Semester–V

L T P C

15PHU505A **NUMERICAL METHODS**

5 - - 5

Scope: This paper explains the different numerical methods of calculations which is of very much importance in the analysis of many problems in Physics.

Objectives: Numerical methods is very important where large number of calculations are involved, and the original calculations from first principles is very difficult and complicated. In this paper, different methods are introduced for carrying out complicated calculations.

UNIT I

Principle of least squares - fitting a straight line - linear regression - fitting a parabola - fitting an exponential curve.

UNIT II

Bisection method - method of successive approximations - RegulaFalsi method - Newton-Raphson method - Horner's method - Euler's method - modified Euler's method - RungeKutta method (II & IV).

UNIT III

Gauss elimination method - Gauss-Jordan method - Gauss-Seidel method - computation of inverse of a matrix using Gauss elimination method - method of triangularisation.

UNIT IV

First differences - difference tables - properties of the operator A.E.D.

Linear interpolation: Newton forward interpolation formula and backward interpolation formula - Bessel's Formula.

Interpolation with unequal intervals: Lagrange's interpolation formula.

UNIT V

Trapezoidal rule - Simpson's 1/3 rule and 3/8 rule - practical applications - Weddle's rule - Gaussian Quadrature formulae.

Text Text Book

E Balagurusamy 1st edition 2014 numerical methods Tata Mcgraw hills

REFERENCES

Venkatraman, M.K., 1977, Numerical Methods in Science and Engineering, National publishing Company, Chennai.

Shastry, S.S, 2007, Introductory Methods of Numerical Analysis, Prentice Hall of India, Pvt. Ltd., New Delhi.

M K Jain, R K Jain, SRK Iyenger 6th edition 2014 Numerical methods for Scientific and Engineering Computation, New Age Publishers.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Class : III – B.Sc. Physics	**Semester : V**	**L T P C**
Subject: Numerical Methods	**Subject Code: 15PHU505A**	**5 0 0 5**

UNIT-II

Bisection method - method of successive approximations - RegulaFalsi method - Newton- Raphson method - Horner's method - Euler's method - modified Euler's method - RungeKutta method (II & IV).

SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Introduction

The solution of the equation of the form $f(x) = 0$ occurs in the field of science, engineering and other applications. If $f(x)$ is a polynomial of degree two or more ,we have formulae to find solution. But, if $f(x)$ is a transcendental function, we do not have formulae to obtain solutions. When such type of equations are there, we have some methods like Bisection method, Newton-Raphson Method and the method of false position. Those methods are solved by using a theorem in theory of equations, *i.e., If $f(x)$* is continuous in the interval (a,b) and if f(a) and f(b) are of opposite signs, then the equation f(x) = 0 will haveatleast one real root between a and b.

Bisection Method

Let us suppose we have an equation of the form $f(x) = 0$ in which solution lies between in the range (a,b). Also $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are opposite signs, then there existatleast one real root between a and b.Let f(a) be positive and f(b) negative. Which implies atleast one root exits between a and b. We assume that root to be $x_o= (a+b)/2$. Check the sign of $f(x_o)$. If $f(x_0)$is negative , the root lies between a and x_o. If $f(x_0)$is positive , the root lies between x_o and b. Subsequently any one of this case occur.

$$X_1= X_0+a /2 \quad (or) \quad x_0+b/2$$

When $f(x_1)$ is negative , the root lies between xo and x1 and let the root be $x_2=(x_0 +x_1) / 2$.
Again $f(x_2)$ negative then the root lies between x_o and x_2, let $x_3 = (x_0+x_2)/2$ and so on. Repeat the process x_o, x_1,x_2, …. Whose limit of convergence is the exact root.

Steps:

1. Find a and b in which f(a) and f(b) are opposite signs for the given equation using trial and error method.
2. Assume initial root as $x_o= (a+b)/2$.
3. If $f(x_0)$is negative , the root lies between a and x_o and take the root as $x_1 = (x_o+a)/2$.
4. If $f(x_o)$ is positive , then the root lies between x_o and b and take the root as $x_1= (x_o +b)/ 2$.
5. If $f(x_1)$ is negative , the root lies between x_o and $x_1$and let the root be $x_2= (x_0 +x_1) / 2$.
6. If $f(x_2)$ is negative , the root lies between x_o and $x_1$and let the root be $x_3= (x_0 +x_2) / 2$.
7. Repeat the process until any two consecutive values are equal and hence the root.

Example:

Find the positive root of $x^3 – x = 1$ correct to four decimal places by bisection method.

Solution:

Let $f(x) = x^3 – x – 1$

$f(0) = 0^3 - 0 -1 = -1 = -ve$

$f(1) = 1^3 - 1 -1 = -1 = -ve$

$f(2) = 2^3 - 2 -1 = 5 = +ve$

So root lies between 1 and 2 , we can take (1+2) /2 as initial root and proceed.

i.e., f(1.5) = 0.8750 = +ve

and f(1) = -1 = -ve

So root lies between 1 and 1.5 ,

Let x_o = (1+1.5) /2 as initial root and proceed.

f(1.25) = - 0.2969

So root lies between *x1* *bet*ween 1.25 and 1.5

Now x_1 = (1.25 + 1.5) /2 = *1.3750*

f(1.375) = 0.2246 = +ve

So root lies between x_2 *bet*ween 1.25 and 1.375

Now x_2 =(*1.25 + 1.375)/2* = *1.3125*

f(1.3125) = -0.051514 = -ve

Therefore, root lies between 1.375and 1.3125

Now x_3 = (1.375 + 1.3125) / 2 = *1.3438*

f(1.3438) = 0.082832 = +ve

So root lies between 1.3125 and 1.3438

Now x_4 = (1.3125 + 1.3438) / 2 = *1.3282*

f(1.3282) = 0.014898 = +ve

So root lies between 1.3125 and 1.3282

Now x_5 = (1.3125 + 1.3282) / 2 = *1.3204*

f(1.3204) = - 0.018340 = -ve

So root lies between 1.3204 and 1.3282

Now x_6 = (1.3204 + 1.3282) / 2 = *1.3243*

f(1.3243) = -ve

So root lies between 1.3243 and 1.3282

Now x_7 = (1.3243 + 1.3282) / 2 = *1.3263*

$f(1.3263) = +ve$

So root lies between 1.3243 and 1.3263

Now $x_8 = (1.3243 + 1.3263) / 2 = 1.3253$

$f(1.3253) = +ve$

So root lies between 1.3243 and 1.3253

Now $x_9 = (1.3243 + 1.3253) / 2 = 1.3248$

$f(1.3248) = +ve$

So root lies between 1.3243 and 1.3248

Now $x_{10} = (1.3243 + 1.3248) / 2 = 1.3246$

$f(1.3246) = -ve$

So root lies between 1.3248 and 1.3246

Now $x_{11} = (1.3248 + 1.3246) / 2 = 1.3247$

$f(1.3247) = -ve$

So root lies between 1.3247 and 1.3248

Now $x_{12} = (1.3247 + 1.3247) / 2 = 1.32475$

Therefore , the approximate root is 1.32475

Example

Find the positive root of $x - \cos x = 0$ by bisection method.

Solution :

Let $f(x) = x - \cos x$

$f(0) = 0 - \cos (0) = 0 - 1 = -1 = -ve$

$f(0.5) = 0.5 - \cos (0.5) = -0.37758 = -ve$

$f(1) = 1 - \cos (1) = 0.42970 = +ve$

So root lies between 0.5 and 1

Let x_o = (0.5 +1) /2 as initial root and proceed.

f(0.75) = *0.75 – cos (0.75)* = *0.018311* = *+ve*

So root lies between 0.5 and 0.75

x_1 = (0.5 +0.75) /2 =0.625

f(0.625) = *0.625 – cos (0.625)* = - *0.18596*

So root lies between 0.625 and 0.750

x_2 = (0.625 +0.750) /2 = 0.6875

f(0.6875) = - *0.085335*

So root lies between 0.6875 and 0.750

$x3$ = (0.6875 +0.750) /2 = 0.71875

f(0.71875) = 0.71875-cos(0.71875) = - *0.033879*

So root lies between 0.71875 and 0.750

$x4$ = (0.71875 +0.750) /2 = 0.73438

f(0.73438) = -0.0078664 = -ve

So root lies between 0.73438 and 0.750

$x5$ = 0.742190

f(0.742190) = 0.0051999 = + ve

$x6$ = (0.73438 +0.742190) /2 = 0.73829

f(0.73829) = -0.0013305

So root lies between 0.73829 and 0.74219

$x7$ = (0.73829+0.74219) = 0.7402

f(0.7402) = 0.7402-cos(0.7402) = 0.0018663

So root lies between 0.73829 and 0.7402

$x8$ = 0.73925

f(0.73925) = 0.00027593

$x9 = 0.7388$

The root is 0.7388.

Newton-Raphsonmethod (or Newton's method)

Let us suppose we have an equation of the form $f(x) = 0$ in which solution is lies between in the range (a,b). Also $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are opposite signs, then there exist atleast one real root between a and b.

Let $f(a)$ be positive and $f(b)$ negative. Which implies atleast one root exits between a and b. We assume that root to be either a or b, in which the value of f(a) or f(b) is very close to zero. That number is assumed to be initial root. Then we iterate the process by using the following formula until the value is converges.

$$X_{n+1} = Xn - \frac{f(X_n)}{f'(Xn)}$$

Steps:

1. Find a and b in which $f(a)$ and $f(b)$ are opposite signs for the given equation using trial and error method.
2. Assume initial root as $X_o = a$ i.e., if $f(a)$ is very close to zero or $Xo = b$ if $f(a)$ is very close to zero
3. Find X1 by using the formula

$$X_1 = Xo - \frac{f(X_o)}{f'(Xo)}$$

4. Find X_2 by using the following formula

$$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$$

5. Find $X_3, X_4, \ldots X_n$ until any two successive values are equal.

Example:

Find the positive root of f(x) = 2x3- 3x-6 =0 by Newton – Raphson method correct to five decimal places.

Solution:

Let $f(x) = 2x^3 - 3x - 6$; $f'(x) = 6x^2 - 3$

$f(1)$ = 2-3-6 = -7 = -ve

$f(2)$ = 16 – 6-6 = 4 = +ve

So, a root between 1 and 2 . In which 4 is closer to 0 Hence we assume initial root as 2.

Consider $x_0 = 2$

So $X_1 = X_0 - f(X_0)/f'(X_0)$

$= X_0 - ((2X_03 - 3X_0 - 6) / 6\alpha_0 - 3) = (4X_03 + 6)/(6X_02 - 3)$

$X_{i+1} = (4X_i3 + 6)/(6X_i2 - 3)$

$X_1 = (4(2)^2+6)/(6(2)^2 - 3) = 38/21 = 1.809524$

$X_2 = (4(1.809524)^3+6)/(6(1.809524)^2 - 3) = 29.700256/16.646263 = 1.784200$

$X_3 = (4(1.784200)^3+6)/(6(1.784200)^2 - 3) = 28.719072/16.100218 = 1.783769$

$X_4 = (4(1.783769)^3+6)/(6(1.783769)^2 - 3) = 28.702612/16.090991 = 1.783769$

Example:

Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$ correct to 5 decimal places.

Solution :

Let $f(x) = x^3 - 6x + 4$; $f(0) = 4$ = +ve; $f(1) = -1$ = -ve

So a root lies between 0 and 1

$f(1)$ is nearer to 0. Therefore we take initial root as $X_0 = 1$

$f'(x) = 3x^2 - 6$

$$= x - \frac{f(x)}{f'(x)}$$

$$= x - (3x^3 - 6x + 4)/(3x^2 - 6)$$

$= (2x^3 - 4)/(3x^2 - 6)$

$X_1 = (2X_0 3 - 4)/(3X_0 2 - 6) = (2-4)/(3-6) = 2/3 = 0.66666$

$X_2 = (2(2/3)^3 - 4)/(3(2/3)^2 - 6) = 0.73016$

$X_3 = (2(0.73015873)^3 - 4)/(3(0.73015873)^2 - 6)$

$= (3.22145837/ 4.40060469)$

$= 0.73205$

$X_4 = (2(0.73204903)^3 - 4)/(3(0.73204903)^2 - 6)$

$= (3.21539602/ 4.439231265)$

$= 0.73205$

The root is 0.73205 correct to 5 decimal places.

Method of False Position (or RegulaFalsi Method)

Consider the equation *f(x) = 0* and *f(a)* and *f(b) are*of opposite signs. Also let a< b.

The graph *y = f(x) will* Meet the x-axis at some point between A(a, f(a)) and B (b,f(b)). The equation of the chord joining the two points A(a, f(a)) and B (b,f(b)) is

$$\frac{y - f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

The x- Coordinate of the point of intersection of this chord with the x-axis gives an approximate value for the of f(x) = 0. Taking y = 0 in the chord equation, we get

$$\frac{- f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

$x[f(a) - f(b)] - a f(a) + a f(b) = -a f(a) + b f(b)$

$x[f(a) - f(b)] = b f(a) - a f(b)$

This x_1 gives an approximate value of the root f(x) = 0. $(a < x_1 < b)$

Now $f(x_1)$ and f(a)are of opposite signs or $f(x_1)$ and f(b) are opposite signs.

If $f(x_1)$, f(a) <0 . then x2 lies between x_1 and a.

Therefore $$x_2 = \frac{a\, f(x_1) - x_1\, f(b)}{f(x_1) - f(a)}$$

This process of calculation of ($x_3, x_4, x_5, \ldots$) is continued till any two successive values are equal and subsequently we get the solution of the given equation.

Steps:

1. Find *a* and *b* in which *f(a)* and *f(b)* are opposite signs for the given equation using trial and error method.
2. Therefore root lies between *a* and *b* if *f(a)* is very close to zero select and compute x_1 by using the following formula:

$$x_1 = \frac{a\, f(b) - b\, f(a)}{f(b) - f(a)}$$

3.If $f(x_1)$, *f(a)* <0 . then root lies between x_1 and *a*. Compute x_2 by using the following formula:

$$x2 = \frac{a\, f(x_1) - x_1\, f(b)}{f(x_1) - f(a)}$$

4. Calculate the values of ($x_3, x_4, x_5, \ldots$) by using the above formula until any two successive values are equal and subsequently we get the solution of the given equation.

. **Example:**

Solve for a positive root of x^3-4x+1=0 by and RegulaFalsi method

Solution :

Let f(x) = x^3-4x+1 = 0

$f(0) = 0^3 - 4(0) + 1 = 1 = +ve$

$f(1) = 1^3 - 4(1) + 1 = -2 = -ve$

So a root lies between 0 and 1

We shall find the root that lies between 0 and 1.

Here a=0, b=1

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{(0 \times f(1) - 1 \times f(0))}{(f(1) - f(0))}$$

$$= \frac{-1}{(-2 - 1)}$$

$$= 0.333333$$

$f(x_1) = f(1/3) = (1/27) - (4/3) + 1 = -0.2963$

Now f(0) and f(1/3) are opposite in sign.

Hence the root lies between 0 and 1/3.

$$x_2 = \frac{(0 \times f(1/3) - 1/3 \times f(0))}{(f(1/3) - f(0))}$$

$x_2 = (-1/3)/(-1.2963) = 0.25714$

Now $f(x_2) = f(0.25714) = -0.011558 = -ve$

So the root lies between 0 and 0.25714

$x_3 = (0 \times f(0.25714) - 0.25714 \times f(0)) / (f(0.25714) - f(0))$

$= -0.25714/-1.011558 = 0.25420$

$f(x_3) = f(0.25420) = -0.0003742$

So the root lies between 0 and 0.25420

x_4= (0xf(0.25420)- 0.25420 xf(0))/ (f(0.25420) – f(0))

= -0.25420 / -1.0003742 = 0.25410

$f(x_4)$ = f(0.25410) = - 0.000012936

The root lies between 0 and 0.25410

x_5= (0xf(0.25410) - 0.25410 xf(0))/ (f(0.25410) – f(0))

= -0.25410 / -1.000012936 = 0.25410

Hence the root is 0.25410.

Example:

Find an approximate root of x $\log_{10}x$ – 1.2 = 0 by False position method.

Solution :

Let f(x) = x $\log_{10}x$ – 1.2

f(1) = -1.2 = -ve; f(2) = 2x 0.30103 -1.2 = - 0.597940

f(3) = 3x0.47712 – 1.2 = 0.231364 = +ve

So, the root lies between 2 and 3.

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2 \times 0.23136 - 3 \times (-0.59794)}{0.23136 + 0.597}$$

= 2.721014

$f(x_1)$ = f(2.7210) = - 0.017104

The root lies between x_1 and 3.

$$x_2 = \frac{x_1 \, xf(3) - 3 \, xf(x_1)}{f(3) - f(x_1)} = \frac{2.721014 \times 0.231364 - 3 \times (-0.017104)}{0.23136 + 0.017104}$$

= 2.740211

$f(x_2) = f(2.7402) = 2.7402 \times \log(2.7402) - 1.2$

$= -0.00038905$

So the root lies between 2.740211 and 3

$$x_3 = \frac{2.7402 \times f(3) - 3 \times f(2.7402)}{f(3) - f(2.7402)} = \frac{2.7402 \times 0.231336 + 3 \times (0.00038905)}{0.23136 + 0.00038905}$$

$$= \frac{0.63514}{0.23175} = 2.740627$$

$f(2.7406) = 0.00011998$

So the root lies between 2.740211 and 2.740627

$$x_4 = \frac{2.7402 \times f(2.7406) - 2.7406 \times f(2.7402)}{f(2.7406) - f(2.7402)}$$

$$= \frac{2.7402 \times 0.00011998 + 2.7406 \times 0.00038905}{0.00011998 + 0.00038905}$$

$$= \frac{0.0013950}{0.00050903}$$

$= 2.7405$

Hence the root is 2.7405

Horner's Method

This numerical methods is employed to determine both the commensurable and the incommensurable real roots of a numerical polynomial equation. Firstly, we find the integral part of the root and then by every iteration. We find each decimal place value in succession.

Suppose a positive root of f(x) = 0 lies between a and a+1.
Let that root be a,a1a2a3….

First diminish the root of f(x)-0 by the integral part a and let $\phi1(x) = 0$ possess the root 0.a1a2a3…

Secondly , multiply the roots of $\phi1(x) = 0$ by 10 and let $\phi2(x) = 0$ possess the root a1.a2a3…as a root.

Thirdly, find the value od a1 and then diminish the roots by a1 and let $\phi3(x) = 0$ possess a root 0.a2a3…

Now repeating the process we find a2,a3,a4…. each time.

Example:

Find the positive root of $x^3 + 3x - 1 = 0$, correct to two decimal places by Horner's method.

Solution:
Let $f(x) = x^3 + 3x - 1 = 0$

$f(0)$ = -ve $f(1)$ = * ve.

The positive root lies between 0 and 1.

Let it be 0.a1a2a3….

Since the integral part is zero, diminishing the root by the integral part is not necessary. Therefore multiply the roots by 10.

Therefore $\phi 1(x) = x^3 + 300x - 1000 = 0$ has root a1.a2a3…

$\phi 1(3)$ = -ve, $\phi 1(4)$ = +ve

Therefore a1=3

Now, the root is 3.a2a3…

Therefore, diminish root of $\phi 1(x) = 0$ by 3

By synthetic division method, we get

$\phi 2(x) = x^3 + 9x^2 + 327x - 73 = 0$ has root 0.a2a3…

Multiply the roots of $\phi 2(x) = 0$ by 10.

$\phi 3(x) = x^3 + 90x^2 + 32700x - 73000 = 0$ has root a2.a3a4…
Now, $\phi 3(2)$ = -ve, $\phi 3(3)$ = +ve

Therefore a2=2

Now diminish the roots of $\phi 3(x)$ by 2.

By synthetic division method, we get

$\phi 4(x) = x^3 + 96x^2 + 33072x - 7232 = 0$ has root 0.a3a4…

Multiply the roots of $\phi4(x) = 0$ by 10.

$\phi5(x) = x^3 +960x^2 + 3307200x -7232000 =0$ has root a3.a4…

Now, $\phi5(2)$ = -ve, $\phi5(3)$ = +ve

Therefore a3=2

Hence the root is 0.322.

Graeffe's Root SquaringMethod

This is a direct method to find the roots of any polynomial equation with real coefficients. The basic idea behind this method is to separate the roots of the equations by squaring the roots. This can be done by separating even and odd powers of **x** in

$$P_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + . . . + a_{n-1}x + a_n= 0$$

and squaring on both sides. Thus we get,

$$(x^n + a_2 x^{n-2} + a_4 x^{n-4} + . . .)^2= (x^n + a_1 x^{n-1} + a_3 x^{n-3} + . . .)^2$$

$$x^{2n}{}_n - (a_1^2- 2a_2)x^{2n-2} + (a_2^2- 2a_1a_3 + 2a_4)x^{2n-4} +. . . +(-1)^n a_n^2 = 0$$

substituting **y** for $-x^2$ we have

$$y^n +b_1y^{n-1} + . . . + b_{n-1}y + b_n = 0$$

where

$$b_1 = a_1^2- 2a_2$$

$$b_2 = a_2^2- 2a_1a_3 + 2a_4$$

.

.

.

$$b_n = a_n^2$$

Thus all the b_i's ($i = 0, 1, 2, . . . n$) are known in terms of a_i's. The roots of this equation are $-s_1^2,. -s_2^2, . . ., s_n^2$ where $s_1, s_2 , . . ., s_n$ are the roots of $P_n (x) = 0$.

A typical coefficient b_k of $b_i , i = 1, 2, . . .n$ is obtained by following .The terms alternate in sign starting with a +ve sign. The first term is the square of the coefficient a_k .

The second term is twice the product of the nearest neighbouring coefficients a_{i-1} and a_{i+1}. The third is twice the product of the next neighbouring coefficients a_{i-2} and a_{i+2}. This procedure is continued until there are no available coefficients to form the cross products.

This procedure can be repeated many times so that the final equation
$x^n + B_1 x^{n-1} + \ . \ . \ . + B_{n-1}x + B_n = 0$ has the roots $R_1, R_2, \ldots, R_n$ such that $R_i = -s_i^{(2^m)}$
, $i = 1, 2, \ldots, m$

if we repeat the process for **m** times.

If we assume $|s_1| > |s_2| > \ldots |s_n|$ then $|R_1| >> |R_2| >> \ldots >> |R_n|$

that is the roots R_i are very widely separated for large **m**.

Now we have $-B_1 = \Sigma R_i \Sigma R_1$
$B_2 = \Sigma R_i R_j \Sigma R_1 R_2$
$-B_3 = \Sigma R_i R_j R_k \Sigma R_1 R_2 R_3$

.

.

.

$(-1)^n B_n = R_1 R_2 \ldots R_n$

which gives $R_i = -B_i / B_{i-1}$, $i = 1, 2, \ldots n$

where $B_0 = 1$.

since $|s_i|^{2^m} = |R_i|$ $i = 1, 2, \ldots n$

$\Sigma |s_i| = |R_i|^{2^{-m}}$ $i = 1, 2, \ldots n$

This determines the absolute values of the roots and substitution in the original equation will give the sign of the roots.

Example :

Find the roots of $x^3 - 7x^2 + 14x - 8 = 0$

a[]	1	-7	14	-8
b[]	1	21	84	64
roots =	**4.583**	**2**	**0.873**	
b[]	1	273	4368	4096
roots =	**4.065**	**2**	**0.984**	
b[]	1	65793	1.68E7	1.68E7
roots =	**4.002**	**2**	**0.9995**	

Thus the absolute values of the roots are **4, 2, 1**.

Since **f(1) = 0**, **f(2) = 0** and **f(4) = 0**, the signs of the roots **1, 2** and **4** are all positive.

UNIT-I

SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Part B (5x8=40 Marks)

Possible Questions

1. Find the positive root of x – cos x = 0 by using bisection method.

2.Find the positive root of $e^x = 3x$ by using Bisection method.

3. Solve the equation $x^3 + x^2 - 1 = 0$ for the positive root by iteration method.

4. Find the real root of the equation cos x = 3x – 1 correct to 4 decimal places by iteration method.

5. Find an approximate root of $x \log_{10} x = 1.2$ by False position method.

6. Find an approximate root of $x^3 - 4x + 1 = 0$ by False position method.

7. Find the real positive root of 3x-cos x –1 =0 by Newton's method correct to 3 decimal places.

8.Find the positive root of $x^3 + 3x - 1 = 0$,correct to two decimal places, by Horner's method.

9. Find all the roots of the equation $2x^3 + x^2 - 2x - 1 = 0$ by Graeffe's method (four squaring) .

10. Find all the roots of the equation $x^3 - 9x^2 + 18x - 6 = 0$ by Graeffe's method (root squaring, three times).

-

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Class : III – B.Sc. Physics **Semester : V** **L T P C**

Subject: Numerical Methods **Subject Code: 15PHU505A** **5 0 0 5**

UNIT-III

SOLUTIONS OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS

INTRODUCTION

We will study here a few methods below deals with the solution of simultaneous Linear Algebraic Equations

GAUSS ELIMINATION METHOD (DIRECT METHOD).

This is a direct method based on the elimination of the unknowns by combining equations such that the *n* unknowns are reduced to an equation upper triangular system which could be solved by back substitution.

Consider the *n* linear equations in *n* unknowns, viz.

$$a_{11}x_1+a_{12}x_2+....+a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+....+a_{2n}x_n=b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1+a_{n2}x_2+.....+a_{nn}x_n=b_n \quad ....(1)$$

Where a_{ij} and $b_i$are known constants and x_i's are unknowns.

The system (1) is equivalent to $AX=B$ .....(2)

Where $A = \begin{pmatrix} a_{11}\ a_{12}\ ....a_{1n} \\ a_{21}\quad a_{22}\,.........\ a_{2n} \\ ....\ ....\ .... \\ a_{n1}\quad a_{n2}\ .....\ a_{nn} \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Now our aim is to reduce the augmented matrix (A,B) to upper triangular matrix.

$$(A,B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \quad ..(3)$$

Now, multiply the first row of (3) (if $a_{11} \neq 0$) by $-\frac{a_{i1}}{a_{11}}$ and add to the ith row of (A,B), where i=2,3,…,n. By thia, all elements in the first column of (A,B) except a_{11} are made to zero. Now (3) is of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & \cdots & b_{2n} & c_2 \\ \cdots & \cdots & \cdots & & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} & c_n \end{array}\right) \quad …….(4)$$

Now take the pivot b_{22}. Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second row of (4) by $-\frac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row (i=3,4,…,n). Now all elements below b_{22} are reduced to zero. Now (4) reduces to

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} & c_2 \\ 0 & 0 & c_{23} & \cdots & c_{3n} & d_3 \\ \cdots & \cdots & \cdots & \cdots & & \vdots \\ 0 & 0 & c_{n3} & \cdots & c_{nn} & d_n \end{array}\right) \quad ……(5)$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get (A,B) after all these operations as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \ldots\ldots\ldots\ldots & & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \ldots\ldots\ldots\ldots & & b_{2n} & c_2 \\ 0 & 0 & c_{23} & c_{34}\ldots\ldots\ldots & & c_{3n} & d_3 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots\ldots & c_{nn} & d_n \end{array}\right) \quad \ldots(6)$$

From, (6) the given system of linear equations is equivalent to

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots.+a_{1n}x_n=b_1$$

$$b_{22}x_2+b_{23}x_3+\ldots.+b_{2n}x_n=c_2$$

$$c_{33}x_3+\ldots.+c_{3n}x_n=d_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$\alpha_{nn}x_n=k_n$$

Going from the bottom of these equation, we solve for $x_n=\frac{k_n}{\alpha_{nn}}$. Using this in the penultimate equation, we get $x_{n-1}$and so. By this back substitution method for we solve x_n , x_{n-1}, $x_{n-2}, \ldots\ldots x_2, x_1$.

GAUSS – JORDAN ELIMINATION METHOD (DIRECT METHOD)

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system $AX=B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making the matrix A not above the leading diagonal of A also as zeros. By this way, the system $AX=B$ will reduce to the form.

$$\left(\begin{array}{cccccc|c} a_{11} & 0 & 0 & 0\ldots\ldots\ldots\ldots & & a_{1n} & b_1 \\ 0 & b_{22} & 0 & 0\ldots\ldots\ldots\ldots & & b_{2n} & c_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots & d_3 \\ 0 & 0 & 0 & 0 & \ldots\ldots\ldots\ldots & \alpha_{nn} & k_n \end{array}\right) \ldots(7)$$

From (7)

$x_n = \frac{k_n}{\alpha_{nn}}$, ……, $x_2 = \frac{c_2}{b_{22}}$, $x_n = \frac{b_1}{a_{11}}$

Note: By this method, the values of $x_1, x_2, \ldots x_n$ are got immediately without using the process of back substitution.

Example 1. *Solve the system of equations by (i) Gauss elimination method (ii) Gauss – Jordan method.*

x+2y+z=3, 2x+3y+3z=10, 3x-y+2z=13.

Solution. (By Gauss method)

This given system is equivalent to

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

$A\ X = B$

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now, we will make the matrix A upper triangluar.

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) R_2+(-2)R_1,\ R_3+(-3)R_1$$

Now, take b_{22}=-1 as the pivot and make b_{32}as zero.

$$(A,B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] R_{32}(-7) \ \ldots\ldots\ldots.(2)$$

From this, we get

$x+2y+z = 3$, $-y+z = 4$, $-8z = -24$

$\therefore$ $z = 3$, $y = -1$, $x = 2$ by back substitution.

$x = 2$, $y = -1$, $z = 3$

Solution. (Gauss – Jordan method)

In stage 2, make the element, in the position (1,2), also zero.

$$(A,B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_{12}(2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_3\left(\frac{1}{8}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_{13}(3),\ R_{23}(1)$$

i.e., $x = 2$, $y = -1$, $z = 3$

METHOD OF TRIANGULARIZATION (OR METHOD OF FACTORIZATION) (*DIRECT METHOD*)

This method is also called as *decomposition* method. In this method, the coefficient matrix A of the system $AX = B$, decomposed or factorized into theproduct of a lower triangular matrix L and an upper triangular matrix$U$. we will explain this method in the case of three equations in three unknowns.

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system is equivalent to $AX = B$

Where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

And an upper triangular matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$ so that

$LUX = B$ Let $UX = Y$ And hence $LY = B$

That is, $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\therefore y_1 = b$, $l_{21}y_1+y_2 = b_2$, $l_{31}y_1+l_{32}y_2+y_3 = b_3$

By forward substitution, y_1, y_2, y_3 can be found out if L is known.

From (4), $\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$, $u_{22}x_2 + u_{23}x_3 = y_2$ and $u_{33}x_3 = y_3$

From these, x_1, x_2, x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known. Now L and U can be found from $LU = A$

i.e., $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

i.e.,

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$l_{31}u_{11}$ $l_{31}u_{12}+l_{32}u_{22}$ $l_{31}u_{13}+l_{32}u_{23}+u_{33}$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3 l's and 6 u's.

That is, L and U re known. Hence X is found out. Going into details, we get $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$. That is the elements in the first rows of U are same as the elements in the first of A.

Also, $l_{21}u_{11} = a_{21}$ $l_{21}u_{12}+u_{22} = a_{22}$ $l_{21}u_{13}+u_{23} = a_{23}$

$$l_{21} = \frac{a_{21}}{a_{11}},\quad u_{22} = a_{22} - \frac{a_{21}}{a_{11}}.a_{12} \text{ and } u_{23} = a_{23} - \frac{a_{21}}{a_{11}}.a_{13}$$

again, $l_{31}u_{11} = a_{31}$, $l_{31}u_{12}+l_{32}u_{22} = a_{32}$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = a_{33}$

$$\text{solving,}\quad l_{31} = \frac{a_{31}}{a_{11}},\ l_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}}.a_{12}}{a_{22} - \frac{a_{21}}{a_{11}}.a_{12}}$$

$$u_{33} = \left(a_{33} - \frac{a_{31}}{a_{11}}.a_{13}\right) - \left(\frac{a_{32} - \frac{a_{31}}{a_{11}}.a_{12}}{a_{22} - \frac{a_{21}}{a_{11}}.a_{12}}\right) a_{33} - \frac{a_{31}}{a_{11}}.a_{13}$$

Therefore L and U are known.

Example 2 *By the method of triangularization, solve the following system.*

$5x - 2y + z = 4,\ \ 7x + y - 5z = 8,\ \ 3x + 7y + 4z = 10.$

Solution. The system is equivalent to

$$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$A \quad X \quad = \quad B$

Now, let $LU = A$

That is, $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$$

Multiplying and equating coefficients,

$$u_{11} = 5,\quad u_{12} = -2,\quad u_{13} = 1$$

$l_{21}u_{11} = 7 \quad l_{21}u_{12}+u_{22} = 1 \quad l_{21}u_{13}+u_{23} = -5$

$l_{21} = \frac{7}{5}$, $u_{22} = 1 - \frac{7}{5}.(-2) = \frac{19}{5}$ and

$u_{23} = -5 - \frac{7}{5}.(1) = -\frac{32}{5}$

Again equating elements in the third row,

$l_{31}u_{11} = 3$, $l_{31}u_{12}+l_{32}u_{22} = 7$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = 4$

$$\therefore \quad l_{31} = \frac{3}{5}, \quad l_{32} = \frac{7 - \frac{3}{5}.(-2)}{\frac{19}{5}} = \frac{41}{19}$$

$$u_{33} = 4 - \frac{3}{5}.(1) - \frac{41}{19}\left(-\frac{32}{5}\right) = 4 - \frac{3}{5} + \frac{1312}{95}$$

$$= \frac{1635}{95} = \frac{327}{19}$$

Now L and U are known. Since $LUX = B$, $LY = B$ where $UX = Y$.

From $LY = B$,

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$y_1 = 4$, $\frac{7}{5} y_1+y_2 = 8$, $\frac{3}{5}y_1+ \frac{41}{19} y_2+y_3 = 10$

$$y_2 = 8 - \frac{28}{5} = \frac{12}{5}$$

$$y_3 = 10 - \frac{12}{5} - \frac{41}{19} \times \frac{12}{5} = 10 - \frac{12}{5} - \frac{492}{95} = \frac{46}{19}$$

$UX = Y$ gives $\begin{pmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{pmatrix}$

$5x - 2y + z = 4$

$\frac{19}{5} y - \frac{32}{5} z = \frac{12}{5}$

$\frac{327}{19} z = \frac{46}{19}$

$z = \frac{46}{327}$

$\frac{19}{5} y = \frac{12}{5} + \frac{32}{5}\left(\frac{46}{327}\right)$

$y = \frac{284}{327}$

$5x = 4 + 2y - z = 4 + 2\left(\frac{568}{327}\right) - \frac{46}{327}$

$\therefore \quad x = \frac{366}{327}$

$\therefore \quad x = \frac{366}{327}, \; y = \frac{284}{327}, \; z = \frac{46}{327}$

GAUSS – SEIDEL METHOD OF ITERATION:

This is only a refinement of Guass – Jacobi method. As before,

$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$

$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

We start with the initial values y^0, z^0 for y and z and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)})$$

While using the second equation, we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of x^0 as in Jacobi's method, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(0)})$$

Now, having known $x^{(1)} and\ y^{(1)}$, use $x^{(1)}$ for x $and\ y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ are the rth iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1y^{(r)} - c_1z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2x^{(r+1)} - c_2z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3x^{(r+1)} - b_3y^{(r+1)})$$

This process of iteration is continued until the convergence assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss – seidel method is very fast when compared to Gauss – Jacobi method. The rate of convergence in Gauss – Seidel method is roughly two times than that of Gauss – Jacobi method. As we saw the sufficient condition already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is, *the method of iteration will converge if in each equation of the given system, the absolute value of the largest* coefficient is greater than the sum of the absolute values of all the remaining coefficients. (The largest coefficients must be the coefficients for different unknowns).

Example 3 Solve the following system by Gauss – Jacobi and Gauss – Seidel methods:

10x-5y-2z = 3 ; 4x-10y+3z = -3 ; x+6y+10z = -3.

Solution: Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix}$ is diagonally dominant, since $|10| > |-5| + |-2|$.

$|-10| > |4| + |3|$.

$|10| > |1| + |6|$

Gauss – Jacobi method, solving for *x*, *y*, *z* we have

$x = \frac{1}{10}(3+5y+2z)$ ………………. (1)

$y = \frac{1}{10}(3+4x+3z)$ ………………. (2)

$z = \frac{1}{10}(-3-x-6y)$ ………………. (3)

First iteration: Let the initial values be (0, 0, 0).

Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}(3+5(0)+2(0)) = 0.3$$

$$y^{(1)} = \frac{1}{10}(3+4(0)+3(0)) = 0.3$$

$$z^{(1)} = \frac{1}{10}(-3-(0)-6(0)) = -0.3$$

Second iteration: using these values in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}(3+5(0.3)+2(-0.3)) = 0.39$$

$y^{(2)} = \frac{1}{10}(3+4(0.3)+3(-0.3)) = 0.33$ $z^{(2)} = \frac{1}{10}(-3-(0.3)-6(0.3)) = -0.51$

Third iteration: using these values of $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ in (1), (2), (3), we get,

$x^{(3)} = \frac{1}{10}(3+ 5(0.33) + 2(-0.51)) = 0.363$

$y^{(3)} = \frac{1}{10}(3+ 4(0.39)+ 3(-0.51)) = 0.303$

$z^{(3)} = \frac{1}{10}(-3- (0.39)- 6(0.33)) = - 0.537$

Fourth iteration:

$x^{(4)} = \frac{1}{10}(3+ 5(0.303) + 2(-0.537)) = 0.3441$

$y^{(4)} = \frac{1}{10}(3+ 4(0.363)+ 3(-0.537)) = 0.2841$

$z^{(4)} = \frac{1}{10}(-3- (0.363)- 6(0.303)) = -0.5181$

Fifth iteration:

$x^{(5)} = \frac{1}{10}(3+ 5(0.2841) + 2 (-0.5181)) = 0.33843$

$y^{(5)} = \frac{1}{10}(3+ 4(0.3441)+ 3(-0.5181)) = 0.2822$

$z^{(5)} = \frac{1}{10}(-3- (0.3441)- 6(0.2841)) = - 0.50487$

Sixth iteration:

$x^{(6)} = \frac{1}{10}(3+ 5(0.2822)+ 2 (-0.50487)) = 0.340126$

$y^{(6)} = \frac{1}{10}(3+ 4(0.33843)+ 3(-0.50487)) = 0.283911$

$z^{(6)} = \frac{1}{10}(-3- (0.33843)- 6(0.2822)) = - 0.503163$

Seventh iteration:

$x^{(7)} = \frac{1}{10}(3+ 5(0.283911)+2(-0.503163))$ =0.3413229

$y^{(7)} = \frac{1}{10}(3+ 4(0.340126)+3(-0.503163))$ = 0.2851015

$z^{(7)} = \frac{1}{10}(-3- (0.340126)- 6(0.283911))$ = - 0.5043592

Eighth iteration:

$x^{(8)} = \frac{1}{10}(3+ 5(0.2851015)+2 (-0.5043592))$

=0.34167891

$y^{(8)} = \frac{1}{10}(3+ 4(0.3413229)+ 3(-0.5043592))$

= 0.2852214

$z^{(8)} = \frac{1}{10}(-3-(0.3413229)- 6(0.2851015))$

= - 0.50519319

Ninth iteration:

$x^{(9)} = \frac{1}{10}(3+ 5(0.2852214)+ 2 (-0.50519319))$

= 0.341572062

$y^{(9)} = \frac{1}{10}(3+ 4(0.34167891)+ 3(-0.50519319))$

= 0.285113607

$z^{(9)} = \frac{1}{10}(-3- (0.34167891)- 6(0.2852214))$ = - 0.505300731

Hence, correct to 3 decimal places, the values are

x = 0.342, y = 0.285, z = -0.505

Gauss – seidel method: Initial values : y = 0, z = 0.

First iteration: $x^{(1)} = \frac{1}{10}(3 + 5(0) + 2(0)) = 0.3$

$y^{(1)} = \frac{1}{10}(3 + 4(0.3) + 3(0)) = 0.42$

$z^{(1)} = \frac{1}{10}(-3 - (0.3) - 6(0.42)) = -0.582$

Second iteration:

$x^{(2)} = \frac{1}{10}(3 + 5(0.42) + 2(-0.582)) = 0.3936$

$y^{(2)} = \frac{1}{10}(3 + 4(0.3936) + 3(-0.582)) = 0.28284$

$z^{(2)} = \frac{1}{10}(-3 - (0.3936) - 6(0.28284)) = -0.509064$

Third iteration:

$x^{(3)} = \frac{1}{10}(3 + 5(0.28284) + 2(-0.509064)) = 0.3396072$ $y^{(3)} = \frac{1}{10}(3+4(0.3396072)+3(-0.509064)) = 0.28312368$

$z^{(3)} = \frac{1}{10}(-3 - (0.3396072) - 6(0.28312368))$

$= -0.503834928$

Fourth iteration:

$x^{(4)} = \frac{1}{10}(3 + 5(0.28312368) + 2(-0.503834928))$

$= 0.34079485$

$y^{(4)} = \frac{1}{10}(3 + 4(0.34079485) + 3(-0.503834928))$

$= 0.285167464$

$z^{(4)} = \frac{1}{10}(-3 - (0.34079485) - 6(0.285167464))$

$= -0.50517996$

Fifth iteration:

$x^{(5)} = \frac{1}{10}(3+ 5(0.285167464) + 2(-0.50517996))$

$= 0.34155477$

$y^{(5)} = \frac{1}{10}(3+ 4(0.34155477)+ 3(-0.50517996))$

$= 0.28506792$

$z^{(5)} = \frac{1}{10}(-3- (0.34155477)– 6(0.28506792))$

$= - 0.505196229$

Sixth iteration:

$x^{(6)} = \frac{1}{10}(3+ 5(0.28506792) + 2(-0.505196229))$

$= 0.341494714$

$y^{(6)} = \frac{1}{10}(3+ 4(0.341494714)+ 3(-0.505196229))$

$= 0.285039017$

$z^{(6)} = \frac{1}{10}(-3- (0.341494714)– 6(0.28506792))$

$= - 0.5051728$

Seventh iteration:

$x^{(7)} = \frac{1}{10}(3+ 5(0.285039017) + 2(-0.5051728))$

$= 0.3414849$

$y^{(7)} = \frac{1}{10}(3+ 4(0.3414849)+ 3(- 0.5051728))$

$= 0.28504212$

$z^{(7)} = \frac{1}{10}(-3- (0.3414849)– 6(0.28504212))$

$= - 0.5051737$

The values at each iteration by both methods are tabulated below:

Iteration	Gauss - jacobi method			Gauss – seidel method		
	x	*y*	*z*	*x*	*y*	*z*
1	0.3	0.3	-0.3	0.3	0.42	-0.582
2	0.39	0.33	-0.51	0.3936	0.2828	-0.5090
3	0.363	0.303	-0.537	0.3396	0.2831	-0.5038
4	0.3441	0.2841	-0.5181	0.3407	0.2851	-0.5051
5	0.3384	0.2822	-0.5048	0.3415	0.2850	-0.5051
6	0.3401	0.2839	-0.5031	0.3414	0.2850	-0.5051
7	0.3413	0.2851	-0.5043	0.3414	0.2850	-0.5051
8	0.3416	0.2852	-0.5051			
9	0.3411	0.2851	-0.5053			

The values correct to 3 decimal places are

$$x = 0.342, \ \ y = 0.285, \ \ z = -0.505$$

UNIT-II

SOLUTIONS OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS

Part -B (5x8=40 Marks)

Possible Questions

1. Solve the following system by Gauss elimination method.

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

2. Solve the following system by Gauss elimination method

$$x + y + z + w = 2$$
$$2x - y + 2z - w = -5$$
$$3x + 2y + 3z + 4w = 7$$
$$x - 2y - 3z + 2w = 5$$

3. Solve the following system by Gauss Jordan method

$$10x + y + z = 12$$
$$x + 10y + z = 12$$
$$x + y + 10z = 12$$

4. Solve the following system by Gauss Jordan method

$$x + y + 2z = 4$$
$$3x + y - 3z = -4$$
$$2x - 3y - 5z = -5$$

5. Solve the following system by triangularisation method.

$$5x - 2y + z = 4$$
$$7x + y - 5z = 8$$
$$3x + 7y + 4z = 10$$

6. Solve the following system by triangularisation method.

$$5x - 2y + z = 4$$
$$7x + y - 5z = 8$$
$$3x + 7y + 4z = 10$$

7. Solve the following system of equations by Crout's method.

$$2x + 3y + z = -1$$
$$5x + y + z = 9$$
$$3x + 2y + 4z = 11$$

8.Solve the following system of equations by Gauss-Jacobi method

$10x - 5y - 2z = 3$

$4x - 10y + 3z = -3$

$x + 6y + 10z = -3$

9. Solve the following system of equations by Gauss-seidal method

$28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$

10. Solve the following system of equations by Gauss-Seidel method

$8x - 3y + 2z = 20$

$4x + 11y - z = 33$

$6x + 3y + 12z = 35$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Class : III – B.Sc. Physics **Semester : V** **L T P C**

Subject : Numerical Methods **Subject Code: 15PHU505A 5 0 0 5**

UNIT IV

First differences - difference tables - properties of the operator A.E.D. Linear interpolation: Newton forward interpolation formula and backward interpolation formula - Bessel's Formula. Interpolation with unequal intervals: Lagrange's interpolation formula.

First Differences:

Let y=f(x) be a give function of x and let y0,y1,y2……yn be the values of y corresponding to x0,x1,x2…..xn

The values of x. the independent variable x is called the argument and the corresponding dependent value y is called the entyr. In general the difference between any two consecutive values of x need not be same or equal.

Forward, backward, and central differences

Only three forms are commonly considered: forward, backward, and central differences.

A **forward difference** is an expression of the form

$$\Delta_h[f](x) = f(x+h) - f(x).$$

Depending on the application, the spacing *h* may be variable or constant.

A **backward difference** uses the function values at *x* and *x* − *h*, instead of the values at *x* + *h* and *x*:

$$\nabla_h[f](x) = f(x) - f(x-h).$$

Finally, the **central difference** is given by

$$\delta_h[f](x) = f(x+\tfrac{1}{2}h) - f(x-\tfrac{1}{2}h).$$

Relation with derivatives

The derivative of a function *f* at a point *x* is defined by the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

If *h* has a fixed (non-zero) value, instead of approaching zero, then the right-hand side is

$$\frac{f(x+h) - f(x)}{h} = \frac{\Delta_h[f](x)}{h}.$$

Hence, the forward difference divided by *h* approximates the derivative when *h* is small. The error in this approximation can be derived from Taylor's theorem. Assuming that *f* is continuously differentiable, the error is

$$\frac{\Delta_h[f](x)}{h} - f'(x) = O(h) \quad (h \to 0).$$

The same formula holds for the backward difference:

$$\frac{\nabla_h[f](x)}{h} - f'(x) = O(h).$$

However, the central difference yields a more accurate approximation. Its error is proportional to square of the spacing (if *f* is twice continuously differentiable):

$$\frac{\delta_h[f](x)}{h} - f'(x) = O(h^2).$$

The main problem with the central difference method, however, is that oscillating functions can yield zero derivative. If f(nh)=1 for n uneven, and f(nh)=2 for n even, then f'(nh)=0 if it is calculated with the central difference scheme. This is particularly troublesome if the domain of f is discrete.

Higher-order differences

In an analogous way one can obtain finite difference approximations to higher order derivatives and differential operators. For example, by using the above central difference formula for $f'(x + h / 2)$ and $f'(x - h / 2)$ and applying a central difference formula for the derivative of f' at *x*, we obtain the central difference approximation of the second derivative of *f*:

2nd Order Central

$$f''(x) \approx \frac{\delta_h^2[f](x)}{h^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Similarly we can apply other differencing formulas in a recursive manner. **2nd Order Forward**

$$f''(x) \approx \frac{\Delta_h^2[f](x)}{h^2} = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}.$$

More generally, the n^{th}-order forward, backward, and central differences are respectively given by:

$$\Delta_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n-i)h),$$

$$\nabla_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x - ih),$$

$$\delta_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\left(x + \left(\frac{n}{2} - i\right) h\right).$$

Note that the central difference will, for odd *n*, have *h* multiplied by non-integers. This is often a problem because it amounts to changing the interval of discretization. The problem may be remedied taking the average of $\delta^n[f](x - h/2)$ and $\delta^n[f](x + h/2)$.

The relationship of these higher-order differences with the respective derivatives is very straightforward:

$$\frac{d^n f}{dx^n}(x) = \frac{\Delta_h^n[f](x)}{h^n} + O(h) = \frac{\nabla_h^n[f](x)}{h^n} + O(h) = \frac{\delta_h^n[f](x)}{h^n} + O(h^2).$$

Higher-order differences can also be used to construct better approximations. As mentioned above, the first-order difference approximates the first-order derivative up to a term of order *h*. However, the combination

$$\frac{\Delta_h[f](x) - \frac{1}{2}\Delta_h^2[f](x)}{h} = -\frac{f(x+2h) - 4f(x+h) + 3f(x)}{2h}$$

approximates $f'(x)$ up to a term of order h^2. This can be proven by expanding the above expression in Taylor series, or by using the calculus of finite differences, explained below.

If necessary, the finite difference can be centered about any point by mixing forward, backward, and central differences.

Relations between Difference operators

1. We note that

$$Ef(x) = f(x+h) \quad = [f(x+h) - f(x)] + f(x) \quad = \Delta f(x) + f(x) = (\Delta + 1)f(x).$$

Thus,

$$\boxed{E \equiv 1 + \Delta} \quad \text{or} \quad \Delta \equiv E - 1.$$

2. Further, $\nabla(E(f(x)) = \nabla(f(x+h)) = f(x+h) - f(x).$ Thus,

$$(1-\nabla)Ef(x) = E(f(x)) - \nabla(E(f(x)) = f(x+h) - [f(x+h) - f(x)] = f(x).$$

Thus $E \equiv 1 + \Delta,$ gives us

$$(1-\nabla)(1+\Delta)f(x) = f(x) \quad \text{for all} \quad x.$$

So we write,

$$(1+\Delta)^{-1} = 1 - \nabla \quad \text{or} \quad \boxed{\nabla = 1 - (1+\Delta)^{-1},} \quad \text{and}$$

$$(1-\nabla)^{-1} = 1 + \Delta = E.$$

Similarly,

$$\Delta = (1-\nabla)^{-1} - 1.$$

3. Let us denote by $E^{\frac{1}{2}} f(x) = f(x + \frac{h}{2}).$ Then, we see that

$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2}) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x).$$

Thus,

$$\boxed{\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}.}$$

Recall,

$$\delta^2 f(x) = f(x+h) - 2f(x) + f(x-h) = [f(x+h) + 2f(x) + f(x-h)] - 4f(x) = 4(\mu^2 - 1) f(x).$$

So, we have,

$$\boxed{\mu^2 \equiv \frac{\delta^2}{4} + 1} \quad \text{or} \quad \boxed{\mu \equiv \sqrt{1 + \frac{\delta^2}{4}}}.$$

That is, the action of $\sqrt{1+\frac{\delta^2}{4}}$ is same as that of μ.

4. We further note that,

$$\Delta f(x) = f(x+h) - f(x) = \frac{1}{2}[f(x+h) - 2f(x) + f(x-h)] + \frac{1}{2}[f(x+h) - f(x-h)]$$

$$= \frac{1}{2}\delta^2(f(x)) + \frac{1}{2}[f(x+h) - f(x-h)]$$

5. and

$$\delta\mu f(x) = \delta\left[\frac{1}{2}\left\{f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right\}\right] = \frac{1}{2}[\{f(x+h) - f(x)\} + \{f(x) - f(x-h)\}]$$

$$= \frac{1}{2}\left[f(x+h) - f(x-h)\right].$$

6.

$$\Delta f(x) = \left[\frac{1}{2}\delta^2 + \delta\mu\right] f(x),$$

7.

8.In view of the above discussion, we have the following table showing the relations between various difference operators:

	E	Δ	∇	δ
E	E	$\Delta + 1$	$(1-\nabla)^{-1}$	$\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} + 1$
Δ	$E - 1$	Δ	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{1}{4}\delta^2}$
∇	$1 - E^{-1}$	$1-(1+\nabla)^{-1}$	∇	$-\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{1}{4}\delta^2}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ

Difference of a polynomial:

Theorem:

The n^{th} difference of a polynomial of n^{th}degree are constants.

Proof

We have a polynomial f(x), where, in fact, the x's are specific values

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_{n-2} x^2 + a_{n-1} x + a_n \quad [2.01]$$

Suppose the steps along the x axis are h. The next f(x) value at x+h is:

$$f(x+h) = a_0 (x+h)^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_{n-2} (x+h)^2 + a_{n-1} (x+h) + a_n$$

[2.02]

We recall, by definition:

$$\Delta f(x) = f(x+h) - f(x) \quad [2.03]$$

That is, the difference is equation 2.02 minus equation 2.01:

$$\Delta f(x) = \{a_0 (x+h)^n - a_0 x^n\} + \{a_1 (x+h)^{n-1} - a_1 x^{n-1}\} + \{a_2 (x+h)^{n-2} - a_2 x^{n-}$$

$$\ldots\{a_{n-2} (x+h)^2 - a_{n-2} x^2\} + \{a_{n-1} (x+h) - a_{n-1} x\}\{a_n - a_n\}$$

[2.03]

If we expand the left-hand parts of each term, we find:

$$= a_0\{x^n + n.x^{n-1} h \ldots - x^n\} + a_1\{x^{n-1} + (n-1) x^{n-2} h - x^{n-1}\} + a_2\{x^{n-2} + (n-2$$

$$\ldots a_{n-2}\{x^2 + 2xh + h^2 - x^2\} + a_{n-1}\{(x+h) - x\} + \{a_n - a_n\}$$

[2.04]

The last term, a_n cancels, leaving a new constant, $a_{n-1}h$, (which will cancel out in the 2nd difference):

$$= a_0\{n.x^{n-1} h \ldots\} + a_1\{(n-1) x^{n-2} h\} + a_2\{(n-2) x^{n-3} h \ldots\} + \ldots a_{n-2} (2xh + h^2) +$$

[2.05]

Therefore for a polynomial of degree n, step h

$$\Delta f(x) = n a_0 x^{n-1}.h + \text{terms of degree n}-2 \text{ and lower} \quad [2.06]$$

This is reminiscent of:

$$\frac{d}{dx}(x^n) = n x^{n-1} \quad [2.07]$$

Applying 2.06 again, we get:

$$\Delta^2 f(x) = n(n-1)\, a_0\, x^{n-2}.h^2 + \text{terms of degree n}-3\text{ and lower} \quad [2.07]$$

If we apply the formula 2.06 n times, we have:

$$\Delta^n f(x) = a_0\, n(n-1)(n-2)\ldots 1.\, h^n$$

Or

$$\Delta^n f(x) = a_0\, n!h^n$$

Note:

1. Of course, because this is a constant (it is independent of x), the n+1 difference and further differences will be zero, so:
   $$\Delta^{n+1} f(x) = 0$$
2. When h=1, we can write for a polynomial of degree n:
   $$\Delta^n f(x) = a_0\, n!$$

Factorial Polynomial:

A factorial polynomial looks like this:

$$f(k) \text{ or } k^{(2)} = k(k-1)$$

$$f(k) \text{ or } k^{(3)} = k(k-1)(k-2)$$

In general a factorial polynomial of degree n, (y_k or k^n) is:

$$k^{(n)} = k(k-h)(k-2h)\ldots(k-nh)(k-(n-1)h) \quad [1.01]$$

We assume that n is an integer greater than zero (A natural number).
We can call this k to the n falling (because there is a rising version!) with step h.
k to the n+1 falling is:

$$(k+1)^{(n)} = (k+1)\, k(k-h)(k-2h)\ldots(k-nh)(k-(n-1)h-h)$$

Which, simplifying the last term:

$$(k+1)^{(n)} = (k+1)\, k(k-h)(k-2h)\ldots(k-nh) \quad [1.02]$$

$k^{(0)}$ is defined as 1

Finding the First Difference

By definition, the first difference for the factorial polynomial, $k^{(n)}$, is

$\Delta k^{(n)} = (k+1)^{(n)} - k^{(n)}$ [1.03]

Substituting our values from 1.01 and 1.02 for k^{n+1} and k^{n} in 1.03:

$\Delta k^{(n)} = \left[(k+1)\,k\,(k-h)\,(k-2\,h)\ldots(k-n\,h+h)\,(k-n\,h)\right] - \left[k\,(k-h)\,(k-2\,h)\ldots\right.$

[1.04]

Factorising gives us:

$\Delta k^{(n)} = k\,(k-h)\,(k-2\,h)\ldots(k-n\,h)\,(k-n\,h)\left[(k+h)-(k-(n-1)\,h)\right]$

[1.05]

And further simplifying the final term by cancelling the x's and rounding up the h's:

$\Delta k^{(n)} = k\,(k-h)\,(k-2\,h)\ldots(k-n\,h+h)\,(k-n\,h)\left[n\,h\right]$ [1.06]

We note that, substituting n-1 for n in 1.06:

$k^{(n-1)} = k\,(k-h)\,(k-2\,h)\ldots(k-n\,h)\,(k-(n-1+1)\,h)$

Simplifying the final factor:

$k^{(n-1)} = k\,(k-h)\,(k-2\,h)\ldots(k-n\,h+h)\,(k-n\,h)$ [1.07]

First Difference and General Formula for n>0

From 1.06 substituting 1.07, we have:

$\Delta k^{(n)} = n \cdot h \cdot k^{(n-1)}$ [1.08]

So we can determine any of the differences using 1.08, for instance:

$$\Delta^2 k^{(n)} = n \cdot (n-1) \cdot h^2 \cdot k^{(n-2)}$$

$$\Delta^3 k^{(n)} = n \cdot (n-1) \cdot (n-2) \cdot h^3 \cdot k^{(n-3)}$$

In general, the mth difference is:

$\Delta^m k^{(n)} = n \cdot (n-1) \cdot (n-2)\ldots(n-m+1) \cdot h^m \cdot k^{(n-m)}$ [1.09]

This is reminiscent of differentiating using the infinitesimal calculus.

$\frac{d}{d\,x}\left(x^n\right) = n\,x^{n-1}$ [1.10]

1.08 also reminds us of similar result for regular polynomials, repeated below:

$\Delta\, f\,(x) = n\,a_0\,x^{n-1}.h + \text{terms of degree n}-2\text{ and lower}$

With regular polynomials, the difference isn't so neat as that with factorial polynomials. However, we can convert regular polynomials to factorials and obtain clearer results for differences.

Often, the factorial polynomials we use have a step of 1, or h=1, so:
$k^{(n)}$=k(k-1)(k-2)...(k-n)(k-n+1) [1.11]
And the mth difference when h=1 is:

$$\Delta^m k^{(n)} = n \cdot (n-1) \cdot (n-2) \ldots (n-m+1) \cdot k^{(n-m)}$$ ■[1.12]

UNIT-III

FINITE DIFFERENCES

Part-B (5x8=40 Marks)

Possible Questions

1.Find y(-1) if y(0) = 2, y(1) = 9, y(2) = 28, y(3) = 65, y(4) = 126, y(5) = 217.

2.Findthe 7^{th}term of thesequence 2,9,28,65,126,217 and also. Findthe General term.

3.i) ExplaintheRelationbetween Δ,E and D

ii) Find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0, ...

4. Find $\Delta^3 f(x)$ if

i) f(x) = (3x+1)(3x+4)………(3x+19)

ii) f(x) = x(3x+1)(3x+4)………(3x+19)

5.Evaluate i) $\Delta^n(e^{ax+b})$ ii) Δ^n[sin(ax+b)]

iii) Δ^n[cos(ax+b)] iv)Δ[log(ax+b)]

6.Express i) $x^4 + 3x^3 - 5x + 6x - 7$

ii) $x^3 + x^2 + x + 1$ in factorial polynomials and get their successive differences taking h = 1.

7. Estimate the production for 1964 & 1966 from the following data

Year :	1961	1962	1963	1964	1965	1966	1967
Production :	200	220	260	-	350	-	430

8. Prove that n^{th}difference of a polynomial of the n^{th} degree are constants.

9. The following table gives the values of y which is a polynomial of degree 5. It is known that y = f(3) is in error. Correct the error.

x :	0	1	2	3	4	5	6
y :	1	2	33	254	1025	3126	7777

10. Ify = f(x) is a polynomial of degree 3 and the following table gives the values of x & y. Locate and correct the wrong values of y.

x :	0	1	2	3	4	5	6
y :	4	10	30	75	160	294	490

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Class : III – B.Sc. Physics	**Semester : V**	**L T P C**
Subject: Numerical Methods	**Subject Code: 15PHU505A**	**5 0 0 5**

UNIT IV

First differences - difference tables - properties of the operator A.E.D. Linear interpolation: Newton forward interpolation formula and backward interpolation formula - Bessel's Formula. Interpolation with unequal intervals: Lagrange's interpolation formula.

Text Text Book

E Balagurusamy 1st edition 2014 numerical methods Tata Mcgraw hills

REFERENCES

Venkatraman, M.K., 1977, Numerical Methods in Science and Engineering, National publishing Company, Chennai.
Shastry, S.S, 2007, Introductory Methods of Numerical Analysis, Prentice Hall of India, Pvt. Ltd., New Delhi.
M K Jain, R K Jain, SRK Iyenger 6th edition 2014 Numerical methods for Scientific and Engineering Computation, New Age Publishers.

UNIT-IV

INTERPOLATION

Introduction

Interpolation means the process of computing intermediate values of a function a given set of tabular values of a function. Suppose the following table represents a set of values of x and y.

x : x_1 x_2 x_3............x_n

y : y_1 y_2 y_3...........y_n

We may require the value of $y = y_i$ for the given $x = x_i$, where x lies between x_0 to x_n

Let y = $f(x)$ be a function taking the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the *values* $x_0, x_1, x2, \ldots\ldots\ldots x_n$. Now we are trying to find $y = y_i$ for the given $x = x_i$ under assumption that the function f(x) is not known. In such cases , we replace f(x) by simple fan arbitrary function and let Φ(x) denotes an arbitrary function which satisfies the set of values given in the table above . The function Φ(x) is called interpolating function or smoothing function or interpolation formula.

Newton's forward interpolationformula(or) Gregory-Newton forward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots\ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots\ldots, x_n$.

Let suppose that the values of x i.e., $x_0, x_1, x_2 \ldots\ldots\ldots, x_n$ are equidistant .

$x_1 = x_0 + h$; $x_2 = x_1 + h$; and so on $x_n = x_{n-1} + h$;

Therefore $xi = x0 + i\,h$, where $i = 1,2, \ldots, n$

Let $P_n(x)$ be a polynomial of the n^{th} degree in which x is such that

$y_I = f(x_i) = P_n(x_i)$, $I = 0,1,2, \ldots. n$

Let us assume Pn(x) in the form given below

$P_n(x) = a_0 + a_1(x - x_0)^{(1)} + a_2(x - x_0)^{(2)} + \ldots\ldots\ldots + a_r(x - x_0)^{(r)} + \ldots\ldots +$

$+ \ldots\ldots \quad + \; a_n (x - x_0)^{(n)} \ldots\ldots (1)$

This polynomial contains the n + 1 constants $a_0, a_1, a_2, \ldots\ldots\ldots a_n$ can be found as follows :

$P_n(x_0) = y_0 = a_0$ (setting x = x0, in (1))

Similarly $y_1 = a_0 + a_1 (x_1 - x_0)$

$y_2 = a_0 + a_1 (x_2 - x_0) + a_2 (x_2 - x_0)$

From these, we get the values of $a_0, a_1, a_2, \ldots\ldots a_n$

i.e.,

Therefore, $a_0 = y_0$

$\Delta y_0 = y_1 - y0 = a_1 (x_1 - x_0)$

$= a_1 h$

$\Rightarrow a_1 = \Delta y_0 / h$

lly $\Rightarrow a_2 = (\Delta y_1 - \Delta y_0) / 2h^2 = \Delta^2 y_0 / 2!\, h^2$

lly $\Rightarrow a_3 = \Delta^3 y_0 / 3!\, h^3$

Putting these values in (1), we get

$$P_n(x) = = y_0 + (x - x_0)^{(1)}\Delta y_0 /h + (x - x_0)^{(2)}\Delta^2 y_0/(2!\,h^2) + \ldots\ldots + (x - x_0)^{(r)}\Delta^r y_0/(r!\,h^r) + \ldots\ldots + (x - x_0)^{(n)}\Delta^r y_0/(n!\,h^n)$$

By substituting $\frac{x - x_0}{h} = u$ *, the above equation becomes*

$$y(x_0 + uh) = y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \quad \ldots$$

By substituting $u = u^{(1)}$,

$u(u-1) = u^{(2)}$,

$u(u-1)(u-2) = u^{(3)}$, *… in the above equation, we get*

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y0 + \ldots + \frac{u^{(n)}}{n!}\Delta^n y_0$$

The above equation is known as **Gregory-Newton forward formula or Newton's forward interpolation formula.**

Note : 1. This formula is applicable only when the interval of difference is uniform.

2. This formula apply forward differences of y_0, hence this is used to interpolate the values of y nearer to beginning value of the table (i.e., x lies between x0 to ^{x}1 or x1 to x_2)

Example.

Find the values of y at x = 21 from the following data.

x: 20 23 26

x: 0.3420 0.3907

0.4384

29

0.4848

Solution.

Step 1. Since x = 21 is nearer to beginning of the table. Hence we apply Newton's forward formula.

Step 2. Construct the difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
20	0.3420	(0.3420-0.3907)		
		0.0487	(0.0477-0.0487)	
23	0.3907		-0.001	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

Step 3. Write down the formula and put the various values :

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y0 + \ldots + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u^{(1)} = (x - x_0)/h = (21 - 20)/3 = 0.3333$

$u(2) = u(u-1) = (0.3333)(0.6666)$

$$P_n(x=21) = y(21) = 0.3420 + (0.3333)(0.0487) + (0.3333)(-0.6666)(-0.001)$$
$$+ (0.3333)(-0.6666)(-1.6666)(-0.0003)$$

$= \mathbf{0.3583}$

Example: . From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46.

Age	x:	45	50	55	60	65
Premium	y:	114.84	96.16	83.32	74.48	68.48

Solution.

Step 1. Since $x = 46$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$.. Hence we apply Newton's forward formula.

Step 2. Construct the difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.12		4.00		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6.00			
65	68.48				

Step 3. Write down the formula and put the various values :

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y0 + \ldots. + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u = (x - x_0) / h$ = (46 – 45) / 5 = 01/5 = 0.2

$$P_n(x=46) = y(46) = 114.84 + [0.2\,(-18.68)] + [0.2\,(-0.8)\,(5.84)/\,3]$$

$$+ [0.2\,(-0.8)\,(-1.8)(-1.84)/6\,]$$

$$+ [0.2\,(-0.8)\,(-1.8)(-2.8)(0.68)]$$

= 114.84 – 3.7360 – 0.4672 – 0.08832 – 0.228

= **110.5257**

Example . From the following table , find the value of tan 45^0 15'

x^0:	45	46	47	48	49	50
tan x^0:	1.0	1.03553	1.07237	1.11061	1.15037	1.19175

Solution.

Step 1. Since $x = 45^o\ 15'$is nearer to beginning of the table and the values of x is equidistant i.e., $h = 1$. Hence we apply Newton's forward formula.

Step 2. Construct the difference table to find various Δ's

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
45^0	1.0000					
		0.03553				
460	1.03553		0.00131			
		0.03684		0.00009		
47^0	1.07237		0.00140		0.00003	
		0.03824		0.00012		-0.00005
48^0	1.11061		0.00152		-0.00002	
		0.03976		0.00010		
49^0	1.15037		0.00162			
		0.04138				
50^0	1.19175					

Step 3. Write down the formula and substitute the various values :

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y0 + \ldots. + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u = (45^o\,15' - 45^0) / 1^0$

$= 15' / 1^0$

$= 0.25$ ……………(since $1^0 = 60'$)

$$y(x = 45^o 15') = P_5(45^o 15') = 1.00 + (0.25)(0.03553) + (0.25)(-0.75)(0.00131)/2$$
$$+(0.25)(-0.75)(-1.75)(0.00009)/6$$
$$+(0.25)(-0.75)(-1.75)(-2.75)(0.0003)/24$$
$$+(0.25)(-0.75)(-1.75)(-2.75)(-3.75)(-0.00005)/120$$

$= 1.000 + 0.0088825 - 0.0001228 + 0.0000049$

$= \mathbf{1.00876}$

Newton's backward interpolationformula(or) Gregory-Newton backward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots\ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots\ldots, x_n$.

Let suppose that the values of x i.e., $x_0, x_1, x_2 \ldots\ldots\ldots, x_n$ are equidistant .

$x_1 = x_0 + h$; $x_2 = x_1 + h$; and so on $x_n = x_{n-1} + h$;

Therefore $xi = x0 + i\,h$, where $i = 1, 2, \ldots, n$

Let $P_n(x)$ be a polynomial of the n^{th} degree in which x is such that

$y_I = f(x_i) = P_n(x_i), \quad I = 0, 1, 2, \ldots. n$

$$P_n(x) = a_0 + a_1(x - x_n)^{(1)} + a_2(x - x_n)(x - x_{n-1})^{)} + \ldots\ldots\ldots$$
$$+ a_n(x - x_n)(x - x_{n-1}) \quad \ldots(x - x_1) \ldots\ldots(1)$$

Let us assume Pn(x) in the form given below

$$P_n(x) = a_0 + a_1(x - x_n)^{(1)} + a_2(x - x_n)^{(2)} + \ldots\ldots\ldots + a_r(x - x_n)^{(r)} + \ldots\ldots$$
$$+ \ldots\ldots + a_n(x - x_n)^{(n)} \ldots\ldots(1.1)$$

This polynomial contains the n + 1 constants $a_0, a_1, a_2, \ldots\ldots\ldots a_n$ can be found as follows :

$P_n(x_n) = \quad y_n \;= a_0$ (setting x = xn, in (1))

Similarly $y_{n-1} = a_0 + a_1(x_{n-1} - x_n)$

$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)$

From these, we get the values of $a_0, a_1, a_2, \ldots\ldots a_n$

Therefore, $a_0 = \; y_n$

$˘yn \;= y_n - yn\text{-}1 \;= a_1(x_{n-1} - x_n)$

$= a_1 h$

$=> a_1 \;= y_n /h$

lly $=> a_2 \;= (˘y_1 - ˘y_n) /2h^2 = ˘^2 y_n / 2!\, h^2$

lly $=> a_3 \;= ˘^3 / 3!\, h^3$

Putting these values in (1), we get

$$P_n(x) = \; = y_n + (x - x_n)^{(n)} \nabla y_n / h \; + (x - x_n)^{(2)} \nabla^2 y_n / (2!\, h^2) + (x - x_n)^{(r)} \nabla^r y_n / (r!\, h^r) + \ldots\ldots + (x - x_n)^{(n)} \nabla^n y_n / (n!\, h^n)$$

By substituting $\dfrac{x - x_n}{h} = v$ *, the above equation becomes*

$$y(x_n + vh) = y_n + v \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \ldots\ldots$$

By substituting $v = v^{(1)}$,

$v(v+1) = v^{(2)}$,

$v(v+1)(v+2) = v^{(3)}$, *… in the above equation, we get*

$$P_n(x) = P_n y(x_n + vh) = y_n + v^{(1)} \nabla y_n + \frac{v^{(2)}}{2!} \nabla^2 y_n + \frac{v^{(3)}}{3!} \nabla^3 y_n + \ldots + \frac{v^{(r)}}{r!} \nabla^r y_n + \ldots + \frac{v^{(n)}}{n!} \Delta^n y_n$$

The above equation is known as **Gregory-Newton backward formula or Newton's backward interpolation formula.**

Note : 1. This formula is applicable only when the interval of difference is uniform.

2. This formula apply backward differences of y_n, hence this is used to interpolate the values of y nearer to the end of a set tabular values. (i.e., x lies between xn to xn- 1 and xn-1 to x_{n-2})

Example: Find the values of y at x = 28 from the following data.

x:	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

Solution.

Step 1. Since x = 28 is nearer to beginning of the table. Hence we apply Newton's backward formula.

Step 2. Construct the difference table

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$
20	0.3420	(0.3420-0.3907)		
		0.0487	(0.0477-0.0487)	
23	0.3907		-0.001	
		0.0477		
26	0.4384			-0.0003
			-0.0013	
29	0.4848	0.0464		

Step 3. Write down the formula and put the various values :

$$P_3(x) = P_3 y(x_n + vh) = y_n + v^{(1)} \nabla y_n + \frac{v^{(2)} \nabla^2 y_n}{2!} + \frac{v^{(3)} \nabla^3 y_n}{3!}$$

Where $v^{(1)} = (x - x_n) / h = (28 - 29) / 3 = -0.3333$

$v^{(2)} = v(v+1) = (-0.333)(0.6666)$

$v^{(3)} = v(v+1)(v+2) = (-0.333)(0.6666)(1.6666)$

$$P_n(x=28) = y(28) = 0.4848 + (-0.3333)(0.0464) + (-0.3333)(0.6666)(-0.0013)/2$$

$$+(-0.3333)(0.6666)(1.6666)(-0.0003)/6$$

$$= 0.4848 - 0.015465 + 0.0001444 + 0.0000185$$

$$= \mathbf{0.4695}$$

Example: From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 63.

Age	x:	45	50	55	60	65
Premium	y:	114.84	96.16	83.32	74.48	68.48

Solution.

Step 1. Since $x = 63$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$.. Hence we apply Newton's backward formula.

Step 2. Construct the difference table

x	y	∇y_0	$\nabla^2 y0$	$\nabla^3 y_0$	$\nabla^4 y_0$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.12		4.00		-
				1.16	
		-8.84			
60	74.48		2.84		
		-6.00			
65	68.48				
					0.68

Step 3. Write down the formula and put the various values :

$$P_3 (x)= P_3 y(x_n+ vh) = y_n + v^{(1)}\nabla y_n + \frac{v^{(2)}}{2!}\nabla^2 y_n + \frac{v^{(3)}}{3!}\nabla^3 y_n + \frac{v^{(4)}}{4!}\nabla^4 y_n$$

Where $v^{(1)} = (x - x_n) / h = (63 - 65) / 5 = -2/5 = -0.4$

$v(2) = v(v+1) = (-0.4)(1.6)$

$v(3) = v(v+1)(v+2) = (-0.4)(1.6)(2.6)$

$v(4) = v(v+1)(v+2))(v+3) = (-0.4)(1.6)(2.6)(3.6)$

$$P_4 (x=63)= y(63) = 68.48 + [(-0.4)(-6.0)] + [(-0.4)(1.6)(2.84)/ 2]$$

$$+ [(-0.4)(1.6)(2.6)(-1.16)/6]$$

$+ [(-0.4)\ (1.6)\ (2.6)(3.6)\ (0.68)/24\ ]$

$= 68.48 + 2.40 - 0.3408 + 0.07424 – 0.028288$

$=$ **70.5852**

Example: From the following table , find the value of tan 49^0 15'

x^0:	45	46	47	48	49	50
tan x^0:	1.0	1.03553	1.07237	1.11061	1.15037	1.19175

Solution.

Step 1. Since $x = 49^o 45'$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 1$. Hence we apply Newton's backward formula.

Step 2. Construct the difference table to find various Δ's

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
45^0	1.0000					
		0.03553				
46	1.03553		0.00131			
		0.03684		0.00009		
47^0	1.07237		0.00140		0.00003	
		0.03824		0.00012		-0.00005
48^0	1.11061		0.00152		-0.00002	
		0.03976		0.00010		
49^0	1.15037		0.00162			
		0.04138				
50^0	1.19175					

Step 3. Write down the formula and substitute the various values :

$$P_5(x) = P_5 y(x_n + vh) = y_n + v^{(1)} \nabla y_n + \frac{v^{(2)}}{2!} \nabla^2 y_n + \frac{v^{(3)}}{3!} \nabla^3 y_n + \frac{v^{(4)}}{4!} \nabla^4 y_n + \frac{v^{(5)}}{5!} \nabla^5 y_n$$

Where $v = (49^o 45' - 50^0) / 1^0$

$= - 15' / 1^0$

$= - 0.25$ ……………(since $1^0 = 60'$)

$v(2) = v(v+1) \quad = (-0.25))(0.75)$

$= (-0.25) (0.75)(1.75)$

$v(3) = v(v+1) (v+2)$

$v(4) = v(v+1) (v+2)) (v+3) \quad = (-0.25)(0.75) (1.75) (2.75)$

$y (x=49^o 15') = P_5 (49^o 15') \quad =1.19175 + (-0.25)(0.04138) + (-0.25)(0.75) (0.00162)/2$

$+(-0.25) (0.75)(1.75) (0.0001)/6$

$+(-0.25)(0.75) (1.75) (2.75) (-0.0002)/24$

$+(-0.25)(0.75) (1.75)(2.75) (3.75) (-0.00005)/120$

$= 1.19175 - 0.010345 - 0.000151875 +0 .000005+ \ldots$

$=\mathbf{1.18126}$

Lagrange's Interpolation Formula

Interpolation means the process of computing intermediate values of a function a given set of tabular values of a function. Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	$x3$	………..	x_n
y:	y_0	y_1	y_2	$y3$	………..	y_n

We may require the value of $y = y_i$ for the given $x = x_i$, where x lies between x_0 *to* x_n

Let y = *f(x)* be a function taking the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the *values* x_0, x_1, $x2$, ……….. x_n. Now we are trying to find $y = y_i$ for the given $x = x_i$ under assumption that the function f(x) is not known. In such cases, x_i '*s* are not equally spaced we use *Lagrange's interpolation formula.*

Newton's Divided Difference Formula:

The divided difference $f[x_0, x_1, x_2, \ldots, x_n]$, sometimes also denoted $[x_0, x_1, x_2, \ldots, x_n]$, on $n+1$ points X_0, x_1, ..., x_n of a function $f(x)$ is defined by $f[x_0] \equiv f(x_0)$ and

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_0, \ldots, x_{n-1}] - f[x_1, \ldots, x_n]}{x_0 - x_n}$$

for $n \geq 1$. The first few differences are

$$f[x_0, x_1] = \frac{f_0 - f_1}{x_0 - x_1}$$

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}$$

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_0, \ldots, x_{n-1}] - f[x_1, \ldots, x_n]}{x_0 - x_n}.$$

Defining

$\pi_n(x) \equiv (x - x_0)(x - x_1)\cdots(x - x_n)$ and taking the derivative

$\pi_n'(x_k) = (x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)$ gives the identity

$$f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} \frac{f_k}{\pi_n'(x_k)}.$$

Lagrange's interpolationformula(for unequal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots\ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots\ldots, x_n$.

Let suppose that the values of x *i.e.*, $x_0, x_1, x_2 \ldots\ldots\ldots, x_n$. are not equidistant .

$$y_I = f(x_i) \quad I = 0,1,2, \ldots. N$$

Now, there are (n+1) paired values$(x_i,y_i,)$, $I = 0,1,2, \ldots n$ *and hence* $f(x)$ *can be represented by a polynomial function of degree n in x.*

Let us consider f(x) as follows

$$f(x) = a_0(x - x_1)(x - x_2)(x - x_3)\ldots(x - x_n)$$
$$+ a_1 (x - x_0)(x - x_2)(x - x_3) \ldots (x - x_n)$$
$$+ a_2 (x - x_0)(x - x_3)(x - x_4) \ldots (x - x_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+a_n (x - x_0)(x - x_2)(x - x_3) \ldots(x - x_{n-1}) \quad \ldots\ldots\ldots(1)$$

Substituting $x= x_0, y=y_0$, *in the above equation*

$$y_0 = a_0(x - x_1)(x - x_2)(x - x_3)\ldots(x - x_n)$$

which implies $a_0 = y_0 / (x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \ldots (x_0 - x_n)$

Similarly $a_1 = y_1 / (x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_n)$

$a_2 = y_2 / (x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \ldots (x_2 - x_n)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$a_n = y_n (x_n - x_0)(x_n - x_2)(x_n - x_3) \ldots (x_n - x_{n-1})$

Putting these values in (1), we get

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \ldots (x_0 - x_n)} y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3) \ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_n)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n)}{(x_2 - x_0)(x_2 - x_2)(x_1 - x_3) \ldots (x_1 - x_n)} y_2$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3) \ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_2)(x_n - x_3) \ldots (x_n - x_{n-1})} y_n$$

The above equation is called ***Lagrange's interpolation formula*** for unequal intervals.

Note : 1. This formula is will be more useful when the interval of difference is not uniform.

Example. Using Lagrange's interpolation formula, find *y(10) from the following table*

x	:	*5*	*6*	*9*	*11*

y : 3 13 14 16

Solution:

Step 1. Write down the Lagrange's formula :

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}\, y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\, y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_2)(x_1-x_3))}\, y_2$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_2)}{(x_3-x_0)(x_3-x_2)(x_3-x_3)}\, y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}\,(12)$$

(13)

$$+ \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}$$

$$+ \frac{(x-5)(x-6)(x-11)\ (14)}{(9-5)(9-6)(9-11)}$$

$$+ \frac{(x-5)(x-6)(x-19)}{(11-5)(11-6)(11-9)}(16)$$

Putting x = 10 in the above equation

$$Y(10) = f(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13)$$

$$\frac{(5)(4)(1))}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

= 14.666

UNIT-IV
INTERPOLATION
Part- B (5x8=40 Marks)

Possible Questions

1. The population of a town is as follows.

Year (x) :	1941	1951	1961	1971	1981	1991
Population in Lakhs (y) :	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976.

2. Using inverse interpolation formula, find the value of x when y=13.5.

x :	93.0	96.2	100.0	104.2	108.7
y :	11.38	12.80	14.70	17.07	19.91

3.Find the polynomial of least degree passing the points (0, -1), (1, 1), (2, 1), (3, -2).

4. Find the values of y at X=21 and X=28 from the following data.

X :	20	23	26	29
Y :	0.3420	0.3907	0.4384	0.4848

5. From the data given below, find the number of students whose weight is between 60 and 70.

Weight in Ibs. :	0-40	40-60	60-80	80-100	100-120
No. of students :	250	120	100	70	50

6. Using Lagrange’s interpolation formula find the value corresponding to x = 10 from the following table.

x :	5	6	9	11
y :	12	13	14	16

7. Find the missing value of the table given below. What assumption have you made to find it?

Year :	1917	1918	1919	1920	1921
Export(in tons) :	443	384	-	397	467

8. Using Newton's divided difference formula, find the values of f(2), f(8) and f(15) given the following table.

x :	4	5	7	10	11	13
f(x) :	48	100	294	900	1210	2028

9. From the following table of half-yearly premium for policies maturing at different ages. Estimate the premium for policies maturing at age 46 & 63.

Age x :	45	50	55	60	65
Premium y :	114.84	96.16	83.32	74.48	68.48

10.Write the procedure for Lagrange’s Interpolation Fomula.