18BEBME201 /18BTBT201/18BEEC201/18BTFT201 MATHEMATICS – II 3 1 0 4

COURSE OBJECTIVES:

- To develop the use of matrix algebra techniques that is needed by engineers for practical applications.
- To acquaint the student with the concepts of vector calculus needed for problems in all engineering disciplines.
- To develop an understanding of the standard techniques of complex variable theory so as to enable the student to apply them with confidence, in application areas such as fluid dynamics and flow of the electric current.
- To make the student appreciate the purpose of using transforms to create a new domain in which it is easier to handle the problem that is being investigated.

INTENDED OUTCOMES:

The students will learn:

- To Evaluate complex integrals using the Cauchy integral formula and the residue Theorem
- To Appreciate how complex methods can be used to prove some important theoretical results.
- To Evaluate line, surface and volume integrals in simple coordinate systems
- To Calculate grad, div and curl in Cartesian and other simple coordinate systems, and establish identities connecting these quantities
- To Use Gauss, Stokes and Greens theorems to simplify calculations of integrals and prove simple results.

UNIT I : MATRICES 12

Eigen values and Eigenvectors of a real matrix, Characteristic equation, Properties of eigenvalues and eigenvectors, Cayley-Hamilton theorem, Diagonalization of matrices , Reduction of a quadratic form to canonical form by orthogonal transformation, Nature of quadratic forms. Simple Problems using Scilab.

UNIT II: VECTOR CALCULUS 12

Gradient and directional derivative, Divergence and Curl, Irrotational and Solenoidal vector fields, Line integral over a plane curve, Surface integral, Area of a curved surface, Volume integral, Green's, Gauss divergence and Stoke's theorems, Verification and application in evaluating line, surface and volume integrals.

UNIT III :ANALYTIC FUNCTION 12

Analytic functions, Necessary and sufficient conditions for analyticity, Properties, Harmonic conjugates, Construction of analytic function, Conformal mapping, Mapping by Functions $w = z+c$, cz, $1/z$, z^2 , Bilinear transformation.

UNIT IV: COMPLEX INTEGRATION 12

Line integral, Cauchy's integral theorem, Cauchy's integral formula, Taylor's and Laurent's series, Singularities, Residues, Residue theorem, Application of residue theorem for evaluation of real integrals, Use of circular contour and semicircular contour with no pole on real axis.

UNIT V: LAPLACE TRANSFORMS 12

Existence conditions, Transforms of elementary functions, Transform of unit step function and unit impulse function, Basic properties, Shifting theorems, Transforms of derivatives and integrals, Initial and final value theorems, Inverse transforms, Convolution theorem , Transform of periodic functions, Application to solution of linear ordinary differential equations with constant coefficients.

Total: 60

SUGGESTED TEXT/REFERENCE BOOKS

S. NO.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAROF PUBLICATION
1.	Hemamalini. P.T	Engineering Mathematics	McGraw Hill Education (India) Private Limited, New Delhi.	2014 &2017
2.	Erwin kreyszig	Advanced Engineering Mathematics, 9th Edition,	John Wiley & Sons	2014
3.	B.S. Grewal	Higher Engineering Mathematics, 43rd Edition	Khanna Publishers	2014
4.	Ramana B.V	Higher Engineering Mathematics	Tata McGraw Hill	2010
5.	Glyn James	Advanced Modern Engineering Mathematics	Pearson Education	2007
6.	Jain R.K. and Iyengar S.R.K	Advanced Engineering Mathematics , 3rd Edition	Narosa Publications	2007
7.	Bali N., Goyal M. and Watkins C	Advanced Engineering Mathematics, 7th Edition	Firewall Media (An imprint of Lakshmi Publications Pvt., Ltd)	2009
8.	O’Neil, P.V	Advanced Engineering Mathematics	Cengage Learning India Pvt., Ltd	2007

KARPAGAM ACADEMY OF HIGHER EDUCATION
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DEPARTMENT OF SCIENCE AND HUMANITIES

LECTURE PLAN

Subject : MATHEMATICS II

Code : 18BEBME201/18BTBT201/18BEECE201/18BTFT201

Unit No.	List of Topics	No. of Hours
UNIT I	**MATRICES**	
	Introduction of Matrix Algebra	1
	Characteristic Equation	1
	Problems based on Characteristic Equation - Eigen values and Eigen vectors	1
	Problems based on Characteristic Equation - Eigen values and Eigen vectors	1
	Tutorial 1: Characteristic Equation - Eigen values and Eigen vectors	1
	Properties of eigenvalues and eigenvectors	1
	Problems based on Properties	1
	Cayley – Hamilton theorem	1
	Problems based on Cayley – Hamilton theorem	1
	Diagonalization of matrices	1
	Reduction of a quadratic form to canonical form by orthogonal transformation	1
	Reduction of a quadratic form to canonical form by orthogonal transformation	1
	Nature of quadratic forms	1
	Tutorial 2: Cayley – Hamilton theorem and Canonical form through orthogonal reduction	1
	TOTAL	**14**
UNIT – II	**VECTOR CALCULUS**	
	Introduction – Vector Calculus	1
	Gradient and directional derivative	1
	Divergence and Curl	1
	Irrotational and Solenoidal vector fields	1
	Irrotational and Solenoidal vector fields, scalar potential	1
	Vector Integration-, Line integral over a plane curve	1
	Surface integral, Area of a curved surface	1
	Volume integral	1
	Tutorial 3 – Irrotational and solenoidal, Green's theorem	1
	Gauss divergence theorem - Statement , Problems	1
	Gauss divergence theorem - Problems	1
	Stoke's theorem - Statement , Problems	1
	Stoke's theorem - Problems	1
	Tutorial 4 – Gauss divergence and Stoke' theorem,	1
	TOTAL	**14**
	ANALYTIC FUNCTION	
	Introduction – Analytic Function	1
	Necessary and Sufficient conditions for an analytic function-Cauchy-Riemann equations –Cartesian form	1
	Necessary and Sufficient conditions for an analytic function-Cauchy-Riemann equations –Cartesian form	1
	Cauchy-Riemann equations – Polar form	1

UNIT – III	Harmonic functions and its conjugate	1
	Tutorial 5-Cauchy-Riemann equations Harmonic functions	1
	Properties of analytic functions	1
	Construction of an Analytic Function Milne-Thomson method	1
	Construction of an Analytic Function Milne-Thomson method	1
	Conformal mapping: The transformations w = z+a, az	1
	Conformal mapping: The transformations w = 1/z, Z^2	1
	Bilinear transformation	1
	Bilinear transformation	1
	Tutorial 6 - Conformal mapping, Bilinear transformation	1
	TOTAL	**14**
UNIT – IV	**COMPLEX INTEGRATION**	
	Introduction - Complex Integration, Line integral	1
	Problems solving using Cauchy's integral theorem	1
	Problems solving using Cauchy's integral formula	1
	Taylor's Series Problems	1
	Taylor's Series Problems	1
	Laurent series problems	1
	Laurent series problems	
	Tutorial 7 - Taylor's and Laurent's series problems	1
	Theory of Residues	1
	Cauchy's residue theorem	1
	Applications of Residue theorem to evaluate real integrals.	1
	Applications of Residue theorem to evaluate real integrals.	1
	Use of circular contour and semicircular contour with no pole on real axis.	1
	Tutorial 8 - Cauchy's residue theorem, Applications	1
	TOTAL	**14**
UNIT – V	**LAPLACE TRANSFORMS**	
	Introduction – Transforms, Existence conditions	1
	Transforms of elementary functions	1
	Transform of unit step function and unit impulse function	1
	Basic properties	1
	Transforms of derivatives and integrals	1
	Initial and final value theorems	1
	Tutorial 9 - Basic properties , Transforms of derivatives and integrals	1
	Inverse Laplace transforms, Convolution theorem	1
	Inverse Laplace transforms, Convolution theorem	1
	Transform of periodic functions	1
	Transform of periodic functions-Problems	1
	Application to solution of linear ordinary differential equations with constant coefficients using Laplace transforms	1
	Application to solution of linear ordinary differential equations with constant coefficients using Laplace transforms	1
	Tutorial 10 - Solution of Ordinary Differential Equations, Transform of periodic functions	1
	TOTAL	**14**
	GRAND TOTAL	**70**

Staff in charge HoD

Part I

Unit I Matrices

Characteristic equation; Eigen values and Eigen vectors of a real matrix; Properties; Cayley–Hamilton theorem (excluding proof); Orthogonal transformation of a symmetric matrix to diagonal form; Quadratic forms; Reduction to canonical form through orthogonal reduction.

Unit II Three-Dimensional Analytical Geometry

Direction ratios of the Line Joining two points; The plane; Plane through the intersection of two lines; The straight line; The plane and the straight line; Shortest distance between two skew lines; Equation of a sphere.

Unit III Geometrical Applications of Differential Calculus

Curvature in Cartesian coordinates; Centre and radius of curvature; Circle of curvature; Evolutes; Envelopes; Evolutes as envelope of normals.

Unit IV Functions of Several Variables

Partial derivatives; Euler's theorem for homogeneous functions; Total derivatives; Differentiation of implicit functions; Jacobians; Maxima and minima of functions of two or more variables; Method of Lagrangian multipliers.

Unit V Differential Equations

Equations of the first order and higher degree; Linear differential equations of second and higher order with constant coefficients; Euler's homogeneous linear differential equations; Mathematica software demonstration.

Unit I

Matrices

1 Matrices

Chapter Outline

- Introduction
- Definition of a Matrix
- Special type of Matrices
- Properties of Matrix Addition and Scalar Multiplication
- Properties of Matrix Transposition
- Determinants
- Simultaneous Linear Equations

1.1 ❑ INTRODUCTION

Matrices were invented about a century ago in connection with the study of simple changes and movements of geometric figures in coordinate geometry.

J J Sylvester was the first to use the Latin word "*matrix*" in 1850 and later on in 1858, **Arthur Cayley** developed the theory of matrices in a systematic way.

Matrices are powerful tools of modern mathematics and their study is becoming important day by day due to their wide applications in almost every branch of science and especially in physics (atomic) and engineering. These are used by sociologists in the study of dominance within a group, by demographers in the study of births and deaths, mobility and class structure, etc., by economists in the study of inter-industry economics, by statisticians in the study of 'design of experiments' and 'multivariate analysis', by engineers in the study of 'network analysis' used in electrical and communication engineering.

Matrix is an essential tool for engineers and scientists to solve a large number of problems in the branches of engineering such as in (i) electrical engineering, where the problems with electrical circuits are modelled with the help of matrix equations; (ii) structural engineering, where the problems are modelled in the form of matrix equations and then solved; (iii) a neural network, where a set of matrices represents a neural network and its activity can be explained with the help of matrix operations and also the knowledge gathered from a set of observations is stored in matrix form; (iv) image processing, where an image is considered as a big matrix and the templates for image processing operators like edge detection, thinning, filtering etc are basically matrices and the image-processing operations are directly or indirectly matrix operations; (v) graph theory, where a graph is represented by a matrix and the problem related to the graph can be solved using matrix algebra; (vi) control engineering, where the control problems are modelled using matrix or matrix differential equations; (vii) compiler design, where the grammar of a programming language may be expressed in terms of Boolean matrices and then the precedence of the operators used is the operator precedence grammar are computed; (viii) automata, where state transitions can be expressed using matrix theory.

Rectangular Array

Before we come to the formal definition of 'matrices' and to understand the same, let us consider the following example:

In an inter-university debate, a student can speak either of the five languages: Hindi, English, Bangla, Marathi and Tamil. A certain university, say, A sent 25 students of which 7 offered to speak in Hindi, 8 in English, 2 in Bangla, 5 in Marathi and the rest in Tamil; another university, say B, sent 20 students of which 10 spoke in Hindi, 7 in English and 3 in Marathi. Out of 25 students from the third university, say C, 5 spoke in Hindi, 10 in English, 6 in Bangla and 4 in Tamil.

The information given in the above example can be put in a compact way if we present it in a tabular form as follows:

University	*Number of speakers in*				
	Hindi	*English*	*Bangla*	*Marathi*	*Tamil*
A	7	8	2	5	3
B	10	7	0	3	0
C	5	10	6	0	4

The numbers in the above arrangement form is known as a **rectangular array**. In this array, the lines down the page are called **columns** whereas those across the page are called **rows**. Any particular number in this arrangement is known as an **entry** or an **element**. Thus, in the above arrangement, we find that there are 3 rows and 5 columns and we observe that there are 5 elements in each row and so the total number of elements = 3×5, i.e., 15.

If the data given in the above arrangement is written without lines enclosed by a pair of square brackets, i.e., in the form $\begin{bmatrix} 7 & 8 & 2 & 5 & 3 \\ 10 & 7 & 0 & 3 & 0 \\ 5 & 10 & 6 & 0 & 4 \end{bmatrix}$ then this is called a matrix.

1.2 ❑ DEFINITION OF A MATRIX

A system of any mn numbers arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix (which is read as m by n matrix).

For example, $\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mn} \end{bmatrix}$ ← row is an $m \times n$ matrix where the symbols (↓ Column)

a_{ij} represent any numbers (a_{ij} lies in the ith row and jth column) and $\begin{bmatrix} 1 & 5 & 2 \\ 3 & -6 & 4 \end{bmatrix}$ is a 2×3 matrix.

➢ **Note**

(i) A matrix may be represented by the symbols $[a_{ij}]$, (a_{ij}), $||a_{ij}||$. Generally, the first system is adopted.

(ii) Each of the mn numbers constituting an $m \times n$ matrix is known as an **element of the matrix.**
The elements of a matrix may be scalar or vector quantities.

(iii) When $m = n$, the matrix is square, and is called a matrix of order n or an n – **square matrix.**

(iv) The plural of 'matrix' is 'matrices'.

1.3 ❑ SPECIAL TYPES OF MATRICES

Row Matrix

Any $1 \times n$ matrix which has only one row is called a row matrix or a row vector.

The matrix $A = [a_{11}, a_{12} \ldots a_{1n}]$ is a row matrix.

Column Matrix

Any $m \times 1$ matrix which has only one column is called a column matrix or a column vector.

The matrix $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix}$ is a column matrix.

Null Matrix or Zero Matrix

If the elements of a matrix are all zero, it is called a null or zero matrix. A zero matrix of order $m \times n$ is denoted by $0_{m,n}$ or simply by 0. A zero matrix may be rectangular or square.

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices which are square and rectangular respectively.

Diagonal Matrix

A square matrix with all the elements equal to zero except those in the leading diagonal is called a diagonal matrix.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix.

Scalar Matrix

A diagonal matrix all of whose diagonal elements are equal is called a scalar matrix.

For example, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 3.

Unit Matrix

A square matrix of order n which has unity for all its elements in the leading diagonal and whose all other elements are zero is called the unit matrix or the identity matrix of order n and is denoted by I_n. In other words, if each diagonal element of a scalar matrix is unity, the matrix is called a unit matrix.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 2 and 3 respectively.

Triangular Matrices (Echelon Form)

A square matrix in which all the elements below the leading diagonal are zero is called an **upper triangular matrix**. A square matrix in which all the elements above the leading diagonal are zero is called a **lower triangular matrix**.

For example, $\begin{bmatrix} a_{11} & 0 & . & . & 0 \\ a_{21} & a_{22} & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix}$ is lower triangular and $\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ 0 & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & a_{nn} \end{bmatrix}$ is upper triangular.

Transpose of a Matrix

The matrix got from any given matrix A by interchanging its rows and columns is called the transpose of A and is denoted by A' or A^T.

For example, if $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 5 & 6 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 \\ -1 & 5 \\ 3 & 6 \end{bmatrix}$ clearly $(A')' = A$.

Conjugate of a Matrix

If A is an $m \times n$ matrix then the $m \times n$ matrix obtained by replacing each element of A by its complex conjugate is called the conjugate matrix of A and is denoted by $\overline{A}$.

Thus, if $A = [a_{ij}]$ then $\overline{A} = \left[\overline{a_{ij}}\right]$ where $\overline{a_{ij}}$ is the complex conjugate of a_{ij}.

For example, if $A = \begin{bmatrix} 3+i & 5-i & 7 \\ 6 & 3+i & 2-i \\ 2+7i & 8 & 9 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 3-i & 5+i & 7 \\ 6 & 3-i & 2+i \\ 2-7i & 8 & 9 \end{bmatrix}$

➢ **Note**

(i) If the elements of A are over the field of real numbers then the conjugate of A coincides with A, i.e., $\overline{A} = A$.

(ii) The conjugate of the conjugate of a matrix coincides with itself, i.e., $\overline{(\overline{A})} = A$.

Symmetric Matrices

A square matrix $A = [a_{ij}]$ is said to be **symmetric** if $A = A^T$, i.e., $a_{ij} = a_{ji}$, and **skew-symmetric** if $A = -A^T$, i.e., $a_{ij} = -a_{ji}$, where i and j vary from 1 to n.

The matrices $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ are respectively symmetric and skew-symmetric.

➢ **Note**

In a symmetric matrix, all the elements placed symmetrically about the main diagonal are equal and in a skew-symmetric matrix, they differ by a multiple of –1.

Hermitian Matrices and Skew-Hermitian Matrices

A square matrix $A = [a_{ij}]$ is said to be **Hermitian** if $a_{ij} = \overline{a_{ji}}$, i.e., the (i, j)th element is the conjugate complex of the (j, i)th element.

A square matrix $A = [a_{ij}]$ is said to be **skew-Hermitian** if $a_{ij} = -\overline{a_{ji}}$, i.e., (i, j)th element is the negative conjugate of the (j, i)th element.

For example, $\begin{bmatrix} 1 & 1-4i \\ 1+4i & 2 \end{bmatrix}$ and $\begin{bmatrix} 3i & 2+i \\ -2+i & i \end{bmatrix}$ are respectively, Hermitian and skew-Hermitian matrices.

Trace of a Square Matrix

The sum of the main diagonal elements of a square matrix A is called the trace of A and is denoted by tr A.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} \text{ then}$$

trace $(A) = tr\ A = a_{11} + a_{22} + \ldots + a_{nn}$

➢ **Note**

(i) If A and B are of the same order then $tr(A + B) = tr\ A + tr\ B$
(ii) If A be of order $m \times n$ and B of order $n \times m$, then $tr\ AB = tr\ BA$.

1.4 ❑ PROPERTIES OF MATRIX ADDITION AND SCALAR MULTIPLICATION

Property (i) $A + B = B + A$
Property (ii) $(A + B) + C = A + (B + C)$
Property (iii) $\alpha(A + B) = \alpha A + \alpha B$
Property (iv) $(\alpha + \beta)A = \alpha A + \beta A$
Property (v) $(\alpha\beta)\ A = \alpha(\beta A)$

Thus, the matrix addition is **commutative** [Property (i)] and **associative [**Property (ii)**]**; and the scalar multiplication of a matrix is **distributive** over matrix addition [Property (iii)].

1.5 ❑ PROPERTIES OF MATRIX TRANSPOSITION

If A and B are two matrices, and 'α' is a scalar then
Property (i) $(A^T)^T = A$
Property (ii) $(A + B)^T = A^T + B^T$
Property (iii) $(\alpha A)^T = \alpha A^T$
Property (iv) $(AB)^T = B^T A^T$

1.6 ❑ DETERMINANTS

With each square matrix A, we can associate a determinant which is denoted by the symbol $|A|$ or det A or Δ. When A is a square matrix of order n, the corresponding determinant $|A|$ is said to be a determinant of order n. A matrix is just an arrangement and has no numerical value. A determinant has numerical value. In fact, every square matrix has its determinant and while finding inverse, rank, etc., of a matrix or solving the linear equations by matrix method, we come across it.

Further, $\begin{bmatrix} 2 & 5 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 5 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 5 \\ 6 & 2 \end{bmatrix}$ and $\begin{bmatrix} 9 & 6 \\ 5 & 2 \end{bmatrix}$ are different matrices but the corresponding determinants have the same value (–12). In matrices, numbers are enclosed by brackets or parenthesis or double bars. In determinants, numbers are enclosed by a pair of vertical lines (bars).

Determinants were first introduced for solving linear systems and have important engineering applications in systems of differential equations, electrical networks, Eigen-value problems, and so on. Many complicated expressions occurring in electrical and mechanical systems can be simplified by expressing them in the form of determinants.

The differences between matrices and determinants are as follows:

Matrices	*Determinants*
1. Number of rows and number of columns can be equal or unequal.	1. Number of rows and number of columns are equal.
2. Elements are enclosed by brackets or parentheses or double bars.	2. Elements are enclosed by a pair of vertical lines (bars).
3. A matrix has no numerical value.	3. A determinant has a numerical value.
4. Matrices are arrangements. By interchanging rows and columns in a matrix, a new matrix is obtained.	4. Even after interchanging rows and columns in a determinant, the value of the determinant is unaltered.

Properties of Determinants

The following properties can be used in evaluating determinants.

(i) A determinant is unaltered if the corresponding rows and columns are interchanged.

(ii) If each element of a row or column be multiplied by a constant, the value of the determinant is multiplied by the same constant.

(iii) If two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed.

(iv) If two rows (or columns) are identical, the value of the determinant is zero.

(v) A determinant is unaltered if the elements of any row (or column) be multiplied by a constant and added to the corresponding element of any other row (or column).

(vi) The determinant of a diagonal matrix is equal to the product of the elements in the diagonal.

(vii) The determinant of the product of two matrices is equal to the product of the determinants of the two matrices,

i.e., $|AB| = |A| \cdot |B|$

Minors of a Matrix

The determinant of every square submatrix of a given matrix A is called a minor of the matrix A.

For example, if $A = \begin{bmatrix} 5 & 2 & 10 \\ -1 & 3 & 7 \\ 6 & 4 & 6 \end{bmatrix}$

Some of the minors are $\begin{vmatrix} 5 & 2 & 10 \\ -1 & 3 & 7 \\ 6 & 4 & 6 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ -1 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 7 \\ 4 & 6 \end{vmatrix}$, 3, 6, etc.

Singular and Nonsingular Matrices

A square matrix A is said to be **singular** if its determinant is zero.

A square matrix A is said to be **nonsingular** if its determinant is not equal to zero. For example,

consider $$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(6 - 16) - 2(18 - 8) + 3(12 - 2) \\ &= -10 - 20 + 30 \\ &= 0 \end{aligned}$$

$\therefore$ A is a singular matrix.

Consider $$B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} |B| &= 2(6 - 1) - 1(4 - 1) + 3(2 - 3) \\ &= 10 - 3 - 3 \\ &= 4 \end{aligned}$$

Since $|B| = 4 \neq 0$, B is a nonsingular matrix.

Adjoint of a Square Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The adjoint of A is defined to be the transpose of the co-factor matrix of A and is denoted by adj$A$.

$$\text{adj}A = (A_{ij})^T, \text{ where } A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\therefore \quad \text{adj}A = (A_{ij})^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Reciprocal Matrix or Inverse of a Matrix

- **Definition**

If A be any matrix then a matrix B, if it exists such that $AB = BA = I$, B is called the inverse of A; I being a unit matrix.

For the products AB, BA to be both defined and equal, it is necessary that A and B are both square matrices of the same order. Thus, nonsquare matrices cannot possess inverses. Also, we can at once show that the inverse of a matrix, in case it exists, must be unique.

Nonsingular and Singular Matrices

A square matrix A is said to be nonsingular or singular according as $|A| \neq 0$ or $|A| = 0$.

Thus, only nonsingular matrices possess inverses.

➢ **Note**

(i) If A, B be two nonsingular matrices of the same order then the product AB is nonsingular and $(AB)^{-1} = B^{-1} A^{-1}$.
(ii) If A be a nonsingular matrix and k a positive integer then $A^{-k} = (A^k)^{-1}$.
(iii) The operations of transposing and inverting are commutative,
i.e., $(A^T)^{-1} = (A^{-1})^T$
(iv) The operations of conjugate transpose and inverse are commutative,
i.e., $(A^\theta)^{-1} = (A^{-1})^\theta$.

Orthogonal Matrix

A square matrix A is said to be orthogonal if $AA^T = A^T A = I$

But we know that $A \cdot A^{-1} = A^{-1} \cdot A = I$

Hence, we note that $A^T = A^{-1}$.

Hence, an orthogonal matrix can also be defined as follows:

A square matrix A is said to be orthogonal if $A^T = A^{-1}$

For example, if $$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

then $$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is orthogonal.

Rank of a Matrix

A number r is defined as the rank of an $m \times n$ matrix A provided,

(i) A has at least one minor of order r which does not vanish, and
(ii) there is no minor of order $(r + 1)$ which is not equal to zero.

➢ **Note**

(i) The rank of a matrix A is denoted by $\rho(A)$ (or) simply $R(A)$.
(ii) The rank of a zero matrix by definition is 0 (i.e.) $\rho(0) = 0$.
(iii) The rank of a matrix remains unaltered by the application of elementary row or column operations, i.e., all equivalent matrices have the same rank.

(iv) From the definition of rank of a matrix, we conclude that:
  (a) If a matrix A does not possess any minor of order $(r + 1)$ then $\rho(A) \leq r$.
  (b) If at least one minor of order r of the matrix A is not equal to zero then $\rho(A) \geq r$.

(v) If every minor of order p of a matrix A is zero then every minor of order higher than p is definitely zero.

Idempotent Matrix

A matrix such that $A^2 = A$ is called an idempotent matrix.

For example, if $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Periodic Matrix

A matrix A will be called a periodic matrix if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer, for which $A^{k+1} = A$, then k is said to be the period of A. If we choose $k = 1$, we get $A^2 = A$ and we call it the idempotent matrix.

Nilpotent Matrix

A matrix A will be called a nilpotent matrix if $A^k = 0$ (null matrix) where k is a positive integer; if however k is the least positive integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

For example, if $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$,

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Here, A is a nilpotent matrix whose index is 2.

Involuntary Matrix

A matrix A will be called an involuntary matrix if $A^2 = I$ (unit matrix). Since $I^2 = I$ always, the unit matrix is involuntary.

Equal Matrices

Two matrices are said to be equal if
(i) they are of the same order, and
(ii) the elements in the corresponding positions are equal.

Thus, if $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$

Here, $A = B$.

1.7 ❑ SIMULTANEOUS LINEAR EQUATIONS

The concepts and operations in matrix algebra are extremely useful in solving simultaneous linear equations.

Let the equations be

$$a_1x + a_2y + a_3z = d_1 \quad b_1x + b_2y + b_3z = d_2 \quad c_1x + c_2y + c_3z = d_3$$

$$\Rightarrow \quad \begin{bmatrix} a_1x & a_2y & a_3z \\ b_1x & b_2y & b_3z \\ c_1x & c_2y & c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\therefore \quad AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Hence, to solve linear equations, write down the coefficient matrix A and find its inverse A^{-1}. Then find $A^{-1}B$. This gives the value X which is the solution for the given linear equations.

Consistency of a System of Simultaneous Linear Equations

A system of simultaneous linear equations is $AX = B$ in matrix form. Consider the coefficient matrix A. Augment A by writing the constants vector as the last column. The resulting matrix is called an **augmented matrix** and is denoted by $(A : B)$ or $(A \vdots B)$ or simply $[A, B]$.

A system of simultaneous linear equations is **consistent** if the ranks of the coefficient matrix and the augmented matrix are equal,

i.e., $\rho(A) = \rho(A : B)$ (or) $R[A] = R[A, B]$.

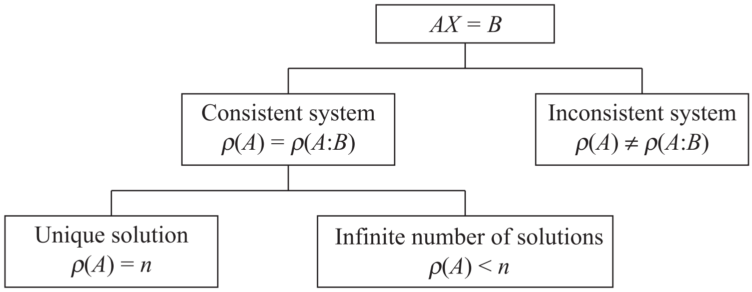
There are two possibilities:

(i) When $\rho(A) = \rho(A : B) = n$ (the number of unknowns), the system has a **unique solution.**

(ii) When $\rho(A) = \rho(A : B) < n$ (the number of unknowns), the system has **infinite solutions.** Let $\rho(A) = \rho(A : B) = r < n \cdot (n - r)$ of the unknowns are to be assigned values arbitrarily and the remaining r unknowns can then be obtained in terms of those $(n - r)$ values.

On the contrary, a system of simultaneous linear equations is **inconsistent** if the ranks of the coefficient matrix and the augmented matrix are not equal, i.e., $\rho(A) \neq \rho(A : B)$

These different possibilities are presented in a chart as follows:



2 Eigen Values, Eigen Vectors and the Characteristic Equation

Chapter Outline

- Introduction
- Characteristic Equation of a Matrix
- Important Properties of Eigen Values
- Linear Dependence and Independence of Vectors
- Properties of Eigen Vectors

2.1 ❑ INTRODUCTION

In this chapter, we shall discuss mainly square matrices A and throughout the ensuing discussion, any new facts and developments will be based on the determination of a vector X (to be called characteristic vector or Eigen vector) and a scalar λ (to be called characteristic value or Eigen value) such that $AX = \lambda X$. Based on these concepts of Eigen values and Eigen vectors, we shall indicate the conditions on A under which a nonsingular matrix P can be selected such that $P^{-1}AP$ is diagonal, i.e., A is similar to a diagonal matrix.

2.2 ❑ CHARACTERISTIC EQUATION OF A MATRIX

Characteristic Matrix

For a given matrix A, $A - \lambda I$ matrix is called the characteristic matrix, where λ is a scalar and I is the unit matrix.

Let $$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 3 & 1-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Characteristic Polynomial

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call the characteristic polynomial of the matrix A.

For example,

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 3 & 1-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (2 - \lambda)(\lambda^2 - 3\lambda) - 2(-3\lambda + 5) + 1(\lambda + 5)$$
$$= -\lambda^3 + 5\lambda^2 + \lambda - 5$$

Characteristic Equation

The equation $|A - \lambda I| = 0$ is known as the characteristic equation of A and its roots are called the **characteristic roots** or **latent roots** or **Eigen values** or **characteristic values** or **latent values** or **proper values** of A.

Spectrum of A

The set of all Eigen values of the matrix A is called the spectrum of A.

Eigen-value Problem

The problem of finding the Eigen values of a matrix is known as an Eigen-value problem.

Characteristic Vector

Any nonzero vector X is said to be a characteristic vector of a matrix A if there exists a number λ such that $AX = \lambda X$, where λ is a characteristic root of a matrix A.

2.3 ❑ IMPORTANT PROPERTIES OF EIGEN VALUES

(i) Any square matrix A and its transpose A^T have the same Eigen values.

(ii) The sum of the Eigen values of a matrix is equal to the trace of the matrix.
[**Note**: The sum of the elements on the principal diagonal of a matrix is called the trace of the matrix.]

(iii) The product of the Eigen values of a matrix A is equal to the determinant of A.

(iv) If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the Eigen values of A then the Eigen values of

(a) KA are $k\lambda_1, k\lambda_2 \ldots k\lambda_n$

(b) A^m are $\lambda_1^m, \lambda_2^m \ldots \lambda_n^m$

(c) A^{-1} are $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2} \cdots \dfrac{1}{\lambda_n}$.

(iv) The Eigen values of a real symmetric matrix (i.e. a symmetric matrix with real elements) are real.

2.4 ❑ LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

n-dimensional Vector or n-vector

An ordered set of n elements x_i of a field F written as

$$A = [x_1, x_2 \dots x_n] \tag{2.1}$$

is called an n-dimensional vector or n-vector over F and the elements $x_1, x_2 \dots x_n$ are called the first, second … nth components of A.

We find it more convenient to write the components of a vector in a column as

$$A^T = [x_1, x_2, x_3 \dots x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \tag{2.2}$$

Equation (2.1) is called a **row-vector** and Eq. (2.2) is called a **column-vector**.

Linear Dependence and Independence of Vectors

The vectors $A_1 = [x_{11}, x_{12}, x_{13} \dots x_{1m}]$, $A_2 = [x_{21}, x_{22}, x_{23} \dots x_{2m}] \dots A_n = [x_{n1}, x_{n2}, x_{n3} \dots x_{nm}]$ are called **linearly dependent** over F if there exists a set of n elements $\lambda_1, \lambda_2 \dots \lambda_n$ of F, λ_i's being not all zero, such that $\lambda_1 A_1 + \lambda_2 A_2 + \dots \lambda_n A_n = 0$.

Otherwise the n-vectors are called **linearly independent** over F.

2.5 ❑ PROPERTIES OF EIGEN VECTORS

(i) The Eigen vector X of a matrix A is not unique.

(ii) If $\lambda_1, \lambda_2 \dots \lambda_n$ be distinct Eigen values of an $n \times n$ matrix then the corresponding Eigen vectors $X_1, X_2 \dots X_n$ form a linearly independent set.

(iii) If two or more Eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to the equal roots.

(iv) Two Eigen vectors X_1 and X_2 are called orthogonal vectors if $X_1^T X_2 = 0$

(v) Eigen vectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

Applications

The Eigen-value and Eigen-vector method is useful in many fields because it can be used to solve homogeneous linear systems of differential equations with constant coefficients. Furthermore, in chemical engineering, many models are formed on the basis of systems of differential equations that are either linear or can be linearized and solved using the Eigen-value, Eigen-vector method. In general, most ordinary differential equations can be linearized and, therefore, solved by this method. Initial-value problems can also be solved by using the Eigen-value and Eigen-vector method.

Eigen-value analysis is also used in the designing of car stereo systems so that the sounds are directed appropriately for the listening pleasure of both the drivers and the passengers. Eigen-value analysis can indicate what needs to be changed to reduce the vibration of the car due to the music being played.

Oil companies frequently use Eigen-value analysis to explore land for oil. Oil, dirt and other substances give rise to linear systems which have different Eigen values, so Eigen-value analysis can give a good indication of where oil reserves are located.

Eigen values and Eigen vectors are used widely in science and engineering, particularly in physics. Rigid physical bodies have a preferred direction of rotation, about which they can rotate freely. For example, if someone were to throw a football, it would rotate around its axis while flying through the air. If someone were to hit the ball in the air, the ball would be likely to flop in a less simple way. Although this may seem like common sense, even rigid bodies with more complicated shapes will have preferred directions of rotation. These are called **axes of inertia**, and they are calculated by finding the Eigen vectors of a matrix called the **inertia tensor**. The Eigen values are also important and they are called **moments of inertia**.

SOLVED EXAMPLES

Example 1 Find the characteristic roots of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2-0 & 3-0 \\ 0-0 & 2-\lambda & 3-0 \\ 0-0 & 0-0 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)^2$$

$\therefore$ the characteristic equation of the matrix A is $(1 - \lambda)(2 - \lambda)^2 = 0$ and its roots are 1, 2, 2. **Ans.**

Example 2 Find the characteristic roots and corresponding characteristic vectors for the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution The characteristic equation is $|A - \lambda I| = 0$,

i.e.,
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$

$\Rightarrow$ $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\Rightarrow$ $\lambda(-\lambda^2 + 18\lambda - 45) = 0$

$\Rightarrow$ $\lambda = 0, 3, 15$ are the characteristic roots of the matrix.

The characteristic vector X is obtained from $(A - \lambda I)X = 0$.

Case (i) $\lambda = 0$

If x, y, z are the components of a characteristic vector corresponding to the characteristic root 0, we have

$$(A - 0I)X = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$8x - 6y + 2z = 0$

$-6x + 7y - 4z = 0$

$2x - 4y + 3z = 0$

$\therefore$ $$\frac{x}{21-16} = \frac{-y}{-18+8} = \frac{z}{24-8}$$

$\Rightarrow$ $$\frac{x}{5} = \frac{-y}{-10} = \frac{z}{10}$$

i.e., $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

$$(A - 3I)X = 0 \Rightarrow \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e., $$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $5x - 6y + 2z = 0$

$-6x + 4y - 4z = 0$

$2x - 4y = 0$

$\therefore \quad \dfrac{x}{0-16} = \dfrac{-y}{0+8} = \dfrac{z}{24-8}$

$\Rightarrow \quad \dfrac{x}{-16} = \dfrac{-y}{8} = \dfrac{z}{16}$

$\Rightarrow \quad \dfrac{x}{-2} = \dfrac{y}{-1} = \dfrac{z}{2}$

$\therefore \quad X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Case (iii) $\lambda = 15$

$$(A - 15I)\, X = 0 \Rightarrow \begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e., $\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\Rightarrow \quad -7x - 6y + 2z = 0$

$-6x - 8y - 4z = 0$

$2x - 4y - 12z = 0$

$\therefore \quad \dfrac{x}{96-16} = \dfrac{-y}{72+8} = \dfrac{z}{24+16}$

$\Rightarrow \quad \dfrac{x}{80} = \dfrac{-y}{80} = \dfrac{z}{40}$

$\therefore \quad \dfrac{x}{2} = \dfrac{y}{-2} = \dfrac{z}{1}$

$\therefore \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Hence, $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ **Ans.**

➢ **Note**

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the characteristic equation is given by $|A - \lambda I| = 0$ or $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$ where $D_1 = a_{11} + a_{22} + a_{33}$ (sum of the diagonals of A (or) trace of a matrix A)

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

= sum of the second-order minors of A whose principal diagonals lie along the principal diagonal of A.

$D_3 = |A|$ = determinant of A.

Example 3 Find the characteristic roots and corresponding characteristic vectors of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. **[KU Nov. 2010]**

Solution The characteristic equation is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

where $D_1 = 6 + 3 + 3 = 12$

$$D_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$= (18 - 4) + (18 - 4) + (9 - 1)$

$= 14 + 14 + 8$

$= 36$

$$D_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 6(9 - 1) + 2(-6 + 2) + 2(2 - 6)$

$= 48 - 8 - 8$

$= 32$

$\therefore$ the characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ and the roots are 2, 2, 8.

Case (i) $\lambda = 2$ (twice)

$$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e.,

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$

$4x - 2y + 2z = 0$

$-2x + y - z = 0$

$2x - y + z = 0$

which are equivalent to a single equation . There is one equation in three unknowns.

$\therefore$ taking two of the unknowns, say $x = 1$ and $y = 0$, we get $z = -2$ and taking $x = 0$ and $y = 1$, we get $z = 1$.

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 8$

$$(A-8I)X = 0 \Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e.,
$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$
$$2x - y - 5z = 0$$

$\therefore$
$$\frac{x}{25-1} = \frac{-y}{10+2} = \frac{z}{2+10}$$
$$\frac{x}{24} = \frac{y}{-12} = \frac{z}{12}$$

$\therefore$
$$X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ **Ans.**

Example 4 The matrix A is defined as $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. Find the Eigen values of $3A^3 + 5A^2 - 6A + 2I$.

Solution The characteristic equation is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)(3-\lambda)(-2-\lambda) = 0$

i.e., $\lambda = 1, 3, -2$

Eigen values of $A^3 = 1, 27, -8$

Eigen values of $A^2 = 1, 9, 4$

Eigen values of $A = 1, 3, -2$

Eigen values of $I = 1, 1, 1$

$\therefore$ Eigen values of $3A^3 + 5A^2 - 6A + 2I$

First Eigen value $= 3(1)^3 + 5(1)^2 - 6(1) + 2 = 4$

Second Eigen value $= 3(27) + 5(9) - 6(3) + 2(1) = 110$

Third Eigen value $= 3(-8) + 5(4) - 6(-2) + 2(1) = 10$

$\therefore$ Required Eigen values are 4, 110, 10. **Ans.**

Example 5 Find the Eigen values and Eigen vectors of the matrix

$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. **[KU May 2010]**

Solution The characteristic equation is given by $|A - \lambda I| = 0$.

i.e.,
$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow$ $(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 1, 2, 3$

To find Eigen vectors for the corresponding Eigen values, we will consider the matrix equation $(A - \lambda I)X = 0$.

Case (i) $\lambda = 1$

$$(A - \lambda I)X = 0 \quad \Rightarrow \quad \begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $-z = 0$

$\Rightarrow$ $x + y + z = 0$

$\Rightarrow$ $2x + 2y + 2z = 0$

Let $x = 1 \quad \Rightarrow \quad y = -1$

$\therefore$
$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Case (ii) $\lambda = 2$

$$(A - \lambda I)X = 0 \quad \Rightarrow \quad \begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $-x - z = 0$

$x + z = 0$

$2x + 2y + z = 0$

$\therefore$
$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{2}$$

$\therefore$
$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Case (iii) $\lambda = 3$

$$(A - \lambda I)X = 0 \quad \Rightarrow \quad \begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $-2x - z = 0$

$x - y + z = 0$

$2x + 2y = 0$

$\therefore$
$$\frac{x}{-2} = \frac{-y}{-2} = \frac{z}{4}$$

$$\therefore \qquad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Hence, the Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ **Ans.**

EXERCISE

Part A

1. If 1, 5 are the Eigen values of a matrix A, find the value of det A.
2. Find the constants a and b such that the matrix $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 and –2 as its Eigen values.
3. If the sum of two Eigen values and trace of a 3×3 matrix A are equal, find $|A|$.
4. What do you understand by the characteristic equation of the matrix A?
5. What is Eigen-value problem?
6. Find latent vectors of the matrix $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.
7. Define linearly dependent and linearly independent set of vectors.
8. Show that the set of vectors $X_1 = [1, 2, 3]$, $X_2 = [1, 0, 1]$ and $X_3 = [0, 1, 0]$ are linearly independent.
9. Prove that the set of vectors $X_1 = [1, 2, 3]$, $X_2 = [1, 0, 1]$ and $X_3 = [0, 1, 0]$ are linearly independent.
10. Define spectrum of a matrix.
11. Prove that any square matrix A and its transpose A^T have the same Eigen values.
12. Find the sum and product of the Eigen values of the matrix $A = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$.
13. Given $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, find the Eigen values of A^2.
14. Find the sum of the squares of the Eigen values of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.
15. Find the sum of the Eigen values of the inverse $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.
16. If A and B are 2 square matrices then what can you say about the characteristic roots of the matrices AB and BA?

17. If two of the Eigen values of a 3×3 matrix, whose determinant equals 4, are –1 and +2, what will be the third Eigen value of the matrix?

18. The matrix A is defined as $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Find the Eigen values of A^2.

19. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, find the Eigen values of $A^3 + 5A = 8I$.

20. The Eigen values of a matrix A are 1, – 2, 3. Find the Eigen values of $3I - 2A + A^2$.

Part B

1. Find the Eigen values of the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$. (**Ans.** 0, 1, – 2)

2. The matrix A is defined as $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$. Find the Eigen values of $3A^3 + 5A^2 + 6A + I$. (**Ans.** 15, –15, –53)

3. Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$$\left(\textbf{Ans.}\ -1, 1, 2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right)$$

4. Show that the vectors [1, 2, 0], [8, 13, 0] and [2, 3, 0] are linearly dependent.
5. Show the set of vectors [1, 1, 1], [1, 2, 3] and [2, 3, 8] are linearly independent.
6. Given that $A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$, verify that the sum and product of the Eigen values of A are equal to the trace of A and $|A|$ respectively.

7. Find the Eigen values and Eigen vectors of (adj$A$), where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

$$\left(\textbf{Ans.}\ 1, 4, 4, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right).$$

8. Verify that the Eigen vectors of the real symmetric matrix $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$ are orthogonal in pairs.

(**Hint:** Prove that $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$)

9. Find the Eigen values and Eigen vectors of the following matrices:

(i) $\begin{bmatrix}2 & -2 & 2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$ $\left(\textbf{Ans.} -2, 2, 2, \begin{bmatrix}-4\\ -1\\ 7\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}\right)$

(ii) $\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$ $\left(\textbf{Ans.} 1, 1, 5, \begin{bmatrix}1\\ 2\\ -5\end{bmatrix}, \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\right)$

(iii) $\begin{bmatrix}4 & 2 & -2\\ -5 & 3 & 2\\ -2 & 4 & 1\end{bmatrix}$ $\left(\textbf{Ans.} 1, 2, 5, \begin{bmatrix}2\\ 1\\ 4\end{bmatrix}, \begin{bmatrix}1\\ 1\\ 2\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right)$

(iv) $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 3 & 2 & 3\end{bmatrix}$ $\left(\textbf{Ans.} 0, 1, 5, \begin{bmatrix}0\\ -1\\ 1\end{bmatrix}, \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}4\\ 5\\ 11\end{bmatrix}\right)$

(v) $\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ **[KU April 2012]**

$\left(\textbf{Ans.} 5, -3, -3, \begin{bmatrix}1\\ 2\\ -1\end{bmatrix}, \begin{bmatrix}2\\ -1\\ 0\end{bmatrix}, \begin{bmatrix}3\\ 0\\ 1\end{bmatrix}\right)$

10. Find the Eigen values and Eigen vectors of (adj$A$), given that the matrix $A=\begin{bmatrix}2 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 2\end{bmatrix}$ **[KU May 2010]**

$\left(\textbf{Ans.} 1, 2, 3, \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}\right)$

3 Cayley–Hamilton Theorem

Chapter Outline

- Introduction
- Cayley–Hamilton Theorem

3.1 ❑ INTRODUCTION

This theorem provides an alternative method for finding the inverse of a matrix, and any positive integral power of A can be expressed as a linear combination of those of lower degree.

3.2 ❑ CAYLEY–HAMILTON THEOREM

Every square matrix satisfies its own characteristic equation.

Application

The Cayley–Hamilton theorem can be used to find

- The power of a matrix, and
- The inverse of an $n \times n$ matrix A, by expressing these as polynomials in A of degree $< n$.

SOLVED EXAMPLES

Example 1 Verify that the matrix $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation and, hence, find A^4. **[KU May 2010, AU Jan. 2010]**

Solution The characteristic equation is $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

i.e., $$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

According to Cayley–Hamilton theorem, to prove $A^3 - 6A^2 + 8A - 3I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

Hence, $A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the given matrix A satisfies its own characteristic equation, i.e., $A^3 - 6A^2 + 8A - 3I = 0$

Multiplying on both sides by A, we get

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$
$$A^4 = 6A^3 - 8A^2 + 3A$$

$$A^4 = \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$ **Ans.**

Example 2 Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and, hence, find A^{-1} and A^4. **[KU Nov. 2010]**

Solution The characteristic equation is $|A - \lambda I| = 0$,

i.e., $$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$

To prove $A^3 - 3A^2 - 9A - 5I = 0$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$$

$$\therefore \quad A^3 - 3A^2 - 9A - 5I = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix} - \begin{bmatrix} 27 & 24 & 24 \\ 24 & 27 & 24 \\ 24 & 24 & 27 \end{bmatrix} - \begin{bmatrix} 9 & 18 & 18 \\ 18 & 9 & 18 \\ 18 & 18 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the Cayley–Hamilton theorem is verified.

$$A^3 - 3A^2 - 9A - 5I = 0 \qquad (1)$$

To find A^{-1}

$\div$ by $A \Rightarrow A^2 - 3A - 9I - 5A^{-1} = 0$

i.e., $-5A^{-1} = -A^2 + 3A + 9I$

$$-5A^{-1} = \begin{bmatrix} -9 & -8 & -8 \\ -8 & -9 & -8 \\ -8 & -8 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & 6 \\ 6 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 9 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix}$$

$$-5A^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = -\frac{1}{5}\begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

To find A^4, multiply (1) by A

$$A^4 - 3A^3 - 9A^2 - 5A = 0$$

i.e., $A^4 = 3A^3 + 9A^2 + 5A$

$$= \begin{bmatrix} 123 & 126 & 126 \\ 126 & 123 & 126 \\ 126 & 126 & 123 \end{bmatrix} + \begin{bmatrix} 81 & 72 & 72 \\ 72 & 81 & 72 \\ 72 & 72 & 81 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 209 & 208 & 208 \\ 208 & 209 & 208 \\ 208 & 208 & 209 \end{bmatrix}$$ **Ans.**

EXERCISE

Part A

1. State Cayley–Hamilton theorem.
2. Give two uses of the Cayley–Hamilton theorem.
3. If $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, write A^2 in terms of A and I, using Cayley–Hamilton theorem.
4. Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$.
5. Using Cayley–Hamilton theorem, find the inverse of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
6. Verify Cayley–Hamilton theorem for $\begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}$.
7. Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.
8. Using Cayley–Hamilton theorem, find the inverse of $\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$
9. The Cayley–Hamilton theorem is used to find ____________
   (a) Eigen values (b) Eigen vectors
   (c) inverse and higher powers of A (d) quadratic form

Part B

1. Using Cayley–Hamilton theorem, find A^4 if $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

   $\left(\textbf{Ans.} \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix} \right)$

2. Using Cayley–Hamilton theorem, find the inverse of the matrix

   $A = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}$ $\left(\textbf{Ans.} \begin{bmatrix} 8 & 0 & -3 \\ -43 & 1 & 17 \\ 3 & 0 & -1 \end{bmatrix} \right)$

3. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and, hence, obtain the inverse of the given matrix. **[KU April 2011]**

$$\left(\textbf{Ans. } \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0;\ A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}\right)$$

4. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the equation is satisfied by A. (**Ans.** $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$)

5. Using Cayley–Hamilton theorem, find the inverse of (i) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

$$\left(\textbf{Ans. (i) } \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \text{ (ii) } \frac{1}{50}\begin{bmatrix} -8 & 20 & -7 \\ -40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}\right)$$

6. Find the characteristic equation of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. Verify Cayley–Hamilton theorem for this matrix. Hence, find A^{-1}.

$$\left(\textbf{Ans. } A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}\right)$$

7. Use Cayley–Hamilton theorem to find the inverse of the matrix
$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\left(\textbf{Ans. } A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right)$

8. Using Cayley–Hamilton theorem, find A^{-1} given that $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$

$$\left(\textbf{Ans. } A^{-1} = -\frac{1}{5}\begin{bmatrix} 4 & -5 & -2 \\ 7 & -10 & -1 \\ -2 & 0 & 1 \end{bmatrix}\right)$$

9. Using Cayley–Hamilton theorem, find the inverse of the matrix

$$A = \begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}.$$

$$\left(\textbf{Ans. } A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \\ -1 & 1 & -1 \end{bmatrix} \right)$$

10. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and show that the equation is also satisfied by A. (**Ans.** $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$)

11. Verify Cayley–Hamilton theorem and hence find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}.$$

$$\left(\textbf{Ans. } \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{21}{10} & \frac{-7}{20} & \frac{-2}{5} \\ \frac{-9}{10} & \frac{3}{10} & \frac{1}{5} \end{bmatrix} \right)$$

4 Diagonalization of Square Matrices

Chapter Outline

- Introduction
- Diagonalization of Square Matrices
- Diagonalization by Orthogonal Transformation or Orthogonal Reduction

4.1 ❑ INTRODUCTION

Two square matrices A and B are said to be **similar** if there exists a nonsingular matrix C such that $B = C^{-1}AC$. The transformation A to $C^{-1}AC$ is called **similarity transformation**. The determinant, rank and Eigen values are preserved under similarity transformation. A matrix is said to be diagonalizable if it is similar to a diagonal matrix. The determinant of a diagonal matrix is simply the product of the diagonal elements; the rank is the number of nonzero diagonal elements and the Eigen values are the diagonal elements. Hence, it is very easy to deal with diagonal matrices.

4.2 ❑ DIAGONALIZATION OF SQUARE MATRICES

The process of finding a matrix M such that $M^{-1}AM = D$, where D is a diagonal matrix, is called diagonalization of the matrix A. As $M^{-1}AM = D$ is a similarity transformation, the matrices A and D are similar and, hence, A and D have the same Eigen values. The Eigen values of D are its diagonal elements. Thus, if we find a matrix M such that $M^{-1}AM = D$, D is a diagonal matrix whose diagonal elements are the Eigen values of A. A square matrix which is not diagonalizable is called **defective**.

Application

The direct application of diagonalization is that it gives us an easy way to compute large powers of a matrix A. The Eigen values of a system determine sometimes whether the system is stable or not. This has all to do with diagonalizing matrices. In quantum mechanical and quantum chemical computations, matrix diagonalization is one of the most frequently applied numerical processes.

➢ **Note**

(i) M is called the modal matrix of A whose elements are the Eigen vectors of A.
(ii) For this diagonalization process, A need not necessarily have distinct Eigen values. Even if two or more Eigen values of A are equal, the process holds good provided the Eigen vectors of A are linearly independent.

4.3 ❑ DIAGONALIZATION BY ORTHOGONAL TRANSFORMATION OR ORTHOGONAL REDUCTION

The process of finding a normalized modal matrix N such that $N^{-1}AN = D$ where D is a diagonal matrix is called orthogonal transformation or orthogonal reduction. The elements of N are the normalized Eigen vectors of A and it can be proved that N is an orthogonal matrix (i.e. $N^{-1} = N^T$). It is important to note that diagonalization by orthogonal transformation is possible only for a real symmetric matrix.

SOLVED EXAMPLES

Example 1 Reduce the matrix $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$ to diagonal form. **[AU Jan. 2010]**

Solution Let $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

Here, $D_1 = 17$, $D_2 = 42$, $D_3 = 0$.

$\therefore$ the characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$.

i.e., $$\lambda(\lambda^2 - 17\lambda + 42) = 0$$
$$\lambda(\lambda - 14)(\lambda - 3) = 0$$

$\Rightarrow$ $$\lambda = 0, 14, 3$$

$\therefore$ the Eigen values are 0, 14, 3.

To find the Eigen vectors, $[A - \lambda I]X = 0$.

i.e., $$\begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(10-\lambda)x_1 - 2x_2 - 5x_3 = 0$$
$$-2x_1 + (2-\lambda)x_2 + 3x_3 = 0$$
$$-5x_1 + 3x_2 + (5-\lambda)x_3 = 0$$

$\lambda = 0$ gives $10x_1 - 2x_2 - 5x_3 = 0$; $-2x_1 + 2x_2 + 3x_3 = 0$; $-5x_1 + 3x_2 + 5x_3 = 0$.
Consider first two equations, which gives $x_1 = 1$, $x_2 = -5$, $x_3 = 4$.

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

$\lambda = 14$ gives

$$\begin{aligned} -4x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 - 12x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 - 9x_3 &= 0 \end{aligned}$$

Considering first two equations gives $x_1 = -3$, $x_2 = 1$, $x_3 = 2$.

$$\therefore \quad X_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

$\lambda = 3$ gives

$$\begin{aligned} 7x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 - x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}$$

$\Rightarrow x_1 = 1$, $x_2 = 1$, $x_3 = 1$

$$\therefore \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad M = \begin{bmatrix} 1 & -3 & 1 \\ -5 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$M^{-1} = \dfrac{1}{|M|} \text{Adj} M$ provided $|M| \neq 0$

$$|M| = -42$$

To find Adj$M$,
Co-factor of 1 = –1, Co-factor of –3 = 9, Co-factor of 1 = –14, Co-factor of 1 = –14, Co-factor of 1 = –3, Co-factor of –5 = 5
Co-factor of 4 = –4, Co-factor of 2 = –6, Co-factor of 1 = –14

$$\therefore \quad \text{Adj}M = \begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}$$

$$\Rightarrow \quad M^{-1} = -\frac{1}{42} \begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}$$

Consider

$$M^{-1}AM = -\frac{1}{42}\begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 & 1 \\ -5 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= -\frac{1}{42}\begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}\begin{bmatrix} 0 & -42 & 3 \\ 0 & 14 & 3 \\ 0 & 28 & 3 \end{bmatrix}$$

$$= -\frac{1}{42}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -588 & 0 \\ 0 & 0 & -126 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Proved.

Example 2 Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by orthogonal transformation.

[KU April 2011]

Solution The characteristic equation is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2-\lambda)(\lambda^2 - 2\lambda - 3) - (-\lambda - 1) - (-\lambda - 1) = 0$

$\Rightarrow$ $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$\Rightarrow$ $(\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$

$\therefore$ The Eigen values are –1, 1, 4.
The Eigen vectors are given by $(A - \lambda I)X = 0$.
when $\lambda = -1$

The Eigen vector is given by $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow$ $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

When $\lambda = 1$, the Eigen vector is given by $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow$ $X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

When $\lambda = 4$, the Eigen vector is given by $\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\Rightarrow \qquad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Hence, the modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore$ normalized modal matrix is,

$$N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

To prove $N^{-1} AN = D$, since N is an orthogonal matrix, it satisfies $N^{-1} = N^T$.
$\therefore$ it is enough to prove that $N^{-1} AN = D$.
Consider

$$N^{-1}AN = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

Proved.

EXERCISE

Part A

1. When are two matrices said to be similar?
2. Define diagonalizing a matrix.
3. What is the difference between diagonalization of a matrix by similarity and orthogonal transformations?
4. Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
5. Is it possible to diagonalize the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$?

   [**Ans:** The Eigen values $\lambda = 0, 0$ but there is only one Eigen vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So the matrix cannot be diagonalized.]
6. What type of matrices can be diagonalized using (i) similarity transformation, and (ii) orthogonal transformation?
7. In the orthogonal transformation $N^T AN = D$, D refers to a/an __________ matrix.
   (i) diagonal (ii) orthogonal
   (iii) symmetric (iv) skew-symmetric
8. In a modal matrix, the columns are the Eigen vectors of __________
   (i) A^{-1} (ii) A^2 (iii) A (iv) adj A
9. If $X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0,$ the Eigen vectors are said to be __________
   (i) dependent (ii) pairwise orthogonal
   (iii) skew-symmetric (iv) independent
10. If A is an orthogonal matrix, show that A^{-1} is also orthogonal.

Part B

1. Find the modal matrix of the following matrices.

   (i) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

   $\left[\textbf{Ans.} \text{ (i)} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ (ii)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right]$

2. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.
   (**Ans.** $A + 5I$)

3. Show that $A^T = A^{-1}$ for $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$.

4. Diagonalize the following matrices:

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$$\left[\textbf{Ans.}\ \text{(i)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \text{(ii)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(iii)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right]$$

5. A square matrix A is defined by $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Find the modal matrix M and the resulting diagonal matrix D of A.

$$\left(\textbf{Ans.}\ M = \begin{bmatrix} -1 & 1+\sqrt{5} & 1-\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}\right)$$

6. Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Find a matrix M such that $M^{-1} AM$ is a diagonal matrix.

$$\left(\textbf{Ans.}\ M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)$$

7. Obtain the modal matrix and diagonalize the following matrices:

(i) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$$\left[\textbf{Ans.}\ \text{(i)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{(ii)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right]$$

8. Diagonalize the matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$.

$$\left(\textbf{Ans.} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right)$$

9. Diagonalize $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ by similarity transformation.

$$\left(\textbf{Ans.} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}\right)$$

10. Diagonalize the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$.

$$\left(\textbf{Ans.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

11. Diagonalize the following matrices by orthogonal transformation:

(i) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$\left[\textbf{Ans.} \text{ (i) } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ (ii) } \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$

12. Diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation.

$\left(\textbf{Ans.} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$

13. Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ by orthogonal transformation.

$\left(\textbf{Ans.} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right)$

14. Diagonalize $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by orthogonal transformation.

$\left(\textbf{Ans.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)$ **[AU May 2011]**

5 Quadratic Forms

Chapter Outline

- Definition
- Quadratic Forms Expressed in Matrices
- Linear Transformation of Quadratic Form
- Canonical Form
- Index and Signature of the Quadratic Form
- Nature of Quadratic Forms
- Determination of the Nature of Quadratic Form (QF) without Reduction to Canonical Form

5.1 ❑ DEFINITION

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

For example,

(i) $ax^2 + 2hxy + by^2$

(ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$

(iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

5.2 ❑ QUADRATIC FORM EXPRESSED IN MATRICES

Quadratic form can be expressed as a product of matrices.

Quadratic form = X^TAX.

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (symmetric matrix)

X^T is the transpose of X.

$$X^T AX = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \quad a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \quad a_{13}x_1 + a_{23}x_2 + a_{33}x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{21}x_1x_2 + a_{31}x_1x_3 + a_{12}x_1x_2 + a_{22}x_2^2 + a_{32}x_2x_3 + a_{13}x_1x_3 + a_{23}x_2x_3 + a_{33}x_3^2$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{23} + a_{32})x_2x_3 + (a_{31} + a_{13})x_1x_3$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3$$

(As $a_{21} = a_{12}$, $a_{32} = a_{23}$, $a_{31} = a_{13}$ in a symmetric matrix, in general, $a_{ij} = a_{ji} = \frac{1}{2}$ coefficient of x_{ij} if $i \neq j$.)

5.3 ❑ LINEAR TRANSFORMATION OF QUADRATIC FORM

Let the given quadratic form in n variables be X^TAX where A is a symmetric matrix.

Consider the linear transformation $X = PY$.

Then $X^T = (PY)^T = Y^TP^T$.

$\therefore$ $X^TAX = (Y^TP^T)A(PY) = Y^T(P^TAP)Y = Y^TBY$

where $B = P^TAP$.

Therefore, Y^TBY is also a quadratic form in n variables. Hence, it is a linear transformation of the quadratic form X^TAX under the linear transformation $X = PY$ and $B = P^TAP$.

5.4 ❑ CANONICAL FORM

If a real quadratic form be expressed as a sum or difference of the squares of new variables by means of any real nonsingular linear transformation then the latter quadratic expression is called a canonical form of the given quadratic form.

5.5 ❑ INDEX AND SIGNATURE OF THE QUADRATIC FORM

When the quadratic form X^TAX is reduced to the canonical form, it will contain only r terms, if the rank of A is r. The terms in the canonical form may be positive, zero or negative.

The number (p) of positive terms in the canonical form is called the **index** of the quadratic form.

Number of positive terms – Number of negative terms, i.e., $p - (r - p) = 2p - r$ is called **signature** of the quadratic form.

5.6 ❑ NATURE OF QUADRATIC FORMS

Definite, Semi-definite and Indefinite Real Quadratic Forms

Let X^TAX be a real quadratic form in n – variables $x_1, x_2, \ldots x_n$ with rank r and index p.

Then we say that the quadratic form is

(i) positive definite if $r = n, p = r$.
(ii) negative definite if $r = n, p = 0$.
(iii) positive semi-definite if $r < n, p = r$.
(iv) negative semi-definite if $r < n, p = 0$.

If the canonical form has both positive and negative terms, the quadratic form is said to be indefinite.

Examples:

(i) $x_1^2 + x_2^2$ is positive definite.
(ii) $-x_1^2 - x_2^2$ is negative definite.
(iii) $(x_1 - x_2)^2$ is positive semi-definite.
(iv) $-(x_1 - x_2)^2$ is negative semi-definite.
$x_1^2 - x_2^2$ is indefinite.

➢ **Note**

If X^TAX is positive definite then $|A| > 0$.

5.7 ❑ DETERMINATION OF THE NATURE OF QUADRATIC FORM (QF) WITHOUT REDUCTION TO CANONICAL FORM

Consider the quadratic form

$$X^T AX = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $D_1 = |a_{11}|,\ D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The QF is

(i) positive definite if $D_i > 0$ for $i = 1, 2, 3$;
(ii) negative definite if $D_2 > 0$ and $D_1 < 0, D_3 < 0$;
(iii) positive semi-definite if $D_i > 0$ and at least one $D_i = 0$;
(iv) negative semi-definite if some of the determinants are zero in case (ii); and
(v) indefinite in all other cases.

Criteria for the Nature of Quadratic Form (or Value Class) in Terms of Nature of Eigen Values

Value Class	*Nature of Eigen Values*
Positive definite	Positive Eigen values
Positive semi-definite	Positive Eigen values and at least one is zero
Negative definite	Negative Eigen values
Negative semi-definite	Negative Eigen values and at least one is zero
Indefinite	Positive as well as negative Eigen values

SOLVED EXAMPLES

Example 1 Discuss the nature of the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$. **[KU April 2011]**

Solution The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$D_1 = |8| = 8 > 0,\ D_2 = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 20 > 0 \text{ and } D_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 0$$

$\therefore$ the QF is positive semi-definite. **Ans.**

Example 2 Write down the matrix of the quadratic form $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_2 + 5x_2x_3$

Solution

$$x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3 \qquad (1)$$

Coefficient of $x_1^2 = 1 = a_{11}$,

Coefficient of $x_2^2 = 2 = a_{22}$,

Coefficient of $x_3^2 = -7 = a_{33}$,

$\frac{1}{2}$ coefficient of $x_1x_2 = \frac{1}{2}(-4) = -2 = a_{12}$

$\frac{1}{2}$ coefficient of $x_1x_3 = \frac{1}{2}(8) = 4 = a_{13}$

$\frac{1}{2}$ coefficient of $x_2x_3 = \frac{1}{2}(5) = \frac{5}{2} = a_{23}$

$\therefore$ Eq. (1) can be expressed as X^TAX, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & \frac{5}{2} \\ 4 & \frac{5}{2} & -7 \end{bmatrix}$$

$$\therefore \text{ given quadratic form} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & \frac{5}{2} \\ 4 & \frac{5}{2} & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ **Ans.**

Example 3 Write down the quadratic form corresponding to the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}.$$

Solution Quadratic form = $X^T AX$

$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1 + 2x_2 + 5x_3 \quad 2x_1 + 3x_3 \quad 5x_1 + 3x_2 + 4x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 2x_1x_2 + 5x_3x_1 + 2x_1x_2 + 3x_2x_3 + 5x_1x_3 + 3x_2x_3 + 4x_3^2$$

$$= x_1^2 + 4x_3^2 + 4x_1x_2 + 10x_1x_3 + 6x_2x_3.$$ **Ans.**

Example 4 Reduce the quadratic forms $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 4x_2x_3 + 18x_3x_1$ and $2x_1^2 + 5x_2^2 + 4x_1x_2 + 2x_3x_1$ simultaneously to canonical forms by a real nonsingular transformation. **[KU May 2010]**

Solution The matrix of the first quadratic form is $A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$

The matrix of the second quadratic form is $B = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

The characteristic equation is $|A - \lambda B| = 0$.

i.e.,
$$\begin{vmatrix} 6-2\lambda & 2-2\lambda & 9-\lambda \\ 2-2\lambda & 3-5\lambda & 2 \\ 9-\lambda & 2 & 14 \end{vmatrix} = 0$$

$\Rightarrow \quad 5\lambda^3 - \lambda^2 - 5\lambda + 1 = 0$

i.e., $\quad (\lambda - 1)(5\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \quad \lambda = -1, \frac{1}{5}, 1$

When $\lambda = -1$, $(A - \lambda B)X = 0$, given the equations,

$$8x_1 + 4x_2 + 10x_3 = 0;\ 4x_1 + 8x_2 + 2x_3 = 0;\ 10x_1 + 2x_2 + 14x_3 = 0$$

by solving, $X_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

When $\lambda = \frac{1}{5}$, $(A - \lambda B)X = 0$ gives

$$28x_1 + 8x_2 + 44x_3 = 0;\ 8x_1 + 10x_2 + 10x_3 = 0;\ 44x_1 + 10x_2 + 70x_3 = 0$$

by solving, $X_2 = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

When $\lambda = 1$, $(A - \lambda B)X = 0$ gives

$$4x_1 + 8x_3 = 0;\ -2x_2 + 2x_3 = 0;\ 8x_1 + 2x_2 + 14x_3 = 0$$

$\Rightarrow \quad X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

Since X_1, X_2, X_3 are not pairwise orthogonal, consider the modal matrix P.

Now, $P = \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

$$P^T AP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}\begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the quadratic form X^TAX is reduced to the canonical form $y_1^2 + y_2^2 + y_3^2$.

Now

$$P^T BP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the quadratic form X^TBX is reduced to the canonical form $y_1^2 + 5y_2^2 + y_3^2$. **Ans.**

Example 5 Reduce $6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_3x_1$ into canonical form. Find its nature, rank, index and signature.

[KU Nov. 2010, AU Jan. 2010, KU April 2012]

Solution The matrix of the quadratic form is $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$

The characteristic roots are given by $|A-\lambda I|=0$

i.e., $$\begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{vmatrix}=0$$

$\Rightarrow$ $$\lambda^3-12\lambda^2+36\lambda-32=0$$

$\therefore$ the Eigen values are $\lambda=8,2,2$

The Eigen vectors are obtained by $(A-\lambda I)X=0$

When $\lambda=8$, $(A-\lambda I)X=0$ gives

$$\begin{bmatrix}-2 & -2 & 2\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$\Rightarrow$ $$-2x_1-2x_2+2x_3=0;\ -2x_1-5x_2-x_3=0;\ 2x_1-x_2-5x_3=0$$

$\Rightarrow$ $$X_1=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

When $\lambda=2$, $(A-\lambda I)X=0$ reduces to a single equation $2x_1-x_2+x_3=0$

Putting $x_1=0$, we get $X_2=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$

Again, by putting $x_2=0$, we get $X_3=\begin{bmatrix}1\\ 0\\ -2\end{bmatrix}$

Now $X_1=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}, X_2=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$ and $X_3=\begin{bmatrix}1\\ 0\\ -2\end{bmatrix}$

Here, X_1, X_2, X_3 are not pairwise orthogonal.

(i.e., $X_1^TX_2=0, X_2^TX_3\neq 0, X_3^T X_1=0$)

X_3 is orthogonal to X_2, only when $X_3=\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}$, so that $X_1^TX_2=X_2^TX_3=X_3^TX_1=0$

$$\therefore \text{ the normalized modal matrix is } P=\begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Consider

$$P^T AP=\begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the quadratic form X^TAX is transformed to the canonical form $8y_1^2+2y_2^2+2y_3^2$
Here, rank of the quadratic form = 3, index = 3, signature = 3.
$\therefore$ it is positive definite. **Ans.**

EXERCISE

Part A

1. If the canonical form of a quadratic form is $5y_1^2+6y_2^2$ then the rank is ______.
   (i) 5 (ii) 0 (iii) 2 (iv) 1
2. The nonsingular linear transformation used to transform the quadratic form to canonical form is ______
   (i) $X=N^TY$ (ii) $X=NY$ (iii) $Y=NX$ (iv) $Y=X$
3. Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$.
4. Define a quadratic form and give an example in two and three variables.
5. What do you mean by canonical form of a quadratic form?
6. Define index and signature of a quadratic form.
7. Discuss the nature of the quadratic form $2x^2+5y^2+3z^2+4xy$.
8. Discuss the nature of the quadratic form $2xy+2yz+2zx$.
9. Determine the nature of the following quadratic forms without reducing them to canonical forms:

(i) $x_1^2+3x_2^2+6x_3^2+2x_1x_2+2x_2x_3+4x_3x_1$

(ii) $2x_1^2+x_2^2-3x_3^2+12x_1x_2-8x_2x_3-4x_3x_1$

10. Find the index and signature of the quadratic form, $2x_1^2-5x_2^2+7x_3^2$.
11. State the conditions for a quadratic form to be positive definite and positive semi-definite.
12. Write down the matrices of the following quadratic forms:
    (i) $2x^2+3y^2+6xy$
    (ii) $2x^2+5y^2-6z^2-2xy-yz+8zx$
    (iii) $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$
    (iv) $x_1^2+2x_2^2+3x_3^2+4x_4^2+2x_1x_2+4x_1x_3-6x_1x_4-4x_2x_3+8x_2x_4-12x_3x_4$
13. Write down the quadratic forms corresponding to the following matrices.

(i) $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

14. Write down the matrix of the QF

$$3x_1^2+5x_2^2+5x_3^2-2x_1x_2+2x_2x_3+6x_3x_1$$

15. Define pairwise orthogonal.

Part B

1. Reduce the QF $8x_1^2+7x_2^2+3x_3^2-12x_1x_2-8x_2x_3+4x_3x_1$ to the canonical form through an orthogonal transformation and, hence, show that it is positive definite. Find also a nonzero set of values for x_1, x_2, x_3 that will make the QF zero.

$$\left(\textbf{Ans. } P=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}; Q=3y_2^2+15y_3^2;\ x_1=1,\ x_2=2,\ x_3=2\right)$$

2. Reduce the QF $10x_1^2+2x_2^2+5x_3^2+6x_2x_3-10x_3x_1-4x_1x_2$ to a canonical form by orthogonal reduction. Find also a set of nonzero values of x_1, x_2, x_3 which will make the QF zero.

$$\left(\textbf{Ans. } P=\begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}; Q=3y_2^2+14y_3^2;\ x_1=1,\ x_2=-5,\ x_3=4\right)$$

3. Find the value of λ so that the quadratic form $\lambda(x_1^2+x_2^2+x_3^2)+2x_1x_2-2x_2x_3+2x_3x_1$ may be positive definite. (**Ans.** $\lambda > 2$)
4. Reduce the following quadratic forms to canonical forms or to sum of squares by orthogonal transformation. Write also rank, index and signature.
   (i) $3x^2+5y^2+3z^2-2xy-2yz+2zx$
   (ii) $2x_1^2+2x_2^2+2x_3^2-2x_1x_2+2x_1x_3-2x_2x_3$
   (iii) $3x^2-2x^2-z^2-4xy+8xz+12yz$
   (iv) $x^2+3y^2+3z^2-2yz$
   [**Ans.** (i) $2y_1^2+3y_2^2+6y_3^2$; rank = 3, index = 3, signature = 3
   (ii) $4y_1^2+y_2^2+y_3^2$; rank = 3, index = 3, signature = 3
   (iii) $3y_1^2+6y_2^2-9y_3^2$; rank = 3, index = 2, signature = 1
   (iv) $y_1^2+2y_2^2+4y_3^2$; rank = 3, index = 3, signature = 3]
5. Reduce the QF $2x_1x_2+2x_1x_3-2x_2x_3$ to the canonical form by an orthogonal transformation. (**Ans.** $y_1^2+y_2^2-2y_3^2$)
6. Reduce the QF $x_1^2+3x_2^2+3x_3^2-2x_2x_3$ into the canonical by an orthogonal transformation. (**Ans.** $y_1^2+2y_2^2+4y_3^2$)
7. Reduce the QF y^2+2xy into the canonical form by an orthogonal reduction and state the nature of the QF. (**Ans.** $-y_1^2+y_2^2+y_3^2$; indefinite)
8. Discuss the nature of the following quadratic forms:
   (i) $2x^2+3z^2+2xy$
   (ii) $11x_1^2+14x_1y_1+14x_1z_1+8y_1z_1$
   (iii) $x^2+4xy+6xz-y^2+2yz+4z^2$
   [**Ans.** (i) Positive definite (ii) Indefinite (iii) Positive semi-definite]
9. Reduce the following quadratic forms to canonical forms by orthogonal transformation. State the nature.
   (i) $16x_1x_2-x_3^2$
   (ii) $7x_1^2+6x_2^2+5x_3^2-4x_1x_2-4x_2x_3$
   (iii) $x_1^2+2x_2^2+3x_3^2+4x_1x_2+4x_2x_3$
   [**Ans.** (i) $8y_1^2-y_3^2-8y_3^2$; indefinite (ii) $9y_1^2+6y_2^2+3y_3^2$; positive definite
   (iii) $5y_1^2+2y_2^2-y_3^2$; indefinite]
10. Find the nature of the following:
    (i) $3x^2-2y^2-z^2-4xy+8xz+12yz$
    (ii) $6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_3x_1$
    (iii) $5x^2+26y^2+10z^2+4yz+14xz+6xy$
    [**Ans.**(i) Indefinite (ii) Positive definite (iii) Positive semi-definite]

15 Gradient, Divergence, Curl and Directional Derivative

Chapter Outline

- Introduction
- Partial Differentiation of Vectors
- Scalar and Vector Fields
- Gradient of a Scalar Field
- Properties of Gradient
- Divergence of a Vector Field
- Properties of Divergence and Curl
- Directional Derivative of a Scalar Point Function

15.1 ❑ INTRODUCTION

Vector calculus is a branch of mathematics concerned with multivariate real analysis, i.e., differentiation and integration of vectors in two or more dimensions. It consists of a suite of formula and problem-solving techniques very useful for physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

In vector algebra, we mostly deal with constant vectors, i.e., vectors constant in magnitude and fixed in direction. In vector calculus, we deal with variable vectors, i.e., vectors varying in magnitude or direction or both. Vector calculus is used to model a vast range of engineering phenomena including electrostatic charges, electromagnetic fields, air flow around aircraft, cars and other solid objects, fluid flow around ships and heat flow in nuclear reactors. This chapter starts by explaining what is meant by operators, gradient, divergence and curl. These are used to carry out various differentiation operations in such fields.

15.2 ❑ PARTIAL DIFFERENTIATION OF VECTORS

Consider the vector field $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ where each component v_1, v_2 and v_3 is a function of x, y and z. We can partially differentiate the vector with respect to x as follows:

$$\frac{\partial \vec{v}}{\partial x} = \frac{\partial v_1}{\partial x}\vec{i} + \frac{\partial v_2}{\partial x}\vec{j} + \frac{\partial v_3}{\partial x}\vec{k}$$

This is a new vector with a magnitude and direction different from those of $\vec{v}$.

Partial differentiation with respect to y and z is defined in a similar way as the higher derivatives.

For example,

$$\frac{\partial^2 \vec{v}}{\partial x^2} = \frac{\partial^2 v_1}{\partial x^2}\vec{i} + \frac{\partial^2 v_2}{\partial y^2}\vec{j} + \frac{\partial^2 v_3}{\partial z^2}\vec{k}$$

15.3 ❑ SCALAR AND VECTOR FIELDS

A variable quantity whose value at any point in a region of space depends upon the position of the point is called a **point function**. There are two types of point functions.

Scalar Point Function

A function $\phi(x, y, z)$ is called a **scalar point function** if it associates a scalar with every point in space. The temperature distribution in a heated body, density of a body and potential due to gravity, atmospheric pressure in space are the examples of a scalar point function.

- **Example**

The temperature distribution in a medium or the distribution of atmospheric pressure in space are some examples of scalar point functions.

Vector Point Function

If a function $v(x, y, z)$ defines a vector at every point of a region then $\vec{v}\,(x, y, z)$ is called a vector point function.

- **Example**

The velocity of a moving fluid at any instant or the gravitational force are some examples of vector point functions.

15.4 ❑ GRADIENT OF A SCALAR FIELD

Given a scalar function of x, y, z.

$$\phi = \phi(x, y, z)$$

we can differentiate it partially with respect to each of its independent variables to find $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$.

Then the vector $\frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k}$ turns out to be particularly important. We call this vector the **gradient** of ϕ and denote it by $\nabla\phi$ or grad ϕ.

An alternative form of writing $\nabla\phi$ is as three components $\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$,

i.e.,
$$\nabla\phi = \text{grad } \phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

The process of forming a gradient applies only to a scalar field and the result is always a vector field.

It is often useful to write $\nabla\phi$ in the form $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi$.

where the quantity in brackets is called a **vector operator** and is regarded as operating on the scalar ϕ.

Thus, the vector operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \qquad \text{(read \textbf{nabla} or \textbf{del})}$$

Physical Interpretation of $\nabla\phi$

Consider the scalar field $\phi(x, y, z)$ as describing the temperature throughout a region. This temperature will vary from point to point. At a particular point, it can be shown that $\nabla\phi$ is a vector pointing in the direction in which the rate of temperature increase is greater. $|\nabla\phi|$ is the magnitude of the rate of increase in that direction. Similarly, the rate of temperature decreases greatly in the direction of $-\nabla\phi$.

15.5 ❑ PROPERTIES OF GRADIENT

(a) If ϕ is a constant scalar point function then $\nabla\phi = \overline{0}$.

(b) If ϕ_1 and ϕ_2 are two scalar point functions then

- (i) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$
- (ii) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1, c_2 are constants.
- (iii) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$
- (iv) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}, \phi_2 \neq 0$

15.6 ❑ DIVERGENCE OF A VECTOR FIELD

Given a vector field $\vec{v} = \vec{v}(x, y, z)$)

If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ taking each component in turn and differentiate it partially with respect to x, y and z respectively, we get $\frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial y}, \frac{\partial v_3}{\partial z}$.

If we add the calculated quantities, the result turns out to be a very useful scalar quantity known as the **divergence of** $\vec{v}$, i.e., divergence of $\vec{v}$ = div $\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$.

Alternatively, the notation $\nabla \cdot \vec{v}$ is often used.

$$\therefore \quad \nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (\vec{v})$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (v_1, v_2, v_3)$$

Interpreting the · as a scalar product, we find

$$\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

We note that the process of finding the divergence is always performed on a vector field and the result is always a scalar field.

$$\text{i.e.,} \quad \text{div } \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Physical Interpretation of $\nabla \cdot \vec{v}$

If the vector field $\vec{v}$ represents a fluid velocity field then simply speaking, the divergence of $\vec{v}$ evaluated at a point represents the rate at which fluid is flowing away from or towards that point. If the fluid is flowing away from a point then either the fluid density must be decreasing there or there must be some source providing a supply of new fluid.

If the divergence of a flow is zero at all points then outflow from any point must be matched by an equal in flow to balance this. Such a vector field is said to be **solenoidal**.

15.7 ❑ CURL OF A VECTOR FIELD

A third differential operator is known as **curl**. It is defined rather like a vector product.

$$\text{curl } \vec{v} = \nabla \times \vec{v}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (v_1, v_2, v_3)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

This determinant is evaluated in the usual way except that we must regard $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ as operators, not multipliers.

Thus, for example,

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_1 & v_2 \end{vmatrix} \quad \text{means} \quad \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

Explicitly, we have

$$\text{curl } \vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\vec{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\vec{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\vec{k}$$

➢ **Note**

(i) div $\vec{v}$ gives the rate of outflow per unit volume at a point of the fluid.

(ii) If div $\vec{v} = 0$ everywhere in some region R of space then $\vec{v}$ is called the **solenoidal** vector point function.

(iii) curl $\vec{v}$ is a vector which measures the extent to which individual particles of the fluid are spinning or rotating.

(iv) If curl $\vec{v} = \vec{0}$ then $\vec{v}$ is said to be an **irrotational** vector. Otherwise, it is named a **rotational** vector.

15.8 ❑ PROPERTIES OF DIVERGENCE AND CURL

(i) div $\vec{F}$ is a scalar function and curl $\vec{F}$ is a vector quantity.

(ii) For a constant vector $\vec{a}$, div $\vec{a} = 0$, curl $\vec{a} = \vec{0}$

(iii) $\text{div}(\vec{a} + \vec{b}) = \text{div}\,\vec{a} + \text{div}\,\vec{b}$ or $\nabla \cdot (\vec{a} + \vec{b}) = \nabla \cdot \vec{a} + \nabla \cdot \vec{b}$

(iv) $\text{curl}(\vec{a} + \vec{b}) = \text{curl}\,\vec{a} + \text{curl}\,\vec{b}$ or $\nabla \times (\vec{a} + \vec{b}) = \nabla \times \vec{a} + \nabla \times \vec{b}$

(v) If $\vec{a}$ is a vector function and ϕ is a scalar function then
$\text{div}(\phi\vec{a}) = \phi\,\text{div}\,\vec{a} + (\text{grad}\,\phi)\cdot\vec{a}$ or $\nabla \cdot (\phi\vec{a}) = \phi(\nabla \cdot \vec{a}) + (\nabla\phi)\cdot\vec{a}$

(vi) If $\vec{a}$ is a vector function and ϕ is a scalar function then
$\text{curl}(\phi\vec{a}) = (\text{grad}\,\phi)\times\vec{a} + \phi\,\text{curl}\,\vec{a}$ or $\nabla \times (\phi\vec{a}) = (\nabla\phi)\times\vec{a} + \phi(\nabla \times \vec{a})$

(vii) $\nabla(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a})$

(viii) $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$

(ix) $\nabla\times(\vec{a}\times\vec{b}) = (\nabla\cdot\vec{b})\vec{a} - (\nabla\cdot\vec{a})\vec{b} + (\vec{b}\,.\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$

15.9 ❑ DIRECTIONAL DERIVATIVE OF A SCALAR POINT FUNCTION

The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi\cdot\vec{d}$ and is called the directional derivative of ϕ in the direction of $\vec{d}$.

Let P and Q be two neighbouring points whose position vectors with respect to the origin O be $\vec{r}$ and $\vec{r} + \overrightarrow{\Delta r}$ respectively, so that $\overrightarrow{PQ} = \overrightarrow{\Delta r}$ and $PQ = \Delta r$.

Let ϕ and $\phi + \Delta\phi$ be the values of a scalar point function ϕ at the points P and Q respectively.

Then $\dfrac{d\phi}{dr} = \underset{\Delta r \to 0}{Lt}\left(\dfrac{\Delta\phi}{\Delta r}\right)$ is called the **directional derivative** of ϕ in the direction OP,

i.e., $\dfrac{d\phi}{dr}$ gives the rate of change of ϕ with respect to the distance measured in the direction of $\vec{r}$.

In particular, $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ are the directional derivatives of ϕ at $P(x, y, z)$ in the directions of the coordinate axes.

SOLVED EXAMPLES

Example 1 Find the divergence and curl of the vector $\vec{v} = (xyz)\vec{i} + (3x^2y)\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point (2, –1, 1).

Solution

$$\text{div } \vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot ((xyz)\vec{i} + (3x^2y)\vec{j} + (xz^2 - y^2z)\vec{k})$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

$\therefore$ $$(\text{div } \vec{v})_{(2,-1,1)} = -1 + 12 + 4 - 1 = 14$$

$$\text{curl } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & (xz^2 - y^2z) \end{vmatrix}$$

$$= \vec{i}(-2yz - 0) - \vec{j}(z^2 - xy) + \vec{k}(6xy - xz)$$

$\therefore$ $$(\text{curl } \vec{v})_{(2,-1,1)} = \vec{i}(2) - \vec{j}(1+2) + \vec{k}(-12-2)$$

$$= 2\vec{i} - 3\vec{j} - 14\vec{k}$$ **Ans.**

Example 2 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, show that

(i) grad $r = \frac{\vec{r}}{r}$ (ii) grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

(iii) $\nabla r^n = nr^{n-2}\vec{r}$ (iv) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector

Solution $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \quad \text{or} \quad r^2 = x^2 + y^2 + z^2$$

Differentiating partially with respect to, x we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

(i) grad $r = \nabla r = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)r$

$$= \left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$$

$$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}[x\vec{i} + y\vec{j} + z\vec{k}] = \frac{\vec{r}}{r}$$

(ii) grad $\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$

$$= \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \vec{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \vec{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right) + \vec{j}\left(\frac{-1}{r^2}\cdot\frac{y}{r}\right) + \vec{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{\vec{r}}{r^3}$$

(iii) $\nabla r^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)r^n$

$$= \vec{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \vec{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \vec{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \vec{i}\left(nr^{n-1}\frac{x}{r}\right) + \vec{j}\left(nr^{n-1}\frac{y}{r}\right) + \vec{k}\left(nr^{n-1}\frac{z}{r}\right)$$

$$= nr^{n-2}[x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= nr^{n-2}\vec{r}$$

(iv) Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ where a_1, a_2, a_3 are constants.

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

$\therefore$ $$\nabla(\vec{a}\cdot\vec{r}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \vec{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \vec{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \vec{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= \vec{i}(a_1) + \vec{j}(a_2) + \vec{k}(a_3)$$

$$= \vec{a}$$

$\therefore$ $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ **Proved.**

Example 3 Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point (2, –1, 1) in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$.

Solution The directional derivative

$$= \nabla\phi \cdot \text{unit vector in the direction of } \vec{i} + 2\vec{j} + 2\vec{k}$$

$$= \nabla\phi \cdot \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \nabla\phi \cdot \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(y^2) + \vec{j}(2xy + z^3) + \vec{k}(3z^2y)$$

Hence, $(\nabla\phi)_{(2,-1,1)} = \vec{i} - 3\vec{j} - 3\vec{k}$

∴ the required directional derivative

$$= (\vec{i} - 3\vec{j} - 3\vec{k}) \cdot \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$= -\frac{11}{3}$$ **Ans.**

Example 4 Find the directional derivative of $\phi = xy^2z^3$ at (1, 1, 1) along the normal surface $x^2 + xy + z^3 = 3$ at (1, 1, 1). **[KU Nov. 2010]**

Solution The equation of the surface $x^2 + xy + z^3 = 3$ is identified with $\psi(x, y, z) = c$.

∴ $\psi(x, y, z) = x^2 + xy + z^3$ and $c = 3$.

The direction of the normal to this surface is the same as that of $\nabla\psi$.

Now, $\nabla\psi = \vec{i}(2x + y) + \vec{j}(x) + \vec{k}(3z^2)$.

∴ $$(\nabla\psi)_{(1,1,1)} = 3\vec{i} + \vec{j} + 3\vec{k} = \vec{b} \text{ (say)}$$

$$\phi = xy^2z^3$$

$$\nabla\phi = \vec{i}(y^2z^3) + \vec{j}(2xyz^3) + \vec{k}(3xy^2z^2)$$

$$(\nabla\phi)_{(1,1,1)} = \vec{i} + 2\vec{j} + 3\vec{k}$$

Directional derivative of ϕ in the direction of $\vec{b} = \nabla\phi \cdot \hat{b}$

$$= \frac{\nabla\phi \cdot \vec{b}}{|\vec{b}|} \left(\text{as } \hat{b} = \frac{\vec{b}}{|\vec{b}|}, \vec{b} \neq \vec{0} \right)$$

$$= \frac{(\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (3\vec{i} + \vec{j} + 3\vec{k})}{\sqrt{9 + 1 + 9}} = \frac{3 + 2 + 9}{\sqrt{19}} = \frac{14}{\sqrt{19}} \text{ units}$$ **Ans.**

Example 5 Find the values of the constants a, b, c so that $\vec{F} = (axy + bz^3)\vec{i} + (3x^2 - cz)\vec{j} + (3xz^2 - y)\vec{k}$ may be irrotational. For these values of a, b, c, find also the scalar potential of $\vec{F}$. **[KU Nov. 2011]**

Solution Given $\vec{F}$ irrotational.

$$\nabla \times \vec{F} = 0$$

$$\therefore \quad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (axy + bz^3) & (3x^2 - cz) & (3xz^2 - y) \end{vmatrix} = 0$$

i.e.,
$$\vec{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - cz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(axy + bz^3)\right] +$$
$$\vec{k}\left[\frac{\partial}{\partial x}(3x^2 - cz) - \frac{\partial}{\partial y}(axy + bz^3)\right] = 0$$

$$\vec{i}(-1 + c) - \vec{j}(3z^2 - 3bz^2) + \vec{k}(6x - ax) = 0$$

$\therefore\ c - 1 = 0,\ 3z^2 - 3bz^2 = 0,\ 6x - ax = 0$

$\Rightarrow a = 6,\ b = 1,\ c = 1$

Using these values of a, b, c,

$$\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k} \qquad (1)$$

Let ϕ be the scalar potential of $\vec{F}$.

$$\therefore \quad \vec{F} = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

By comparing with (1) we get

$$\frac{\partial\phi}{\partial x} = 6xy + z^3,\ \frac{\partial\phi}{\partial y} = 3x^2 - z,\ \frac{\partial\phi}{\partial z} = 3xz^2 - y$$

Integrating partially with respect to the concerned variables, we get

$\phi = 3x^2y + xz^3 + a$ function independent of x (2)

$\phi = 3x^2y - zy + a$ function independent of y (3)

$\phi = xz^3 - yz + a$ function independent of z (4)

From (2), (3) and (4), we get

$$\phi = 3x^2y + xz^3 - yz + c$$ **Ans.**

EXERCISE

Part A

1. Define divergence and curl of a vector point function.
2. Evaluate
   (i) div $(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k})$ at the point (1, 2, 3)
   (ii) div $[(xy\sin z)\vec{i} + (y^2\sin x)\vec{j} + (z^2\sin xy)\vec{k}]$ at the point $\left(0, \dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

3. If $\vec{F}=(x+y+1)\vec{i}+\vec{j}-(x+y)\vec{k}$, show that $\vec{F}\cdot \text{curl } \vec{F}=0$.
4. Find the divergence and curl of the vectors:
   (i) $\vec{v}=(xyz)\vec{i}+3x^2y\vec{j}+(xz^2-y^2z)\vec{k}$
   (ii) $\vec{R}=(x^2+yz)\vec{i}+(y^2+zx)\vec{j}+(z^2+xy)\vec{k}$
5. If $\vec{P}=5t^2\vec{i}+t^3\vec{j}-t\vec{k}$ and $\vec{Q}=2\sin t\vec{i}-\cos t\vec{j}+5t\vec{k}$, find (i) $\frac{d}{dt}(\vec{P}\cdot\vec{Q})$, and (ii) $\frac{d}{dt}(\vec{P}\times\vec{Q})$.
6. A particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time variable. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.
7. Find $\nabla\phi$ if $\phi = \log(x^2 + y^2 + z^2)$.
8. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).
9. Evaluate div $\vec{F}$ and curl $\vec{F}$ at the point (1, 2, 3) given
   (i) $\vec{F}=x^2yz\vec{i}+xy^2z\vec{j}+xyz^2\vec{k}$ (ii) $\vec{F}=\text{grad}\,(x^3y+y^3z+z^3x-x^2y^2z^2)$
10. Show that each of the following vectors are solenoidal:
    (i) $(-x^2+yz)\vec{i}+(4y-z^2x)\vec{j}+(2xz-4z)\vec{k}$
    (ii) $3y^4z^2\vec{i}+4x^3z^2\vec{j}+3x^2y^2\vec{k}$
11. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A}\times\vec{B}$ is solenoidal.
12. Compute the gradient of the scalar function and evaluate it at the given point.
    (i) $x^3 - 3x^2y^2 + y^3$, (1, 2)
    (ii) $x\sin(yz)+y\sin(xz)+z\sin(xy), \left(0, \frac{\pi}{4}, 1\right)$
    (iii) In $(x^2 + y^2 + z^2)$, (3, –4, 5)
    (iv) $(x^2+y^2+z^2)^{\frac{1}{2}}, (1,1,1)$
    (v) $x^3+y^3\sin 4y+z^2, \left(1, \frac{\pi}{3}, 1\right)$
13. Prove the following properties of gradient (f and g are scalar functions):
    (i) $\nabla(fg) = f\nabla g + g\nabla f$ (ii) $\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f - f\nabla g}{g^2}, g\neq 0$
14. Compute div $\vec{v}$, curl $\vec{v}$ and verify that div $(\text{curl }\vec{v})=0$.
    (i) $\vec{v}=x\vec{i}+2y\vec{j}+z\vec{k}$
    (ii) $\vec{v}=(x^2+y^2+z^2)^{\frac{3}{2}}(x\vec{i}+y\vec{j}+z\vec{k})$
    (iii) $\vec{v}=(x^2-y^2)\vec{i}+4xy\vec{j}+(x^2-xy)\vec{k}$
    (iv) $\vec{v}=xe^{-y}\vec{i}+2ze^{-y}\vec{j}+xy^2\vec{k}$
15. Compute grad f and verify that curl (grad f) = 0.
    (i) $f(x, y, z) = 16xy^3z^2$
    (ii) $f(x, y, z) = e^{x+y+z}$
    (iii) $f(x, y, z) = x\sin(x + y + z)$

16. Show that $\vec{v}\cdot \text{curl}\ \vec{v}=0$ if $\vec{v}=-(x+y+2)\vec{i}-2\vec{j}+(x+y)\vec{k}$.

17. If $\vec{a}$ is a constant vector and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, prove the following identities:

    (i) $\text{div}\ (\vec{a}\times\vec{r})=0$

    (ii) $\text{curl}\ (\vec{a}\times\vec{r})=2\vec{a}$

    (iii) $\nabla\cdot(\vec{a}\times\vec{v})=-\vec{a}\cdot(\nabla\times\vec{v}), \vec{v}$ is any vector

    (iv) $\vec{a}\times \text{curl}\,\vec{r}=0$

    (v) $\nabla\cdot[(\vec{r}\cdot\vec{r})a]=2(\vec{r}\cdot\vec{a})$

18. If $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $r=|\vec{r}|$, show that $\text{div}\left(\frac{\vec{r}}{r^3}\right)=0$.

19. Find the directional derivative of $f(x, y, z) = xy^2 + 4xyz + z^2$ at the point (1, 2, 3) in the direction of $3\vec{i}+4\vec{j}-5\vec{k}$.

20. Give the physical interpretation of $\nabla\cdot\vec{v}$.

21. Define solenoidal and irrotational vectors.

22. Prove that the vector $\vec{F} = (3x + 2y + 4z)\vec{i} + (2x + 5y + 4z)\vec{j} + (4x + 4y - 8z)\vec{k}$ is both solenoidal and irrotational.

23. If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, find grad (div $\vec{F}$)

24. Find 'a' such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

25. If $\vec{v} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + \lambda z)\vec{k}$ is solenoidal, find λ.

Part B

1. If $\vec{v}=\frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}}$, show that $\nabla\cdot\vec{v}=\frac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla\times\vec{v}=\vec{0}$.

2. If $\vec{A}=(3xz^2)\vec{i}-(yz)\vec{j}+(x+2z)\vec{k}$, find curl (curl $\vec{A}$)

    [**Ans.** $-6x\vec{i}+(6z-1)\vec{k}$]

3. Show that the vector field $\vec{V} = (\sin y + z)\vec{i} + (x\cos y - z)\vec{j} + (x - y)\vec{k}$ is irrotational.

4. Find the constants a, b, c so that $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational. (**Ans.** $a = 4, b = 2, c = -1$)

5. Show that $\vec{E}=\frac{\vec{r}}{r^2}$ is irrotational.

6. If $\vec{E}$ and $\vec{H}$ are irrotational, prove that $\vec{E}\times\vec{H}$ is solenoidal.

7. For a solenoidal vector $\vec{F}$, prove that curl curl curl curl $\vec{F}=\nabla^4\vec{F}$.

8. Find the directional derivative of $\nabla\cdot(\nabla\phi)$ at the point (1, –2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$. $\left(\textbf{Ans.}\ \frac{1724}{\sqrt{21}}\right)$

9. If $\frac{d\vec{u}}{dt}=\vec{w}\times\vec{u}$ and $\frac{d\vec{v}}{dt}=\vec{w}\times\vec{v}$, prove that $\frac{d}{dt}(\vec{u}\times\vec{v})=\vec{w}\times(\vec{u}\times\vec{v})$.

10. Show that $\nabla\cdot(\phi\nabla\psi-\psi\nabla\phi)=\phi\nabla^2\psi-\psi\nabla^2\phi$.

11. Calculate (i) curl (grad f), given $f(x, y, z) = x^2 + y^2 - z$, and (ii) curl (curl $\vec{A}$) given $\vec{A} = x^2y\vec{i} + y^2z\vec{j} + z^2y\vec{k}$. [**Ans.** (i) 0 (ii) $2(x+z)\vec{j} + 2y\vec{k}$]
12. If $u = x^2yz$, $v = xy - 3z^2$, find (i) $\nabla(\nabla u \cdot \nabla v)$, and (ii) $\nabla \cdot (\nabla u \cdot \nabla v)$.

[**Ans.** (i) $2(y^3 + 3x^2y - 6xy^2)z\vec{i} + 2(3xy^2 + x^3 - 6x^2y)z\vec{j} + 2(xy^2 + x^3 - 3x^2y)y\vec{k}$ (ii) 0]

13. Show that the vector field $\vec{v}$ is irrotational and find a scalar function $f(x, y, z)$ such that $\vec{v} = \nabla f$.

    (i) $(y^2 - x^2 + y)\vec{i} + x(2y+1)\vec{j}$ $\left[\textbf{Ans. } x(y^2+y) - \frac{x^3}{3} + c\right]$

    (ii) $e^{xy}(y\vec{i} + x\vec{j}) + 2e^z\vec{k}$ (**Ans.** $e^{xy} + 2e^z + c$)

    (iii) $\cos\,(x^2 + y^2 + z^2)(x\vec{i} + y\vec{j} + z\vec{k})$ $\left[\textbf{Ans. } \frac{1}{2}\sin(x^2 + y^2 + z^2) + c\right]$

14. Let $f(x, y, z)$ be a solution of the Laplace equation $\nabla^2 f = 0$. Then show that ∇f is a vector which is both irrotational and solenoidal.
15. If $\phi_1 = x + y + z$, $\phi_2 = x + y$ and $\phi_3 = -(2xz + 2yz + z^2)$, show that $\nabla\phi_1 \cdot (\nabla\phi_2 \times \nabla\phi_3) = 0$.
16. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1). $\left[\textbf{Ans. } \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)\right]$

Unit V

Vector Differentiation

14 Vector Differentiation

Chapter Outline

- Introduction
- Types of Vectors
- Components of a Vector
- Product of Two Vectors

14.1 ❑ INTRODUCTION

The development of the concept of vectors was influenced by the works of the German mathematician **H G Grassmann** (1809–1877) and the Irish mathematician **W R Hamilton** (1805–1865). It is interesting to note that both were linguists specialised in Sanskrit literature.

The best features of Quaternion calculus and Cartesian geometry were united largely through the efforts of the American mathematician J B Gibbs (1839–1903) and Q Heariside (1850–1925) of England and a new subject called **vector algebra** was created. The term *vector* was due to Hamilton and it was derived from the Latin word "to carry". The theory of vectors was also based on Grassman's theory of extension.

Vectors are the ideal tools for the fruitful study of many ideas in geometry and physics. Vector algebra is widely used in the study of certain types of problems in geometry, mechanics, engineering and other branches of applied mathematics.

The physical quantities may be divided into two groups: (i) scalars, and (ii) vectors.

Certain physical quantities are fully described by a single number: for example, the mass of a stone, the speed of a car, etc. Such quantities are called **scalars**.

A scalar quantity, or simply a scalar, has magnitude but is not related to any definite direction in space. Examples of such quantities are mass, volume, density, temperature, work, quantity of heat, electric charge and potential. To specify a scalar, we need a unit quantity of the same type and the ratio (m) which the given quantity bears to this unit so that it may be expressed as m times the unit. The number m is called the **measure** of the quantity in terms of the chosen unit. It is the measures

d, m, V, v, E of density, mass, volume, speed and energy respectively that enter into the equations of physics and mechanics.

On the other hand, some quantities are not fully described until a direction is specified in addition to the number.

For example, a velocity of 20 metres per second due east is different from a velocity of 20 metres per second due north. These quantities are called **vectors**. There are many engineering applications in which vector and scalar quantities play important roles.

A **vector quantity,** or simply a **vector,** has magnitude and is related to a definite direction in space; thus it is an arrow or directed line segment. For example, speed, potential, work and energy are scalars, while velocity, momentum, electric and magnetic forces, the position of a robot and the state-space representation of a system can all be described by vectors (Fig. 14.1).

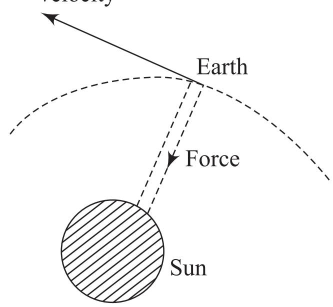


Fig. 14.1

A vector (arrow) has a tail called **initial point** and a tip called **terminal point**. We denote vectors by boldface letters ***a, b,*** *v* or $\vec{a}, \vec{b}$ and $\vec{v}$.

The line segment AB of 3-unit length in Fig. 14.2 can represent a vector in the direction shown by the arrow on AB. This vector is denoted by $\overrightarrow{AB}$. Note that $\overrightarrow{AB} \neq \overrightarrow{BA}$. The vector $\overrightarrow{AB}$ is directed from A to B, but $\overrightarrow{BA}$ is directed from B to A. $\overrightarrow{AB}$ is also denoted by $\vec{a}$.

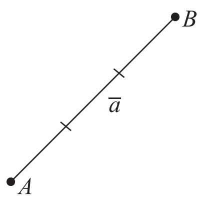


Fig. 14.2

Magnitude of a Vector

The **modulus** or **magnitude** of a vector $\vec{a} = \overrightarrow{AB}$ is a positive number which is the measure of its length. The length (or magnitude) of a vector $\vec{a}$ (length of the arrow) is also called the **norm** (or Euclidean norm) of $\vec{a}$ and is denoted by $|\vec{a}|$.

Thus, $|\vec{a}| = a; |\vec{b}| = b; |\vec{c}| = c$, etc., $|\overrightarrow{AB}| = AB; |\overrightarrow{CD}| = CD$, etc.,

14.2 ❑ TYPES OF VECTORS

Unit Vector

A vector whose modulus is unity is called a unit vector. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ (read as '*a* cap'). Thus, $|\hat{a}| = 1$.

The unit vectors parallel to $\vec{a}$ are $\pm\hat{a}$.

➢ **Note**

$\vec{a} = |\vec{a}|\hat{a}$ [i.e., any vector = (its modulus) × (unit vector in that direction)

$$\Rightarrow \qquad \hat{a} = \frac{\vec{a}}{|\vec{a}|}; \quad [\vec{a} \neq \vec{0}]$$

∴ in general,
Unit vector in any direction = Vector in that direction/Modulus of the vector

Equal Vectors

Equal vectors are those vectors which have equal magnitude, same direction (parallel) and same sense (arrow).

Like and Unlike Vectors

Like vectors are those vectors which have same direction (parallel) and same sense (arrow). The magnitude may be different.

Unlike vectors are those vectors which have same direction (parallel) and opposite sense (arrow). The magnitude may be different.

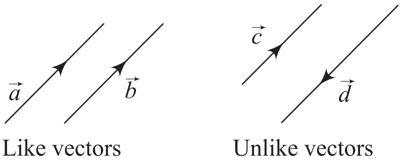


Fig. 14.3

Zero Vector, or Null Vector

Zero vector, or null vector, or a void vector is that vector whose magnitude is zero. The zero vector is denoted by $\vec{0}$.

Vectors other than the null vector are called **proper vectors**.

Co-initial Vectors

Vectors having the same initial point are called co-initial vectors.

- **Co-terminus Vectors**

Vectors having the same terminal point are called co-terminus vectors.

Collinear Vectors, or Parallel Vectors

Vectors are said to be collinear or parallel if they have the same line of action or have the lines of action parallel to one another.

Coplanar Vectors

Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane.

Negative Vector

The negative of vector is a vector whose magnitude is equal to that of the given vector with same direction (parallel) but opposite sense (arrow).

Thus, if $\overrightarrow{AB} = \vec{a}$ then $\overrightarrow{BA} = -\vec{a}$.

Reciprocal of a Vector

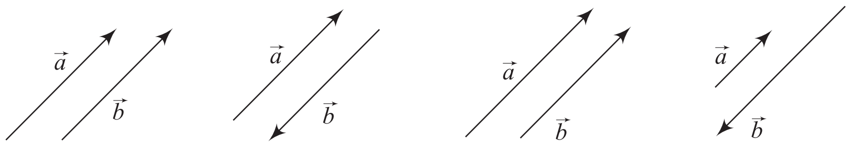
Let $\vec{a}$ be a nonzero vector. The vector which has the same direction as that of $\vec{a}$ but has magnitude reciprocal to that of $\vec{a}$ is called the reciprocal of $\vec{a}$ and is written as $|(\vec{a})^{-1}| = \frac{1}{a}$.

Free Vector and Localized Vector

When we are at liberty to choose the origin of the vector at any point then it is said to be a free vector. But when it is restricted to a certain specified point then the vector is said to be localized vector.

Equality of Vectors

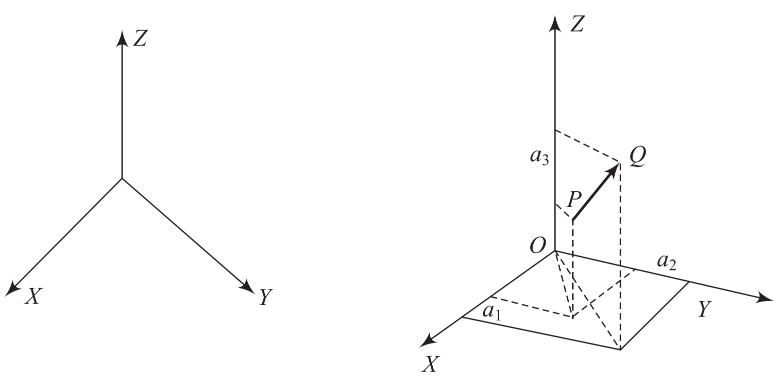
Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, if they have the same length and the same direction. In Fig. 14.4, vectors $\vec{a}$ and $\vec{b}$ are equal even though their locations differ. Hence, a vector can be arbitrarily translated, that is, its initial point can be chosen arbitrarily.



Equal vectors | Vectors having the same length but different directions | Vectors having the same direction but different lengths | Vectors having different lengths and different directions

Fig. 14.4

14.3 ❑ COMPONENTS OF A VECTOR



Cartesian coordinate system | Components of a vector

Fig. 14.5

Choose the *XYZ* Cartesian coordinate system in space. If a given vector $\vec{a}$ has initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$, the three numbers

$$a_1 = x_2 - x_1, \quad a_2 = y_2 - y_1, \quad a_3 = z_2 - z_1$$

are called the **components** of the vector $\vec{a}$ with respect to that coordinate system. It is simply represented as $\vec{a} = [a_1, a_2, a_3]$.

Position Vector

Consider a Cartesian coordinate system. The **position vector** $\vec{r}$ of a point $A(x, y, z)$ is the vector with the origin $O(0, 0, 0)$ as the initial point, and A as the terminal point.

Thus, $\vec{r} = [x, y, z]$.

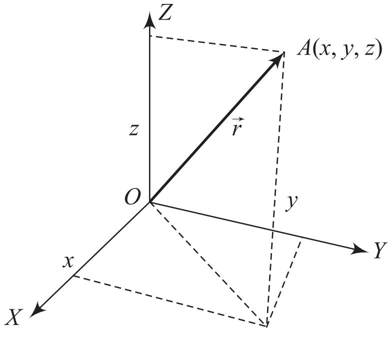


Fig. 14.6

Addition of Vectors (Vector Addition)

The sum $\vec{a}+\vec{b}$ of two vectors $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ is obtained by adding the corresponding components,

i.e., $$\vec{a}+\vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

Geometrically, the vectors are placed as the initial point of $\vec{b}$ at the terminal point of $\vec{a}$. Then $\vec{a}+\vec{b}$ is the vector drawn from the initial point of $\vec{a}$ to the terminal point of $\vec{b}$.

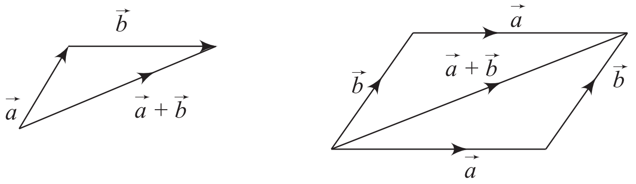


Fig. 14.7

- **Basic Properties of Vector Addition**

(i) **Commutative** $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) **Associative** $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

(iii) $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$

(iv) $\vec{a}+(-\vec{a})=\vec{0}$, where $-\vec{a}$ denotes the vector having the length $|\vec{a}|$ and the direction opposite to that of $\vec{a}$.

Scalar Multiplication

The product $c\vec{a}$ of any vector $\vec{a} = [a_1, a_2, a_3]$ and any scalar c (real number) is the vector obtained by multiplying each component of $\vec{a}$ by c.

i.e., $$c\vec{a} = C[a_1, a_2, a_3] = [ca_1, ca_2, ca_3]$$

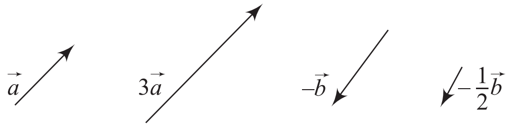


Fig. 14.8

➢ **Note**

(i) If $\vec{a} \neq \vec{0}$ then $c\vec{a}$ with $c > 0$ has the direction of $\vec{a}$, and with $c < 0$ the direction is opposite to $\vec{a}$.
(ii) The length of $c\vec{a}$ is given by $|c\vec{a}| = |c||\vec{a}|$ and $c\vec{a} = 0$ if $\vec{a} = \vec{0}$ or $c = 0$ or both.

- **Basic Properties of Scalar Multiplication**

For any scalars c and k and for any vectors $\vec{a}$ and $\vec{b}$:

(i) $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
(ii) $(c + k)\vec{a} = c\vec{a} + k\vec{a}$
(iii) $c(k\vec{a}) = (ck)\vec{a} = ck\vec{a}$
(iv) $1.\ \vec{a} = \vec{a}$
(v) $0.\ \vec{a} = \vec{0}$
(vi) $(-1)\ \vec{a} = -\vec{a}$

Vector Subtraction

Subtraction of one vector from another is performed by adding the corresponding negative vector,

i.e., if we seek $\vec{a} - \vec{b}$, we form $\vec{a} + (-\vec{b})$.

Orthogonal Vectors

If the angle between two vectors $\vec{a}$ and $\vec{b}$ is 90°, that is $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}$ and $\vec{b}$ are said to be **orthogonal.**

- **Example**

Position vectors provide a useful means of determining the position of a robot. There are many different types of robots but a common type uses a series of rigid links connected together by flexible joints. Usually, the mechanism is anchored at one point.

The anchor point is X and the tip of the robot is situated at Y. The final link is sometimes called the hand of the robot. The hand often has rotating and gripping facilities and its size, relative to the rest of the robot, is usually quite small. Each of the robot links can be represented by a vector. The vector $\vec{d}$ corresponds to the hand. A common requirement in robotics is to calculate the position of the tip of the hand to ensure that it does not collide with other objects. This can be achieved by defining a set of Cartesian coordinates with the origin at the anchor point of the robot X. Each of the link vectors can then be represented in terms of these coordinates.

For example, in Fig. 14.9,

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k},$$
$$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}, \vec{d} = d_1\vec{i} + d_2\vec{j} + d_3\vec{k}$$

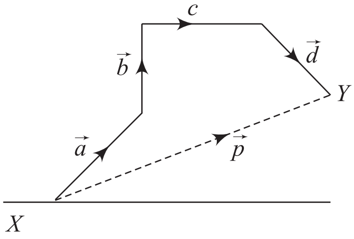


Fig. 14.9

The position of the tip of the hand can be calculated by adding these vectors together.

$\therefore \qquad \vec{p} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$

$$= (a_1 + b_1 + c_1 + d_1)\vec{i} + (a_2 + b_2 + c_2 + d_2)\vec{j} + (a_3 + b_3 + c_3 + d_3)\vec{k}$$

14.4 ❑ PRODUCT OF TWO VECTORS

The product of two vectors results in two different ways: one, a number and the other, a vector. So there are two types of products of two vectors, namely, scalar product and vector product. They are written as $\bar{a} \cdot \bar{b}$ and $\bar{a} \times \bar{b}$.

Inner Product (Dot Product or Scalar Product)

Now, we shall define a multiplication of two vectors that gives a scalar as the product.

The **inner product** or **dot product** $\vec{a} \cdot \vec{b}$ (read "$\vec{a}$ dot $\vec{b}$") of two vectors $\vec{a}$ and $\vec{b}$ is the product of their length times the cosine of their angle.

$$\left.\begin{array}{l} \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \gamma \text{ if } \vec{a} \neq 0, \vec{b} \neq 0 \\ \vec{a} \cdot \vec{b} = 0 \text{ if } \vec{a} = 0 \text{ or } \vec{b} = 0 \end{array}\right\} \tag{14.1}$$

The angle γ, $0 \le \gamma \le \pi$ between $\vec{a}$ and $\vec{b}$ is measured when the vectors have their initial points coinciding.

If $\qquad \vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$ then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{14.2}$$

Since the cosine in (14.1) may be positive, zero or negative, so may be the inner product (Fig. 14.10).

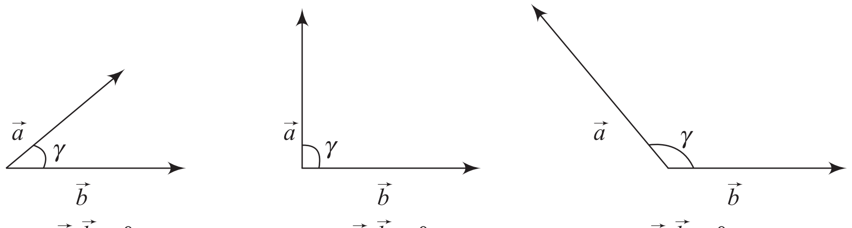


Fig. 14.10 *Angle between vectors and value of inner product*

➢ **Note**

(i) A vector $\vec{a}$ is said to be **orthogonal** to a vector $\vec{b}$ if $\vec{a} \cdot \vec{b} = 0$.
(ii) Zero vector is orthogonal to every vector.
(iii) For nonzero vectors, $\vec{a} \cdot \vec{b} = 0$ if and only if (iff) cos $\gamma = 0$. Thus, $\gamma = \pi/2$.

- **Example: Work Done by a Force as Dot Product**

Consider a body on which a constant force $\vec{p}$ acts. Let the body be given a displacement $\vec{d}$. Then the work done by $\vec{p}$ in the displacement is defined as

$$W = |\vec{p}||\vec{d}| \cos \gamma = \vec{p} \cdot \vec{d},$$

If $\gamma < 90°$ then $W > 0$ (Fig. 11).

If $\vec{p}$ and $\vec{d}$ are orthogonal then the work(W) = 0.

If $\gamma > 90°$ then $W < 0$ which means that in the displacement one has to do work against the force.

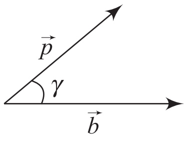


Fig. 14.11

➢ **Note**

(i) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (the scalar product is commutative)

(ii) $k(\vec{a}\cdot\vec{b}) = (k\vec{a}\cdot\vec{b})$, where k is a scalar.

(iii) $(\vec{a}+\vec{b})\cdot\vec{c} = (\vec{a}\cdot\vec{c}) + (\vec{b}\cdot\vec{c})$ (distributive)

(iv) If $\vec{a}$ and $\vec{b}$ are parallel vectors then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$

(v) If $\vec{a}$ and $\vec{b}$ are orthogonal vectors then $\vec{a}\cdot\vec{b} = 0$.

(vi) $\vec{a}\cdot\vec{a} = a^2$

(vii) $\vec{i}\cdot\vec{i} = 1, \vec{j}\cdot\vec{j} = 1, \vec{k}\cdot\vec{k} = 1$
$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = 0$
$\vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{j} = 0$
$\vec{k}\cdot\vec{i} = \vec{i}\cdot\vec{k} = 0$

(viii) $(m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(m\vec{b})$

(ix) The angle between two vectors $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

(x) $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Vector Product (Cross Product)

The vector or cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined to be a vector such that

(i) its magnitude is $|\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

(ii) its direction is perpendicular to both vectors $\vec{a}$ and $\vec{b}$

(iii) it forms a right-handed system

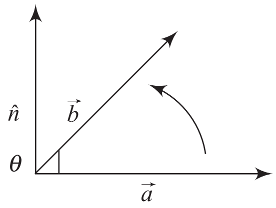
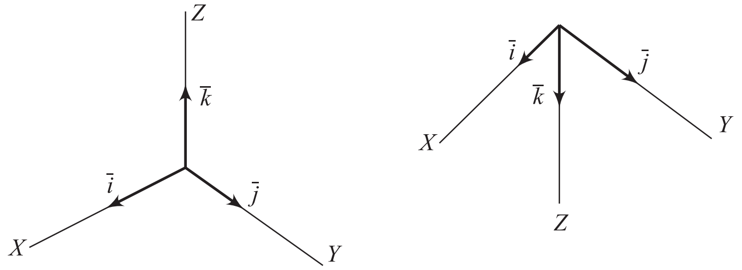


Fig. 14.12

Let $\hat{n}$ be a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. $\vec{a}, \vec{b}$ and $\hat{n}$ are forming a right-handed system, then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\cdot\hat{n}$

➢ **Note**

A Cartesian coordinate system is called **right-handed** if the corresponding unit vectors $\vec{i}, \vec{j}, \vec{k}$ in the positive directions of the axes form a right-handed triple as in Fig. 14.13(a).



(a) Right-handed (b) Left-handed

Fig. 14.13

The system is called **left-handed** if the sense of $\vec{k}$ is reversed as in Fig. 14.13(b). In applications, we prefer right-handed systems.

(i) $\vec{a}\times\vec{b}=\vec{b}\times\vec{a}$ (vector product is not commutative)

(ii) $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (distributive)

(iii) $k(\vec{a}\times\vec{b})=(k\vec{a})\times\vec{b}=\vec{a}\times(k\vec{b})$ where k is a scalar.

(iv) $|\vec{a}\times\vec{b}|$ is the area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

(v) Two vectors are parallel if $\vec{a}\times\vec{b}=0$. In particular, $\vec{a}\times\vec{a}=0$.

(vi) $\vec{i}\times\vec{i}=0, \vec{j}\times\vec{j}=0\ \vec{k}\times\vec{k}=0$

$\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\vec{k}$

$\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\vec{i}$

$\vec{k}\times\vec{i}=-\vec{i}\times\vec{k}=\vec{j}$

(vii) $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$

(viii) If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Applications of Vector Products

(i) If A be any point on AB whose position vector with respect to O is $\vec{r}$ then $\vec{r}\times\vec{F}$ represents the moment or torque of $\vec{F}$ (acting along AB) about O.

(ii) If $\vec{F}$ represents a force and $\vec{d}$ is the displacement of its point of application, $\vec{F}\cdot\vec{d}$ represents the work done by the force.

Unit VIII

Vector Integration

Chapter 20: Line Integral, Surface Integral and Integral Theorems

20 Line Integral, Surface Integral and Integral Theorems

Chapter Outline

20.1 ❑ INTRODUCTION

In multiple integrals, we generalized integration from one variable to several variables. Our goal in this chapter is to generalize integration still further to include integration over curves or paths and surfaces. We will define integration not just of functions but also of vector fields. Integrals of vector fields are particularly important in applications involving the "field theories" of physics, such as the theory of electromagnetism, heat transfer, fluid dynamics and aerodynamics.

In this chapter, we shall define line integrals and surface integrals. We shall see that a line integral is a natural generalization of a define integral and a surface integral is a generalization of a double integral. Line integrals can be transformed into double integrals or into surface integrals and conversely. Triple integrals can be transformed into surface integrals and vice versa. These transformations are of great practical importance. Theorems of Green, Gauss and Stokes serve as powerful tools in many applications as well as in theoretical problems.

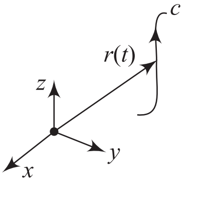


Fig. 20.1

In this chapter, we study the three main theorems of Vector Analysis: Green's Theorem, Stokes' Theorem and the Divergence Theorem. This is a fitting conclusion to the text because each of these theorems is a vector generalization of the Fundamental Theorem of calculus. This chapter is thus the culmination of efforts to extend the concepts and methods of single-variable calculus to the multivariable setting. However, far from being a terminal point, vector analysis the gateway to the field theories of mathematics physics and engineering. This includes, first and foremost, the theory of electricity and magnetism as expressed by the famous *Maxwell's equations*. It also includes fluid dynamics, aerodynamics, analysis of continuous matter, and at a more advanced level, fundamental physical theories such as general relativity and the theory of elementary particles.

Curves

Curves in space are important in calculus and in physics (for instance, as paths of moving bodies).

A curve C in space can be represented by a vector function

$$\vec{r}(t) = [x(t), y(t), z(t)] \quad (20.1)$$
$$= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

where x, y, z are Cartesian coordinates. This is called a **parametric representation** of the curve (Fig. 20.1), t is called the **parameter** of the representation. To each value t_0 of t, there corresponds a point of C with position vector $\vec{r}(t_0)$, that is with coordinates $x(t_0)$, $y(t_0)$ and $z(t_0)$.

The parameter t may be time or something else. Equation (20.1) gives the **orientation** of C, a direction of travelling along C, so that t increasing is called the **positive sense** on C given by (20.1) and that of decreasing t is the **negative sense**.

- **Examples**

Straight line, ellipse, circle, etc.

The concept of a line integral is a simple and natural generalization of a definite integral

$$\int_a^b f(x)dx \quad (20.2)$$

In (20.2), we integrate the **integrand** $f(x)$ from $x = a$ to $x = b$ along the x-axis. In a line integral, we integrate a given function, called the integrand, along a curve C in space (or in the plane).

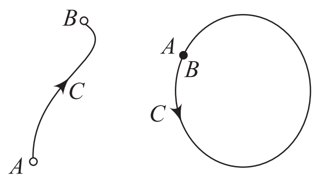
Hence, curve integral would be a better turn, but line integral is standard.

We represent a curve C by a parametric representation

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, (a \le t \le b)$$

We call C the **path of integration**, $A{:}\vec{r}(a)$ its **initial point** and $B{:}\vec{r}(b)$, its **terminal point**. The curve C is now oriented. The direction from A to B, in which t increases, is called the positive direction on C. We can indicate the direction by an arrow [Fig. 20.2(a)].

The points A and B may coincide [Fig. 20.2(b)]. Then C is called a **closed path**.



(a) **(b)**

Fig. 20.2

➢ **Note**

(i) A **plane curve** is a curve that lies in a plane in space.
(ii) A curve that is not plane is called a **twisted curve**.

20.2 ❑ INTEGRATION OF VECTORS

If two vector functions $\vec{F}(t)$ and $\vec{G}(t)$ be such that $\frac{d\vec{G}(t)}{dt} = \vec{F}(t)$, then $\vec{G}(t)$ is called an integral of $\vec{F}(t)$ with respect to the scalar variable t and we write $\int \vec{F}(t)dt = \vec{G}(t)$. If $\vec{C}$ be an arbitrary constant vector, we have $\vec{F}(t) = \frac{d\vec{G}(t)}{dt} = \frac{d}{dt}[\vec{G}(t) + \vec{C}]$, then $\int \vec{F}(t)dt = \vec{G}(t) + \vec{C}$. This is called the indefinite integral of $\vec{F}(t)$ and its definite integral is $\int_a^b \vec{F}(t)dt = [\vec{G}(t) + \vec{C}]_a^b = \vec{G}(b) - \vec{G}(a)$.

20.3 ❑ LINE INTEGRAL

Any integral which is to be evaluated along a curve is called a **line integral**. Consider a continuous vector point function $\vec{F}(R)$ which is defined at each point of the curve C in space. Divide C into n parts at the points $A = p_0, p_1 \dots p_{i-1}, p_i \dots p_n = B$

Let their position vectors be $\vec{R}_0, \vec{R}_1 \dots \vec{R}_{i-1}, \vec{R}_i \dots \vec{R}_n$

Let $\vec{v}_i$ be the position vector of any point on the arc $P_{i-1} P_i$

Now consider the sum $S = \sum_{i=0}^{n} \vec{F}(\vec{v}_i) \cdot \delta\vec{R}_i$ where $\delta\overrightarrow{R_i} = \overrightarrow{R_i} - \overrightarrow{R_{i-1}}$.

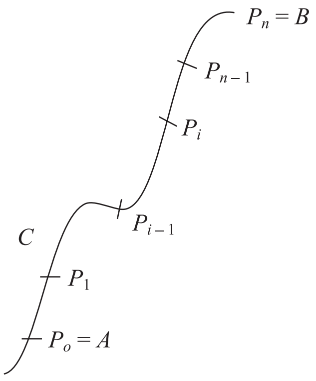


Fig. 20.3

The limit of this sum as $n - \infty$ in such a way that $|\delta\overrightarrow{R_i}| \to 0$, provided it exists, is called the **tangential line integral** of $\vec{F}(\vec{R})$ along C which is a scalar and is symbolically written as

$$\int_C \vec{F}(\vec{R}) \cdot \overrightarrow{dR} \text{ or } \int_C \vec{F} \cdot \frac{d\vec{R}}{dt} \cdot dt$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\vec{F}(\vec{R}) = f(x,y,z)\vec{i} + \phi(x,y,z)\vec{j} + \psi(x,y,z)\vec{k}$ and $d\vec{R} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

then $\int_C \vec{F}(\vec{R}) \cdot d\vec{R} = \int_C (fdx + \phi dy + \psi dz)$.

Two other types of line integrals are $\int_C \vec{F} \times d\vec{R}$ and $\int_C fd\vec{R}$ which are both vectors.

20.4 ❑ CIRCULATION

In fluid dynamics, if $\vec{F}$ represents the velocity of a fluid particle then the line integral $\int_C \vec{F}\cdot d\vec{r}$ is called the circulation of $\vec{F}$ around the curve. When the circulation of $\vec{F}$ around every closed curve in a region E vanishes, $\vec{F}$ is said to be **irrotational** in E.

Conservative Vector

If the value of $\int_A^B \vec{F}\cdot\overrightarrow{dr}$ does not depend on the curve C, but only on the terminal points A and B, $\vec{F}$ is called a **conservative vector**.

A force field $\vec{F}$ is said to be **conservative** if it is derivable from a potential function ϕ, i.e., $\vec{F} =$ grad ϕ. Then curl $(\vec{F})$ = curl $(\nabla\phi) = 0$.
$\therefore$ if $\vec{F}$ is **conservative** then $\text{curl}(\vec{F}) = 0$ and there exists a scalar potential function ϕ such that $\vec{F} = \nabla\phi$.

20.5 ❑ APPLICATIONS OF LINE INTEGRALS

Work Done by a Force

Let $\vec{v}\,(x, y, z) = v_1(x, y, z)\,\vec{i} + v_2(x, y, z)\,\vec{j} + v_3(x, y, z)\,\vec{k}$ be a vector function defined and continuous at every point on C. Then, the integral of the tangential component of $\vec{v}$ along the curve C from a point P on to the point Q is given by

$$\int_P^Q \vec{v}\cdot\overrightarrow{dr} = \int_{C_1} \vec{v}\cdot d\vec{r} = \int_{C_1} v_1 dx + v_2 dy + v_3 dz$$

where C_1 is the part of C, whose initial and terminal points are P and Q.

Let $\vec{v} = \vec{F}$, variable force acting on a particle which moves along a curve C. Then the work done W by the force $\vec{F}$ in displacing the particle from the point P to the point Q along the curve C is given by

$$W = \int_P^Q \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}$$

where C_1 is the part of C whose initial and terminal points are P and Q.

Suppose $\vec{F}$ is a conservative vector field; then $\vec{F}$ can be written as $\vec{F}$ = grad ϕ, where ϕ is a scalar potential.
Then, the work done

$$W = \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (\text{grad}\,\phi)\cdot d\vec{r}$$

$$= \int_{C_1} \left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right] = \int_P^Q d\phi = \left[\phi(x, y, z)\right]_P^Q$$

$\therefore$ work done depends only on the initial and terminal points of the curve C_1, i.e., the work done is independent of the path of integration. The units of work depend on the units of $|\vec{F}|$ and on the units of distance.

➢ **Note**

(i) **Condition for $\vec{F}$ to be conservative**

If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = 0$.

It is possible only when $\vec{F} = \nabla\phi$. which $\Rightarrow$ $\vec{F}$ is conservative.

$\therefore$ if $\vec{F}$ is an irrotational vector, it is conservative.

(ii) If $\vec{F}$ is irrotational (and, hence, conservative) and C is a closed curve then

$\oint_C \vec{F} \cdot d\vec{r} = 0$. [$\because$ $\phi(A) = \phi(B)$, as A and B coincide].

20.6 ❑ SURFACES

A surface S may be represented by $F(x, y, z) = 0$.

The parametric representation of S is of the form

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

and the continuous functions $u = \phi(t)$ and $v = \phi(t)$ of a real parameter t represent a curve C on this surface S.

If S has a unique normal at each of its points whose direction depends continuously on the points of S then the surface S is called a **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions then it is called a **piecewise smooth surface**. For example, the surface of a sphere is smooth while the surface of a cube is piecewise smooth.

If a surface S is smooth from any of its points P, we may choose a unit normal vector $\vec{n}$ of S at P. The direction of $\vec{n}$ is then called the **positive normal direction of S at *P***. A surface S is said to be **orientable** or **two-sided**, if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface. If the positive direction of the normal is reversed as we move around a curve on S passing through P then the surface is **non-orientable** (i.e., one-sided) (Fig. 20.4).

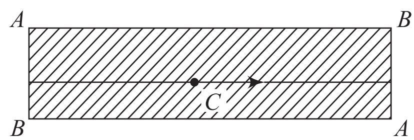


Fig. 20.4

● **Example**

A sufficiently small portion of a smooth surface is always orientable (Fig. 20.5).

A Mobius strip is an example of a non-orientable surface. A model of a Mobius strip can be made by taking a long rectangular piece of paper, making a half-twist and sticking the shorter sides together so that the two points A and the two points B coincide; then the surface generated is non-orientable.

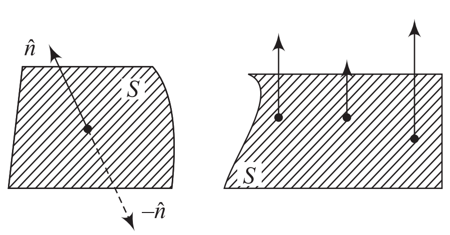


Fig. 20.5

20.7 ❑ SURFACE INTEGRALS

Any integral which is to be evaluated over a surface is called a **surface integral**.

Let S be a two-sided surface, one side of which is considered arbitrarily as the positive side.

Let $\vec{F}$ be a vector point function defined at all points of S. Let ds be the typical elemental surface area in S surrounding the point $P(x, y, z)$.

Let $\hat{n}$ be the unit vector normal to the surface S at $P(x, y, z)$, drawn in the positive side (or outward direction).

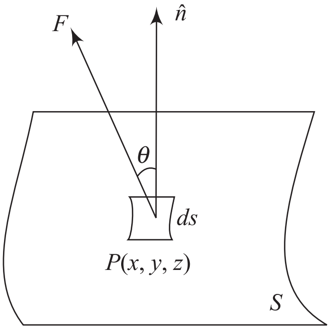


Fig. 20.6

Let θ be the angle between $\vec{F}$ and $\hat{n}$.

$\therefore$ the normal component of $\vec{F} = \vec{F} \cdot \hat{n} = F \cos\theta$.

The integral of this normal component through the elemental surface area ds over the surface S is called the **surface integral** of $\vec{F}$ over S and denoted as $\int_S F \cos\theta \cdot ds$ or $\int_S \vec{F} \cdot \hat{n} ds$.

If $d\vec{s}$ is a vector whose magnitude is ds and whose direction is that of $\hat{n}$, then $\overrightarrow{ds} = \hat{n} \cdot ds$. $\therefore$ $\int_S \vec{F} \cdot \hat{n} ds$ can also be written as $\int_S \vec{F} \cdot \overrightarrow{ds}$.

➢ **Note**

(i) If S in a closed surface, the outer surface is usually chosen as the positive side.

(ii) $\int_S \phi d\vec{s}$ and $\int_S \vec{F} \times d\vec{s}$ where ϕ is a scalar point function are also surface integrals.

(iii) The surface integral $\int_S \vec{F} \cdot d\vec{s}$ is also denoted as $\iint_S \vec{F} \cdot d\vec{s}$.

(iv) If $\vec{F}$ represents the velocity of a fluid particle then the total outward flux of $\vec{F}$ across a closed surface S is the surface integral $\int_S \vec{F} \cdot d\vec{s}$.

(v) When the flux of $\vec{F}$ across every closed surface S in a region E vanishes, $\vec{F}$ is said to be a **solenoidal vector point function** in E.

(vi) It may be noted that $\vec{F}$ could equally well be taken as any other physical quantity such as gravitational force, electric force, magnetic force, etc.

20.8 ❑ VOLUME INTEGRALS

Any integral which is to be evaluated over a volume is called a **volume integral**.

If V is a volume bounded by a surface S then the triple integrals $\iiint_V \phi dv$ and $\iiint_V \vec{F} dv$ are called volume integrals. The first of these is a scalar and the second is a vector.

20.9 ❑ INTEGRAL THEOREMS

The following three theorems in vector calculus are of importance from theoretical and practical considerations:

(i) Green's theorem in a plane

(ii) Stokes' theorem

(iii) Gauss' divergence theorem

Green's theorem provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R. Green's theorem is also called the **first fundamental theorem** of integral vector calculus.

Stokes' theorem transforms line integrals into surface integrals and conversely. This theorem is a generalization of Green's theorem. It involves the curl.

Gauss' divergence theorem transforms surface integrals into a volume integral. It is named Gauss' divergence theorem because it involves the divergence of a vector function.

We shall give the statements of the above theorems (without proof) and apply them to solve problems.

Green's Theorem in a Plane

If C is a simple closed curve enclosing a region R in the xy-plane and $P(x, y)$, $Q(x, y)$ and its first-order partial derivatives are continuous in R then $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ where C is described in the anticlockwise direction.

Stokes' Theorem (Relation between Line Integral and Surface Integral)

Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} \cdot ds$

Gauss' Divergence Theorem or Gauss' Theorem of Divergence (Relation between Surface Integral and Volume Integral)

The surface integral of the normal component of a vector function $\vec{F}$ taken around a closed surface S is equal to the integral of the divergence of $\vec{F}$ taken over the volume V enclosed by the surface S.

Mathematically, $\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V \text{div}\, \vec{F} \cdot dv$.

SOLVED EXAMPLES

Example 1 If $\vec{F} = 3xy\vec{i}, - y^2\vec{j}$, evaluate $\int_C \vec{F} \cdot dr$, where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution Let $x = t$, then the parametric equations of the parabola $y = 2x^2$ are $x = t$, $y = 2t^2$.
At the point (0, 0), $x = 0$ and so $t = 0$.
At the point (1, 2), $x = 1$ and so $t = 1$.
If $\vec{r}$ is the position vector of any point (x, y) in C, then

$$\vec{r} = x\vec{i} + y\vec{j}$$
$$= t\vec{i} + 2t^2\vec{j}$$

Also in terms of t, $\vec{F} = 3t(2t^2)\vec{i} - (2t^2)^2\vec{j}$

$$= 6t^3\vec{i} - 4t^4\vec{j}$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$
$$= \int_0^1 (6t^3\vec{i} - 4t^4\vec{j})\cdot(\vec{i} + 4t\vec{j})dt$$
$$= \int_0^1 (6t^3 - 16t^5)dt$$
$$= \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1$$
$$= \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$ **Ans.**

Example 2 Evaluate $\iint_S \vec{A}\cdot\hat{n}ds$ where $\vec{A} = (x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant. **[KU May 2010]**

Solution A vector normal to the surface S is given by

$$\nabla(2x + y + 2z) = 2\vec{i} + \vec{j} + 2\vec{k}$$

$\therefore$ $\hat{n} = a$ unit vector normal to the surface S

$$= \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\vec{k}\cdot\hat{n} = \vec{k}\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = \frac{2}{3}$$

$$\therefore \quad \iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$$

where R is the projection of S

Now, $$\vec{A}\cdot\hat{n} = [(x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}]\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$$
$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$
$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right)$$

$\left(\text{since on the plane } 2x + y + 2z = 6,\ z = \dfrac{6-2x-y}{2}\right)$

$$= \frac{2}{3}y(y + 6 - 2x - y)$$

$$= \frac{4}{3}y(3-x)$$

Hence, $\displaystyle\iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$.

$$= \iint_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}dx\,dy$$

$$= \int_0^3\int_0^{6-2x} 2y(3-x)dy\,dx$$

$$= \int_0^3 2(3-x)\left(\frac{y^2}{2}\right)_0^{6-2x} dx$$

$$= \int_0^3 (3-x)(6-2x)^2\,dx$$

$$= 4\int_0^3 (3-x)^3\,dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3$$

$$= 81$$ **Ans.**

Example 3 If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$ then evaluate $\displaystyle\iiint \nabla\cdot\vec{F}\cdot dV$, where V is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution $\nabla\cdot\vec{F} = \dfrac{\partial}{\partial x}(2x^2 - 3z) + \dfrac{\partial}{\partial y}(-2xy) + \dfrac{\partial}{\partial z}(-4x)$

$$= 4x - 2x = 2x$$

$\therefore$ $\displaystyle\iiint_V \nabla\cdot F\cdot dv = \iiint_V 2x\,dx\,dy\,dz$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x[z]_0^{4-2x-2y}\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2x(4-2x-2y)\,dy\,dx = \int_0^2 \int_0^{2-x} [4x(2-x)-4xy]\,dy\,dx$$

$$= \int_0^2 [4x(2-x)y - 2xy^2]_0^{2-x} \cdot dx$$

$$= \int_0^2 [4x(2-x)^2 - 2x(2-x)^2]\,dx$$

$$= \int_0^2 2x(2-x)^2\,dx$$

$$= 2\int_0^2 (4x-4x^2+x^3)\cdot dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left[8 - \frac{32}{3} + 4\right] = \frac{8}{3}$$ **Ans.**

Example 4 Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ and the curve C is the rectangle in the xy-plane bounded by $y = 0$, $y = b$, $x = 0$, $x = a$.

Solution In the xy-plane, $z = 0$

$$\vec{r} = x\vec{i} + y\vec{j},\ d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+y^2)dx - 2xydy \quad (1)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \quad (2)$$

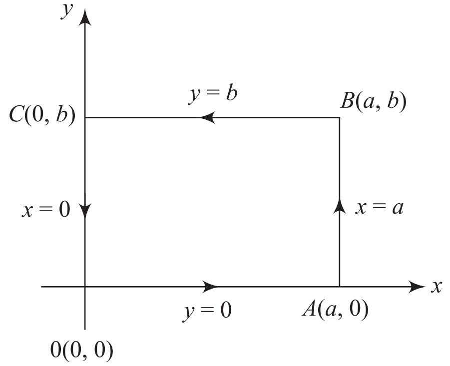


Fig. 20.7

Along OA, $y = 0$; $dy = 0$ and x varies from 0 to a
Along AB, $x = a$; $dx = 0$ and y varies from 0 to b

Along BC, $y = b$; $dy = 0$ and x varies from a to 0
Along CO, $x = 0$; $dx = 0$ and y varies from b to 0
Hence, from (1) and (2),

$$\int_C \vec{F}\cdot\overrightarrow{dr} = \int_{x=0}^{a} x^2\,dx - \int_{y=0}^{b} 2ay\,dy + \int_{x=a}^{0}(x^2+b^2)\,dx + \int_b^0 0\cdot dy$$

$$= \left(\frac{x^3}{3}\right)_0^a - (ay^2)_0^b + \left(\frac{x^3}{3}+b^2x\right)_a^0 + 0$$

$$= \left(\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2\right) = -2ab^2$$ **Ans.**

Example 5 Find the work done by the force $\vec{F} = (2xy+z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ when it moves a particle from (1, –2, 1) to (3, 1, 4) along any path. **[AU Dec. 2011]**

Solution Since the equation of the path is not given, the work done by the force $\vec{F}$ depends only on the terminal points.

Consider $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy+z^3) & x^2 & 3xz^2 \end{vmatrix}$

$$= \vec{i}[0-0] - \vec{j}[3z^2-3z^2] + \vec{k}[2x-2x] = 0$$

$\Rightarrow$ $\vec{F}$ is irrotational

Hence, $\vec{F}$ is conservative

Since $\vec{F}$ is irrotational, we have $\vec{F} = \nabla\phi$

It is easy to see that $\phi = x^2y + xz^3 + C$

$\therefore$ work done by $\vec{F} = \displaystyle\int_{(1,-2,1)}^{(3,1,4)} \vec{F}\cdot d\vec{r}$

$$= \int_{(1,-2,1)}^{(3,1,4)} \nabla\phi\cdot\overrightarrow{dr} = \int_{(1,-2,1)}^{(3,1,4)} d\phi \quad [\text{as } \nabla\phi\cdot\overrightarrow{dr} = d\phi]$$

$$= [\phi]_{(1,-2,1)}^{(3,1,4)}$$

$$= [x^2y + xz^3 + C]_{(1,-2,1)}^{(3,1,4)}$$

$$= (201 + C) - (-1 + C) = 202$$ **Ans.**

Example 6 Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = e^x \sin y\vec{i} + e^x \cos y\vec{j}$; and C is the rectangle whose vertices are $(0,0), (1,0), \left(1, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.

Solution

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\therefore \quad \vec{F}\cdot d\vec{r} = e^x \sin y \cdot dx + e^x \cos y \cdot dy$$

Now along OA, $y = 0$; $dy = 0$
along AB, $x = 1$; $dx = 0$

along BC, $y = \dfrac{\pi}{2}$; $dy = 0$

along CO, $x = 0$; $dx = 0$
$\therefore$ circulation round the rectangle $OABC$ is

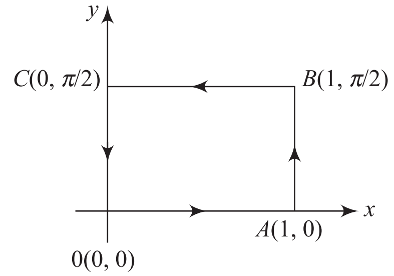


Fig. 20.7

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (e^x \sin y\,dx + e^x \cos y\,dy)$$

$$= \int_{OA} o + \int_{AB} e^1 \cos y\,dy + \int_{BC} e^x \sin\frac{\pi}{2}dx + \int_{CO} \cos y\,dy$$

$$= 0 + \int_0^{\frac{\pi}{2}} e \cos y \cdot dy + \int_1^0 e^x \sin\frac{\pi}{2}dx + \int_{\frac{\pi}{2}}^0 \cos y\,dy$$

$$= [e \sin y]_0^{\frac{\pi}{2}} + [e^x]_1^0 + [\sin y]_{\frac{\pi}{2}}^0 = e + (1-e) - 1 + 0 = 0 \quad \textbf{Ans.}$$

Example 7 Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Solution Total work done

$$= \int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\vec{i} - 5z\vec{j} + 10x\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C [3xydx - 5zdy + 10xdz]$$

$$= \int_{t=1}^{2} [3(t^2+1)(2t^2)d(t^2+1) - 5t^3 d(2t^2) + 10(t^2+1)d(t^3)]$$

$$= \int_{t=1}^{2} [6t^2(t^2+1)(2tdt) - 20t^4 dt + 30t^2(t^2+1)dt]$$

$$= \int_{t=1}^{2} [12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2]dt$$

$$= \int_{t=1}^{2} [12t^5 + 10t^4 + 12t^3 + 30t^2]dt$$

$$= 12\left[\frac{t^6}{6}\right]_1^2 + 10\left[\frac{t^5}{5}\right]_1^2 + 12\left[\frac{t^4}{4}\right]_1^2 + 30\left[\frac{t^3}{3}\right]_1^2$$

$$= 12\left[\frac{2^6}{6} - \frac{1}{6}\right] + 10\left[\frac{2^5}{5} - \frac{1}{5}\right] + 12\left[\frac{2^4}{4} - \frac{1^4}{4}\right] + 30\left[\frac{2^3}{3} - \frac{1^3}{3}\right]$$

$$= 12 \cdot \frac{63}{6} + 10 \cdot \frac{31}{5} + 12 \cdot \frac{15}{4} + 30 \cdot \frac{7}{3}$$

$$= 126 + 62 + 45 + 70$$

$$= 303$$

Ans.

Example 8 If $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$, evaluate $\iint_S \vec{F} \cdot \hat{n} ds$ where S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. **[AU Dec. 2009]**

Solution The surface of the cube consists of the following six faces:

(a) Face *LMND*
(b) Face *TQPO*
(c) Face *QPNM*
(d) Face *TODL*
(e) Face *TQMl*
(f) Face *ODNP*

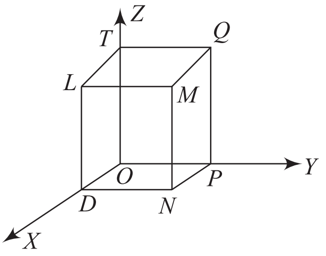


Fig. 20.8

Now, for the face *LMND*:

$$\hat{n} = \vec{i}, x = OD = 1$$

Hence, $$\iint_S F \cdot \hat{n} ds = \iint_{LMND} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{i}\, dy\, dz$$

$$= \iint_{LMND} 4xz\, dy\, dz = 4 \int_{LMND} z\, dy\, dz \qquad (\because x = 1)$$

$$= 4 \int_{z=0}^{1} \int_{y=0}^{1} z\, dy\, dz = 4\left[\left(\frac{z^2}{2}\right)_0^1 (y)_0^2\right] = 2 \qquad (1)$$

For the face *TQPO*: $\hat{n} = -\vec{i}, x = 0$

Hence, $$\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iint_{TQPO} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{i}) dy\, dz$$

$$= \iint_{TQPO} (-4xz) dy\, dz = 0 \qquad (\because x = 0) \qquad (2)$$

For the face *OPNM*: $\hat{n} = \vec{j}, y = 1$

Hence, $$\iint_S \vec{F} \cdot \hat{n} ds = \iint_{QPNM} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{j}\, dx\, dz$$

$$= \iint_{QPNM} (-y^2\, dx\, dz) = \iint_{QPNM} -dx\, dz \quad (\because y = 1)$$

$$= -\int_{z=0}^{1}\int_{x=0}^{1} dx\,dz = -[x]_0^1[z]_0^1 = -1 \tag{3}$$

For the face *TODL*: $\hat{n} = -\vec{j}, y = 0$

Hence, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{TODL} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j})dx\,dz$

$$= \iint_{TODL} (y^2 dxdz) = 0 \quad (\because y = 0) \tag{4}$$

For the face *TQML*: $\hat{n} = \vec{k}, z = 1$

Hence, $\iint_{TQML} \vec{F}\cdot\hat{n}\,ds = \iint_{TQML} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\,dx\,dy$.

$$= \iint_{TQML} yz\,dx\,dy = \iint_{TQML} y\,dx\,dy \qquad (\because z = 1)$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} y\,dx\,dy = [x]_0^1\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} \tag{5}$$

For the face *ODNP*: $\hat{n} = -\vec{k}, z = 0$

Hence, $\iint_{ODNP} \vec{F}\cdot\hat{n}\,ds = \iint_{ODNP} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{k})\cdot dx\,dy$

$$= \iint_{ODNP} (-yz)dx\,dy = 0, \quad (\because z = 0) \tag{6}$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$$ **Ans.**

Example 9 Verify Stokes' theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}$ over the surface of a cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the *XOY* plane (open at the bottom). **[KU May 2010]**

Solution Consider the surface of the cube as shown in the figure. Bounding path is *OABCO* shown by arrows.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}]\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (y - z + 2)dx + (yz + 4)dy - xz\,dz$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \tag{1}$$

Along *OA*, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2dx = (2x)_0^2 = 4$$

Along *AB*, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4dy = 4(y)_0^2 = 8$$

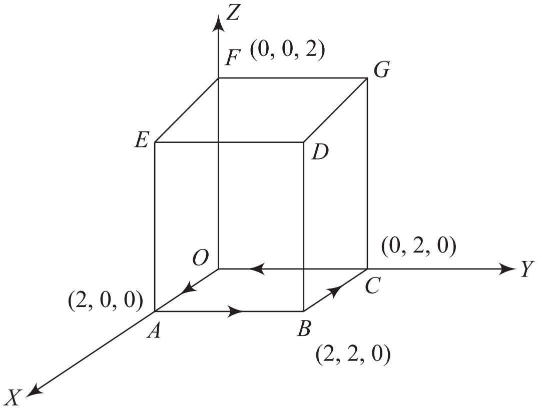


Fig. 20.9

Along *BC*, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_0^2 (2 - 0 + 2)dx = (4x)_2^0 = -8$$

Along *CO*, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int (y - 0 + 2) \times 0 + (0 + 4)dy - 0$$

$$= 4\int dy = 4(y)_2^0 = -8$$

On putting the values of these integrals in (1), we get

$$\int_C \vec{F} \cdot d\vec{r} = 4 + 8 - 8 = -4$$

To obtain surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= (0 - y)\vec{i} - (-z + 1)\vec{j} + (0 - 1)\vec{k} = -y\vec{i} + (z - 1)\vec{j} - \vec{k}$$

Here, we have to integrate over the five surfaces, *ABDE*, *OCGF*, *BCGD*, *OAEF*, *DEFG*.
Over the surface *ABDE*: $x = 2$, $\hat{n} = \vec{i}$, $ds = dydz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{i}\, dy\, dz$$

$$= \iint_S -y\, dydz = -\int_0^2 y\, dy \int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$$

Over the surface *OCGF*: $x = 0$, $\hat{n} = -\vec{i}$, $ds = dy\, dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{i}) dy\, dz$$

$$= \iint_S y\, dy\, dz = \int_0^2 y\, dy \int_0^2 dz = \left[\frac{y^2}{2}\right]_0^2 = 4$$

Over the surface *BCGD*: $y = 2$, $\hat{n} = \vec{j}$, $ds = dx\, dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{j}\, dx\, dz$$

$$= \iint_S (z-1) dx\, dz$$

$$= \int_0^2 dx \int_0^2 (z-1)\, dz$$

$$= [x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *OAEF*: $y = 0$, $\hat{n} = -\vec{j}$, $ds = dx\, dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{j}) dx\, dz$$

$$= -\iint_S (z-1) dx\, dz$$

$$= -\int_0^2 dx \int_0^2 (z-1) dz$$

$$= -[x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *DEFG*: $z = 2$, $\hat{n} = \vec{k}$, $ds = dx\,dy$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{k}\, dx\, dy$$

$$= -\iint dx\, dy = -\int_0^2 dx \int_0^2 dy$$

$$= -[x]_0^2 [y]_0^2 = -4$$

Total surface integral = –4 + 4 + 0 + 0 – 4 = –4

Thus $\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \int_C \vec{F} \cdot d\vec{r} = -4$

which verifies Stokes' theorem. **Verified.**

Example 10 Verify Green's theorem in the plane for $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where *C* is a square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

Solution Given integrand is of the form $Mdx + Ndy$, where $M = x^2 - xy^3$, $N = y^2 - 2xy$.

Now to verify Green's theorem, we have to verify that

$$\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \iint_R (-2y + 3xy^2)dx\, dy \qquad (1)$$

Consider $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where the curve *C* is divided into four parts, hence the line integral along *C* is nothing but the sum of four line integrals along four lines *OA*, *AB*, *BC* and *CO*.

Along *OA* : $y = 0$, $dy = 0$ and x varies from 0 to 2.

Hence, $$\int_{OA} [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \int_{x=0}^{2} x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

Along *AB* : $x = 2$, $dx = 0$, and y varies from 0 to 2.

Hence, $$\int_{AB} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_0^2 (y^2 - 4y)dy = \left(\frac{y^3}{3} - 4\frac{y^2}{2}\right)_0^2$$

$$= \left(\frac{8}{3}\right) - 8 = -\frac{16}{3}$$

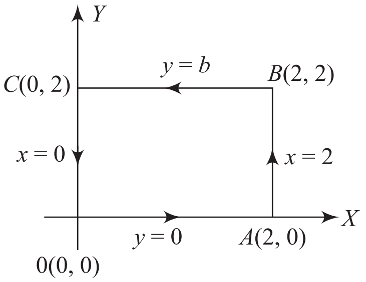


Fig. 20.10

Along *BC*: $y = 2$, $dy = 0$ and x varies from 2 to 0.

Hence, $$\int_{BC} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_{x=2}^{0} (x^2 - 8x)dx = \left(\frac{x^3}{3} - 8\frac{x^2}{2}\right)_2^0$$

$$= 0 - 0 - \frac{8}{3} + 16 = \frac{40}{3}$$

Along CO : $x = 0$, $dx = 0$ and y varies from 2 to 0

Hence, $\int_{CO} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$

$$= \int_{y=2}^{0} y^2 dy = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$$

$$\therefore \quad \int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8 \qquad (2)$$

Now consider

$$\iint_R (-2y + 3xy^2)dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2} (-2y + 3xy^2)dy\,dx$$

$$= \int_{x=0}^{2} \left(-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_{x=0}^{2} \left[-4 + 3x\left(\frac{8}{3}\right)\right] dx = \left(-4x + 8\frac{x^2}{2}\right)_0^2$$

$$= -8 + 16 + 0 = 8 \qquad (3)$$

From (2) and (3), we observe that the relation (1) is true.
Hence, Green's theorem is verified. **Ans.**

Example 11 Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. **[KU Nov. 2010]**

Solution For verification of the divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$

$$\therefore \quad \iiint_V \text{div}\, \vec{F} dv$$

$$= \int_o^c \int_o^b \int_o^a 2(x + y + z)\,dx\,dy\,dz$$

$$= \int_o^c \int_o^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dy\,dz$$

$$= \int_0^c \int_o^b 2\left(\frac{a^2}{2} + ya + za\right) dy\,dz$$

$$= \int_o^c 2\left[\left(\frac{a^2}{2}y + \frac{y^2 a}{2} + azy\right)\right]_o^b dz$$

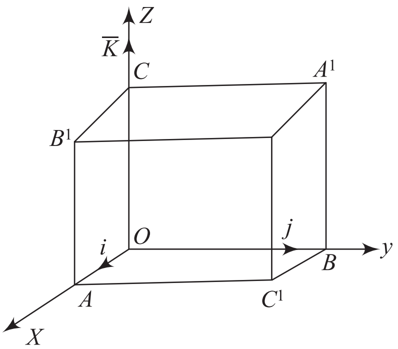


Fig. 20.11

$$= 2\int_o^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right)dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + \frac{abz^2}{2}\right]_o^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a + b + c) \quad (1)$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelepiped into 6 parts.

S_1 : Face $OAC'B$

S_2 : Face $CB'PA'$

S_3 : Face $OBA'C$

S_4 : Face $AC'PB'$

S_5 : Face $OCB'A$

S_6 : Face $BA'PC'$

Also,

$$\iint_S \vec{F}\cdot\hat{n}ds = \iint_{S_1} \vec{F}\cdot\hat{n}ds + \iint_{S_2} \vec{F}\cdot\hat{n}ds + \iint_{S_3} \vec{F}\cdot\hat{n}ds + \iint_{S_4} \vec{F}\cdot\hat{n}ds + \iint_{S_5} \vec{F}\cdot\hat{n}ds + \iint_{S} \vec{F}\cdot\hat{n}ds \quad (2)$$

On $S_1 : z = 0,\ \hat{n} = -\vec{k}$, $ds = dx\, dy$

so that $\vec{F}\cdot\hat{n} = (x^2\vec{i} + y^2\vec{j} - xy\vec{k})\cdot(-\vec{k}) = xy$

$$\therefore \quad \iint_{S_1} \vec{F}\cdot\hat{n}ds = \int_o^b\int_o^a xy\,dx\,dy = \int_o^b \left(y\frac{x^2}{2}\right)_o^a dy$$

$$= \frac{a^2}{2}\int_o^b y\,dy = \frac{a^2b^2}{4} \quad (3)$$

On $S_2 : z = c,\ \hat{n} = \vec{k}$, $ds = dx\, dy,\ \vec{F} = (x^2 - cy)\vec{i} + (y^2 - cx)\vec{j} + (c^2 - xy)\vec{k}$.

so that $\vec{F}\cdot\hat{n} = [(x^2 - cy)\vec{i} + (y^2 - (x)\vec{j}) + (c^2 - xy)\vec{k}]\cdot\vec{k} = c^2 - xy$.

$$\therefore \quad \iint_{S_2} \vec{F}\cdot\hat{n}ds = \int_o^b\int_o^a (c^2 - xy)\,dx\,dy = \int_o^b \left(c^2a - \frac{a^2}{2}y\right)dy$$

$$= abc^2 - \frac{a^2b^2}{4} \quad (4)$$

On $S_3 : x = 0,\ \hat{n} = -\vec{i}, \vec{F} = -yz\vec{i} + y^2\vec{j} + z^2\vec{k}, dz = dy\,dz$

so that $\vec{F}\cdot\hat{n} = (-yz\vec{i} + y^2\vec{j} + z^2\vec{k})\cdot(-\vec{i}) = yz, ds = dy\,dz$

$$\therefore \quad \iint_{S_3} \vec{F}\cdot\hat{n}ds = \int_o^c\int_o^b yz\,dy\,dz = \int_o^c \frac{b^2}{2}z\,dz = \frac{b^2c^2}{4} \quad (5)$$

On $S_4 : x = a,\ \hat{n} = \vec{i}, \vec{F} = (a^2 - yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}$

so that $\vec{F}\cdot\hat{n} = [(a^2 - yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}]\cdot\vec{i}$

$= a^2 - yz,\ ds = dy\,dz$

$$\therefore \qquad \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_o^c\int_o^b (a^2 - yz)\,dy\,dz = \int_o^c \left(a^2 b - \frac{b^2}{2}z\right)dz$$

$$= a^2bc - \frac{b^2c^2}{4} \qquad (6)$$

On $S_5 : y = 0,\ \hat{n} = -\vec{j}, \vec{F} = x^2\vec{i} - zx\vec{j} + z^2\vec{k}, ds = dxdz$

so that $\vec{F}\cdot\hat{n} = (x^2\vec{i} - zx\vec{j} + z^2\vec{k})\cdot(-\vec{j}) = zx$

$$\therefore \qquad \iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_o^a\int_o^c zx\,dz\,dx = \int_o^a \frac{c^2}{2}x\,dx = \frac{a^2c^2}{4} \qquad (7)$$

On $S_6 : y = b,\ \hat{n} = \vec{j}, \vec{F} = (x^2 - bz)\vec{i} + (b^2 - zx)\vec{j} + (z^2 - bx)\vec{k}$

$ds = dxdz$

so that $\vec{F}\cdot\hat{n} = [(x^2 - bz)\vec{i} + (b^2 - zx)\vec{j} + (z^2 - bx)\vec{k}]\cdot\vec{j}$

$= b^2 - zx.$

$$\therefore \qquad \iint_{S_6} \vec{F}\cdot\hat{n} = \int_o^a\int_o^c (b^2 - zx)\,dz\,dx$$

$$= \int_o^a \left(b^2c - \frac{c^2}{2}x\right)\cdot dx = ab^2c - \frac{a^2c^2}{4} \qquad (8)$$

By using (3), (4), (5), (6), (7) and (8), in (2), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

$$= abc(a + b + c) \qquad (9)$$

The equalities (1) and (9) verify the divergence theorem. **Ans.**

Example 12 Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by (i) $y = \sqrt{x}$, $y = x^2$ and (ii) $x = 0$, $y = 0$, $x + y = 1$. **[AU July 2010, June 2012 ; KU Nov. 2011, KU April 2013]**

Solution

(i) $y = \sqrt{x}$, i.e., $y^2 = x$ and $y = x^2$ are two parabolas intersecting at 0(0, 0) and $A(1, 1)$.

Here, $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y, \frac{\partial Q}{\partial x} = -6y$$

$$\therefore \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$$

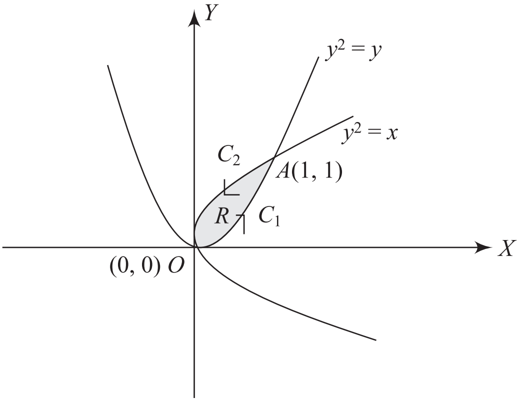


Fig. 20.12

If R is the region bounded by C then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dy\,dx = \int_0^1 10\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\,dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\left[\frac{3}{10}\right] = \frac{3}{2} \qquad (1)$$

Also, $\int_C P\,dx + Q\,dy = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy)$

Along C_1, $x^2 = y$. $\therefore$ $2x\,dx = dy$ and the limits of x are from 0 to 1.

$\therefore \qquad \int_{C_1} (P\,dx + Q\,dy)$

$$= \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x \cdot x^2) \cdot 2x\,dx \text{ (since } x^2 = y)$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= [x^3 + 2x^4 - 4x^5]_0^1 = -1$$

Along C_2, $y^2 = x$. $\therefore$ $2y\,dy = dx$ and the limits of y are from 1 to 0.

$$\int_{C_2}(P\,dx + Q\,dy)$$

$$= \int_1^0 (3y^4 - 8y^2)2y\,dy + (4y - 6y^2 \cdot y)\cdot dy$$

$$\therefore \quad = \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$$\therefore \quad \int_C (P\,dx + Q\,dy) = -1 + \frac{5}{2} = \frac{3}{2} \qquad (2)$$

The equalities of (1) and (2) verify Green's theorem in the plane. **Ans.**

(ii) Here, $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx$$

$$= \int_0^1 5[y^2]_0^{1-x}\,dx$$

$$= 5\int_0^1 (1-x)^2\,dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

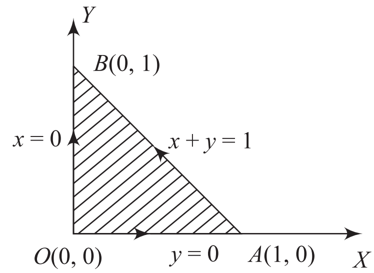


Fig. 20.13

$$= \frac{-5}{3}(0-1) = \frac{5}{3} \qquad (1)$$

Along OA, $y = 0 \therefore dy = 0$ and the limits of x are from 0 to 1.

$$\therefore \quad \int_{OA} P\,dx + Q\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$$

Along AB, $y = 1 - x$. $\therefore dy = -dx$ and the limits of x are from 1 to 0.

$$\therefore \quad \int_{AB} P\,dx + Q\,dy = \int_1^0 [3x^2 - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)](-dx)$$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)dx$$

$$= \int_1^0 (-12 + 26x - 11x^2)\cdot dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

Along BO, $x = 0$. $\therefore dx = 0$ and the limits of y are from 1 to 0

$$\therefore \quad \int_{BO} P\,dx + Q\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$\therefore$ line integral along C (i.e., along $OABO$) $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$

i.e., $$\int_C (P\,dx + Q\,dy) = \frac{5}{3} \qquad (2)$$

The equality of (1) and (2) verifies Green's theorem in the plane. **Verified.**

Example 13 Evaluate $\int_C (e^x dx + 2y\,dy - dz)$ by using Stokes' theorem, where C is the curve $x^2 + y^2 = 4$, $z = 2$. **[AU May 2010]**

Solution

$$\int_C (e^x\,dx + 2y\,dy - dz)$$
$$= \int_C (e^x\vec{i} + 2y\vec{j} - \vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$
$$= \int_C \vec{F}\cdot\overrightarrow{dr} \text{ where } \vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$
$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$
$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

$\therefore$ by Stokes' theorem, $$\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\cdot ds$$
$$= 0 \text{ (since curl } \vec{F} = 0)$$ **Ans.**

Example 14 Find the work done by the force $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, when it moves a particle along the arc of the curve $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ from $t = 0$ to $t = 2n$. **[AU Dec. 2007]**

Solution From the vector equation of the curve C, we get the parametric equations of the curve as $x = \cos t$, $y = \sin t$, $z = t$.

Work done by the force $\vec{F} = \int_C \vec{F}\cdot\overrightarrow{dr}$

$$= \int_C (z\vec{i} + x\vec{j} + y\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$
$$= \int_C (z\,dx + x\,dy + y\,dz)$$
$$= \int_0^{2\pi} [t(-\sin t) + \cos^2 t + \sin t]dt$$
$$= \left[t\cos t - \sin t + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) - \cos t\right]_0^{2\pi}$$
$$= (2\pi + \pi - 1) - (-1)$$
$$= 3\pi$$ **Ans.**

Example 15 Verify Stokes' theorem for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\vec{k}$ where S is the open surface of the rectangular parallelepiped formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ above the XOY-plane. **[AU Dec. 2007]**

Solution Stokes' theorem is given by

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

Here, curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -zx \end{vmatrix}$

$$= 2y\vec{i} + z\vec{j} + x\vec{k} \quad \therefore \int_C (xy\,dx - 2yz\,dy - zx\,dz) - \iint_S (2y\vec{i} + z\vec{j} + x\vec{k})\cdot\hat{n}\,ds \qquad (1)$$

The open cuboid S is made up of the five faces $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ and is bounded by the rectangle $OAC'B$ lying on the XOY plane. LHS of (1) is

$$= \int_{OAC'B} (xy\,dx - 2yzdy - zxdz)$$

$$= \int_{OAC'B} xy\,dx$$

(since the boundary C lies on the XOY plane, $z = 0$)

$$= \int_{OA} xy\,dx + \int_{AC'} xy\,dx + \int_{C'B} xy\,dx + \int_{BO} xy\,dx$$

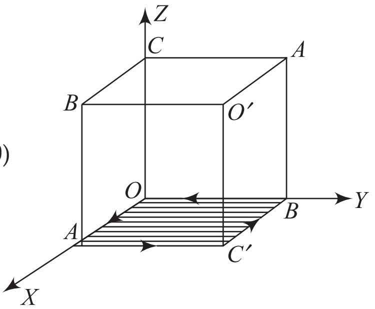


Fig. 20.14

Along OA, $y = 0$, $dy = 0$
Along AC', $x = 1$, $dx = 0$
Along $C'B$, $y = 2$, $dy = 0$
Along BO, $x = 0$, $dx = 0$

$$\therefore \qquad \int_{OAC'B} xy\,dx = 0 + 0 + \int_{C'B} xy\,dx + 0 = \int_1^0 2x\,dx$$

(as along $C'B$, x varies from 1 to 0).

$$= -1 \qquad (2)$$

RHS of (1) is

$$\iint_S (2y\vec{i} + z\vec{j} + x\vec{k})\cdot\hat{n}\,ds = \iint_{O'C'AB'} (2y\vec{i} + z\vec{j} + x\vec{k})\cdot\hat{n}\,ds + \iint_{A'BOC} (2y\vec{i} + z\vec{j} + x\vec{k})\cdot\hat{n}\,ds$$

$$+ \iint_{A'BC'O'} (2y\vec{i} + z\vec{j} + x\vec{k})\cdot\hat{n}\,ds + \iint_{COAB'} (2y\vec{i} + z\vec{j} + x\vec{k})\cdot\hat{n}\,ds$$

$$+ \iint_{A'O'B'C} (2y\vec{i} + z\vec{j} + x\vec{k})\cdot\hat{n}\,ds$$

$$= \int_0^3\int_0^2 2y\,dy\,dz - \int_0^3\int_0^2 2y\,dy\,dz - \int_0^1\int_0^3 z\,dz\,dx$$
$$- \int_0^1\int_0^3 z\,dz\,dx - \int_0^2\int_0^1 x\,dx\,dy$$
$$= -\int_0^2\int_0^1 x\,dx\,dy = -\int_0^2\left(\frac{x^2}{2}\right)_0^1 dy = -1 \quad (3)$$

From (2) and (3), Stokes' theorem is verified. **Verified.**

Example 16 Verify the divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by $x = \pm1$, $y = \pm1$, $z = \pm1$. **[AU Dec. 2007, KU Nov. 2011]**

Solution Gauss' divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\text{div}\,\vec{F})\,dv \quad (1)$$

LHS of $(1) = \iint_{x=1} x^2\,ds + \iint_{x=-1} -x^2\,ds + \iint_{y=1} z\,ds + \iint_{y=-1} -z\,ds + \iint_{z=1} yz\,ds + \iint_{z=-1} -yz\,ds = 0 \quad (2)$

RHS of $(1) = \iiint_V (\text{div}\,\vec{F})\cdot dv$

$$= \iiint_V (2x + y)\,dx\,dy\,dz$$
$$= \int_{-1}^1\int_{-1}^1\int_{-1}^1 (2x + y)\,dx\,dy\,dz$$
$$= \int_{-1}^1\int_{-1}^1 2y\,dy\,dz = 0 \quad (3)$$

From (2) and (3), Gauss' divergence theorem is verified. **Verified.**

Example 17 Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot\vec{dr}$, where $\vec{F} = (\sin x - y)\vec{i} - \cos x\vec{j}$ and C is the boundary of the triangle whose vertices are $(0, 0), \left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$. **[KU Nov. 2011]**

Solution By Stokes' theorem, we have $\int_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\cdot ds$.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x - y & -\cos x & 0 \end{vmatrix}$$
$$= (\sin x + 1)\vec{k}$$

∴ the given line integral

$$= \iint_R (1 + \sin x)\,dx\,dy$$

$$= \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (1 + \sin x)\,dx\,dy$$

$$= \int_0^1 \left[x - \cos x\right]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \left[\frac{\pi}{2} - \frac{\pi y}{2} + \cos\frac{\pi y}{2}\right] dy$$

$$= \left[\frac{\pi}{2}y - \frac{\pi y^2}{4} + \frac{2}{\pi}\sin\frac{\pi y}{2}\right]_0^1$$

$$= \frac{\pi}{4} + \frac{2}{\pi}$$

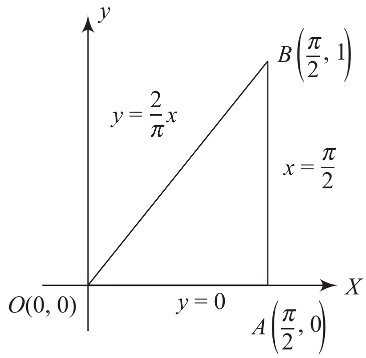


Fig. 20.15

$\therefore$ $$\int_C \vec{F} \cdot \overrightarrow{dr} = \frac{\pi}{4} + \frac{2}{\pi}$$ **Ans.**

EXERCISE

Part A

1. State Green's theorem in a plane.
2. Give the relation between a line integral and a surface integral.
3. State Gauss' divergence theorem.
4. Deduce Green's theorem in a plane from Stokes' theorem.
5. In Gauss' divergence theorem, surface integral is equal to _______ integral.
6. The integral of $\vec{F} \cdot d\vec{r}$ is
   (i) line integral (ii) zero
   (iii) surface integral (iv) one
7. Using Green's theorem, prove that the area enclosed by a simple closed curve C is $\frac{1}{2}\int (x\,dy - y\,dx)$.
8. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F} \cdot \overrightarrow{dr}$ where C is the part of the curve $y = x^3$ between $x = 1$ and $x = 2$.
9. If $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F} \cdot \overrightarrow{dr}$ along the straight line $y = x$ from (0, 0) to (1, 1).
10. If C is a simple closed curve and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, prove that $\int_C \vec{r} \cdot \overrightarrow{dr} = 0$.
11. Evaluate $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ where C is the circle given by $x^2 + y^2 + z^2 = 1$ and $z = 0$.
12. Use the integral theorems to prove $\nabla \cdot (\nabla \times \vec{F}) = 0$.

13. Evaluate $\int_C (x\,dy - y\,dx)$, where C is the circle $x^2 + y^2 = a^2$.
14. Evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and C is the curve $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, t varying from –1 to 1.

Part B

1. If a force $\vec{F} = 2x^2y\vec{i} + 3xy\vec{j}$ displaces a particle in the xy plane from (0, 0) to (1, 4) along a curve $y = 4x^2$, find the work done. $\left(\textbf{Ans. } \frac{104}{5}\right)$
2. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to (1, 1) along a parabola $y^2 = x$. $\left(\textbf{Ans. } \frac{2}{3}\right)$
3. Verify Green's theorem in a plane with respect to $\int_C (x^2dx + xy\,dy)$, where C is the boundary of the square formed by $x = 0, y = 0, x = a, y = a$. **[AU Dec. 2009]** $\left(\textbf{Ans. } \frac{a^3}{2}\right)$
4. Use Green's theorem to evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is the square formed by the lines $y = \pm 1, x = \pm 1$. **(Ans.** 0**)**
5. Use divergence theorem to evaluate $\iint_S (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k})\cdot\hat{n}ds$ where S is the closed surface bounded by the XOY-plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane. **(Ans.** πa^4**)**
6. Verify Stokes' theorem for $\vec{F} = (x^2 + y - 4)\vec{i} + 3xy\vec{j} + (2xz + z^2)\vec{k}$ over the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above the XOY plane. **(Ans.** -16π**)**
7. Use the divergence theorem to evaluate $\int_S \vec{A}\cdot\overrightarrow{ds}$ where $\vec{A} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2 = y^2 + z^2 = a^2$. $\left(\textbf{Ans. } \frac{12\pi a^5}{5}\right)$
8. Use the divergence theorem to evaluate $\iint_S x^3dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where S is the surface of the region bounded by the closed cylinder $x^2 + y^2 = a^2$, $(0 \le z \le b)$ $z = 0$ and $z = b$. $\left(\textbf{Ans. } \frac{5\pi a^4 b}{4}\right)$
9. Using Green's theorem, evaluate $\int_C [(y - \sin x)dx + \cos x\,dy]$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$. $\left[\textbf{Ans. } -\left(\frac{\pi^2 + 8}{4\pi}\right)\right]$
10. Evaluate $\int_C [(x^2 + y^2)dx - 2xy\,dy]$ where C is the rectangle bounded by $y = 0$, $x = 0, y = b, x = a$ using Green's theorem. **(Ans.** $-2ab^2$**)**
11. Verify Stokes' theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **(Ans.** $-\pi$**)**
12. Verify Stokes' theorem for $\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is the boundary. **(Ans.** 9π**)**

13. Find the area of $x^{2/3} + y^{2/3} = a^{2/3}$ using Green's theorem. $\left(\textbf{Ans. } \frac{3\pi a^2}{8}\right)$

14. Using Stokes' theorem, evaluate $\int_C (xy\,dx + xy^2 dy)$ taking C to be a square with vertices (1, 1), (–1, 1), (–1, –1) and (1, –1). $\left(\textbf{Ans. } \frac{4}{3}\right)$

15. Verify Gauss' divergence theorem for $\vec{F} = y\vec{i} + x\vec{j} + z^3\vec{k}$ over the cylindrical region $x^2 + y^2 = 9$, $z = 0$, $z = 6$. **(Ans.** 1944π**)**

Unit IX

Analytic Functions

21 Complex Numbers

Chapter Outline

21.1 ❑ INTRODUCTION

Quite often, it is believed that complex numbers arose from the need to solve quadratic equations. In fact, contrary to this belief, these numbers arose from the need to solve cubic equations. In the sixteenth century, Cardano was possibly the first to introduce $a+\sqrt{-b}$, a complex number, in algebra. Later, in the eighteenth century, Euler introduced the notation i for $\sqrt{-1}$ and visualized complex numbers as points with rectangular coordinates, but he did not give a satisfactory foundation for complex numbers. However, Euler defined the complex exponential and proved the identity $e^{i\varphi} = (\cos \varphi + i \sin \varphi)$, thereby establishing connection between trigonometric and exponential functions through complex analysis.

We know that there is no square root of negative numbers among real numbers.

However, algebra itself and its applications require such an extension of the concept of a number for which the extraction of the square root of a negative number would be possible.

We have repeatedly encountered the notion of extension of a number. Fractional numbers are introduced to make it possible to divide one integral number by another, negative numbers are introduced to make it possible to subtract a large number from a smaller one and irrational numbers become necessary in order to describe the result of measurement of the length of a segment in the case when the segment is incommensurable with the chosen unit of length.

The square root of the number –1 is usually denoted by the letter i and numbers of the form $a + ib$ where a and b are ordinary real numbers known as **complex numbers**.

The necessity of considering complex numbers first arose in the sixteenth century when several Italian mathematicians discovered the possibility of algebraic solutions of third-degree equations.

The theoretical and applied values of complex numbers are far beyond the scope of algebra. The theory of functions of a complex variable, which was much advanced in the nineteenth century, proved to be a very valuable apparatus for the investigation of almost all the divisions of theoretical physics, such, for instance, as the theory of oscillations, hydrodynamics, the divisions of the theory of elementary particles, etc.

Many engineering problems may be treated and solved by methods involving complex numbers and complex functions. There are two kinds of such problems. The first of them consists of elementary problems for which some acquaintances with complex numbers are sufficient. This includes many applications to electric circuits or mechanical vibrating systems. The second kind consists of more advanced problems for which we must be familiar with the theory of complex analytic functions. Interesting problems in heat conduction, fluid flow and electrostatics belong to this category.

21.2 ❑ COMPLEX NUMBERS

A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ (i is pronounced as **iota**) is called a **complex number**. x is called the **real part** of $x + iy$ and is written as $\text{Re}(x + iy)$ and y is called the **imaginary part** and is written as $\text{Im}(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be **conjugates** of each other.

Properties

(i) If $x_1 + iy_1 = x_2 + iy_2$ then $x_1 - iy_1 = x_2 - iy_2$

(ii) Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are said to be equal when $\text{Re}(x_1 + iy_1) = \text{Re}(x_2 + iy_2)$, i.e., $x_1 = x_2$ and $\text{Im}(x_1 + iy_1) = \text{Im}(x_2 + iy_2)$ i.e., $y_1 = y_2$

(iii) **Algebra of Complex Numbers**

The arithmetic operations on complex numbers follow the usual rules of elementary algebra of real numbers with the definition $i^2 = -1$. If $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$ are any two complex numbers then we define the following arithmetic operations.

Addition

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Division Let $z_2 \neq 0$. Then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \left[\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right] + i\left[\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right]$$

i.e., sum, difference, product and quotient of any two complex numbers is itself a complex number.

(iv) Every complex number $x + iy$ can always be expressed in the form $r(\cos\theta + i\sin\theta)$.
i.e., $re^{i\theta}$ (Exponential form).

➢ **Note**

(i) The number $r = +\sqrt{x^2 + y^2}$ is called the **module** of $x + iy$ and is written as mod $(x + iy)$ or $|x + iy|$. The angle θ is called the **amplitude** or **argument** of $x + iy$ and is written as amp $(x + iy)$ or arg$(x + iy)$. Evidently, the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the **principal value of the amplitude.**

(ii) $\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')

(iii) If the conjugate of $z = x + iy$ be $\bar{z}$ then

(a) $\text{Re}(z) = \frac{1}{2}(z + \bar{z}), \text{Im}(z) = \frac{1}{2i}(z - \bar{z})$

(b) $|z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2} = |\bar{z}|$

(c) $z\bar{z} = |z|^2$

(d) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(e) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(f) $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}, \overline{z_2} \neq 0$

(iv) **De Moivre's Theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

21.3 ❑ COMPLEX FUNCTION

Recall from calculus that a real function f defined on a set S of real numbers is a rule that assigns to every x in S a real number $f(x)$, called the **value** of f at x. Now in the complex region, S is a set of complex numbers. A **function** f defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z.

We write $w = f(z)$. Here, z varies in S and is called a **complex variable**. The set S is called the **domain** of f.

If to each value of z, there corresponds one and only one value of w then w is said to be a **single-valued function** of z; otherwise, it is a **multi-valued function**. For example, $w = \frac{1}{z}$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

➢ **Note**

(i) If $z = x + iy$ then $f(z) = u + iv$ (a complex number).

(ii) Since $e^{iy} = \cos y + i \sin y$, $e^{-iy} = \cos y - i \sin y$, the circular functions are

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2}, \text{ and so on}$$

∴ circular functions of the complex variable z are given by $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$,

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\tan z = \dfrac{\sin z}{\cos z}$ with cosec z, sec z and cot z as their respective reciprocals.

(iii) **Euler's Theorem**

$$e^{iz} = \cos z + i \sin z$$

(iv) **Hyperbolic Functions**

If x be real or complex, $\dfrac{e^x - e^{-x}}{2} = \sin hx$ (named hyperbolic sine of x)

$\dfrac{e^x + e^{-x}}{2} = \cos hx$ (named hyperbolic cosine of x)

Also, we define,

$$\tan hx = \frac{\sin hx}{\cos hx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cot hx = \frac{1}{\tan hx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\sec hx = \frac{1}{\cos hx} = \frac{2}{e^x + e^{-x}}$$

$$\text{cosec } hx = \frac{1}{\sin hx} = \frac{2}{e^x - e^{-x}}$$

(v) **Relations between Hyperbolic and Circular Functions**

$$\sin ix = i \sin hx$$
$$\cos ix = \cos hx$$
$$\tan ix = i \tan hx$$

(vi) $\cos h^2x - \sin h^2x = 1$, $\sec h^2x + \tan h^2x = 1$

$\cot h^2x - \text{cosec } h^2x = 1$

(vii) $\sin h(x \pm y) = \sin hx \cos hy \pm \cos hx \sin hy$

$\cos h(x \pm y) = \cos hx \cos hy \pm \sinh x \sinh y$

$$\tan h(x \pm y) = \frac{\tan hx \pm \tan hy}{1 + \tan hx \tan hy}$$

(viii) $\sin h2x = 2 \sin hx \cosh x$

$\cos h2x = \cos h^2x + \sin h^2x = 2 \cos h^2x - 1 = 1 + 2 \sin h^2x$

$\tan h2x = \dfrac{2 \tan hx}{1 + \tan h^2x}$

(ix) $\sin h3x = 3\sin hx + 4\sin h^3 x$

$\cos h3x = 4\cos h^3 x - 3\cos hx$

$\tan h3x = \dfrac{3\tan hx + \tan h^3 x}{1 + 3\tan h^2 x}$

(x)
$$\sin hx + \sin hy = 2\sin h\frac{x+y}{2}\cos h\frac{x-y}{2}$$
$$\sin hx - \sin hy = 2\cos h\frac{x+y}{2}\sin h\frac{x-y}{2}$$
$$\cos hx + \cos hy = 2\cos h\frac{x+y}{2}\cos h\frac{x-y}{2}$$
$$\cos hx - \cos hy = 2\sin h\frac{x+y}{2}\sin h\frac{x-y}{2}$$

(xi) $\cos h^2 x - \sin h^2 x = 1$

(xii) Complex trigonometric functions satisfy the same identities as real trigonometric functions.

$$\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z$$
$$\sin^2 z + \cos^2 z = 1$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$
$$\sin 2z = 2\sin z\cos z \quad \text{and} \quad \cos 2z = \cos^2 z - \sin^2 z$$
$$\sin \bar{z} = \overline{\sin z}$$

$\sin(z + 2n\pi) = \sin z$, n is any integer

$\cos(z + 2n\pi) = \cos z$, n is any integer

(xiii) **Inverse Trigonometric and Hyperbolic Functions**

Complex inverse trigonometric functions are defined by the following:

$$\cos^{-1} z = -i\log(z + \sqrt{z^2+1})$$
$$\sin^{-1} z = -i\log(iz + \sqrt{1-z^2})$$
$$\tan^{-1} z = -\frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right) = \frac{i}{2}\log\frac{i+z}{i-z},\ z \neq \pm i$$
$$\operatorname{cosec}^{-1} z = \sin^{-1}\left(\frac{1}{z}\right) = -i\log\left(\frac{1+\sqrt{z^2-1}}{z}\right),\ z \neq 0$$
$$\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right) = -i\log\left(\frac{1+\sqrt{1-z^2}}{z}\right),\ z \neq 0$$
$$\cot^{-1} z = \tan^{-1}\left(\frac{1}{z}\right) = \frac{-i}{2}\log\left(\frac{z+i}{z-i}\right),\ z \neq \pm i$$

Complex inverse hyperbolic functions are defined by the following:

$$\cosh^{-1} z = \log(z + \sqrt{z^2 - 1}),\ \sinh^{-1} z = \log(z + \sqrt{z^2 + 1})$$

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\ z \neq \pm 1$$

$$\operatorname{cosech}^{-1} z = \sinh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right),\ z \neq 0$$

$$\operatorname{sech}^{-1} z = \cosh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right),\ z \neq 0$$

$$\coth^{-1} z = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right),\ z \neq \pm 1$$

21.4 ❑ LIMIT OF A FUNCTION

A function $f(z)$ is said to have the **limit** 'b' as z approaches a point 'a', written $\lim_{z\to a} f(z) = b$, if f is defined in a neighborhood of 'a' (except perhaps at 'a' itself) and if the values of f are close to 'b' for all z close to 'a', i.e., the number b is called the **limit** of the function $f(z)$ as $z \to a$, if the absolute value of the difference $f(z) - b$ remains less than any preassigned positive number $\in$ every time the absolute value of the difference $z - a$ for $z \neq a$, is less than some positive number δ (dependent on $\in$).

More briefly, the number b is the limit of the function $f(z)$ as $z \to a$, if the absolute value $|f(z) - b|$ is arbitrarily small when $|z - a|$ is sufficiently small.

21.5 ❑ DERIVATIVE

A function $f(z)$ is said to be **differentiable** at a point $z = z_0$ if the limit $\lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists. This limit is then called the derivative of $f(z)$ at the point $z = z_0$ and is denoted by $f'(z_0)$.

If we write $z = z_0 + \Delta z$ then

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

21.6 ❑ ANALYTIC FUNCTIONS

A function defined at a point z_0 is said to be **analytic** at z_0, if it has a derivative at z_0 and at every point in some neighborhood of z_0. It is said to be analytic in a region R, if it is analytic at every point of R. Analytic functions are otherwise named **holomorphic** or **regular** functions.

A point at which a function $f(z)$ is not analytic is called a **singular point** or **singularity** of $f(z)$.

21.7 ❑ CAUCHY–RIEMANN EQUATIONS

The necessary condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain R is that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ must exist and satisfy the Cauchy–Riemann equations, namely,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The sufficient condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain R is that the four partial derivatives u_x, u_y, v_x and v_y exist, are continuous and satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ at each point of R.

➢ **Note**

(i) The two partial differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called the **Cauchy–Riemann equations** and they may be written as $u_x = v_y$ and $u_y = -v_x$

(ii) The Cauchy–Riemann equations are referred as C-R equations

(iii) C-R equations in polar form are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

21.8 ❑ HARMONIC FUNCTION

A real function of two variables x and y that possesses continuous second-order partial derivatives and satisfies the Laplace equation is called a **harmonic function**.

If u and v are harmonic functions such that $u + iv$ is analytic then each is called the **conjugate harmonic function** of the other.

➢ **Note**

(i) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the **Laplacian operator** and is denoted by ∇^2.

(ii) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$ is known as **Laplace equation** in two dimensions.

21.9 ❑ PROPERTIES OF ANALYTIC FUNCTIONS

Property 1

The real and imaginary parts of an analytic function $f(z) = u + iv$ satisfy the Laplace equation in two dimensions.

• **Proof**

Since $f(z) = u + iv$ is an analytic function, it satisfies C-R equations,

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and} \qquad (21.1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{21.2}$$

Differentiating both sides of (21.1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{21.3}$$

Differentiating both sides of (21.2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \partial x} \tag{21.4}$$

By adding (21.3) and (21.4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\text{since } \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}, \text{ when they are continuous}\right)$$

$\Rightarrow u$ satisfies Laplace equation.
Now differentiating both sides of (21.1) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \tag{21.5}$$

Differentiating both sides of (21.2) partially with respect to x we get

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2} \tag{21.6}$$

Subtracting (21.5) and (21.6),

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

i.e.,
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore v$ satisfies Laplace equation.
Hence, if $f(z)$ is analytic then both real and imaginary parts satisfy Laplace's equation.

➢ **Note**

If $f(z) = u + iv$ is analytic then u and v are harmonic. Conversely, when u and v are any two harmonic functions then $f(z) = u + iv$ need not be analytic.

Property 2

If $f(z) = u + iv$ is an analytic function then the curves of the family $u(x, y) = C_1$ cut orthogonally the curves of the family $v(x, y) = C_2$ where C_1 and C_2 are constants.

- **Proof**

Given $u(x, y) = C_1$
Taking differentials on both sides, we get

$$du = 0$$

i.e.,
$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1$$ (say), where m_1 is the slope of the curve $u(x, y) = C_1$ at (x, y)

From the second curve $v(x, y) = C_2$, we get $\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} = m_2$, where m_2 is the slope of the curve $v(x, y) = C_2$ at (x, y).

Now,
$$m_1 m_2 = \frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

$$= \frac{\left(\frac{\partial v}{\partial y}\right)}{-\left(\frac{\partial v}{\partial x}\right)} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$ (as $f(z)$ is analytic, it satisfies C-R equation)

$\Rightarrow m_1 m_2 = -1$

Hence, the curves cut each other orthogonally.

Here, the two families are said to be **orthogonal trajectories** of each other.

21.10 ❑ CONSTRUCTION OF ANALYTIC FUNCTIONS (MILNE–THOMSON METHOD)

To find f(z) when u is given

We know that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

i.e., $$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(By C-R equations)} \qquad (21.7)$$

Let $$\frac{\partial u(x, y)}{\partial x} = \phi_1(x, y) \text{ and then calculate } \phi_1(z, 0) \qquad (21.8)$$

and $$\frac{\partial u(x, y)}{\partial y} = \phi_2(x, y) \text{ and then calculate } \phi_2(z, 0) \qquad (21.9)$$

Substituting (21.8) and (21.9) in (21.7), we get

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

Integrating, we get $\int f^1(z)dz = \int \phi_1(z, 0)dz - i\int \phi_2(z, 0)dz$

i.e., $$f(z) = \int \phi_1(z, 0)dz - i\int \phi_2(z, 0)dz.$$

To find f(z) when v is given

We know that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \tag{21.10}$$

Let $$\frac{\partial v(x, y)}{\partial y} = \phi_1(z, 0) \tag{21.11}$$

and $$\frac{\partial v(x, y)}{\partial x} = \phi_2(z, 0) \tag{21.12}$$

Substituting (21.11) and (21.12) in (21.10), we get

$$f'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

Integrating, we get $\int f'(z)dz = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$

i.e., $$f(z) = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$$

21.11 ❑ APPLICATIONS

Irrotational Flows

A flow in which the fluid particles do not rotate about their own axes while flowing is said to be irrotational.

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u = \frac{-\partial \phi}{\partial x}, v = \frac{-\partial \phi}{\partial y} \tag{21.13}$$

In two-dimensional flow, the stream function ψ always exists such that

$$u = \frac{-\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x} \tag{21.14}$$

From (21.13) and (21.14), we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x} \tag{21.15}$$

which are the well-known **Cauchy–Riemann equations**. Hence, $\phi + i\psi$ is an analytic function of $z = x + iy$. Moreover, ϕ and ψ are known as conjugate functions.

On multiplying and rewriting, (21.15) gives

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0 \tag{21.16}$$

showing that the families of curves given by ϕ = constant and ψ = constant intersect orthogonally. Thus, the curves of equi-velocity potential and the stream lines intersect orthogonally.

Differentiating the equation given in (21.15) with respect to x and y respectively, we get $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y}$ and $\frac{\partial^2 \phi}{\partial y^2} = \frac{-\partial^2 \psi}{\partial x \partial y}$. (21.17)

Since $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$, (21.17) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{21.18}$$

Again differentiating Eq. (21.15) with respect to y and x respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial y^2} \text{ and } \frac{\partial^2\phi}{\partial x\partial y}=\frac{-\partial^2\psi}{\partial x^2}$$

Subtracting these, $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$ (21.19)

Equations (21.18) and (21.19) show that ϕ and ψ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.

Complex Potential

Let $w = \phi + i\psi$ be taken as a function of $x + iy$

Thus, suppose that $w = f(z)$

i.e., $\phi + i\psi = f(x = iy)$ (21.20)

Differentiating (21.20) with respect to x and y respectively, we get

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=f'(x+iy)$$

(21.21)

and $\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}=i\,f'(x+iy)$

or $\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}=i\left(\frac{\partial\phi}{\partial x}+\frac{i\partial\psi}{\partial x}\right)$ by (21.22)

Equating real and imaginary parts, we get

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=\frac{-\partial\psi}{\partial x}$$

which are C-R equations. Then w is an analytic function of z and w is known as the complex potential.

Conversely, if w is an analytic function of z then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion. The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called **equipotential lines** and **stream lines** respectively.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **equipotential lines** and **lines of force**.

In heat-flow problems, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **isothermals** and **heat-flow lines.**

SOLVED EXAMPLES

Example 1 Prove that the function $f(z) = |z|^2$ is differentiable only at the origin.

Solution Given $f(z) = |z|^2$

i.e., $u + iv = |x + iy|^2 = [\sqrt{x^2 + y^2}\,]^2$ (as $z = x + iy$ and $f(z) = u + iv$)

$= x^2 + y^2$

$$\Rightarrow \qquad u = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$

$$v = 0$$

$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

If $f(z)$ is differentiable then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \qquad 2x = 0 \quad \Rightarrow \quad x = 0$$

Also, $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \qquad 2y = 0 \quad \Rightarrow \quad y = 0$$

$\therefore$ C-R equations are satisfied only when $x = 0, y = 0$

Hence, $f(z) = |z|^2$ is differentiable only at the origin (0, 0). **Proved.**

Example 2 Prove that the function $f(z) = z\bar{z}$ is not analytic except at $z = 0$.

Solution Given $f(z) = z\bar{z}$

i.e., $$u + iv = (x + iy)(x - iy)$$

$$u + iv = x^2 + y^2$$

Equating real and imaginary parts.

$$u = x^2 + y^2$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$

$$v = 0$$

$$\Rightarrow \qquad \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C-R equations are not satisfied

$\therefore f(z) = z\bar{z}$ is not analytic except at $z = 0$. **Proved.**

Example 3 Show that (i) an analytic function with a constant real part is a constant, and (ii) an analytic function with a constant modulus is also a constant.

[KU Nov. 2010, April 2012; AU Nov. 2010, Nov. 2011]

Solution Let $f(z) = u + iv$ be an analytic function.

(i) Let $u = C_1$ (a constant)

Then $\frac{\partial u}{\partial x} = u_x = 0$ and $\frac{\partial u}{\partial y} = u_y = 0$.

Since $f(z)$ is an analytic function, by C-R equations $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow \qquad v_y = 0$ and $v_x = 0$.

As $v_x = 0$ and $v_y = 0$, v must be independent of x and y and must be a constant C_2.

$\therefore \ f(z) = u + iv = C_1 + iC_2$ which is a constant.

(ii) Let $f(z) = u + iv$ be an analytic function.

Given $|f(z)| = \sqrt{u^2 + v^2} = k$ (a constant)

Differentiating partially with respect to x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and
$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Since $f(z)$ is an analytic function, it satisfies C-R equations.

$\therefore$ the above two equations may be written as,

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$

and
$$v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

By solving, we get $\frac{\partial u}{\partial x} = u_x = 0$ and $\frac{\partial u}{\partial y} = u_y = 0$.

By C-R equations, it implies that $\frac{\partial v}{\partial x} = v_x = 0$ and $\frac{\partial v}{\partial y} = v_y = 0$.

Thus, $f(z) = u + iv$ is a constant. **Proved.**

Example 4 If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

[AU May 2006, KU Nov. 2011, KU April 2013]

Solution Let $f(z) = u(x, y) + iv(x, y)$

Then $|f(z)|^2 = u^2 + v^2$ and $|f'(z)|^2 = u_x^2 + v_x^2$

To prove $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2) = 4(u_x^2 + v_x^2)$

Now, $\frac{\partial}{\partial x}(u^2) = 2uu_x$ and $\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x)$

$$= 2[uu_{xx} + u_x u_x] = 2uu_{xx} + u_x^2]$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2[uu_{yy} + u_y^2]$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2u[u_{xx} + u_{yy}] + 2[u_x^2 + u_y^2]$$

$$= 2[u_x^2 + u_y^2] \qquad (\text{since } u_{xx} + u_{yy} = 0) \qquad (1)$$

Again,
$$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x^2]$$

and
$$\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(v^2) = 2v(v_{xx}+v_{yy})+2(v_x^2+v_y^2)$$

$$= 2(v_x^2+v_y^2) \qquad (\text{since } v_{zz}+v_{yy}=0) \qquad (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 2[u_x^2+u_y^2+v_x^2+v_y^2]$$

$$= 2[u_x^2+v_x^2+v_x^2+u_x^2] \qquad (\text{by using C-R equations}) \quad = 4[u_x^2+v_x^2].$$

Hence, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ **Proved.**

Example 5 Show that if $f(z)$ is a regular function of z then $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$. **[AU May 2012]**

Solution $\text{Log}\,|f(z)| = \frac{1}{2}\log|f(z)|^2 = \frac{1}{2}\log(u^2+v^2)$

$$\therefore \quad \frac{\partial}{\partial x}\log|f(z)| = \frac{1}{2}\left[\frac{2uu_x+2v\cdot v_x}{u^2+v^2}\right] = \frac{uu_x+vv_x}{u^2+v^2}$$

$$\frac{\partial^2}{\partial x^2}\log|f(z)| = \frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_{xx}+v_x^2)-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{1}{u^2+v^2}[uu_x+vv_{xx}+u_x^2+v_x^2] - \frac{2}{(u^2+v^2)^2}(uu_x+vv_x)^2 \qquad (1)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\log|f(z)| = \frac{1}{u^2+v^2}[uu_{yy}+vv_{yy}+u_y^2+v_y^2] - \frac{2}{(u^2+v^2)^2}(uu_y+vv_y)^2 \qquad (2)$$

Adding (1) and (2), we get $-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|$

$$= \frac{1}{u^2+v^2}[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+u_x^2+v_x^2+u_y^2+v_y^2] - \frac{2}{(u^2+v^2)^2}[(uu_x+vv_x)^2+(uu_y+vv_y)^2]$$

$$= \frac{1}{(u^2+v^2)}[2(u_x^2+v_x^2)] - \frac{2}{(u^2+v^2)^2}[(uu_x+vv_x)^2+(-uv_x+vu_x)^2]$$

$$= \frac{2(u_x^2+v_x^2)}{u^2+v^2} - \frac{2}{(u^2+v^2)^2}[u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)]$$

$$= \frac{2(u_x^2+v_x^2)}{u^2+v^2} - \frac{2(u^2+v^2)(u_x^2+v_x^2)}{(u^2+v^2)^2}$$

$$= 0$$

Proved.

Example 6 Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and determine its conjugate. Also find $f(z)$. **[KU May 2010, KU April 2013]**

Solution Given $u = \frac{1}{2}\log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}; \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}; \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

Hence, u satisfies Laplace's equation.
$\therefore$ u is harmonic.

To find conjugate of *u*

We know that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \frac{-\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)} = \frac{x\,dy - y\,dx}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}d\left(\frac{y}{x}\right)$$

$$\int dv = \int \frac{d(y/x)}{1+(y/x)^2}$$

i.e.,

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore$ the required analytic function is $f(z) = u + iv$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

i.e., $f(z) = \log z$ **Ans.**

Example 7 If $u(x, y) = e^x(x \cos y - y \sin y)$, find $f(z)$ so that $f(z)$ is analytic.

Solution Given $u = e^x(x \cos y - y \sin y)$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \cos y(xe^x + e^x) - y \sin y e^x$$

$$\therefore \quad \phi_1(z, 0) = ze^z + e^z \quad (1)$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = -xe^x \sin y - e^x(\sin y + y\cos y)$$

$\therefore$ $$\phi_2(z, 0) = 0 \quad (2)$$

By Milne–Thomson method,

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= ze^z + e^z + 0$$
$$= e^z(z + 1)$$

$\therefore$ $$f(z) = \int e^z(z + 1)dz$$
$$= ze^z - e^z + e^z + C$$

i.e., $$f(z) = ze^z + C$$ **Ans.**

Example 8 Find the analytic function $f(z) = u + iv$ given that $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

[AU May 2006]

Solution Given $u + iv = f(z)$ (1)

$\therefore$ $iu - v = i\,f(z)$ (2)

Adding (1) and (2), we get

$$(u - v) + i(u + v) = (1 + i)f(z)$$

Let $u - v = U$,

$$u + v = V \quad \text{and} \quad F(z) = (1 + i)\,f(z)$$

$$\frac{\partial V}{\partial x} = \frac{(\cos h2y - \cos 2x)2\cos 2x - \sin 2x \cdot 2\sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(x, y) = \frac{\partial V}{\partial x}$$

$$= \frac{2\cos 2x\cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x\cosh 2y - 2}{(\cos h2y - \cos 2x)^2}$$

$$\phi_1(x, y) = \frac{\partial V}{\partial y} = \frac{-\sin 2x(2\sin h2y)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{-2\sinh 2y\sin 2x}{(\cos h2y - \cos 2x)^2}$$

By Milne–Thomson method, we have

$$F'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

$$\phi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$\phi_1(z, 0) = 0$$

and $$F'(z) = i\frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= i\frac{-2}{1 - \cos 2z} = i\frac{-1}{\frac{1 - \cos 2z}{2}}$$

$$= i\frac{-1}{\sin^2 z} = -i\,\mathrm{cosec}^2 z$$

$$\therefore \quad f(z) = -\frac{i}{1+i}\int \text{cosec}^2 z\, dz$$

$$\text{i.e.,} \quad f(z) = \frac{i+1}{2}\cot z + C$$ **Ans.**

Example 9 Find the analytic function $f(z) = u + iv$ if $u + v = \frac{x}{x^2+y^2}$ and $f(1) = 1$.

[AU Nov. 2010]

Solution Given $u + iv = f(z)$ (1)

$iu - v = if(z)$ (2)

Adding (1) and (2), we get

$(u - v) + i(u + v) = (1 + i)f(z)$

i.e., $U + iV = F(z)$ (3)

where $U = u - v,\ V = u + v = \frac{x}{x^2+y^2},\ F(z) = (1 + i)f(z)$ (4)

$$V = \frac{x}{x^2+y^2}$$

$$\phi_1(x, y) = \frac{\partial V}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \quad \phi_1(z, 0) = 0 \quad (5)$$

$$\phi_2(x, y) = \frac{\partial V}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\therefore \quad \phi_2(z, 0) = \frac{-z^2}{z^4} = -\frac{1}{z^2} \quad (6)$$

By Milne's method, we have

$$F'(z) = \phi_1(z_1 0) + i\phi_2(z, 0)$$

$$= 0 - i\frac{1}{z^2}$$

$$F(z) = -i\int \frac{1}{z^2}dz$$

$$\therefore \quad = -i\left(-\frac{1}{z}\right) + C$$

$$F(z) = \frac{i}{z} + C \quad (7)$$

But $F(z) = (1 + i) f(z)$ [from (4) and (8)]

From (7) and (8), we get

$$(1+i)\, f(z) = \frac{i}{z} + C$$

$$f(z) = \frac{i}{z(1+i)} + \frac{C}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)z} + C_1, \text{ where } C_1 = \frac{C}{1+i}$$

$$f(z) = \frac{1+i}{2z} + C_1$$

Given $f(1) = 1$

i.e., $$f(1) = \frac{1+i}{2} + C_1 = 1$$

$\Rightarrow$ $$C_1 = 1 - \frac{(1+i)}{2} = \frac{1-i}{2}$$

$\therefore$ $$f(z) = \frac{1+i}{2z} + \frac{1-i}{2}$$ **Ans.**

Example 10 Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. **[AU Nov. 2010]**

Solution

Let $z = x + iy$ (1)

$\therefore$ $\bar{z} = x - iy$ (2)

From (1) and (2), we get

$$x = \frac{z+\bar{z}}{2}, \ y = \frac{z-\bar{z}}{2i} = \frac{-i}{2}(z-\bar{z})$$

Now, $$\frac{\partial x}{\partial z} = \frac{1}{2}, \ \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$$

$$\frac{\partial y}{\partial z} = \frac{-i}{2}, \ \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

Now, $$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad (3)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad (4)$$

$\therefore$ $$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$\Rightarrow$ $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$ **Proved.**

Example 11 If $f(z) = u + iv$ is analytic, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$. **[AU Nov. 2010]**

Solution We know that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\partial \bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\partial \bar{z}}\cdot\frac{1}{2}\log|f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}\log[f'(z)\,f'(\bar{z})]$$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}[\log f'(z)+\log f'(\bar{z})]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right]=0$$

Proved.

Example 12 If $u = x^2 - y^2$ and $v = -\dfrac{y}{x^2+y^2}$, prove that both u and v satisfy Laplace's equation but that $u + iv$ is not a regular function of z. **[KU Nov. 2011]**

Solution Given $u = x^2 - y^2$

Then $$\frac{\partial u}{\partial x} = u_x = 2x; \frac{\partial^2 u}{\partial x^2} = u_{xx} = 2; \frac{\partial u}{\partial y} = u_y = -2y; \frac{\partial^2 u}{\partial y^2} = u_{yy} = -2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

i.e., u satisfies Laplace's equation.

$$v = -\frac{y}{x^2+y^2}$$

Then $$\frac{\partial v}{\partial x} = v_x = \frac{2xy}{(x^2+y^2)^2};\ v_{xx} = 2y\left[\frac{(x^2+y^2)\cdot - x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = v_y = -\left[\frac{(x^2+y^2)\cdot 1 - 2y^2}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = v_{yy} = \frac{(x^2+y^2)^2\,2y - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

i.e., v satisfies Laplace's equation.
Now, $u_x \neq v_y$ and $u_y \neq -v_x$
i.e., C-R equations are not satisfied by u and v.
Hence, $u + iv$ is not an analytic (regular) function of z. **Ans.**

Example 13 Show that the function $u(x, y) = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find a function $v(x, y)$ such that $u + iv$ is an analytic function.

[AU June 2010]

Solution Let $f(z) = u + iv$ be an analytic function with $u(x, y) = 3x^2y + x^2 - y^3 - y^2$

Then
$$\frac{\partial u}{\partial x} = u_x = 6xy + 2x; \frac{\partial^2 u}{\partial x^2} = u_{xx} = 6y + 2;$$
$$\frac{\partial u}{\partial y} = u_y = 3x^2 - 3y^2 - 2y; \frac{\partial^2 u}{\partial y^2} = u_{yy} = -6y - 2$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence, $u(x, y)$ is a harmonic function.

$$dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy = \frac{-\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = -u_y dx + u_x dy$$

$\therefore dv = (-3x^2 + 2y + 3y^2)dx + (6xy + 2x)dy$ where the RHS is a perfect differential equation.

$$dv = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$
$$= -\int (3x^2 - 3y^2 - 2y)\,dx + \int (6xy + 2x)\,dy$$

$$\therefore \quad v = (3xy^2 + 2xy - x^3) + C$$
$$\therefore \quad f(z) = u + iv = 3x^2y + x^2 - y^3 - y^2 + i(3xy^2 + 2xy - x^3 + C)$$
$$= -i[x^3 + 3x^2(iy) + 3xi^2y^2 + i^3y^3] + [x^2 + 2xiy + i^2y^2] + iC$$
$$= -i[x + iy]^3 + [x + iy]^2 + iC$$
$$\therefore \quad f(z) = iz^3 + z^2 + iC$$

Ans.

EXERCISE

Part A

1. Define analytic function of a complex variable.
2. State any two properties of an analytic function.
3. Define a harmonic function with an example.
4. Verify whether the function $\phi(x, y) = e^x \sin y$ is harmonic or not.
5. Find the constant 'a' so that $u(x, y) = ax^2 - y^2 + xy$ is harmonic.
6. Is $f(z) = z^3$ analytic? Justify.
7. What do you mean by a conjugate harmonic function? Find the conjugate harmonic of x.
8. Show that an analytic function with a constant real part is constant.
9. Write down the necessary condition for $w = f(z) = f(re^{i\theta})$ to be analytic.
10. Show that the function $u = \tan^{-1}\left(\frac{y}{x}\right)$ is harmonic.
11. Show that xy^2 cannot be the real part of an analytic function.
12. $f(z) = u + iv$ is such that u and v are harmonic. Is $f(z)$ analytic always? Justify.

13. State C-R equations in Cartesian coordinates.
14. Prove that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is a harmonic function.
15. Show that the function $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ satisfies Cauchy–Riemann equations.
16. Show that the real part u of an analytic function satisfies the equation $\nabla^2 u = 0$.
17. Check whether the function $\frac{1}{z}$ is analytic or not.
18. Test the analyticity of the function $2xy + i(x^2 - y^2)$.
19. State the basic difference between the limit of a function of a real variable and that of a complex variable.
20. Find the analytic function $f(z) = u + iv$, given that (i) $u = y^2 - x^2$, (ii) $v = \sin hx \sin y$, and (iii) $u = \dfrac{x}{x^2 + y^2}$.

Part B

1. Prove that the following functions are not differentiable (and, hence, not analytic) at the origin.

   (i) $f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

   (ii) $f(z) = \begin{cases} \dfrac{xy^2(x + iy)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

2. Prove that for the following function, C-R equations are satisfied at the origin but $f(z)$ is not analytic there.

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

3. Show that $f(z) = \sin \bar{z}$ is not an analytic function of z.
4. Find whether the Cauchy–Riemann equations are satisfied for the following functions where $w = f(z)$.

   (i) $w = 2xy + i(x^2 - y^2)$ **(Ans.** No**)**

   (ii) $w = \dfrac{x - iy}{x^2 + y^2}$ **(Ans.** No**)**

   (iii) $w = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$ **(Ans.** Yes**)**

   (iv) $w = \cos x \sin hy$ **(Ans.** Yes**)**

   (v) $w = z^3 - 2z^2$ **(Ans.** Yes**)**

5. Show that an analytic function with a constant imaginary part is constant.
6. Show that $u + iv = \dfrac{x - iy}{x - iy + a}$, where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

7. Find an analytic function $w = u + iv$ whose real part is given by
   (i) $u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$ **[Ans.** $e^{-x}(x - iy)^2 (\cos y - i \sin y)$**]**
   (ii) $u = \dfrac{x}{x^2 + y^2}$ $\left(\textbf{Ans. } \dfrac{1}{z} + C\right)$
   (iii) $u = e^x(x \cos y - y \sin y)$ **(Ans.** $ze^z + C$**)**
   (iv) $u = x^4 - 6x^2y^2 + y^4$ **(Ans.** $z^4 + C$**)**
   (v) $u = -\sin x \sin hy$ **(Ans.** $-i \cos z + iC$**)**
8. Find an analytic function $w = u + iv$ whose imaginary part is given by
   (i) $v = e^x(x \cos y + y \sin y)$ **(Ans.** $ize^{-z} + C$**)**
   (ii) $v = -2 \sin x(e^y - e^{-y})$ **(Ans.** $\log z + C$**)**
   (iii) $v = \dfrac{\sin x \sin hy}{\cos 2x + \cos h2y}$ $\left(\textbf{Ans. } \dfrac{1 + \sec z}{2}\right)$
   (iv) $v = x^2 - y^2 + 2xy - 3x - 2y$ **[Ans.** $z^2 - 2z + i(z^2 - 3z)$**]**
   (v) $v = x^3 - 3x^2y + 2x + 1 + y^3 - 3xy^2$ **[Ans.** $(i - 1)z^3 + 2z + C$**]**
9. Show that the following functions are harmonic and find their harmonic conjugates.
   (i) $u = \cos x \cos hy$ **(Ans.** $-\sin x \sin hy + C$**)**
   (ii) $u = e^x(\cos y - \sin y)$ **(Ans.** Not harmonic**)**
   (iii) $u = e^{-x}(y \cos y - x \sin y)$ (**Ans.** $e^x(x \cos y + y \sin y) + C$)
   (iv) $u = e^x \cos y$ (**Ans.** $e^x \sin y + C$)
   (v) $u = 2xy + 3xy^2 - 2y^3$ **(Ans.** Not harmonic**)**
10. Find $f(z) = u + iv$, if $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, given that $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$.
    $\left[\textbf{Ans. } f(z) = \cot\left(\dfrac{z}{2}\right) + \left(\dfrac{1 - i}{2}\right)\right]$
11. Find $f(z) = u + iv$ if $2u - 3v = 3y^2 - 2xy - 3x^2 + 3y - x$ and $f(0) = 0$.
    (**Ans.** $f(z) = iz^2 - z$**)**
12. If $f(z) = u + iv$ is a regular function of z, then show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) | f(z)|^p = p^2 | f(z)|^{p-2} | f'(z)|^2$.
13. If $u = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$, find $f(z)$ such that $f(z)$ is analytic.
    (Ans. $f(z) = \cot z + C$**)**
14. Show that $\phi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ can represent the velocity potential in an incompressible fluid flow. Also find the corresponding stream function and complex potential. $\left[\textbf{Ans. } \psi = 2xy - \dfrac{y}{x^2 + y^2} + C;\ f(z) = z^2 + \dfrac{1}{z} + iC\right]$

22 Conformal Mapping

Chapter Outline

- Introduction
- Conformal Transformation
- Conformal Mapping by Elementary Transformations
- Some Standard Transformations
- Bilinear Transformation

22.1 ❑ INTRODUCTION

Many physical problems involving ideal fluid flow, steady-state heat flow, electrostatics, magnetism, current flow etc., can be solved using conformal mapping techniques. These problems generally involve Laplacian in three-dimensional coordinates and also divergence and are of three-dimensional vector functions.

Geometrical Representation

To draw the curve of a complex variable (x, iy), we take two axes, i.e., the first one is the real axis and the other is the imaginary axis. A number of points (x, y) are plotted on the z-plane, by taking different values of z (different values of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called an **Argand diagram**.

Let $w = f(z) = f(x + iy) = u + iv$.

To draw a curve of w, we take the u-axis and v-axis. By plotting different points (u, v) on the w-plane and joining them, we get a curve C on the w-plane.

Transformation

For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this **transformation or mapping of z-plane into w-plane.** If a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.

If the point $P(x, y)$ moves along the curve C in the z-plane, the point $P'(u, v)$ will move along a corresponding curve C_1 in the w-plane. We then say that a curve C in the z-plane is mapped into the corresponding curve C_1 in the w-plane by the relation $w = f(z)$.

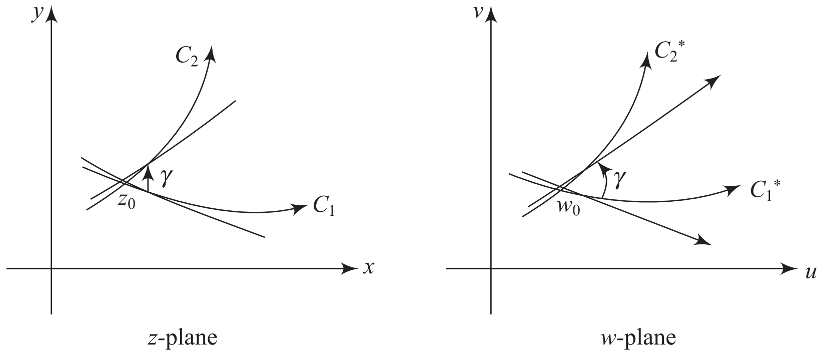


Fig. 22.1

22.2 ❑ CONFORMAL TRANSFORMATION (OR CONFORMAL MAPPING)

A mapping $w = f(z)$ is said to be **conformal** if the angle between any two smooth curves C_1, C_2 in the z-plane intersecting at the point z_0 is equal in magnitude and sense to the angle between their images C_1^*, C_2^* in the w-plane at the point $w_0 = f(z_0)$.

Thus, **conformal mapping** preserves angles both in magnitude and sense (which is also known as conformal mapping of the first kind). If only the magnitude of the angle is preserved, then the mapping is known as **isogonal mapping** (or conformal mapping of the second kind).

Conformal mapping is used to map complicated regions conformally onto simpler, standard regions such as circular disks, half-planes and strips for which the boundary-value problems are easier.

Given two mutually orthogonal one-parameter family of curves, say $\phi(x, y) = C_1$ and $\phi(x, y) = C_2$. Their image curves in the w-plane $\phi(u, v) = C_3$ and $\phi(u, v) = C_4$ under a conformal mapping are also mutually orthogonal. Thus, conformal mapping preserves the property of mutual orthogonality of a system of curves in the plane.

➢ **Note**

(i) **Critical point** of a function $w = f(z)$ is a point z_0, where $f'(z_0) \neq 0$.

(ii) A mapping $w = f(z)$ is conformal at each point z_0 where $f(z)$ is analytic and $f'(z_0) \neq 0$

(iii) An analytic function $f(z)$ is conformal everywhere except at its critical points where $f'(z) \neq 0$.

(iv) Solutions of Laplace's equation are invariant under conformal transformation.

(v) Conjugate functions remain conjugate functions after conformal transformation. This is the main reason for the great importance of conformal transformations in applications.

22.3 ❑ CONFORMAL MAPPING BY ELEMENTARY TRANSFORMATIONS

General linear transformation, or simply transformation, is defined by the function

$$w = f(z) = az + b \qquad (22.1)$$

where $a \neq 0$ and b are arbitrary complex constants. The function maps conformally the extended complex z-plane onto the extended w-plane, since this function is analytic and $f'(z) = a \neq 0$ for any z. If $a = 0$ (22.1) reduces to a constant function.

22.4 ❑ SOME STANDARD TRANSFORMATIONS

Translation

The transformation $w = z + c$, where c is a complex constant, represents a translation. Consider the transformation $w = z + c$, where $c = a + ib$.

i.e., $\quad u + iv = (x + iy) + (a + ib)$

$\Rightarrow \quad u = x + a \quad$ and $\quad v = y + b$

i.e., $\quad x = u - a \quad$ and $\quad y = v - b$

On substituting the values of x and y in the equation of the curve to be transformed, we get the equation of the image in the w-plane.

The point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + a, y + b)$ in the w-plane. Similarly, other points of the z-plane are mapped onto the w-plane. Thus, if the w-plane is superposed on the z-plane, the figure of the w-plane is shifted through a vector c.

In other words, the transformation is a mere **translation** of the axes.

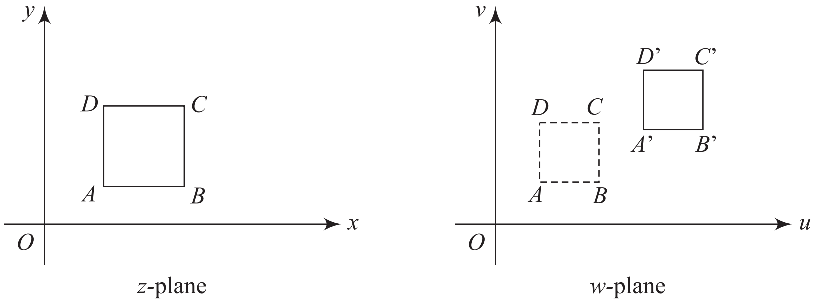


Fig. 22.2

Magnification and Rotation

Consider the transformation $w = cz \qquad (22.2)$

where c, z, w are all complex numbers.

Let $z = re^{i\theta}$, $w = \text{R}e^{i\phi}$, $c = ae^{i\alpha}$

Substituting these values in (22.2), we have

$$\text{R}e^{i\phi} = (ae^{i\alpha})(re^{i\theta}) = ar\, e^{i(\theta + \alpha)}$$

i.e., $\quad R = ar \quad$ and $\quad \phi = \theta + \alpha$

Thus, we see that the transformation $w = cz$ corresponds to a rotation together with magnification.

Algebraically, $\quad w = cz \quad$ or $\quad u + iv = (a + ib)(x + iy)$

$\quad u + iv = ax - by + i(ay + bx)$

$\Rightarrow \quad u = ax - by \quad$ and $\quad v = ay + bx.$

On solving these equations, we can get the values of x and y.

i.e., $$x = \frac{au + bv}{a^2 + b^2};\ y = \frac{-bu + av}{a^2 + b^2}$$

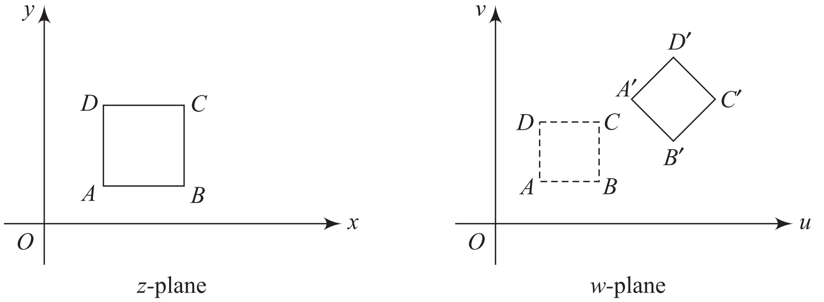


Fig. 22.3

On putting the values of x and y in the equation of the curve to be transformed, we get the equation of the image.

Inversion and Reflection **[KU April 2012]**

Consider the transformation $w = \frac{1}{z}$ (22.3)

$$z = re^{i\theta} \quad \text{and} \quad w = \mathrm{R}e^{i\phi}$$

Substituting these values in (22.3), we get

$$\mathrm{R}e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\Rightarrow \qquad R = \frac{1}{r} \text{ and } \phi = -\theta$$

Thus, the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'\left(\frac{1}{r}, -\theta\right)$ in the w-plane. Hence, the transformation is an inversion of z followed by reflection into the real axis. The points inside the unit circle $|z| = 1$ map onto points outside it, and points outside the unit circle into points inside it.

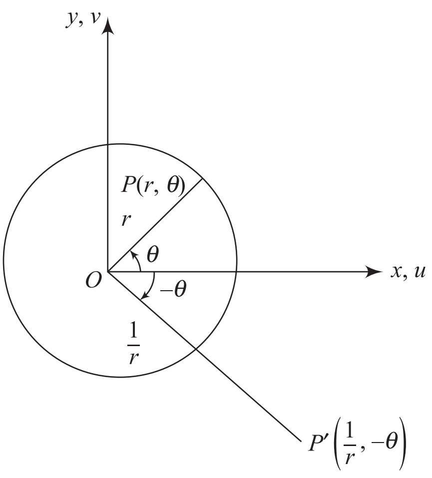


Fig. 22.4

Now consider the transformation $w = \frac{1}{z}$ or $z = \frac{1}{w}$.

i.e., $$x + iy = \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{(u + iv)(u - iv)} = \frac{u - iv}{u^2 + v^2}$$

$$\therefore \qquad x = \frac{u}{u^2+v^2},\ y = \frac{-v}{u^2+v^2}$$

Let the circle $a(x^2 + y^2) + bx + cy + d = 0$ (22.4)
be in the z-plane.
If $a \neq 0$, (22.4) represents a circle and if $a = 0$, it represents a straight line.
On substituting the values of x and y in (22.4), we get

$$\frac{a}{u^2+v^2} + \frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} + d = 0$$

$$\Rightarrow \qquad d(u^2 + v^2) + bu - cv + a = 0 \qquad (22.5)$$

If $d \neq 0$ Eq. (22.5) represents a circle and if $d = 0$ it represents a straight line.
The various cases are discussed as follows:

- **When $a \neq 0$, $d \neq 0$**

The transformation $w = \frac{1}{z}$ transforms circles not passing through the origin into circles not passing through the origin.

- **When $a \neq 0$, $d = 0$**

The transformation $w = \frac{1}{z}$ transforms circles passing through the origin in the z-plane and maps into the straight lines not passing through the origin in the w-plane.

- **When $a = 0$, $d \neq 0$**

The transformation $w = \frac{1}{z}$ transforms straight lines in the z-plane not passing through the origin into circles through the origin in the w-plane.

- **When $a = 0$, $d = 0$**

The transformation $w = \frac{1}{z}$ transforms straight lines through the origin in the z-plane into straight lines through the origin in the w-plane.

22.5 ❑ BILINEAR TRANSFORMATION (OR MÖBIUS TRANSFORMATION)

The transformation $w = f(z) = \frac{az+b}{cz+d}$ (22.8)

where a, b, c, d are complex or real constants subject to $ad - bc \neq 0$ is known as bilinear transformation.
Differentiating (22.8), we get

$$\frac{dw}{dz} = \frac{(cz+d)a - (az+b)c}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2}$$

If $ad - bc \neq 0$ then $\frac{dw}{dz} \neq 0$ for any z and, therefore, bilinear transformation is conformal for all z, i.e., it maps the z-plane conformally onto the w-plane

If $ad - bc = 0$ then $\frac{dw}{dz} = 0$ for any z. Then every point of the z-plane is critical and the function is not conformal.

From (22.8), we get $w(cz + d) = az + b$,

i.e.,
$$cwz + dw - az - b = 0 \tag{22.9}$$

Equation (22.9) is linear in z and linear in w or bilinear in z and w. Bilinear transformation is also known as **linear fractional transformation** or **Mobius transformation.**

For a choice of the constants a, b, c, d, we get special cases of bilinear transformation as

(i) $w = z + b$ (Translation)
(ii) $w = az$ (Rotation)
(iii) $w = az + b$ (Linear transformation)
(iv) $w = \frac{1}{z}$ (Inversion in the unit circle)

Thus, bilinear transformation can be considered as a combination of these transformations.

Fixed Points (or Invariant Points)

Fixed (or invariant) points of a function $w = f(z)$ are points which are mapped onto themselves, i.e., $w = f(z) = z$.

- **Example**

$w = z$ has every point as a fixed point.
$w = \bar{z}$ infinitely many.
$w = \frac{1}{z}$ has two.
$w = z + b$ has no fixed point.

The fixed points of the bilinear transformation $w = \frac{az+b}{cz+d}$ are given by $\frac{az+b}{cz+d} = z$.

As this is quadratic in z, we will get two fixed points for the bilinear transformation.

Cross-ratio

The **cross-ratio,** or **anharmonic ratio,** of four numbers z_1, z_2, z_3, z_4 is the linear function given by $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$.

➢ **Note**

(i) The cross-ratio of four points is invariant under a bilinear transformation, i.e., if w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation, then $\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_1 - w_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$.

(ii) The bilinear transformation that maps three given points z_2, z_3, z_4 onto three given points w_2, w_3, w_4 is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w-w_3)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z-z_3)}$$

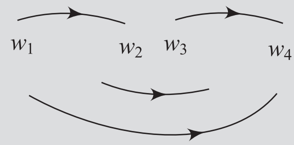


Fig. 22.5

SOLVED EXAMPLES

Example 1 Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$.

Solution Let $z = x + iy$; $w = u + iv$

Given $\quad w = z + 3 + 2i$

i.e., $\quad u + iv = (x + iy) + (3 + 2i)$

$\Rightarrow \quad u = x + 3;\ v = y + 2$

Given the circle $|z| = 2$

i.e., $\quad x^2 + y^2 = 4$

i.e., $\quad (u-3)^2 + (v-2)^2 = 4$

Hence, the circle $x^2 + y^2 = 4$ maps into $(u-3)^2 + (v-2)^2 = 4$ in the w-plane which is also a circle with centre at (3, 2) and radius of 2 units. **Ans.**

Example 2 Find the image of the triangular region in the z-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ under the transformation $w = 2z$. **[KU May 2010]**

Solution Given $w = 2z$. i.e., $u + iv = 2(x + iy)$

$\therefore \quad u = 2x \quad$ and $\quad v = 2y$

When $x = 0$, $u = 0$, the line $x = 0$ is transformed into the line $u = 0$ in the w-plane.

When $y = 0$, $v = 0$, the line $y = 0$ is transformed into the line $v = 0$ in the w-plane.

When $x + y = 1$, we get

$$\frac{u}{2}+\frac{v}{2}=1$$

$\Rightarrow \quad u + v = 2$

$\therefore$ the line $x + y = 1$ is transformed into the line $u + v = 2$ in the w-plane.

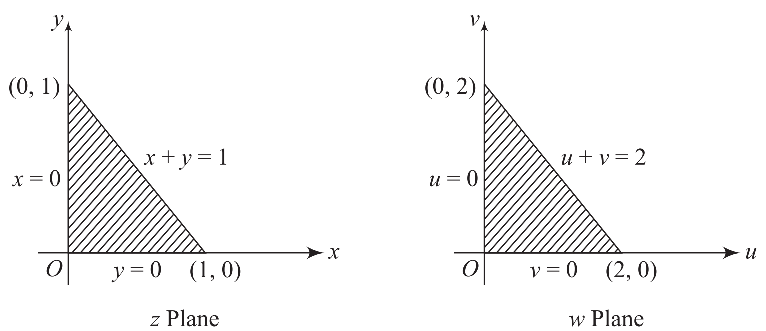


Fig. 22.6

Example 3 Find the image of the circle $|z-1|=1$ in the complex plane under the mapping $w=\frac{1}{z}$.

Solution The given transformation is $w=\frac{1}{z}$

i.e., $$z=\frac{1}{w}$$

The equation of the circle is $|z-1|=1$

i.e., $$|x+iy-1|=1$$

$$(x-1)^2+y^2=1 \quad \Rightarrow \quad x^2-2x+y^2=0 \qquad (1)$$

Now, $w=u+iv$

$\therefore$ $$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

$\therefore$ $$x=\frac{u}{u^2+v^2} \qquad (2)$$

and $$y=\frac{-v}{u^2+v^2} \qquad (3)$$

Substituting (2) and (3) in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2-2\left(\frac{u}{u^2+v^2}\right)+\left(\frac{-v}{u^2+v^2}\right)^2=0$$

i.e., $$u^2-2u(u^2+v^2)+v^2=0$$

$$(u^2+v^2)(1-2u)=0$$

$\Rightarrow$ $$1-2u=0 \qquad (\text{since } u^2+v^2\neq 0)$$

i.e., $2u-1=0$ which is a straight line in the w-plane. Hence, the circle $|z-1|=1$ is mapped into a straight line under the transformation $w=\frac{1}{z}$. **Ans.**

Example 4 Find the image of the infinite strips (i) $\frac{1}{4}<y<\frac{1}{2}$; and (ii) $0<y<\frac{1}{2}$ under the transformation $w=\frac{1}{z}$. **[KU April 2013]**

Solution Let $w=u+iv$, $z=x+iy$.

Given $$w=\frac{1}{z}$$

i.e., $$u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$$

i.e., $$u=\frac{x}{x^2+y^2} \qquad (1)$$

$$v=\frac{-y}{x^2+y^2} \qquad (2)$$

Now, $\frac{u}{v} = \frac{-x}{y}$.

i.e., $$x = \frac{-uy}{v} \qquad (3)$$

Substituting (3) in (2), we get

$$v = \frac{-y}{\frac{u^2y^2}{v^2} + y^2} = \frac{-v^2}{(u^2+v^2)\cdot y}$$

or $$y = \frac{-v}{u^2+v^2} \qquad (4)$$

(i) Consider a strip $\frac{1}{4} < y < \frac{1}{2}$.

When $y = \frac{1}{4}$,

From (4), $\frac{1}{4} = \frac{-v}{u^2+v^2}$

i.e., $u^2 + v^2 + 4v = 0$ or $u^2 + (v+2)^2 = 4$.

which is a circle whose centre is at (0, –2) in the w-plane and radius is 2 units.

When $y = \frac{1}{2}$,

From (4), $\frac{-v}{u^2+v^2} = \frac{1}{2}$

i.e., $u^2 + (v+1)^2 = 1$.

which is a circle whose centre is at (0, –1) in the w-plane and the radius is 1 unit.

Hence, the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region common to the circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w-plane.

(ii) Consider a strip $0 < y < \frac{1}{2}$.

When $y = 0$,

from (4), we get $v = 0$.

When $y = \frac{1}{2}$,

from (4), we get $\frac{1}{2} = \frac{-v}{u^2+v^2}$.

i.e., $u^2 + v^2 + 2v = 0$

i.e., $u^2 + (v+1)^2 - 1 = 0$

which is a circle whose centre is at (0, –1) in the w-plane and radius is 1 unit.

$\therefore$ the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half-plane. **Ans.**

Example 5 Find the invariant points of the transformation $w=-\dfrac{2z+4i}{iz+1}$.

Solution The invariant points of the transformation are given by $z=-\dfrac{2z+4i}{iz+1}$

$\Rightarrow$ $iz^2+3z+4i=0$

i.e., $z^2-3iz+4=0$

i.e., $(z-4i)(z+i)=0$

i.e., $z=4i,-i$ are the invariant points. **Ans.**

Example 6 Find the image of $|z+2i|=2$ under the transformation $w=\dfrac{1}{z}$.

[AU May 2010]

Solution The given transformation is $w=\dfrac{1}{z}$

i.e., $z=\dfrac{1}{w}$

Given $|z+2i|=2$

$|x+iy+2i|=2$

i.e., $|x+i(y+2)|=2$

$\Rightarrow$ $x^2+(y+2)^2=4$

i.e., $x^2+y^2+4y=0$ (1)

Now, $w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$$

i.e., $x+iy=\dfrac{u-iv}{u^2+v^2}$

$\Rightarrow$ $x=\dfrac{u}{u^2+v^2}$, (2)

and $y=\dfrac{-v}{u^2+v^2}$ (3)

Substituting (2) and (3) in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2+4\left(\frac{-v}{u^2+v^2}\right)=0$$

$$u^2+v^2-4v(u^2+v^2)=0$$

$$(u^2+v^2)(1-4v)=0$$

$\Rightarrow$ $1-4v=0$ (as $u^2+v^2\neq 0$)

which is a straight line in the w-plane. **Ans.**

Example 7 Find the bilinear transformation that maps the points $z_1=-i$, $z_2=0$, $z_3=i$ into the points $w_1=-1$, $w_2=i$, $w_3=1$ respectively. **[AU Oct. 2009, KU Nov. 2010]**

Solution Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \qquad (1)$$

Given $z_1=-i,\ z_2=0,\ z_3=0;\ w_1=-1,\ w_2=i,\ w_3=1$ (2)

Substituting (2) in (1), we get

$$\frac{(w+1)(i-1)}{(-1-i)(1-w)}=\frac{(z+i)(0-i)}{(-i-0)(i-z)}$$

i.e.,
$$\frac{(w+1)}{(w-1)}\frac{(i-1)(i-1)}{(i+1)(i-1)}=\frac{-(z+i)}{(z-i)}$$

i.e.,
$$\frac{w+1}{w-1}\cdot\frac{-2i}{-2}=\frac{-(z+i)}{(z-i)}$$

$$\frac{w+1}{w-1}=\frac{i(z+i)}{z-i}$$

By componendo and dividendo,

$$\frac{(w+1)+(w-1)}{(w+1)-(w-1)}=\frac{i(z+i)+(z-i)}{i(z+i)-(z-i)}$$

$$\frac{2w}{2}=\frac{z(1+i)-(1+i)}{z(i-1)-(1-i)}$$

$$w=\frac{(1+i)(z-1)}{(i-1)(z+1)}$$

$$=\frac{(1+i)(-i-1)}{(i-1)(-i-1)}\cdot\frac{(z-1)}{(z+1)}$$

$\Rightarrow$
$$w=-\left(\frac{z-1}{z+1}\right)$$
Ans.

Example 8 Find the bilinear transformation which maps the points $z_1=-1,\ z_2=0,\ z_3=1$ into the points $w_1=0,\ w_2=i,\ w_3=3i$ respectively.

[AU Nov. 2010, KU April 2012]

Solution Let the bilinear translation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \qquad (1)$$

Given $z_1=-1,\ z_2=0,\ z_3=1;\ w_1=0,\ w_2=i,\ w_3=3i$ (2)

Substituting (2) in (1), we get

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)}=\frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{w(-2i)}{-i(3i-w)}=\frac{(z+1)}{1-z}$$

$$\frac{-2iw}{(w-3i)i}=-\left(\frac{z+1}{z-1}\right)$$

i.e., $$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2w(z-1) = (z+1)(w-3i)$$
$$= zw - 3iz + w - 3i$$

⇒ $$w[2(z-1) - (z+1)] = -3i(z+1)$$

or $$w = -3i\frac{(z+1)}{z-3}$$ **Ans.**

Example 9 Show that under the mapping $w = \frac{i-z}{i+z}$, the image of the circle $x^2 = y^2 < 1$ is the entire half of the w-plane to the right of the imaginary axis.

[AU Nov. 2011]

Solution Given $w = \frac{i-z}{i+z}$

i.e., $(i + z)w = i - z$

$iw + zw = i - z$

i.e., $z(w + 1) = i(1 - w)$

⇒ $$z = \frac{i(1-w)}{1+w}$$

Also given $x^2 + y^2 < 1$

i.e., $|z| < 1$, i.e., $\left|\frac{i(1-w)}{1+w}\right| < 1$

i.e., $|i|\ |1 - w| < |1 + w|$, i.e., $|1 - u - iv| < |1 + u + iv|$ [as $|i| = 1$]

i.e., $(1 - u)^2 + v^2 < (1 + u)^2 + v^2$

i.e., $1 + u^2 - 2u + v^2 < 1 + u^2 + 2u + v^2$

⇒ $4u > 0$

or $u > 0$

Hence, the circle $x^2 + y^2 < 1$, i.e., $|z| < 1$ is mapped into the entire half of the w-plane to the right of the imaginary axis.

When $|z| = 1$ i.e., $x^2 + y^2 = 1$ which is the unit circle, we get $u = 0$ which is the imaginary axis of the w-plane. **Proved.**

EXERCISE

Part A

1. Define conformal mapping.
2. When is a transformation said to be isogonal? Prove that the mapping $w = \bar{z}$ is isogonal.
3. Define critical point of a transformation.

4. Find the images of the circle $|z| = a$ under the transformations (i) $w = z + 2 + 3i$, and (ii) $w = 2z$.
5. Under the transformation $w = iz + i$, show that the half-plane $x > 0$ maps into the half-plane $w > 1$.
6. Find the invariant point of the bilinear transformation $w = \dfrac{1+z}{1-z}$.
7. Find the fixed points of $w = \dfrac{3z-4}{z-1}$.
8. Define Mobius transformation.
9. Find the invariant point of the transformation $w = \dfrac{1}{z-2i}$.
10. Find the image of $x^2 + y^2 = 4$ under the transformation $w = 3z$.
11. Find the image of the circle $|z - \alpha| = r$ by the mapping $w = z + c$ where c is a constant.
12. Find the fixed points of the transformation $w = \dfrac{1}{z+2i}$.
13. Find the invariant points of the transformation $w = \dfrac{1+z}{1-z}$.
14. Find the image of the circle $|z| = 3$ under the transformation $w = 2z$.
15. Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$.
16. Find the image of the real axis of the z-plane by the transformation $w = \dfrac{1}{z+i}$.
17. Define cross-ratio of four points in a complex plane.
18. Prove that a bilinear transformation has at most two fixed points.

Part B

1. For the mapping $w = \dfrac{1}{z}$, find the image of the family of circles $x^2 + y^2 = ax$, where a is real. $\left(\textbf{Ans. } u = \dfrac{1}{a}, \text{ is a straight line}\right)$
2. Determine the region of the w-plane into which the region bounded by $x = 1$, $y = 1$, $x + y = 1$ is mapped by the transformation $w = z^2$.
   (**Ans.** $4u + v^2 = 4$, $4u - v^2 = -4$, $u^2 = 2$, $v^2 = 1$)
3. Determine the images of the regions under $w = \dfrac{1}{z}$. (i) $x > 1, y > 0$ (ii) $0 < y < \dfrac{1}{2c}$.
   $\left[\textbf{Ans. (i) } \left|w - \dfrac{1}{2}\right| < \dfrac{1}{2} \text{ (ii) } u^2 + (v+c)^2 > c^2\right]$
4. Find an analytic function $w = f(z)$ which maps the half-plane $x \geq 0$ onto the region $u \geq 2$ such that $z = 0$ corresponds to $w = 2 + i$.
   (*Hint*: $w_1 = z$, $w_2 = w_1 + 2$, $w = w_2 + \text{i}$)
   (**Ans.** $w = z + 2 + i$)
5. Determine and plot the images of the regions under the transformation $w = z^2$.
   (i) $|z| = 2$ (ii) $|\arg z| \leq \dfrac{\pi}{2}$ (iii) $\dfrac{1}{2} < |z| < 2, \text{Re } z \geq 0$
   $\left[\textbf{Ans. (i) } 1w > 4 \text{ (ii) } |\arg w| \leq \pi \text{ (iii) } \dfrac{1}{4} < |w| < 4, -\pi \leq \phi \leq \pi\right]$

6. Find the invariant (fixed) points of the transformation:

   (i) $w = \dfrac{z-1}{z+1}$ (ii) $w = z^2$ (iii) $w = \dfrac{2z-5}{z+4}$ (iv) $w = (z-i)^2$

   $$\left[\textbf{Ans.}\ \text{(i) } z = \pm i \ \text{ (ii) } z = 0, 1 \ \text{ (iii) } z = -1 + 2i \ \text{ (iv) } z = \frac{(1+2i) \pm \sqrt{1+4i}}{2}\right]$$

7. Find the bilinear transformation that maps z_1, z_2, z_3 onto w_1, w_2, w_3 respectively.
   (i) $z = -1, 0, 1$ onto $w = 0, i, 3i$
   (ii) $z = 0, -i, -1$ onto $w = i, 1, 0$
   (iii) $z = 1, i, -1$ onto $w = 2, i, -2$
   (iv) $z = \infty, i, 0$ onto $w = 0, i, \infty$
   (v) $z = 1, 0, -1$ onto $w = i, 1, \infty$

   $$\left[\begin{array}{l}\textbf{Ans.}\ \text{(i) } w = \dfrac{-3i(z+1)}{z-3},\ \text{(ii) } w = -i\left(\dfrac{z+1}{z-1}\right)\ \text{(iii) } w = \dfrac{-6z+2i}{iz-3} \\ \quad \text{(iv) } w = -\dfrac{1}{z}\ \text{(v) } w = \dfrac{(-1+2i)z+1}{z+1}\end{array}\right]$$

8. Verify that the equation $w = \dfrac{1+iz}{1+z}$ maps the exterior of the circle $|z| = 1$ into the upper half-plane $v > 0$.
9. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. **(Ans.** i, $2i$**)**
10. Show that the transformation $w = \dfrac{i(1-z)}{1+z}$ maps the circle $|z| = 1$ into the real axis of the w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane.
11. Show that the transformation $w = \dfrac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$.
12. Show that transformation $w = \dfrac{i-z}{i+z}$ maps the circle $|z| = 1$ onto the imaginary axis of the w-plane. Find also the images of the interior and exterior of this circle.

Unit X

Complex Integration

23 Complex Integration

Chapter Outline

- Introduction
- Line Integral in a Complex Plane
- Line Integral
- Basic Properties of Line Integrals
- Simply Connected Region and Multiply Connected Region
- Evaluation of Complex Integrals
- Cauchy's Integral Theorem
- Extension of Cauchy's Integral Theorem to Multiply Connected Regions
- Cauchy's Integral Formula
- Cauchy's Integral Formula for the Derivation of an Analytic Function

23.1 ❑ INTRODUCTION

Integration of functions of a complex variable plays a very important role in many areas of science and engineering. The advantage of complex integration is that certain complicated real integrals can be evaluated and properties of analytical functions can be established. Using integration, we shall prove a very important result in the theory of analytic functions:

If a function $f(z)$ is analytic in a domain D then it possesses derivatives of all orders in D, that is $f'(z)$, $f''(z)$ … are all analytic functions in D.

Such a result does not exist in the real-variable theory. Also, the complex-integration approach can be used to evaluate many improper integrals of a real variable, which cannot be evaluated using real integral calculus. The concept of definite integral for functions of a real variable does not directly extend to the case of complex variables.

In the case of a real variable, the path of integration in the definite integral $\int_a^b f(x)dx$ is along a straight line. In complex integration, the path could be along any curve from $z = a$ to $z = b$.

23.2 ❑ LINE INTEGRAL IN COMPLEX PLANE

- **Continuous Arc**

The set of points (x, y) defined by $x = \phi(t)$, $y = \psi(t)$, with parameter t in the interval (a, b), defines a continuous arc provided ϕ and ψ are continuous functions.

- **Smooth Arc**

If ϕ and ψ are differentiable, the arc is said to be smooth.

- **Simple Curve**

It is a curve having no self-intersections, i.e., no two distinct values of t correspond to the same point (x, y).

- **Closed Curve**

It is one in which end points coincide, i.e., $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$.

- **Simple Closed Curve**

It is a curve having no self-intersections and with coincident end points.

- **Contour**

It is a continuous chain of a finite number of smooth arcs.

- **Closed Contour**

It is a piecewise smooth closed curve without points of self-intersection.

23.3 ❑ LINE INTEGRAL

Definite integral or complex line integral or simply line integral of a complex function $f(z)$ from z_1 to z_2 along a curve C is defined as

$$\begin{aligned}\int_C f(z)dz &= \int_C (u+iv)(dx+i\,dy)\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\end{aligned}$$

Here, C is known as path of integration. If it is a closed curve, the line integral is denoted by $\oint_C$.

When the direction is in positive sense, it is indicated as $\int_{C+}$ or simply, $\int_C$ while negative direction is denoted by $\int_C$. Counter integral is an integral along a closed contour.

23.4 ❑ BASIC PROPERTIES OF LINE INTEGRALS

(i) Linearity: $\int_C (k_1 f(z) + k_2 g(z))\,dz = k_1 \int_C f(z)dz + k_2 \int_C g(z)dz$

(ii) Sense reversal: $\int_a^b f(z)dz = -\int_b^a f(z)dz$

(iii) Partitioning of path: $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$
where the curve C consists of the curves C_1 and C_2.

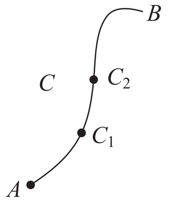


Fig. 23.1

➢ **Note**

Although real definite integrals are interpreted as area, no such interpretation is possible for complex definite integrals.

23.5 ❑ SIMPLY CONNECTED REGION AND MULTIPLY CONNECTED REGION

A simply connected region R is a domain such that every simple closed path in R contains only points of R.

- **Example**

Interior of a circle, rectangle, triangle, ellipse, etc.

A multiply connected region is one that is not simply connected.

- **Example**

Annulus region, region with holes.

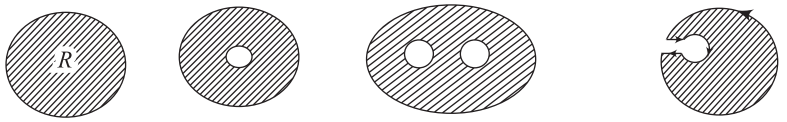


Fig. 23.2

23.6 ❑ EVALUATION OF A COMPLEX INTEGRAL

To evaluate the integral $\int_C f(z)dz$, we have to express it in terms of real variables.

Let $f(z) = u + iv$ where $z = x + iy$, $dz = dx + idy$

$$\therefore \quad \int_C f(z)dz = \int_C (u + iv)dz$$
$$= \int_C (u + iv)(dx + idy)$$
$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

23.7 ❑ CAUCHY'S INTEGRAL THEOREM OR CAUCHY'S FUNDAMENTAL THEOREM

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\int_C f(z)dz = 0$.

- **Proof**

Let the region enclosed by a curve C be R and let

$$f(z) = u + iv,\ z = x + iy,\ dz = dx + i\,dy$$

$$\int_C f(z)dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy \quad \text{(by Green's theorem)}$$

Replacing $-\dfrac{\partial v}{\partial x}$ by $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ by $\dfrac{\partial u}{\partial x}$, we get

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)dx\,dy$$

$$= 0 + i0 = 0$$

or $\int_C f(z)dz = 0$

➢ **Note**

(i) Cauchy's integral theorem is also known as Cauchy's theorem.
(ii) Cauchy's theorem without the assumption that f' is continuous is known as the **Cauchy–Goursat theorem.**
(iii) Simple connectedness is essential.

23.8 ❑ EXTENSION OF CAUCHY'S INTEGRAL THEOREM TO MULTIPLY CONNECTED REGIONS

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

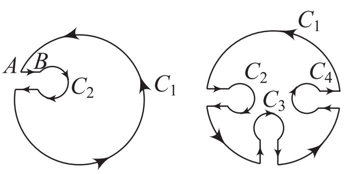


Fig. 23.3

- **Proof**

By Cauchy's integral theorem, we know that $\int_C f(z)dz = 0$ where the path of integration is along AB and the curve C_2 in clockwise direction, and BA and along C_1 in anticlockwise direction,

i.e., $$\int_{AB} f(z)dz + \int_{C_2} f(z)dz + \int_{BA} f(z)dz + \int_{C_1} f(z)dz = 0$$

or $$\int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0 \ \left(\text{since } \int_{AB} f(z)dz = -\int_{BA} f(z)dz\right)$$

Reversing the direction of the integral around C_2, we get

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

➢ **Note**

By introducing as many cross-cuts as the number of inner boundaries, we can give the proof in a similar manner for the extension of Cauchy's integral theorem.

23.9 ❑ CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve C and if a is any point within C then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$.

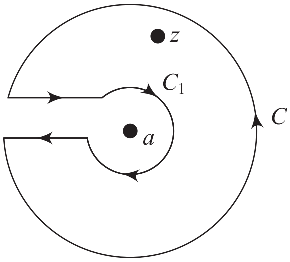


Fig. 23.4

• **Proof**

Consider the function $\frac{f(z)}{z-a}$, which is analytic at all points within C except $z = a$.

With a point a as centre and radius r, draw a small circle C_1 lying entirely within C. Now, $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1;

Hence, by Cauchy's integral theorem for a multiply connected region, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)-f(a)+f(a)}{z-a}dz$$

$$= \int_{C_1} \frac{f(z)-f(a)}{z-a}dz + f(a)\int_{C_1} \frac{dz}{z-a} \qquad (23.1)$$

For any point on C_1

Now, $$\int_{C_1} \frac{f(z)-f(a)}{z-a}dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}} ire^{i\theta}d\theta$$

[as $z - a = re^{i\theta}$ and $dz = ire^{i\theta}\,d\theta$]

$$= \int_0^{2\pi} [f(a+re^{i\theta})-f(a)]id\theta = 0$$ (where r tends to zero]

$$\int_{C_1} \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = \int_0^{2\pi} id\theta = i[0]_0^{2\pi i} = 2\pi i$$

Putting the values of the integrals of RHS in (23.1), we have

$$\int_C \frac{f(z)}{z-a}dz = 0 + f(a)(2\pi i)$$

or $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

23.10 ❑ CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

If a function $f(z)$ is analytic in a region R then its derivative at any point $z = a$ of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

where C is any closed curve in R surrounding the point $z = a$.

- **Proof**

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \qquad (23.2)$$

Differentiating (23.2) with respect to a, we get

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right)\cdot dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

Similarly,

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz$$

$$f^n(a) = \frac{n!}{2\pi !}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

SOLVED EXAMPLES

Example 1 Use Cauchy's integral formula to evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)}dz$, where C is the circle $|z| = 4$.

[AU June 2009, April 2011; KU Nov. 2011]

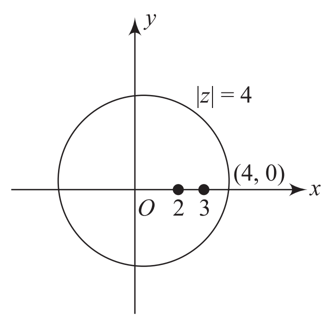


Fig. 23.5

Solution

$$\frac{1}{(z-2)(z-3)} = \frac{1}{(z-3)} - \frac{1}{(z-2)}$$

∴ given integral

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3}dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}dz$$

$$= \int_C \frac{f(z)}{(z-3)}dz - \int_C \frac{f(z)}{(z-2)}dz \qquad (1)$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic on and inside C.

The points $z = 2$ and $z = 3$ lie inside C.

$\therefore$ by Cauchy's integral formula, from (1), we get,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$$

$$= 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=3} - 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=2}$$

$$= 2\pi i(\sin 9\pi + \cos 9\pi) - 2\pi i(\sin 4\pi + \cos 4\pi)$$

$$= -2\pi i - 2\pi i = -4\pi i$$ **Ans.**

Example 2 Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$, using Cauchy's integral formula. **[AU May 2012]**

Solution $|z-2| = \frac{1}{2}$ is the circle with centre at $z = 2$ and radius equal to $\frac{1}{2}$.

The point $z = 2$ lies inside the circle $|z-2| = \frac{1}{2}$

The given integral can be rewritten as

$$\int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2} dz = \int_C \frac{f(z)}{(z-2)^2} dz \text{ (say)}$$

$f(z) = \frac{z}{z-1}$ is analytic on and inside C and the point $z = 2$ lies inside C.

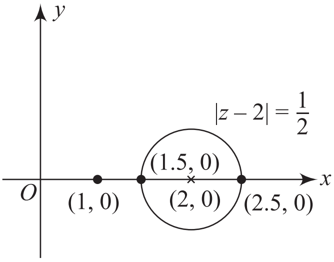


Fig. 23.6

$\therefore$ by Cauchy's integral formula,

$$\int_C \frac{z}{(z-1)(z-2)^2} dz = \frac{2\pi i}{1!} f'(2)$$

$$= 2\pi i \left\{ \frac{d}{dz}\left(\frac{z}{z-1}\right) \right\}_{z=2}$$

$$= 2\pi i \left\{ \frac{-1}{(z-1)^2} \right\}_{z=2} = -2\pi i$$ **Ans.**

Example 3 Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $|z + 1 + i| = 2$ using Cauchy's integral formula. **[AU Nov. 2011]**

Solution $|z + 1 + i| = 2$ is the circle whose centre is $-1 - i$ and radius is 2 units.

Consider $\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$

$\therefore$ the integral is not analytic at $z = -1 - 2i$ and $-1 + 2i$.

The point $z = -1 - 2i$ lies inside C.

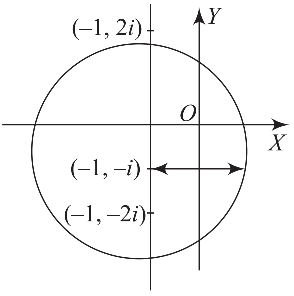


Fig. 23.7

We rewrite the given integral as

$$\int_C \frac{\left(\dfrac{z+4}{z+1-2i}\right)}{z+1+2i}dz = \int_C \frac{f(z)}{z-(-1-2i)}dz \text{ (say)}$$

$f(z)$ is analytic on and inside C and the point $(-1, -2i)$ lies inside C.

$\therefore$ by Cauchy's integral formula,

$$\int_C \frac{z+4}{z^2+2z+5}dz = 2\pi i\, f(-i-2i)$$

$$= 2\pi i \left\{\frac{-1-2i+4}{-1-2i+1-2i}\right\}$$

$$= \frac{-\pi}{2}(3-2i)$$

Ans.

EXERCISE

Part A

1. The value of the integral $\int_C \frac{dz}{z^2-2z}$ where C is the circle $|z-2| = 1$, traversed in the counter-clockwise sense is
   (i) $-\pi i$ (ii) $2\pi i$ (iii) πi (iv) 0
2. The value of the integral $\int_C \frac{z^2-z+1}{z-1}dz$, where C is the circle $|z| = \frac{1}{2}$ is
   (i) 0 (ii) πi (iii) $-\pi i$ (iv) $-2\pi i$
3. What is the value of $\int_C e^z\, dz$ if $c : |z| = 1$?
4. State Cauchy's integral formula.
5. Evaluate $\int_C \frac{dz}{z-2}$ where C is the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).
6. Evaluate $\int_C \frac{3z^2+7z+1}{(z-3)}dz$ where $C : |z| = 2$.
7. Evaluate $\int_C \frac{dz}{z^2-5z+6}$ where C is the circle $|z-1| = \frac{1}{2}$.
8. State Cauchy's formula for the first derivative of an analytic function.
9. State Cauchy's fundamental theorem.
10. Evaluate $\int_C \frac{z\,dz}{z-2}$ where $C : |z| = 1$.
11. Evaluate $\int_C \frac{2}{(z+3)}dz$ where $C : |z| = 2$.
12. Evaluate $\int_C \frac{1}{2z-3}dz$ where $C : |z| = 1$.

13. Evaluate $\dfrac{1}{2\pi i}\int_C \dfrac{z^2+5}{z-3}dz$ where C is $|z| = 4$ using Cauchy's integral formula.

14. Evaluate $\int_C \dfrac{dz}{(z-3)^2}$ where $C : |z| = 1$.

15. State the Cauchy–Goursat theorem.

Part B

1. Evaluate $\int_C \dfrac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i| = 2$. $\left(\textbf{Ans. } \dfrac{-2\pi i}{9}\right)$

2. Evaluate $\int_C \dfrac{4-3z}{z(z-1)(z-2)}dz$ using Cauchy's integral formula. where C is the circle $|z| = \dfrac{3}{2}$. **(Ans.** $2\pi i$**)**

3. Find the value of $\int_C \dfrac{2z^2+z}{z^2-1}dz$. **(Ans.** $3\pi i$**)**

4. Evaluate the following:

   (i) $\int_C \dfrac{dz}{(z^2+4)^2}$, where C is $|z-i| = 2$

   (ii) $\int_C \dfrac{z^3+z+1}{z^2-7z+6}dz$ where C is the ellipse $4x^2 + 9y^2 = 1$

   (iii) $\int_C \dfrac{z^3+1}{z^2-3iz}dz$ where C is $|z| = 1$. $\left[\textbf{Ans. } \text{(i) } \dfrac{\pi}{16}, \text{(ii) } 0, \text{(iii) } -\dfrac{2\pi}{3}\right]$

5. Evaluate $\int_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}dz$ where C is $|z| = 3$. **(Ans.** $-4\pi i$**)**

6. If $f(a) = \int_C \dfrac{4z^2+z+5}{z-a}dz$ where C is $|z| = 2$, find the values of $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$. **(Ans.** $20\pi i$; $2\pi(i-1)$; $-14\pi i$; $16\pi i$**)**

7. Evaluate $\int_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1). **(Ans.** $-1 + i$**)**

8. Evaluate $\int_C \dfrac{z^2+1}{z^2-1}dz$ where (i) $C : |z-1| = 1$, (ii) $C : |z+1| = 1$, and (iii) $C : |z-i| = 1$. **[Ans.** (i) $2\pi i$ (ii) $-2\pi i$ (iii) 0**]**

9. Evaluate $\int_C \dfrac{\sin 2z}{(z+3)(z+1)^2}dz$ where C is the rectangle with vertices at $3 + i$, $-2 + i$, $-2 - i$, $3 - i$. $\left[\textbf{Ans. } \pi i \dfrac{(4\cos 2 + \sin 2)}{2}\right]$

10. Evaluate $\int_C \dfrac{z^4-3z^2+6}{(z+i)^3}dz$ where $C : |z| = 2$. **(Ans.** $-18\pi i$**)**

24 Taylor and Laurent Series Expansions

Chapter Outline

- Introduction
- Taylor's Series
- Laurent's Series

24.1 ❑ INTRODUCTION

Power Series

A power series in powers of $(z - z_0)$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0) + \cdots \tag{24.1}$$

Here, a_0, a_1, a_2 ... are complex (or real) constants known as coefficients of the series. z is a complex variable and z_0 is called the centre of the series. Equation (24.1) is also known as the power series about the point z_0.

Power series in powers of z is

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

obtained as a particular case with $z_0 = 0$ in (24.1). The **region of convergence** of a series is the set of all points z for which the series converges.

Three distinct possibilities exist regarding the region of convergence of a power series (24.1).

(i) The series converges only at the point $z = z_0$.

(ii) The series converges everywhere inside a circular disk $|z - z_0| < R$ and diverges everywhere outside the disk $|z - z_0| > R$. Here, R is known as the **radius of convergence** and the circle $|z - z_0| = R$ as the **circle of convergence.**

➢ **Note**

(i) The series may converge or diverge at the points on the circle of convergence.

(ii) **Geometric Series:** $\sum_{m=0}^{\infty} z^m = 1 + z + z^2 + \cdots$ converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. (i.e., $R = 1$)

(iii) **Power series:** $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z. (i.e., $R = \infty$)

Power series play an important role in complex analysis, since they represent analytic functions and conversely every analytic function has a power series representation called Taylor series similar to Taylor series in real calculus.

Analytic functions can also be represented by another type of series called **Laurent series,** which consist of positive and negative integral powers of the independent variable. They are useful for evaluating complex and real integrals.

24.2 ❑ TAYLOR'S SERIES (TAYLOR'S THEOREM)

If a function $f(z)$ is analytic at all points inside a circle C with its centre at the point a and radius R then at each point z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

• **Proof**

Take any point z inside C. Draw a circle C_1 with centre a, enclosing the point z. Let w be a point on the circle C_1.

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)}$$

$$= \frac{1}{(w-a)} \frac{1}{\left(1 - \frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

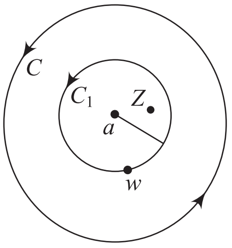


Fig. 24.1

Applying the binomial theorem,

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right]$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^{n+1}} + \cdots \qquad (24.2)$$

As $|z-a| < |w-a|$ or $\frac{|z-a|}{|w-a|} < 1$,

so the series converges uniformly. Hence, the series is integrable.

Multiplying (24.2) by $f(w)$,

$$\frac{f(w)}{w-z}=\frac{f(w)}{w-a}+(z-a)\frac{f(w)}{(w-a)^2}+(z-a)^2\frac{f(w)}{(w-a)^3}+\cdots+(z-a)^n\frac{f(w)}{(w-a)^{n+1}}+\cdots$$

On integrating with respect to w, we get

$$\int_{C_1}\frac{f(w)}{w-z}dw=\int_{C_1}\frac{f(w)}{w-a}dw+(z-a)\int_{C_1}\frac{f(w)}{(w-a)^2}dw+\cdots$$
$$+(z-a)^n\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw+\cdots \qquad (24.3)$$

We know that

$$\int_{C_1}\frac{f(w)}{(w-z)}dz=2\pi i\, f(z),\ \int_{C_1}\frac{f(w)}{w-a}dw=2\pi i\, f(a)$$

$$\int_{C_1}\frac{f(w)}{(w-a)^2}dw=2\pi i\, f'(a)\text{ , and so on.}$$

Substituting these values in (24.3), we get

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^n(a)}{n!}(z-a)^n+\cdots$$

➢ **Note**

(i) Putting $a=0$ in the Taylor's series, we get $f(z)=f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\cdots$

This series is called the **McLaurin's series** of $f(z)$.

(ii) **Standard McLaurin's Series**

(a) $e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$ for $|z|<\infty$

(b) $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$ for $|z|<\infty$

(c) $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}\cdots$ for $|z|<\infty$

(d) $\sin hz=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$ for $|z|<\infty$

(e) $\cos hz=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots$ for $|z|<\infty$

(f) $(1-z)^{-1}=1+z+z^2+z^3+\cdots$ for $|z|<1$

(g) $(1+z)^{-1}=1-z+z^2-z^3+\cdots$ for $|z|<1$

(h) $(1-z)^{-2}=1+2z+3z^2+\cdots$ for $|z|<1$

(iii) Expansion of a function $f(z)$ about a singular point $z=h$ means, expansion of $f(z)$ in powers of $(z-h)$.

24.3 ❑ LAURENT'S SERIES (LAURENT'S THEOREM)

If $f(z)$ is analytic on C_1 and C_2 and the annular region bounded by the two concentric circles C_1 and C_2 of radii r_1 and $r_2 (r_2 < r_1)$ and with centre at a then for all in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots$$

where
$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, n = 0, 1, 2, 3...$$
$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, n = 1, 2, 3...$$

- **Proof**

By introducing a cross-cut AB, the multiply connected region R is converted to a simply connected region. Now, $f(z)$ is analytic in this region.

Now by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}$$
$$\int_{AB} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_{BA} \frac{f(w)}{w-z} dw$$

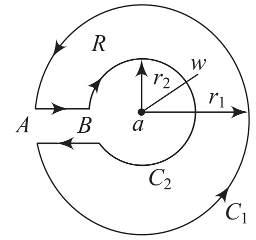


Fig. 24.2

Integral along c_2 is clockwise, so it is negative.

$$\therefore \quad f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw \tag{24.4}$$

For the first integral, $\frac{f(w)}{w-z}$ can be expended exactly as in Taylor's series since w lies on C_1,

$$|z-a| \le |w-a| \text{ or } \frac{|z-a|}{|w-a|} \le 1$$

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-a} dw + \frac{(z-a)}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^2} dw$$
$$+ \frac{(z-a)^2}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^3} dw + \cdots$$
$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \tag{24.5}$$
$$\left[\text{as } a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw\right]$$

In the second integral, w lies on C_2

$$\therefore \quad |w-a| < |z-a| \text{ or } \frac{|w-a|}{|z-a|} < 1$$

So here,
$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)} = \frac{-1}{(z-a)} \cdot \frac{1}{1 - \frac{w-a}{z-a}}$$

$$= -\frac{1}{z-a}\left[1-\frac{w-a}{z-a}\right]^{-1}$$

$$= -\frac{1}{z-a}\left[1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\cdots+\left(\frac{w-a}{z-a}\right)^{n+1}+\cdots\right]$$

Multiplying by $\frac{-f(w)}{2\pi i}$, we get

$$-\frac{1}{2\pi i}\frac{f(w)}{w-z} = \frac{1}{2\pi i}\frac{f(w)}{z-a}+\frac{1}{2\pi i}\frac{(w-a)}{(z-a)^2}f(w)+\frac{1}{2\pi i}\frac{(w-a)^2}{(z-a)^3}f(w)+\cdots$$

$$= \left(\frac{1}{z-a}\right)\frac{1}{2\pi i}f(w)+\frac{1}{(z-a)^2}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-1}}+\frac{1}{(z-a)^3}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-2}}+\cdots$$

Integrating, we have

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw = \frac{1}{(z-a)}\frac{1}{2\pi i}\int_{C_2}f(w)dw+\frac{1}{(z-a)^2}\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-1}}dw$$

$$+\frac{1}{(z-a)^3}\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-2}}dw+\cdots$$

$$= \frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+\cdots \tag{24.6}$$

$$\left[\text{as } b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw\right]$$

Substituting the values of both integrals from (24.5) and (24.6) in (24.4), we get

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + b_1(z-a)^{-1} + b_2(z-a)^{-2} + \cdots$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

➢ **Note**

(i) If $f(z)$ is analytic at all points inside C_1 (i.e., no singular points inside C_2) then by Cauchy's theorem, $b_n = 0$ for all $n - 1 \geq 0$. Hence, the Laurent series reduces to Taylor series. Thus, Laurent's series expansion about an analytic point a is Taylor series expansion about a.

(ii) The region of convergence of Laurent's series is the annulus region $R_1 < |z - a| < R_2$.

(iii) If $f(z)$ has more than one singular point then several (more than one) Laurent series expansions can be obtained about the same singular point by appropriately considering analytic regions about (centred) at a.

(iv) The part $\sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of $(z - a)$ is called the **analytic part** of the Laurent's series, while $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z - a)$ is called the **principal part** of the Laurent's series.

SOLVED EXAMPLES

Example 1 Obtain Taylor's series expansion to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z|<2$. **[KU Nov. 2010]**

Solution Let $f(z)=\frac{z^2-1}{(z+2)(z+3)}$

$$=1+\frac{-5z-7}{(z+2)(z+3)} \qquad (1)$$

Consider $$\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put $z=-3 \Rightarrow B=-8$

Put $z=-2 \Rightarrow A=3$

$\therefore$ $$\frac{-5z-7}{(z+2)(z+3)}=\frac{3}{z+2}-\frac{8}{z+3}$$

$\therefore$ $$(1) \Rightarrow f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$

Given $|z|<2$, i.e., $\frac{|z|}{2}<1$, so clearly $\frac{|z|}{3}<1$

i.e., $\left|\frac{z}{2}\right|<1$ and $\left|\frac{z}{3}\right|<1$

$\therefore$ $$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

By using binomial theorem,

$$f(z)=1+\frac{3}{2}\left[1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right]$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n}-\frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n}$$

$$=1+\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right]z^n$$ **Ans.**

Example 2 Expand $\frac{1}{(z-1)(z-2)}$ in Laurent's series valid for $|z|<1$ and $1<|z|<2$. **[AU Nov. 2010]**

Solution Let $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

(i) Given $|z| < 1$ obviously $\dfrac{|z|}{2} < 1$, i.e., $\left|\dfrac{z}{2}\right| < 1$

$$\therefore \quad \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= -\frac{1}{2\left(1-\dfrac{z}{2}\right)} + \frac{1}{1-z}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right] + [1+z+z^2+\cdots]$$

i.e., $$f(z) = \frac{1}{2} + \frac{3z}{4} + \frac{7}{8}z^2 + \cdots$$

(ii) Given $1 < |z| < 2$

$$1 < |z| \Rightarrow \frac{1}{|z|} < 1, \text{ i.e., } \left|\frac{1}{z}\right| < 1$$

$$|z| < 2 \Rightarrow \frac{|z|}{2} < 1, \text{ i.e., } \left|\frac{z}{2}\right| < 1$$

$$\therefore \quad f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$ **Ans.**

Example 3 If $0 < |z-1| < 2$, express $f(z) = \dfrac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of $z-1$. **[AU April 2011]**

Solution Let $z - 1 = u$
$\therefore$ $0 < |z-1| < 2$ becomes $0 < |u| < 2$

Now, $$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$z = A(z-3) + B(z-1)$$

Put $z=1, \Rightarrow A=-\frac{1}{2}$

Put $z=3, \Rightarrow B=\frac{3}{2}$

$$\therefore \quad \frac{z}{(z-1)(z-3)}=\frac{-\frac{1}{2}}{z-1}+\frac{\frac{3}{2}}{z-3}$$

(or) $$\frac{u+1}{u(u-2)}=-\frac{1}{2u}+\frac{3}{2(u-2)} \quad (\text{as } z-1=u \Rightarrow z=u+1)$$

So instead of expanding $\frac{z}{(z-1)(z-3)}$ in powers of $(z-1)$, it is enough to expand $\frac{u+1}{u(u-2)}$ in powers of u.

$$\frac{u+1}{u(u-2)}=-\frac{1}{2u}+\frac{3}{2(u-2)}$$

Since $|u|<2$, we have $\frac{|u|}{2}<1$. i.e., $\left|\frac{u}{2}\right|<1$.

$$\therefore \quad \frac{u+1}{u(u-2)}=\frac{-1}{2u}-\frac{3}{4\left(1-\frac{u}{2}\right)}$$

$$=\frac{-1}{2u}-\frac{3}{4}\left(1-\frac{u}{2}\right)^{-1}$$

$$=\frac{-1}{2u}-\frac{3}{4}\left[1+\frac{u}{2}+\left(\frac{u}{2}\right)^2+\cdots\right]$$

$$=\frac{-1}{2u}-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n$$

$$\therefore \quad \frac{z}{(z-1)(z-3)}=\frac{-1}{2(z-1)}-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$ **Ans.**

Example 4 Obtain the Laurent's expansion for $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ which are valid in (i) $1<|z|<4$, and (ii) $|z|>4$. **[AU Nov. 2011]**

Solution Let $f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}$

$$\Rightarrow \quad f(z)=1+\frac{-5z-8}{(z+1)(z+4)} \quad (1)$$

(since the degrees of z in both numerator and in denominator are equal, divide it)

Consider $$\frac{-5z-8}{(z+1)(z+4)}=\frac{A}{(z+1)}+\frac{B}{(z+4)}$$

$$-5z-8 = A(z+4)+B(z+1)$$

Put $\quad z=-1 \quad \Rightarrow \quad A=-1$

Put $\quad z=-4 \quad \Rightarrow \quad B=-4$

$$\therefore \quad \frac{-5z-8}{(z+1)(z+4)} = \frac{-1}{(z+1)} - \frac{-4}{(z+4)} \qquad (2)$$

Substituting (2) in (1), we get

$$f(z) = 1 - \frac{1}{(z+1)} - \frac{4}{(z+4)}$$

(i) Given $1 < |z| < 4$

$$1 < |z| \;\Rightarrow\; \frac{1}{|z|} < 1, \text{ i.e., } \left|\frac{1}{z}\right| < 1$$

$$|z| < 4 \;\Rightarrow\; \frac{|z|}{4} < 1, \text{ i.e., } \left|\frac{z}{4}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - 4\frac{1}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \left[-\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \left[-\frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{z^n} - \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{z}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z^n} - \left(\frac{z}{4}\right)^n\right]$$

(ii) Given $|z| > 4$

$$\frac{4}{|z|} < 1, \text{ i.e., } \left|\frac{4}{z}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{1}{1+z} - \frac{4}{z+4}$$

$$= 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{z\left(1+\frac{4}{z}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right] - \frac{4}{z}\left[1 - \frac{4}{z} + \left(\frac{4}{z}\right)^2 - \cdots\right]$$

$$= 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n} - \frac{4}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{z}\right)^n$$

$$= 1 - \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{z^{n+1}} + \left(\frac{4}{z}\right)^{n+1}\right]$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}}(1+4^{n+1})$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n(1+4^n)\cdot\frac{1}{z^n} \qquad \textbf{Ans.}$$

Example 5 Find the Laurent's series of $f(z) = \frac{1}{z(1-z)}$ valid in the region (i) $|z+1| < 1$, (ii) $1 < |z+1| < 2$, and (iii) $|z+1| > 2$. **[KU May 2010, Nov. 2011]**

Solution Let $z + 1 = u$ or $z = u - 1$

$$\therefore \qquad f(z) = \frac{1}{z(1-z)} = \frac{1}{(u-1)(2-u)} = \frac{1}{u-1} + \frac{1}{2-u} \qquad (1)$$

(i) Given $|z+1| < 1 \Rightarrow |u| < 1$

$$\therefore \qquad f(z) = \frac{-1}{1-u} + \frac{1}{2\left(1-\frac{u}{2}\right)}$$

$$= -(1-u)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-1}$$

$$= -[1+u+u^2+\cdots] + \frac{1}{2}\left[1+\left(\frac{u}{2}\right)+\left(\frac{u}{2}\right)^2+\cdots\right]$$

$$= -\sum_{n=0}^{\infty}u^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\left(-1+\frac{1}{2^{n+1}}\right)u^n$$

i.e.,
$$f(z) = \sum_{n=0}^{\infty}\left(-1+\frac{1}{2^{n+1}}\right)(z+1)^n$$

(ii) Given $1 < |z+1| < 2$. i.e., $1 < |u| < 2$

$$1 < |u| \Rightarrow \frac{1}{|u|} < 1, \text{ i.e., } \left|\frac{1}{u}\right| < 1$$

$$|u| < 2 \Rightarrow \frac{|u|}{2} < 1 \text{ i.e., } \left|\frac{u}{2}\right| < 1$$

Consider (1), $f(z) = \dfrac{1}{u-1} + \dfrac{1}{2-u}$

$$= \frac{1}{u\left(1 - \frac{1}{u}\right)} + \frac{1}{2\left(1 - \frac{u}{2}\right)}$$

$$= \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} + \frac{1}{2}\left(1 - \frac{u}{2}\right)^{-1}$$

$$= \frac{1}{u}\left[1 + \frac{1}{u} + \frac{1}{u^2} + \cdots\right] + \frac{1}{2}\left[1 + \left(\frac{u}{2}\right) + \left(\frac{u}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{u^{n+1}} + \sum_{n=0}^{\infty}\frac{u^n}{2^{n+1}}$$

i.e.,
$$f(z) = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z+1)^n$$

(iii) $|z+1| > 2$, i.e., $|u| > 2 \Rightarrow \left|\dfrac{2}{u}\right| < 1$

$\therefore$
$$f(z) = \frac{1}{u\left(1 - \frac{1}{u}\right)} - \frac{1}{u\left(1 - \frac{2}{u}\right)}$$

$$= \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1 - \frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[1 + \frac{1}{u} + \frac{1}{u^2} + \cdots\right] - \frac{1}{u}\left[1 + \frac{2}{u} + \left(\frac{2}{u}\right)^2 + \cdots\right]$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} - \frac{1}{u}\sum_{n=0}^{\infty}\frac{2^n}{u^n}$$

$$= \sum_{n=0}^{\infty}(1 - 2^n)\frac{1}{u^{n+1}}$$

or
$$f(z) = \sum_{n=0}^{\infty}(1 - 2^n)\frac{1}{(z+1)^{n+1}}$$
Ans.

EXERCISE

Part A

1. Define radius and circle of convergence of power series.
2. State Taylor's theorem and Laurent's theorem.
3. State McLaurin's series.
4. Give some standard McLaurin's series.
5. What do you mean by analytic part and principal part of Laurent's series of a function of z?
6. Expand $\frac{1}{z(z-1)}$ as Laurent's series about $z = 0$ in the annulus $0 < |z| < 1$.
7. Find the Laurent's series expansion of $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$.
8. Expand $f(z) = e^z$ in a Taylor's series about $z = 0$.
9. Expand $\cos z$ at $z = \frac{\pi}{4}$ in a Taylor's series.
10. In the power series $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$, z_0 is called the ______ of the series.

Part B

1. Find the Taylor's series expansion of $f(z) = \frac{z}{z(z+1)(z+2)}$ about $z = i$.

   State also the region of convergence of the series.

   $$\left[\textbf{Ans. } \sum_{n=0}^{\infty}(-1)^n\left\{\frac{2}{(2+i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n\right]$$

2. Find the Laurent's series expansion of $f(z) = \frac{z^2-1}{z^2+5z+6}$ valid in the region (i) $|z| < 2$, (ii) $2 < |z| < 3$, and (iii) $|z| > 3$ **[KU April 2013]**

   $$\left[\begin{array}{l}\textbf{Ans. } \text{(i) } 1 + \sum_{n=0}^{\infty}(-1)^n\left\{\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right\}z^n \text{ (ii) } 1 + 3\sum(-1)^n\frac{2^n}{z^{n+1}} - 8\sum(-1)^n\frac{z^n}{3^{n+1}} \\ \text{(iii) } 1 + \sum(-1)^n\{3.2^n - 8.3^n\}1/z^{n+1}\end{array}\right]$$

3. Find the Laurent's series expansion of $f(z) = \frac{z}{(z-1)(z-2)}$, valid in the region (i) $|z+2| < 3$, (ii) $3 < |z+2| < 4$, and (iii) $|z+2| > 4$.

   $$\left[\begin{array}{l}\textbf{Ans. } \text{(i) } \sum_{n=0}^{\infty}\left[-\frac{1}{2.4^n} + \frac{1}{3^{n+1}}\right](z+2)^n \text{ (ii) } -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z+2)^n}{4^n} - \sum_{n=0}^{\infty}\frac{3^n}{(z+2)^{n+1}} \\ \text{(iii) } \sum_{n=0}^{\infty}(2.4^n - 3^n)\cdot\frac{1}{(z+2)z^{n+1}}\end{array}\right]$$

4. Expand $\dfrac{z^2-6z-1}{(z-1)(z+2)(z-3)}$ in $3 < |z+2| < 5$.

$$\left[\textbf{Ans. } \frac{2}{z+2}+\frac{3}{(z+2)^2}+\frac{3^2}{(z+2)^3}+\cdots+\frac{1}{5}\left[1+\frac{z+2}{5}+\frac{(z+2)^2}{5^2}+\frac{(z+2)^3}{5^3}+\cdots\right]\right]$$

5. Find Laurent's series of $f(z)=\dfrac{e^z}{z(1-z)}$ about $z = 1$. Find the region of convergence.

$$\left[\begin{array}{c}\textbf{Ans. } f(z)=\dfrac{1}{e}\left[-\dfrac{1}{z-1}-\dfrac{3}{2}(z-1)+\dfrac{1}{3}(z-1)^2+\cdots\right]\\ \text{Region of convergence is } |z-1|<1\end{array}\right]$$

6. Obtain the Laurent's series expansion for $f(z)=\dfrac{1}{z(z-1)}$ for (i) $0 < |z| < 1$, and (ii) $0 < |z-1| < 1$. $\left[\textbf{Ans. } \text{(i) } -\dfrac{1}{z}(1+z+z^2+\cdots) \text{ (ii) } \dfrac{1}{z-1}(1-(z-1)+(z-1)^2\ldots)\right]$

7. Find Laurent's series about the indicated singularity. (i) $\dfrac{e^{2z}}{(z-1)^3}, z=1$ (ii) $\dfrac{z}{(z+1)(z+2)}, z=-2$ (iii) $\dfrac{1}{z^2(z-3)^2}, z=3$

$$\left[\begin{array}{l}\textbf{Ans. } \text{(i) } \dfrac{e^2}{(z-1)^3}+\dfrac{2e^2}{(z-1)^2}+\dfrac{2e^2}{(z-1)}+\dfrac{4e^2}{3}+\dfrac{2e^2}{3}(z-1)+\cdots\\ \text{(ii) } \dfrac{2}{2+z}+1+(z+2)+(z+2)^2+\cdots\\ \text{(iii) } \dfrac{1}{9(z-3)^2}-\dfrac{2}{27(z-3)}+\dfrac{1}{27}-\dfrac{4(z-3)}{243}+\cdots\end{array}\right]$$

25 Theory of Residues

Chapter Outline

25.1 ❑ INTRODUCTION

The residue theorem is a very powerful and elegant theorem in complex integration. Using the residue theorem, many complicated real integrals can be evaluated. It is also used to sum a real convergent series and to find the inverse of a Laplace transform.

25.2 ❑ CLASSIFICATION OF SINGULARITIES

A point at which a function $f(z)$ is not analytic is known as a **singular point** or **singularity** of the function.

- **Example**

The function $f(z) = \dfrac{1}{z-5}$ has a singular point at $z - 5 = 0$ or $z = 5$.

If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$ then $z = a$ is said to be an **isolated singularity** of the function $f(z)$. Otherwise, it is called **non-isolated**.

- **Example**

(i) The function $\dfrac{1}{(z-2)(z-7)}$ has two isolated singular points, namely, $z = 2$ and $z = 7$ [since $(z-2)(z-7) = 0$ or $z = 2, 7$].

(ii) The function $\dfrac{1}{\sin\frac{\pi}{z}}$ is not analytic at the points where $\sin\frac{\pi}{z} = 0$, i.e., at the points $\frac{\pi}{z} = n\pi$.

i.e., at the points $z = \frac{1}{n}(n = 1, 2, 3\ldots)$.

Thus, $z = 1, \frac{1}{2}, \frac{1}{3}\ldots, z = 0$ are the points of singularity. But $z = 0$ is the non-isolated singularity of the function $\dfrac{1}{\sin\frac{\pi}{z}}$ because in the neighbourhood $z = 0$, there are infinite number of other singularities $z = \frac{1}{n}$, where n is very large.

Let a function $f(z)$ have an isolated singular point $z = a$. $f(z)$ can be expanded in a Laurent's series expansion around $z = a$ as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2}$$

$$+ \cdots + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}} + \cdots$$

In some cases, it may happen that the coefficients $b_{m+1} = b_{m+2} = \ldots = 0$,

Then the series reduces to

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}$$

Then $z = a$ is said to be **a pole of order m** of the function $f(z)$.

When $m = 1$, the pole is said to be a **simple pole**.

In this case, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \dfrac{b_1}{(z-a)}$.

If the number of terms of negative powers in the above expansion are infinite then $z = a$ is called an **essential singular point** of $f(z)$.

If a single-valued function $f(z)$ is not defined at $z = a$, but $\lim_{z\to a} f(z)$ exists then $z = a$ is called a **removable singularity**.

- **Example**

$z = 0$ is a removable singularity of $f(z) = \dfrac{\sin z}{z}$, since $f(0)$ is not defined, but $\lim_{z\to 0}\left(\dfrac{\sin z}{z}\right) = 1$.

25.3 ❑ RESIDUES

Residue of an analytic function $f(z)$ at an isolated singular point $z = a$ is the coefficient say b_1 of $(z - a)^{-1}$ in the Laurent's series expansion of $f(z)$ about a. Residue of $f(z)$ at a is denoted by $\operatorname{Res}_{z=a} f(z)$. From Laurent's series, we know that the coefficient b_1 is given by $b_1 = \dfrac{1}{2\pi i}\int_C f(z)\,dz.$

Thus, the residue of $f(z)$ at $z = a$, $= \operatorname{Res}_{z=a} f(z) = b_1 = \dfrac{1}{2\pi i}\int_C f(z)\,dz.$

where C is any closed contour enclosing a (and such that f is analytic on and within C).

Calculation of Residue at Simple Pole

(i) If $f(z)$ has a simple pole at $z = a$, then $\operatorname{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)\,f(z)$.

(ii) Suppose $f(z) = \dfrac{P(z)}{Q(z)}$ has a simple pole at a such that $P(a) \neq 0$.

Then $\operatorname{Res}_{z=a} f(z) = \operatorname{Res}_{z=a} \dfrac{P(z)}{Q'(z)} = \dfrac{P(a)}{Q'(a)}$

Calculation of Residue at a Multiple Pole

If $f(z)$ has a pole of order n at $z = a$, then

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{(n-1)!}\lim_{z\to a}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}$$

25.4 ❑ CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic within and on a simple closed curve C except at a finite number of poles within C then $\oint_C f(z)dz = 2\pi i$ (sum of residues at the poles within C).

Proof Let $C_1, C_2, C_3 \ldots C_n$ be the non-intersecting circles with centre at $a_1, a_2 \ldots a_n$ respectively and radii so small that they lie entirely within the closed curve C. Then $f(z)$ is analytic in the multiply connected legion lying between the curves C and $C_1, C_2 \ldots C_n$. Applying Cauchy's theorem,

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$
$$= 2\pi i \operatorname{Res}_{z=a_1} f(z) + 2\pi i \operatorname{Res}_{z=a_2} f(z) \cdots + 2\pi i \operatorname{Res}_{z=a_n} f(z)$$
$$= 2\pi i\left[\operatorname{Res}_{z=a_1} f(z) + \operatorname{Res}_{z=a_2} f(z) \cdots + \operatorname{Res}_{z=a_n} f(z)\right]$$

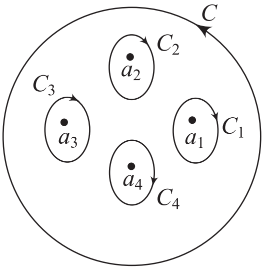


Fig. 25.1

$\therefore \oint_c f(z)dz = 2\pi i$ (sum of residues at the poles within C)

25.5 ❑ EVALUATION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated using Cauchy's theorem of residues. For finding the integrals, we take a closed curve C, find the poles of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

Then using Cauchy's theorem of residues, we have $\int_C f(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at the poles within C)

We call the curve a contour and the process of integration along a contour as contour integration.

Type 1

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where f is a rational function of $\cos\theta$ and $\sin\theta$

In this type of integrals, put $z = e^{i\theta}$

Differentiating with respect to θ, we get,

$$dz = ie^{i\theta}\, d\theta, \text{ i.e., } d\theta = \frac{dz}{iz}$$

We know that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

i.e., $$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and $$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\therefore \quad \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \int_C f\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz}$$

where C is the unit circle $|z| = 1$

$$= \frac{1}{i}\int_C f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{dz}{z}$$

$$= \int_C \phi(z)\, dz \text{ (say)}$$

Clearly, $\phi(z)$ is a rational function of z.

Hence, by the residue theorem, $\int_C \phi(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at its poles inside C).

Type 2

Consider the integral $\int_C \phi(z)dz$, where C is the positively oriented semicircle Γ, $|z| = R$, Im $z \geq 0$ together with the line segment $L : [-R, R]$. Such integrals can be evaluated by integrating $f(z)$ round a contour C consisting of a semicircle Γ of radius R large enough to include all the poles of $f(z)$

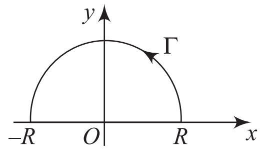


Fig. 25.2

and the part of the real axis from $x = -R$ to $x = R$. Here, the only singularities of $f(z)$ in the upper half-plane are poles.

When $\phi(z)$ has singularities on the real axis then $\int_C \phi(z)dz = \int_{-R}^{R} \phi(x)dx + \int_\Gamma \phi(z)dz$.

By the residue theorem, we have $\int_C \phi(z)dz = 2\pi i$ (sum of the residues of the function $\phi(z)$ at its poles in the upper half-plane).

i.e., $\int_{-R}^{R} \phi(x)dx + \int_\Gamma \phi(z)dz = 2\pi i$ (sum of the residues of the function $\phi(z)$ at its poles within C).

Putting $R \to \infty$ we get, $\int_{-\infty}^{\infty} \phi(x)dx,$ provided $\int_\Gamma \phi(z)dz \to 0$.

Type 3

Integrals of the form $\int_{-\infty}^{\infty} (\sin ax)\, f(x)dx$ or $\int_{-\infty}^{\infty} (\cos ax)\, f(x)dx$. $a > 0$ where $f(z)$ is such that $f(z) \to 0$ as $z \to \infty$ and it does not have a pole on the real axis.

SOLVED EXAMPLES

Example 1 Find the residue of $f(z) = \dfrac{1}{(z^2+1)^2}$ about each singularity.

Solution Given $f(z) = \dfrac{1}{(z^2+1)^2} = \dfrac{1}{[(z-i)(z+i)]^2}$

$$= \frac{1}{(z-i)^2(z+i)^2}$$

Here, $z = i, -i$ are poles of order 2.

Now,
$$[\text{Res}\, f(z)]_{z=i} = \underset{z\to i}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}[(z-i)^2 f(z)]$$

$$= \underset{z\to i}{\text{Lt}} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{1}{(z-i)^2(z+i)^2}\right]$$

$$= \underset{z\to i}{\text{Lt}} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$$

$$= \underset{z\to i}{\text{Lt}} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$= \frac{-i}{4}$$

$$[\text{Res } f(z)]_{z=-i} = \underset{z \to -i}{\text{Lt}} \frac{1}{1!} \frac{d}{dz}[(z+i)^2 f(z)]$$

$$= \underset{z \to -i}{\text{Lt}} \frac{d}{dz}\left[(z+i)^2 \cdot \frac{1}{(z-i)^2 (z+i)^2}\right]$$

$$= \underset{z \to -i}{\text{Lt}} \frac{d}{dz}\left[\frac{1}{(z-i)^2}\right]$$

$$= \underset{z \to -i}{\text{Lt}} \frac{-2}{(z-i)^3} = \frac{-2}{8i} = \frac{i}{4}$$ **Ans.**

Example 2 Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$ where C is the circle $|z - i| = 2$.

[AU June 2009, May 2012]

Solution Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

Here, $z = -1$ is a pole of order 2.

And $z = 2$ is a simple pole.

Clearly, $z = 2$ lies outside the circle $|z - i| = 2$

$\therefore \quad [\text{Res } f(z)]_{z=2} = 0$

Now,

$$[\text{Res } f(z)]_{z=-1} = \underset{z \to -1}{\text{Lt}} \frac{1}{1!} \frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \underset{z \to -1}{\text{Lt}} \frac{d}{dz}\left[(z+1)^2 \cdot \frac{(z-1)}{(z+1)^2 (z-2)}\right]$$

$$= \underset{z \to -1}{\text{Lt}} \frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \underset{z \to -1}{\text{Lt}} \left[\frac{(z-2)-(z-1)}{(z-2)^2}\right]$$

$$= \underset{z \to -1}{\text{Lt}} \left[\frac{-2+1}{(z-2)^2}\right] = \underset{z \to -1}{\text{Lt}} \left[-\frac{1}{(z-2)^2}\right]$$

$$= \frac{-1}{(-1-2)^2} = -\frac{1}{9}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C \frac{z-1}{(z+1)^2(z-2)} dz = 2\pi i \text{ [sum of the residues]}$$

$$= 2\pi i\left(-\frac{1}{9}\right) = \frac{-2\pi i}{9}$$ **Ans.**

Example 3 Evaluate $\int_C \frac{dz}{c(z^2+9)^3}$, where C is $|z - i| = 3$ by using Cauchy's residue theorem.

[KU Nov. 2011]

Solution Let $f(z) = \frac{1}{(z^2+9)^3}$

The singularities of $f(z)$ are obtained by $z^2 + 9 = 0$
$\Rightarrow z = \pm 3i$, of which $z = 3i$ lies inside the circle $|z - i| = 3$
$z = 3i$ is a triple pole of $f(z)$.

$$\therefore \qquad [\text{Res } f(z)]_{z=3i} = \frac{1}{2!}\left[\frac{d^2}{dz^2}\frac{1}{(z+3i)^3}\right]_{z=3i}$$

$$= \frac{1}{2!}\left[\frac{12}{(z+3i)^5}\right]_{z=3i}$$

$$= \frac{6}{6^5 i^5} = \frac{1}{1296i}$$

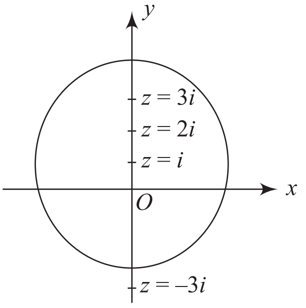


Fig. 25.3

By Cauchy's residue theorem,

$$\int_C \frac{dz}{(z^2+9)^3} = 2\pi i \times \frac{1}{1296i} = \frac{\pi}{648}$$ **Ans.**

Example 4 Show that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, a > b > 0$.

[KU May 2010; AU Nov. 2010, Nov. 2011, April 2013]

Solution Let $z = e^{i\theta}$

$$\Rightarrow \qquad d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\therefore \qquad \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{a + \frac{1}{2}b\left(z + \frac{1}{z}\right)} \text{ where C is } |z| = 1$$

$$= \frac{1}{i}\int_C \frac{dz}{z\left[a + \frac{1}{2}b\left(z + \frac{1}{z}\right)\right]}$$

$$= \frac{1}{i}\int_C \frac{dz}{z\left[\frac{2az + bz^2 + b}{2z}\right]}$$

i.e.,
$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{i}\int_C \frac{dz}{bz^2 + 2az + b}$$

$$= \frac{2}{i}\int_C f(z)dz \qquad (1)$$

The poles of $f(z)$ are given by the roots of $bz^2 + 2az + b = 0$

$$\therefore \qquad z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

i.e., $$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a-\sqrt{a^2-b^2}}{b}$$

Let $$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}; \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Since $a > b > 0$, $|\beta| > 1$

But the modulus of the product of the roots $|\alpha\beta| = 1$ (since if $az^2 + b + c = 0$, product of the roots $|\alpha\beta| = \frac{c}{a}$).

Since $|\beta| > 1$ and $|\alpha\beta| = 1$, we get $|\alpha| < 1$ so that $z = \alpha$ is the only simple pole inside C.

Since $z = \alpha$ and $z = \beta$ are the roots of $bz^2 + 2az + b = 0$, we can write $bz^2 + 2az + b = b(z - \alpha)(z - \beta)$

Hence, $$f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$$

Now, $$[\text{Res } f(z)]_{z=\alpha} = \underset{z\to\alpha}{\text{Lt}} (z-\alpha)\cdot f(z)$$

$$= \underset{z\to\alpha}{\text{Lt}} (z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \underset{z\to\alpha}{\text{Lt}} \frac{1}{b(z-\beta)} = \frac{1}{b(\alpha-\beta)}$$

$$= \frac{1}{b\left[\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right]}$$

$$= \frac{1}{b\frac{2\sqrt{a^2-b^2}}{b}}$$

$$= \frac{1}{2\sqrt{a^2-b^2}}$$

From (1), since $|\beta| > 1$,
β lies outside the circle $|z| = 1$

$\therefore$ $$[\text{Res } f(z)]_{z=\beta} = 0$$

Hence, (1) $\Rightarrow$ $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\int_C f(z)dz$$

$$= \frac{2}{i}[2\pi i \times (\text{sum of the residues})]$$

$$= \frac{2}{i}\cdot 2\pi i\left[\frac{1}{2\sqrt{a^2-b^2}}\right]$$

$\therefore$ $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$ **Ans.**

Example 5 Evaluate $\int_0^{\pi} \frac{a d\theta}{a^2 + \sin^2\theta}, a > 0$. **[KU Nov. 2010]**

Solution Let

$$I = \int_0^{\pi} \frac{a d\theta}{a^2 + \sin^2\theta}$$

$$= \int_0^{\pi} \frac{a d\theta}{a^2 + \left(\frac{1 - \cos 2\theta}{2}\right)}$$

$$= \int_0^{\pi} \frac{2a d\theta}{2a^2 + 1 - \cos 2\theta}$$

Put $2\theta = \phi \quad \Rightarrow \quad 2d\theta = d\phi$

When $\theta = 0$, $\phi = 0$ and when $\theta = \pi$, $\phi = 2\pi$

$$\therefore \qquad I = \int_0^{2\pi} \frac{2a\left(\frac{d\phi}{2}\right)}{2a^2 + 1 - \cos\phi}$$

$$= \int_0^{2\pi} \frac{a d\phi}{2a^2 + 1 - \cos\phi} \qquad (1)$$

Put $z = e^{i\phi}$, then $d\phi = \frac{dz}{iz}$

$$\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Then

$$(1) \Rightarrow I = \int_C \frac{a \cdot \frac{dz}{iz}}{\left[2a^2 + 1 - \frac{1}{2}\left(z + \frac{1}{z}\right)\right]}$$

where C is the unit circle $|z| = 1$

$$= \frac{a}{i}\int_C \frac{dz}{\left[2a^2 + 1 - \frac{1}{2}\left(\frac{z^2 + 1}{z}\right)\right]}$$

$$= \frac{a}{i}\int_C \frac{dz}{\left[\frac{4a^2 z + 2z - z^2 - 1}{2z}\right]}$$

$$= \frac{2a}{i}\int_C \frac{dz}{(4a^2 + 2) - z^2 - 1}$$

$$= -\frac{2a}{i}\int_C \frac{dz}{z^2 - (4a^2 + 2)z + 1}$$

$$= 2ai\int_C \frac{dz}{z^2 - (4a^2 + 2)z + 1}$$

$\therefore I = \int_C f(z)dz$, where $f(z) = \dfrac{2ai}{z^2 - (4a^2 + 2)z + 1}$

The poles of $f(z)$ are the solutions of

$$z^2 - (4a^2 + 2)z + 1 = 0$$

$$z^2 - (4a^2 + 2)z + 1 = 0$$

$$\therefore \quad z = \frac{(4a^2+2) \pm \sqrt{(4a^2+2)^2 - 4}}{2}$$

$$= \frac{2(2a^2+1) \pm 4a\sqrt{a^2+1}}{2}$$

$$= (2a^2+1) \pm 2a\sqrt{a^2+1}$$

$$\Rightarrow \quad z = (2a^2+1) + 2a\sqrt{a^2+1} \text{ or } (2a^2+1) - 2a\sqrt{a^2+1}$$

Let $\alpha = (2a^2+1) + 2a\sqrt{a^2+1}$ and $\beta = (2a^2+1) - 2a\sqrt{a^2+1}$

Since α, β are the roots of $z^2 - (4a^2 + 2)z + 1 = 0$, the product of the roots $\alpha\beta = 1$

Since $a > 0$, $\alpha > 1$ also, $\beta < 1$.

$\therefore$ out of the two poles α and β, $z = \beta$ lies within the unit circle $|z| = 1$ (since $|\beta| < 1$)

Now,

$$[\text{Res}\, f(z)]_{z=\beta} = \underset{z\to\beta}{\text{Lt}} (z-\beta)\cdot f(z)$$

$$= \underset{z\to\beta}{\text{Lt}} (z-\beta)\cdot \frac{2ai}{(z-\alpha)(z-\beta)}$$

$$= \frac{2ai}{\beta - \alpha}$$

$$= \frac{2ai}{(2a^2+1-2a\sqrt{a^2+1}) - (2a^2+1-2a\sqrt{a^2+1})}$$

$$= \frac{2ai}{-4a\sqrt{a^2+1}} = \frac{-i}{2\sqrt{a^2+1}}$$

$$\therefore \quad I = \int_C f(z)dz$$

$$= 2\pi i \text{ [sum of the residues of } f(z) \text{ at its poles]}$$

$$= 2\pi i\left[\frac{-i}{2\sqrt{a^2+1}}\right]$$

$$\therefore \quad \int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{a^2+1}}$$

Ans.

Example 6 Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx, a > 0, b > 0$.

[KU May 2010, Nov. 2011]

Solution Let $\int_C \phi(z)dz = \int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz$

where C consists of the semicircle Γ and the bounding diameter $[-R, R]$.

Now,

$$\int_C \phi(z)dz = \int_{-R}^{R} \phi(x)dx + \int_{\Gamma} \phi(z)dz \tag{1}$$

Now, $$\phi(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$
$$= \frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

Here, the poles are $z = ia, -ia, ib, -ib$

Here, $z = ia$ and $z = ib$ lie in the upper half-plane while $z = -ia$ and $z = -ib$ lie in the lower half-plane.

We have to find the residues of $\phi(z)$ at each of its poles which lies in the upper half-plane.

$$\therefore \quad [\text{Res}\, f(z)]_{z=ia} = \underset{z\to ia}{\text{Lt}}\,(z-ia)\cdot\phi(z)$$
$$= \underset{z\to ia}{\text{Lt}}\,(z-ia)\frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}$$
$$= \underset{z\to ia}{\text{Lt}}\,\frac{z^2}{(z-ia)(z^2+b^2)}$$
$$= \underset{z\to ia}{\text{Lt}}\,\frac{(ia)^2}{(ia+ia)((ia)^2+b^2)}$$
$$= \frac{-a^2}{2ia(-a^2+b^2)}$$
$$= \frac{a}{2i(a^2-b^2)}$$

$$[\text{Res}\, f(z)]_{z=ib} = \underset{z\to ib}{\text{Lt}}\,(z-ib)\phi(z)$$
$$= \underset{z\to ib}{\text{Lt}}\,(z-ib)\frac{z^2}{(z^2+a^2)(z+ib)(z-ib)}$$
$$= \underset{z\to ib}{\text{Lt}}\,\frac{z^2}{(z^2+a^2)(z+ib)}$$
$$= \frac{(ib)^2}{[(ib)^2+a^2][ib+ib]}$$
$$= \frac{-b^2}{(a^2-b^2)2ib} = \frac{-b}{2i(a^2-b^2)}$$

In (1), making $R\to\infty$, we get
$$\int_C \phi(z)dz = \int_{-\infty}^{\infty}\phi(x)dx + \int_\Gamma \phi(z)dz$$

When $R\to\infty$, $|z|\to\infty$ and $\phi(z)\to 0$

$$\therefore \quad \int_C \phi(z)dz = \int_{-\infty}^{\infty}\phi(x)dx \quad \text{[from (1)]}$$

$$\therefore \quad \int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty}\frac{z^2dz}{(z^2+a^2)(z^2+b^2)}$$
$$= 2\pi i$$

[sum of the residues of $\phi(z)$ at each pole in the upper half-plane]

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right]$$

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)}\right] = 2\pi i\left[\frac{a-b}{2i(a-b)(a+b)}\right]$$

$$\Rightarrow \quad \int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$ **Ans.**

Example 7 Evaluate $\int_0^{\infty}\frac{dx}{x^4+1}$. **[KU Nov. 2010]**

Solution Consider $\int_0^{\infty}\frac{dx}{x^4+1}$

$$\int_0^{\infty}\frac{dx}{x^4+1} = \int_0^{\infty}\frac{dx}{z^4+1}$$

i.e., $$2\int_0^{\infty}\frac{dx}{x^4+1} = \int_{-\infty}^{\infty}\frac{dx}{z^4+1}$$

The poles are the roots of $z^4 + 1 = 0$

i.e., $z^4 = -1$

$\Rightarrow$ $z = (-1)^{\frac{1}{4}}$

$$= \left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right] \text{ where } n = 0, 1, 2, 3$$

When $n = 0$, $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

When $n = 1$, $z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = e^{\frac{i3\pi}{4}}$

When $n = 2$, $z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = e^{\frac{i5\pi}{4}}$

When $n = 3$, $z = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = e^{\frac{i7\pi}{4}}$

Hence, the poles are $z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$.

Out of these poles, $z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}$ lies in the upper half-plane.

$\therefore$ $$[\text{Res}\,\phi(z)]_{z=e^{\frac{i\pi}{4}}} = \underset{z\to e^{\frac{i\pi}{4}}}{\text{Lt}}\frac{z-e^{\frac{i\pi}{4}}}{z^4+1}$$

$$= \underset{z\to e^{\frac{i\pi}{4}}}{\text{Lt}}\frac{1}{4z^3} = \frac{1}{4\left(e^{\frac{i\pi}{4}}\right)^3} \text{ (applying L'Hospital's rule)}$$

$$= \frac{1}{4e\frac{i3\pi}{4}}$$

$$[\text{Res}\,\phi(z)]_{z=e^{\frac{i3\pi}{4}}} = \underset{z\to e^{\frac{i3\pi}{4}}}{\text{Lt}} \frac{z-e^{\frac{i3\pi}{4}}}{z^4+1}$$

$$= \underset{z\to e^{\frac{i3\pi}{4}}}{\text{Lt}} \frac{1}{4z^3} = \frac{1}{4\left(e^{\frac{i3\pi}{4}}\right)^3}$$

$$= \frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$\therefore \quad 2\int_0^\infty \frac{dx}{x^4+1} = \int_{-\infty}^\infty \frac{dz}{z^4+1}$$

$= 2\pi i$ [sum of the residues at each pole in the upper half-plane]

$$= 2\pi i\left[\frac{1}{4e^{\frac{i3\pi}{4}}} + \frac{1}{4e^{\frac{i9\pi}{4}}}\right]$$

$$= \frac{\pi i}{2}\left[e^{-\frac{i3\pi}{4}} + e^{-\frac{i9\pi}{4}}\right]$$

$$= \frac{\pi i}{2}\left[\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + \left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right)\right]$$

$$= \frac{\pi i}{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = \frac{\pi i}{2}\left[\frac{-2i}{\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

$$\therefore \quad \int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dz}{z^4+1} = \frac{1}{2}\frac{\pi}{\sqrt{2}}$$ **Ans.**

EXERCISE

Part A

1. Define essential singularity with an example.
2. Define removable singularity with an example.
3. Define simple pole and multiple pole of a function $f(z)$. Give one example for each.
4. Define the residue of a function at an isolated singularity.
5. State the formula for finding the residue of a function at a multiple pole.
6. Find the residues at the isolated singularities of each of the following:

   (i) $\dfrac{z}{(z+1)(z-2)}$ (ii) $\dfrac{ze^z}{(z-1)^2}$ (iii) $\dfrac{z\sin z}{(z-\pi)^3}$

7. Evaluate the following integrals using Cauchy's residue theorem:

   (i) $\displaystyle\int_C \frac{z+1}{z(z-1)}dz$ where $C: |z| = 2$

(ii) $\int_C \frac{e^{-z}}{z^2} dz$ where C : |z| = 1

8. Explain how to convert $\int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta$ into a contour integral, where *f* is a rational function.

9. Obtain the poles of $\frac{z+4}{z^2+2z+5}$.

10. By using residue theorem, find the value of $\int_C \frac{z-2}{z-1} dz$ where *C* is |*z*| = 2.

11. Find the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ at $z = -2$.

12. Find the singularities of $f(z) = \frac{z+4}{z^2+2z+2}$.

13. Find the residue of $f(z) = \frac{z}{z^2+1}$ about $z = i$.

14. Find the residue of $f(z) = \frac{1}{(z^2+a^2)^2}$ at $z = ai$

15. Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

16. Find the poles of $f(z) = \frac{1}{\sin\frac{1}{z-a}}$.

17. Find the singularities of the function $f(z) = \frac{\cot \pi z}{(z-a)^3}$.

18. Give the forms of the definite integrals which can be evaluated using the infinite semicircular contour above the real axis.

19. Define Cauchy's residue theorem.

20. Find the residue of $\frac{1}{(z^3-1)^2}$ at $z = 1$.

Part B

1. Evaluate the following using Cauchy's residue theorem:

(i) $\int_C \frac{1-2z}{z(z-1)(z-2)} dz, C: |z| = \frac{3}{2}$

(ii) $\int_C \frac{2z-1}{z(z+2)(2z+1)} dz, C: |z| = 1$

(iii) $\int_C \frac{e^{-z}}{z^2} dz, C: |z| = 1$

(iv) $\int_C \frac{12z-7}{(z-1)^2(2z+3)} dz, C: |z+i| = \sqrt{3}$

$\left[\textbf{Ans.} \text{ (i) } 3\pi i \text{ (ii) } \frac{5\pi i}{3} \text{ (iii) } -2\pi i \text{ (iv) } 4\pi i\right]$

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

3. Evaluate $\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta}$. $\left(\textbf{Ans. } \frac{2\pi}{15}\right)$

4. Evaluate $\int_0^{\infty} \frac{dx}{x^4+a^4}$. $\left(\textbf{Ans. } \frac{\pi}{a^3\cdot\sqrt{2}}\right)$

5. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

6. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$. $\left(\textbf{Ans. } \frac{\pi}{4a^3}, a>0\right)$

7. Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx$. $\left(\textbf{Ans. } \frac{1}{2}\pi e^{-a}\right)$

8. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx$. $\left(\textbf{Ans. } \frac{\pi}{a}e^{-a}\right)$

9. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$.

10. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

11. Evaluate the integral $\int_0^{\infty} \frac{x^2}{x^4+1}dx$ using contour integration.

12. Evaluate $\int_0^{\infty} \frac{\cos x}{(1+x^2)^2}dx$. $\left(\textbf{Ans. } \frac{\pi}{2e}\right)$

UNIT V

LAPLACE TRANSFORMS

CHAPTER I

1.1 Introduction

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer **Oliver Heaviside (1850-1925)** to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by **Bromwich** and **Carson** during 1916-17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called **Laplace transforms**(Pierre Simon Marquis De Laplace, French Mathematician (1749-1827) used such transforms much earlier in 1799, while developing the theory of probability).

Laplace transform is useful since

(i) Particular solution is obtained without first determining the general solution.

(ii) non homogeneous equation are solved without obtaining the complementary integral.

(iii) Laplace transform is applicable not only to continuous functions but also to piecewise continuous functions, complicated periodic functions, step functions and impulse functions.

Before the advent of calculators and computers, logarithms were extensively used to replace multiplication (or division) of two large numbers by addition (or subtraction) of two numbers. The crucial idea which made the Laplace transform, a very powerful technique is that it replaces operations of calculus by operations of algebra.

Laplace transformation when applied to the initial value problem consisting of a single or a system of linear, ordinary differential equations, converts it into a single or a system of linear, algebraic equations in terms of the Laplace transform of the dependent variable. This equation is called the **subsidiary equation**. The initial conditions are automatically absorbed during the derivation of this algebraic equation. The solution of this algebraic equation gives the expression for the Laplace transform of the dependent variable. Taking the inverse Laplace transformation, we find the solution of the original initial value problem.

In the case of partial differential equations in terms of two independent variables, the Laplace transformation is applied with respect to one of the variables, usually the variable t(time). The resulting ordinary differential equation in terms of the second variable is solved by the usual methods of solving ordinary differential equations. The inverse laplace transform of this solution gives the solution of the given partial differential equation.

One of the important applications of Laplace transformation is the solution of the mathematical models of physical systems in which the right hand side of the differential equation, representing the driving force is discontinuous or acts for a short time only or is a periodic function (which is not necessarily a since or a cosine function).

1.2 Laplace transform

Let $f(t)$ be a given function defined for all $t \geq 0$. Laplace transform of $f(t)$ denoted by $L(f(t))$ or Simply $L(f)$ is defined as

$$L(f(t)) = \int_0^\infty e^{-st} f(t) dt = F(s) \tag{1}$$

L is known as Laplace transform operator. The original given function $f(t)$ known as determining function depends on t, while the new function to be determined $F(s)$, called as generating function, depends only on s (because the improper integral on the R.H.S of (1) is integrated with respect to t).

$F(s)$ in (1) is known as the Laplace transform of $f(t)$. Equation (1) is known as **direct transform,** or simply **transform** in which $f(t)$ is given and $F(s)$ is to be determined.

Thus Laplce transform transforms one class of complicated functions $f(t)$ to produce another class of simpler functions $F(s)$.

1.3 Applications

Laplace transform is very useful in obtaining solution of linear differential equations, both ordinary and partial, solution of system of simultaneous differential equations, solution of integral equations, solution of linear difference equations and in the evaluation of definite integrals.

1.4 Sufficient conditions for the existence of Laplace transform of $\mathbf{f(t)}$

The Laplace transform of $f(t)$ exists, when the following sufficient conditions are satisfied.

Piece-wise or sectional continuity

A function $f(x)$ is called **sectionally continuous** or piece-wise continuous in any interval $[a,b]$ if it is continuous and has finite left and right hand limits in every subinterval $[a_1, b_1]$ as shown in the graph of the function $f(x)$.

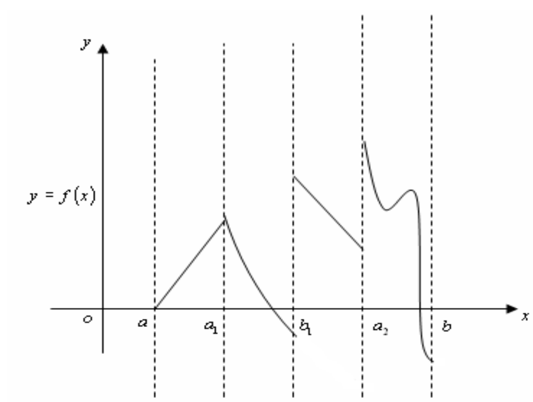


Fig. 1

Functions of exponential order

A function $f(x)$ is said to be of **exponential order** 'a' as $x \to \infty$ if $\underset{x\to\infty}{Lt}\ e^{-ax}\ f(x) =$ finite quantity.

Example:

(a) Since $\underset{t\to\infty}{Lt}\ \dfrac{t^2}{e^{3t}} =$ finite, $f(t) = t^2$ is of exponential order say 3 .

(b) Since $\underset{t\to\infty}{Lt}\ \dfrac{e^{t^2}}{e^{\alpha t}} =$ not finite, $f(t) = e^{t^2}$ is not of exponential order.

1.5 Laplace transforms of some elementary functions.

1. $L(1) = \dfrac{1}{s}, (s > 0)$

2. $L(t^n) = \dfrac{n!}{s^{n+1}}$, when $n = 0,1,2$...........

   or $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$, when $n = 0,1,2$........

3. $L(e^{at}) = \dfrac{1}{s-a}, (s > a)$

4. $L(\sin at) = \dfrac{a}{s^2 + a^2}, (s > 0)$

5. $L(\cos at) = \dfrac{s}{s^2 + a^2}, (s > 0)$

6. $L(\sin hat) = \dfrac{a}{s^2 - a^2}, (s > |a|)$

7. $L(\cos hat) = \dfrac{s}{s^2 - a^2}, (s > |a|)$

Proof

1. $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$\therefore\ L(1) = \int_0^\infty 1 e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$$

$$= -\frac{1}{s}[0-1] = \frac{1}{s}$$

Hence $L(1) = \dfrac{1}{s}$

In general $L(k) = \dfrac{K}{s}$, where $s > 0$ and k is a constant.

2. $L[t^n]=\int_0^\infty e^{-st} f(t^n)dt$

Putting $st = x$ or $t=\frac{x}{s}$ or $dt=\frac{dx}{s}$

Thus we have $L(t^n)=\int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$

i.e., $L(t^n)=\frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n dx$

or $L(t^n)=\frac{n!}{s^{n+1}}$ [since $\Gamma(n+1)=\int_0^\infty e^{-x}x^n dx$ and $\Gamma(n+1)=n!$]

3. $L(e^{at})=\int_0^\infty e^{-st}\, e^{at} dt$

$$=\int_0^\infty e^{-st+at}dt$$

$$=\int_0^\infty e^{-(s-a)t}\, dt=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$=-\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$=-\frac{1}{(s-a)}(0-1) \quad =\frac{1}{s-a}$$

4. $L(\sin at)=\int_0^\infty e^{-st}\sin at\, dt$

$$=\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at-a\cos at)\right]_0^\infty$$

$$=\frac{a}{s^2+a^2}$$

(or)

$L(\sin at)=L\left(\frac{e^{iat}-e^{-iat}}{2i}\right)$. (as $\sin at=\frac{e^{iat}-e^{-iat}}{2i}$)

$$=\frac{1}{2i}\left[L\left(e^{iat}-e^{-iat}\right)\right]$$

$$=\frac{1}{2i}\left[L\left(e^{iat}\right)-L\left(e^{-iat}\right)\right]$$

$$=\frac{1}{2i}\left[\frac{1}{s-ia}-\frac{1}{s+ia}\right]=\frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right]=\frac{a}{s^2+a^2}$$

5. $L(\cos at)=\int_0^\infty e^{-st}.\cos at\, dt$

$$=\left[\frac{e^{-st}}{s^2+a^2}(-s\cos at-a\sin at)\right]_0^\infty$$

$$= -\frac{1}{s^2 + a^2}(-s)$$

$$\therefore\ L(\cos at) = \frac{s}{s^2 + a^2}$$

6. $L(\sinh at) = \int_0^\infty e^{-st} \sinh at\, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right).dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$\therefore\ L(\sinh at) = \frac{a}{s^2 - a^2}$$

7. $L(\cosh at) = \int_0^\infty e^{-st} \cosh at\, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} + e^{-at}}{2}\right).dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-st} e^{at}\, dt + \int_0^\infty e^{-st} e^{-at}\, dt\right]$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt + \int_0^\infty e^{-(s+a)t}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right] = \frac{s}{s^2 - a^2}$$

$$\therefore\quad L(\cosh at) = \frac{s}{s^2 - a^2}.$$

1.6 Laplace transforms of some special functions

Heaviside's unit step function

The function

$$u(t-a) = \begin{cases} 0, & if \quad t < a \\ 1, & if \quad t > a \quad where \quad a > 0 \end{cases}$$

is called Heaviside's unit step function and is denoted by $u_a(t)$ or $u(t-a)$.

In particular when $a = 0$,

$$u(t) = \begin{cases} 0 & if \quad t < 0 \\ 1 & if \quad t > 0 \end{cases}$$

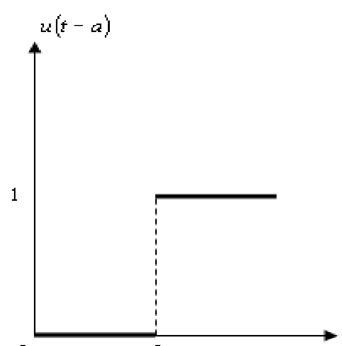


Fig. 2

Multiplying a given function $f(t)$ with the unit step function $u(t-a)$, several effects can be produced as shown in the following figure.

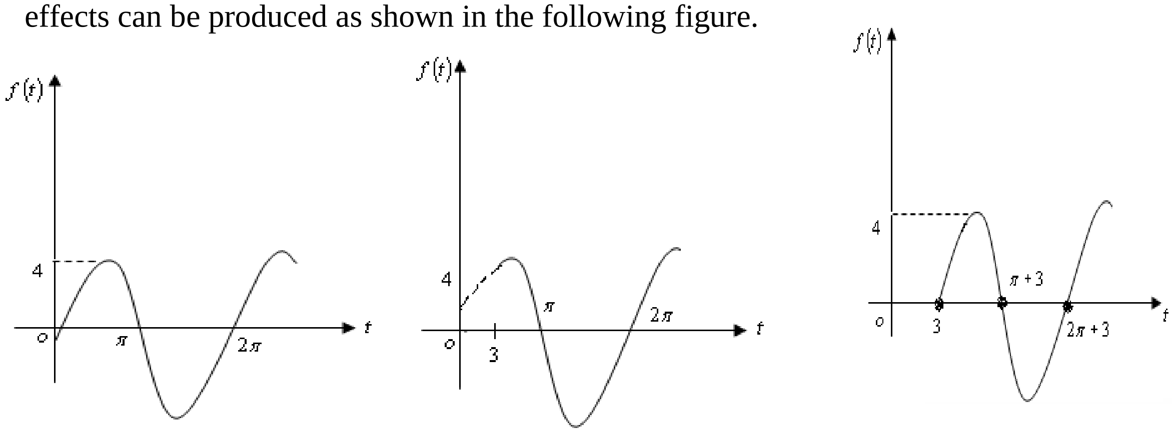


Fig. 3

$\mathbf{f(t) = 4\sin t}$	$\mathbf{f(t)\ u(t-3)}$	$\mathbf{f(t-3)u(t-3)}$
Given function	**Switching off and on**	**Shifted to the right by 3 units**

Unit impulse function (or Dirac's Delta function)

When a large force acts for a short time, then the product of the force and the time is called impulse in Fluid Mechanics.

Impulse of a forces $f(t)$ in the interval $(a, a+\in)$

$$= \int_a^{a+\in} f(t)dt\ .$$

Now define the function

$$f_\in(t-a) = \begin{cases} 0 & for \quad t < a \\ \dfrac{1}{\in} & for \quad a \le t \le a+\in \\ 0 & for \quad t > a \end{cases}$$

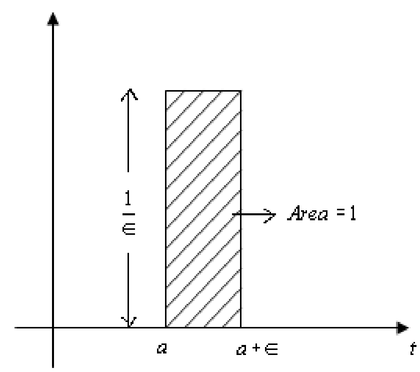


Fig. 4

This can also be represented interms of two unit step functions as follows.

$$f_\in(t-a) = \frac{1}{\in}[u(t-a) - u(t-(a+\in))]$$

Note that

$$\int_0^\infty f_\in(t-a)dt = \int_0^a 0 + \int_a^{a+\in} \frac{1}{\in} dt + \int_{a+\in}^\infty 0 = 1$$

Thus the Impulse $I_\in$ is 1

Taking Laplace transform

$$L[f_\in(t-a)] = \frac{1}{\in} L[u(t-a) - u(t-(a+\in))]$$

$$= \frac{1}{\in s}\left[e^{-as} - e^{-(a+\in)s}\right] = e^{-as}\,\frac{\left(1 - e^{-\in S}\right)}{\in s}$$

Dirac delta function (or unit impulse function) denoted by $\delta(t-a)$ is defined as the limit of $f_{\in}(t-a)$ as $\in \to 0$.

i.e., $\delta(t-a) = \underset{\in \to 0}{Lt}\, f_{\in}(t-a)$.

Laplace transform of unit step function

$$L(u_a(t)) = \int_0^{\infty} e^{-st}\, u_a(t)\,dt$$

$$= \int_0^{\infty} e^{-st}\, u_a(t)\,dt + \int_a^{\infty} e^{-st}\, u_a(t)\,dt$$

$$= \int_0^{\infty} e^{-st}\, dt \text{ (by the definition of } u_a(t))$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^{\infty} \quad = \frac{e^{-as}}{s}, \text{ assuming that } s > 0$$

In particular $L(u_0(t)) = \frac{1}{s} = L(1)$.

Laplace transform of Dirac delta function

$$L(\delta(t-a)) = \underset{\in \to 0}{Lt}\, L[f_{\in}(t-a)]$$

$$= \underset{\in \to 0}{Lt}\, e^{-as}\,\frac{\left(1 - e^{-\in s}\right)}{\in s}$$

$$\therefore \; L(\delta(t-a)) = e^{-as}.$$

1.7 Properties of Laplace transforms

1. **Linearity Property**

   If a, b, c be any constants and f, g, h any functions of t, then

   $$L[a\,f(t) + bg(t) - c\,h(t)] = a\,L(f(t)) + b\,L(g(t)) - c\,L(h(t))$$

   L.H.S

   $$L[a\,f(t) + bg(t) - c\,h(t)] = \int_0^{\infty} e^{-st}[a\,f(t) + bg(t) - ch(t)]dt$$

   $$= a\int_0^{\infty} e^{-st}\, f(t)dt + b\int_0^{\infty} e^{-st}\, g(t)dt - c\int_0^{\infty} e^{-st} h(t)dt$$

   $$= a\,L(f(t)) + b\,L(g(t)) - c\,L(h(t)).$$

   This result can easily be generalized.

   Because of the above property of L, it is called a **linear operator**.

2. **First shifting property** (or) (Translation on the s-axis or shifting on the s-axis)

   If $L(f(t)) = F(s)$, then $L\left(e^{at} f(t)\right) = F(s-a)$.

L.H.S

$$L\left(e^{at} f(t)\right) = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

i.e., $L\left(e^{at}\ f(t)\right) = F(s-a)$ (since $L\ f(t) = F(s)$)

similarly we can prove

$L\left(e^{-at}\ f(t)\right) = F(s+a)$,

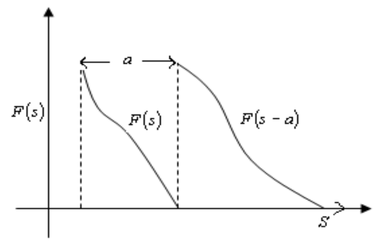


Fig. 5
Translation on the s-axis
(first shifting theorem)

3. **Second Shifting Property** (or Translation on the $t-$axis)

If $L(f(t)) = F(s)$, then $L[f(t-a).u(t-a)] = e^{-as}.F(s)$

L.H.S

$$L[f(t-a)u(t-a)] = \int_0^\infty e^{-st}[f(t-a)u(t-a)].dt$$

$$= \int_0^a e^{-st}\ f(t-a)\ 0\,dt + \int_a^\infty e^{-st}\ f(t-a).1.dt$$

$$= \int_a^\infty e^{-st}\ f(t-a)dt$$

$= \int_0^\infty e^{-s(x+a)}\ f(x)dx$. (by putting $t-a = x, dt = dx$.

when $t = a, x = 0$ when $t = \infty, x = \infty$)

$$= e^{-sa}\int_0^\infty e^{-sx} f(x)dx$$

$= e^{-as}\int_0^\infty e^{-st} f(t)dt$ by changing the dummy variable x as t.

i.e., $L[f(t-a)u(t-a)] = e^{-as}F(s)$.

4**. Change of scale property**

If $L(f(t)) = F(s)$, then $L(f(at)) = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

L.H.S

$$L(f(at)) = \int_0^\infty e^{-st} f(at).dt$$

Put $at = u$ then $dt = \dfrac{du}{a}$

$$= \int_0^\infty e^{-\frac{su}{a}} f(u)\frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u).du \quad = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}.u} f(u).du = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Note

Application of first shifting property leads to the following results:

1) $L\left(e^{at}\right)=\dfrac{1}{s-a}, \qquad \because\ L(1)=\dfrac{1}{s}$

2) $L\left(e^{at}t^{n}\right)=\dfrac{n!}{(s-a)^{n+1}}, \qquad \because\ L\left(t^{n}\right)=\dfrac{n!}{s^{n+1}}$

3) $L\left(e^{at}\sin bt\right)=\dfrac{b}{(s-a)^{2}+b^{2}}, \qquad \because\ L(\sin bt)=\dfrac{b}{s^{2}+b^{2}}$

4) $L\left(e^{at}\cos bt\right)=\dfrac{s-a}{(s-a)^{2}+b^{2}}, \qquad \because\ L(\cos bt)=\dfrac{s}{s^{2}+b^{2}}$

5) $L\left(e^{at}\sinh bt\right)=\dfrac{b}{(s-a)^{2}+b^{2}}, \qquad \because\ L(\sinh bt)=\dfrac{b}{s^{2}-b^{2}}$

6) $L\left(e^{at}\cosh bt\right)=\dfrac{s-a}{(s-a)^{2}-b^{2}}, \qquad \because\ L(\cosh bt)=\dfrac{s}{s^{2}-b^{2}}$

where in each case $s>a$.

Periodic function

A function $f(t)$ is said to be a periodic function of period $T>0$ if $f(t)=f(t+T)=f(t+2T)=...........f(t+nT)$.

Examples: $\sin t$ and $\cos t$ are periodic functions of period 2π.

Geometrically, this implies that the graph of the function $y=f(t)$ repeats itself after every interval of length T.

The following are some examples of periodic functions.

(i) **Triangular wave**

$$f(t)=\begin{cases}\dfrac{t}{a}, & 0\le t<a\\[2mm] \dfrac{2a-t}{a}, & a\le t\le 2a\end{cases}.$$

$$f(t+T)=f(t+2a)=f(t).$$

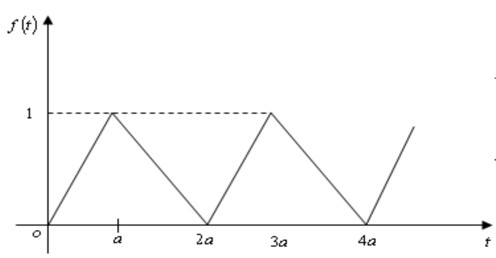


Fig. 6
Triangular wave

(ii) **Square wave**

$$f(t)=\begin{cases}k, & 0\le t<a\\ -k, & a\le t\le 2a\end{cases}$$

$$f(t+T)=f(t+2a)=f(t)$$

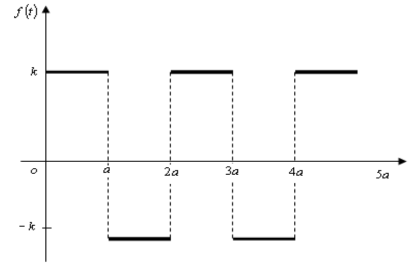


Fig. 7 Square wave

(iii) **Square wave**

$$f(t)=\begin{cases} k & , \quad 0\le t<a \\ 0 & , \quad a\le t\le 2a \end{cases}$$

$$f(t+T)=f(t+2a)=f(t)$$

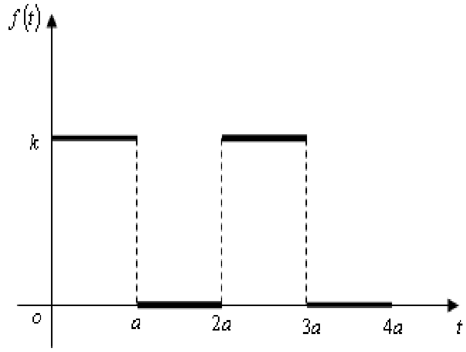


Fig. 8 Square Wave

(iv) **Sawtooth wave**

$$f(t)=t, \quad 0\le t<a.$$

$$f(t+T)=f(t+a)=f(t).$$

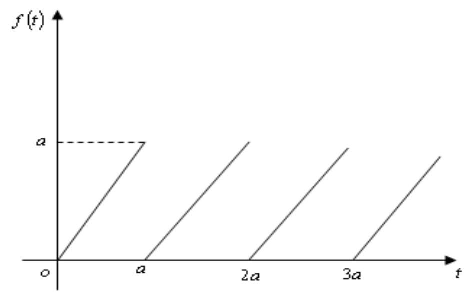


Fig. 9 Sawtooth wave

1.8 Laplace transform of periodic function:

If $f(t)$ is a periodic function with period T, i.e., $f(t+T)=f(t)$, then

$$L(f(t))=\frac{1}{1-e^{-ST}}\int_O^T e^{-st}\, f(t)dt.$$

Proof

We have $L(f(t))=\int_0^\infty e^{-st} f(t)dt$.

$$=\int_0^T e^{-st} f(t)dt+\int_T^{2T} e^{-st}\, f(t)dt+\int_{2T}^{3T} e^{-st} f(t)dt+..........$$

In the second integral put $t=u+T$, in the third integral put $t=u+2T$ and so on. Then

$$L(f(t))=\int_0^T e^{-st} f(t)dt+\int_0^T e^{-s(u+T)} f(u+T)du+\int_0^T e^{-s(u+2T)} f(u+2T)du+..........$$

$$=\int_0^T e^{-st} f(t)dt+e^{-ST}\int_0^T e^{-st} f(u)du+e^{-2ST}\int_0^T e^{-st} f(u)du+..........$$

(since $f(u)=f(u+T)=f(u+2T)$...............)

$$=\int_0^T e^{-st} f(t)dt+e^{-ST}\int_0^T e^{-st} f(t)dt+e^{-2ST}\int_0^T e^{-su} f(t)dt+..........$$

$$=\left(1+e^{-ST}+e^{-2ST}+.........\right)\int_0^T e^{-st} f(t)dt$$

$$\therefore\ L(f(t))=\frac{1}{1-e^{-ST}}\int_0^T e^{-st} f(t)dt.$$

1.9 Laplace Transform of Derivatives

If $L(f(t)) = F(s)$, then $L(f'(t)) = sF(s) - f(0)$.

Proof

$$L(f'(t)) = \int_0^\infty e^{-st} f'(t).dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)dt$$. (using integration by parts)

Now assuming $f(t)$ to be such that $\underset{t \to \infty}{Lt}\, e^{-st}\, f(t) = 0$

Thus $L(f'(t)) = -f(0) + s\int_0^\infty e^{-st} f(t)dt$

i.e., $L(f'(t)) = s\,F(s) - f(0)$

Similarly, $L(f''(t)) = s^2\, F(s) - sf(0) - f'(0)$

$$L(f^n(t)) = s^n L\, f(t) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots\ldots - f^{n-1}(0).$$

1.10 Laplace Transform of $t^n f(t)$. (Multiplication by t^n)

If $L(f(t)) = F(s)$, then $L(t^n\, f(t)) = (-1)^n \dfrac{d^n}{ds^n}(F(s))$, where $n = 1, 2 \ldots\ldots\ldots\ldots$

Proof

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds}(F(s)) = \frac{d}{ds}\left[\int_0^\infty e^{-st}\, f(t)dt\right] = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)dt$$

$$= \int_0^\infty \left(-te^{-st}\right) f(t)dt = \int_0^\infty e^{-st}(-t\, f(t)).dt$$

$$= L(-t\, f(t)) \text{ or } L(t\,.\,f(t)) = (-1)^1 \frac{d}{ds}(F(s))$$

Similarly $L(t^2 f(t)) = (-1)^2 . \dfrac{d^2}{ds^2}(F(s))$

$$L(t^3 f(t)) = (-1)^3 . \frac{d^3}{ds^3}(F(s))$$

…………………………………

……... …………….....................

…………………………………

$$L(t^n . f(t)) = (-1)^n . \frac{d^n}{ds^n}(F(s)).$$

1.11 Laplace Transform of $\frac{1}{t}f(t)$ (Division by t)

If $L(f(t)) = F(s)$ then $L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} F(s).ds$, provided $\underset{t\to 0}{Lt}\left[\frac{1}{t} f(t)\right]$ exists.

Proof

$L(f(t)) = F(s) = \int_0^{\infty} e^{-st} f(t)dt$

Integrating on both sides with respect to s, we get,

$$\int_0^{\infty} F(s)ds = \int_s^{\infty}\left[\int_0^{\infty} e^{-st}\ f(t)\ ds\right].dt$$

$$= \int_0^{\infty}\int_s^{\infty} f(t)e^{-st}\ ds\,dt \text{ . (changing the order of integration)}$$

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}\ ds\right].dt$$

$$= \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt \quad = \int_0^{\infty} e^{-st}\ \frac{f(t)}{t} dt = L\left(\frac{f(t)}{t}\right)$$

Hence $L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} F(s).ds$.

In many problems of electrical engineering, we encounter integro-differential equations. Consider a series electric circuit. Using the kirchoff's second law, we obtain that the flow of current satisfies the integro-differential equation.

$$L\frac{di}{dt} + Ri + \frac{1}{c}\int_0^t i\,d\tau = E_0 \cos\omega t$$

Many other integro-differential equations arise in the theory of electrical circuits. If Laplace transform method is to be applied, we need the formula for the Laplace transform of an integral. Such a formula is presented as follows.

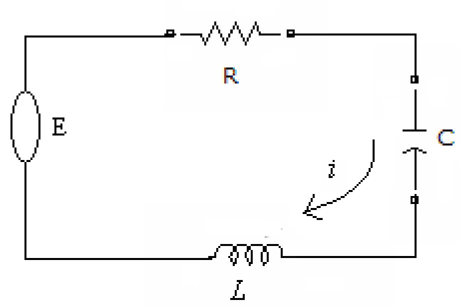


Fig. 10 Series electric circuit
C : Capacitance, E : impressed voltage
L : inductance, R : resistance

1.12 Laplace Transform of integrals

If $L(f(t)) = F(s)$, then $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$.

Proof

Let $\phi(t) = \int_0^t f(t)\,dt$ then $\phi'(t) = f(t)$ and $\phi(0) = 0$

We know that

$$L(\phi'(t)) = s\,L(\phi(t)) - \phi(0)$$
$$= s\,L(\phi(t)) \text{ (since } \phi(0) = 0\text{)}$$

or $L(\phi(t)) = \frac{1}{s}L(\phi'(t))$

subsisting the values of $\phi(t)$ and $\phi'(t)$, we get

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

i.e., $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$**.**

Example 1

Find the Laplace transform of $e^{at} - e^{bt}$.

Solution

$$L\left[e^{at} - e^{bt}\right] = L\left(e^{at}\right) - L\left(e^{bt}\right)$$
$$= \frac{1}{s-a} - \frac{1}{s-b} = \frac{a-b}{(s-a)(s-b)}.$$ **Ans.**

Example 2

Find the Laplace transform of $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$.

Solution

$$L\left[3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\right]$$
$$= 3L\left(t^4\right) - 2L\left(t^3\right) + 4L\left(e^{-3t}\right) - 2L(\sin 5t) + 3L(\cos 2t)$$
$$= 3.\frac{4!}{s^5} - 2.\frac{3!}{s^4} + 4.\frac{1}{s+3} - 2.\frac{5}{s^2+5^2} + 3.\frac{s}{s^2+2^2}.$$ **Ans.**

Example 3

Find the Laplace transform of $\left[3t^5 - 2t^4 + 4e^{-5t} - 3\sin 6t + 4\cos 4t\right]e^{2t}$.

Solution

$$L\left[3t^5 - 2t^4 + 4e^{-5t} - 3\sin 6t + 4\cos 4t\right]$$
$$= 3L\left(t^5\right) - 2L\left(t^4\right) + 4L\left(e^{-5t}\right) - 3L(\sin 6t) + 4L(\cos 4t)$$

$$=3.\frac{5!}{s^6}-2.\frac{4!}{s^5}+4.\frac{1}{s+5}-3.\frac{6}{s^2+36}+4.\frac{s}{s^2+16}$$

Applying first shifting theorem,

$$L\{[3t^5-2t^4+4e^{-5t}-3\sin 6t+4\cos 4t]e^{2t}\}$$

$$=\frac{360}{s^6}-\frac{48}{s^5}+\frac{4}{s+5}-\frac{18}{s^2+36}+\frac{4s}{s^2+16} \text{ with } s \text{ replaced by } s-2$$

$$=\frac{360}{(s-2)^6}-\frac{48}{(s-2)^5}+\frac{4}{(s+3)}-\frac{18}{(s-2)^2+36}+\frac{4(s-2)}{(s-2)^2+16}.$$ **Ans.**

Example 4

Find the Laplace transform of (i) $e^{-3t}(2\cos 5t-3\sin 5t)$ (ii) $e^{2t}\cos^2 t$ (iii) $e^{4t}\sin 2t\cos t$.

Solution

(i) $L\{e^{-3t}(2\cos 5t-3\sin 5t)\}=2L(e^{-3t}\cos 5t)-3L(e^{-3t}\sin 5t)$

$$=2.\frac{s+3}{(s+3)^2+5^2}-3.\frac{5}{(s+3)^2+5^2}=\frac{2s-9}{s^2+6s+34}$$

(ii) Since $L(\cos^2 t)=\frac{1}{2}L(1+\cos 2t)=\frac{1}{2}\left\{\frac{1}{s}+\frac{s}{s^2+4}\right\}$

$\therefore$ By shifting property, we get

$$L(e^{2t}\cos^2 t)=\frac{1}{2}\left\{\frac{1}{s-2}+\frac{s-2}{(s-2)^2+4}\right\}$$

(iii) Since $L(\sin 2t\cos t)=\frac{1}{2}L(\sin 3t+\sin t)$

$$=\frac{1}{2}\left\{\frac{3}{s^2+3^2}+\frac{1}{s^2+1^2}\right\}$$

$\therefore$ By shifting property, we obtain

$$L(e^{4t}\sin 2t\cos t)=\frac{1}{2}\left\{\frac{3}{(s-4)^2+9}+\frac{1}{(s-4)^2+1}\right\}.$$ **Ans.**

Example 5

Find the Laplace transform of

$$f(t)=\begin{cases}1, & 0<t\le 1\\ t, & 1<t\le 2\\ 0, & t>2\end{cases}$$

Solution

$$L(f(t))=\int_0^1 e^{-st}.1.dt+\int_1^2 e^{-st}.t\,dt+\int_2^\infty e^{-st}(0).dt$$

$$=\left[\frac{e^{-st}}{-s}\right]_0^1+\left[t.\frac{e^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_1^2$$

$$=\frac{1-e^{-s}}{s}+\left\{\left[-\frac{2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}\right]-\left[\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}\right]\right\}$$

$$=\frac{1}{s}-\frac{2e^{-2s}}{s}+\frac{e^{-s}}{s^2}-\frac{e^{-2s}}{s^2}.$$ **Ans.**

Example 6

Find the Laplce transform of $t^2\ \cos at$.

Solution

$$L(\cos at)=\frac{s}{s^2+a^2}$$

$$L\left(t^2\cos at\right)=(-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right]$$

$$=\frac{d}{ds}\left[\frac{\left(s^2+a^2\right).1-s(2s)}{\left(s^2+a^2\right)^2}\right]\ =\frac{d}{ds}\left[\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right]$$

$$=\frac{\left(s^2+a^2\right)^2(-2s)-\left(a^2-s^2\right).2\left(s^2+a^2\right)(2s)}{\left(s^2+a^2\right)^4}$$

$$=\frac{-2s^3-2a^2s-4a^2s+4s^3}{\left(s^2+a^2\right)^3}$$

$$=\frac{2s\left(s^2-3a^3\right)}{\left(s^2+a^2\right)^3}.$$ **Ans.**

Example 7

Obtain the Laplce transform of $t^2\ e^t.\sin 4t$.

Solution

$$L(\sin 4t)=\frac{4}{s^2+16},L\left(e^t.\sin 4t\right)=\frac{4}{(s-1)^2+16}$$

$$\therefore\ L\left(t\,e^t\sin 4t\right)=\frac{-d}{ds}\frac{4}{\left(s^2-2s+17\right)}$$

$$=\frac{4(2s-2)}{\left(s^2-2s+17\right)^2}$$

$$L\left(t^2\,e^t\sin 4t\right)=-4\frac{d}{ds}\frac{2s-2}{\left(s^2-2s+17\right)^2}$$

$$= -4\frac{(s^2-2s+17)^2.2-(2s-2)2(s^2-2s+17)(2s-2)}{(s^2-2s+17)^4}$$

$$= -4\frac{(2s^2-4s+34-8s^2+16s-8)}{(s^2-2s+17)^3}$$

$$= -4\frac{(-6s^2+12s+26)}{(s^2-2s+17)^3} = \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}.$$ **Ans.**

Example 8

Find the Laplace transform of $\frac{\sin 2t}{t}$.

Solution

Here $\underset{t\to 0}{Lt}\left(\frac{\sin 2t}{t}\right)$ exists.

$$L(\sin 2t) = \frac{2}{s^2+4}$$

$$\therefore L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}.ds = 2.\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2} = \cot^{-1}\frac{s}{2}.$$ **Ans.**

Example 9

Find the Laplace transform of $t^2u(t-3)$.

Solution

$$t^2.u(t-3) = \left[(t-3)^2 + 6(t-3) + 9\right]u(t-3)$$

$$= (t-3)^2u(t-3) + 6(t-3).u(t-3) + 9u(t-3)$$

$$L(t^2.u(t-3)) = L(t-3)^2.u(t-3) + 6L(t-3)u(t-3) + 9Lu(t-3)$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right].$$ **Ans.**

Example 10

Evaluate (i) $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}dt\right\}$

(ii) $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}dt\right\}$

(iii) $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,dt\,dt\right\}$.

Solution

We know that $L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore \quad L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1}ds \quad = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$\therefore\ L\left\{\int_0^t \frac{\sin t}{t}dt\right\} \ = \frac{1}{s}\cot^{-1} s$$

Thus by shifting property, $L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}dt\right)\right\} \ = \dfrac{1}{s+1}\cot^{-1}(s+1)$.

(ii) Since $L\left(\dfrac{\sin t}{t}\right) \ = \cot^{-1} s$

$$\therefore\ L\left(e^{-t}\,\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and $L\left\{\int_0^t e^{-t}\,\dfrac{\sin t}{t}\,dt\right\} \ = \dfrac{1}{s}\cot^{-1}(s+1)$

Hence $L\left\{t.\int_0^t e^{-t}\,\dfrac{\sin t}{t}\,dt\right\} \ = \dfrac{-d}{ds}\left\{\dfrac{\cot^{-1}(s+1)}{s}\right\}$

$$= -\frac{s\left[\dfrac{-1}{1+(s+1)^2}\right] - \cot^{-1}(s+1)}{s^2}$$

$$= \frac{s + (s^2+2s+2) - \cot^{-1}(s+1)}{s^2(s^2+2s+2)}.$$

(iii) Since $L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore\ L(t\ \sin t) = -\frac{d}{ds}\,\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$

Thus $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t).dt\,dt\,dt\right\}$.

$$= \frac{1}{s^3}L(t\sin t) \ = \frac{1}{s^3}.\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}.$$ **Ans.**

Example 11

Find $L\left[\dfrac{e^{at} - \cos 6t}{t}\right]$ and $L\left[t.e^{-t}\sin t\right]$. [AU APR 2011, AU NOV 2011].

Solution

Consider $\underset{t\to 0}{L}\left[\dfrac{e^{at} - \cos 6t}{t}\right]$

Since the limit exists, we can find $L\left[\frac{e^{at}-\cos 6t}{t}\right]$

$$\therefore \quad L\left[\frac{e^{at}-\cos 6t}{t}\right]=\int_s^\infty L\left(e^{at}-\cos 6t\right).ds$$

$$=\int_0^\infty \frac{1}{s-a}.ds-\int_0^\infty \frac{s}{s^2+36}.ds$$

$$=\left[\log(s-a)-\frac{1}{2}\log\left(s^2+36\right)\right]_s^\infty$$

$$=\log\left[\frac{s-a}{\left(s^2+36\right)^{1/2}}\right]_s^\infty$$

$$=\log\left[\frac{\left(1-\frac{a}{s}\right)}{\left(1+\frac{36}{s^2}\right)^{1/2}}\right]_s^\infty$$

$$=\log(1)-\log\left[\left(\frac{s-a}{s}\right)\times\frac{s}{\left(s^2+36\right)^{1/2}}\right]$$

$$=\log\left[\frac{\left(s^2+36\right)^{1/2}}{s-a}\right].$$

(ii) To find $L\left[t.e^{-t}\sin t\right]$

We know that $L(\sin t)=\frac{1}{s^2+1}$

$$\therefore \; L\left(e^{-t}\sin t\right)=\frac{1}{(s+1)^2+1}$$

$$\therefore \; L\left[t.e^{-t}\sin t\right]=-\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right]$$

$$=-\left[\frac{-(2s+2)}{\left(s^2+2s+2\right)^2}\right]=\frac{2(s+1)}{(s+1)^4}=\frac{2}{(s+1)^3}.$$ **Ans.**

Example 12

Find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$ [AU MAY 2012].

Solution

$$L\left(e^{-at}-e^{-bt}\right)=\frac{1}{s+a}-\frac{1}{s+b}$$

Now $L\left[\dfrac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty\left(\dfrac{1}{s+a}-\dfrac{1}{s+b}\right)ds$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\log\left[\frac{s+a}{s+b}\right] = \log\left[\frac{\left(1+\frac{a}{s}\right)}{\left(1+\frac{b}{s}\right)}\right]_s^\infty$$

$\therefore\ L\left(e^{-at}-e^{-bt}\right)=\log\left[\dfrac{s+b}{s+a}\right].$ **Ans.**

Example 13

Evaluate $\int_0^\infty te^{-2t}\cos t\,dt$. [AU MAY 2012]

Solution

$\int_0^\infty t\,e^{-2t}\cos t\,dt = \int_0^\infty e^{-2t}(t\cos t)dt$

$=L(t\cos t)$ and here $s=2$

$$=(-1)\frac{d}{ds}L(\cos t)$$

$$=(-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$=-\left[\frac{s^2+1-s(2s)}{(s^2+1)^2}\right] = -\left[\frac{-s^2+1}{(s^2+1)^2}\right]$$

$=\dfrac{s^2-1}{(s^2+1)^2}.$ **Ans.**

Example 14

Find the Laplace transform of $e^{-2t}\ t\ \sin 2t$ (or) $L\left(e^{-2t}\ t\ \sin 2t\right)$. [KU NOV 2011]

Solution

We know that $L(\sin 2t)=\dfrac{2}{s^2+4}$

$\therefore\ L\left(e^{-2t}\sin 2t\right)=\dfrac{2}{(s+2)^2+4} = \dfrac{2}{s^2+4s+8}$

Then $L\left(t\,e^{-2t}\sin 2t\right)=-\dfrac{d}{ds}\left[\dfrac{2}{s^2+4s+8}\right]$

$$= -\left[\frac{-2(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= \frac{4(s+2)}{(s^2+4s+8)^2}.$$ **Ans.**

Example 15

Find the Laplace transform of the function (Half wave rectifier)

$$f(t) = \begin{cases} \sin \omega t & for \quad 0 < t < \dfrac{\pi}{\omega} \\ 0 & for \quad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}.$$

Solution

Since $f(t)$ is a periodic function with period $2\pi/\omega$, we have

$$L(f(t)) = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)dt$$

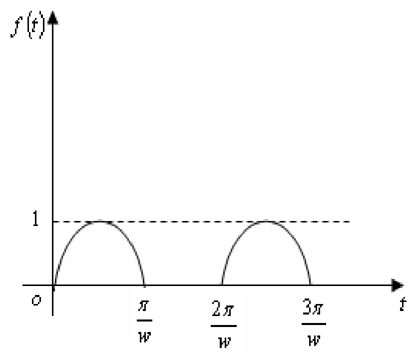


Fig. 11

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin \omega t\, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}(0)dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\pi/\omega} e^{-st}\sin \omega t\, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin \omega t - \omega \cos \text{cot})\right]_0^{\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{\omega e^{-\frac{\pi s}{\omega}} + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left[1+e^{-\frac{\pi s}{\omega}}\right]}{(s^2+\omega^2)\left[1-e^{-\frac{2\pi s}{\omega}}\right]}$$

$$= \frac{\omega}{(s^2+\omega^2)\left[1-e^{-\frac{\pi s}{\omega}}\right]}.$$ **Ans.**

Example 16

Find the transform of the function defined by(triangular wave function)

$$f(t)=\begin{cases} t & 0<t<a \\ 2a-t & a<t<2a \end{cases}$$

where $f(t+2a)=f(t)$ [AU OCT 2009, AU DEC 2009, APR 2011, KU NOV 2011].

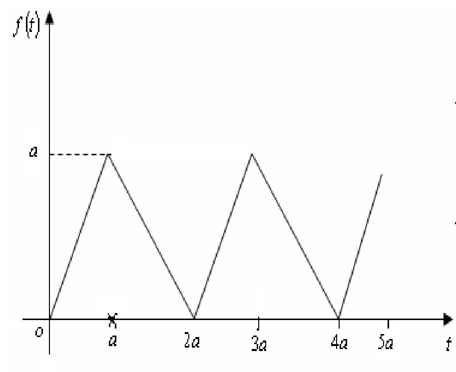


Fig. 12

Solution

The given function is periodic of period $2a$.

$$L(f(t))=\frac{1}{1-e^{-ST}}\int_0^T e^{-st}\ f(t)dt$$

$$=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}\ f(t)dt$$

$$=\frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}.t\ dt+\int_a^{2a} e^{-st}(2a-t).dt\right]$$

$$=\frac{1}{1-e^{-2as}}\left\{\left[t.\frac{e^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^a+\left[(2a-t)\frac{e^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$=\frac{1}{1-e^{-2as}}\left[-\frac{a}{s}e^{-as}-\frac{1}{s^2}e^{-as}+\frac{1}{s^2}+\frac{1}{s^2}e^{-2as}+\frac{a}{s}e^{-as}-\frac{1}{s^2}e^{-as}\right]$$

$$=\frac{1}{s^2}.\frac{1}{1-e^{-2as}}\left[1-2e^{-as}+e^{-2as}\right]$$

$$=\frac{1}{s^2}\frac{(1-e^{-as})^2}{(1-e^{-as})(1+e^{-as})}\quad=\frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})}$$

Multiply and divide by $e^{\frac{as}{2}}$

$$\therefore\ L(f(t))=\frac{1}{s^2}\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}=\frac{1}{s^2}\tan h\left(\frac{as}{2}\right).$$ **Ans.**

Example 17

Find the Laplace transform of the rectangular wave given by

$$f(t)=\begin{cases} 1 & , & 0<t<b \\ -1 & , & b<t<2b \end{cases}$$ with $f(t+2b)=f(t)$. [AU NOV 2010, AU NOV 2011]

Solution

The given function is periodic of period $2b$

Now $L(f(t))=\frac{1}{1-e^{-ST}}\int_0^T e^{-st}f(t)dt$

$$=\frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}(1)dt + \int_b^{2b} e^{-st}(-1)dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - 1\left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[-\frac{1}{s}\left(e^{-bs}-1\right)+\frac{1}{s}\left(e^{-2bs}-e^{-bs}\right)\right]$$

$$= \frac{1}{1-e^{-2bs}}\frac{1}{s}\left[e^{-2bs}-2e^{-bs}+1\right]$$

$$= \frac{\left(1-e^{-bs}\right)^2}{s\left(1-e^{-bs}\right)\left(1+e^{-bs}\right)}$$

$$= \frac{1}{s}\frac{\left(1-e^{-bs}\right)}{\left(1+e^{-bs}\right)}$$

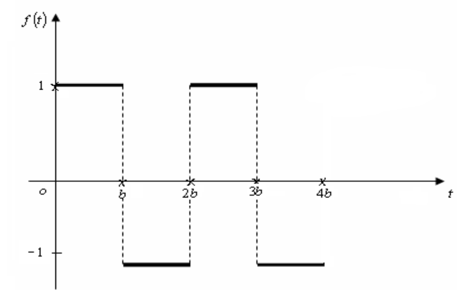


Fig. 13

Multiply and divide by $e^{\frac{bs}{2}}$

Then $L(f(t)) = \frac{1}{s}\frac{e^{\frac{bs}{2}} - e^{-\frac{bs}{2}}}{e^{\frac{bs}{2}} + e^{-\frac{bs}{2}}} = \frac{1}{s}\tan h\left(\frac{bs}{2}\right).$ **Ans.**

Example 18

Find the Laplace transform of the periodic function defined by the sawtooth wave. $f(t) = t, 0 \le t \le a,\ f(t+a) = f(t).$

Solution

$$L(f(t)) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\ f(t)dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a t\,e^{-st}\,dt. \ \text{(since } f(t+a) = f(t)\text{)}$$

$$= \frac{1}{1-e^{-as}}\left[-\left(\frac{t}{s}+\frac{1}{s^2}\right)e^{-st}\right]_0^a$$

$$= \frac{1}{1-e^{-as}}\left[-\left(\frac{a}{s}+\frac{1}{s^2}\right)e^{-as}+\frac{1}{s^2}\right]$$

$$= \frac{1}{1-e^{-as}}\left[-\frac{a}{s}e^{-as}+\frac{1}{s^2}\left(1-e^{-as}\right)\right]$$

$$= \frac{1}{s^2} - \frac{ae^{-as}}{s\left(1-e^{-as}\right)}, s > 0.$$ **Ans.**

1.13 Inverse Laplace transform

If $L(f(t)) = F(s)$ then $f(t)$ is known as the inverse Laplace transform or inverse transform or simply inverse of $F(s)$ and is denoted by $L^{-1}(F(s))$.

Thus $f(t) = L^{-1}(F(s))$. (1)

L^{-1} is known as the inverse laplace transform operator and is such that $LL^{-1} = L^{-1}L = 1$.

In, (1), $F(s)$ is given (known) and $f(t)$ is to be determined.

Note

Inverse laplace transform of $F(s)$ need not exist for all $F(s)$.

Some important formulae

1. $L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3..........$
3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
4. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos hat$
5. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sin hat$
6. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$
7. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
8. $L^{-1}F(s-a) = e^{at} f(t)$
9. $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt$
10. $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$
11. $L^{-1}\left(\frac{1}{(s-a)^2-b^2}\right) = \frac{1}{b}e^{at}\sinh bt$
12. $L^{-1}\left(\frac{s-a}{(s-a)^2-b^2}\right) = e^{at}\cosh bt$
13. $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at)$
14. $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a}t\sin at$

15. $L^{-1}\left(\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}\right)=t\cos at$

16. $L^{-1}\left(\dfrac{s^2}{\left(s^2+a^2\right)^2}\right)=\dfrac{1}{2a}\left[\sin at+at\cos at\right]$

17. $L^{-1}\left(-\dfrac{d}{ds}F(s)\right)=t\,f(t)$

18. Linearity property

$L^{-1}\left(aF(s)+bG(s)\right)=aL^{-1}\left(F(s)\right)+b\,L^{-1}\left(G\,(s)\right)$

19. Multiplication by s

$L^{-1}\left(s.F(s)\right)=\dfrac{d}{dt}\,f(t)+\,f(0)\,\delta(t)$

20. Division by s

$L^{-1}\left(\dfrac{F(s)}{s}\right)=\int_0^t L^{-1}\left(F(s)\right).dt=\int_0^t f(t)dt$

21. First shifting property

If $L^{-1}\left(F(s)\right)=\,f(t)$, then $L^{-1}\left(F(s+a)\right)=e^{-at}\,L^{-1}\left(F(s)\right)$

22. Second shifting property

$L^{-1}\left(e^{-as}\,F(s)\right)=\,f(t-a)u(t-a)$

23. Inverse Laplace transform of integrals

$L^{-1}\left[\int_s^\infty F(s).ds\right]=\dfrac{f(t)}{t}=\dfrac{1}{t}L^{-1}\left(F(s)\right)$

(or)

$L^{-1}\left(F(s)\right)=t\,L^{-1}\left[\int_s^\infty F(s).ds\right].$

Example 1

Find $L^{-1}\left[\log\left(\dfrac{s^2+1}{(s-1)^2}\right)\right].$

Solution

Let $f(t)=L^{-1}\left[\log\left(\dfrac{s^2+1}{(s-1)^2}\right)\right]$

$\Rightarrow\quad L\left(f(t)\right)=\log\left(s^2+1\right)-\log\left(s-1\right)^2$

Then $L\left(t.f(t)\right)=-\dfrac{d}{ds}\left[\log\left(s^2+1\right)-\log\left(s-1\right)^2\right]$

$$=-\left[\frac{2s}{s^2+1}-\frac{2(s-1)}{(s-1)^2}\right]\;=\frac{2}{(s-1)}-2\frac{s}{s^2+1}$$

$$\therefore \quad t\,f(t) = L^{-1}\left[\frac{2}{s-1}\right] - 2\,L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= 2.e^t - 2\cos t$$

$$\therefore \quad f(t) = \frac{2}{t}\left[e^t - \cos t\right].$$ **Ans.**

Example 2

Find the inverse Laplace transforms of the following

(i) $\log\left(\frac{s+1}{s-1}\right)$ (ii) $\log\left(\frac{s^2+1}{s(s+1)}\right)$ (iii) $\cot^{-1}\left(\frac{s}{2}\right)$ (iv) $\tan^{-1}\left(\frac{2}{s^2}\right)$. [KU NOV 2011]

Solution

(i) If $f(t) = L^{-1}\log\left(\frac{s+1}{s-1}\right)$

We know that $t.f(t) = L^{-1}\left\{-\frac{d}{ds}F(s)\right\}$

$$\therefore \quad t\,f(t) = L^{-1}\left\{-\frac{d}{ds}.\log\left(\frac{s+1}{s-1}\right)\right\} = -L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log(s-1)\right\}$$

$$= -L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{s-1}\right) = -e^{-t} + e^t = 2\sin ht$$

Thus $f(t) = \frac{1}{t}2\sin ht$.

(ii) If $f(t) = L^{-1}\log\left(\frac{s^2+1}{s(s+1)}\right)$

$$t.f(t) = L^{-1}\left\{\frac{-d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\}$$

$$= -L^{-1}\left\{\frac{d}{ds}\log(s^2+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log s\right\} + L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}$$

$$= -L^{-1}\left(\frac{2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -2\cos t + 1 + e^{-t}$$

Thus $f(t) = \frac{1}{t}\left(1 + e^{-t} - 2\cos t\right)$.

(iii) If $f(t) = L^{-1}\cot^{-1}\left(\frac{s}{2}\right)$

$$t.f(t) = L^{-1}\left\{\frac{-d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\}$$

$$= L^{-1}\left(\frac{2}{s^2+2^2}\right) = \sin 2t$$

Thus $f(t) = \frac{1}{t}\sin 2t$.

(iv) If $f(t) = L^{-1}\left(\tan^{-1}\frac{2}{s^2}\right)$

$$t.f(t) = L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$= L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\} = L^{-1}\left\{\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= e^t\sin t - e^{-t}\sin t = 2\sin ht \sin t.$$ **Ans.**

Example 3

Obtain inverse Laplace transform of

(i) $\frac{2s-5}{9s^2-25}$ (ii) $\frac{s-2}{6s^2+20}$ (iii) $\frac{3s}{2s+9}$ (iv) $\frac{1}{s(s+a)}$ (v) $\frac{s^3+3}{s(s^2+9)}$

(vi) $\frac{1}{(s+2)^5}$ (vii) $\frac{s}{s^2+4s+13}$ (viii) $\frac{1}{9s^2+6s+1}$ (ix) $\frac{e^{-\pi s}}{(s+3)}$ (x) $\frac{e^{-s}}{(s+1)^3}$.

Solution

(i) $L^{-1}\left[\frac{2s-5}{9s^2-25}\right] = L^{=1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right]$

$$= L^{=1}\left[\frac{2s}{9\left(s^2-\frac{25}{9}\right)} - \frac{5}{9\left(s^2-\frac{25}{9}\right)}\right]$$

$$= L^{-1}\left[\frac{2s}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]} - \frac{5}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}\right]$$

$$= \frac{2}{9}\cos h\frac{5}{3}t - \frac{1}{3}L^{-1}\left[\frac{\frac{5}{3}}{s^2-\left(\frac{5}{3}\right)^2}\right]$$

$$= \frac{2}{9}\cos h\frac{5}{3}t - \frac{1}{3}\sin\frac{5t}{3}.$$

(ii) $L^{-1}\left[\frac{s-2}{6s^2+20}\right] = L^{-1}\left[\frac{s}{6s^2+20}\right] - L^{-1}\left[\frac{2}{6s^2+20}\right]$

$$= \frac{1}{6}L^{-1}\left[\frac{s}{s^2+\frac{10}{3}}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s^2+\frac{10}{3}}\right]$$

$$= \frac{1}{6}\cos\sqrt{\frac{10}{3}}.t - \frac{1}{3}\sqrt{\frac{3}{10}}.L^{-1}\left[\frac{\sqrt{\frac{10}{3}}}{s^2+\frac{10}{3}}\right]$$

$$= \frac{1}{6}\cos\sqrt{\frac{10}{3}}.t - \frac{1}{\sqrt{30}}\sin\sqrt{\frac{10}{3}}.t.$$

(iii) $L^{-1}\left[\frac{3}{2s+9}\right] = \frac{3}{2}L^{-1}\left[\frac{1}{s+\frac{9}{2}}\right] = \frac{3}{2}e^{-\frac{9}{2}t}$

$$L^{-1}\left[\frac{3s}{2s+9}\right] = \frac{3}{2}\frac{d}{dt}\left(e^{-\frac{9}{2}t}\right) + \frac{3}{2}e^{-\frac{9}{2}(0)}$$

$$= -\frac{27}{4}e^{-\frac{11}{2}t+\frac{3}{2}}.$$

(iv) $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left(\frac{1}{s+a}\right)dt$$

$$= \int_0^t e^{-at}\,dt = \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{e^{-at}}{-a} + \frac{1}{a} = \frac{1}{a}\left[1 - e^{-at}\right].$$

(v) $L^{-1}\left[\frac{s^2+3}{s\left(s^2+9\right)}\right] = L^{-1}\left[\frac{s^2+9-6}{s\left(s^2+9\right)}\right] = L^{-1}\left[\frac{1}{s} - \frac{6}{s\left(s^2+9\right)}\right]$

$$= 1 - 2\int_0^t \sin 3t\, dt$$

$$= 1 - \int_0^t L^{-1}\left(\frac{6}{s^2+9}\right) ds$$

$$= 1 + 2.\frac{1}{3}\left(\cos 3t\right)_0^t$$

$$= 1 + \frac{2}{3}\cos 3t - \frac{2}{3}$$

$$= \frac{2}{3}\cos 3t + \frac{1}{3} = \frac{1}{3}\left[2\cos 3t + 1\right].$$

(vi) $L^{-1}\left[\frac{1}{s^5}\right] = \frac{t^4}{4!}$

then $L^{-1}\left[\frac{1}{(s+2)^5}\right] = e^{-2t}.\frac{t^4}{4!}$

(vii)

$$L^{-1}\left[\frac{s}{s^2+4s+13}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+3^2}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - L^{-1}\left[\frac{2}{(s+2)^2+3^2}\right]$$

$$= e^{-2t}.L^{-1}\left[\frac{s}{s^2+3^2}\right] - e^{-2t}.L^{-1}\frac{2}{3}\left[\frac{3}{s^2+3^2}\right]$$

$$= e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t.$$

(viii) $L^{-1}\left[\frac{1}{9s^2+6s+1}\right] = L^{-1}\left[\frac{1}{(3s+1)^2}\right]$

$$= \frac{1}{9}L^{-1}\left[\frac{1}{\left(s+\frac{1}{3}\right)^2}\right]$$

$$= \frac{1}{9}e^{-\frac{t}{3}}L^{-1}\left(\frac{1}{s^2}\right) = \frac{1}{9}e^{-\frac{t}{3}}.t = \frac{te^{-\frac{t}{3}}}{9}.$$

(ix) $L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$

$\therefore\ L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right] = e^{-3(t-\pi)}u(t-\pi).$

(x) $L^{-1}\left(\frac{1}{s^3}\right)=\frac{t^2}{2!}$

$\therefore \quad L^{-1}\left[\frac{1}{(s+1)^3}\right]=e^{-t}.\frac{t^2}{2!}$

then $\quad L^{-1}\left[\frac{e^{-s}}{(s+1)^3}\right]=e^{-(t-1)}.\frac{(t-1)^2}{2!}u(t-1).$ **Ans.**

Example 4

Find the inverse Laplace transform of $\frac{s+4}{s(s-1)(s^2+4)}$.

Solution

Let us first resolve $\frac{s+4}{s(s-1)(s^2+4)}$ into partial fractions

$$\frac{s+4}{s(s-1)(s^2+4)}=\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+4}$$

$$s+4=A(s-1)(s^2+4)+Bs(s^2+4)+(Cs+D)s(s-1) \qquad (1)$$

Putting $s=0, \quad \Rightarrow \quad A=-1$

Putting $s=1, \quad \Rightarrow \quad B=1$

Equating the coefficients of s^3 on both sides of (1), we get

$0=A+B+C \quad \Rightarrow \quad C=0$

Equating the coefficients of s on both sides of (1), we get

$1=4A+4B-D \quad \Rightarrow \quad D=-1$

On putting the values of A,B,C,D, we get

$$\frac{s+4}{s(s-1)(s^2+4)}=-\frac{1}{s}+\frac{1}{s-1}-\frac{1}{s^2+4}$$

$$\therefore \quad L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right]=L^{-1}\left[-\frac{1}{s}+\frac{1}{s-1}-\frac{1}{s^2+4}\right]$$

$$=-L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$=-1+e^t-\frac{1}{2}\sin 2t.$$ **Ans.**

Example 5

Find the inverse transform of $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$.

Solution

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6}=\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

$$2s^2-6s+5=A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$$

$$\Rightarrow \quad A=\frac{1}{2}, B=-1, C=\frac{5}{2}$$

$$\therefore\ L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right)=\frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right)-1L^{-1}\left(\frac{1}{s-2}\right)+\frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$=\frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}.$$ **Ans.**

Example 6

Find $L^{-1}\left[\frac{1}{(s+2)(s^2+2s+2)}\right]$.

Solution

$$\frac{1}{(s+2)(s^2+2s+2)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+2s+2}$$

$$1=A(s^2+2s+2)+(Bs+C)(s+2)$$

Put $s=-2, \quad \Rightarrow \quad A=\frac{1}{2}$

Equating the coefficients of s^2 on both sides,

$$0=A+B \quad \Rightarrow \quad B=-A=-\frac{1}{2}$$

Equating the coefficients of s on both sides,

$$0=2A+2B+C \quad \Rightarrow \quad C=-2A-2B=0$$

Now $\frac{1}{(s+2)(s^2+2s+2)}=\frac{\frac{1}{2}}{s+2}+\frac{-\frac{1}{2}s}{s^2+2s+2}$

$$\therefore\quad L^{-1}\left[\frac{1}{(s+2)(s^2+2s+2)}\right]=\frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)-\frac{1}{2}L^{-1}\left(\frac{s+1-1}{(s+1)^2+1}\right)$$

$$=\frac{1}{2}e^{-2t}-\frac{1}{2}L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=\frac{1}{2}e^{-2t}-\frac{1}{2}e^{-t}\cos t+\frac{1}{2}e^{-t}\sin t$$

$$=\frac{1}{2}e^{-2t}-\frac{1}{2}e^{-t}(\cos t-\sin t).$$ **Ans.**

Example 7

Find $L^{-1}\left[\dfrac{s}{s^4+s^2+1}\right]$.

Solution

$$\frac{s}{s^4+s^2+1}=\frac{s}{\left(s^2+1\right)^2-s^2}=\frac{s}{\left(s^2-s+1\right)\left(s^2+s+1\right)}$$

$$=\frac{1}{2}\left[\frac{1}{\left(s^2-s+1\right)}-\frac{1}{\left(s^2+s+1\right)}\right]$$

$$L^{-1}\left[\frac{s}{\left(s^4+s^2+1\right)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{\left(s^2-s+1\right)}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{\left(s^2+s+1\right)}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$=\frac{1}{2}\left[\frac{2}{\sqrt{3}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}.t-\frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$=\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}.t\sin h\left(\frac{t}{2}\right).$$ **Ans.**

EXERCISE

PART A

1. Define Laplace transform.
2. State the conditions for the existence of Laplace transform of a function.
3. State change of scale property, first shifting property, second shifting property in Laplace transformation.
4. Find the Laplace transform of unit step function.
5. Find the Laplace transform of unit impulse function.
6. Find $L(f(t))$, if $f(t)=\begin{cases}\sin t & for \quad 0<t<\pi\\ t & for \quad t>\pi\end{cases}$
7. State the formula for the Laplace transform of a periodic function.
8. State the relation between the Laplace transforms of $f(t)$ and $t.f(t)$.
9. Find the relation between the inverse Laplace transform of $F(s)$ and its integral.
10. Find the inverse Laplace transform of $\log\left(\dfrac{s}{s-1}\right)$.

11. Find the laplace transform of $\frac{1-\cos at}{t}$.

12. If $L(f(t)) = \frac{1}{s(s+1)}$ find $f(0)$ and $f(\infty)$.

13. Find $L(\cos 4t \sin 2t)$.

14. Find the inverse Laplace transform of $\frac{1}{s(s^2+a^2)}$.

15. Find $L\left[\int_0^t e^{-t}\, dt\right]$

16. Find $L^{-1}\left[\frac{1}{\sqrt{s+2}}\right]$.

17. If $L(f(t)) = \frac{1}{s(s+a)}$, find $f(0)$.

18. State the sufficient conditions for the existence of Laplace transform of $f(t)$.

19. If $L(f(t)) = F(s)$, prove that $L(f(at)) = \frac{1}{a}F\left(\frac{s}{a}\right)$.

20. Find $L(e^{-at} \sin bt)$.

21. Find $L^{-1}\left[\frac{1}{(s+2)^3}\right]$.

22. Find $L(\sin^2 t)$.

23. Find $L^{-1}\left(\frac{s+2}{s^2+4s+8}\right)$.

24. Find $L\left(\frac{1-e^t}{t}\right)$.

25. Define periodic function with an example.

26. Find $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$.

27. Find $L(e^{-2t} \sin 3t)$.

28. If $L(f(t)) = F(s)$, then find $L\left(f\left(\frac{t}{2}\right)\right)$.

29. Find $L^{-1}\left(\frac{s}{(s+3)^2}\right)$.

30. Find the inverse Laplace transform of $\frac{s+2}{s^2+2s+2}$.

31. Find the Laplace transform of $e^{-2t}(1+t)^2$

32. If $L(f(t)) = F(s)$, what is $L(e^{-at} f(t))$.

33. Write a function for which laplace transformation does not exist. Explain why laplace transform does not exist.

34. Find $L(t\sin 2t)$.

35. Find the Laplace transform of $\frac{\sin 2t}{t}$.

PART B

1. Find the Laplace transform of the following
   (i) $\sin^3 2t$ (ii) $e^{-t}\cos^2 t$ (iii) $\sin 2t\cos 3t$ (iv) $\sin h^3 t$

   (v) $f(t)=\begin{cases} t^2 & 0<t<2 \\ t-1 & 2<t<3 \\ 7 & t>3 \end{cases}$

(Ans. (i) $\frac{48}{(s^2+4)(s^2+36)}$ (ii) $\frac{1}{2s+2}+\frac{s+1}{2s^2+4s+10}$ (iii) $\frac{2(s^2-5)}{(s^2+1)(s^2+25)}$

(iv) $\frac{6}{(s^2-1)(s^2-9)}$ (v) $\frac{2}{s^3}-\frac{e^{-2s}}{s^3}(2+3s+3s^2)+\frac{e^{-3s}}{s^2}(5s-1)$**).**

2. Find the Laplace transform of the following.
   (i) $t\cos t$ (ii) $t^2\sin t$ (iii) $te^{at}\sin at$ (iv) $\int_0^t e^{-2t}.t\sin^3 t.dt$

   (v) $t^2 e^{-2t}\cos t$.

(Ans. (i) $\frac{s^2-1}{(s^2+1)^2}$ (ii) $\frac{2(3s^2-1)}{(s^2+1)^3}$ (iii) $\frac{2a(s-a)}{(s^2-2as+2a^2)^2}$

(iv) $\frac{3(s+2)}{2s}\left[\frac{1}{[(s+2)^2+9]^2}-\frac{1}{[(s+2)^2+1]^2}\right]$ (v) $\frac{2(s^3+10s^2+25s+22)}{(s^2+4s+5)^3}$**)**

3. Find the Laplace transform of the following (i) $\frac{1}{t}(\cos at-\cos bt)$

   (ii) $\frac{1}{t}\sin^2 t$ (iii) $\frac{1}{t}(e^{-t}\sin t)$ (iv) $\sin tu(u-4)$ (v) $e^t.u(t-1)$.

(Ans. (i) $-\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$ (ii) $\frac{1}{4}\log\frac{s^2+4}{s^2}$ (iii) $\cot^{-1}(s+1)$

(iv) $\frac{e^{-4s}}{s^2+1}(\cos 4+s\sin 4)$ (v) $\frac{e^{-(s-1)}}{s-1}$**).**

4. Find the Laplace transform of the following.
   (i) $f(t)=t^2,\ 0<t<2,\quad f(t+2)=f(t)$

   (ii) $f(t)=\begin{cases} \cos\omega t, & 0<t<\pi/\omega \\ 0, & \pi/\omega<t<2\pi/\omega \end{cases}$

(iii) $f(t)=\begin{cases} t & , \quad 0<t<1 \\ 0 & , \quad 1<t<2, \qquad f(t+2)=f(t) \end{cases}$

(iv) $f(t)=\begin{cases} \dfrac{2t}{T} & , \quad 0 \le t \le \dfrac{T}{2} \\ \dfrac{2}{T}(T-t) & , \quad \dfrac{T}{2} \le t \le T \quad , \quad f(t+T)=f(t) \end{cases}$

(v) $f(t)=\begin{cases} E & , \quad 0 \le t \le \dfrac{T}{2} \\ -E & , \quad \dfrac{T}{2} \le t \le T \quad , \quad f(t+T)=f(t) \end{cases}$

(Ans. (i) $\dfrac{2-e^{-2s}-4se^{-2s}-4s^2e^{-2s}}{s^3(1-e^{-2s})}$ (ii) $\dfrac{s}{(s^2+w^2)(1-e^{-\frac{\pi s}{w}})}$ (iii) $\dfrac{1-e^{-s}(s+1)}{s^2(1-e^{-2s})}$

(iv) $\dfrac{2}{Ts^2}\tanh\dfrac{sT}{4}-\dfrac{1}{s(e^{\frac{sT}{2}}+1)}$ (v) $\dfrac{E}{s}\tanh(\dfrac{sT}{4})$ **)**

5. Find the inverse Laplace transform of the following.

(i) $\dfrac{1}{s^2-9}$ (ii) $\dfrac{s}{s^2+9}$ (iii) $\dfrac{1}{(s+3)^2-4}$ (iv) $\dfrac{s+2}{(s+2)^2-25}$

(v) $\dfrac{1}{2s-7}$. **(Ans.** (i) $\dfrac{1}{3}\sin h3t$ (ii) $\cos 3t$ (iii) $\dfrac{1}{2}e^{-3t}\sin h2t$ (iv) $e^{-2t}\times\cos h5t$ (v) $\dfrac{1}{2}e^{\frac{7}{2}t}$ **)**

6. Find the inverse Laplace transform of the following.

(i) $\dfrac{3(s^2-2)^2}{2s^5}$ (ii) $\dfrac{5s-10}{9s^2-16}$ (iii) $\dfrac{2s}{3s+6}$ (iv) $\dfrac{s^2+4}{s^2+9}$

(v) $\dfrac{1}{(s-3)^2}$. **(Ans.** (i) $\dfrac{3}{2}-3t^2+\dfrac{1}{2}t^4$ (ii) $\dfrac{5}{9}\cos h\dfrac{4}{3}t-\dfrac{5}{6}\sin h\dfrac{4}{3}t$

(iii) $\dfrac{2}{3}\left(-2e^{-2t}+1\right)$ (iv) $-\dfrac{5}{3}\sin 3t+1$ (v) $e^{3t}.t$ **)**

7. Find the inverse Laplace transform of the following.

(i) $\dfrac{1}{2s(s-3)}$ (ii) $\dfrac{1}{s(s^2+a^2)}$ (iii) $\dfrac{1}{s^3(s^2+1)}$ (iv) $\dfrac{s}{(s+3)^2+4}$

(v) $\dfrac{s-4}{4(s-3)^2+16}$. **(Ans.** (i) $\dfrac{1}{2}\left[\dfrac{e^{3t}}{3}-1\right]$ (ii) $\dfrac{1-\cos at}{a^2}$ (iii) $\dfrac{t^2}{2}+\cos t-1$

(iv) $e^{-3t}\left(\cos 2t-\dfrac{3}{2}\sin 2t\right)$ (v) $\dfrac{1}{4}e^{3t}\cos 2t-\dfrac{1}{8}e^{3t}\sin 2t$ **).**

8. Obtain inverse Laplace transform of the following.

(i) $\dfrac{e^{-s}}{(s+2)^3}$ (ii) $\dfrac{e^{-\pi s}}{s^2+1}$ (iii) $\log\left(1+\dfrac{1}{s^2}\right)$ (iv) $\dfrac{s+1}{(s^2+6s+13)^2}$

(v) $\dfrac{1}{2}\log\left\{\dfrac{s^2+b^2}{(s-a)^2}\right\}$.

(Ans. (i) $e^{-(t-2)}\dfrac{(t-2)^2}{2}u(t-2)$ (ii) $-\sin t\,u(t-\pi)$ (iii) $\dfrac{2}{t}(1-\cos\omega t)$

(iv) $\dfrac{e^{-3t}}{8}[2t\sin 2t+2t\cos 2t-\sin 2t]$ (v) $\dfrac{1}{t}(e^{-at}-\cos bt)$**).**

9. Find the inverse Laplace transform of (i) $\dfrac{s^2+2s+6}{s^3}$ (ii) $\dfrac{s+2}{s^2-4s+13}$

(iii) $\dfrac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ (iv) $\dfrac{16}{(s^2+2s+5)^2}$ (v) $\dfrac{1}{(s-2)(s^2+1)}$

(Ans. (i) $1+2t+3t^2$ (ii) $e^{2t}\cos 3t+\dfrac{4}{3}e^{2t}\sin 3t$ (iii) $2e^{-t}+5e^{2t}-\dfrac{3}{2}e^{\frac{t}{2}}$

(iv) $e^{-t}(\sin 2t-2t\cos 2t)$ (v) $\dfrac{1}{5}e^{2t}-\dfrac{1}{5}\cos t-\dfrac{2}{5}\sin t$ **).**

CHAPTER II

CONVOLUTION THEOREM, APPLICATIONS OF LAPLACE TRANSFORM

2.1 Introduction

Convolution is used to find inverse Laplace transforms in solving differential equations and integral equations.

Suppose two Laplace transforms $F(s)$ and $G(s)$ are given. Let $f(t)$ and $g(t)$ be their inverse Laplace transforms respectively. i.e., $f(t)=L^{-1}(F(s))$ and $g(t)=L^{-1}(G(s))$. Then the inverse $h(t)$ of the product of transforms $H(s)=F(s).G(s)$ can be calculated from the known inverse $f(t)$ and $g(t)$.

Convolution

The convolution or convolution integral of two functions $f(t)$ and $g(t), t\geq 0$ is defined as the integral $\int_0^t f(u)g(t-u)du$.

i.e., $(f*g)(t)=f(t)*g(t)=\int_0^t f(u)g(t-u)du$.

$f*g$ is called the **convolution** or **faltung** of f and g and can be regarded as a "generalized product" of these functions.

2.2 Convolution Theorem

If $f(t)$ and $g(t)$ are two functions of t and $L(f(t))=F(s)$ and $L(g(t))=G(s)$ for $t\geq 0$ then

$L[f(t)*g(t)]=F(s).G(s)$ (or) $L^{-1}[F(s).G(s)]=f(t)*g(t)$.

Proof

By definition

$$L[f(t)*g(t)]=\int_0^\infty e^{-st}(f(t)*g(t)).dt$$

$$=\int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)du\right].dt$$

by the definition of convolution,

$$=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du\,dt \quad (1)$$

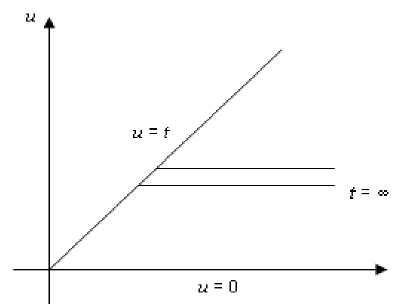


Fig. 14

The region of integration for the double integral (1) is bounded by the lines $u=0, u=t, t=0$ and $t=\infty$. Changing the order of integration in (1), we get

$$L[f(t)*g(t)]=\int_0^\infty\int_u^\infty e^{-st}f(u)g(t-u)dt\,du \quad (2)$$

In the inner integral in (2), on putting $t-u=v$, we get

$$L[f(t)*g(t)] = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)\, g(v)\,dv\,du$$

$$= \int_0^\infty e^{-su}.f(u)\left[\int_0^\infty e^{-sv}\, g(v)dv\right]du$$

$$= \int_0^\infty e^{-su}.f(u)du \int_0^\infty e^{-sv}\, g(v)dv$$

$$= \int_0^\infty e^{-st}.f(t)dt. \int_0^\infty e^{-st}\, g(t)dt.$$

(on changing the dummy variables u and v)

i.e., $L[f(t)*g(t)] = L(f(t)).L(g(t))$.

2.3 Initial value theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L(f(t)) = F(s)$, then $\underset{t\to 0}{Lt}(f(t)) = \underset{s\to\infty}{Lt}(s\,F(s))$.

Proof

We know that $L(f'(t)) = s\,F(s) - f(0)$

$\therefore \quad s\,F(s) = L(f'(t)) + f(0)$

$$= \int_0^\infty e^{-st}\, f'(t)dt + f(0)$$

$\therefore \quad \underset{s\to\infty}{Lt}(s\,F(s)) = \underset{s\to\infty}{Lt}\int_0^\infty e^{-st} f'(t)dt + f(0)$

$$= \int_0^\infty \underset{s\to\infty}{Lt}\left(e^{-st} f'(t)\right)dt + f(0)$$

i.e., $\underset{s\to\infty}{Lt}(s\,F(s)) = f(0) = \underset{t\to 0}{Lt}(f(t))$

$\therefore \ \underset{t\to 0}{Lt}(f(t)) = \underset{s\to\infty}{Lt}(s.F(s))$

2.4 Final value theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L(f(t)) = F(s)$ then $\underset{t\to\infty}{Lt}(f(t)) = \underset{s\to 0}{Lt}(s.F(s))$.

Proof

We know that $L(f'(t)) = s\,F(s) - f(0)$

$\therefore \quad s\,F(s) = L(f'(t)) + f(0)$

$$= \int_0^\infty e^{-st} f'(t).dt + f(0)$$

$\therefore \ \underset{s\to 0}{Lt}(s.F(s)) = \underset{s\to 0}{Lt}\int_0^\infty e^{-st} f'(t)dt + f(0)$

$$= \int_0^\infty \underset{s\to 0}{Lt}\left(e^{-st} f'(t)\right)dt + f(0)$$

$$= \int_0^\infty f'(t)dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= \underset{t\to\infty}{Lt}\, f(t) - f(0) + f(0)$$

$$\therefore\ \underset{t\to\infty}{Lt}\left(f(t)\right) = \underset{s\to 0}{Lt}\left(s.F(s)\right).$$

Example 1

Apply convolution theorem to Evaluate $L^{-1}\left(\dfrac{s}{\left(s^2+a^2\right)^2}\right)$.

[AU JUNE 2010, AU MAY 2012]

Solution

Let $F(s) = \dfrac{1}{\left(s^2+a^2\right)} \quad \Rightarrow \quad L^{-1}\left(F(s)\right) = f(t) = \dfrac{1}{a}\sin at$

$G(s) = \dfrac{s}{\left(s^2+a^2\right)} \ \Rightarrow\ L^{-1}\left(G(s)\right) = g(t) = \cos at$

Now by convolution theorem,

$$L^{-1}\left(F(s).G(s)\right) = \int_{u=0}^{t} f(u)\, g(t-u).du$$

$$= \frac{1}{a}\int_{u=0}^{t} \sin au \cos a(t-u).du$$

$$= \frac{1}{2a}\int_{u=0}^{t} \left[\sin\left(au+at-au\right) + \sin\left(au-at+au\right)\right]du$$

$$= \frac{1}{2a}\int_{u=0}^{t} \left[\sin at + \sin a(2u-t)\right]du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos a\left(2u-t\right)\right]_{u=0}^{t}$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\cos at - 0 + \frac{1}{2a}\cos at\right]$$

$$= \frac{t\sin at}{2a}.$$ **Ans.**

Example 2

Apply convolution theorem to evaluate $L^{-1}\left[\dfrac{1}{(s+3)(s-1)}\right]$ [AU APR 2011].

Solution

Let $F(s) = \dfrac{1}{s+3} \ \Rightarrow\ L^{-1}\left(F(s)\right) = f(t) = e^{-3t}$

$G(s) = \dfrac{1}{s-1} \ \Rightarrow\ L^{-1}\left(G(s)\right) = g(t) = e^{t}$

By convolution theorem

$$L^{-1}\left[\frac{1}{(s+3)(s-1)}\right]=\int_{u=0}^{t}e^{-3u}e^{(t-u)}du=e^{t}\int_{u=0}^{t}e^{-3u}.e^{-u}.du$$

$$=e^{t}\int_{u=0}^{t}e^{-4u}.du\ =e^{t}\left(\frac{e^{-4u}}{-4}\right)_{u=0}^{t}$$

$$=\frac{1}{4}e^{t}\left(1-e^{-4t}\right).$$ **Ans.**

Example 3

Evaluate $L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right]$ by convolution theorem. [KU NOV 2011]

Solution

$$L^{-1}\left(\frac{1}{s^2+1}\right)=\sin t\ ;\ L^{-1}\left(\frac{1}{s^2+4}\right)=\frac{\sin 2t}{2}$$

$\therefore$ By convolution theorem, we get

$$L^{-1}\left[\frac{1}{s^2+1}.\frac{1}{s^2+4}\right]=\int_0^t \sin u.\frac{\sin 2(t-u)}{2}du$$

$$=\frac{1}{6}\int_0^t[\cos(3u-2t)-\cos(2t-u)]du$$

$$=\frac{1}{6}\left[\frac{\sin(3u-2t)}{3}-\frac{\sin(2t-u)}{-1}\right]_0^t$$

$$=\frac{1}{6}\left[\frac{1}{3}(\sin t-\sin 2t)+(\sin t-\sin 2t)\right]$$

$$=\frac{1}{6}\left[\frac{4}{3}\sin t.-\frac{4}{3}\sin 2t\right]$$

$$=\frac{2}{9}(\sin t-\sin 2t).$$ **Ans.**

Example 4

By using convolution theorem, find the inverse laplace transform of $\frac{1}{(s+1)(s+2)}$.

Solution

$$L^{-1}\left(\frac{1}{s+1}\right)=e^{-t}\ \ ;\ L^{-1}\left(\frac{1}{s+2}\right)=e^{-2t}$$

$\therefore$ By convolution theorem, we get

$$L^{-1}\left[\frac{1}{s+1}.\frac{1}{s+2}\right]=\int_0^t e^{-u}.e^{-2(t-u)}.du$$

$$=e^{-2t}\int_0^t e^{u}.du=e^{-2t}\left(e^{t}-1\right)=e^{-t}-e^{-2t}.$$ **Ans.**

2.5 Application to Differential Equations

The Laplace transform method of solving differential equations yields particular solutions with out the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter method and is especially useful for solving linear differential equations with constant coefficients and a few integral and intergo-differential equations.

Working procedure

1. Take the Laplace transform on both sides of the differential equation. Apply the formula and the given initial conditions.
2. Transpose the terms with minus signs to the right.
3. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.
4. Resolve this function of s into partial fractions and take the inverse transform on both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Note

(i) $L(y(t)) = \bar{y}(s)$

(ii) $L(y^n(t)) = s^n \bar{y}(s) - s^{n-1} y(0) - s^{n-2} y(0) - \ldots\ldots\ldots\ldots y^{n-1}(0)$.

Example 1

Solve the Differential equation $(D^2 + 4D + 3)y = e^{-t}$. Given $y = 1, \frac{dy}{dt} = 1$ at $t = 0$ using Laplace transforms. [AU NOV 2011]

Solution

Given differential equation is $y'' + 4y' + 3y = e^{-t}$, where $y' = \frac{dy}{dt}$

Taking Laplace transform on both sides,

$$s^2 \bar{y}(s) - s\,y(0) - y'(0) + 4[s\,\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow \quad (s^2 + 4s + 3)\bar{y}(s) - s(1) - 1 - 4 = \frac{1}{s+1}$$

$$\Rightarrow \quad (s^2 + 4s + 3)\bar{y}(s) = s + 5 + \frac{1}{s+1}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{s^2 + 6s + 6}{(s+1)(s^2 + 4s + 3)} = \frac{s^2 + 6s + 6}{(s+1)(s+1)(s+3)}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{s^2 + 6s + 6}{(s+1)^2 (s+3)} \qquad (1)$$

Consider $\frac{s^2 + 6s + 6}{(s+1)^2 (s+3)} = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

$$\Rightarrow \quad s^2 + 6s + 6 + A(s+1)^2 + B(s+3)(s+1) + C(s+3)$$

Put $s=-1 \quad \Rightarrow \quad c=\frac{1}{2}$

Put $s=-3 \quad \Rightarrow \quad A=-\frac{3}{4}$

Equating the coefficients of s^2,

$$1=A+B \quad \Rightarrow \quad B=1-A=1+\left(\frac{3}{4}\right)=\frac{7}{4}$$

$$\therefore \quad (1) \Rightarrow \bar{y}(s)=\frac{-(3/4)}{s+3}+\frac{(7/4)}{s+1}+\frac{(1/2)}{(s+1)^2}$$

Taking inverse transform on both sides,

$$L^{-1}(\bar{y}(s))=y(t)=L^{-1}\left[\frac{(-3/4)}{s+3}\right]+\frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$=-\frac{3}{4}e^{-3t}+\frac{7}{4}e^{-t}+\frac{1}{2}t\,e^{-t}.$$ **Ans.**

Example 2

Solve the equation $(D^2+4D+13)y=e^{-t}\sin t$, $y=0$ and $Dy=0$ at $t=0$, where $D=\frac{d}{dt}$. [AU JUNE 2009]

Solution

Given differential equation is $y''+4y'+13y=e^{-t}\sin t$.

Taking Laplace transforms and using the given initial conditions, we get

i.e., $(s^2+4s+13)\bar{y}(s)=\dfrac{1}{s^2+2s+2}$

$$\therefore \quad \bar{y}(s)=\frac{1}{(s^2+2s+2)(s^2+4s+13)}$$

$$=\frac{As+B}{s^2+2s+2}=\frac{Cs+D}{s^2+4s+13}$$

$$=\frac{1}{85}\left[\frac{-2s+7}{s^2+2s+2}+\frac{2s-3}{s^2+4s+13}\right]$$

$$=\frac{1}{85}\left[\frac{-2(s+1)+9}{(s+1)^2+1}+\frac{2(s+2)-7}{(s+2)^2+9}\right]$$

$$\therefore \quad y(t)=\frac{1}{85}\left[e^{-t}(-2\cos t+9\sin t)+e^{-2t}\left(2\cos 3t-\frac{7}{3}\sin 3t\right)\right].$$ **Ans.**

Example 3

Using Laplace transform, find the solution of the initial value problem $y''+9y=9u(t-3), y(0)=y'(0)=0$, where $u(t-3)$ is the unit step function.

Solution

Given $y''+9y=9u(t-3)$

Taking Laplace transform on both sides,

$$s^2\bar{y}(s)-s\,y(0)-y'(0)+9\bar{y}(s)=\frac{9e^{-3s}}{s} \qquad (1)$$

Putting the values of $y(0)=0$ and $y'(0)=0$ in (1), we get

$$s^2\bar{y}(s)+9\bar{y}(s)=\frac{9e^{-3s}}{s}$$

$$(s^2+9)\bar{y}(s)=\frac{9e^{-3s}}{s}$$

$$\bar{y}(s)=\frac{9e^{-3s}}{s(s^2+9)}$$

$$\Rightarrow \qquad y(t)=L^{-1}\left[\frac{9e^{-3s}}{s(s^2+9)}\right]$$

$$L^{-1}\left[\frac{3}{s^2+9}\right]=\sin 3t$$

and $\quad 3L^{-1}\left[\frac{3}{s(s^2+9)}\right] \;=3\int_0^t \sin 3t\,dt \quad =-(\cos 3t)_0^t \quad =1-\cos 3t$

$\therefore \quad y(t)=L^{-1}\left[\frac{9e^{-3s}}{s(s^2+9)}\right]$ gives

$y(t)=[1-\cos 3(t-3)]u(t-3)$. **Ans.**

Example 4

A resistance R in series with inductance L is connected with e.m.f $E(t)$. The current i is given by $L\frac{di}{dt}+Ri=E(t)$.

If the switch is connected at $t=0$ and disconnected at $t=a$, find the current i interms of t.

Solution

Conditions under which current i flows are $i=0$ at $t=0$,

$$E(t)=\begin{cases} E \quad , \quad 0<t<a \\ 0 \quad , \quad t>a \end{cases}$$

Given equation is $L\frac{di}{dt}+Ri=E(t)$ (1)

Taking Laplace transform of (1), we get.

$$L\left[s\bar{i}-i(0)\right]+R\bar{i}=\int_0^\infty e^{-st}E(t)dt$$

$$L(s\bar{i})+R\bar{i}=\int_0^\infty e^{-st}E(t)dt. \quad (\text{since } i(0)=0)$$

$$(Ls+R)\bar{i}=\int_0^\infty e^{-st}\,E.dt=\int_0^a e^{-st}\,Edt+\int_a^\infty e^{-st}(0)dt$$

$$=E\left[\frac{e^{-st}}{-s}\right]_0^a+0 \quad =\frac{E}{s}\left[1-e^{-as}\right] \quad =\frac{E}{s}-\frac{E}{s}e^{-as}$$

$$\bar{i}=\frac{E}{s(Ls+R)}-\frac{E\,e^{-as}}{s\,(Ls+R)}$$

On inversion, we obtain

$$i=L^{-1}\left[\frac{E}{s(Ls+R)}\right]-L^{-1}\left[\frac{E\,e^{-as}}{s(Ls+R)}\right] \qquad (2)$$

Consider $L^{-1}\left[\frac{E}{s(Ls+R)}\right]$

$$L^{-1}\left[\frac{E}{s(Ls+R)}\right]=\frac{E}{L}L^{-1}\left[\frac{1}{s\left(s+\frac{R}{L}\right)}\right]$$

$$=\frac{E}{L}.\frac{L}{R}.L^{-1}\left[\frac{1}{s}-\frac{1}{s+\frac{R}{L}}\right] \quad \text{(Resolving into partial fractions)}$$

$$=\frac{E}{R}\left[1-e^{-\frac{R}{L}.t}\right]$$

and $L^{-1}\left[\frac{E\,e^{-as}}{s(Ls+R)}\right]=\frac{E}{R}\left[1-e^{-\frac{R}{L}(t-a)}\right]u(t-a)$. (By second shifting theorem)

On substituting the values of the inverse transform in (2), we get.

$$i=\frac{E}{R}\left[1-e^{-\frac{R}{L}t}\right]-\frac{E}{R}\left[1-e^{-\frac{R}{L}(t-a)}\right]u(t-a)$$

Hence $i=\frac{E}{R}\left[1-e^{-\frac{R}{L}t}\right]$ for $0<t<a,[u(t-a)=0]$

$$i=\frac{E}{R}\left[1-e^{-\frac{R}{L}t}\right]-\frac{E}{R}\left[1-e^{-\frac{R}{L}(t-a)}\right] \quad [\text{for } t>a, u(t-a)=1$$

$$\therefore \quad i=\frac{E}{R}\left[e^{-\frac{R}{L}(t-a)}-e^{-\frac{R}{L}t}\right] \quad =\frac{E}{R}e^{-\frac{R}{L}t}\left[e^{\frac{Ra}{L}}-1\right].$$ **Ans.**

Example 5

Using Laplace transforms solve $y''+5y'+6y=2, y'(0)=0, y(0)=0$. [KU NOV 2010]

Solution

Given $y''+5y'+6y=2$

Taking Laplace transforms on both sides

$L(y''(t))+5L(y'(t))+6L(y(t))=L(2)$.

$s^2\overline{y}(s)-s\,y(0)-y'(0)+5[s\,\overline{y}(s)-y(0)]+6\overline{y}(s) \ =\frac{2}{s}$

Given $y(0)=0$ and $y'(0)=0$

$\therefore \quad s^2\overline{y}(s)+5s\overline{y}(s)+6\overline{y}(s)=\frac{2}{s}$

$(s^2+5s+6)\overline{y}(s)=\frac{2}{s}$

$\therefore \quad \overline{y}(s)=\frac{2}{s(s^2+5s+6)}$

i.e., $\overline{y}(s)=\frac{2}{s(s+2)(s+3)}$

$\therefore \quad y(t)=L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$

By using partial fraction,

$\frac{2}{s(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+3}$

$2=A(s+2)(s+3)+Bs(s+3)+Cs(s+2)$

Put $s=-2 \quad \Rightarrow \quad B=-1$

Put $s=-3 \quad \Rightarrow \quad C=\frac{2}{3}$

Put $s=0 \quad \Rightarrow \quad A=\frac{1}{3}$

$\therefore \quad L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]=L^{-1}\left[\frac{1}{3s}\right]-L^{-1}\left[\frac{1}{s+2}\right]+\frac{2}{3}L^{-1}\left[\frac{1}{s+3}\right]$

i.e., $y(t)=\frac{1}{3}-e^{-2t}+\frac{2}{3}e^{-3t}$. **Ans.**

Example 6

Solve $y''-3y'+2y=4t, \quad y(0)=1, \quad y'(0)=-1$ using Laplace transforms. [KU NOV 2011]

Solution

Given $y''-3y'+2y=4t$

Taking Laplace transforms on both sides, we get

$$L(y'')-3L(y')+2L(y)=4L(t)$$

$$s^2\bar{y}(s)-s\,y(0)-y'(0)-3\left[s\,\bar{y}(s)-y(0)\right]+2\bar{y}(s)=\frac{4}{s^2}$$

$$s^2\bar{y}(s)-s+1-3\left[s\,\bar{y}(s)-1\right]+2\bar{y}(s)=\frac{4}{s^2}$$

$$\left(s^2-3s+2\right)\bar{y}(s)-s+1+3=\frac{4}{s^2}$$

$$\left(s^2-3s+2\right)\bar{y}(s)-(s-4)=\frac{4}{s^2}$$

$$\left(s^2-3s+2\right)\bar{y}(s)=\frac{4}{s^2}+(s-4)$$

$$\Rightarrow\quad \bar{y}(s)=\frac{4}{s^2\left(s^2-3s+2\right)}+\frac{s-4}{\left(s^2-3s+2\right)}$$

$$\therefore\quad y(t)=L^{-1}\left[\frac{4}{s^2\left(s^2-3s+2\right)}\right]+L^{-1}\left[\frac{s-4}{s^2-3s+2}\right]$$

$$=L^{-1}\left[\frac{16s+18}{9s^2}+\frac{(-5s+19)}{9\left(s^2-3s+2\right)}\right]+L^{-1}\left[\frac{-2}{s-2}+\frac{3}{s-1}\right]$$

$$=\frac{16}{9}L^{-1}\left(\frac{1}{s}\right)+\frac{18}{9}L^{-1}\left(\frac{1}{s^2}\right)-\frac{5}{9}L^{-1}\left(\frac{s}{s^2-3s+2}\right)+\frac{19}{9}L^{-1}\left(\frac{1}{s^2-3s+2}\right)$$
$$+L^{-1}\left(\frac{-2}{s-2}\right)+3.L^{-1}\left(\frac{1}{s-1}\right)$$

$$=\frac{16}{9}+2t-\frac{5}{9}\left[L^{-1}\left(\frac{2}{s-2}\right)-L^{-1}\left(\frac{1}{s-1}\right)\right]+\frac{19}{9}\left[L^{-1}\left(\frac{1}{s-2}\right)-L^{-1}\left(\frac{1}{s-1}\right)\right]-2e^{2t}+3e^{t}$$

$$=\frac{16}{9}+2t-\frac{5}{9}\left[2e^{2t}-e^{t}\right]+\frac{19}{9}\left[e^{2t}-e^{t}\right]+\frac{19}{9}\left[e^{2t}-e^{t}\right]-2e^{2t}+3e^{t}$$

$$=\frac{16}{9}+2t+e^{2t}\left(-\frac{10}{9}+\frac{19}{9}-2\right)+e^{t}\left(\frac{5}{9}-\frac{19}{9}+3\right)$$

$$\therefore\quad y(t)=\frac{16}{9}+2t-e^{2t}+\frac{13}{9}e^{t}.$$ **Ans.**

EXERCISE

PART A

1. State the initial value theorem in Laplace transforms.
2. State the final value theorem in Laplace transforms.
3. Define the convolution product of two functions and prove that it is commutative.

4. State convolution theorem in Laplace transforms.
5. Verify initial value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$.

PART B

1. Obtain the inverse Laplace transform by convolution. (i) $\dfrac{s^2}{(s^2 + a^2)^2}$

(ii) $\dfrac{1}{(s^2+1)^3}$ (iii) $\dfrac{1}{s^2(s^2-a^2)}$ (iv) $\dfrac{s}{(s^2+4)(s^2+9)}$ (v) $\dfrac{10}{(s+1)(s^2+4)}$

(vi) $\dfrac{1}{s^2(s+1)^3}$ (vii) $\dfrac{s^2}{(s^2+4)^2}$ (viii) $\dfrac{1}{s(s^2+4)}$ (ix) $\dfrac{1}{s(s^2-a^2)}$ (x) $\dfrac{s^2}{s^4-a^4}$.

(**Ans.** (i) $\dfrac{1}{2}t\cos at + \dfrac{1}{2a}\sin at$ (ii) $\dfrac{1}{8}(3-t^2)\sin t - 3t\cos t$

(iii) $\dfrac{1}{a^3}[-at + \sin hat]$ (iv) $\dfrac{1}{5}[\cos 2t - \cos 5t]$ (v) $2e^{-t} + \sin 2t - 2\cos 2t$

(vi) $\dfrac{e^{-t}}{2}[t^2 + 4t + 6] + t - 3$ (vii) $\dfrac{1}{4}(\sin 2t + 2t\cos 2t)$ (viii) $\dfrac{1}{4}(1 - \cos 2t)$

(ix) $\dfrac{1}{a^2}(\cos h\,at - 1)$ (x) $\dfrac{1}{2a}(\sin h\,at + \sin at)$**)**.

2. Solve the following differential equations by Laplace transform.

(i) $\dfrac{d^2y}{dx^2} + y = 0,$ where $y = 1, \dfrac{dy}{dx} = 1$ at $x = 0$.

(ii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$ where $y = 2, \dfrac{dy}{dx} = -4$ at $x = 0$.

(iii) $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$ given $y = \dfrac{dy}{dx} = 0, \dfrac{d^2y}{dx^2} = 6$ at $x = 0$.

(iv) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 1 - 2x$ given $y = 0, \dfrac{dy}{dx} = 4$ at $x = 0$.

(v) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x + e^{2x}$ where $y = 1, \dfrac{dy}{dx} = -1$ at $x = 0$.

(**Ans.** (i) $y = \sin x + \cos x$ (ii) $y = e^{-x}(2\cos 2x - \sin 2x)$

(iii) $y = e^x - 3e^{-x} + 2e^{-2x}$ (iv) $y = e^x - e^{-2x} + x$

(v) $y = 3 + 2x + \dfrac{1}{2}e^{3x} - 2e^{2x} - \dfrac{1}{2}e^x$**)**

UNIT I

Questions	opt1	opt2	opt3	opt4	Answer
The sum of the main diagonal elements of a matrix is called------	trace of a matrix	quadratic form	eigen value	canonical form	trace of a matrix
The orthogonal transformation used to diagonalise the symmetric matrix A is----	$N^T AN$	$N^T A$	NAN^{-1}	NA	$N^T AN$
If $\lambda_1, \lambda_2, \lambda_3,\ldots\ldots\ldots \lambda_n$ are the eigen values of A ,then $k\lambda_1, k\lambda_2, k\lambda_3,\ldots\ldots\ldots, k\lambda_n$ are the eigen values of --------------	kA	kA^2	kA^{-1}	A^{-1}	kA
Diagonalisation of a matrix by orthogonal reduction is true only for a ----- matrix.	diagonal	triangular	real symmetric	scalar	real symmetric
If atleast one of the eigen values of A is zero, then det A = -----	0	1	10	5	0
det (A- λI) represents------	characteristic polynomial	characteristic equation	quadratic form	canonical form	characteristic polynomial
If λ_1, λ_2, λ_3,……… λ_n are the eigen values of A ,then 1/λ_1,1/ λ_2, 1/λ_3,……… 1/λ_n are the eigen values of --------	A^-1	A	A^n	2A	A^-1
If λ_1, λ_2, λ_3,……… λ_n are the eigen values of A ,then λ_1^p, λ_2^p, λ_3^p,……… λ_n^p are the eigen values of	A^-1	A^2	A^-p	A^p	A^p
The eigen values of a --------------------- matrix are its diagonal elements	diagonal	symmetric	skew-matrix	triangular	triangular
In an orthogonal transformation $N^T AN = D$, D refers to a ---------- matrix.	diagonal	orthogonal	symmetric	skew-symmetric	diagonal

In a modal matrix, the columns are the eigen vectors of----------	A^{-1}	A^2	A	adj A	A
If the eigen values of $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ are 0,3 & 15, then its nature is----------	positive definite	positive semidefinite	indefinite	negative definite	positive semidefinite
The elements of the matrix of the quadratic form $x_1^2 + 3x_2^2 + 4x_1x_2$ are ------------	$a_{11} = 1, a_{12} = 2, a_{21} = 2, a_{22} = 3$	$a_{11} = -1, a_{12} = -2, a_{21} = 2, a_{22} = 3$	$a_{11} = 1, a_{12} = 4, a_{21} = 4, a_{22} = 3$	$a_{11} = 1, a_{12} = 4, a_{21} = 3, a_{22} = 1$	$a_{11} = 1, a_{12} = 2, a_{21} = 2, a_{22} = 3$
If the sum of two eigen values and trace of a 3x3 matrix A are equal, then det A = ---------	λ_1 λ_2 λ_3	0	1	2	0
If 1,5 are the eigen values of a matrix A, then det A = -------	5	0	25	6	5
If the canonical form of a quadratic form is $5y_1^2 + 6y_2^2$, then the rank is --------	4	0	2	1	2
The eigen vector is also known as-------	latent value	latent vector	column value	orthogonal value	latent vector
If 1,3,7 are the eigen values of A, then the eigen values of 2A are ------------	1,3,7	1,9,21	2,6,14	1,9,49	2,6,14
If the eigen values of 2A are 2, 6, 8 then eigen values of A are __________	1,3,4	2,6,8	1,9,16	12,4,3	1,3,4
The number of positive terms in the canonical form is called the ________ of the quadratic form.	rank	index	Signature	indefinite	index

If all the eigenvalues of A are positive then it is called as_______	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Positive definite
If all the eigenvalues of A are negative then it is called as_______	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Negative definite
A homogeneous polynomial of the second degree in any number of variables is called the ______	characteristic polynomial	characteristic equation	quadratic form	canonical form	quadratic form
The Set of all eigen values of the matrix A is called the ___________ of A	rank	index	Signature	spectrum	spectrum
A Square matrix A and its transpose have _______ eigen values.	different	Same	Inverse	Transpose	Same
The sum of the __________ of a matrix A is equal to the sum of the principal diagonal elements of A.	characteristic polynomial	characteristic equation	eigen values	eigen vectors	eigen values
The product of the eigenvalues of a matrix A is equal to_________	Sum of main diagonal	Determinant of A	Sum of minors of Main diagonal	Sum of the cofactors of A	Determinant of A
The eigenvectors of a real symmetric are _______	equal	unequal	real	symmetric	real
If the eigen values of 2A are 2, 6, 8, then eigen values of A are _________	**1,3,4**	2,6,8	1,9,16	12,4,3	**1,3,4**

The eigen values of a triangular matrix are --------	main diagonal elements	first row elements	first column elements	last column element	main diagonal elements
The main diagonal elements of a triangular matrix are -----------	characteristic polynomial	characteristic equation	eigen values	eigen vectors	eigen values
The main diagonal elements are the eigen values of the -------matrix.	square	symmetric	non symmetric	triangular	triangular
If atleast one of the eigen values of A is zero, then det A = ___	0	1	10	5	0
If the eigen values of A are 2, 3, 4 then the eigen values of A^{-1} is	1/2 , 1/3, 1/4	2,3,4	-2,-3,-4	(-1/2,-1/3,-1/4)	1/2 , 1/3, 1/4
If the sum of two eigen values of matrix A are equal to the trace of the matrix, then the determinant of A is____________	1	2	0	3	0
Sum of the principal diagonal elements ________	product of eigen values	product of eigen vectors	sum of eigen values	product of eigen values	sum of eigen values
If 1 and 2 are the eigen values of a matrix A, then the eigen values of A^2 are _____	2,3	3,5	1,4	1,2	1,4
The eigen vector is also known as ______	latent square	column vector	row vector	latent vector	latent vector
If all the eigen values of a matrix are distinct, then the corresponding eigen vectors______	linearly dependent	unique	not unique	linearly independent	linearly independent
A matrix is called symmetric if and only if ----------	A=A^T	A=A^-1	A=-A^T	A=A	A=A^T

If a matrix A is equal to A^T then A is a -------- matrix.	symmetric	non symmetric	skew-symmetric	singular	symmetric
A matrix is called skew-symmetric if and only if ---------	A=A^T	A=A^-1	A=-A^T	A=A	A=-A^T
If a matrix A is equal to -A^T then A is a -------- matrix.	symmetric	non symmetric	skew-symmetric	singular	skew-symmetric
A matrix is called orthogonal if and only if -----	A^T=A^-1	A^T=-A^-1	A^T=A^-2	A^T=-A^-2	A^T=A^-1
A matrix is called ------------if and only if A^T=A^-1.	orthogonal	square	non symmetric	triangular	orthogonal
The equation det (A-λI) = 0 is used to find ----------	characteristic polynomial	characteristic equation	eigen values	eigen vectors	characteristic equation
If the characteristic equation of a matrix A is λ^2 – 2 = 0, then the eigen values are -------	2,2	(-2,-2)	(2^(1/2),-2^(1/2))	(2i,-2i)	(2^(1/2),-2^(1/2))
If 1,3,7 are the eigen values of A, then the eigen values of 2A are ------------	1,3,7	1,9,21	2,6,14	1,9,49	2,6,14
If 1,5 are the eigen values of a matrix A, then det A = -------	5	0	25	6	5
Eigen value of the characteristic equation λ^2-4 = 0 is	2, 4	2, -4	2, -2	2, 2	2,-2
Eigen value of the characteristic equation λ^3-6λ^2+11λ-6 = 0 is	1,2,3	1, -2,3	1,2,-3	1,-2,-3	1,2,3
Largest Eigen value of the characteristic equation λ^3-3λ^2+2λ = 0 is	1	0	2	4	2
Smallest Eigen value of the characteristic equation λ^3-7λ^2+36 = 0 is	-3	3	-2	6	-2

Sum of the principal diagonal elements =	product of eigen values	product of eigen vectors	sum of eigen values	sum of eigen vectors	sum of eigen values
Product of the eigen values =	(-|A|)	1/|A|	(-1/|A|)	|A|	|A|
A Square matrix A and its transpose have ________ eigen values.	different	Same	Inverse	Transpose	Same
If 1 and 2 are the eigen values of a 2X2 matrix A, then the eigen values of A^2 is	2, 4	3,4	5,6	1, 4	1, 4
If 1 and 2 are the eigen values of a 2X2 matrix A, then the eigen values of A^-1is	2,1/2	1,1/2	1,2	4,1/2	1,1/2
If a real symmetric matrix of order 2 has ---------then the matrix is a scalar matrix.	equal eigen vectors	different eigen vectors	equal eigen values	different eigen values	equal eigen values
If A and B are nxn matrices and B is a non singular matrix then A and B^-1AB have	same eigen vectors	different eigen vectors	same eigen values	different eigen values	same eigen values
Every square matrix satisfies its own ----------	characteristic polynomial	characteristic equation	orthogonal transformation	canonical form	characteristic equation
In a modal matrix, the columns are the -----------	eigen vectors of A	eigen vectors of adj A	eigen vectors of inverse ofA	eigen values of A	eigen vectors of A
Cayley -Hamilton theorem is used to find ------------	inverse and higher powers of A	eigen values	eigen vectors	quadratic form	inverse and higher powers of A
If the canonical form of a quadratic form is 5y12 - 6 y22 , then the index is --------	4	0	2	1	1

The non –singular linear transformation used to transform the quadratic form to canonical form is ----------	X= NTY	X= NY	Y= NX	NXA	X= NY
The eigen vector is also known as-------	latent value	latent vector	column value	orthogonal value	latent vector
The sum of the __________ of a matrix A is equal to the sum of the principal diagonal elements of A.	characteristic polynomial	characteristic equation	eigen values	eigen vectors	eigen values
The product of the eigenvalues of a matrix A is equal to_________	Sum of main diagonal	Determinant of A	Sum of minors of Main diagonal	Sum of the cofactors of A	Determinant of A
The eigenvectors of a real symmetric are _______	equal	unequal	real	symmetric	real
When the quadratic form is reduced to the canonical form, it will contain only r terms, if the _____ of A is r.	rank	index	Signature	spectrum	rank
The excess of the number of positive terms over the number of negative terms in the canonical form is called the ___________ of the quadratic form.	rank	index	Signature	spectrum	Signature
If all the eigen values of A are less than zero and atleast one eigen value is zero then the quadratic form is said to be ___________	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Negative semidefinite

If all the eigen values of A are greater than zero and atleast one eigen value is zero then the quadratic form is said to be ____________	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Positive semidefinite
If the quadratic form has both positive and negative terms then it is said to be ____________	Positive definite	Negative definite	Positive semidefinite	indefinite	indefinite

UNIT II

Questions	opt1	opt2	opt3	opt4	Answer
If ∇.**F**=0 then **F** is	irrotational	solenoidal	rotational	curl	solenoidal
If ∇×**F**=0 then **F** is	irrotational	solenoidal	rotational	curl	irrotational
Any motion in which the curl of the velocity vector is zero is said to be ___	irrotational	solenoidal	rotational	curl	irrotational
A function is said to be _______ if it associates a scalar with every point in space.	Scalar function	Vector function	Point function	vector point function	Scalar function
A variable quantity whose value at any point in a region of space depends upon the position of the point is called a ___	Scalar function	Vector function	Point function	vector point function	Point function
A function is said to be _______ if it associates with vector in every point in space.	Scalar function	Vector function	Point function	vector point function	Vector function
If the divergence of a flow is zero at all points then it is said to be ________	rotational	irrotational	solenoidal	conservative	solenoidal
______ gives the rate of outflow per unit volume at a point of the fluid.	curl **V**	div **V**	curl **V=0**	div **V=0**	div **V**
If div **V**=0 everywhere in some region R of space then **V** is called the _____ vector point function.	rotational	irrotational	solenoidal	conservative	solenoidal
_______ is a vector which measures the extent to which individual particles of the fluid are spnning or rotating.	curl **V**	div **V**	curl **V=0**	div **V=0**	curl **V**
div **F** is a ________ function.	point	vector	scalar	rotational	scalar
If curl **V**=0 then **V** is said to be an ____________.	rotational	irrotational	solenoidal	conservative	irrotational
If **r**=x**I**+y**J**+z**K** then div **r**=________	0	1	2	3	3
If **r**=x**I**+y**J**+z**K** then curl **r**=________	0	1	2	3	0

div (curl **V**)=	0	div **V**	curl **V**	**V**	0
curl (grad ϕ)=	**0**	div **V**	curl **V**	ϕ	**0**
Two surfaces are said to cut orthogonally at a point of intersection, if the respective normals at that point are ______.	parallel	perpendicular	equal	zero	perpendicular
A sufficiently small portion of a smooth surface is always _______	plane	smooth	twisted	orientable	orientable
A curve that is not plane is called a _______ curve.	plane	point	twisted	closed	twisted
Any integral which is to be evaluated over a surface is called a ___	Line integral	Volume integral	surface integral	closed integral	surface integral
When the circulation of **F** around every closed curve in a region vanishes, then **F** is said to be _______ in that region.	rotational	irrotational	solenoidal	conservative	irrotational
A force field **F** is said to be ____________ if it is derivable from a potential function ϕ such that **F =** grad ϕ.	rotational	irrotational	solenoidal	conservative	conservative
If **F** is ______ then cur **F**=0.	rotational	irrotational	solenoidal	conservative	conservative
If S has a unique normal at each of its points whose direction depends continuously on the point of S then the surface S is called a ______ surface.	Orientable	smooth	plane	twisted	smooth
_________ provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Green's Theorem
________ is also called the first fundamental theorem of integral vector calculus.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Green's Theorem

_________ transforms line integrals into surface integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Stoke's Theorem
_______ transforms surface integrals into a volume integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Gauss Theorem
_________ is stated as surface integral of the component of curl **F** along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function **F** taken along the closed curve C.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Stoke's Theorem
________ is stated as the surface integral of the normal component of a vector function **F** taken around a closed surface S is equal to the integral of the divergence of **F** taken over the volume V enclosed by the surface S.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	Gauss Theorem
If ∇ϕ is solenoidal, then ∇^2(ϕ)=	ϕ	1	0	-1	0
If (3x-2y+z)**I**+(4x+ay-z)**J**+(x-y-2z)**K** is solenoidal then a=	0	1	-1	2	-1
If ϕ=x+y+z-8 then grad ϕ is ____	**I**+**J**+**K**	**I**+**J**-**K**	**I**-**J**+**K**	0	**I**+**J**+**K**
If ϕ=x^2+y^2+z^2-8 then grad ϕ at(2,2,2) is ____	4**I**+4**J**+4**K**	4**I**+4**J**-4**K**	4**I**-4**J**+4**K**	0	4**I**+4**J**+4**K**
If ϕ =x^2+y^2+z^2-8 then grad ϕ at(2,0,2) is ____	4**I**+4**K**	4**J**+4**K**	4**I**+4**J**	0	4**I**+4**K**
If **F** = (x+2y+az)**I**+(bx-3y-z)**J**+(4x+cy+2z)**K** is irrotational, then the values of a,b and c are _______	a=2, b=4, c=-1	a=-1, b=2, c=4	a=4, b=2, c=1	a=4, b=2, c=-1	a=4, b=2, c=-1
If **F**= xy**I**-yz**J**-zx**K** then curl **F** =	x**I**+y**J**+z**K**	x**I**-y**J**-z**K**	y**I**+z**J**+x**K**	y**I**+z**J**-x**K**	y**I**+z**J**-x**K**
If **F**= xy**I**-yz**J**-zx**K** then div **F** =	x**I**+y**J**+z**K**	x**I**-y**J**-z**K**	y**I**-z**J**-x**K**	y**I**+z**J**-x**K**	y**I**-z**J**-x**K**

If **F**= xy**I**-yz**J**-zx**K** then at (1,1,1), div **F** =	**I**+**J**+**K**	**I**-**J**+**K**	**I**-**J**-**K**	**I**+**J**-**K**	**I**-**J**-**K**
If **F**= x^2-y^2+2z^2 then at (1,2,3), div **F** =	**2I**+4**J**+12**K**	**2I**-4**J**+12**K**	**2I-4J-6K**	**2I**+4**J**-12**K**	**2I**-4**J**+12**K**
div F is a _________ function.	point	vector	scalar	rotational	scalar
If curl V=0 then V is said to be an ____________.	rotational	irrotational	solenoidal	conservative	irrotational
If **F**= x^2+y^2+2z^2 then grad F at (2,0,2)is-----	4i+4k	4j+4k	4i+4j	0	4i+4k
If **F** is an irrotational vector, it is _______	rotational	irrotational	solenoidal	conservative	conservative
A _______ _curve that lies in a plane in space.	plane	point	twisted	closed	plane
If F is conservative then cur F=0 and there exists a scalar potential function ϕ such that _____	rotational	irrotational	solenoidal	conservative	**F** = grad ϕ.
Any integral which is to be evaluated along a curve is called a ___	Line integral	Volume integral	surface integral	closed integral	Line integral
Any integral which is to be evaluated over a volume is called a ___	Line integral	Volume integral	surface integral	closed integral	Volume integral
If F is conservative then cur F=0 and there exists a scalar potential function f such that ____	rotational	irrotational	solenoidal	conservative	F = grad f.
The integral of vector F.dr is -----.	line integral	zero	surface integral	one	line integral
The integral of vector F.dr is is conservative if the terminal points A and B _______	Coinside	split	different	deviate	Coinside
Greens theorem is called the _____theorem of integral vector calculus.	second fundamental	first fundamental	third fundamental	fourth fundamental	first fundamental
If del x F then vector F is ________	conservative	non conservative	curl	solenoidal	conservative

If a force moves a particle from one place to another place in any curve then integral of vector F.dr is called ---------- by that force.	work done	rest taken	conservative	displacement	work done
If a force--------a particle from one place to another place in any curve then integral of vector F.dr is called work done by that force.	moves	still	constant	idle	moves
If S is not smooth but can be divided into finitely many smooth portions then it is called a _______ surface.	Orientable	smooth	piecewise smooth	twisted	piecewise smooth
If F is an irrotational vector, it is _______	rotational	irrotational	solenoidal	conservative	conservative
A force field F is said to be _____________ if it is derivable from a potential function f such that F = grad f.	rotational	irrotational	solenoidal	conservative	conservative

UNIT III

S.No	Questions	Opt 1	Opt 2	Opt 3	Opt 4	Answer
1	An example of single valued function of z is ________.	w = z^2	w = z^(1/2)	w=SQRT(z)	w=z^-1	w = z^2
2	An example of multiple valued function of z is _______.	w = z^2	w = z^(1/2)	w=SQRT(z)	w=z^-1	w = z^(1/2)
3	The distance between two points z and z_0 is	$\|z-z_0\|$	$\|z+z_0\|$	z	z_0	**$\|z-z_0\|$**
4	A circle of radius 1 with centre at origin can be represented by _______.	|z|>1	|z| < 1	|z|=1	|z| = 0	|z| = 1
7	If f(z) is differentiable at z_0 then f(z) is ________ at z_0.	discontinuous	continuous	regular	irregular	**continuous**
8	A function is said to be _________ at a point if its derivative exists not only at point but also in some neighborhood of that point.	entire function	integral function	analytic	continuous	**analytic**
9	A function which is analytic everywhere in the finite plane is called ___________.	analytic function	holomorphic function	regular function	entire function	**entire function**
11	The necessary condition for f(z) to be analytic is_________	$u_x = v_y$ and $v_x = -u_y$	$u_x = -v_y$ and $v_x = u_y$	$u_x = v_y$ and $v_x = u_y$	$u_x = -v_y$ and $v_x = -u_y$	**$u_x = v_y$ and $v_x = -u_y$**
12	A real function of two variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called __________.	analytic function	regular function	holomorphic function	harmonic function	**harmonic function**
13	If u and v are harmonic functions such that u+iv is analytic then each is called the _________ of the other.	conjugate harmonic	analytic	entire function	not analytic	**conjugate harmonic**

14	A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is called _______ at that point.	Conformal	isogonal	entire function	unconformal	**Conformal**
15	A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be __________ at that point.	Conformal	isogonal	entire function	unconformal	**isogonal**
16	A mapping w = f(z) is said to be conformal at $z = z_0$ if __________.	$f'(z_0) = 0$	$f'(z_0) = f(z)$	$f'(z_0) \neq 0$	$f'(z_0) \neq f(z)$	**$f'(z_0) \neq 0$**
17	The point at which the mapping w = f(z) is not conformal, that is, f'(z) = 0 is called ________ of the mapping.	common	fixed	invariant	critical	**critical**
18	A _________ point of a mapping w = f(z) are points that are mapped onto themselves, are kept fixed under the mapping.	common	fixed	critical	variant	**fixed**
19	The transformation w = a+z where a is a complex constant, represents a __________.	translation	magnification	rotation	reflection	**translation**
20	The transformation _________ where a is a complex constant represents a translation.	w = az	w = az+b	w = a+z	w = 1/z	**w = a+z**
21	The transformation _________ where a is a real constant represents magnification.	w = a+z	w = 1/z	w = az+b	w = az	w = az
22	The transformation w = az where a is a real constant represents __________.	translation	magnification	reflection	inversion	**magnification**
23	In general linear transformation, w = az+b where a and b are complex constants represents _______.	magnification	rotation	translation	magnification, rotation and translation	**magnification, rotation and translation**
24	The transformation w=(az+b)/(cz+d), where a, b, c, d are complex numbers is called a ______.	Linear transformation	bilinear transformation	fractional transformation	translation	**bilinear transformation**
25	A bilinear transformation is also called a _______.	linear transformation	inversion	fractional transformation	linear fractional transformation	**linear fractional transformation**
26	The value of i =	SQRT(-1)	SQRT(1)	-1	1	**SQRT(-1)**
27	_________ represents the interior of the circle excluding its circumference.	$\|z - z_0\| >$ delta	$\|z - z_0\| <$ delta	$\|z - z_0\| \geq$ delta	$\|z - z_0\| \leq$ delta	**$\|z - z_0\| <$** delta
28	_________ represents the interior of the circle including its circumference.	$\|z - z_0\| >$ delta	$\|z - z_0\| <$ delta	$\|z - z_0\| \geq$ delta	$\|z - z_0\| \leq$ delta	**$\|z - z_0\| \leq$ delta**

29	_________ represents the exterior of the circle.	$\lvert z - z_0 \rvert >$ delta	$\lvert z - z_0 \rvert <$ delta	$\lvert z - z_0 \rvert \geq$ delta	$\lvert z - z_0 \rvert \leq$ delta	**$\lvert z - z_0 \rvert >$ delta**
30	Cauchy-Riemann equations are necessary conditions for a function w = f(z) to be an ______.	entire function	integral function	analytic function	continuous function	**analytic function**
31	Cauchy-Riemann equations are	$u_x = v_y$ and $v_x = -u_y$	$u_x = -v_y$ and $v_x = u_y$	$u_x = v_y$ and $v_x = u_y$	$u_x = -v_y$ and $v_x = -u_y$	**$u_x = v_y$ and $v_x = -u_y$**
32	The real and imaginary parts of an analytic function f(z) = u+iv satisfies the ______ equation in two dimensions.	Cauchy-Riemann	Homogeneous	Laplace	Euler	**Laplace**
33	An analytic function with a constant real part is ___________.	a variable	a constant	an analytic function	an entire function	**a constant**
34	An analytic function with a constant modulus is ___________.	a variable	a constant	an analytic function	an entire function	**a constant**
35	A fixed point is also called as _________.	invariant points	critical points	common point	origin	**invariant points**
36	The fixed point of w=(5z+4)/(z+5) is	2,1	1,-1	-2, 2	0, 1	**-2, 2**
37	The critical point of z=(2z+1)/(z+2) is	1, 1	1, -1	1,2	0,1	**1, -1**
38	Solutions of Laplace's equation are ________ under conformal transformation	common	fixed	invariant	critical	**invariant**
39	If f(z) is analytic, and f'(z)=0 everywhere, then f(z) is _____	a variable	a constant	an analytic function	an entire function	**a constant**
40	An analytic function with a constant imaginary part is __________.	a variable	a constant	an analytic function	an entire function	**a constant**
41	If u+iv is analytic, then v-iu is _______	entire function	integral function	analytic	continuous	**analytic**
44	w=z has every point as a _______ point	fixed	critical	invariant	common	**fixed**
45	w=1/z has _______ fixed points	1	2	3	4	**2**
46	w=z+b has ________ fixed points	0	1	2	3	**0**

UNIT IV

Questions	opt1	opt2	opt3	opt4	Answer
A curve is called a_____________ if it does not intersect itself	Simple closed curve	multiple curve	simply connected region	multiple connected region	Simple closed curve
A curve is called ________ if it is not a simple closed curve	connected region	multiple curve	simply connected region	multiple connected region	multiple curve
If f(z) is analytic in a simply connected domain D and C is any simple closed path then ∫(from c)f(z)dz =	1	2πi	0	πi	0
If f(z) is analytic inside on a simple closed curve C and a be any point inside C then ∫(from c)f(z)dz /(z-a)=	2πi f(a)	2πi	0	πi	2πi f(a)
The value of ∫(from c) [(3z^2+7z+1)/(z+1)] dz where C is |z| = 1/2 is	2πi	-6πi	πi	πi/2	-6πi
The value of ∫(from c) (cos πz/z-1) dz if C is |z| =2	2πi	-2πi	πi	πi/3	-2πi
The value of ∫(from c) (1/z-1) dz if C is |z| =2	2πi	3πi	πi	πi/4	2πi
The value of ∫(from c) (1/z-3) dz if C is |z| =1	3πi	πi	πi/4	0	0
The value of ∫(from c) (1/(z-3)^3) dz if C is |z| =2	3πi	πi	πi/5	0	0
The Taylor's series of f(z) about the point z=0 is called ___________series	Maclaurin's	Laurent's	Geometric	Arithmetic	Maclaurin's
The value of ∫(from c) (1/z+4) dz if C is |z| =3	3πi	πi	πi/4	0	0
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____ part	regular	principal	real	imaginary	regular
In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____ part	regular	principal	real	imaginary	principal
The poles of the function f(z) = z/((z-1)(z-2)) are at z = _______	1, 2	2,3	1,-1	3,4	1, 2
The poles of cotz are______	2nπ	nπ	3nπ	4nπ	nπ
The poles of the function f(z) = cos z/((z+3)(z-4)) are at z = ______	- 3, 4	2,3	1,-1	3,4	- 3, 4
The isolated singular point of f(z) = z/((z-4)(z-5))	1,2	2,3	0,2	4,5	4,5
The isolated singular point of f(z) = z/((z(z-3))	1,3	2,4	0,3	4,5	0,3
A simple pole is a pole of order _______	1	2	3	4	1
The order of the pole z= 2 for f(z) = z/((z+1)(z-2)^2)	1	2	3	4	2
Residue of (cosz / z) at z = 0 is	0	1	2	4	1

The residue at z = 0 of ((1 + e^z) / (zcosz+sinz)) is	0	1	2	4	1
The residue of f(z) = cot z at z= 0 is_____	0	1	2	4	1
The singularity of f(z) = z / ((z-3)^3) is______	0	1	2	3	3
A point z=a is said to be a _______point of f(z), if f(z) is not analytic at z=a	Singular	isolated singular	removable	essential singular	Singular
A point z=a is said to be a _______point of f(z), if f(z) is analytic except at z=a	Singular	isolated singular	removable	essential singular	isolated singular
In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____point	Singular	isolated singular	removable singular	essential singular	essential singular
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____point	Singular	isolated singular	removable singular	essential singular	removable singular
In contour integration, cos θ=______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz	(z^2+1)/2z
In contour integration, sin θ=______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz	(z^2-1)/2iz

Unit V					
Questions	opt1	opt2	opt3	opt4	Answer
The operator L that transforms f(t) into F(s) is called the -------- operator.	Fourier	Hankel	Laplace operator	Z	Laplace operator
The Laplace transform is said to exist if the integral is --------- for some value of s; otherwise it does not exist.	discontinuous	divergent	closed	convergent	convergent
and is of exponentialorder 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	unifromly continuous	piecewise continuous	convergent	divergent	piecewise continuous
is of exponentialorder 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	closed interval [0,1]	Half open interval [0,1)	infinite interval in (0,∞)	finite interval in (0,∞)	finite interval in (0,∞)
If f(t) is piecewise continuous on every finite interval in (0,∞) and is of ------- 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	exponential order	quadratic order	cubic order	n th order	exponential order
If f(t) is piecewise continuous on every finite interval in (0,∞) and is ofexponentialorder'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a. This condition is	necessary	non sufficient	Sufficient	both necessary and sufficient	Sufficient
L[1] =	n! / s^(n+1)	1/s , s > 0	1/(t+1)	1/ (s-a)	1/s , s > 0
L[t^n] =	2/(s-1)	n!	s^(n+1)	s^(n+1)	n! / s^(n+1)
L[e^(at)] =	1/ (s-a)	1/s , s > 0	s^(n+1)	a/(s-a)	1/ (s-a)
L[e^(-at)] =	F(s-a)	s^2 F(s)-s f(0)- f '(0)	1/ (s+a)	n! / s^(n+1)	1/ (s+a)
L[sinat]=	a/(s^2 +a^2)	1/(s^2 +a^2)	(s^2 +a^2)	a/(s^3+a^3)	a/(s^2 +a^2)
L[cosat]=	n! / s^(n+1)	s^(n+1)	t^(n+1)	s/(s^2 +a^2)	s/(s^2 +a^2)
L[coshat]=	s/(s^2 -a^2)	1/(s^3 - a^3)	s/(s^2 +a^2)	1/a F(s/a)	s/(s^2 -a^2)

L[af(t) + bg(t)]=	aF(s)+bG(s)	aF(s)-bG(s)	bF(s)-aG(s)	bF(s) * aG(s)	aF(s)+bG(s)
L[af(t) + bg(t)]= aF(s)+bG(s) is called ------property	quasi linear	non-linear	Linearity	homogenous	Linearity
Lineraity property is	L[af(t) + bg(t)]= aF(s) * bG(s)	L[af(t) + bg(t)]= aF(s)+bG(s)	1/a F(s/a)	L[af(t) + bg(t)]= aF(s)-bG(s)	L[af(t) + bg(t)]= aF(s)+bG(s)
If L[f(t)]=F(s) then L[e^at f(t)]=	aF(s)+bG(s)	F(s+a)	1-s	F(s-a)	F(s-a)
First Shifting property is if L[f(t)] = F(s) then ------	L[e^at f(t)]=F(s-a)	L[f(at)]= 1/a F(s/a)	s^2 F(s)-s f(0)- f '(0)	s^(n+1)	L[e^at f(t)]=F(s-a)
If L[f(t)]=F(s) then L[e^at f(t)]=F(s-a) is called ---- property	linear	convolution	First shifting property	non homogenous	First shifting property
If L[f(t)]= F(s) then L[f(at)]=1/a F(s/a) is called ________ property.	Change of scale	convolution	First shifting property	non homogenous	Change of scale
If L[f(t)]= F(s) then L[f(at)]=	F(s/a)	1/a F(s/a)	F(s-a)	a F(s/a)	1/a F(s/a)
______is called the change of scale property	L[f(at)]= t-1	L[f(at)]= 1/(s^3 - a^3)	L[f(at)]= 1/a F(s/a)	L[e^at f(t)]=F(s-a)	L[f(at)]= 1/a F(s/a)
Change of scale property is -----	L[f(at)]= 1/a F(s/a)	L[f(at)]= F(s/a)	L[f(at)]= F(a/s)	L[f(at)]= a F(s/a)	L[f(at)]= 1/a F(s/a)
If L[f(t)]= F(s) then L[f ' (t)] =	F(s)-f(0)	s F(s)-+(0)	s F(s)-f(0)	F(s)+f(0)	s F(s)-f(0)

If L[f(t)]= F(s) then L[f " (t)] =	s^2 F(s)-s f(0)	s^2 F(s)-s f(0)- f '(0)	s^2 F(s)-s f(0)+ f '(0)	s^2 F(s)+s f(0)+ f '(0)	s^2 F(s)-s f(0)- f '(0)
L[5 (t^3)] =	1	1/s , s > 0	3/ (s^4)	30/ (s^4)	30/ (s^4)
L[6 t] =	6	6/(s^2)	6/s	6-s	6/(s^2)
L[2 e ^ (-6 t)] =	2/(s+6)	2	2/(s-6)	2/s	2/(s+6)
L[7] =	7/s	1/s , s > 0	(-7/s)	7	7/s
L[10 sin2t]=	20/(s^2-4)	2/(s^2+4)	2/(s^2-4)	20/(s^2+4)	20/(s^2+4)
L[7 cosh3t]=	7s/(s^2-9)	7/(s^2-9)	s/(s^2-9)	7s/(s^2+9)	7s/(s^2-9)
The inverse laplace transform of 1/s is =	0	-1	s+a	1	1
The inverse laplace transform of 1/(s-a) is =	e^(-at)	1/e^(at)	e^(at)	1/e^(-at)	e^(at)
The inverse laplace transform of 1/(s+a) is =	e^(-at)	1/e^(at)	1/e^(-at)	e^(at)	e^(-at)
If L[f(t)]=F(s) then f(t) is called -------- laplace transform of F(s)	Linear	non-linear	inverse	quasi linear	inverse
If L is linear then -------- is Linear.	L+1	L^(-1)	1/L	(-1/L)	L^(-1)
If L is linear then L inverse is --------	non-linear	Linear	divergent	quasi linear	Linear
The convolution of f*g of f(t) and g(t) is defined as	(f*g)(t)=∫ (from 0 to t) f(u) g(t+u) du	(f*g)(t)=∫ (from 0 to t) f(u) du	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du	(f*g)(t)=∫ (from 0 to t) g(t-u) du	(f*g)(t)=∫ (from 0 to t) f(u) g(t-u) du
________is called the convolution theorem.	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du	(f*g)(t)=1-t	(f*g)(t)=e^(-at)	(f*g)(t)= L^(-1)(1)	from 0 to t f(u) g(t-u) du
A function f(t) is said to be -----with period T>0 if f(t+T)=f(t) for all t	even	projection	odd	peroidic	periodic
L[k] =	k/s	k/s , s > 0	(-1/s)	k	k/s

L[sinhat]=	a/(s^2 -a^2)	1/(s^3 - a^3)	a/(s^2 +a^2)	1/a F(s/a)	a/(s^2 -a^2)
L[e^(8t)] =	1/ (s-8)	1/s , s > 0	n! / s^(n+1)	8/(s-8)	1/ (s-8)