19BEBME101 /19BTBT101/19BEEC101/19BTFT101 MATHEMATICS – I 3 1 0 4

COURSE OBJECTIVES:

- The goal of this course is for students to gain proficiency in calculus computations. In calculus, we use three main tools for analyzing and describing the behavior of functions: limits, derivatives, and integrals.
- To familiarize the student with functions of several variables. This is needed in many branches of engineering.
- To acquaint the student with mathematical tools needed in evaluating multiple integrals and their usage.
- To make the student acquire sound knowledge of techniques in solving ordinary differential equations that model engineering problems.

COURSE OUTCOMES:

- Understanding of the ideas of limits and continuity and an ability to calculate with them and apply them.
- Improved facility in algebraic manipulation.
- Fluency in integration using standard methods, including the ability to find an appropriate method for a given integral.
- Understanding the ideas of differential equations and facility in solving simple standard examples.

UNIT I :DIFFERENTIAL CALCULUS 12

Representation of functions, New functions from old functions, Limit of a function, Limits at infinity, Continuity , Derivatives, Differentiation rules, Polar coordinate system , Differentiation in polar coordinates, Maxima and Minima of functions of one variable.

UNIT II : FUNCTIONS OF SEVERAL VARIABLES 12

Partial derivatives, Homogeneous functions and Euler's theorem, Total derivative, Differentiation of implicit functions, Change of variables, Jacobians, Partial differentiation of implicit functions, Taylor's series for functions of two variables, Errors and approximations, Maxima and minima of functions of two variables, Lagrange's method of undetermined multipliers.

UNIT III : INTEGRAL CALCULUS 12

Definite and Indefinite integrals, Substitution rule, Techniques of Integration, Integration by parts, Trigonometric integrals, Trigonometric substitutions, Integration of rational functions by partial fraction, Integration of irrational functions, Improper integrals.

UNIT IV: MULTIPLE INTEGRALS 12

Double integrals, Change of order of integration, Double integrals in polar coordinates, Area enclosed by plane curves, Triple integrals , Volume of solids, Change of variables in double and triple integrals.

UNIT V : DIFFERENTIAL EQUATIONS **12**

Method of variation of parameters, Method of undetermined coefficients, Homogenous equation of Euler's and Legendre's type, System of simultaneous linear differential equations with constant coefficients.

Total: 60

TEXT &REFERENCE BOOKS

S. NO.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAROF PUBLICATION
1.	Hemamalini. P.T	Engineering Mathematics	McGraw Hill Education (India) Private Limited, New Delhi.	2014 &2017
2.	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning	2008
3.	Narayanan S. and Manicavachagom Pillai T. K.	Calculus Volume I and II	S. Viswanathan Publishers Pvt. Ltd	2007
4.	Erwin kreyszig	Advanced Engineering Mathematics, 9th Edition,	John Wiley & Sons	2014
5.	B.S. Grewal	Higher Engineering Mathematics, 43rdEdition	Khanna Publishers	2014
6.	Ramana B.V	Higher Engineering Mathematics, 11th Reprint,.	Tata McGraw Hill New Delhi,	2010
7.	Jain R.K. and Iyengar S.R.K	Advanced Engineering Mathematics , 3rd Edition	Narosa Publications	2007
8.	Bali N., Goyal M. and Watkins C	Advanced Engineering Mathematics, 7th Edition	Firewall Media (An imprint of Lakshmi Publications Pvt., Ltd)	2009
9.	Greenberg M.D., 5th Reprint, 2009.	Advanced Engineering Mathematics, 2nd Edition,5th Reprint	Pearson Education	2009
10.	O'Neil, P.V	Advanced Engineering Mathematics	Cengage Learning India Pvt., Ltd	2007

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed University Established Under Section 3 of UGC Act, 1956)

COIMBATORE-641 021

DEPARTMENT OF SCIENCE AND HUMANITIES

FACULTY OF ENGINEERING

I B.E / B.TECH - I Semester

LESSON PLAN

SUBJECT : MATHEMATICS – I

SUB.CODE : 19BEBME101 / 19BTBT101 / 19BEEC101/19BTFT101

S.NO	Topics covered	No. of hours
	UNIT I - DIFFERENTIAL CALCULUS	
1	Introduction – Differential Calculus	1
2	Representation of a function	1
3	Representation of a function - Problems	1
4	Limit of a function - Problems	1
5	Limit of a function	1
6	Limits at infinity	1
7	Continuity	1
8	Tutorial 1 – Representation of a function, Limits and Continuity	1
9	Derivatives	1
10	Differentiation rules	1
11	Differentiation - Problems	1
12	Differentiation - Problems	1
13	Differentiation in polar coordinates	1
14	Maxima and Minima of functions of one variable	1
15	Maxima and Minima of functions of one variable - Problems	1
16	Tutorial 2 – Differentiation, Derivatives and Maxima and Minima of functions	1
	Total	**16**
	UNIT II – FUNCTIONS OF SEVERAL VARIABLES	
17	Introduction – Partial Derivatives	1
18	Homogeneous functions and Euler's theorem	1
19	Homogeneous functions and Euler's theorem	1
20	Total derivative	1
21	Differentiation of implicit functions	1
22	Change of variables	1
23	Jacobians	1
24	Partial differentiation of implicit functions	1
25	Tutorial 3 – Euler's theorem, Differentiation of implicit functions	1
26	Taylor's series for functions of two variables	1
27	Taylor's series for functions of two variables	1
28	Errors and approximations	1
29	Maxima and minima of functions of two variables	1
30	Lagrange's method of undetermined multipliers	1
31	Lagrange's method of undetermined multipliers – Problems	1
32	Tutorial 4 - Taylor's series, Lagrange's method of undetermined	1

	multipliers	
	Total	**16**
UNIT III - INTEGRAL CALCULUS		
33	Introduction – Integral Calculus	1
34	Definite Integrals	1
35	Indefinite integrals	1
36	Indefinite integrals	1
37	Substitution rule	1
38	Techniques of Integration	1
39	Techniques of Integration – Problems	1
40	Integration by parts	1
41	Tutorial 5 – Definite, Indefinite integrals, Techniques of Integration	1
42	Trigonometric integrals	1
43	Trigonometric substitutions	1
44	Integration of rational functions by partial fraction	1
45	Integration of rational functions by partial fraction	1
46	Integration of irrational functions	1
47	Improper integrals	1
48	Tutorial 6 – Integration of rational, irrational functions and improper integrals	1
	Total	**16**
UNIT IV - MULTIPLE INTEGRALS		
49	Introduction – Multiple Integrals	1
50	Double integrals	1
51	Double integrals	1
52	Change of order of integration	1
53	Change of order of integration - Problems	1
54	Double integrals in polar coordinates	1
55	Area enclosed by plane curves	1
56	Area enclosed by plane curves - Problems	1
57	Tutorial 7 - Double integrals, Change of order of integration	1
58	Triple integrals	1
59	Volume of solids	1
60	Volume of solids	1
61	Change of variables in double integrals	1
62	Change of variables in double integrals - Problems	1
63	Change of variables in triple integrals	1
64	Tutorial 8 – Volume of solids and Change of variables in double and triple integrals	1
	Total	**16**
UNIT V - DIFFERENTIAL EQUATIONS		
65	Introduction – Differential equations	1
66	Method of variation of parameters	1
67	Method of variation of parameters - Problems	1
68	Method of variation of parameters - Problems	1
69	Method of variation of parameters	1
70	Method of undetermined coefficients	1
71	Method of undetermined coefficients - Problems	1
72	Method of undetermined coefficients - Problems	1
73	Tutorial 9 - Method of variation of parameters, Method of undetermined coefficients	1

74	Homogenous equation of Euler’s and Legendre’s type	1
75	Homogenous equation of Euler’s and Legendre’s type	1
76	Homogenous equation of Euler’s and Legendre’s type	1
77	System of simultaneous linear differential equations with constant coefficients	1
78	System of simultaneous linear differential equations with constant coefficients - Problems	1
79	System of simultaneous linear differential equations with constant coefficients - Problems	1
80	Tutorial 10 - Homogenous equation of Euler’s and Legendre’s type, System of simultaneous linear differential equations with constant coefficients	1
	Total	**16**
	TOTAL	**70+10=80**

SUGGESTED READINGS

1. Hemamalini. P.T, (2014)&(2017). Engineering Mathematics, McGraw Hill Education (India) Private, Limited, New Delhi.
2. James Stewart, (2008).Calculus with Early Transcendental Functions, Cengage Learning,
3. Narayanan S. and Manicavachagom Pillai T. K., (2007).Calculus Volume I and II, S. Viswanathan Publishers Pvt. Ltd,
4. Erwin kreyszig, (2014).Advanced Engineering Mathematics, 9th Edition,John Wiley & Sons,
5. B.S. Grewal, (2014)Higher Engineering Mathematics, 43rd Edition, Khanna Publishers,
6. Ramana B.V, (2010), Higher Engineering Mathematics, 11th Reprint,., Tata McGraw Hill New Delhi,
7. Jain R.K. and Iyengar S.R.K, (2007). Advanced Engineering Mathematics , 3rd Edition, Narosa Publications,
8. Bali N., Goyal M. and Watkins C, (2009). Advanced Engineering Mathematics, 7th Edition, Firewall Media (An imprint of Lakshmi Publications Pvt., Ltd),
9. Greenberg M.D.,5th Reprint, (2009). Advanced Engineering Mathematics, 2nd Edition,5th ReprintPearson Education,
10. O’Neil, (2007).P.V, Advanced Engineering Mathematics, Cengage Learning India Pvt., Ltd,.

Staff- Incharge **HoD**

UNIT - I

DIFFERENTIAL CALCULUS

Representation of functions

(i) Function : A function f from a set D to a set E is a rule that assigns a unique element $f(x) \in E$ to each element $x \in D$.

The set D of all possible input values is called the domain of the function. The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A symbol that represents an arbitrary number in the domain of a function f is called an independant variable. A symbol that represents a number in the range of f is called a dependent variable.

(ii) Real-Valued functions :

A function whose domain and Co-domain are subsets of the set of all real numbers, is known as real-valued function.

(iii) Explicit functions

If x and y be so related that y can be expressed explicitly in terms of x, then y is called explicit function of x.

Example : $y = x+2$

(iv) Implicit functions:

If x and y be so related that y cannot be expressed explicitly in terms of x, then y is called implicit function of x.

Eg: $x^2+y^2+xy=0$.

(v) Domain, Co-domain, range and image

Let $f: A \to B$ then

Set A is called the domain of the function

Set B is called co-domain.

The set of all the images of all the elements of A under the function f is called the range of f and is denoted by $f(A)$.

Thus range of f is $f(A) = \{f(x) : x \in A\}$ clearly, $f(A) \subseteq B$

If $x \in A$, $y \in B$ and $y = f(x)$, then y is called the image of x under f.

(vi) Graph of functions

If f is a function with domain D, then its graph is the set of ordered pairs $\{x, f(x) \mid x \in D\}$.

(vii) Even function and odd function

If a function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$

for every number x in its domain

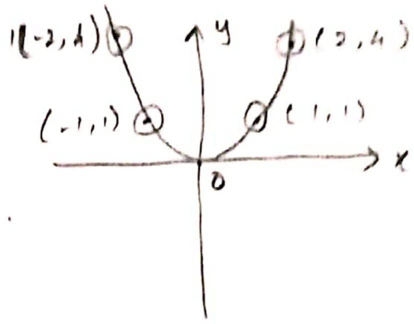
(viii) Increasing and Decreasing functions

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ then f is said to be increasing on

If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ then f is said to be decreasing o

1. Find the domain and range and sketch the graph of the function $f(x) = x^2$



Given $f(x) = x^2$ (ie) $y = x^2$

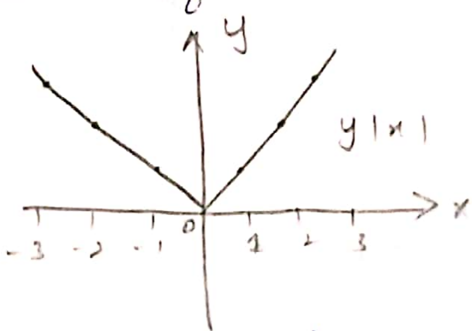
Domain x	0	1	-1	2	-2	...
Range y	0	1	1	4	4	

So the domain of f is the set of all real numbers R (ie) $(-\infty, \infty)$

The graph shows that the range is $[0, \infty)$, $[\because x^2 \geq 0]$

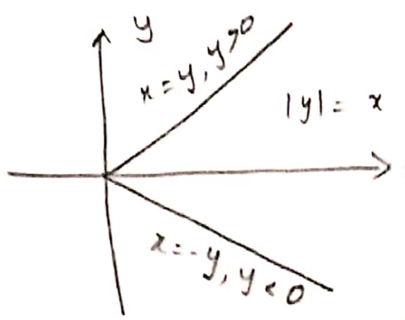
2. Sketch the graph of the absolute value function $f(x) = |x|$.

Let $y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



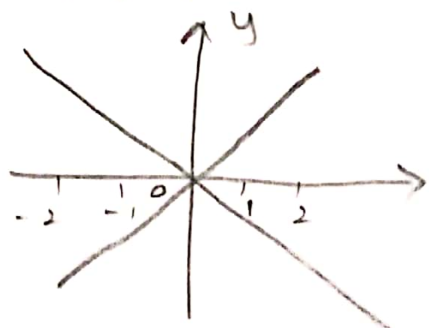
3. Graph the following equations and explain they are not graph of functions of x.

a) $|y| = x$



For each positive value of x there are two values of y.

b) $y^2 = x^2$



For each value of $x \neq 0$ there are two values of y.

4. Find the domain and sketch the graph of the function

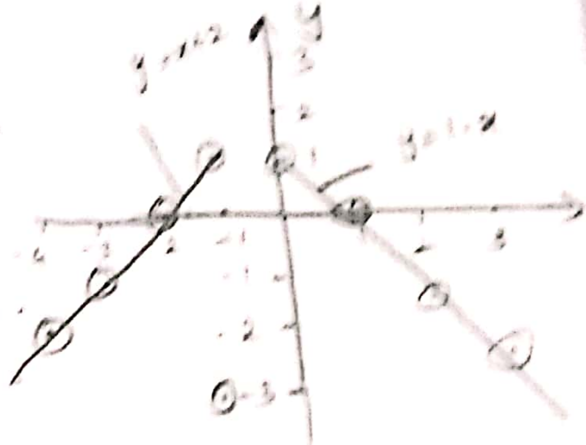
$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

Given $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

$y = x+1 \ , \ x < 0$

$y = 1-x \ , \ x \geq 0$

$x < 0$	-1	-2	-3	-4	...
$y = x+1$	1	0	-1	-2	
$x \geq 0$	0	1	2	3	...
$y = 1-x$	1	0	-1	-2	



So the domain is $(-\infty, \infty)$

1. Determine whether each of the following functions is even, odd or neither even nor odd.

1. $f(x) = x^3 + x$ — Odd
2. $f(x) = x^2 + 1$ — even
3. $f(x) = x + 1$ — Neither even nor odd
4. $f(x) = \dfrac{x}{x^2+1}$ — Odd
5. $f(x) = \dfrac{x}{x+1}$ — Neither even nor odd
6. $f(x) = 1 + 3x^2 - x^4$ — even
7. $f(x) = 3$ — even
8. $f(x) = \dfrac{1}{x^2 - 1}$ — even
9. $g(x) = 2x + 1$ — Neither even nor odd
10. $f(x) = 1 - \cos x$ — Even
11. $f(x) = x \cos x$ — Odd
12. $f(x) = e^{x^2}$ — Even

Note: $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

Find the domain and the range of each function.

a) $f(x) = 1+x^2$

Let $y = 1+x^2$

$y-1 = x^2$

$x^2 \geq 0 \Rightarrow y-1 \geq 0$

$\Rightarrow y \geq 1$

So the domain is $(-\infty, \infty)$ and the range is $[1, \infty)$

b) $f(x) = \sqrt{5x+10}$

$y = \sqrt{5x+10}$

$5x+10 \geq 0$ [$\because$ square root of a negative no is not defined as a real number]

$5x \geq -10$

$x \geq -2$

So the domain is $[-2, \infty)$ and the range is $[0, \infty)$.

c) $f(x) = \dfrac{4}{3-x}$

$y = \dfrac{4}{3-x}$ division by zero is not allowed for $x=3$, we get $3-x=0$.

So the domain is $(-\infty, 3) \cup (3, \infty)$

and the range is $(-\infty, 0) \cup (0, \infty)$

Find the domain of each function

a) $f(x) = \dfrac{x+4}{x^2-9}$

$y = \dfrac{x+4}{x^2-9}$

$x^2-9=0 \Rightarrow x = \pm 3$, division by zero is not allowed

So the domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

b) $f(x) = \sqrt[3]{2x-1}$

$y = \sqrt[3]{2x-1}$

$y^3 = 2x-1$

So the domain is $(-\infty, \infty)$

c) $f(x) = \sqrt{x+2}$

$y = \sqrt{x+2}$

$x+2 \geq 0$

$x \geq -2$

The Domain is $[-2, \infty)$

d) $f(x) = \dfrac{1}{\sqrt[4]{x^2-5x}}$

$y = \dfrac{1}{\sqrt[4]{x^2-5x}}$

For $x = 0$, we get $x^2 - 5x = 0 - 0 = 0$

$x = 5$, we get $x^2 - 5x = 25 - 25 = 0$

So, the domain is $(-\infty, 0) \cup (5, \infty)$

[$\because$ fourth root of a negative number is not defined as real no

so $(0, 5)$ not allowed]

e) $f(x) = \dfrac{1}{x^2-x}$

$y = \dfrac{1}{x^2-x}$

$x^2 - x = 0 \Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ or $x = 1$

So the domain is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Limit of a function:

A function $f(x)$ tends to a definite l as x tends to 'a' if the difference between $f(x)$ and l can be made as small as we like by making x approach sufficiently near 'a' and we write

$$\lim_{x \to a} f(x) = l$$

Left hand limit of $f(x)$:

The left hand limit of $f(x)$ as x approaches a is equal to L.

(i.e) $\lim_{x \to a^-} f(x) = L$. Here $x \to a^-$ means $x < a$.

Right hand limit of $f(x)$:

The Right hand limit of $f(x)$ as x approaches a is equal to L.

(ie) $\lim_{x \to a^+} f(x) = L$. Here $x \to a^+$ mean $x > a$.

Note: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

Definition: Infinite Limits.

Let f be a function defined on both sides of a, except possibly at a itself.

1) Then $\lim_{x \to a} f(x) = \infty$ means that $f(x)$ can be arbitrarily large by taking x sufficiently close to a, but not equal to a.

2) Then $\lim_{x \to a} f(x) = -\infty$ means that $f(x)$ can be arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

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Evaluate the following limits

a) $\lim_{x \to 5} (2x^2 - 3x + 4) = 2(5)^2 - 3(5) + 4 = 39$

b) Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} (x+1) = 1+1 = 2$$

c) Find $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$

d) Find $\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2}$

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}$$

$$= \lim_{t \to 0} \frac{t^2}{t^2 \sqrt{t^2+9} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

e) Find $\lim_{t \to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$

$$\lim_{t \to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} = \lim_{t \to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t} + \sqrt{1-t}}{\sqrt{1+t} + \sqrt{1-t}}$$

$$= \lim_{t \to 0} \frac{2t}{t(\sqrt{1+t} + \sqrt{1-t})} = \frac{2}{\sqrt{1} + \sqrt{1}} = \frac{2}{2} = 1$$

f) $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$

$$\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) = \lim_{t \to 0} \frac{1}{t} \left[\frac{1 - \sqrt{1+t}}{\sqrt{1+t}} \right]$$

$$= \lim_{t \to 0} \frac{1}{t} \frac{(1 - \sqrt{1+t})}{\sqrt{1+t}} \times \frac{(1 + \sqrt{1+t})}{(1 + \sqrt{1+t})} = \lim_{t \to 0} \left[\frac{1 - (1+t)}{\sqrt{1+t}\,(1 + \sqrt{1+t})} \right]$$

$$= \frac{1}{t} \left[\frac{-t}{(\sqrt{1+t})(1 + \sqrt{1+t})} \right] = \frac{-1}{\sqrt{1}(1+\sqrt{1})} = \frac{-1}{2}$$

g) Find $\lim_{x \to -4} \dfrac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$$\lim_{x \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \lim_{x \to -4}\left(\frac{\frac{x+4}{4x}}{4+x}\right) = \lim_{x \to -4}\left(\frac{1}{4x}\right) = \frac{1}{4(-4)} = \frac{-1}{16}$$

h) Find $\lim_{x \to 1} h(x)$ where $h(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$

Here $h(x)$ is defined at $x = 1$ and $h(1) = \pi$ the value of a limit as x approaches 1 does not depend on 1.

Since $h(x) = x+1$ and $x \neq 1$

$$\lim_{x \to 1} h(x) = \lim_{x \to 1} x+1 = 1+1 = 2.$$

i) Prove that $\lim_{x \to 0} |x| = 0$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$ for $|x| = x,\ x > 0$

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$ for $|x| = -x,\ x < 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$$

$$\therefore \lim_{x \to 0} |x| = 0$$

j) Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

$$f(x) = \frac{|x|}{x}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$$

$\lim_{x \to 0^+} \neq \lim_{x \to 0^-}$ $\quad$ $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Note : (i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$ for all rational values of n

(ii) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ measured in radians

(iii) $\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$ for all rational Values of n.

1. Find $\lim_{x \to a} \frac{x^{5/8} - a^{5/8}}{x^{1/3} - a^{1/3}}$

$$\lim_{x \to a} \frac{x^{5/8} - a^{5/8}}{x^{1/3} - a^{1/3}} = \lim_{x \to a} \left[\frac{x^{5/8} - a^{5/8}}{x - a} \cdot \frac{x - a}{x^{1/3} - a^{1/3}} \right]$$

$$= \lim_{x \to a} \left[\frac{x^{5/8} - a^{5/8}}{x - a} \cdot \frac{1}{\frac{x^{1/3} - a^{1/3}}{x - a}} \right]$$

$$= 5/8 (a^{5/8 - 1}) \cdot \frac{1}{1/3\, a^{1/3 - 1}} = \frac{15}{8} a^{5/8 - 1 - 1/3 + 1} = \frac{15}{8} a^{7/24}$$

2. Find $\lim_{\theta \to 0} \frac{\tan\theta}{\theta}$

$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta} = (1)\frac{1}{\cos 0} = (1)(1) = 1$$

3. Find $\lim_{x \to 0} \frac{1 - \cos x}{x}$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{2 \sin^2 x/2}{x} = \lim_{x \to 0} \left[\frac{\sin x/2}{x/2} \right]^2 \lim_{x \to 0} \frac{x}{2} = (1)(0) = 0$$

4. Find $\lim_{x \to 0} (1 + x)^{1/x}$

Let $1/n = x$

$$\lim_{x \to 0} (1 + x)^{1/x} = \lim_{n \to \infty} (1 + 1/n)^n = e$$

5. Find $\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2}$.

$$\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{x \to \pi/2} \frac{2\cos^2 x}{(\pi - 2x)^2} = \lim_{x \to \pi/2} \frac{2 \sin^2(\pi/2 - x)}{2^2 (\pi/2 - x)^2}$$

$$= \lim_{x \to \pi/2} \frac{1}{2} \left[\frac{\sin(\pi/2 - x)}{\pi/2 - x} \right]^2 = \lim_{(x - \pi/2) \to 0} \frac{1}{2} \left[\frac{\sin(x - \pi/2)}{(x - \pi/2)} \right]^2$$

Let $\theta = x - \pi/2$

$$\lim_{\theta \to 0} \frac{1}{2} \left(\frac{\sin\theta}{\theta} \right)^2 = 1/2 (1)^2 = \frac{1}{2}$$

Continuity

A function f is continuous at a number a if
$\lim_{x \to a} f(x) = f(a)$.

Note : 1

If f is continuous at a, then

1. $f(a)$ should exist (ie) a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists both on the left and right
3. $\lim_{x \to a} f(x) = f(a)$.

Note : 2

The function $f(x)$ is said to be discontinuous at $x = a$ if one or more of the above three conditions are not satisfied.

1. Show that $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$.

Given $f(x) = 3x^2 + 2x - 1$

$$\underset{x \to 2^-}{Lt} f(x) = \lim_{h \to 0} f(2-h)$$

$$= \lim_{h \to 0} \left[3(2-h)^2 + 2(2-h) - 1 \right]$$

$$= 3(2)^2 + 2(2) - 1 = 15$$

$$\underset{x \to 2^+}{Lt} f(x) = \lim_{h \to 0} f(2+h)$$

$$= \lim_{h \to 0} \left[3(2+h)^2 + 2(2+h) - 1 \right]$$

$$= 3(2)^2 + 2(2) - 1 = 15.$$

$$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x) = 15$$

Hence $f(x)$ is continuous at $x = 2$.

2. Verify whether the function is continuous at $x = -2$ for $f(x) = \frac{1}{x+2}$.

$f(x) = \frac{1}{x+2}$, $x = -2$

$f(-2) = \frac{1}{-2+2} = \frac{1}{0} = \infty$ = undefined

$\therefore$ $f(x)$ is discontinuous at $x = -2$.

3. Use continuity to evaluate the limit $\lim_{x \to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$

Given: $\lim_{x \to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$

$$= \frac{\lim_{x \to 4} (5+\sqrt{x})}{\lim_{x \to 4} \sqrt{5+x}} = \frac{5+\sqrt{\lim_{x \to 4} x}}{\sqrt{5+\lim_{x \to 4} x}}$$

$$= \frac{5+\sqrt{4}}{\sqrt{5+4}} = \frac{5+2}{\sqrt{9}} = \frac{7}{3}$$

1. $y = xe^x$, $y = x^4 e^x$ find y' & y''

2. $y = (x^3 + 2x)e^x$

3. $y = (1 - e^x)(x + e^x)$

4. $y = x^4 - \sin x$

5. $y = x^3 \sin x$

6. $y = \dfrac{\sin x}{x}$

7. $y = \dfrac{\cos x}{1 - \sin x}$

8. $y = x e^x \operatorname{cosec} x$

9. $y = (1 - x^2)^{10}$

10. $y = x e^x \log x$

11. $y = \sqrt{\cos \sqrt{x}}$

12. $y = \log(\sin x)$

13. $y = e^{\sqrt{x}}$

14. $y = \sin^5 x$

15. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

16. $y = \dfrac{1}{(t^4 + 1)^3}$

Maxima & Minima

If a Continuous function increases upto a Certain value and then decreases, that value is called a maximum value of the function. III^{ly} if a constant function, decreases upto a Certain value and then increases, that value is Called minimum value of the function.

Definition :

(i) $f(x)$ is maximum at $x=a$ if $f'(a)=0$ & $f''(a)$ is $-ve$.

(ii) $f(x)$ is minimum at $x=a$ if $f'(a)=0$ & $f''(a)$ is $+ve$.

Procedure for finding maxima and Minima

1) Write the given function $f(x)$

2) Find $f'(x)$ and equate it to zero. Solve this eqn & let the roots are $a, b, c, \ldots$

3) Find $f''(x)$ and substitute in it by terms $x = a, b, c \ldots$

If $f''(a)$ is $-ve$, $f(x)$ is maximum at $x=a$.

If $f''(a)$ is $+ve$, $f(x)$ is minimum at $x=a$.

4) Sometimes $f''(x)$ may be difficult to find out or $f''(x)$ may be zero at $x=a$. In such cases, see if

$f'(x)$ Changes sign from $+ve$ to $-ve$ as x passes through a, then $f(x)$ is max at $x=a$.

5) If $f'(x)$ Changes Sign from $-ve$ to $+ve$ as x passes through a, $f(x)$ is min at $x=a$.

If $f'(x)$ doesnot Change sign while passing through $x=a$, $f(x)$ is neither max nor min at $x=a$.

1. Find the maxima & minima of the function $2x^3 - 3x^2 - 36x + 10$

let $f(x) = 2x^3 - 3x^2 - 36x + 10$

$f'(x) = 6x^2 - 6x - 36 = 0$

$\Rightarrow x = -2, 3$

$f''(x) = 12x - 6$

$f''(x)_{at\, x=-2} = -30 = -ve$

$f(x)$ is max at $x = -2$.

$f(-2) = 54$

$f''(x)_{at\, x=3} = 30 = +ve$

$f(x)$ is min at $x = 3$

$f(3) = -71$

2. Find the max & min values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval $(0, 2)$

FUNCTIONS OF SEVERAL VARIABLES

Partial Derivatives

Let $z = f(x, y)$ be a function of two variables x and y. The derivative of z w.r.to x treating y as constant, is called the partial derivative of z w.r.to x.

Notation: $\frac{\partial z}{\partial x} = p$; $\frac{\partial z}{\partial y} = q$; $\frac{\partial^2 z}{\partial x^2} = r$; $\frac{\partial^2 z}{\partial x \partial y} = s$; $\frac{\partial^2 z}{\partial y^2} = t$

Homogeneous functions

An expression in x and y in which the sum of the indices of the variable x and y in each term is the same is called a homogeneous function.

In general, a homogeneous function of degree n in x and y can be written as $f(x, y) = x^n g\left(\frac{y}{x}\right)$

Euler's theorem:

If u be a homogeneous function of degree n in x and y then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu.$$

Proof:

Since u is a homogeneous function of degree n in x and y, we can write

$$u = x^n g(y/x)$$

$$\frac{\partial u}{\partial x} = x^n g'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + g\left(\frac{y}{x}\right) n x^{n-1}$$

$$x \frac{\partial u}{\partial x} = -y x^{n-1} g'\left(\frac{y}{x}\right) + n x^n g\left(\frac{y}{x}\right) \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial y} = x^n \cdot g'\left(\frac{y}{x}\right)(1/x) = x^{n-1} g'\left(\frac{y}{x}\right)$$

$$y \frac{\partial u}{\partial y} = y x^{n-1} g'\left(\frac{y}{x}\right) \quad \text{——— (2)}$$

Adding (1) & (2) we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n x^n g\left(\frac{y}{x}\right) = nu.$$

1. If $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

$$u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$$

$$\tan u = \left[\frac{x^3+y^3}{x+y}\right] = \frac{x^3(1+(y/x)^3)}{x(1+y/x)} = x^2 f(y/x)$$

$\therefore \tan u$ is a homogeneous function of degree 2 in x and y.

$\therefore$ By Euler's theorem

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\sin u\cos u = \sin 2u.$$

2. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove the following

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$; (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u$.

$$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$\sin u = \left(\frac{x^2+y^2}{x+y}\right) = \frac{x^2[1+(y/x)^2]}{x[1+y/x]} = x \cdot f(y/x)$$

$\therefore \sin u$ is a homogeneous function of degree 1 in x and y.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 1 \cdot \sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u. \quad \text{——} (1)$$

Partially diff (1) w.r.t x and y we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \sec^2 u\frac{\partial u}{\partial x} \quad \text{——} (2)$$

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \sec^2 u \cdot \frac{\partial u}{\partial y} \quad \text{——} (3)$$

(2) $\times x$ + (3) $\times y$ we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\tan u=\sec^2 u\tan u.$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sec^2 u\tan u-\tan u$$

$$=\tan u\,(\sec^2 u-1)$$

$$=\tan u\,(\tan^2 u)$$

$$=\tan^3 u.$$

If u is a homogeneous function of degree n in x and y

Prove (i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

Since u is a homogeneous function of degree n in x and y, by Euler's theorem, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \quad \text{—— (1)}$$

Diff (1) partially w.r.to x we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\cdot\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \quad \text{—— (2)}$$

Diff (1) partially w.r.to y we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \quad \text{—— (3)}$$

(2) $\times x$ + (3) $\times y$ we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=n(n-1)u$$

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4. If z be a function of x and y, where $x = e^{u} + e^{-v}$ and $y = e^{-u} - e^{v}$, prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot e^{u} + \frac{\partial z}{\partial y}\cdot(-e^{-u}) = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= -e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y} \quad \text{--- (2)}$$

From (1) & (2), $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^{u} + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} + e^{v})\frac{\partial z}{\partial y}$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

5. If $u = f(x-y, y-z, z-x)$, Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Let $\alpha = x-y$, $\beta = y-z$, $\gamma = z-x$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} + \frac{\partial u}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial x} = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \gamma} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial u}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} + \frac{\partial u}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial y} = -\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial z} + \frac{\partial u}{\partial \beta}\cdot\frac{\partial \beta}{\partial z} + \frac{\partial u}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial z} = -\frac{\partial u}{\partial \beta} + \frac{\partial u}{\partial \gamma} \quad \text{--- (3)}$$

From (1), (2) & (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

6. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} =$

7. If $u = x^2 y z^3$ where $x = t$, $y = t^3$, $z = e^{2t}$ find $\frac{du}{dt}$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$= \ldots + 3x^2yz^2(2e^{2t}) = e^{6t}(5t^4 + 6t^5)$

8 If $u = \log(\tan x + \tan y + \tan z)$ Show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

$$u = \log(\tan x + \tan y + \tan z)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 z$$

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \sin 2x \cdot \frac{\sec^2 x}{\tan x + \tan y + \tan z} + \sin 2y \cdot \frac{\sec^2 y}{\tan x + \tan y + \tan z} + \sin 2z \cdot \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{1}{\tan z + \tan y + \tan z}\left[2\frac{\sin x}{\cos x} + 2\frac{\sin y}{\cos y} + 2\frac{\sin z}{\cos z}\right]$$

$$= 2(1) = 2$$

9. If $z = f(x, y)$ where $x = u^2 - v^2$, $y = 2uv$ prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{4(u^2+v^2)}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(2u) + \frac{\partial z}{\partial y}(2v)$$

$$\frac{\partial z}{\partial u} = 2\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]z \Rightarrow \frac{\partial}{\partial u} = 2\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left[\frac{\partial z}{\partial u}\right]$$

$$= 2\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right] 2\left[u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right]$$

$$= 4\left[u^2\frac{\partial^2 z}{\partial x^2} + 2uv\frac{\partial^2 z}{\partial x \partial y} + v^2\frac{\partial^2 z}{\partial y^2}\right] \quad \text{——— (1)}$$

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$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(+2u)$$

$$\frac{\partial z}{\partial v} = 2\left[-v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right]z$$

$$\Rightarrow \frac{\partial}{\partial v} = 2\left[-v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right]$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left[\frac{\partial z}{\partial v}\right] = 4\left[v^2\frac{\partial^2 z}{\partial x^2} + 2uv\frac{\partial^2 z}{\partial x \partial y} + u^2\frac{\partial^2 z}{\partial y^2}\right] \quad (2)$$

From (1) & (2)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{4(u^2+v^2)}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

10. If the tranformations are $u = e^x \cos y$ and $v = e^x \sin y$ and that f is a function of u and v and also x and y

Prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$

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Jacobians

If $u=f(x,y)$, $v=g(x,y)$ are two continuous functions of the variables x and y such that the first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous then $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u and v w.r.t x and y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J(u,v)$.

In general if $u, u_2, \ldots, u_n$ are functions of $x_1, x_2, \ldots, x_n$ then the Jacobian of $u_1, u_2, \ldots, u_n$ w.r.t $x_1, x_2, \ldots, x_n$ is defined as

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Properties of Jacobians

(i) If u and v are functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

(ii) If u and v are functions of r and s, where r and s are functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$.

(iii) If u, v, w are functionally dependent functions of three independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

1. If $u = 2xy$, $v = x^2 - y^2$, Evaluate $\frac{\partial(u,v)}{\partial(x,y)}$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2$$
$$= -4[x^2 + y^2]$$

2. If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$ and $y = r\sin\theta$ compute $\frac{\partial(u,v)}{\partial(r,\theta)}$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= -4(y^2 + x^2)\, r(\cos^2\theta + \sin^2\theta)$$

$$= -4r^3 \qquad [\because x^2 + y^2 = r^2]$$

3. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & z/x & y/x \\ z/y & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix} = \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & xz & xy \\ zy & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{x^2y^2z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

4. If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

$x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Now, $x^2 + y^2 = r^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, $|||^{ly}$ $\frac{\partial r}{\partial y} = \frac{y}{r}$

$\theta = \tan^{-1}(y/x)$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{x}{r^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1.$$

Taylor's Series Expansion of a function of two Variables

If $f(x,y)$ and all its partial derivatives are finite and Continuous at all points (a,b) then

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + \cdots$$

Note:

Taylor's series of $f(x,y)$ near the point $(0,0)$ is Mclaurin's series of $f(x,y)$.

1. Expand e^x as Taylor's series at the point $(0,0)$.

	At $(0,0)$	
$f(x,y) = e^x$	$f = 1$	
$f_x = e^x$	$f_x = 1$	
$f_y = 0$	$f_y = 0$	$h = x - a = x - 0 = x$
		$k = y - b = y - 0 = y$.
$f_{xx} = e^x$	$f_{xx} = 1$	
$f_{xy} = 0$	$f_{xy} = 0$	
$f_{yy} = 0$	$f_{yy} = 0$	

By Taylor's series

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots$$

Taylor's Series Expansion of a function of two Variables

If $f(x,y)$ and all its partial derivatives are finite and Continuous at all points (a,b) then

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \dots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + \dots$$

Note:

Taylor's series of $f(x,y)$ near the point $(0,0)$ is Maclaurin's Series of $f(x,y)$.

1. Expand e^x as Taylor's series at the point $(0,0)$.

	At $(0,0)$
$f(x,y) = e^x$	$f = 1$
$f_x = e^x$	$f_x = 1$
$f_y = 0$	$f_y = 0$
$f_{xx} = e^x$	$f_{xx} = 1$
$f_{xy} = 0$	$f_{xy} = 0$
$f_{yy} = 0$	$f_{yy} = 0$

$h = x - a = x - 0 = x$

$k = y - b = y - 0 = y.$

By Taylor's series

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \dots$$

2. Expand $e^x \log(1+y)$ as Taylor's series at $(0,0)$

Function	value at $(0,0)$
$f(x,y) = e^x \log(1+y)$	$f(0,0) = 0$
$f_x = e^x \log(1+y)$	$f_x = 0$
$f_y = e^x (1+y)^{-1}$	$f_y = 1$
$f_{xx} = e^x \log(1+y)$	$f_{xx} = 0$
$f_{xy} = \dfrac{e^x}{1+y}$	$f_{xy} = 1$
$f_{yy} = \dfrac{-e^x}{(1+y)^2}$	$f_{yy} = -1$
$f_{xxx} = e^x \log(1+y)$	$f_{xxx} = 0$
$f_{xxy} = e^x (1+y)^{-1}$	$f_{xxy} = 1$
$f_{xyy} = -e^x (1+y)^{-2}$	$f_{xyy} = -1$
$f_{yyy} = 2e^x (1+y)^{-3}$	$f_{yyy} = 2$

$h = x - a = x - 0 = x$

$k = y - b = y - 0 = y$

By Taylor series expansion,

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)] + \dots$$

$$e^x \log(1+y) = 0 + \left(\frac{x(0) + y(1)}{1!}\right) + \frac{x^2(0) + 2xy(1) + y^2(-1)}{2!} + \frac{x^3(0) + x^2y(1) + xy^2(-1) + y^3(2)}{3!} + \dots$$

$$= \frac{y}{1!} + \frac{2xy - y^2}{2!} + \frac{x^2y - xy^2 + 2y^3}{3!} + \dots$$

UNIT - I

MULTIPLE INTEGRALS

Double integral - Cartesian Co-ordinates - Polar Co-ordinates - Change of order of integration. Triple integration in Cartesian Co-ordinates. Area as double integrals

Introduction

Algebraic and transcendental functions together constitute the elementary functions. Special functions are functions other than the elementary functions such as Gamma, Beta functions. Special functions also include Bessel, Legendre, Laguerre, Hermite, Chebyshev polynomials, error function, Sine integral, exponential integral, Fresnel integrals etc.

Double integration in Cartesian Co-ordinates

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum $f(x_1)\delta x_1 + f(x_2)\delta x_2 + \dots + f(x_n)\delta x_n$ where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \dots$ tends to zero.

i.e $\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \delta x \to 0}} [f(x_1)\delta x_1 + f(x_2)\delta x_2 + \dots + f(x_n)\delta x_n]$

Double integral over region R may be evaluated by two successive integrations. If A is described as $f_1(x) \le y \le f_2(x)$, $[y_1 \le y \le y_2]$ and $x_1 \le x \le x_2$ then $\iint_R f(x,y)\,dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$

Problems

1) Evaluate $\int_0^1\int_1^2 x(x+y)\,dy\,dx$

Soln $\int_0^1\int_1^2 x(x+y)\,dy\,dx = \int_0^1\int_1^2 (x^2+xy)\,dy\,dx = \int_0^1 \left[x^2y + \frac{xy^2}{2}\right]_1^2 dx$

$= \int_0^1 \left(2x^2+2x-x^2-\frac{x}{2}\right)dx = \int_0^1 \left(x^2+\frac{3}{2}x\right)dx$

$= \left[\frac{x^3}{3}+\frac{3x^2}{4}\right]_0^1 = \frac{1}{3}+\frac{3}{4} = \frac{13}{12}$

2) Evaluate $\int_2^3\int_1^2 \frac{1}{xy}\,dx\,dy$

Soln $\int_2^3\int_1^2 \frac{1}{xy}\,dx\,dy = \int_2^3 \frac{1}{y}\,dy \cdot \int_1^2 \frac{1}{x}\,dx = (\log y)_2^3 \cdot (\log x)_1^2$

$= (\log 3 - \log 2)(\log 2 - \log 1)$

$= \log\left(\frac{3}{2}\right) \cdot \log 2.$

3) Evaluate $\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Soln $\int_0^5\int_0^{x^2} (x^3+xy^2)\,dx\,dy = \int_0^5\int_0^{x^2}(x^3+xy^2)\,dy\,dx = \int_0^5 \left[x^3y + \frac{xy^3}{3}\right]_0^{x^2} dx$

$= \int_0^5 \left[x^5 + \frac{x^7}{3}\right] dx = \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5 = 5^6\left[\frac{1}{6} + \frac{25}{24}\right] = 5^6\left[\frac{29}{24}\right]$

4) Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$.

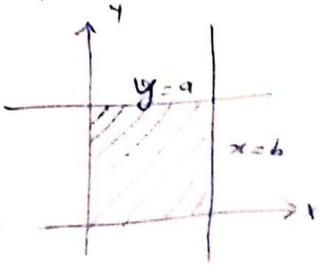
Soln $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2+y^2}\,dx$

$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx.$

$= \frac{\pi}{4}\left\{\log\left[x+\sqrt{1+x^2}\right]\right\}_0^1 = \frac{\pi}{4}\left\{\log(1+\sqrt{2})\right\}$

5) Describe the region of integration. $\int_0^a\int_0^b \frac{xy}{\sqrt{1-x^2-y^2}}\,dx\,dy$

Soln The boundaries of the region of integration are given by the limits of integration



Given y values from 0 to a ie $y=0$ to $y=a$

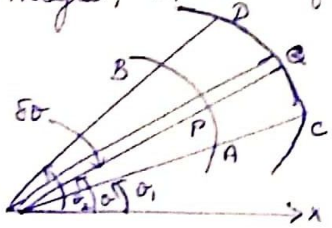
x values from 0 to b ie $x=0$ to $x=b$

∴ The region of integration is the rectangle whose sides are $x=0$, $x=b$, $y=0$ and $y=a$.

Double integration in Polar Co-ordinates

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$, we first integrate w.r.t r between limits $r=r_1$ and $r=r_2$. Keeping θ fixed and the resulting expression is integrated w.r.t θ from θ_1 to θ_2.

In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.



Here, AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$. PQ is wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r,\theta)\,dr$ indicates that the integration is along PQ from P and Q while the integration w.r.t θ corresponds to the turning of PQ from AC to BD. Thus the whole region of integration is the area ACDB. The order of integration may be changed with appropriate change in the limits.

Problems

1) Evaluate $\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2\,d\theta\,dr$.

Soln. $I = \int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2\cos\theta} d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2} 8\cos^3\theta\,d\theta$.

$$= \frac{8}{3}\cdot 2\int_0^{\pi/2}\cos^3\theta\,d\theta = \frac{16}{3}\cdot\frac{2}{3}\cdot\frac{1}{1} = \frac{32}{9}$$

2) Evaluate $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$

Soln $I = \int_0^{\pi}\left(\int_0^{a\cos\theta} r\,dr\right)\sin\theta\,d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a\cos\theta}\sin\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$

$$= \frac{a^2}{2}\int_0^{\pi}\cos^2\theta\,-d(\cos\theta) = -\frac{a^2}{2}\left(\frac{\cos^3\theta}{3}\right)_0^{\pi}$$

$$= -\frac{a^2}{6}\left[\cos^3\pi - \cos^3 0\right] = -\frac{a^2}{6}\left[-1-1\right] = a^2/3$$

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3. Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$

Sol: $I = \int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{8}$

4. Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$

Soln: $I = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^3}{3}\right)_{a(1-\cos\theta)}^{a} d\theta = \frac{1}{3}\int_0^{\pi/2}\left[a^3 - a^3(1-\cos\theta)^3\right]d\theta$

$= \frac{1}{3}\int_0^{\pi/2}\left[a^3 - a^3(1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\right]d\theta$

$= \frac{a^3}{3}\int_0^{\pi/2}\left[3\cos\theta - 3\cos^2\theta + \cos^3\theta\right]d\theta = \frac{a^3}{3}\left[(3\sin\theta)_0^{\pi/2} - 3\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{2}{3}\right]$

$= \frac{a^3}{3}\left[3 - \frac{3\pi}{4} + \frac{2}{3}\right] = \frac{a^3}{36}\left[44 - 9\pi\right]$

Change of order of integration

Change of order of integration is done to make the evaluation of the integral easier.

When all the limits are constants, we can change the order of integration as we like, the only point to be remembered is that the limits of x are to be retained for x.

But when the limits for inner integration are functions of a variable, the change in the order of integration will result in changes in the limit of integration. i.e the double integral $\int_c^d\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$ will take the form $\int_a^b\int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$ when the order of integration is changed. This process of converting a given double integral by changing the order of integration

<u>Procedure to evaluate the double integration by changing its order of integration</u>

(i) Using the limits of the given double integration sketch the region of integration

(ii) Find the intersecting points from the curves and mark them.

(iii) If the limit of the inner integral is a function of x, we have to change the limit of inner integral as a function of y and viceversa.

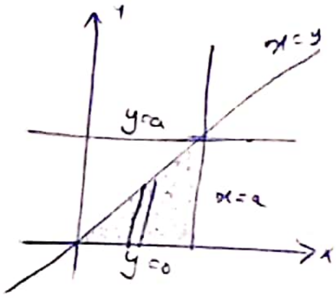
(iv) Find the new limits for inner and outer integrals using the region of integration

(v) Evaluate the given double integral as usual.

Problems

1) Change the order of integration and then evaluate $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$.

Soln. The region of integration R is defined by $y \le x \le a$ & $0 \le y \le a$.



In the region R, y varies from 0 to x.
x varies from 0 to a.

$$\therefore \int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^a \int_0^x \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

$$= \int_0^a x \left(\log\left(y+\sqrt{y^2+x^2}\right)\right)_0^x dx$$

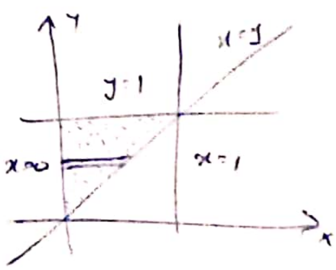
$$= \int_0^a x \left[\log\left(x+\sqrt{2x^2}\right) - \log\left(0+\sqrt{0+x^2}\right)\right] dx$$

$$= \int_0^a x \left[\cancel{\log x} + \log(1+\sqrt{2}) - \cancel{\log x}\right] dx$$

$$= \log(1+\sqrt{2}) \left(\frac{x^2}{2}\right)_0^a = \frac{a^2 \log(1+\sqrt{2})}{2}$$

2) Change the order of integration and then evaluate $\int_0^1 \int_x^1 \frac{x}{x^2+y^2}\,dx\,dy$.

Soln. The region of integration R is defined by $x \le y \le 1$ & $0 \le x \le 1$.



In the region R, x varies from 0 to y.
y varies from 0 to 1.

$$\therefore \int_0^1 \int_x^1 \frac{x}{x^2+y^2}\,dx\,dy = \int_0^1 \int_0^y \frac{x}{x^2+y^2}\,dx\,dy = \int_0^1 \frac{1}{2}\left[\log(x^2+y^2)\right]_0^y dy$$

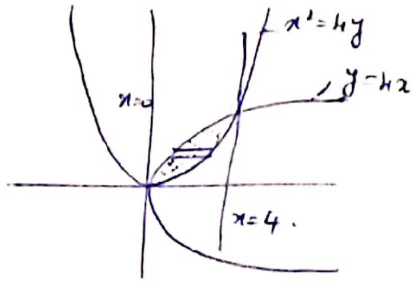
$$= \frac{1}{2}\int_0^1 \left(\log 2y^2 - \log(0+y^2)\right) dy = \frac{1}{2}\int_0^1 \log 2\,dy = \frac{\log 2}{2}$$

3) Change the order of integration and then evaluate $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\, dx$.

Soln. The region of integration is defined by $y = x^2/4$, $y = 2\sqrt{x}$, $x = 0$ & $x = 4$

In Region R, x varies from $y^2/4$ to $2\sqrt{y}$

y varies from 0 to 4.



$$\therefore \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\, dx = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\, dy = \int_0^4 (y)_{y^2/4}^{2\sqrt{y}} dy.$$

$$= \int_0^4 (2y^{1/2} - y^2/4)\, dy = \left[\frac{4y^{3/2}}{3} - \frac{y^3}{12} \right]_0^4$$

$$= \frac{4\times 8}{3} - \frac{64}{12} = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} //.$$

4) Change the order of integration and then evaluate $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$.

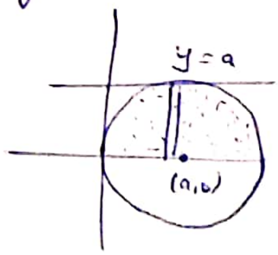
Soln The region of integration is defined by $x = a - \sqrt{a^2-y^2}$, $x = a+\sqrt{a^2-y^2}$, $y = 0$ and $y = a$.

$x = a - \sqrt{a^2-y^2}$, $x = a + \sqrt{a^2-y^2} \Rightarrow (x-a)^2 = a^2 - y^2$

$\Rightarrow (x-a)^2 + y^2 = a^2$.

In Region R, y varies from 0 to $\sqrt{a^2-(x-a)^2}$

x varies from 0 to $2a$.



$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy = \int_0^{2a} \int_0^{\sqrt{a^2-(x-a)^2}} xy\, dy\, dx$$

$$= \int_0^{2a} x \left[\frac{y^2}{2} \right]_0^{\sqrt{a^2-(x-a)^2}} dx = \frac{1}{2} \int_0^{2a} x\left(a^2 - (x-a)^2\right) dx.$$

$$= \frac{1}{2} \int_0^{2a} x\,(2ax - x^2)\, dx = \frac{1}{2} \int_0^{2a} (2ax^2 - x^3)\, dx$$

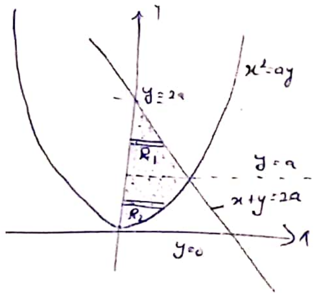
$$= \frac{1}{2} \left(\frac{2ax^3}{3} - \frac{x^4}{4} \right)_0^{2a} = \frac{1}{2} \left[\frac{16a^4}{3} - \frac{16a^4}{4} \right] = 8a^4 \cdot \frac{1}{12}$$

$$= \frac{2a^4}{3} //.$$

5) Change the order of integration in $\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy$ and hence evaluate the same

Soln.

The region of integration is defined by $y = \frac{x^2}{a}$, $y = 2a - x$, $x = 0$ & $x = a$



Divide the region R into two regions R_1 & R_2 by drawing a line $y = a$ parallel to y-axis.

In region R_1, x varies from 0 to $2a - y$

y varies from a to $2a$

In region R_2, x varies from 0 to $\sqrt{ay}$

y varies from 0 to a

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy = \int_a^{2a}\int_0^{2a-y} xy\,dx\,dy + \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$$

$$= \int_a^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy + \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_a^{2a} (4a^2y - 4ay^2 + y^3)\,dy + \frac{1}{2}\int_0^a ay^2\,dy$$

$$= \frac{1}{2}\left[\frac{4a^2y^2}{2} - \frac{4ay^3}{3} + \frac{y^4}{4}\right]_a^{2a} + \frac{1}{2}\,a\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{1}{2}\left[8a^4 - \frac{32a^4}{3} + \frac{16a^4}{4} - 2a^4 + \frac{4a^4}{3} - \frac{a^4}{4} + \frac{a^4}{3}\right]$$

$$= \frac{1}{2}\left[6a^4 - 9a^4 + \frac{65a^4}{4}\right] = \frac{1}{2}\left[\frac{65a^4}{4} - 3a^4\right]$$

$$= \frac{a^4}{2}\left[\frac{53}{4}\right] = \frac{3a^4}{8}$$

Area as double Integral

Plane area enclosed by one or more curves can be expressed as a double integral both in cartesian co-ordinates and in polar coordinates

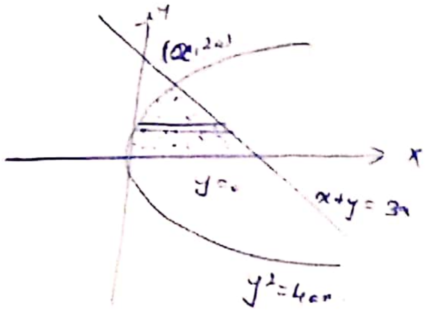
In cartesian co-ordinates, $A = \iint_R dx\,dy$

In Polar co-ordinates, $A = \iint_R r\,dr\,d\theta$

Problems

1) Find the area enclosed by the curve $y^2=4ax$ and the lines $x+y=3a$, $y=0$

Soln.



$$\text{Area} = \iint_R dx\,dy$$

In region R,

x varies from $\frac{y^2}{4a}$ to $-y+3a$

y varies from 0 to $2a$.

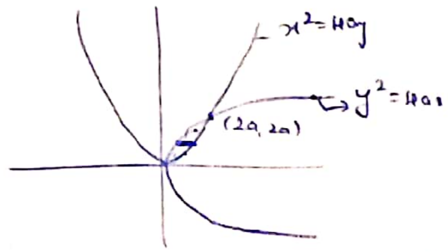
$$\therefore \text{Area} = \int_0^{2a}\int_{y^2/4a}^{-y+3a} dx\,dy = \int_0^{2a} (x)_{y^2/4a}^{-y+3a}\,dy$$

$$= \int_0^{2a}\left[-y+3a-\frac{y^2}{4a}\right]dy = \left[-\frac{y^2}{2}+3ay-\frac{y^3}{12a}\right]_0^{2a}$$

$$= -2a^2+6a^2-\frac{8a^3}{12a} = 4a^2-\frac{2a^2}{3} = \frac{10a^2}{3}.$$

2) Find the area enclosed by the parabola $y^2=4ax$ and $x^2=4ay$.

Soln.



$$\text{Area} = \iint_R dx\,dy.$$

In region R,

x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$

y varies from 0 to $2a$

$x^2 = 4a \cdot 2\sqrt{ax}$

$x^{3/2} = (2a)^{3/2}$

$x = 2a,\ y = 2a$

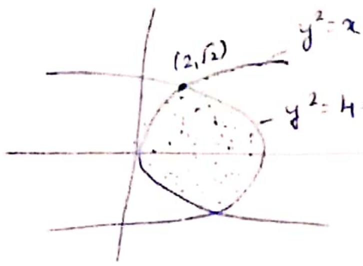
$$\therefore \text{Area} = \int_0^{2a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{2a} (x)_{y^2/4a}^{2\sqrt{ay}}\,dy$$

$$= \int_0^{2a}\left(2\sqrt{a}\,y^{1/2}-\frac{y^2}{4a}\right)dy = \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2}-\frac{y^3}{12a}\right]_0^{2a}$$

$$= \frac{4\sqrt{a}\,2\sqrt{2}\,a\sqrt{a}}{3}-\frac{8a^3}{12a} = \frac{8\sqrt{2}\,a^2}{3}-\frac{8a^2}{12} = \frac{a^2}{3}\left[8\sqrt{2}-2\right]$$

3) Find the area bounded by parabola $y^2=4-x$ and $y^2=x$ by double integration.

Soln.



$$\text{Area} = \iint_R dx\,dy$$

In region R,

x varies from y^2 to $4-y^2$.

y varies from 0 to $\sqrt{2}$

$x = 4-x$

$2x = 4$

$\boxed{x=2}$

$y^2 = 2$

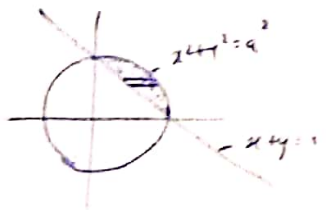
$y = \sqrt{2}$

Since the area is symmetry about x axis

$$\text{Area} = 2\int_0^{\sqrt{2}}\int_{y^2}^{4-y^2} dx\,dy = 2\int_0^{\sqrt{2}} (4-y^2-y^2)\,dy = 2\int_0^{\sqrt{2}} (4-2y^2)\,dy$$

$$= 4\left[2y - \frac{y^3}{3}\right]_0^{\sqrt{2}} = 4\left[\frac{6\sqrt{2}-2\sqrt{2}}{3}\right] = \frac{16\sqrt{2}}{3}$$

4) Find the area bounded between the circle $x^2+y^2=a^2$ and the line $x+y=a$ lying in the first quadrant, by double integration.



Soln

$$\text{Area} = \iint_R dx\,dy$$

In region R, x varies from $a-y$ to $\sqrt{a^2-y^2}$

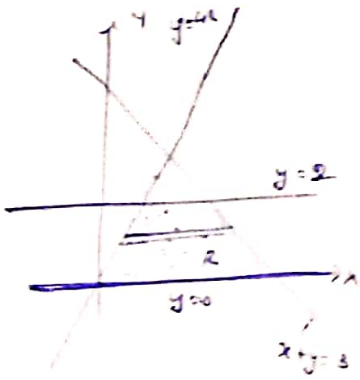
y varies from 0 to a.

$$\text{Area} = \int_0^a\int_{a-y}^{\sqrt{a^2-y^2}} dx\,dy = \int_0^a \left(\sqrt{a^2-y^2} - a + y\right)dy = \left(\cancel{\frac{y}{2}\sqrt{a^2-y^2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right) - ay + \frac{ay^2}{2}\right)_0^a$$

$$= \frac{a^2}{2}\sin^{-1}(1) - a^2 + \frac{a^2}{2} = \frac{a^2\pi}{4} - \frac{a^2}{2} = \left(\frac{\pi - 2}{4}\right)a^2$$

$$= \frac{a^2(\pi-2)}{4}$$

5) Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the area bounded by the curves $y=4x$, $x+y=3$, $y=0$; $y=2$.



Soln

In region R, x varies from $y/4$ to $3-y$

y varies from 0 to 2

$$\iint_R (x^2+y^2)\,dx\,dy = \int_0^2\int_{y/4}^{3-y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^2 \left(\frac{x^3}{3} + y^2x\right)_{y/4}^{3-y} dy = \int_0^2\left\{\left[\frac{(3-y)^3}{3} + y^2(3-y)\right] - \left(\frac{y^3}{64\times 3} + \frac{y^3}{4}\right)\right\}dy$$

$$= \int_0^2\left[\frac{(3-y)^3}{3} + 3y^2 - y^3 - \frac{y^3}{192} - \frac{y^3}{4}\right]dy = \int_0^2\left[\frac{(3-y)^3}{3} + 3y^2 - \frac{145y^3}{192}\right]dy$$

$$= \left(\frac{(3-y)^4}{-12} + y^3 - \frac{241y^4}{192\times 4}\right)_0^2 + \frac{81}{12} = -\frac{1}{12} + 8 - \frac{241\times 16}{\underset{48}{192}\times 4} + \frac{81}{12}$$

-192
-1
193
$+48$
241

$$= \frac{384 - 4 - 241 + 384}{48} = \frac{384 - 245}{48} = \frac{463}{48}$$

Triple Integrals

Triple integration in Cartesian co-ordinates

A triple integral of a function defined over a region R is denoted by $\iiint_R f(x,y,z)\,dx\,dy\,dz$ or $\iiint_R f(x,y,z)\,dV$ or $\iiint_R f(x,y,z)\,d(x,y,z)$

Problems.

1) Evaluate $\int_0^a\int_0^b\int_0^c xyz\,dz\,dy\,dx$.

Soln. Let $I = \int_0^a\int_0^b\int_0^c xyz\,dz\,dy\,dx = \int_0^a x\,dx \cdot \int_0^b y\,dy \cdot \int_0^c z\,dz$

$$= \left(\frac{x^2}{2}\right)_0^a \cdot \left(\frac{y^2}{2}\right)_0^b \cdot \left(\frac{z^2}{2}\right)_0^c = \frac{a^2b^2c^2}{8}$$

2) Evaluate $\int_0^a\int_0^b\int_0^c e^{x+y+z}\,dz\,dy\,dx$.

Soln. Let $I = \int_0^a e^x\,dx \cdot \int_0^b e^y\,dy \cdot \int_0^c e^z\,dz = (e^a-1)(e^b-1)(e^c-1)$.

3) Evaluate $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

Soln. $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^x \cdot e^y \cdot e^z\,dz\,dy\,dx = \int_0^{\log 2}\int_0^x e^x e^y\left(e^{x+\log y}-1\right)dy\,dx$

$$= \int_0^{\log 2}\int_0^x \left[e^{2x}\cdot ye^y - e^x e^y\right]dy\,dx = \int_0^{\log 2} e^{2x}\int_0^x ye^y\,dy\,dx - \int_0^{\log 2} e^x\int_0^x e^y\,dy\,dx$$

$$= \int_0^{\log 2} e^{2x}\left[ye^y - e^y\right]_0^x dx - \int_0^{\log 2} e^x(e^x-1)\,dx$$

$$= \int_0^{\log 2}\left(xe^{3x} - e^{3x} + \cancel{e^{2x}} - \cancel{e^{2x}} + e^x\right)dx$$

$$= \int_0^{\log 2}\left[(x-1)e^{3x} + e^x\right]dx = \left[\frac{(x-1)e^{3x}}{3} - \frac{e^{3x}}{9} + e^x\right]_0^{\log 2}$$

$$= \left[\frac{(\log 2-1)\,8}{3} - \frac{8}{9} + 2\right] - \left[\frac{-1}{3} - \frac{1}{9} + 1\right] = \frac{8}{3}\log 2 - \frac{8}{3} - \frac{8}{9} + 2 + \frac{1}{3} + \frac{1}{9}$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

4. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx.$

Soln

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(\sqrt{1-x^2-y^2})^2 - z^2}}\, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1 = \frac{\pi}{2}\left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}\sin^{-1}(0)\right]$$

$$= \pi^2/8.$$

5. Find the volume of the tetrahedron bounded by the planes $x=0, y=0,$ $z=0,$ $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Soln.

$$\text{Volume} = \iiint_V dV = \int_0^c \int_0^{b[1-\frac{z}{c}]} \int_0^{a[1-\frac{y}{b}-\frac{z}{c}]} dx\, dy\, dz$$

$$= \int_0^c \int_0^{b[1-z/c]} (x)_0^{a[1-\frac{y}{b}-\frac{z}{c}]} dy\, dz$$

$$= a\int_0^c \int_0^{b[1-\frac{z}{c}]} \left[1 - \frac{y}{b} - \frac{z}{c}\right] dy\, dz$$

$$= a\int_0^c \left(y - \frac{y^2}{2b} - \frac{zy}{c}\right)_0^{b(1-z/c)} dz = a\int_0^c \left[b(1-z/c) - \frac{b}{2}(1-z/c)^2 - \frac{zb}{c}(1-z/c)\right] dz$$

$$= ab\int_0^c \left(1 - \cancel{\frac{z}{c}} - \frac{1}{2} + \cancel{\frac{z}{c}} - \frac{z^2}{2c^2} - \frac{z}{c} + \frac{z^2}{c^2}\right) dz$$

$$= ab\int_0^c \left(\frac{1}{2} - \frac{z}{c} + \frac{z^2}{2c^2}\right) dz = ab\left(\frac{z}{2} - \frac{z^2}{2c}\right)_0^c \frac{z^3}{6c^2}\Big)_0^c$$

$$= ab\left(\frac{c}{2} - \frac{c}{2} + \frac{c}{6}\right)$$

$$= abc\left(\frac{1}{6}\right) = \frac{abc}{6}.$$

6) Evaluate $\iiint_V (x+y+z)\, dx\, dy\, dz$, where the region V is bounded by $x+y+z=a$ $(a>0)$ $x=0, y=0, z=0$.

Soln

$$\text{volume} = \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z)\, dz\, dy\, dz = \int_0^a \int_0^{a-x} \left(xz + yz + \frac{z^2}{2}\right)_0^{a-x-y} dy\, dx$$

$$= \int_0^a \int_0^{a-x} \left[x(a-x-y) + y(a-x-y) + \frac{(a-x-y)^2}{2} \right] dy\, dx$$

$$= \int_0^a \int_0^{a-x} \left[ax - x^2 - xy + ay - xy - y^2 + \frac{(a-x-y)^2}{2} \right] dy\, dx$$

$$= \int_0^a \left(axy - x^2 y - 2x\frac{y^2}{2} + \frac{ay^2}{2} - \frac{y^3}{3} - \frac{(a-x-y)^3}{6} \right)_0^{a-x} dx$$

$$= \int_0^a \left[ax(a-x) - x^2(a-x) - x(a-x)^2 + \frac{a}{2}(a-x)^2 - \frac{(a-x)^3}{3} - \frac{(a-x-\cancel{(a-x)})^3}{6} \right] dx$$

$$a^3 + 3ax^2 - 3a^2x - x^3$$

$$= \left[\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} - \frac{ax^3}{3} + \frac{x^4}{4} - \frac{a^2x^2}{2} + \frac{2ax^3}{3} - \frac{x^4}{4} - \frac{1}{3}\left[a^3x + \frac{3ax^3}{3} - \frac{3a^2x^2}{2} - \frac{x^4}{4} \right] \right) \right]_0^a$$

$$= \left[\frac{a^4}{2} - \frac{a^4}{3} - \frac{a^4}{3} + \cancel{\frac{a^4}{4}} - \cancel{\frac{a^4}{2}} + \cancel{\frac{2a^4}{3}} - \cancel{\frac{a^4}{4}} - \frac{a^4}{3} - \frac{a^4}{3} + \frac{a^4}{2} - \frac{a^4}{12} \right] = \left(\frac{6-8-1}{12} \right) a^4$$

$$= \frac{3a^4}{8}$$

7) Evaluate $\iiint_V dx\,dy\,dz$, where V is the finite region of space, (tetrahedron) formed by the planes $x=0, y=0, z=0$ and $2x+3y+4z=12$.

Soln.

$$I = \iiint_V dx\,dy\,dz = \int_0^6 \int_0^{\frac{1}{3}[12-2x]} \int_0^{\frac{1}{4}[12-2x-3y]} dz\, dy\, dx$$

$$= \frac{1}{4} \int_0^6 \int_0^{\frac{1}{3}[12-2x]} (12-2x-3y)\, dy\, dx$$

$$= \frac{1}{4} \int_0^6 \left(12y - 2xy - \frac{3y^2}{2} \right)_0^{\frac{1}{3}[12-2x]} dx$$

$$= \frac{1}{4} \int_0^6 \left(4[12-2x] - \frac{2}{3}x[12-2x] - \frac{1}{6}[144-48x+4x^2] \right) dx$$

$$= \frac{1}{4} \int_0^6 \left(48 - 8x - \cancel{8x} + \frac{4x^2}{3} - 24 + \cancel{8x} - \frac{2}{3}x^2 \right) dx$$

$$= \frac{1}{4} \int_0^6 \left(24 - 8x + \frac{2}{3}x^2 \right) dx = \frac{1}{4} \left[24x - 4x^2 + \frac{2x^3}{9} \right]_0^6$$

$$= \frac{1}{4} [144 - 144 + 48]$$

$$= \frac{48}{4}$$

$$= 12$$

$\frac{14 \times 36}{3}$

$\frac{2 \times 36 \times 6}{9}$

Equations of the first order and higher degree

The general form of the differential equation of the first order and n^{th} degree is

$$\left(\frac{dy}{dx}\right)^n + f_1(x,y)\left(\frac{dy}{dx}\right)^{n-1} + f_2(x,y)\left(\frac{dy}{dx}\right)^{n-2} + \dots + f_{n-1}(x,y)\frac{dy}{dx} + f_n(x,y) = 0.$$

If we denote $\frac{dy}{dx}$ by p for convenience, the general equation becomes

$$p^n + f_1(x,y)p^{n-1} + f_2(x,y)p^{n-2} + \dots + f_{n-1}(x,y)p + f_n(x,y) = 0 \quad \text{—— (1)}$$

To solve (1) it is to identified as an equation of any one of the following types

1. Equations solvable for p.
2. Equations solvable for y.
3. Equations solvable for x.
4. clairaut's equations

Equations Solvable for p:

If equation (1) is of this type, then the L.H.S of (1) can be resolved into n linear factors. Then (1) becomes $(p-F_1)(p-F_2)\dots(p-F_n)=0$, from which we get $p=F_1$, $p=F_2$, … $, p=F_n$, where $F_1, F_2, \dots, F_n$ are functions of x and y.

Each of these n equations is of the first order and first degree and can be solved. by ~~methods~~.

Let the solutions of the above n components equations be $\phi_1(x,y,c)=0$, $\phi_2(x,y,c)=0, \dots, \phi_n(x,y,c)=0$. Then the general solution of (1) is got by combining the above solutions and given as $\phi_1(x,y,c)\,\phi_2(x,y,c)\dots\phi_n(x,y,c)=0$.

Equations solvable for y.

If the given differential equation is of this type then y can be expressed explicitly as a single valued function of x and p.

(ie) the equation of this type can be re-written as

$$y = f(x, p) \quad \text{—— ①}$$

Differentiating ① with respect to x, we get

$$p = \phi\left(x, p, \frac{dp}{dx}\right) \quad \text{—— ②}$$

The solution of ② be $\psi(x, p, c) = 0$ —— ③

where c is an arbitrary constant.

If we eliminate p between ① and ③, the eliminant is the general solution of the given equation.

If p cannot be easily eliminated between ① & ③, they jointly provide the required solution in terms of the parameter p

III'ly Equations Solvable for x.

Clairaut's equations.

An equation of the form $y = px + f(p)$ —— ① is called Clairaut's equation.

Clairaut's equation is only a particular case of type-2 equation.

Differentiating ① w.r.t 'x'.

$$p = p + \{x + f'(p)\} \frac{dp}{dx}$$

$$\frac{dp}{dx} = 0 \quad \text{—— ②} \qquad \text{or} \qquad f'(p) + x = 0 \quad \text{—— ③}$$

Solving ② we get $p = c$

Eliminating p between ① & ④, we get the general solution of ① as $y = cx + f(c)$.

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Thus the general solution of a Clairaut's equation is obtained by replacing p by c in the given equation.

Eliminating p between (1) & (3), we get solution of (1). This solution does not contain any arbitrary constant. Also it cannot be obtained as a particular case of the general solution. This solution is called the singular solution of the equation (1).

Problems:

1. Solve $p^2 - 3p + 2 = 0$

Since the given eqn is quadratic in (p), we have

$$p = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 2 \text{ or } 1$$

$$\frac{dy}{dx} = 2 \quad \text{or} \quad \frac{dy}{dx} = 1$$

$$dy = 2\,dx \qquad dy = dx$$

Integrating we get

$$\int dy = 2\int dx \quad \text{or} \quad \int dy = \int dx$$

$$y = 2x + c_1 \qquad y = x + c_2$$

$$y - 2x - c_1 = 0 \qquad y - x - c_2 = 0$$

Hence the solution is $(y - 2x - c_1)(y - x - c_2) = 0$.

$$\left(\frac{dy}{dx}\right)^2 - 6\left(\frac{dy}{dx}\right) + 8 = 0$$

$$p^2 - 6p + 8 = 0$$

$$(p-2)(p-4) = 0$$

$$p = 2 \text{ or } p = 4$$

$$\frac{dy}{dx} = 2, \quad \frac{dy}{dx} = 4$$

$$y = 2x + c \qquad y = 4x + c$$

$$(y - 2x - c)(y - 4x - c) = 0$$

2. Solve $x = p^2 + y$

Given $x = p^2 + y$ —— (1)

Diff (1) w.r.t 'y' we get

$$\frac{dx}{dy} = 2p\frac{dp}{dy} + 1 \qquad \left[\because p = \frac{dy}{dx}\right]$$

$$\frac{1}{p} = 2p\frac{dp}{dy} + 1$$

$$2p\frac{dp}{dy} = \frac{1}{p} - 1 = \frac{1-p}{p}$$

$$\frac{2p^2}{1-p}\,dp = dy$$

Integrating we get, $2\int \frac{p^2}{1-p}\,dp = \int dy$

$$2\int \left(-p-1+\frac{1}{1-p}\right)dp = \int dy$$

$$-2\int \left(p+1+\frac{1}{p-1}\right)dp = y \qquad -p+p^2-1+p+1$$

$$-2\left[\frac{p^2}{2}+p+\log(p-1)\right] = y+c$$

(ie) $y = c-2\left[\frac{p^2}{2}+p+\log(p-1)\right]$ —— ②

Here eliminating 'p' from ① & ② is difficult.

Hence $x = p^2+y$

$$y = c-2\left[\frac{p^2}{2}+p+\log(p-1)\right]$$

Constitute the solution of the given differential equation.

3. Solve $y = 3x+\log p$.

Given $y = 3x+\log p$ —— ①

Diff ① w.r.t x we get

$p^3 - 2xyp + 4y^2 = 0$

$$p = \frac{dy}{dx} = 3+\frac{1}{p}\cdot\frac{dp}{dx}$$

$y = px - 1/p^2$

$$\frac{dp}{dx} = p(p-3)$$

$\frac{dy}{dx} = p -$

$$\frac{dp}{p(p-3)} = dx$$

Integrating we get $\int \frac{dp}{p(p-3)} = \int dx$

$$-1/3 \log p + \int \left(\frac{-1/3}{p}+\frac{1/3}{p-3}\right)dp = \int dx$$

$$-1/3\log p + 1/3\log(p-3) = x+c$$

$$\log\frac{p-3}{p} = 3x+c_1$$

$$\frac{p-3}{p} = e^{3x+c_1}$$

$$1-c_2e^{3x} = 3/p$$

$p = \frac{3}{1-c_2e^{3x}}$, Sub ② in ① we get

$y = 3x+\log\frac{3}{1-c_2e^{3x}}$ —— ②

4. Solve $y = xp + p^2$

Given $y = xp + p^2$ —— ①

The general solution is $y = cx + c^2$ —— ②

To find Singular Solution:

Diff ② w.r.t 'c' we get

$0 = x + 2c$ —— ③

Eliminating c between ② & ③.

From ③, $c = -x/2$

Substituting ④ in ②, we get, $y = (-x/2)x + (-x/2)^2$

$= \left(\frac{-x^2}{2}\right) + \frac{x^2}{4} = \frac{-x^2}{4}$.

(ie) $x^2 + 4y = 0$ gives the Singular solution of ①.

Also $x^2 + 4y = 0$ gives the envelope of the family of Straight lines given by equation ②.

Linear differential equations of second and higher order with Constant Coefficients:

The general form of a linear differential equation of the n^{th} order with constant coefficient is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X$$ —— ①

where $(a_0 \neq 0)$, $a_1, a_2, \ldots a_n$ are Constants and X is a function of x

If we use the differential operator Symbol

$$D \equiv \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \ \ldots, \ D^n = \frac{d^n}{dx^n}.$$

equation ① becomes

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = X$$ —— ②

When $X = 0$, ② becomes, $f(D)y = 0$ —— ③

③ is called the homogeneous equation corresponding to equation ②.

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General solution of eqn ② is $y = u + v$, where $y = u$ is the general solution of ③, that contains n arbitrary constants and $y = v$ is a particular solution of ②, that contains no arbitrary constants. u is called the complementary function (C.F) and v is called the particular integral (P.I.) of the solution of eqn ②.

Complementary Function:

To find the C.F of the solution of eqn ②, we require the general solution of eqn ③.

To get the solution of $f(D)y = 0$ or

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_n) y = 0$$

The auxilary equation is

$$f(m) = 0 \quad \text{or} \quad a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$

The auxilary equation is an nth degree algebraic equation in m.

The solution of eqn ③ depends on the nature of roots of the A.E.

Case (i): The roots of the A.E are real and distinct.

Let the roots of the A.E be $m_1, m_2, \dots m_n$.

Then $C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

Case (ii): The A.E has got real roots, some of which are equal

Let the roots of the A.E be $m_1, m_1, m_3, m_4, \dots, m_n$.

Then $C.F = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

Case (iii): Two roots of the A.E are complex.

Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

$\therefore C.F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

Case (iv): Two pairs of complex roots of the A.E are equal

$m_1 = m_3 = \alpha + i\beta$ and $m_2 = m_4 = \alpha - i\beta$

$$C.F = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$$

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Particular Integral:

P.I = $\frac{1}{}$ The particular integral of the solution of the equation $f(D)y = X$ is the function V, where $y = V$ is a particular solution of (3) containing no arbitrary constants.

The P.I depends on the value of the R.H.S function X and is defined as $P.I = \frac{1}{f(D)} X$, where $\frac{1}{f(D)}$ is the inverse operator of $f(D)$.

1. If $X = e^{ax}$, where α is a constant,

$$P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(\alpha)} e^{\alpha x}, \quad f(\alpha) \neq 0$$

2. $X = \sin ax$ or $\cos ax$, where a is a constant.

$$P.I = \frac{1}{\phi(D^2)} \begin{matrix}\sin ax \\ \text{or} \\ \cos ax\end{matrix} = \frac{1}{\phi(-a^2)} \sin ax \text{ or } \cos ax, \quad \phi(-a^2) \neq 0$$

3. $X = x^m$, where m is a positive integer

$$P.I = \frac{1}{f(D)} x^m = \frac{1}{a_0 D^K} [1 \pm \phi(D)]^{-1} x^m$$

4. $X = e^{ax} . V(x)$ where $V(x)$ is any function

$$P.I = \frac{1}{f(D)} e^{ax} V(x) = e^{ax} . \frac{1}{f(D+a)} V(x)$$

5. $X = x . V(x)$, where $V(x)$ is of the form $\sin ax$ or $\cos ax$

$$P.I = \frac{1}{f(D)} . x V(x)$$

$$= x \frac{1}{f(D)} V(x) - \frac{f'(D)}{\{f(D)\}^2} V(x)$$

6. X is any other function of x.

$$P.I = \frac{1}{f(D)} X = \frac{X}{(D-m_1)(D-m_2) \cdots (D-m_n)}$$

Problems:

1. Solve the equation $(D^2-4D+3)y = \sin 3x + x^2$

The A.E is $m^2-4m+3=0$ $\quad (m-1)(m-3)=0$

$m=1,3$

$C.F = c_1e^x + c_2e^{3x}$

$$P.I = \frac{1}{D^2-4D+3}(\sin 3x + x^2)$$

$$= \frac{1}{D^2-4D+3}\sin 3x + \frac{1}{D^2-4D+3}x^2$$

$$= \frac{1}{-9-4D+3}\sin 3x + \frac{1}{3\left(1-\frac{D(4-D)}{3}\right)}x^2$$

$$= \frac{1}{2(2D+3)}\sin 3x + \frac{1}{3}\left(1-\frac{D(4-D)}{3}\right)^{-1}x^2$$

$$= \frac{-1}{2}\frac{(2D-3)}{4D^2-9}\sin 3x + \frac{1}{3}\left[1+\frac{D(4-D)}{3}+\frac{D^2(4-D)^2}{9}\right]x^2$$

$$= \frac{1}{90}(2D-3)\sin 3x + \frac{1}{3}\left(1+\frac{4}{3}D+\frac{13}{9}D^2\right)x^2$$

$$= \frac{1}{90}(6\cos 3x - 3\sin 3x) + \frac{1}{3}\left(x^2+\frac{8}{3}x+\frac{26}{9}\right)$$

$$= \frac{1}{30}(2\cos 3x - \sin 3x) + \frac{1}{3}\left(x^2+\frac{8}{3}x+\frac{26}{9}\right)$$

The general solution $= C.F + P.I$ $\qquad (1-x)^{-n} = 1+nx+\frac{n(n-1)}{2!}x^2 + \cdots$

2. Solve $(D^3-3D^2+3D-1)y = e^{-x}x^3$.

A.E is $m^3-3m^2+3m-1=0$ $\quad (m-1)^3=0$ $\quad m=1,1,1$

$C.F = (c_1x^2+c_2x+c_3)e^x$

$$P.I = \frac{1}{(D-1)^3}e^{-x}x^3 = e^{-x}\cdot\frac{1}{(D-2)^3}x^3 = -\frac{1}{8}e^{-x}\cdot\left(1-\frac{D}{2}\right)^{-3}\cdot x^3$$

$$= -\frac{1}{8}e^{-x}\cdot\frac{1}{1\cdot 2}\left(1\cdot 2 + 2\cdot 3\frac{D}{2} + 3\cdot 4\frac{D^2}{4} + 4\cdot 5\frac{D^3}{8}\right)x^3$$

$$= -\frac{1}{16}e^{-x}\left(2+3D+3D^2+\frac{5}{2}D^3\right)x^3 = \frac{-1}{16}e^{-x}(2x^3+9x^2+18x+15)$$

The general solution $= C.F + P.I$

3. Solve $(D^2-4D+13)y = e^{2x}\cos 3x$

A.E is $m^2-4m+13=0$; $(m-2)^2=-9$

$\therefore$ The roots are $m = 2\pm 3i$

$C.F = e^{2x}(A\cos 3x + B\sin 3x)$

$$P.I = \frac{1}{D^2-4D+13}\, e^{2x}.\cos 3x$$

$$= e^{2x}. \frac{1}{(D+2)^2-4(D+2)+13}\cos 3x.$$

$$= e^{2x}. \frac{1}{D^2+9}\cos 3x = e^{2x}. \frac{x}{2}\, \frac{\sin 3x}{3}$$

$$= \frac{1}{6}. x. e^{2x}. \sin 3x$$

$\therefore$ The general solution is $y = C.F + P.I$

4. Solve the equation $(D^2+4)y = x^2\cos 2x$

A.E is $m^2+4=0$

The roots are $m = \pm i2$

$\therefore C.F = A\cos 2x + B\sin 2x$

$$P.I = \frac{1}{D^2+4}\, R.P \text{ of } x^2 e^{i2x}$$

$$= R.P \text{ of } e^{i2x}. \frac{1}{(D+i2)^2+4}x^2$$

$$= R.P \text{ of } e^{i2x}. \frac{1}{D^2+4iD}.x^2$$

$$= R.P \text{ of } \frac{e^{i2x}}{4iD}\left(1-\frac{iD}{4}\right)^{-1}x^2$$

$$= R.P \text{ of } \frac{e^{i2x}}{4iD}\left(1+\frac{iD}{4}-\frac{D^2}{16}-\frac{iD^3}{64}\right)x^2$$

$$= R.P \text{ of } \frac{-i}{4}e^{i2x}\left(\frac{x^3}{3}+\frac{i}{4}x^2-\frac{x}{8}\frac{-i}{32}\right)$$

$$= R.P \text{ of } \frac{1}{4}(\sin 2x - i\cos 2x)\left\{\left(\frac{x^3}{3}-\frac{x}{8}\right)+\frac{i}{4}\left(x^2-\frac{1}{8}\right)\right\}$$

$$= \frac{1}{4}\left\{\left(\frac{x^3}{3}-\frac{x}{8}\right)\sin 2x + \frac{1}{4}\left(x^2-\frac{1}{8}\right)\cos 2x\right\}$$

General Solution $y = C.F + P.I$

Euler's Homogeneous Linear Differential Equations.

The equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad \text{—— (1)}$$

where $a_0, a_1, \ldots, a_n$ are constants and X is a function of x is called Euler's homogeneous linear differential equation.

Equation (1) can be reduced to a linear differential equation with constant co-efficients by changing the independent variable from x to t by means of the transformation

$$x = e^t \quad \text{or} \quad t = \log x.$$

(ie) $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \cdot \frac{dy}{dt}$$

$$x \frac{dy}{dx} = \frac{dy}{dt} \quad \text{—— (2)}$$

Denoting $\frac{d}{dx}$ by D and $\frac{d}{dt}$ by θ.

(2) gives $xD = \theta$ and

III^ly
$$x^2D^2 = \theta^2 - \theta = \theta(\theta-1)$$
$$x^3D^3 = \theta(\theta-1)(\theta-2)$$
$$x^4D^4 = \theta(\theta-1)(\theta-2)(\theta-3) \quad \text{and so on.}$$

The more general form of Euler's homogeneous equation is

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = X \quad \text{—— (3)}$$

Equation (3) can be reduced to a linear differential equation with constant coefficients by the substitution $ax+b = e^t$.

Equation (3) is called Legendre's linear differential equation.

Problems

1 Solve the equation $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=x^2+\frac{1}{x^2}$

The given equation is $(x^2D^2+4xD+2)y=x^2+\frac{1}{x^2}$ where $D\equiv\frac{d}{dx}$

Put $x=e^t$ or $t=\log x$ and denote $\frac{d}{dt}$ by θ

(ie) $[\theta(\theta-1)+4\theta+2]y=e^{2t}+e^{-2t}$

$(\theta^2+3\theta+2)y=e^{2t}+e^{-2t}$

A.E is $m^2+3m+2=0$

(ie) $(m+1)(m+2)=0$

$\therefore$ The roots are $m=-1,-2$

$C.F=Ae^{-t}+Be^{-2t}=\frac{A}{x}+\frac{B}{x^2}$

$P.I=\frac{1}{(\theta+1)(\theta+2)}(e^{2t}+e^{-2t})$

$=\frac{1}{12}e^{2t}-\frac{1}{\theta+2}e^{-2t}$

$=\frac{1}{12}e^{2t}-te^{-2t}=\frac{1}{12}x^2-\frac{1}{x^2}\log x$

General solution $=C.F+P.I$

2. Solve $(x^2D^2+xD+1)y=\sin(2\log x).\sin(\log x)$

Putting $x=e^t$ or $t=\log x$ and denoting $\frac{d}{dt}$ by θ, the given equation becomes

$[\theta(\theta-1)+\theta+1]y=\sin 2t\sin t$

$(\theta^2+1)y=\frac{1}{2}(\sin 3t+\sin t)$

A.E is $m^2+1=0$

The roots are $m=\pm i$

$\therefore C.F=A\cos t+B\sin t=A\cos(\log x)+B\sin(\log x)$

$P.I=\frac{1}{\theta^2+1}\cdot\frac{1}{2}(\sin 3t+\sin t)$

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$= \frac{1}{2}\left\{ -\frac{1}{8}\sin 3t + \frac{t}{2}(-\cos t) \right\}$

$= \frac{-1}{16}\sin(3\log x) - \frac{1}{4}\log x \cos(\log x)$

$\therefore$ General solution is $y = C.F + P.I$

Questions	**opt1**	**opt2**	**opt3**	**opt4**	**opt5**	**opt6**	**Answer**
The function f(x)=|x| is	continuous for all x	discontinuous at x=0 only	continuous at x = 0 only	discontinuous for all x			continuous at x = 0 only
Which of the following is continuous at x = 0 ?	f(x) = 1/x	f(x) = |x| /	f(x) = |x|	x = x / |x|			f(x) = |x|
If f is finitely derivable at c, then f is also _____ at c	discontinuous	continuous	derivative	limit			continuous
A function f is said to be ___ in an interval [a, b] if it is continuous at each and every point of the interval	discontinuous	continuous	derivative	limit			continuous
A function f is said to be continuous in an interval [a, b] if it is____ at each and every point of the interval	discontinuous	continuous	derivative	limit			continuous
The exponential function is ___ at all points of R	discontinuous	continuous	derivative	limit			continuous
If x and y be so related that y can be expressed explicitly in terms of x , then y is called_____ function of x	implicit	explicit	even	odd			explicit
If x and y be so related that y cannot be expressed explicitly in terms of x , then y is called_____ function of x	implicit	explicit	even	odd			implicit
A function, whose domain and co-domain aresubsets of the set of all real numbers, is known as _________ function	implicit	explicit	real valued	piecewise continuous			real valued
The set of all the images of all the elements of A under the function f is called the ______ of f.	domain	codomain	range	image			range
Which of the following is continuous function?	e^x	sin x	cos x	e^x, sinx, cosx			e^x, sinx, cosx
Every differentiable function is ____	constant	discontinu	algebraic	continuous			continuous
Every polynomical function of degree n is ___	constant	discontinu	algebraic	continuous			continuous
The derivative of (log x) is	1/x	x	x^2	0			1/x
The derivative of (e^x) is	1/x	x	x^2	e^x			e^x
The derivative of constant is	1/x	0	x^2	x			0
The derivative of (sin x) is	cos x	0	x^2	x			cos x
The derivative of (cos x) is	(cos x)	(- sin x)	tan x	(-x)			(- sin x)
The derivative of (tan x) is	(cos x)	(- sin x)	tan x	(sec^2 x)			(sec^2 x)

The derivative of (cosec x) is	(-cos x)	(- cosec x. cot x)	tan x	(sec^2x)			(- cosec x. cot x)
The derivative of (sec x) is	(sec x tan x)	(- cosec x. cot x)	tan x	(sec^2x)			(sec x tan x)
The derivative of (cot x) is	(-cos x)	(- cosec^2 x)	tan x	(sec^2x)			(- cosec^2 x)
The derivative of (x^3) is	3x^2	3x^3	3x	3			3x^2
The derivative of (5x) is	5x	5	1	0			5
The derivative of (10) is	0	2	3	10			0
The derivative of (5x^2) is	10	0	10x	5x			10x
The derivative of (e^3x) is	6 e^3x	3 e^x	3 e^3x	e^3x			3 e^3x
The derivative of (sin 4x) is	(4cos 4x)	(- 4sin x)	tan4 x	(cos 4x)			(4cos 4x)
The derivative of (cos 2x) is	(- 2sin x)	(- 2sin 2x)	tan x	(- sin 2x)			(- 2sin 2x)
The derivative of (cos 5x) is	(- 5sin x)	(- 5sin 5x)	tan x	(- sin 5x)			(- 5sin 5x)
Find the first derivative of 6x^3	18x^2	18x	18	6x^2			18x^2
Find the second derivative of 6x^3	36	18x^2	36x	18x			36x
Find the third derivative of 6x^3	36	18x^2	36x	18x			36
Find the first derivative of (x^3+2)	x^2+2	x^2	3x^2	3x			3x^2
Find the second derivative of (x^3+2)	x^2+2	6x	3x^2	3x			6x
Find the third derivative of (x^3+2)	x^2+2	6x	3x^2	6			6
Find the first derivative of (log x+2)	1/x	x	x^2	0			1/x
Find the first derivative of (e^x+2x)	e^x	e^x+2	e^x	0			e^x+2
Find the second derivative of (e^x+2x)	e^x	e^x+2	e^x	0			e^x
Find the first derivative of (kx)	kx	x	k	1			k
Find the second derivative of (kx)	kx	x	k	0			0
Find the derivative of y = (x^2) with respect to x	x	2x	x^2	0			2x
Find the derivative of y = (sin 10x) with respect to x	10 cos 10x	(-5 cos 10x)	cos 10x	10 cos x			10 cos 10x

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
The partial differentiation is a function of_________ or more variables .	two	zero	one	three			two
If z=f(x,y) then x and y are ______ function of another variable t	continuous	differential	two	one			continuous
If f(x,y)=0 then xand y are said to be an _________ function	implicit	extremum	explicit	differential			implicit
The concept of jacobian is used when we change the variables in________	multiple integrals	single integrals	diffenential	function			multiple integrals
The jacobian were introduced by_________	C.G.Jacobi	johon	Gauss	Euler			C.G.Jacobi
f(a,b) is said to be extreme value of f(x,y) if it is either a______________	maximum or	zero	minimum	maximum			maximum or minimum
Every extremum value is a stationary value but a stationary value need not be an ___________ value.	infimum	minimum	maximum	extremum			extremum
F is differentiable and where not all of its first differential derivatives vanish simultaneously then the functions u1,u2…..un are said to be functionally_____	independent	dependent	explicit	implicit			dependent
f(a,b) is a maximum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood ________	f(a b)>f(a+h b+k)	f(a b)<f(a+h b+k)	f(a b)<0	f(a b)>0			f(a b)>f(a+h b+k)
f(a,b) is a minimum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood ________	f(a b)>f(a+h b+k)	f(a b)<f(a+h b+k)	f(a b)<0	f(a b)>0			f(a b)<f(a+h b+k)
The necessary condition for maxima is __________	dho f/dho x (a b)=0	dho f/dho x (a b)= 1	dho f/dho y (a b)=5	dho f/dho y (a b)=1			dho f/dho x (a b)=0
The necessary condition for minimum is ____________	dho f/dho x (a b)=0	dho f/dho y (a b)=0	dho f/dho x (a b)=1	dho f/dho y (a b)=1			dho f/dho y (a b)=0
f(a,b) is said to be a stationary value of f(x,y) if (x,y) is__________	dho f/dho x (a b)=0 and dho f/dho y (a b)=0	dho f/dho x (a b)=1	dho f/dho y (a b)=0	dho f/dho y (a b)=1			dho f/dho x (a b)=0 and dho f/dho y (a b)=0
If f(a,b) is said to be _________of f(x,y) if it is either maximum or minimum.	extremum value	boundary value	end	power			extremum value
If u be a _____ of degree n in x and y.	linear	homogeneous	non-homogeneous	polynonmial			homogeneous
The _______ differentiation is a function of two or more variables.	ODE	PDE	partial	total			partial
The ______ were introduced by C.G.Jacobi.	jacobian	millian	taylor	Gauss			jacobian
The concept of ________ is used when we change the variables in multiple integrals	taylor	gauss	maculaurin	jacobian			jacobian

If the function u,v,w of three independent variables x,y,z are not independent then the Jacobian of u,v,w with respect to x,y,z is always equal to	1	0	Infinity	Jacobian of (x y z) with respect ro (u v w)			0
The function f(x)=10+x^6	is a decrasing function of x	has a minimum at x=0	has neither a maximum nor a minimum at x=0	saddle point			has neither a maximum nor a minimum at x=0
The function f(x,y)=2x^2+2xy-y^3 has	only one stationary point at (0 0)	two stationary points at (0 0) and (1/6 & 1/3)	two stationary point at (0 0) and (1 - 1)	not stationary points			two stationary points at (0 0) and (1/6 & 1/3)
If(a/3,a/3) is an extreme point on xy(a-x-y), the maxima is	a^3/27	a/27	a^3/9	a/9			a^3/27
Any function of the type f(x,y)=c is called an _______function	Implicit	Explicit	Constant	composite			Implicit
If u=f(x,y) ,where x=pi(t),y=si(t) then u is a function of t and is called the ____ function	Implicit	Explicit	Constant	composite			composite
The point at which function f(x,y) is either maximum or minimum is known as ______ point	Stationary	Saddle point	extremum	implicit			Stationary
If rt-s^2>0 and r<0 at (a,b) the f(x,y) is maximum at (a,b) and the_______ value of the function(a,b)	Maximum	Minimum	maximum or minimum	zero			Maximum
If rt-s^2>0 and r>0 at (a,b) the f(x,y) is minimum at (a,b) and the_______ value of the function(a,b)	Maximum	Minimum	maximum or minimum	zero			Minimum
If rt-s^2>0 at (a,b) the f(x,y) is neither maximum nor minimum at (a,b) such point is known as _______ point	Stationary	Saddle point	extremum	implicit			Saddle point
If f(x,y) is a function of two variables x,y then _________	lim f(x y)=1	lim f(x y)=0	lim f(x y)>0	lim f(x y)<0			lim f(x y)=1

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
The function f(x) is integrated with respect to x between the limits a and b, then the integral is known as …………………	Definite Integral	Indefinite Integral	Finite Integral	Infinte integral			Definite Integral
The function f(x) is integrated with respect to x without limits, then the integral is known as ……………..	Definite Integral	Indefinite Integral	Finite Integral	Infinte integral			Indefinite Integral
The definite intergral is a…………………	Differentiation	function	Number	limit			Number
The indefinite intergral is a…………………	Number	function	Differentiation	limit			function
∫ dx=.............	x+C	1	0	x^2			x+C
∫cdx=…………….	cx+C	0	1	x+C			cx+C
∫ 5dx=.............	x+C	5x+C	x^2+C	5+C			5x+C
∫ x^n dx=.............	x^(n+1)/ (n+1)+ C	x^(n-1)/ (n-1)+ C	nx^ (n-1)+ C	(n+1) x^(n+1)+ C			x^(n+1)/ (n+1)+ C
∫xdx=…..	x^2+C	x^2/2+C	x^3/2+C	x^2/2+C			x^2/2+C
∫ x^ (2) dx=.............	(x^(2)/2)+C	(x^(3)/3)+C	x+C	2x+C			(x^(3)/3)+C
∫ 3x^(2) dx=.............	3x^(2)+C	x+C	x^2+C	x^(3) +C			x^(3) +C
∫ (1/x) dx=.............	1+ C	log x+C	(-1)+C	(-log x)+ C			log x+C
∫ e^(x) dx=.............	(-e^x)+ C	e^(-x) + C	(-e^(-x))+C	e^x + C			e^x + C
∫ e^(-x) dx=.............	(-e^x)+ C	e^(-x) + C	(-e^(-x))+C	e^x + C			(-e^(-x))+C
∫ e^(2x) dx=.............	(-e^2x)/2+ C	e^(-2x)/2 + C	(-e^(-2x))/2+C	e^2x/2+ C			e^2x/2 + C
∫ e^(-2x) dx=.............	(-e^(-2x))/2+ C	e^(-2x)/2 + C	(-e^(-2x))/2+C	e^(-2x)/2+ C			(-e^(-2x))/2+C
∫ cosx dx=.............	sinx + C	cosx + C	(-cosx)+C	(-sinx)+C			sinx + C
∫ sinx dx=.............	sinx + C	cosx + C	(-cosx)+C	(-sinx)+C			(-cosx)+C
∫ cosmx dx=.............	(sinmx)/m + C	(cosmx)/m + C	(-cosmx)/m+C	(-sinmx)/m+C			(sinmx)/m+ C
∫ sinnx dx=.............	(sinnx)/n + C	(cosnx)/n + C	(-cosnx)/n+C	(-sinnx)/n+C			(-cosnx)/n+C
∫ cos2x dx=.............	(sin2x)/2 + C	(cos2x)/2 + C	(-cosx)/2+C	(-sinx)/2+C			(sin2x)/2 + C
∫ sin3x dx=.............	(sin3x)/3 + C	(cos3x)/3 + C	(-cos3x)/3+C	(-sin3x)/3+C			(-cos3x)/3+C
∫ Sec ^ (2) x dx=.............	secx.tanx+C	tanx+C	tan^(2) x +C	Secx+C			tanx+C
∫ Cosec ^ (2) x dx=.............	cosecx.tanx+C	cotx+C	(-cot(x)) +C	cosecx+C			(-cot(x)) +C
∫ Secx. tanx dx=.............	secx.tanx+C	tanx+C	tan^(2) x +C	Secx+C			Secx+C
∫ cosecx. cotx dx=.............	cosecxcotx+C	cotx+C	(-cosec x) +C	Secx+C			(-cosec x) +C
∫dx/(a^2-x^2)=……………	1/2a log (a+x/a-x)	1/a tan^-1(x/a)	1/2a log (x-a/x+a)	sin^-1(x/a)			1/2a log (a+x/a-x)

∫dx/(x^2-a^2)=……………	sin^-1(x/a)	1/2a log (x-a/x+a)	1/2a log (a+x/a-x)	1/a tan^-1(x/a)			1/2a log (x-a/x+a)
∫dx/(x^2+a^2)=……………	1/2a log (a+x/a-x)	sin^-1(x/a)	1/a tan^-1(x/a)	1/2a log (x-a/x+a)			1/a tan^-1(x/a)
∫dx/√(a^2-x^2)=……………	1/2a log (x-a/x+a)	1/a tan^-1(x/a)	1/2a log (a+x/a-x)	sin^-1(x/a)			sin^-1(x/a)
If u and v are differentiable functions then ∫ u dv =..........	uv-∫ v du	uv+ ∫ v du	(-uv)+ ∫ v du	(-uv)- ∫ v du			uv-∫ v du
∫(limit 1 to ∞)(1/x)dx=.............	∞	0	1	5			∞
∫(limit 0 to 1) x^ (2) dx =.............	1	(1/3)	(1/2)	0			(1/3)
∫(limit 0 to 2)xdx=…..	-2	5	2	1			2

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
The triple integral ∫∫∫ dv gives the ___________ over the region v	area	volume	Direction	weight			volume
The value of ∫∫ dx dy , inner integral limt varies from 1 to 2 and the outer integral limit varies from 0 to 1	0	1	2	3			1
∫∫∫ dx dy dz, the inner integral limit varies from 0 to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1	2	4	6	8			6
When the limits are not given, the integral is named as ___________	Definite integral	Indefinite integral	volume integral	Surface integral			Indefinite integral
The Double integral ∫∫ dx dy gives __________ of the region R	area	modulus	Direction	weight			area
The value of ∫∫ (x+y) dx dy , inner integral limt varies from 0 to 1 and the outer integral limit varies from 0 to 1	0	1	2	3			1
The value of ∫∫∫ x^2 yz dx dy dz, the inner integral limit varies from 1 to 2, the central integral limit varios from 0 to 2 and outer integral limit various from 0 to 1	7/3	1/3	2/3	3			7/3
Evaluate ∫∫ 4xy dx dy, the inner integral limit varies from 0 to 1 and outer integral limit varies from 0 to 2	10	4	5	1			4
The value of ∫∫ d xdy ⁄xy, the inner integral limit varies from 0 to b and the outer limit varies from 0 to a	0	1	ab	loga log b			loga log b
If the limits are given in the integral , the the integral is name as_____________	Definite integral	Infinite integral	volume integral	Surface integral			Definite integral
The value of ∫∫(x^2+3y^2) dy dx , the inner integral limit varies from 0 to 1, the outer integral limit varies from 0 to 3	10	15	12	30			12
The value od ∫∫∫ dxdy d, the inner integral limit varies from 0to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1	6	1	16	12			6
If the limits are not given in the integral , the the integral is name as_____________	Definite integral	Infinite integral	volume integral	Surface integral			Infinite integral
The value of ∫∫(x^2+y^2) dy dx ,the inner integral limit varies from 0 to x, the outer integral limit varies from 0 to 1	1	1⁄3	2⁄3	3⁄2			1⁄3
The value of ∫∫dy dx, the inner integral limit various from 0 to x ,the outer integral limit varies from -a to a	0	1	2	3			0
The Double integral ∫∫ dx dy gives __________ of the region R	area	modulus	Direction	weight			area
The value of ∫∫∫ dx dy dz, the inner integral limit varies from 0 to a , the central integral limit varies from 0 to a and the outer integral limit varies from 0 to a	0	a^3	a^2	a^4			a^3
The value of ∫∫(x+y) dx dy , the inner integral limit varies from 0 to 1 and the outer integral limit varies from 0 to 2	0	1⁄3	2⁄3	3⁄2			1⁄3
The extension of double integral is nothing but _________ integral	Single	Line	volume integral	Triple			Triple
Evaluate ∫x^2⁄2 dx, the limit varies from 0 to 1	2	1⁄6	1⁄10	34			1⁄6
Evaluate ∫42y dy, the limit varies from 0 to 10	10	2100	2000	100			2100
The value of ∫∫2 xy dy dx, the inner integral limit varies from 0 to x and the outer integral limit varies from 1 to 2	15⁄4	9⁄2	3⁄2	4⁄3			15⁄4
The value of ∫∫dy dx, the inner integral limit varies from 2 to 4 ,the outer integral limit varies from 1 to 5	8	2	4	5			8
The value of ∫∫xy dy dx, the inner integral limit varies from 0 to 3 , the outer integral limit varies from 0 to 4	12	36	1⁄2	4			36

The value of ∫∫dy dx, the inner integral limit varies from 0 to 2 , the outer integral limit varies from 0 to 1	2	1	3⁄2	4			2
The value of ∫∫dx dy, the inner integral limit varies from y to 2 , the outer integral limit varies from 0 to 1	1⁄2	1	3⁄2	4			3⁄2
The value of ∫∫dx dy, the inner integral limit varies from 2 to 4 , the outer integral limit varies from 1 to 2	2	6	3	1			2
When a function f(x) is integrated with respect to x between the limits a and b, we get _________	Definite integral	infinite integralv	volume integral	Surface integral			Definite integral
In two dimensions the x and y axes divide the entire xy- plane into ___________ quadrants	1	2	3	4			2
In three dimensions the xy and yz and zx planes divide the entire space into ___________ parts called octants	3	2	8	4			8
Evaluate ∫(2x+3) dx, the integral limitvaries from 0 to 2	10	42	51	1			10

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
An equation involving one dependent variable and its derivatives with respect to one independent variable is called ___________	Ordinary Differential Equation	Partial Differential Equation	Difference Equation	Integral Equation			Ordinary Differential Equation
The roots of the Auxillary equation of Differential equation, (D^2-2D+1)y=0 are	(0 1)	(3 2)	(1 2)	(1 1)			(1 1)
The order of the (D^2+D)y=0 is	2	1	0	-1			2
The roots of the Auxillary equation of Differential equation, (D^4-1)y=0 are	(1 1 1 1)	(1 1 -1 1)	(1 -1 1 -1)	(1 -1 i -i)			(1 -1 i -i)
The degree of the (D^2+2D+2) y=0 is	1	3	0	2			1
The particular integral of (D^2-2D+1)y=e^x is__________	((x^2)/2) e^x	(x/2) e^x	((x^2)/4) e^x	((x^3)/3) e^x			((x^2)/2) e^x
The roots of the Auxillary equation of Differential equation (D^2-4D+4)y=0 are	(2 1)	(2 2)	(2 -2)	(-2 2)			(2 2)
The P.I of the Differential equation (D^2 -3D+2)y=12 is________	1 / 2	1 / 7	6	10			6
If f(D)=D^2 -2, (1/f(D))e^2x=______	(1 / 2) e^x	(1 / 4) e^2x	(1 / 2) e^(-2x)	(1 / 2) e^2x			(1 / 2) e^2x
If f(D)=D^2 +5, (1/f(D)) sin 2x =______	sin x	cos x	sin 2x	-sin 2x			sin 2x
The particular integral of (D^2 +19D+60)y= e^x is______	(-e^(-x))/80	(e^(-x))/80	(e^x)/80	(-e^x)/80			(e^x)/80
The particular integral of (D^2+25) y= cosx is _________	(cosx)/24	(cosx)/25	(-cosx)/24	(-cosx)/25			(cosx)/24
The particular integral of (D^2+25) y= sin4x is __________	(-sin4x)/9	(sin4x)/9	(sin4x)/41	(-sin4x)/41			(sin4x)/9
The particular integral of (D^2+1) y= sinx is ___________	xcosx/2	(-xcosx)/2	(-xsinx)/2	xsinx/2			(-xcosx)/2
The particular integral of (D^2 -9D+20)y=e^(2x) is___________	e^(2x) /6	e^(2x) /(-6)	e^(2x) /12	e^(2x) /(-12)			e^(2x) /6
The particular integral of (D^2-1) y= sin2x is _________	(-sin2x)/5	sin2x/5	sin2x/3	(-sin2x)/3			(-sin2x)/5
The particular integral of (D^2+2) y= cosx is_________	(-cosx)	(-sinx)	cosx	sinx			cosx
The particular integral of (D^2- 7D-30)y= 5 is________	(1/30)	(-1/30)	(1/6)	(-1/6)			(-1/6)
The particular integral of (D^2- 12D-45)y= -9 is________	(-1/5)	(1/5)	(1/45)	(-1/45)			(1/5)
The particular integral of (D^2- 11D-42)y=21 is_______	(-1/42)	(1/42)	(1/2)	(-1/2)			(-1/2)
The particular integral of (D^2+1) y= 2 is ___________	1	2	-1	-2			2
solve (D^2+2D+1) y=0	y=(AX+B)e^(-1)x	y=(AX+B)e^(-2)x	y=(AX^2+B)e^(-1)x	y=(AX-B)e^(-1)x			y=(AX+B)e^(-1)x
The ______ of a PDE is that of the highest order derivative occurring in it	degree	power	order	ratio			order
The degree of the a PDE is ______of the higest order derivative	power	ratio	degree	order			power
C.F+P.I is called ------------------- solution	singular	complete	general	particular			general
Particular integral is the solution of -------------	f(a,b)=F(x,y)	f(1,0)=0	[1/f(D,D')]F(x,y)	f(a,b)=F(u,v)			[1/f(D,D')]F(x,y)
Which is independent varible in the equation z= 10x+5y	x&y	z	x,y,z	x alone			x&y
Which is dependent varible in the equation z=2x+3y	x	z	y	x&y			z
The relation between the independent and the dependent variables which satisfies the PDE is called-------	solution	complet solution	general solution	singular solution			solution