KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Under Section3 of UGC Act 1956)
COIMBATORE-641 021
FACULTY OF ENGINEERING
DEPARTMENT OF SCIENCE AND HUMANITIES
SYLLABUS

B.E Electronics and Communication Engineering / B.E Biomedical Engineering

2018-2019

18BEECEA/18BEBME301B **Semester-III**

Mathematics-III **4H-4C**

(Linear Algebra And Partial Differential Equations)

Instruction Hours/week: L:4 T:0 P:0 **Marks:** Internal:**40** External:**60** Total:**100**

End Semester Exam:3 Hours

COURSE OBJECTIVES:

- To introduce the basic notions of groups, rings, fields which will then be used to solve related problems.
- To understand the concepts of vector space, linear transformations and diagonalization.
- To apply the concept of inner product spaces in orthogonalization.
- To understand the procedure to solve partial differential equations.
- To give an integrated approach to number theory and abstract algebra, and provide a firm basis for further reading and study in the subject.

COURSE OUTCOMES:

Upon successful completion of the course, students should be able to:

1. Explain the fundamental concepts of advanced algebra and their role in modern mathematics and applied contexts.
2. Demonstrate accurate and efficient use of advanced algebraic techniques.
3. Demonstrate their mastery by solving non - trivial problems related to the concepts and by proving simple theorems about the statements proven by the text.
4. Able to solve various types of partial differential equations.
5. Able to solve engineering problems using Fourier series.
6. Able to apply the fundamental concepts in their respective engineering fields

UNIT I VECTOR SPACES

Vector spaces – Subspaces – Linear combinations and linear system of equations – Linear independence and linear dependence – Bases and dimensions.

UNIT II LINEAR TRANSFORMATION AND DIAGONALIZATION

Linear transformation - Null spaces and ranges - Dimension theorem - Matrix representation of a linear transformations - Eigenvalues and eigenvectors - Diagonalizability.

UNIT III INNER PRODUCT SPACES

Inner product, norms - Gram Schmidt orthogonalization process - Adjoint of linear operations - Least square approximation.

UNIT IV PARTIAL DIFFERENTIAL EQUATIONS

Formation – Solutions of first order equations – Standard types and equations reducible to standard types – Singular solutions – Lagrange‘s linear equation – Integral surface passing through a given curve – Classification of partial differential equations - Solution of linear equations of higher order with constant coefficients – Linear non-homogeneous partial differential equations.

UNIT V FOURIER SERIES SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Dirichlet‘s conditions – General Fourier series – Half range sine and cosine series - Method of separation of variables – Solutions of one dimensional wave equation and one-dimensional heat equation – Steady state solution of two-dimensional heat equation – Fourier series solutions in Cartesian coordinates.

Total: 60

Suggested readings:

1. Grewal B.S., ―Higher Engineering Mathematics‖, Khanna Publishers, New Delhi, 43rd Edition, 2014.
2. Friedberg, A.H., Insel, A.J. and Spence, L., ―Linear Algebra‖, Prentice Hall of India, New Delhi, 2004.
3. Burden, R.L. and Faires, J.D, "Numerical Analysis", 9th Edition, Cengage Learning, 2016.
4. James, G. ―Advanced Modern Engineering Mathematics‖, Pearson Education, 2007.
5. Kolman, B. Hill, D.R., ―Introductory Linear Algebra‖, Pearson Education, New Delhi, First Reprint, 2009.
6. Kumaresan, S., ―Linear Algebra – A Geometric Approach‖, Prentice – Hall of India, New Delhi, Reprint, 2010.
7. Lay, D.C., ―Linear Algebra and its Applications‖, 5th Edition, Pearson Education, 2015.
8. O‘Neil, P.V., ―Advanced Engineering Mathematics‖, Cengage Learning, 2007.
9. Strang, G., ―Linear Algebra and its applications‖, Thomson (Brooks/Cole), New Delhi, 2005.
10. Sundarapandian, V. ―Numerical Linear Algebra‖, Prentice Hall of India, New Delhi, 2008.

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SYLLABUS

B.E Electronics and Communication Engineering / B.E Biomedical Engineering **2018-2019**

18BECE301/18BEBME301 **Semester-III**

MATHEMATICS – III
(Linear Algebra And Partial Differential Equations) **4H-4C**

Instruction Hours/week: L:3 T:1 P:0 **Marks:** Internal**:40** External**:60** Total**:100**
End Semester Exam:3 Hours

COURSE OBJECTIVES:

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Vector spaces – Subspaces – Linear combinations and linear system of equations – Linear independence and linear dependence – Bases and dimensions.

UNIT II Linear Transformation And Diagonalization

Linear transformation - Null spaces and ranges - Dimension theorem - Matrix representation of a linear transformations - Eigenvalues and eigenvectors - Diagonalizability.

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Formation – Solutions of first order equations – Standard types and equations reducible to standard types – Singular solutions – Lagrange‘s linear equation – Integral surface passing through a given curve – Classification of partial differential equations - Solution of linear equations of higher order with constant coefficients – Linear non-homogeneous partial differential equations.

UNIT V Fourier Series Solutions Of Partial Differential Equations

Dirichlet‘s conditions – General Fourier series – Half range sine and cosine series - Method of separation of variables – Solutions of one dimensional wave equation and one-dimensional heat equation – Steady state solution of two-dimensional heat equation – Fourier series solutions in Cartesian coordinates.

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FACULTY OF ENGINEERING
DEPARTMENT OF SCIENCE AND HUMANITIES
LECTURE PLAN

Subject : MATHEMATICS – III (Linear Algebra and Partial Differential Equation)
Code : 18BEBME301A /18BEECE301B

Unit No.	List of Topics	No. of Hours
Unit I	**Vector Spaces**	
	Introduction of Vector Spaces -Definition, examples	1
	Vector space-Simple Properties	1
	Theorems and problems based on vector space.	1
	Theorems based on homomorphism	1
	Subspaces-Definition, examples	1
	Theorems with proof on subspaces	1
	Linear combinations and linear system of equations	1
	Linear combinations and linear system of equations	1
	Linear Independence and linear dependence	1
	Properties and problems based on linear independence	1
	Bases and Dimension of a vector space	1
	Bases and Dimension of a vector space	1
	TOTAL	**12**
Unit II	**Linear Transformation and Diagonalization**	
	Introduction of Linear transformation-examples and properties.	1
	Null spaces and ranges	1
	Dimension theorem	1
	Dimension theorem	
	Matrix representation of a linear transformation-Problems	1
	Linear transformation corresponding to a matrix-problems	1
	Linear transformation corresponding to a matrix-problems	1
	Eigen values and eigenvectors	1
	Eigen values and eigenvectors	1
	Diagonalizability	1
	Diagonalizability	1
	Diagonalizability	1
	TOTAL	**12**
Unit III	**Inner Product Spaces**	
	Introduction of Inner Product Spaces	1
	Inner product space, norms- definition and examples	1
	Problems based on Inner product space, norms	1
	Problems based on Inner product space, norms	1
	Gram-Schmidt process for constructing orthonormal basis	1
	Gram-Schmidt process for constructing orthonormal basis-problems	1
	Gram-Schmidt process for constructing orthonormal basis-problems	1
	Adjoint of linear operations	1
	Adjoint of linear operations	1
	Least square approximation	1
	Problems based on Least square approximation	1
	Problems based on Least square approximation	1
	TOTAL	**12**
Unit IV	**Partial Differential Equations**	
	Introduction to Partial Differential Equations	1
	Formation of PDE by eliminating arbitrary constants	1

	Formation of PDE by eliminating arbitrary functions	1
	Solution of PDE of first order (Standard type)	1
	Solution of PDE of first order (Standard type)	1
	Solution of PDE of first order (Standard type)	1
	Lagrange's linear equation	1
	Integral surface passing through a given curve	1
	Integral surface passing through a given curve	1
	Solution of linear equations of higher order with constant coefficients	1
	Solution of linear equations of higher order with constant coefficients	1
	Linear non-homogeneous partial differential equations.	1
	TOTAL	**12**
Unit V	**Fourier Series Solutions of Partial Differential Equations**	1
	Introduction to Fourier series and Dirichlet's conditions	1
	General Fourier series	1
	General Fourier series	1
	Half range sine series	1
	Half range cosine series	1
	Classification of partial differential equations and Method of separation of variables	1
	Solutions of one dimensional wave equation	1
	Solutions of one dimensional wave equation	1
	Solutions of one dimensional heat equation	1
	Solutions of one dimensional heat equation	1
	Solutions of one dimensional heat equation	1
	Steady state solution of two-dimensional heat equation	1
	TOTAL	**12**
	GRAND TOTAL	**60**

Staff- Incharge **HoD**

MA8352- LINEAR ALGEBRA AND PARTIAL DIFFERENTIAL EQUATIONS

II year ECE- III Semester

UNIT I –VECTOR SPACES

CLASS NOTES

VECTOR SPACES

A vector space (or linear space) V over a field F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y in V there is a unique element x + y in V, and for each element 'a' in F and each element 'x' in V there is a unique element 'ax' in V, such that the following conditions (Axioms) hold. In the list below, let x, y and z be arbitrary vectors in *V*, and *a* and *b* scalars in *F*.

Axiom	Meaning
Associativity of addition	(x + y) + z = x + (y + z), $\forall\, x, y, z \in V$
Commutativity of addition	x + y = y + x, $\forall\, x, y \in V$
Identity element of addition	There exists an element **0** ∈ *V*, called the *zero vector*, such that **x** + **0** = x for all x ∈ *V*.
Inverse elements of addition	For every **x** ∈ *V*, there exists an element −x ∈ *V*, called the *additive inverse* of x, such that x + (−x) = **0**.
Compatibility of scalar multiplication with field multiplication	*a*(*b*x) = (*ab*)x
Identity element of scalar multiplication	1x = x, where 1 denotes the multiplicative identity in *F*.
Distributivity of scalar multiplication with respect to vector addition	*a*(x + y) = *a***x** + *ay*
Distributivity of scalar multiplication with respect to field addition	(*a* + *b*)x = *ax* + *bx*

Elements of *V* are commonly called *vectors*. Elements of *F* are commonly called *scalars*.

NOTE:

- When the scalar field *F* is the real numbers **R**, the vector space is called a *real vector space*.
- When the scalar field is the complex numbers **C**, the vector space is called a *complex vector space*.
- These two cases are the ones used most often in engineering.
- The general definition of a vector space allows scalars to be elements of any fixed field *F*. The notion is then known as an *F-vector spaces* or a *vector space over F*.

Coordinate spaces

- The simplest example of a vector space over a field *F* is the field itself, equipped with its standard addition and multiplication.
- More generally, a vector space can be composed of *n*-tuples (sequences of length *n*) of elements of *F*, such as $(a_1, a_2, ..., a_n)$, where each a_i is an element of *F*.
- A vector space composed of all the *n*-tuples of a field *F* is known as a *coordinate space*, usually denoted F^n.
- The case *n* = 1 is the above-mentioned simplest example, in which the field *F* is also regarded as a vector space over itself.

Complex numbers and other field extensions

- The set of complex numbers **C**, i.e., numbers that can be written in the form *x* + *iy* for real numbers *x* and *y* where *i* is the imaginary unit, form a vector space over the reals with the usual addition and multiplication: $(x + iy) + (a + ib) = (x + a) + i(y + b)$ and $c \cdot (x + iy) = (c \cdot x) + i(c \cdot y)$ for real numbers x, y, a, b and c.

Function spaces

- Functions from any fixed set Ω to a field F also form vector spaces, by performing addition and scalar multiplication pointwise. That is, the sum of two functions f and g is the function $(f + g)$ given by $(f + g)(w) = f(w) + g(w)$, and similarly for multiplication.
- Such function spaces occur in many geometric situations, when Ω is the real line or an interval, or other subsets of **R**.

Exmples of Vector Spaces:

Example 1:

(1) Let V and W be vector spaces over a field F. Let $Z = \{(v, w) : v \in V \text{ and } w \in W\}$. **Then Z is a vector space over F with the operations**
$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and $c(v_1, w_1) = (cv_1, cw_1)$

Proof:

(1) For all x,y in Z,

$$x = (v_1, w_1),\ y = (v_2, w_2)$$
$$x + y = (v_1 + v_2, w_1 + w_2)$$
$$= (v_2 + v_1, w_2 + w_1) \quad \because \text{V and W are vector spaces over F (commutativity)}$$
$$= (v_2, w_2) + (v_1, w_1)$$
$$= y + x$$

(2) For all x,y, z in Z where $z = (v_3, w_3)$

$$(x + y) + z = (v_1 + v_2, w_1 + w_2) + (v_3, w_3)$$
$$= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3)$$
$$= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \quad \because \text{V and W are vector spaces over F}$$
$$= (v_1, w_1) + (v_2 + v_3, w_2 + w_3)$$
$$= x + (y + z)$$

(3) There exists a zero vector $(0_V, 0_W)$ in Z where 0_V and 0_W are the zero vectors of V and W respectively such that

$$x + (0_V, 0_W) = (v_1, w_1) + (0_V, 0_W)$$
$$= (v_1 + 0_V, w_1 + 0_W)$$
$$= (v_1, w_1) \qquad (\because v_1 + 0_V = v_1 \text{ and } w_1 + 0_W = w_1)$$
$$= x,\ \forall x \in Z.$$

(4) For each element x in Z, there exists an element y in Z such that

$$x + y = (0_V, 0_W)$$
$$(u_1, v_1) + (u_2, v_2) = (0_V, 0_W)$$
$$(u_2, v_2) = (0_V, 0_W) - (u_1, v_1))$$
$$= (0_V - u_1, 0_W - v_1)$$
$$y = (u_2, v_2) = (-u_1, -v_1)$$

(5) For each element x in Z ,

$1.x = 1.(u_1, v_1) = (1.u_1, 1.v_1) \quad \because 1.u_1 = u_1 \ \& \ 1.v_1 = v_1$

$= (u_1, v_1) = x$

(6) For each pair of elements a, b ∈ F and x ∈ Z

$$\begin{aligned}(ab)x &= (ab)(u_1, v_1)\\ &= ((ab)u_1, (ab)v_1)\\ &= (a(bu_1), a(bv_1))\\ &= a(bu_1, bv_1)\\ &= a(bx)\end{aligned}$$

(7) For each element a ∈ F and each pair of elements x, y ∈ Z ,

$$\begin{aligned}a(x+y) &= a(v_1+v_2, w_1+w_2)\\ &= (a(v_1+v_2), a(w_1+w_2))\\ &= (av_1+av_2, aw_1+aw_2)\\ &= (av_1, aw_1)+(av_2, aw_2)\\ &= ax+ay\end{aligned}$$

(8) For each pair of elements a, b ∈ F and x ∈ Z

$$\begin{aligned}(a+b)x &= (a+b)(v_1, w_1)\\ &= ((a+b)v_1, (a+b)w_1)\\ &= (av_1+bv_1, aw_1+bw_1)\\ &= (av_1, aw_1)+(bv_1, bw_1)\\ &= a(v_1, w_1)+b(v_1, w_1)\\ &= ax+bx\end{aligned}$$

∴ Z is a Vector space over F. This space is called the direct sum of V and W.

Example 2:

For $n \geq 0$, the set P_n of polynomials of degree at most n consists of all polynomials of the form $a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ where the coefficients $a_0, \cdots a_n$ and the variable t are real numbers is a vector space.

Proof:

(i) Let $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ and $q(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$, then p + q is defined by

$$(p+q)(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$$

p + q is a polynomial of degree less than or equal to n.

∴p(t),q(t) ∈ P_n ⇒ (p + q)(t) ∈ P_n

The scalar multiple cp is the polynomial defined by

$$(cp)(t) = cp(t) = (ca_0) + (ca_1)t + (ca_2)t^2 + \cdots + (ca_n)t^n$$

⟹cp is a polynomials of degree less than or equal to n.

⟹ cp ∈ P_n

It follows from the properties of the real numbers.

(ii) Let $r(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$.

$$((p+q)+r)(t) = (p+q)(t) + r(t)$$
$$= \left[(a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2 + \cdots + (a_n+b_n)t^n\right] + (c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n)$$
$$= (a_0+b_0+c_0) + (a_1+b_1+c_1)t + (a_2+b_2+c_2)t^2 + \cdots + (a_n+b_n+c_n)t^n$$
$$= (a_0+(b_0+c_0)) + (a_1+(b_1+c_1))t + (a_2+(b_2+c_2))t^2 + \cdots + (a_n+(b_n+c_n))t^n$$
$$= (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + \left[(b_0+c_0) + (b_1+c_1)t + (b_2+c_2)t^2 + \cdots + (b_n+c_n)t^n\right]$$
$$= p(t) + (q+r)(t)$$
$$= (p+(q+r))(t) \quad \forall\, p(t), q(t), r(t) \in P_n$$

(iii) If all the coefficients are zero, p is called zero polynomial.

The zero polynomial is included in P_n even though its degree is not defined.

(iv) Clearly zero polynomial acts as the zero vector. Finally $(-1)p$ acts as the negative of p.

(v) $1\,p(t) = p(t) \quad \forall\, p(t) \in P_n$

(vi) $(ab)p(t) = (ab)(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)$
$$= (aba_0) + (aba_1)t + (aba_2)t^2 + \cdots + (aba_n)t^n$$
$$= (a)(ba_0 + ba_1 t + ba_2 t^2 + \cdots + ba_n t^n)$$
$$= (a)(bp(t)) \quad \forall\, p(t) \in P_n, a, b \in R$$

(viii) $(a+b)(p(t)) = (a+b)(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)$
$$= (aa_0 + aa_1 t + aa_2 t^2 + \cdots + aa_n t^n)$$
$$+ (ba_0 + ba_1 t + ba_2 t^2 + \cdots + ba_n t^n)$$
$$= ap(t) + bp(t) \quad \forall\, p(t) \in P_n, a, b \in R$$

Thus $P_n(t)$ is the vector space.

Example for not a Vector Space:

Example 3:

Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$. Show that V is not a Vector Space.

Proof:

(1) For all x,y in V,

$x = (a_1, a_2),\ y = (b_1, b_2)$

$$x + y = (a_1 + b_1, a_2 b_2)$$
$$= (b_1 + a_1, b_2 a_2)$$
$$= y + x$$

(2) For all x,y, z in V where $z = (c_1, c_2)$

$$(x+y)+z = [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2)$$
$$= [(a_1 + b_1, a_2 b_2)] + (c_1, c_2)$$
$$= ((a_1 + b_1) + c_1, (a_2 b_2) c_2)$$
$$= (a_1 + b_1 + c_1, a_2 b_2 c_2)$$
$$= (a_1 + (b_1 + c_1), a_2 (b_2 c_2))$$
$$= (a_1, a_2) + (b_1 + c_1, b_2 c_2)$$
$$= (a_1, a_2) + [(b_1, b_2) + (c_1, c_2)]$$
$$= x + (y + z)$$

(3) There exists a zero vector (b_1, b_2) in V such that

$$x + (b_1, b_2) = (a_1, a_2)$$
$$(a_1, a_2) + (b_1, b_2) = (a_1, a_2)$$
$$(a_1 + b_1, a_2 b_2) = (a_1, a_2)$$
$$a_1 + b_1 = a_1 \;\; \& \;\; a_2 b_2 = a_2$$
$$\Rightarrow b_1 = 0 \;\; \& \;\; b_2 = 1$$

$\therefore (0,1) \in V$ is the zero vector.

(4) For each element x in V, there exists an element y in V such that

$$x + y = (0,1)$$
$$(a_1, a_2) + (b_1, b_2) = (0,1)$$
$$(a_1 + b_1, a_2 b_2) = (0,1)$$
$$a_1 + b_1 = 0 \;\; \& \;\; a_2 b_2 = 1$$
$$\Rightarrow b_1 = -a_1 \;\; \& \;\; b_2 = \frac{1}{a_2} \notin R \text{ if } a_2 = 0$$

Hence V is not a vector space over R.

Theorem 1 (Cancellation Law for Vector Addition):

If x, y, and z are vectors in a vector space V such that x + z = y + z, then x = y

Proof: From the definition of Vector space, there exists a vector v in V such that z + v = 0.

Thus x = x + 0 = x + (z + v)

= (x + z) + v

= (y + z) + v

= y + (z + v)

= y + 0 = y.

Theorem 2: In any vector space V, the following statements are true:

(i) 0x = 0 for each $x \in V$

(ii) (- a)x = - (ax) = a(-x)

(iii) a0 = 0 for each $a \in F$.

Proof:

(i) Consider 0 x + 0 x = (0 + 0) x (∵ (a + b)x = ax + bx, $\forall$ a, b $\in$ F and x $\in$ V
= 0 x (∵ 0 + 0 = 0)
= 0 x + 0 (By additive identity of vector space)
= 0 + 0 x (By commutativity of addition)
i.e., 0 x + 0 x = 0 + 0 x
∴ 0 x = 0 (By cancellation Law for vector Addition).

(ii) The vector − (a x) is the unique element of V such that a x + (−(a x)) = 0

Thus if a x + (−a) x = 0, we have −(a x) = (−a) x

But a x + (−a) x = [a + (−a)] x (∵ (a + b)x = ax + bx, $\forall$ a, b $\in$ F and x $\in$ V)

= 0 x

= 0 (by (i))

∴ (−a) x = − (a x).

In particular (−1) x = − (1 x) = − x

a(− x) = a ((−1) x) = [a(−1) x] = (− a) x

∴ (−a) x = − (a x) = a(− x).

(iii) Since $a(x + y) = ax + ay, \forall a \in F$ and $x \in V$, we have,

a 0 + a 0 = a (0 + 0)

= a 0

= a 0 + 0 (By additive identity of vector space)

a 0 + a 0 = 0 + a 0 (By commutativity of addition)

⟹a 0 = 0 (By cancellation Law for vector Addition).

$\therefore a\,0 = 0 \;\forall a \in F$

SUBSPACES

A nonempty subset W of a vector space V overa field is called a subspace of V if W is a vector space over F with the operations as in V. Let V be a vector space and W be a subset of V. Then W is a subspace of V if and only if the following conditions are hold:

(i) The zero vector of V is in W.

(ii) W is closed under vector addition. i.e., $u + v \in W, \;\forall\, u, v \in W$

(iii) W is closed under multiplication by scalars. i.e., $cu \in W, \forall\, c \in F,\; u \in W$

Examples:

1. The set of all diagonal matrices is a subspace of $M_{mxn}(F)$.
2. Let n be a non-negative integer and $P_n(F)$ consists of all polynomials in P(F) having degree less than or equal to n. Then $P_n(F)$ is a subspace of P(F).

NOTE:

- Subspaces of *V* are vector spaces (over the same field) in their own right.
- A linear subspace of dimension 1 is a **vector line**.
- A linear subspace of dimension 2 is a **vector plane**.
- A linear subspace that contains all elements but one of a basis of the ambient space is a **vector**

hyperplane.

Example 4: Is the following set a subspace of $\mathbf{R}^2$?

$$A = \{(x,\ 3x+1):\ x \in \mathbf{R}\}$$

Solution:

To establish that A is a subspace of $\mathbf{R}^2$, it must be shown that A is closed under addition and scalar multiplication. If a counterexample to even one of these properties can be found, then the set is not a subspace. In the present case, it is very easy to find such a counterexample. For instance, both **u** = (1, 4) and **v** = (2, 7) are in A, but their sum, **u** + **v** = (3, 11), is not. In order for a vector **v** = (v_1, v_2) to be in A, the second component (v_2) must be 1 more than three times the first component (v_1). Since $11 \neq 3(3) + 1$, $(3, 11) \notin A$. Therefore, the set A is not closed under addition, so A cannot be a subspace. Also **u** = (1, 4) is in A, the scalar multiple 2 **u** = (2, 8) is not in A.

Example 5: Show that if V is a subspace of $\mathbf{R}^n$, then V must contain the zero vector.

Solution:

First, choose any vector **v** in V. Since V is a subspace, it must be closed under scalar multiplication. By selecting 0 as the scalar, the vector 0 **v**, which equals **0**, must be in V. [Another method proceeds like this: If **v** is in V, then the scalar multiple (−1) **v** = − **v** must also be in V. But then the sum of these two vectors, **v** + (− **v**) = **0**, must be in V, since V is closed under addition.]

Example 6: Does the plane P given by the equation $2x + y - 3z = 0$ form a subspace of $\mathbf{R}^3$?

One way to characterize P is to solve the given equation for y,

$y = 3z - 2x$ and write

$$P = \{(x,\ 3z-2x,\ z):\ x, z \in \mathbf{R}\}$$

If $\mathbf{p}_1 = (x_1, 3z_1 - 2x_1, z_1)$ and $\mathbf{p}_2 = (x_2, 3z_2 - 2x_2, z_2)$ are points in P, then their sum,

$$\mathbf{p}_1 + \mathbf{p}_2 = (x_1 + x_2,\ 3(z_1+z_2) - 2(x_1+x_2),\ z_1 + z_2)$$

is also in P, so P is closed under addition.

Furthermore, if $\mathbf{p} = (x, 3z - 2x, z)$ is a point in P, then any scalar multiple, $k\mathbf{p} = (kx,\ 3(kz) - 2(kx),\ kz)$

is also in P, so P is also closed under scalar multiplication.

Therefore, P does indeed form a subspace of $\mathbf{R}^3$. Note that P contains the origin.

Note:

By contrast, the plane $2x + y - 3z = 1$, although parallel to P, is *not* a subspace of $\mathbf{R}^3$ because it does not contain (0, 0, 0). In fact, a plane in $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$ if and only if it contains the origin.

Example 7:

Let H be the set of points inside and on the unit circle in the xy-plane . That is, let

$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$. **Find a specific example- two vectors and a scalar- to show that H is not a subspace of R^2 .**

Proof:

Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$.

$$\text{Let } u = \left(\frac{1}{2}, \frac{1}{2}\right) \ \& \ v = \left(\frac{1}{4}, \frac{1}{4}\right)$$

$$\because \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{2}{4} = \frac{1}{2} < 1 \text{ and}$$

$$\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{2}{16} = \frac{1}{8} < 1, \text{ we have}$$

$u, v \in H$

$$\text{Now } u + v = \left(\frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{4}, \frac{1}{4}\right) = \left(\frac{3}{4}, \frac{3}{4}\right)$$

$$\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{18}{16} > 1$$

Hence $u + v \notin H$

$$\text{Let } c = 4 \text{ be any scalar and } u = \left(\frac{1}{2}, \frac{1}{2}\right) \in H$$

$$\text{Now } cu = 4\left(\frac{1}{2}, \frac{1}{2}\right) = (2, 2)$$

Since $2^2 + 2^2 = 8 > 1$

$\therefore cu \notin H$.

Theorem 3:

Given v_1 and v_2 in a vector space V and let H = span{v_1, v_2). Then H is a subspace of V.

Proof:

The zero vector is in H, since $0 = 0v_1 + 0v_2$. To show that H is closed under vector addition, take two arbitrary vectors in H, say, $u = s_1v_1 + s_2v_2$ and $w = t_1v_1 + t_2v_2$

For the vector space V,

$$u + w = (s_1v_1 + s_2v_2) + (t_1v_1 + t_2v_2)$$
$$=(s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

So u + w is in H. Furthermore, if c is any scalar, $cu = c(s_1v_1 + s_2v_2) = (cs_1)v_1 + (cs_2)v_2$

which shows that cu is in H and H is closed under scalar multiplication.

Thus H is a subspace of V.

Theorem 4:

Any intersection of subspaces of a vector space V is a subspace of V.

Proof:

Let C be a collection of subspaces of V, and let W denote the intersection of the subspaces in C.

Since every subspace contains the zero vector, 0 ∈ W.

Let a ∈ F and x, y ∈ W. Then x and y are contained in each subspace in C. Because each subspace in C is closed under addition and scalar multiplication, it follows that x + y and ax are contained in each subspace in C.

Hence x + y and a x are also contained in W, so that W is a subspace of V. (Because, if V is a vector space and W is a subset of V, then W is a subspace of V if and only if the following conditions hold for the operations defined in V. (a) 0 ∈ W. (b) x + y ∈ W whenever x ∈ W and y ∈ W. (c) cx ∈ W whenever c ∈ F and x ∈ W.)

Theorem 5:

Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Proof:

Assume that $W_1 \cup W_2$ is a subspace of V. To prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Assume the contrary that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then there exists elements $x \in W_1$ but $x \notin W_2$ and $y \in W_2$ but $y \notin W_1$. Therefore, x and $y \in W_1 \cup W_2$.

Since $W_1 \cup W_2$ is a subspace of V then $x + y \in W_1 \cup W_2$.

Case 1: Take $x + y \in W_1$.

Now $x + y \in W_1$ and $-x \in W_1$ then $-x + x + y \in W_1 \Rightarrow y \in W_1$. This is a contradiction.

Case 2: Take $x + y \in W_2$.

Now $x + y \in W_2$ and $-y \in W_2$ then $x + y + (-y) \in W_2 \Rightarrow x \in W_1$. This is a contradiction.

Therefore, $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Conversely, assume that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. To prove that $W_1 \cup W_2$ is a subspace of V.

Let $W_1 \subseteq W_2$. Then $W_1 \cup W_2 = W_2$, a subspace of V.

Let $W_2 \subseteq W_1$. Then $W_1 \cup W_2 = W_1$, a subspace of V.

Define sum of V:

Let W_1 and W_2 be subspaces of a vector space V. The sum of W_1 and W_2 is defined as $W_1 + W_2 = \{x + y \,/\, x \in W_1 \text{ and } y \in W_2\}$.

Direct sum of two subspaces:

A vector space V is called the direct sum of W_1 and W_2, if W_1 and W_2 are subspaces of V such that (i) $W_1 + W_2 = V$ and (ii) $W_1 \cap W_2 = \phi$.

Theorem 6:

Let W_1 denote the set of all polynomials $f(x)$ in $P(F)$ such that in the representation

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, we have $a_i = 0$ whenever i is even.

Likewise let W_2 denote the set of all polynomials g(x) in $P(F)$ such that in the representation $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, we have $b_i = 0$ whenever i is odd. Then $P(F) = W_1 \oplus W_2$.

Proof:

Clearly, $W_1 \cap W_2 = \{0\}$

$$
\begin{aligned}
P(F) &= \{a_1 x + a_3 x^3 + a_5 x^5 + \cdots + a_{2n-1}x^{2n-1} : a_i \in F\} \\
&\quad + \{b_0 + b_2 x^2 + b_4 x^4 + \cdots + b_{2n}x^{2n} : b_i \in F\} \\
&= \{b_0 + a_1 x + b_2 x^2 + a_3 x^3 + b_4 x^5 + \cdots + a_{2n-1}x^{2n-1} + b_{2n}x^{2n} : a_i, b_i \in F\} \\
&= W_1 \oplus W_2.
\end{aligned}
$$

LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

In mathematics, a **linear combination** is an expression constructed from a set of terms by multiplying each term by a constant and adding the results (e.g. a linear combination of *x* and *y* would be any expression of the form *ax* + *by*, where *a* and *b* are constants).

Linear combination:

Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$ in F such that $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$.

In this case we also say that v is a linear combination of $u_1, u_2, \ldots, u_n$ and call $a_1, a_2, \ldots, a_n$ the co-efficients of the linear combination.

In any vector space V, 0v = 0 for each v in V. Thus the zero vector is a linear combination of any nonempty subset of V.

Span of S: Let V be a vector space over a field F and $S \subseteq V$.The span of S, denoted span (S), is the set consisting of all linear combinations of the vector in S. In particular span $(\phi)=\{0\}$.

Example 8:

Show that $3x^3 - 2x^2 + 7x + 8$ **can be expressed as a linear combinations of** $x^3 - 2x^2 - 5x - 3$ **and** $3x^3 - 5x^2 - 4x - 9$

Proof:

Let $u_1 = x^3 - 2x^2 - 5x - 3$; $u_2 = 3x^3 - 5x^2 - 4x - 9$

determine the scalars a_1, a_2 such that

$$
\begin{aligned}
3x^3 - 2x^2 + 7x + 8 &= a_1 u_1 + a_2 u_2 \\
&= a_1(x^3 - 2x^2 - 5x - 3) + a_2(3x^3 - 5x^2 - 4x - 9) \\
&= (a_1 + 3a_2)x^3 + (-2a_1 - 5a_2)x^2 + (-5a_1 - 4a_2)x + (-3a_1 - 9a_2)
\end{aligned}
$$

Therefore,

$$a_1 + 3a_2 = 3$$
$$-2a_1 - 5a_2 = -2$$
$$-5a_1 - 4a_2 = 7$$
$$-3a_1 - 9a_2 = 8$$

Solving the above system by elimination method

$$\left(\begin{array}{cc|c} 1 & 3 & 3 \\ -2 & -5 & -2 \\ -5 & -4 & 7 \\ -3 & -9 & 8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 11 & 22 \\ 0 & 0 & 17 \end{array}\right) \begin{array}{l} R_2 \leftrightarrow R_2 + 2R_1 \\ R_3 \leftrightarrow R_3 + 5R_1 \\ R_4 \leftrightarrow R_4 + 3R_1 \end{array}$$

The reduced equations are

$$a_1 + 3a_2 = 3$$
$$a_2 = 4$$
$$11a_2 = 22$$
$$0 = 17$$

The last equation is impossible.

Hence, $3x^3 - 2x^2 + 7x + 8$ cannot be expressed as a linear combinations of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$.

Example 9:

Show that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ **generate $M_{2\times2}(F)$.**

Proof:

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be any arbitrary matrix in $M_{2\times2}(F)$. Let a, b, c and d by any scalarsso that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow a = a_{11},\ b = a_{12},\ c = a_{21}, \text{ and } d = a_{22}$$

$\therefore$the given matrices generate $M_{2\times2}(F)$.

Example 10:

Prove that the vector $(2,6,8)$ **can be expressed as a linear combinations of**

$(1,2,1)$**,** $(-2,-4,-2)$**,** $(0,2,3)$**,** $(2,0,-3)$**,** $(-3,8,16)$

Proof:

Let $u_1 = (1,2,1)$; $u_2 = (-2,-4,-2)$; $u_3 = (0,2,3)$; $u_4 = (2,0,-3)$; $u_5 = (-3,8,16)$.

determine the scalars a_1, a_2, a_3, a_4, a_5 such that

$$\begin{aligned}(2,6,8) &= a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 \\ &= a_1(1,2,1) + a_2(-2,-4,-2) + a_3(0,2,3) + a_4(2,0,-3) + a_5(-3,8,16) \\ &= (a_1 - 2a_2 + 2a_4 - 3a_5, 2a_1 - 4a_2 + 2a_3 + 8a_5, a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5)\end{aligned}$$

Therefore,

$$\begin{aligned} a_1 - 2a_2 + 0a_3 + 2a_4 - 3a_5 &= 2 \\ 2a_1 - 4a_2 + 2a_3 + 0a_4 + 8a_5 &= 6 \\ a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5 &= 8 \end{aligned}$$

Solving the above system by elimination method

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2 \\ 2 & -4 & 2 & 0 & 8 & 6 \\ 1 & -2 & 3 & -3 & 16 & 8 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2 \\ 0 & 0 & 2 & -4 & 14 & 2 \\ 0 & 0 & 3 & -5 & 19 & 6 \end{array}\right) \begin{array}{l} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & -2 & 7 & 1 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right) R_3 \leftrightarrow 2R_3 - 3R_2$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & -2 & 7 & 1 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right) \begin{array}{l} R_1 \leftrightarrow R_1 - R_3 \\ R_2 \leftrightarrow R_2 + R_3 \end{array}$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right) \begin{array}{l} R_1 \leftrightarrow R_1 - R_3 \\ R_2 \leftrightarrow R_2 + R_3 \end{array}$$

The reduced equations are

$$\begin{aligned} a_1 - 2a_2 + a_5 &= -4 \\ a_3 + 3a_5 &= 7 \\ 2a_4 - 4a_5 &= 6 \end{aligned}$$

Take $a_2 = 0$ and $a_5 = 0$ we get, $a_1 = -4$, $a_3 = 7$, $a_4 = 3$.

$\therefore (2,6,8) = -4(1,2,1) + 7(0,2,3) + 3(2,0,-3)$

Thus, $(2,6,8)$ can be expressed as a linear combinations of $(1,2,1)$, $(-2,-4,-2)$, $(0,2,3)$, $(2,0,-3)$, $(-3,8,16)$

Generator Set:

A subset S of a vector space V generates (or spans) V if span(s) = V. Here S is called generator set of V.

Example 11:

Prove that the vectors (1, 1, 0), (1, 0, 1) and (0, 1, 1) generate R^3.

Proof:

The vectors (1, 1, 0), (1, 0, 1) and (0, 1, 1) generate R^3 since any arbitrary vector (a_1, a_2, a_3) in R^3 is a linear combination of the three given vectors; in fact,for the scalars r, s, and t

We have, $r(1, 1, 0) + s(1, 0, 1) + t(0, 1, 1) = (a_1, a_2, a_3)$

$$r + s = a_1 --- (1)$$

Equating we get, $r + t = a_2 --- (2)$

$$s + t = a_3 --- (3)$$

Adding above, $2(r + s + t) = a_1 + a_2 + a_3$

$$r + s + t = \frac{a_1 + a_2 + a_3}{2}$$

$$r + a_3 = \frac{a_1 + a_2 + a_3}{2} \quad by(3)$$

$$\therefore r = \frac{1}{2}(a_1 + a_2 - a_3)$$

Similarly, $s = \frac{1}{2}(a_1 - a_2 + a_3)$, and $t = \frac{1}{2}(-a_1 + a_2 + a_3)$.

Thus, $(a_1, a_2, a_3) = \frac{(a_1 + a_2 - a_3)}{2}(1,1,0) + \frac{(a_1 - a_2 + a_3)}{2}(1,0,1) + \frac{(-a_1 + a_2 + a_3)}{2}(0,1,1)$

$\therefore$(1, 1, 0), (1, 0, 1) and (0, 1, 1) generate R^3.

Example 12:

Show that W is in the subspace of R^4 spanned by v_1, v_2, v_3, where

$$w = \begin{bmatrix} 9 \\ -4 \\ -4 \\ 7 \end{bmatrix}, v_1 = \begin{bmatrix} 8 \\ -4 \\ -3 \\ 9 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 3 \\ -2 \\ -8 \end{bmatrix}, v_3 = \begin{bmatrix} -7 \\ 6 \\ -5 \\ -18 \end{bmatrix}$$

Proof:

To prove $w \in span\{v_1, v_2, v_3\}$, we must find some scalars a, b, and c (not all zero) in R such that $w = av_1 + bv_2 + cv_3$

$$\begin{bmatrix} 9 \\ -4 \\ -4 \\ 7 \end{bmatrix} = a\begin{bmatrix} 8 \\ -4 \\ -3 \\ 9 \end{bmatrix} + b\begin{bmatrix} -4 \\ 3 \\ -2 \\ -8 \end{bmatrix} + c\begin{bmatrix} -7 \\ 6 \\ -5 \\ -18 \end{bmatrix}$$

$$\Rightarrow 8a - 4b - 7c = 9 \rightarrow (1)$$

$$-4a + 3b + 6c = -4 \rightarrow (2)$$

$$-3a - 2b - 5c = -4 \rightarrow (3)$$

$$9a - 8b - 18c = 7 \rightarrow (4)$$

$$(1) \Rightarrow 8a - 4b - 7c = 9$$

$$(2) \times 2 \Rightarrow -8a + 6b + 12c = -8$$

Adding, we get $2b + 5c = 1 \rightarrow (5)$

$$(4) \Rightarrow 9a - 8b - 18c = 7$$

$$(3) \times 3 \Rightarrow -9a - 6b - 15c = -12$$

$$\text{Adding, we get } -14b - 33c = -5$$

$$\Rightarrow 14b + 33c = 5$$

$$(5) \times 7 \Rightarrow 14b + 35c = 7$$

$$\text{subtracting, we get } -2c = -2$$

$$\Rightarrow c = 1$$

substituting $c = 1$ in (1) & (2), we have

$$8a - 4b = 16 \rightarrow (6)$$

$$-4a + 3b = -10 \rightarrow (7)$$

$$(6) \Rightarrow 8a - 4b = 16$$

$$(7) \times 2 \Rightarrow -8a + 6b = -20$$

Adding,

$$2b = -4$$

$$\Rightarrow b = -2$$

Substituting $c = 1$ and $b = -2$ in (1), we get $a = 1$.

$$\therefore a = 1, b = -2, c = 1$$

$$w = v_1 - 2v_2 + v_3$$

Hence ,W is in the subspace of R^4 spanned by v_1, v_2, v_3

Theorem 7:

The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S

Proof:

Case (i): If S = ∅, then span(∅) = $\{0\}$, which is a subspace that is contained in any subspace of V.

Case(ii): If S ≠ ∅, then S contains a vector z, so 0 z = 0 is in span (S).

Let x, y ∈ span (S). Then there exist vectors

$u_1, u_2, \cdots u_m, v_1, v_2, \cdots v_n$ in S and scalars $a_1, a_2, \cdots a_m, b_1, b_2, \cdots b_n$ such that

$x = a_1u_1 + a_2u_2 + \cdots + a_mu_m$ and

$y = b_1v_1 + b_2v_2 + \cdots + b_nv_n$.

Then $x + y = a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + b_2v_2 + \cdots + b_nv_n$

and for any scalar c,

$cx = (ca_1)u_1 + (ca_2)u_2 + \cdots + (ca_m)u_m$ are clearly linear combinations of vectors in S_i

so $x + y \in span(S)$ and $cx \in span(S)$.

Thus span(S) is a subspace of V.

Now let W denote any subspace of V that contains S. Now to prove span(S)⊆ W.

Let W ∈ span(S), then W is of the form

$w = c_1w_1 + c_2w_2 + \cdots + c_kw_k$ for some vectors $w_1, w_2 \cdots w_k$ in S and some scalars $c_1, c_2 \cdots c_k$. since $S \subseteq W$, we have $w_1, w_2 \cdots w_k \in W$.

$\therefore W = c_1w_1 + c_2w_2 + \cdots + c_kw_k \in W$

Since W is an arbitrary vector in span(S), belong to W, it follows that span(S) $\subseteq$ W.

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

Linear dependence set: A subset S of a vector space V is called linearly dependent set if there exist a finite number of distinct vectors $u_1, u_2, \ldots\ldots, u_n$ in S and scalars $a_1, a_2, \ldots.., a_n$, not all zero, such that

$a_1u_1 + a_2u_2 + \ldots\ldots + a_nu_n = 0$.

Linear independence set: A subset S of a vector space that is not linearly dependent then it is called linearly independent

Example 13:

Show that the vectors u =(1, 1, 0), v =(1, 3, 2), w =(4, 9, 5) are linearly dependent

Proof:

Let u = (1, 1, 0), v = (1, 3, 2), w = (4, 9, 5). Then u, v, w are linearly dependent, because

3u + 5v – 2w = 3(1, 1, 0) + 5(1, 3, 2) – 2(4, 9, 5) = (0, 0, 0) = 0.

Example 14:

Show that the vectors u =(1, 2, 3), v =(2, 5, 7), w =(1, 3, 5) are linearly independent

Proof:

To show that the vectors u = (1, 2, 3), v = (2, 5, 7), w = (1, 3, 5) are linearly independent, we form the vector equation au + bv + cw = 0, where a, b, c are unknown scalars. This yields

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}2\\5\\7\end{bmatrix} + z\begin{bmatrix}1\\3\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

or a + 2b + c = 0 --- (1)

2a + 5b + 3c = 0 --- (2)

3a + 7b + 5c = 0 ---(3)

(2) - (1)x2 =>b + c = 0 ---(4)

(3) - (1)x3 =>b – 2c = 0 ---(5)

(4) – (5) => 3c = 0---(6)

Back – substitution yields a = 0, b = 0, c = 0. We have shown that au + bv +cw = 0 implies

a = 0, b = 0, c = 0. Therefore, u, v, w are linearly independent.

Example 15:

In $M_{2\times3}(R)$, show that the set $\left\{\begin{pmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{pmatrix}, \begin{pmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{pmatrix}, \begin{pmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{pmatrix}\right\}$ **is linearly dependent**

Proof:

Let $a\begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix} + b\begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix} + c\begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$a - 3b - 2c = 0;\ -3a + 7b + 3c = 0;\ 2a + 4b + 11c = 0;\ -4a + 6b - c = 0;$

$0.a - 2b - 3c = 0;\ 5a - 7b + 2c = 0$

Use Gauss Elimination Method

Write the augmented matrix for the first three equations

$$\begin{bmatrix} 1 & -3 & -2 & 0 \\ -3 & 7 & 3 & 0 \\ 2 & 4 & 11 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 10 & 15 & 0 \end{bmatrix} \quad R_2 = R_2 + 3R_1, R_3 = R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 + 5R_2$$

By back substitution

$a - 3b - 2c = 0 \rightarrow (1)$

$-2b - 3c = 0 \rightarrow (2)$

$(2) \Rightarrow -2b = 3c \Rightarrow c = \dfrac{-2b}{3}$

Let us take b=3

then c=-2, a=5

Hence the given set is linearly dependent because

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 16:

Determine whether $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(R)$ is linearly dependent or not.

Solution:

Let $u = x^3 + 2x^2,\ v = -x^2 + 3x + 1,\ w = x^3 - x^2 + 2x - 1$

Consider au + bv + cw = 0

$a\left(x^3 + 2x^2\right) + b\left(-x^2 + 3x + 1\right) + c\left(x^3 - x^2 + 2x - 1\right) = 0$

$(a+c)x^3+(2a-b-c)x^2+(3b+2c)x+(b-c)=0$

$a+c=0 \rightarrow (1)$

$2a-b-c=0 \rightarrow (2)$

$3b+2=0 \rightarrow (3)$

$b-c=0 \rightarrow (4)$

$(4) \Rightarrow b=c$

$\text{Sub } b=c \text{ in } (3) \Rightarrow b=0$

$\Rightarrow a=0$

$\therefore a=b=c=0$

Hence u, v, w are linearly independent.

Theorem 8:

Let S be a linearly independent subset of a vector space V and let $v \in V$ such that $v \notin S$.

Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof:

If $v \in \text{span}(S)$, then v is a linear combination of vectors from S.

i.e.there exists $u_1, \ldots, u_n \in S$ and $a_1, \ldots, a_n \in F$ such that $v = a_1u_1 + \ldots + a_nu_n$.

$a_1u_1 + \ldots + a_nu_n + (-1)v = 0$.

Set $a_{n+1} = -1$ and $u_{n+1} = v$.

$\sum_{i=1}^{n+1} a_iu_i = 0$, where not all $a_i = 0$ because $a_{n+1} \neq 0$.

$\therefore S \cup \{v\}$ is linearly dependent.

Converse part, If $S \cup \{v\}$ is linearly dependent, then $\exists u_1, \ldots, u_n \in S$ and $a_1, \ldots, a_n, a_{n+1} \in F$ such that

$a_1u_1 + \ldots + a_nu_n + a_{n+1}v = 0$, where not all $a_i = 0$.

$a_1u_1 + \ldots + a_nu_n = (-1)a_{n+1}v$

$v = (-a_1/a_{n+1})u_1 + \ldots + (-a_n/a_{n+1})u_n$, where $(-a_i/a_{n+1}) \in F \ \forall i$.

v is a linear combination of vectors $u_1, \ldots, u_n$ from S.

$\Rightarrow v \in \text{span}(S)$

BASES AND DIMENSION

Basis: A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Example:

1) For the vector space F^n, $\{e_1, e_2, \ldots, e_n\}$ is a basis.

   Where $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$.

2)For the vector space vector space of all polynomials of degree $\leq n$, $P_n(F)$, $\{1, x, x^2, \ldots, x^n\}$ is a basis.

3) For the vector space of all polynomials of any degreeP(F), $\{1, x, x^2, x^3, \ldots\}$ is a basis.

Example 17:

Prove that the vectors $u_1=(2,-3,1)$, $u_2=(1,4,-2)$, $u_3=(-8,12,-4)$, $u_4=(1,37,-17)$and $u_5=(-3,-5,8)$ generate R^3. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for R^3.

Proof:

Let us select any non-zero vector from the set

$S = \{u_1, u_2, u_3, u_4, u_5\}$, say (2, −3,1) to be a vector in the basis.

Since $(-8,12,-4) = -4(2,-3,1)$, the set $\{(2,-3,1), (-8,12,-4)\}$ is linearly dependent.

Hence we do not include (−8, 12, −4), in our basis. Whereas (1, 4, −2) is not a multiple of (2,−3, 1) and viceversa, so the set {(2,−3,1), (1,4, −2)} is linearly independent. So we include (1,4,−2) in our basis.

Now consider the set {(2,−3,1), (1,4, −2), (1,37,−17)} by adjoining (1,37,−17).

We include (1,37,−17) in our basis or exclude it from the basis according to whether the set is linearly independent or dependent. Since $u_4 = -3u_1 + 7u_2$, the set $\{u_1, u_2, u_4\}$ is linearly dependent,so we exclude u_4 from our basis. Next let us include $u_5 = (-3, -5, 8)$ to the set {(2,−3,1), (1,4, −2). Now consider the set $\{u_1, u_2, u_5\}$.

To check $\{u_1, u_2, u_5\}$ is linearly independent or not.

Consider $a(2,-3,1) + b(1,4,-2) + c(-3,-5,8) = (0,0,0)$

$2a + b - 3c = 0 \quad \rightarrow (1)$

$-3a + 4b - 5c = 0 \rightarrow (2)$

$a - 2b + 8c = 0 \quad \rightarrow (3)$

Consider the augmented matrix

$$\begin{pmatrix} 2 & 1 & -3 & 0 \\ -3 & 4 & -5 & 0 \\ 1 & -2 & 8 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & -3 & 0 \\ 0 & 11 & -19 & 0 \\ 0 & -5 & 13 & 0 \end{pmatrix} R_2 = 2R_2 + 3R_1 \ \& \ R_3 = 2R_3 - R_1$$

$$\sim \begin{pmatrix} 2 & 1 & -3 & 0 \\ 0 & 11 & -19 & 0 \\ 0 & 0 & 48 & 0 \end{pmatrix} R_3 = 11R_3 - R_1$$

$\Rightarrow 2a + b - 3c = 0$

$1b - 19c = 0$

$48c = 0$

Back substitution yields $a = b = c = 0$

$\therefore$ the set $\{u_1, u_2, u_5\}$ is linearly independent.

Hence it forms a basis for R^3.

Theorem 9:

Let V be a vector space and $\beta = \{u_1, \ldots, u_n\}$ be a subset of V.Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors in β.

Proof:

Assume that β is a basis for V then β is a linearly independent set and span(β) = V

Let v $\in$ V then v $\in$span(β). Therefore v can be expressed as a linear combinations of vectors in β.

i.e. there exists scalars $a_1, \ldots, a_n \in$ F such that $v = \sum_{i=1}^{n} a_i u_i$

To prove uniqueness, suppose there exists scalars $b_1, \ldots, b_n \in$ F such that $v = \sum_{i=1}^{n} b_i u_i$.

Now, v – v = $\sum_{i=1}^{n} (a_i - b_i) u_i$. Since β is a linearly independent then $a_i - b_i = 0 \Rightarrow a_i = b_i$.

$\therefore$ each v$\in$V is uniquely expressed as a linear combination of vectors in β.

Converse part: Assume that each v $\in$ V can be uniquely expressed as a linear combination of vectors in β, then there exists scalars $a_1, \ldots, a_n \in$ F such that $v = \sum_{i=1}^{n} a_i u_i$ **.**

Therefore, 0 $\in$ V. Has only trivial representation given by $0 = \sum_{i=1}^{n} c_i u_i$ **.**

Hence β is linearly independent.

Further, each v $\in$ V can be uniquely expressed as a linear combination of vectors in β. $\therefore$ V$\subseteq$span(β).

Also, Span(β)$\subseteq$ V.

$\Rightarrow$ Span(β) = V.

Hence β is a basis for V.

Theorem 10:

Let H and K be subspaces of a vector space V. Then dim(H $\cap$ K) $\leq$ dim H

Proof:

Let $\{v_1,\ldots,v_p\}$ be a basis for H $\cap$ K. Since $\{v_1,\ldots,v_p\}$ is a linearly independent subset of H, hence $\{v_1,\ldots v_p\}$ can be expanded , if necessary, to a basis for H. Since the dimension of a subspace is just a number of vectors in a basis, it follows that dim(H $\cap$ K) = p $\leq$ dim H.

Finite dimensional vector space:

A vector space is called finite dimensional if it has a basis consisting of finite number of vectors. The unique number of vectors in each basis for V is called the **dimension of V** and is denoted by dim(V).

A vector space that is not finite dimensional is called infinite dimensional.

NOTE:

- Vector space $\{0\}$ has dimension zero.
- Vector space $P_n(F)$ has dimension n+1.
- Vector space $M_{mxn}(F)$ has dimension mn
- Over the field of complex numbers, the vector space of complex numbers has dimension 1. (A basis is $\{1\}$).
- Over the field of real numbers, the vector space of complex numbers has dimension 2. (A basis is $\{1, i\}$).
- Every vector space has a basis.

- The dimension of the coordinate space F^n is n
- The dimension of the polynomial ring $F[x]$ is countably infinite, a basis is given by 1, x, x^2, ...
- The dimension of more general function spaces, such as the space of functions on some (bounded or unbounded) interval, is infinite.
- The dimension of the solution space of a homogeneous ordinary differential equation equals the degree of the equation.
- Expressed in terms of elements, the span is the subspace consisting of all the linear combinations of elements of S.
- In a vector space of finite dimension n, a vector hyperplane is a subspace of dimension $n - 1$.

Example 18:

Find the dimensions of the subspace $H = \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix}$ **: a, b, c, d in R**

Solution:

Clearly H is the set of all linear combinations of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

$v_1 \neq 0$, v_2 is not a multiple of v_1, but v_3 is a multiple of v_2. By the spanning Set theorem, we may discard v_3 and still have a set that spans H. Finally, v_4 is not a linear combination of v_1 and v_2. So $\{v_1, v_2, v_4\}$ is linearly independent and hence is a basis for H. Thus dim H = 3.

Example 19:

Let V be a space of 2×2 matrices over R and let the sub-space generated by

$$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$$

Show that (i) $\left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\}$ forms a basis set (ii) $\dim(W) = 2$.

Solution:

The basis set of V(R) is

$$S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The co-ordinate of vectors x_1, x_2, x_3, x_4 relatived to the basis S_1 are (1, -5, -4, 2), (1, 1, -1, 5), (2, -4, -5, 7), (1, -7, -5, 1) reapecively.

Thus form the matrix whose rows are given vectors

Consider the augmented matrix

$$A = \begin{pmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{pmatrix} \qquad \begin{bmatrix} \text{Operating } R_{21}(-1) \\ R_{31}(-2) \\ R_{41}(-1) \end{bmatrix}$$

$$\sim \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{bmatrix} \text{Operating } R_{32}(-1) \\ R_{42}(\frac{1}{3}) \end{bmatrix}$$

$$\sim \begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The non – zero rows $(1,-5,-4,2)$ and $(0,6,3,3)$ of the above echelon matrix form a basis.

Hence the set of corresponding matrices is

$$S_1 = \left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\},$$

which forms a basis set W.

Hence $\dim(W) = 2$.

Theorem 11:
Replacement Theorem:
Let V be a vector space that is spanned by a set G containing n vectors. Let $L \subseteq V$ be a linearly independent subset containing m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V

Proof:
The proof is by mathematical induction on m.
Start with $m = 0$. In that case $L = \emptyset$, the empty set, and so taking H = G gives the desired result.
Let's assume this theorem is true for some integer $m \geq 0$. We prove that the theorem is true for m + 1.
Let $L = \{v_i\}_{j=1}^{m+1}$ and define it as a linearly independent subset of V consisting of $m + 1$ vectors.

Since any subset of a linearly independent set is linearly independent as well ($S1 \subseteq S2 \subseteq V$), then $\{v_i\}_{j=1}^{m+1}$ is linearly independent also.

It then says to use the induction hypothesis to say that $m \geq n$

The next step is to say that there is therefore another subset, $\{u_k\}_{k=1}^{n-m}$ of G such that $\{u_k\}_{k=1}^{n-m}$ spans V. That being the case there are scalars $\{a_j\}_{j=1}^{m+1}$ and $\{b_k\}_{k=1}^{n-m}$ which we can multiply by the vectors vj and uk.

Then add the two sets of vectors, yielding

$$\sum_{j=1}^{m+1} a_j v_j + \sum_{j=1}^{m+1} b_k u_k = v_{m+1} \quad . . . (*)$$

Note that $n - m > 0$ -- otherwise v_{m+1} is linearly dependent (contradiction). But then it says not only is $n > m$ but $n > m+1$.

Moreover, some b_i, say b_1 is nonzero, for otherwise we obtain the same contradiction. Solving (*) for u_1 gives

$$u_1 = (-b_1^{-1} a_1)v_1 + \cdots + (-b_1^{-1} a_m)v_m + (b_1^{-1})v_{m+1} + (-b_1^{-1} b_2)u_2 + \cdots + (-b_1^{-1} b_{n-m})u_{n-m}$$

Let $H = \{u_2,\ldots,u_{n-m}\}$. Then $u_1 \in \text{span}(L \cup H)$ and because $v_1,\ldots,v_m,u_2,\ldots,u_{n-m}$ are clearly in $\text{span}(L \cup H)$, it follows that $L' \cup H' = \{v_1,\ldots,v_m,u_1,\ldots,u_{n-m}\} \subseteq \text{span}(L \cup H)$.

Because $L' \cup H'$ generates V, $\text{span}(L \cup H)$ generates V.

Since H is a subset of G which contains $(n - m) - 1 = n - (m+1)$ vectors the theorem is true for $m + 1$. This completes the induction.

The Lagrange Interpolation Formula:

Let $a_0, a_1, \cdots, a_n$ *be distinct scalars in an infinite field* F. *The polynomials* $f_0(x), f_1(x), \cdots, f_n(x)$ *defined by*

$$f_i(x) = \frac{(x - c_0)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_n)}{(c_i - c_0)\cdots(c_i - c_{i-1})(c_i - c_{i+1})\cdots(c_i - c_n)} = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - c_k}{c_i - c_k}$$

are called the Lagrange polynomials (associated with $c_0, c_1, \cdots c_n$*).*

Example 20:

Find an approximate polynomial for f(x) using Lagrange's interpolation for the following data

x	0	1	2	5
f(x)	2	3	12	147

Solution:

The Lagrange's interpolation formula

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_{01})(x_1 - x_2)(x_1 - x_3)} y$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{1}{20}[-x^3 + 166x^2 - 184x - 40]$$

Example 21:

Using Lagrange's formula find the polynomial for the following data

x	0	1	2	4
f(x)	2	3	12	147

Solution:

$$y(x) = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(x) = f(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(2) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3)$$

$$+\frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(12) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(147)$$

$$f(x) = \frac{(x-1)(x^2-6x+8)}{(-8)}(2) + \frac{(x)(x^2-6x+8)}{(3)}(3)$$

$$+\frac{(x)(x^2-5x+4)}{(-4)}(12) + \frac{(x)(x^2-3x+2)}{(24)}(147)$$

$$= \frac{x^3-7x^2+14x-8}{-4} + (x^3-6x^2+8x) + (-3x^3+15x^2-12x) + \frac{49x^3-147x^2+98x}{8}$$

$$= x^3\left[-\frac{1}{4}+1-3+\frac{49}{8}\right] + x^2\left[\frac{7}{4}-6+15-\frac{147}{8}\right] + x\left[-\frac{14}{4}+8-12+\frac{98}{8}\right] + \left[\frac{8}{4}\right]$$

$$= \frac{31}{8}x^3 - \frac{61}{8}x^2 + \frac{38}{8}x + 2$$

simplifying further we get

$$f(x) = \frac{1}{8}\left[31x^3 - 61x^2 + 38x + 16\right]$$

Example 22:

Using Lagrange's interpolation formula, find y(10) given that y(5)=12,y(6)=13,y(9)=14 and y(11)=16.

Solution:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_{01})(x_1-x_2)(x_1-x_3)}y +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+\frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

$put\ x = 10$

$y(10) = 14.6666$

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
The set of all linear combinations of finite sets of elements of S is called the ___ of S.	linear dependent	spanning set	linear span	linear combination			linear span
The vector space {0} then the dimension is ___.	0	1	2	3			0
The ____ of two subspaces of a vector space is a subspace.	union	intersection	complement	rank			intersection
The intersection of any number of subspaces of a vectors space V is a __ of V.	subspace	basis	dimension	rank			subspace
Row equivalence matrices have the same ___ space.	column	null	row	kernel			row
The nonzero rows of a matrix in echelon form are ____.	linearly dependent	linearly independent	linearly span	linearly combination			linearly independent
Any subset of a linearly independent set is ___.	linearly dependent	linearly independent	linearly span	linearly combination			linearly independent
A set S of vectors is a ___ of V if it satisfies span and linearly independent.	subspace	basis	dimension	rank			basis
___ denotes the column space of A	Ker A	Im A	dim A	Rank A			Im A
Let V be a vector space then any n+1 or more vectors in V are___.	linearly dependent	linearly independent	linearly span	linearly combination			linearly dependent
The ___ of T is defined to be the dimension of images.	rank	kernel	basis	linear map			rank
Let V be a vector space of finite dimension n. Then any n+1 or more vectors in V are___	linearly dependent	linearly independent	linearly span	linearly combination			linearly dependent
Let V be a vector space of finite dimension n. Then any ___ or more vectors in V are linearly dependent.	n+1	n	n-1	n+2			n+1
Let V be a vector space of finite dimension n. Then any ___ set S with n elements is a basis of V.	linearly dependent	linearly independent	linearly span	linearly combination			linearly independent
Let V be a vector space of finite dimension n. Then any linearly independent set S with n elements is a ____ of V.	linearly dependent	basis	linearly span	linearly combination			basis
Let V be a vector space of finite dimension n. Then any spanning set T of V with n elements is a ____ of V.	linearly dependent	basis	linearly span	linearly combination			basis
Let V be a vector space of finite dimenstion n. Then any ____T of V with n elements is a basis of V.	linearly dependent	spanning set	linearly span	linearly combination			spanning set
The sum of two vectors is a ___	scalar	vector	unit	inner product			vector
The product of a scalar and a vector is a ___	scalar	vector	unit	inner product			vector
{0} and V are subspaces of any vector space V. They are called the ___ subspaces of V	scalar	vector	unit	trivial			trivial
Let V be a vector space and A and B are subspaces of V then __ is a subspace of V	A+B	A-B	A*B	A/B			A+B
Let V be a vector space and A and B are subspaces of V then A is a subspace of ___	A+B	A-B	A*B	A/B			A+B
Let V be a vector space and A and B are subspaces of V then B is a subspace of ___	A+B	A-B	A*B	A/B			A+B
Let S be a non-empty subset of a vector space V. Then the set of all ___ of finite sets of elements of S is called the linear span of S.	linearly dependent	linearly independent	linear span	linear combinations			linear combinations

The Linear span is denoted by___	dim V	dim S	L(S)	S	L(S)
Let V be a vector space over a field F and S be a non-empty subset of V. Then L(S) is a ___ of V.	linear span	linear independent	linear dependent	subspace	subspace
L[L(S)] = __	dim V	dim S	L(S)	S	L(S)
Any vector space is an abelian group with respect to vector___	addition	subtraction	multiplication	division	addition
In R, let S = {1}. Then L(S) =	S	C	R	Q	R
In C, let S = {1, i}. Then L(S) =	S	C	R	{a+bi}	C

MA8352- LINEAR ALGEBRA AND PARTIAL DIFFERENTIAL EQUATIONS

II year ECE- III Semester

Important Problems

UNIT II LINEAR TRANSFORMATION AND DIAGONALIZATION

Linear transformation on a vector space:

Let V and W be vector spaces over F. We call a function $T : V \to W$ a linear transformation from V to W if for all x,y $\in$ V and c $\in$ F, we have

a) T(x+y) = T(x) + T(y) and
b) T(cx) = cT(x).

Properties of a function T:

Let $T : V \to W$ be a linear transformation from V to W. If for all x,y $\in$ V and c $\in$ F, we have

1. If T is linear, then T(0) = 0.
2. T is linear if and only if T(cx + y) = cT(x) + T(y)
3. If T is linear, then T(x – y) = T(x) – T(y)
4. T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$

Example 1:

Show that T: $R^2 \to R^2$ defined by $T(a_1,a_2) = (2a_1 + a_2, a_1)$ is linear.

Proof:

Let x, y $\in R^2$ and c $\in$ R, where X = (b_1, b_2), y = (d_1, d_2)

Since we know that T is linear if and only if T(x + y) = cT (x) + T(y)

Now (x + y) = $(cb_1 + d_1 , cb_2 + b_2)$

T(cx + y) = $(2(cb_1 + d_1) + cb_2 + d_2 , cb_1 + d_1)$

Also cT(x) + T(y) = $c(2b_1 +b_2 , b_1) + (2d_1 + d_2 , d_1)$

= $(2cb_1 + cb_2 + 2d_1 + d_2 , cb_1 + d_1)$

= $(2(cb_1 + d_1) + cb_2 + d_2 , cb_1 + d_1)$

T(cx + y) = cT(x) + T(y)

Example 2:

For any angle θ, define T $T_\theta : R^2 \to R^2$ by the rule: $T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq (0, 0)$, and $T_\theta(0,0) = (0,0)$. Then $T_\theta : R^2 \to R^2$ is a linear transformation that is called the rotation by θ.

Example 3:

Define $T : R^2 \to R^2$ by $T(a_1, a_2) = (a_1, -a_2)$. T is called the reflection about the x-axis.

Example 4:

Define $T : R^2 \to R^2$ by $T(a_1, a_2) = (a_1, 0)$. T is called the projection on the x-axis.

Null space of T:

Let V and W be vector spaces and let $T : V \to W$ be linear. The null space or kernel, N(T) is the set of all vectors x in V such that T(x) = 0. i.e, N(T) = {x $\in$ V : T(x) = 0}.

Range of T:

The Range or image R(T) is the subset of W consisting of all images under T of vectors in V. i.e , R(T) = {T(x) : x $\in$ V}.

Example 4: Let $T: R^3 \to R^2$ **be the linear transformation defined by** $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$ **,**

Find N(T) and R(T) .

Solution :

Let V and W be vector spaces and let T : V→W be linear.

The null space or kernel, N(T) is the set of all vectors x in V such that T(x) = 0

i.e, N(T) = {x ϵ V : T(x) = 0}.

The Range or image R(T) is the subset of W consisting of all images under T of vectors in V

i.e , R(T) = {T(x) : x ϵ V}.

Given : $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

N(T) = {x ϵ V : T(x) = 0}.

$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = (0,0)$

$\Rightarrow a_1 = a_2$ and $a_3 = 0$

$\therefore N(T) = \{(a_1, a_1, 0) : a \in R\}$

$\therefore R(T) = \{T(x) : x \in V\}$

$= \{(a_1 - a_2, 2a_3) : a_1, a_2, a_3 \in R\}$

$= R^2$

Example 5:

For the following linear operator T on a vector space V and ordered basis β compute $[T]_\beta$

$V = R^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$ and $\beta = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$

Solution:

$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$

$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 - 12 \\ 17 - 20 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 - 18 \\ 34 - 30 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$[T]_\beta = \begin{bmatrix} -2 & 2 \\ -3 & 4 \end{bmatrix}$

Example 6:

T(1) = 0, Zero Polynomial

T(t) = 1, Constant Polynomial

$T(t^2) = 2t$

$[T(1)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ [T(t)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ [T(t^2)]_\beta = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$[T]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Theorem 1 (Dimension Theorem):

Let V and W be vector spaces,and let $T : V \to W$ be linear. If V is finite-dimensional, then

nullity(T) + rank(T) = dim(V).

Proof: Suppose that dim(V) = n, dim(N(T)) = k, and $\{v_1, v_2 \cdots v_k\}$ is a basis for N(T).

We know that if W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V. So we may extend $\{v_1, v_2 \cdots v_k\}$ to a basis $\beta = \{v_1, v_2 \cdots v_n\}$ for V.

We claim that $S = \{T(v_{k+1}), T(v_{k+2}), \cdots T(v_n)\}$ is a basis for R(T).

First we prove that S generates R(T). Since $T(V_i) = 0$ for $1 \le i \le k$, we have

$$R(T) = span(\{T(v_1), T(v_2), \cdots T(v_n)\})$$

$$= span(\{T(v_{k+1}), T(v_{k+2}), \cdots T(v_n)\}) = span(S).$$

Now we prove that S is linearly independent.

Suppose that $\sum_{i=k+1}^{n} b_i T(v_i) = 0$ for $b_{k+1}, b_{k+2}, \cdots b_n \in F$.

Since T is linear, we have

$$T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0.$$

So $\sum_{i=k+1}^{n} b_i v_i \in N(T)$.

Hence there exist $c_1, c_2, \cdots c_k \in F$ such that

$$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i \text{ or } \sum_{i=1}^{k} (-c_i) v_i + \sum_{i=k+1}^{n} b_i v_i = 0.$$

Since β is a basis for V, we have $b_i = 0$ for all i. Hence S is linearly independent. This shows that $T(v_{k+1}), T(v_{k+2}), \cdots T(v_n)$ are distinct; ∴ rank(T) = n – k.

Hence the proof.

Note:

The **rank-nullity theorem** states that the rank and the **nullity** (the dimension of the kernel) sum to the number of columns in a given matrix. If there is a matrix with rows and columns over a field, then This can be generalized further to linear maps: if is a linear map, then

The rank-nullity theorem is further generalized by consideration of the fundamental subspaces and the fundamental theorem of linear algebra.

The rank-nullity theorem is useful in calculating either one by calculating the other instead, which is useful as it is often much easier to find the rank than the nullity (or vice versa).

Example 7:

Prove that the transformation $T : M_{2\times3}(F) \to M_{2\times2}(F)$ **defined by**

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

(i) T is linear

(ii) Find bases for N(T) and R(T)

(iii) compute the nullity and rank (T) and verify the dimension theorem

(iv) use appropriate theorems to determine whether T is one to one or on to .

Solution :

Given : $T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$

Let , $x = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ and $y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \in M_{2\times 3}(F)$ and $c \in F$

$$cx + y = \begin{pmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} & ca_{13} + b_{13} \\ ca_{21} + b_{21} & ca_{22} + b_{22} & ca_{23} + b_{23} \end{pmatrix}$$

$$\therefore T(cx + y) = \begin{pmatrix} 2(ca_{11} + b_{11}) - (ca_{12} + b_{12}) & (ca_{13} + b_{13}) + 2(ca_{12} + b_{12}) \\ 0 & 0 \end{pmatrix}$$

Also , $cT(x) + T(y) = c\begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 2(a_{11} \cdot c + b_{11}) - (ca_{12} + b_{12}) & (ca_{13} + b_{13}) + 2(ca_{12} + b_{12}) \\ 0 & 0 \end{pmatrix}$$

$$\therefore T(cx + y) = cT(x) + T(y)$$

Hence T is linear .

(ii) $N(T) = \{x \in M_{2\times 3}(F); T(x) = 0\}$

given $T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow 2a_{11} - a_{12} = 0$----------(1)

$a_{13} + 2a_{12} = o$ --------(2)

$\Rightarrow 2a_{11} = a_{12}$

$a_{13} = -2a_{12} = -4a_{11}$

$$\therefore N(T) = \left\{ \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \right\}$$

$\therefore$

$$\begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + a_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Basis for N(T) = $\left\{ \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

$R(T) = \{T(x); x \in M_{2\times 3}(F)\}$

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

Let $2a_{11}-a_{12} = s$ and $a_{13} + 2a_{12} = t$

$$\text{Then } R(T) = \left\{ \begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix} = s\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \text{ Basis for } R(T) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

(iii) Nullity of T = dim (N(T)) = 4

Rank (T) = dim(R(T)) = 2

Dim ($M_{2x3}(F)$) = 6

Nullity + Rank (T) = dim ($M_{2x3}(F)$)

$\therefore$ Dimension theorem is verified.

(iv) Since N(T) $\neq \{0\}$,by theorem

Let V and W be vector spaces and let $T : V \to W$ be linear. Then T is one to one if and only if

$N(T) = \{0\}$

$\therefore$ T is not one – one not onto.

Example 8:

For the following transformation $T : R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$ show that

(i) T is a linear transformation

(ii) Find basis for both N(T) and R(T)

(iii) Compute the nullity and rank(T) and verify the dimension theorem

(iv) Finally use the appropriate theorems to determine whether T is one to one or on to.

Solution:

Consider the transformation $T : R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

Let x = (a_1, a_2) & y = $(b_1, b_2) \in R^2$ and c $\in$ R.

$cx + y = (ca_1 + b_1, ca_2 + b_2)$

$T(cx + y) = T(ca_1 + b_1, ca_2 + b_2)$

$= (ca_1 + b_1 + ca_2 + b_2,\ 0,\ 2(ca_1 + b_1) - (ca_2 + b_2))$

Also

$cT(x) + T(y) = c(a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2)$

$= (ca_1 + ca_2 + b_1 + b_2, 0, 2ca_1 - ca_2 + 2b_1 - b_2)$

$= (ca_1 + b_1 + ca_2 + +b_2, 0, 2(ca_1 + b_1) - (ca_2 + b_2))$

$\therefore T(cx + y) = cT(x) + T(y)$

Hence T is linear.

$N(T) = \{x \in R^2 : T(x) = 0\}$

$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2) = (0, 0, 0)$

$\Rightarrow a_1 + a_2 = 0 \ \&\ 2a_1 - a_2 = 0$

$\Rightarrow a_2 = -a_1 \ \&\ a_2 = 2a_1$

This is possible only when $a_1 = a_2 = 0$.

$\Rightarrow N(\mathrm{T}) = \{0\}$

Hence the basis for $N(T) = \varphi$

$R(T) = \{T(x) : x \in R^2\}$

$= \{(a_1 + a_2, 0, 2a_1 - a_2) : a_1, a_2 \in R\}$

$(a_1 + a_2, 0, 2a_1 - a_2) = a_1(1, 0, 2) + a_2(1, 0, -1)$

Basis for $R(T) = \{(1, 0, 2), (1, 0, -1)\}$

nullity $= \dim(N(T)) = 0$.

$\mathrm{Rank}(T) = \dim(R(T)) = 2$

$\dim(V) = \dim(R^2) = 2$

$\mathrm{nullity}(T) + \mathrm{rank}(\mathrm{T}) = \dim(\mathrm{V})$.

$\mathrm{N(T)} = \{0\}$ if and only if T is one to one.

$R(T) \neq \mathrm{R}^3$

$\therefore$ it is not onto.

Theorem 2:

Let V and W be vector spaces and let $T : V \to W$ be linear. Then T is one to one if and only if $N(T) = \{0\}$

Proof:

Suppose that T is one-to-one and x ∈N(T). Then T(x) = 0 = T(0). Since T is one-to-one, we have x = 0. Hence N(T) = {0}.Now assume that N(T) = {0}, and suppose that T(x) = T(y). Then 0 = T(x) – T(y) = T(x – y)

∴ x – y ∈N(T) = {0}. So x – y = 0, or x = y. This means that T is one-to-one.

Theorem 3:

Let V and W be vector spaces of equal (finite) dimension and let $T : V \to W$ be linear. Then the following are equivalent.

(i) T is one to one
(ii) T is onto

(iii) rank(T) = dim(V).

Proof:

From the dimension theorem, we have nullity(T) + rank(T) = dim(V).

Now by the above theorem, we have that T is one-to-one if and only if N(T) = {0}, if and only if nullity(T) = 0, if and only if rank(T) = dim(V), if and only if rank(T) = dim(W), and if and only if dim(R(T)) = dim(W). This equality is equivalent to R(T) = W, the definition of T being onto.

Example 9:

Let $T : P_2(R) \to P_3(R)$ be the linear transformation defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt.$

Now R(T) = span ({T(1), T(x), T(x^2)}) = span({3x, 2 + (3/2)x^2, 4x + x^3}).

Since ({3x, 2 + (3/2)x^2, 4x + x^3} is linearly independent, rank(T) = 3. Since dim(P_3(R)) = 4, T is not onto. From the dimension theorem, nullity (T) + 3 = 3. So nullity(T) = 0, and therefore N(T) = {0}.

We know that if V and W are vector spaces and $T : V \to W$ is linear, Then T is one to one if and only if $N(T) = \{0\}$. Hence T is one-to-one.

Example 10:

For the following transformation $T : R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$ show that

(v) T is a linear transformation
(vi) Find basis for both N(T) and R(T)
(vii) Compute the nullity and rank(T) and verify the dimension theorem

Finally use the appropriate theorems to determine whether T is one to one or on to.

Solution:

Consider the transformation $T : R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

Let x = (a_1, a_2) & y = (b_1, b_2) ∈ R^2 and c ∈ R.

$cx + y = (ca_1 + b_1, ca_2 + b_2)$

$T(cx + y) = T(ca_1 + b_1, ca_2 + b_2)$

$= (ca_1 + b_1 + ca_2 + b_2,\ 0,\ 2(ca_1 + b_1) - (ca_2 + b_2))$

$Also$

$cT(x) + T(y) = c(a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2)$

$= (ca_1 + ca_2 + b_1 + b_2, 0, 2ca_1 - ca_2 + 2b_1 - b_2)$

$= (ca_1 + b_1 + ca_2 + + b_2, 0, 2(ca_1 + b_1) - (ca_2 + b_2)$

$\therefore T(cx + y) = cT(x) + T(y)$

Hence T is linear.

$N(T) = \{x \in R^2 : T(x) = 0\}$

$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2) = (0,0,0)$

$\Rightarrow a_1 + a_2 = 0 \ \& \ 2a_1 - a_2 = 0$

$\Rightarrow a_2 = -a_1 \ \& \ a_2 = 2a_1$

This is possible only when $a_1 = a_2 = 0.$

$\Rightarrow N(\mathrm{T}) = \{0\}$

Hence the basis for $N(T) = \varphi$

$R(T) = \{T(x) : x \in R^2\}$

$= \{(a_1 + a_2, 0, 2a_1 - a_2) : a_1, a_2 \in R\}$

$(a_1 + a_2, 0, 2a_1 - a_2) = a_1(1,0,2) + a_2(1,0,-1)$

Basis for $R(T) = \{(1,0,2),(1,0,-1)\}$

$nullity = \dim(N(T)) = 0.$

$Rank(T) = \dim(R(T)) = 2$

$\dim(V) = \dim(R^2) = 2$

$nullity(T) + \mathrm{rank(T)} = \dim(\mathrm{V}).$

$\mathrm{N(T)} = \{0\}$ *if and only if T is one to one.*

$R(T) \neq \mathrm{R}^3$

$\therefore$ *it is not onto.*

Matrix representation of a linear transformation

In linear algebra, linear transformations can be represented by matrices. If T is a linear transformation mapping R^n to R^m and $\vec{x}$ is a column vector with n entries, then

$$T(\vec{x}) = A\vec{x}$$

for some m × n matrix A, called the **transformation matrix** of T.

DIAGONALIZATION

Characteristic Polynomial:

Let A ϵ $M_{n\times n}$ (F). The polynomial F (t))= det (A – t I_n) is called the characteristic polynomial of A.

Characteristic Equation: Let A ϵ $M_{n\times n}$ (F). The equation $\det(A - tI_n) = 0$ is called the Characteristic equation of A.

Eigen Value and Eigen Vector:

Let T be a linear operator in a vector space V. A non zero vector v ϵ V is called an eigen vector of T if there exists a scalar λ such that T (v) = λ v. The scalar λ is called the eigen value corresponding to the eigen vector v.

Example 11:

Find the characteristic equation and the eigen values of $\begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$

Solution:

The characteristic polynomial is $\det(A - tI_2) = 0$

$$\det(A - tI_2) = \begin{vmatrix} 1-t & -4 \\ 4 & 2-t \end{vmatrix} = 0$$

$$\Rightarrow (1-t)(2-t) + 16 = 0$$

$$t^2 - 3t + 18 = 0$$

$$t = \frac{3 \pm \sqrt{9-72}}{2} = \frac{3 \pm \sqrt{-63}}{2}$$

Since t is complex, A has no real eigen values.

The matrix A is acting on a real vector space R^2 and there is no non-zero vector X in R^2 such that $Ax = \lambda x$

EXAMPLE 12:

Find the characteristic equation of $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

The characteristic equation is $\det(A - tI_4) = 0$

$$\det(A - tI_4) = \begin{bmatrix} 5-t & -2 & 6 & -1 \\ 0 & 3-t & -8 & 0 \\ 0 & 0 & 5-t & 4 \\ 0 & 0 & 0 & 1-t \end{bmatrix} = 0$$

$$(5-t)(3-t)(5-t)(1-t) = 0$$

$$t^4 - 14t^3 + 68t^2 - 130t + 75 = 0$$

EXAMPLE 13:

For the following linear operator T on a vector space V and ordered basis β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigen vector of T.

$V = R^3,\ T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}$ **and** $\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$

Solution:

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{pmatrix} \quad \& \quad \beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(0)+2(1)-2(1) \\ -4(0)-3(1)+2(1) \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Similarly,

$$T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\therefore [T]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since $T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ *and* $T\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

The eigen values are -1, 1, -1 and eigen vectors are $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

$\therefore \beta$ is a basis consisting of eigenvectors of T.

EXAMPLE 14:

Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$. Let β be the standard ordered basis for $P_2(R)$ and let $A = [T]_\beta$. Find the eigen values and eigen vector of A.

Solution:

Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$. Let $\beta = \{1, x, x^2\}$ be the standard ordered basis for $P_2(R)$

$T(1) = 1.1 + 0.x + 0.x^2$

$T(x) = x + (x + 1) = 1 + 2x = 1.1 + 2.x + 0.x^2$

$T(x^2) = x^2 + (x + 1)\, 2x = 2x + 3x^2 = 0.1 + 2.x + 3.x^2$

Let $A = [T]_\beta$. Then $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

The characteristic polynomial of T is

$$\det(A - tI_3) = \det\begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix}$$

$$= (1-t)(2-t)(3-t)$$

$$= -(t-1)(t-2)(t-3)$$

Hence λ is an eigenvalue of T (or A) if and only if λ = 1, 2, or 3.

Eigenvalue for $\lambda_1 = 1$:

$$(A - \lambda_1 \mathrm{I})\,\mathrm{X} = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0$$

$$x_2 + 2x_3 = 0$$

$$2x_3 = 0$$

$$x_3 = 0,\ x_2 = 0$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvalue for $\lambda_2 = 2$:

$$(A - \lambda_2 \mathrm{I})\,\mathrm{X} = 0$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + x_2 = 0 \Rightarrow x_2 = x_1$$

$$2x_3 = 0 \Rightarrow x_3 = 0$$

$$x_3 = 0,\ take\ x_1 = x_2 = 1$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvalue for $\lambda_3 = 3$:

$$(A - \lambda_3 \mathrm{I})\,\mathrm{X} = 0$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 = 0 \Rightarrow x_2 = 2x_1$$

$$-x_2 + 2x_3 = 0 \Rightarrow x_2 = 2x_3$$

$x_1 = 1,\ x_2 = 2, x_3 = 1$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

EXAMPLE 15:

Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Solution:

<u>To find the eigen values:</u>

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

We compute the characteristic polynomial:

$$\det(A - tI_2) = \det\begin{bmatrix} 1-t & 1 \\ 4 & 1-t \end{bmatrix}$$

$$= t^2 - 2t - 3 = (t-3)(t+1).$$

$\because$ t is an eigen value of A if and only if $\det(A - tI_n) = 0$, $\det(A - tI_2) = 0$

$\Rightarrow t = 3, -1$

Hence the only eigen values of A are 3 and −1.

<u>To find the eigen vectors:</u>

Let $\lambda_1 = 3$ and $\lambda_2 = -1$.

We begin by finding all the eigenvectors corresponding to $\lambda_1 = 3$.

$$B_1 = A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

Then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$

is an eigenvector corresponding to $\lambda_1 = 3$ if and only if $x \neq 0$ and $x \in N(L_{B_1})$;

that is, $x \neq 0$ and $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Clearly the set of all solutions to this equation is

$$\left\{ t\begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in R \right\}.$$

Hence x is an eigen vector corresponding to $\lambda_1 = 3$ if and only if

$x = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for some $t \neq 0$.

Now suppose that x is an eigen vector of A corresponding to $\lambda_2 = -1$. Let

$B_2 = A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$

Then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N(L_{B_2})$

if and only if x is a solution to the system $2x_1 + x_2 = 0$; $4x_1 + 2x_2 = 0$.

$N(L_{B_2}) = \left\{ t\begin{pmatrix} 1 \\ -2 \end{pmatrix} : t \in R \right\}.$

Thus x is an eigen vector of A corresponding to $\lambda_2 = -1$ if and only if

$x = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for some $t \neq 0$.

We observe that

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$

is a basis for R^2 consisting of eigenvectors of A. Thus L_A, and hence A, is diagonalizable.

Eigen Space of T:

Let T be a linear operator on a vector space V and let λ be an eigen value of T. Define $E_\lambda = \{x \in v : T(x) = \lambda x\} = N(T - \lambda Iv)$. The set E_λ is called eigen space of T corresponding to the eigen value λ.

Diagonalizablity of a linear operaot T:

A linear operator T on a finite dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called diagonalizable if L_A is diagonalizable

EXAMPLE 16:

Check whether the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_{2\times 2}$ **is diagonalizable or not .**

Solution :

The characteristic polynomial of A (and hence of L_A) is $\det(A - tI_2) = 0$

$$|A - tI_2| = \begin{vmatrix} 1-t & 1 \\ 1 & 1-t \end{vmatrix} = (1-t)^2 - 1 = 0$$

$\Rightarrow t^2 - 2t = 0$

$\Rightarrow t(t-2) = 0$

The eigen values of L_A are 0 and 2 .

Since , L_A is a linear operator on R^2 , by the corollary which states " Let T be a linear operator on an n – dimensional vector spaces V . If T has n distinct eigenvalues , then T is diagonizable ,"

$\therefore$ We conclude that L_A (and hence A) is diagonizable .

EXAMPLE 17:

Let T be a linear operator on $P_2(R)$ **defined by T(f(x)) =** $f'(x)$. **Determine whether T is diagonizable or not .**

Proof :

T(f(x)) = $f'(x)$. Consider a standard ordered basis for $P_2(R)$.

i.e; $\beta = \{1, x, x^2\}$

T (1) = 0 = $0.1 + 0.x + 0.x^2$

T (x) = 1 = $1.1 + 0.x + 0.x^2$

T (x^2) = 2 x = $0.1 + 2.x + 0.x^2$

The matrix of linear transformation is A $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

The characteristic polynomial of T is

Det$(A - t_3 I) = \det\begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 2 \\ 0 & 0 & -t \end{pmatrix} = -t^3$

Thus T has only one eigenvalue , namely $\lambda = 0$, with multiplicity 3.

$E_\lambda = N(T - \lambda I) = N(T) = \{x \in P_2(R) : T(x) = 0\}$

$i.e; N(T) = \{x \in P_2(R) : f'(x) = 0\}$

$\therefore E_\lambda = N(T)$ is the subspace of $P_2(R)$ consisting of constant polymonials . so $\{1\}$ is a basis for E_λ , and therefore dim (E_λ) = 1. So there is no basis for $P_2(R)$ consisting of eigen vectors of T and therefore T is not diagonalizable.

EXAMPLE 18:

Test for diagonalizability of the linear transformation T on $P_2(R)$ defined by

$T(f(x)) = f(1) + f'(0)x + \left(f'(0) + f''(0)\right)x^2$

Solution:

Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f(1) + f'(0)x + \left(f'(0) + f''(0)\right)x^2$

We write test T for diagonalizability.

Let α denote the standard ordered basis for $P_2(R)$ and B = $[T]_\alpha$

Then B = $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

The characteristic polynomial of B, and hence of T, is $-(t-1)^2(t-2)$, which splits.

Hence diagonalization condition (1) is satisfied. Also B has the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with multiplicities 2 and 1, respectively. condition (2) is satisfied for λ_2 because it has multiplicity 1. So we need only to verify condition (2) for $\lambda_1 = 1$.

For this case,

$$3\text{-rank}(B-\lambda_1 I)=3\text{-rank}\begin{pmatrix}0&1&1\\0&0&0\\0&1&1\end{pmatrix}=3-1=2,$$

which is equal to the multiplicity of λ_1. $\therefore$ T is diagonalizable.

We now find an ordered basis γ for R^3 of eigenvectors of B. We consider each eigenvalue separately. The eigenspace corresponding to $\lambda_1 = 1$ is

$$E_{\lambda_1} = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in R^3 : \begin{pmatrix}0&1&1\\0&0&0\\0&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0\right\},$$

which is the solution space for the system

$x_2 + x_3 = 0$,

and has $\gamma_1 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}$ as a basis.

The eigenspace corresponding to $\lambda_2=2$ is

$$E_{\lambda_2} = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in R^3 : \begin{pmatrix}-1&1&1\\0&-1&0\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0\right\},$$

which is the solution space for the system

$-x_1+x_2+x_3=0$

$x_2=0$,

and has $\gamma_2 = \left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ as a basis.

$$Let\ \gamma = \gamma_1 \cup \gamma_2 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\-1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}.$$

Then γ is an ordered basis for R^3 consisting of eigenvectors of B.

Finally, observe that the vectors in γ are the coordinate vectors relative to α of the vectors in the set $\beta=\{1,-x+x^2,1+x^2\}$, which is an ordered basis for $P_2(R)$ consisting of eigenvectors of T. Thus

$$[T]_\beta = \begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}.$$

EXAMPLE 19:

Let T be the linear operator on R^3 defined by $T\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}4a_1 + a_3\\2a_1 + 3a_2 + 2a_3\\a_1 + 4a_3\end{pmatrix}$ **Show that T is diagonalizable**

Solution:

Let T be the linear operator on R^3 defined by $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{pmatrix}$

We determine the eigenspace of T corresponding to each eigenvalue. Let β be the standard ordered basis for R^3. Then

Let T be the linear operator on

$$[T]_\beta = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

and hence the polynomial of T is defined by

$$\det\left([T]_\beta - tI\right) = \det\begin{pmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{pmatrix} = -(t-5)(t-3)^2.$$

So the eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = 3$ with multiplicities 1 and 2, respectively.

$$T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{pmatrix}$$

$$E_{\lambda_1} = N(T - \lambda_1 I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3 : \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

E_{λ_1} is the solution space of the system of linear equations

$-x_1 + x_3 = 0$

$2x_1 - 2x_2 + 2x_3 = 0$

$x_1 - x_3 = 0.$

It is clear that $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for E_{λ_1}. Hence $\dim(E_{\lambda_1}) = 1$.

Similarly, $E_{\lambda_2} = N(T - \lambda_2 I)$ is the solution space of the system

$x_1 + x_3 = 0$

$2x_1 + 2x_3 = 0$

$x_1 + x_3 = 0.$

Since the unknown x_2 does not appear in this system, we assign it a parametric value, say $x_2 = s$, and solve the system for x_1 and x_3, introducing another parameter t. The result is the general solution to the system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ for\ s,t \in R.$$

It follows that

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for E_{λ_2}, *and* $\dim(E_{\lambda_2}) = 2$.

In this case, the multiplicity of each eigenvalue λ_i is equal to the dimension of the corresponding eigenspace E_{λ_i} .We observe that the union of the two bases just derived, namely,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is linearly independent and hence is a basis for R^3 consisting of eigenvectors of T. Consequently, T is diagonalizable.

EXAMPLE 20:

For $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \in M_{n\times n}(F)$

- **(i)** **Determine all the eigen values of A**
- **(ii)** **Find the set of eigen vectors corresponding to** λ
- **(iii)** **If possible find a basis for** F^n **consisting of eigen vectors of A**
- **(iv)** **If successful in finding such a basis determine an invertible matrix Q and a diagonal matrix D such that** $Q^{-1}AQ = D$

Solution:

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$$

We compute the characteristic polynomial:

$$\det(A - tI_3) = \det\left[\begin{pmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{pmatrix}\right]$$

$$= (2-t)\left[(1-t)(-1-t)\right] - 1(-2(1-t)$$

$$= (2-t)(-(1-t^2)) + 2(1-t)$$

$$= -(2-t)(1-t)(1+t) + 2(1-t)$$

$$= (1-t)\left[2 - (2-t)(1+t)\right]$$

$$= (1-t)\left[2 - (2 - t^2 + t)\right] = (1-t)(t^2 - t) = (1-t)(t-1)\,t$$

$\therefore$ t is an eigen value of A if and only if $\det(A - tI_n) = 0$, $\det(A - tI_3) = 0$

$\Rightarrow (1-t)(t-1)\,t = 0$

$\Rightarrow t = 0,1,1$

Hence the only eigen values of A are 0 and 1.

To find the eigen vectors:

Let $\lambda_1 = 0$ and $\lambda_2 = 1$.

We begin by finding all the eigenvectors corresponding to $\lambda_1 = 0$.

$$B_1 = A - \lambda_1 I = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3$$

is an eigenvector corresponding to $\lambda_1 = 0$ if and only if $X_1 \neq 0$ *and* $X_1 \in N(L_{B_1})$;

$$\text{that is, } X_1 \neq 0 \text{ and } \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Clearly the set of all solutions to this equation is

$$\left\{ t\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} : t \in R \right\} \qquad \left(\begin{array}{c} \because \text{ By cross multiplication rule,} \\ \begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & -1 & 2 & 0 \\ 1 & -4 & 4 & 1 \end{array} \\ x_1 = 1, x_2 = 4, x_3 = 2 \end{array} \right)$$

Hence X_1 is an eigen vector corresponding to $\lambda_1 = 0$ if and only if

$$X_1 = t\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \quad \text{for some } t \neq 0.$$

$$\text{when } t = 1, \ X_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

Next we find two distinct eigenvectors corresponding to λ_2& $\lambda_3 = 1$

$$B_2 = A - \lambda_2 I = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3$$

is an eigenvector corresponding to $\lambda_2 = 1$ if and only if $X_2 \neq 0, X_3 \neq 0$

and $X_2, X_3 \in N(L_{B_2})$;

$$\text{that is, } X_2 \neq 0, \ X_3 \neq 0 \text{ and } \begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Clearly the set of all solutions to this equation is

$\because\ x_1 - x_3 = 0$

$x_1 = x_3$ & x_2 is any arbitrary vector.

Let $x_2 = 1$

If $x_1 = x_3 = 1$,

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

If $x_1 = x_3 = 0$,

$$X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Let } \beta = \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$|Q| = 1(-1) - 1(-2) = -1 + 2 = 1$$

$$Q^{-1} = \frac{1}{|Q|} Adj\,Q = \begin{pmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & -1 & -3 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$D = Q^{-1} A Q = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

``

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
The rank nullity theorem is dim V = ___.	rank(T)+ nullity(T)	rank(T)- nullity(T)	rank(T).nullity(T)	basis			rank(T)+nullity(T)
The kernel of T is named as_____.	dim (Im T)	dim (ker T)	dim V	linear transformation			dim (ker T)
___ denotes the null space of A	Ker A	Rank A	Im A	dim A			Ker A
Let V and W be vector space over a field F, then T from V to W defined by T(v)=0 for all v belongs to V is a ___ linear transformation	scalar	vector	identity	reflection			trivial
Let V and W be vector space over a field F, then T from V to W defined by T(v)=v for all v belongs to V is a ___ linear transformation	scalar	vector	identity	trivial			identity
The eigenvectors of a real symmetric are _______.	**equal**	**unequal**	real	symmetric			real
Diagonalisation of a matrix by orthogonal reduction is true	diagonal	triangular	real	scalar			real symmetric
If the sum of two eigen values of matrix A are equal to the trace of the matrix, then the determinant of A is___________	1	-1	0	2			0
Sum of the principal diagonal elements ________	product of eigen values	product of eigen vectors	sum of eigen values	product of eigen values			sum of eigen values
Let V and W be a linear transformation, then dimension of Null space of T is -----	Nullity (T)	Rank A	Im A	dim A			Nullity (T)
A square matrix A is --------, if $L_{\_A}$ is diagonalizable.	Eigen space	Diagonalizable	orthogonal	kernel			Diagonalizable
___ denotes the column space of A	Ker A	Rank A	Im A	dim A			Im A
The ___ of T is defined to be the dimension of images	rank	kernel	basis	linear map			rank
Let V and W be vector space over a field F, then T from V to W defined by T(v)=v for all v belongs to V is a ___ linear transformation	identity	vector	scalar	trivial			identity
Let V and W be vector space over a field F, then T from V to W defined by T(v)=v for all v belongs to V .if Ker(T)=0, then T is ___	one-one	onto	not onto	one-one and onto			one-one
Any ____matrix A can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix.	row	cloumn	zero	square			square

MA8352- LINEAR ALGEBRA AND PARTIAL DIFFERENTIAL EQUATIONS

II year ECE- III Semester

UNIT- 3 INNER PRODUCT SPACE

CLASS NOTES

Inner Product space:

Let V be a vector space over F. An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V then

i). $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

ii). $\langle cx, y\rangle = c\langle x, y\rangle$

iii). $\langle x, y\rangle = \langle y, x\rangle$

iv). $\langle x, x\rangle > 0$ if $x \neq 0$

EXAMPLE 1:

Let $V = M_{n\times n}(F)$ and define for A, B in V, $\langle A, B\rangle = tr(B^* A)$ Then this is an inner product

PROOF:

For $A, B, C \in V$ and $a \in F$

a). $\langle A+B, C\rangle = tr\left(C^*(A+B)\right)$

$= tr\left(C^*A + C^*B\right)$

$= tr\left(C^*A\right) + tr\left(C^*B\right)$

$= \langle A, C\rangle + \langle B, C\rangle$

b). $\overline{\langle A, B\rangle} = \overline{tr\left(B^*A\right)} = \overline{\sum_{i=1}^{n}\left(B^*A\right)_{ii}}$

$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n}\overline{(B)^*_{ik}}\,\overline{(A)_{ki}}\right)$

$= \sum_{i=1}^{n}\sum_{k=1}^{n}(B)_{ki}\,\overline{(A)_{ki}} = \sum_{i=1}^{n}\sum_{k=1}^{n}\overline{(A)_{ki}}\,(B)_{ki}$

$= \sum_{i=1}^{n}\sum_{k=1}^{n}\left(A^*\right)_{ik}(B)_{ki} = \sum_{i=1}^{n}\left(A^*B\right)_{ii}$

$= tr\left(A^*B\right) = \langle B, A\rangle$

c). $\langle aA, B\rangle = tr\left(B^*aA\right) = a\,tr\left(B^*A\right) = a\langle A, B\rangle$

d). $\langle A, A\rangle = tr\left(A^*A\right) = \sum_{i=1}^{n}\left(A^*A\right)_{ii}$

$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(A^*\right)_{ik}(A)_{ki}\right) = \sum_{i=1}^{n}\sum_{k=1}^{n}\overline{A_{ki}}A_{ki}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\left|A_{ki}\right|^2$$

If $A \neq 0$, $A_{ki} \neq 0$ for some k and i

Therfore, $A \neq 0$ then $\langle A, A\rangle = \sum_{i=1}^{n}\sum_{k=1}^{n}\left|A_{ki}\right|^2 > 0$

Hence $\langle .,.\rangle$ is an inner product on V.

EXAMPLE 2:

In an Euclidean inner product find cosine of the angle between the vectors $u = (2,3,5)$ and

$v=(1,-4,3)$

Solution:

$$\cos\theta=\frac{\langle u,v\rangle}{\|u\|\|v\|}$$

$$\langle u,v\rangle=2-12+15=5,\ \|u\|=\sqrt{4+9+25}=\sqrt{38},\ \|v\|=\sqrt{1+16+9}=\sqrt{26}$$

$$\cos\theta=\frac{\langle u,v\rangle}{\|u\|\|v\|}=\frac{5}{\sqrt{38}\sqrt{26}}$$

EXAMPLE 3:

Compute the angle between two vectors (x,y) and $(-y,x)$ in an Euclidean inner product space R^2 .

Solution:

$$\cos\theta=\frac{\langle u,v\rangle}{\|u\|\|v\|}$$

$$\langle u,v\rangle=0,\ \|u\|=\sqrt{x^2+y^2},\ \|v\|=\sqrt{x^2+y^2}$$

$$\cos\theta=\frac{\langle u,v\rangle}{\|u\|\|v\|}=\frac{0}{x^2+y^2}=0$$

$$\Rightarrow\theta=\frac{\pi}{2}.$$

THEOREM 1:

Let V be an inner product space. Then for $x,y,z\in V$ and $c\in F$ the following statements are true

a). $\langle x,y+z\rangle=\langle x,y\rangle+\langle x,z\rangle$

b). $\langle x,cy\rangle=\bar{c}\langle x,y\rangle$

c). $\langle x,0\rangle=\langle 0,x\rangle=0$

d). $\langle x,x\rangle=0$ **if and only if** $x=0$

e). $\langle x,y\rangle=\langle x,z\rangle$ **for all** $x\in V$**, then** $y=z$

Proof:

a). $\langle x,y+z\rangle=\overline{\langle y+z,x\rangle}$

$=\overline{\langle y,x\rangle+\langle z,x\rangle}$

$=\overline{\langle y,x\rangle}+\overline{\langle z,x\rangle}$

$=\langle x,y\rangle+\langle x,z\rangle$

b). $\langle x,cy\rangle=\overline{\langle cy,x\rangle}=\bar{c}\overline{\langle y,x\rangle}=\bar{c}\langle x,y\rangle$

c). $\langle x,0\rangle=\langle x,a+(-a)\rangle=\langle x,a\rangle+\langle x,-a\rangle$

$=\langle x,a\rangle-\langle x,a\rangle=0$ where $a\in V$

Similarly $\langle 0,x\rangle=0$

d). Assume that $\langle x,x\rangle=0$ ----(1)

Then $x=0$. Otherwise (i.e. if $x\neq 0$), $\langle x,x\rangle>0$ contradicts (1)

Assume that $x=0$. Then by (c)

$\langle x,x\rangle=\langle 0,0\rangle=0$

e). $\langle x,y\rangle=\langle x,z\rangle$. Then $\langle x,y\rangle-\langle x,z\rangle=0$

$\Rightarrow\langle x,y-z\rangle=0,\ \forall x\in V$

Taking $x=y-z,\ \langle y-z,y-z\rangle=0$

By (d), $y-z=0\Rightarrow y=z$.

EXAMPLE 4:

Let S consist of the following vectors in R^4: $u_1=(1,1,0,-1)$, $u_2=(1,2,1,3)$, $u_3=(1,1,-9,2)$, $u_4=(16,-3,1,3)$. Find the coordinates of any arbitrary vector $v=(a,b,c,d)$ in R^4 relative to basis S

Solution:

$u_1.u_2=1+2+0-3=0$

$u_1.u_2=1+1+0-2=0$

$u_1.u_4=16-13+0-3=0$

$u_2.u_3=1+2-9+6=0$

$u_2.u_4=16-26+1+9=0$

$u_3.u_4=16-13-9+6=0$

Thus S is orthogonal and s is linearly independent. Accordingly S is a basis for R^4 because any four linearly independent vectors form a basis of R^4

$$k_1=\frac{\langle v,u_1\rangle}{\langle u_1,u_1\rangle}=\frac{a+b-d}{3}$$

$$k_2=\frac{\langle v,u_2\rangle}{\langle u_2,u_2\rangle}=\frac{a+2b+c+3d}{15}$$

$$k_3=\frac{\langle v,u_3\rangle}{\langle u_3,u_3\rangle}=\frac{a+b-9c+2d}{87}$$

$$k_4=\frac{\langle v,u_4\rangle}{\langle u_4,u_4\rangle}=\frac{16a-13b+c+3d}{435}$$

Norm of a vector in an inner product space:

Let V in an inner product space. For $x\in V$, we define the norm or length of x by

$\|x\|=\sqrt{\langle x,x\rangle}$

EXAMPLE 5:

Find norm and distance between the vectors $u=(1,0,1)$ and $v=(-1,1,0)$.

SOLUTION:

$\|u\|^2=1^2+0^2+1^2=2,\ \|v\|^2=(-1)^2+1^2+0^2=2$

The distance between two vectors u and v is defined by $d(u,v)=\|u-v\|$

$\|u-v\|^2=\langle u-v,u-v\rangle=\langle u,u\rangle-\langle u,v\rangle-\langle v,u\rangle+\langle v,v\rangle$

$\|u-v\|^2=2+1+1+2=6$

$\|u-v\|=\sqrt{6}$

EXAMPLE 6:

Consider $f(t)=3t-5$ **and** $g(t)=t^2$ **in the polynomial space** $P(t)$ **with inner product** $\langle f,g\rangle=\int_0^1 f(t)g(t)\,dt$ **hen find** $\|f\|$ **and** $\|g\|$**.**

SOLUTION:

$f(t)f(t)=9t^2-30t+25,\quad g(t)g(t)=t^4$

$$\|f\|^2=\langle f,f\rangle=\int_0^1\left(9t^2-30t+25\right)dt=\left[3t^3-15t^2+25t\right]_0^1=13$$

$$\|g\|^2=\langle g,g\rangle=\int_0^1 t^4dt=\left[\frac{t^5}{5}\right]_0^1=\frac{1}{5}$$

THEOREM 2:

Let V **be an inner product space over** F **. Then for all** $x,y\in V$ **and** $c\in F$ **, Then the following statements are true**

1. $\|cX\|=|c|\|X\|$

2. $\|x\|=0$ **if and only if** $x=0$**.**

3. Cauchy Schwarz Inequality $|\langle x,y\rangle|\le\|x\|\|y\|$

4. Triangle Inequality $\|x+y\|\le\|x\|+\|y\|$

Solution:

1. $\|cx\|^2=\langle cx,cx\rangle=c\bar{c}\langle x,x\rangle=|c|^2\|x\|^2$

2. $\|x\|=0\Leftrightarrow\|x\|^2=0\Leftrightarrow\langle x,x\rangle=0\Leftrightarrow x=0$

3. case(i). let $y=0$. Then $|\langle x,y\rangle|=|\langle x,0\rangle|=0$

and $\|x\|\|y\|=\|x\|\|0\|=0$.

Therefore $|\langle x,y\rangle|=\|x\|\|y\|$

Case(ii). Let $y\neq 0$. For any $c\in F$, we have

$0\le\|x-cy\|^2=\langle x-cy,x-cy\rangle$

$=\langle x,x\rangle-\langle x,cy\rangle-\langle cy,x\rangle+\langle cy,cy\rangle$

$=\|x\|^2-\bar{c}\langle x,y\rangle-c\langle x,y\rangle+c\bar{c}\langle y,y\rangle$

Take $c=\dfrac{\langle x,y\rangle}{\|y\|^2}$.

Then $0\le\|x\|^2-\dfrac{\overline{\langle x,y\rangle}}{\|y\|^2}\langle x,y\rangle-\dfrac{\langle x,y\rangle}{\|y\|^2}\overline{\langle x,y\rangle}+\dfrac{|\langle x,y\rangle|^2}{\|y\|^4}\|y\|^2$

$0\le\|x\|^2\|y\|^2-|\langle x,y\rangle|^2-|\langle x,y\rangle|^2+|\langle x,y\rangle|^2$

$\Rightarrow|\langle x,y\rangle|^2\le\|x\|^2\|y\|^2$

$\Rightarrow |\langle x, y\rangle| \le \|x\|\|y\|$

4. we have $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$\|x+y\|^2 = \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2$

$\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$

$\|x+y\|^2 \le \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$

$\|x+y\|^2 \le (\|x\| + \|y\|)^2$

$\|x+y\| \le \|x\| + \|y\|$

EXAMPLE 7:

Using Euclidean inner product on R^3 show that $u = (-3,1,0)$ and $v = (2,-1,3)$ satisfy Cauchy Schwartz inequality.

Solution:

$\|x\|^2 = \langle x, x\rangle = 9+1+0 = 10$

$\|x\| = \sqrt{10}$

$\|y\|^2 = \langle y, y\rangle = 4+1+9 = 15$

$\|y\| = \sqrt{15}$

$\|x\|\|y\| = \sqrt{10}\sqrt{15} = \sqrt{150}$

$\langle x, y\rangle = -6-1 = -7$

$|\langle x, y\rangle| = 7$

$\Rightarrow |\langle x, y\rangle| \le \|x\|\|y\|$

Definition: Orthogonal

A vector $u \in V$ is said to be orthogonal to $v \in V$ if $\langle u, v\rangle = 0$.

Definition: Orthogonal Subspaces

Two subspaces $V \& W$ of a vector space are orthogonal if every vector v in V is perpendicular to every vector w in W. ie. $\langle v, w\rangle = 0, \forall v \in V, w \in W$.

Definition: Orthonormal Set

A subset S of V is called an orthonormal set if

i). $\|x\| = 1, \forall x \in X$

ii). $\langle x, y\rangle = 0 \ \forall x, y \in S$

Definition: Orthonormal Basis

A basis of an inner product space that consists of mutually orthogonal unit vectors is called an Orthonormal basis.

EXAMPLE 8:

Find the value of a if the vectors $(2, a)$ and $(6, 4)$ are orthogonal vectors in R^2.

SOLUTION:

Let $u=(2,a)$ and $v=(6,4)$

Since the vectors are orthogonal $\langle u,v\rangle=0$

$12+4a=0$

$4a=-12 \Rightarrow a=-3$

EXAMPLE 9:

Find k so that $u=(1,2,k,3)$ and $v=(3,k,7,-5)$ in R^4 are orthogonal.

SOLUTION:

$\langle u,v\rangle=(1,2,k,3).(3,k,7,-5)=3+2k+7k-15=9k-12$

Then set $\langle u,v\rangle=9k-12=0$

$\Rightarrow k=\frac{4}{3}$.

EXAMPLE 10:

If $v=(1,2,1)$ and $u=(2,1,2)$ find proj(v,u).

SOLUTION:

$\text{Proj}(v,u)=\frac{\langle v,u\rangle}{\langle u,u\rangle}u$

$\text{Proj}(v,u)=\frac{6}{9}(2,1,2)=\left(\frac{4}{3},\frac{2}{3},\frac{4}{3}\right)$

EXAMPLE 11:

Prove that in an inner product space V, for any $u,v\in V$, $\|u+v\|^2+\|u-v\|^2=2\|u\|^2+2\|v\|^2$.

SOLUTION:

$\|u+v\|^2+\|u-v\|^2=\langle u+v,u+v\rangle+\langle u-v,u-v\rangle$

$=\langle u,u\rangle+\langle u,v\rangle+\langle v,u\rangle+\langle v,v\rangle+\langle u,u\rangle-\langle u,v\rangle-\langle v,u\rangle+\langle v,v\rangle$

$=\|u\|^2+\|v\|^2+\|u\|^2+\|v\|^2=2\|u\|^2+2\|v\|^2$

EXAMPLE 12:

Suppose u,v & w are vectors in an inner product space such that $\langle u,v\rangle=2$, $\langle u,w\rangle=-3$, $\langle v,w\rangle=5$, $\|u\|=1$, $\|v\|=2$, $\|w\|=7$.

Evaluate(i). $\langle u+v,v+w\rangle$ (ii). $\langle 2u-w,3u+2w\rangle$

SOLUTION:

(i). $\langle u+v,v+w\rangle=\langle u,v\rangle+\langle u,w\rangle+\langle v,v\rangle+\langle v,w\rangle$

$=2-3+4+5=8$

(ii). $\langle 2u-w,3u+2w\rangle=6\langle u,u\rangle+4\langle u,w\rangle-3\langle w,u\rangle-2\langle w,w\rangle$

$=6\|u\|^2+4(-3)-3(-3)-2\|w\|^2=6-12+9-2(49)=-95$

EXAMPLE 13:

If u and v are orthonormal vectors in an inner product space V then find $\|u+v\|$.

SOLUTION:

$\|u+v\|^2=\langle u+v,u+v\rangle=\langle u,u\rangle+\langle u,v\rangle+\langle v,u\rangle+\langle v,v\rangle$

$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$ since vectors are orthonormal $\langle u,v\rangle = 0$ $= \|u\|^2 + \|v\|^2$
EXAMPLE 14: **In Euclidean inner product space R^2 verify $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$ for the vectors $u=(3,-2), v=(4,5)$ and $w=(-1,6)$.**
SOLUTION: $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$ $\langle u,v\rangle = 12-10=2,\ \langle u,w\rangle = -3-12=-15$ $RHS = \langle u,v\rangle + \langle u,w\rangle = 2-15=-13$ $v+w=(3,11)$ $LHS = \langle u,v+w\rangle = 9-22=-13$ Hence verified
EXAMPLE 15: **Find the norm of the vector $u=(1,1,-1)$ and $v=(-1,1,0)$ in R^3 with respect to the inner product defined by $\langle u,v\rangle = u_1v_1+2u_2v_2+3u_3v_3$ where $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$**
SOLUTION: $\|u\|^2 = 1+2+3=6,\ \|v\|^2 = 1+2+0=3$ $\Rightarrow \|u\| = \sqrt{6},\ \|v\| = \sqrt{3}$
THEOREM 3: **Let V be an inner product space and $S=\{v_1,v_2,v_3,...v_k\}$ be an orthogonal subset of V consisting of non-zero vectors. If $y \in \text{span}(s)$ then $y=\sum_{i=1}^{k}\frac{\langle y,v_i\rangle}{\|v_i\|^2}v_i$** **Proof:** Let $y \in \text{span}(s)$ and $S=\{v_1,v_2,v_3,...v_k\}$ $\Rightarrow$ there exists scalars $a_1,a_2,a_3,...a_k$ such that $y=a_1v_1+a_2v_2+a_3v_3+...+a_kv_k=\sum_{i=1}^{k}a_iv_i \quad -----(1)$ For $1 \le j \le k$, $\langle y,v_j\rangle = \left\langle \sum_{i=1}^{k}a_iv_i, v_j\right\rangle = a_1\langle v_1,v_j\rangle + ... + a_j\langle v_j,v_j\rangle + ... + a_n\langle v_n,v_j\rangle$ $\langle y,v_i\rangle = a_j\|v_j\|^2$ since S is an orthogonal set $\Rightarrow a_j = \frac{\langle y,v_j\rangle}{\|v_j\|^2} \quad -----(2)$ Using (2) in (1) $y=\sum_{i=1}^{k}\frac{\langle y,v_i\rangle}{\|v_i\|^2}v_i$.
THEOREM 4: **(Gram – Schmidt orthogonalization process)** **Let V be an inner product space and $S=\{w_1,w_2,...,w_n\}$ be a linearly independent subset of V.**

Define $S'=\{v_1,v_2,...,v_n\}$ **where** $v_1=w_1$ **and** $v_k=w_k-\sum_{j=1}^{k-1}\frac{<w_k,v_j>}{\|v_j\|^2}v_j$, **for** $2\le k\le n$**. Then** S' **is an orthogonal set of non-zero vectors such that** $span(S')=span(S)$**.**

Proof:

The proof is by induction on n.

Let $S_k=\{w_1,w_2,...,w_k\}$, for $k=1,2,3,...,n$

Since $\{w_1\}$ is linearly independent, $v_1=w_1\ne 0$.

Clearly, $\{v_1\}$ is orthogonal and $span(v_1)=span(w_1)$.

Therefore the theorem is valid for $n=1$.

Assume that the theorem is valid for $n=k-1$.

i.e., $\{v_1,v_2,...,v_{k-1}\}$ is an orthogonal set of non-zero vectors and

$span(v_1,v_2,...,v_{k-1})=span(w_1,w_2,...,w_{k-1})$-----------(1)

Now, we prove that the theorem is valid for $n=k$.

$$v_k=w_k-\sum_{j=1}^{k-1}\frac{<w_k,v_j>}{\|v_j\|^2}v_j\Rightarrow v_k\ne 0---------(2)$$

Now for $m\le k-1$,

$$\langle v_k,v_m\rangle=\left\langle w_k-\sum_{j=1}^{k-1}\frac{\langle w_k,v_j\rangle}{\|v_j\|^2}v_j,v_m\right\rangle$$

$$\langle v_k,v_m\rangle=\langle w_k,v_m\rangle-\left\langle\sum_{j=1}^{k-1}\frac{\langle w_k,v_j\rangle}{\|v_j\|^2}v_j,v_m\right\rangle$$

$\langle v_k,v_m\rangle=\langle w_k,v_m\rangle-\frac{\langle w_k,v_m\rangle}{\|v_m\|^2}\langle v_m,v_m\rangle$ since $\{v_1,v_2,...,v_{k-1}\}$ is orthogonal.

$$\langle v_k,v_m\rangle=\langle w_k,v_m\rangle-\frac{\langle w_k,v_m\rangle}{\|v_m\|^2}\|v_m\|^2=0$$

$\Rightarrow\{v_1,v_2,...,v_k\}$ is an orthogonal set of non-zero vectors.

Further from (1)

$span(v_1,v_2,...,v_{k-1},v_k)=span(w_1,w_2,...,w_{k-1},v_k)$

$=span(w_1,w_2,...,w_{k-1},w_k)$ by (2)

Therefore the theorem is true for $n=k$.

Hence by induction the theorem is valid for all $n\ge 1$.

EXAMPLE 16:

Let R^3 **have the Euclidean inner product. Use the Gram-Schmidt process to convert basis** $B=\{u_1,u_2,u_3\}$ **where** $u_1=(1,0,1),u_2=(-1,1,0),u_3=(-3,2,0)$ **into an orthogonal basis.**

Solution:

$v_1=w_1=(1,0,1)$

$$v_2=w_2-\sum_{j=1}^{1}\frac{\langle w_2,v_j\rangle}{\|v_j\|^2}v_j$$

$$=w_2-\frac{\langle w_2,v_1\rangle}{\|v_1\|^2}v_1$$

$=(-1,1,0)+\frac{1}{2}(1,0,1)\because \|v_1\|^2=1+0+1=2$

$v_2=\left(-\frac{1}{2},1,\frac{1}{2}\right)$

$$v_3=w_3-\sum_{j=1}^{2}\frac{\langle w_3,v_j\rangle}{\|v_j\|^2}v_j$$

$$=(-3,2,0)-\left\{\frac{\langle w_3,v_1\rangle}{\|v_1\|^2}v_1+\frac{\langle w_3,v_2\rangle}{\|v_2\|^2}v_2\right\}\because \|v_2\|^2=\frac{1}{4}+1+\frac{1}{4}=\frac{3}{2}$$

$$=(-3,2,0)-\left\{-\frac{3}{2}(1,0,1)+\frac{7\times 2}{2\times 3}\left(-\frac{1}{2},1,\frac{1}{2}\right)\right\}$$

$v_3=\left(\frac{-1}{3},\frac{-1}{3},\frac{1}{3}\right).$

Therefore $\{v_1,v_2,v_3\}=\left\{(1,0,1),\left(-\frac{1}{2},1,\frac{1}{2}\right),\left(\frac{-1}{3},\frac{-1}{3},\frac{1}{3}\right)\right\}$ is an orthogonal basis.

EXAMPLE 17:

Let V be the vector space of polynomials $f(t)$ with inner product $\langle f,g\rangle=\int_{-1}^{1}f(t)g(t)dt$.

Apply the Gram-schmidt orthogonalization process to $\{1,t,t^2,t^3\}$ to find an orthonormal basis $\{f_0,f_1,f_3,f_4\}$ with integer coefficients for $P_3(t)$.

Solution:

Let $w_1=1,w_2=t,w_3=t^2$

$v_1=w_1=1$

$$v_2=w_2-\frac{\langle w_2,v_1\rangle}{\|v_1\|^2}v_1,$$

$$\langle w_2,v_1\rangle=\int_{-1}^{1}tdt=\left(\frac{t^2}{2}\right)_{-1}^{1}=0$$

$$\|v_1\|^2=\langle v_1,v_1\rangle=\int_{-1}^{1}dt=2$$

$v_2=t-0=t$

$$v_3=w_3-\frac{\langle w_3,v_1\rangle}{\|v_1\|^2}v_1-\frac{\langle w_3,v_2\rangle}{\|v_2\|^2}v_2$$

$$\langle w_3,v_1\rangle=\int_{-1}^{1}t^2dt=\left(\frac{t^3}{3}\right)_{-1}^{1}=\frac{2}{3}$$

$$\langle w_3,v_2\rangle=\int_{-1}^{1}t^3dt=\left[\frac{t^4}{4}\right]_{-1}^{1}=0$$

$$\|v_2\|^2 = \langle v_2, v_2\rangle = \int_{-1}^{1} t.tdt = \int_{-1}^{1} t^2 dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$v_3 = t^2 - \frac{2/3}{2} - 0 = t^2 - \frac{1}{3}$$

$$v_4 = w_4 - \frac{\langle w_4, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_4, v_2\rangle}{\|v_2\|^2} v_2 - \frac{\langle w_4, v_3\rangle}{\|v_3\|^2} v_3$$

$$\langle w_4, v_1\rangle = \int_{-1}^{1} t^3 dt = \left[\frac{t^4}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

$$\langle w_4, v_2\rangle = \int_{-1}^{1} t^4 dt = \left[\frac{t^5}{5}\right]_{-1}^{1} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$\langle w_4, v_3\rangle = \int_{-1}^{1} t^3\left(t^2 - \frac{1}{3}\right) dt = \int_{-1}^{1}\left(t^5 - \frac{t^3}{3}\right) dt$$

$$\langle w_4, v_3\rangle = \left[\frac{t^6}{6} + \frac{t^4}{12}\right]_{-1}^{1} = \left[\frac{1}{6} + \frac{1}{12} - \frac{1}{6} - \frac{1}{12}\right] = 0$$

$$v_4 = t^3 - \frac{2}{5} \times \frac{3}{2} t = t^3 - \frac{3}{5} t$$

$$\|v_3\|^2 = \langle v_3, v_3\rangle = \int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt$$

$$= \int_{-1}^{1}\left(t^4 + \frac{1}{9} - \frac{2t^2}{3}\right) dt = \left[\frac{t^5}{5} + \frac{t}{9} - \frac{2t^3}{9}\right]_{-1}^{1}$$

$$= \left[\frac{1}{5} + \frac{1}{9} - \frac{2}{9} + \frac{1}{5} + \frac{1}{9} - \frac{2}{9}\right]$$

$$\|v_3\|^2 = \frac{8}{45}$$

$$\|v_4\|^2 = \langle v_4, v_4\rangle = \int_{-1}^{1}\left(t^3 - \frac{3}{5} t\right)^2 dt$$

$$\|v_4\|^2 = \int_{-1}^{1}\left(t^3 - \frac{3}{5} t\right)^2 dt = \int_{-1}^{1}\left(t^6 + \frac{9}{25} t^2 - \frac{6}{5} t^4\right) dt$$

$$= \left[\frac{t^7}{7} + \frac{9}{25}\frac{t^3}{3} - \frac{6}{5}\frac{t^5}{5}\right]_{-1}^{1} = \frac{24}{525}$$

Therefore $\{v_1, v_2, v_3, v_4\} = \left\{1, t, t^2 - \frac{1}{3}, t^3 - \frac{3}{5} t\right\}$ is an orthogonal basis.

Now $\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}, \frac{v_4}{\|v_4\|}\right\} = \left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} t, \frac{3\sqrt{5}}{2\sqrt{2}}\left(t^2 - \frac{1}{3}\right), \frac{\sqrt{525}}{\sqrt{24}}\left(t^3 - \frac{3}{5} t\right)\right\}$ is an orthonormal basis for

$P_3(R)$.

EXAMPLE 19:

Obtain an orthonormal basis T with respect to standard inner product for the subspace of R^3 generated by $S=\{(1,0,3),(2,1,1)\}$ such that $L(T)=L(S)$.

Solution:

$v_1 = w_1 = (1,0,3)$

$$v_2 = w_2 - \sum_{j=1}^{1} \frac{\langle w_2, v_j \rangle}{\|v_j\|^2} v_j$$

$$= w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$\|v_1\|^2 = 1^2 + 0 + 3^2 = 10$

$$v_2 = (2,1,1) - \frac{(2+0+3)}{10}(1,0,3)$$

$$= (2,1,1) - \frac{5}{10}(1,0,3)$$

$$= (2,1,1) - \frac{1}{2}(1,0,3)$$

$$v_2 = \left(\frac{3}{2},1,-\frac{1}{2}\right)$$

$$\|v_2\|^2 = \frac{9}{4} + 1 + \frac{1}{4} = \frac{7}{2}$$

Therefore $\{v_1, v_2\} = \left\{(1,0,3),\left(\frac{3}{2},1,-\frac{1}{2}\right)\right\}$ is an orthogonal basis.

Now $\left\{\frac{v_1}{\|v_1\|},\frac{v_2}{\|v_2\|}\right\} == \left\{\left(\frac{1}{\sqrt{10}},0,\frac{3}{\sqrt{10}}\right),\left(\frac{3\sqrt{2}}{2\sqrt{7}},\frac{\sqrt{2}}{\sqrt{7}},-\frac{\sqrt{2}}{2\sqrt{7}}\right)\right\}$ is an orthonormal basis for R^3.

Define Orthogonal Complement.

Let S be a subspace of V (an inner product space). Then the set $S^{\perp} = \{x \in V : \langle x, y \rangle = 0 \ \forall y \in S\}$ is called orthogonal Complement of S.

EXAMPLE 18:

Let $w=(1,2,3,1)$ be a vector in R^4. Find an orthogonal basis for $W^{\perp}$.

SOLUTION:

Find a nonzero solution of $x+2y+3z+t=0$, say $v_1=(0,0,1,-3)$

Now find a nonzero solution of the system $x+2y+3z+t=0$, $z-3t=0$ say $v_2=(0,-5,3,1)$

Lastly find a nonzero solution of the system $x+2y+3z+t=0$, $-5y+3z+t=0$, $z-3t=0$ say $v_3=(-14,2,3,1)$. Thus v_1,v_2,v_3 form an orthonormal basis of $W^{\perp}$.

THEOREM 5:

Let V be an finite dimensional inner product space and let T be a linear operator on V.

Then there exists a unique function $T^*:V\to V$ such that $\langle T(x),y\rangle=\langle x,T^*(y)\rangle,\ (x,y\in V)$

Further T^* is linear.

PROOF:

Let $y\in V$. Then define $g:V\to F$ by $g(x)=\langle T(x),y\rangle,\ (x\in V)$------(1)

Claim: g is linear

For $x_1,x_2\in V$ and $c\in F$

$g(cx_1+x_2)=\langle T(cx_1+x_2),y\rangle$

$=c\langle T(x_1),y\rangle+\langle T(x_2),y\rangle$

$=cg(x_1)+g(x_2)$

$\Rightarrow g$ is linear.

By known theorem given any linear transformation g, there exists a unique vector $y'\in V$

Such that $g(x)=\langle x,y'\rangle,\ (x\in V)$-------(2)

From (1) and (2) given $y\in V$ there exists unique $y'\in V$

Such that $\langle T(x),y\rangle=\langle x,y'\rangle,\ (x\in V)$

Now define $T^*:V\to V$ as $T^*(y)=y'$ with $\langle T(x),y\rangle=\langle x,T^*(y)\rangle$

Claim T^* is linear

For $y_1,y_2\in V$ and $c\in F$,

$\langle x,T^*(cy_1+y_2)\rangle=\langle T(x),cy_1+y_2\rangle$

$=\langle T(x),cy_1\rangle+\langle T(x),y_2\rangle$

$=\bar{c}\langle T(x),y_1\rangle+\langle T(x),y_2\rangle$

$=\bar{c}\langle x,T^*(y_1)\rangle+\langle x,T^*(y_2)\rangle$

$=\langle x,cT^*(y_1)\rangle+\langle x,T^*(y_2)\rangle$

$=\langle x,cT^*(y_1)+T^*(y_2)\rangle,\ \forall x\in V$

$T^*(cy_1+y_2)=cT^*(y_1)+T^*(y_2)$

$\Rightarrow T^*$ is linear

Uniqueness of T^*

Suppose that $U:V\to V$ is linear such that $\langle T(x),y\rangle=\langle x,U(y)\rangle,\ \forall x,y\in V$

Then $\langle x,T^*(y)\rangle=\langle x,U(y)\rangle,\ \forall x,y\in V$

$\Rightarrow T^*(y)=U(y)$ is Unique.

THEOREM 6;

Let V be an inner Product space and let T and U be linear operators on V. Then

a. $(T+U)^*=T^*+U^*$

b. $(cT)^*=\bar{c}T^*$ **for any** $c\in F$

c. $(TU)^*=U^*T^*$

d. $T^{**}=T$

e. $I^*=I$

Proof:

a). Let $x,y\in V$

$\langle x,(T+U)^*(y)\rangle=\langle (T+U)(x),y\rangle$

$=\langle T(x),y\rangle+\langle U(x),y\rangle$

$= \langle x, T^*(y) \rangle + \langle x, U^*(y) \rangle$

$= \langle x, T^*(y) + U^*(y) \rangle, \ \forall x, y \in V$

$(T+U)^*(y) = (T^* + U^*)(y), \ \forall y \in V$

$\Rightarrow (T+U)^* = (T^* + U^*)$

b). $\langle x, (cT)^*(y) \rangle = \langle (cT)(x), y \rangle$

$= \langle cT(x), y \rangle$

$= c\langle T(x), y \rangle$

$= \langle T(x), \bar{c}y \rangle$

$= \langle x, T^*(\bar{c}y) \rangle$

$= \langle x, \bar{c}T^*(y) \rangle, \ \forall x, y \in V$

$= \langle x, (\bar{c}T^*)(y) \rangle, \ \forall x, y \in V$

$\Rightarrow (cT)^*(y) = (\bar{c}T^*)(y), \ \forall y \in V$

$\Rightarrow (cT)^* = \bar{c}T^*$

c). $\langle x, (TU)^*(y) \rangle = \langle (TU)(x), y \rangle$

$= \langle T(U(x)), y \rangle$

$= \langle U(x), T^*(y) \rangle$

$= \langle x, U^*(T^*(y)) \rangle$

$= \langle x, (U^*T^*)(y) \rangle, \ \forall x, y \in V$

$\Rightarrow (TU)^*(y) = (U^*T^*)(y), \ \forall y \in V$

$\Rightarrow (TU)^* = U^*T^*$

d). $\langle x, T^{**}(y) \rangle = \langle T^*(x), y \rangle = \overline{\langle y, T^*(x) \rangle}$

$= \overline{\langle T(y), x \rangle} = \langle x, T(y) \rangle, \ \forall x, y \in V$

$\Rightarrow T^{**}(y) = T(y), \ \forall y \in V$

$\Rightarrow T^{**} = T$

e). $\langle x, I^*(y) \rangle = \langle I(x), y \rangle$

$= \langle x, y \rangle = \langle x, I(y) \rangle, \ \forall x, y \in V$

$\Rightarrow I^*(y) = I(y), \ \forall y \in V$

$\Rightarrow I^* = I$

Questions	opt 1	opt 2	opt 3	opt 4	opt5	opt6	Answer
The inner product of (0, v) is ____	1	(-1)	0	v			0
Any element ‘x’ in vector space V then the element ‘x’ is called a unit vector if the norm of ‘x’ is ____.	1	(-1)	0	x			1
If x =(2, 1+i, i) and y =(2-i, 2,1+2i) be a vector in C^3 then norm of x+y is---------------.	sqrt37	5	5	3i			sqrt37
A subset S of V is orthogonal if any two distinct vectors in S are------------------.	orthogonal	parallel	normal	linear			orthogonal
The inner product of (0, u) is ____.	1	(-1)	0	u			0
If x =(2, 1+i, i) be a vector in C^3 then norm of x is---------------.	8	5	sqrt(7)	7i			sqrt(7)
Let T be a linear operator on an inner product space V, and if norm of T(x) is equal to norm of x then T is -------------	on to	one to one	one to one and on to	into			one to one
A subset S of V is orthonormal if S is orthogonal and contains entirely of -------- vectors.	unit	zero	row	column			unit
______.	product	space	row space	space			product
reciprocal of its length is called-------------.	mal set	lization	orthonormalization	ng			ng
If x =(2, 2+i) be a vector in C^2 then norm of x is---------------.	7	5	3	3i			3
The process of------------ a non zero vector by the reciprocal of its length is called normalizing.	multiplying	adding	subtracting	dividing			multiplying
Standard inner product is called the ----------------product.	dot	cross	vector	curl			dot
Let T be a linear operator on an inner product space V, and if T is one to one then norm of T(x) is equal to -------------	norm of x	norm of V	norm ofT+V	norm of x+V			norm of x
An inner product space is called an _____ space.	Euclidean	unitary	null	Euclidean or unitary			Euclidean or unitary
Let V be an inner product space. The vectors x and y in V are orthogonal if inner product of x,y is equal to----------.	zero	1	2	linear			**zero**
A subset S of V is -------------- if any two distinct vectors in S are orthogonal.	orthogonal	parallel	normal	linear			orthogonal
If T is linear then -------------------- is linear.	parallel operator T	adjoint operator T	product operator T	del operator			adjoint operator T

UNIT-IV

PARTIAL DIFFERENTIAL EQUATION

Notations:

$$p = \frac{\partial z}{\partial x} \;,\quad q = \frac{\partial z}{\partial y} \;,\quad r = \frac{\partial^2 z}{\partial x^2} \;,\quad s = \frac{\partial^2 z}{\partial x \partial y} \;,\quad t = \frac{\partial^2 z}{\partial y^2}$$

1. No of arbitrary constants eliminated = No of independent variables, then we get first order PDE. (Use p & q Only)
2. No. of arbitrary constants > No of independent variables, then we get second or higher order PDE. (Use p ,q, r, s & t)
3. In elimination of arbitrary functions, the order of PDE = No. of arbitrary functions eliminated.(arbitrary functions=1 use p & q Only and arbitrary functions=2 use p, q, r, s& t)
4. Elimination of arbitrary functions from $\phi(u,v) = 0, \quad the \quad solution \quad is \quad \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$.

5.

Type	Given form	Complete integral	Singular Integral	General solution
I	$F(p,q) = 0$	Trial solution z = ax + by + c …. (2), p = a, q = b. Sub in (1) then write “b” interms of “a” sub. in (2), we get **C.I.**	Differentiate (3) p.w.r.to “c”. 0 = 1 [absurd], there is no **S.I.**	Put c = f(a) in (3), we get eqn. (4). Diff. (4) p.w.r.to “a” and then eliminate “a” between the equation we get **G.I.**
II	Clairaut’s Form $z = px + qy + f(p,q)$	Put p = a, q = b … (2) in eqn. (1), we get **C.I.**	Apply Con : $\frac{\partial z}{\partial a} = 0, \frac{\partial z}{\partial b} = 0$ we get …….. (i) ……. (ii) eliminate a & b	Put $b = \phi(a)$ in (3), we get eqn. (4), Diff. (4) p.w.r.to “a” and then eliminate “a” between the

			from (3), (i) & (ii), we get **S.I.**	equation we get **G.I.**
III	$F(z,p,q)=0$	z is a function of u. u = x + ay, p = $\frac{dz}{du}$, q = $a\frac{dz}{du}$ sub. in (1) and then solving we get $\frac{dz}{du}=\phi(z,a)\Rightarrow\frac{dz}{\phi(z,a)}=du\cdots(2)$ $\int\frac{dz}{\phi(z,a)}=\int du$, *we get the C.I.* $f(z,a)=u+b=x+ay+b\cdots(3)$	No **S.I.**	
IV	$F_1(x,p)= F_2(y,q)$	$f_1(x,p)=f_2(y,q)=a$. Write 'p' interms of 'a & x' and 'q' interms of 'a & y'. Then dz = p dx + q dy …. (2), integrate we get **C.I.**	No **S.I.**	
V	$F(x^m p, y^n q)=0$ $F(z, x^m p, y^n q)=0$	**Case 1 : If** $m\neq 1$ *and* $n\neq 1$ *Put* $X=x^{1-m}$ *and* $Y=y^{1-m}$ $\frac{\partial X}{\partial x}=(1-m)x^{-m}$, $\frac{\partial Y}{\partial y}=(1-n)y^{-n}$ *Let* *and* $P=\frac{\partial Z}{\partial X}$, $Q=\frac{\partial Z}{\partial Y}$ $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$, $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$ $p=P(1-m)x^{-m}$, $q=Q(1-n)y^{-n}$ $\therefore x^m p=P(1-m)$, $y^n q=Q(1-n)$ Then it can be reduced to type 1and 3.	**Case 2 : If** $m=n=1$ *Put* $\log x=X$ *and* $\log y=Y$ $\frac{\partial X}{\partial x}=\frac{1}{x}$, $\frac{\partial Y}{\partial y}=\frac{1}{y}$ *Let* $P=\frac{\partial Z}{\partial X}$, $Q=\frac{\partial Z}{\partial Y}$ $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$, $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$ $p=P\frac{1}{x}$, $q=Q\frac{1}{y}$ $\therefore xp=P$, $yq=Q$ Then it can be reduced to type 1and 3	
VI	$F(z^m p, z^m q)=0$ $F(x, z^m p, z^m q)=0$	**Case 1 : If** $m\neq -1$ *Put* $Z=z^{(m+1)}$ *and* $P=\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}=(m+1)z^m p$ The $\frac{P}{m+1}=z^m p$, $\frac{Q}{m+1}=z^m q$ n it can be reduced to type 1and 4	**Case 2 : If** $m=-1$ *Put* $Z=\log z$ $P=\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}=\frac{1}{z}p = z^{-1}p$ *and* $Q=z^{-1}q$ Then it can be reduced to type 1and 4	

6. **Lagrange's Linear Equations:** In the form of Pp+Qq=R, The subsidiary equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

1. **Method of Grouping:**

In the subsidiary equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ if the variables can be separated in any pair of equations, then we get a solution of the form $u(x,y)=a$ and $v(x,y)=b$.

2. **Method of Multipliers:**

Choose any three nmultipliers l,m,n which may be constants or function of x, y, z we have

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}.$$

If it is possible to choose l,m,n such that $lP+mQ+nR=0$ then $ldx+mdy+ndz=0$. If $ldx+mdy+ndz$ is an exact differential then, on integration, we get a solution u = a. Similarly v = b .

7. **Classification of p.d.e of the second order:**

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

1. $B^2-4AC<0-elliptic$ equation $\quad eg: u_{xx}+u_{yy}=0 \quad$ (*laplace equation*)
2. $B^2-4AC=0-Parabolic$ equation $\quad eg: \alpha^2 u_{xx}=u_t \quad 1-D \quad$ (*heat flow equation*)
3. $B^2-4AC>0-Hyperbolic$ equation $\quad eg: \alpha^2 u_{xx}=u_{tt} \quad 1-D \quad$ (*Wave equation*)

8. **Homogeneous linear differential equation of third order is**

$$(a_0 D^3+a_1 D^2 D'+a_2 DD'^2+a_3 D'^3)z=F(x,y), where\ D=\frac{\partial}{\partial x}, D'=\frac{\partial}{\partial y}$$

Method for finding the solutions:

Complementary function

S.No	Case	CF
Homogeneous equation – Replace D = m and D' = 1.		
1	**If the roots are real (or imaginary)** $m_1\neq m_2\neq m_3$	$z=f_1(y+m_1x)+f_2(y+m_2x)+f_3(y+m_3x)$
2	All the roots are equal say $m_1=m_2=m_3=m$	$z=f_1(y+m\ x)+xf_2(y+m\ x)+x^2 f_3(y+m\ x)$

Non Homogeneous equation – Replace D = h and D' = k		
1	Form a quadratic equation interms of 'h', then h = $m_1 k+\alpha_1$ *and* $m_2 k+\alpha_2$	$z=e^{\alpha_1 x} f_1(y+m_1 x)+e^{\alpha_2 x} f_2(y+m_2 x)$

Particular integral:

S.No	Case	PI
1	$F(x,y)=e^{ax+by}$	$PI=\frac{1}{f(D,D')}e^{ax+by}=\frac{1}{f(a,b)}e^{ax+by}$, $f(a,b)\neq 0.$
2	$F(x,y)=x^m y^n$	$PI=\frac{1}{f(D,D')}x^m y^n=\left[f(D,D')\right]^{-1}x^m y^n$ Expand $\left[f(D,D')\right]^{-1}$ using binomial expansion
3	$F(x,y)=\sin(mx+ny)$ *or* $\cos(mx+ny)$	$PI=\frac{1}{f(D,D')}\sin(mx+ny)$ *or* $\cos(mx+ny)$ Re *place* $D^2=-m^2, D'^2=-n^2, DD'=-mn$
4	$F(x,y)=e^{ax+by}\phi(x,y)$	$PI=\frac{1}{f(D,D')}e^{ax+by}\phi(x,y)$ $=e^{ax+by}\frac{1}{f(D+a,D'+b)}\phi(x,y)$
5	Other than in rule (i),(ii),(iii),(iv)	$PI=\frac{1}{(D-m_1D')(D-m_2D')}e^{ax+by}F(x,y)$ $\frac{1}{(D-m_1D')}F(x,y)=\int F(x,c-mx)dx$ *Where* $y=c-mx$

If $f(D,D')=0$, multiply Nr by x differentiate Dr with respect to D & do the same Procedure.

Note:

D - Differentiation Nr - Numerator Dr - Denominator

$\frac{1}{D}=\underset{\text{y is constant}}{\int dx}$ $\frac{1}{D'}=\underset{\text{x is constant}}{\int dy}$

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
In a PDE, there will be one dependent variable and ____ independent variables	only one	two or more	no	infinite number of			two or more
The ______ of a PDE is that of the highest order derivative occurring in it	degree	power	order	ratio			order
The degree of the a PDE is ______of the higest order derivative	power	ratio	degree	order			power
Afirst order PDE is obtained if______	Number of arbitrary constants is equal Number of independent variables	Number of arbitrary constants is lessthan Number of independent variables	Number of arbitrary constants is greater than Number of independent variables	Number of arbitrary constants is not equal to Number of independent variables			Number of arbitrary constants= Number of independent variables
In the form of PDE, f(x,y,z,a,b)=0. What is the order?	1	2	3	4			1
What is form of the z=ax+by+ab by eliminating the arbitrary constants?	z=qx+py+pq	z=px+qy+pq	z=px+qy+p	z=py+qy+q			z=px+qy+pq
General solution of PDE F(x,y,z,p,q)=0 is any arbitray function F of specific functions u,v is____satisfying given PDE	F(u,v)=0	F(x,y,z)=0	F(x,y)=0	F(p,q)=0			F(u,v)=0
The PDE of the first order can be written as--------------	F(x,y,s,t)	F(x,y,z,p,q)=0	F(x,y,z,1,3,2)=0	F(x,y)=0			F(x,y,z,p,q)=0
The complete solution of clairaut's equation is _____	z=bx+ay+f(a,b)	z=ax+by+f(a,b)	z=ax+by	z=f(a,b)			z=ax+by+f(a,b)
The Clairaut's equation can be written in the form ---------	z=px+qy+f(p,q)	z=(p-1)x+qy+f(x,y)	z=Pp+Qq	Pq+Qp=r			z=px+qy+f(p,q)
From the PDE by eliminating the arbitrary function from z=f(x^2 -y^2) is	xp+yq=0	p=-(x/y)	q=yp/x	yp+xq=0			yp+xq=0
Which of the following is the type f(z,p,q)=0 ?	p(1+q)=qx	p(1+q)=qz	p(1+q)=qy	p=2x f(y+2x)			p(1+q)=qz

The equation (D^2 z+2xy(Dz)^2+D'=5 is of order ____and degree____	2 and 2	2 and 1	1 and 1	0 and 1	2 and 1
The complementry function of (D^2 -4DD'+4D'^2)z=x+y is	f(y+2x)+xg(y+2x)	f(y+x)+xg(y+2x)	f(y+x)+xg(y+x)	f(y+4x)+xg(y+4x)	f(y+2x)+xg(y+2x)
The solution of xp+yq=z is _____	f(x^2,y^2)=0	f(xy,yz)	f(x,y)=0	f(x/y ,y/z)=0	f(x/y ,y/z)=0
The solution of p+q=z is ____	f(xy,ylogz)=0	f(x+y, y+logz)=0	f(x-y, y-logz)=0	f(x-y,y+logz)=0	f(x-y, y-logz)=0
A solution which contains the maximum possible number of arbitrary functions is called------------- integral.	singular	complete	general	particular	general
The lagrange's linear equation can be written in the form -------------	Pq+Qp=r	Pq+Qp=R	Pp+Qq=R	F(x,y)=0	Pp+Qq=R
The complete solution of the PDE 2p+3q =1 is -----------------	z=ax+[(1-2a)/3]y+c	z=ax+y+c	z=ax+(1-2x)/y+c	z=ax+b	z=ax+[(1-2a)/3]y+c
The complete solution of the PDE pq=1 is ------------------	z=ax+(1/a)y+b	z=ax+y+b	z=ax+ay/b+c	z=ax+b	z=ax+(1/a)y+b
The solution got by giving particular values to the arbitrary constants in a complete integral is called a --------	general	singular	particular	complete	particular
The general solution of Lagrange's equation is denoted as-------	f(u,v)=0	zx	f (x,y)	F(x,y,s,t)=0	f(u,v)=0
The subsidiary equations are px+qy=z is --------------	dx/y=dy/z=dz/x	dx/x=dy/y=dz/z	xdx=ydy=zdz	dz/z=dx/y=dy/x	dx/x=dy/y=dz/z
The general solution of equation p+q=1 is -------------	f(xyz,0)	f(x-y,y-z)	f(x-y,y+z)	F(x,y,s,t)=0	f(x-y,y-z)
The separable equation of the first order PDE can be written in the form of ------------	f(x,y)=g(x,y)	f(a,b)=g(x,y)	f(x,p)=g(y,q)	f(x)=g(a)	f(x,p)=g(y,q)
Complementary function is the solution of ------------------------	f(a,b)	f(1,0)=0	f(D,D')z=0	f(a,b)=F(x,y)	f(D,D')z=0
C.F+P.I is called ------------------- solution	singular	complete	general	particular	general
Particular integral is the solution of -------------	f(a,b)=F(x,y)	f(1,0)=0	[1/f(D,D')]F(x,y)	f(a,b)=F(u,v)	[1/f(D,D')]F(x,y)

Which is independent varible in the equation z= 10x+5y	x&y	z	(x,y,z)	x alone	x&y
Which is dependent varible in the equation z=2x+3y	x	z	y	x&y	z
Which of the following is the type f(z,p,q)=0	p(1+q)=qx	p(1+q)=qz	p(1+q)=qy	p=2xf'(x^2)-(y^2))	p(1+q)=qz
Which is complete integral of z=px+qy+(p^2)(q^2)	z=ax+by+(a^2)(b^2)	z=a+b+ab	z=ax+by+ab	z=a+f(a)x	z=ax+by+(a^2)(b^2)
The complete integral of PDE of the form F(p,q)=0 is	z=ax+f(a)y+c	z=ax+f(a)+b	z=a+f(a)x	z=ax+f(a)	z=ax+f(a)y+c
The relation between the independent and the dependent variables which satisfies the PDE is called-------	solution	complet solution	general solution	singular solution	solution
A solution which contains the maximum possible number of arbitrary constant is called---------	general	complete	solution	singular	complete
The equations which do not contain x & y explicitly can be written in the form---------------	f(z,p,q)=0	f(p,q)=0	(p,q)=0	f(x,p,q)=0	f(z,p,q)=0
The subsidiary equations of the lagranges equation 2y(z-3)p + (2x-z)q = y(2x-3)	dx/2y(z-3) = dy/(2x-z) =dz/y(2x-3)	dx/(2x-z) =dy/2y(z-3) =dz/y(2x-3)	dx/2y=dz/(z-3)	dx/2y=dz/(z-3)=dy/2x	dx/2y(z-3) =dy/(2x-z) =dz/y(2x-3)
A PDE ., the partial derivatives occuring in which are of the first degree is said to be --------------	linear	non-linear	order	degree	linear
A PDE., the partial derivatives occuring in which are of the 2 or more than 2 degree is said to be------	linear	non-linear	order	degree	non-linear
If z=(x^2+a)(y^2+b) then differentiating z partially with respect to x is -----	2x	3x(y^2+b)	2x(y^2+b)	3x+y	2x(y^2+b)
If z=ax+by+ab then differentiating z partially with respect to y is -----	a	a+b	0	b	b
The complete solution of the PDE p=2qx is ----------------	z=ax+ay+c	ax+b	z = ax^2+ay+c	z= ax+(b+c)	z = ax^2+ay+c

The general solution of px-qy=xz is	f(u,v)=0	f(xy,x-logz)=0	f(x-y,y-z)=0	f(x-y,y+z)=0	f(xy,x-logz)=0
If z= f(x^2+y^@) then differentiating z partially with respect to x is -----	p=2xf ’ (x^2+y^2)	p=2xf(x^2+y^2)	p=2xf’(x^2- y^2)	p(1+q)=qy	p=2xf ’ (x^2+y^2)
If z= f(x^2+y^2 +z^2) thendifferentiating z partially with respect to y is -----	q=2xf(x^2+y^2)	q=(2y+2zz') f'(x^2+y^2 +z^2)	q=2y	q=0	q=(2y+2zz') f'(x^2+y^2 +z^2)
The solution of differentiating z partially with respect to x twice gives -----	ax	ax+by+c	ax+b	ax=p	ax+b
The auxiliary equation of (D^2-4DD'+4 D'^2)z=0 is	m^2-4m+4=0	m^2+4m+4=0	m^2-4m-4=0	m^2+4m-4=0	m^2-4m+4=0
The auxiliary equation of (D^3-7DD'^2-6D'^3)z=0 is	m^3+7m+6=0	m^3-7m-6=0	m^3-7m+6=0	m^3+7m-6=0	m^3-7m-6=0
The auxiliary equation of (D^3+DD'^2 -D^2D' -D'^3)z=0 is	m^3-m^2+m-1=0	$m^3+m^2+m-1=0$	m^3-m^2+m+1=0	m^3-m^2-m-1=0	m^3-m^2+m-1=0
The auxiliary equation of (D^2-4DD'+4 D'^2)z=e^x is	m^2+4m+4=0	m^2-4m-4=0	m^2+4m-4=0	none	none
The auxiliary equation of(D^3+7DD'^2+6D'^3)z=cos ax is	m^3+7m+6=0	m^3-7m-6=0	m^3-7m+6=0	m^+7m-6=0	m^3+7m+6=0
The roots of the partial differential equation (D^2-4DD’+4 D'^2)z=0 are	(2,1)	(2,2)	(2,-2)	(2,-2)	(2,2)
Theroots of the partial differential equation (D^3-7DD'^2-6D'^3)z=0 are	(1,2,3)	(2,1,3)	(2,3, -1)	(3,-1,-2)	(3,-1,-2)
The roots of the partial differential equation (D^3 -D^2D’ +DD'^2 -D'^3)z =0 are	(1,i,- i)	(1,1,i)	(i,i,1)	(1,1,1)	(1,i,- i)
The roots of the partial differential equation (D^3 -D^2D’ -DD'^2 +D'^3)z z =0 are	(1,1,1)	(1,1,-1)	(1,-1,-1)	(-1,-1,-1)	(1,-1,-1)
The roots of the partial differential equation (D^2-2DD’+D'^2)z=0 are	(0,1)	(i,-1)	(1,2)	(1,1)	(1,1)

The particular integral of e^$^{(ax+by)}$/ (D-(aD'/b))^2 is ------	e^(ax+by)	(x^^{2}/2) e^(ax+by)	ax-by+c	ax+by	(x^^{2}/2)e^(ax+by)
The particular integral of e^$^{(ax+by)}$/ (D-(aD'/b)) is ---------	ax-by+c	e^(ax+by)	ax+by	xe^(ax+by)	xe^(ax+by)

Basic Formulas

$\frac{d}{dx}(\sin nx) = n\cos nx$	$\int \sin nx\, dx = -\frac{\cos nx}{n}$
$\frac{d}{dx}(\cos nx) = -n\sin nx$	$\int \cos nx\, dx = \frac{\sin nx}{n}$
$\frac{d}{dx}(e^{nx}) = ne^{nx}$	$\int e^{nx}\, dx = \frac{e^{nx}}{n}$
$\frac{d}{dx}(x^n) = n\,x^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1}$
$\cos(-\theta) = \cos\theta$	$\sin(-\theta) = -\sin\theta$
$\cos n\pi = (-1)^n$; $\cos 2n\pi = 1$	$\sin n\pi = 0$ for all values of n
$\cos\left(n\pi \pm \frac{\pi}{2}\right) = 0$	$\sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n ; \sin\left(n\pi - \frac{\pi}{2}\right) = -(-1)^n$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$	$\sin(A+B) = \sin A\cos B + \cos A\sin B$ $\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\sin A\cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$	$\cos A\sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$
$\cos A\cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$	$\sin A\sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$
$\sin^2(\theta) = \frac{1-\cos 2\theta}{2}$	$\cos^2(\theta) = \frac{1+\cos 2\theta}{2}$
$\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$	$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$
$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$	$\int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}$
$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$	$\int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}$

Bernoulli's: $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots$

To identify u and v **use ILATE which first comes taken as** u

Where E : Exponential, T : Trigonometry , A : Algebra , L : Logarithmic , I : Inverse

Unit-V Fourier Series Solutions of Partial Differential Equations

1. Dirichlet's conditions

(i) f(x) is periodic, single-valued and finite function.

(ii) f(x) has a finite number of finite discontinuities in any one period and has no infinite discontinuity.

(iii) f(x) has at the most a finite number of maxima and minima.

2. Odd and Even function

Odd and Even function cases arises only when the function is defined in $(-l,l)$ *and* $(-\pi,\pi)$

Odd function	Even function
f(-x)= - f(x)	f(-x)= f(x)
Odd*Even ; Odd*Even Even *Odd ; Even *Odd	Odd*Odd Even*Even
Example : x,x^3,sin x, x cosx	Example : x^2, cos x, $\sin^2 x$, $\lvert x\rvert$,x sin x,
$a_0 = a_n = 0$	$b_n = 0$

3. Fourier series: l Form

<table>
<tr><td colspan="4">Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$</td></tr>
<tr><td rowspan="2">$(0,2l)$</td><td colspan="3">$(-l,l)$</td></tr>
<tr><td>Even Function</td><td>Odd function</td><td>Neither Even Nor Odd</td></tr>
<tr><td>$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$</td><td>$a_0 = \frac{2}{l}\int_o^{l} f(x)dx$</td><td>$a_0 = 0;$</td><td>$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$</td></tr>
<tr><td>$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$</td><td>$a_n = \frac{2}{l}\int_o^{l} f(x)\cos\frac{n\pi x}{l}dx$</td><td>$a_n = 0;$</td><td>$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$</td></tr>
<tr><td>$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$</td><td>$b_n = 0$</td><td>$b_n = \frac{2}{l}\int_o^{l} f(x)\sin\frac{n\pi x}{l}dx$</td><td>$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$</td></tr>
</table>

4. Fourier series: π Form

Fourier series: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$			
$(0,2\pi)$	$(-\pi,\pi)$		
	Even Function	**Odd function**	**Neither Even Nor Odd**
$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx$	$a_0=\frac{2}{\pi}\int_o^{\pi} f(x)dx$	$a_0=0;$	$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$
$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$	$a_n=\frac{2}{\pi}\int_o^{\pi} f(x)\cos nx\, dx$	$a_n=0;$	$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$
$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$	$b_n=0$	$b_n=\frac{2}{\pi}\int_o^{\pi} f(x)\sin nx\, dx$	$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

5. Half Range Fourier Series

Cosine series		**Sine series**	
l **Form**	π **Form**	l **Form**	π **Form**
$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$	$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$	$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$
$a_0=\frac{2}{l}\int_o^{l} f(x)dx$ $a_n=\frac{2}{l}\int_o^{l} f(x)\cos\frac{n\pi x}{l}dx$	$a_0=\frac{2}{\pi}\int_o^{\pi} f(x)dx$ $a_n=\frac{2}{\pi}\int_o^{\pi} f(x)\cos nx\, dx$	$b_n=\frac{2}{l}\int_o^{l} f(x)\sin\frac{n\pi x}{l}dx$	$b_n=\frac{2}{\pi}\int_o^{\pi} f(x)\sin nx\, dx$

SOLUTION OF ONE DIMENSIONAL AND TWO DIMENSIONAL PDE

Equation	Possible solutions
1-D Wave Equation: $y_{tt} = \alpha^2 y_{xx}$ or $\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ where $\alpha^2 = \frac{T}{M} = \frac{Tension}{Mass}$	$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$ $y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ $y(x,t) = (Ax + B)(Ct + D)$
1-D heat equations : $y_t = \alpha^2 y_{xx}$ or $\frac{\partial y}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ where $\alpha^2 = \frac{k}{pc} = \frac{Thermal\ conductivity}{density\ of\ the\ material \times specific\ heat}$	$y(x,t) = (Ax + B)C$ $y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)Ce^{\alpha^2\lambda^2 t}$ $y(x,t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\alpha^2\lambda^2 t}$
Two dimensional heat flow equation (plate), In steady state 2-D heat equation is $u_{xx} + u_{yy} = 0$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y)$ $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ $u(x,y) = (Ax + B)(Cy + D)$

ONE DIMENSIONAL WAVE EQUATION

S.NO	*Equation*		*Boundary conditions*	*Correct Solution*	*Most general solution*
1	$y_{tt} = a^2 y_{xx}$	*Displacement given*	i) $y(0,t) = 0$ for all t ii) $y(l,t) = 0$ for all t iii) $\frac{\partial y(x,0)}{\partial t} = 0$ iv) $y(x,0) = f(x)$	$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$	$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$

		Initial velocity given	$i)\ y(0,t)=0 \quad forall\ t$ $ii)\ y(l,t)=0 \quad forall\ t$ $iii)\ y(x,0)=0$ $iv)\ \frac{\partial y(x,0)}{\partial t}=f(x)$	$y(x,t)=$ $(A\cos\lambda x+B\sin\lambda x)$ $(C\cos\lambda at+D\sin\lambda at)$	$y(x,t)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

ONE DIMENSIONAL HEAT EQUATION

The steady state temperature distribution on the rod : $u(x)=\left(\frac{b-a}{l}\right)x+a$

a : temp.at end x=0 ; b= temp.at end x=l ; l= Length of the rod.

S.NO	Equation	Boundary conditions	Correct Solution	Most general solution
1	$y_t=\alpha^2 y_{xx}$	$i)\ y(0,t)=0 \quad forall\ t\geq 0$ $ii)\ y(l,t)=0 \quad forall\ t\geq 0$ $iii)\ y(x,0)=f(x)$	$y(x,t)=$ $(A\cos\lambda x+B\sin\lambda x)$ $e^{-\alpha^2\lambda^2 t}$	$y(x,t)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}$

SQUARE PLATE

s.no	*Equation*	*Boundary conditions*	*Correct Solution*	*Most general solution*
1	**Upper horizontal edge (y=a)**	$i)\ u(0,y)=0$ $ii)\ u(a,y)=0$ $iii)\ u(x,0)=0$ $iv)\ u(x,a)=f(x)$	$u(x,y)=$ $(A\cos\lambda x+B\sin\lambda x)$ $(Ce^{\lambda y}+De^{-\lambda y})$	$u(x,y)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$
2	**Lower horizontal edge (y=0)**	$i)\ u(0,y)=0$ $ii)\ u(a,y)=0$ $iii)\ u(x,a)=0$ $iv)\ u(x,0)=f(x)$		
3	**Vertical edge**	$i)\ u(x,0)=0$ $ii)\ u(x,a)=0$ $iii)\ u(0,y)=0$ $iv)\ u(a,y)=f(y)$	$u(x,y)=$ $(A\cos\lambda y+B\sin\lambda y)$ $(Ce^{\lambda x}+De^{-\lambda x})$	$u(x,y)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{a}\sinh\frac{n\pi x}{a}$

RECTANGULAR PLATE

FINITE

S.NO	Equation	Boundary conditions	Correct Solution	Most general solution
1	Upper horizontal edge (y=b)	$i)\ u(0,y)=0$ $ii)\ u(a,y)=0$ $iii)\ u(x,0)=0$ $iv)\ u(x,b)=f(x)$	$u(x,y)=$ $(A\cos\lambda x+B\sin\lambda x)$ $(Ce^{\lambda y}+De^{-\lambda y})$	$u(x,y)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{b}$
2	Lower horizontal edge (y=0)	$i)\ u(0,y)=0$ $ii)\ u(a,y)=0$ $iii)\ u(x,b)=0$ $iv)\ u(x,0)=f(x)$		$u(x,y)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} e^{\frac{-n\pi y}{a}}$ $(e^{\frac{2n\pi y}{a}}-e^{\frac{2n\pi y}{a}})$
3	Vertical edge	$i)\ u(x,0)=0$ $ii)\ u(x,b)=0$ $iii)\ u(0,y)=0$ $iv)\ u(a,y)=f(y)$	$u(x,y)=$ $(A\cos\lambda y+B\sin\lambda y)$ $(Ce^{\lambda x}+De^{-\lambda x})$	$u(x,y)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{b}$

INFINITE PLATE

S.NO	Equation	Boundary conditions	Correct Solution	Most general solution
4	Horizontal infinite plate	$i)\ u(0,y)=0$ $ii)\ u(l,y)=0$ $iii)\ u(x,\infty)=0$ $iv)\ u(x,0)=f(x)$	$u(x,y)=$ $(A\cos\lambda x+B\sin\lambda x)$ $(Ce^{\lambda y}+De^{-\lambda y})$	$u(x,y)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{\frac{-n\pi y}{l}}$
5	Vertical infinite plate	$i)\ u(x,0)=0$ $ii)\ u(x,l)=0$ $iii)\ u(\infty,y)=0$ $iv)\ u(0,y)=f(y)$	$u(x,y)=$ $(A\cos\lambda y+B\sin\lambda y)$ $(Ce^{\lambda x}+De^{-\lambda x})$	$u(x,y)=$ $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{l} e^{\frac{-n\pi x}{l}}$

Questions	opt 1	opt 2	opt 3	opt 4	opt5	opt 6	Answer
If a function satisfies the condition f(-x) = f(x) then which is true?	$a_{0} = 0$	$a_{n} = 0$	$a_{0} = a_{n} = 0$	$b_{n} = 0$			$b_{n} = 0$
If a function satisfies the condition f(-x) = -f(x) then which is true?	$a_0 = 0$	$a_n = 0$	$a_{0} = a_{n} = 0$	$b_{n} = 0$			$a_{0} = a_{n} = 0$
Which of the following is an odd function?	sin x	cos x	x^{2}	x^4			sin x
Which of the following is an even function?	x^{3}	cos x	sin x	$\sin^{2}x$			cos x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	f(-x) = f(-x)			f(-x) = - f(x)
The function f(x) is said to be an even function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	f(-x) = f(-x)			f(-x) = f(x)
∫f(x) dx = 2∫f(x) dx between the limits -a to a if f(x) is ------	even	continuous	odd	discontinues			even
∫f(x) dx = 0 between the limits -a to a if f(x) is ------	even	continuous	odd	discontinues			odd
If a periodic function f(x) is odd, it's Fourier expansion contains no ------ terms.	coefficient a_n	sine	coefficient a_0	cosine			cosine
If a periodic function f(x) is even, it's Fourier expansion contains no ------ terms.	cosine	sine	coefficient a_{0}	coefficient a_{n}			sine
In dirichlet condition, the function f(x) has only a ----- number of maxima and minima.	uncountable	continuous	infinite	finite			finite
In Fourier series, the function f(x) has only a finite number of maxima and minima. This condition is known as -------	Dirichlet	Kuhn Tucker	Laplace	Cauchy			Dirichlet
In dirichlet condition, the function f(x) has only a ----- number of discontinuities .	uncountable	continuous	infinite	finite			finite
In Fourier series, the expansion $f(x) = a_0/2 + \sum (a_n \cos nx + b_n \sin nx)$ is possible only if in the interval $c_1 \leq x \leq c_2$ the function f(x) satisfies ---condition.	kuhn-Tucker	Laplace	Dirichlet	Cauchy			Dirichlet

If the periodic function f(x) is even, then it's Fourier co- efficient -------- is zero.	a_0	a_1	b_n	a_0 & a_n	b_n
If the periodic function f(x) is odd, then it's Fourier co-efficient -------- is zero.	a_0 & a_n	a_1	b_n	b_1	a_0 & a_n
1/pi∫f(x) cos nx dx gives the Fouier coefficient -----------	a_0	b_1	b_n	a_n	a_n
1/pi∫f(x) dx gives the Fourier coefficient	a_0	a_n	b_n	b_1	a_0
1/pi∫f(x)sin nx dx gives the Fouier coefficient -----------	a_0	a_n	b_n	b_1	b_n
The period of cos nx where n is the positive integer is	2*pi/n	pi/2n	2*pi	n*pi	2*pi/n
The Fourier co efficient a_0 for the function defined by f(x) = x for 0< x< pi is	pi	pi/2	2*pi	0	pi
If the function f (x) = -π in the interval –pi< x< 0, the coefficient a_0 is	pi^2/3	(2*pi^2)/3	2*pi/3	2 *pi	2*pi
If the function f(x) = x sin x, in –pi< x<pi then Fourier coefficient	$b_n = 0$	$a_0 = 1$	a_0 = (pi^2)/3	$a_0 = -1$	$b_n = 0$
For the cosine series, which of the Fourier coefficient will vanish?	a_n	b_n	a_1	Both a_0 and a_n	b_n
For the sine series, which of the Fourier coefficient variables will be vanish?	b_n	a_n	Both a_0 and a_n	a_0	Both a_0 and a_n
For a function f(x) = x^3, in –pi< x< pi the Fourier coefficient	$b_n = 0$	$a_n = 1$	$a_0 = 1$	$a_0 = a_n = 0$	$a_0 = a_n = 0$
F(x)=x cos x is an ------ function.	an odd function	even function	neither odd or even	both even and odd	an odd function
If f(x) = x, in –pi< x< pi then Fourier co efficient	b_n = 0	a_n = π	a_0 = a_n = 0	a_n = 1	a_0 = a_n = 0
F(x)=e^x is in –pi< x< pi.	an odd function	even function	neither odd or even	Both even and odd	neither odd or even
Which of the coefficients in the Fourier series of the function $f(x) = x^2$ in -pi < x< pi will vanish	a_0	a_0 and a_n	b_n	a_n	b_n

If f(-x) = -f(x), then the function f(x) is said to be -----	odd	Continuous	even	discontinuous	odd
If f(-x) = f(x), then the function f(x) is said to be -----	odd	continuous	even	discontinuous	even
The function x sin x is a ------- function in –pi< x< pi.	even	odd	continuous	discontinuous	even
The function x cos x is a ------- function in –pi< x< pi.	even	odd	continuous	discontinuous	odd
If f(x)= x in 0<x<2*pi and f(x)=f(x+2*pi) then the sum of the fourier series of f(x) at x=2*pi is------	2*pi	2	0	pi	2*pi
If f(x)= x^2 in 0<x<2*pi and f(x)=f(x+2*pi) then the sum of the fourier series of f(x) at x= 0 is------	2*pi^2	0	6*pi	4*pi	2*pi^2
For any peroidic function f(x) in –pi< x< pi the point x=- pi is a ----- point.	Continous	discontinous	intermediate	Continous and discontinous	discontinous
For any peroidic function f(x) in 0< x<2* pi the point x= pi is a ----- point.	Continous and discontinous	intermediate	Continous	discontinous	Continous
For any peroidic function f(x) in 0≤ x≤ pi the point x= 0 is a ----- point.	discontinous	Continous and discontinous	Continous	intermediate	Continous
Partial differential equation of second order is said to Elliptic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	B^2-4AC<0
Partial differential equation of second order is said to Parabolic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	B^2-4AC=0
Partial differential equation of second order is said to Hyperbolic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	B^2-4AC>0
Two dimensional Laplace Equation is ----------	u_xx+u_yy=1	u_xx+u_yy=0	u_x=u_y	u_x+u_y=0	u_xx+u_yy=0
One dimensional heat Equation is ----------	u_xx=(1/a^2)u_t	u_xx=[(1/a^2)u_t]+10	u_xx=u_tt	u_xx+u_tt=0	u_xx=(1/a^2)u_t

One dimensional wave Equation is ----------	u_xx=(1/a^2)u_t	u_xx+u_yy=0	u_xx=(1/a^2)u_t^2	u_xx=u_t	u_xx=(1/a^2)u_t^2
The Possion equation is of the form -----	y(x,t)=f(x-at)+g(x+at)	u_xx=(1/a^2)u_t	u_xx=(1/a^2)u_tt	u_xx+u_yy=f(x,y)	u_xx+u_yy=f(x,y)
The steady state temperature of a rod of length l whose ends are kept at 30 and 40 is	u(x)= 10x/l + 30	u(x)= 40x/l	u(x)= 30x/l	None	u(x)= 10x/l + 30
The temperature distribution of the plate in the steady state is ---------	u_xx=(1/a^2)u_t	u_xx+u_yy=0	u_xx=(1/a^2)u_t^2	u_xx=u_t	u_xx+u_yy=0
Two dimensional heat Equation is known as ---------- equation.	partial	Radio	laplace	Poisson	laplace
In one dimensional heat flow equation ,if the temperature function u is independent of time, then the solution is------	u(x)= ax + b	u(x,t)= a(x,t)	u(t) = at + b	u(t) = at - b	u(x)= ax + b
f_xx+2f_xy+4f_yy=0 is a _______	Elliptic	Hyperbolic	Parabolic	circle	Elliptic
f_xx=2f_yy is a ---------	Elliptic	Hyperbolic	Parabolic	circle	Hyperbolic
f_xx-2f_xy+f_yy=0 is a -----------	Hyperbolic	Elliptic	Parabolic	circle	Parabolic
The diffusivity of substance is------	k/pc	pc	k	pc/k	k/pc
Heat flows from a ------- temperature	higher to lower	lower to higher	normal	high	higher to lower
The Amount of heat required to produce a given temperature change in a bodies propostional to the ------- of the body and to the temperature change.	temperature	heat	mass	wave	mass
The rate at which heat flows through an area is------ to the area and to the temperature gradient normal to the area.	equal	not equal	lessthan	proportional	proportional
In steady state conditions the temperature at any particular point does not vary with ---	Time	temperature	mass	none	Time
The wave equation is a linear and ------ equation	non homogeneous	homogeneous	quadratic	none	homogeneous

In method of separation of variables we assume the solution in the form of -----	u(x,y)=X(x)	u(x,t)=X(x)T(t)	u(x,0)=u(x,y)	u(x,y)=X(y)Y(x)	u(x,t)=X(x)T(t)
u(x,t)=(Acosax+Bsinax)Ce^(-(b^2))(a^2)t) is the possible solution of ------ equation	heat	wave	laplace	none	heat
y=(Ax+B)(Ct+D) is the possible solution of ------ equation	heat	wave	laplace	none	wave
If the heat flow is one dimensional ,then the -------- is a function x and t only	heat	light	temperature	wave	temperature
The stream lines are parallel to the X-axis ,then the rate of change of the temperature in the direction of the Y-axis will be ---------.	one	two	zero	five	zero
The boundary condition with non zero value on the R.H.S of the wave equation should be taken as the -------------- boundary condition.	First	Second	Last	none	Last
In one dimensional heat equation u_t= (α^2)u_xx, What does α^2 stands for?	k/pc	pc	k	pc/k	k/pc
If B^2-4AC = 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced	parabolic
If B^2-4AC > 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced	hyperbolic
If B^2-4AC < 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced	elliptic
The flow is two dimensional the temperature at any point of the plane is ------- of Z-coordinates.	linear	independent	dependent	none	independent
u(x,y)=(Acosλx+Bsinλx)(Ce^(λy)De^(-λy)) is the possible solution of the _______equation.	heat	wave	laplace	none	laplace

UNIT II

Foutier Series

S.NO.	Questions	opt 1	opt 2	opt 3	opt 4	opt5
1	Which of the following functions has the period 2π?	cos x	sin nx	tan nx	tan x	
2	1/π ∫ f(x) sinnx dx between the limits c to c+2π gives the Fourier coefficient_____	a_0	a_n	b_n	b_1	
3	If f(x) = -x for -π< x< 0 then its Fourier coefficient a_0 is_______-	(π^2)/2	π/2	π/3	π	
4	If a function satisfies the condition f(-x) = f(x) then which is true?	$a_0 = 0$	$a_n = 0$	$a_0 = a_n = 0$	$b_n = 0$	
5	If a function satisfies the condition f(-x) = -f(x) then which is true?	$a_0 = 0$	$a_n = 0$	$a_0 = a_n = 0$	$b_n = 0$	
6	Which of the following is an odd function?	sin x	cos x	$x^{\wedge 2}$	x^4	
7	Which of the following is an even function?	$x^{\wedge 3}$	cos x	sin x	$\sin^{\wedge 2} x$	
8	The function f(x) is said to be an odd function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	f(-x) = f(- x)	
9	The function f(x) is said to be an even function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	f(-x) = f(- x)	
10	∫f(x) dx = 2∫f(x) dx between the limits -a to a if f(x) is ------	even	continuous	odd	discontinues	
11	∫f(x) dx = 0 between the limits -a to a if f(x) is ------	even	continuous	odd	discontinues	

12	If a periodic function f(x) is odd, it's Fourier expansion contains no ------ terms.	coefficient a_n	sine	coefficient a_0	cosine	
13	If a periodic function f(x) is even, it's Fourier expansion contains no ------ terms.	cosine	sine	coefficient a_0	coefficient a_n	
14	In dirichlet condition, the function f(x) has only a ----- number of maxima and minima.	uncountable	continuous	infinite	finite	
15	In Fourier series, the function f(x) has only a finite number of maxima and minima. This condition is known as	Dirichlet	Kuhn Tucker	Laplace	Cauchy	
16	In dirichlet condition, the function f(x) has only a ----- number of discontinuities .	uncountable	continuous	infinite	finite	
17	The Fourier series of f(x) is given by ----	$a_0/2 + \sum (a_n$ cosnx+	$a_0/2 + \sum (a_n$ cos nx-	$a_n/2 + \sum (a_n$ sin nx+ bn	$a_0/2 + \sum (a_0$ sin nπx/ l)	
18	In Fourier series, the expansion $f(x) = a_0/2 + \sum (a_n \cos nx + b_n \sin nx)$ is possible only if in the interval	kuhn-Tucker	Laplace	Dirichlet	Cauchy	
19	If the periodic function f(x) is even, then the Fourier expansion is of the form ---	$a_0/2 + \sum a_n \sin(n\pi x/l)$	$a_0/2 + \sum a_n \cos(n\pi x/l)$	$a_n/2 + \sum a_n$ cos(	$a_0/2 + \sum a_0 \sin(n\pi x/l)$	
20	If the periodic function f(x) is even, then it's Fourier co- efficient an is of the form ---	2/ l ∫f(x) sin(nπx/ l) dx	2/ l ∫f(x) cos (nπx/ l) dx	1/ l ∫f(x) / l dx	∫f(x) dx	
21	If the periodic function f(x) is even, then it's Fourier co- efficient a_0 is of the form ---	2/ l ∫f(x) dx	1/ l ∫f(x) dx	2/ l ∫f(x)/ l dx	∫f(x) dx	
22	If the periodic function f(x) is odd, then it's Fourier co- efficient bn is of the form ---	2/ l ∫f(x) cos (nπx/ l) dx	2/ l ∫f(x) sin(nπx/ l) dx	∫f(x) dx	1/ l ∫f(x) / l dx	
23	If the periodic function f(x) is even, then it's Fourier co- efficient -------- is zero.	a_0	a_1	b_n	a_0 & a_n	
24	If the periodic function f(x) is odd, then it's Fourier co- efficient -------- is zero.	a_0 & a_n	a_1	b_n	b_1	
25	If the periodic function f(x) is even, then the Fourier expansion is of the form ---	∑b_n sin nπx/ l	∑b_n sin nπx/ l	∑ b_n cos nπx/ l	a_0/2+∑ a_n cos (nπx/ l)	

26	If the periodic function f(x) is odd, then the Fourier expansion is of the form ---	$\sum b_n \sin n\pi x/l$	$\sum a_n \sin n\pi x/l$	$\sum b_n \cos n\pi x/l$	$\sum a_n \cos n\pi x/l$	
27	$1/\pi\int f(x) \cos nx\, dx$ gives the Fouier coefficient -----------	a_0	b_1	b_n	a_n	
28	$1/\pi\int f(x)\, dx$ gives the Fourier coefficient	a_0	a_n	b_n	b_1	
29	$1/\pi\int f(x)\sin nx\, dx$ gives the Fouier coefficient -----------	a_0	a_n	b_n	b_1	
30	The period of cos nx where n is the positive integer is	$2\pi/n$	$\pi/2n$	2π	$n\pi$	
31	The Fourier co efficient a_0 for the function defined by f(x) = x for $0< x< \pi$ is	π	$\pi/2$	2π	0	
32	If the function f (x) = $-\pi$ in the interval $-\pi< x< 0$, the coefficient a_0 is	π^2/3	(2π^2)/3	$2\pi/3$	2π	
33	If the function f(x) = x sin x, in $-\pi< x< \pi$ then Fourier coefficient	$b_n = 0$	$a_0 = 1$	$a_0 =$ (π^2)/3	$a_0 = -1$	
34	For the cosine series, which of the Fourier coefficient will vanish?	a_n	b_n	a_1	Both a_0 and a_n	
35	For the sine series, which of the Fourier coefficient variables will be vanish?	b_n	a_n	Both a_0 and a_n	a_0	
36	For a function f(x) = x^3, in $-\pi< x< \pi$ the Fourier coefficient	$b_n = 0$	$a_n = 1$	$a_0 = 1$	$a_0 = a_n = 0$	
37	F(x)=x cos x is an ------ function.	an odd function	even function	neither odd or even	both even and odd	
38	If f(x) = x, in $-\pi< x< \pi$ then Fourier co efficient	b_n = 0	a_n = π	a_0 = a_n = 0	a_n = 1	
39	F(x)=e^x is in $-\pi< x< \pi$.	an odd function	even function	neither odd or even	Both even and odd	

40	Which of the coefficients in the Fourier series of the function f(x) = x^2 in $-\pi < x < \pi$ will vanish	a_0	a_0 and a_n	b_n	a_n	
41	If f(-x) = -f(x), then the function f(x) is said to be -----	odd	Continuous	even	discontinuous	
42	If f(-x) = f(x), then the function f(x) is said to be -----	odd	continuous	even	discontinuous	
43	The function x sin x is a ------- function in $-\pi < x < \pi$.	even	odd	continuous	discontinuous	
44	The function x cos x is a ------- function in $-\pi < x < \pi$.	even	odd	continuous	discontinuous	
45	The formula for finding the fourier coefficient a_0 in Harmonic analysis is ----	(2/N)Σ y cos nx	(2/N)Σ y	(2/N)Σ y sin nx	a_0 /2 + ∑ (a_n cosnx+	
46	The formula for finding the fourier coefficient a_n in Harmonic analysis is ----	(2/N)Σ y cos nx	(2/N)Σ y	a_0 /2 + ∑ (a_n cosnx+	(2/N)Σ y sin nx	
47	The formula for finding the fourier coefficient b_n in Harmonic analysis is ----	(2/N)Σ y	(2/N)ΣΣ y sin nx	(2/N)Σ y cos nx	$a_0 /2$ + ∑ (a_n cosnx+	
48	The term $a_1$cos x+ b_1 sin x is called the----- harmonic.	second	first	third	end	
49	The term ------------- is called the first harmonic in Furier Series expansion.	$a_1$cosn x+ b_1 sin x	$a_1$cos2 x+ b_1 sin x	$a_1$cos x+ b_1 sin2 x	$a_1$cos x+ b_1 sin x	
50	If f(x)= x in $0<x<2\pi$ and $f(x)=f(x+2\pi)$ then the sum of the fourier series of f(x) at $x=2\pi$ is------	2π	2	0	π	
51	If f(x)= x^2 in $0<x<2\pi$ and $f(x)=f(x+2\pi)$ then the sum of the fourier series of f(x) at x= 0 is------	2π^2	0	6π	4π	
52	For any peroidic function f(x) in $-\pi < x < \pi$ the point $x=-\pi$ is a ----- point.	Continous	discontinous	intermediate	Continous and discontinous	
53	For any peroidic function f(x) in $0< x<2\pi$ the point $x=\pi$ is a ----- point.	Continous and discontinous	intermediate	Continous	discontinous	

54	For any peroidic function f(x) in $0\leq x\leq \pi$ the point x= 0 is a ----- point.	discontinous	Continous and discontinous	Continous	intermediate	
55	The process of finding the Fourier series for a function given by ------------ at equally spaced points is known as harmonic analysis	initial value	numerical value	final value	fundamental value	
56	The process of finding the Fourier series for a function given by numerical values at ----------- points is known as harmonic analysis	equally spaced	unequally spaced	intermediate	both equally and unequally	
57	The process of finding the Fourier series for a function given by numerical values at equally spaced points is known as	mathematical analysis	complex analysis	real analysis	harmonic analysis.	
58	The complex form of Fourier series of f(x) in (c,c+2*l*) *is*	$f(x)=\sum cn\ e^{\wedge}(^{in\pi x/l})$	$\sum b_n \sin n\pi x/l$	$f(x)=\sum cn\ e^{\wedge}(^{in\pi x/l})$	$f(x)=\sum cn\ e^{\wedge}(-^{in\pi x/l})$	
59	The Euler constant c_n in the complex form of fourier series of f(x) in (c,c+2*l*) is	c_n=1/2*l* ∫f(x) e^(-	c_n= e^(inx/*l*)	∑c_n e^(inπx/*l*)	∑c_n e^(-inπx/*l*)	
60	$y^{\wedge 2}=1/(b-a)\int(f(x))^{\wedge 2}\,dx$ is called the ----of the function.	Root mean square value	parsevals identity	Harmonic	euler	
1	Partial differential equation of second order is said to Elliptic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	
2	Partial differential equation of second order is said to Parabolic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	
3	second order is said to Hyperbolic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	
4	Two dimensional Laplace Equation is ----------	u_xx+u_yy=1	u_xx+u_yy=0	u_x=u_y	u_x+u_y=0	
5	One dimensional heat Equation is -----------	u_xx=(1/α^2)u_t	u_xx=[(1/α^2)u_t]+10	u_xx=u_tt	u_xx+u_tt=0	
6	One dimensional wave Equation is -----------	u_xx=(1/α^2)u_t	u_xx+u_yy=0	u_xx=(1/α^2)u_t^2	u_xx=u_t	
7	The D'Alembert's solution of the One dimensional wave Equation is-----	x-αt)+ψ(x+αt)	y(x,t)=0	u_xx=(1/α^2)u_t	u_xx=(1/α^2)u_t^2	

8	The Possion equation is of the form -----	x-αt)+ψ(x+αt)	u_xx=(1/α^2)u_t	u_xx=(1/α^2)u_tt	u_xx+u_yy=f(x,y)
9	The steady state temperature of a rod of length l whose ends are kept at 30 and 40 is	u(x)= 10x/l + 30	u(x)= 40x/l	u(x)= 30x/l	None
10	The temperature distribution of the plate in the steady state is ---------	u_xx=(1/α^2)u_t	u_xx+u_yy=0	u_xx=(1/α^2)u_t^2	u_xx=u_t
11	Two dimensional heat Equation is known as ----------equation.	partial	Radio	laplace	Poisson
12	equation ,if the temperature function u is independent of time, then the solution is------	u(x)= ax + b	u(x,t)= a(x,t)	u(t) = at + b	u(t) = at - b
13	f_xx+2f_xy+4f_yy=0 is a _______	Elliptic	Hyperbolic	Parabolic	circle
14	f_xx=2f_yy is a ---------	Elliptic	Hyperbolic	Parabolic	circle
15	f_xx-2f_xy+f_yy=0 is a -----------	Hyperbolic	Elliptic	Parabolic	circle
16	The diffusivity of substance is------	k/pc	pc	k	pc/k
17	Heat flows from a ------- temperature	higher to lower	lower to higher	normal	high
18	produce a given temperature change in a bodies propostional to the ------- of the body and to the	temperature	heat	mass	wave
19	through an area is------ to the area and to the temperature gradient normal to the area.	equal	not equal	lessthan	proportional
20	In steady state conditions the temperature at any particular point does not vary with ---	Time	temperature	mass	none
21	The wave equation is a linear and ------ equation	non homogeneous	homogeneous	quadratic	none

22	In method of separation of variables we assume the solution in the form of -----	u(x,y)=X(x)	u(x,t)=X(x)T(t)	u(x,0)=u(x,y)	u(x,y)=X(y)Y(x)
23	u(x,t)=(Acosλx+Bsinλx)Ce^(-(α^2))(λ^2)t) is the possible solution of ------ equation	heat	wave	laplace	none
24	y=(Ax+B)(Ct+D) is the possible solution of ------ equation	heat	wave	laplace	none
25	If the heat flow is one dimensional ,then the -------- is a function x and t only	heat	light	temperature	wave
26	X-axis ,then the rate of change of the temperature in the direction of the Y-axis will be ---------.	one	two	zero	five
27	To solve y_tt=(α^2)yxx, we need -------- boundary conditions.	if t>=0; y(l,t)=0 if t>=0	if t>0; y(t)=0 if t=0	y(x,t)=0 if t>0	none
28	zero value on the R.H.S of the wave equation should be taken as the -------------- boundary condition.	First	Second	Last	none
29	In one dimensional heat equation u_t= (α^2)u_xx, What does α^2 stands for?	k/pc	pc	k	pc/k
30	The possible solution of wave equation is ----------	y=(Ax+B)(Ct+D)	cosλx+Bsinλx)(Ce^(λy)+De	u(x,t)=Acosλx+Bsinλx	u(x,t)=Acosλx-Bsinλx
31	The possible solution of heat equation is ----------	cosλx+Bsinλx)Ce^(-	cosλx+Bsinλx)(Ce^(λy)+De	u(x,t)=Acosλx+Bsinλx	u(x,t)=Acosλx-Bsinλx
32	If B^2-4AC = 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced
33	If B^2-4AC > 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced
34	If B^2-4AC < 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced
35	The laplace equation in the polar coordinates is of the form-----	u_r+u_θ=0	u_xx=(1/α^2)u_t^2	u_xx=(1/α^2)u_t	(r^2)u_rr+ru_r+u_θθ=0

36	The flow is two dimensional the temperature at any point of the plane is ------- of Z-coordinates.	linear	independent	dependent	none
37	u(x,y)=(Acosλx+Bsinλx)(Ce^(λy)De^(-λy)) is the possible solution of the _______equation.	heat	wave	laplace	none
38	U(r,θ)=(A log r+B)(Cθ+D) is the possible solution of ------ equation	heat	wave	laplace	none

opt6	opt 5	opt 6	Answer
			cos x
			b_n
			π
			$b_{\_n} = 0$
			$a_{\_0} = a_{\_n} = 0$
			sin x
			cos x
			f(-x) = -f(x)
			f(-x) = f(x)
			even
			odd

			cosine
			sine
			finite
			Dirichlet
			finite
			$a_0/2 + \sum (a_n$ cosnx+
			Dirichlet
			$a_0/2 + \sum a_n \cos(n\pi x/ l)$
			$2/l \int f(x) \cos(n\pi x/l)\, dx$
			$2/l \int f(x)\, dx$
			$2/l \int f(x) \sin(n\pi x/l)\, dx$
			b_n
			a_0 & a_n
			a_0/2+∑ a_n cos (nπx/ l)

			$\sum b_n \sin n\pi x/ l$
			a_n
			a_0
			b_n
			$2\pi/n$
			π
			2π
			$b_n = 0$
			b_n
			Both a_0 and a_n
			$a_0 = a_n = 0$
			an odd function
			a_0 = a_n = 0
			neither odd or even

			b_n
			odd
			even
			even
			odd
			(2/N)Σ y
			(2/N)Σ y cos nx
			(2/N)Σ y sin nx
			first
			$a_1$cos x+ b_1 sin x
			2π
			2π^2
			discontinous
			Continous

			Continous
			numerical value
			equally spaced
			harmonic analysis.
	f(x)= ∑cn e^(inπ x/ l)		f(x)= ∑cn e^(inπ x/ l)
	c_n=1/2l ∫f(x) e^(-		c_n=1/2l ∫f(x) e^(-
	Root mean square value		Root mean square value

B^2-4AC<0

B^2-4AC=0

B^2-4AC>0

u_xx+u_yy=0

u_xx=(1/α^2)u_t

u_xx=(1/α^2)u_t^2

x-αt)+ψ(x+αt)

$u_{xx}+u_{yy}=f(x,y)$

$u(x)=10x/l+30$

$u_{xx}+u_{yy}=0$

laplace

$u(x)=ax+b$

Elliptic

Hyperbolic

Parabolic

k/pc

higher to lower

mass

proportional

Time

homogeneous

u(x,t)=X(
x)T(t)

heat

wave

temperat
ure

zero
if t>=0;
y(l,t)=0
if t>=0

Last

k/pc

y=(Ax+B
)(Ct+D)
cosλx+B
sinλx)Ce
^(-

parabolic

hyperboli
c

elliptic
(r^2)u_rr
+ru_r+u_
θθ=0

independ
ent

laplace

laplace