In (20.2), we integrate the **integrand** $f(x)$ from $x = a$ to $x = b$ along the x-axis. In a line integral, we integrate a given function, called the integrand, along a curve C in space (or in the plane).

Hence, curve integral would be a better turn, but line integral is standard.

We represent a curve C by a parametric representation

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, (a \le t \le b)$$

We call C the **path of integration**, $A{:}\vec{r}(a)$ its **initial point** and $B{:}\vec{r}(b)$, its **terminal point**. The curve C is now oriented. The direction from A to B, in which t increases, is called the positive direction on C. We can indicate the direction by an arrow [Fig. 20.2(a)].

The points A and B may coincide [Fig. 20.2(b)]. Then C is called a **closed path**.

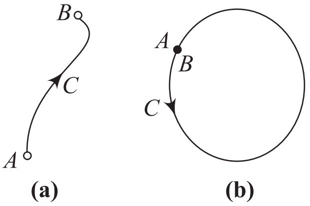

Fig. 20.2

➢ **Note**

(i) A **plane curve** is a curve that lies in a plane in space.
(ii) A curve that is not plane is called a **twisted curve**.

20.2 ❑ INTEGRATION OF VECTORS

If two vector functions $\vec{F}(t)$ and $\vec{G}(t)$ be such that $\dfrac{d\vec{G}(t)}{dt} = \vec{F}(t)$, then $\vec{G}(t)$ is called an integral of $\vec{F}(t)$ with respect to the scalar variable t and we write $\int \vec{F}(t)dt = \vec{G}(t)$. If $\vec{C}$ be an arbitrary constant vector, we have $\vec{F}(t) = \dfrac{d\vec{G}(t)}{dt} = \dfrac{d}{dt}[\vec{G}(t) + \vec{C}]$, then $\int \vec{F}(t)dt = \vec{G}(t) + \vec{C}$. This is called the indefinite integral of $\vec{F}(t)$ and its definite integral is $\displaystyle\int_a^b \vec{F}(t)dt = [\vec{G}(t) + \vec{C}]_a^b = \vec{G}(b) - \vec{G}(a)$.

20.3 ❑ LINE INTEGRAL

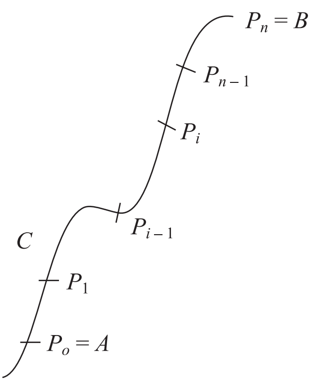

Fig. 20.3

Any integral which is to be evaluated along a curve is called a **line integral**. Consider a continuous vector point function $\vec{F}(R)$ which is defined at each point of the curve C in space. Divide C into n parts at the points $A = p_0, p_1 \ldots p_{i-1}, p_i \ldots p_n = B$

Let their position vectors be $\vec{R}_0, \vec{R}_1 \ldots \vec{R}_{i-1}, \vec{R}_i \ldots \vec{R}_n$

Let $\vec{v}_i$ be the position vector of any point on the arc $P_{i-1} P_i$

Now consider the sum $S = \displaystyle\sum_{i=0}^{n} \vec{F}(\vec{v}_i) \cdot \delta\vec{R}_i$ where $\overrightarrow{\delta R_i} = \overrightarrow{R_i} - \overrightarrow{R_{i-1}}$.

The limit of this sum as $n - \infty$ in such a way that $|\overrightarrow{\delta R_i}| \to 0$, provided it exists, is called the **tangential line integral** of $\vec{F}(\vec{R})$ along C which is a scalar and is symbolically written as

$$\int_C \vec{F}(\vec{R}) \cdot \overrightarrow{dR} \text{ or } \int_C \vec{F} \cdot \frac{d\vec{R}}{dt} \cdot dt$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\quad \vec{F}(\vec{R}) = f(x, y, z)\vec{i} + \phi(x, y, z)\vec{j} + \psi(x, y, z)\vec{k}$ and $d\vec{R} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

then $\displaystyle\int_C \vec{F}(\vec{R}) \cdot d\vec{R} = \int_C (fdx + \phi dy + \psi dz)$.

Two other types of line integrals are $\displaystyle\int_C \vec{F} \times d\vec{R}$ and $\displaystyle\int_C f d\vec{R}$ which are both vectors.

20.4 ❑ CIRCULATION

In fluid dynamics, if $\vec{F}$ represents the velocity of a fluid particle then the line integral $\int_C \vec{F}\cdot d\vec{r}$ is called the circulation of $\vec{F}$ around the curve. When the circulation of $\vec{F}$ around every closed curve in a region E vanishes, $\vec{F}$ is said to be **irrotational** in E.

Conservative Vector

If the value of $\int_A^B \vec{F}\cdot \overrightarrow{dr}$ does not depend on the curve C, but only on the terminal points A and B, $\vec{F}$ is called a **conservative vector**.

A force field $\vec{F}$ is said to be **conservative** if it is derivable from a potential function ϕ, i.e., $\vec{F} =$ grad ϕ. Then curl $(\vec{F})$ = curl $(\nabla\phi) = 0$.
$\therefore$ if $\vec{F}$ is **conservative** then curl$(\vec{F}) = 0$ and there exists a scalar potential function ϕ such that $\vec{F} = \nabla\phi$.

20.5 ❑ APPLICATIONS OF LINE INTEGRALS

Work Done by a Force

Let $\vec{v}\,(x, y, z) = v_1(x, y, z)\,\vec{i} + v_2(x, y, z)\,\vec{j} + v_3(x, y, z)\,\vec{k}$ be a vector function defined and continuous at every point on C. Then, the integral of the tangential component of $\vec{v}$ along the curve C from a point P on to the point Q is given by

$$\int_P^Q \vec{v}\cdot\overrightarrow{dr} = \int_{C_1} \vec{v}\cdot d\vec{r} = \int_{C_1} v_1 dx + v_2 dy + v_3 dz$$

where C_1 is the part of C, whose initial and terminal points are P and Q.

Let $\vec{v} = \vec{F}$, variable force acting on a particle which moves along a curve C. Then the work done W by the force $\vec{F}$ in displacing the particle from the point P to the point Q along the curve C is given by

$$W = \int_P^Q \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}$$

where C_1 is the part of C whose initial and terminal points are P and Q.

Suppose $\vec{F}$ is a conservative vector field; then $\vec{F}$ can be written as $\vec{F}$ = grad ϕ, where ϕ is a scalar potential.
Then, the work done

$$W = \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (\text{grad}\,\phi)\cdot d\vec{r}$$

$$= \int_{C_1} \left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right] = \int_P^Q d\phi = \left[\phi(x, y, z)\right]_P^Q$$

$\therefore$ work done depends only on the initial and terminal points of the curve C_1, i.e., the work done is independent of the path of integration. The units of work depend on the units of $|\vec{F}|$ and on the units of distance.

➢ **Note**

(i) **Condition for $\vec{F}$ to be conservative**

If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = 0$.

It is possible only when $\vec{F} = \nabla\phi$. which $\Rightarrow$ $\vec{F}$ is conservative.

$\therefore$ if $\vec{F}$ is an irrotational vector, it is conservative.

(ii) If $\vec{F}$ is irrotational (and, hence, conservative) and C is a closed curve then $\oint_C \vec{F}\cdot d\vec{r} = 0$. [$\because$ $\phi(A) = \phi(B)$, as A and B coincide].

20.6 ❑ SURFACES

A surface S may be represented by $F(x, y, z) = 0$.

The parametric representation of S is of the form

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

and the continuous functions $u = \phi(t)$ and $v = \phi(t)$ of a real parameter t represent a curve C on this surface S.

If S has a unique normal at each of its points whose direction depends continuously on the points of S then the surface S is called a **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions then it is called a **piecewise smooth surface**. For example, the surface of a sphere is smooth while the surface of a cube is piecewise smooth.

If a surface S is smooth from any of its points P, we may choose a unit normal vector $\vec{n}$ of S at P. The direction of $\vec{n}$ is then called the **positive normal direction of S at *P***. A surface S is said to be **orientable** or **two-sided**, if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface. If the positive direction of the normal is reversed as we move around a curve on S passing through P then the surface is **non-orientable** (i.e., one-sided) (Fig. 20.4).

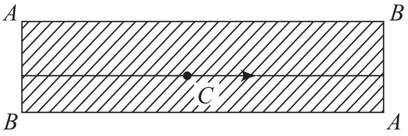

Fig. 20.4

- **Example**

A sufficiently small portion of a smooth surface is always orientable (Fig. 20.5).

A Mobius strip is an example of a non-orientable surface. A model of a Mobius strip can be made by taking a long rectangular piece of paper, making a half-twist and sticking the shorter sides together so that the two points A and the two points B coincide; then the surface generated is non-orientable.

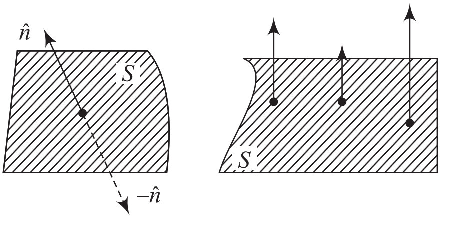

Fig. 20.5

20.7 ❑ SURFACE INTEGRALS

Any integral which is to be evaluated over a surface is called a **surface integral**.

Let S be a two-sided surface, one side of which is considered arbitrarily as the positive side.

Let $\vec{F}$ be a vector point function defined at all points of S. Let ds be the typical elemental surface area in S surrounding the point $P(x, y, z)$.

Let $\hat{n}$ be the unit vector normal to the surface S at $P(x, y, z)$, drawn in the positive side (or outward direction).

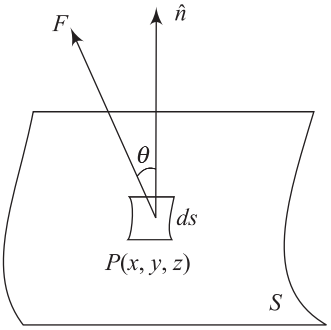

Fig. 20.6

Let θ be the angle between $\vec{F}$ and $\hat{n}$.

$\therefore$ the normal component of $\vec{F} = \vec{F}\cdot\hat{n} = F\cos\theta$.

The integral of this normal component through the elemental surface area ds over the surface S is called the **surface integral** of $\vec{F}$ over S and denoted as $\int_S F\cos\theta \cdot ds$ or $\int_S \vec{F}\cdot\hat{n}\,ds$.

If $d\vec{s}$ is a vector whose magnitude is ds and whose direction is that of $\hat{n}$, then $\overrightarrow{ds} = \hat{n}\cdot ds$. $\therefore \int_S \vec{F}\cdot\hat{n}\,ds$ can also be written as $\int_S \vec{F}\cdot\overrightarrow{ds}$.

➢ **Note**

(i) If S in a closed surface, the outer surface is usually chosen as the positive side.

(ii) $\int_S \phi\, d\vec{s}$ and $\int_S \vec{F}\times d\vec{s}$ where ϕ is a scalar point function are also surface integrals.

(iii) The surface integral $\int_S \vec{F}\cdot d\vec{s}$ is also denoted as $\iint_S \vec{F}\cdot d\vec{s}$.

(iv) If $\vec{F}$ represents the velocity of a fluid particle then the total outward flux of $\vec{F}$ across a closed surface S is the surface integral $\int_S \vec{F}\cdot d\vec{s}$.

(v) When the flux of $\vec{F}$ across every closed surface S in a region E vanishes, $\vec{F}$ is said to be a **solenoidal vector point function** in E.

(vi) It may be noted that $\vec{F}$ could equally well be taken as any other physical quantity such as gravitational force, electric force, magnetic force, etc.

20.8 ❑ VOLUME INTEGRALS

Any integral which is to be evaluated over a volume is called a **volume integral**.

If V is a volume bounded by a surface S then the triple integrals $\iiint_V \phi\, dv$ and $\iiint_V \vec{F}\, dv$ are called volume integrals. The first of these is a scalar and the second is a vector.

20.9 ❑ INTEGRAL THEOREMS

The following three theorems in vector calculus are of importance from theoretical and practical considerations:

(i) Green's theorem in a plane
(ii) Stokes' theorem
(iii) Gauss' divergence theorem

Green's theorem provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R. Green's theorem is also called the **first fundamental theorem** of integral vector calculus.

Stokes' theorem transforms line integrals into surface integrals and conversely. This theorem is a generalization of Green's theorem. It involves the curl.

Gauss' divergence theorem transforms surface integrals into a volume integral. It is named Gauss' divergence theorem because it involves the divergence of a vector function.

We shall give the statements of the above theorems (without proof) and apply them to solve problems.

Green's Theorem in a Plane

If C is a simple closed curve enclosing a region R in the xy-plane and $P(x, y)$, $Q(x, y)$ and its first-order partial derivatives are continuous in R then $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$ where C is described in the anticlockwise direction.

Stokes' Theorem (Relation between Line Integral and Surface Integral)

Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \cdot ds$

Gauss' Divergence Theorem or Gauss' Theorem of Divergence (Relation between Surface Integral and Volume Integral)

The surface integral of the normal component of a vector function $\vec{F}$ taken around a closed surface S is equal to the integral of the divergence of $\vec{F}$ taken over the volume V enclosed by the surface S.

Mathematically, $\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V \text{div } \vec{F} \cdot dv$.

SOLVED EXAMPLES

Example 1 If $\vec{F} = 3xy\vec{i}, - y^2\vec{j}$, evaluate $\int_C \vec{F} \cdot dr$, where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution Let $x = t$, then the parametric equations of the parabola $y = 2x^2$ are $x = t$, $y = 2t^2$.

At the point (0, 0), $x = 0$ and so $t = 0$.

At the point (1, 2), $x = 1$ and so $t = 1$.

If $\vec{r}$ is the position vector of any point (x, y) in C, then

$$\vec{r} = x\vec{i} + y\vec{j}$$
$$= t\vec{i} + 2t^2\vec{j}$$

Also in terms of t, $\vec{F} = 3t(2t^2)\vec{i} - (2t^2)^2\vec{j}$

$$= 6t^3\vec{i} - 4t^4\vec{j}$$

$$\therefore \qquad \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$
$$= \int_0^1 (6t^3\vec{i} - 4t^4\vec{j})\cdot(\vec{i} + 4t\vec{j})dt$$
$$= \int_0^1 (6t^3 - 16t^5)dt$$
$$= \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1$$
$$= \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$ **Ans.**

Example 2 Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = (x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant. **[KU May 2010]**

Solution A vector normal to the surface S is given by

$$\nabla(2x + y + 2z) = 2\vec{i} + \vec{j} + 2\vec{k}$$

$\therefore$ $\hat{n} = a$ unit vector normal to the surface S

$$= \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\vec{k}\cdot\hat{n} = \vec{k}\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = \frac{2}{3}$$

$$\therefore \qquad \iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$$

where R is the projection of S

Now, $$\vec{A}\cdot\hat{n} = [(x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}]\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$$
$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$
$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right)$$

$$\left(\text{since on the plane } 2x + y + 2z = 6,\ z = \frac{6-2x-y}{2}\right)$$

$$= \frac{2}{3}y(y + 6 - 2x - y)$$

$$= \frac{4}{3}y(3-x)$$

Hence, $\iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$.

$$= \iint_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}dx\,dy$$

$$= \int_0^3\int_0^{6-2x} 2y(3-x)dy\,dx$$

$$= \int_0^3 2(3-x)\left(\frac{y^2}{2}\right)_0^{6-2x} dx$$

$$= \int_0^3 (3-x)(6-2x)^2\,dx$$

$$= 4\int_0^3 (3-x)^3\,dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3$$

$$= 81$$ **Ans.**

Example 3 If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$ then evaluate $\iiint \nabla\cdot\vec{F}\cdot dV$, where V is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution $$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x = 2x$$

$\therefore$ $$\iiint_V \nabla\cdot F\cdot dv = \iiint_V 2x\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x[z]_0^{4-2x-2y}\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2x(4-2x-2y)\,dy\,dx = \int_0^2 \int_0^{2-x} [4x(2-x)-4xy]\,dy\,dx$$

$$= \int_0^2 [4x(2-x)y - 2xy^2]_0^{2-x} \cdot dx$$

$$= \int_0^2 [4x(2-x)^2 - 2x(2-x)^2]\,dx$$

$$= \int_0^2 2x(2-x)^2\,dx$$

$$= 2\int_0^2 (4x - 4x^2 + x^3)\cdot dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left[8 - \frac{32}{3} + 4\right] = \frac{8}{3}$$ **Ans.**

Example 4 Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ and the curve C is the rectangle in the xy-plane bounded by $y = 0$, $y = b$, $x = 0$, $x = a$.

Solution In the xy-plane, $z = 0$

$$\vec{r} = x\vec{i} + y\vec{j},\ d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+y^2)dx - 2xydy \quad (1)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \quad (2)$$

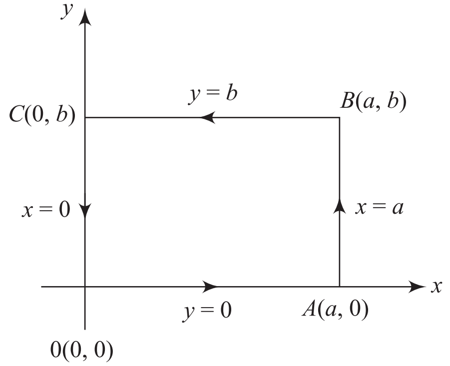

Fig. 20.7

Along OA, $y = 0$; $dy = 0$ and x varies from 0 to a
Along AB, $x = a$; $dx = 0$ and y varies from 0 to b

Along BC, $y = b$; $dy = 0$ and x varies from a to 0
Along CO, $x = 0$; $dx = 0$ and y varies from b to 0
Hence, from (1) and (2),

$$\int_C \vec{F}\cdot\overrightarrow{dr} = \int_{x=0}^{a} x^2\,dx - \int_{y=0}^{b} 2ay\,dy + \int_{x=a}^{0}(x^2+b^2)\,dx + \int_b^0 0\cdot dy$$

$$= \left(\frac{x^3}{3}\right)_0^a - (ay^2)_0^b + \left(\frac{x^3}{3}+b^2x\right)_a^0 + 0$$

$$= \left(\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2\right) = -2ab^2$$ **Ans.**

Example 5 Find the work done by the force $\vec{F} = (2xy+z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ when it moves a particle from (1, –2, 1) to (3, 1, 4) along any path. **[AU Dec. 2011]**

Solution Since the equation of the path is not given, the work done by the force $\vec{F}$ depends only on the terminal points.

Consider $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy+z^3) & x^2 & 3xz^2 \end{vmatrix}$

$$= \vec{i}[0-0] - \vec{j}[3z^2-3z^2] + \vec{k}[2x-2x] = 0$$

$\Rightarrow$ $\vec{F}$ is irrotational

Hence, $\vec{F}$ is conservative

Since $\vec{F}$ is irrotational, we have $\vec{F} = \nabla\phi$

It is easy to see that $\phi = x^2y + xz^3 + C$

$\therefore$ work done by $\vec{F} = \displaystyle\int_{(1,-2,1)}^{(3,1,4)} \vec{F}\cdot d\vec{r}$

$$= \int_{(1,-2,1)}^{(3,1,4)} \nabla\phi\cdot\overrightarrow{dr} = \int_{(1,-2,1)}^{(3,1,4)} d\phi \quad [\text{as } \nabla\phi\cdot\overrightarrow{dr} = d\phi]$$

$$= [\phi]_{(1,-2,1)}^{(3,1,4)}$$

$$= [x^2y + xz^3 + C]_{(1,-2,1)}^{(3,1,4)}$$

$$= (201 + C) - (-1 + C) = 202$$ **Ans.**

Example 6 Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = e^x \sin y\vec{i} + e^x \cos y\vec{j}$; and C is the rectangle whose vertices are $(0,0), (1,0), \left(1, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.

Solution

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\therefore \quad \vec{F} \cdot d\vec{r} = e^x \sin y \cdot dx + e^x \cos y \cdot dy$$

Now along OA, $y = 0$; $dy = 0$
along AB, $x = 1$; $dx = 0$

along $BC, y = \dfrac{\pi}{2}; dy = 0$

along CO, $x = 0$; $dx = 0$
$\therefore$ circulation round the rectangle $OABC$ is

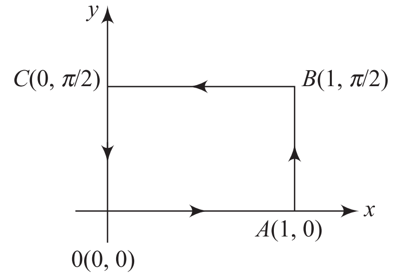

Fig. 20.7

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (e^x \sin y\, dx + e^x \cos y\, dy)$$

$$= \int_{OA} o + \int_{AB} e^1 \cos y\, dy + \int_{BC} e^x \sin\frac{\pi}{2} dx + \int_{CO} \cos y\, dy$$

$$= 0 + \int_0^{\frac{\pi}{2}} e \cos y \cdot dy + \int_1^0 e^x \sin\frac{\pi}{2} dx + \int_{\frac{\pi}{2}}^0 \cos y\, dy$$

$$= [e \sin y]_0^{\frac{\pi}{2}} + [e^x]_1^0 + [\sin y]_{\frac{\pi}{2}}^0 = e + (1 - e) - 1 + 0 = 0 \qquad \textbf{Ans.}$$

Example 7 Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Solution Total work done

$$= \int_C \vec{F} \cdot d\vec{r} = \int_C (3xy\vec{i} - 5z\vec{j} + 10x\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C [3xydx - 5zdy + 10xdz]$$

$$= \int_{t=1}^{2} [3(t^2+1)(2t^2)d(t^2+1) - 5t^3 d(2t^2) + 10(t^2+1)d(t^3)]$$

$$= \int_{t=1}^{2} [6t^2(t^2+1)(2tdt) - 20t^4 dt + 30t^2(t^2+1)dt]$$

$$= \int_{t=1}^{2} [12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2]dt$$

$$= \int_{t=1}^{2} [12t^5 + 10t^4 + 12t^3 + 30t^2]dt$$

$$= 12\left[\frac{t^6}{6}\right]_1^2 + 10\left[\frac{t^5}{5}\right]_1^2 + 12\left[\frac{t^4}{4}\right]_1^2 + 30\left[\frac{t^3}{3}\right]_1^2$$

$$= 12\left[\frac{2^6}{6} - \frac{1}{6}\right] + 10\left[\frac{2^5}{5} - \frac{1}{5}\right] + 12\left[\frac{2^4}{4} - \frac{1^4}{4}\right] + 30\left[\frac{2^3}{3} - \frac{1^3}{3}\right]$$

$$= 12 \cdot \frac{63}{6} + 10 \cdot \frac{31}{5} + 12 \cdot \frac{15}{4} + 30 \cdot \frac{7}{3}$$

$$= 126 + 62 + 45 + 70$$

$$= 303$$

Ans.

Example 8 If $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$, evaluate $\iint_S \vec{F} \cdot \hat{n} ds$ where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. **[AU Dec. 2009]**

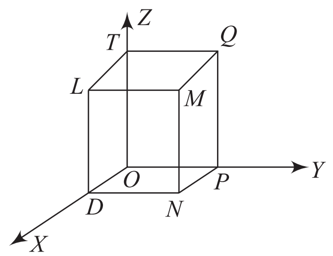

Fig. 20.8

Solution The surface of the cube consists of the following six faces:

(a) Face *LMND*
(b) Face *TQPO*
(c) Face *QPNM*
(d) Face *TODL*
(e) Face *TQMl*
(f) Face *ODNP*

Now, for the face *LMND*:

$$\hat{n} = \vec{i}, x = OD = 1$$

Hence, $$\iint_S F \cdot \hat{n} ds = \iint_{LMND} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{i}\, dy\, dz$$

$$= \iint_{LMND} 4xz\, dy\, dz = 4 \int_{LMND} z\, dy\, dz \qquad (\because x = 1)$$

$$= 4 \int_{z=0}^{1} \int_{y=0}^{1} z\, dy\, dz = 4\left[\left(\frac{z^2}{2}\right)_0^1 (y)_0^2\right] = 2 \qquad (1)$$

For the face *TQPO*: $\hat{n} = -\vec{i}, x = 0$

Hence, $$\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iint_{TQPO} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{i}) dy\, dz$$

$$= \iint_{TQPO} (-4xz) dy\, dz = 0 \qquad (\because x = 0) \qquad (2)$$

For the face *OPNM*: $\hat{n} = \vec{j}, y = 1$

Hence, $$\iint_S \vec{F} \cdot \hat{n} ds = \iint_{QPNM} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{j}\, dx\, dz$$

$$= \iint_{QPNM} (-y^2\, dx\, dz) = \iint_{QPNM} -dx\, dz \qquad (\because y = 1)$$

$$= -\int_{z=0}^{1}\int_{x=0}^{1} dx\,dz = -[x]_0^1[z]_0^1 = -1 \tag{3}$$

For the face *TODL*: $\hat{n} = -\vec{j}, y = 0$

Hence, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{TODL} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j})dx\,dz$

$$= \iint_{TODL} (y^2 dxdz) = 0 \quad (\because y = 0) \tag{4}$$

For the face *TQML*: $\hat{n} = \vec{k}, z = 1$

Hence, $\iint_{TQML} \vec{F}\cdot\hat{n}\,ds = \iint_{TQML} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\,dx\,dy$.

$$= \iint_{TQML} yz\,dx\,dy = \iint_{TQML} y\,dx\,dy \qquad (\because z = 1)$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} y\,dx\,dy = [x]_0^1\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} \tag{5}$$

For the face *ODNP*: $\hat{n} = -\vec{k}, z = 0$

Hence, $\iint_{ODNP} \vec{F}\cdot\hat{n}\,ds = \iint_{ODNP} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{k})\cdot dx\,dy$

$$= \iint_{ODNP} (-yz)dx\,dy = 0, \quad (\because z = 0) \tag{6}$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$$ **Ans.**

Example 9 Verify Stokes' theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}$ over the surface of a cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the *XOY* plane (open at the bottom). **[KU May 2010]**

Solution Consider the surface of the cube as shown in the figure. Bounding path is *OABCO* shown by arrows.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}]\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (y - z + 2)dx + (yz + 4)dy - xz\,dz$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \tag{1}$$

Along OA, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^2 2dx = (2x)_0^2 = 4$$

Along AB, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^2 4dy = 4(y)_0^2 = 8$$

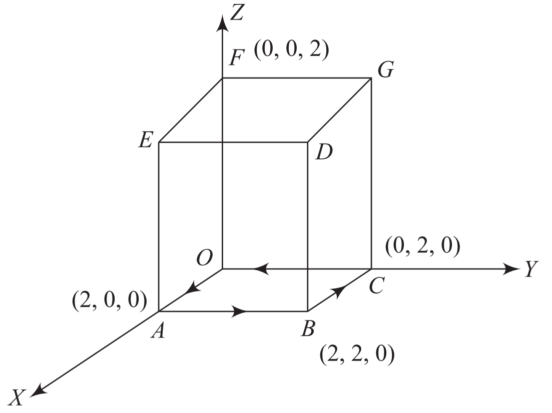

Fig. 20.9

Along BC, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^2 (2-0+2)dx = (4x)_2^0 = -8$$

Along CO, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{CO} \vec{F}\cdot d\vec{r} = \int (y-0+2)\times 0 + (0+4)dy - 0$$

$$= 4\int dy = 4(y)_2^0 = -8$$

On putting the values of these integrals in (1), we get

$$\int_C \vec{F}\cdot d\vec{r} = 4 + 8 - 8 = -4$$

To obtain surface integral

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= (0-y)\vec{i} - (-z+1)\vec{j} + (0-1)\vec{k} = -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

Here, we have to integrate over the five surfaces, *ABDE*, *OCGF*, *BCGD*, *OAEF*, *DEFG*.
Over the surface *ABDE*: $x = 2$, $\hat{n} = \vec{i}$, $ds = dydz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{i}\, dy\, dz$$

$$= \iint_S -y\, dydz = -\int_0^2 y\, dy \int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$$

Over the surface *OCGF*: $x = 0$, $\hat{n} = -\vec{i}$, $ds = dy\, dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{i}) dy\, dz$$

$$= \iint_S y\, dy\, dz = \int_0^2 y\, dy \int_0^2 dz = \left[\frac{y^2}{2}\right]_0^2 = 4$$

Over the surface *BCGD*: $y = 2$, $\hat{n} = \vec{j}$, $ds = dx\, dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{j}\, dx\, dz$$

$$= \iint_S (z-1) dx\, dz$$

$$= \int_0^2 dx \int_0^2 (z-1)\, dz$$

$$= [x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *OAEF*: $y = 0$, $\hat{n} = -\vec{j}$, $ds = dx\, dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{j}) dx\, dz$$

$$= -\iint_S (z-1) dx\, dz$$

$$= -\int_0^2 dx \int_0^2 (z-1) dz$$

$$= -[x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *DEFG*: $z = 2$, $\hat{n} = \vec{k}$, $ds = dx\,dy$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{k}\,dx\,dy$$

$$= -\iint dx\,dy = -\int_0^2 dx \int_0^2 dy$$

$$= -[x]_0^2 [y]_0^2 = -4$$

Total surface integral $= -4 + 4 + 0 + 0 - 4 = -4$

Thus $\iint_S \text{curl } \vec{F} \cdot \hat{n}\,ds = \int_C \vec{F} \cdot d\vec{r} = -4$

which verifies Stokes' theorem. **Verified.**

Example 10 Verify Green's theorem in the plane for $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where C is a square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

Solution Given integrand is of the form $Mdx + Ndy$, where $M = x^2 - xy^3$, $N = y^2 - 2xy$.

Now to verify Green's theorem, we have to verify that

$$\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \iint_R (-2y + 3xy^2)dx\,dy \tag{1}$$

Consider $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where the curve C is divided into four parts, hence the line integral along C is nothing but the sum of four line integrals along four lines *OA*, *AB*, *BC* and *CO*.

Along *OA* : $y = 0$, $dy = 0$ and x varies from 0 to 2.

Hence, $$\int_{OA} [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \int_{x=0}^{2} x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

Along *AB* : $x = 2$, $dx = 0$, and y varies from 0 to 2.

Hence, $$\int_{AB} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_0^2 (y^2 - 4y)dy = \left(\frac{y^3}{3} - 4\frac{y^2}{2}\right)_0^2$$

$$= \left(\frac{8}{3}\right) - 8 = -\frac{16}{3}$$

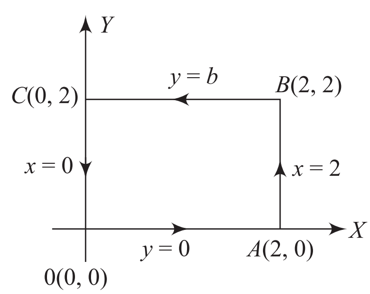

Fig. 20.10

Along *BC*: $y = 2$, $dy = 0$ and x varies from 2 to 0.

Hence, $$\int_{BC} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_{x=2}^{0} (x^2 - 8x)dx = \left(\frac{x^3}{3} - 8\frac{x^2}{2}\right)_2^0$$

$$= 0 - 0 - \frac{8}{3} + 16 = \frac{40}{3}$$

Along CO : $x = 0$, $dx = 0$ and y varies from 2 to 0

Hence, $\int_{CO} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$

$$= \int_{y=2}^{0} y^2 dy = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$$

$\therefore$ $$\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8 \quad (2)$$

Now consider

$$\iint_R (-2y + 3xy^2)dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2} (-2y + 3xy^2)dy\,dx$$

$$= \int_{x=0}^{2} \left(-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_{x=0}^{2} \left[-4 + 3x\left(\frac{8}{3}\right)\right]dx = \left(-4x + 8\frac{x^2}{2}\right)_0^2$$

$$= -8 + 16 + 0 = 8 \quad (3)$$

From (2) and (3), we observe that the relation (1) is true.
Hence, Green's theorem is verified. **Ans.**

Example 11 Verify divergence theorem for $\vec{F} = (x^2 - yz)\,\vec{i} + (y^2 - zx)\,\vec{j} + (z^2 - xy)\,\vec{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. **[KU Nov. 2010]**

Solution For verification of the divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$

$\therefore$ $$\iiint_V \text{div}\,\vec{F}dv$$

$$= \int_o^c\int_o^b\int_o^a 2(x + y + z)\,dx\,dy\,dz$$

$$= \int_o^c\int_o^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dy\,dz$$

$$= \int_0^c\int_o^b 2\left(\frac{a^2}{2} + ya + za\right)dy\,dz$$

$$= \int_o^c 2\left[\left(\frac{a^2}{2}y + \frac{y^2 a}{2} + azy\right)\right]_o^b dz$$

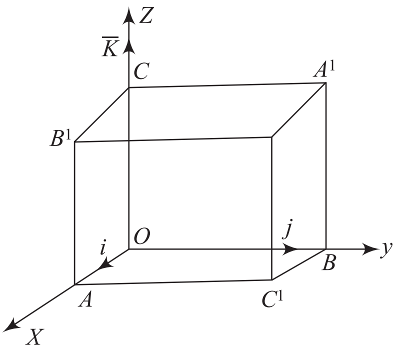

Fig. 20.11

$$= 2\int_o^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right)dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + \frac{abz^2}{2}\right]_o^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a + b + c) \tag{1}$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelepiped into 6 parts.

S_1 : Face $OAC'B$

S_2 : Face $CB'PA'$

S_3 : Face $OBA'C$

S_4 : Face $AC'PB'$

S_5 : Face $OCB'A$

S_6 : Face $BA'\,PC'$

Also,
$$\iint_S \vec{F}\cdot\hat{n}ds = \iint_{S_1} \vec{F}\cdot\hat{n}ds + \iint_{S_2} \vec{F}\cdot\hat{n}ds + \iint_{S_3} \vec{F}\cdot\hat{n}ds + \iint_{S_4} \vec{F}\cdot\hat{n}ds + \iint_{S_5} \vec{F}\cdot\hat{n}ds + \iint_{S} \vec{F}\cdot\hat{n}ds \tag{2}$$

On $S_1 : z = 0,\ \hat{n} = -\vec{k}$, $ds = dx\,dy$

so that $\vec{F}\cdot\hat{n} = (x^2\vec{i} + y^2\vec{j} - xy\vec{k})\cdot(-\vec{k}) = xy$

$$\therefore \quad \iint_{S_1} \vec{F}\cdot\hat{n}ds = \int_o^b\int_o^a xy\,dx\,dy = \int_o^b\left(y\frac{x^2}{2}\right)_o^a dy$$

$$= \frac{a^2}{2}\int_o^b y\,dy = \frac{a^2b^2}{4} \tag{3}$$

On $S_2 : z = c,\ \hat{n} = \vec{k}$, $ds = dx\,dy$, $\vec{F} = (x^2 - cy)\,\vec{i} + (y^2 - cx)\,\vec{j} + (c^2 - xy)\,\vec{k}$.

so that $\vec{F}\cdot\hat{n} = [(x^2 - cy)\vec{i} + (y^2 - (x)\vec{j}) + (c^2 - xy)\vec{k}]\cdot\vec{k} = c^2 - xy$.

$$\therefore \quad \iint_{S_2} \vec{F}\cdot\hat{n}ds = \int_o^b\int_o^a (c^2 - xy)\,dx\,dy = \int_o^b\left(c^2a - \frac{a^2}{2}y\right)dy$$

$$= abc^2 - \frac{a^2b^2}{4} \tag{4}$$

On $S_3 : x = 0,\ \hat{n} = -\vec{i}, \vec{F} = -yz\vec{i} + y^2\vec{j} + z^2\vec{k}, dz = dy\,dz$

so that $\vec{F}\cdot\hat{n} = (-yz\vec{i} + y^2\vec{j} + z^2\vec{k})\cdot(-\vec{i}) = yz, ds = dy\,dz$

$$\therefore \quad \iint_{S_3} \vec{F}\cdot\hat{n}ds = \int_o^c\int_o^b yz\,dy\,dz = \int_o^c \frac{b^2}{2}z\,dz = \frac{b^2c^2}{4} \tag{5}$$

On $S_4 : x = a,\ \hat{n} = \vec{i}, \vec{F} = (a^2 - yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}$

so that $\vec{F}\cdot\hat{n}=[(a^2-yz)\vec{i}+(y^2-az)\vec{j}+(z^2-ay)\vec{k}]\cdot\vec{i}$
$=a^2-yz,\ ds=dy\,dz$

$$\therefore \qquad \iint_{S_4}\vec{F}\cdot\hat{n}ds=\int_o^c\int_o^b(a^2-yz)dy\,dz=\int_o^c\left(a^2b-\frac{b^2}{2}z\right)dz$$

$$=a^2bc-\frac{b^2c^2}{4} \tag{6}$$

On $S_5: y=0,\ \hat{n}=-\vec{j},\vec{F}=x^2\vec{i}-zx\vec{j}+z^2\vec{k},ds=dxdz$

so that $\vec{F}\cdot\hat{n}=(x^2\vec{i}-zx\vec{j}+z^2\vec{k})\cdot(-\vec{j})=zx$

$$\therefore \qquad \iint_{S_5}\vec{F}\cdot\hat{n}ds=\int_o^a\int_o^c zx\,dz\,dx=\int_o^a\frac{c^2}{2}x\,dx=\frac{a^2c^2}{4} \tag{7}$$

On $S_6: y=b,\ \hat{n}=\vec{j},\vec{F}=(x^2-bz)\vec{i}+(b^2-zx)\vec{j}+(z^2-bx)\vec{k}$
$ds=dxdz$

so that $\vec{F}\cdot\hat{n}=[(x^2-bz)\vec{i}+(b^2-zx)\vec{j}+(z^2-bx)\vec{k}]\cdot\vec{j}$
$=b^2-zx.$

$$\therefore \qquad \iint_{S_6}\vec{F}\cdot\hat{n}=\int_o^a\int_o^c(b^2-zx)dz\,dx$$

$$=\int_o^a\left(b^2c-\frac{c^2}{2}x\right)\cdot dx=ab^2c-\frac{a^2c^2}{4} \tag{8}$$

By using (3), (4), (5), (6), (7) and (8), in (2), we get

$$\iint_S\vec{F}\cdot\hat{n}ds=\frac{a^2b^2}{4}+abc^2-\frac{a^2b^2}{4}+\frac{b^2c^2}{4}+a^2bc-\frac{b^2c^2}{4}+\frac{a^2c^2}{4}+ab^2c-\frac{a^2c^2}{4}$$

$$=abc(a+b+c) \tag{9}$$

The equalities (1) and (9) verify the divergence theorem. **Ans.**

Example 12 Verify Green's theorem in the plane for $\int_C(3x^2-8y^2)dx+(4y-6xy)dy$ where C is the boundary of the region defined by (i) $y=\sqrt{x}$, $y=x^2$ and (ii) $x=0$, $y=0$, $x+y=1$. **[AU July 2010, June 2012 ; KU Nov. 2011, KU April 2013]**

Solution

(i) $y=\sqrt{x}$, i.e., $y^2=x$ and $y=x^2$ are two parabolas intersecting at 0(0, 0) and $A(1, 1)$.
Here, $P=3x^2-8y^2$, $Q=4y-6xy$

$$\frac{\partial P}{\partial y}=-16y,\ \frac{\partial Q}{\partial x}=-6y$$

$$\therefore \qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=10y$$

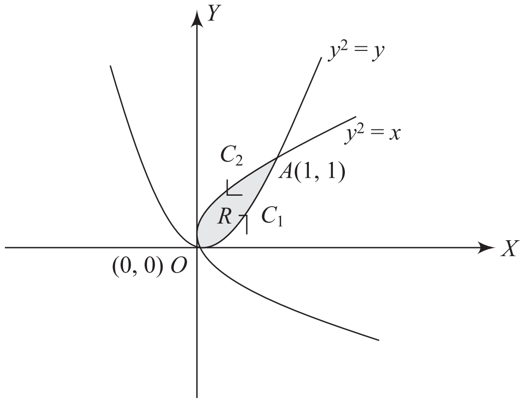

Fig. 20.12

If R is the region bounded by C then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx = \int_0^1 10\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\left[\frac{3}{10}\right] = \frac{3}{2} \qquad (1)$$

Also, $\int_C P\,dx + Q\,dy = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy)$

Along C_1, $x^2 = y$. $\therefore$ $2x\, dx = dy$ and the limits of x are from 0 to 1.

$\therefore$ $\int_{C_1} (P\,dx + Q\,dy)$

$$= \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x \cdot x^2) \cdot 2x\,dx \text{ (since } x^2 = y)$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= [x^3 + 2x^4 - 4x^5]_0^1 = -1$$

Along C_2, $y^2 = x$. $\therefore$ $2y\, dy = dx$ and the limits of y are from 1 to 0.

$$\int_{C_2}(P\,dx+Q\,dy)$$

$$=\int_1^0 (3y^4-8y^2)2y\,dy+(4y-6y^2\cdot y)\cdot dy$$

$$\therefore \qquad =\int_1^0 (4y-22y^3+6y^5)\,dy=\left[2y^2-\frac{11}{2}y^4+y^6\right]_1^0=\frac{5}{2}$$

$$\therefore \qquad \int_C (P\,dx+Q\,dy)=-1+\frac{5}{2}=\frac{3}{2} \qquad (2)$$

The equalities of (1) and (2) verify Green's theorem in the plane. **Ans.**

(ii) Here, $\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$

$$=\int_0^1\int_0^{1-x} 10y\,dy\,dx$$

$$=\int_0^1 5[y^2]_0^{1-x}\,dx$$

$$=5\int_0^1 (1-x)^2\,dx=5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

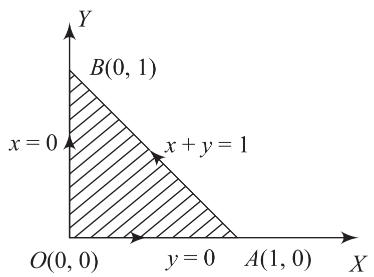

Fig. 20.13

$$=\frac{-5}{3}(0-1)=\frac{5}{3} \qquad (1)$$

Along OA, $y=0$ $\therefore$ $dy=0$ and the limits of x are from 0 to 1.

$$\therefore \qquad \int_{OA} P\,dx+Q\,dy=\int_0^1 3x^2dx=[x^3]_0^1=1$$

Along AB, $y=1-x$. $\therefore$ $dy=-dx$ and the limits of x are from 1 to 0.

$$\therefore \qquad \int_{AB} P\,dx+Q\,dy=\int_1^0 [3x^2-8(1-x)^2]dx+[4(1-x)-6x(1-x)](-dx)$$

$$=\int_1^0 (3x^2-8+16x-8x^2-4+4x+6x-6x^2)dx$$

$$=\int_1^0 (-12+26x-11x^2)\cdot dx$$

$$=\left[-12x+13x^2-\frac{11}{3}x^3\right]_1^0=-\left[-12+13-\frac{11}{3}\right]=\frac{8}{3}$$

Along BO, $x=0$. $\therefore$ $dx=0$ and the limits of y are from 1 to 0

$$\therefore \qquad \int_{BO} P\,dx+Q\,dy=\int_1^0 4y\,dy=[2y^2]_1^0=-2$$

$\therefore$ line integral along C (i.e., along $OABO$) $=1+\frac{8}{3}-2=\frac{5}{3}$

i.e., $$\int_C (P\,dx + Q\,dy) = \frac{5}{3} \tag{2}$$

The equality of (1) and (2) verifies Green's theorem in the plane. **Verified.**

Example 13 Evaluate $\int_C (e^x dx + 2y\,dy - dz)$ by using Stokes' theorem, where C is the curve $x^2 + y^2 = 4$, $z = 2$. **[AU May 2010]**

Solution

$$\begin{aligned}
&\int_C (e^x\,dx + 2y\,dy - dz)\\
&\quad = \int_C (e^x\vec{i} + 2y\vec{j} - \vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})\\
&\quad = \int_C \vec{F}\cdot\overrightarrow{dr} \text{ where } \vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}
\end{aligned}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

$\therefore$ by Stokes' theorem, $\displaystyle\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\cdot ds$

$$= 0 \text{ (since curl } \vec{F} = 0)$$ **Ans.**

Example 14 Find the work done by the force $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, when it moves a particle along the arc of the curve $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ from $t = 0$ to $t = 2n$. **[AU Dec. 2007]**

Solution From the vector equation of the curve C, we get the parametric equations of the curve as $x = \cos t$, $y = \sin t$, $z = t$.

Work done by the force $\vec{F} = \int_C \vec{F}\cdot\overrightarrow{dr}$

$$\begin{aligned}
&= \int_C (z\vec{i} + x\vec{j} + y\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})\\
&= \int_C (z\,dx + x\,dy + y\,dz)\\
&= \int_0^{2\pi} [t(-\sin t) + \cos^2 t + \sin t]\,dt\\
&= \left[t\cos t - \sin t + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) - \cos t\right]_0^{2\pi}\\
&= (2\pi + \pi - 1) - (-1)\\
&= 3\pi
\end{aligned}$$ **Ans.**

Example 15 Verify Stokes' theorem for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\vec{k}$ where S is the open surface of the rectangular parallelepiped formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ above the XOY-plane. **[AU Dec. 2007]**

Solution Stokes' theorem is given by

$$\int_C \vec{F} \cdot \overrightarrow{dr} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds$$

Here, curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -zx \end{vmatrix}$

$$= 2y\vec{i} + z\vec{j} + x\vec{k} \quad \therefore \int_C (xy\,dx - 2yz\,dy - zx\,dz) - \iint_S (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\,ds \qquad (1)$$

The open cuboid S is made up of the five faces $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ and is bounded by the rectangle $OAC'B$ lying on the XOY plane. LHS of (1) is

$$= \int_{OAC'B} (xy\,dx - 2yzdy - zxdz)$$

$$= \int_{OAC'B} xy\,dx$$

(since the boundary C lies on the XOY plane, $z = 0$)

$$= \int_{OA} xy\,dx + \int_{AC'} xy\,dx + \int_{C'B} xy\,dx + \int_{BO} xy\,dx$$

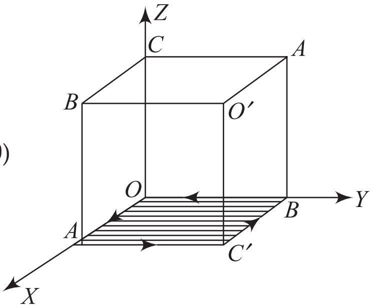

Fig. 20.14

Along OA, $y = 0$, $dy = 0$
Along AC', $x = 1$, $dx = 0$
Along $C'B$, $y = 2$, $dy = 0$
Along BO, $x = 0$, $dx = 0$

$$\therefore \qquad \int_{OAC'B} xy\,dx = 0 + 0 + \int_{C'B} xy\,dx + 0 = \int_1^0 2x\,dx$$

(as along $C'B$, x varies from 1 to 0).

$$= -1 \qquad (2)$$

RHS of (1) is

$$\iint_S (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\,ds = \iint_{O'C'AB'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\,ds + \iint_{A'BOC} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\,ds$$
$$+ \iint_{A'BC'O'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\,ds + \iint_{COAB'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\,ds$$
$$+ \iint_{A'O'B'C} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\,ds$$

$$= \int_0^3\int_0^2 2y\,dy\,dz - \int_0^3\int_0^2 2y\,dy\,dz - \int_0^1\int_0^3 z\,dz\,dx$$
$$- \int_0^1\int_0^3 z\,dz\,dx - \int_0^2\int_0^1 x\,dx\,dy$$
$$= -\int_0^2\int_0^1 x\,dx\,dy = -\int_0^2\left(\frac{x^2}{2}\right)_0^1 dy = -1 \qquad (3)$$

From (2) and (3), Stokes' theorem is verified. **Verified.**

Example 16 Verify the divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. **[AU Dec. 2007, KU Nov. 2011]**

Solution Gauss' divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}ds = \iiint_V (\text{div }\vec{F})dv \qquad (1)$$

$$\text{LHS of } (1) = \iint_{x=1} x^2\,ds + \iint_{x=-1} -x^2\,ds + \iint_{y=1} z\,ds + \iint_{y=-1} -z\,ds + \iint_{z=1} yz\,ds + \iint_{z=-1} -yz\,ds = 0 \qquad (2)$$

$$\text{RHS of } (1) = \iiint_V (\text{div }\vec{F})\cdot dv$$
$$= \iiint_V (2x + y)\,dx\,dy\,dz$$
$$= \int_{-1}^1\int_{-1}^1\int_{-1}^1 (2x+y)\,dx\,dy\,dz$$
$$= \int_{-1}^1\int_{-1}^1 2y\,dy\,dz = 0 \qquad (3)$$

From (2) and (3), Gauss' divergence theorem is verified. **Verified.**

Example 17 Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot\vec{dr}$, where $\vec{F} = (\sin x - y)\vec{i} - \cos x\vec{j}$ and C is the boundary of the triangle whose vertices are $(0,0), \left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$. **[KU Nov. 2011]**

Solution By Stokes' theorem, we have $\int_C \vec{F}\cdot\vec{dr} = \iint_S \text{ curl } \vec{F}\cdot\hat{n}\cdot ds$.

$$\text{curl }\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin x - y & -\cos x & 0 \end{vmatrix}$$
$$= (\sin x + 1)\vec{k}$$

$\therefore$ the given line integral

$$= \iint_R (1 + \sin x)\,dx\,dy$$

$$= \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (1 + \sin x)\,dx\,dy$$

$$= \int_0^1 \left[x - \cos x\right]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \left[\frac{\pi}{2} - \frac{\pi y}{2} + \cos\frac{\pi y}{2}\right] dy$$

$$= \left[\frac{\pi}{2} y - \frac{\pi y^2}{4} + \frac{2}{\pi}\sin\frac{\pi y}{2}\right]_0^1$$

$$= \frac{\pi}{4} + \frac{2}{\pi}$$

$\therefore$ $$\int_C \vec{F} \cdot \overrightarrow{dr} = \frac{\pi}{4} + \frac{2}{\pi}$$ **Ans.**

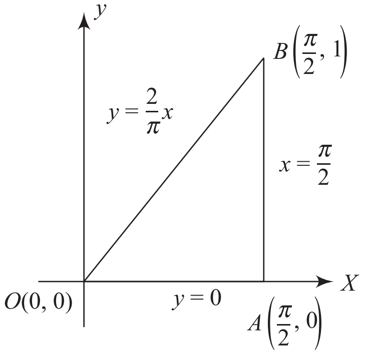

Fig. 20.15

EXERCISE

Part A

1. State Green's theorem in a plane.
2. Give the relation between a line integral and a surface integral.
3. State Gauss' divergence theorem.
4. Deduce Green's theorem in a plane from Stokes' theorem.
5. In Gauss' divergence theorem, surface integral is equal to _______ integral.
6. The integral of $\vec{F} \cdot d\vec{r}$ is
   (i) line integral (ii) zero
   (iii) surface integral (iv) one
7. Using Green's theorem, prove that the area enclosed by a simple closed curve C is $\frac{1}{2}\int (x\,dy - y\,dx)$.
8. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F} \cdot \overrightarrow{dr}$ where C is the part of the curve $y = x^3$ between $x = 1$ and $x = 2$.
9. If $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F} \cdot \overrightarrow{dr}$ along the straight line $y = x$ from (0, 0) to (1, 1).
10. If C is a simple closed curve and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, prove that $\int_C \vec{r} \cdot \overrightarrow{dr} = 0$.
11. Evaluate $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ where C is the circle given by $x^2 + y^2 + z^2 = 1$ and $z = 0$.
12. Use the integral theorems to prove $\nabla \cdot (\nabla \times \vec{F}) = 0$.

13. Evaluate $\int_C (x\,dy - y\,dx)$, where C is the circle $x^2 + y^2 = a^2$.
14. Evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and C is the curve $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, t varying from –1 to 1.

Part B

1. If a force $\vec{F} = 2x^2y\vec{i} + 3xy\vec{j}$ displaces a particle in the xy plane from (0, 0) to (1, 4) along a curve $y = 4x^2$, find the work done. $\left(\textbf{Ans. } \dfrac{104}{5}\right)$
2. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to (1, 1) along a parabola $y^2 = x$. $\left(\textbf{Ans. } \dfrac{2}{3}\right)$
3. Verify Green's theorem in a plane with respect to $\int_C (x^2dx + xy\,dy)$, where C is the boundary of the square formed by $x = 0$, $y = 0$, $x = a$, $y = a$. **[AU Dec. 2009]** $\left(\textbf{Ans. } \dfrac{a^3}{2}\right)$
4. Use Green's theorem to evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is the square formed by the lines $y = \pm 1$, $x = \pm 1$. **(Ans.** 0**)**
5. Use divergence theorem to evaluate $\iint_S (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k})\cdot\hat{n}\,ds$ where S is the closed surface bounded by the XOY-plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane. **(Ans.** πa^4**)**
6. Verify Stokes' theorem for $\vec{F} = (x^2 + y - 4)\vec{i} + 3xy\vec{j} + (2xz + z^2)\vec{k}$ over the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above the XOY plane. **(Ans.** -16π**)**
7. Use the divergence theorem to evaluate $\int_S \vec{A}\cdot\overrightarrow{ds}$ where $\vec{A} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2 = y^2 + z^2 = a^2$. $\left(\textbf{Ans. } \dfrac{12\pi a^5}{5}\right)$
8. Use the divergence theorem to evaluate $\iint_S x^3dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where S is the surface of the region bounded by the closed cylinder $x^2 + y^2 = a^2$, $(0 \le z \le b)$ $z = 0$ and $z = b$. $\left(\textbf{Ans. } \dfrac{5\pi a^4 b}{4}\right)$
9. Using Green's theorem, evaluate $\int_C [(y - \sin x)dx + \cos x\,dy]$ where C is the triangle bounded by $y = 0, x = \dfrac{\pi}{2}, y = \dfrac{2x}{\pi}$. $\left[\textbf{Ans. } -\left(\dfrac{\pi^2 + 8}{4\pi}\right)\right]$
10. Evaluate $\int_C [(x^2 + y^2)dx - 2xy\,dy]$ where C is the rectangle bounded by $y = 0$, $x = 0$, $y = b$, $x = a$ using Green's theorem. **(Ans.** $-2ab^2$**)**
11. Verify Stokes' theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **(Ans.** $-\pi$**)**
12. Verify Stokes' theorem for $\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is the boundary. **(Ans.** 9π**)**

13. Find the area of $x^{2/3} + y^{2/3} = a^{2/3}$ using Green's theorem. $\left(\textbf{Ans. } \dfrac{3\pi a^2}{8}\right)$

14. Using Stokes' theorem, evaluate $\int_C (xy\,dx + xy^2 dy)$ taking C to be a square with vertices (1, 1), (–1, 1), (–1, –1) and (1, –1). $\left(\textbf{Ans. } \dfrac{4}{3}\right)$

15. Verify Gauss' divergence theorem for $\vec{F} = y\vec{i} + x\vec{j} + z^3\vec{k}$ over the cylindrical region $x^2 + y^2 = 9$, $z = 0$, $z = 6$. **(Ans.** 1944π**)**

Objective type questions	Opt 1	Opt2	Opt3	Opt4	Answer
The triple integral ∫∫∫ dv gives the __________ over the region v	area	volume	Direction	weight	volume
The value of ∫∫ dx dy , inner integral limt varies from 1 to 2 and the outer integral limit varies from 0 to 1	0	1	2	3	1
∫∫∫ dx dy dz, the inner integral limit varies from 0 to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1	2	4	6	8	6
When the limits are not given, the integral is named as __________	Definite integral	Infinite integral	volume integral	Surface integral	Infinite integral
The Double integral ∫∫ dx dy gives _________ of the region R	area	modulus	Direction	weight	
The value of ∫∫ (x+y) dx dy , inner integral limt varies from 0 to 1 and the outer integral limit varies from 0 to 1	0	1	2	3	1
The value of ∫∫∫ x^2 yz dx dy dz, the inner integral limit varies from 1 to 2, the central integral limit varios from 0 to 2 and outer integral limit various from 0 to 1	7/3	1/3	2/3	3	7/3
Evaluate ∫∫ 4xy dx dy, the inner integral limit varies from 0 to 1 and outer integral limit varies from 0 to 2	10	4	5	1	4
The value of ∫∫ d xdy ⁄xy, the inner integral limit varies from 0 to b and the outer limit varies from 0 to a	0	1	ab	loga log b	loga log b
If the limits are given in the integral , then the integral is name as____________	Definite integral	Infinite integral	volume integral	Surface integral	Definite integral
The value of ∫∫(x^2+3y^2) dy dx , the inner integral limit varies from 0 to 1, the outer integral limit varies from 0 to 3	10	15	12	30	12
central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1	6	1	16	12	6
If the limits are not given in the integral , the the integral is name as____________	Definite integral	Infinite integral	volume integral	Surface integral	Infinite integral
The value of ∫∫(x^2+y^2) dy dx ,the inner integral limit varies from 0 to x, the outer integral limit varies from 0 to 1	1	1⁄3	2⁄3	3⁄2	1⁄3
The value of ∫∫dy dx, the inner integral limit various from 0 to x ,the outer integral limit varies from -a to a	0	1	2	3	0
The Double integral ∫∫ dx dy gives _________ of the region R	area	modulus	Direction	weight	area
The value of ∫∫∫ dx dy dz, the inner integral limit varies from 0 to a , the central integral limit varies from 0 to a and the outer integral limit varies from 0 to a	0	a^3	a^2	a^4	a^3
The value of ∫∫(x+y) dx dy , the inner integral limit varies from 0 to 1 and the outer integral limit varies from 0 to 1	0	1	2	3	1
The concept of line integral as a generalization of the concept of ________ integral	Single	Double	change of order	Triple	Double
The extension of double integral is nothing but ________ integral	Single	Line	volume integral	Triple	Triple
The concept of _________ integral as a generalization of the concept of double integral	Single	Surface	Line	Triple	Line
Evaluate ∫x^2⁄2 dx, the limit varies from 0 to 1	2	1⁄6	1⁄10	34	1⁄6
Evaluate ∫42y dy, the limit varies from 0 to 10	10	2100	2000	100	2100
The value of ∫∫2 xy dy dx, the inner integral limit varies from 0 to x and the outer integral limit varies from 1 to 2	15⁄4	9⁄2	3⁄2	4⁄3	15⁄4
The value of ∫∫dy dx, the inner integral limit varies from 2 to 4 ,the outer integral limit varies from 1 to 5	8	2	4	5	8
The value of ∫∫xy dy dx, the inner integral limit varies from 0 to 3 , the outer integral limit varies from 0 to 4	12	36	1⁄2	4	12
The value of ∫∫dy dx, the inner integral limit varies from 0 to 2 , the outer integral limit varies from 0 to 1	2	1	3⁄2	4	2
The value of ∫∫dx dy, the inner integral limit varies from y to 2 , the outer integral limit varies from 0 to 1	1⁄2	1	3⁄2	4	3⁄2
The value of ∫∫dx dy, the inner integral limit varies from 2 to 4 , the outer integral limit varies from 1 to 2	2	6	3	1	2
When a function f(x) is integrated with respect to x between the limits a and b, we get _________	Definite integral	infinite integralv	volume integral	Surface integral	Definite integral
In two dimensions the x and y axes divide the entire xy- plane into ___________ quadrants	1	2	3	4	2
In three dimensions the xy and yz and zx planes divide the entire space into ___________ parts called octants	3	2	8	4	8
Evaluate ∫(2x+3) dx, the integral limitvaries from 0 to 2	10	42	51	1	10
_________ provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Stoke's Theorem
________ is also called the first fundamental theorem of integral vector calculus.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Green's Theorem
_________ transforms line integrals into surface integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Green's Theorem
_______ transforms surface integrals into a volume integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Gauss Theorem
_________ is stated as surface integral of the component of curl F along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function F taken along	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Stoke's Theorem
________ is stated as the surface integral of the normal component of a vector function F taken around a closed surface S is equal to the	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Gauss Theorem

[Differential equations].

UNIT: 1

[First order Ordinary differential Equations]

Differential equation:

* A differential equation is an equation which involves differential co-efficients.

Ordinary differential equations: (O.D.E).

* An ordinary differential equation is that in which all the differential co-efficients has a single independent variable.

Ex: $\frac{dy}{dx} = 2x.$

Partial differential equations: (P.D.E)

* A Partial differential equation is that in which there are two or more independent variable.

Ex: $x\frac{du}{dx} + y\frac{du}{dy} = 2u.$

Exact differential equation:

* A differential equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if its left hand number is the exact differential of some function $u(x,y)$.

ie) $du = M\,dx + N\,dy = 0$.

$\therefore$ The solution is $u(x,y) = c$

Theorem :

* The Necessary and Sufficient condition for that differential equation $M dx + N dy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Necessary condition:

* The equation $M dx + N dy = 0$ will be exact if $M dx + N dy = du$. where 'u' is the some function of x and y.

* $\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Sufficient condition:

* If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then $M dx + N dy = 0$ is exact

Methods of solution:

* The equation $Mdx + Ndy = 0$ becomes

$d\left[u + \int f(y)\,dy\right] = 0$, Integrating $\int d\left[u + \int f(y)\,dy\right] = 0$,

$\therefore$ ~~The solution~~ $u + \int f(y)\,dy = 0$.

$$u = \int_{y\ \text{constant}} M\,dx$$

$f(y)$ = terms of N not containing x.

$\therefore$ The solution of $Mdx + Ndy = 0$ is $\displaystyle\int_{(y\ \text{constant})} M\,dx + \int \left(\text{term of } N \text{ not containing } x\right) dy = c$

Provided $\dfrac{\partial u}{\partial y} = \dfrac{\partial N}{\partial x}$.

Example: 1

Solve $\underbrace{\left(y^2 e^{xy^2} + 4x^3\right) dx}_{M\,dx} + \underbrace{\left(2xy\,e^{xy^2} - 3y^2\right) dy}_{N\,dy} = 0$.

Here,

$M = y^2 e^{xy^2} + 4x^2$; $N = 2xy\,e^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2 \cdot e^{xy^2} \cdot 2xy$$
$$= 2y\,e^{xy^2} + 2xy^3 \cdot e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2y\left(e^{xy^2} + x \cdot e^{xy^2} \cdot y^2\right)$$
$$= 2y\,e^{xy^2} + 2xy^3\,e^{xy^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{constant})} M\,dx + \int \left(\text{terms of } N \text{ not containing } x\right) dy = c$$

$$\int (y^2 e^{xy^2} + 4x^3)\,dx + \int (-3y^2)\,dy = c$$

$$\int y^2 e^{xy^2} dx + \int 4x^3 dx - \int 3y^2 dy = c$$

$$\cancel{y^2} \cdot \frac{e^{xy^2}}{\cancel{y^2}} + \cancel{4}\frac{x^4}{\cancel{4}} - \cancel{3}\frac{y^3}{\cancel{3}} = c$$

$$\boxed{e^{xy^2} + x^4 - y^3 = c.}$$

— x —

Solve $\left[y\left(1+\frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]\, dy = 0.$

$M\,dx$ $\quad N\,dy$

Soln:

$$M = y\left(1+\frac{1}{x}\right) + \cos y \quad ; N = x + \log x - x \sin y.$$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y \quad ; \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

The equation is exact and its solution is

$$\underset{(y \text{ constant})}{\int M\,dx} + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int \left(y\left(1+\frac{1}{x}\right) + \cos y\right) dx + \int (0)\, dy = c$$

$$y\left[\int dx + \int \frac{1}{x} dx\right) + \int \cos y\, dx\Big]$$

$$\boxed{y\,[x + \log x] + \cos y\, x = c}$$

— x —

Solve $(1+2xy\cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)\,dy = 0$.

$\quad\quad M\,dx \quad\quad\quad N\,dy$

Soln:

$$M = 1+2xy\cos x^2 - 2xy \;;\; N = \sin x^2 - x^2$$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x \;;\; \frac{\partial N}{\partial x} = \cos x^2 \cdot 2x - 2x$$

$$= 2x\cos x^2 - 2x.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is exact and its solution is

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$
(y constant)

$$\int (1+2xy\cos x^2 - 2xy)\,dx + \int (0)\,dy = c$$

$$\int dx + y\int \cos x^2 \, 2x\,dx - \int 2xy\,dx = c$$

$$x + y\int d(\sin x^2) - \cancel{2}y\frac{x^2}{\cancel{2}} = c$$

$$\boxed{x + y\sin x^2 - yx^2 = c}$$

— x —

Solve $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$$\frac{(\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx}{(\sin x + x\cos y + x)\,dx} = 0$$

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0.$$

$\quad\quad M\,dx \quad\quad\quad N\,dy$

$M = y\cos x + \sin y + y \quad\quad N = \sin x + x\cos y + x$

$\partial M/\partial y = \cos x + \cos y + 1 \quad\quad \partial N/\partial x = \cos x + \cos y + 1$

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$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\underset{(y\ \text{constant})}{\int M\,dx} + \int \left(\begin{array}{c}\text{terms of } N \text{ not} \\ \text{containing } x\end{array}\right) dy = \cancel{0}\ c$$

$$\int (y\cos x + \sin y + y)\,dx + \int (0)\,dy = c$$

$$y\sin x + (\sin y + y)\,x = c$$

$$\boxed{y\sin x + x\sin y + xy = c}$$

— x —

Linear equation:

* A differential equation is ~~saind~~ said to be linear if it dependent variable and its differential co-efficient aquire only in the first degree

* The Standard form of a Linear equation of the first order, commonly known as Leibnitz's linear equation.

$$\frac{dy}{dx} + Py = Q$$

where, P, Q are the functions of x

$$y\,e^{\int P dx} = \int Q\,e^{\int P dx}\,dx + c$$

$$I.F = e^{\int e p dx}$$

$$\boxed{y\,(I.F) = \int Q\,(I.F)\,dx + c}$$

Solve the linear equation $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

Soln:-

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2.$$

$\div$ by $(x+1)$

$$\frac{dy}{dx} - \frac{y}{(x+1)} = e^{3x}(x+1)$$

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{-1}{x+1} \quad ; \quad Q = e^{3x}(x+1)$$

$$I.F = e^{\int P dx} = e^{\int \left(-\frac{1}{x+1}\right) dx}$$

$$= e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)}$$

$$= e^{\log(x+1)^{-1}}$$

$$= (x+1)^{-1} = \frac{1}{x+1}$$

$$\boxed{I.F = \frac{1}{x+1}}$$

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x} \cancel{(x+1)} \frac{1}{\cancel{(x+1)}} dx + c$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x} dx + c$$

$$y\left(\frac{1}{x+1}\right) = \frac{e^{3x}}{3} + c$$

$$\boxed{y = \left(\frac{e^{3x}}{3} + c\right)(x+1)}$$

—— x ——

Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

$$\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

which is Leibnitz's linear equation

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{\sqrt{x}} \quad ; \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2} dx = \frac{x^{-1/2+1}}{-1/2+1} = \frac{x^{1/2}}{1/2} = 2x^{1/2} = 2\sqrt{x}.$$

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$= \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}} dx + c$$

$$= \int \frac{1}{\sqrt{x}} e^0 dx + c$$

$$= \int \frac{dx}{\sqrt{x}} \cdot 1 + c$$

$$\boxed{y e^{2\sqrt{x}} = 2\sqrt{x} + c.}$$

— x —

Solve

$$\frac{dx}{dy} + Px = Q$$

$P, Q \rightarrow$ functions of y.

$$I.F = e^{\int P dy}$$

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + c.$$

Solve $(y \log y)\, dx + (x - \log y)\, dy = 0$

soln:

$$y \log y \, dx = -(x - \log y)\, dy$$

$$\frac{dx}{dy} = \frac{\log y - x}{y \log y} = \frac{1}{y} \frac{(\log y - x)}{\log y}$$

$$= \frac{1}{y}\left[1 - \frac{x}{\log y}\right].$$

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$\frac{dx}{dy} + Px = Q$$

$$P = \frac{1}{y \log y} \quad ; \quad Q = \frac{1}{y}.$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{y \log y} dy}$$

$$= e^{\int \frac{1/y \, dy}{\log y}} = e^{\log(\log y)}$$

$$= \log y.$$

$$x e^{\int P dy} = \int Q \cdot e^{\int P dy} dy + c.$$

$$x \log y = \int \frac{1}{y} \log y \, dy + c$$

$$= \int t' \, dt + c$$

$\log y = t$

$\frac{1}{y} dy = dt.$

$$x \log y = \frac{t^2}{2} + c$$

$$x \log y = \frac{1}{2} (\log y)^2 + c$$

$$\boxed{x = \frac{1}{2} \log y + c (\log y)^{-1}.}$$

—— x ——

Solve: $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$.

soln:
$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\frac{dx}{dy} + Px = Q$$

$$P = \frac{1}{1+y^2} \quad ; \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}\,dy} = e^{\tan^{-1}y}$$

$$x\,e^{\int P\,dy} = \int Q \cdot e^{\int P\,dy}\,dy + c$$

$$x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y}\,dy + c$$

$$= \int t\,e^t \cdot dt + c$$

$$= t\,e^t - \int e^t\,dt + c$$

$$= t\,e^t - e^t + c$$

$$= e^t(t-1) + c$$

$\int u\,dv = uv - \int v\,du$

$u = t$, $dv = e^t$

$du = dt$, $v = \int e^t\,dt = e^t$

$u - D$

$v - I$

$$\boxed{x\,e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c}$$

——x——

Put $t = \tan^{-1}y$

$$dt = \frac{1}{1+y^2}\,dy$$

$$x = (\tan^{-1}y - 1) + c\,e^{-\tan^{-1}y}$$

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Bernoulli's Equation:

$$\frac{dy}{dx} + Py = Qy^n \rightarrow (1)$$

To solve (1)

($\div$) both sides by y^n

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Put $y^{1-n} = z$.

$$(1-n)\, y^{\cancel{1}-n-\cancel{1}} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

(or) $$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

which is Leibnitz's linear in z & can be solved easily.

— x —

solve

$$x\frac{dy}{dx} + y = x^3y^6.$$

Soln:

$\div$ by x

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^6$$

$\div$ by y^6

$$y^{-6}\frac{dy}{dx} + \frac{y}{x\cdot y^6} = x^2$$

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \rightarrow (1)$$

Put $z = y^{-5}$

$$\frac{dz}{dx} = -5y^{-6}\frac{dy}{dx} \Rightarrow y^{-6}\frac{dy}{dx} = \frac{1}{-5}\frac{dz}{dx}$$

Sub $\frac{dy}{dx}$ in (1)

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$$

$\div$ by $\left(-\frac{1}{5}\right)$ $$\frac{dz}{dx} + \frac{z/x}{-1/5} = \cancel{x^2/{-5}}\ \frac{x^2}{-1/5}$$

$$\frac{dz}{dx} - \frac{5}{x}z = -5x^2$$

which is Leibnitz's linear equation in z

$$\frac{dz}{dx} + Pz = Q$$

$$P = -5/x \ ; \quad Q = -5x^2.$$

$$I.F = e^{\int P dx} = e^{\int -5/x\, dx} = e^{-5\int \frac{dx}{x}} = e^{-5\log x}$$

$$= e^{\log x^{-5}} = x^{-5}$$

$$z e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$$

$$z x^{-5} = \int -5x^2 \cdot x^{-5} dx + c$$

$$z x^{-5} = -5\int x^{-3} dx + c$$

$$= -5\left(\frac{x^{-3+1}}{-3+1}\right) + c$$

$$z x^{-5} = -5\frac{x^{-2}}{-2} + c$$

$$y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c$$

$(\div)$ by $y^{-5}x^{-5}$

$$1 = \frac{\frac{5}{2}x^{-2} + c}{x^{-5}y^{-5}}$$

$$1 = \left(\frac{5}{2} + cx^2\right)x^3y^5.$$

— x —.

Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

Soln:

$$xy(1+xy^2) = \frac{dx}{dy}$$

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\frac{dx}{dy} - xy = x^2y^3.$$

$\div$ by x^2

$$x^{-2}\frac{dx}{dy} - \frac{xy}{x^{\cancel{2}}} = \frac{\cancel{x^2}y^3}{\cancel{x^2}}$$

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \longrightarrow ①$$

put $x^{-1} = z$

$$-1x^{-1-1}\frac{dx}{dy} = \frac{dz}{dy}$$

$$-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$$

① becomes,

$$-\frac{dz}{dy} - yz = y^3$$

$$\frac{dz}{dy} + yz = -y^3 \longrightarrow ②$$

which is Leibnitz's linear equation in z

$$\frac{dz}{dy} + Pz = Q\cancel{y}.$$

Here $P = y$

$$I.F = e^{\int P dy} = e^{\int y dy} = e^{y^2/2}.$$

$\therefore$ The solution is $z(I.F) = \int Q(I.F)\,dy + c$

$$z e^{y^2/2} = \int (-y^3) e^{y^2/2} dy + c$$

$$= -\int y^2 e^{y^2/2} y \, dy + c$$

$$= -\int 2t e^t dt + c$$

$$= -2 \int t e^t dt + c$$

$$= -2 \left[t e^t - \int e^t dt \right] + c$$

$$= -2 \left[t e^t - e^t \right] + c$$

$$= -2 (t-1) e^t + c$$

$$= -2 \left[\frac{y^2}{2} - 1 \right] e^{y^2/2} + c$$

$$z = (-y^2+2) e^{y^2/2} + c e^{-y^2/2}$$

$$\frac{1}{x} = (2-y^2) e^{y^2/2} + c e^{-y^2/2}$$

$\frac{y^2}{2} = t$

$y^2 = 2t$

$\int u \, dv = uv - \int$

$u = t$; $dv =$

$du = 1$; v

Put $\frac{y^2}{2} = t$

$\frac{\cancel{2} y \, dy}{\cancel{2}} = dt$

— x —

solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

$\div$ by $\cos^2 y$

$$\frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{2x \sin y \cancel{\cos y}}{\cos^{\cancel{2}} y} = \frac{x^3 \cancel{\cos^2 y}}{\cancel{\cos^2 y}}$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \rightarrow ①$$

Put $\tan y = z$

$$\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ ① becomes

$$\frac{dz}{dx} + 2xz = x^3 \rightarrow ②$$

$$\frac{dz}{dx} + Pz = Q$$

which is Leibnitz's linear equation in z

Put $P = 2x$; $Q = x^3$

$$I.F = e^{\int P dx} = e^{\int 2x dx} = e^{\cancel{2}x^2/\cancel{2}} = e^{x^2}$$

The solution is

$$z(I.F) = \int Q(I.F) dx + c$$

$$z e^{x^2} = \int x^3 e^{x^2} dx + c$$

$$z e^{x^2} = \int x^2 x e^{x^2} dx + c$$

$u = t ; dv = e^t$

$du = 1 ; v = e^t$

$uv - \int v du.$

$$= \int \underset{u}{t} \underset{dv}{e^t} dt + c$$

$$= \frac{1}{2} [t e^t - \int e^t] + c$$

$$= \frac{1}{2} [t e^t - e^t] + c$$

Put $x^2 = t$

$2x dx = dt$

$x dx = \frac{dt}{2}$.

$$= \frac{1}{2} (t-1) e^t + c$$

$$z e^{x^2} = \frac{1}{2} [x^2 - 1] e^{x^2} + c$$

$\div$ by e^{x^2}

$$z = \frac{1}{2} (x^2 - 1) + c e^{-x^2}$$

— x —

Equations of first order and higher degree.

* The general form of the differential equation of the first ~~degree~~ order and n^{th} degree.

$$\left(\frac{dy}{dx}\right)^n + f_1(x,y) \left(\frac{dy}{dx}\right)^{n-1} + f_2(x,y) \left(\frac{dy}{dx}\right)^{n-2} + \dots\dots + f_{n-1}(x,y) \left(\frac{dy}{dx}\right) + f_n(x,y) = 0$$

If $\frac{dy}{dx} = P$

$$p^n + f_1(x,y)\,p^{n-1} + f_2(x,y)\,p^{n-2} + \dots + f_{n-1}(x,y)\,P + f_n(x,y).$$

Since equation ① is the first order its general solution will contain only one orbitary constant

To solve ① is to be identified as an equation any one of the types

* Solvable for P
* Solvable for y
* Solvable for x
* ~~solvable~~ clairaut's form.

* A differential equation of the first order but of nth degree is of the form

$$p^n + f_1(x,y)\,p^{n-1} + f_2(x,y)\,p^{n-2} + \dots + f_{n-1}(x,y)\,P + f_n(x,y) = 0$$

L.H.S of ① can be resolved in n linear factors

then ① becomes

$$(P-F_1)(P-F_2)\dots\dots(P-F_n) = 0$$

$$P = F_1,\; P = F_2,\; P = F_n.$$

$$\phi_1(x,y,c) = 0 \;;\; \phi_2 = (x,y,c) = 0 \dots\dots \phi_n(x,y,c$$

The general solution is obtained.

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$\phi_1(x,y,c)\ \phi_2(x,y,c) \cdots \phi_n(x,y,c) = 0.$

— x —

solve $\left(\frac{dy}{dx}\right)^2 - 6\left(\frac{dy}{dx}\right) + 8 = 0.$

soln:

Put $\frac{dy}{dx} = P$

The given equation is $P^2 - 6P + 8 = 0.$

$$(P-4)(P-2) = 0$$

$$P = 4 \text{ (or) } P = 2$$

$\frac{dy}{dx} = 4$ (or) $\frac{dy}{dx} = 2$

Integrating

$\int dy = \int 4\,dx$

$y = 4x + c$

$(y - 4x - c) = 0$

Integrating

$\int dy = \int 2\,dx$

$y = 2x + c$

$(y - 2x - c) = 0$

$$(y-4x-c)\ (y-2x-c) = 0$$

— x —

solve

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

Put $\frac{dy}{dx} = P$

$$P - \frac{1}{P} = \frac{x}{y} - \frac{y}{x}$$

$$\frac{P^2 - 1}{P} = \frac{x}{y} - \frac{y}{x}.$$

$$P^2 - 1 = P\left(\frac{x}{y} - \frac{y}{x}\right).$$

$$p^2 - p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$

$$\left(p + \frac{x}{y}\right)\left(p - y/x\right) = 0$$

-1

$\frac{x}{y}$ $\quad$ $-y/x$

$$p + x/y = 0 \quad \text{(or)} \quad p - y/x = 0$$

$$p = -\frac{x}{y} \quad \text{(or)} \quad p = y/x$$

$$p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$$

$$\left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$$

-1

y/x $\quad$ $-x/y$

$$p + \frac{y}{x} = 0 \quad \text{(or)} \quad p - \frac{x}{y} = 0$$

$$p = -\frac{y}{x} \quad \text{(or)} \quad p = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{-y}{x} \quad \text{(or)} \quad \frac{dy}{dx} = \frac{x}{y}$$

$$x\,dy = -y\,dx \quad \text{(or)} \quad y\,dy = x\,dx$$

$x\,dy + y\,dx = 0$ $\qquad$ $x\,dx - y\,dy = 0$

Integrating $\qquad$ Integrating

$\int d(x.y) = 0$ $\qquad$ $\int (x\,dx - y\,dy) = 0$

$xy = c$ $\qquad$ $\frac{x^2}{2} - \frac{y^2}{2} = c$

$(xy - c) = 0$; $\qquad$ $x^2 - y^2 - c = 0$.

$$(xy - c)(x^2 - y^2 - c) = 0.$$

— x —.

Solve $p^2 + 2py \cot x = y^2$

$p^2 + 2py \cot x - y^2 = 0$

v.Imp

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$; $b = 2y \cot x$; $c = -y^2$

$$P = \frac{-2y\cot x \pm \sqrt{(2y\cot x)^2 - 4(1)(-y^2)}}{2(1)}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm 2\sqrt{y^2\cot^2 x + y^2}}{2}$$

$$= \frac{\cancel{2}\left(-y\cot x \pm \sqrt{y^2\cot^2 x + y^2}\right)}{\cancel{2}}$$

$$= -y\cot x \pm \sqrt{y^2\cot^2 x + y^2}$$

$$P = -y\cot x \pm \sqrt{y^2(1+\cot^2 x)}$$

$$= -y\cot x \pm y\sqrt{\operatorname{cosec}^2 x}$$

$$\frac{dy}{dx} = -y\cot x \pm y\operatorname{cosec} x$$

$$= y\operatorname{cosec} x - y\cot x$$

$$\frac{dy}{dx} = y(\operatorname{cosec} x - \cot x)$$

$$= y\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$\sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\frac{dy}{dx} = y\left(\frac{1-\cos x}{\sin x}\right) = y\left(\frac{\cancel{2}\sin^{\cancel{2}} x/2}{\cancel{2}\sin\cancel{\frac{x}{2}}\cos\frac{x}{2}}\right)$$

$$\frac{dy}{dx} = y \tan\frac{x}{2}$$

$\int \tan x = \log$

$$\frac{dy}{y} = \tan\frac{x}{2}\,dx$$

$$\int dy/y = \int \tan\frac{x}{2}\,dx$$

$$\log y = \frac{\log \sec\left(\frac{x}{2}\right)}{\frac{1}{2}} + \log c$$

$$\log y = 2\log \sec\left(\frac{x}{2}\right) + \log c$$

$$= \log \sec^2\frac{x}{2} + \log c$$

$$\log y = \log\left(c \sec^2\frac{x}{2}\right)$$

$$y = c\sec^2\frac{x}{2} = c\,\frac{1 \cdot 2}{1+\cos x}$$

$$y(1+\cos x) = 2c$$

$\int \tan x = \log \sec x$

$\frac{d}{dx}(\log x) = \frac{1}{x}$

$\log x = \int \frac{1}{x}\,dx$

$1+\cos x = 2\cos^2\frac{x}{2}$

$\log ab = \log a + \log b$

— x —

Type 1:

Integrating faction

equation reducable to exact equation

* Differential equation which are not exact

we can sometime made exact after multiplying by a suitable [f(x,y)] called the integrating factor.

* Integrating factor found by Inspection.

Example:

solve $y\,dx - x\,dy = 0$

$$y\,dx - x\,dy \rightarrow (1)$$

$$M\,dx - N\,dy = 0$$

$$M = y \; ; \quad N = -x.$$

$$\frac{dM}{\partial y} = 1 \; ; \quad \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Multiplying (1) by $\frac{1}{y^2}$

$$\frac{y\,dx - x\,dy}{y^2} = 0$$

$$d\left(\frac{x}{y}\right) = 0.$$

which is exact

(1) $\times \frac{1}{x^2}$, $\quad \frac{y\,dx - x\,dy}{x^2} = 0$

$$d\left(y/x\right) = 0$$

Multiply (1) by $1/xy$.

$$\frac{y\,dx - x\,dy}{xy} = 0$$

$$\frac{\cancel{y}\,dx}{x\cancel{y}} - \frac{\cancel{x}\,dy}{\cancel{x}y} = 0$$

$$\int \frac{dx}{x} - \int \frac{dy}{y} = 0$$

$$\log x - \log y = \log c$$

$\cancel{\log(x-y) = \log c}$ $\quad \log(x/y) = \log c$

$\cancel{x}$ $\quad \frac{x}{y} = c$

$$x = cy$$

$\therefore \frac{1}{y^2}, \frac{1}{x^2}, \frac{1}{xy}$ are integrating factor of (1)

—x—

Integrating factor of a homogeneous equation

solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

$$M\,dx + N\,dy = 0$$

$$M = x^2y - 2xy^2$$

$$N = -(x^3 - 3x^2y)$$

This equation is homogeneous in x and y.

$$\text{Integrating factor} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x + (3x^2y - x^3)y}$$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 + 3x^2y^2 - \cancel{x^3y}}$$

$$I.F = \frac{1}{x^2y^2}$$

Multiplying by $\frac{1}{x^2y^2}$

$$\frac{1}{x^2y^2}[x^2y - 2xy^2]\,dx - \frac{1}{x^2y^2}[x^3 - 3x^2y]\,dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

$$M = \frac{1}{y} - \frac{2}{x} \; ; \; N = -\left(\frac{x}{y^2} - \frac{3}{y}\right)$$

$$\frac{\partial M}{\partial y} = -1/y^2 \; ; \; \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ which is exact

The solution is $\int_{(y \text{ constant})} M\,dx + \int \left(\text{terms of } N \text{ not containing } x\right) dy = c$

$$M = \frac{1}{y} - \frac{2}{x} \; ; \; N = -\left(\frac{x}{y^2} - \frac{3}{y}\right)$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \; ; \; \frac{\partial N}{\partial x} = \frac{-1}{y^2}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y}\, dy = c$$

$$\frac{1}{y}\int dx - 2\int \frac{1}{x}\, dx + 3\int \frac{1}{y}\, dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c.$$

—x—.

Type 3:

I.F for an equation of the type

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$$

If the equation $M\,dx + N\,dy = 0$ be of this form, $\frac{1}{Mx - Ny}$ is an I.F $(Mx - Ny) \neq 0$

—x—.

Ex

Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0.$

This is of the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$$

$$M = (1 + xy)\,y \; ; \; N = (1 - xy)\,x.$$

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$$I.F = \frac{1}{Mx - Ny} = \frac{1}{(1+xy)yx - (1-xy)xy}$$

$$= \frac{1}{\cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2}$$

$$= \frac{1}{2x^2y^2}$$

Multiplying by $\frac{1}{2x^2y^2}$

$$\frac{1}{2x^2y^2}(1+xy)y\,dx + \frac{1}{2x^2y^2}(1-xy)x\,dy = 0$$

$$\frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

$$M = \frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right) \;;\; N = \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2x^2y^2} \;;\; \frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact

$\int M\,dx + \int$ (terms of N not containing x) $dy = C$

(y constant)

$$\frac{1}{2}\int\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \int \frac{1}{2}\left(-\frac{1}{y}\right)dy = C$$

$$\frac{1}{2}\left(\frac{-1}{xy}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = C$$

$x^n = \frac{x^{n+1}}{n+1}$

$\int x^{-2}dx = \frac{x^{-2+1}}{-2+1}$

$= \frac{x^{-1}}{-1} = \frac{-1}{x}$

$\div$ by $\frac{1}{2}$ $\quad -\frac{1}{xy} + \log x - \log y = C$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = C$$

— x —

In the equation

$$M\,dx + N\,dy = 0$$

a) if $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}$ be a function of x only $= f(x)$,

then $e^{\int f(x)\,dx}$ is an I.F

b) if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ be a function of y only $= F(y)$ then $e^{\int F(y)\,dy}$ is an I.F.

— x —

solve

$$\left(xy^2 - e^{1/x^3} \right) dx - x^2 y\,dy = 0.$$

$$M = xy^2 - e^{1/x^3} \quad ; N = -x^2 y$$

$$\frac{\partial M}{\partial y} = 2xy \quad ; \frac{\partial N}{\partial x} = -2xy$$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2 y} = \frac{4\cancel{xy}}{-\cancel{x^2 y}} = \frac{-4}{x}$$

which is a function of x only.

$$I.F = e^{\int -4/x\,dx} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4}$$

Multiple by x^{-4}

$$x^{-4}\left(xy^2 - e^{1/x^3} \right) dx - x^{-4}\left(x^2 y\,dy \right) = 0$$

$$\left(x^{-3} y^2 - x^{-4} e^{1/x^3} \right) dx + x^{-2} y\,dy = 0$$

$$M = x^{-3} y^2 - x^{-4} e^{1/x^3} \quad ; N = x^{-2} y$$

$$\frac{\partial M}{\partial y} = 2y x^{-3} \quad ; \frac{\partial N}{\partial x} = 2x^{-3} y.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact

The solution is

$\int M dx + \int$ (terms of N not containing x) dy = c
(y constant)

$$\int (x^{-3}y^2 - x^{-4} e^{1/x^3}) dx + \int 0 = c$$

$$-y^2 \frac{x^{-2}}{2} + \frac{1}{3}\int e^{-x^3}(-3x^{-4}) dx = c$$

$$\frac{1}{3} e^{-x^3} - \frac{1}{2} y^2/x^2 = c$$

—x—

Solve $(xy^3+y)dx + 2(x^2y^2 + x + y^4) dy = 0.$

$M = xy^3 + y \qquad N = 2(x^2y^2 + x + y^4)$

$$\frac{\partial M}{\partial y} = x3y^2 + 1 \;;\; \frac{\partial N}{\partial x} = 2[2xy^2 + 1]$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is not exact

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2(2xy^2+1) - (3xy^2+1)}{xy^3+y} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3+y}$$

$$= \frac{xy^2+1}{xy^3+y} = \frac{xy^2+1}{y(xy^2+1)} = \frac{1}{y}$$

∴ which is function of y alone.

$$I.F = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiple by y

$$y(xy^3+y)dx + 2y(x^2y^2 + x + y^4) dy = 0$$

$$(xy^4 + y^2) dx + (2x^2y^3 + 2xy + 2y^5) dy = 0$$

$M = xy^4 + y^2$ $\qquad$ $N = 2x^2y^3 + 2xy + 2y^5$

$\frac{\partial M}{\partial y} = 4y^3x + 2y$ $\qquad$ $\frac{\partial N}{\partial x} = 4xy^3 + 2y$,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact.

The solution is

$$\underset{(y\ \text{constant})}{\int M\,dx} + \int \left(\begin{array}{c}\text{terms of } N \text{ not}\\ \text{containing } x\end{array}\right) dy = c$$

$$\int (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$$

$$\frac{x^2}{2}y^4 + y^2x + \frac{\cancel{2}y^6}{\cancel{6}\,3} = c$$

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

—x—.

Equation not of first degree

Solvable for y.

$$y = f(x, p)$$

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

Let its should be $\quad F(x, p, c) = 0$

$$x = F_1(p, c)\ ;\ y = F_2(p, c)$$

$$y - 2px = \tan^{-1}(xp^2)$$

$$y = 2px + \tan^{-1}(xp^2). \rightarrow (1)$$

Diff ① with respect to x on both sides.

$$\frac{dy}{dx} = 2\left(P \cdot 1 + \frac{dp}{dx}x\right) + \frac{1}{1+(xp^2)^2}\left(x \cdot 2p\frac{dp}{dx} + p^2\right)$$

$$p = 2\left(p + x\frac{dp}{dx}\right) + \frac{1}{1+x^2p^4}\left(2px\frac{dp}{dx} + p^2\right)$$

$$p = \left[\left(2p + 2x\frac{dp}{dx}\right) + \frac{1}{1+x^2p^4}\, p\left(2x\frac{dp}{dx} + p\right)\right].$$

$$p = \left(p + 2x\frac{dp}{dx}\right)\left(p + \frac{p}{1+x^2p^4}\right)$$

$$\cancel{p} = \left(p + 2x\frac{dp}{dx}\right)\cancel{p}\left(1 + \frac{1}{1+x^2p^4}\right)$$

$$\left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{1}{x^2p^4}\right) = 0$$

$$p + 2x\frac{dp}{dx} = 0$$

$$2x\frac{dp}{dx} = -p$$

$$2\frac{dp}{p} = -\frac{dx}{x}$$

$$2\frac{dp}{p} + \frac{dx}{x} = 0$$

$$2\int\frac{dp}{p} + \int\frac{dx}{x} = 0$$

$$2\log p + \log x = \log c$$

$$\log p^2 + \log x = \log c$$

$$\log(xp^2) = c$$

$$\boxed{xp^2 = c}$$

$$p^2 = c/x$$

$$P = \sqrt{c/x} \rightarrow ②$$

Eliminate P from ① & ②

$$y = 2\sqrt{\frac{c}{x}}\, x + \tan^{-1} c$$

$$= 2\sqrt{c}\, \frac{\sqrt{x}\,\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} + \tan^{-1} c$$

$$y = 2\sqrt{cx} + \tan^{-1} c$$

— x —

Solve $y = 2px - p^2$

$$\frac{dy}{dx} = P = 2\left(P \cdot 1 + x\frac{dp}{dx}\right) - 2p\frac{dp}{dx}$$

$$P = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$2P - P + 2(x - P)\frac{dp}{dx} = 0$$

$$P + 2(x - P)\frac{dp}{dx} = 0$$

$$P = -2(x - P)\, dp/dx$$

$$P\frac{dx}{dp} = -2x + 2P$$

$$P\, dx/dp + 2x = 2P$$

$$\frac{dx}{dp} + \frac{2x}{P} = \frac{2\cancel{P}}{\cancel{P}}$$

$$\frac{dx}{dp} + \underbrace{\frac{2x}{P}}_{P} = \underbrace{2}_{Q}$$

$$I.F = e^{\int P dp} = e^{\int 2/p\, dp} = e^{\log P} = e^{\log P^2} = P^2 \quad \boxed{I.F = P^2}$$

$$x(I.F) = \int Q\,(I.F)\,dp + c$$

$\frac{dy}{dx} + Py = Q$

$I.F = e^{\int P dx}$

$$xp^2 = \int 2p^2\,dp + c$$

$$xp^2 = \frac{2p^3}{3} + c \Rightarrow \div p^2; \quad x = \frac{2p^3}{3p^2} + \frac{c}{p^2}$$

$$\boxed{x = \frac{2p}{3} + cp^{-2}.}$$

— x —

solve $y + px = x^4 p^2$

soln: $y = -px + x^4 p^2$.

diff with respect to x

$$\frac{dy}{dx} = P = -\left(P.1 + x\frac{dp}{dx}\right) + \left(4x^3p^2 + x^4\, 2p\frac{dp}{dx}\right)$$

$$P + P + x\frac{dp}{dx} - 4x^3p^2 - 2x^4 p\frac{dp}{dx} = 0$$

$$2p - 4x^3p^2 + x\frac{dp}{dx} - 2x^4 p\frac{dp}{dx} = 0$$

$$2p(1-2x^3p) + (1-2x^3p)\,x\,dp/dx = 0.$$

$$(1-2x^3p)(2p + x\,dp/dx) = 0$$

Discarding the factor $(1-2x^3p)$

$$2p + x\frac{dp}{dx} = 0$$

$$x\frac{dp}{dx} = -2p \Rightarrow \cancel{\frac{dp}{dx}}\ \frac{dp}{p} = -2\frac{dx}{x}$$

Integrating

$$\int dp/p = -2\int dx/x$$

$$\log p = -2\log x + \log c$$

$$\log p + 2\log x = \log c$$

$$\log p + \log x^2 = \log c \Rightarrow px^2 = c \Rightarrow p = c/x^2$$

sub p in ①

$$y = -\frac{c}{x^2}.x + x^4\left(\frac{c}{x^2}\right)^2$$

$$= -c/x + \cancel{x^4}\frac{c^2}{\cancel{x^4}}$$

$$\boxed{y = \frac{-c}{x} + c^2}$$

Equation solvable of x

* The equation of this type $x = f(y, p) \rightarrow ①$

Differentiating ① with respect to y. | $\frac{dy}{dx} = p$

$$x = f(y, p) \rightarrow ①$$

Diff ① with respect to y,

$$\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \rightarrow ②$$

② is the differential equation of first order in p and y

Solution of ② is $\phi(y, p, c) = 0 \rightarrow ③$

* Eliminate p from equation ① & ③ gives the required equation.

— x —

(x) v. Imp

Solve

$$y = 2px + y^2p^3 \rightarrow (*)$$

$$y - y^2p^3 = 2px$$

$$\Rightarrow x = \frac{y - y^2p^3}{2p}$$

Solving for x, $x = \frac{1}{2}\left[\frac{y}{p} - y^2p^2\right] \rightarrow ①$

Diff ① with respect to y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} + y\left(\frac{-1}{p^2}\right)\frac{dp}{dy} - \left(2y \cdot p^2 + y^2 2p\frac{dp}{dy}\right)\right]$$

$$\frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - y^2\, 2p\frac{dp}{dy}\right].$$

$$\frac{1}{p} = \frac{1}{2}\cdot\frac{1}{p}\left(1 - \frac{y}{p}\frac{dp}{dy} - 2p^2 p y - y^2\, 2p\cdot p\frac{dp}{dy}\right]$$

$$2p = p\left(1 - y/p\ dp/dy - 2yp^3 - 2y^2p^2\frac{dp}{dy}\right).$$

$$2p = p - y\frac{dp}{dy} - 2yp^4 - y^2\,2p^3\frac{dp}{dy}$$

$$\boxed{\begin{array}{l} 2p - p \\ = p \end{array}}$$

$$p + y\frac{dp}{dy} + 2yp^4 + 2y^2p^3\frac{dp}{dy} = 0.$$

$$(p + 2yp^4) + (y + 2y^2p^3)\frac{dp}{dy} = 0$$

$$p(1+2yp^3) + (1+2yp^3)y\frac{dp}{dy} = 0$$

$$(1+2yp^3)\left(p + y\frac{dp}{dy}\right) = 0$$

Discarding the factor $(1+2yp^3)$, we get

$$p + y\frac{dp}{dy} = 0$$

$$y\frac{dp}{dy} = -p$$

$$\frac{\cancel{dp}}{\cancel{dy}} \cancel{+} + \frac{\cancel{dy}}{\cancel{y}} \qquad \frac{dp}{p} = -\frac{dy}{y}$$

$$\frac{dp}{p} + \frac{dy}{y} = 0,$$

Integrating,

$$\int\frac{dp}{p} + \int\frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = c/y$$

sub p in ⊛

$$y = 2px + y^2p^3$$

$$y = \frac{2cx}{y} + y^2\left(\frac{c}{y}\right)^3$$

$$y = \frac{2cx}{y} + y^2 \frac{c^3}{y^3}$$

$$y = \frac{2cx + c^3}{y}$$

$$\boxed{y^2 = 2cx + c^3.}$$

— x —

(5)

solve $P = \tan\left(x - \frac{P}{1+P^2}\right)$.

$$\tan^{-1} p = x - \frac{p}{1+p^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu - u}{v^2}$$

$$x = \tan^{-1} p + \frac{p}{1+p^2}$$

$\tan^{-1} x = \frac{1}{1+x^2}$

Diff with respect to y

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p\left(0+2p\frac{dp}{dy}\right)}{(1+p^2)^2}$$

$$\frac{1}{p} = \left[\frac{1}{(1+p^2)} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$= \left[\frac{(1+p^2) + (1+p^2) - 2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{2 + \cancel{2p^2} - \cancel{2p^2}}{(1+p^2)^2}\frac{dp}{dy}$$

$$1/p = \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$dy = \frac{2p}{(1+p^2)^2}\,dp$$

Integrating

$$\int dy = \int \frac{2p}{(1+p^2)^2}\,dp$$

$1+p^2=t$

$2p\,dp = dt$

$$y = \int \frac{dt}{t^2}$$

$$= \int t^{-2}\,dt$$

$$= \frac{t^{-2+1}}{-2+1} = \frac{t^{-1}}{-1}$$

$$= \frac{-1}{t}$$

$$y = \frac{-1}{1+p^2} + c$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$$\boxed{y = c - \frac{1}{1+p^2}}$$

— x —

(*) 2m

Clairaut's type equation:

* An equation of the form $y = px + f(p) \rightarrow ①$

is known as clairaut's equation.

Differentiating ① with respect to x we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + [x + f'(p)]\frac{dp}{dx}$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$

$$\frac{dp}{dx} = 0$$

Integrate , $P = c$

Puting $P = c$ in ①

$$\boxed{y = cx + f(c)}$$

* Thus the solution clairaut's equation is obtain by writing c for p.

— x —

Ex. 2m

Solve $(y - px)(p-1) = p$

The given equation is

$$(y - px)(p-1) = p$$

$$y - px = \frac{p}{p-1}$$

$$y = px + \frac{p}{p-1}$$

$$y = px + f(p)$$

which is clairaut's equation.

Putting $P = c$ we get the solution is

$$\boxed{y = cx + \frac{c}{c-1}}$$

* Thus the solution clairaut's equation is obtain by writing c for P

— x —

Ex. Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$.

The given equation is $e^{4x}(p-1) + e^{2y}p^2 = 0$

$$y = px + f(p)$$

$$e^{lx} \qquad e^{my}$$

$$X = e^{kx} \qquad Y = e^{ky}$$

$K \rightarrow$ H.C.F of l & m.

Putting $X = e^{2x}$ $\quad Y = e^{2y}$

$$dX = 2e^{2x}\,dx \quad ; \quad dY = 2e^{2y}\,dy.$$

$$p = \frac{dy}{dx} = \frac{dY/2e^{2y}}{dX/2e^{2x}} = \frac{dY}{2e^{2y}} \times \frac{2e^{2x}}{dX}$$

$$= \frac{X}{Y}\frac{dY}{dX} = \frac{X}{Y}P$$

$$\boxed{p = \frac{X}{Y}P}$$

The given equation is

$$X^2\left(\frac{X}{Y}P - 1\right) + Y\left(\frac{XP}{Y}\right)^2 = 0.$$

$$X^2\left(\frac{XP-Y}{Y}\right) + \frac{\cancel{Y}X^2P^2}{Y^2} = 0$$

$$\frac{X^2}{Y}\left[XP - Y + P^2\right] = 0$$

$$XP - Y + P^2 = 0$$

$$PX + P^2 = Y$$

$$\boxed{Y = PX + P^2}$$

which is of clairaut's equation

$$Y = cX + c^2$$

$$e^{2y} = ce^{2x} + c^2.$$

—x—.

Solve $(Px-y)(Py+x) = 2P$

V. Imp The given equation is

$$(Px-y)(Py+x) = 2P \longrightarrow (1)$$

Putting $X = x^2$; $Y = y^2$

$$dX = 2x\,dx \; ; \; dY = 2y\,dy$$

$$\frac{dX}{2x} = dx \; ; \; \frac{dY}{2y} = dy$$

$$P = \frac{dy}{dx} = \frac{dY/2y}{dX/2x} = \frac{dY}{2y} \times \frac{2x}{dX}$$

$$= \frac{x}{y}\frac{dY}{dX}$$

$$= \frac{\sqrt{X}}{\sqrt{Y}}\frac{dY}{dX}$$

$$\boxed{P = \frac{\sqrt{X}}{\sqrt{Y}}\,P}$$

$\frac{dY}{dX} = P$

$x^2 = X$

$x = X^{1/2}$

$x = \sqrt{X}$

where $\frac{dY}{dX} = P$

$\therefore$ The equation (1) is

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\left(\frac{\sqrt{X}P\sqrt{X} - \sqrt{Y}\sqrt{Y}}{\sqrt{Y}}\right)(\sqrt{X}P + \sqrt{X}) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\left(\frac{XP-Y}{\sqrt{Y}}\right)\sqrt{X}(P+1) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\frac{\sqrt{X}}{\sqrt{Y}}(XP-Y)(P+1) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\frac{(xP-y)}{\;}\frac{(P+1)}{\;} = 2\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}P\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}$$

$$(xP-y)(P+1) = 2P$$

$$(Px-y) = \frac{2P}{P+1}$$

$$Px - \frac{2P}{P+1} = y$$

$$y = Px - \frac{2P}{P+1}$$

$$\boxed{y = Px + f(p)}$$

which is a clairaut's equation

By Putting $P=c$, we get the solution is

$$y = cx - \frac{2c}{c+1}$$

$$y^2 = cx^2 - \frac{2c}{c+1}$$

—x—

Solve

$$(Px-y)(Py+x) = a^2 p.$$

The given equation is

$$(Px-y)(Py+x) = a^2 p \longrightarrow (1)$$

Putting $X = x^2$; $Y = y^2$

$$dX = 2x\,dx \;;\; dY = 2y\,dy.$$

$$\frac{dX}{2x} = dx \;;\; \frac{dY}{2y} = dy.$$

$$P = \frac{dy}{dx} = \frac{dy/\cancel{dy}\,2y}{dx/\cancel{dx}\,2x} = \frac{dy}{\cancel{2}y} \times \frac{\cancel{2}x}{dx}$$

$$= \frac{x}{y}\frac{dy}{dx}$$

$$= \frac{\sqrt{x}}{\sqrt{y}}\frac{dy}{dx}$$

$$P = \frac{\sqrt{x}}{\sqrt{y}}\,p.$$

The equation ① is

$$\left(\frac{\sqrt{x}}{\sqrt{y}}\,p\sqrt{x} - \sqrt{y}\right)\left(\frac{\sqrt{x}}{\sqrt{y}}\,p\sqrt{y} + \sqrt{x}\right) = \alpha^2\frac{\sqrt{x}}{\sqrt{y}}\,p$$

$$\left(\frac{\sqrt{x}\,p\sqrt{x} - \sqrt{y}\sqrt{y}}{\sqrt{y}}\right)\left(\sqrt{x}\,p + \sqrt{x}\right) = \alpha^2\frac{\sqrt{x}}{\sqrt{y}}\,p$$

$$\left(\frac{xp - y}{\sqrt{y}}\right)\sqrt{x}\,(p+1) = \alpha^2\frac{\sqrt{x}}{\sqrt{y}}\,p$$

$$(xp - y)(p+1) = \alpha^2\frac{\sqrt{x}}{\sqrt{\cancel{x}}}\,p \times \frac{\sqrt{\cancel{x}}}{\sqrt{\cancel{x}}}$$

$$(xp - y)(p+1) = \alpha^2 p$$

$$(xp - y) = \frac{\alpha^2 p}{p+1}$$

$$px - \frac{\alpha^2 p}{p+1} = y$$

$$y = px - \frac{\alpha^2 p}{p+1}$$

$$\boxed{y = P(x) + f(P)}$$

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Objective type questions	Opt 1	Opt2	Opt3	Opt4	Answer
The necessary and sufficient condition for the differential equation to be exact is	M_x = N_y	M_y = N_x	M_x = N_x	M_y = N_y	M_y = N_x
The equation is known is dy/dx+Py=Q y^2	Euler equation	Bernoulli's Equation	Legendre equation	Homogeneous	Bernoulli's Equation
The integrating factor of dy/dx+y/x=x^2	x	y	logx	0	x
The solution of Mdx+Ndy=0 is posses an	Infinite no of integrating factor	finite no of integrating factor	none of these	one integrating factor	Infinite no of integrating factor
A differential equation is said to be ______ if the dependent variable and its derivative occur only in the first degree and are not multiplied together	Linear	nonlinear	quadratic	PDE	Linear
The order of d^2y/dx^2+y=x^2-2 is	0	1	2	3	2
The integrating factor of dy/dx+ysinx = 0 is	e^-cosx	ye^-cosx	logx	e^sinx	e^-cosx
The integrating factor of dy/dx-ycotx = sinx is	sinx	- sinx	cosx	- cosx	- sinx
The solution of y=(x-a)p-p^2	y = (x-a)c-c^2	y = (x-a)c+c^2	0	-1	y = (x-a)c-c^2
An equation of the form y=px+f(p) is known as	linear	Bernoulli's Equation	exact	Clairaut's equation	Clairaut's equation
The order of d^2y/dx^2+y=0 is	2	1	0	-1	2
The clairaut's form of p=tan(px-y)	y=cx+tan^-1 c	y=cx-tan^-1 c	c=tan(cx-y)	c=tan(px+y)	y=cx-tan^-1 c
An equation involving one dependent variable and its derivatives with respect to one independent variable is called __________	ODE	PDE	Partial	Total	ODE
The _______ is differentiation of a function of two or more variables	ODE	PDE	Partial	Total	PDE
A differential equation is said to be linear if the dependent variable and its derivative occur only in the ______degree and are not multiplied together	first	second	third	first and second	first
The highest derivative of the differential equation is________	Order	Degree	Power	second degree	Order
The powerof the hightest derivative of the differential equation is called____	Order	Degree	Power	second degree	Degree
The order of y''-y'+7=x^2+4 is	0	1	2	3	2
The order of y'''+xy'+7x=0 is	0	1	2	3	3
The degree of the (d^2y/dx^2)^2+(dy/dx)^3+3y=0	0	1	2	3	2
The degree of the (d^2y/dx^2)^3+(dy/dx)^3+7y=0	0	1	2	3	3
The order and degree of (d^3/dx^3)^2+dy/dx+9y=0	3,2	2,3	1,2	2,1	3,2
The standard form of a linear equation of the first order	dy/dx+Py=Q	dy/dx+py=Q	dy/dx+Py=q	5dy/dx+Py=Q	dy/dx+Py=Q
The integrating factor of linear equation of the form dx/dy+Px=Q is	e^integral Qdx	e^integral Pdy	e^integral Qdx	e^Qdx	e^integral Pdy
The integrating factor of linear equation of the form dy/dx+Py=Q is	e^integral Qdy	e^integral Pdx	e^integral Qdx	e^Qdx	e^integral Pdx
The integrating factor of dy/dx+ysinx=0 is	e^(-cosx)	e^(-cosx)y	logx	e^(sinx)	e^(-cosx)
The integrating factor of dy/dx-ycotx=0 is	cos x	(-cos x)	cosec x	sin x	cosec x
If the given equationMdx+Ndy=0 is homogenous and Mx+Ny≠0 then the integrating factor is ___	1/(Nx-My)	1/(Mx+Ny)	1/(Mx-Ny)	1/(Nx+My)	1/(Mx+Ny)
The solution of Mdx+Ndy is	intergral y constant Mdx+integral of terms of N not containing x dy	intergral y constant Mdx+integral of terms not containing x dx	intergral y constant Ndx+integral of terms not containing x dx	intergral y constant Mdx+integral of terms not containing y dx	intergral y constant Mdx+integral of terms of N not containing x dy
If Mdx+Ndy=0 be a homogeneous equation in x and y, then______is an integrating factor(Mx+Ny≠0)	1/(Mx+Ny)	1/(Mx-Ny)	Mdy+Ndx	Mdy-Ndx	1/(Mx+Ny)
If Mdx+Ndy=0 be a homogeneous equation in x and y, then______is an integrating factor(Mx-Ny≠0)	1/(Mx+Ny)	1/(Mx-Ny)	Mdy+Ndx	Mdy-Ndx	1/(Mx-Ny)

Ordinary differential equation of higher order.

* consider The general linear differential equation with constant co-efficient of the form

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + K_n y = x$$

* $K_1, K_2, \dots K_n$ are constant

Replace

$$\frac{d}{dx} \rightarrow D$$

$$D^n y + K_1 D^{n-1} y + K_2 D^{n-2} y + \dots + K_n y = X$$

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \dots + K_n) y = X$$

The general solution of the equation ① is

$$y = C.F + P.I$$

C.F → complementary function.

P.I → Particular integral.

Rules for finding C.F

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \dots + K_n) y \; X$$

Auxiliary equation is replace $D \rightarrow m$

By solving we get the roots

S. NO	Roots	complementary function (C.F)
1.	If two roots are real & distinct $m_1 \neq m_2$	$y = Ae^{m_1 x} + Be^{m_2 x}$
2	If two roots are real & equal $m_1 = m_2 = m$	$y = (Ax+B)e^{mx}$
3	If two roots are real & imaginary $(\alpha \pm i\beta)$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Rules for finding P.I

$$\Rightarrow f(D)y = X$$

$$P.I = \frac{1}{f(D)} X$$

where x is the function of x

R.H.S = 0

There is only C.F no P.I

Solve $(D^2+5D+6)y=0$

Given $(D^2+5D+6)y=0$

Replace $D \rightarrow m$

Auxiliary equation is $m^2+5m+6=0$

$(m+2)(m+3)=0$

$m=-2,-3$

The roots are real & distinct.

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$m_1 = -2 ; m_2 = -3$

$m_1 \neq m_2$.

~~∴ C.F =~~ ∴ C.F is $y = A e^{m_1 x} + B e^{m_2 x}$

$$\boxed{y = A e^{-2x} + B e^{-3x}}$$

— x —

Solve

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

soln:

$$\frac{d}{dx} \to D$$

$$D^2y + 6Dy + 9y = 0$$

$$(D^2 + 6D + 9)y = 0$$

Replace $D \to m$

Auxiliary equation is $m^2 + 6m + 9 = 0$

$$(m+3)(m+3) = 0$$

$$m = -3, -3$$

$m_1 = -3 ; m_2 = -3$.

$m_1 = m_2$

The roots are real & equal

$m_1 = m_2 = m$

∴ C.F is $y = (Ax + B) e^{mx}$

$$\boxed{y = (Ax + B) e^{-3x}}$$

— x —

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Solve $(D^2+D+1)\,y=0$

Soln: Replace: $D \rightarrow m$

Auxiliary equation is $m^2+m+1=0$

$a=1;\ b=1,\ c=1.$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$$

$$=\frac{-1\pm i\sqrt{3}}{2}$$

∴ The roots are real & imaginary

$$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

$$\alpha=-1/2\ ;\ \beta=\sqrt{3}/2$$

∴ C.F is $y=e^{-1/2\,x}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)$

—x—.

Solve $(D^2+D+1)y = x^2$

Soln:

Auxiliary equation is $m^2+m+1=0$

$a=1;\ b=1;\ c=1$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

The roots are real & imaginary

$\alpha = -1/2$

$\beta = \frac{\sqrt{3}}{2}$

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$C.F = e^{-1/2 x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

To find P.I

$$P.I = \frac{1}{D^2+D+1}x^2$$

$$= \frac{1}{1+D^2+D}x^2$$

$1+x$

$$= [1+(D^2+D)]^{-1}x^2$$

$$(1+x)^{-1} = 1-x+x^2-x^3+\dots$$

$$= [1-(D^2+D)+(D^2+D)^2-(D^2+D^3)+\dots]x$$

$$= x^2-(D^2+D)(x^2)+(D^4+2D^3+D^2)x^2$$

$$= x^2 - D^2(x^2) - D(x^2) + D^2(x^2)$$

$$= x^2 - \cancel{2} - 2x + \cancel{2}$$

$D(x^2) = 2x$
$D^2(x^2) = 2$
$D^3(x^2) = 0$

$$P.I = x^2 - 2x$$

$$\therefore y = C.F + P.I$$

$$= e^{-1/2 x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + x^2 - 2x$$

— x —

Find P.I of $(D^2+5D+6)y = x^2$

$$P.I = \frac{1}{D^2+5D+6}x^2$$

$$= \frac{1}{6\left(\frac{D^2+5D}{6}+1\right)}x^2$$

$$= \frac{1}{6}\left[1+\left(\frac{D^2+5D}{6}\right)\right]^{-1}x^2 \quad (1+x)^{-1}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2+5D}{6}\right)+\left(\frac{D^2+5D}{6}\right)^2 - \cdots\right](x^2)$$

$$= \frac{1}{6}\left[x^2 - \left(\frac{D^2+5D}{6}\right)(x^2) + \left(\frac{D^4+10D^3+25D^2}{36}\right)x^2\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{6}\left(D^2(x^2) + 5D(x^2)\right) + \frac{1}{36}\left[D^4(x^2) + 10D^3(x^2) + 25D^2(x^2)\right]\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{6}(2+5(2x)) + \frac{1}{\cancel{36}_{18}}(25\times\cancel{2})\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{6}(2+10x) + \frac{25}{18}\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{3} - \frac{\cancel{10}^{5}x}{\cancel{6}_{3}} + \frac{25}{18}\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{3} - \frac{5x}{3} + \frac{25}{18}\right]$$

$$\frac{1}{3}\times\frac{6}{6} = \frac{6}{18}$$

$$\frac{25}{18}\times\frac{1}{1} = \frac{25}{18}$$

$$P.I = \frac{1}{6}\left[x^2 - \frac{5x}{3} + \frac{19}{18}\right].$$

$$\frac{25-6}{18} = \frac{19}{18}$$

—x—

R.H.S $= e^{ax}\cos bx$ (or) $e^{ax}\sin bx$ (or) $e^{ax}x^n$.

$$P.I = \frac{1}{f(D)}\, e^{ax}\cos bx \text{ (or) } e^{ax}\sin bx \text{ (or) } e^{ax}x^n.$$

Replace $D \to D+a$.

—x—

$$(D^2+2D+5)y = e^x \sin 2x$$

Auxiliary equation is $m^2+2m+5=0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i.$$

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$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$C.F = e^{-x}(A\cos 2x + B\sin 2x)$$

To find P.I

$$P.I = \frac{1}{D^2+2D+5} e^{x}\sin 2x \qquad a=1$$

Replace $D \to D+a = D+1$

$$= \frac{1}{(D+1)^2+2(D+1)+5} e^{x}\sin 2x$$

$$= e^{x}\frac{1}{D^2+2D+1+2D+2+5}\sin 2x$$

$$= e^{x}\frac{1}{D^2+4D+8}\sin 2x$$

Replace $D^2 \to -a^2 = -4$

$$= e^{x}\frac{1}{-4+4D+8}\sin 2x$$

$$= e^{x}\frac{1}{4D+4}\sin 2x$$

$$= \frac{e^{x}}{4}\frac{1}{D+1}\sin 2x$$

$$= \frac{e^{x}}{4}\cdot\frac{1}{(D+1)(D-1)}(D-1)\sin 2x$$

$$= \frac{e^{x}}{4}\frac{(D-1)\sin 2x}{D^2-1}$$

$$= \frac{e^x}{4} \left(\frac{D \sin 2x - \sin 2x}{-4-1} \right) \qquad D^2 \to -a^2 = -4$$

$$= \frac{e^x}{4(-5)} (2\cos 2x - \sin 2x)$$

$$P.I = \frac{e^x}{-20} (2\cos 2x - \sin 2x).$$

— x —

~~Solve $(D^2 + 4D + 3)y = xe^{3x}$~~

— x —

Solve $(D^2 + 4D + 3)y = x e^{3x}$

soln:

$$P.I = \frac{1}{D^2+4D+3} x e^{3x}$$

Replace $D \to D+a = D+3$

$$= e^{3x} \frac{1}{(D+3)^2 + 4(D+3) + 3} x$$

$$= e^{3x} \frac{1}{D^2+9+6D+4D+12+3} x$$

$$= e^{3x} \frac{1}{D^2+10D+24} x$$

$$= e^{3x} \frac{1}{24\left(\frac{D^2+10D}{24}+1\right)} x$$

$$= \frac{e^{3x}}{24} \left[1 + \left(\frac{D^2+10D}{24}\right)\right]^{-1}$$

$D(x) = 1$
$D^2(x) = 0$

$$= \frac{e^{3x}}{24} \left[1 - \left(\frac{D^2+10D}{24}\right) + \left(\frac{D^2+10D}{24}\right)^2 \dots\right] x$$

$$= \frac{e^{3x}}{24}\left[x - \frac{1}{24}\left(D^2(x) + 10D(x)\right)\right]$$ (Neglecting higher Powers)

$$= \frac{e^{3x}}{24}\left[x - \frac{1}{24}10(1)\right] \Rightarrow \frac{e^{3x}}{24}\left[x - \frac{\cancel{10}^{5}}{\cancel{24}_{12}}\right]$$

$$= \frac{e^{3x}}{24}\left(x - 5/12\right).$$

—x—

R.H.S $= x^n \sin ax$ (or) $x^n \cos ax$

$$P.I = \frac{1}{f(D)} x^n \sin ax$$

$$= \text{Imaginary Part of } \frac{1 \cdot e^{iax}}{f(D)} x^n.$$

Replacing $D \rightarrow D+ia$

$$= \text{Imaginary Part of } 1 \cdot e^{iax} \frac{1}{f(D+ia)} x^n.$$

—x—

Solve $(D^2 - 2D + 1)y = x \sin x$.

Soln:

Auxiliary equation is $m^2 - 2m + 1 = 0$.

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

The roots are real & equal

$$C.F = (Ax+B)e^{mx}$$

$$C.F = (Ax+B)e^{x}$$

To find P.I

$$P.I = \frac{1}{D^2-2D+1} x\sin x$$

$$= \text{Imaginary Part of } \frac{1}{D^2-2D+1} e^{ix} x.$$

Replace $D \rightarrow D+i$

$$= \text{Imaginary Part of } e^{ix} \frac{1}{(D+i)^2 - 2(D+i)+1} x$$

$$= \text{Imaginary Part of } e^{ix} \frac{1}{D^2+i^2+2iD-2D-2i+1} x$$

$$= \text{Imaginary Part of } e^{ix} \frac{1}{D^2-\cancel{1}+2iD-2D-2i+\cancel{1}}$$

$$= I.P \text{ of } e^{ix} \frac{1}{-2i\left[1-\left(\frac{D^2+2iD-2D}{2i}\right)\right]} x$$

$$= I.P \text{ of } e^{ix} \frac{1}{-2i} \left[1-\left(\frac{D^2+2iD-2D}{2i}\right)\right]^{-1} x$$

$$= I.P \text{ of } e^{ix} \frac{1}{-2i} \left[1+\left(\frac{D^2+2iD-2D}{2i}\right)\right] x$$

[neglecting higher Power]

$$= I.P \text{ of } e^{ix} -\frac{1}{2i} \left[x+\left(\frac{D^2(x)+2iD(x)-2D(x)}{2i}\right)\right]$$

$$= I.P \text{ of } e^{ix} -\frac{1}{2i} \left[x+0+\frac{2i(1)-2(1)}{2i}\right]$$

$$= I.P \text{ of } e^{ix} -\frac{1}{2i} \left[x+\frac{i-1}{i}\right]$$

$D(x)=1$
$D^2(x)=0$

$$= I.P \text{ of } e^{ix} -\frac{1}{2i} \left[(x+1) - \frac{1}{i}\right]$$

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$$= I.P \text{ of } \frac{-e^{ix}}{2i}\left[\frac{i(x+1)-i}{i}\right]$$

$$= I.P \text{ of } -\frac{e^{-ix}}{2}\left[\frac{i(x+1)-i}{-i}\right]$$ $i^2=-1$

$$= I.P \text{ of } \frac{e^{-ix}}{2}\left[i(x+1)-i\right].$$

$$= I.P \text{ of } \frac{1}{2}\left[\cos x + i\sin x\right)\left[i(x+1)-i\right]$$

$$= \frac{1}{2}\left[\cos x(x+1) - \sin x\right].$$

—x—

(x) 8m

$$(D^2+6D+8)y = e^{-2x} + \cos 2x$$

A.E is $m^2+6m+8=0$

$(m+4)(m+2)=0$

$m=-4,-2$

∴ The roots are real & distinct

$$C.F = Ae^{m_1x} + Be^{m_2x}$$ $m_1=-4$, $m_2=-2$

$$C.F = Ae^{-4x} + Be^{-2x}$$

To Find P.I1

$$P.I_1 = \frac{1}{D^2+6D+8} e^{-2x}$$

Replace $D \rightarrow -2$

$$= \frac{1}{(-2)^2+6(-2)+8} e^{-2x}$$

$$= \frac{1}{4-12+8} e^{-2x}$$

$$= \frac{1}{12-12} e^{-2x}$$

$$= \frac{x}{2D+6} e^{-2x}$$

$$= \frac{x}{2(-2)+6} e^{-2x}$$

Replace $D \to -2$

$$= \frac{x}{-4+6} e^{-2x}$$

$$\boxed{P.I_1 = \frac{x}{2} e^{-2x}}$$

To find $P.I_2$

$$P.I_2 = \frac{1}{D^2+6D+8} \cos 2x$$

$a = 2$

Replace $D^2 \to -a^2 = -2^2 = -4$

$$= \frac{1}{-4+6D+8} \cos 2x$$

$$= \frac{1}{6D+4} \cos 2x$$

$$= \frac{1}{(6D+4)} \times \frac{(6D-4)}{(6D-4)} \cos 2x$$

$$= \frac{(6D-4)}{(6D)^2-4^2} \cos 2x$$

$$= \frac{(6D-4) \cos 2x}{36D^2-16}$$

$a = 2$

Replace $D^2 \to -a^2 = -4.$

$$= \frac{6D(\cos 2x) - 4\cos 2x}{36(-4) - 16}$$

$$= \frac{6(-\sin 2x \cdot 2) - 4\cos 2x}{-144 - 16}$$

$$= \frac{-12\sin 2x - 4\cos 2x}{-160}$$

$$= \frac{\cancel{-4}(3\sin 2x + \cos 2x)}{\cancel{-160}\ 40}$$

$$\boxed{P.I_2 = \frac{1}{40}(3\sin 2x + \cos 2x)}$$

$\therefore$ The general solution is $y = C.F + P.I_1 + P.I_2$

$$y = Ae^{-4x} + Be^{-2x} + \frac{x}{2}e^{-2x} + \frac{1}{40}(3\sin 2x + \cos 2x).$$

— x —

$$(D^3 + 2D^2 + D)y = e^{2x} + \sin x.$$

Soln:

A.E is $m^3 + 2m^2 + m = 0$

$$m(m^2 + 2m + 1) = 0$$

$$m = 0, \quad m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$m = -1, -1$$

$$C.F = Ae^{m_1 x} + (Bx + C)e^{m_2 x}$$

$$C.F = Ae^{0x} + (Bx + C)e^{-x}$$

$$\boxed{C.F = A + (Bx + C)e^{-x}}$$

To find P.I1

$$P.I_1 = \frac{1}{D^3 + 2D^2 + D} e^{2x}$$

Replace $D \rightarrow a = 2$

$$= \frac{1}{2^3 + 2(2)^2 + 2} e^{2x}$$

$$= \frac{1}{8 + 2(4) + 2} e^{2x}$$

$$= \frac{1}{18} e^{2x}$$

$$= \frac{e^{2x}}{18}$$

$$\boxed{P.I_1 = \frac{e^{2x}}{18}}$$

To find P.I2

$$P.I_2 = \frac{1}{D^3 + 2D^2 + D} \sin x$$

Replace $D^2 \rightarrow -a^2 = -1^2 = -1$

$$= \frac{1}{D.D^2 + 2D^2 + D} \sin x$$

$$= \frac{1}{D(-1) + 2(-1) + D} \sin x$$

$$= \frac{1}{-\cancel{D} - 2 + \cancel{D}} \sin x$$

$$\boxed{P.I_2 = \sin x / -2}$$

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$\therefore$ The general solution is $y = C.F + P.I_1 + P.I_2$

~~$y \neq Ae^{+4x}$~~

$$y = A + (Bx + C)e^{-x} + \frac{e^{2x}}{18} + \left(\frac{\sin x}{-2}\right)$$

$$y = A + (Bx+C)e^{-x} + \frac{e^{2x}}{18} - \frac{\sin x}{2}.$$

— x —

(4) V.Imp

Solve $(D^2-2D+1)y = (e^x+1)^2$.

$(D^2-2D+1)y = (e^{2x}+2e^x+1)$.

Soln:

A.E is $m^2-2m+1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

The roots are real & equal;

$C.F = (Ax+B)e^{mx}$

$$\boxed{C.F = (Ax+B)e^x.}$$

To find $\underline{P.I_1}$

$$P.I_1 = \frac{1}{(D^2-2D+1)} e^{2x}$$

Replace $D \rightarrow a = 2$.

$$= \frac{1}{2^2-2(2)+1} e^{2x}.$$

$$= \frac{e^{2x}}{4-4+1}$$

$$\boxed{P.I_1 = e^{2x}}$$

To find $P.I_2$

$$P.I_2 = \frac{1}{D^2-2D+1} 2e^x$$

Replace $D \to a = 1$

$$= 2 \cdot \frac{1}{1-2(1)+2} e^x$$

$$= 2 \frac{1}{1} e^{x}$$

$$\boxed{P.I_2 = 2e^x.}$$

To find $P.I_3$

$$P.I_3 = \frac{1}{D^2-2D+1} e^{0x}$$

Replace $D \to a = 0$

$$= \frac{1}{0-0+1} e^{0x} = e^{0x} = 1$$

$$\boxed{P.I_3 = 1}$$

$\therefore$ The general solution is $y = C.F + P.I_1 + P.I_2 + P.I_3$

$$y = (Ax+B)e^x + e^{2x} + 2e^x + 1.$$

— x —

Solve: $(D^2-4D+3)y = \sin 3x \cos 2x$

(*) v. Imp

-3-1

The Auxiliary equation is

$$m^2-4m+3=0$$

$$(m-3)(m-1)=$$

$$m=3,1.$$

The roots are real & distinct.

$$\boxed{C.F = Ae^{3x}+Be^{x}}$$

~~To~~ find ~~P.I~~

$$\cancel{P.I} = \frac{X}{\cancel{(D^2-4D+3)}}$$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$\sin(A-B) = \sin A\cos B - \cos A \sin B$

$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$

$\frac{1}{2}[\sin(A+B)+\sin(A-B)] = \sin A\cos B$

$$R.H.S = \sin 3x \cos 2x$$

$$= \frac{1}{2}[\sin(3x+2x)+\sin(3x-2x)]$$

$$= \frac{1}{2}[\sin 5x + \sin x].$$

$$P.I = \frac{1}{D^2-4D+3}\frac{1}{2}[\sin 5x + \sin x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

Replace $D^2 \to -a^2$
$= -5^2$
$= -25$

Replace $D^2 \to -a^2$
$= -1^2$
$= -1$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-4D-22}\sin 5x + \frac{1}{-4D+2}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-4D+22}{(-4D-22)(-4D+22)}\sin 5x + \frac{1}{(-4D+2)}\frac{(4D-2)}{(4D-2)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-4D(\sin 5x)+22\sin 5x}{(-4D)^2-(22)^2} + \frac{4D\sin x - 2\sin x}{(-4D)^2-(2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{-4(5)\cos 5x + 22\sin 5x}{16D^2-484} + \frac{4\cos x - 2\sin x}{16D^2-4}\right]$$

$D^2 \rightarrow -25$ $\qquad D^2 = -1$

$$= \frac{1}{2}\left[\frac{-20\cos 5x + 22\sin 5x}{16(-25)-484} + \frac{4\cos x - 2\sin x}{-16-4}\right]$$

$$= \frac{1}{2}\left[\frac{-20\cos 5x + 22\sin 5x}{-884} + \frac{4\cos x - 2\sin x}{-20}\right]$$

$$= \frac{1}{2}\left[2\left[\frac{(-10\cos 5x + 11\sin 5x)}{-442}\right] + 2\left(\frac{2\cos x - \sin x}{-10}\right)\right]$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\left[\frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}\right]$$

$$P.I = \frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}$$

The general solution is $y = C.F + P.I$

$$y = Ae^{3x} + Be^{x} + \left(\frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}\right).$$

$$\therefore y = Ae^{3x} + Be^{x} + \left[\frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}\right].$$

Method of variation of Parameters

consider $\frac{d^2y}{dx^2} + M^2y = X$.

$C.F = Af_1 + Bf_2$

$W = f_1 f_2' - f_1' f_2$

$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

$A = \int \frac{-f_2 X}{f_1 f_2' - f_1' f_2} dx$

$B = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$.

— x —

v. Imp

using the method of variation of Parameters

solve

$\frac{d^2y}{dx^2} + 4y = \tan 2x$

X

$(D^2+4)y = \tan 2x$

Auxiliary equation is

$m^2+4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i$

The roots are real & imaginary

$\alpha = 0$
$\beta = 2$

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$= c_1\cos 2x + c_2 \sin 2x$
$\quad\; c_1 f_1 \qquad c_2 f_2$.

$f_1 = \cos 2x \quad ; \quad f_2 = \sin 2x$

$f_1' = -\sin 2x \cdot 2 \quad ; \quad f_2' = \cos 2x \cdot 2$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = \cos 2x \cdot 2\cos 2x - (-2\sin 2x \cdot \sin 2x)$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

The general soln/: is $y = C.F + P.I$ where $P.I = A f_1 + B f_2$.

$$A = -\int \frac{f_2 \times}{W} dx$$

$$= -\int \frac{\sin 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2}\int \frac{\sin 2x \cdot \sin 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{(1-\cos^2 2x)}{\cos 2x} dx.$$

$$= -\frac{1}{2}\int \left(\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right) dx \qquad \because \int \sec x = \log(\sec x + \tan x)$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x)\, dx$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + c$$

$$A = -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x + c\right]$$

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$$B = \int \frac{f_1 X}{W} dx$$

$$= \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= \frac{1}{2} \int \cancel{\cos 2x} \frac{\sin 2x}{\cancel{\cos 2x}} dx$$

$$= \frac{1}{2} \int \sin 2x \, dx$$

$$= \frac{1}{2} \left(\frac{-\cos 2x}{2} \right)$$

$$B = -\frac{1}{4} \cos 2x.$$

The solution is $P.I = Af_1 + Bf_2$

$$= \left[-\frac{1}{4} \left[\log (\sec 2x + \tan 2x) - \sin 2x \right] \cos 2x + \left[-\frac{1}{4} \cos 2x \sin 2x \right] \right]$$

$$= -\frac{1}{4} \left[\log (\sec 2x + \tan 2x) \cos 2x - \cancel{\sin 2x \cos 2x} + \cancel{\sin 2x \cos 2x} \right.$$

$$\therefore -\frac{1}{4} \left[\log (\sec 2x + \tan 2x) \cos 2x \right].$$

$\therefore \; y = C.F + P.I$

$$y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \left[\log (\sec 2x + \tan 2x) \cos 2x \right]$$

— x —

14m

(4) Solve $[(3x+2)^2 D^2 + 3(3x+2)D - 36]y = 3x^2+4x+1$

soln.:

Let $(3x+2) = e^t$

$\therefore t = \log(3x+2)$

$x = a^t$

$t = \log x$

$\cancel{xD = \theta}$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$\cancel{Dy = \frac{dy}{dt}}$

$$Dy = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{1 \cdot 3}{(3x+2)} = \frac{3}{3x+2}$$

$$Dy = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{3}{3x+2}$$

$$D^2y = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{3}{3x+2}\right)$$

$$= \frac{d^2y}{dt^2} \cdot \frac{3}{3x+2} + \frac{dy}{dt}\left[\frac{(3x+2)0 - 3(3)}{(3x+2)^2}\right]$$

unwanted

$$D^2y = \frac{d^2y}{dt} \cdot \frac{3}{3x+2} + \frac{dy}{dt}\left[\frac{0-9}{(3x+2)^2}\right].$$

$\cancel{(3x+2)D = 3\theta}$

Another method.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{3}{3x+2} \cdot \frac{dy}{dt}\right)\frac{3}{3x+2}$$

$$= \frac{3}{3x+2} \cdot \frac{d^2y}{dt^2} + \left(\frac{-9}{(3x+2)^2}\right)\frac{dy}{dt}$$

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14M ④

$$(3x+2)D = 3\theta$$

$$(3x+2)^2D^2 = 9\theta(\theta-1)$$

$$(ax+b)D = a\theta$$
$$(ax+b)^2D^2 = a^2\theta(\theta-1)$$

$$3x+2 = e^t$$
$$3x = e^t - 2$$
$$x = \frac{e^t-2}{3}$$

$$[9\theta(\theta-1) + 3(3\theta) - 36]y = 3\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1$$

$$[9\theta^2 - \cancel{9\theta} + \cancel{9\theta} - 36]y = \cancel{3}\left[\frac{e^{2t}+4-4e^t}{\cancel{9}\,3}\right] + 4\left(\frac{e^t-2}{3}\right) + 1$$

$$= \frac{e^{2t}+4-4e^t}{3} + \frac{4e^t-8}{3} + 1$$

$$= \frac{1}{3}\left[e^{2t}+4-\cancel{4e^t}+\cancel{4e^t}-8+3\right]$$

$$(9\theta^2-36)y = \frac{1}{3}\left[e^{2t}-1\right]$$

$$9(\theta^2-4)y = \frac{1}{3}\left[e^{2t}-1\right]$$

$$(\theta^2-4)y = \frac{1}{3\times 9}\left[e^{2t}-1\right]$$

$$(\theta^2-4)y = \frac{1}{27}\left[e^{2t}-1\right]$$

$$(\theta^2-4)y = \frac{e^{2t}-1}{27}$$

Auxiliary equation is $m^2-4=0$

$$m^2=4$$

$$m=\pm 2$$

$$m = 2, -2$$

The roots are real & distinct

$$C.F = Ae^{2t} + Be^{-2t}$$

To find P.I

$$P.I = \frac{1}{\theta^2-4}\left[\frac{e^{2t}-1}{27}\right]$$

$$= \frac{1}{27}\left[\frac{1}{\theta^2-4}e^{2t} - \frac{1}{\theta^2-4}e^{0t}\right]$$

Replace $\theta \to 2$ Replace $\theta \to 0$

$$= \frac{1}{27}\left[\frac{1}{2^2-4}e^{2t} - \frac{1}{0-4}e^{0t}\right]$$

$$= \frac{1}{27}\left[\frac{1}{4-4}e^{2t} - \frac{1}{(-4)}e^{0t}\right]$$

$$= \frac{1}{27}\left[\frac{t}{2\theta}e^{2t} - \frac{1}{(-4)}e^{0t}\right]$$

Replace $\theta \to 2$

$$= \frac{1}{27}\left[\frac{t}{2(2)}e^{2t} + \frac{1}{4}e^{0t}\right]$$

$$= \frac{1}{27}\left[\frac{t}{4}e^{2t} + \frac{1}{4}\cdot 1\right]$$

$$= \frac{1}{27(4)}\left[te^{2t}+1\right]$$

$$P.I = \frac{1}{108}\left[te^{2t}+1\right]$$

The general solution is $y = C.F + P.I$

$$y = Ae^{2t} + Be^{-2t} + \frac{1}{108}\left[te^{2t}+1\right]$$

$$= Ae^{2\log(3x+2)} + Be^{-2\log(3x+2)} + \frac{1}{108}\left[\log(3x+2)\, e^{2\log(3x+2)} + 1\right]$$

$$= A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}\left[\log(3x+2)(3x+2)^2 + 1\right]$$

$$= A(3x+2)^2 + \frac{B}{(3x+2)^2} + \frac{1}{108}\left[\log(3x+2)\cdot(3x+2)^2 + 1\right].$$

— x —

Power series solution

Legendre Polynomials

The differential equation $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

$$y_1 = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{24(2n-1)(2n-3)}x^{n-4} + \cdots\right]$$

$$P_n(x) = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{24(2n-1)(2n-3)}x^{n-4} + \cdots\right]$$

$\therefore P_n(x)$ is a terminating series

$$P_n(1) = 1$$

Rodrigue's formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Put $n = 0, 1, 2, \ldots$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2-1) = \frac{1}{2}(2x-0) = x.$$

$$P_2(x) = \frac{1}{2^2(2)} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{8} \frac{d^2}{dx^2} (x^4-2x^2+1)$$

$$P_2(x) = \frac{1}{2}(3x^2-1).$$

$$P_3(x) = \frac{1}{2^3 3!} \cdot \frac{d^3}{dx^3} (x^2-1)^3$$

$$= \frac{1}{48} \frac{d^3}{dx^3} (x^6-3x^4+3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

$$P_6(x) = \frac{1}{6}\left[231x^6-351x^4+105x^2-5\right]$$

$$x^4 = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$$

We know that $P_0(x) = 1$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_4(x) = \frac{1}{8}\left[35x^4-30x^2+3\right]$$

$$\frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$$

$$= \frac{1}{35}\left[\cancel{8}\frac{1}{\cancel{8}}(35x^4-30x^2+3) + \overset{10}{\cancel{20}}\cdot\frac{1}{\cancel{2}}(3x^2-1)+7\right.$$

$$= \frac{1}{35}\left[35x^4 - 3\cancel{0}x^2 + 3 + 3\cancel{0}x^2 - 10 + 7\right]$$

$$= \frac{1}{35}\left[35x^4 - \cancel{10} + \cancel{10}\right] \Rightarrow \frac{\cancel{35}x^4}{\cancel{35}} = x^4$$

—x—

f(P4) 1/7

V. Imp $f(x) = x^3 - 5x^2 + x + 2$

We know that $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$2P_3(x) = 5x^3 - 3x$$

$$2P_3(x) + 3x = 5x^3$$

$$\frac{2P_3(x) + 3x}{5} = x^3$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x.$$

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - 5x^2 + x + 2$$

$$= \left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + x + 2$$

$$= \frac{2}{5}P_3(x) + \frac{3}{5}x - \frac{10}{3}P_2(x) - \frac{5}{3} + x + 2.$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \left(\frac{3}{5}+1\right)x + \frac{1}{3}$$

$$= \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{8}{5} x + \frac{1}{3}.$$

$$= \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{8}{5} P_1(x) + \frac{1}{3} P_0 x.$$

Since: $P_1(x) = x$; $P_0(x) = 1$

— x —

(10) v. Imp:
Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre Polynomial.

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$2P_2(x) = 3x^2 - 1$

$\frac{2}{3} P_2(x) + 1 = x^2$

$x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} x$

soln:

$$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$$

$$8P_4(x) = 35x^4 - 30x^2 + 3.$$

$$8P_4(x) + 30x^2 - 3 = 35x^4$$

$$x^4 = \frac{1}{35}[8P_4(x) + 30x^2 - 3].$$

$$\therefore f(x) = \frac{1}{35}[8P_4(x) + 30x^2 - 3] + 3x^3 - x^2 + 5x - 2.$$

$$= \frac{1}{35}[8P_4(x) + 30x^2 - 3] + 3\left[\frac{2}{5} P_3(x) + \frac{3}{5} x\right] - x^2 + 5x - 2$$

$$= \frac{1}{35} 8P_4(x) + \frac{30}{35} x^2 - \frac{3}{35} + \frac{6}{5} P_3(x) + \frac{9}{5} x - x^2 + 5x - 2$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) + \left(\frac{30}{35} - 1\right) x^2 + \left(\frac{9}{5} + 5\right) x - \left(\frac{3}{35} + 2\right)$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{5}{35}\left(\frac{2}{3} P_2(x) + \frac{1}{3}\right) + \frac{34}{5} x - \frac{73}{35}$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{1}{7}\left(\frac{2}{3}\right) P_2(x) - \frac{1}{7}\cdot\frac{1}{3} + \frac{34}{5}[P_1(x)] - \frac{73}{35}.$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \left(\frac{1}{21} + \frac{73}{35}\right)\cdot(1)$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \left(\frac{5+219}{105}\right) P_0(x).$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \frac{224}{105} P_0$$

—x—

Orthogonality Property of Legendre Polynomials

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad (m \neq n)$$

Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2) y = 0 \qquad (*) \quad 2m$$

Bessel function of I kind

$$y_1 = \frac{x^n}{2^n \Gamma(n+1)} \sum_{K=0}^{n} \frac{(-1)^K \Gamma(n+1)}{2^{2K} K!\, \Gamma(n+K+1)} x^{2K} = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\, \Gamma(n+K+1)} \left(\frac{x}{2}\right)^{n+}$$

$$J_{-n}(x) = \sum_{K=0}^{n} \frac{(-1)^K}{K!\,\Gamma(-n+K+1)} \left(\frac{x}{2}\right)^{-n+2K}$$

when n is an integer, the two function $J_n(x)$ & $J_{-n}(x)$

$$J_{-n}(x) = (-1)^n . J_n(x).$$

— x —

Find $J_0(x)$ & $J_1(x)$

$$J_n(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(n+K+1)} \left(\frac{x}{2}\right)^{n+2K}$$

$$J_0(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(K+1)} \left(\frac{x}{2}\right)^{2K}$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,K!} \left(\frac{x}{2}\right)^{2K} \qquad [\because \Gamma(K+1) = K!]$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{(K!)^2} \left(\frac{x}{2}\right)^{2K}$$

$$= 1 - \frac{1}{1}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

$$J_1(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(1+K+1)} \left(\frac{x}{2}\right)^{1+2K}.$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(K+2)} \left(\frac{x}{2}\right)^{1+2K}$$

$\Gamma(K+1) = K!$

$\Gamma(K+1+1) = (K+1)!$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,(K+1)!} \left(\frac{x}{2}\right)^{1+2K},$$

$$= \frac{1}{1}\frac{x}{2} - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3 + \frac{(-1)^2}{2!\,(2+1)!}\left(\frac{x}{2}\right)^5 \cdots$$

$$= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6}$$

$$J_0(0) = 1$$

$$J_1(0) = 0.$$

— x —

V.Imp. Qn.

Find $J_{1/2}(x)$

soln:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$n = 1/2$.

$$J_{\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\frac{1}{2}+k+1)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}+2k}$$

$\because \Gamma(k+1) = k!$
$\Gamma(k+1+1) = (k+1)!$

$$= \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{1!\,\Gamma(1+\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}+2} + \frac{1}{2!\,\Gamma(2+\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}+4} \cdots$$

$$= \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^{1/2}\left(\frac{x}{2}\right)^{2} + \frac{1}{2!\,\Gamma(7/2)}(x/2)^{1/2}\left(\frac{x}{2}\right)^{4} \cdots$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma(1/2)} - \frac{1}{\cancel{2\times 5}\cdot\frac{3}{\cancel{2}}\cdot\frac{1}{2}\,\Gamma(1/2)}\left(\frac{x}{2}\right)^{4}\cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma(1/2)}\left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!}\cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}}\cdot\frac{2}{x}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}\cdots\right]$$

$\boxed{\Gamma(1/2) = \sqrt{\pi}}$

$$= \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{2}}\cdot\sqrt{\pi}}\cdot\frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{x}}\cdot\sqrt{x}} = \sqrt{\frac{2}{\pi x}}\,\sin x.$$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$\Gamma(5/2) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right).$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x$$

— x —.

— x —.

~~Solve $p = \tan x = \frac{p}{1+p^2}$~~

Find $J_{-1/2}(x)$

Soln:

$$J_n(x) = \sum_{K=0}^{\infty}\frac{(-1)^K}{K!\,\Gamma(n+K+1)}\left(\frac{x}{2}\right)^{n+2K}.$$

$n = -1/2$

$$J_{-1/2}(x) = \sum_{K=0}^{\infty}\frac{(-1)^K}{K!\,\Gamma(-1/2+K+1)}\left(\frac{x}{2}\right)^{-1/2+2K}$$

$$= \sum_{K=0}^{\infty}\frac{(-1)^K}{K!\,\Gamma(K+1/2)}\left(\frac{x}{2}\right)^{-1/2+2K}$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{x}{2}\right)^{-1/2} - \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{3/2} + \frac{1}{2!\,\Gamma(5/2)}\left(x/2\right)^{7/2}\cdots$$

$$=\left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\Gamma(1/2)}-\frac{1}{1/2\,\Gamma(1/2)}\left(\frac{x}{2}\right)^{2}+\frac{1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(}\left(x\right.\right.$$

$$=(2/x)^{1/2}\frac{1}{\Gamma(1/2)}\left[1-\frac{x^2}{\frac{1}{\cancel{2}}\cdot\cancel{2}}+\frac{1}{2\cdot\frac{3}{2}\cdot1/\cancel{2}}\cdot\frac{x^4}{\cancel{16}\,2}\cdots\right]$$

$$=\frac{\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\right]$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x.$$

$\Gamma(n+1)=n\Gamma(n)$

$\Gamma(3/2)=\Gamma\left(\frac{1}{2}+1\right)$

$=\frac{1}{2}\Gamma(1/2$

$\Gamma(5/2)=\Gamma(3/2+$

$=\frac{3}{2}\Gamma(3/2)$

~~$=3/2\cdot1/2\,\Gamma$~~

$=\frac{3}{2}\,\frac{1}{2}\,\Gamma(1/$

——x——

(x) Im. (x).

Prove that $\frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$

Proof:

We know that

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$x^n\cdot J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\,x^n\cdot x^{n+2k}}{2^{n+2k}\,k!\,\Gamma(n+k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{2n+2k}}{2^{n+2k}\,k!\,\Gamma(n+k+1)}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k)\, x^{2n+2k-1}}{2^{n+2k}\, k!\, \Gamma(n+k+1)}$$

$x^{2n} = x^n x^n$

$$= \sum_{k=0}^{\infty} \frac{x^n (-1)^k \cancel{2}\cancel{(n+k)}\, x^{n+2k-1}}{\cancel{2}\cdot 2^{n+2k-1}\, k!\, \cancel{(n+k)}\, \Gamma(n+k)}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n+k)} \left(\frac{x}{2}\right)^{n+2k-1}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n-1+k+1)} \left(\frac{x}{2}\right)^{n-1+2k}$$

$$= x^n J_{n-1}(x)$$

$\frac{d}{dx}(x^n) = nx^n$

$x^{2n} = x^n \cdot x^n$

$1 = 2^0 = 2^{1-1} = 2^1 \cdot 2^{-1}$

$\Gamma(n+1) = n\Gamma(n)$

$$\therefore \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$\therefore$ Hence it is Proved

— x —

Similarly.

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

— x —

Proove that

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

$1 = x^0 = x^{n-n}$

Proof

L.H.S

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = \frac{d}{dx}\left[x^{n-n} x^1 . J_n(x) . J_{n+1}(x)\right]$$

$$= \frac{d}{dx}\left[x^{-n} J_n(x) . x^{n+1} J_{n+1}(x)\right]$$

$$= x^{-n} J_n(x) \frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] + x^{n+1} J_{n+1}(x).$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] \to ①$$

Now, we know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

change n to n+1

$$\frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_n(x)$$

Also we know that

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

∴ From ①,

$$\frac{d}{dx}\left[x J_n(x) . J_{n+1}(x)\right] = x^{-n} J_n(x)\, x^{n+1} J_n(x) + x^{n+1} J_{n+1}(x)\left[-x^{-n} J_{n+1}(x)\right]$$

Scanned by CamScanner

$$= x^{-n+n+1} J_n^2(x) - x^{n+1-n} J_{n+1}^2(x)$$

$$= x J_n^2(x) - x J_{n+1}^2(x)$$

$$= x\left[J_n^2(x) - J_{n+1}^2(x)\right].$$

Objective type questions	opt1	opt2	opt3	opt4	Answer
The solution of the differential equation (D^2 + 5D+6)y=0 is.............	A e^ (-2x)+ B e^ (-3x)	A e^ (2x)+ B e^ (3x)	A e^ (-2x)+ B e^ (3x)	A e^ (2x)+ B e^ (-3x)	A e^ (-2x)+ B e^ (-3x)
The solution of the differential equation (D^2 + 6D+9)y=0 is.............	(A+Bx) e^ (3x)	(A+Bx) e^ (x)	(A+Bx) e^ (-2x)	(A+Bx) e^ (-3x)	(A+Bx) e^ (-3x)
The solution of the differential equation (D^2 - 4D+4)y=0 is.............	(A+Bx) e^ (3x)	(A+Bx) e^ (-2x)	(A+Bx) e^ (-3x)	(A+Bx) e^ (2x)	(A+Bx) e^ (2x)
The particular integral of (D^2 -3D+2)y=12 is.................	(1/5)	(1/6)	(1/4)	(1/3)	(1/6)
The complementary function of (D^2 -2D+1)y=x sinx is.............	(A+Bx) e^ (-x)	(A+Bx) e^ (x)	(A+Bx) e^ (-2x)	(A+Bx) e^ (2x)	(A+Bx) e^ (x)
If f(D)= D^ (2)- 2, 1/f(D) e^(-2x) is...........	0.5 e^ (2x)	-0.5 e^ (2x)	0.5 e^ (-2x)	0.5 e^ (3x)	0.5 e^ (2x)
The particular integral of (D^2+4) y= cos2x is	(x cos2x)/2	(sin2x)/2	(sin2x)/2	(x sin2x)/4	(x sin2x)/4
If (D^2 +4)y=0 is a linear differential equation then general solution is	A cos2x+ B sin4x	Acos2x+Bsin2x	Asin2x+Bcos4x	Asin4x+Bsin4x	Acos2x+Bsin2x
If (D^2 − 6D+13) y = 0 is a linear differential equation then G.S. is -------	e^(3x) (A cos2x+ B sin2x)	e^(3x) (A cos4x+ B sin4x)	e^(3x) (A cos2x+ B sin2x)	e^(2x) (A cos2x+ B sin2x)	e^(3x) (A cos2x+ B sin2x)
The solution of the differential equation (D^2 - 4D+3)y=0 is.............	A e^ (x)+ B e^ (3x)	A e^ (-x)+ B e^ (3x)	A e^ (x)+ B e^ (-3x)	A e^ (2x)+ B e^ (-3x)	A e^ (x)+ B e^ (3x)
The solution of the differential equation (D^2 +3D+2)y=0 is.............	A e^ (x)+ B e^ (2x)	A e^ (-x)+ B e^ (2x)	A e^ (-x)+ B e^ (x)	A e^ (-x)+ B e^ (-2x)	A e^ (-x)+ B e^ (-2x)
The particular integral of (D^2 +3D+2)y= 2 e^(x) is.............	e^(x)/3	(-e^(x))/3	e^(x)/6	(-e^(x))/6	e^(x)/3
The particular integral of (D^2+4) y= e^(x) is	1/5* e^(x)	1/5* e^(-x)	1/6* e^(x)	1/6* e^(x)	1/5* e^(-x)
If the roots of the auxilliary equation are real and distinct then the C.F is...	Ae^(m1x)+Be^(m2x)	(A+Bx) e^ (m1x)	e^(αx) (Acosβx+Bsinβx)	(A+Bx) e^ (m2x)	Ae^(m1x)+Be^(m2x)
If the roots of the auxilliary equation are real and equal then the C.F is...	Ae^(m1x)+Be^(m2x)	e^(αx) (Acosβx+Bsinβx)	(A+Bx) e^ (mx)	(A+Bx) e^ (-mx)	(A+Bx) e^ (mx)
If the roots of the auxilliary equation are complex then the C.F is...	Ae^(m1x)+Be^(m2x)	e^(-αx) (Acosβx+Bsinβx)	(A+Bx) e^ (mx)	e^(αx) (Acosβx+Bsinβx)	e^(αx) (Acosβx+Bsinβx)
The particular integral of (D^2 +10D+24)y= e^(-x) is.................	(1/35) e^(-x)	(-1/35)e^(-x)	(-1/25)e^(-x)	(1/25)e^(-x)	(1/25)e^(-x)
The particular integral of (D^2+9) y= cos2x is	cos2x/13	(-cos2x)/13	(-cos2x)/5	cos2x/5	cos2x/5
The particular integral of (D^2+9) y= cos3x is	x cos3x/2	(-x cos3x)/2	(xcos3x)/6	(-xcos3x)/6	(xcos3x)/6
The particular integral of (D^2 +12D+27)y= e^(-x) is..........	(1/16) e^ (-x)	(-1/16) e^ (-x)	(1/16) e^ (x)	(-1/16) e^ (x)	(1/16) e^ (-x)
The solution of the differential equation (D^2 +19D+60)y=0 is.............	A e^ (15x)+ B e^ (4x)	A e^ (-15x)+ B e^ (4x)	A e^ (15x)+ B e^ (-4x)	A e^ (-15x)+ B e^ (-4x)	A e^ (-15x)+ B e^ (-4x)
The solution of the differential equation (D^2 +13D+40)y=0 is.............	A e^ (5x)+ B e^ (8x)	A e^ (5x)+ B e^ (-8x)	A e^ (-5x)+ B e^ (-8x)	A e^ (-5x)+ B e^ (8x)	A e^ (-5x)+ B e^ (-8x)
The solution of the differential equation (D^2 - 9D+20)y=0 is.............	A e^ (-5x)+ B e^ (4x)	A e^ (5x)+ B e^ (-4x)	A e^ (5x)+ B e^ (4x)	A e^ (-5x)+ B e^ (-4x)	A e^ (5x)+ B e^ (4x)
The solution of the differential equation (D^2 +D-72)y=0 is.............	A e^ (-8x)+ B e^ (-9x)	A e^ (-8x)+ B e^ (9x)	A e^ (8x)+ B e^ (9x)	A e^ (8x)+ B e^ (-9x)	A e^ (8x)+ B e^ (-9x)
The solution of the differential equation (D^2- 11D-42)y=0 is.............	A e^ (14x)+ B e^ (-3x)	A e^ (-14x)+ B e^ (-3x)	A e^ (-14x)+ B e^ (3x)	A e^ (14x)+ B e^ (3x)	A e^ (14x)+ B e^ (-3x)
The solution of the differential equation (D^2- 12D-45)y=0 is.............	A e^ (15x)+ B e^ (3x)	A e^ (-15x)+ B e^ (3x)	A e^ (15x)+ B e^ (-3x)	A e^ (-15x)+ B e^ (-3x)	A e^ (15x)+ B e^ (-3x)
The solution of the differential equation (D^2- 7D-30)y=0 is.............	A e^ (-10x)+ B e^ (-3x)	A e^ (10x)+ B e^ (-3x)	A e^ (10x)+ B e^ (3x)	A e^ (-10x)+ B e^ (3x)	A e^ (10x)+ B e^ (-3x)
The particular integral of (D^2 +19D+60)y= e^x is.............	(-e^(-x))/80	(e^(-x))/80	(e^x)/80	(-e^x)/80	(e^x)/80
The particular integral of (D^2+25) y= cosx is	(cosx)/24	(cosx)/25	(-cosx)/24	(-cosx)/25	cosx/24
The particular integral of (D^2+1) y= sinx is	xcosx/2	(-xcosx)/2	(-xsinx)/2	xsinx/2	(-xcosx)/2
The particular integral of (D^2 -9D+20)y=e^(2x) is.............	e^(2x) /6	e^(2x) /(-6)	e^(2x) /12	e^(2x) /(-12)	e ^ (2x) /6
The particular integral of (D^2-1) y= sin2x is	(-sin2x)/5	sin2x/5	sin2x/3	(-sin2x)/3	(-sin2x)/5
The particular integral of (D^2- 7D-30)y= 5 is.............	(1/30)	(-1/30)	(1/6)	(-1/6)	(-1/6)
The solution of the differential equation (D^2- 11D-42)y=21 is.............	(-1/42)	(1/42)	(1/2)	(-1/2)	A e^ (14x)+ B e^ (-3x)
Which one is Bessel's Equation of order n	x^2d^y/dx^2-xdy/dx+(x^2-n^2)y=0	x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0	x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0	x^2d^y/dx^2+xdy/dx+(x^2+n^2)y=0	x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0
In this equation x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0 is called ________	Legendre's Equation	Cauchy's equation	Partial Equation	Bessel's Equation	Bessel's Equation
Bessel's Equation is x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0 of order ______	n+1	n-1	n	2n-1	n
Which one is Bessel's Equation of order 0	xd^2y/dx^2+dy/dx+xy=0	xd^2y/dx^2+dy/dx-xy=0	xd^2y/dx^2-dy/dx+xy=0	x^2d^2y/dx^2+dy/dx+xy=0	xd^2y/dx^2+dy/dx+xy=0
Bessel's Equation is xd^y/dx^2+dy/dx+xy=0 of order ______	0	1	2	3	0
d/dx[x^n J_n(x)] is equal to______	x^n J_n+1(x)	x^n-1 J_n-1(x)	x^n J_n-1(x)	_x^-n J_n+1(x)	x^n J_n-1(x)
d/dx[x^(-n) J_n(x)]is equal to______	x^n J_n+1(x)	x^n-1 J_n-1(x)	x^n J_n-1(x)	_x^-n J_n+1(x)	_x^-n J_n+1(x)
J_n(x)=	(x/2n)[J_n+1 (x)+J_n+1 (x)]	(x/2n)[J_n-1 (x)+J_n+1 (x)]	(x/2n)[J_n-1 (x)+J_n-1 (x)]	(x/2n)[J_n-1 (x)-J_n+1 (x)]	(x/2n)[J_n-1 (x)+J_n+1 (x)]
x^-n J_n+1(x)=	_d/dx[x^(-n) J_n(x)]	_d/dx[x^(n) J_n(x)]	d/dx[x^(-n) J_n(x)]	_d/dx[x^(-n) J_(-n)(x)]	_d/dx[x^(-n) J_n(x)]
J'_n (x)=	(1/2)[J_(n+1)(x)-J_(n+1)(x)]	(1/2)[J_(n-1)(x)-J_(n-1)(x)]	(1/2)[J_(n-1)(x)-J_(n+1)(x)]	(1/4)[J_(n-1)(x)-J_(n+1)(x)]	(1/2)[J_(n-1)(x)-J_(n+1)(x)]
J'_(n)(x)=	(n/x)[J_(n)(x)-J_(n+1)(x)]	(n/x^2)[J_(n)(x)-J_(n+1)(x)]	(n/x)[J_(n)(x)-J_(n-1)(x)]	(2n/x)[J_(n) (x)-J_(n-1) (x)]	(n/x)[J_(n)(x)-J_(n+1)(x)]
J_n+1 (x)=	(n/x)[J_(n)(x)-J_(n+1)(x)]	(n/x^2)[J_(n)(x)-J_(n+1)(x)]	(n/x)[J_(n)(x)-J_(n-1)(x)]		(2n/x)[J_(n) (x)-J_(n-1)(x)]
J_-(1/2) (x)=	sqrt(2/pi) cosx	sqrt(4/pi) cosx	sqrt(2/pi) sinx	sqrt(4/pi) sinx	sqrt(2/pi) cosx
J_(1/2) (x)=	sqrt(2/pi) cosx	sqrt(4/pi) cosx	sqrt(2/pi) sinx	sqrt(4/pi) sinx	sqrt(2/pi) sinx

Unit IX

Analytic Functions

21 Complex Numbers

Chapter Outline

- Introduction
- Complex Numbers
- Complex Function
- Limit of a Function
- Derivative
- Analytic Function
- Cauchy–Riemann Equations
- Harmonic Function
- Properties of Analytic Functions
- Construction of Analytic Function (Milne–Thomson Method)

21.1 ❑ INTRODUCTION

Quite often, it is believed that complex numbers arose from the need to solve quadratic equations. In fact, contrary to this belief, these numbers arose from the need to solve cubic equations. In the sixteenth century, Cardano was possibly the first to introduce $a+\sqrt{-b}$, a complex number, in algebra. Later, in the eighteenth century, Euler introduced the notation i for $\sqrt{-1}$ and visualized complex numbers as points with rectangular coordinates, but he did not give a satisfactory foundation for complex numbers. However, Euler defined the complex exponential and proved the identity $e^{i\varphi} = (\cos\varphi + i\sin\varphi)$, thereby establishing connection between trigonometric and exponential functions through complex analysis.

We know that there is no square root of negative numbers among real numbers.

However, algebra itself and its applications require such an extension of the concept of a number for which the extraction of the square root of a negative number would be possible.

We have repeatedly encountered the notion of extension of a number. Fractional numbers are introduced to make it possible to divide one integral number by another, negative numbers are introduced to make it possible to subtract a large number from a smaller one and irrational numbers become necessary in order to describe the result of measurement of the length of a segment in the case when the segment is incommensurable with the chosen unit of length.

The square root of the number –1 is usually denoted by the letter i and numbers of the form $a + ib$ where a and b are ordinary real numbers known as **complex numbers**.

The necessity of considering complex numbers first arose in the sixteenth century when several Italian mathematicians discovered the possibility of algebraic solutions of third-degree equations.

The theoretical and applied values of complex numbers are far beyond the scope of algebra. The theory of functions of a complex variable, which was much advanced in the nineteenth century, proved to be a very valuable apparatus for the investigation of almost all the divisions of theoretical physics, such, for instance, as the theory of oscillations, hydrodynamics, the divisions of the theory of elementary particles, etc.

Many engineering problems may be treated and solved by methods involving complex numbers and complex functions. There are two kinds of such problems. The first of them consists of elementary problems for which some acquaintances with complex numbers are sufficient. This includes many applications to electric circuits or mechanical vibrating systems. The second kind consists of more advanced problems for which we must be familiar with the theory of complex analytic functions. Interesting problems in heat conduction, fluid flow and electrostatics belong to this category.

21.2 ❑ COMPLEX NUMBERS

A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ (i is pronounced as **iota**) is called a **complex number**. x is called the **real part** of $x + iy$ and is written as $\mathrm{Re}(x + iy)$ and y is called the **imaginary part** and is written as $\mathrm{Im}(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be **conjugates** of each other.

Properties

(i) If $x_1 + iy_1 = x_2 + iy_2$ then $x_1 - iy_1 = x_2 - iy_2$

(ii) Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are said to be equal when $\mathrm{Re}(x_1 + iy_1) = \mathrm{Re}(x_2 + iy_2)$, i.e., $x_1 = x_2$ and $\mathrm{Im}(x_1 + iy_1) = \mathrm{Im}(x_2 + iy_2)$ i.e., $y_1 = y_2$

(iii) **Algebra of Complex Numbers**

The arithmetic operations on complex numbers follow the usual rules of elementary algebra of real numbers with the definition $i^2 = -1$. If $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$ are any two complex numbers then we define the following arithmetic operations.

Addition

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Division Let $z_2 \neq 0$. Then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \left[\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right] + i\left[\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right]$$

i.e., sum, difference, product and quotient of any two complex numbers is itself a complex number.

(iv) Every complex number $x + iy$ can always be expressed in the form $r(\cos\theta + i\sin\theta)$.
i.e., $re^{i\theta}$ (Exponential form).

➢ **Note**

(i) The number $r = +\sqrt{x^2 + y^2}$ is called the **module** of $x + iy$ and is written as mod $(x + iy)$ or $|x + iy|$. The angle θ is called the **amplitude** or **argument** of $x + iy$ and is written as amp $(x + iy)$ or $\arg(x + iy)$. Evidently, the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the **principal value of the amplitude.**

(ii) $\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')

(iii) If the conjugate of $z = x + iy$ be $\bar{z}$ then

(a) $\text{Re}(z) = \frac{1}{2}(z + \bar{z}), \text{Im}(z) = \frac{1}{2i}(z - \bar{z})$

(b) $|z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2} = |\bar{z}|$

(c) $z\bar{z} = |z|^2$

(d) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(e) $\overline{z_1z_2} = \overline{z_1} \cdot \overline{z_2}$

(f) $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}, \overline{z_2} \neq 0$

(iv) **De Moivre's Theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

21.3 ❑ COMPLEX FUNCTION

Recall from calculus that a real function f defined on a set S of real numbers is a rule that assigns to every x in S a real number $f(x)$, called the **value** of f at x. Now in the complex region, S is a set of complex numbers. A **function** f defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z.

We write $w = f(z)$. Here, z varies in S and is called a **complex variable**. The set S is called the **domain** of f.

If to each value of z, there corresponds one and only one value of w then w is said to be a **single-valued function** of z; otherwise, it is a **multi-valued function**. For example, $w = \frac{1}{z}$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

➢ **Note**

(i) If $z = x + iy$ then $f(z) = u + iv$ (a complex number).

(ii) Since $e^{iy} = \cos y + i \sin y$, $e^{-iy} = \cos y - i \sin y$, the circular functions are

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2}, \text{ and so on}$$

$\therefore$ circular functions of the complex variable z are given by $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$,

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\tan z = \dfrac{\sin z}{\cos z}$ with cosec z, sec z and cot z as their respective reciprocals.

(iii) **Euler's Theorem**

$$e^{iz} = \cos z + i \sin z$$

(iv) **Hyperbolic Functions**

If x be real or complex, $\dfrac{e^x - e^{-x}}{2} = \sin hx$ (named hyperbolic sine of x)

$\dfrac{e^x + e^{-x}}{2} = \cos hx$ (named hyperbolic cosine of x)

Also, we define,

$$\tan hx = \frac{\sin hx}{\cos hx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cot hx = \frac{1}{\tan hx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\sec hx = \frac{1}{\cos hx} = \frac{2}{e^x + e^{-x}}$$

$$\text{cosec } hx = \frac{1}{\sin hx} = \frac{2}{e^x - e^{-x}}$$

(v) **Relations between Hyperbolic and Circular Functions**

$$\sin ix = i \sin hx$$
$$\cos ix = \cos hx$$
$$\tan ix = i \tan hx$$

(vi) $\cos h^2x - \sin h^2x = 1$, $\sec h^2x + \tan h^2x = 1$

$\cot h^2x - \text{cosec } h^2x = 1$

(vii) $\sin h(x \pm y) = \sin hx \cos hy \pm \cos hx \sin hy$

$\cos h(x \pm y) = \cos hx \cos hy \pm \sinh x \sinh y$

$$\tan h(x \pm y) = \frac{\tan hx \pm \tan hy}{1 + \tan hx \tan hy}$$

(viii) $\sin h2x = 2 \sin hx \cosh x$

$\cos h2x = \cos h^2x + \sin h^2x = 2 \cos h^2x - 1 = 1 + 2 \sin h^2x$

$\tan h2x = \dfrac{2 \tan hx}{1 + \tan h^2x}$

(ix) $\sin h3x = 3\sin hx + 4\sin h^3x$

$\cos h3x = 4\cos h^3x - 3\cos hx$

$$\tan h3x = \frac{3\tan hx + \tan h^3x}{1+3\tan h^2x}$$

(x)

$$\sin hx + \sin hy = 2\sin h\frac{x+y}{2}\cos h\frac{x-y}{2}$$

$$\sin hx - \sin hy = 2\cos h\frac{x+y}{2}\sin h\frac{x-y}{2}$$

$$\cos hx + \cos hy = 2\cos h\frac{x+y}{2}\cos h\frac{x-y}{2}$$

$$\cos hx - \cos hy = 2\sin h\frac{x+y}{2}\sin h\frac{x-y}{2}$$

(xi) $\cos h^2x - \sin h^2x = 1$

(xii) Complex trigonometric functions satisfy the same identities as real trigonometric functions.

$$\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z$$

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2\sin z\cos z \quad \text{and} \quad \cos 2z = \cos^2 z - \sin^2 z$$

$$\sin\bar{z} = \overline{\sin z}$$

$\sin(z + 2n\pi) = \sin z$, n is any integer

$\cos(z + 2n\pi) = \cos z$, n is any integer

(xiii) **Inverse Trigonometric and Hyperbolic Functions**

Complex inverse trigonometric functions are defined by the following:

$$\cos^{-1} z = -i\log(z + \sqrt{z^2+1})$$

$$\sin^{-1} z = -i\log(iz + \sqrt{1-z^2})$$

$$\tan^{-1} z = -\frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right) = \frac{i}{2}\log\frac{i+z}{i-z},\ z \neq \pm i$$

$$\operatorname{cosec}^{-1} z = \sin^{-1}\left(\frac{1}{z}\right) = -i\log\left(\frac{1+\sqrt{z^2-1}}{z}\right),\ z \neq 0$$

$$\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right) = -i\log\left(\frac{1+\sqrt{1-z^2}}{z}\right),\ z \neq 0$$

$$\cot^{-1} z = \tan^{-1}\left(\frac{1}{z}\right) = \frac{-i}{2}\log\left(\frac{z+i}{z-i}\right),\ z \neq \pm i$$

Complex inverse hyperbolic functions are defined by the following:

$$\cosh^{-1} z = \log(z + \sqrt{z^2 - 1}),\ \sinh^{-1} z = \log(z + \sqrt{z^2 + 1})$$

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\ z \neq \pm 1$$

$$\text{cosech}^{-1} z = \sinh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right),\ z \neq 0$$

$$\text{sech}^{-1} z = \cosh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right),\ z \neq 0$$

$$\coth^{-1} z = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right),\ z \neq \pm 1$$

21.4 ❑ LIMIT OF A FUNCTION

A function $f(z)$ is said to have the **limit** 'b' as z approaches a point 'a', written $\lim_{z \to a} f(z) = b$, if f is defined in a neighborhood of 'a' (except perhaps at 'a' itself) and if the values of f are close to 'b' for all z close to 'a', i.e., the number b is called the **limit** of the function $f(z)$ as $z \to a$, if the absolute value of the difference $f(z) - b$ remains less than any preassigned positive number $\in$ every time the absolute value of the difference $z - a$ for $z \neq a$, is less than some positive number δ (dependent on $\in$).

More briefly, the number b is the limit of the function $f(z)$ as $z \to a$, if the absolute value $|f(z) - b|$ is arbitrarily small when $|z - a|$ is sufficiently small.

21.5 ❑ DERIVATIVE

A function $f(z)$ is said to be **differentiable** at a point $z = z_0$ if the limit $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists. This limit is then called the derivative of $f(z)$ at the point $z = z_0$ and is denoted by $f'(z_0)$.

If we write $z = z_0 + \Delta z$ then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

21.6 ❑ ANALYTIC FUNCTIONS

A function defined at a point z_0 is said to be **analytic** at z_0, if it has a derivative at z_0 and at every point in some neighborhood of z_0. It is said to be analytic in a region R, if it is analytic at every point of R. Analytic functions are otherwise named **holomorphic** or **regular** functions.

A point at which a function $f(z)$ is not analytic is called a **singular point** or **singularity** of $f(z)$.

21.7 ❑ CAUCHY–RIEMANN EQUATIONS

The necessary condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain R is that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ must exist and satisfy the Cauchy–Riemann equations, namely,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The sufficient condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain R is that the four partial derivatives u_x, u_y, v_x and v_y exist, are continuous and satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ at each point of R.

➢ **Note**

(i) The two partial differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called the **Cauchy–Riemann equations** and they may be written as $u_x = v_y$ and $u_y = -v_x$

(ii) The Cauchy–Riemann equations are referred as C-R equations

(iii) C-R equations in polar form are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

21.8 ❑ HARMONIC FUNCTION

A real function of two variables x and y that possesses continuous second-order partial derivatives and satisfies the Laplace equation is called a **harmonic function**.

If u and v are harmonic functions such that $u + iv$ is analytic then each is called the **conjugate harmonic function** of the other.

➢ **Note**

(i) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the **Laplacian operator** and is denoted by ∇^2.

(ii) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$ is known as **Laplace equation** in two dimensions.

21.9 ❑ PROPERTIES OF ANALYTIC FUNCTIONS

Property 1

The real and imaginary parts of an analytic function $f(z) = u + iv$ satisfy the Laplace equation in two dimensions.

• **Proof**

Since $f(z) = u + iv$ is an analytic function, it satisfies C-R equations,

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \tag{21.1}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{21.2}$$

Differentiating both sides of (21.1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{21.3}$$

Differentiating both sides of (21.2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \partial x} \tag{21.4}$$

By adding (21.3) and (21.4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\text{since } \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}, \text{ when they are continuous}\right)$$

$\Rightarrow$ u satisfies Laplace equation.
Now differentiating both sides of (21.1) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \tag{21.5}$$

Differentiating both sides of (21.2) partially with respect to x we get

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2} \tag{21.6}$$

Subtracting (21.5) and (21.6),

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

i.e.,
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ v satisfies Laplace equation.
Hence, if $f(z)$ is analytic then both real and imaginary parts satisfy Laplace's equation.

➢ **Note**

If $f(z) = u + iv$ is analytic then u and v are harmonic. Conversely, when u and v are any two harmonic functions then $f(z) = u + iv$ need not be analytic.

Property 2

If $f(z) = u + iv$ is an analytic function then the curves of the family $u(x, y) = C_1$ cut orthogonally the curves of the family $v(x, y) = C_2$ where C_1 and C_2 are constants.

- **Proof**

Given $u(x, y) = C_1$
Taking differentials on both sides, we get

$$du = 0$$

i.e.,
$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\therefore\ \frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1$$ (say), where m_1 is the slope of the curve $u(x, y) = C_1$ at (x, y)

From the second curve $v(x, y) = C_2$, we get $\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} = m_2$, where m_2 is the slope of the curve $v(x, y) = C_2$ at (x, y).

Now, $$m_1 m_2 = \frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

$$= \frac{\left(\frac{\partial v}{\partial y}\right)}{-\left(\frac{\partial v}{\partial x}\right)} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$ (as $f(z)$ is analytic, it satisfies C-R equation)

$\Rightarrow m_1 m_2 = -1$

Hence, the curves cut each other orthogonally.

Here, the two families are said to be **orthogonal trajectories** of each other.

21.10 ❑ CONSTRUCTION OF ANALYTIC FUNCTIONS (MILNE–THOMSON METHOD)

To find f(z) when u is given

We know that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

i.e., $$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(By C-R equations)} \tag{21.7}$$

Let $$\frac{\partial u(x, y)}{\partial x} = \phi_1(x, y) \text{ and then calculate } \phi_1(z, 0) \tag{21.8}$$

and $$\frac{\partial u(x, y)}{\partial y} = \phi_2(x, y) \text{ and then calculate } \phi_2(z, 0) \tag{21.9}$$

Substituting (21.8) and (21.9) in (21.7), we get

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

Integrating, we get $\int f^1(z)dz = \int \phi_1(z, 0)dz - i \int \phi_2(z, 0)dz$

i.e., $$f(z) = \int \phi_1(z, 0)dz - i\int \phi_2(z, 0)dz.$$

To find f(z) when v is given

We know that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \tag{21.10}$$

Let $$\frac{\partial v(x, y)}{\partial y} = \phi_1(z, 0) \tag{21.11}$$

and $$\frac{\partial v(x, y)}{\partial x} = \phi_2(z, 0) \tag{21.12}$$

Substituting (21.11) and (21.12) in (21.10), we get

$$f'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

Integrating, we get $\int f'(z)dz = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$

i.e., $$f(z) = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$$

21.11 ❑ APPLICATIONS

Irrotational Flows

A flow in which the fluid particles do not rotate about their own axes while flowing is said to be irrotational.

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u = \frac{-\partial \phi}{\partial x}, v = \frac{-\partial \phi}{\partial y} \tag{21.13}$$

In two-dimensional flow, the stream function ψ always exists such that

$$u = \frac{-\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x} \tag{21.14}$$

From (21.13) and (21.14), we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x} \tag{21.15}$$

which are the well-known **Cauchy–Riemann equations**. Hence, $\phi + i\psi$ is an analytic function of $z = x + iy$. Moreover, ϕ and ψ are known as conjugate functions.

On multiplying and rewriting, (21.15) gives

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0 \tag{21.16}$$

showing that the families of curves given by ϕ = constant and ψ = constant intersect orthogonally. Thus, the curves of equi-velocity potential and the stream lines intersect orthogonally.

Differentiating the equation given in (21.15) with respect to x and y respectively, we get $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y}$ and $\frac{\partial^2 \phi}{\partial y^2} = \frac{-\partial^2 \psi}{\partial x \partial y}$. (21.17)

Since $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$, (21.17) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{21.18}$$

Again differentiating Eq. (21.15) with respect to y and x respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial y^2} \text{ and } \frac{\partial^2\phi}{\partial x\partial y}=\frac{-\partial^2\psi}{\partial x^2}$$

Subtracting these, $\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}=0$ (21.19)

Equations (21.18) and (21.19) show that ϕ and ψ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.

Complex Potential

Let $w = \phi + i\psi$ be taken as a function of $x + iy$

Thus, suppose that $w = f(z)$

i.e., $\phi + i\psi = f(x = iy)$ (21.20)

Differentiating (21.20) with respect to x and y respectively, we get

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=f'(x+iy)$$

(21.21)

and $\dfrac{\partial\phi}{\partial y}+i\dfrac{\partial\psi}{\partial y}=i\,f'(x+iy)$

or $\dfrac{\partial\phi}{\partial y}+i\dfrac{\partial\psi}{\partial y}=i\left(\dfrac{\partial\phi}{\partial x}+\dfrac{i\partial\psi}{\partial x}\right)$ by (21.22)

Equating real and imaginary parts, we get

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=\frac{-\partial\psi}{\partial x}$$

which are C-R equations. Then w is an analytic function of z and w is known as the complex potential.

Conversely, if w is an analytic function of z then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion. The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called **equipotential lines** and **stream lines** respectively.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **equipotential lines** and **lines of force**.

In heat-flow problems, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **isothermals** and **heat-flow lines.**

SOLVED EXAMPLES

Example 1 Prove that the function $f(z) = |z|^2$ is differentiable only at the origin.

Solution Given $f(z) = |z|^2$

i.e., $u+iv=|x+iy|^2=[\sqrt{x^2+y^2}\,]^2$ (as $z = x + iy$ and $f(z) = u + iv$)

$$=x^2+y^2$$

$\Rightarrow \quad u = x^2 + y^2$

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$

$$v = 0$$

$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

If $f(z)$ is differentiable then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$\Rightarrow \quad 2x = 0 \quad \Rightarrow \quad x = 0$

Also, $\quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\Rightarrow \quad 2y = 0 \quad \Rightarrow \quad y = 0$

$\therefore$ C-R equations are satisfied only when $x = 0$, $y = 0$

Hence, $f(z) = |z|^2$ is differentiable only at the origin (0, 0). **Proved.**

Example 2 Prove that the function $f(z) = z\bar{z}$ is not analytic except at $z = 0$.

Solution Given $f(z) = z\bar{z}$

i.e., $\quad u + iv = (x + iy)(x - iy)$

$$u + iv = x^2 + y^2$$

Equating real and imaginary parts.

$$u = x^2 + y^2$$

$\Rightarrow \quad \dfrac{\partial u}{\partial x} = 2x, \dfrac{\partial u}{\partial y} = 2y$

$$v = 0$$

$\Rightarrow \quad \dfrac{\partial v}{\partial x} = 0, \dfrac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C-R equations are not satisfied

$\therefore f(z) = z\bar{z}$ is not analytic except at $z = 0$. **Proved.**

Example 3 Show that (i) an analytic function with a constant real part is a constant, and (ii) an analytic function with a constant modulus is also a constant.

[KU Nov. 2010, April 2012; AU Nov. 2010, Nov. 2011]

Solution Let $f(z) = u + iv$ be an analytic function.

(i) Let $u = C_1$ (a constant)

Then $\dfrac{\partial u}{\partial x} = u_x = 0$ and $\dfrac{\partial u}{\partial y} = u_y = 0$.

Since $f(z)$ is an analytic function, by C-R equations $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow \quad v_y = 0$ and $v_x = 0$.

As $v_x = 0$ and $v_y = 0$, v must be independent of x and y and must be a constant C_2.

$\therefore \ f(z) = u + iv = C_1 + iC_2$ which is a constant.

(ii) Let $f(z) = u + iv$ be an analytic function.

Given $|f(z)| = \sqrt{u^2 + v^2} = k$ (a constant)

Differentiating partially with respect to x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and $$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Since $f(z)$ is an analytic function, it satisfies C-R equations.

$\therefore$ the above two equations may be written as,

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$

and $$v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

By solving, we get $\frac{\partial u}{\partial x} = u_x = 0$ and $\frac{\partial u}{\partial y} = u_y = 0$.

By C-R equations, it implies that $\frac{\partial v}{\partial x} = v_x = 0$ and $\frac{\partial v}{\partial y} = v_y = 0$.

Thus, $f(z) = u + iv$ is a constant. **Proved.**

Example 4 If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

[AU May 2006, KU Nov. 2011, KU April 2013]

Solution Let $f(z) = u(x, y) + iv(x, y)$

Then $|f(z)|^2 = u^2 + v^2$ and $|f'(z)|^2 = u_x^2 + v_x^2$

To prove $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2) = 4(u_x^2 + v_x^2)$

Now, $\frac{\partial}{\partial x}(u^2) = 2uu_x$ and $\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x)$

$$= 2[uu_{xx} + u_x u_x] = 2uu_{xx} + u_x^2]$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2[uu_{yy} + u_y^2]$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2u[u_{xx} + u_{yy}] + 2[u_x^2 + u_y^2]$$

$$= 2[u_x^2 + u_y^2] \qquad (\text{since } u_{xx} + u_{yy} = 0) \tag{1}$$

Again, $$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x^2]$$

and $$\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$$

$\therefore \qquad \left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)(v^2)=2v(v_{xx}+v_{yy})+2(v_x^2+v_y^2)$

$$=2(v_x^2+v_y^2) \qquad (\text{since } v_{zz}+v_{yy}=0) \qquad (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)=2[u_x^2+u_y^2+v_x^2+v_y^2]$$

$$=2[u_x^2+v_x^2+v_x^2+u_x^2] \qquad (\text{by using C-R equations}) \quad =4[u_x^2+v_x^2].$$

Hence, $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$ **Proved.**

Example 5 Show that if $f(z)$ is a regular function of z then $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\log|f(z)|$ $=0.$ **[AU May 2012]**

Solution $\operatorname{Log}|f(z)|=\dfrac{1}{2}\log|f(z)|^2=\dfrac{1}{2}\log(u^2+v^2)$

$$\therefore \qquad \frac{\partial}{\partial x}\log|f(z)|=\frac{1}{2}\left[\frac{2uu_x+2v\cdot v_x}{u^2+v^2}\right]=\frac{uu_x+vv_x}{u^2+v^2}$$

$$\frac{\partial^2}{\partial x^2}\log|f(z)|=\frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_{xx}+v_x^2)-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$=\frac{1}{u^2+v^2}[uu_x+vv_{xx}+u_x^2+v_x^2]-\frac{2}{(u^2+v^2)^2}(uu_x+vv_x)^2 \qquad (1)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\log|f(z)|=\frac{1}{u^2+v^2}[uu_{yy}+vv_{yy}+u_y^2+v_y^2]-\frac{2}{(u^2+v^2)^2}(uu_y+vv_y)^2 \qquad (2)$$

Adding (1) and (2), we get $-\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\log|f(z)|$

$$=\frac{1}{u^2+v^2}[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+u_x^2+v_x^2+u_y^2+v_y^2]-\frac{2}{(u^2+v^2)^2}$$
$$[(uu_x+vv_x)^2+(uu_y+vv_y)^2]$$

$$=\frac{1}{(u^2+v^2)}[2(u_x^2+v_x^2)]-\frac{2}{(u^2+v^2)^2}[(uu_x+vv_x)^2+(-uv_x+vu_x)^2]$$

$$=\frac{2(u_x^2+v_x^2)}{u^2+v^2}-\frac{2}{(u^2+v^2)^2}[u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)]$$

$$=\frac{2(u_x^2+v_x^2)}{u^2+v^2}-\frac{2(u^2+v^2)(u_x^2+v_x^2)}{(u^2+v^2)^2}$$

$$=0$$ **Proved.**

Example 6 Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and determine its conjugate. Also find $f(z)$. **[KU May 2010, KU April 2013]**

Solution Given $u = \frac{1}{2}\log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}; \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}; \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

Hence, u satisfies Laplace's equation.
$\therefore$ u is harmonic.

To find conjugate of *u*

We know that
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \frac{-\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)} = \frac{x\,dy - y\,dx}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}d\left(\frac{y}{x}\right)$$

$$\int dv = \int \frac{d(y/x)}{1+(y/x)^2}$$

i.e.,
$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore$ the required analytic function is $f(z) = u + iv$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

i.e.,
$$f(z) = \log z$$ **Ans.**

Example 7 If $u(x, y) = e^x(x\cos y - y\sin y)$, find $f(z)$ so that $f(z)$ is analytic.

Solution Given $u = e^x(x\cos y - y\sin y)$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \cos y(xe^x + e^x) - y\sin y e^x$$

$$\therefore \quad \phi_1(z, 0) = ze^z + e^z \quad (1)$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = -xe^x \sin y - e^x(\sin y + y\cos y)$$

$\therefore$ $$\phi_2(z, 0) = 0 \tag{2}$$

By Milne–Thomson method,

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= ze^z + e^z + 0$$
$$= e^z(z + 1)$$

$\therefore$ $$f(z) = \int e^z(z+1)dz$$
$$= ze^z - e^z + e^z + C$$

i.e., $$f(z) = ze^z + C$$ **Ans.**

Example 8 Find the analytic function $f(z) = u + iv$ given that $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

[AU May 2006]

Solution Given $u + iv = f(z)$ (1)

$\therefore$ $iu - v = i\,f(z)$ (2)

Adding (1) and (2), we get

$$(u - v) + i(u + v) = (1 + i)f(z)$$

Let $u - v = U$,

$$u + v = V \quad \text{and} \quad F(z) = (1 + i)\,f(z)$$

$$\frac{\partial V}{\partial x} = \frac{(\cos h2y - \cos 2x)2\cos 2x - \sin 2x \cdot 2\sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(x, y) = \frac{\partial V}{\partial x}$$

$$= \frac{2\cos 2x\cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x\cosh 2y - 2}{(\cos h2y - \cos 2x)^2}$$

$$\phi_1(x, y) = \frac{\partial V}{\partial y} = \frac{-\sin 2x(2\sin h2y)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{-2\sinh 2y\sin 2x}{(\cos h2y - \cos 2x)^2}$$

By Milne–Thomson method, we have

$$F'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

$$\phi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$\phi_1(z, 0) = 0$$

and $$F'(z) = i\frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= i\frac{-2}{1 - \cos 2z} = i\frac{-1}{\frac{1 - \cos 2z}{2}}$$

$$= i\frac{-1}{\sin^2 z} = -i\,\mathrm{cosec}^2 z$$

$\therefore \qquad f(z) = -\dfrac{i}{1+i}\displaystyle\int \text{cosec}^2 z\, dz$

i.e., $\qquad f(z) = \dfrac{i+1}{2}\cot z + C$ **Ans.**

Example 9 Find the analytic function $f(z) = u + iv$ if $u + v = \dfrac{x}{x^2+y^2}$ and $f(1) = 1$.

[AU Nov. 2010]

Solution Given $u + iv = f(z)$ (1)

$iu - v = if(z)$ (2)

Adding (1) and (2), we get

$(u - v) + i(u + v) = (1 + i)f(z)$

i.e., $\qquad U + iV = F(z)$ (3)

where $\qquad U = u - v,\ V = u + v = \dfrac{x}{x^2+y^2},\ F(z) = (1 + i)\, f(z)$ (4)

$$V = \frac{x}{x^2+y^2}$$

$$\phi_1(x, y) = \frac{\partial V}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$\therefore \qquad \phi_1(z, 0) = 0$ (5)

$$\phi_2(x, y) = \frac{\partial V}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$\therefore \qquad \phi_2(z, 0) = \dfrac{-z^2}{z^4} = -\dfrac{1}{z^2}$ (6)

By Milne's method, we have

$$F'(z) = \phi_1(z_1 0) + i\phi_2(z, 0)$$

$$= 0 - i\frac{1}{z^2}$$

$$F(z) = -i\int \frac{1}{z^2}\, dz$$

$\therefore \qquad = -i\left(-\dfrac{1}{z}\right) + C$

$F(z) = \dfrac{i}{z} + C$ (7)

But $F(z) = (1 + i)\, f(z)$ [from (4) and (8)]

From (7) and (8), we get

$$(1+i)\, f(z) = \frac{i}{z} + C$$

$$f(z) = \frac{i}{z(1+i)} + \frac{C}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)z} + C_1, \text{ where } C_1 = \frac{C}{1+i}$$

$$f(z) = \frac{1+i}{2z} + C_1$$

Given $f(1) = 1$

i.e., $$f(1) = \frac{1+i}{2} + C_1 = 1$$

$\Rightarrow$ $$C_1 = 1 - \frac{(1+i)}{2}$$
$$= \frac{1-i}{2}$$

$\therefore$ $$f(z) = \frac{1+i}{2z} + \frac{1-i}{2}$$ **Ans.**

Example 10 Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. **[AU Nov. 2010]**

Solution

Let $$z = x + iy \qquad (1)$$

$\therefore$ $$\bar{z} = x - iy \qquad (2)$$

From (1) and (2), we get

$$x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i} = \frac{-i}{2}(z-\bar{z})$$

Now, $$\frac{\partial x}{\partial z} = \frac{1}{2}, \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$$
$$\frac{\partial y}{\partial z} = \frac{-i}{2}, \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

Now, $$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} \qquad (3)$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} \qquad (4)$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$\therefore$ $$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$\Rightarrow$ $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$ **Proved.**

Example 11 If $f(z) = u + iv$ is analytic, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$. **[AU Nov. 2010]**

Solution We know that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$\therefore \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\partial\bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\cdot\frac{1}{2}\log|f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\log[f'(z)\,f'(\bar{z})]$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}[\log f'(z)+\log f'(\bar{z})]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right] = 0$$

Proved.

Example 12 If $u = x^2 - y^2$ and $v = -\dfrac{y}{x^2+y^2}$, prove that both u and v satisfy Laplace's equation but that $u + iv$ is not a regular function of z. **[KU Nov. 2011]**

Solution Given $u = x^2 - y^2$

Then $$\frac{\partial u}{\partial x} = u_x = 2x;\ \frac{\partial^2 u}{\partial x^2} = u_{xx} = 2;\ \frac{\partial u}{\partial y} = u_y = -2y;\ \frac{\partial^2 u}{\partial y^2} = u_{yy} = -2$$

$$\therefore \qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

i.e., u satisfies Laplace's equation.

$$v = -\frac{y}{x^2+y^2}$$

Then $$\frac{\partial v}{\partial x} = v_x = \frac{2xy}{(x^2+y^2)^2};\ v_{xx} = 2y\left[\frac{(x^2+y^2)\cdot - x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = v_y = -\left[\frac{(x^2+y^2)\cdot 1 - 2y^2}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = v_{yy} = \frac{(x^2+y^2)^2\,2y - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\therefore \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0$$

i.e., v satisfies Laplace's equation.
Now, $u_x \neq v_y$ and $u_y \neq -v_x$
i.e., C-R equations are not satisfied by u and v.
Hence, $u + iv$ is not an analytic (regular) function of z. **Ans.**

Example 13 Show that the function $u(x, y) = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find a function $v(x, y)$ such that $u + iv$ is an analytic function.

[AU June 2010]

Solution Let $f(z) = u + iv$ be an analytic function with $u(x, y) = 3x^2y + x^2 - y^3 - y^2$

Then
$$\frac{\partial u}{\partial x} = u_x = 6xy + 2x; \frac{\partial^2 u}{\partial x^2} = u_{xx} = 6y + 2;$$
$$\frac{\partial u}{\partial y} = u_y = 3x^2 - 3y^2 - 2y; \frac{\partial^2 u}{\partial y^2} = u_{yy} = -6y - 2$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence, $u(x, y)$ is a harmonic function.

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy = \frac{-\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -u_y dx + u_x dy$$

$\therefore dv = (-3x^2 + 2y + 3y^2)dx + (6xy + 2x)dy$ where the RHS is a perfect differential equation.

$$dv = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$$
$$= -\int (3x^2 - 3y^2 - 2y)\, dx + \int (6xy + 2x)\, dy$$

$$\therefore \quad v = (3xy^2 + 2xy - x^3) + C$$

$$\therefore \quad f(z) = u + iv = 3x^2y + x^2 - y^3 - y^2 + i(3xy^2 + 2xy - x^3 + C)$$
$$= -i[x^3 + 3x^2(iy) + 3xi^2y^2 + i^3y^3] + [x^2 + 2xiy + i^2y^2] + iC$$
$$= -i[x + iy]^3 + [x + iy]^2 + iC$$

$$\therefore \quad f(z) = iz^3 + z^2 + iC$$ **Ans.**

EXERCISE

Part A

1. Define analytic function of a complex variable.
2. State any two properties of an analytic function.
3. Define a harmonic function with an example.
4. Verify whether the function $\phi(x, y) = e^x \sin y$ is harmonic or not.
5. Find the constant 'a' so that $u(x, y) = ax^2 - y^2 + xy$ is harmonic.
6. Is $f(z) = z^3$ analytic? Justify.
7. What do you mean by a conjugate harmonic function? Find the conjugate harmonic of x.
8. Show that an analytic function with a constant real part is constant.
9. Write down the necessary condition for $w = f(z) = f(re^{i\theta})$ to be analytic.
10. Show that the function $u = \tan^{-1}\left(\frac{y}{x}\right)$ is harmonic.
11. Show that xy^2 cannot be the real part of an analytic function.
12. $f(z) = u + iv$ is such that u and v are harmonic. Is $f(z)$ analytic always? Justify.

13. State C-R equations in Cartesian coordinates.
14. Prove that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is a harmonic function.
15. Show that the function $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ satisfies Cauchy–Riemann equations.
16. Show that the real part u of an analytic function satisfies the equation $\nabla^2 u = 0$.
17. Check whether the function $\frac{1}{z}$ is analytic or not.
18. Test the analyticity of the function $2xy + i(x^2 - y^2)$.
19. State the basic difference between the limit of a function of a real variable and that of a complex variable.
20. Find the analytic function $f(z) = u + iv$, given that (i) $u = y^2 - x^2$, (ii) $v = \sin hx \sin y$, and (iii) $u = \dfrac{x}{x^2+y^2}$.

Part B

1. Prove that the following functions are not differentiable (and, hence, not analytic) at the origin.

   (i) $f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

   (ii) $f(z) = \begin{cases} \dfrac{xy^2(x + iy)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

2. Prove that for the following function, C-R equations are satisfied at the origin but $f(z)$ is not analytic there.

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

3. Show that $f(z) = \sin \bar{z}$ is not an analytic function of z.
4. Find whether the Cauchy–Riemann equations are satisfied for the following functions where $w = f(z)$.

   (i) $w = 2xy + i(x^2 - y^2)$ **(Ans.** No**)**

   (ii) $w = \dfrac{x - iy}{x^2 + y^2}$ **(Ans.** No**)**

   (iii) $w = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$ **(Ans.** Yes**)**

   (iv) $w = \cos x \sin hy$ **(Ans.** Yes**)**

   (v) $w = z^3 - 2z^2$ **(Ans.** Yes**)**
5. Show that an analytic function with a constant imaginary part is constant.
6. Show that $u + iv = \dfrac{x - iy}{x - iy + a}$, where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

7. Find an analytic function $w = u + iv$ whose real part is given by
   (i) $u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$ **[Ans.** $e^{-x}(x - iy)^2 (\cos y - i \sin y)$**]**
   (ii) $u = \dfrac{x}{x^2+y^2}$ $\left(\textbf{Ans. } \dfrac{1}{z} + C\right)$
   (iii) $u = e^x(x \cos y - y \sin y)$ **(Ans.** $ze^z + C$**)**
   (iv) $u = x^4 - 6x^2y^2 + y^4$ **(Ans.** $z^4 + C$**)**
   (v) $u = -\sin x \sin hy$ **(Ans.** $-i \cos z + iC$**)**
8. Find an analytic function $w = u + iv$ whose imaginary part is given by
   (i) $v = e^x(x \cos y + y \sin y)$ **(Ans.** $ize^{-z} + C$**)**
   (ii) $v = -2 \sin x(e^y - e^{-y})$ **(Ans.** $\log z + C$**)**
   (iii) $v = \dfrac{\sin x \sin hy}{\cos 2x + \cos h2y}$ $\left(\textbf{Ans. } \dfrac{1+\sec z}{2}\right)$
   (iv) $v = x^2 - y^2 + 2xy - 3x - 2y$ **[Ans.** $z^2 - 2z + i(z^2 - 3z)$**]**
   (v) $v = x^3 - 3x^2y + 2x + 1 + y^3 - 3xy^2$ **[Ans.** $(i - 1)z^3 + 2z + C$**]**
9. Show that the following functions are harmonic and find their harmonic conjugates.
   (i) $u = \cos x \cos hy$ **(Ans.** $-\sin x \sin hy + C$**)**
   (ii) $u = e^x(\cos y - \sin y)$ **(Ans.** Not harmonic**)**
   (iii) $u = e^{-x}(y \cos y - x \sin y)$ (**Ans.** $e^x(x \cos y + y \sin y) + C$)
   (iv) $u = e^x \cos y$ (**Ans.** $e^x \sin y + C$)
   (v) $u = 2xy + 3xy^2 - 2y^3$ **(Ans.** Not harmonic**)**
10. Find $f(z) = u + iv$, if $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, given that $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$.
    $\left[\textbf{Ans. } f(z) = \cot\left(\dfrac{z}{2}\right) + \left(\dfrac{1-i}{2}\right)\right]$
11. Find $f(z) = u + iv$ if $2u - 3v = 3y^2 - 2xy - 3x^2 + 3y - x$ and $f(0) = 0$.
    (**Ans.** $f(z) = iz^2 - z$**)**
12. If $f(z) = u + iv$ is a regular function of z, then show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$.
13. If $u = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, find $f(z)$ such that $f(z)$ is analytic.
    (Ans. $f(z) = \cot z + C$**)**
14. Show that $\phi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$ can represent the velocity potential in an incompressible fluid flow. Also find the corresponding stream function and complex potential. $\left[\textbf{Ans. } \psi = 2xy - \dfrac{y}{x^2+y^2} + C;\ f(z) = z^2 + \dfrac{1}{z} + iC\right]$

S.No	Questions	Opt 1	Opt 2	Opt 3	Opt 4	Answer
1	An example of single valued function of z is ________.	w = z^2	w = z^(1/2)	w=SQRT(z)	w=z^-1	w = z^2
2	An example of multiple valued function of z is _______.	w = z^2	w = z^(1/2)	w=SQRT(z)	w=z^-1	w = z^(1/2)
3	The distance between two points z and z_0 is	$\|z-z_0\|$	$\|z+z_0\|$	z	z_0	$\|z-z_0\|$
4	A circle of radius 1 with centre at origin can be represented by _______.	|z|>1	|z| < 1	|z|=1	|z| = 0	|z| = 1
7	If f(z) is differentiable at z_0 then f(z) is ________ at z_0.	discontinuous	continuous	regular	irregular	continuous
8	A function is said to be _________ at a point if its derivative exists not only at point but also in some neighborhood of that point.	entire function	integral function	analytic	continuous	analytic
9	A function which is analytic everywhere in the finite plane is called ___________.	analytic function	holomorphic function	regular function	entire function	entire function
11	The necessary condition for f(z) to be analytic is_________	$u_x = v_y$ and $v_x = -u_y$	$u_x = -v_y$ and $v_x = u_y$	$u_x = v_y$ and $v_x = u_y$	$u_x = -v_y$ and $v_x = -u_y$	$u_x = v_y$ and $v_x = -u_y$
12	A real function of two variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called __________.	analytic function	regular function	holomorphic function	harmonic function	harmonic function
13	If u and v are harmonic functions such that u+iv is analytic then each is called the _________ of the other.	conjugate harmonic	analytic	entire function	not analytic	conjugate harmonic
14	A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is called _______ at that point.	Conformal	isogonal	entire function	unconformal	Conformal
15	A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be __________ at that point.	Conformal	isogonal	entire function	unconformal	isogonal
16	A mapping w = f(z) is said to be conformal at z = z_0 if __________.	$f'(z_0) = 0$	$f'(z_0) = f(z)$	$f'(z_0) \neq 0$	$f'(z_0) \neq f(z)$	$f'(z_0) \neq 0$
17	The point at which the mapping w = f(z) is not conformal, that is, f'(z) = 0 is called ________ of the mapping.	common	fixed	invariant	critical	critical
18	A _________ point of a mapping w = f(z) are points that are mapped onto themselves, are kept fixed under the mapping.	common	fixed	critical	variant	fixed
19	The transformation w = a+z where a is a complex constant, represents a __________.	translation	magnification	rotation	reflection	translation
20	The transformation _________ where a is a complex constant represents a translation.	w = az	w = az+b	w = a+z	w = 1/z	w = a+z
21	The transformation _________ where a is a real constant represents magnification.	w = a+z	w = 1/z	w = az+b	w = az	w = az
22	The transformation w = az where a is a real constant represents __________.	translation	magnification	reflection	inversion	magnification
23	In general linear transformation, w = az+b where a and b are complex constants represents _______.	magnification	rotation	translation	magnification, rotation and translation	magnification, rotation and translation
24	The transformation w=(az+b)/(cz+d), where a, b, c, d are complex numbers is called a ______.	Linear transformation	bilinear transformation	fractional transformation	translation	bilinear transformation
25	A bilinear transformation is also called a _______.	linear transformation	inversion	fractional transformation	linear fractional transformation	linear fractional transformation
26	The value of i =	SQRT(-1)	SQRT(1)	-1	1	SQRT(-1)
27	_________ represents the interior of the circle excluding its circumference.	$\|z – z_0\|$ > delta	$\|z – z_0\|$ < delta	$\|z – z_0\|$ ≥ delta	$\|z – z_0\|$ ≤ delta	$\|z – z_0\|$ < delta
28	_________ represents the interior of the circle including its circumference.	$\|z – z_0\|$ > delta	$\|z – z_0\|$ < delta	$\|z – z_0\|$ ≥ delta	$\|z – z_0\|$ ≤ delta	$\|z – z_0\|$ ≤ delta
29	_________ represents the exterior of the circle.	$\|z – z_0\|$ > delta	$\|z – z_0\|$ < delta	$\|z – z_0\|$ ≥ delta	$\|z – z_0\|$ ≤ delta	$\|z – z_0\|$ > delta
30	Cauchy-Riemann equations are necessary conditions for a function w = f(z) to be an _____.	entire function	integral function	analytic function	continuous function	analytic function
31	Cauchy-Riemann equations are	$u_x = v_y$ and $v_x = -u_y$	$u_x = -v_y$ and $v_x = u_y$	$u_x = v_y$ and $v_x = u_y$	$u_x = -v_y$ and $v_x = -u_y$	$u_x = v_y$ and $v_x = -u_y$
32	The real and imaginary parts of an analytic function f(z) = u+iv satisfies the ______ equation in two dimensions.	Cauchy-Riemann	Homogeneous	Laplace	Euler	Laplace
33	An analytic function with a constant real part is __________.	a variable	a constant	an analytic function	an entire function	a constant
34	An analytic function with a constant modulus is __________.	a variable	a constant	an analytic function	an entire function	a constant
35	A fixed point is also called as _________.	invariant points	critical points	common point	origin	invariant points
36	The fixed point of w=(5z+4)/(z+5) is	2,1	1,-1	-2, 2	0, 1	-2, 2
37	The critical point of z=(2z+1)/(z+2) is	1, 1	1, -1	1,2	0,1	1, -1
38	Solutions of Laplace's equation are ________ under conformal transformation	common	fixed	invariant	critical	invariant
39	If f(z) is analytic, and f'(z)=0 everywhere, then f(z) is _____	a variable	a constant	an analytic function	an entire function	a constant
40	An analytic function with a constant imaginary part is __________.	a variable	a constant	an analytic function	an entire function	a constant
41	If u+iv is analytic, then v-iu is _______	entire function	integral function	analytic	continuous	analytic
44	w=z has every point as a _______ point	fixed	critical	invariant	common	fixed
45	w=1/z has _______ fixed points	1	2	3	4	2
46	w=z+b has ________ fixed points	0	1	2	3	0

Unit X

Complex Integration

23 Complex Integration

Chapter Outline

23.1 ❑ INTRODUCTION

Integration of functions of a complex variable plays a very important role in many areas of science and engineering. The advantage of complex integration is that certain complicated real integrals can be evaluated and properties of analytical functions can be established. Using integration, we shall prove a very important result in the theory of analytic functions:

If a function $f(z)$ is analytic in a domain D then it possesses derivatives of all orders in D, that is $f'(z)$, $f''(z)$ … are all analytic functions in D.

Such a result does not exist in the real-variable theory. Also, the complex-integration approach can be used to evaluate many improper integrals of a real variable, which cannot be evaluated using real integral calculus. The concept of definite integral for functions of a real variable does not directly extend to the case of complex variables.

In the case of a real variable, the path of integration in the definite integral $\int_a^b f(x)dx$ is along a straight line. In complex integration, the path could be along any curve from $z = a$ to $z = b$.

23.2 ❑ LINE INTEGRAL IN COMPLEX PLANE

- **Continuous Arc**

The set of points (x, y) defined by $x = \phi(t)$, $y = \psi(t)$, with parameter t in the interval (a, b), defines a continuous arc provided ϕ and ψ are continuous functions.

- **Smooth Arc**

If ϕ and ψ are differentiable, the arc is said to be smooth.

- **Simple Curve**

It is a curve having no self-intersections, i.e., no two distinct values of t correspond to the same point (x, y).

- **Closed Curve**

It is one in which end points coincide, i.e., $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$.

- **Simple Closed Curve**

It is a curve having no self-intersections and with coincident end points.

- **Contour**

It is a continuous chain of a finite number of smooth arcs.

- **Closed Contour**

It is a piecewise smooth closed curve without points of self-intersection.

23.3 ❑ LINE INTEGRAL

Definite integral or complex line integral or simply line integral of a complex function $f(z)$ from z_1 to z_2 along a curve C is defined as

$$\begin{aligned}\int_C f(z)dz &= \int_C (u+iv)(dx+i\,dy)\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\end{aligned}$$

Here, C is known as path of integration. If it is a closed curve, the line integral is denoted by $\oint_C$.

When the direction is in positive sense, it is indicated as $\int_{C+}$ or simply, $\int_C$ while negative direction is denoted by $\int_C$. Counter integral is an integral along a closed contour.

23.4 ❑ BASIC PROPERTIES OF LINE INTEGRALS

(i) Linearity: $\int_C (k_1 f(z) + k_2 g(z))\,dz = k_1 \int_C f(z)dz + k_2 \int_C g(z)dz$

(ii) Sense reversal: $\displaystyle\int_a^b f(z)dz = -\int_b^a f(z)dz$

(iii) Partitioning of path: $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$
where the curve C consists of the curves C_1 and C_2.

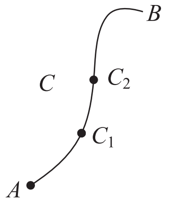


Fig. 23.1

➢ **Note**

Although real definite integrals are interpreted as area, no such interpretation is possible for complex definite integrals.

23.5 ❑ SIMPLY CONNECTED REGION AND MULTIPLY CONNECTED REGION

A simply connected region R is a domain such that every simple closed path in R contains only points of R.

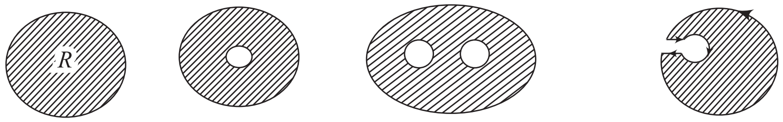
- **Example**

Interior of a circle, rectangle, triangle, ellipse, etc.

A multiply connected region is one that is not simply connected.

- **Example**

Annulus region, region with holes.



Simply connected region	Doubly connected region	Triply connected region	Simply connected region (or) Multiply connected region converted into simply connected region by cross-cuts.

Fig. 23.2

23.6 ❑ EVALUATION OF A COMPLEX INTEGRAL

To evaluate the integral $\int_C f(z)dz$, we have to express it in terms of real variables.

Let $\quad f(z) = u + iv$ where $z = x + iy$, $dz = dx + idy$

$$\therefore \quad \int_C f(z)dz = \int_C (u + iv)dz$$
$$= \int_C (u + iv)(dx + idy)$$
$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

23.7 ❑ CAUCHY'S INTEGRAL THEOREM OR CAUCHY'S FUNDAMENTAL THEOREM

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\int_C f(z)dz = 0$.

- **Proof**

Let the region enclosed by a curve C be R and let

$$f(z) = u + iv,\ z = x + iy,\ dz = dx + i\,dy$$

$$\int_C f(z)dz = \int_C (u + iv)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy \quad \text{(by Green's theorem)}$$

Replacing $-\frac{\partial v}{\partial x}$ by $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$, we get

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)dx\,dy$$

$$= 0 + i0 = 0$$

or $\int_C f(z)dz = 0$

➢ **Note**

(i) Cauchy's integral theorem is also known as Cauchy's theorem.
(ii) Cauchy's theorem without the assumption that f' is continuous is known as the **Cauchy–Goursat theorem.**
(iii) Simple connectedness is essential.

23.8 ❑ EXTENSION OF CAUCHY'S INTEGRAL THEOREM TO MULTIPLY CONNECTED REGIONS

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

- **Proof**

By Cauchy's integral theorem, we know that $\int_C f(z)dz = 0$ where the path of integration is along AB and the curve C_2 in clockwise direction, and BA and along C_1 in anticlockwise direction,

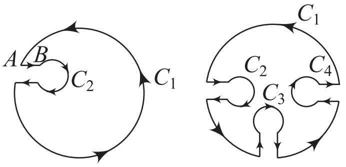


Fig. 23.3

i.e., $\int_{AB} f(z)dz + \int_{C_2} f(z)dz + \int_{BA} f(z)dz + \int_{C_1} f(z)dz = 0$

or $\int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0$ (since $\int_{AB} f(z)dz = -\int_{BA} f(z)dz$)

Reversing the direction of the integral around C_2, we get

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

➢ **Note**

By introducing as many cross-cuts as the number of inner boundaries, we can give the proof in a similar manner for the extension of Cauchy's integral theorem.

23.9 ❑ CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve C and if a is any point within C then $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{z-a}dz$.

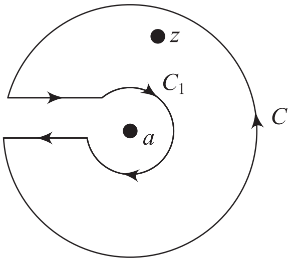


Fig. 23.4

- **Proof**

Consider the function $\dfrac{f(z)}{z-a}$, which is analytic at all points within C except $z = a$.

With a point a as centre and radius r, draw a small circle C_1 lying entirely within C. Now, $\dfrac{f(z)}{z-a}$ is analytic in the region between C and C_1;

Hence, by Cauchy's integral theorem for a multiply connected region, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)-f(a)+f(a)}{z-a}dz$$

$$= \int_{C_1} \frac{f(z)-f(a)}{z-a}dz + f(a)\int_{C_1} \frac{dz}{z-a} \tag{23.1}$$

For any point on C_1

Now, $$\int_{C_1} \frac{f(z)-f(a)}{z-a}dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}} ire^{i\theta}d\theta$$

[as $z-a = re^{i\theta}$ and $dz = ire^{i\theta}\, d\theta$]

$$= \int_0^{2\pi} [f(a+re^{i\theta})-f(a)]id\theta = 0$$ (where r tends to zero]

$$\int_{C_1} \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = \int_0^{2\pi} id\theta = i[0]_0^{2\pi i} = 2\pi i$$

Putting the values of the integrals of RHS in (23.1), we have

$$\int_C \frac{f(z)}{z-a}dz = 0 + f(a)(2\pi i)$$

or $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

23.10 ❑ CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

If a function $f(z)$ is analytic in a region R then its derivative at any point $z = a$ of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

where C is any closed curve in R surrounding the point $z = a$.

- **Proof**

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \tag{23.2}$$

Differentiating (23.2) with respect to a, we get

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right)\cdot dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

Similarly,
$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz$$

$$f^n(a) = \frac{n!}{2\pi !}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

SOLVED EXAMPLES

Example 1 Use Cauchy's integral formula to evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)}dz$, where C is the circle $|z| = 4$.

[AU June 2009, April 2011; KU Nov. 2011]

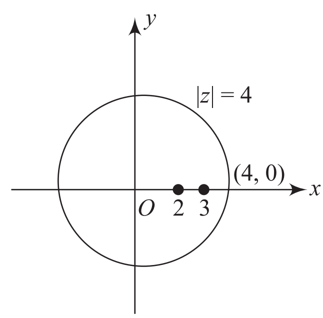


Fig. 23.5

Solution

$$\frac{1}{(z-2)(z-3)} = \frac{1}{(z-3)} - \frac{1}{(z-2)}$$

$\therefore$ given integral

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3}dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}dz$$

$$= \int_C \frac{f(z)}{(z-3)}dz - \int_C \frac{f(z)}{(z-2)}dz \tag{1}$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic on and inside C.

The points $z = 2$ and $z = 3$ lie inside C.

$\therefore$ by Cauchy's integral formula, from (1), we get,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$$
$$= 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=3} - 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=2}$$
$$= 2\pi i(\sin 9\pi + \cos 9\pi) - 2\pi i(\sin 4\pi + \cos 4\pi)$$
$$= -2\pi i - 2\pi i = -4\pi i$$ **Ans.**

Example 2 Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$, using Cauchy's integral formula. **[AU May 2012]**

Solution $|z-2| = \frac{1}{2}$ is the circle with centre at $z = 2$ and radius equal to $\frac{1}{2}$.

The point $z = 2$ lies inside the circle $|z-2| = \frac{1}{2}$

The given integral can be rewritten as

$$\int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2} dz = \int_C \frac{f(z)}{(z-2)^2} dz \text{ (say)}$$

$f(z) = \frac{z}{z-1}$ is analytic on and inside C and the point $z = 2$ lies inside C.

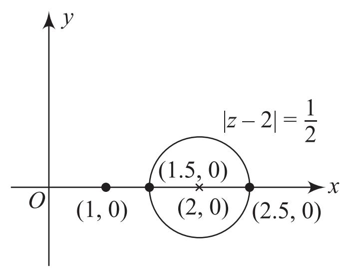


Fig. 23.6

$\therefore$ by Cauchy's integral formula,

$$\int_C \frac{z}{(z-1)(z-2)^2} dz = \frac{2\pi i}{1!} f'(2)$$
$$= 2\pi i \left\{ \frac{d}{dz}\left(\frac{z}{z-1}\right) \right\}_{z=2}$$
$$= 2\pi i \left\{ \frac{-1}{(z-1)^2} \right\}_{z=2} = -2\pi i$$ **Ans.**

Example 3 Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $|z+1+i| = 2$ using Cauchy's integral formula. **[AU Nov. 2011]**

Solution $|z+1+i| = 2$ is the circle whose centre is $-1-i$ and radius is 2 units.

Consider $\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$

$\therefore$ the integral is not analytic at $z = -1-2i$ and $-1+2i$.

The point $z = -1-2i$ lies inside C.

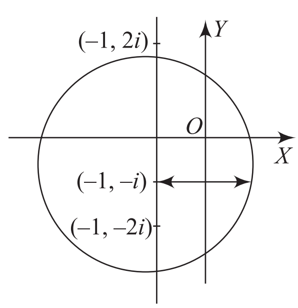


Fig. 23.7

We rewrite the given integral as

$$\int_C \frac{\left(\dfrac{z+4}{z+1-2i}\right)}{z+1+2i}\,dz = \int_C \frac{f(z)}{z-(-1-2i)}\,dz \text{ (say)}$$

$f(z)$ is analytic on and inside C and the point $(-1, -2i)$ lies inside C.

$\therefore$ by Cauchy's integral formula,

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\, f(-i-2i)$$

$$= 2\pi i\left\{\frac{-1-2i+4}{-1-2i+1-2i}\right\}$$

$$= \frac{-\pi}{2}(3-2i)$$ **Ans.**

EXERCISE

Part A

1. The value of the integral $\int_C \frac{dz}{z^2-2z}$ where C is the circle $|z-2| = 1$, traversed in the counter-clockwise sense is
   (i) $-\pi i$ (ii) $2\pi i$ (iii) πi (iv) 0
2. The value of the integral $\int_C \frac{z^2-z+1}{z-1}\,dz$, where C is the circle $|z| = \frac{1}{2}$ is
   (i) 0 (ii) πi (iii) $-\pi i$ (iv) $-2\pi i$
3. What is the value of $\int_C e^z\,dz$ if $c : |z| = 1$?
4. State Cauchy's integral formula.
5. Evaluate $\int_C \frac{dz}{z-2}$ where C is the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).
6. Evaluate $\int_C \frac{3z^2+7z+1}{(z-3)}\,dz$ where $C : |z| = 2$.
7. Evaluate $\int_C \frac{dz}{z^2-5z+6}$ where C is the circle $|z-1| = \frac{1}{2}$.
8. State Cauchy's formula for the first derivative of an analytic function.
9. State Cauchy's fundamental theorem.
10. Evaluate $\int_C \frac{z\,dz}{z-2}$ where $C : |z| = 1$.
11. Evaluate $\int_C \frac{2}{(z+3)}\,dz$ where $C : |z| = 2$.
12. Evaluate $\int_C \frac{1}{2z-3}\,dz$ where $C : |z| = 1$.

13. Evaluate $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}dz$ where C is $|z| = 4$ using Cauchy's integral formula.
14. Evaluate $\int_C \frac{dz}{(z-3)^2}$ where $C: |z| = 1$.
15. State the Cauchy–Goursat theorem.

Part B

1. Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i| = 2$. $\left(\textbf{Ans. } \frac{-2\pi i}{9}\right)$
2. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$ using Cauchy's integral formula. where C is the circle $|z| = \frac{3}{2}$. **(Ans.** $2\pi i$**)**
3. Find the value of $\int_C \frac{2z^2+z}{z^2-1}dz$. **(Ans.** $3\pi i$**)**
4. Evaluate the following:
   (i) $\int_C \frac{dz}{(z^2+4)^2}$, where C is $|z-i| = 2$
   (ii) $\int_C \frac{z^3+z+1}{z^2-7z+6}dz$ where C is the ellipse $4x^2 + 9y^2 = 1$
   (iii) $\int_C \frac{z^3+1}{z^2-3iz}dz$ where C is $|z| = 1$. $\left[\textbf{Ans. } \text{(i) } \frac{\pi}{16}, \text{(ii) } 0, \text{(iii) } -\frac{2\pi}{3}\right]$
5. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}dz$ where C is $|z| = 3$. (**Ans.** $-4\pi i$)
6. If $f(a) = \int_C \frac{4z^2+z+5}{z-a}dz$ where C is $|z| = 2$, find the values of $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$. (**Ans.** $20\pi i$; $2\pi(i-1)$; $-14\pi i$; $16\pi i$)
7. Evaluate $\int_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1). **(Ans.** $-1+i$**)**
8. Evaluate $\int_C \frac{z^2+1}{z^2-1}dz$ where (i) $C: |z-1| = 1$, (ii) $C: |z+1| = 1$, and (iii) $C: |z-i| = 1$. **[Ans.** (i) $2\pi i$ (ii) $-2\pi i$ (iii) 0**]**
9. Evaluate $\int_C \frac{\sin 2z}{(z+3)(z+1)^2}dz$ where C is the rectangle with vertices at $3+i$, $-2+i$, $-2-i$, $3-i$. $\left[\textbf{Ans. } \pi i\frac{(4\cos 2+\sin 2)}{2}\right]$
10. Evaluate $\int_C \frac{z^4-3z^2+6}{(z+i)^3}dz$ where $C: |z| = 2$. **(Ans.** $-18\pi i$**)**

Questions	opt1	opt2	opt3	opt4	Answer
A curve is called a____________ if it does not intersect itself	Simple closed curve	multiple curve	simply connected region	multiple connected region	Simple closed curve
A curve is called ________ if it is not a simple closed curve	connected region	multiple curve	simply connected region	multiple connected region	multiple curve
If f(z) is analytic in a simply connected domain D and C is any simple closed path then ∫(from c)f(z)dz =	1	2πi	0	πi	0
If f(z) is analytic inside on a simple closed curve C and a be any point inside C then ∫(from c)f(z)dz /(z-a)=	2πi f(a)	2πi	0	πi	2πi f(a)
The value of ∫(from c) [(3z^2+7z+1)/(z+1)] dz where C is |z| = 1/2 is	2πi	-6πi	πi	πi/2	-6πi
The value of ∫(from c) (cos πz/z-1) dz if C is |z| =2	2πi	-2πi	πi	πi/3	-2πi
The value of ∫(from c) (1/z-1) dz if C is |z| =2	2πi	3πi	πi	πi/4	2πi
The value of ∫(from c) (1/z-3) dz if C is |z| =1	3πi	πi	πi/4	0	0
The value of ∫(from c) (1/(z-3)^3) dz if C is |z| =2	3πi	πi	πi/5	0	0
The Taylor's series of f(z) about the point z=0 is called ___________series	Maclaurin's	Laurent's	Geometric	Arithmetic	Maclaurin's
The value of ∫(from c) (1/z+4) dz if C is |z| =3	3πi	πi	πi/4	0	0
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____ part	regular	principal	real	imaginary	regular
In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____ part	regular	principal	real	imaginary	principal
The poles of the function f(z) = z/((z-1)(z-2)) are at z = ______	1, 2	2,3	1,-1	3,4	1, 2
The poles of cotz are______	2nπ	nπ	3nπ	4nπ	nπ
The poles of the function f(z) = cos z/((z+3)(z-4)) are at z = _____	- 3, 4	2,3	1,-1	3,4	- 3, 4
The isolated singular point of f(z) = z/((z-4)(z-5))	1,2	2,3	0,2	4,5	4,5
The isolated singular point of f(z) = z/((z(z-3))	1,3	2,4	0,3	4,5	0,3
A simple pole is a pole of order ______	1	2	3	4	1
The order of the pole z= 2 for f(z) = z/((z+1)(z-2)^2)	1	2	3	4	2
Residue of (cosz / z) at z = 0 is	0	1	2	4	1
The residue at z = 0 of ((1 + e^z) / (zcosz+sinz)) is	0	1	2	4	1
The residue of f(z) = cot z at z= 0 is_____	0	1	2	4	1
The singularity of f(z) = z / ((z-3)^3) is______	0	1	2	3	3
A point z=a is said to be a _______point of f(z), if f(z) is not analytic at z=a	Singular	isolated singular	removable	essential singular	Singular
A point z=a is said to be a _______point of f(z), if f(z) is analytic except at z=a	Singular	isolated singular	removable	essential singular	isolated singular
In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____point	Singular	isolated singular	removable singular	essential singular	essential singular
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____point	Singular	isolated singular	removable singular	essential singular	removable singular
In contour integration, cos θ=_______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz	(z^2+1)/2z
In contour integration, sin θ=_______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz	(z^2-1)/2iz