B.E Electrical and Electronics Engineering **2019-2020**

19BEEE201 **Semester-II**

Mathematics – II **4H-4C**

(Linear Algebra, Transform Calculus and Numerical Method)

Instruction Hours/week: L:3 T:1 P:0 **Marks:** Internal**:40** External**:60** Total**:100**

End Semester Exam:3 Hours

Course Objectives

- The objective of this course is to familiarize the prospective engineers with techniques in Linear Algebra, Transform calculus and Numerical methods.
- The syllabus is designed to develop the use of Matrix algebra techniques which is needed by Engineers for practical applications.
- It aims to equip the students in numerical methods to solve engineering problems, Fundamentals of numerical methods/algorithms to solve systems of different mathematical equations will be introduced.
- To learn numerical methods to obtain approximate solutions to mathematical problem.
- To learn Basic concepts of Laplace transforms.

Course Outcomes

The students will learn:

1. To solve the problems in engineering using Matrix algebra Techniques.
2. Derive numerical methods for various mathematical operations and tasks such as interpolation, differentiation and integration.
3. To analyze and evaluate the accuracy of solution for ordinary differential equations.
4. To implement numerical methods to solve Partial differential equations.
5. To solve problems using Laplace Transforms.
6. To improve facility in numerical manipulation.

UNIT I - Matrices

Inverse and rank of a matrix, rank-nullity theorem; System of linear equations; Symmetric, skew-symmetric and orthogonal matrices; Determinants; Eigenvalues and eigenvectors; Diagonalization of matrices; Cayley-Hamilton Theorem, Orthogonal transformation. Simple Problems using Scilab.

UNIT II - Numerical Methods

Solution of polynomial and transcendental equations – Bisection method, Newton-Raphson method and Regula-Falsi method. Finite differences, Interpolation using Newton’s forward and backward difference formulae. Central difference interpolation: Gauss’s forward and backward formulae. Numerical integration: Trapezoidal rule and Simpson’s 1/3rd and 3/8 rules.

UNIT III - Numerical Methods

Ordinary differential equations: Taylor's series, Euler and modified Euler's methods. RungeKutta method of fourth order for solving first and second order equations. Milne's And Adam's predicator-corrector methods.

UNIT IV -Numerical Methods

Partial differential equations: Finite difference solution two Dimensional Laplace equation and Poisson equation, Implicit and explicit methods for one Dimensional heat equation(Bender-Schmidt and Crank-Nicholson methods), Finite difference Explicit method for wave equation.

UNIT V - Transform Calculus

Laplace Transform, Properties of Laplace Transform, Laplace transform of periodic functions. Finding inverse Laplace transform by different methods, convolution theorem. Evaluation of Integrals by Laplace transform, solving ODEs and PDEs by Laplace Transform method.

SUGGESTED READINGS

1. P.Kandasamy,K.Thilagavathy., K.Gunavathy (2008), Numerical Methods,S.Chand Ltd,.
2. B.S. Grewal, (2010), Higher Engineering Mathematics, 36th Edition, Khanna Publishers
3. D. Poole, (2005),Linear Algebra: A Modern Introduction, 2nd Edition,Brooks/Cole.
4. N.P. Bali and Manish Goyal, (2008), A text book of Engineering Mathematics, Laxmi Publications.
5. Hemamalini. P.T, (2014), Engineering Mathematics, McGraw Hill Education (India) Private Limited, New Delhi.
6. V. Krishnamurthy, V. P. Mainra and J. L. Arora,(2005), An introduction to Linear Algebra, Affiliated East-West press.

KARPAGAM ACADEMY OF HIGHER EDUCATION
COIMBATORE-21.
FACULTY OF ENGINEERING
DEPARTMENT OF SCIENCE AND HUMANITIES

1st year-B.EElectrical and Electronics Engineering

LECTURE PLAN

Subject: Mathematics – II(Linear Algebra, Transform Calculus and Numerical Method)
Code : 19BEEE201

Unit No.	List of Topics	No. of Hours
UNIT – I	**Matrices**	
	Inverse and rank of a matrix	1
	Rank-nullity theorem	1
	System of linear equations	1
	Symmetric, skew-symmetric and orthogonal matrices	1
	Determinants	1
	Eigenvalues and eigenvectors	1
	Tutorial. 1: Symmetric, skew-symmetric and orthogonal matrices, Eigenvalues and eigenvectors	1
	Diagonalization of matrices	1
	Diagonalization of matrices	1
	Cayley-Hamilton Theorem	1
	Orthogonal transformation.	1
	Orthogonal transformation.	1
	Simple Problems using Scilab	1
	Tutorial 2: Problems based on Diagonalization, Cayley-Hamilton Theorem, Orthogonal transformation	1
	TOTAL	**14**
UNIT – II	**Numerical Methods**	
	Solution of polynomial and transcendental equations	1
	Introduction and Problems for Bisection method	1
	Introduction and Problems for Newton-Raphson method	1
	Introduction and Problems for Regula-Falsi method	1
	Introduction and Problems for Finite differences	1
	Interpolation using Newton’s forward difference formulae	1
	Tutorial 3: Problems based on types of numerical methods	1
	Interpolation using Newton’s backward difference formulae	1
	Introduction and Problems for Central difference interpolation	1
	Introduction for Gauss’s forward and backward formulae	1
	Problems based on Gauss’s forward and backward formulae	1
	Numerical integration: Trapezoidal rule	1
	Numerical integration: Simpson’s 1/3rd and 3/8 rules.	1
	Tutorial 4: Problems based on types of numerical methods	1
	TOTAL	**14**
UNIT – III	**Numerical Methods**	
	Introduction to using numerical methods in ordinary differential equations	1
	Introduction to Taylor’s series	1
	Problems based on Taylor’s series	1
	Problems based on Taylor’s series	1
	Introduction to Euler and modified Euler’s methods.	1
	Problems based on Euler and modified Euler’s methods.	1
	Problems based on Euler and modified Euler’s methods.	1
	Tutorial 5: Problems based on types numerical methods in ODE’s	1
	Introduction to Runge-Kutta method	1

	Problems based on RungeKutta method	1
	Introduction to Milne's And Adam's predicator-corrector methods	1
	Problems based on Milne's And Adam's predicator-corrector methods	1
	Problems based on Milne's And Adam's predicator-corrector methods	1
	Tutorial 6: Problems based on types numerical methods ODE's	1
	TOTAL	**14**
UNIT – IV	**Numerical Methods**	
	Introduction to using numerical methods in partial differential equations	1
	Introduction to Finite difference scheme	1
	Solution of 2-D Laplace equation using Finite difference scheme	1
	Solution of 2-D Laplace equation using Finite difference scheme	1
	Solution of 2-D Poisson equation using Finite difference scheme	1
	Implicit and explicit methods for one Dimensional heat equation: Bender-Schmidt method and Crank-Nicholson methods	1
	Introduction to Bender-Schmidt method.	1
	Problems based on Bender-Schmidt method	1
	Tutorial 7: Problems based on numerical methods in PDE's	1
	Introduction to Crank-Nicholson methods	1
	Problems based on Crank-Nicholson methods	1
	Introduction to Finite difference Explicit method for wave equation.	1
	Problems based on Finite difference Explicit method for wave equation.	1
	Tutorial 8: Problems based on numerical methods in PDE's	1
	TOTAL	**14**
UNIT – V	**Transform Calculus**	
	Introduction of Laplace Transform	1
	Properties of Laplace Transform	1
	Laplace transform of periodic functions	1
	Laplace transform of periodic functions	1
	Finding inverse Laplace transform by different methods	1
	Finding inverse Laplace transform by different methods	1
	convolution theorem	1
	Tutorial 9: Problems based on Laplace Transform and convolution theorem	1
	Evaluation of Integrals by Laplace transform	1
	Evaluation of Integrals by Laplace transform	1
	Solving ODEs and PDEs by Laplace Transform method.	1
	Solving ODEs and PDEs by Laplace Transform method.	1
	Tutorial 10: Problems based on Solving ODEs and PDEs by Laplace Transform method.	1
	Discussion of previous years ESE Questions	1
	TOTAL	**14**
	TOTAL NO. OF HOURS	**70**

FACULTY IN-CHARGE **HOD**

UNIT - I

MATRICES

Characteristic Polynomial :

The determinant $|A-\lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

Characteristic Equation :

Let A be any square matrix of order n. The characteristic equation of A is $|A-\lambda I|=0$.

Eigen values :

Let A be a square matrix, the characteristic equation of A is $|A-\lambda I|=0$. The roots of the characteristic equation are called Eigen values of A.

Eigen vector :

Let A be a square matrix. If there exists a non-zero vector X such that $AX = \lambda X$, then the vector X is called an Eigen vector of A corresponding to the Eigen value λ.

Note :

1) The characteristic equation of 2×2 matrix is

$$\lambda^2 - S_1\lambda + S_2 = 0$$

S_1 = Sum of main diagonal elements

$S_2 = |A|$

2) The characteristic equation of 3×3 matrix is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = Sum of main diagonal elements

S_2 = Sum of the minors of main diagonal elements

$S_3 = |A|$

Problems :

1) Find the Eigen values and Eigen vectors of the matrix $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Soln:

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 1 - 1 = 0$

$S_2 = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$

$\therefore \lambda^2 - 0\lambda - 4 = 0$

$\lambda^2 - 4 = 0$

$\lambda^2 = 4$

$\lambda = \pm 2$

The eigen values are $-2, 2$

To find: Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x_1 + x_2 &= 0 \\ 3x_1 + (-1-\lambda)x_2 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i): $\lambda = -2$

$3x_1 + x_2 = 0 \longrightarrow (2)$

$3x_1 + x_2 = 0 \longrightarrow (3)$

Solve (2) & (3)

$3x_1 + x_2 = 0$

$3x_1 = -x_2$

$\frac{x_1}{1} = \frac{x_2}{-3}$

$\therefore$ The eigen vector is $X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Case (ii): $\lambda = 2$

$-x_1 + x_2 = 0 \longrightarrow (4)$

$3x_1 - 3x_2 = 0 \longrightarrow (5)$

Solve (4) & (5),

$-x_1 + x_2 = 0$

$x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

$\therefore$ The eigen vector is $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2) Find the Eigen values and Eigen vectors of (3)

$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Soln :

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 1+2+3 = 6$

$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$

$= (6-2) + (3+2) + (2-0)$

$= 4+5+2 = 11$

$S_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

$= 1(6-2) - 0 - 1(2-4)$

$= 4+2 = 6$.

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

To find : Eigen values

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ \hline 6 & -5 \\ -3 & -2 \end{array}$$

$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda = 1, 2, 3$

$\therefore$ The Eigen values are 1, 2, 3

To find : Eigen vector

$(A - \lambda I)X = 0$

$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

A

$$\left.\begin{aligned}(1-\lambda)x_1+0x_2-x_3&=0\\ x_1+(2-\lambda)x_2+x_3&=0\\ 2x_1+2x_2+(3-\lambda)x_3&=0\end{aligned}\right\}\longrightarrow(1)$$

Case (i): $\lambda = 1$

$$0x_1+0x_2-x_3=0 \longrightarrow (2)$$

$$x_1+x_2+x_3=0 \longrightarrow (3)$$

$$2x_1+2x_2+2x_3=0 \longrightarrow (4).$$

Solve (2) & (3),

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x_1}{0+1}=\frac{x_2}{-1+0}=\frac{x_3}{0-0}$$

$$\frac{x_1}{1}=\frac{x_2}{-1}=\frac{x_3}{0}$$

$$\therefore X_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

Case (ii): $\lambda = 2$

$$-x_1+0x_2-x_3=0 \longrightarrow (5)$$

$$x_1+0\cdot x_2+x_3=0 \longrightarrow (6)$$

$$2x_1+2x_2+x_3=0 \longrightarrow (7)$$

Solve (6) & (7),

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{array}$$

$$\frac{x_1}{0-2}=\frac{x_2}{2-1}=\frac{x_3}{2-0}$$

$$\frac{x_1}{-2}=\frac{x_2}{1}=\frac{x_3}{2}$$

$$\therefore X_2=\begin{bmatrix}-2\\1\\2\end{bmatrix}$$

⑤

Case (iii) : $\lambda = 3$

$-2x_1 + 0x_2 - x_3 = 0 \longrightarrow (8)$

$x_1 + x_2 + x_3 = 0 \longrightarrow (9)$

$2x_1 + 2x_2 + 0 \cdot x_3 = 0 \longrightarrow (10)$

Solve (9) & (10),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -1 & & 1 & & 1 & & -1 \\ 2 & & 0 & & 2 & & 2 \end{array}$$

$$\frac{x_1}{0-2} = \frac{x_2}{2-0} = \frac{x_3}{2+2}$$

$$\frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\therefore X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ The eigen ~~values~~ vectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} +2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

3) Find all the Eigen values and Eigen vectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Soln :

Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = -2 + 1 + 0 = -1$

$$s_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (0-12) + (0-3) + (-2-4)$$

$$= -12 - 3 - 6 = -21$$

$$s_3 = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$(1-\lambda)x_1 + 0x_2 - x_3 = 0$

$-2(0-12) - 2(0-6) - 3(-4+1)$

(6) $= -2(0-12) - 2(0-6) - 3(-3)$

$= -2(-12) - 2(-6) - 3(-3)$

$= 24 + 12 + 9 = 45$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ \hline -15 & -2 \\ -5 & 3 \end{array}$$

$(\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$

$(\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\lambda = -3, -3, 5$

$\therefore$ The Eigen values are $-3, -3, 5$.

To find : Eigen vector

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} (-2-\lambda)x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + (1-\lambda)x_2 - 6x_3 = 0 \\ -x_1 - 2x_2 + (-\lambda)x_3 = 0 \end{array}\right\} \longrightarrow (1)$$

Case (i): $\lambda = 5$

$-7x_1 + 2x_2 - 3x_3 = 0 \longrightarrow (2)$

$2x_1 - 4x_2 - 6x_3 = 0 \longrightarrow (3)$

$-x_1 - 2x_2 - 7x_3 = 0 \longrightarrow (4)$

Solve (2) & (3),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & -3 & & -7 & & 2 \\ -4 & & -6 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

(7)

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Case (ii) : $\lambda = -3$

$x_1 + 2x_2 - 3x_3 = 0 \longrightarrow (5)$

$2x_1 + 4x_2 - 6x_3 = 0 \longrightarrow (6)$

$-x_1 - 2x_2 + 3x_3 = 0 \longrightarrow (7)$

Equations are same

$(5) \Rightarrow x_1 + 2x_2 - 3x_3 = 0$

Put $x_1 = 0$.

$(5) \Rightarrow 2x_2 - 3x_3 = 0$

$2x_2 = 3x_3$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Case (iii) : $\lambda = -3$

doubt $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$x_1 + 2x_2 - x_3 = 0 \longrightarrow (8)$

$0x_1 + 3x_2 + 2x_3 = 0 \longrightarrow (9)$.

Solve (8) & (9)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & & -1 & & 1 & & 2 \\ 3 & & 2 & & 0 & & 3 \end{array}$$

$$\frac{x_1}{4+3} = \frac{x_2}{0-2} = \frac{x_3}{3-0} \qquad \therefore X_3 = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$$

$\therefore$ The Eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$

4) Find the Eigen values and Eigen vectors of

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Soln:

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda$

$s_1 = 2+2+1 = 5$

$$s_2 = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$= (2-0) + (2-0) + (4-1)$

$= 2+2+3 = 7.$

$$s_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 2(2-0) - 1(1-0) + 1(0-0)$

$= 4 - 1 = 3.$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

$\therefore \lambda = 3, 1, 1.$

To find : Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 + x_3 &= 0 \\ x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = 3$

$-x_1 + x_2 + x_3 = 0 \longrightarrow (2)$

$x_1 - x_2 + x_3 = 0 \longrightarrow (3)$

$0x_1 + 0x_2 - 2x_3 = 0 \longrightarrow (4)$

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Solve (2) & (3)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 1 & & -1 & 1 \\ -1 & & 1 & & 1 & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{1+1} = \frac{x_3}{1-1}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case (ii) : $\lambda = 1$

$$x_1 + x_2 + x_3 = 0 \longrightarrow (5)$$

$$x_1 + x_2 + x_3 = 0 \longrightarrow (6)$$

$$0x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \longrightarrow (7)$$

Put $x_1 = 0$ in (5)

$$0 + x_2 + x_3 = 0$$

$$x_2 = -x_3$$

$$\frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) : $\lambda = 1$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$x_1 + x_2 + 0x_3 = 0 \longrightarrow (8)$$

$$0x_1 - x_2 + x_3 = 0 \longrightarrow (9)$$

Solve (8) & (9),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 0 & & 1 & 1 \\ -1 & & 1 & & 0 & -1 \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-1-1 \\ 1-2-1 \\ 0+0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$\therefore X_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$\therefore$ The Eigen vectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

5) Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Soln :

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 1 + 5 + 1 = 7$

$S_2 = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$

$= (5-1) + (1-9) + (5-1)$

$= 4 - 8 + 4 = 0$.

$S_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$

$= 1(5-1) - 1(1-3) + 3(1-15)$

$= 1(4) - 1(-2) + 3(-14)$

$= 4 + 2 - 42 = -36$

$\therefore \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ \hline 18 & -9 \\ -6 & -3 \end{array}$$

$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$

$\lambda = -2, \lambda = 3, 6$.

$\therefore \lambda = -2, 3, 6$.

To find : Eigen vectors

$(A - \lambda I)X = 0$

(ii)

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x_1 + x_2 + 3x_3 &= 0 \\ x_1 + (5-\lambda)x_2 + x_3 &= 0 \\ 3x_1 + x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = -2$

$3x_1 + x_2 + 3x_3 = 0 \longrightarrow (2)$

$x_1 + 7x_2 + x_3 = 0 \longrightarrow (3)$

$3x_1 + x_2 + 3x_3 = 0 \longrightarrow (4)$

Solve (2) & (3),

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) : $\lambda = 3$

$-2x_1 + x_2 + 3x_3 = 0 \longrightarrow (5)$

$x_1 + 2x_2 + x_3 = 0 \longrightarrow (6)$

$3x_1 + x_2 - 2x_3 = 0 \longrightarrow (7).$

Solve (5) & (6),

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) : $\lambda = 6$

$-5x_1 + x_2 + 3x_3 = 0 \longrightarrow (8)$

$x_1 - x_2 + x_3 = 0 \longrightarrow (9)$

$3x_1 + x_2 - 5x_3 = 0 \longrightarrow (10)$.

Solve (8) & (9),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & -5 & & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ The Eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Orthogonal matrix :

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$ since $A^{-1}A = AA^{-1} = I$, it follows that a matrix A is orthogonal if $A^T = A^{-1}$.

1) show that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Soln :

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal.

Diagonalization of the matrix :

Working Rule :

Step : 1

To find characteristic equation

Step : 2

To find Eigen values

Step : 3

To find Eigen vectors

Step : 4

Check whether the Eigen vectors are orthogonal.

Step : 5

To form normalized matrix N

Step : 6

To calculate N^T

Step : 7

Calculate $D = N^T A N$

[Diagonal elements and Eigen values are same].

Problems :

1) Diagonalize the matrix $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of orthogonal transformation.

Soln :

(14) Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3$

$s_1 = 2+1+1 = 4$

$s_2 = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$= (1-4) + (2-1) + (2-1)$

$= -3 + 1 + 1 = -1.$

$s_3 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$

$= 2(1-4) - 1(1-2) - 1(-2+1)$

$= 2(-3) - 1(-1) - 1(-1)$

$= -6 + 1 + 1 = -4$

$\therefore \lambda^3 - 4\lambda^2 - \lambda + 4 = 0.$

$\lambda = 4, 1, -1.$

To find : Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 - x_3 &= 0 \\ x_1 + (1-\lambda)x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \rightarrow (1)$$

Case (i) : $\lambda = 4$

$-2x_1 + x_2 - x_3 = 0 \rightarrow (2)$

$x_1 - 3x_2 - 2x_3 = 0 \rightarrow (3)$

$-x_1 - 2x_2 - 3x_3 = 0 \rightarrow (4)$

Solve (2) & (3),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & -1 & & -2 & & 1 \\ -3 & & -2 & & 1 & & -3 \end{array}$$

(15)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Case (ii): $[\lambda = 1]$

$$+2x_1 + x_2 - x_3 = 0 \longrightarrow (5)$$

$$x_1 + 0x_2 - 2x_3 = 0 \longrightarrow (6)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \longrightarrow (7)$$

Solve (5) & (6)

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & 1 & 1 \\ 0 & -2 & 1 & 0 \end{array}$$

$$\frac{x_1}{-2-0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

Case (iii): $\lambda = -1$

$$3x_1 + x_2 - x_3 = 0 \longrightarrow (8)$$

$$x_1 + 2x_2 - 2x_3 = 0 \longrightarrow (9)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \longrightarrow (10)$$

Solve (8) & (9),

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 2 \end{array}$$

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

To find : orthogonal

$$X_1^T X_2 = (-1 \ -1 \ \ 1)\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

$$= 2 - 1 - 1 = 0$$

$$X_2^T X_3 = (-2 \ \ 1 \ -1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= 0 + 1 - 1 = 0$$

$$X_3^T X_1 = (0 \ \ 1 \ \ 1)\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0.$$

To form Normalized matrix, $N = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$N^T = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2) Diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by means of orthogonal transformation. (17)

Soln :

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 3+3+3 = 9$

$s_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$

$= (9-1) + (9-1) + (9-1)$

$= 8 + 8 + 8 = 24$

$s_3 = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(9-1) - (3+1) + 1(-1-3)$

$= 24 - 4 - 4 = 16$

$\therefore \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$.

$\lambda = 1, 4, 4$.

To find : Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (3-\lambda)x_1 + x_2 + x_3 &= 0 \\ x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = 1$

$2x_1 + x_2 + x_3 = 0 \longrightarrow (2)$

$x_1 + 2x_2 - x_3 = 0 \longrightarrow (3)$

$x_1 - x_2 + 2x_3 = 0 \longrightarrow (4)$

Solve (2) & (3)

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) : $\lambda = 4$

$$-x_1 + x_2 + x_3 = 0 \longrightarrow (5)$$

$$x_1 - x_2 - x_3 = 0 \longrightarrow (6)$$

$$x_1 - x_2 - x_3 = 0 \longrightarrow (7)$$

Put $x_1 = 0$ in (6)

$$0 - x_2 - x_3 = 0$$

$$-x_2 = x_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Case (iii) : $\lambda = 4$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$

$$0x_1 + x_2 - x_3 = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0+1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\therefore X_3 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

To find : Orthogonal

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad X_3 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

$$X_1^T X_2 = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0+1-1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = 0-1+1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} -2 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 2-1-1 = 0$$

$\therefore$ Eigen vectors are orthogonal.

To form : Normalized vector

$$N = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Quadratic form

A homogeneous polynomial of second degree in any number of variables is called quadratic form.

Problems :

1) Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$.

Soln :

$$A = \begin{array}{c|c} & \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix} \end{array}$$

2) Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$.

Soln :

$$A = \begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} x \\ y \\ z \end{matrix} & \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix} \end{array}$$

3) Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

Soln :

$$A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

$$Q = X^T A X$$

$$= (x_1 \ x_2 \ x_3) \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 0x_1^2 + x_2^2 + 2x_3^2 + 10x_1x_2 + 12x_2x_3 - 2x_1x_3$$

4) Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Soln:

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

$$Q = X^T A X$$

$$= [x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + x_2^2 + 0x_3^2 + 0x_1x_2 + 2x_1x_3 - 2x_2x_3$$

Nature of the Quadratic form:

Let $D_1 = a_{11}$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Note:

1) Index → No. of +ve terms
2) Signature → No. of +ve terms − No. of −ve terms
3) Rank → No. of non-zero diagonal elements

S.No	Nature	Condition
1)	Positive definite	$D_n > 0$ (+ve) (or) All the eigen values are +ve
2)	Negative definite	$D_n < 0$ (−ve) (or) All the eigen values are −ve
3)	Positive semi-definite	$D_n > 0$ & Atleast one value is zero. (or)

(20)

(23)

		All the eigen values ≥ 0 & atleast one value is zero
4)	Negative semi-definite	$D_n < 0$ & atleast one value is zero. (or) All the eigen values ≤ 0 & atleast one value is zero.
5)	Indefinite	All other cases.

Problems:

1) Prove that the quadratic form $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3$ is indefinite.

Soln:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$D_1 = 1$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 1(6-1) - 1(3+1) - 1(1+2)$$

$$= 1(5) - 1(4) - 1(3)$$

$$= 5 - 4 - 3 = -2$$

$\therefore$ The nature is indefinite.

2) Discuss the nature of the quadratic form $2x_1x_2 + 2x_2x_3 - 2x_1x_3$ without reducing to the canonical form.

Soln:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$D_1 = 0$

$$D_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$D_3 = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= 0 - 1(0+1) - 1(1-0)$$

$$= -1 - 1 = -2.$$

$\therefore$ The nature is negative semi-definite.

3) Find the index, signature and the nature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$.

Soln:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$D_1 = 1$

$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$$

$$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= 1(-6-0) + 0 + 0 = -6.$$

$\therefore$ The nature is indefinite.

Index $= 2$

Signature $= 2 - 1 = 1$

Reduction of a Quadratic form to canonical form

Working Rule:

1) Construct the quadratic form to matrix form 'A'.
2) Diagonalize the matrix A
3) Canonical form $= Y^T D Y$

Problems:

1) Reduce the Quadratic form

$10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_1x_3 - 4x_1x_2$

to a canonical form through an orthogonal trans

-formation. Also find index, signature, Rank and nature.

Soln :

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 10 + 2 + 5 = 17$

$$S_2 = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix}$$

$= (10-9) + (50-25) + (20-4)$

$= 1 + 25 + 16 = 42$

$$S_3 = \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{vmatrix}$$

$= 10(10-9) + 2(-10+15) - 5(-6+10)$

$= 10(1) + 2(5) - 5(4)$

$= 10 + 10 - 20 = 0.$

$\therefore \lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\lambda(\lambda^2 - 17\lambda + 42) = 0$

$\lambda = 0, 14, 3.$

To find : Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (10-\lambda)x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 + (2-\lambda)x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 + (5-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = 0$

$$10x_1 - 2x_2 - 5x_3 = 0 \longrightarrow (2)$$
$$-2x_1 + 2x_2 + 3x_3 = 0 \longrightarrow (3)$$
$$-5x_1 + 3x_2 + 5x_3 = 0 \longrightarrow (4)$$

Solve (2) & (3)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -2 & & -5 & & 10 & & -2 \\ 2 & & 3 & & -2 & & 2 \end{array}$$

$$\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Case (ii) : $\lambda = 14$

$$-4x_1 - 2x_2 - 5x_3 = 0 \longrightarrow (5)$$
$$-2x_1 - 12x_2 + 3x_3 = 0 \longrightarrow (6)$$
$$-5x_1 + 3x_2 - 9x_3 = 0 \longrightarrow (7)$$

Solve (5) & (6)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -2 & & -5 & & -4 & & -2 \\ -12 & & 3 & & -2 & & -12 \end{array}$$

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$\frac{x_1}{-6} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{2}$$

(31)

$$\therefore X_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Case (iii) : $\lambda = 3$

$$7x_1 - 2x_2 - 5x_3 = 0 \longrightarrow (8)$$
$$-2x_1 - x_2 + 3x_3 = 0 \longrightarrow (9)$$
$$-5x_1 + 3x_2 + 2x_3 = 0 \longrightarrow (10)$$

Solve (8) & (9)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & & -5 & & 7 & & -2 \\ -1 & & 3 & & -2 & & -1 \end{array}$$

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\therefore X_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

To verify : The Eigen vectors are orthogonal.

$$X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} \quad X_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$X_1^T X_2 = (1 \ -5 \ 4)\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3-5+8 = 0$$

$$X_2^T X_3 = (-3 \ 1 \ 2)\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = 3-1-2 = 0$$

$$X_3^T X_1 = (-1 \ -1 \ -1)\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = -1+5-4 = 0$$

$\therefore$ Eigen vectors are orthogonal.

To form : Normalized vector

$$N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \\ \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

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$$N^T = \begin{bmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{1}{\sqrt{42}} & -\frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{3}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & -\frac{1}{\sqrt{3}} \\ \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix})$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To form : Canonical form.

Canonical form $= Y^T D Y$

$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 0y_1^2 + 14y_2^2 + 3y_3^2$$

Index $= 2$

Signature $= 2 - 0 = 2$

Rank $= 2$.

$\therefore$ The nature is positive semi-definite.

2) Reduce the Quadratic form $2x^2+5y^2+3z^2+4xy$ to canonical form by orthogonal reduction and state its nature.

Soln :

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - S_3 = 0$

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$S_1 = 2+5+3 = 10$

$S_2 = \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix}$

$= (15-0) + (6-0) + (10-4)$

$= 15+6+6 = 27$

$S_3 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

$= 2(15-0) - 2(6-0) + 0$

$= 30-12 = 18.$

$\therefore \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

$\lambda = 1, 3, 6.$

To find : Eigen vector

$(A-\lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 + 0\cdot x_3 &= 0 \\ 2x_1 + (5-\lambda)x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i): $\lambda = 3$

$-x_1 + 2x_2 + 0x_3 = 0 \longrightarrow (2)$

$2x_1 + 2x_2 + 0x_3 = 0 \longrightarrow (3)$

$0x_1 + 0x_2 + 0x_3 = 0 \longrightarrow (4)$

Solve (2) & (3)

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & 0 & -1 & 2 \\ 2 & 0 & 2 & 2 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0+0} = \frac{x_3}{-2-4}$$

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$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-6}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) : $\lambda = 6$

$$-4x_1 + 2x_2 + 0 \cdot x_3 = 0 \longrightarrow (5)$$

$$2x_1 - x_2 + 0x_3 = 0 \longrightarrow (6)$$

$$0x_1 + 0x_2 - 3x_3 = 0 \longrightarrow (7)$$

Solve (6) & (7)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -1 & & 0 & & 2 & & -1 \\ 0 & & -3 & & 0 & & 0 \end{array}$$

$$\frac{x_1}{3-0} = \frac{x_2}{0+6} = \frac{x_3}{0+0}$$

$$\frac{x_1}{3} = \frac{x_2}{6} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Case (iii) : $\lambda = 1$

$$x_1 + 2x_2 + 0x_3 = 0 \longrightarrow (8)$$

$$2x_1 + 4x_2 + 0x_3 = 0 \longrightarrow (9)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \longrightarrow (10)$$

Solve (8) & (10)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & 0 & & 1 & & 2 \\ 0 & & 2 & & 0 & & 0 \end{array}$$

$$\frac{x_1}{4-0} = \frac{x_2}{0-2} = \frac{x_3}{0-0}$$

$$\frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{0}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$\therefore X_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

To find : The vectors are orthogonal.

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = X_2 \quad X_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$X_1^T X_2 = (0 \quad 0 \quad -1) \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0+0+0 = 0$$

$$X_2^T X_3 = (1 \quad 2 \quad 0) \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2-2+0 = 0$$

$$X_3^T X_1 = (2 \quad -1 \quad 0) \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0+0+0 = 0$$

$\therefore$ The Eigen vectors are orthogonal.

To form : Normalized matrix

$$N = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ -1 & 0 & 0 \end{bmatrix}$$

$$N^T = \begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(31)

To find : Canonical form

Canonical form = $Y^T D Y$

$$= [y_1\ y_2\ y_3]\begin{bmatrix}3 & 0 & 0\\0 & 6 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

$$= 3y_1^2 + 6y_2^2 + y_3^2$$

Index = 3

Signature = 3

Rank = 3.

$\therefore$ The Nature is positive definite.

3) Reduce the quadratic form $6x^2+3y^2+3z^2-4xy-2yz+4xz$ into a canonical form by an orthogonal reduction. Hence find its rank and nature.

Soln :

$$A = \begin{bmatrix}6 & -2 & 2\\-2 & 3 & -1\\2 & -1 & 3\end{bmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 6+3+3 = 12$

$$s_2 = \begin{vmatrix}3 & -1\\-1 & 3\end{vmatrix} + \begin{vmatrix}6 & 2\\2 & 3\end{vmatrix} + \begin{vmatrix}6 & -2\\-2 & 3\end{vmatrix}$$

$$= (9-1)+(18-4)+(18-4)$$

$$= 8+14+14 = 36$$

$$s_3 = \begin{vmatrix}6 & -2 & 2\\-2 & 3 & -1\\2 & -1 & 3\end{vmatrix}$$

$$= 6(9-1)+2(-6+2)+2(2-6)$$

$$= 6(8)+2(-4)+2(-4)$$

$$= 48-8-8 = 32.$$

(32) $\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 8, 2, 2.$

To find : Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2y_2 + 2z_3 &= 0 \\ -2x_1 + (3-\lambda)y_2 - z_3 &= 0 \\ 2x_1 - y_2 + (3-\lambda)z_3 &= 0 \end{aligned}\right\} \longrightarrow (1).$$

Case (i) : $\lambda = 8$

$$-2x_1 - 2y_2 + 2z_3 = 0 \longrightarrow (2)$$
$$-2x_1 - 5y_2 - z_3 = 0 \longrightarrow (3)$$
$$2x_1 - y_2 - 5z_3 = 0 \longrightarrow (4)$$

Solve (2) & (3)

$$\begin{array}{cccccc} & x_1 & & y_2 & & z_3 \\ -2 & & 2 & & -2 & & -2 \\ -5 & & -1 & & -2 & & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{y_2}{-4-2} = \frac{z_3}{10-4}$$

$$\frac{x_1}{12} = \frac{y_2}{-6} = \frac{z_3}{6}$$

$$\frac{x_1}{2} = \frac{y_2}{-1} = \frac{z_3}{1}$$

$$\therefore X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii) : $\lambda = 2$

$$4x_1 - 2y_2 + 2z_3 = 0 \longrightarrow (5)$$
$$-2x_1 + y_2 - z_3 = 0 \longrightarrow (6)$$
$$2x_1 - y_2 + z_3 = 0 \longrightarrow (7)$$

Put $x = 0$ in (7) (33)

$$0 - y + z = 0$$
$$-y = -z$$
$$\frac{y}{-1} = \frac{z}{-1}$$
$$Y = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

Case (iii) : $\lambda = 2$

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$2x - y + z = 0 \rightarrow (8)$$
$$0x - y - z = 0 \rightarrow (9)$$

Solve (8) & (9)

$$\begin{array}{cccc} & x & y & z \\ -1 & 1 & 2 & -1 \\ -1 & -1 & 0 & -1 \end{array}$$

$$\frac{x}{1+1} = \frac{y}{0+2} = \frac{z}{-2+0}$$
$$\frac{x}{2} = \frac{y}{2} = \frac{z}{-2}$$
$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$
$$\therefore Z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

To verify : Eigen vectors are orthogonal.

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \quad Z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$X^T Y = (2 \ -1 \ 1) \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = 0 + 1 - 1 = 0$$

$$Y^T Z = (0 \ -1 \ -1) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 - 1 + 1 = 0$$

(3)

$$Z^T X = (1 \quad 1 \quad -1)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2-1-1 = 0.$$

$\therefore$ Eigen vectors are orthogonal.

To find : Normalized matrix.

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To find : Canonical form.

Canonical form $= Y^T D Y$

$$= [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 3y_3^2$$

Rank = 3

Nature is positive definite.

of determinant of A.

Properties of Eigen values :

i) Sum of the eigen values = Sum of the diagonal elements = Trace

ii) Product of the eigen values = $|A|$

iii) The eigen values of diagonal matrix (or) upper triangular matrix (or) lower triangular matrix are the diagonal elements.

iv) A and A^T have the same eigen values.

Proof :

Let λ be an eigen value of A then

$$|A - \lambda I| = 0$$

$$(A - \lambda I)^T = A^T - (\lambda I)^T$$
$$= A^T - \lambda I^T$$
$$= A^T - \lambda I$$
$$|A - \lambda I|^T = |A^T - \lambda I|$$
$$|A^T - \lambda I| = 0$$

$\therefore \lambda$ is an eigen value of A^T.

v) If λ is an eigen value of A, then $k\lambda$ is an eigen value of kA.

Proof :

Let λ be an eigen value of A then

$$AX = \lambda X$$
$$k(AX) = k(\lambda X)$$
$$(kA)X = (k\lambda)X$$

$\therefore k\lambda$ is an eigen value of kA.

vi) If λ is an eigen value of A, then λ^k is an eigen value of A^k.

Proof :

Let λ be an eigen value of A then

$$AX = \lambda X$$
$$A(AX) = A(\lambda X)$$
$$A^2 X = (A\lambda) X$$
$$= (\lambda A) X$$
$$A^2 X = \lambda (AX)$$
$$= \lambda(\lambda X)$$
$$A^2 X = \lambda^2 X$$

Similarly, λ^k is an eigen value of A^k.

vii) If λ is an eigen value of A then $\frac{1}{\lambda}$ is an eigen value of A^{-1} provided A is an non-singular

Proof:

Let λ be an eigen value of A.

$$AX = \lambda X$$
$$A^{-1}(AX) = A^{-1}(\lambda X)$$
$$A^{-1}AX = A^{-1}\lambda X$$
$$IX = \lambda A^{-1} X$$
$$X = \lambda A^{-1} X$$
$$\frac{1}{\lambda} X = A^{-1} X.$$

$\therefore \frac{1}{\lambda}$ is an eigen value of A^{-1}.

Note:

$\lambda \rightarrow$ Eigen value of A

$\frac{1}{\lambda} \rightarrow$ Eigen value of A^{-1}

$\frac{|A|}{\lambda} \rightarrow$ Eigen value of adj A.

Problems:

1) If the sum of the eigen values and trace of a 3×3 matrix A are equal then find the value

of determinant of A.

Soln.

Given A is a 3×3 matrix.

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen values.

WKT, Sum of the eigen values = Trace

$$\lambda_1+\lambda_2+\lambda_3 = \text{Trace}$$

$$\text{Trace} + \lambda_3 = \text{Trace}$$

$$\lambda_3 = 0$$

$|A|$ = Product of eigen values $= \lambda_1\lambda_2\lambda_3$

$|A| = 0 \quad [\because \lambda_3 = 0]$

2) Find the sum and product of the eigen values of the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$

Soln:

Sum of the eigen values = Sum of the diagonal elements

$$= 2+3-6 = -1$$

Product of the eigen value $= |A|$

$$= 2(-18-1) - 1(-6-2) + 2(1-6)$$

$$= 2(-19) - 1(-8) + 2(-5)$$

$$= -38+8-10 = -40.$$

3) Find the eigen value of $-6A$, A^3 and A^{-1} where $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Soln:

Given, A is an upper triangular matrix.

The eigen values of A is $3, 2, 5$

The eigen values of $-6A$ is $-18, -12, -30$

The eigen values of A^3 is $27, 8, 125$

The eigen values of A^{-1} is $\frac{1}{\lambda} \Rightarrow \frac{1}{3}, \frac{1}{2}, \frac{1}{5}$

Q (38) 4) If 2, -1, -3 are the eigen values of the matrix A then find the eigen values of $A^2 - 2I$.

Soln:

The eigen values of A is 2, -1, -3

The eigen values of A^2 is 4, 1, 9

The eigen values of I is 1, 1, 1.

The eigen values of 2I is 2, 2, 2

The eigen values of $A^2 - 2I$ is 2, -1, 7.

5) If the eigen values of matrix A of order 3x3 are 2, 3, 1 then the eigen values of adj A.

Soln:

The eigen values of A are 2, 3, 1.

$|A|$ = Product of eigen values = 6

The eigen vector of adj A $= \frac{|A|}{\lambda}$

$= \frac{6}{2}, \frac{6}{3}, \frac{6}{1}$

$= 3, 2, 6.$

6) If 3 and 6 are two eigen values of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ write down all the eigen values of A in rows.

Soln:

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen values of A.

Given, $\lambda_1 = 3$, $\lambda_2 = 6$.

WKT, Sum of eigen values = Sum of the diagonal elements

$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1$

$3 + 6 + \lambda_3 = 7$

$\lambda_3 = -2$

The eigen values of A is 3, 6, -2

The eigen values of A^{-1} is $\frac{1}{3}, \frac{1}{6}, -\frac{1}{2}$.

7) The product of two eigen values of matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. find the 3^{rd} eigen value.

Soln :

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen value of A.

Given, $\lambda_1 \lambda_2 = 16$.

WKT, Product of eigen value $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$16\lambda_3 = 6(8) + 2(-4) + 2(-4)$$

$$16\lambda_3 = 48 - 8 - 8$$

$$16\lambda_3 = 32$$

$$\lambda_3 = 2.$$

8) One of the eigen value of $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ is -9 find the other two eigen values.

Soln :

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen values of A.

Given, $\lambda_1 = -9$

WKT, Sum of the eigen values = Sum of the diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 7 - 8 - 8$$

$$-9 + \lambda_2 + \lambda_3 = -9$$

$$\lambda_2 + \lambda_3 = 0$$

$$\lambda_3 = -\lambda_2 \rightarrow (1)$$

WKT, Product of the eigen values $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = 7(64-1) - 4(-32+4) - 4(-4+32)$$

$$-9\lambda_2(-\lambda_2) = 7(63) - 4(-28) - 4(28)$$

$$9\lambda_2^2 = 441$$

$$\lambda_2^2 = 49$$

$$\lambda_2 = \pm 7$$

(10)

$\therefore \lambda_3 = \pm 7$

$\therefore$ The eigen values of A are $-9, \pm 7, \pm 7$.

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

Problems:

1) Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ and hence find A^{-1} & A^4.

Soln:

Given, $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 2 + 2 + 2 = 6$

$|A| = s_3 = 2\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + 0 - 1\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}$

$= 2(4-0) + 0 - 1(0+2)$

$= 8 - 2 = 6$.

$s_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

$= (4-0) + (4-1) + (4-0)$

$= 4 + 3 + 4 = 11$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

By Cayley - Hamilton theorem,

$A^3 - 6A^2 + 11A - 6 = 0$.

$A^2 = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix}$ $\quad \therefore A^3 = \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix}$

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$A^3 - 6A^2 + 11A - 6I$

$$= \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 30 & 0 & -24 \\ 0 & 24 & 0 \\ -24 & 0 & 30 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 0 & 22 & 0 \\ -11 & 0 & 22 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 11A - 6I = 0$

To find : A^{-1}

$A^3 - 6A^2 + 11A - 6I = 0$

$\times$ by A^{-1}

$A^2 - 6A + 11I - 6A^{-1} = 0$

$6A^{-1} = A^2 - 6A + 11I$

$A^{-1} = \frac{1}{6}\left[A^2 - 6A + 11I\right]$

$$= \frac{1}{6}\left\{\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 12 & 0 & -6 \\ 0 & 12 & 0 \\ -6 & 0 & 12 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}\right\}$$

$$= \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

To find : A^4

$A^3 - 6A^2 + 11A - 6I = 0$

$\times$ by A

$A^4 - 6A^3 + 11A^2 - 6A = 0$

$A^4 = 6A^3 - 11A^2 + 6A$

$$= \begin{bmatrix} 84 & 0 & -78 \\ 0 & 48 & 0 \\ -78 & 0 & 84 \end{bmatrix} - \begin{bmatrix} 55 & 0 & -44 \\ 0 & 44 & 0 \\ -44 & 0 & 55 \end{bmatrix} + \begin{bmatrix} 12 & 0 & -6 \\ 0 & 12 & 0 \\ -6 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}$$

2) Using Cayley Hamilton theorem find A^{-1} & A^4 if $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Soln:

Given, $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 1+3+1 = 5$

$s_2 = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$

$= (3-0) + (1-0) + (3+2)$

$= 3+1+5 = 9$

$s_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$

$= 1(3-0) - 2(-1-0) - 2(2-0)$

$= 1(3) - 2(-1) - 2(2) = 3+2-4 = 1.$

$\therefore \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0.$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 9A - I = 0$

$A^2 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$ $\quad A^3 = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$

To find: A^{-1}

$A^3 - 5A^2 + 9A - I = 0$

$\times$ by A^{-1}

$A^2 - 5A + 9I - A^{-1} = 0$

$A^{-1} = A^2 - 5A + 9I$

$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & -10 \\ -5 & 15 & 0 \\ 0 & -10 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

To find : A^4

$$A^3 - 5A^2 + 9A - I = 0$$

$\times$ by A

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$= \begin{bmatrix} 65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} - \begin{bmatrix} -9 & 108 & -36 \\ -36 & 63 & 18 \\ 18 & -72 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 82 & -40 & -23 \end{bmatrix}$$

3) Using Cayley Hamilton theorem to find the value of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

i) $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

ii) $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

Soln:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 2 + 1 + 2 = 5$$

$$s_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0) + (4-1) + (2-0)$$

$$= 2 + 3 + 2 = 7$$

$$s_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1)$$

$$= 4 - 1 = 3$$

(44)

$$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0.$$

i)

$$
\begin{array}{r|l}
 & A^5 + 8A + 35I \\
\hline
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I \\
 & \underline{A^8 - 5A^7 + 7A^6 - 3A^5} \\
 & 8A^4 - 5A^3 + 8A^2 - 2A + I \\
 & \underline{8A^4 - 40A^3 + 56A^2 - 24A} \\
 & 35A^3 - 48A^2 + 22A + I \\
 & \underline{35A^3 - 175A^2 + 245A - 105I} \\
 & 127A^2 - 223A + 106I
\end{array}
$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + 8A + 35) + (127A^2 - 223A + 106I)$$

$$= 127A^2 - 223A + 106I$$

$$= 127\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 223\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 106\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 635 & 508 & 508 \\ 0 & 127 & 0 \\ 508 & 508 & 635 \end{bmatrix} - \begin{bmatrix} 446 & 223 & 223 \\ 0 & 223 & 0 \\ 223 & 223 & 446 \end{bmatrix} + \begin{bmatrix} 106 & 0 & 0 \\ 0 & 106 & 0 \\ 0 & 0 & 106 \end{bmatrix}$$

$$= \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}$$

ii)

$$
\begin{array}{r|l}
 & A^5 + A \\
\hline
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\
 & \underline{A^8 - 5A^7 + 7A^6 - 3A^5} \\
 & A^4 - 5A^3 + 8A^2 - 2A + I \\
 & \underline{A^4 - 5A^3 + 7A^2 - 3A} \\
 & A^2 + A + I
\end{array}
$$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$

$= A^2 + A + I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

4) Find A^n using cayley Hamilton theorem taking $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ hence find A^3.

Soln:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 1 + 3 = 4$

$S_2 = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

$\therefore \lambda^2 - 4\lambda - 5 = 0$

$\lambda = -1, 5$

To find : A^n

When λ^n is divided by $\lambda^2 - 4\lambda - 5$

Let the quotient be $Q(\lambda)$ & remainder be $a\lambda + b$.

$\lambda^n = (\lambda^2 - 4\lambda - 5)Q(\lambda) + (a\lambda + b)$

Put $\lambda = -1$

$(-1)^n = [(-1)^2 + 4(-1) - 5]Q(-1) + a(-1) + b$

$(-1)^n = -a + b \longrightarrow (1)$

4(b)

Put $\lambda = 5$

$$5^n = [(5)^2 - 4(5) - 5]\, Q(5) + a(5) + b$$

$$5^n = 5a + b \rightarrow (2)$$

solve (1) & (2),

$$(-1)^n = -a + b$$

$$\underline{\overset{(-)}{}5^n = \overset{(-)}{}5a \overset{(-)}{+} b}$$

$$(-1)^n - 5^n = -6a$$

$$a = \frac{(-1)^n - 5^n}{-6}$$

Sub in (1),

$$(-1)^n = \frac{(-1)^n - 5^n}{6} + b$$

$$b = (-1)^n - \frac{(-1)^n - 5^n}{6}$$

$$= (-1)^n - \frac{(-1)^n}{6} + \frac{5^n}{6}$$

$$= \frac{5(-1)^n}{6} + \frac{5^n}{6} = \frac{5(-1)^n + 5^n}{6}$$

$$A^n = (A^2 - 4A - 5)\, Q(A) + aA + b$$

$$A^n = aA + b$$

$$A^n = \frac{(-1)^n - 5^n}{-6} A + \frac{5(-1)^n + 5^n}{6}$$

$$= \frac{[5^n - (-1)^n]}{6} A + \frac{5(-1)^n + 5^n}{6}$$

$$A^3 = \frac{5^3 - (-1)^3}{6} A + \frac{5(-1)^3 + 5^3}{6}$$

$$= \frac{125 + 1}{6} A + \frac{-5 + 125}{6}$$

$$= \frac{126 A}{6} + \frac{120}{6} = 21A + 20I.$$

$$\therefore A^3 = 21A + 20I$$

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$$A^3 = 21\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 84 \\ 42 & 63 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix}$$

Questions	opt1	opt2	opt3	opt4	Answer
The sum of the main diagonal elements of a matrix is called------	trace of a matrix	quadratic form	eigen value	canonical form	trace of a matrix
The orthogonal transformation used to diagonalise the symmetric matrix A is----	$N^T AN$	$N^T A$	NAN^{-1}	NA	$N^T AN$
If $\lambda_1, \lambda_2, \lambda_3,\ldots\ldots\ \lambda_n$ are the eigen values of A ,then $k\lambda_1, k\lambda_2, k\lambda_3,\ldots\ldots, k\lambda_n$ are the eigen values of --------------	kA	kA^2	kA^{-1}	A^{-1}	kA
Diagonalisation of a matrix by orthogonal reduction is true only for a ----- matrix.	diagonal	triangular	real symmetric	scalar	real symmetric
If atleast one of the eigen values of A is zero, then det A = -----	0	1	10	5	0
det (A- λI) represents------	characteristic polynomial	characteristic equation	quadratic form	canonical form	characteristic polynomial
If λ_1, λ_2, λ_3,……… λ_n are the eigen values of A ,then 1/λ_1,1/ λ_2, 1/λ_3,……… 1/λ_n are the eigen values of --------	A^-1	A	A^n	2A	A^-1
If λ_1, λ_2, λ_3,……… λ_n are the eigen values of A ,then λ_1^p, λ_2^p, λ_3^p,……… λ_n^p are the eigen values of	A^-1	A^2	A^-p	A^p	A^p
The eigen values of a --------------------- matrix are its diagonal elements	diagonal	symmetric	skew-matrix	triangular	triangular
In an orthogonal transformation $N^T AN = D$, D refers to a ---------- matrix.	diagonal	orthogonal	symmetric	skew-symmetric	diagonal
In a modal matrix, the columns are the eigen vectors of----------	A^{-1}	A^2	A	adj A	A
If the eigen values of $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ are 0,3 & 15, then its nature is-----------	positive definite	positive semidefinite	indefinite	negative definite	positive semidefinite
The elements of the matrix of the quadratic form $x_1^2 + 3x_2^2 + 4x_1x_2$ are ------------	$a_{11} = 1, a_{12} = 2$, $a_{21} = 2$, $a_{22} = 3$	$a_{11} = -1$, $a_{12} = -2$, $a_{21} = 2$, $a_{22} = 3$	$a_{11} = 1$, $a_{12} = 4$, $a_{21} = 4$, $a_{22} = 3$	$a_{11} = 1$, $a_{12} = 4$, $a_{21} = 3$, $a_{22} = 1$	$a_{11} = 1, a_{12} = 2$, $a_{21} = 2$, $a_{22} = 3$
If the sum of two eigen values and trace of a 3x3 matrix A are equal, then det A = ---------	λ_1 λ_2 λ_3	0	1	2	0
If 1,5 are the eigen values of a matrix A, then det A = -------	5	0	25	6	5
If the canonical form of a quadratic form is $5y_1^2 + 6y_2^2$, then the rank is --------	4	0	2	1	2
The eigen vector is also known as-------	latent value	latent vector	column value	orthogonal value	latent vector
If 1,3,7 are the eigen values of A, then the eigen values of 2A are ------------	1,3,7	1,9,21	2,6,14	1,9,49	2,6,14

If the eigen values of 2A are 2, 6, 8 then eigen values of A are __________	1,3,4	2,6,8	1,9,16	12,4,3	1,3,4
The number of positive terms in the canonical form is called the ________ of the quadratic form.	rank	index	Signature	indefinite	index
If all the eigenvalues of A are positive then it is called as_______	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Positive definite
If all the eigenvalues of A are negative then it is called as_______	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Negative definite
A homogeneous polynomial of the second degree in any number of variables is called the ______	characteristic polynomial	characteristic equation	quadratic form	canonical form	quadratic form
The Set of all eigen values of the matrix A is called the ____________ of A	rank	index	Signature	spectrum	spectrum
A Square matrix A and its transpose have _______ eigen values.	different	Same	Inverse	Transpose	Same
The sum of the __________ of a matrix A is equal to the sum of the principal diagonal elements of A.	characteristic polynomial	characteristic equation	eigen values	eigen vectors	eigen values
The product of the eigenvalues of a matrix A is equal to_________	Sum of main diagonal	Determinant of A	Sum of minors of Main diagonal	Sum of the cofactors of A	Determinant of A
The eigenvectors of a real symmetric are _______	equal	unequal	real	symmetric	real
If the eigen values of 2A are 2, 6, 8, then eigen values of A are __________	**1,3,4**	2,6,8	1,9,16	12,4,3	**1,3,4**
The eigen values of a triangular matrix are --------	main diagonal elements	first row elements	first column elements	last column elements	main diagonal elemen
The main diagonal elements of a triangular matrix are -----------	characteristic polynomial	characteristic equation	eigen values	eigen vectors	eigen values
The main diagonal elements are the eigen values of the -------matrix.	square	symmetric	non symmetric	triangular	triangular
If atleast one of the eigen values of A is zero, then det A = ___	0	1	10	5	0
If the eigen values of A are 2, 3, 4 then the eigen values of A^{-1} is	1/2 , 1/3, 1/4	2,3,4	-2,-3,-4	(-1/2,-1/3,-1/4)	1/2 , 1/3, 1/4
If the sum of two eigen values of matrix A are equal to the trace of the matrix, then the determinant of A is	1	2	0	3	0
Sum of the principal diagonal elements ________	product of eigen values	product of eigen vectors	sum of eigen values	product of eigen values	sum of eigen values

If 1 and 2 are the eigen values of a matrix A, then the eigen values of A^2 are ____	2,3	3,5	1,4	1,2	1,4
The eigen vector is also known as _____	latent square	column vector	row vector	latent vector	latent vector
If all the eigen values of a matrix are distinct, then the corresponding eigen vectors______	linearly dependent	unique	not unique	linearly independent	linearly independent
A matrix is called symmetric if and only if ----------	A=A^T	A=A^-1	A=-A^T	A=A	A=A^T
If a matrix A is equal to A^T then A is a -------- matrix.	symmetric	non symmetric	skew-symmetric	singular	symmetric
A matrix is called skew-symmetric if and only if ---------	A=A^T	A=A^-1	A=-A^T	A=A	A=-A^T
If a matrix A is equal to -A^T then A is a -------- matrix.	symmetric	non symmetric	skew-symmetric	singular	skew-symmetric
A matrix is called orthogonal if and only if -----	A^T=A^-1	A^T=-A^-1	A^T=A^-2	A^T=-A^-2	A^T=A^-1
A matrix is called ------------if and only if A^T=A^-1.	orthogonal	square	non symmetric	triangular	orthogonal
The equation det (A-λI) = 0 is used to find ----------	characteristic polynomial	characteristic equation	eigen values	eigen vectors	characteristic equation
If the characteristic equation of a matrix A is λ^2 – 2 = 0, then the eigen values are -------	2,2	(-2,-2)	(2^(1/2),-2^(1/2))	(2i,-2i)	(2^(1/2),-2^(1/2))
If 1,3,7 are the eigen values of A, then the eigen values of 2A are ------------	1,3,7	1,9,21	2,6,14	1,9,49	2,6,14
If 1,5 are the eigen values of a matrix A, then det A = -------	5	0	25	6	5
Eigen value of the characteristic equation λ^2-4 = 0 is	2, 4	2, -4	2, -2	2, 2	2,-2
Eigen value of the characteristic equation λ^3-6λ^2+11λ-6 = 0 is	1,2,3	1, -2,3	1,2,-3	1,-2,-3	1,2,3
Largest Eigen value of the characteristic equation λ^3-3λ^2+2λ = 0 is	1	0	2	4	2
Smallest Eigen value of the characteristic equation λ^3-7λ^2+36 = 0 is	-3	3	-2	6	-2
Sum of the principal diagonal elements =	product of eigen values	product of eigen vectors	sum of eigen values	sum of eigen vectors	sum of eigen values
Product of the eigen values =	(-|A|)	1/|A|	(-1/|A|)	|A|	|A|
A Square matrix A and its transpose have _______ eigen values.	different	Same	Inverse	Transpose	Same
If 1 and 2 are the eigen values of a 2X2 matrix A, then the eigen values of A^2 is	2, 4	3,4	5,6	1, 4	1, 4
If 1 and 2 are the eigen values of a 2X2 matrix A, then the eigen values of A^-1is	2,1/2	1,1/2	1,2	4,1/2	1,1/2
If a real symmetric matrix of order 2 has ---------then the matrix is a scalar matrix.	equal eigen vectors	different eigen vectors	equal eigen values	different eigen values	equal eigen values

If A and B are nxn matrices and B is a non singular matrix then A and B^-1AB have	same eigen vectors	different eigen vectors	same eigen values	different eigen values	same eigen values
Every square matrix satisfies its own ----------	characteristic polynomial	characteristic equation	orthogonal transformation	canonical form	characteristic equation
In a modal matrix, the columns are the -----------	eigen vectors of A	eigen vectors of adj A	eigen vectors of inverse ofA	eigen values of A	eigen vectors of A
Cayley -Hamilton theorem is used to find ------------	inverse and higher powers of A	eigen values	eigen vectors	quadratic form	inverse and higher powers of A
If the canonical form of a quadratic form is 5y12 - 6 y22 , then the index is --------	4	0	2	1	1
The non –singular linear transformation used to transform the quadratic form to canonical form is ----------	X= NTY	X= NY	Y= NX	NXA	X= NY
The eigen vector is also known as-------	latent value	latent vector	column value	orthogonal value	latent vector
The sum of the ___________ of a matrix A is equal to the sum of the principal diagonal elements of A.	characteristic polynomial	characteristic equation	eigen values	eigen vectors	eigen values
The product of the eigenvalues of a matrix A is equal to__________	Sum of main diagonal	Determinant of A	Sum of minors of Main diagonal	Sum of the cofactors of A	Determinant of A
The eigenvectors of a real symmetric are ________	equal	unequal	real	symmetric	real
When the quadratic form is reduced to the canonical form, it will contain only r terms, if the _____ of A is r.	rank	index	Signature	spectrum	rank
The excess of the number of positive terms over the number of negative terms in the canonical form is called the ____________ of the quadratic form.	rank	index	Signature	spectrum	Signature
If all the eigen values of A are less than zero and atleast one eigen value is zero then the quadratic form is said to be ____________	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Negative semidefinite
If all the eigen values of A are greater than zero and atleast one eigen value is zero then the quadratic form is said to be ____________	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Positive semidefinite
If the quadratic form has both positive and negative terms then it is said to be ____________	Positive definite	Negative definite	Positive semidefinite	indefinite	indefinite

2.1 Bisection Method

The bisection method is the easiest to numerically implement and almost always works. The main disadvantage is that convergence is slow. If the bisection method results in a computer program that runs too slow, then other faster methods may be chosen; otherwise it is a good choice of method.

We want to construct a sequence $x_0, x_1, x_2, \ldots$ that converges to the root $x = r$ that solves $f(x) = 0$. We choose x_0 and x_1 such that $x_0 < r < x_1$. We say that x_0 and x_1 bracket the root. With $f(r) = 0$, we want $f(x_0)$ and $f(x_1)$ to be of opposite sign, so that $f(x_0)f(x_1) < 0$. We then assign x_2 to be the midpoint of x_0 and x_1, that is $x_2 = (x_0 + x_1)/2$, or

$$x_2 = x_0 + \frac{x_1 - x_0}{2}.$$

The sign of $f(x_2)$ can then be determined. The value of x_3 is then chosen as either the midpoint of x_0 and x_2 or as the midpoint of x_2 and x_1, depending on whether x_0 and x_2 bracket the root, or x_2 and x_1 bracket the root. The root, therefore, stays bracketed at all times. The algorithm proceeds in this fashion and is typically stopped when the increment to the left side of the bracket (above, given by $(x_1 - x_0)/2$) is smaller than some required precision.

2.2 Newton's Method

This is the fastest method, but requires analytical computation of the derivative of $f(x)$. Also, the method may not always converge to the desired root.

We can derive Newton's Method graphically, or by a Taylor series. We again want to construct a sequence $x_0, x_1, x_2, \ldots$ that converges to the root $x = r$. Consider the x_{n+1} member of this sequence, and Taylor series expand $f(x_{n+1})$ about the point x_n. We have

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n)f'(x_n) + \ldots.$$

To determine x_{n+1}, we drop the higher-order terms in the Taylor series, and assume $f(x_{n+1}) = 0$. Solving for x_{n+1}, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Starting Newton's Method requires a guess for x_0, hopefully close to the root $x = r$.

Newton's forward interpolation formula (equal intervals).

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \frac{x-x_0}{h}$

① Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence evaluate y at $x=5$.

x	4	6	8	10
y	1	3	8	10

Soln

$u = \frac{x-x_0}{h}$, $h=2$

$u = \frac{x-4}{2}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		-6
8	8		-3	
		2		

The Newton's forward interpolation form. is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \left(\frac{x-4}{2}\right)(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2} \times 3$$

$$+ \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)\left(\frac{x-4}{2}-2\right)}{3!} \times -6$$

$$= 1 + (x-4) + \frac{3(x-4)(x-6)}{8} - \frac{(x-4)(x-6)(x-8)}{8}$$

$$= \frac{1}{8}\Big[8 + 8x - 32 + 3[x^2 - 10x + 24]$$
$$- [x^3 - 18x^2 + 104x - 192]\Big]$$

$$y = \frac{1}{8}[-x^3 + 21x^2 - 126x + 240].$$

Put $x = 5$

$$y(5) = \frac{1}{8}[-5^3 + 21 \times 5^2 - 126 \times 5 + 240]$$

$$\boxed{y(5) = 1 \cdot 25}$$

Newton's Backward Interpolation formula

$$y = y_0 + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

Where $v = \dfrac{x - x_n}{h}$

Use Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data.

$f(-0.75) = -0.07181250$ $f(-0.5) = -0.024750$

$f(-0.25) = 0.33493750$, $f(0) = 1.10100$.

Hence find $f(-\frac{1}{3})$.

Soln.

$v = \dfrac{x - x_n}{h}$ & $h = 0.25$

$v = \dfrac{x - 0}{0.25} = \dfrac{x}{0.25}$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
-0.75	-0.07181250			
		0.0470625		
			0.312625	
-0.50	-0.024750			0.09375
		0.3596875		
			0.406375	
-0.25	0.33493750			
		0.7660625		
0	1.10100			

The Newton's backward interpolation formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

$$= 1.10100 + \left(\frac{x}{0.25}\right)(0.7660625)$$
$$+ \left(\frac{x}{0.25}\right)\left(\frac{x}{0.25}+1\right)(0.406375)$$
$$+ \frac{\left(\frac{x}{0.25}\right)\left(\frac{x}{0.25}+1\right)\left(\frac{x}{0.25}+2\right)}{3!}(0.09375)$$

$$= 1.10100 + (-1.33333)(0.7660625)$$
$$+ \frac{(-1.33333)(-0.33333)}{2}(0.406375)$$
$$+ \frac{(-1.33333)(-0.33333)(-0.66666)}{6}(0.09375)$$

$$= 1.10100 - 1.021414 + 0.090304426$$
$$+ 0.0046295$$

$$y(-1/3) = 0.165260.$$

Numerical Differentiation and Integration

Numerical differentiation :

It is the Process of finding the Values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ & $\frac{d^3y}{dx^3}$, ... for some particular Value of x.

① find the first derivatives of $f(x)$ at $x=2$ for the data $f(-1) = -21$, $f(1) = 15$, $f(2) = 12$ $f(3) = 3$. using Newton's divided difference formula.

Soln

x	-1	1	2	3
y	-21	15	12	3

The Newton's divided difference formula is

$$y = y_0 + (x-x_0) \Delta y_0 + (x-x_0)(x-x_1) \Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2) \Delta^3 y_0 + \cdots$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

$$y = -21 + (x+1)\,18 + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2)(1)$$

$$= -21 + 18x + 18 - 7(x^2-1) + (x^2-1)(x-2)$$

$$= -21 + 18x + 18 - 7x^2 + 7 + x^3 - 2x^2 - x + 2$$

$$y = x^3 - 9x^2 + 17x + 6$$

$$y' = 3x^2 - 18x + 17$$

$$y'(2) = -7.$$

Numerical Integration

Trapezoidal rule

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\Big[(\text{Sum of first and last ordinate}) + 2(\text{Sum of remaining ordinates})\Big]$$

$$h = \frac{b-a}{n}$$

Simpson's 1/3 rule

$$I = \int_a^b f(x)\,dx = \frac{h}{3}\Big[(\text{first} + \text{Last}) + 4(\text{Sum of odd ordinates}) + 2(\text{Sum of even ordinates})\Big]$$

$$h = \frac{b-a}{n} - [\text{multiples of } 2]$$

Simpson's 3/8 rule

$$I = \frac{3h}{8}\Big[(\text{first Last}) + 2(\text{Sum of multiples of } 3) + 3(\text{Sum of non-multiples of } 3)\Big]$$

$$h = \frac{b-a}{n} \quad [\text{multiples of } 3]$$

(1) Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ taking 8 intervals.

Soln

$$h = \frac{b-a}{n} = \frac{1+1}{8} = \frac{2}{8} = 0.25$$

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
y	0.5	0.65	0.8	0.9412	1	0.9412	0.8	0.64	0.5

$$I = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{0.25}{2}\left[(0.5 + 0.5) + 2(0.65 + 0.8 + 0.9412 + 1 + 0.9412 + 0.8 + 0.64)\right]$$

$$= \frac{0.25}{2}\left[1 + 2(5.7624)\right]$$

$$= \frac{0.25}{2}\left[12.5248\right]$$

$$= 1.5656$$

2) Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ with $h = 1/6$ by Trapezoidal rule.

Soln

$f(x) = \frac{1}{1+x^2}$ $h = \frac{1}{6}$

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	1
y	1	$36/37$	$9/10$	$4/5$	$9/13$	$36/61$	$1/2$

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{(1/6)}{2}\left[(1+1/2) + 2\left(\frac{36}{37} + \frac{9}{10} + \frac{4}{5} + \frac{9}{13} + \frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 2(3.9554)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 7.9108\right]$$

$$= 0.7842$$

(5) Dividing the range into 10 equal parts find the value of $\int_0^{\pi/2} \sin x \, dx$ by Simpson's $1/3$ rule.

Soln

$f(x) = \sin x \qquad h = \frac{b-a}{n} = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$

x	0	$\pi/20$	$2\pi/20$	$3\pi/20$	$4\pi/20$	$5\pi/20$	$6\pi/20$	$7\pi/20$	$8\pi/20$
$f(x)$	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511

$$I = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{\pi/20}{3}\left[(0+1) + 4(0.1564 + 0.4540 + 0.7071 + 0.8910) + 2(0.3090 + 0.5878 + 0.8090)\right]$$

$$= \pi/60 * 19.0986 = 1$$

Unit-II **Interpolation**

Questions	opt1	opt2	opt3	opt4	Answer
The process of computing the value of the function inside the given range is called ________	Interpolation	extrapolation	reduction	expansion	Interpolation
If the point lies inside the domain [x_0, x_n], then the estimation of f(y) is called _________	Interpolation	extrapolation	reduction	expansion	Interpolation
The process of computing the value of the function outside the given range is called ________	Interpolation	extrapolation	reduction	expansion	extrapolation
If the point lies outside the domain [x_0, x_n], then the estimation of f(y) is called	Interpolation	extrapolation	reduction	expansion	extrapolation
Interpolation is the process of computing _______ values of a function from a given set of tabular values of a function	positive	negative	constant	intermediate	intermediate
The estimation of values between well-known discrete points are called .	Interpolation	extrapolation	reduction	expansion	Interpolation
______ is the process of finding the most appropriate estimate for missing data.					
For making the most probable estimate the changes in the series are must be ______ within a period.	uniform	Normal	Exponentially	periodic	uniform
For making the most probable estimate the frequency distribution must be ______.	Normal	uniform	periodic	Exponentially	Normal
Lagrange's interpolation formula can be used when the values of independent variable x are ____	equally – spaced	unequally – spaced	both equally andunequally – spaced	positive	both equally andunequally – spaced
To find the unknown value of x for some y, which lies at the unequal intervals we use ------------------- formula.	Newton's forward	Newton's backward	Newtons divided difference	inverse interpolation	Newtons divided difference
If the values of the variable y are given, then the method of finding the unknown variable x is called ----------------	Newton's forward	Newton's backward	interpolation	inverse interpolation	inverse interpolation
In Newton's backward difference formula, the value of n is calculated by -------.	$n = (x–x_{\_n}) / h$	$n = (x_{\_n}–x) / h$	$n = (x–x_0) / h$	$n = (x_{\_0}–x) / h$	$n = (x–x_{\_n}) / h$
In Newton's forward difference formula, the value of n is calculated by -------.	$n = (x–x_{\_n}) / h$	$n = (x_{\_n}–x) / h$	$n = (x–x_0) / h$	$n = (x_{\_0}–x) / h$	$n = (x–x_0) / h$
In the forward difference table y_0 is called element.	leading	ending	middle	positive	leading
In the forward difference table forward symbol ((y_0)), forward symbol(^2(y_0)) , ... are called difference.	leading	ending	middle	positive	leading
The difference of first forward difference is called ________.	divided difference	2nd forward difference	3rd forward difference	4th forward difference	2nd forward difference
_________ Formula can be used for interpolating the value of f(x) near the end of the tabular values.	Newton's forward	Newton's backward	Lagrange	stirling	Newton's backward

Gregory Newton forward interpolation formula is also called as Gregory Newton forward formula.	Elimination	iteration	difference	distance	difference
Gregory Newton backward interpolation formula is also called as Gregory Newton backward formula	Elimination	iteration	difference	distance	difference
Gregory Newton backward interpolation formula is also called as Gregory Newton backward formula .	Elimination	iteration	difference	distance	difference
The divided differences are _________ in their arguments.	constant	symmetrical	varies	singular	symmetrical
In Gregory Newton forward interpolation formula 1st two terms of this series give the result for the interpolation.	Ordinary linear	ordinary differential	parabolic	central	Ordinary linear
Gregory Newton forward interpolation formula 1st three terms of this series give the result for the interpolation.	Ordinary linear	ordinary differential	parabolic	central	parabolic
Gregory Newton forward interpolation formula is mainly used for interpolating the values of y near the _____________ of the set of tabular values.	beginning	end	centre	side	beginning
Gregory Newton backward interpolation formula is mainly used for interpolating the values of y near the of the set of tabular values.	beginning	end	centre	side	end
From the definition of divided difference (u-u_0)/(x-x_0) we have =	(y,y_0)	(x,y)	(x_0, y_0)	(x,x_0)	(x_0, y_0)
If f(x) =0, then the equation is called __________	Homogenous	non-homogenous	first order	second order	Homogenous
If the values $x_0 = 0$, $y_0 = 0$ and h = 1 are given for Newton's forward method, then the value of x is ------------.	0	1	n	X	n
The n th order difference of a polynomial of n th degree is ------------.	constant	zero	polynomial in first degree	polynomial in n-1 degree	constant
What will be the first difference of a polynomial of degree four?	Polynomial of degree one	Polynomial of degree two	Polynomial of degree three	Polynomial of degree four	Polynomial of degree three
A function which satisfies the difference equation is a ________of the difference equation.	Solution	general solution	complementary solution	particular solution	Solution
The degree of the difference equation is ________	The highest powers of y's	The difference between the arguments of y	The difference between the constant	The highest value of x	The highest powers of y's
The degree of the difference equation is ________	2	0	1	3	1
The difference between the highest and lowest subscripts of y are called ______ of the difference equation	degree	order	power	value	order

E-1=	backward difference operator	forward difference operator	μ	δ	forwarddifference operator
Which of the following is the central difference operator?	E		μ	δ	δ
1+(forward difference operator)=	backward difference symbol	E	μ	δ	E
μ is called the ______ operator	Central	average	backward	displacement	average
The other name of shifting operator is ______ operator	Central	average	backward	displacement	displacement
The difference of constant functions are___________	0	1	2	3	0
The nth order divided difference of x_n will be a polynomial of degree .	0	1	2	3	2
The operator forward symbol is __________	homogenous	heterogeneous	linear	a variable	linear

Unit - IV

Initial Value Problem for ordinary differential Equation

Method - 1

Taylor Series:

The taylor Series formula is

$$y = y_0 + (x-x_0)\frac{y_0'}{1!} + (x-x_0)^2\frac{y_0''}{2!} + (x-x_0)^3\frac{y_0'''}{3!} + \cdots$$

1. Use taylor series method to find $y(0.1)$ and $y(0.2)$. Given that $\frac{dy}{dx} = 3e^x + 2y$

$y(0) = 0$;

Soln: Given $\frac{dy}{dx} = y' = 3e^x + 2y$; $y(0) = 0$;

The taylor series formula is,

$$y = y_0 + (x-x_0)\frac{y_0'}{1!} + (x-x_0)^2\frac{y_0''}{2!} + (x-x_0)^3\frac{y_0'''}{3!} + \frac{(x-x_0)^4}{}$$

x	0	x_0
y	0	y_0
$y' = 3e^x + 2y$	3	y_0'
$y'' = 3e^x + 2y'$	9	y_0''
$y''' = 3e^x + 2y''$	21	y_0'''
$y^{iv} = 3e^x + 2y'''$	45	y_0^{iv}

$$y = 0 + (x-0)\cdot 3/1! + (x-0)^2 \cdot 9/2 + (x-0)^3 \frac{21}{6} + (x-0)^4 \cdot \frac{45}{24};$$

$$y = 3x + 9/2\, x^2 + 7/2\, x^3 + 15/8\, x^4$$

$y(0.1) = 0.3487$.

$y(0.2) = 0.8110$.

2. use taylor series method, solve $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ at $x = 0.1, 0.2, 0.3$.

soln:

The taylor series formula is,

$$y = y_0 + (x-x_0)\frac{y_0'}{1!} + (x-x_0)^2\frac{y_0''}{2!} + (x-x_0)^3\frac{y_0'''}{3!} + (x-x_0)^4\frac{y_0^{iv}}{4!}$$

$y' = x^2 - y$; $y(0) = 1$

x	0	x_0
y	1	y_0
$y' = x^2 - y$	-1	y_0'
$y'' = 2x - y'$	1	y_0''
$y''' = 2 - y''$	1	y_0'''
$y^{iv} = -y'''$	-1	y_0^{iv}

$$y = 1 + (x-0)\left(\frac{-1}{1!}\right) + (x-0)^2\frac{1}{2!} + (x-0)^3 \times \frac{1}{3!} + (x-0)^4\left(\frac{-1}{4!}\right)$$

$$y = 1 - x + x^2/2 + x^3/6 - \frac{x^4}{24}.$$

$y(0.1) = \cancel{0.9102}.\ 0.9052$

$= \cancel{7/6\, x^4 + 4/3\, x^3}$

$y = 7/6\, x^4 + 4/3\, x^3 + x^2 + x + 1$

$y(0.1) = 1.1115$

$y(0.2) = 1.2525$

4. Obtain y by <u>taylor series</u> method given that $y' = xy + 1$; $y(0) = 1$; for $x = 0.1$; $x = 0.2$; correct to four decimal places.

Soln: The formula is,

$$y = y_0 + (x - x_0)\frac{y_0'}{1!} + (x - x_0)^2 \frac{y_0''}{2!} + (x - x_0)^3 \frac{y_0'''}{3!} +$$

$$(x - x_0)^4 \frac{y_0^{iv}}{4!} + \cdots$$

x	0	x_0
y	1	y_0
$y' = xy + 1$	1	y_0'
$y'' = y + xy'$	1	y_0''
$y''' = y' + y' + xy''$	2	y_0'''
$y^{iv} = y'' + y'' + y'' + xy'''$	3	y_0^{iv}

$y = 1 + (x - 0)\frac{1}{1!} + (x - 0)^2 \frac{1}{2} + (x - 0)^3 \frac{2}{6} +$

$(x - 0)^4\ 3$

Method-II : Euler's method :

Consider $\frac{dy}{dx} = f(x,y)$

The Euler's formula is,

$$y_{n+1} = y_n + h f(x_n, y_n) \text{ (or)}$$

$$y_{n+1} = y_n + h y_n'$$

1. Solve $y' = \frac{y-x}{y+x}$, $y(0)=1$ at $x=0.1$

by taking $h=0.02$; by using

Euler's method.

Soln:

$$y' = \frac{y-x}{y+x} \quad ; \quad y(0)=1$$

The Euler's formula is,

$$y_{n+1} = y_n + h f(x_n, y_n)$$

(or)

$$y_{n+1} = y_n + h \cdot y_n'$$

x	0	0.02	0.04	0.06	0.08	0.
y	1	1.02	1.0392	1.0577	1.0756	1.
$y' = \frac{y-x}{y+x}$	1	0.9615	0.9259	0.8926	0.8615	0

$n=0$;

$$y_1 = y_0 + h y_0' = 1 + 0.02 \times 1 = 1.02$$

$n=1$;

$$y_2 = y_1 + h y_1' = 1.02 + 0.02 \times 0.9615 = 1.0392$$

$n=2$;

$$y_3 = y_2 + h \cdot y_2' = 1.0392 + 0.02 \times 0.9259 = 1.0577$$

$n=3$;

$$y_4 = y_3 + h y_3' = \cancel{0.9}\ 1.0577 + 0.02 \times 0.8926$$

$$y_4 = 1.0756$$

$n=4$;

$$y_5 = y_4 + h y_4' = 1.0756 + 0.02 \times 0.8615$$

$$y_5 = 1.0928$$

~~$n=5$;~~

~~$y_6 = y_5 + h y_5' = 1.0928 + 0.$~~

2. Using Euler's method to find $y(0.4)$ Gi..

$\frac{dy}{dx} = x+y$, $y(0) = 1$. Taking $h = 0.2$.

Soln:

Given $\frac{dy}{dx} = x+y$, $y(0)=1$.

The Euler's formula is $y_{n+1} = y_n + h y_n'$

x	0	0.2	0.4
y	1	1.2	1.48
$y' = x+y$	1	1.4	1.88

$n=0 \Rightarrow y_1 = y_0 + h y_0' = 1 + (0.2 \times 1) = 1.2$

$n=1 \Rightarrow y_2 = y_1 + h y_1' = 1.2 + (0.2 \times 1.4) = 1.48$.

3. Using Euler's method find the solution of the initial value problem (IVP) $\frac{dy}{dx} = \log_e(x+y)$ $y(0)=2$ at $x=0.6$ by assuming $h=0.2$.

Soln:

Given $y' = \log_{10}(x+y)$; $y(0)=2$.

The Euler's formula is $y_{n+1} = y_n + h y_n'$

x	0	0.2	0.4	0.6
y	2	2.0602	2.1810	2.2117.
$y' = \log_{10}(x+y)$	0.3010	0.3541	0.4033	0.4490.

$n=0 \Rightarrow y_1 = y_0 + h y_0' = 2 + (0.2 \times 0.3010) = 2.0602$.

$n=1 \Rightarrow y_2 = y_1 + h y_1' = 2.0602 + (0.2 \times 0.3541) = 2.1810$.

$n=2 \Rightarrow y_3 = y_2 + h y_2' = 2.1810 + (0.2 \times 0.4033) = 2.2117$.

using Runge-kutta method of order 4; find y value when $x = 1.2$ in steps of 0.1 given that $y' = x^2 + y^2$, $y(1) = 1.5$.

soln:

The Runge-kutta formula is

$k_1 = h \cdot f(x, y)$

$k_2 = h \cdot f(x + h/2, y + k_1/2)$

$k_3 = h \cdot f(x + h/2, y + k_2/2)$

$k_4 = h \cdot f(x + h, y + k_3)$

given $y' = x^2 + y^2$

here, $f(x, y) = x^2 + y^2$; $h = 0.1$

x	1	1.1	1.2
y	1.5	y_1 1.8955	y_2 2.5044

To find y_1

$x = 1$; $y = 1.5$.

$k_1 = h f(x, y) = 0.1 \times f(1, 1.5)$.

$= 0.1 \times 3.25 = 0.325$.

$k_2 = h \cdot f(x + h/2, y + k_1/2) = 0.1 \times f(1.05, 1.662$

$= 0.1 \times 3.8664 = 0.3866$.

$k_3 = h \cdot f(x + h/2, y + k_2/2) = 0.1 f(1.05, 1.6933)$

$= 0.1 \times 3.9698 = 0.3970$.

$k_4 = h \cdot f(x + h, y + k_3) = 0.1 \times f(1.1, 1.8970$

$= 0.4809$.

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1.5 + \frac{1}{6}[0.325 + 2 \times 0.3866 + 2 \times 0.3970 + 0.4809]$

$y_1 = 1.8955$

$f(x, y) = x^2 + y^2$

$k_1 = h \cdot f(x, y) = 0.1 \times f(1.1, 1.8955)$

$= 0.1 \times 4.8029 = 0.4803.$

$k_2 = h\, f(x + h/2,\ y + k_1/2) = \cancel{0.1\times}$

$= 0.1\, f(1.150,\ 2.1357)$

$= 0.1 \times 5.8837 = 0.5884.$

$k_3 = h\, f(x + h/2,\ y + k_2/2)$

$= 0.1 \times f(1.150,\ 2.1097)$

$= 0.1 \times 6.1173 = 0.6117.$

$k_4 = h\, f(x + h,\ y + k_3)$

$= 0.1 \times f(1.2,\ 2.5072)$

$= 0.7726.$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.8955 + \frac{1}{6}\left[0.4803 + 2 \times 0.5884 + 2 \times 0.6117 + 0.7726\right]$

2. Find $y(0.7)$ & $y(0.8)$ given that $y' = y - x^2$ $y(0.6) = 1.7379$ by using RK method of 4th order.

Soln:

$k_1 = h \cdot f(x, y)$

$k_2 = h \cdot f(x + h/2, y + k_1/2)$

$k_3 = h \cdot f(x + h/2, y + k_2/2)$

$k_4 = h \cdot f(x + h, y + k_3)$

given $y' = y - x^2$

Here $f(x, y) = y - x^2$; $h = 0.1$

	x_0	x_1	x_2
x	0.6	0.7	0.8
y	1.7379	1.8763	2.0145.

To find y_1 :

$x = 0.6$; $y = 1.7379$.

$k_1 = h \cdot f(x, y) = 0.1 \times f(0.6, 1.7379)$

$= 0.1378$.

$k_2 = 0.1 \times f(0.6 + 0.1/2, 1.7379 + 0.1378/2)$

$K_2 = \cancel{0.0240}$ 0.1384

$K_3 = 0.1 \times f\left[0.6 + \frac{0.1}{2}, 1.7879 + 0.1384/2\right]$

$= 0.1 \times f(0.65, 1.8071)$

$= 0.1385.$

$K_4 = 0.1 \times f(0.7, 1.8764)$

$= 0.1386.$

$y_1 = \overset{1.7379}{\cancel{y_0}} + 1/6\ (0.1378 + 0.1384 \times 2 + 0.1385 \times 2 + 0.1386)$

$= 1.8763$

To find y_2.

$x = 0.7$; $y = 1.8763$.

$K_1 = 0.1 \times f(0.7, 1.8763) = 0.1386$

$K_2 = 0.1 \times f(0.75, 1.9456) = 0.1383$

$K_3 = 0.1 \times f(0.75, 1.9455) = 0.1383$.

$K_4 = 0.1 \times f(0.8, 2.0146) = 0.1375$

$y_2 = y_1 + 1/6\left[K_1 + 2K_2 + 2K_3 + K_4\right)$.

$= 1.8763 + 1/6\left(0.1386 + 2 \times 0.1383 + 2 \times 0.1383 + \overset{0.1375}{\cancel{0.1402}}\right)$

3/14. Milne's Predictor - corrector Method.

Consider $\frac{dy}{dx} = f(x, y)$

P: $y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$

C: $y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$

1. By using Milne's predictor - corrector formula to find $y(0.4)$ & $y(0.5)$. G.T $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}$, $y(0) = 1$; $y(0.1) = 1.06$; $y(0.2) = 1.12$; $y(0.3) = 1.21$

Soln: The Milne's predictor - corrector formula is,

$$P:- \; y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right] \quad \text{--- (1)}$$

$$C: \; y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right] \quad \text{--- (2)}$$

x	0 x_0	0.1 x_1	0.2 x_2	0.3 x_3	0.4 x_4	
y	1 y_0	1.06 y_1	1.12 y_2	1.21 y_3	1.2771 y_4	1.2797
$y' = \frac{(1+x^2)y^2}{2}$	y'_0 0.5	y'_1 0.5674	y'_2 0.6523	y'_3 0.7979	0.9460	0.94[illegible]

Put $n=3$ in (1).

$$P: y_4 = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

$$= 1 + \frac{4 \times 0.1}{3}\left(2 \times 0.5674 - 0.6523 + 2 \times 0.797\right.$$

$$P: y_4 = 1.2771.$$

put $n=3$ in eqn (2).

$$C: y_4 = y_2 + h/3\left(y'_2 + 4y'_3 + y'_4\right)$$

$$= 1.12 + \frac{0.1}{3}\left(0.6523 + 4 \times 0.7979 + 0.9460\right]$$

$$C: y_4 = 1.2797.$$

unit-III

Numerical differentiation and Integration

Questions	opt1	opt2	opt3	opt4	Answer
_________ Formula can be used for interpolating the value of f(x) near the end of the tabular values.	Newton’s forward	Newton’s backwa	Lagrange	stirling	Newton’s backward
_________ Formula can be used for interpolating the value of f(x) near the beginning of the tabular values.	Newton’s forward	Newton’s backwa	Lagrange	stirling	Newton’s forward
In Numerical integration, the length of all intervals is in ----------- distances.	Greater than the other	less than the	equal	not equal	equal
When the function is given in the form of table values instead of giving analytical expression we use ___________.	numerical differentiation	numerical elimination	approximation	addition	numerical differentiation
_________ is the process of computing the value of the definite integral from the set of numerical values of the integrand.	numerical differentiation	numerical integration	Simpsons rule	Trapezoidal rule	numerical integration
Numerical integration is the process of computing the value of a ________ from a set of numerical values of the integrand.	indefinite integral	definite integral	expression	equation	definite integral
Numerical evaluation of a definite integral is called --------	integration	differentiation	interpolation	triangularisation	integration
What is the value of h if a=0,b=2 and n=2.	1	2	3	4	1
Integral (f(x) dx)=(h/2) [Sum of the first and last ordinates + 2(sum of the remaining ordinates)] is called _____	Constant rule	Simpsons rule	Trapezoidal rule	Rombergs rule	Trapezoidal rule
If the given integral is approximated by the sum of ‘n’ trapezoids, then the rule is called as ----------------.	Newton's method	Trapezoidal rule	simpson's rule	none	Trapezoidal rule

What is the formula for finding the length interval h in trapezoidal tule?	h=(b-a)/n	h=(b/a)/n	h=(b*a)/n	h=(b+a)/n	h=(b-a)/n
The accuracy of the result using the Trapezoidal rule can be improved by --------	Increasing the interval h	Decreasing the	Increasing the number of	altering the given	Decreasing the length
The order of error in Trapezoidal rule is -------------.	h	h^2	h^3	h^4	h^2
Simpson's rule is exact for a ----------------- even though it was derived for a Quadratic.	cubic	less than cubic	linear	quadratic	linear
The order of error in Simpson's rule is ----------	h	h^2	h^3	h^4	h^4
For what type of functions, Simpsons rule and direct integration will give the same result?	parabola	hyperbola	ellipse	cardiod	parabola
Simpson's rule gives exact result if the entire curve y=f(x) itself is a __________.	parabola	hyperbola	ellipse	cardiod	parabola
To apply Simpsons one third rule the number of intervals must be_________.	odd	even	equally spaced	unequal	even
The end point coordinates y_0 and y_n are included in the Simpsons 1/3 rule, so it is called _________ formula.	Newton's	open	closed	Gauss	closed
Simpson's one-third rule on numerical integration is called a --------- formula.	closed	open	semi closed	semi opened	closed
The order of error in Simpson's formula is ________.	1	2	3	4	4
In two point Gaussian quadrature Formula n =	1	2	3	4	2
In Simpsons 1/3rd rule, the number of ordinates must be _________.	odd	even	0	3	odd

In three point Gaussian quadrature Formula n = ________.	1	2	3	4	3
Two point Gaussian quadrature Formula requires only _______ functional evaluations and gives a good estimate of the value of the integral.	1	2	3	4	2
_________ formula is based on the concept that the accuracy of numerical integration can be improved by choosing the sampling wisely , rather than on the basis of equal spacing	Newtons	elimination	Gauss quadrature	hermite	Gauss quadrature
Gauss Quadrature formula is also called as _________.	Newton's	Gauss-Legendre	Gauss-seidal	Gauss-Jordan	Gauss-Legendre
The 2 point Gauss-quadrature is exact for the polynomial up to degree _________.	1	2	3	4	3
The 3 point Gauss-quadrature is exact for the polynomial up to degree _________.	1	5	3	4	5
Integrating f(x)=5x^4 in the interval [-1,1] using Gaussion two point formula gives______.	1/2	9/5	10/9	5/9	10/9
The modified Eulers method is based on the ___________ of points	sum	multiplication	average	subratction	average
__________ prior values are required to predict the next value in Milne's method	1	2	3	4	4
__________ prior values are required to predict the next value in Adams method	1	2	3	4	3
The Eulers method is used only when the slope at point ____________ in computing is y(n+1)	(x(n), y)	(x, y(n))	(x(n), y(n))	(0, 0)	(x(n), y(n))

The Runge Kutta method agrees with Taylor series solution upto the ________ terms	h^2	h^3	h^4	h^r	h^r
Runge Kutta method agree with ________ solution upto the terms h^4	Taylor Series	Eulers	Milnes	Adams	Taylor Series
________ method is better than Taylor's series method	Runge Kutta	Milnes	Adams	Eulers	Runge Kutta
Taylors series method belongs to ____________ method	Single step	multi step	step by step	liminatio n	Single step
If all the n conditions are specified at the initial point only then it is called a problem	Initial value	final value	boundary value	semi defined	Initial value
The problem dy/dx = f(x,y) with the initial condition y(x(0)) = y(0)is problem	initial value	final value	boundary value	multistep	initial value
The solution of an ODE means finding an explicit expression for y, in terms of a ______ number of elementary functions of x.	finite	infinite	positive	negative	finite
The solution of an ODE is known as ____________ solution	infinite	open-form	closed-form	negative form	closed-form
The differential equation of the 2^{nd} order can be solved by reducing it to a ____ differential equation	lower order	higher-order	partial	simultan eous	lower order
The Eulers method is used only when the slope at point (x(n), y(n)) in computing is	y(n+1)	y(n-1)	(dy/dx)(n+1)	(dy/dx)(n -1)	y(n+1)
The Eulers method is used only when the slope at point ____________ in computing is y(n+1)	(x(n),y)	(x, y(n))	(x(n), y(n))	(0, 0)	(x(n), y(n))
The modified Eulers method is a ____________ method of predictor-corrector type	Self-correcting	Self-starting	Self-evaluating	Self-predicting	Self-starting
The modified Eulers method has greater accuracy than method	Taylor's	Picard's	Euler's	Adam's	Taylor's
The formula y(n+1) = y(n) + hf(x(n), y(n)) is ________formula	Euler's	modified Euler's	Picard's	Taylor's	Euler's
Modified Eulers method is the Runge-kutta method of order	1^{st}	2^{nd}	3^{rd}	4^{th}	2^{nd}
Modified Eulers method is same as the ________ method of 2^{nd} order	Eulers	Taylors	Picards	Runge Kutta	Runge Kutta

The process used in Eulers method is very slow and to obtain reasonable accuracy we need to take a value of h	Smaller	Larger	negative	Positive	Smaller
The process used in Eulers method is very slow and to obtain reasonable accuracy we need to take a smaller value of	h	h^2	h^3	h^4	h
The ________ formula is given by y(i+1) = y(i) +hf (x(i), y(i))	Taylors	predictor	Corrector	Eulers	Eulers
The predictor formula and ________ formula are one and the same	Taylors	Eulers	Modified Eulers	Eulers	Eulers
The ________ formula is given by y(i+1) = y(i) + h/2[f(x(i), y(i)) + f(x(i+1), y(i+1))], i = 1,2,3.....	Taylors	predictor	Corrector	Picards	Corrector
The ________ formula is used to predict the value y(i+1) of y at x(i+1)	Predictor	Corrector	Corrector	Picards	Predictor
The ________ formula is used to improve the value of y(i+1)	Predictor	Corrector	Taylors	Picards	Corrector
In predictor corrector methods, ____ prior values of y are needed to evaluate the value of y at x(i+1)	1	2	3	4	4
In ________ methods, 4 prior values of y are needed to evaluate the value of y at x(i+1)	Taylor's	predictor	Predictor-corrector	Euler's	Predictor-corrector
In predictor corrector methods 4 prior values of ________ are needed to evaluate of values of are needed to evaluate of value of y at x(i+1)	y	y^2	y^3	y^4	y

UNIT - V

BOUNDARY VALUE PROBLEM IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATION.

Finite difference Method:

Replace x by x_k

y by y_k

y' by $\frac{y_{k+1} - y_k}{h}$

y'' by $\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$

where, $h = \frac{b-a}{n}$

1. Solve $y'' = x+y$ with the boundary condition $y(0) = y(1) = 0$.

Soln:

x	0	0.25	0.5	0.75	1
y	0	-0.0349	-0.0564	-0.05	0

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25.$

$y'' = x+y.$

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} = x_k + y_k.$$

$$y_{k-1} - 2y_k + y_{k+1} = h^2 x_k + h^2 y_k.$$

$$y_{k-1} - 2y_k + y_{k+1} - h^2 y_k = h^2 x_k$$

$$y_{k-1} + y_k(-2-h^2) + y_{k+1} = h^2 x_k$$

$$y_{k-1} - 2.0625\, y_k + y_{k+1} = 0.0625\, x_k$$

$k=1$;

$$y_0 - 2.0625\, y_1 + y_2 = 0.0625\, x_1$$

$$-2.0625\, y_1 + y_2 = 0.0156 \quad \text{—} (1).$$

$k=2$;

$$y_1 - 2.0625\, y_2 + y_3 = 0.0625\, x_2$$

$$y_1 - 2.0625\, y_2 + y_3 = 0.0313 \quad \text{—} (2)$$

$k=3$;

$$y_2 - 2.0625\, y_3 + y_4 = 0.0625\, x_3$$

$$y_2 - 2.0625\, y_3 = 0.0469 \quad \text{—} (3).$$

solve (1), (2) & (3)

$$y_1 = -0.0349; \quad y_2 = -0.0564; \quad y_3 = -0.0501;$$

24/3/14

Classification of partial differential Equation

Consider,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

$B^2 - 4AC < 0$ The P.D.E is elliptic.

$B^2 - 4AC = 0$ The P.D.E is parabolic

$B^2 - 4AC > 0$ The P.D.E is hyperbolic.

One dimensional heat equation :

The One dimensional heat eqn is

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t} \quad (\text{or}) \quad U_{xx} = a U_t$$

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0.$$

$A = 1$; $B = 0$; $C = 0$

$B^2 - 4AC = 0 - 4 \times 1 \times 0$

$= 0$

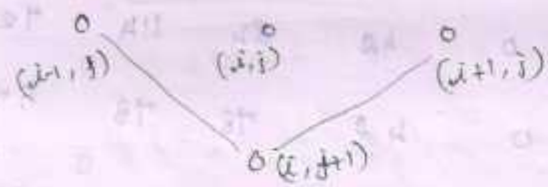
The one dimensional heat eqn is parabolic

There are two methods to solve one dimensional heat equations

i) Bender-Schmidt formula (Explicit)

ii) Crank-Nicolson method (Implicit)

Bender-Schmidt formula:



$$u_{i,j+1} = \frac{u_{i-1\,j} + u_{i+1\,j}}{2}$$

Here, $k = \frac{ah^2}{2}$

1. Solve $u_t = u_{xx}$ in $0 < x < 5$, $t > 0$ given that $u(0,t) = 0$, $u(5,t) = 0$, $u(x,0) = x^2(25 - x^2)$ Compute u upto 3sec. with $\Delta x = 1$ by using Bender Schmidt formula.

soln:

Given $u_t = u_{xx} \Rightarrow a = 1$

$h = \Delta x = 1$

$k = \frac{ah^2}{2} = \frac{1 \times 1}{2} = 0.5$.

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}$$

t \ x	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0

2. Solve $u_{xx} = 32u_t$, $h = 0.25$ for $t \geq 0$, $0 < x < 1$, with $u(0,t) = 0$, $u(x,0) = 0$; $u(1,t) = t$;

soln:

$u_{xx} = 32 u_t \qquad a = 32.$

$h = 0.25.$

$$k = \frac{ah^2}{2} = \frac{32 \times 0.25}{2} = 1$$

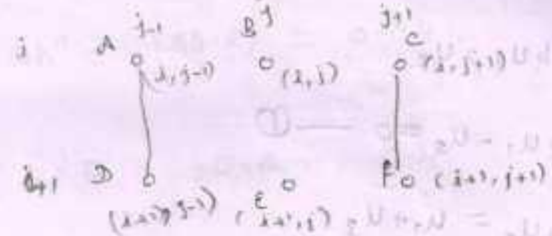
$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}$$

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	1.625	4
5	0	0.25	0.875	2.25	5.

25/3/14. Crank – Nicolson's Method (Implicit method):

Consider, $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t}$ (one dimensional heat eqn).

$k = ah^2$



$$4u_E = u_A + u_C + u_D + u_F$$

1. Using Crank – Nicolson's scheme solve

$$16\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ t > 0.$$

Subjected to $u(x, 0) = 0$; $u(0, t) = 0$; $u(1, t) = 100t$. Compute u for one step in t-direction. taking $h = 1/4$

Sol:

$$16\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$a = 16.$

$h = 0.25$

$k = ah^2 = 16 \times (0.25)^2 = 1$

t/x	0	0.25	0.5	0.75	
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

$$4u_1 = u_2$$

$$4u_1 - u_2 = 0 \quad \text{——} (1)$$

$$4u_2 = u_1 + u_3$$

$$u_1 - 4u_2 + u_3 = 0 \quad \text{——} (2)$$

$$4u_3 = u_2 + 100$$

$$-u_2 + 4u_3 = 100 \quad \text{——} (3)$$

solve (1), (2) & (3).

$$u_1 = 1.7857$$

$$u_2 = 7.1429$$

$$u_3 = 26.7857$$

2. find $u(x,t)$ for one time step the Equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(x,0) = \sin(\pi x)$; $u(0,t) = u(1,t) = 0$. Take $h = 0.2$ use implicit method.

Soln:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$a = 1$

$h = 0.2$.

$k = ah^2 = 1 \times (0.2)^2 = 0.04$.

t/x	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.04	0	u_1	u_2	u_3	u_4	0

$4u_1 = u_2 + 0.9511$

$4u_1 - u_2 = 0.9511$ — (1)

$4u_2 = u_1 + u_3 + 1.5389$.

$\cancel{-u_1 + 4u_2}$ $u_1 - 4u_2 + u_3 = -1.5389$ — (2)

$4u_3 = u_2 + u_4 + 1.5389$.

$u_2 - 4u_3 + u_4 = -1.5389$ — (3)

$4u_4 = 0.9511 + u_3$.

~~$u_3 + 4u_4 = 0.9511$ — (4)~~.

$u_4 = \frac{u_3}{4} + 0.2378$ — (4).

Sub ④ in ⑧

$u_2 - 4u_3 + u_4 = -1.5389.$

$u_2 - 4u_3 + \frac{u_3}{4} + 0.2378 = -1.5389.$

$u_2 - \frac{15}{4}u_3 = -1.4767.$

$u_2 - 3.75u_3 = -1.4767$ —— ⑤

Solve ~~eqn~~ ①, ②, ⑤.

$u_1 = 0.3993$

$u_2 = 0.6461$

$u_3 = 0.6461$

④ $\Rightarrow$ $u_4 = \frac{0.6461}{4} + 0.2378 = 0.3993.$

$u_4 = 0.3993.$

27/3/14.

3. Solve by crank nicolson's method, eqn $u_{xx} = u_t$ subjected to $u(x,0) = 0$; $u(0,t) = 0$; $u(1,t) = t$ for two time step.

soln :

$u_{xx} = u_t$.

$a = 1$

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$.

$k = ah^2 = 1 \times 0.25^2 = 0.0625$.

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.0625	0	0.0011	0.0045	0.0167	0.0625
0.125	0	0.0059	0.0191	0.0528	0.125

$4u_1 = u_2$.

$4u_1 - u_2 = 0$ —— (1)

$4u_2 = u_1 + u_3$

$u_1 - 4u_2 + u_3 = 0$ —— (2)

$4u_3 = u_2 + 0.0625$.

$-u_2 + 4u_3 = 0.0625$ —— (3) .

solve by (1), (2), (3)

$u_1 = 0.0011$; $u_2 = 0.0045$; $u_3 = 0.0167$.

$4u_4 = u_5 + 0.0045$.

$4u_4 - u_5 = 0.0045$ —— (4)

$4u_5 = u_4 + u_6 + 0.0178$

$u_4 - 4u_5 + u_6 = -0.0178$ — (5)

$4u_6 = u_5 + 0.1920$

$-u_5 + 4u_6 = 0.1920$ — (6)

solve by (4), (5), & (6)

$u_4 = 0.0059 \quad u_5 = 0.0191 \quad u_6 = 0.0528$

One dimentional wave Equation:

The One dimentional wave Equation

is, $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}$; $k = ah$

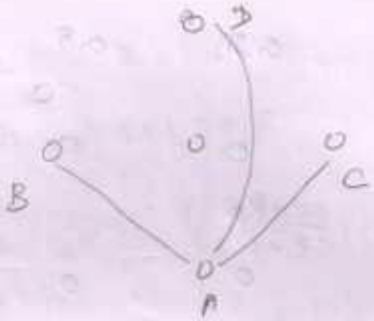
$U_{xx} = a^2 U_{tt}$

$U_{xx} - a^2 U_{tt} = 0$

$A = 1$; $B = 0$; $C = -a^2$

$B^2 - 4AC = 0 + 4a^2 = 4a^2 > 0$

The P.D.E is hyperbolic.



The formula is,

$$U_A = U_B + U_C - U_D$$

1. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0<x<1$, $t>0$

Given $u(x,0)=0$; $\frac{\partial u}{\partial t}(x,0)=0$; $u(0,t)=0$; $u(1,t)=100\sin(\pi t)$. compute $u(x,t)$ for 4 times steps with $h=0.25$

Soln:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$a^2=1$

$a=1.$

$h=0.25$

$k=ah = 1\times 0.25 = 0.25.$

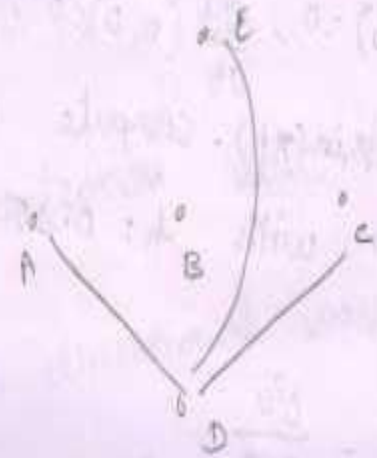
t \ x	0	0.25	0.5	0.75	1
0					
0.25					
0.5					
0.75					

$u_1 = \frac{0+0}{2} + K.A$

$u_1 = \frac{\text{add side two values}}{2} + K.A$

22/5/14.

$t \backslash x$	0	0.25	0.5	0.75	
0	0	0	0	0	
0.25	0	$\frac{0+0+k \cdot 0}{2}$ u_1 0	u_2 0	u_3 0	70
0.5	0	0	0	70.7107	10
0.75	0	0	70.7107	100	70.
1	0	70.7107	100	70.7107	0



$$U_D = U_A + U_C - U_E$$

1/4/14. Laplace and poisson Equation.

The laplace Equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$U_{xx} + U_{yy} = 0 \quad (or) \quad \nabla^2 u = 0.$$

The poisons Equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

(or)

$$U_{xx} + U_{yy} = f(x,y)$$

(or)

$$\nabla^2 u = f(x,y)$$

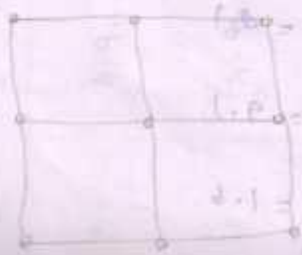
Here $A = 1$; $B = 0$; $C = 1$

$$B^2 - 4AC = 0 - 4 \times 1 \times 1$$

$$= -4 < 0.$$

Hence, Laplace and poisson equation are elliptic

Standard Diagonal five point formula,

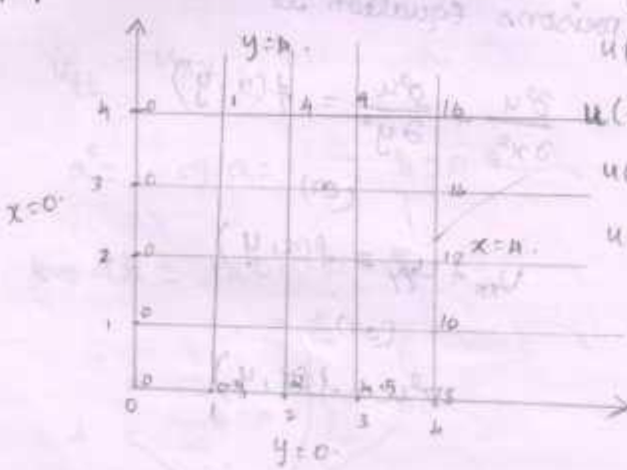


(+) SFPF : $U_E = \frac{U_B + U_D + U_F + U_H}{4}$.

(x) DFPF : $U_E = \frac{U_A + U_C + U_G + U_I}{4}$

1. By Liebmann iteration method solve $u_{xx}+u_{yy}$ over the ~~surface~~ square region of side 4 satisfying $u(0,y)=0$ $0\le y\le 4$; $u(4,y)=8+2y$ $u(x,0)=x^2/2$ $0\le x\le 4$, $u(x,4)=x^2$, $0\le x\le 4$. Com[pute] the values at the interior points with $h=k=$

Soln:



$u(0,y)=0$

$u(4,y)=8+$

$u(x,0)=x^2/2$

$u(x,4)=x^2$

Rough values:

SFPF : $u_5 = \frac{0+4+12+2}{4} = 4.5$

DFPF : $u_1 = \frac{0+4+0+u_5}{4} = 2.1$

DFPF : $u_3 = \frac{4+16+12+u_5}{4} = 9.1$

DFPF : $u_7 = \frac{0+u_5+0+2}{4} = 1.6$

DFPF : $u_9 = \frac{u_5+12+2+8}{} = 6.6$

$u_1 = \frac{1+u_2+u_4}{4}$	$u_2 = \frac{4+u_1+u_3+u_5}{4}$	$u_3 = \frac{18+u_2+u_6}{4}$	$u_4 = \frac{u_1+u_5+u_7}{4}$	$u_5 = \frac{u_2+u_4+u_6+u_8}{4}$	$u_6 = \frac{12+u_5+u_3+u_9}{4}$	$u_7 = \frac{0.5+u_4+u_8}{4}$	$u_8 = \frac{2+u_5+u_7+u_9}{4}$	$u_9 = \frac{14.5+u_6+u_8}{4}$
2.1	4.9.	9.1	2.1	4.5	8.1	1.6	3.7.	6.6
2.	4.9.	9.	2	~~2.8~~ 4.7.	8.1.	~~2.7~~ 1.6	3.7.	6.6.
2.	4.9.	9	2.1	4.7.	8.1	1.6	3.7.	6.6.
2	4.9	9	2.1	4.7	8.1	1.6	3.7	6.6

Questions	opt1	opt2	opt3	opt4	Answer
If B^2-4AC = 0, then the differential equation is said to be______.	parabolic	elliptic	hyperbolic	equally spaced	parabolic
If B^2-4AC > 0, then the differential equation is said to be______.	parabolic	elliptic	hyperbolic	equally spaced	hyperbolic
If B^2-4AC < 0, then the differential equation is said to be______.	parabolic	elliptic	hyperbolic	equally s	elliptic
The differential equation is said to be parabolic, if	B^2-4AC	B^2-4AC > 0	B^2-4AC < 0	B^2-4AC =0	B^2-4AC
The differential equation is said to be elliptic, if	B^2-4AC	B^2-4AC > 0	B^2-4AC < 0	B^2-4AC =0	B^2-4AC < 0
The differential equation is said to be hyperbolic, if	B^2-4AC	B^2-4AC > 0	B^2-4AC < 0	B^2-4AC =0	B^2-4AC > 0
The differential equation is said to be _____________ in a region R if B^2 - 4AC >0 at all points of a region.	Parabolic	elliptic	hyperbolic	rectangu	hyperbolic
The differential equation is said to be _____________ in a region R if B^2 - 4AC < 0 at all points of a region.	Parabolic	elliptic	hyperbolic	rectangu	elliptic
The differential equation is said to be _____________ in a region R if B^2- 4AC = 0 at all points of the region.	Parabolic	elliptic	hyperbolic	rectangular hyperbolic	Parabolic
One dimensional heat equation is the example of _______ equation.	Laplace	Poisson	Parabolic	Hyperbolic	Parabolic
One dimensional wave equation is the example of _______ equation.	elliptic	rectangular l	Parabolic	Hyperbolic	Hyperbolic
Two dimensional heat equation is the example of _______ equation.	elliptic	rectangular l	Parabolic	Hyperbolic	elliptic
Poisson equation is an example of ________equ	Parabolic	elliptic	hyperbolic	rectangular	elliptic
______________ equation is an example of parabalic equation.	One dimens	One dimensi	Poisson	Laplace	One dimensional heat
___________equation is an example of hyperbolic equation.	One dimens	One dimensi	Poisson	Laplace	One dimensional wave
__________ equation is an example of elliptic eq	One dimens	One dimensi	Poisson	Laplace	Poisson
_________ equation is the example of elliptic equation.	One dimens	One dimensi	Poisson	Laplace	Laplace
(f(x+h)-f(x))/h is known as the __________	difference q	average	derivative	f(x)	difference quotient
The equation del^2(u) = 0 is ________ equation.	Laplace	Poisson	Heat	Wave	Laplace

[x f(xx)+yf(yy)]=0 x>0, y>0 is _____ type of equation.	elliptic	Poisson	Parabolic	Hyperbo	elliptic
[f(xx)-2f(yy)]=0, x>0, y>0 is _____ type of equation.	elliptic	Poisson	Parabolic	Hyperbolic	Hyperbolic
The equation del^ 2(u) = f(x, y) is known as__________ equation	Poisson	Newtons	Jacobis	Gaussian	Poisson
_____________ process is used to solve two dimensional heat equations	Explicit	Bender-Schmidt	Crank-Nicolson	Liebmanns iteration	Liebmanns iteration
One dimensional heat equation can be solved using ________ method.	Newtons	Crank-Nicolson	elimination	Liebman	Crank-Nicolson
One dimensional heat equation can be solved using ________ method.	Newtons	Bender-Schmidt	elimination	Liebmanns iteration	Bender-Schmidt
One dimensional wave equationcan be solved using ________ method.	Explicit	Bender-Schn	Crank-Nico	Liebman	Explicit
Poisson equationcan be solved using ________ I	Explicit	Bender-Schn	Crank-Nico	Liebman	Liebmanns iteration
Liebmanns iteration process is used to solve ----------- equations.	One dimens	One dimensi	two dimen	Paraboli	two dimensional heat
_____________ equation can be solved using Crank-Nicolson method.	One dimens	two dimensi	One dimen	Poisson	One dimensional heat
__________equation can be solved using Bender-Schmidt method.	One dimens	two dimensi	One dimen	Poisson	One dimensional heat
___________ equation can be solved using Explicit method.	two dimens	One dimensi	One dimen	Poisson	One dimensional wave
__________ equation can be solved using Liebm	Parabolic	One dimensi	Poisson	One dim	Poisson
Crank-Nicolson method is also called as _______	Explicit	Implicit	eliminatior	reductio	Implicit
Bender-Schmidt method is also called as ______	Explicit	Implicit	eliminatior	reductio	Explicit
Liebmanns iteration process is used to solve ----------- equations.	Parabolic	elliptic	hyperbolic	rectangular hyperbolic	elliptic
_____________ equations can be solved using Crank-Nicolson method.	Parabolic	elliptic	hyperbolic	rectangular hyperbolic	Parabolic

__________equations can be solved using Bender-Schmidt method.	Parabolic	elliptic	hyperbolic	rectangular hyperbolic	Parabolic
___________ equatios can be solved using Explicit method.	Parabolic	elliptic	hyperbolic	rectangular hyperbolic	hyperbolic
Diagonal five point formula and standard five p	Parabolic	elliptic	hyperbolic	rectangular hyperbolic	elliptic
The number of conditions required to solve Laplace equation is______.	4	3	2	1	4
The number of conditions required to solve Poisson equation is______.	4	3	2	1	4
The number of conditions required to solve One dimensional heat equation is______.	4	3	2	1	3
The number of conditions required to solve one dimensional wave equation is______.	4	3	2	1	4
The error in solving Poisson equation by _____________ methods is of order h^2.	Difference	iteration	eliminatior	interpol	Difference
The error in solving __________equation by difference method is of order h^2.	Newton's	Jacobi's	Poisson	Gaussiar	Poisson
The error in solving Poisson's equation by difference methods is of order________.	h	h^2	h^3	h^4	h^2
The _____________ formula is used to complete the improved value of u,	Newtons	elimination	Liebmanns iteration	reduction	Liebmanns iteration
The value of u can be improved by _____________ process	Newtons	elimination	Liebmanns iteration	reduction	Liebmanns
The value of u is obtained at any _____________ lattice points which is the arithmetic mean of the values of u at 4 lattice points near to it	interior	exterior	positive	negative	interior
The value of $u_{i,j}$ in the difference equation are defined only at the __________points	equal	unequal	apex	lattice	lattice
The points of intersection of these families of lines are called _________ points	equal	unequal	apex	lattice	lattice
the solution decrease with the increasing value of _____________	k	a	(ka)/h	k/h	(ka)/h

If (ka)/h < 1, it is stable but the accuracy of the solution decrease with the increasing value of _____________	k	a	k/h	(ka)/h	(ka)/h
Schmidt method belongs to _____________ type	explicit	implicit	elliptic	hyperbo	explicit
The value of $u_{i,j}$ is the average of its value at the ________ neighbouring diagonal mesh points	2	3	4	5	4
The value of u(i,j) is the ___________of its values at the four neighbouring diagonal mesh points	sum	difference	average	product	average
The value of u(i,j) is the average of its values at the four neighbouring _____________ mesh points	Square	rectangle	diagonal	column	diagonal
The mesh points are also called _____________	grid point	starting poin	Ending poi	bisectior	grid point
The points of intersection of the dividing lines are called _____________	bisection	mesh points	vertex	end poin	mesh poin

ıts

UNIT V

LAPLACE TRANSFORMS

1.1 Introduction

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer **Oliver Heaviside (1850-1925)** to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by **Bromwich** and **Carson** during 1916-17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called **Laplace transforms**(Pierre Simon Marquis De Laplace, French Mathematician (1749-1827) used such transforms much earlier in 1799, while developing the theory of probability).

Laplace transform is useful since

(i) Particular solution is obtained without first determining the general solution.

(ii) non homogeneous equation are solved without obtaining the complementary integral.

(iii) Laplace transform is applicable not only to continuous functions but also to piecewise continuous functions, complicated periodic functions, step functions and impulse functions.

Before the advent of calculators and computers, logarithms were extensively used to replace multiplication (or division) of two large numbers by addition (or subtraction) of two numbers. The crucial idea which made the Laplace transform, a very powerful technique is that it replaces operations of calculus by operations of algebra.

Laplace transformation when applied to the initial value problem consisting of a single or a system of linear, ordinary differential equations, converts it into a single or a system of linear, algebraic equations in terms of the Laplace transform of the dependent variable. This equation is called the **subsidiary equation**. The initial conditions are automatically absorbed during the derivation of this algebraic equation. The solution of this algebraic equation gives the expression for the Laplace transform of the dependent variable. Taking the inverse Laplace transformation, we find the solution of the original initial value problem.

In the case of partial differential equations in terms of two independent variables, the Laplace transformation is applied with respect to one of the variables, usually the variable t(time). The resulting ordinary differential equation in terms of the second variable is solved by the usual methods of solving ordinary differential equations. The inverse laplace transform of this solution gives the solution of the given partial differential equation.

One of the important applications of Laplace transformation is the solution of the mathematical models of physical systems in which the right hand side of the differential equation, representing the driving force is discontinuous or acts for a short time only or is a periodic function (which is not necessarily a since or a cosine function).

1.2 Laplace transform

Let $f(t)$ be a given function defined for all $t \geq 0$. Laplace transform of $f(t)$ denoted by $L(f(t))$ or Simply $L(f)$ is defined as

$$L(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s) \tag{1}$$

L is known as Laplace transform operator. The original given function $f(t)$ known as determining function depends on t, while the new function to be determined $F(s)$, called as generating function, depends only on s (because the improper integral on the R.H.S of (1) is integrated with respect to t).

$F(s)$ in (1) is known as the Laplace transform of $f(t)$. Equation (1) is known as **direct transform,** or simply **transform** in which $f(t)$ is given and $F(s)$ is to be determined.

Thus Laplce transform transforms one class of complicated functions $f(t)$ to produce another class of simpler functions $F(s)$.

1.3 Applications

Laplace transform is very useful in obtaining solution of linear differential equations, both ordinary and partial, solution of system of simultaneous differential equations, solution of integral equations, solution of linear difference equations and in the evaluation of definite integrals.

1.4 Sufficient conditions for the existence of Laplace transform of $\mathbf{f(t)}$

The Laplace transform of $f(t)$ exists, when the following sufficient conditions are satisfied.

Piece-wise or sectional continuity

A function $f(x)$ is called **sectionally continuous** or piece-wise continuous in any interval $[a,b]$ if it is continuous and has finite left and right hand limits in every subinterval $[a_1,b_1]$ as shown in the graph of the function $f(x)$.

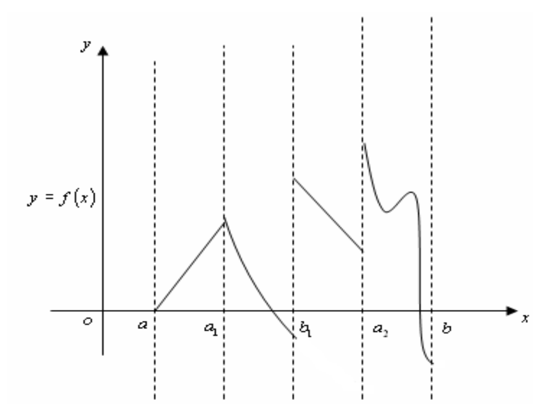


Fig. 1

Functions of exponential order

A function $f(x)$ is said to be of **exponential order** 'a' as $x \to \infty$ if $\underset{x\to\infty}{Lt}\, e^{-ax}\ f(x) =$ finite quantity.

Example:

(a) Since $\underset{t\to\infty}{Lt}\,\dfrac{t^2}{e^{3t}} =$ finite, $f(t)=t^2$ is of exponential order say 3 .

(b) Since $\underset{t\to\infty}{Lt}\,\dfrac{e^{t^2}}{e^{\alpha t}} =$ not finite, $f(t)=e^{t^2}$ is not of exponential order.

1.5 Laplace transforms of some elementary functions.

1. $L(1)=\dfrac{1}{s},(s>0)$
2. $L(t^n)=\dfrac{n!}{s^{n+1}}$, when $n=0,1,2$...........

   or $L(t^n)=\dfrac{\Gamma(n+1)}{s^{n+1}}$, when $n=0,1,2$........
3. $L(e^{at})=\dfrac{1}{s-a},(s>a)$
4. $L(\sin at)=\dfrac{a}{s^2+a^2},(s>0)$
5. $L(\cos at)=\dfrac{s}{s^2+a^2},(s>0)$
6. $L(\sin hat)=\dfrac{a}{s^2-a^2},(s>|a|)$
7. $L(\cos hat)=\dfrac{s}{s^2-a^2},(s>|a|)$

Proof

1. $L[f(t)]=\int_0^\infty e^{-st} f(t)dt$

$$\therefore\ L(1)=\int_0^\infty 1e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty=-\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$$

$$=-\frac{1}{s}[0-1]=\frac{1}{s}$$

Hence $L(1)=\dfrac{1}{s}$

In general $L(k)=\dfrac{K}{s}$, where $s>0$ and k is a constant.

2. $L[t^n]=\int_0^\infty e^{-st} f(t^n)dt$

Putting $st = x$ or $t = \frac{x}{s}$ or $dt = \frac{dx}{s}$

Thus we have $L(t^n)=\int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$

i.e., $L(t^n)=\frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n dx$

or $L(t^n)=\frac{n!}{s^{n+1}}$ [since $\Gamma(n+1)=\int_0^\infty e^{-x}x^n dx$ and $\Gamma(n+1)=n!$]

3. $L(e^{at})=\int_0^\infty e^{-st}\, e^{at} dt$

$$=\int_0^\infty e^{-st+at}dt$$

$$=\int_0^\infty e^{-(s-a)t}\, dt =\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$=-\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$=-\frac{1}{(s-a)}(0-1) \quad =\frac{1}{s-a}$$

4. $L(\sin at)=\int_0^\infty e^{-st}\sin at\, dt$

$$=\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$$=\frac{a}{s^2+a^2}$$

(or)

$L(\sin at)=L\left(\frac{e^{iat}-e^{-iat}}{2i}\right).$ (as $\sin at = \frac{e^{iat}-e^{-iat}}{2i}$)

$$=\frac{1}{2i}\left[L(e^{iat}-e^{-iat})\right]$$

$$=\frac{1}{2i}\left[L(e^{iat})-L(e^{-iat})\right]$$

$$=\frac{1}{2i}\left[\frac{1}{s-ia}-\frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2}$$

5. $L(\cos at)=\int_0^\infty e^{-st}.\cos at\, dt$

$$=\left[\frac{e^{-st}}{s^2+a^2}(-s\cos at - a\sin at)\right]_0^\infty$$

$$= -\frac{1}{s^2 + a^2}(-s)$$

$$\therefore\ L(\cos at) = \frac{s}{s^2 + a^2}$$

6. $L(\sin hat) = \int_0^\infty e^{-st} \sin hat\, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right).dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$\therefore\ L(\sin hat) = \frac{a}{s^2 - a^2}$$

7. $L(\cos hat) = \int_0^\infty e^{-st} \cos hat\, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} + e^{-at}}{2}\right).dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-st}e^{at}\,dt + \int_0^\infty e^{-st}e^{-at}\,dt\right]$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt + \int_0^\infty e^{-(s+a)t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right] = \frac{s}{s^2 - a^2}$$

$$\therefore\quad L(\cos hat) = \frac{s}{s^2 - a^2}.$$

1.6 Laplace transforms of some special functions

Heaviside's unit step function

The function

$$u(t-a) = \begin{cases} 0, & if \quad t < a \\ 1, & if \quad t > a \quad where \quad a > 0 \end{cases}$$

is called Heaviside's unit step function and is denoted by $u_a(t)$ or $u(t-a)$.

In particular when $a = 0$,

$$u(t) = \begin{cases} 0 & if \quad t < 0 \\ 1 & if \quad t > 0 \end{cases}$$

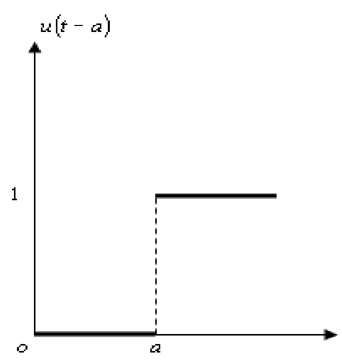


Fig. 2

Multiplying a given function $f(t)$ with the unit step function $u(t-a)$, several effects can be produced as shown in the following figure.

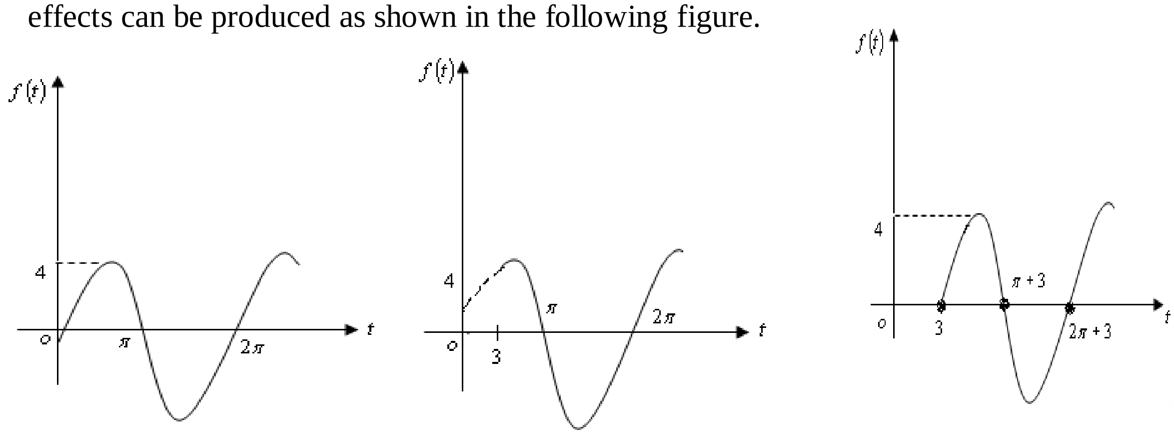


Fig. 3

$\mathbf{f(t) = 4\sin t}$	$\mathbf{f(t)\ u(t-3)}$	$\mathbf{f(t-3)u(t-3)}$
Given function	**Switching off and on**	**Shifted to the right by 3 units**

Unit impulse function (or Dirac's Delta function)

When a large force acts for a short time, then the product of the force and the time is called impulse in Fluid Mechanics.

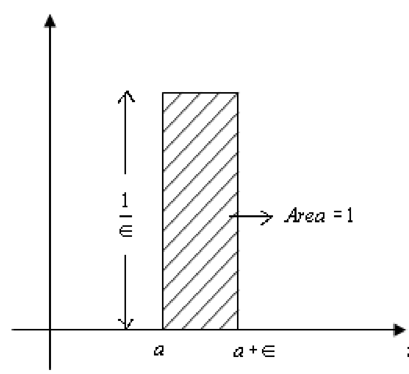


Fig. 4

Impulse of a forces $f(t)$ in the interval $(a, a+\in)$

$$=\int_a^{a+\in} f(t)dt .$$

Now define the function

$$f_\in(t-a)=\begin{cases}0 & for \quad t<a\\ \dfrac{1}{\in} & for \quad a\le t\le a+\in\\ 0 & for \quad t>a\end{cases}$$

This can also be represented interms of two unit step functions as follows.

$$f_\in(t-a)=\frac{1}{\in}[u(t-a)-u(t-(a+\in))]$$

Note that

$$\int_0^\infty f_\in(t-a)dt=\int_0^a 0+\int_a^{a+\in}\frac{1}{\in}dt+\int_{a+\in}^\infty 0=1$$

Thus the Impulse $I_\in$ is 1

Taking Laplace transform

$$L[f_\in(t-a)]=\frac{1}{\in}L[u(t-a)-u(t-(a+\in))]$$

$$=\frac{1}{\in s}\left[e^{-as}-e^{-(a+\in)s}\right]=e^{-as}\frac{\left(1-e^{-\in s}\right)}{\in s}$$

Dirac delta function (or unit impulse function) denoted by $\delta(t-a)$ is defined as the limit of $f_{\in}(t-a)$ as $\in\to 0$.

i.e., $\delta(t-a)=\underset{\in\to 0}{Lt}\, f_{\in}(t-a)$.

Laplace transform of unit step function

$$L(u_a(t))=\int_0^{\infty}e^{-st}\,u_a(t)dt$$

$$=\int_0^{\infty}e^{-st}\,u_a(t)dt+\int_a^{\infty}e^{-st}\,u_a(t)dt$$

$$=\int_0^{\infty}e^{-st}\,dt \text{ (by the definition of } u_a(t))$$

$$=\left[\frac{e^{-st}}{-s}\right]_a^{\infty}=\frac{e^{-as}}{s}, \text{ assuming that } s>0$$

In particular $L(u_0(t))=\frac{1}{s}=L(1)$.

Laplace transform of Dirac delta function

$$L(\delta(t-a))=\underset{\in\to 0}{Lt}\,L[f_{\in}(t-a)]$$

$$=\underset{\in\to 0}{Lt}\,e^{-as}\frac{\left(1-e^{-\in s}\right)}{\in s}$$

$\therefore\ L(\delta(t-a))=e^{-as}$.

1.7 Properties of Laplace transforms

1. **Linearity Property**

If a,b,c be any constants and f,g,h any functions of t, then

$$L[a\,f(t)+bg(t)-c\,h(t)]=a\,L(f(t))+b\,L(g(t))-c\,L(h(t))$$

L.H.S

$$L[a\,f(t)+bg(t)-c\,h(t)]=\int_0^{\infty}e^{-st}[a\,f(t)+bg(t)-ch(t)]dt$$

$$=a\int_0^{\infty}e^{-st}f(t)dt+b\int_0^{\infty}e^{-st}g(t)dt-c\int_0^{\infty}e^{-st}h(t)dt$$

$$=a\,L(f(t))+b\,L(g(t))-c\,L(h(t)).$$

This result can easily be generalized.

Because of the above property of L, it is called a **linear operator**.

2. **First shifting property** (or) (Translation on the s-axis or shifting on the s-axis)

If $L(f(t))=F(s)$, then $L\left(e^{at}f(t)\right)=F(s-a)$.

L.H.S

$$L\left(e^{at}f(t)\right)=\int_0^\infty e^{-st}e^{at}f(t)dt$$

$$=\int_0^\infty e^{-(s-a)t}f(t)dt$$

i.e., $L\left(e^{at}\ f(t)\right)=F(s-a)$ (since $L\ f(t)=F(s)$)

similarly we can prove

$L\left(e^{-at}\ f(t)\right)=F(s+a),$

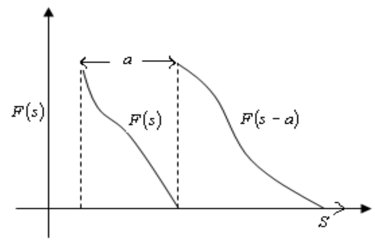


Fig. 5
Translation on the s-axis (first shifting theorem)

3. **Second Shifting Property** (or Translation on the $t-$axis)

If $L(f(t))=F(s)$, then $L[f(t-a).u(t-a)]=e^{-as}.F(s)$

L.H.S

$$L[f(t-a)u(t-a)]=\int_0^\infty e^{-st}[f(t-a)u(t-a)].dt$$

$$=\int_0^a e^{-st}\ f(t-a)\ 0\,dt+\int_a^\infty e^{-st}\ f(t-a).1.dt$$

$$=\int_a^\infty e^{-st}\ f(t-a)dt$$

$$=\int_0^\infty e^{-s(x+a)}\ f(x)dx.$$ (by putting $t-a=x, dt=dx$.

when $t=a, x=0$ when $t=\infty, x=\infty$)

$$=e^{-sa}\int_0^\infty e^{-sx}f(x)dx$$

$$=e^{-as}\int_0^\infty e^{-st}f(t)dt$$ by changing the dummy variable x as t.

i.e., $L[f(t-a)u(t-a)]=e^{-as}F(s)$.

4. **Change of scale property**

If $L(f(t))=F(s)$, then $L(f(at))=\frac{1}{a}F\left(\frac{s}{a}\right)$

L.H.S

$$L(f(at))=\int_0^\infty e^{-st}f(at).dt$$

Put $at=u$ then $dt=\frac{du}{a}$

$$=\int_0^\infty e^{-\frac{su}{a}}f(u)\frac{du}{a}$$

$$=\frac{1}{a}\int_0^\infty e^{-su/a}f(u).du\quad=\frac{1}{a}\int_0^\infty e^{-\frac{s}{a}.u}f(u).du=\frac{1}{a}F\left(\frac{s}{a}\right)$$

Note

Application of first shifting property leads to the following results:

1) $L\left(e^{at}\right)=\dfrac{1}{s-a}, \quad \because\ L(1)=\dfrac{1}{s}$

2) $L\left(e^{at}t^{n}\right)=\dfrac{n!}{(s-a)^{n+1}}, \quad \because\ L\left(t^{n}\right)=\dfrac{n!}{s^{n+1}}$

3) $L\left(e^{at}\sin bt\right)=\dfrac{b}{(s-a)^{2}+b^{2}}, \quad \because\ L(\sin bt)=\dfrac{b}{s^{2}+b^{2}}$

4) $L\left(e^{at}\cos bt\right)=\dfrac{s-a}{(s-a)^{2}+b^{2}}, \quad \because\ L(\cos bt)=\dfrac{s}{s^{2}+b^{2}}$

5) $L\left(e^{at}\sinh bt\right)=\dfrac{b}{(s-a)^{2}+b^{2}}, \quad \because\ L(\sinh bt)=\dfrac{b}{s^{2}-b^{2}}$

6) $L\left(e^{at}\cosh bt\right)=\dfrac{s-a}{(s-a)^{2}-b^{2}}, \quad \because\ L(\cosh bt)=\dfrac{s}{s^{2}-b^{2}}$

where in each case $s>a$.

Periodic function

A function $f(t)$ is said to be a periodic function of period $T>0$ if $f(t)=f(t+T)=f(t+2T)=\ldots\ldots\ldots\ldots f(t+nT)$.

Examples: $\sin t$ and $\cos t$ are periodic functions of period 2π.

Geometrically, this implies that the graph of the function $y=f(t)$ repeats itself after every interval of length T.

The following are some examples of periodic functions.

(i) **Triangular wave**

$$f(t)=\begin{cases}\dfrac{t}{a}, & 0\le t<a\\[2mm] \dfrac{2a-t}{a}, & a\le t\le 2a\end{cases}.$$

$$f(t+T)=f(t+2a)=f(t).$$

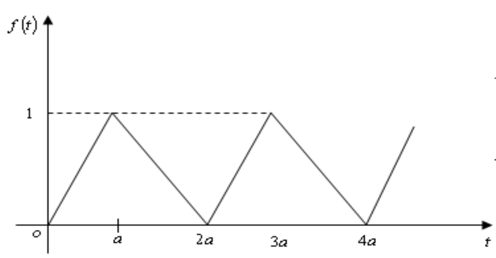


Fig. 6
Triangular wave

(ii) **Square wave**

$$f(t)=\begin{cases}k, & 0\le t<a\\ -k, & a\le t\le 2a\end{cases}$$

$$f(t+T)=f(t+2a)=f(t)$$

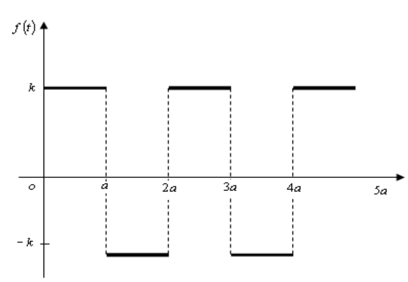


Fig. 7 Square wave

(iii) **Square wave**

$$f(t)=\begin{cases} k & , \quad 0 \le t < a \\ 0 & , \quad a \le t \le 2a \end{cases}$$

$$f(t+T)=f(t+2a)=f(t)$$

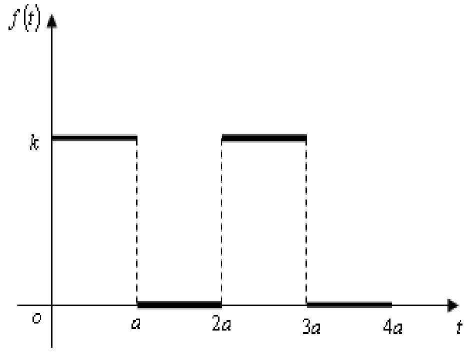


Fig. 8 Square Wave

(iv) **Sawtooth wave**

$$f(t)=t, \quad 0 \le t < a.$$

$$f(t+T)=f(t+a)=f(t).$$

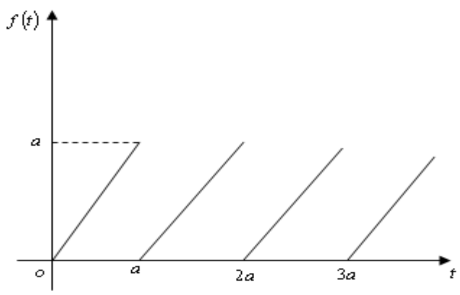


Fig. 9 Sawtooth wave

1.8 Laplace transform of periodic function:

If $f(t)$ is a periodic function with period T, i.e., $f(t+T)=f(t)$, then

$$L(f(t))=\frac{1}{1-e^{-ST}}\int_O^T e^{-st} f(t)dt.$$

Proof

We have $L(f(t))=\int_0^\infty e^{-st} f(t)dt$.

$$=\int_0^T e^{-st} f(t)dt+\int_T^{2T} e^{-st} f(t)dt+\int_{2T}^{3T} e^{-st} f(t)dt+\text{..........}$$

In the second integral put $t=u+T$, in the third integral put $t=u+2T$ and so on. Then

$$L(f(t))=\int_0^T e^{-st} f(t)dt+\int_0^T e^{-s(u+T)} f(u+T)du+\int_0^T e^{-s(u+2T)} f(u+2T)du+\text{..........}.$$

$$=\int_0^T e^{-st} f(t)dt+e^{-ST}\int_0^T e^{-st} f(u)du+e^{-2ST}\int_0^T e^{-st} f(u)du+\text{..........}$$

(since $f(u)=f(u+T)=f(u+2T)$...............)

$$=\int_0^T e^{-st} f(t)dt+e^{-ST}\int_0^T e^{-st} f(t)dt+e^{-2ST}\int_0^T e^{-su} f(t)dt+\text{..........}$$

$$=\left(1+e^{-ST}+e^{-2ST}+\text{.........}\right)\int_0^T e^{-st} f(t)dt$$

$$\therefore\ L(f(t))=\frac{1}{1-e^{-ST}}\int_0^T e^{-st} f(t)dt.$$

1.9 Laplace Transform of Derivatives

If $L(f(t)) = F(s)$, then $L(f'(t)) = sF(s) - f(0)$.

Proof

$$L(f'(t)) = \int_0^\infty e^{-st} f'(t).dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt \text{. (using integration by parts)}$$

Now assuming $f(t)$ to be such that $\underset{t\to\infty}{Lt}\, e^{-st}\ f(t) = 0$

Thus $L(f'(t)) = -f(0) + s\int_0^\infty e^{-st} f(t)dt$

i.e., $L(f'(t)) = s\,F(s) - f(0)$

Similarly, $L(f''(t)) = s^2\ F(s) - sf(0) - f'(0)$

$L(f^n(t)) = s^n L\ f(t) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) .... - f^{n-1}(0)$.

1.10 Laplace Transform of $\mathbf{t^n f(t)}$. (Multiplication by $\mathbf{t^n}$)

If $L(f(t)) = F(s)$, then $L(t^n\ f(t)) = (-1)^n \dfrac{d^n}{ds^n}(F(s))$, where $n = 1, 2...........$

Proof

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds}(F(s)) = \frac{d}{ds}\left[\int_0^\infty e^{-st}\ f(t)dt\right] = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)dt$$

$$= \int_0^\infty \left(-te^{-st}\right) f(t)dt = \int_0^\infty e^{-st}(-t\ f(t)).dt$$

$$= L(-t\ f(t)) \text{ or } L(t.f(t)) = (-1)^1 \frac{d}{ds}(F(s))$$

Similarly $L(t^2 f(t)) = (-1)^2 . \dfrac{d^2}{ds^2}(F(s))$

$$L(t^3 f(t)) = (-1)^3 . \frac{d^3}{ds^3}(F(s))$$

....

....

....

$$L(t^n . f(t)) = (-1)^n . \frac{d^n}{ds^n}(F(s)).$$

1.11 Laplace Transform of $\frac{1}{t}f(t)$ (Division by t)

If $L(f(t)) = F(s)$ then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s).ds$, provided $\underset{t\to 0}{Lt}\left[\frac{1}{t}f(t)\right]$ exists.

Proof

$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt$

Integrating on both sides with respect to s, we get,

$$\int_0^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st}\ f(t)\ ds\right].dt$$

$$= \int_0^\infty \int_s^\infty f(t)e^{-st}\ ds\, dt \text{. (changing the order of integration)}$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\ ds\right].dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt \quad = \int_0^\infty e^{-st}\ \frac{f(t)}{t}dt = L\left(\frac{f(t)}{t}\right)$$

Hence $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s).ds$.

In many problems of electrical engineering, we encounter integro-differential equations. Consider a series electric circuit. Using the kirchoff's second law, we obtain that the flow of current satisfies the integro-differential equation.

$$L\frac{di}{dt} + Ri + \frac{1}{c}\int_0^t i\,d\tau = E_0 \cos\omega t$$

Many other integro-differential equations arise in the theory of electrical circuits. If Laplace transform method is to be applied, we need the formula for the Laplace transform of an integral. Such a formula is presented as follows.

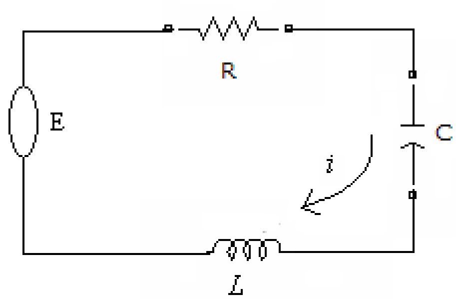


Fig. 10 Series electric circuit
C : Capacitance, E : impressed voltage
L : inductance, R : resistance

1.12 Laplace Transform of integrals

If $L(f(t)) = F(s)$, then $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$.

Proof

Let $\phi(t) = \int_0^t f(t)dt$ then $\phi'(t) = f(t)$ and $\phi(0) = 0$

We know that

$$L(\phi'(t)) = s\,L(\phi(t)) - \phi(0)$$
$$= s\,L(\phi(t)) \text{ (since } \phi(0) = 0 \text{)}$$

or $L(\phi(t)) = \frac{1}{s}L(\phi'(t))$

subsisting the values of $\phi(t)$ and $\phi'(t)$, we get

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

i.e., $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$.

Example 1

Find the Laplace transform of $e^{at} - e^{bt}$.

Solution

$$L\left[e^{at} - e^{bt}\right] = L\left(e^{at}\right) - L\left(e^{bt}\right)$$
$$= \frac{1}{s-a} - \frac{1}{s-b} = \frac{a-b}{(s-a)(s-b)}.$$ **Ans.**

Example 2

Find the Laplace transform of $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$.

Solution

$$L\left[3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\right]$$
$$= 3L\left(t^4\right) - 2L\left(t^3\right) + 4L\left(e^{-3t}\right) - 2L(\sin 5t) + 3L(\cos 2t)$$
$$= 3.\frac{4!}{s^5} - 2.\frac{3!}{s^4} + 4.\frac{1}{s+3} - 2.\frac{5}{s^2+5^2} + 3.\frac{s}{s^2+2^2}.$$ **Ans.**

Example 3

Find the Laplace transform of $\left[3t^5 - 2t^4 + 4e^{-5t} - 3\sin 6t + 4\cos 4t\right]e^{2t}$.

Solution

$$L\left[3t^5 - 2t^4 + 4e^{-5t} - 3\sin 6t + 4\cos 4t\right]$$
$$= 3L\left(t^5\right) - 2L\left(t^4\right) + 4L\left(e^{-5t}\right) - 3L(\sin 6t) + 4L(\cos 4t)$$

$$=3.\frac{5!}{s^6}-2.\frac{4!}{s^5}+4.\frac{1}{s+5}-3.\frac{6}{s^2+36}+4.\frac{s}{s^2+16}$$

Applying first shifting theorem,

$$L\left\{\left[3t^5-2t^4+4e^{-5t}-3\sin 6t+4\cos 4t\right]e^{2t}\right\}$$

$$=\frac{360}{s^6}-\frac{48}{s^5}+\frac{4}{s+5}-\frac{18}{s^2+36}+\frac{4s}{s^2+16} \text{ with } s \text{ replaced by } s-2$$

$$=\frac{360}{(s-2)^6}-\frac{48}{(s-2)^5}+\frac{4}{(s+3)}-\frac{18}{(s-2)^2+36}+\frac{4(s-2)}{(s-2)^2+16}.$$ **Ans.**

Example 4

Find the Laplace transform of (i) $e^{-3t}(2\cos 5t-3\sin 5t)$ (ii) $e^{2t}\cos^2 t$ (iii) $e^{4t}\sin 2t\cos t$.

Solution

(i) $L\left\{e^{-3t}(2\cos 5t-3\sin 5t)\right\}=2L\left(e^{-3t}\cos 5t\right)-3L\left(e^{-3t}\sin 5t\right)$

$$=2.\frac{s+3}{(s+3)^2+5^2}-3.\frac{5}{(s+3)^2+5^2}=\frac{2s-9}{s^2+6s+34}$$

(ii) Since $L\left(\cos^2 t\right)=\frac{1}{2}L(1+\cos 2t)=\frac{1}{2}\left\{\frac{1}{s}+\frac{s}{s^2+4}\right\}$

$\therefore$ By shifting property, we get

$$L\left(e^{2t}\cos^2 t\right)=\frac{1}{2}\left\{\frac{1}{s-2}+\frac{s-2}{(s-2)^2+4}\right\}$$

(iii) Since $L(\sin 2t\cos t)=\frac{1}{2}L(\sin 3t+\sin t)$

$$=\frac{1}{2}\left\{\frac{3}{s^2+3^2}+\frac{1}{s^2+1^2}\right\}$$

$\therefore$ By shifting property, we obtain

$$L\left(e^{4t}\sin 2t\cos t\right)=\frac{1}{2}\left\{\frac{3}{(s-4)^2+9}+\frac{1}{(s-4)^2+1}\right\}.$$ **Ans.**

Example 5

Find the Laplace transform of

$$f(t)=\begin{cases}1, & 0<t\le 1\\ t, & 1<t\le 2\\ 0, & t>2\end{cases}$$

Solution

$$L(f(t))=\int_0^1 e^{-st}.1.dt+\int_1^2 e^{-st}.t\,dt+\int_2^\infty e^{-st}(0).dt$$

$$=\left[\frac{e^{-st}}{-s}\right]_0^1+\left[t.\frac{e^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_1^2$$

$$=\frac{1-e^{-s}}{s}+\left\{\left[-\frac{2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}\right]-\left[\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}\right]\right\}$$

$$=\frac{1}{s}-\frac{2e^{-2s}}{s}+\frac{e^{-s}}{s^2}-\frac{e^{-2s}}{s^2}.$$ **Ans.**

Example 6

Find the Laplce transform of $t^2\ \cos at$.

Solution

$$L(\cos at)=\frac{s}{s^2+a^2}$$

$$L(t^2\cos at)=(-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right]$$

$$=\frac{d}{ds}\left[\frac{(s^2+a^2).1-s(2s)}{(s^2+a^2)^2}\right]\ =\frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$=\frac{(s^2+a^2)^2(-2s)-(a^2-s^2).2(s^2+a^2)(2s)}{(s^2+a^2)^4}$$

$$=\frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3}$$

$$=\frac{2s(s^2-3a^3)}{(s^2+a^2)^3}.$$ **Ans.**

Example 7

Obtain the Laplce transform of $t^2\ e^t.\sin 4t$.

Solution

$$L(\sin 4t)=\frac{4}{s^2+16},L(e^t.\sin 4t)=\frac{4}{(s-1)^2+16}$$

$$\therefore\ L(t\,e^t\sin 4t)=\frac{-d}{ds}\frac{4}{(s^2-2s+17)}$$

$$=\frac{4(2s-2)}{(s^2-2s+17)^2}$$

$$L(t^2\ e^t\sin 4t)=-4\frac{d}{ds}\frac{2s-2}{(s^2-2s+17)^2}$$

$$=-4\frac{\left(s^2-2s+17\right)^2.2-(2s-2)2\left(s^2-2s+17\right)(2s-2)}{\left(s^2-2s+17\right)^4}$$

$$=-4\frac{\left(2s^2-4s+34-8s^2+16s-8\right)}{\left(s^2-2s+17\right)^3}$$

$$=-4\frac{\left(-6s^2+12s+26\right)}{\left(s^2-2s+17\right)^3} \quad =\frac{8\left(3s^2-6s-13\right)}{\left(s^2-2s+17\right)^3}.$$ **Ans.**

Example 8

Find the Laplace transform of $\frac{\sin 2t}{t}$.

Solution

Here $\underset{t\to 0}{Lt}\left(\frac{\sin 2t}{t}\right)$ exists.

$L(\sin 2t)=\frac{2}{s^2+4}$

$$\therefore L\left(\frac{\sin 2t}{t}\right)=\int_s^\infty \frac{2}{s^2+4}.ds = 2.\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$=\left[\tan^{-1}\infty-\tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2}-\tan^{-1}\frac{s}{2} = \cot^{-1}\frac{s}{2}.$$ **Ans.**

Example 9

Find the Laplace transform of $t^2u(t-3)$.

Solution

$$t^2.u(t-3)=\left[(t-3)^2+6(t-3)+9\right]u(t-3)$$

$$=(t-3)^2u(t-3)+6(t-3).u(t-3)+9u(t-3)$$

$$L\left(t^2.u(t-3)\right)=L(t-3)^2.u(t-3)+6L(t-3)u(t-3)+9\,L\,u(t-3)$$

$$=e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right].$$ **Ans.**

Example 10

Evaluate (i) $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\}$

(ii) $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$

(iii) $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,dt\,dt\right\}$.

Solution

We know that $L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore \quad L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1} ds \quad = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$\therefore \; L\left\{\int_0^t \frac{\sin t}{t} dt\right\} \; = \frac{1}{s}\cot^{-1} s$$

Thus by shifting property, $L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t} dt\right)\right\} \; = \dfrac{1}{s+1}\cot^{-1}(s+1)$.

(ii) Since $L\left(\dfrac{\sin t}{t}\right) \; = \cot^{-1} s$

$$\therefore \; L\left(e^{-t}\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and $L\left\{\int_0^t e^{-t}\dfrac{\sin t}{t} dt\right\} \; = \dfrac{1}{s}\cot^{-1}(s+1)$

Hence $L\left\{t.\int_0^t e^{-t}\dfrac{\sin t}{t} dt\right\} \; = \dfrac{-d}{ds}\left\{\dfrac{\cot^{-1}(s+1)}{s}\right\}$

$$= -\frac{s\left[\dfrac{-1}{1+(s+1)^2}\right] - \cot^{-1}(s+1)}{s^2}$$

$$= \frac{s+(s^2+2s+2) - \cot^{-1}(s+1)}{s^2(s^2+2s+2)}.$$

(iii) Since $L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore \; L(t \sin t) = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$

Thus $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t).dt\,dt\,dt\right\}$.

$$= \frac{1}{s^3} L(t\sin t) \; = \frac{1}{s^3}.\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}.$$ **Ans.**

Example 11

Find $L\left[\dfrac{e^{at}-\cos 6t}{t}\right]$ and $L\left[t.e^{-t}\sin t\right]$. [AU APR 2011, AU NOV 2011].

Solution

Consider $\underset{t\to 0}{L}\left[\dfrac{e^{at}-\cos 6t}{t}\right]$

Since the limit exists, we can find $L\left[\dfrac{e^{at}-\cos 6t}{t}\right]$

$$\therefore \quad L\left[\frac{e^{at}-\cos 6t}{t}\right]=\int_s^\infty L\left(e^{at}-\cos 6t\right).ds$$

$$=\int_0^\infty \frac{1}{s-a}.ds-\int_0^\infty \frac{s}{s^2+36}.ds$$

$$=\left[\log(s-a)-\frac{1}{2}\log\left(s^2+36\right)\right]_s^\infty$$

$$=\log\left[\frac{s-a}{\left(s^2+36\right)^{1/2}}\right]_s^\infty$$

$$=\log\left[\frac{\left(1-\frac{a}{s}\right)}{\left(1+\frac{36}{s^2}\right)^{1/2}}\right]_s^\infty$$

$$=\log(1)-\log\left[\left(\frac{s-a}{s}\right)\times\frac{s}{\left(s^2+36\right)^{1/2}}\right]$$

$$=\log\left[\frac{\left(s^2+36\right)^{1/2}}{s-a}\right].$$

(ii) To find $L\left[t.e^{-t}\sin t\right]$

We know that $L(\sin t)=\dfrac{1}{s^2+1}$

$$\therefore \; L\left(e^{-t}\sin t\right)=\frac{1}{(s+1)^2+1}$$

$$\therefore \; L\left[t.e^{-t}\sin t\right]=-\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right]$$

$$=-\left[\frac{-(2s+2)}{\left(s^2+2s+2\right)^2}\right]=\frac{2(s+1)}{(s+1)^4}=\frac{2}{(s+1)^3}.$$ **Ans.**

Example 12

Find $L\left[\dfrac{e^{-at}-e^{-bt}}{t}\right]$ [AU MAY 2012].

Solution

$$L\left(e^{-at}-e^{-bt}\right)=\frac{1}{s+a}-\frac{1}{s+b}$$

Now $L\left[\dfrac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \left(\dfrac{1}{s+a}-\dfrac{1}{s+b}\right)ds$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\log\left[\frac{s+a}{s+b}\right] = \log\left[\frac{\left(1+\frac{a}{s}\right)}{\left(1+\frac{b}{s}\right)}\right]_s^\infty$$

$\therefore\ L\left(e^{-at}-e^{-bt}\right) = \log\left[\dfrac{s+b}{s+a}\right]$. **Ans.**

Example 13

Evaluate $\int_0^\infty te^{-2t}\cos t\,dt$. [AU MAY 2012]

Solution

$$\int_0^\infty t\,e^{-2t}\cos t\,dt = \int_0^\infty e^{-2t}(t\cos t)dt$$

$$= L(t\cos t) \text{ and here } s=2$$

$$=(-1)\frac{d}{ds}L(\cos t)$$

$$=(-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$=-\left[\frac{s^2+1-s(2s)}{\left(s^2+1\right)^2}\right] = -\left[\frac{-s^2+1}{\left(s^2+1\right)^2}\right]$$

$$=\frac{s^2-1}{\left(s^2+1\right)^2}.$$ **Ans.**

Example 14

Find the Laplace transform of $e^{-2t}\,t\,\sin 2t$ (or) $L\left(e^{-2t}\,t\,\sin 2t\right)$. [KU NOV 2011]

Solution

We know that $L(\sin 2t) = \dfrac{2}{s^2+4}$

$\therefore\ L\left(e^{-2t}\sin 2t\right) = \dfrac{2}{(s+2)^2+4} = \dfrac{2}{s^2+4s+8}$

Then $L\left(t\,e^{-2t}\sin 2t\right) = -\dfrac{d}{ds}\left[\dfrac{2}{s^2+4s+8}\right]$

$$= -\left[\frac{-2(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= \frac{4(s+2)}{(s^2+4s+8)^2}.$$ **Ans.**

Example 15

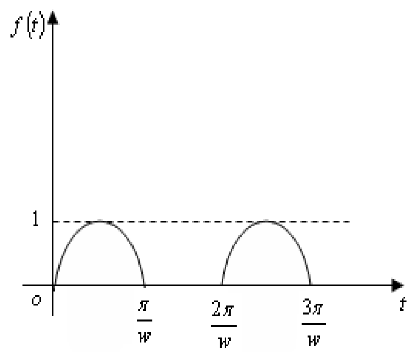
Find the Laplace transform of the function (Half wave rectifier)

$$f(t) = \begin{cases} \sin \omega t & for \quad 0 < t < \dfrac{\pi}{\omega} \\ 0 & for \quad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}.$$

Solution

Since $f(t)$ is a periodic function with period $2\pi/\omega$, we have

$$L(f(t)) = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t) dt$$



$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}(0) dt\right]$$

Fig. 11

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \mathrm{cot})\right]_0^{\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{\omega e^{-\frac{\pi s}{\omega}} + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left[1+e^{-\frac{\pi s}{\omega}}\right]}{(s^2+\omega^2)\left[1-e^{-\frac{2\pi s}{\omega}}\right]}$$

$$= \frac{\omega}{(s^2+\omega^2)\left[1-e^{-\frac{\pi s}{\omega}}\right]}.$$ **Ans.**

Example 16

Find the transform of the function defined by(triangular wave function)

$$f(t)=\begin{cases} t & 0<t<a \\ 2a-t & a<t<2a \end{cases}$$

where $f(t+2a)=f(t)$ [AU OCT 2009, AU DEC 2009, APR 2011, KU NOV 2011].

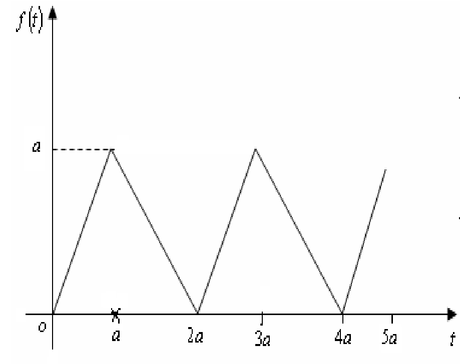


Fig. 12

Solution

The given function is periodic of period $2a$.

$$L(f(t))=\frac{1}{1-e^{-ST}}\int_0^T e^{-st}\,f(t)dt$$

$$=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}\,f(t)dt$$

$$=\frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}.t\,dt+\int_a^{2a} e^{-st}(2a-t).dt\right]$$

$$=\frac{1}{1-e^{-2as}}\left\{\left[t.\frac{e^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^a+\left[(2a-t)\frac{e^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$=\frac{1}{1-e^{-2as}}\left[-\frac{a}{s}e^{-as}-\frac{1}{s^2}e^{-as}+\frac{1}{s^2}+\frac{1}{s^2}e^{-2as}+\frac{a}{s}e^{-as}-\frac{1}{s^2}e^{-as}\right]$$

$$=\frac{1}{s^2}.\frac{1}{1-e^{-2as}}\left[1-2e^{-as}+e^{-2as}\right]$$

$$=\frac{1}{s^2}\frac{(1-e^{-as})^2}{(1-e^{-as})(1+e^{-as})}\quad=\frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})}$$

Multiply and divide by $e^{\frac{as}{2}}$

$$\therefore\ L(f(t))=\frac{1}{s^2}\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}=\frac{1}{s^2}\tan h\left(\frac{as}{2}\right).$$

Ans.

Example 17

Find the Laplace transform of the rectangular wave given by

$$f(t)=\begin{cases} 1 \quad , & 0<t<b \\ -1 \quad , & b<t<2b \end{cases}$$

with $f(t+2b)=f(t)$**.** [AU NOV 2010, AU NOV 2011]

Solution

The given function is periodic of period $2b$

Now $L(f(t))=\dfrac{1}{1-e^{-ST}}\displaystyle\int_0^T e^{-st}f(t)dt$

$$=\frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}(1)dt+\int_b^{2b} e^{-st}(-1)dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b-1\left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[-\frac{1}{s}\left(e^{-bs}-1\right)+\frac{1}{s}\left(e^{-2bs}-e^{-bs}\right)\right]$$

$$=\frac{1}{1-e^{-2bs}}\frac{1}{s}\left[e^{-2bs}-2e^{-bs}+1\right]$$

$$=\frac{\left(1-e^{-bs}\right)^2}{s\left(1-e^{-bs}\right)\left(1+e^{-bs}\right)}$$

$$=\frac{1}{s}\frac{\left(1-e^{-bs}\right)}{\left(1+e^{-bs}\right)}$$

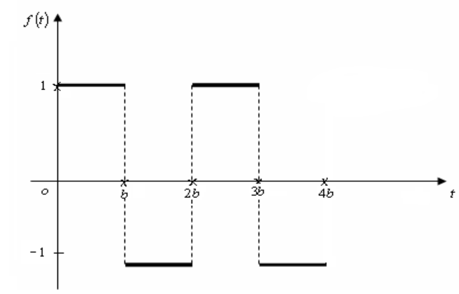


Fig. 13

Multiply and divide by $e^{\frac{bs}{2}}$

Then $L(f(t))=\frac{1}{s}\frac{e^{\frac{bs}{2}}-e^{-\frac{bs}{2}}}{e^{\frac{bs}{2}}+e^{-\frac{bs}{2}}} \quad =\frac{1}{s}\tan h\left(\frac{bs}{2}\right).$ **Ans.**

Example 18

Find the Laplace transform of the periodic function defined by the sawtooth wave. $f(t)=t, 0\le t\le a,\ f(t+a)=f(t).$

Solution

$$L(f(t))=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}\ f(t)dt$$

$$=\frac{1}{1-e^{-as}}\int_0^a t\,e^{-st}\,dt. \text{ (since } f(t+a)=f(t))$$

$$=\frac{1}{1-e^{-as}}\left[-\left(\frac{t}{s}+\frac{1}{s^2}\right)e^{-st}\right]_0^a$$

$$=\frac{1}{1-e^{-as}}\left[-\left(\frac{a}{s}+\frac{1}{s^2}\right)e^{-as}+\frac{1}{s^2}\right]$$

$$=\frac{1}{1-e^{-as}}\left[-\frac{a}{s}e^{-as}+\frac{1}{s^2}\left(1-e^{-as}\right)\right]$$

$$=\frac{1}{s^2}-\frac{ae^{-as}}{s\left(1-e^{-as}\right)}, s>0.$$ **Ans.**

1.13 Inverse Laplace transform

If $L(f(t))=F(s)$ then $f(t)$ is known as the inverse Laplace transform or inverse transform or simply inverse of $F(s)$ and is denoted by $L^{-1}(F(s))$.

Thus $f(t) = L^{-1}(F(s))$. (1)

L^{-1} is known as the inverse laplace transform operator and is such that $LL^{-1} = L^{-1}L = 1$.

In, (1), $F(s)$ is given (known) and $f(t)$ is to be determined.

Note

Inverse laplace transform of $F(s)$ need not exist for all $F(s)$.

Some important formulae

1. $L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3..........$
3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
4. $L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cos hat$
5. $L^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{a}\sin hat$
6. $L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a}\sin at$
7. $L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$
8. $L^{-1}F(s-a) = e^{at} f(t)$
9. $L^{-1}\left(\frac{1}{(s-a)^2 + b^2}\right) = \frac{1}{b}e^{at}\sin bt$
10. $L^{-1}\left(\frac{s-a}{(s-a)^2 + b^2}\right) = e^{at}\cos bt$
11. $L^{-1}\left(\frac{1}{(s-a)^2 - b^2}\right) = \frac{1}{b}e^{at}\sinh bt$
12. $L^{-1}\left(\frac{s-a}{(s-a)^2 - b^2}\right) = e^{at}\cosh bt$
13. $L^{-1}\left(\frac{1}{(s^2 + a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at)$
14. $L^{-1}\left(\frac{s}{(s^2 + a^2)^2}\right) = \frac{1}{2a}t\sin at$

15. $L^{-1}\left(\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}\right)=t\cos at$

16. $L^{-1}\left(\dfrac{s^2}{\left(s^2+a^2\right)^2}\right)=\dfrac{1}{2a}\left[\sin at+at\cos at\right]$

17. $L^{-1}\left(-\dfrac{d}{ds}F(s)\right)=t\,f(t)$

18. Linearity property

$L^{-1}(aF(s)+bG(s))=aL^{-1}(F(s))+b\,L^{-1}(G\,(s))$

19. Multiplication by s

$L^{-1}(s.F(s))=\dfrac{d}{dt}\,f(t)+\,f(0)\,\delta(t)$

20. Division by s

$L^{-1}\left(\dfrac{F(s)}{s}\right)=\int_0^t L^{-1}(F(s)).dt=\int_0^t f(t)dt$

21. First shifting property

If $L^{-1}(F(s))=\,f(t)$, then $L^{-1}(F(s+a))=e^{-at}\,L^{-1}(F(s))$

22. Second shifting property

$L^{-1}\left(e^{-as}\,F(s)\right)=\,f(t-a)u(t-a)$

23. Inverse Laplace transform of integrals

$L^{-1}\left[\int_s^\infty F(s).ds\right]=\dfrac{f(t)}{t}=\dfrac{1}{t}L^{-1}(F(s))$

(or)

$L^{-1}(F(s))=t\,L^{-1}\left[\int_s^\infty F(s).ds\right]$.

Example 1

Find $L^{-1}\left[\log\left(\dfrac{s^2+1}{(s-1)^2}\right)\right]$.

Solution

Let $f(t)=L^{-1}\left[\log\left(\dfrac{s^2+1}{(s-1)^2}\right)\right]$

$\Rightarrow\quad L(f(t))=\log\left(s^2+1\right)-\log(s-1)^2$

Then $L(t.f(t))=-\dfrac{d}{ds}\left[\log\left(s^2+1\right)-\log(s-1)^2\right]$

$$=-\left[\frac{2s}{s^2+1}-\frac{2(s-1)}{(s-1)^2}\right]\;=\frac{2}{(s-1)}-2\frac{s}{s^2+1}$$

$$\therefore \ t\,f(t) = L^{-1}\left[\frac{2}{s-1}\right] - 2\,L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= 2.e^t - 2\cos t$$

$$\therefore \ f(t) = \frac{2}{t}\left[e^t - \cos t\right].$$ **Ans.**

Example 2

Find the inverse Laplace transforms of the following

(i) $\log\left(\frac{s+1}{s-1}\right)$ (ii) $\log\left(\frac{s^2+1}{s(s+1)}\right)$ (iii) $\cot^{-1}\left(\frac{s}{2}\right)$ (iv) $\tan^{-1}\left(\frac{2}{s^2}\right)$. [KU NOV 2011]

Solution

(i) If $f(t) = L^{-1}\log\left(\frac{s+1}{s-1}\right)$

We know that $t.f(t) = L^{-1}\left\{-\frac{d}{ds}F(s)\right\}$

$$\therefore \ t\,f(t) = L^{-1}\left\{-\frac{d}{ds}.\log\left(\frac{s+1}{s-1}\right)\right\} = -L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log(s-1)\right\}$$

$$= -L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{s-1}\right) = -e^{-t} + e^t = 2\sin ht$$

Thus $f(t) = \frac{1}{t}2\sin ht$.

(ii) If $f(t) = L^{-1}\log\left(\frac{s^2+1}{s(s+1)}\right)$

$$t.f(t) = L^{-1}\left\{\frac{-d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\}$$

$$= -L^{-1}\left\{\frac{d}{ds}\log(s^2+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log s\right\} + L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}$$

$$= -L^{-1}\left(\frac{2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -2\cos t + 1 + e^{-t}$$

Thus $f(t) = \frac{1}{t}\left(1 + e^{-t} - 2\cos t\right)$.

(iii) If $f(t) = L^{-1}\cot^{-1}\left(\frac{s}{2}\right)$

$$t.f(t) = L^{-1}\left\{\frac{-d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\}$$

$$= L^{-1}\left(\frac{2}{s^2+2^2}\right) = \sin 2t$$

Thus $f(t) = \frac{1}{t}\sin 2t$.

(iv) If $f(t) = L^{-1}\left(\tan^{-1}\frac{2}{s^2}\right)$

$$t.f(t) = L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$= L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\} = L^{-1}\left\{\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$= e^t\sin t - e^{-t}\sin t = 2\sin ht\sin t$. **Ans.**

Example 3

Obtain inverse Laplace transform of

(i) $\frac{2s-5}{9s^2-25}$ (ii) $\frac{s-2}{6s^2+20}$ (iii) $\frac{3s}{2s+9}$ (iv) $\frac{1}{s(s+a)}$ (v) $\frac{s^3+3}{s(s^2+9)}$

(vi) $\frac{1}{(s+2)^5}$ (vii) $\frac{s}{s^2+4s+13}$ (viii) $\frac{1}{9s^2+6s+1}$ (ix) $\frac{e^{-\pi s}}{(s+3)}$ (x) $\frac{e^{-s}}{(s+1)^3}$.

Solution

(i) $L^{-1}\left[\frac{2s-5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9s^2-25}-\frac{5}{9s^2-25}\right]$

$$= L^{-1}\left[\frac{2s}{9\left(s^2-\frac{25}{9}\right)}-\frac{5}{9\left(s^2-\frac{25}{9}\right)}\right]$$

$$= L^{-1}\left[\frac{2s}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}-\frac{5}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}\right]$$

$$=\frac{2}{9}\cos h\frac{5}{3}t-\frac{1}{3}L^{-1}\left[\frac{\frac{5}{3}}{s^2-\left(\frac{5}{3}\right)^2}\right]$$

$$=\frac{2}{9}\cos h\frac{5}{3}t-\frac{1}{3}\sin\frac{5t}{3}.$$

(ii) $L^{-1}\left[\frac{s-2}{6s^2+20}\right] = L^{-1}\left[\frac{s}{6s^2+20}\right]-L^{-1}\left[\frac{2}{6s^2+20}\right]$

$$=\frac{1}{6}L^{-1}\left[\frac{s}{s^2+\frac{10}{3}}\right]-\frac{1}{3}L^{-1}\left[\frac{1}{s^2+\frac{10}{3}}\right]$$

$$=\frac{1}{6}\cos\sqrt{\frac{10}{3}}.t-\frac{1}{3}\sqrt{\frac{3}{10}}.L^{-1}\left[\frac{\sqrt{\frac{10}{3}}}{s^2+\frac{10}{3}}\right]$$

$$=\frac{1}{6}\cos\sqrt{\frac{10}{3}}.t-\frac{1}{\sqrt{30}}\sin\sqrt{\frac{10}{3}}.t.$$

(iii) $L^{-1}\left[\frac{3}{2s+9}\right]=\frac{3}{2}L^{-1}\left[\frac{1}{s+\frac{9}{2}}\right]=\frac{3}{2}e^{-\frac{9}{2}t}$

$$L^{-1}\left[\frac{3s}{2s+9}\right]=\frac{3}{2}\frac{d}{dt}\left(e^{-\frac{9}{2}t}\right)+\frac{3}{2}e^{-\frac{9}{2}(0)}$$

$$=-\frac{27}{4}e^{-\frac{11}{2}t+\frac{3}{2}}.$$

(iv) $L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}$

$$L^{-1}\left[\frac{1}{s(s+a)}\right]=\int_0^t L^{-1}\left(\frac{1}{s+a}\right)dt$$

$$=\int_0^t e^{-at}\,dt=\left[\frac{e^{-at}}{-a}\right]_0^t$$

$$=\frac{e^{-at}}{-a}+\frac{1}{a}=\frac{1}{a}\left[1-e^{-at}\right].$$

(v) $L^{-1}\left[\frac{s^2+3}{s\left(s^2+9\right)}\right]=L^{-1}\left[\frac{s^2+9-6}{s\left(s^2+9\right)}\right]=L^{-1}\left[\frac{1}{s}-\frac{6}{s\left(s^2+9\right)}\right]$

$$=1-2\int_0^t \sin 3t\,dt$$

$$=1-\int_0^t L^{-1}\left(\frac{6}{s^2+9}\right)ds$$

$$=1+2.\frac{1}{3}\left(\cos 3t\right)_0^t$$

$$=1+\frac{2}{3}\cos 3t-\frac{2}{3}$$

$$=\frac{2}{3}\cos 3t+\frac{1}{3}\ =\frac{1}{3}\left[2\cos 3t+1\right].$$

(vi) $L^{-1}\left[\frac{1}{s^5}\right]=\frac{t^4}{4!}$

then $L^{-1}\left[\frac{1}{\left(s+2\right)^5}\right]=e^{-2t}.\frac{t^4}{4!}$

(vii)

$$L^{-1}\left[\frac{s}{s^2+4s+13}\right]=L^{-1}\left[\frac{s+2-2}{\left(s+2\right)^2+3^2}\right]=L^{-1}\left[\frac{s+2}{\left(s+2\right)^2+3^2}\right]-L^{-1}\left[\frac{2}{\left(s+2\right)^2+3^2}\right]$$

$$=e^{-2t}.L^{-1}\left[\frac{s}{s^2+3^2}\right]-e^{-2t}.L^{-1}\,\frac{2}{3}\left[\frac{3}{s^2+3^2}\right]$$

$$=e^{-2t}\cos 3t-\frac{2}{3}e^{-2t}\sin 3t\,.$$

(viii) $L^{-1}\left[\frac{1}{9s^2+6s+1}\right]=L^{-1}\left[\frac{1}{\left(3s+1\right)^2}\right]$

$$=\frac{1}{9}L^{-1}\left[\frac{1}{\left(s+\frac{1}{3}\right)^2}\right]$$

$$=\frac{1}{9}e^{-\frac{t}{3}}L^{-1}\left(\frac{1}{s^2}\right)\ =\frac{1}{9}e^{-\frac{t}{3}}.t=\frac{te^{-\frac{t}{3}}}{9}.$$

(ix) $L^{-1}\left[\frac{1}{s+3}\right]=e^{-3t}$

$\therefore\quad L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right]=e^{-3\left(t-\pi\right)}u\left(t-\pi\right).$

(x) $L^{-1}\left(\frac{1}{s^3}\right)=\frac{t^2}{2!}$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+1)^3}\right]=e^{-t}.\frac{t^2}{2!}$$

then $\quad L^{-1}\left[\frac{e^{-s}}{(s+1)^3}\right]=e^{-(t-1)}.\frac{(t-1)^2}{2!}u(t-1)$. **Ans.**

Example 4

Find the inverse Laplace transform of $\frac{s+4}{s(s-1)(s^2+4)}$.

Solution

Let us first resolve $\frac{s+4}{s(s-1)(s^2+4)}$ into partial fractions

$$\frac{s+4}{s(s-1)(s^2+4)}=\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+4}$$

$$s+4=A(s-1)(s^2+4)+Bs(s^2+4)+(Cs+D)s(s-1) \qquad (1)$$

Putting $s=0, \quad \Rightarrow \quad A=-1$

Putting $s=1, \quad \Rightarrow \quad B=1$

Equating the coefficients of s^3 on both sides of (1), we get

$0=A+B+C \quad \Rightarrow \quad C=0$

Equating the coefficients of s on both sides of (1), we get

$1=4A+4B-D \quad \Rightarrow \quad D=-1$

On putting the values of A,B,C,D, we get

$$\frac{s+4}{s(s-1)(s^2+4)}=-\frac{1}{s}+\frac{1}{s-1}-\frac{1}{s^2+4}$$

$$\therefore \quad L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right]=L^{-1}\left[-\frac{1}{s}+\frac{1}{s-1}-\frac{1}{s^2+4}\right]$$

$$=-L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$=-1+e^t-\frac{1}{2}\sin 2t .$$ **Ans.**

Example 5

Find the inverse transform of $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$.

Solution

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6}=\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

$$2s^2-6s+5=A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$$

$$\Rightarrow \quad A=\frac{1}{2}, B=-1, C=\frac{5}{2}$$

$$\therefore \; L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right)=\frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right)-1L^{-1}\left(\frac{1}{s-2}\right)+\frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$=\frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}.$$

Ans.

Example 6

Find $L^{-1}\left[\frac{1}{(s+2)(s^2+2s+2)}\right]$.

Solution

$$\frac{1}{(s+2)(s^2+2s+2)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+2s+2}$$

$$1=A(s^2+2s+2)+(Bs+C)(s+2)$$

Put $s=-2, \quad \Rightarrow \quad A=\frac{1}{2}$

Equating the coefficients of s^2 on both sides,

$$0=A+B \quad \Rightarrow \quad B=-A=-\frac{1}{2}$$

Equating the coefficients of s on both sides,

$$0=2A+2B+C \quad \Rightarrow \quad C=-2A-2B=0$$

Now $\frac{1}{(s+2)(s^2+2s+2)}=\frac{\frac{1}{2}}{s+2}+\frac{-\frac{1}{2}s}{s^2+2s+2}$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+2)(s^2+2s+2)}\right]=\frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)-\frac{1}{2}L^{-1}\left(\frac{s+1-1}{(s+1)^2+1}\right)$$

$$=\frac{1}{2}e^{-2t}-\frac{1}{2}L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=\frac{1}{2}e^{-2t}-\frac{1}{2}e^{-t}\cos t+\frac{1}{2}e^{-t}\sin t$$

$$=\frac{1}{2}e^{-2t}-\frac{1}{2}e^{-t}(\cos t-\sin t).$$

Ans.

Example 7

Find $L^{-1}\left[\dfrac{s}{s^4+s^2+1}\right]$.

Solution

$$\frac{s}{s^4+s^2+1}=\frac{s}{\left(s^2+1\right)^2-s^2}=\frac{s}{\left(s^2-s+1\right)\left(s^2+s+1\right)}$$

$$=\frac{1}{2}\left[\frac{1}{\left(s^2-s+1\right)}-\frac{1}{\left(s^2+s+1\right)}\right]$$

$$L^{-1}\left[\frac{s}{\left(s^4+s^2+1\right)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{\left(s^2-s+1\right)}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{\left(s^2+s+1\right)}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$=\frac{1}{2}\left[\frac{2}{\sqrt{3}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}.t-\frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$=\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}.t\sin h\left(\frac{t}{2}\right).$$ **Ans.**

EXERCISE

PART A

1. Define Laplace transform.
2. State the conditions for the existence of Laplace transform of a function.
3. State change of scale property, first shifting property, second shifting property in Laplace transformation.
4. Find the Laplace transform of unit step function.
5. Find the Laplace transform of unit impulse function.
6. Find $L(f(t))$, if $f(t)=\begin{cases}\sin t & for \quad 0<t<\pi \\ t & for \quad t>\pi\end{cases}$
7. State the formula for the Laplace transform of a periodic function.
8. State the relation between the Laplace transforms of $f(t)$ and $t.f(t)$.
9. Find the relation between the inverse Laplace transform of $F(s)$ and its integral.
10. Find the inverse Laplace transform of $\log\left(\dfrac{s}{s-1}\right)$.

11. Find the laplace transform of $\frac{1-\cos at}{t}$.

12. If $L(f(t))=\frac{1}{s(s+1)}$ find $f(0)$ and $f(\infty)$.

13. Find $L(\cos 4t\sin 2t)$.

14. Find the inverse Laplace transform of $\frac{1}{s(s^2+a^2)}$.

15. Find $L\left[\int_0^t e^{-t}\,dt\right]$

16. Find $L^{-1}\left[\frac{1}{\sqrt{s+2}}\right]$.

17. If $L(f(t))=\frac{1}{s(s+a)}$, find $f(0)$.

18. State the sufficient conditions for the existence of Laplace transform of $f(t)$.

19. If $L(f(t))=F(s)$, prove that $L(f(at))=\frac{1}{a}F\left(\frac{s}{a}\right)$.

20. Find $L(e^{-at}\sin bt)$.

21. Find $L^{-1}\left[\frac{1}{(s+2)^3}\right]$.

22. Find $L(\sin^2 t)$.

23. Find $L^{-1}\left(\frac{s+2}{s^2+4s+8}\right)$.

24. Find $L\left(\frac{1-e^t}{t}\right)$.

25. Define periodic function with an example.

26. Find $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$.

27. Find $L(e^{-2t}\sin 3t)$.

28. If $L(f(t))=F(s)$, then find $L\left(f\left(\frac{t}{2}\right)\right)$.

29. Find $L^{-1}\left(\frac{s}{(s+3)^2}\right)$.

30. Find the inverse Laplace transform of $\frac{s+2}{s^2+2s+2}$.

31. Find the Laplace transform of $e^{-2t}(1+t)^2$

32. If $L(f(t))=F(s)$, what is $L(e^{-at}f(t))$.

33. Write a function for which laplace transformation does not exist. Explain why laplace transform does not exist.
34. Find $L(t\sin 2t)$.
35. Find the Laplace transform of $\dfrac{\sin 2t}{t}$.

PART B

1. Find the Laplace transform of the following
(i) $\sin^3 2t$ (ii) $e^{-t}\cos^2 t$ (iii) $\sin 2t\cos 3t$ (iv) $\sin h^3 t$
(v) $f(t)=\begin{cases} t^2 & 0<t<2 \\ t-1 & 2<t<3 \\ 7 & t>3 \end{cases}$

(Ans. (i) $\dfrac{48}{(s^2+4)(s^2+36)}$ (ii) $\dfrac{1}{2s+2}+\dfrac{s+1}{2s^2+4s+10}$ (iii) $\dfrac{2(s^2-5)}{(s^2+1)(s^2+25)}$
(iv) $\dfrac{6}{(s^2-1)(s^2-9)}$ (v) $\dfrac{2}{s^3}-\dfrac{e^{-2s}}{s^3}(2+3s+3s^2)+\dfrac{e^{-3s}}{s^2}(5s-1)$**).**

2. Find the Laplace transform of the following.
(i) $t\cos t$ (ii) $t^2\sin t$ (iii) $te^{at}\sin at$ (iv) $\int_0^t e^{-2t}.t\sin^3 t.dt$
(v) $t^2 e^{-2t}\cos t$.

(Ans. (i) $\dfrac{s^2-1}{(s^2+1)^2}$ (ii) $\dfrac{2(3s^2-1)}{(s^2+1)^3}$ (iii) $\dfrac{2a(s-a)}{(s^2-2as+2a^2)^2}$
(iv) $\dfrac{3(s+2)}{2s}\left[\dfrac{1}{[(s+2)^2+9]^2}-\dfrac{1}{[(s+2)^2+1]^2}\right]$ (v) $\dfrac{2(s^3+10s^2+25s+22)}{(s^2+4s+5)^3}$**)**

3. Find the Laplace transform of the following (i) $\dfrac{1}{t}(\cos at-\cos bt)$
(ii) $\dfrac{1}{t}\sin^2 t$ (iii) $\dfrac{1}{t}(e^{-t}\sin t)$ (iv) $\sin t u(u-4)$ (v) $e^t.u(t-1)$.

(Ans. (i) $-\dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$ (ii) $\dfrac{1}{4}\log\dfrac{s^2+4}{s^2}$ (iii) $\cot^{-1}(s+1)$
(iv) $\dfrac{e^{-4s}}{s^2+1}(\cos 4+s\sin 4)$ (v) $\dfrac{e^{-(s-1)}}{s-1}$**).**

4. Find the Laplace transform of the following.
(i) $f(t)=t^2,\ 0<t<2,\quad f(t+2)=f(t)$
(ii) $f(t)=\begin{cases} \cos\omega t\ , & 0<t<\pi/\omega \\ 0\ , & \pi/\omega<t<2\pi/\omega \end{cases}$

(iii) $f(t)=\begin{cases} t & , \quad 0<t<1 \\ 0 & , \quad 1<t<2, \end{cases} \quad f(t+2)=f(t)$

(iv) $f(t)=\begin{cases} \dfrac{2t}{T} & , \quad 0 \le t \le \dfrac{T}{2} \\ \dfrac{2}{T}(T-t) & , \quad \dfrac{T}{2} \le t \le T \end{cases} \quad , \quad f(t+T)=f(t)$

(v) $f(t)=\begin{cases} E & , \quad 0 \le t \le \dfrac{T}{2} \\ -E & , \quad \dfrac{T}{2} \le t \le T \end{cases} \quad , \quad f(t+T)=f(t)$

(Ans. (i) $\dfrac{2-e^{-2s}-4se^{-2s}-4s^2e^{-2s}}{s^3(1-e^{-2s})}$ (ii) $\dfrac{s}{(s^2+w^2)(1-e^{-\frac{\pi s}{w}})}$ (iii) $\dfrac{1-e^{-s}(s+1)}{s^2(1-e^{-2s})}$

(iv) $\dfrac{2}{Ts^2}\tanh\dfrac{sT}{4}-\dfrac{1}{s(e^{\frac{sT}{2}}+1)}$ (v) $\dfrac{E}{s}\tanh(\dfrac{sT}{4})$ **)**

5. Find the inverse Laplace transform of the following.

(i) $\dfrac{1}{s^2-9}$ (ii) $\dfrac{s}{s^2+9}$ (iii) $\dfrac{1}{(s+3)^2-4}$ (iv) $\dfrac{s+2}{(s+2)^2-25}$

(v) $\dfrac{1}{2s-7}$. **(Ans.** (i) $\dfrac{1}{3}\sin h3t$ (ii) $\cos 3t$ (iii) $\dfrac{1}{2}e^{-3t}\sin h2t$ (iv) $e^{-2t}\times\cos h5t$ (v) $\dfrac{1}{2}e^{\frac{7}{2}t}$ **)**

6. Find the inverse Laplace transform of the following.

(i) $\dfrac{3(s^2-2)^2}{2s^5}$ (ii) $\dfrac{5s-10}{9s^2-16}$ (iii) $\dfrac{2s}{3s+6}$ (iv) $\dfrac{s^2+4}{s^2+9}$

(v) $\dfrac{1}{(s-3)^2}$. **(Ans.** (i) $\dfrac{3}{2}-3t^2+\dfrac{1}{2}t^4$ (ii) $\dfrac{5}{9}\cos h\dfrac{4}{3}t-\dfrac{5}{6}\sin h\dfrac{4}{3}t$

(iii) $\dfrac{2}{3}\left(-2e^{-2t}+1\right)$ (iv) $-\dfrac{5}{3}\sin 3t+1$ (v) $e^{3t}.t$ **)**

7. Find the inverse Laplace transform of the following.

(i) $\dfrac{1}{2s(s-3)}$ (ii) $\dfrac{1}{s(s^2+a^2)}$ (iii) $\dfrac{1}{s^3(s^2+1)}$ (iv) $\dfrac{s}{(s+3)^2+4}$

(v) $\dfrac{s-4}{4(s-3)^2+16}$. **(Ans.** (i) $\dfrac{1}{2}\left[\dfrac{e^{3t}}{3}-1\right]$ (ii) $\dfrac{1-\cos at}{a^2}$ (iii) $\dfrac{t^2}{2}+\cos t-1$

(iv) $e^{-3t}\left(\cos 2t-\dfrac{3}{2}\sin 2t\right)$ (v) $\dfrac{1}{4}e^{3t}\cos 2t-\dfrac{1}{8}e^{3t}\sin 2t$ **).**

8. Obtain inverse Laplace transform of the following.

(i) $\dfrac{e^{-s}}{(s+2)^3}$ (ii) $\dfrac{e^{-\pi s}}{s^2+1}$ (iii) $\log\left(1+\dfrac{1}{s^2}\right)$ (iv) $\dfrac{s+1}{(s^2+6s+13)^2}$

(v) $\dfrac{1}{2}\log\left\{\dfrac{s^2+b^2}{(s-a)^2}\right\}$.

(Ans. (i) $e^{-(t-2)}\dfrac{(t-2)^2}{2}u(t-2)$ (ii) $-\sin t\, u(t-\pi)$ (iii) $\dfrac{2}{t}(1-\cos\omega t)$

(iv) $\dfrac{e^{-3t}}{8}[2t\sin 2t+2t\cos 2t-\sin 2t]$ (v) $\dfrac{1}{t}\left(e^{-at}-\cos bt\right)$**).**

9. Find the inverse Laplace transform of (i) $\dfrac{s^2+2s+6}{s^3}$ (ii) $\dfrac{s+2}{s^2-4s+13}$

(iii) $\dfrac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ (iv) $\dfrac{16}{(s^2+2s+5)^2}$ (v) $\dfrac{1}{(s-2)(s^2+1)}$

(Ans. (i) $1+2t+3t^2$ (ii) $e^{2t}\cos 3t+\dfrac{4}{3}e^{2t}\sin 3t$ (iii) $2e^{-t}+5e^{2t}-\dfrac{3}{2}e^{\frac{t}{2}}$

(iv) $e^{-t}(\sin 2t-2t\cos 2t)$ (v) $\dfrac{1}{5}e^{2t}-\dfrac{1}{5}\cos t-\dfrac{2}{5}\sin t$ **).**

CHAPTER II

CONVOLUTION THEOREM, APPLICATIONS OF LAPLACE TRANSFORM

2.1 Introduction

Convolution is used to find inverse Laplace transforms in solving differential equations and integral equations.

Suppose two Laplace transforms $F(s)$ and $G(s)$ are given. Let $f(t)$ and $g(t)$ be their inverse Laplace transforms respectively. i.e., $f(t)=L^{-1}(F(s))$ and $g(t)=L^{-1}(G(s))$. Then the inverse $h(t)$ of the product of transforms $H(s)=F(s).G(s)$ can be calculated from the known inverse $f(t)$ and $g(t)$.

Convolution

The convolution or convolution integral of two functions $f(t)$ and $g(t), t\geq 0$ is defined as the integral $\int_0^t f(u)g(t-u)du$.

i.e., $(f*g)(t)= f(t)*g(t)=\int_0^t f(u)g(t-u)du$.

$f*g$ is called the **convolution** or **faltung** of f and g and can be regarded as a "generalized product" of these functions.

2.2 Convolution Theorem

If $f(t)$ and $g(t)$ are two functions of t and $L(f(t))=F(s)$ and $L(g(t))=G(s)$ for $t\geq 0$ then

$L[f(t)*g(t)]=F(s).G(s)$ (or) $L^{-1}[F(s).G(s)]= f(t)*g(t)$.

Proof

By definition

$$L[f(t)*g(t)]=\int_0^\infty e^{-st}(f(t)*g(t)).dt$$

$$=\int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)du\right].dt$$

by the definition of convolution,

$$=\int_0^\infty\int_0^t e^{-st} f(u)g(t-u)du\,dt \quad (1)$$

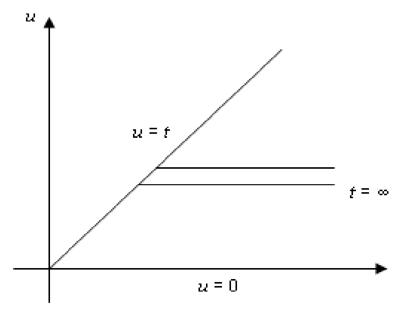


Fig. 14

The region of integration for the double integral (1) is bounded by the lines $u=0, u=t, t=0$ and $t=\infty$. Changing the order of integration in (1), we get

$$L[f(t)*g(t)]=\int_0^\infty\int_u^\infty e^{-st} f(u)g(t-u)dt\,du \quad (2)$$

In the inner integral in (2), on putting $t-u=v$, we get

$$L[f(t)*g(t)] = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)\, g(v) dv\, du$$
$$= \int_0^\infty e^{-su}. f(u)\left[\int_0^\infty e^{-sv}\, g(v) dv\right] du$$
$$= \int_0^\infty e^{-su}. f(u) du \int_0^\infty e^{-sv}\, g(v) dv$$
$$= \int_0^\infty e^{-st}. f(t) dt. \int_0^\infty e^{-st}\, g(t) dt.$$

(on changing the dummy variables u and v)

i.e., $L[f(t)*g(t)] = L(f(t)).L(g(t)).$

2.3 Initial value theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L(f(t)) = F(s)$, then $\underset{t\to 0}{Lt}(f(t)) = \underset{s\to\infty}{Lt}(s\,F(s)).$

Proof

We know that $L(f'(t)) = s\,F(s) - f(0)$

$\therefore \quad s\,F(s) = L(f'(t)) + f(0)$

$$= \int_0^\infty e^{-st}\, f'(t) dt + f(0)$$

$\therefore \quad \underset{s\to\infty}{Lt}(s\,F(s)) = \underset{s\to\infty}{Lt}\int_0^\infty e^{-st} f'(t) dt + f(0)$

$$= \int_0^\infty \underset{s\to\infty}{Lt}\left(e^{-st} f'(t)\right) dt + f(0)$$

i.e., $\underset{s\to\infty}{Lt}(s\,F(s)) = f(0) = \underset{t\to 0}{Lt}(f(t))$

$\therefore \underset{t\to 0}{Lt}(f(t)) = \underset{s\to\infty}{Lt}(s.F(s))$

2.4 Final value theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L(f(t)) = F(s)$ then $\underset{t\to\infty}{Lt}(f(t)) = \underset{s\to 0}{Lt}(s.F(s)).$

Proof

We know that $L(f'(t)) = s\,F(s) - f(0)$

$\therefore \quad s\,F(s) = L(f'(t)) + f(0)$

$$= \int_0^\infty e^{-st} f'(t).dt + f(0)$$

$\therefore \quad \underset{s\to 0}{Lt}(s.F(s)) = \underset{s\to 0}{Lt}\int_0^\infty e^{-st} f'(t) dt + f(0)$

$$= \int_0^\infty \underset{s\to 0}{Lt}\left(e^{-st} f'(t)\right) dt + f(0)$$
$$= \int_0^\infty f'(t) dt + f(0)$$
$$= [f(t)]_0^\infty + f(0)$$

$$= \underset{t\to\infty}{Lt}\, f(t) - f(0) + f(0)$$

$$\therefore\ \underset{t\to\infty}{Lt}\left(f(t)\right) = \underset{s\to 0}{Lt}\left(s.F(s)\right).$$

Example 1

Apply convolution theorem to Evaluate $L^{-1}\left(\dfrac{s}{\left(s^2+a^2\right)^2}\right)$.

[AU JUNE 2010, AU MAY 2012]

Solution

Let $F(s) = \dfrac{1}{\left(s^2+a^2\right)} \quad \Rightarrow \quad L^{-1}\left(F(s)\right) = f(t) = \dfrac{1}{a}\sin at$

$G(s) = \dfrac{s}{\left(s^2+a^2\right)} \ \Rightarrow\ L^{-1}\left(G(s)\right) = g(t) = \cos at$

Now by convolution theorem,

$$L^{-1}\left(F(s).G(s)\right) = \int_{u=0}^{t} f(u)\, g(t-u).du$$

$$= \frac{1}{a}\int_{u=0}^{t} \sin au \cos a(t-u).du$$

$$= \frac{1}{2a}\int_{u=0}^{t} \left[\sin\left(au + at - au\right) + \sin\left(au - at + au\right)\right]du$$

$$= \frac{1}{2a}\int_{u=0}^{t} \left[\sin at + \sin a(2u-t)\right]du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos a\left(2u-t\right)\right]_{u=0}^{t}$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\cos at - 0 + \frac{1}{2a}\cos at\right]$$

$$= \frac{t\sin at}{2a}.$$ **Ans.**

Example 2

Apply convolution theorem to evaluate $L^{-1}\left[\dfrac{1}{(s+3)(s-1)}\right]$ [AU APR 2011].

Solution

Let $F(s) = \dfrac{1}{s+3} \ \Rightarrow\ L^{-1}\left(F(s)\right) = f(t) = e^{-3t}$

$G(s) = \dfrac{1}{s-1} \ \Rightarrow\ L^{-1}\left(G(s)\right) = g(t) = e^{t}$

By convolution theorem

$$L^{-1}\left[\frac{1}{(s+3)(s-1)}\right]=\int_{u=0}^{t}e^{-3u}e^{(t-u)}du=e^{t}\int_{u=0}^{t}e^{-3u}.e^{-u}.du$$

$$=e^{t}\int_{u=0}^{t}e^{-4u}.du\;=e^{t}\left(\frac{e^{-4u}}{-4}\right)_{u=0}^{t}$$

$$=\frac{1}{4}e^{t}\left(1-e^{-4t}\right).$$ **Ans.**

Example 3

Evaluate $L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right]$ by convolution theorem. [KU NOV 2011]

Solution

$$L^{-1}\left(\frac{1}{s^2+1}\right)=\sin t\ ;\ L^{-1}\left(\frac{1}{s^2+4}\right)=\frac{\sin 2t}{2}$$

$\therefore$ By convolution theorem, we get

$$L^{-1}\left[\frac{1}{s^2+1}.\frac{1}{s^2+4}\right]=\int_0^t \sin u.\frac{\sin 2(t-u)}{2}du$$

$$=\frac{1}{6}\int_0^t[\cos(3u-2t)-\cos(2t-u)]du$$

$$=\frac{1}{6}\left[\frac{\sin(3u-2t)}{3}-\frac{\sin(2t-u)}{-1}\right]_0^t$$

$$=\frac{1}{6}\left[\frac{1}{3}(\sin t-\sin 2t)+(\sin t-\sin 2t)\right]$$

$$=\frac{1}{6}\left[\frac{4}{3}\sin t.-\frac{4}{3}\sin 2t\right]$$

$$=\frac{2}{9}(\sin t-\sin 2t).$$ **Ans.**

Example 4

By using convolution theorem, find the inverse laplace transform of $\frac{1}{(s+1)(s+2)}$.

Solution

$$L^{-1}\left(\frac{1}{s+1}\right)=e^{-t}\ \ ;\ L^{-1}\left(\frac{1}{s+2}\right)=e^{-2t}$$

$\therefore$ By convolution theorem, we get

$$L^{-1}\left[\frac{1}{s+1}.\frac{1}{s+2}\right]=\int_0^t e^{-u}.e^{-2(t-u)}.du$$

$$=e^{-2t}\int_0^t e^{u}.du=e^{-2t}\left(e^{t}-1\right)=e^{-t}-e^{-2t}.$$ **Ans.**

2.5 Application to Differential Equations

The Laplace transform method of solving differential equations yields particular solutions with out the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter method and is especially useful for solving linear differential equations with constant coefficients and a few integral and intergo-differential equations.

Working procedure

1. Take the Laplace transform on both sides of the differential equation. Apply the formula and the given initial conditions.
2. Transpose the terms with minus signs to the right.
3. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.
4. Resolve this function of s into partial fractions and take the inverse transform on both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Note

(i) $L(y(t)) = \bar{y}(s)$

(ii) $L(y^n(t)) = s^n \bar{y}(s) - s^{n-1} y(0) - s^{n-2} y(0) - \ldots\ldots\ldots\ldots\ldots y^{n-1}(0)$.

Example 1

Solve the Differential equation $(D^2 + 4D + 3)y = e^{-t}$. Given $y = 1, \frac{dy}{dt} = 1$ at $t = 0$ using Laplace transforms. [AU NOV 2011]

Solution

Given differential equation is $y'' + 4y' + 3y = e^{-t}$, where $y' = \frac{dy}{dt}$

Taking Laplace transform on both sides,

$$s^2 \bar{y}(s) - s\,y(0) - y'(0) + 4[s\,\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow \quad (s^2 + 4s + 3)\bar{y}(s) - s(1) - 1 - 4 = \frac{1}{s+1}$$

$$\Rightarrow \quad (s^2 + 4s + 3)\bar{y}(s) = s + 5 + \frac{1}{s+1}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{s^2 + 6s + 6}{(s+1)(s^2 + 4s + 3)} = \frac{s^2 + 6s + 6}{(s+1)(s+1)(s+3)}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{s^2 + 6s + 6}{(s+1)^2 (s+3)} \tag{1}$$

Consider $\frac{s^2 + 6s + 6}{(s+1)^2 (s+3)} = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

$$\Rightarrow \quad s^2 + 6s + 6 + A(s+1)^2 + B(s+3)(s+1) + C(s+3)$$

Put $s=-1 \quad \Rightarrow \quad c=\frac{1}{2}$

Put $s=-3 \quad \Rightarrow \quad A=-\frac{3}{4}$

Equating the coefficients of s^2,

$$1=A+B \quad \Rightarrow \quad B=1-A=1+\left(\frac{3}{4}\right)=\frac{7}{4}$$

$$\therefore \quad (1) \Rightarrow \bar{y}(s)=\frac{-(3/4)}{s+3}+\frac{(7/4)}{s+1}+\frac{(1/2)}{(s+1)^2}$$

Taking inverse transform on both sides,

$$L^{-1}(\bar{y}(s))=y(t)=L^{-1}\left[\frac{(-3/4)}{s+3}\right]+\frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$=-\frac{3}{4}e^{-3t}+\frac{7}{4}e^{-t}+\frac{1}{2}t\,e^{-t}.$$ **Ans.**

Example 2

Solve the equation $(D^2+4D+13)y=e^{-t}\sin t, y=0$ and $Dy=0$ at $t=0$, where $D=\frac{d}{dt}$. [AU JUNE 2009]

Solution

Given differential equation is $y''+4y'+13y=e^{-t}\sin t$.

Taking Laplace transforms and using the given initial conditions, we get

i.e., $(s^2+4s+13)\bar{y}(s)=\frac{1}{s^2+2s+2}$

$$\therefore \quad \bar{y}(s)=\frac{1}{(s^2+2s+2)(s^2+4s+13)}$$

$$=\frac{As+B}{s^2+2s+2}=\frac{Cs+D}{s^2+4s+13}$$

$$=\frac{1}{85}\left[\frac{-2s+7}{s^2+2s+2}+\frac{2s-3}{s^2+4s+13}\right]$$

$$=\frac{1}{85}\left[\frac{-2(s+1)+9}{(s+1)^2+1}+\frac{2(s+2)-7}{(s+2)^2+9}\right]$$

$$\therefore \quad y(t)=\frac{1}{85}\left[e^{-t}(-2\cos t+9\sin t)+e^{-2t}\left(2\cos 3t-\frac{7}{3}\sin 3t\right)\right].$$ **Ans.**

Example 3

Using Laplace transform, find the solution of the initial value problem $y''+9y=9u(t-3), y(0)=y'(0)=0$, where $u(t-3)$ is the unit step function.

Solution

Given $y''+9y=9u(t-3)$

Taking Laplace transform on both sides,

$$s^2\bar{y}(s)-s\,y(0)-y'(0)+9\bar{y}(s)=\frac{9e^{-3s}}{s} \qquad (1)$$

Putting the values of $y(0)=0$ and $y'(0)=0$ in (1), we get

$$s^2\bar{y}(s)+9\bar{y}(s)=\frac{9e^{-3s}}{s}$$

$$(s^2+9)\bar{y}(s)=\frac{9e^{-3s}}{s}$$

$$\bar{y}(s)=\frac{9e^{-3s}}{s(s^2+9)}$$

$$\Rightarrow \qquad y(t)=L^{-1}\left[\frac{9e^{-3s}}{s(s^2+9)}\right]$$

$$L^{-1}\left[\frac{3}{s^2+9}\right]=\sin 3t$$

and $\quad 3L^{-1}\left[\dfrac{3}{s(s^2+9)}\right] \quad =3\displaystyle\int_0^t \sin 3t\,dt \quad =-(\cos 3t)_0^t \quad =1-\cos 3t$

$$\therefore \qquad y(t)=L^{-1}\left[\frac{9e^{-3s}}{s(s^2+9)}\right] \quad \text{gives}$$

$y(t)=[1-\cos 3(t-3)]u(t-3)$. **Ans.**

Example 4

A resistance R in series with inductance L is connected with e.m.f $E(t)$. The current i is given by $L\dfrac{di}{dt}+Ri=E(t)$.

If the switch is connected at $t=0$ and disconnected at $t=a$, find the current i interms of t.

Solution

Conditions under which current i flows are $i=0$ at $t=0$,

$$E(t)=\begin{cases} E & , \quad 0<t<a \\ 0 & , \quad t>a \end{cases}$$

Given equation is $L\dfrac{di}{dt}+Ri=E(t)$ (1)

Taking Laplace transform of (1), we get.

$$L[s\bar{i}-i(0)]+R\bar{i}=\int_0^\infty e^{-st}E(t)dt$$

$$L(s\bar{i}) + R\bar{i} = \int_0^\infty e^{-st} E(t)\,dt. \quad (\text{since } i(0) = 0)$$

$$(Ls + R)\bar{i} = \int_0^\infty e^{-st} E.dt = \int_0^a e^{-st} E dt + \int_a^\infty e^{-st}(0)\,dt$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^a + 0 \quad = \frac{E}{s}\left[1 - e^{-as}\right] \quad = \frac{E}{s} - \frac{E}{s}e^{-as}$$

$$\bar{i} = \frac{E}{s(Ls + R)} - \frac{E\,e^{-as}}{s(Ls + R)}$$

On inversion, we obtain

$$i = L^{-1}\left[\frac{E}{s(Ls + R)}\right] - L^{-1}\left[\frac{E\,e^{-as}}{s(Ls + R)}\right] \tag{2}$$

Consider $L^{-1}\left[\dfrac{E}{s(Ls + R)}\right]$

$$L^{-1}\left[\frac{E}{s(Ls + R)}\right] = \frac{E}{L} L^{-1}\left[\frac{1}{s\left(s + \frac{R}{L}\right)}\right]$$

$$= \frac{E}{L}.\frac{L}{R}.L^{-1}\left[\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right] \quad \text{(Resolving into partial fractions)}$$

$$= \frac{E}{R}\left[1 - e^{-\frac{R}{L}.t}\right]$$

and $L^{-1}\left[\dfrac{E\,e^{-as}}{s(Ls + R)}\right] = \dfrac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right]u(t - a)$. (By second shifting theorem)

On substituting the values of the inverse transform in (2), we get.

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right]u(t - a)$$

Hence $i = \dfrac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right]$ for $0 < t < a, [u(t - a) = 0]$

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right] \quad [\text{for } t > a, u(t - a) = 1$$

$$\therefore \quad i = \frac{E}{R}\left[e^{-\frac{R}{L}(t-a)} - e^{-\frac{R}{L}t}\right] \quad = \frac{E}{R}e^{-\frac{R}{L}t}\left[e^{\frac{Ra}{L}} - 1\right].$$ **Ans.**

Example 5

Using Laplace transforms solve $y''+5y'+6y=2, y'(0)=0, y(0)=0$. [KU NOV 2010]

Solution

Given $y''+5y'+6y=2$

Taking Laplace transforms on both sides

$L(y''(t))+5L(y'(t))+6L(y(t))=L(2)$.

$s^2\bar{y}(s)-s\,y(0)-y'(0)+5\left[s\,\bar{y}(s)-y(0)\right]+6\bar{y}(s) \;=\dfrac{2}{s}$

Given $y(0)=0$ and $y'(0)=0$

$\therefore \quad s^2\bar{y}(s)+5s\bar{y}(s)+6\bar{y}(s)=\dfrac{2}{s}$

$(s^2+5s+6)\bar{y}(s)=\dfrac{2}{s}$

$\therefore \quad \bar{y}(s)=\dfrac{2}{s(s^2+5s+6)}$

i.e., $\bar{y}(s)=\dfrac{2}{s(s+2)(s+3)}$

$\therefore \quad y(t)=L^{-1}\left[\dfrac{2}{s(s+2)(s+3)}\right]$

By using partial fraction,

$\dfrac{2}{s(s+2)(s+3)}=\dfrac{A}{s}+\dfrac{B}{s+2}+\dfrac{C}{s+3}$

$2=A(s+2)(s+3)+Bs(s+3)+Cs(s+2)$

Put $s=-2 \;\Rightarrow\; B=-1$

Put $s=-3 \;\Rightarrow\; C=\dfrac{2}{3}$

Put $s=0 \;\Rightarrow\; A=\dfrac{1}{3}$

$\therefore \quad L^{-1}\left[\dfrac{2}{s(s+2)(s+3)}\right]=L^{-1}\left[\dfrac{1}{3s}\right]-L^{-1}\left[\dfrac{1}{s+2}\right]+\dfrac{2}{3}L^{-1}\left[\dfrac{1}{s+3}\right]$

i.e., $y(t)=\dfrac{1}{3}-e^{-2t}+\dfrac{2}{3}e^{-3t}$. **Ans.**

Example 6

Solve $y''-3y'+2y=4t, \quad y(0)=1, \quad y'(0)=-1$ using Laplace transforms. [KU NOV 2011]

Solution

Given $y''-3y'+2y=4t$

Taking Laplace transforms on both sides, we get

$$L(y'')-3L(y')+2L(y)=4L(t)$$

$$s^2\bar{y}(s)-s\,y(0)-y'(0)-3\left[s\,\bar{y}(s)-y(0)\right]+2\,\bar{y}(s)=\frac{4}{s^2}$$

$$s^2\bar{y}(s)-s+1-3\left[s\,\bar{y}(s)-1\right]+2\,\bar{y}(s)=\frac{4}{s^2}$$

$$\left(s^2-3s+2\right)\bar{y}(s)-s+1+3=\frac{4}{s^2}$$

$$\left(s^2-3s+2\right)\bar{y}(s)-(s-4)=\frac{4}{s^2}$$

$$\left(s^2-3s+2\right)\bar{y}(s)=\frac{4}{s^2}+(s-4)$$

$$\Rightarrow\quad \bar{y}(s)=\frac{4}{s^2\left(s^2-3s+2\right)}+\frac{s-4}{\left(s^2-3s+2\right)}$$

$$\therefore\quad y(t)=L^{-1}\left[\frac{4}{s^2\left(s^2-3s+2\right)}\right]+L^{-1}\left[\frac{s-4}{s^2-3s+2}\right]$$

$$=L^{-1}\left[\frac{16s+18}{9s^2}+\frac{(-5s+19)}{9\left(s^2-3s+2\right)}\right]+L^{-1}\left[\frac{-2}{s-2}+\frac{3}{s-1}\right]$$

$$=\frac{16}{9}L^{-1}\left(\frac{1}{s}\right)+\frac{18}{9}L^{-1}\left(\frac{1}{s^2}\right)-\frac{5}{9}L^{-1}\left(\frac{s}{s^2-3s+2}\right)+\frac{19}{9}L^{-1}\left(\frac{1}{s^2-3s+2}\right)$$
$$+L^{-1}\left(\frac{-2}{s-2}\right)+3.L^{-1}\left(\frac{1}{s-1}\right)$$

$$=\frac{16}{9}+2t-\frac{5}{9}\left[L^{-1}\left(\frac{2}{s-2}\right)-L^{-1}\left(\frac{1}{s-1}\right)\right]+\frac{19}{9}\left[L^{-1}\left(\frac{1}{s-2}\right)-L^{-1}\left(\frac{1}{s-1}\right)\right]-2e^{2t}+3e^{t}$$

$$=\frac{16}{9}+2t-\frac{5}{9}\left[2e^{2t}-e^{t}\right]+\frac{19}{9}\left[e^{2t}-e^{t}\right]+\frac{19}{9}\left[e^{2t}-e^{t}\right]-2e^{2t}+3e^{t}$$

$$=\frac{16}{9}+2t+e^{2t}\left(-\frac{10}{9}+\frac{19}{9}-2\right)+e^{t}\left(\frac{5}{9}-\frac{19}{9}+3\right)$$

$$\therefore\quad y(t)=\frac{16}{9}+2t-e^{2t}+\frac{13}{9}e^{t}\,.$$ **Ans.**

EXERCISE

PART A

1. State the initial value theorem in Laplace transforms.
2. State the final value theorem in Laplace transforms.
3. Define the convolution product of two functions and prove that it is commutative.

4. State convolution theorem in Laplace transforms.
5. Verify initial value theorem for $f(t)=1+e^{-t}(\sin t+\cos t)$.

PART B

1. Obtain the inverse Laplace transform by convolution. (i) $\dfrac{s^2}{(s^2+a^2)^2}$

(ii) $\dfrac{1}{(s^2+1)^3}$ (iii) $\dfrac{1}{s^2(s^2-a^2)}$ (iv) $\dfrac{s}{(s^2+4)(s^2+9)}$ (v) $\dfrac{10}{(s+1)(s^2+4)}$

(vi) $\dfrac{1}{s^2(s+1)^3}$ (vii) $\dfrac{s^2}{(s^2+4)^2}$ (viii) $\dfrac{1}{s(s^2+4)}$ (ix) $\dfrac{1}{s(s^2-a^2)}$ (x) $\dfrac{s^2}{s^4-a^4}$.

(**Ans.** (i) $\dfrac{1}{2}t\cos at+\dfrac{1}{2a}\sin at$ (ii) $\dfrac{1}{8}(3-t^2)\sin t-3t\cos t$

(iii) $\dfrac{1}{a^3}[-at+\sin hat]$ (iv) $\dfrac{1}{5}[\cos 2t-\cos 5t]$ (v) $2e^{-t}+\sin 2t-2\cos 2t$

(vi) $\dfrac{e^{-t}}{2}[t^2+4t+6]+t-3$ (vii) $\dfrac{1}{4}(\sin 2t+2t\cos 2t)$ (viii) $\dfrac{1}{4}(1-\cos 2t)$

(ix) $\dfrac{1}{a^2}(\cos h\,at-1)$ (x) $\dfrac{1}{2a}(\sin h\,at+\sin at)$**)**.

2. Solve the following differential equations by Laplace transform.

(i) $\dfrac{d^2y}{dx^2}+y=0,$ where $y=1, \dfrac{dy}{dx}=1$ at $x=0$.

(ii) $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+5y=0$ where $y=2, \dfrac{dy}{dx}=-4$ at $x=0$.

(iii) $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2y=0$ given $y=\dfrac{dy}{dx}=0, \dfrac{d^2y}{dx^2}=6$ at $x=0$.

(iv) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2y=1-2x$ given $y=0, \dfrac{dy}{dx}=4$ at $x=0$.

(v) $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=4x+e^{2x}$ where $y=1, \dfrac{dy}{dx}=-1$ at $x=0$.

(**Ans.** (i) $y=\sin x+\cos x$ (ii) $y=e^{-x}(2\cos 2x-\sin 2x)$

(iii) $y=e^{x}-3e^{-x}+2e^{-2x}$ (iv) $y=e^{x}-e^{-2x}+x$

(v) $y=3+2x+\dfrac{1}{2}e^{3x}-2e^{2x}-\dfrac{1}{2}e^{x}$**)**

Unit 5 Laplace Transforms

Questions	opt1	opt2	opt3	opt4	Answer
The operator L that transforms f(t) into F(s) is called the -------- operator.	Fourier	Hankel	Laplace operator	Z	Laplace operator
The Laplace transform is said to exist if the integral is --------- for some value of s; otherwise it does not exist.	discontinuous	divergent	closed	convergent	convergent
If f(t) is --------- on every finite interval in (0,∞) and is of exponentialorder 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	unifromly continuous	piecewise continuous	convergent	divergent	piecewise continuous
If f(t) is piecewise continuous on every -------- and is of exponentialorder 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	closed interval [0,1]	open interval [0,1)	infinite interval in (0,∞)	finite interval in (0,∞)	finite interval in (0,∞)
If f(t) is piecewise continuous on every finite interval in (0,∞) and is of ------- 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	exponential order	quadratic order	cubic order	n th order	exponential order
If f(t) is piecewise continuous on every finite interval in (0,∞) and is ofexponentialorder'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a. This condition is	necessary	non sufficient	Sufficient	both necessary and sufficient	Sufficient
L[1] =	n! / s^(n+1)	1/s , s > 0	1/(t+1)	1/ (s-a)	1/s , s > 0
L[t^n] =	2/(s-1)	n!	s^(n+1)	s^(n+1)	s^(n+1)
L[e^(at)] =	1/ (s-a)	0	s^(n+1)	a/(s-a)	1/ (s-a)
L[e^(-at)] =	F(s-a)	s f(0)- f '(0)	1/ (s+a)	n! / s^(n+1)	1/ (s+a)
L[sinat]=	a/(s^2 +a^2)	1/(s^2 +a^2)	(s^2 +a^2)	a/(s^3+a ^3)	a/(s^2 +a^2)
L[cosat]=	n! / s^(n+1)	s^(n+1)	t^(n+1)	s/(s^2 +a^2)	s/(s^2 +a^2)
L[coshat]=	s/(s^2 - a^2)	1/(s^3 - a^3)	s/(s^2 +a^2)	1/a F(s/a)	s/(s^2 - a^2)
L[af(t) + bg(t)]=	aF(s)+bG(s)	aF(s)-bG(s)	bF(s)-aG(s)	bF(s) * aG(s)	aF(s)+bG(s)
L[af(t) + bg(t)]= aF(s)+bG(s) is called ------property	quasi linear	non-linear	Linearity	homogenous	Linearity

Lineraity property is	L[af(t) + bg(t)]= aF(s) * bG(s)	L[af(t) + bg(t)]= aF(s)+bG(s)	1/a F(s/a)	L[af(t) + bg(t)]= aF(s)-bG(s)	L[af(t) + bg(t)]= aF(s)+bG(s)
If L[f(t)]=F(s) then L[e^at f(t)]=	aF(s)+bG(s)	F(s+a)	1-s	F(s-a)	F(s-a)
First Shifting property is if L[f(t)] = F(s) then ------	L[e^at f(t)]=F(s-a)	L[f(at)]= 1/a F(s/a)	s^2 F(s)-s f(0)- f '(0)	s^(n+1)	L[e^at f(t)]=F(s-a)
If L[f(t)]=F(s) then L[e^at f(t)]=F(s-a) is called ----property	linear	convolution	First shifting property	non homogenous	First shifting property
If L[f(t)]= F(s) then L[f(at)]=1/a F(s/a) is called ________ property.	Change of scale	convolution	First shifting property	non homogenous	Change of scale
If L[f(t)]= F(s) then L[f(at)]=	F(s/a)	1/a F(s/a)	F(s-a)	a F(s/a)	1/a F(s/a)
______is called the change of scale property	L[f(at)]= t-1	L[f(at)]= 1/(s^3 - a^3)	L[f(at)]= 1/a F(s/a)	L[e^at f(t)]=F(s-a)	L[f(at)]= 1/a F(s/a)
Change of scale property is -----	L[f(at)]= 1/a F(s/a)	L[f(at)]= F(s/a)	L[f(at)]= F(a/s)	L[f(at)]= a F(s/a)	L[f(at)]= 1/a F(s/a)
If L[f(t)]= F(s) then L[f ' (t)] =	F(s)-f(0)	s F(s)-+(0)	s F(s)-f(0)	F(s)+f(0)	s F(s)-f(0)
If L[f(t)]= F(s) then L[f '' (t)] =	s^2 F(s)-s f(0)	s^2 F(s)-s f(0)- f '(0)	s^2 F(s)-s f(0)+ f '(0)	s^2 F(s)+s f(0)+ f '(0)	s^2 F(s)-s f(0)- f '(0)
L[5 (t^3)] =	1	1/s , s > 0	3/ (s^4)	30/ (s^4)	30/ (s^4)
L[6 t] =	6	6/(s^2)	6/s	6-s	6/(s^2)
L[2 e ^ (-6 t)] =	2/(s+6)	2	2/(s-6)	2/s	2/(s+6)
L[7] =	7/s	1/s , s > 0	(-7/s)	7	7/s
L[10 sin2t]=	20/(s^2-4)	2/(s^2+4)	2/(s^2-4)	20/(s^2+4)	20/(s^2+4)
L[7 cosh3t]=	7s/(s^2-9)	7/(s^2-9)	s/(s^2-9)	7s/(s^2+9)	7s/(s^2-9)

The inverse laplace transform of 1/s is =	0	-1	s+a	1	1
The inverse laplace transform of 1/(s-a) is =	e^(-at)	1/e^(at)	e^(at)	1/e^(-at)	e^(at)
The inverse laplace transform of 1/(s+a) is =	e^(-at)	1/e^(at)	1/e^(-at)	e^(at)	e^(-at)
If L[f(t)]=F(s) then f(t) is called -------- laplace transform of F(s)	Linear	non-linear	inverse	quasi linear	inverse
If L is linear then -------- is Linear.	L+1	L^(-1)	1/L	(-1/L)	L^(-1)
If L is linear then L inverse is --------	non-linear	Linear	divergent	quasi linear	Linear
The convolution of f*g of f(t) and g(t) is defined as	(f*g)(t)=∫ (from 0 to t) f(u) g(t+u) du	(f*g)(t)=∫ (from 0 to t) f(u) du	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du	(f*g)(t)=∫ (from 0 to t) g(t-u) du	(f*g)(t)=∫ (from 0 to t) f(u) g(t-u) du
________is called the convolution theorem.	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du	(f*g)(t)= 1-t	(f*g)(t)=e^(-at)	(f*g)(t)=L^(-1)(1)	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du
A function f(t) is said to be -----with period T>0 if f(t+T)=f(t) for all t	even	projection	odd	peroidic	periodic
L[k] =	k/s	k/s , s > 0	(-1/s)	k	k/s
L[sinhat]=	a/(s^2 - a^2)	1/(s^3 - a^3)	a/(s^2 +a^2)	1/a F(s/a)	a/(s^2 - a^2)
L[e^(8t)] =	1/ (s-8)	1/s , s > 0	n! / s^(n+1)	8/(s-8)	1/ (s-8)