Semester-II

18BECE201 **Mathematics –II** **4H-4C**
(Differential Equations)

Instruction Hours/week: L:3 T:1 P:0 **Marks:** Internal**:40** External**:60** Total**:100**

End Semester Exam:3 Hours

Course Objectives

- The objective of this course is to make the students acquire sound knowledge and techniques in solving Ordinary differential equations, Partial differential equations and complex integration.

Course Outcomes

The students will learn:

1. To develop fundamentals and basic concepts in Ordinary differential equations, Partial differential equations and complex integration.
1 To solve problems related to engineering applications b using these techniques.

.

UNIT I - First order ordinary differential equations

Exact, linear and Bernoulli’s equations, Euler’s equations, Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and Clairaut’s type.

UNIT II - Ordinary differential equations of higher orders

Second order linear differential equations with variable coefficients, method of variation of parameters, Cauchy-Euler equation; Power series solutions; Legendre polynomials, Bessel functions of the first kind and their properties.

UNIT III - Partial Differential Equations

First order partial differential equations, solutions of first order linear and non-linear PDEs- Solution to homogenous and non-homogenous linear partial differential equations second and higher order by complimentary function and particular integral method.

UNIT IV - Partial Differential Equations

Flows, vibrations and diffusions, second-order linear equations and their classification, Initial and boundary conditions (with an informal description of well posed problems), D’Alemberts solution of wave equation. Boundary-value problems: Solution of boundary-value problems for various linear PDEs in various geometries.

UNIT V - Complex Integration

Contour integrals, Cauchy-Goursat theorem (without proof), Cauchy Integral formula (without proof), zeros of analytic functions, singularities, Taylor’s series, Laurent’s series, Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral involving sine and cosine.

SUGGESTED READINGS

1. Hemamalini. P.T**,** (2014),Engineering Mathematics**,** McGraw Hill Education (India) Private Limited, New Delhi.
2. G.B. Thomas and R.L. Finney**,** (2002),Calculus and Analytic geometry, 9th Edition,Pearson.
3. Erwin kreyszig**,** (2006),Advanced Engineering Mathematics, 9th Edition,John Wiley & Sons.
4. W. E. Boyce and R. C. DiPrima**,(2009),** Elementary Differential Equations and BoundaryValue Problems, 9th Edition**,** Wiley India.
5. S. L. Ross**, (1984),**Differential Equations, 3rd Ed.,**,** Wiley India.
6. Veerarajan T**,** (2008),Engineering Mathematics for first year,Tata McGraw-Hill, New Delhi.
7. E. A. Coddington,(1995), An Introduction to Ordinary Differential Equations, Prentice Hall India.
8. E. L. Ince**, (1958),**Ordinary Differential Equations**,** Dover Publications.
9. G.F. Simmons and S.G. Krantz,(2007),Differential Equations,Tata McGraw Hill.
10. S. J. Farlow, (1993),Partial Differential Equations for Scientists and Engineers, Dover Publications
11. R. Haberman, (1998),Elementary Applied Partial Differential equations with Fourier Series and Boundary Value, Problem4th Ed., Prentice Hall.
12. Ian Sneddon,(1964), Elements of Partial Differential Equations, McGraw Hill
13. J. W. Brown and R. V. Churchill,(2004),Complex Variables and Applications, 7th Ed., McGraw Hill.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

COIMBATORE-641 021

DEPARTMENT OF SCIENCE AND HUMANITIES

FACULTY OF ENGINEERING

I B.E CIVIL ENGINEERING

LECTURE PLAN

Subject : MATHEMATICS – II
(Differential Equations)

Code : 18BECE201

S.NO	Topics covered	No. of hours
	UNIT I First order ordinary differential equations	
1	Introduction of first order differential equations	1
2	Exact, linear and Bernoulli's equations	1
3	Exact, linear and Bernoulli's equations	1
4	Euler's equations	1
5	Tutorial 1 - Problems based on Exact, linear and Bernoulli's equations	1
6	Equations not of first degree:Equations solvable for p	1
7	Equations not of first degree:Equations solvable for p	1
8	Equations solvable for y	1
9	Equations solvable for y	1
10	Equations solvable for x	1
11	Equations solvable for x	1
12	Clairaut's type	1
13	Clairaut's type	1
14	Tutorial 2 - Problems based on Clairaut's type, Equations solving for x and y, p	1
	TOTAL	**14**
	UNIT II Ordinary differential equations of higher orders	
15	Introduction of ordinary differential equations	1
16	Second order linear differential equations with variable coefficients	1
17	Second order linear differential equations with variable coefficients	**1**
18	Second order linear differential equations with variable coefficients	1
19	Second order linear differential equations with variable coefficients	1
20	Second order linear differential equations with variable coefficients	1
21	Tutorial 3- Problems based on second order differential equations with variable coefficients	1
22	Method of variation of parameters	1
23	Cauchy-Euler equation	1
24	Power series solutions; Legendre polynomials	1
25	Power series solutions; Legendre polynomials	1
26	Bessel functions of the first kind and their properties	1
27	Bessel functions of the first kind and their properties	1
28	Tutorial 4 - Problems based on Bessel functions and Legendre polynomials	1
	TOTAL	**14**
	UNIT III Partial Differential Equations	
29	Introduction- of partial differential equations	1
30	First order partial differential equations	1
31	First order partial differential equations	1

32	solutions of first order linear and non-linear PDEs	1
33	solutions of first order linear and non-linear PDEs	1
34	solutions of first order linear and non-linear PDEs	1
35	Tutorial 5 - Problems based on solutions of first order linear and non-linear PDEs	1
36	Solution to homogenous and non-homogenous linear partial differential equations second and higher order by complimentary function and particular integral method	1
37	Solution of homogenous linear partial differential equations	1
38	Solution of homogenous linear partial differential equations	1
39	non-homogenous linear partial differential equations	1
40	non-homogenous linear partial differential equations	1
41	Solution of non-homogenous linear partial differential equations second and higher order	1
42	Tutorial 6 - Problems based on homogenous and non-homogenous linear partial differential equations	1
	TOTAL	**14**
	UNIT IV: Partial Differential Equations	
43	Introduction – Flows, vibrations and diffusions	1
44	second-order linear equations and their classification	1
45	second-order linear equations and their classification	1
46	Initial and boundary conditions (with an informal description of well posed problems),	1
47	Initial and boundary conditions (with an informal description of well posed problems),	1
48	Tutorial 7- Initial and boundary conditions (with an informal description of well posed problems)	1
49	D'Alemberts solution of wave equation	1
50	D'Alemberts solution of wave equation	1
51	Boundary-value problems	1
52	Boundary-value problems	1
53	Solution of boundary-value problems for various linear PDEs in various geometries.	1
54	Solution of boundary-value problems for various linear PDEs in various geometries.	1
55	Solution of boundary-value problems for various linear PDEs in various geometries.	1
56	Tutorial 8 - Solution of boundary-value problems for various linear PDEs in various geometries.	1
	TOTAL	**14**
	UNIT V Complex Integration	
57	Introduction - Complex Integration, Line integral	1
58	Problems solving using Cauchy's integral theorem	1
59	Problems solving using Cauchy's integral formula	1
60	Taylor's Series Problems	1
61	Taylor's Series Problems	1
62	Laurent series problems	1
63	Laurent series problems	1
64	Tutorial 9 - Taylor's and Laurent's series problems	1
65	Theory of Residues	1
66	Cauchy's residue theorem	1
67	Applications of Residue theorem to evaluate real integrals.	1

68	Applications of Residue theorem to evaluate real integrals.	1
69	Use of circular contour and semicircular contour with no pole on real axis.	1
70	Tutorial 10 - Cauchy’s residue theorem, Applications	1
	TOTAL	**14**
	GRAND TOTAL	**70**

Staff- Incharge **HoD**

Mathematics - II

[Differential equations]

UNIT : I

[First order Ordinary differential Equations]

Differential equation:

* A differential equation is an equation which involves differential co-efficients.

Ordinary differential equations: (O.D.E).

* An ordinary differential equation is that in which all the differential co-efficients has a single independent variable.

Ex: $\frac{dy}{dx} = 2x.$

Partial differential equations: (P.D.E)

* A Partial differential equation is that in which there are two or more independent variable.

Ex: $x\frac{du}{dx} + y\frac{du}{dy} = 2u.$

Exact differential equation:

* A differential equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if its left hand number is the exact differential of some function $u(x,y)$.

i.e) $du = M\,dx + N\,dy = 0$.

$\therefore$ The solution is $u(x,y) = c$

Theorem :

* The Necessary and Sufficient condition for that differential equation $M\,dx + N\,dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Necessary condition:

* The equation $M\,dx + N\,dy = 0$ will be exact if $M\,dx + N\,dy = du$. where 'u' is the some function of x and y.

* $\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Sufficient condition:

* If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = 0$ is exact

Methods of solution:

* The equation $Mdx + Ndy = 0$ becomes

$d[u + \int f(y)\,dy] = 0$, Integrating $\int d[u + \int f(y)\,dy] = 0$,

$\because$ ~~The solution~~ $u + \int f(y)\,dy = 0$.

$$u = \int_{y\text{ constant}} M\,dx$$

$f(y)$ = terms of N not containing x.

$\therefore$ The solution of $Mdx + Ndy = 0$ is $\int_{(y\text{ constant})} M\,dx + \int \begin{pmatrix}\text{term of } N \\ \text{not containing } x\end{pmatrix} dy = 0$

Provided $\dfrac{\partial u}{\partial y} = \dfrac{\partial N}{\partial x}$

Example: 1

Solve $\underbrace{(y^2 e^{xy^2} + 4x^3)\,dx}_{M\,dx} + \underbrace{(2xy\,e^{xy^2} - 3y^2)\,dy}_{N\,dy} = 0$.

Here,

$M = y^2 e^{xy^2} + 4x^2$; $N = 2xy\,e^{xy^2} - 3y^2$

$\dfrac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2 \cdot e^{xy^2} \cdot 2xy$

$= 2ye^{xy^2} + 2xy^3 \cdot e^{xy^2}$

$\dfrac{\partial N}{\partial x} = 2y\,(e^{xy^2} + x \cdot e^{xy^2} \cdot y^2)$

$= 2y\,e^{xy^2} + 2xy^3 e^{xy^2}$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ constant})} M\,dx + \int \begin{pmatrix}\text{terms of } N \text{ not} \\ \text{containing } x\end{pmatrix} dy = c$$

$$\int (y^2 e^{xy^2} + 4x^3)\,dx + \int (-3y^2)\,dy = c$$

$$\int y^2 e^{xy^2} dx + \int 4x^3 dx - \int 3y^2 dy = c$$

$$\cancel{y^2} \cdot \frac{e^{xy^2}}{\cancel{y^2}} + \cancel{4}\frac{x^4}{\cancel{4}} - \cancel{3}\frac{y^3}{\cancel{3}} = c$$

$$\boxed{e^{xy^2} + x^4 - y^3 = c.}$$

— x —

Solve $\underbrace{\left[y\left(1+\frac{1}{x}\right) + \cos y\right] dx}_{M\,dx} + \underbrace{\left[x + \log x - x\sin y\right] dy}_{N\,dy} = 0.$

Soln:

$$M = y\left(1+\frac{1}{x}\right) + \cos y \quad ; \quad N = x + \log x - x\sin y.$$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y \quad ; \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

The equation is exact and its solution is

$$\underset{(y\text{ constant})}{\int M\,dx} + \int \left(\text{terms of } N \text{ not containing } x\right) dy = c$$

$$\int \left(y\left(1+\frac{1}{x}\right) + \cos y\right) dx + \int (0)\,dy = c$$

$$y\left[\int dx + \int \frac{1}{x}\,dx\right) + \int \cos y\,dx\right]$$

$$\boxed{y\,[x + \log x] + \cos y \cdot x = c}$$

— x —

Solve $(1+2xy\cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)\,dy = 0$.

$M\,dx \qquad N\,dy$

Soln:

$$M = 1+2xy\cos x^2 - 2xy \; ; \quad N = \sin x^2 - x^2$$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x \; ; \quad \frac{\partial N}{\partial x} = \cos x^2 \cdot 2x - 2x.$$

$$= 2x\cos x^2 - 2x.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is exact and its solution is

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

(y constant)

$$\int (1+2xy\cos x^2 - 2xy)\,dx + \int (0)\,dy = c$$

$$\int dx + y\int \cos x^2\, 2x\,dx - \int 2xy\,dx = c$$

$$x + y\int d(\sin x^2) - \cancel{2}y\frac{x^2}{\cancel{2}} = c$$

$$\boxed{x + y\sin x^2 - yx^2 = c}$$

—x—

Solve $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$$\frac{(\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx}{(\sin x + x\cos y + x)\,dx} = 0$$

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0.$$

$M\,dx \qquad N\,dy$

$M = y\cos x + \sin y + y \qquad N = \sin x + x\cos y + x$

$\partial M/\partial y = \cos x + \cos y + 1 \qquad \partial N/\partial x = \cos x + \cos y + 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\underset{(y \text{ constant})}{\int M\,dx} + \int (\text{terms of } N \text{ not containing } x)\,dy = \cancel{x}\,c$$

$$\int (y\cos x + \sin y + y)\,dx + \int (0)\,dy = c$$

$$y\sin x + (\sin y + y)x = c$$

$$\boxed{y\sin x + x\sin y + xy = c}$$

—— X ——

Linear equation:

* A differential equation is ~~saind~~ said to be linear if it dependent variable and its differential co-efficient acquire only in the first degree

* The Standard form of a Linear equation of the first order, commonly known as Leibnitz's linear equation.

$$\frac{dy}{dx} + Py = Q$$

where, P, Q are the functions of x

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$$

$$I.F = e^{\int P\,dx}$$

$$\boxed{y(I.F) = \int Q(I.F)\,dx + c}$$

Solve the linear equation $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

Soln:-

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2.$$

$\div$ by $(x+1)$

$$\frac{dy}{dx} - \frac{y}{(x+1)} = e^{3x}(x+1)$$

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{-1}{x+1} \quad ; \quad Q = e^{3x}(x+1)$$

$$I.F = e^{\int P dx} = e^{\int\left(\frac{-1}{x+1}\right)dx}$$

$$= e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)}$$

$$= e^{\log(x+1)^{-1}}$$

$$= (x+1)^{-1} = \frac{1}{x+1}$$

$$\boxed{I.F = \frac{1}{x+1}}$$

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$y e^{\int P dx} = \int Q e^{\int P dx}\, dx + c$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x}\cancel{(x+1)}\frac{1}{\cancel{(x+1)}}\,dx + c$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x}\,dx + c$$

$$y\left(\frac{1}{x+1}\right) = \frac{e^{3x}}{3} + c$$

$$\boxed{y = \left(\frac{e^{3x}}{3} + c\right)(x+1)}$$

Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

$$\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

which is Leibnitz's linear equation.

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{\sqrt{x}} \;;\; Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2} dx = \frac{x^{-1/2+1}}{-1/2+1} = \frac{x^{1/2}}{1/2} = 2x^{1/2} = 2\sqrt{x}.$$

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$= \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}} dx + c$$

$$= \int \frac{1}{\sqrt{x}} e^{0} dx + c$$

$$= \int \frac{dx}{\sqrt{x}} \cdot 1 + c$$

$$\boxed{y e^{2\sqrt{x}} = 2\sqrt{x} + c}$$

— x —

Solve

$$\frac{dx}{dy} + Px = Q$$

$P, Q \rightarrow$ functions of y.

$$I.F = e^{\int P dy}$$

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + c.$$

Solve $(y \log y)\, dx + (x - \log y)\, dy = 0$

Soln:

$$y \log y \, dx = -(x - \log y)\, dy$$

$$\frac{dx}{dy} = \frac{\log y - x}{y \log y} = \frac{1}{y} \frac{(\log y - x)}{\log y}$$

$$= \frac{1}{y}\left[1 - \frac{x}{\log y}\right].$$

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$\frac{dx}{dy} + Px = Q$$

$$P = \frac{1}{y \log y} \quad ; \quad Q = \frac{1}{y}.$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{y \log y} dy}$$

$$= e^{\int \frac{1/y \, dy}{\log y}} = e^{\log(\log y)}$$

$$= \log y.$$

$$x e^{\int P dy} = \int Q \cdot e^{\int P dy} dy + c.$$

$$x \log y = \int \frac{1}{y} \log y \, dy + c$$

$\log y = t$

$\frac{1}{y} dy = dt.$

$$= \int t' \, dt + c$$

$$x \log y = \frac{t^2}{2} + c$$

$$\boxed{x \log y = \frac{1}{2} (\log y)^2 + c}$$

$$\boxed{x = \frac{1}{2} \log y + c (\log y)^{-1}.}$$

—— x ——

Solve: $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$.

Soln:

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\frac{dx}{dy} + Px = Q$$

$$P = \frac{1}{1+y^2} \; ; \; Q = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$$x\,e^{\int P\,dy} = \int Q \cdot e^{\int P\,dy}\,dy + c$$

$$x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}\, e^{\tan^{-1}y}\,dy + c$$

$$= \int t\,e^t \cdot dt + c$$

$$= t\,e^t - \int e^t\,dt + c$$

$$= t\,e^t - e^t + c$$

$$= e^t(t-1) + c$$

$\int u\,dv = uv - \int v\,du$

$u = t$, $dv = e^t$

$du = dt$ $\quad v = \int e^t dt = e^t$

$u - D$

$v - I$

$$\boxed{x\,e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c}$$

Put $t = \tan^{-1}y$

$dt = \frac{1}{1+y^2}\,dy$

$x = (\tan^{-1}y - 1) + c\,e^{-\tan^{-1}y}$

Bernoulli's Equation:

$$\frac{dy}{dx} + Py = Qy^n \rightarrow (1)$$

To solve (1)

$(\div)$ both sides by y^n

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Put $y^{1-n} = z$.

$$(1-n)\, y^{\cancel{1}-n-\cancel{1}}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

(or) $$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

which is Leibnitz's linear in z & can be solved easily.

—x—

Solve $$\frac{x\,dy}{dx} + y = x^3y^6.$$

Soln:

$\div$ by x

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^6$$

$\div$ by y^6 $$y^{-6}\frac{dy}{dx} + \frac{y}{x\cdot y^6} = x^2$$

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \rightarrow (1)$$

Put $z = y^{-5}$

$$\frac{dz}{dx} = -5y^{-6}\frac{dy}{dx} \Rightarrow y^{-6}\frac{dy}{dx} = \frac{1}{-5}\frac{dz}{dx}$$

sub $\frac{dy}{dx}$ in (1)

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$$

$\div$ by $\left(-\frac{1}{5}\right)$ $\frac{dz}{dx} + \frac{z/x}{-1/5} = \frac{x^2}{-1/5}$

$$\frac{dz}{dx} - \frac{5}{x} z = -5x^2$$

which is Leibnitz's linear equation in z

$$\frac{dz}{dx} + Pz = Q$$

$$P = -5/x \; ; \; Q = -5x^2.$$

$$I.F = e^{\int P dx} = e^{\int -5/x \, dx} = e^{-5\int \frac{dx}{x}} = e^{-5\log x}$$

$$= e^{\log x^{-5}} = x^{-5}$$

$$z e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$$

$$z x^{-5} = \int -5x^2 \cdot x^{-5} dx + c$$

$$z x^{-5} = -5 \int x^{-3} dx + c$$

$$= -5\left(\frac{x^{-3+1}}{-3+1}\right) + c$$

$$z x^{-5} = -5 \frac{x^{-2}}{-2} + c$$

$$y^{-5} x^{-5} = \frac{5}{2} x^{-2} + c$$

$(\div)$ by $y^{-5} x^{-5}$

$$1 = \frac{5}{2} \frac{x^{-2} + c}{x^{-5} y^{-5}}$$

$$1 = \left(\frac{5}{2} + c x^2\right) x^3 y^5.$$

— x —

Solve $xy(1+xy^2)\dfrac{dy}{dx} = 1$

Soln:

$$xy(1+xy^2) = \frac{dx}{dy}$$

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\frac{dx}{dy} - xy = x^2y^3.$$

$\div$ by x^2

$$x^{-2}\frac{dx}{dy} - \frac{xy}{x^{\cancel{2}}} = \frac{\cancel{x^2}y^3}{\cancel{x^2}}$$

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \rightarrow (1)$$

Put $x^{-1} = z$

$$-1x^{-1-1}\frac{dx}{dy} = \frac{dz}{dy}$$

$$-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$$

(1) becomes,

$$-\frac{dz}{dy} - yz = y^3$$

$$\frac{dz}{dy} + yz = -y^3 \rightarrow (2)$$

which is Leibnitz's linear equation in z

$$\frac{dz}{dy} + Pz = Q\cancel{y}.$$

Here $P=y$

$$I.F = e^{\int P dy} = e^{\int y dy} = e^{y^2/2}$$

$\therefore$ The solution is $z(I.F) = \int Q(I.F)\,dy + c$

Scanned by CamScanner

$$z\, e^{y^2/2} = \int (-y^3)\, e^{y^2/2}\, dy + c$$

$$= -\int y^2\, e^{y^2/2}\, y\, dy + c$$

$$= -\int 2t\, e^t\, dt + c$$

$$= -2\int t\, e^t\, dt + c$$

$$= -2\left[t\, e^t - \int e^t\, dt\right] + c$$

$$= -2\left[t\, e^t - e^t\right] + c$$

$$= -2\,(t-1)\, e^t + c$$

$$= -2\left[\frac{y^2}{2} - 1\right] e^{y^2/2} + c$$

$$z = (-y^2+2)\, e^{y^2/2} + c\, e^{-y^2/2}$$

$$\frac{1}{x} = (2-y^2)\, e^{y^2/2} + c\, e^{-y^2/2}$$

$\frac{y^2}{2} = t$

$y^2 = 2t$

$\int u\,dv = uv - \int v\,du$

$u = t \;;\; dv = e^t$

$du = 1 \;;\; v = e^t$

Put $\frac{y^2}{2} = t$

$\frac{\cancel{2} y\,dy}{\cancel{2}} = dt$

— x —.

solve $\quad \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$

$\div$ by $\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{2x \sin y \cancel{\cos y}}{\cos^{\cancel{2}} y} = \frac{x^3 \cancel{\cos^2 y}}{\cancel{\cos^2 y}}$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \rightarrow (1)$$

Put $\tan y = z$

$$\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (1) becomes

$$\frac{dz}{dx} + 2x z = x^3 \rightarrow (2)$$

$$\frac{dz}{dx} + Pz = Q$$

which is Leibnitz's linear equation in z.

Put $P = 2x$; $Q = x^3$

$$I.F = e^{\int P dx} = e^{\int 2x dx} = e^{\cancel{2}x^2/\cancel{2}} = e^{x^2}$$

The solution is

$$z(I.F) = \int Q(I.F) dx + c$$

$$z e^{x^2} = \int x^3 e^{x^2} dx + c$$

$$z e^{x^2} = \int x^2 x e^{x^2} dx + c$$

$$= \int \cancel{t} \underset{u}{t} \underset{dv}{e^t} dt + c$$

$$= \frac{1}{2} [t e^t - \int e^t] + c$$

$$= \frac{1}{2} [t e^t - e^t] + c$$

$$= \frac{1}{2} (t-1) e^t + c$$

$u = t$; $dv = e^t$

$du = 1$; $v = e^t$

$uv - \int v du$.

Put $x^2 = t$

$2x dx = dt$

$x dx = \frac{dt}{2}$.

$$z e^{x^2} = \frac{1}{2} [x^2 - 1] e^{x^2} + c$$

$\div$ by e^{x^2}

$$z = \frac{1}{2} (x^2 - 1) + c e^{-x^2}$$

— x —

Equations of first order and higher degree.

* The general form of the differential equation of the first ~~degree~~ order and n^{th} degree.

$$\left(\frac{dy}{dx}\right)^n + f_1(x,y) \left(\frac{dy}{dx}\right)^{n-1} + f_2(x,y) \left(\frac{dy}{dx}\right)^{n-2} + \dots + f_{n-1}(x,y) \left(\frac{dy}{dx}\right) + f_n(x,y) = 0$$

If $\frac{dy}{dx} = P$

$$p^n + f_1(x,y)p^{n-1} + f_2(x,y)p^{n-2} + \dots + f_{n-1}(x,y)P + f_n(x,y) = 0 \quad \text{①}$$

Since equation ① is the first order its general solution will contain only one orbitary constant

To solve ① is to be identified as an equation any one of the types

* Solvable for P
* Solvable for y
* Solvable for x
* ~~Solvable~~ clairaut's form.

* A differential equation of the first order but of nth degree is of the form

$$p^n + f_1(x,y)p^{n-1} + f_2(x,y)p^{n-2} + \dots + f_{n-1}(x,y)P + f_n(x,y) = 0$$

L.H.S of ① can be resolved in n linear factors

then ① becomes

$$(P-F_1)(P-F_2)\dots\dots(P-F_n) = 0$$

$$P = F_1, \quad P = F_2, \quad P = F_n.$$

$$\phi_1(x,y,c) = 0 \; ; \; \phi_2 = (x,y,c) = 0, \; \dots, \; \phi_n(x,y,c) = 0$$

The general solution is obtained.

$\phi_1(x,y,c)\ \phi_2(x,y,c) \cdots \phi_n(x,y,c) = 0.$

— x —

solve $\left(\frac{dy}{dx}\right)^2 - 6\left(\frac{dy}{dx}\right) + 8 = 0.$

Soln:

Put $\frac{dy}{dx} = P$

The given equation is $P^2 - 6P + 8 = 0.$

$(P-4)(P-2) = 0$

8

-4 -2

$P = 4$ (or) $P = 2$

$\frac{dy}{dx} = 4$ (or) $\frac{dy}{dx} = 2$

Integrating

$\int dy = \int 4dx$

$y = 4x + c$

$(y - 4x - c) = 0$

Integrating

$\int dy = \int 2dx$

$y = 2x + c$

$(y - 2x - c) = 0$

$(y - 4x - c)(y - 2x - c) = 0$

— x —.

solve

$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Put $\frac{dy}{dx} = P$

$P - \frac{1}{P} = \frac{x}{y} - \frac{y}{x}$

$\frac{P^2 - 1}{P} = \frac{x}{y} - \frac{y}{x}.$

$P^2 - 1 = P\left(\frac{x}{y} - \frac{y}{x}\right).$

$$p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$$

$$\left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$$

-1

$y/x \quad -x/y$

$p + \frac{y}{x} = 0$ (or) $p - \frac{x}{y} = 0$

$p = \frac{-y}{x}$ (or) $p = \frac{x}{y}$

$\frac{dy}{dx} = \frac{-y}{x}$ (or) $\frac{dy}{dx} = \frac{x}{y}$

$x\,dy = -y\,dx$ (or) $y\,dy = x\,dx$

$x\,dy + y\,dx = 0$ $\qquad$ $x\,dx - y\,dy = 0$

Integrating $\qquad$ Integrating

$\int d(x,y) = 0$ $\qquad$ $\int(x\,dx - y\,dy) = 0$

$xy = c$ $\qquad$ $\frac{x^2}{2} - \frac{y^2}{2} = c$

$(xy - c) = 0$; $\qquad$ $x^2 - y^2 - c = 0.$

$$(xy - c)(x^2 - y^2 - c) = 0.$$

—x—.

Solve $p^2 + 2py\cot x = y^2$

(4) $p^2 + 2py\cot x - y^2 = 0$

v.Imp

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$; $b = 2y\cot x$; $c = -y^2$

$$p = \frac{-2y\cot x \pm \sqrt{(2y\cot x)^2 - 4(1)(-y^2)}}{2(1)}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm 2\sqrt{y^2\cot^2 x + y^2}}{2}$$

$$= \frac{\cancel{2}(-y\cot x \pm \sqrt{y^2\cot^2 x + y^2})}{\cancel{2}}$$

$$= -y\cot x \pm \sqrt{y^2\cot^2 x + y^2}$$

$$p = -y\cot x \pm \sqrt{y^2(1 + \cot^2 x)}$$

$$= -y\cot x \pm y\sqrt{\operatorname{cosec}^2 x}$$

$$\frac{dy}{dx} = -y\cot x \pm y\operatorname{cosec} x$$

$$= y\operatorname{cosec} x - y\cot x$$

$$\frac{dy}{dx} = y(\operatorname{cosec} x - \cot x)$$

$$= y\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$\sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\frac{dy}{dx} = y\left(\frac{1 - \cos x}{\sin x}\right) = y\left(\frac{\cancel{2}\sin^{\cancel{2}} x/2}{\cancel{2}\cancel{\sin\frac{x}{2}}\cos\frac{x}{2}}\right)$$

$$\frac{dy}{dx} = y \tan\frac{x}{2}$$

$\int \sin ax = -\frac{\cos ax}{a}$

$$\frac{dy}{y} = \tan\frac{x}{2}\, dx$$

$\int \tan x = \log(\sec x)$.

$$\int dy/y = \int \tan\frac{x}{2}\, dx$$

$$\log y = \frac{\log \sec\left(\frac{x}{2}\right)}{\frac{1}{2}} + \log c$$

$$\log y = 2 \log \sec\left(\frac{x}{2}\right) + \log c$$

$$= \log \sec^2\frac{x}{2} + \log c$$

$$\log y = \log\left(c \sec^2\frac{x}{2}\right)$$

$$y = c\sec^2\frac{x}{2} = c\,\frac{1 \cdot 2}{1+\cos x}$$

$$y(1+\cos x) = 2c$$

$\int \tan x = \log \sec x$

$\frac{d}{dx}(\log x) = \frac{1}{x}$

$\log x = \int \frac{1}{x}\, dx$.

$1+\cos x = 2\cos^2\frac{x}{2}$

$\log ab = \log a + \log b$

— x —

Type 1:

Integrating faction

equation reducable to exact equation

* ~~Differential~~ Differential equation which are not exact

k can sometime we made exact after multiplying by a suitable ~~k~~ $[f(x,y)]$ called the integrating factor.

* Integrating factor found by Inspection.

Example:

Solve $y\,dx - x\,dy = 0$

$$y\,dx - x\,dy \rightarrow (1)$$

$$M\,dx - N\,dy = 0$$

$$M = y \; ; \; N = -x.$$

$$\frac{\partial M}{\partial y} = 1 \; ; \; \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Multiplying (1) by $\frac{1}{y^2}$

$$\frac{y\,dx - x\,dy}{y^2} = 0$$

$$d\left(\frac{x}{y}\right) = 0.$$

which is exact

$$(1) \times \frac{1}{x^2}, \quad \frac{y\,dx - x\,dy}{x^2} = 0$$

$$d\left(y/x\right) = 0$$

Multiply (1) by $1/xy$,

$$\frac{y\,dx - x\,dy}{xy} = 0$$

$$\frac{\cancel{y}\,dx}{x\cancel{y}} - \frac{\cancel{x}\,dy}{\cancel{x}y} = 0$$

$$\int \frac{dx}{x} - \int \frac{dy}{y} = 0$$

$$\log x - \log y = \log c$$

$\cancel{\log(x-y) = \log c}$ $\quad \log(x/y) = \log c$

$$\frac{x}{y} = c$$

$$x = cy$$

$\therefore \frac{1}{y^2}, \frac{1}{x^2}, \frac{1}{xy}$ are integrating factor of (1).

—x—

Type 2

* Integrating factor of a homogeneous equation.

solve $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0.$

$M\,dx+N\,dy=0$

$M=x^2y-2xy^2$

$N=-(x^3-3x^2y)$

This equation is homogeneous in x and y.

$$\text{Integrating factor}=\frac{1}{Mx+Ny}=\frac{1}{(x^2y-2xy^2)x+(3x^2y-x^3)y}$$

$$=\frac{1}{\cancel{x^3y}-2x^2y^2+3x^2y^2-\cancel{x^3y}}$$

$$\text{I.F}=\frac{1}{x^2y^2}$$

Multiplying by $\frac{1}{x^2y^2}$

$$\frac{1}{x^2y^2}[x^2y-2xy^2]\,dx-\frac{1}{x^2y^2}[x^3-3x^2y]\,dy=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$$

$$M=\frac{1}{y}-\frac{2}{x}\ ;\ N=-\left(\frac{x}{y^2}-\frac{3}{y}\right)$$

$$\frac{\partial M}{\partial y}=-1/y^2\ ;\ \frac{\partial N}{\partial x}=\frac{-1}{y^2}$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$\therefore$ which is exact

The solution is $\int\limits_{(y \text{ constant})} M\,dx + \int (\text{term of } N \text{ not containing } x)\,dy = c$

$$M = \frac{1}{y} - \frac{2}{x} \; ; \; N = -\left(\frac{x}{y^2} - \frac{3}{y}\right)$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \; ; \; \frac{\partial N}{\partial x} = \frac{-1}{y^2}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$$\frac{1}{y}\int dx - 2\int \frac{1}{x}\,dx + 3\int \frac{1}{y}\,dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c.$$

—x—.

Type 3:

I.F for an equation of the type

$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$

If the equation $M\,dx + N\,dy = 0.$

be of this form, $\frac{1}{Mx - Ny}$ is an I.F $(Mx - Ny) \neq 0$

—x—.

Ex

Solve $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0.$

This is of the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$$

$$M = (1+xy)\,y \; ; \; N = (1-xy)\,x.$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{(1+xy)yx - (1-xy)xy}$$

$$= \frac{1}{\cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2}$$

$$= \frac{1}{2x^2y^2}$$

multiplying by $\frac{1}{2x^2y^2}$

$$\frac{1}{2x^2y^2}(1+xy)y\,dx + \frac{1}{2x^2y^2}(1-xy)x\,dy = 0$$

$$\frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

$$M = \frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right) \; ; \; N = \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2x^2y^2} \; ; \; \frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact

$\int M\,dx + \int$ (terms of N not containing x) dy = c
(y constant)

$$\frac{1}{2}\int\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \int\frac{1}{2}\left(-\frac{1}{y}\right)dy = c$$

$$\frac{1}{2}\left(\frac{-1}{xy}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$\div$ by $\frac{1}{2}$

$$-\frac{1}{xy} + \log x - \log y = c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$$

$x^n = \frac{x^{n+1}}{n+1}$

$\int x^{-2}dx = \frac{x^{-2+1}}{-2+1}$

$= \frac{x^{-1}}{-1} = \frac{-1}{x}$

— x —

Type 4:

In the equation

$$M\,dx + N\,dy = 0$$

a) if $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}$ be a function of x only $= f(x)$,

then $e^{\int f(x)\,dx}$ is an I.F

b) if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ be a function of y only $= F(y)$ then $e^{\int F(y)\,dy}$ is an I.F.

— x —

solve

$$\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0.$$

$$M = xy^2 - e^{1/x^3} \quad ; N = -x^2y$$

$$\frac{\partial M}{\partial y} = 2xy \quad ; \frac{\partial N}{\partial x} = -2xy$$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = \frac{4\cancel{xy}}{-x^{\cancel{2}}\cancel{y}} = \frac{-4}{x}$$

which is a function of x only.

$$I.F = e^{\int -4/x\,dx} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4}$$

Multiple by x^{-4}

$$x^{-4}\left(xy^2 - e^{1/x^3}\right)dx - x^{-4}\left(x^2y\,dy\right) = 0$$

$$\left(x^{-3}y^2 - x^{-4}e^{1/x^3}\right)dx + x^{-2}y\,dy = 0$$

$$M = x^{-3}y^2 - x^{-4}e^{1/x^3} \quad ; N = x^{-2}y$$

$$\frac{\partial M}{\partial y} = 2yx^{-3} \quad ; \frac{\partial N}{\partial x} = 2x^{-3}y.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact

The solution is

$\int M dx + \int$ (terms of N not containing x) $dy = c$
(y constant)

$$\int (x^{-3}y^2 - x^{-4} e^{1/x^3}) dx + \int 0 = c$$

$$-y^2 \frac{x^{-2}}{2} + \frac{1}{3}\int e^{-x^3}(-3x^{-4}) dx = c$$

$$\frac{1}{3} e^{-x^3} - \frac{1}{2} y^2/x^2 = c.$$

— x —

Solve $(xy^3+y)dx + 2(x^2y^2 + x + y^4) dy = 0.$

$M = xy^3 + y$; $N = 2(x^2y^2 + x + y^4)$

$$\frac{\partial M}{\partial y} = x3y^2 + 1 \; ; \; \frac{\partial N}{\partial x} = 2[2xy^2 + 1]$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is not exact

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2(2xy^2+1) - (3xy^2+1)}{xy^3+y} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3+y}$$

$$= \frac{xy^2+1}{xy^3+y} = \frac{xy^2+1}{y(xy^2+1)} = \frac{1}{y}$$

∴ which is function of y alone.

$$I.F = e^{\int 1/y \, dy} = e^{\log y} = y$$

Multiple by y

$$y(xy^3+y)dx + 2y(x^2y^2 + x + y^4) dy = 0$$

$$(xy^4 + y^2) dx + (2x^2y^3 + 2xy + 2y^5) dy = 0$$

$$M = xy^4 + y^2 \qquad N = 2x^2y^3 + 2xy + 2y^5$$

$$\frac{\partial M}{\partial y} = 4y^3x + 2y \qquad \frac{\partial N}{\partial x} = 4xy^3 + 2y.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact.

The solution is

$$\underset{(y\ \text{constant})}{\int M\,dx} + \int \left(\begin{array}{c}\text{terms of } N \text{ not} \\ \text{containing } x\end{array}\right) dy = c$$

$$\int (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$$

$$\frac{x^2}{2}y^4 + y^2x + \frac{\cancel{2}y^6}{\cancel{6}\,3} = c$$

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

—x—.

<u>Equation</u> <u>not</u> <u>of</u> <u>first</u> <u>degree</u>

<u>Solvable</u> <u>for</u> <u>y</u>.

$$y = f(x, p)$$

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

Let its should be $F(x, p, c) = 0$

$$x = F_1(p, c)\ ;\ y = F_2(p, c)$$

$$y - 2px = \tan^{-1}(xp^2)$$

$$y = 2px + \tan^{-1}(xp^2). \rightarrow ①$$

Diff ① with respect to x on both sides.

$$\frac{dy}{dx} = 2\left(p \cdot 1 + \frac{dp}{dx}x\right) + \frac{1}{1+(xp^2)^2}\left(x \cdot 2p\frac{dp}{dx} + p^2\right)$$

$$p = 2\left(p + x\frac{dp}{dx}\right) + \frac{1}{1+x^2p^4}\left(2px\frac{dp}{dx} + p^2\right)$$

$$p = \left[\left(\overset{p+p}{2p} + 2x\frac{dp}{dx}\right) + \frac{1}{1+x^2p^4}\, p\left(2x\frac{dp}{dx} + p\right)\right].$$

$$p = \left(p + 2x\frac{dp}{dx}\right)\left(p + \frac{p}{1+x^2p^4}\right)$$

$$\cancel{p} = \left(p + 2x\frac{dp}{dx}\right)\cancel{p}\left(1 + \frac{1}{1+x^2p^4}\right)$$

$$\left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{1}{x^2p^4}\right) = 0$$

$$p + 2x\frac{dp}{dx} = 0$$

$$2x\frac{dp}{dx} = -p$$

$$2\frac{dp}{p} = -\frac{dx}{x}$$

$$2\frac{dp}{p} + \frac{dx}{x} = 0$$

$$2\int\frac{dp}{p} + \int\frac{dx}{x} = 0$$

$$2\log p + \log x = \log c$$

$$\log p^2 + \log x = \log c$$

$$\log(xp^2) = c$$

$$\boxed{xp^2 = c}$$

$$p^2 = c/x$$

$$p = \sqrt{c/x} \rightarrow (2)$$

Eliminate p from (1) & (2)

$$y = 2\sqrt{\frac{c}{x}}\, x + \tan^{-1} c$$

$$= 2\sqrt{c}\,\frac{\sqrt{x}\sqrt{x}}{\sqrt{x}} + \tan^{-1} c$$

$$y = 2\sqrt{cx} + \tan^{-1} c$$

—x—

Solve $y = 2px - p^2$

$$\frac{dy}{dx} = p = 2\left(p \cdot 1 + x\frac{dp}{dx}\right) - 2p\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$2p - p + 2(x-p)\frac{dp}{dx} = 0$$

$$p + 2(x-p)\frac{dp}{dx} = 0$$

$$p = -2(x-p)\, dp/dx$$

$$p\frac{dx}{dp} = -2x + 2p$$

$$p\, dx/dp + 2x = 2p$$

$$\frac{dx}{dp} + \frac{2x}{p} = \frac{2\cancel{p}}{\cancel{p}}$$

$$\frac{dx}{dp} + \frac{2x}{p} = 2$$

$$\frac{2}{p} \downarrow P \qquad 2 \downarrow Q$$

$$I.F = e^{\int P dp} = e^{\int 2/p\, dp} = e^{2\log p} = e^{\log p^2} = p^2 \quad \boxed{I.F = p^2}$$

$$x(I.F) = \int Q\,(I.F)\,dp + c$$

$$\frac{dy}{dx} + Py = Q$$
$$I.F = e^{\int P\,dx}$$

$$xp^2 = \int 2p^2\,dp + c$$

$$xp^2 = \frac{2p^3}{3} + c \Rightarrow \div p^2; \quad x = \frac{2p^3}{3p^2} + \frac{c}{p^2}$$

$$\boxed{x = \frac{2p}{3} + cp^{-2}}$$

— x —

Solve $y + px = x^4p^2$

Soln:

$$y = -px + x^4p^2$$

diff with respect to x

$$\frac{dy}{dx} = p = -\left(p.1 + x\frac{dp}{dx}\right) + \left(4x^3p^2 + x^4\,2p\frac{dp}{dx}\right)$$

$$p + p + x\frac{dp}{dx} - 4x^3p^2 - 2x^4p\frac{dp}{dx} = 0$$

$$2p - 4x^3p^2 + x\frac{dp}{dx} - 2x^4p\frac{dp}{dx} = 0$$

$$2p(1-2x^3p) + (1-2x^3p)\,x\,dp/dx = 0.$$

$$(1-2x^3p)(2p + x\,dp/dx) = 0$$

Discarding the factor $(1-2x^3p)$

$$2p + x\frac{dp}{dx} = 0$$

$$x\frac{dp}{dx} = -2p \Rightarrow \cancel{\frac{dp}{dx}}\ \frac{dp}{p} = -2\frac{dx}{x}$$

Integrating

$$\int dp/p = -2\int dx/x$$

$$\log p = -2\log x + \log c$$

$$\log p + 2\log x = \log c$$

$$\log p + \log x^2 = \log c \Rightarrow px^2 = c \Rightarrow p = c/x^2$$

Sub p in ①

$$y = -\frac{c}{x^2}.x + x^4\left(\frac{c}{x^2}\right)^2$$

$$= -c/x + x^4\frac{c^2}{x^4}$$

$$\boxed{y = \frac{-c}{x} + c^2}$$

Equation solvable of x

* The equation of this type $x = f(y, p) \longrightarrow ①$

Differentiating ① with respect to y.

$\frac{dy}{dx} = p$

$$x = f(y, p) \longrightarrow ①$$

Diff ① with respect to y,

$$\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \longrightarrow ②$$

② is the differential equation of first order in p and y

Solution of ② is $\phi(y, p, c) = 0 \longrightarrow ③$

* Eliminate p from equation ① & ③ gives the required equation.

ⓧ v. Imp

Solve

$$y = 2px + y^2p^3 \longrightarrow \circledast$$

$$y - y^2p^3 = 2px$$

$$\Rightarrow x = \frac{y - y^2p^3}{2p}$$

Solving for x, $x = \frac{1}{2}\left[\frac{y}{p} - y^2p^2\right] \longrightarrow ①$

Diff ① with respect to y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} + y\left(\frac{-1}{p^2}\right)\frac{dp}{dy} - \left(2y \cdot p^2 + y^2 2p\frac{dp}{dy}\right)\right]$$

$$\frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - y^2\, 2p\frac{dp}{dy}\right].$$

$$\frac{1}{p} = \frac{1}{2}\cdot\frac{1}{p}\left[1-\frac{y}{p}\frac{dp}{dy} - 2p^2py - y^2\, 2p\cdot p\frac{dp}{dy}\right]$$

$$2p = p\left(1 - y/p\ dp/dy - 2yp^3 - 2y^2p^2\frac{dp}{dy}\right).$$

$$2p = p - y\frac{dp}{dy} - 2yp^4 - y^2 2p^3\frac{dp}{dy}$$

$$\boxed{\begin{array}{c}2p-p\\=p\end{array}}$$

$$p + y\frac{dp}{dy} + 2yp^4 + 2y^2p^3\frac{dp}{dy} = 0.$$

$$(p+2yp^4) + (y+2y^2p^3)\frac{dp}{dy} = 0$$

$$p(1+2yp^3) + (1+2yp^3)y\frac{dp}{dy} = 0$$

$$(1+2yp^3)\left(p + y\frac{dp}{dy}\right) = 0$$

Discarding the factor $(1+2yp^3)$, we get

$$p + y\frac{dp}{dy} = 0$$

$$y\frac{dp}{dy} = -p$$

$$\frac{dp}{p} = -\frac{dy}{y}$$

$$\frac{dp}{p} + \frac{dy}{y} = 0,$$

Integrating,

$$\int\frac{dp}{p} + \int\frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = c/y$$

sub p in ⊛

$$y = 2px + y^2p^3$$

$$y = \frac{2cx}{y} + y^2\left(\frac{c}{y}\right)^3$$

$$y = \frac{2cx}{y} + y^2 \frac{c^3}{y^3}$$

$$y = \frac{2cx + c^3}{y}$$

$$\boxed{y^2 = 2cx + c^3.}$$

solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$

✓

$$\tan^{-1}p = x - \frac{p}{1+p^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$x = \tan^{-1}p + \frac{p}{1+p^2}$$

$$\tan^{-1}x = \frac{1}{1+x^2}\frac{dx}{dy}$$

Diff with respect to y

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p\left(0+2p\frac{dp}{dy}\right)}{(1+p^2)^2}$$

$$\frac{1}{p} = \left[\frac{1}{(1+p^2)} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$= \left[\frac{(1+p^2) + (1+p^2) - 2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{2 + \cancel{2p^2} - \cancel{2p^2}}{(1+p^2)^2}\frac{dp}{dy}$$

$$1/p = \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$dy = \frac{2p}{(1+p^2)^2}\,dp$$

Integrating

$$\int dy = \int \frac{2p}{(1+p^2)^2}\,dp$$

$1+p^2 = t$

$2p\,dp = dt$

$$y = \int \frac{dt}{t^2}$$

$$= \int t^{-2}\,dt$$

$$= \frac{t^{-2+1}}{-2+1} = \frac{t^{-1}}{-1}$$

$t^{-1} = \frac{1}{t} = t^{-1}$

$$= \frac{-1}{t}$$

$$y = \frac{-1}{1+p^2} + c$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$$\boxed{y = c - \frac{1}{1+p^2}}$$

—x—

2m

Clairaut's type equation:

* An equation of the form $y = px + f(p) \rightarrow (1)$ is known as clairaut's equation.

Differentiating (1) with respect to x we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + [x + f'(p)]\frac{dp}{dx}$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$

$$\frac{dp}{dx} = 0$$

Integrate, $P = c$

Puting $P = c$ in ①

$$\boxed{y = cx + f(c)}$$

* Thus the solution clairaut's equation is obtain by writing c for p.

— x —

(2) 2m

Solve $(y - px)(p-1) = p$

The given equation is

$$(y - px)(p-1) = p$$

$$y - px = \frac{p}{p-1}$$

$$y = px + \frac{p}{p-1}$$

$$y = px + f(p)$$

which is clairaut's equation.

Putting $P = c$ we get the solution is

$$\boxed{y = cx + \frac{c}{c-1}}$$

* Thus the solution clairaut's equation is obtain by writing c for P

— x —

(3) Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$.

The given equation is $e^{4x}(p-1) + e^{2y}p^2 = 0$

$$y = px + f(p)$$

e^{lx} $\quad e^{my}$

$X = e^{kx}$ $\quad Y = e^{ky}$

$K \rightarrow$ H.C.F of l & m.

Putting $X = e^{2x}$ $Y = e^{2y}$

$$dX = 2e^{2x}\,dx \quad ; \quad dY = 2e^{2y}\,dy.$$

$$P = \frac{dy}{dx} = \frac{dY/2e^{2y}}{dX/2e^{2x}} = \frac{dY}{2e^{2y}} \times \frac{2e^{2x}}{dX}$$

$$= \frac{X}{Y}\frac{dY}{dX} = \frac{X}{Y}P$$

$$\boxed{p = \frac{X}{Y}P}$$

The given equation is

$$X^2\left(\frac{X}{Y}P - 1\right) + Y\left(\frac{XP}{Y}\right)^2 = 0.$$

$$X^2\left(\frac{XP - Y}{Y}\right) + \frac{\cancel{Y}X^2P^2}{Y^{\cancel{2}}} = 0$$

$$\frac{X^2}{Y}\left[XP - Y + P^2\right] = 0$$

$$XP - Y + P^2 = 0$$

$$PX + P^2 = Y$$

$$\boxed{Y = PX + P^2}$$

which is of clairaut's equation

$$Y = cX + c^2$$

$$e^{2y} = ce^{2x} + c^2.$$

—x—

Solve $(px-y)(py+x)=2p$

V.Imp The given equation is

$$(px-y)(py+x)=2p \longrightarrow (1)$$

Putting $X=x^2$; $Y=y^2$

$$dX = 2x\,dx \; ; \; dY = 2y\,dy$$

$$\frac{dX}{2x} = dx \; ; \; \frac{dY}{2y} = dy.$$

$$p = \frac{dy}{dx} = \frac{dY/2y}{dX/2x} = \frac{dY}{2y} \times \frac{2x}{dX}$$

$$= \frac{x}{y}\,\frac{dY}{dX}$$

$$= \frac{\sqrt{X}}{\sqrt{Y}}\,\frac{dY}{dX}$$

$$\boxed{p = \frac{\sqrt{X}}{\sqrt{Y}}\,P}$$

$\frac{dy}{dx} = p$

$x^2 = X$

$x = X^{1/2}$

$x = \sqrt{X}$

where $\frac{dY}{dX} = P$

$\therefore$ The equation (1) is

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\left(\frac{\sqrt{X}P\sqrt{X} - \sqrt{Y}\sqrt{Y}}{\sqrt{Y}}\right)(\sqrt{X}P + \sqrt{X}) = 2\frac{\sqrt{X}P}{\sqrt{Y}}$$

$$\left(\frac{XP-Y}{\sqrt{Y}}\right)\sqrt{X}(P+1) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\frac{\sqrt{X}}{\sqrt{Y}}(XP-Y)(P+1) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$(xP-y)(P+1) = 2\frac{\sqrt{\cancel{x}}}{\sqrt{\cancel{x}}} P \frac{\sqrt{\cancel{x}}}{\sqrt{\cancel{x}}}$$

$$(xP-y)(P+1) = 2P$$

$$(Px-y) = \frac{2P}{P+1}$$

$$Px - \frac{2P}{P+1} = y$$

$$y = Px - \frac{2P}{P+1}$$

$$\boxed{y = Px + f(p)}$$

which is a clairaut's equation

By Putting $P=c$, we get the solution is

$$y = cx - \frac{2c}{c+1}$$

$$y^2 = cx^2 - \frac{2c}{c+1}$$

—x—

Solve $(Px-y)(Py+x) = a^2P$.

The given equation is

$$(Px-y)(Py+x) = a^2 P \rightarrow (1)$$

Putting $X = x^2$; $Y = y^2$

$dX = 2x\,dx$; $dY = 2y\,dy$.

$\frac{dX}{2x} = dx$; $\frac{dY}{2y} = dy$.

$$P = \frac{dy}{dx} = \frac{dy/\cancel{dy}\,2y}{dx/\cancel{dx}\,2x} = \frac{dy}{\cancel{2}y} \times \frac{\cancel{2}x}{dx}$$

$$= \frac{x}{y}\,\frac{dy}{dx}$$

$$= \frac{\sqrt{x}}{\sqrt{y}}\,\frac{dy}{dx}$$

$$P = \frac{\sqrt{x}}{\sqrt{y}}\,p.$$

The equation ① is

$$\left(\frac{\sqrt{x}}{\sqrt{y}}\,p\sqrt{x} - \sqrt{y}\right)\left(\frac{\sqrt{x}}{\sqrt{y}}\,p\sqrt{y} + \sqrt{x}\right) = \alpha^2 \frac{\sqrt{x}}{\sqrt{y}}\,p$$

$$\left(\frac{\sqrt{x}\,p\sqrt{x} - \sqrt{y}\sqrt{y}}{\sqrt{y}}\right)\left(\sqrt{x}\,p + \sqrt{x}\right) = \alpha^2 \frac{\sqrt{x}}{\sqrt{y}}\,p$$

$$\left(\frac{xp - y}{\sqrt{y}}\right)\sqrt{x}\,(p+1) = \alpha^2 \frac{\sqrt{x}}{\sqrt{y}}\,p$$

$$(xp - y)(p+1) = \alpha^2 \frac{\sqrt{x}}{\cancel{\sqrt{y}}}\,p \times \frac{\cancel{\sqrt{y}}}{\cancel{\sqrt{x}}}$$

$$(xp - y)\cdot(p+1) = \alpha^2 p$$

$$(xp - y) = \frac{\alpha^2 p}{p+1}$$

$$px - \frac{\alpha^2 p}{p+1} = y$$

$$y = px - \frac{\alpha^2 p}{p+1}$$

$$\boxed{y = P x + f(P)}$$

which is a clairaut's equation

By Putting $P = c$, we get the solution is

$$y = cx - \frac{\alpha^2 c}{c+1}$$

$$y^2 = cx^2 - \frac{\alpha^2 c}{c+1}$$

—— x ——

UNIT 1 FIRST ORDER ODE

Questions	opt1	opt2	opt3	opt4	Answer
The necessary and sufficient condition for the differential equation to be exact is	$M_x = N_y$	$M_y = N_x$	$M_x = N_x$	$M_y = N_y$	$M_y = N_x$
The equation is known is dy/dx+Py=Q y^2	Euler equation	Bernoulli's Equation	Legendre equation	Homogeneous	Bernoulli's Equation
The integrating factor of dy/dx+y/x=x^2	x	y	logx	0	x
The solution of Mdx+Ndy=0 is posses an	Infinite no of integrating factor	finite no of integrating factor		one integrating factor	Infinite no of integrating factor
A differential equation is said to be ______ if the dependent variable and its derivative occur only in the first degree and are not multiplied together	Linear	nonlinear	quadratic	PDE	Linear
The order of d^2y/dx^2+y=x^2-2 is	0	1	**2**	3	2
The integrating factor of dy/dx+ysinx = 0 is	e^-cosx	ye^-cosx	logx	e^sinx	e^-cosx
The integrating factor of dy/dx-ycotx = sinx is	sinx	- sinx	cosx	- cosx	- sinx
The solution of y=(x-a)p-p^2	y = (x-a)c-c^2	y = (x-a)c+c^2	0	-1	y = (x-a)c-c^2
An equation of the form y=px+f(p) is known as	linear	Bernoulli's Equation	exact	Clairaut's equation	Clairaut's equation
The order of d^2y/dx^2+y=0 is	**2**	1	0	-1	2
The clairaut's form of p=tan(px-y)	y=cx+tan^-1 c	y=cx-tan^-1 c	c=tan(cx-y)	c=tan(px+y)	y=cx-tan^-1 c
An equation involving one dependent variable and its derivatives with respect to one independent variable is called __________	ODE	PDE	Partial	Total	ODE
The _______ is differentiation of a function of two or more variables	ODE	PDE	Partial	Total	PDE
A differential equation is said to be linear if the dependent variable and its derivative occur only in the ______degree and are not multiplied together	first	second	third	irst and secon	first
The highest derivative of the differential equation is_____	Order	Degree	Power	second degree	Order
The powerof the hightest derivative of the differential equation is called____	Order	Degree	Power	second degree	Degree
The order of y''-y'+7=x^2+4 is	0	1	2	3	2
The order of y'''+xy'+7x=0 is	0	1	2	3	3

The degree of the (d^2y/dx^2)^2+(dy/dx)^3+3y=0	0	1	2	3	2
The degree of the (d^2y/dx^2)^3+(dy/dx)^3+7y=0	0	1	2	3	3
The order and degree of (d^3/dx^3)^2+dy/dx+9y=0	3,2	2,3	1,2	2,1	3,2
The standard form of a linear equation of the first order	dy/dx+Py=Q	dy/dx+py=Q	dy/dx+Py=q	5dy/dx+Py=Q	dy/dx+Py=Q
The integrating factor of linear equation of the form dx/dy+Px=Q is	e^integral Qdx	e^integral Pdy	e^integral Qdx	e^Qdx	e^integral Pdy
The integrating factor of linear equation of the form dy/dx+Py=Q is	e^integral Qdy	e^integral Pdx	e^integral Qdx	e^Qdx	e^integral Pdx
The integrating factor of dy/dx+ysinx=0 is	e^(-cosx)	e^(-cosx)y	logx	e^(sinx)	e^(-cosx)
The integrating factor of dy/dx-ycotx=0 is	cos x	(-cos x)	cosec x	sin x	cosec x
If the given equationMdx+Ndy=0 is homogenous and Mx+Ny≠0 then the integrating factor is ___	1/(Nx-My)	1/(Mx+Ny)	1/(Mx-Ny)	1/(Nx+My)	1/(Mx+Ny)
The solution of Mdx+Ndy is	intergral y constant Mdx+integral of terms of N not containing x dy	intergral y constant Mdx+integral of terms not containing x dx	intergral y constant Ndx+integral of terms not containing x dx	intergral y constant Mdx+integral of terms not containing y dx	intergral y constant Mdx+integral of terms of N not containing x dy
If Mdx+Ndy=0 be a homogeneous equation in x and y, then______is an integrating factor(Mx+Ny≠0)	1/(Mx+Ny)	1/(Mx-Ny)	Mdy+Ndx	Mdy-Ndx	1/(Mx+Ny)
If Mdx+Ndy=0 be a homogeneous equation in x and y, then______is an integrating factor(Mx-Ny≠0)	1/(Mx+Ny)	1/(Mx-Ny)	Mdy+Ndx	Mdy-Ndx	1/(Mx-Ny)

Ordinary differential equation of higher order.

consider

* The general linear differential equation with constant co-efficient of the form

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + K_n y = X$$

* $K_1, K_2, \ldots K_n$ are constant

Replace

$$\frac{d}{dx} \rightarrow D$$

$$D^n y + K_1 D^{n-1} y + K_2 D^{n-2} y + \cdots + K_n y = X$$

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y = X$$

The general solution of the equation ① is

$$y = C.F + P.I$$

C.F → complementary function.

P.I → Particular integral.

<u>Rules for finding C.F</u>

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y \; X$$

Auxiliary equation is replace $D \rightarrow m$

By solving we get the roots

S. NO	Roots	complementary function (C.F)
1.	If two roots are real & distinct $m_1 \neq m_2$	$y = A e^{m_1 x} + B e^{m_2 x}$
2	If two roots are real & equal $m_1 = m_2 = m$	$y = (Ax + B) e^{mx}$
3	If two roots are real & imaginary $(\alpha \pm i\beta)$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Rules for finding P.I

$\Rightarrow f(D)y = X$

$$P.I = \frac{1}{f(D)} X$$

where x is the function of x

R.H.S = 0

There is only C.F no P.I

Solve $(D^2 + 5D + 6)y = 0$

Given $(D^2 + 5D + 6)y = 0$

Replace $D \to m$

Auxiliary equation is $m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$

The roots are real & distinct.

Scanned by CamScanner

$m_1 = -2 \,;\, m_2 = -3$

$m_1 \neq m_2.$

~~$\therefore$ C.F =~~ $\therefore$ C.F is $y = Ae^{m_1 x} + Be^{m_2 x}$

$$\boxed{y = Ae^{-2x} + Be^{-3x}}$$

— x —

Solve

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

soln:

$$\frac{d}{dx} \to D$$

$$D^2y + 6Dy + 9y = 0$$

$$(D^2 + 6D + 9)y = 0$$

Replace $D \to m$

Auxiliary equation is $m^2 + 6m + 9 = 0$

$$(m+3)(m+3) = 0$$

$$m = -3, -3$$

$m_1 = -3 \,;\, m_2 = -3.$

$m_1 = m_2$

The roots are real & equal

$m_1 = m_2 = m$

$\therefore$ C.F is $y = (Ax + B)e^{mx}$

$$\boxed{y = (Ax + B)e^{-3x}}$$

— x —

Solve $(D^2+D+1)\,y=0$

Replace: $D \to m$

Soln:

Auxiliary equation is $m^2+m+1=0$

$a=1;\ b=1,\ c=1.$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$$

$$=\frac{-1\pm i\sqrt{3}}{2}$$

∴ The roots are real & imaginary

$$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

$$\alpha=-1/2\ ;\ \beta=\sqrt{3}/2$$

∴ C.F is $y=e^{-1/2\,x}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)$

—x—.

Solve $(D^2+D+1)y = x^2$

Soln:

Auxiliary equation is $m^2+m+1=0$

$a=1$; $b=1$; $c=1$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

The roots are real & imaginary

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x) \quad \Big| \quad \alpha = -1/2, \ \beta = \frac{\sqrt{3}}{2}$$

$$C.F = e^{-1/2 x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

To find P.I

$$P.I = \frac{1}{D^2+D+1}x^2$$

$$= \frac{1}{1+D^2+D}x^2$$

$1+x$

$$= [1+(D^2+D)]^{-1}x^2$$

$$(1+x)^{-1} = 1-x+x^2-x^3+\dots$$

$$= [1-(D^2+D)+(D^2+D)^2-(D^2+D)^3+\dots]x$$

$$= x^2-(D^2+D)(x^2)+(D^4+2D^3+D^2)x^2$$

$$= x^2 - D^2(x^2) - D(x^2) + D^2(x^2)$$

$$= x^2 - \cancel{2} - 2x + \cancel{2}$$

$D(x^2) = 2x$
$D^2(x^2) = 2$
$D^3(x^2) = 0$

$$P.I = x^2 - 2x$$

$$\therefore y = C.F + P.I$$

$$= e^{-1/2 x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + x^2 - 2x$$

— x —

Find P.I of $(D^2+5D+6)y = x^2$

$$P.I = \frac{1}{D^2+5D+6} x^2$$

$$= \frac{1}{6\left(\frac{D^2+5D}{6}+1\right)} x^2$$

$$= \frac{1}{6}\left[1+\left(\frac{D^2+5D}{6}\right)\right]^{-1} x^2$$

$(1+x)^{-1}$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2+5D}{6}\right) + \left(\frac{D^2+5D}{6}\right)^2 - \cdots\right](x^2)$$

$$= \frac{1}{6}\left[x^2 - \left(\frac{D^2+5D}{6}\right)(x^2) + \left(\frac{D^4+10D^3+25D^2}{36}\right)x^2\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{6}\left(D^2(x^2) + 5D(x^2)\right) + \frac{1}{36}\left[D^4(x^2) + 10D^3(x^2) + 25D^2(x^2)\right]\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{6}(2+5(2x)) + \frac{1}{\underset{18}{\cancel{36}}}(25 \times \cancel{2})\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{6}(2+10x) + \frac{25}{18}\right.$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{3} - \frac{\cancel{10}^{5}x}{\cancel{6}_3} + \frac{25}{18}\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{3} - \frac{5x}{3} + \frac{25}{18}\right]$$

$\frac{1}{3} \times \frac{6}{6} = \frac{6}{18}$

$\frac{25}{18} \times \frac{1}{1} = \frac{25}{18}$

$$P.I = \frac{1}{6}\left[x^2 - \frac{5x}{3} + \frac{19}{18}\right].$$

$\frac{25-6}{18} = \frac{19}{18}$

—x—

$R.H.S = e^{ax}\cos bx$ (or) $e^{ax}\sin bx$ (or) $e^{ax}x^n$.

$P.I = \frac{1}{f(D)}\ e^{ax}\cos bx$ (or) $e^{ax}\sin bx$ (or) $e^{ax}x^n$.

Replace $D \to D+a$.

—x—

$$(D^2+2D+5)y = e^x \sin 2x$$

Auxiliary equation is $m^2+2m+5=0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i.$$

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$C.F = e^{-x}(A\cos 2x + B\sin 2x)$

To find P.I

$$P.I = \frac{1}{D^2+2D+5} e^{x}\sin 2x \qquad a=1$$

Replace $D \rightarrow D+a = D+1$

$$= \frac{1}{(D+1)^2+2(D+1)+5} e^{x}\sin 2x$$

$$= e^{x}\frac{1}{D^2+2D+1+2D+2+5}\sin 2x$$

$$= e^{x}\frac{1}{D^2+4D+8}\sin 2x$$

Replace $D^2 \rightarrow -a^2 = -4$

$$= e^{x}\frac{1}{-4+4D+8}\sin 2x$$

$$= e^{x}\frac{1}{4D+4}\sin 2x$$

$$= \frac{e^{x}}{4}\frac{1}{D+1}\sin 2x$$

$$= \frac{e^{x}}{4}\cdot\frac{1}{(D+1)(D-1)}(D-1)\sin 2x$$

$$= \frac{e^{x}}{4}\frac{(D-1)\sin 2x}{D^2-1}$$

$$= \frac{e^x}{4} \left(\frac{D \sin 2x - \sin 2x}{-4-1} \right) \qquad D^2 \to -a^2 = -4$$

$$= \frac{e^x}{4(-5)} (2\cos 2x - \sin 2x)$$

$$P.I = \frac{e^x}{-20} (2\cos 2x - \sin 2x).$$

— x —

~~Solve $(D^2 + 4D + 3)y = xe^{3x}$~~

— x —

Solve $(D^2 + 4D + 3)y = x e^{3x}$

soln:

$$P.I = \frac{1}{D^2+4D+3} x e^{3x}$$

Replace $D \to D+a = D+3$

$$= e^{3x} \frac{1}{(D+3)^2 + 4(D+3) + 3} x$$

$$= e^{3x} \frac{1}{D^2+9+6D+4D+12+3} x$$

$$= e^{3x} \frac{1}{D^2+10D+24} x$$

$$= e^{3x} \frac{1}{24\left(\frac{D^2+10D}{24}+1\right)} x$$

$D(x) = 1$

$D^2(x) = 0$

$$= \frac{e^{3x}}{24} \left[1 + \left(\frac{D^2+10D}{24}\right)\right]^{-1}$$

$$= \frac{e^{3x}}{24} \left[1 - \left(\frac{D^2+10D}{24}\right) + \left(\frac{D^2+10D}{24}\right)^2 \dots\right] x$$

$$= \frac{e^{3x}}{24}\left[x - \frac{1}{24}\left(D^2(x) + 10D(x)\right)\right]$$ (Neglecting higher Powers)

$$= \frac{e^{3x}}{24}\left[x - \frac{1}{24}10(1)\right] \Rightarrow \frac{e^{3x}}{24}\left[x - \frac{\cancel{10}^{5}}{\cancel{24}_{12}}\right]$$

$$= \frac{e^{3x}}{24}\left(x - 5/12\right).$$

— x —

R.H.S $= x^n \sin ax$ (or) $x^n \cos ax$

$$P.I = \frac{1}{f(D)} x^n \sin ax$$

$$= \text{Imaginary Part of } \frac{1 \cdot e^{iax}}{f(D)} x^n.$$

Replacing $D \to D+ia$

$$= \text{Imaginary Part of } 1 \cdot e^{iax} \frac{1}{f(D+ia)} x^n.$$

— x —

Solve $(D^2 - 2D + 1)y = x \sin x$.

Soln:

Auxiliary equation is $m^2 - 2m + 1 = 0$.

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

The roots are real & equal

$$C.F = (Ax+B)e^{mx}$$

$$C.F = (Ax+B)e^{x}$$

To find P.I

$$P.I = \frac{1}{D^2-2D+1} x\sin x$$

$$= \text{Imaginary Part of } \frac{1}{D^2-2D+1} e^{ix} x.$$

Replace $D \to D+i$

$$= \text{Imaginary Part of } e^{ix} \frac{1}{(D+i)^2 - 2(D+i)+1} x$$

$$= \text{Imaginary Part of } e^{ix} \frac{1}{D^2+i^2+2iD-2D-2i+1} x$$

$$= \text{Imaginary Part of } e^{ix} \frac{1}{D^2-\cancel{1}+2iD-2D-2i+\cancel{1}}$$

$$= I.P \text{ of } e^{ix} \frac{1}{-2i\left[1-\left(\frac{D^2+2iD-2D}{2i}\right)\right]} x$$

$$= I.P \text{ of } e^{ix} \frac{1}{-2i}\left[1-\left(\frac{D^2+2iD-2D}{2i}\right)\right]^{-1} x$$

$$= I.P \text{ of } e^{ix} \frac{1}{-2i}\left[1+\left(\frac{D^2+2iD-2D}{2i}\right)\right] x$$

[neglecting higher Power]

$$= I.P \text{ of } e^{ix} -\frac{1}{2i}\left[x+\left(\frac{D^2(x)+2iD(x)-2D(x)}{2i}\right)\right]$$

$$= I.P \text{ of } e^{ix} -\frac{1}{2i}\left[x+0+\frac{2i(1)-2(1)}{2i}\right]$$

$$= I.P \text{ of } e^{ix} -\frac{1}{2i}\left[x+\frac{i-1}{i}\right]$$

$D(x) = 1$

$D^2(x) = 0$

$$= I.P \text{ of } e^{ix} -\frac{1}{2i}\left[(x+1) - \frac{1}{i}\right]$$

Scanned by CamScanner

$= I.P \text{ of } \frac{-e^{ix}}{2i}\left[\frac{i(x+1)-i}{i}\right]$

$= I.P \text{ of } -\frac{e^{-ix}}{2}\left[\frac{i(x+1)-i}{-i}\right]$ $\quad i^2=-1$

$= I.P \text{ of } \frac{e^{-ix}}{2}\left[i(x+1)-i\right].$

$= I.P \text{ of } \frac{1}{2}\left[(\cos x + i\sin x)\left[i(x+1)-i\right]\right]$

$= \frac{1}{2}\left[\cos x(x+1) - \sin x\right].$

—x—

(8) 8m

$(D^2+6D+8)y = e^{-2x} + \cos 2x$

A.E is $m^2+6m+8=0$

$(m+4)(m+2)=0$

$m=-4,-2$

∴ The roots are real & distinct

$C.F = Ae^{m_1x} + Be^{m_2x}$ $\quad m_1=-4$, $m_2=-2$

$C.F = Ae^{-4x} + Be^{-2x}$

To Find P.I

$P.I_1 = \frac{1}{D^2+6D+8}e^{-2x}$

Replace $D \rightarrow -2$

$= \frac{1}{(-2)^2+6(-2)+8}e^{-2x}$

$= \frac{1}{4-12+8}e^{-2x}$

Scanned by CamScanner

$$= \frac{1}{12-12} e^{-2x}$$

$$= \frac{x}{2D+6} e^{-2x}$$

$$= \frac{x}{2(-2)+6} e^{-2x}$$

Replace $D \rightarrow -2$

$$= \frac{x}{-4+6} e^{-2x}$$

$$\boxed{P.I_1 = \frac{x}{2} e^{-2x}}$$

To find $P.I_2$

$$P.I_2 = \frac{1}{D^2+6D+8} \cos 2x$$

$a = 2$

Replace $D^2 \rightarrow -a^2 = -2^2 = -4$

$$= \frac{1}{-4+6D+8} \cos 2x$$

$$= \frac{1}{6D+4} \cos 2x$$

$$= \frac{1}{(6D+4)} \times \frac{(6D-4)}{(6D-4)} \cos 2x$$

$$= \frac{(6D-4)}{(6D)^2-4^2} \cos 2x$$

$$= \frac{(6D-4)\cos 2x}{36D^2-16}$$

$a = 2$

Replace $D^2 \rightarrow -a^2 = -4$.

$$= \frac{6D(\cos 2x) - 4\cos 2x}{36(-4) - 16}$$

$$= \frac{6(-\sin 2x \cdot 2) - 4\cos 2x}{-144 - 16}$$

$$= \frac{-12\sin 2x - 4\cos 2x}{-160}$$

$$= \frac{\cancel{+4}\,(3\sin 2x + \cos 2x)}{\cancel{+160}\ 40}$$

$$\boxed{P.I_2 = \frac{1}{40}(3\sin 2x + \cos 2x)}$$

$\therefore$ The general solution is $y = C.F + P.I_1 + P.I_2$

$$y = Ae^{-4x} + Be^{-2x} + \frac{x}{2}e^{-2x} + \frac{1}{40}(3\sin 2x + \cos 2x).$$

— x —

$$(D^3 + 2D^2 + D)y = e^{2x} + \sin x.$$

Soln:

A.E is $m^3 + 2m^2 + m = 0$

$m(m^2 + 2m + 1) = 0$

$m = 0, \quad m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1, -1$

$C.F = Ae^{m_1 x} + (Bx + C)e^{m_2 x}$

$C.F = Ae^{0x} + (Bx + C)e^{-x}$

$$\boxed{C.F = A + (Bx+C)e^{-x}}$$

To find $P.I_1$

$$P.I_1 = \frac{1}{D^3+2D^2+D} e^{2x}$$

Replace $D \rightarrow a = 2$

$$= \frac{1}{2^3+2(2)^2+2} e^{2x}$$

$$= \frac{1}{8+2(4)+2} e^{2x}$$

$$= \frac{1}{18} e^{2x}$$

$$= \frac{e^{2x}}{18}$$

$$\boxed{P.I_1 = \frac{e^{2x}}{18}}$$

To find $P.I_2$

$$P.I_2 = \frac{1}{D^3+2D^2+D} \sin x$$

Replace $D^2 \rightarrow -a^2 = -1^2 = -1$

$$= \frac{1}{D.D^2+2D^2+D} \sin x$$

$$= \frac{1}{D(-1)+2(-1)+D} \sin x$$

$$= \frac{1}{-\cancel{D}-2+\cancel{D}} \sin x$$

$$\boxed{P.I_2 = \sin x / -2}$$

Scanned by CamScanner

$\therefore$ The general solution is $y = C.F + P.I_1 + P.I_2$

$y \ne A e^{+4x}$

$$y = A + (Bx + C) e^{-x} + \frac{e^{2x}}{18} + \left(\frac{\sin x}{-2}\right)$$

$$y = A + (Bx + C) e^{-x} + \frac{e^{2x}}{18} - \frac{\sin x}{2}.$$

— x —

(Imp) V.Imp

Solve $(D^2 - 2D + 1) y = (e^x + 1)^2$.

$(D^2 - 2D + 1) y = (e^{2x} + 2e^x + 1).$

Soln:

A.E is $m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

The roots are real & equal;

$C.F = (Ax + B) e^{mx}$

$$\boxed{C.F = (Ax + B) e^{x}.}$$

To find $P.I_1$

$$P.I_1 = \frac{1}{(D^2 - 2D + 1)} e^{2x}$$

Replace $D \to a = 2.$

$$= \frac{1}{2^2 - 2(2) + 1} e^{2x}.$$

$$= \frac{e^{2x}}{4 - 4 + 1}$$

$$\boxed{P.I_1 = e^{2x}}$$

To find $P.I_2$

$$P.I_2 = \frac{1}{D^2-2D+1} 2e^x$$

Replace $D \to a = 1$

$$= 2 \cdot \frac{1}{1-2(1)+2} e^x$$

$$= 2 \frac{1}{1} e^{\cancel{2}x}$$

$$\boxed{P.I_2 = 2e^x.}$$

To find $P.I_3$

$$P.I_3 = \frac{1}{D^2-2D+1} e^{0x}$$

Replace $D \to a = 0$

$$= \frac{1}{0-0+1} e^{0x} = e^{0x} = 1$$

$$\boxed{P.I_3 = 1}$$

$\therefore$ The general solution is $y = C.F + P.I_1 + P.I_2 + P.I_3$

$$y = (Ax+B)e^x + e^{2x} + 2e^x + 1.$$

— x —

Scanned by CamScanner

Solve: $(D^2-4D+3)y = \sin 3x \cos 2x$

v. Imp

The Auxiliary equation is

$$m^2-4m+3=0$$

$$(m-3)(m-1)=$$

$$m=3,1.$$

The roots are real & distinct.

$$\boxed{C.F = Ae^{3x}+Be^{x}}$$

~~To~~ find ~~P.I~~

$$\cancel{P.I} = \frac{X}{\cancel{(D^2-4D+3)}}$$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$\sin(A-B) = \sin A\cos B - \cos A \sin B$

$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$

$\frac{1}{2}[\sin(A+B)+\sin(A-B)] = \sin A \cos B$

$$R.H.S = \sin 3x \cos 2x$$

$$= \frac{1}{2}[\sin(3x+2x)+\sin(3x-2x)]$$

$$= \frac{1}{2}[\sin 5x + \sin x].$$

$$P.I = \frac{1}{D^2-4D+3}\,\frac{1}{2}[\sin 5x+\sin x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

Replace $D^2 \to -a^2$
$= -5^2$
$= -25$

Replace $D^2 \to -a^2$
$= -1^2$
$= -1$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-4D-22}\sin 5x + \frac{1}{-4D+2}\sin x\right]$$

-3-1

$$= \frac{1}{2}\left[\frac{-4D+22}{(-4D-22)(-4D+22)}\sin 5x + \frac{1}{(-4D+2)}\frac{(4D-2)}{(4D-2)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-4D(\sin 5x)+22\sin 5x}{(-4D)^2-(22)^2} + \frac{4D\sin x - 2\sin x}{(-4D)^2-(2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{-4(5)\cos 5x + 22\sin 5x}{16D^2-484} + \frac{4\cos x - 2\sin x}{16D^2-4}\right]$$

$D^2 \to -25$ $\qquad$ $D^2 = -1$

$$= \frac{1}{2}\left[\frac{-20\cos 5x + 22\sin 5x}{16(-25)-484} + \frac{4\cos x - 2\sin x}{-16-4}\right]$$

$$= \frac{1}{2}\left[\frac{-20\cos 5x + 22\sin 5x}{-884} + \frac{4\cos x - 2\sin x}{-20}\right]$$

$$= \frac{1}{2}\left[2\left[\frac{(-10\cos 5x + 11\sin 5x)}{-442}\right] + 2\left(\frac{2\cos x - \sin x}{-10}\right)\right]$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\left[\frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}\right]$$

$$P.I = \frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}$$

The general solution is $y = C.F + P.I$

$$y = Ae^{3x} + Be^{x} + \left(\frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}\right).$$

$$\therefore y = Ae^{3x} + Be^{x} + \left[\frac{-10\cos 5x + 11\sin 5x}{-442} + \frac{2\cos x - \sin x}{-10}\right].$$

Method of variation of Parameters

consider $\frac{d^2y}{dx^2} + M^2y = X$.

$C.F = Af_1 + Bf_2$

$W = f_1 f_2' - f_1' f_2$

$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

$A = \int \frac{-f_2 X}{f_1 f_2' - f_1' f_2} dx$

$B = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx.$

—x—

(X) v. Imp Mm

using the method of variation of Parameters

solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$
X

$(D^2+4)y = \tan 2x$

Auxiliary equation is

$m^2+4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i$

The roots are real & imaginary

$\alpha = 0$
$\beta = 2$

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$= A_1\cos 2x + B_2\sin 2x$
$\quad Af_1 \qquad Bf_2.$

$f_1 = \cos 2x \quad ; \quad f_2 = \sin 2x$

$f_1' = -\sin 2x \cdot 2 \quad ; \quad f_2' = \cos 2x \cdot 2$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = \cos 2x \cdot 2\cos 2x - (-2\sin 2x \cdot \sin 2x)$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

The general soln/: is $y = C.F + P.I$ where $P.I = Af_1 + Bf_2$.

$$A = -\int \frac{f_2 \times}{W} dx$$

$$= -\int \frac{\sin 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2} \int \frac{\sin 2x \cdot \sin 2x}{\cos 2x} dx$$

$$= -\frac{1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2} \int \frac{(1-\cos^2 2x)}{\cos 2x} dx.$$

$$= -\frac{1}{2} \int \left(\frac{1}{\cos 2x} - \frac{\cos^{\cancel{2}} 2x}{\cancel{\cos 2x}} \right) dx \qquad \because \int \sec x = \log(\sec x + \tan x)$$

$$= -\frac{1}{2} \int (\sec 2x - \cos 2x) dx$$

$$= -\frac{1}{2} \left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2} \right] + c$$

$$A \cancel{P} = -\frac{1}{4} \left[\log(\sec 2x + \tan 2x) - \sin 2x + c \right]$$

$$B = \int \frac{f_1 x}{W}\, dx$$

$$= \int \frac{\cos 2x \tan 2x}{2}\, dx$$

$$= \frac{1}{2} \int \cancel{\cos 2x}\, \frac{\sin 2x}{\cancel{\cos 2x}}\, dx$$

$$= \frac{1}{2} \int \sin 2x\, dx$$

$$= \frac{1}{2} \left(\frac{-\cos 2x}{2} \right)$$

$$B = -\frac{1}{4} \cos 2x.$$

The solution is $P.I = Af_1 + Bf_2$

$$= \left[-\frac{1}{4} \left[\log (\sec 2x + \tan 2x) - \sin 2x \right] \cos 2x + \left[-\frac{1}{4} \cos 2x \sin 2x \right] \right]$$

$$= -\frac{1}{4} \left[\log (\sec 2x + \tan 2x) \cos 2x - \cancel{\sin 2x \cos 2x} + \cancel{\sin 2x \cos 2x} \right.$$

$$= -\frac{1}{4} \left[\log (\sec 2x + \tan 2x) \cos 2x \right].$$

$\therefore \; y = C.F + P.I$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \left[\log (\sec 2x + \tan 2x) \cos 2x \right]$$

— x —

Scanned by CamScanner

14m

(4) Solve $[(3x+2)^2 D^2 + 3(3x+2)D - 36]y = 3x^2+4x+1$

soln.

Let $(3x+2) = e^t$

$\therefore t = \log(3x+2)$

$x = e^t$

$t = \log x$

~~$xD = \theta$~~

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

~~$Dy = \frac{dy}{dt}$~~

$$Dy = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{1 \cdot 3}{(3x+2)} = \frac{3}{3x+2}$$

$$Dy = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{3}{3x+2}$$

$$D^2y = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{3}{3x+2}\right)$$

$$= \frac{d^2y}{dt^2} \cdot \frac{3}{3x+2} + \frac{dy}{dt}\left[\frac{(3x+2)0 - 3(3)}{(3x+2)^2}\right]$$

unwanted

$$D^2y = \frac{d^2y}{dt} \cdot \frac{3}{3x+2} + \frac{dy}{dt}\left[\frac{0-9}{(3x+2)^2}\right].$$

~~$(3x+2)D = 3\theta$~~

Another method.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{3}{3x+2} \cdot \frac{dy}{dt}\right)\frac{3}{3x+2}$$

$$= \frac{3}{3x+2} \cdot \frac{d^2y}{dt^2} + \left(\frac{-9}{(3x+2)^2}\right)\frac{dy}{dt}$$

14M (4) $(3x+2)D = 3\theta$

$(3x+2)^2 D^2 = 9\theta(\theta-1)$

$(ax+b)D = a\theta$

$(ax+b)^2 D^2 = a^2\theta(\theta-1)$

$3x+2 = e^t$

$3x = e^t - 2$

$x = \frac{e^t-2}{3}$

$$[9\theta(\theta-1) + 3(3\theta) - 36]y = 3\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1$$

$$[9\theta^2 - \cancel{9\theta} + \cancel{9\theta} - 36]y = \cancel{3}\left[\frac{e^{2t}+4-4e^t}{\cancel{9}\,3}\right] + 4\left(\frac{e^t-2}{3}\right) +$$

$$= \frac{e^{2t}+4-4e^t}{3} + \frac{4e^t-8}{3} + 1$$

$$= \frac{1}{3}\left[e^{2t}+4-\cancel{4e^t}+\cancel{4e^t}-8+3\right]$$

$$(9\theta^2-36)y = \frac{1}{3}\left[e^{2t}-1\right]$$

$$9(\theta^2-4)y = \frac{1}{3}\left[e^{2t}-1\right]$$

$$(\theta^2-4)y = \frac{1}{3\times 9}\left[e^{2t}-1\right]$$

$$(\theta^2-4)y = \frac{1}{27}\left[e^{2t}-1\right]$$

$$(\theta^2-4)y = \frac{e^{2t}-1}{27}$$

Auxiliary equation is $m^2-4=0$

$m^2 = 4$

$m = \pm 2$

$m = 2, -2$

The roots are real & distinct

$$C.F = Ae^{2t} + Be^{-2t}$$

To find P.I

$$P.I = \frac{1}{\theta^2-4}\left[\frac{e^{2t}-1}{27}\right]$$

$$= \frac{1}{27}\left[\frac{1}{\theta^2-4}e^{2t} - \frac{1}{\theta^2-4}e^{0t}\right]$$

Replace $\theta \to 2$ Replace $\theta \to 0$

$$= \frac{1}{27}\left[\frac{1}{2^2-4}e^{2t} - \frac{1}{0-4}e^{0t}\right]$$

$$= \frac{1}{27}\left[\frac{1}{4-4}e^{2t} - \frac{1}{(-4)}e^{0t}\right]$$

$$= \frac{1}{27}\left[\frac{t}{2\theta}e^{2t} - \frac{1}{(-4)}e^{0t}\right]$$

Replace $\theta \to 2$

$$= \frac{1}{27}\left[\frac{t}{2(2)}e^{2t} + \frac{1}{4}e^{0t}\right]$$

$$= \frac{1}{27}\left[\frac{t}{4}e^{2t} + \frac{1}{4}\cdot 1\right]$$

$$= \frac{1}{27(4)}\left[te^{2t}+1\right]$$

$$P.I = \frac{1}{108}\left[te^{2t}+1\right]$$

The general solution is $y = C.F + P.I$

$$y = Ae^{2t} + Be^{-2t} + \frac{1}{108}\left[te^{2t}+1\right]$$

Scanned by CamScanner

$$= Ae^{2\log(3x+2)} + Be^{-2\log(3x+2)} + \frac{1}{108}\left[\log(3x+2)\, e^{2\log(3x+2)} + 1\right]$$

$$= A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}\left[\log(3x+2)(3x+2)^2 + 1\right]$$

$$= A(3x+2)^2 + \frac{B}{(3x+2)^2} + \frac{1}{108}\left[\log(3x+2)\cdot(3x+2)^2 + 1\right].$$

— x —

Power series Solution

Legendre Polynomials

The differential equation $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

$$y_1 = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{24(2n-1)(2n-3)}x^{n-4} + \dots\right]$$

$$P_n(x) = \frac{1\cdot 3\cdot 5 \dots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{24(2n-1)(2n-3)}x^{n-4} + \dots\right]$$

$\therefore P_n(x)$ is a terminating Series

$$P_n(1) = 1$$

Scanned by CamScanner

Rodrigue's formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

Put $n = 0, 1, 2, \ldots$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2-1) = \frac{1}{2}(2x-0) = x.$$

$$P_2(x) = \frac{1}{2^2(2)} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{8} \frac{d^2}{dx^2} (x^4-2x^2+1)$$

$$P_2(x) = \frac{1}{2}(3x^2-1).$$

$$P_3(x) = \frac{1}{2^3 3!} \cdot \frac{d^3}{dx^3} (x^2-1)^3$$

$$= \frac{1}{48} \frac{d^3}{dx^3} (x^6-3x^4+3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

$$P_6(x) = \frac{1}{6}\left[231x^6-351x^4+105x^2-5\right]$$

$$x^4 = \frac{1}{35}\left[8P_4(x)+20P_2(x)+7P_0(x)\right]$$

We know that $P_0(x) = 1$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$$

$$\frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_2(x)\right]$$

$$= \frac{1}{35}\left[\cancel{8}\frac{1}{\cancel{8}}(35x^4 - 30x^2 + 3) + \cancel{20}^{10} \cdot \frac{1}{\cancel{2}}(3x^2-1) + 7\right.$$

$$= \frac{1}{35}\left[35x^4 - 3\cancel{0}x^2 + 3 + 3\cancel{0}x^2 - 10 + 7\right]$$

$$= \frac{1}{35}\left[35x^4 - \cancel{10} + \cancel{10}\right] \Rightarrow \frac{\cancel{35}x^4}{\cancel{35}} = x^4$$

—x—

~~f(P4) 17~~

V. Imp

$$f(x) = x^3 - 5x^2 + x + 2$$

We know that $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$2P_3(x) = 5x^3 - 3x$$

$$2P_3(x) + 3x = 5x^3$$

$$\frac{2P_3(x) + 3x}{5} = x^3$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$$

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - 5x^2 + x + 2$$

$$= \left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + x + 2$$

$$= \frac{2}{5}P_3(x) + \frac{3}{5}x - \frac{10}{3}P_2(x) - \frac{5}{3} + x + 2.$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \left(\frac{3}{5}+1\right)x + \frac{1}{3}$$

$$= \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{8}{5} x + \frac{1}{3}.$$

$$= \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{8}{5} P_1(x) + \frac{1}{3} P_0 x.$$

Since:. $P_1(x) = x$; $P_0(x) = 1$

— x —.

(10) v. Imp:-

Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre Polynomial.

soln:

$$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$$

$$8P_4(x) = 35x^4 - 30x^2 + 3.$$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$2P_2(x) = 3x^2 - 1$

$\frac{2}{3} P_2(x) + 1 = x^2$

$x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} x$

$$8P_4(x) + 30x^2 - 3 = 35x^4$$

$$x^4 = \frac{1}{35}[8P_4(x) + 30x^2 - 3].$$

$$\therefore f(x) = \frac{1}{35}[8P_4(x) + 30x^2 - 3] + 3x^3 - x^2 + 5x - 2.$$

$$= \frac{1}{35}[8P_4(x) + 30x^2 - 3] + 3\left[\frac{2}{5} P_3(x) + \frac{3}{5} x\right] - x^2 + 5x - 2$$

$$= \frac{1}{35} 8P_4(x) + \frac{30}{35} x^2 - \frac{3}{35} + \frac{6}{5} P_3(x) + \frac{9}{5} x - x^2 + 5x - 2$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) + \left(\frac{30}{35} - 1\right) x^2 + \left(\frac{9}{5} + 5\right) x - \left(\frac{3}{35} + 2\right)$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{5}{35}\left(\frac{2}{3} P_2(x) + \frac{1}{3}\right) + \frac{34}{5} x - \frac{73}{35}$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{1}{7}\left(\frac{2}{3}\right) P_2(x) - \frac{1}{7}\,\frac{1}{3}$$

$$+ \frac{34}{5}\left[P_1(x)\right] - \frac{73}{35}.$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) -$$

$$\left(\frac{1}{21} + \frac{73}{35}\right).(1)$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) -$$

$$\left(\frac{5+219}{105}\right) P_0(x).$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \frac{224}{105} P_0$$

—x—

Orthogonality Property of Legendre Polynomials

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad (m \neq n)$$

Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2) y = 0 \qquad \text{(X) 2m}$$

Bessel function of I kind

$$y_1 = \frac{x^n}{2^n \Gamma(n+1)} \sum_{K=0}^{n} \frac{(-1)^K \Gamma(n+1)}{2^{2K} K!\, \Gamma(n+K+1)} x^{2K} = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(n+K+1)} \left(\frac{x}{2}\right)^{n+}$$

$$J_{-n}(x) = \sum_{K=0}^{n} \frac{(-1)^K}{K!\,\Gamma(-n+K+1)} \left(\frac{x}{2}\right)^{-n+2K}$$

when n is an integer, the two function $J_n(x)$ & $J_{-n}(x)$

$$J_{-n}(x) = (-1)^n . J_n(x).$$

— x —

Find $J_0(x)$ & $J_1(x)$

$$J_n(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(n+K+1)} \left(\frac{x}{2}\right)^{n+2K}$$

$$J_0(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(K+1)} \left(\frac{x}{2}\right)^{2K}$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,K!} \left(\frac{x}{2}\right)^{2K} \quad [\because \Gamma(K+1) = K!]$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{(K!)^2} \left(\frac{x}{2}\right)^{2K}$$

$$= 1 - \frac{1}{1}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^2 + \cdots$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

$$J_1(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(1+K+1)} \left(\frac{x}{2}\right)^{1+2K}$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(K+2)} \left(\frac{x}{2}\right)^{1+2K}$$

$\Gamma(K+1) = K!$

$\Gamma(K+1+1) = (K+1)!$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,(K+1)!} \left(\frac{x}{2}\right)^{1+2K}$$

$$= \frac{1}{1}\frac{x}{2} - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3 + \frac{(-1)^2}{2!\,(2+1)!}\left(\frac{x}{2}\right)^5 \ldots$$

$$= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 \ldots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6}$$

$$J_0(0) = 1$$

$$J_1(0) = 0.$$

— x —

x f f v.Imp. Qm.

Find $J_{1/2}(x)$

soln:

$$J_n(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(n+K+1)}\left(\frac{x}{2}\right)^{n+2K}$$

$n = 1/2$

$$J_{\frac{1}{2}}(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(\frac{1}{2}+K+1)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2K}$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(K+\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}+2K}$$

$\because \Gamma(K+1) = K!$
$\Gamma(K+1+1) = (K+1)!$

$$= \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{1!\,\Gamma(1+\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}+2} + \frac{1}{2!\,\Gamma(2+\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}+4} \ldots$$

$$= \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^{1/2}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(7/2)}(x/2)^{1/2}\left(\frac{x}{2}\right)^4 \ldots$$

$$= \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{\frac{1}{2}\Gamma(1/2)} - \frac{1}{\cancel{2\times 5}\cdot\frac{3}{\cancel{2}}\cdot\frac{1}{2}\Gamma(1/2)} \left(\frac{x}{2}\right)^{4} \cdots \right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma(1/2)} \left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} \cdots \right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}} \cdot \frac{2}{x} \left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \right]$$

$\boxed{\Gamma(1/2) = \sqrt{\pi}}$

$$= \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{2}}\cdot\sqrt{\pi}} \cdot \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{x}}\cdot\sqrt{x}} = \sqrt{\frac{2}{\pi x}} \sin x.$$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$\Gamma(5/2) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right).$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

— x —.

— x —.

~~Solve $p = \tan x = \frac{p}{1+p^2}$.~~

Find $J_{-1/2}(x)$

Soln:

$$J_n(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(n+K+1)} \left(\frac{x}{2}\right)^{n+2K}.$$

$n = -1/2$

$$J_{-1/2}(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(-1/2+K+1)} \left(\frac{x}{2}\right)^{-1/2+2K}$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{K!\,\Gamma(K+1/2)} \left(\frac{x}{2}\right)^{-1/2+2K}$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{x}{2}\right)^{-1/2} - \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{3/2} + \frac{1}{2!\,\Gamma(5/2)}\left(x/2\right)^{7/2} \cdots$$

$$=\left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\Gamma(1/2)}-\frac{1}{1/2\,\Gamma(1/2)}\left(\frac{x}{2}\right)^2+\frac{1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(}\ (\frac{x}{}\right.$$

$$=(2/x)^{1/2}\frac{1}{\Gamma(1/2)}\left[1-\frac{x^2}{\frac{1}{\cancel{2}}\cdot\cancel{2}}+\frac{1}{2\cdot\frac{3}{2}\cdot 1/\cancel{2}}\cdot\frac{x^4}{\cancel{16}\,2}\cdots\right]$$

$$=\frac{\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\right]$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x.$$

$\Gamma(n+1)=n\Gamma(n)$

$\Gamma(3/2)=\Gamma\left(\frac{1}{2}+1\right)$

$=\frac{1}{2}\Gamma(1/2$

$\Gamma(5/2)=\Gamma(3/2+$

$=\frac{3}{2}\Gamma(3/2)$

$\cancel{=3/2\cdot 1/2\,\pi}$

$=\frac{3}{2}\,\frac{1}{2}\,\Gamma(1/$

—— x ——

(x) Imp. (x).

Prove that $\frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$

Proof:

We know that

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$x^n.J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\,x^n.x^{n+2k}}{2^{n+2k}\,k!\,\Gamma(n+k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{2n+2k}}{2^{n+2k}\,k!\,\Gamma(n+k+1)}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k)\, x^{2n+2k-1}}{2^{n+2k}\, k!\, \Gamma(n+k+1)}$$ $\quad | x^{2n} = x^n x^n$

$$= \sum_{k=0}^{\infty} x^n \frac{(-1)^k \cancel{2}(\cancel{n+k})\, x^{n+2k-1}}{\cancel{2}\cdot 2^{n+2k-1}\, k!\, (\cancel{n+k})\, \Gamma(n+k)}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n+k)} \left(\frac{x}{2}\right)^{n+2k-1}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n-1+k+1)} \left(\frac{x}{2}\right)^{n-1+2k}$$

$\frac{d}{dx}(x^n) = nx^n$

$x^{2n} = x^n \cdot x^n$

$1 = 2^0 = 2^{1-1} = 2^1 \cdot 2^{-1}$

$\Gamma(n+1) = n\Gamma(n)$

$$= x^n J_{n-1}(x)$$

$$\therefore \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$\therefore$ Hence it is Proved

— x —

Similarly.

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

— x —

Proove that

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

$1 = x^0 = x^{n-n}$

Proof

L.H.S

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = \frac{d}{dx}\left[x^{n-n} x^1 . J_n(x) . J_{n+1}(x)\right]$$

$$= \frac{d}{dx}\left[x^{-n} J_n(x) . x^{n+1} J_{n+1}(x)\right]$$

$$= x^{-n} J_n(x) \frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] + x^{n+1} J_{n+1}(x) .$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] \to (1)$$

Now, we know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

Change n to n+1

$$\frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_n(x)$$

Also we know that

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$\therefore$ From (1).

$$\frac{d}{dx}\left[x J_n(x) . J_{n+1}(x)\right] = x^{-n} J_n(x)\, x^{n+1} J_n(x) + x^{n+1} J_{n+1}(x)\left[-x^{-n} J_{n+1}(x)\right]$$

$$= x^{-n+n+1} J_n^2(x) - x^{n+1-n} J_{n+1}^2(x)$$

$$= x J_n^2(x) - x J_{n+1}^2(x)$$

$$= x\left[J_n^2(x) - J_{n+1}^2(x)\right].$$

—x—

UNIT-2 ODE OF HIGHER ORDERS

Questions	**opt1**	**opt2**	**opt3**	**opt4**	**Answer**
An equation involving one dependent variable and its derivatives with respect to one independent variable is called ___________	Ordinary Differential Equation	Partial Differential Equation	Difference Equation	Integral Equation	Ordinary Differential Equation
The roots of the Auxillary equation of Differential equation, (D^2-2D+1)y=0 are	(0 1)	(3 2)	(1 2)	(1 1)	(1 1)
The order of the (D^2+D)y=0 is	2	1	0	-1	2
The roots of the Auxillary equation of Differential equation, (D^4-1)y=0 are	(1 1 1 1)	(1 1 -1 1)	(1 -1 1 -1)	(1 -1 i -i)	(1 -1 i -i)
The degree of the (D^2+2D+2) y=0 is	1	3	0	2	1
The particular integral of (D^2-2D+1)y=e^x is__________	((x^2)/2) e^x	(x/2) e^x	((x^2)/4) e^x	((x^3)/3) e^x	((x^2)/2) e^x
The roots of the Auxillary equation of Differential equation (D^2-4D+4)y=0 are	(2 1)	(2 2)	(2 -2)	(-2 2)	(2 2)
The P.I of the Differential equation (D^2 -3D+2)y=12 is________	1 / 2	1 / 7	6	10	6
If the roots of the auxilliary equation are real and distinct then the C.F is...	Ae^(m1x)+Be^(m2x)	(A+Bx) e^ (m1x)	e^(αx) (Acosβx+Bsinβx)	(A+Bx) e^ (m2x)	Ae^(m1x)+Be^(m 2x)
If the roots of the auxilliary equation are real and equal then the C.F is...	Ae^(m1x)+Be^(m2x)	e^(αx) (Acosβx+Bsinβx)	(A+Bx) e^ (mx)	(A+Bx) e^ (-mx)	(A+Bx) e^ (mx)
If the roots of the auxilliary equation are complex then the C.F is...	Ae^(m1x)+Be^(m2x)	e^(-αx) (Acosβx+Bsinβx)	(A+Bx) e^ (mx)	e^(αx) (Acosβx+Bsinβx)	e^(αx) (Acosβx+Bsinβx)
If f(D)=D^2 -2, (1/f(D))e^2x=______	(1 / 2) e^x	(1 / 4) e^2x	(1 / 2) e^(-2x)	(1 / 2) e^2x	(1 / 2) e^2x
If f(D)=D^2 +5, (1/f(D)) sin 2x =______	sin x	cos x	sin 2x	-sin 2x	sin 2x
The particular integral of (D^2 +19D+60)y= e^x is______	(-e^(-x))/80	(e^(-x))/80	(e^x)/80	(-e^x)/80	(e^x)/80
The particular integral of (D^2+25) y= cosx is _________	(cosx)/24	(cosx)/25	(-cosx)/24	(-cosx)/25	(cosx)/24
The particular integral of (D^2+25) y= sin4x is __________	(-sin4x)/9	(sin4x)/9	(sin4x)/41	(-sin4x)/41	(sin4x)/9
The particular integral of (D^2+1) y= sinx is ___________	xcosx/2	(-xcosx)/2	(-xsinx)/2	xsinx/2	(-xcosx)/2
The particular integral of (D^2 -9D+20)y=e^(2x) is___________	e^(2x) /6	e^(2x) /(-6)	e^(2x) /12	e^(2x) /(-12)	e^(2x) /6
The particular integral of (D^2-1) y= sin2x is _________	(-sin2x)/5	sin2x/5	sin2x/3	(-sin2x)/3	(-sin2x)/5
The particular integral of (D^2+2) y= cosx is _________	(-cosx)	(-sinx)	cosx	sinx	cosx
The particular integral of (D^2- 7D-30)y= 5 is________	(1/30)	(-1/30)	(1/6)	(-1/6)	(-1/6)
The particular integral of (D^2- 12D-45)y= -9 is________	(-1/5)	(1/5)	(1/45)	(-1/45)	(1/5)
The particular integral of (D^2- 11D-42)y=21 is_______	(-1/42)	(1/42)	(1/2)	(-1/2)	(-1/2)
The particular integral of (D^2+1) y= 2 is ___________	1	2	-1	-2	2
solve (D^2+2D+1) y=0	y=(AX+B)e^(-1)x	y=(AX+B)e^(-2)x	y=(AX^2+B)e^(-1)x	y=(AX-B)e^(-1)x	y=(AX+B)e^(-1)x
The ______ of a PDE is that of the highest order derivative occurring in it	degree	power	order	ratio	order
The degree of the a PDE is ______of the higest order derivative	power	ratio	degree	order	power
C.F+P.I is called ------------------- solution	singular	complete	general	particular	general
Particular integral is the solution of -------------	f(a,b)=F(x,y)	f(1,0)=0	[1/f(D,D')]F(x,y)	f(a,b)=F(u,v)	[1/f(D,D')]F(x,y)
Which is independent varible in the equation z= 10x+5y	x&y	z	x,y,z	x alone	x&y
Which is dependent varible in the equation z=2x+3y	x	z	y	x&y	z
J_-(1/2) (x)=	sqrt(2/pi) cosx	sqrt(4/pi) cosx	sqrt(2/pi) sinx	sqrt(4/pi) sinx	sqrt(2/pi) cosx
J_(1/2) (x)=	sqrt(2/pi) cosx	sqrt(4/pi) cosx	sqrt(2/pi) sinx	sqrt(4/pi) sinx	sqrt(2/pi) sinx
(1-x^2)d^y/dx^2-2xdy/dx+n(n+1)y=0 is called _________	Legendre's Equation	Cauchy's equation	Partial Equation	Bessel's Equation	Legendre's Equation

—x—

UNIT - 3

Partial Differentitional equation. (P.D.E)

Application of P.D.E

* P.D.E is an important mathematical tool for solving engineering Problem in control system, Bio technology, chemical engineering, robotics etc..

* A Partial Differential equation is one which is involve Partial Derivatives. The order of the PDE is the order of the higher derivative occurs in it.

Notation:

* z has a dependant variable which depends on two variable x and y.

$$\frac{dz}{dx} = p \qquad \frac{d^2z}{dx^2} = r$$

$$\frac{dz}{dy} = q \qquad \frac{d^2z}{dx\,dy} = \frac{d^2z}{dy\,dx} = s \; ; \; \frac{d^2z}{dy^2} = t$$

Linear Partial differential equation

* A P.D.E said to be linear if, it is are the first degree in the dependent variable and its Partial derivatives.

* It does not contain the Product of dependent variable and either of its Partial derivatives.

* It does not contain transcendental function.

Ex: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

* A PDE which is not linear.

Ex:

$$\left(\frac{\partial f}{\partial x}\right)^3 + \frac{\partial f}{\partial t} = 0$$

—x—.

Solution of Standard types of First order P.D.E

Type 1:

$$F(p,q) = 0$$

1. Solve $\sqrt{p} + \sqrt{q} = 1$

Given $\sqrt{p} + \sqrt{q} = 1 \rightarrow (1)$

It is of the type $F(p,q) = 0$

Let $z = ax + by + c \rightarrow (2)$ be the Solution of (1)

Diff (2) Partially with respect to x

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

Diff (2) Partially with respect to y

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

Sub p & q in (1)

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1-\sqrt{a})^2$$

Hence the complete solution is

$$z = ax + (1-\sqrt{a})^2 + c.$$

—x—

Solve $pq = 1$

Given $pq = 1 \rightarrow (1)$

It is of the ~~form~~ type $F(p,q) = 0$

Let $z = ax + by + c \rightarrow (2)$ be the solution of (1)

Type 2 :

Clairaut's form $z = px + qy + F(p,q)$

Solve $z = px + qy + p^2 - q^2$

Given

$$z = px + qy + p^2 - q^2 \longrightarrow ①$$

~~Given~~

This is of the form $z = px + qy + \cancel{F(q,}F(p,q)$

~~Let $z = ax + by + c \rightarrow ②$ be the solution of~~

Let $z = ax + by + c \longrightarrow ②$ be the solution of ①

pb

Diff ② Partially with respect to x,

$$\frac{\partial z}{\partial x} = a \quad \Rightarrow \quad p = a$$

Diff ② Partially with respect to y

$$\frac{\partial z}{\partial y} = b \quad \Rightarrow \quad q = b$$

Sub $p = a$ and $q = b$ in ①

$$z = ax + by + a^2 - b^2 \longrightarrow ③.$$

This is the complete solution.

To find the singular integral.

Diff ③ Partially with respect to a.

$$\frac{\partial z}{\partial a} = x + 0 + 2a$$

$$\frac{\partial z}{\partial a} = 0 \quad \Rightarrow \quad x + 2a = 0$$

$$2a = -x$$

$$\boxed{a = -x/2}$$

Scanned with CamScanner

Diff ③ Partially with respect to b.

$$\frac{\partial z}{\partial b} = y + 0 - 2b$$

$$\frac{\partial z}{\partial b} = 0 \Rightarrow \cancel{y} \quad y - 2b = 0$$

$$y = 2b$$

$$\frac{y}{2} = b \Rightarrow \boxed{b = \frac{y}{2}}$$

Sub a & b in ③

$$z = -\frac{x}{2} \cdot x + \frac{y}{2} y + \left(-\frac{x}{2}\right)^2 - \left(y/2\right)^2$$

$$z = -\frac{x^2}{2} + \frac{y^2}{2} - \frac{x^2}{4} - \frac{y^2}{4}$$

$$z = \frac{-2x^2+2y^2+x^2-y^2}{4}$$

(or)

$$4z = -x^2+y^2$$

∴ This is the singular integral

—x—.

v. Imp 14 M

$$z = px + qy + \sqrt{1+p^2+q^2} \quad ②$$

Given $z = px + qy + \sqrt{1+p^2+q^2}$

This is of the form $z = px + qy + F(p,q)$.

We know that $p = a$ & $q = b$

∴ The complete integral is

$$z = ax + by + \sqrt{1+a^2+b^2}$$

To find the singular integral

$f(z, p, q) = 0 \rightarrow (1)$

Let $u = x + ay$ be the solution of (1)

$\Rightarrow \quad u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \; ; \; \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{dx} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot 1$$

$$q = \frac{dz}{dy} = \frac{dz}{du} \cdot \frac{du}{dy} = \frac{dz}{du} \cdot a$$

$$\boxed{p = \frac{dz}{du} \, , \quad q = a\frac{dz}{du}}$$

—x—

Given $\quad p(1+q) = qz \rightarrow (1)$

It is of the form $F(z, p, q) = 0$.

Let $u = x + ay$ be the solution of (1)

$$\frac{du}{dx} = 1 \; ; \; \frac{du}{dy} = a.$$

$$p = \frac{dz}{du} \, , \quad q = a\frac{dz}{du}.$$

$$\frac{\cancel{dz}}{\cancel{du}}\left(1 + a\frac{dz}{du}\right) = a\,\frac{\cancel{dz}}{\cancel{du}} \cdot z$$

$$1 + a\frac{dz}{du} = az.$$

$$a\frac{dz}{du} = az-1$$

$$\frac{a\,dz}{az-1} = du$$

$$du = \frac{a\,dz}{az-1}$$

Integrating on both sides

$$\int du = \int \frac{a\,dz}{az-1}$$

$$u = \log(az-1) + \log c$$

$$x+ay = \log c(az-1).$$

— x —

Type 4

Separable equation:

First order Partial differential equations are Separable

$$f_1(x,p) = f_2(y,q)$$

Put $f_1(x,p) = f_2(y,q) = K$

$$f_1(x,K) = p\ ;\ f_2(y,K) = q.$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = f_1(x,K)\,dx + f_2(y,K)\,dy$$

— x —.

Solve:

$$p - q = x^2 + y^2$$

$$p - x^2 = q + y^2$$

It is of the form $f_1(x, p) = f_2(y, q)$.

$$p - x^2 = k \; ; \quad q + y^2 = k$$

$$p = k + x^2 \; ; \quad q = k - y^2$$

$$dz = p\,dx + q\,dy$$

$$dz = (k + x^2)\,dx + (k - y^2)\,dy$$

Integrating on both sides.

$$\int dz = \int (k + x^2)\,dx + \int (k - y^2)\,dy$$

$$z = kx + \frac{x^3}{3} + ky - \frac{y^3}{3} + c \ldots$$

$$z = k(x + y) + \frac{x^3 - y^3}{3} + c \ldots$$

—x—

<u>Lagrange's</u> <u>linear</u> <u>equation</u>.

* $Pp + Qq = R$. is known as Lagrange's equation where P, Q, R are the functions of x, y, z. To solve this it is enough to solve the subsidiary equation.

Scanned with CamScanner

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Auxiliary

* Methods of Grouping
* Method of Multipliers.

Methods of Grouping

* $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$u(x,y) = a \; ; \quad v(x,y) = b$

$\phi(u,v) = 0.$

—x—

Solve

~~Px + Qy~~ $\quad px + qy = z$

Soln:

It is of the form $Pp + Qq = R$

$P = x \; ; \quad Q = y \; ; \quad R = z$

Subsidiary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$\log x = \log y + \log c_1$

$\log x - \log y = \log c_1$

$\log(x/y) = c_1$

$$\frac{x}{y} = c_1$$

$$\boxed{u = x/y}$$

$$\int \frac{dx}{x} = \int \frac{dz}{z}$$

$$\log x = \log z + \log c_2$$

$$\log x - \log z = \log c_2$$

$$\log\left(\frac{x}{z}\right) = \log c_2$$

$$\frac{x}{z} = c_2$$

$$\boxed{v = \frac{x}{z}}$$

$$\phi(u, v) = 0$$

$$\phi\left(\frac{x}{y}, \frac{x}{z}\right) = 0$$

— x —

Methods of Multiplier.

* choose any three multipliers l, m, n which may be constant as functions of x, y, z.

* It is Possible to choose l, m, n such that $lP + mQ + nR = 0$ then automatically denominator zero $l\,dx + m\,dy + n\,dz = 0$. The multipliers l, m, n are Lagrangian Multipliers.

solve: v.Imp

N/M

$$x(y-z)p+y(z-x)q = z(x-y)$$ (10)

Soln:

It is of the form $Pp+Qq=R$.

$P = x(y-z)$, $Q = y(z-x)$, $R = z(x-y)$

subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

choose 1,1,1 as lagrangian Multipliers,

$$\frac{1.dx+1.dy+1.dz}{1[x[y-z]]+1[y(z-x)]+1[z(x-y)]} = \frac{dx+dy+dz}{0}$$

$$dx+dy+dz = 0$$

$$d(x+y+z) = 0$$

Integrating,

$$\boxed{x+y+z = c_1}$$

Choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as lagrangian Multipliers.

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{\cancel{x}}\cdot\cancel{x}(y-z)+\frac{1}{\cancel{y}}\cdot\cancel{y}(z-x)+\frac{1}{\cancel{z}}\cdot\cancel{z}(x-y)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{z}\,dz=0.$$

Integrating $\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$

$$\log x+\log y+\log z=\log c_2$$

$$\log \overset{(x,y,z)}{\cancel{(x+y+z)}}=\log c_2$$

$$\boxed{xyz=c_2}\quad \boxed{\cancel{x+y+z=c_2}}\quad \boxed{c_2=xyz}$$

The general solution is $\phi(u,v)=0$

$$\phi(x+y+z,\,xyz)=0.$$

V.Imp Sum

Solve

$$x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2).$$

It is of the form $Pp+Qq=R$

$P=x(y^2-z^2)$, $Q=y(z^2-x^2)$, $R=z(x^2-y^2)$.

Subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}.$$

choose x,y,z as lagrangian Multipliers

$$\frac{x\,dx+y\,dy+z\,dz}{x[x(y^2-z^2)]+y[y(z^2-x^2)]+z[z(x^2-y^2)]}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$x\,dx + y\,dy + z\,dz = 0.$$

Integrating,

$$\int (x\,dx + y\,dy + z\,dz) = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$$\boxed{x^2 + y^2 + z^2 = c_1}$$

choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as lagrangian Multiplier.

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}\cdot x(y^2 - z^2) + \frac{1}{y}\cdot y(z^2 - x^2) + \frac{1}{z}\cdot z(x^2 - y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log (x, y, z) = \log c_2$$

$$\boxed{x, y, z = c_2}$$

The general solution is $\phi(u, v) = 0$

$$\phi(x^2 + y^2 + z^2,\ xyz) = 0.$$

Homogeneous linear equation.

* The linear PDE with constant co-efficient in which all the partial derivatives are of same order is called Homogeneous. otherwise it is called non-Homogeneous.

* A homogeneous linear PDE of nth order with constant co-efficient is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots\cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y).$$

where a's are constant

$$f(D, D')z = F(x,y)$$

The solution of $f(D, D')z = 0$ is called complementary function.

To Find P.I

$$P.I = \frac{1}{f(D,D')} F(x,y).$$

$z = C.F + P.I$ is the complete solution.

case (i)

The roots are real & distinct $(m_1 \neq m_2)$

$$z = f_1(y + m_1 x) + f_2(y + m_2 x)$$

case (ii)

The roots are real & equal $(m_1 = m_2 = m)$

Scanned with CamScanner

$z = f_1(y+m_1x) + x f_2(y+m_2x) +$

$x^2(y+m_3x).$

— x —

solve $(D^2 - 4DD' + 3D'^2)z = 0.$

soln:

The auxiliary equation is

$m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$m = 1, 3$

The roots are real & distinct

$z = f_1(y+m_1x) + f_2(y+m_2x)$

$z = f_1(y+x) + f_2(y+3x).$

— x —.

Solve $(D^2 + DD' - 2D'^2)z = 0.$

soln:

The auxiliary equation is

$m^2 + m - 2 = 0$

$(m+2)(m-1) = 0$

$m = 1, -2.$

The roots are real & distinct.

$z = f_1(y+m_1x) + f_2(y+m_2x)$

$z = f_1(y+x) + f_2(y-2x).$

Solve $(D^3 - D^2D' + DD'^2 - D'^3)z = 0.$

The auxiliary equation is

1	1	-1	1	-1
	0	1	0	1
	1	0	1	0

$m^3 - m^2 + m - 1 = 0.$

$m = 1, \quad m^2 + 0m + 1 = 0$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1} = \pm i$

$m = 1, i, -i$

The roots are real & distinct

$z = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x)$

$z = f_1(y + x) + f_2(y + ix) + f_3(y - ix).$

—x—.

Type 2:

R.H.S $= e^{ax + by}$

Replace $D \to a,\ D' \to b$

$$P.I = \frac{1}{f(D, D')} e^{ax + by}$$

—x—

Solve

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}.$$

Soln.

$D^2z - 5DD'z + 6D'^2z = e^{x+y}$

$[D^2 - 5DD' + 6D'^2]z = e^{x+y}$

$D \to \partial/\partial x$

$D' \to \partial/\partial y$

The auxiliary equation is

$D \to m$

$D' \to 1$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

The roots are real & distinct

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$$C.F = f_1(y + 2x) + f_2(y + 3x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$$

Replace $D \to a = 1$; $D' \to b = 1$

$D \to a$

$D' \to b$

$$= \frac{1}{1 - 5 + 6} e^{x+y}$$

$$P.I = \frac{e^{x+y}}{2}$$

$\therefore$ The complete solution is $z = C.F + P.I$

$$z = f_1(y + 2x) + f_2(y + 3x) + \frac{e^{x+y}}{2}$$

— x —

Find P.I of $(D^2 + 4DD')z = e^x$.

Soln:

$$P.I = \frac{1}{D^2 + 4DD'} e^x$$

$D \to a = 1$: $D' \to b = 0$.

$$= \frac{1}{1^2 + 0} e^x$$

e^x

Solve:

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x-y}$$

$$D^2 z - 4DD'z + 4D'^2 z = e^{2x-y}$$

$$(D^2 - 4DD' + 4D'^2) z = e^{2x-y}$$

The auxiliary equation is

$D \to m$

$D' \to 1$

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2, 2$$

4

$\wedge$

$-2 \quad -2$

The roots are real & equal.

$$C.F = f_1(y + m_1 x) + x f_2(y + m_2 x)$$

$$C.F = f_1(y + 2x) + x f_2(y + 2x).$$

To find P.I

$$P.I = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x-y}$$

Replace $D \to a = 2$, $D' \to b = -1$.

$$= \frac{1}{2^2 - 4(2)(-1) + 4(1)^2} e^{2x-y}$$

$$= \frac{1}{4+8+4} e^{2x-y}$$

$$P.I = \frac{e^{2x-y}}{16}$$

The complete solution is $z = C.F + P.I$

$$z = f_1(y+2x) + x f_2(y+2x) + \frac{e^{2x-y}}{16}$$

— x —

$$R.H.S = x^r y^s$$

$$P.I = \frac{1}{f(D,D')} x^r y^s = [f(D,D^{-1})]^{-1} x^n y^s.$$

Solve $[D^2 - 7DD' + 6D'^2] z = xy$

Soln:

The auxiliary equation is

$$m^2 - 7m + 6 = 0$$

$$(m-1)(m-6) = 0$$

$$m = 1, 6$$

The roots are real & distinct.

$$C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

$$C.F = f_1(y+x) + f_2(y+6x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 7DD' + 6D'^2} xy$$

$$= \frac{1}{D^2\left[1 - \frac{7\cancel{D}D'}{D^{\cancel{2}}} + \frac{6D'^2}{D^2}\right]} xy$$

$$= \frac{1}{D^2\left[1-\left(\frac{7D'}{D}-\frac{6D'^2}{D^2}\right)\right]}xy$$

$$= \frac{1}{D^2}\left[1-\left(\frac{7D'}{D}-\frac{6D'^2}{D^2}\right)\right]^{-1}xy$$

$$[1-x]^{-1} = 1+x+x^2+\cdots$$

$$= \frac{1}{D^2}\left[1-\left(\frac{7D'}{D}-\frac{6D'^2}{D^2}\right)\right]xy$$

$$= \frac{1}{D^2}\left[xy-\left(\frac{7D'}{D}-\frac{6D'^2}{D^2}\right)xy\right]$$

$$= \frac{1}{D^2}\left[xy-\left(\frac{7D'}{D}(xy)-\frac{6D'^2}{D^2}(xy)\right)\right]$$

$$= \frac{1}{D^2}\left[xy-\left(\frac{7x}{D}-0\right)\right]$$

$D \to \partial/\partial x$

$D' \to \partial/\partial y$

$$= \frac{1}{D^2}\left[xy-\frac{7x^2}{2}\right]$$

$$= \frac{1}{D}\int\left(xy-\frac{7x^2}{2}\right)$$

$$= \frac{1}{D}\left[\frac{x^2}{2}y-\frac{7}{2}\frac{x^3}{3}\right]$$

$$= \left[\frac{1}{2}\frac{x^3}{3}y-\frac{7}{6}\cdot\frac{x^4}{4}\right]$$

$$P.I = \left[\frac{1}{6}x^3y-\frac{7}{24}x^4\right]$$

Scanned with CamScanner

$$= \frac{4x^3y - 7x^4}{24}$$

$$P.I = \frac{x^3}{24}[4y - 7x]$$

$\therefore$ The complete solution is $z = C.F + P.I$

$$z = f_1(y+x) + f_2(y+6x) + \frac{x^3}{24}[4y-7x].$$

14m.
Imp
Solve

$$[D^2 - DD' - 6D'^2]z = x^2y + e^{3x+y}$$

The auxiliary equation is

$$m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3, -2$$

The roots are real & distinct

$$C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

$$C.F = f_1(y+3x) + f_2(y-2x).$$

To find P.I

$$P.I_1 = \frac{1}{D^2 - DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2\left[1 - \frac{\cancel{D}D'}{D^{\cancel{2}}} - \frac{6D'^2}{D^2}\right]} x^2y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} + \frac{6D'^2}{D^2}\right)\right]^{-1} x^2y$$

$(1-x)^{-1} = 1+x+x^2 \ldots$

$$= \frac{1}{D^2}\left[1-\left(\frac{D'}{D}+\frac{6D'^2}{D^2}\right)\right]x^2y$$

$$= \frac{1}{D^2}\left[x^2y-\left[\frac{D'}{D}(x^2y)+\frac{D'^2}{D^2}(x^2y)\right]\right]$$

$$= \frac{1}{D^2}\left[x^2y-\left(\frac{1}{D}x^2+0\right)\right]$$

$$= \frac{1}{D^2}\left[x^2y-\int x^2dx\right]$$

$$= \frac{1}{D^2}\left[x^2y-\frac{x^3}{3}\right]$$

$$= \frac{1}{D}\left[\frac{x^3}{3}y-\frac{x^4}{3\cdot 4}\right]$$

$$= \left[\frac{x^4}{3\cdot 4}y-\frac{x^5}{3\cdot 4\cdot 5}\right]$$

$$P.I_1 = \frac{x^4}{12}y-\frac{x^5}{60}$$

<u>To find</u> <u>$P.I_2$</u>

$$P.I_2 = \frac{1}{D^2-DD'-6D'^2}e^{3x+y}.$$

$D \to a$

$D' \to b$

Replace $D \to a = 3$; $D' \to b = 1$

$$= \frac{1}{3^2-3(1)-6(1)}e^{3x+y}$$

$$= \frac{1}{9-3-6}e^{3x+y}$$

$$= \frac{1}{9-9} e^{3x+y}$$

$$= \frac{x}{2D-D'} e^{3x+y}$$

$$= \frac{x}{2(3)-1} e^{3x+y}$$

$$P.I_2 = \frac{x}{5} e^{3x+y}.$$

$\therefore$ The complete solution is $Z = C.F + P.I_1 + P.I_2$

$$Z = f_1(y+3x) + f_2(y-2x) + \frac{x^4}{12}y - \frac{x^5}{60} + \frac{x}{5} e^{3x+y}.$$

— x —

R.H.S $\quad \sin(ax+by)$ (or) $\cos(ax+by)$

$D^2 \to -a^2, \quad DD' \to -ab \; ; \quad D'^2 \to -b^2$

— x —

Solve:

$$[D^2 - 2DD' + 2D'^2] Z = \sin(x-y).$$

Auxiliary equation is $\qquad D \to m$

$$m^2 - 2m + 2 = 0 \qquad D' \to 1$$

$a = 1$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad b = -2$$

$c = 2$

$$= \frac{-(-2) \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} = \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$m = 1 \pm i$$

$$m = 1+i \; ; \; 1-i$$

The roots are real & distinct.

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$$C.F = f_1(y + (1+i)x) + f_2(y + (1-i)x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 2DD' + 2D'^2} \sin(x-y)$$

$D^2 \rightarrow -a^2 = -(1)^2 = -1$; $DD' \rightarrow -ab = -(1)(-1) = 1$

$D'^2 \rightarrow -b^2 = -(-1)^2 = -1.$

$$= \frac{1}{-1 - 2(1) + 2(-1)} \sin(x-y)$$

$$= \frac{1}{-1-2-2} \sin(x-y)$$

$$P.I = \frac{1}{-5} \sin(x-y).$$

$\therefore$ The complete solution is $z = C.F + P.I$

$$z = f_1(y + (1+i)x) + f_2(y + (1-i)x) + \left(\frac{1}{-5} \sin(x-y)\right).$$

Solve:

(7) $[D^2 - 2DD' + D'^2] z = \cos(x-3y)$

Soln: The auxiliary equation is

$m^2 - 2m + 1 = 0$ | $D \to m$, $D' \to 1$

$(m-1)(m-1) = 0$

$m = 1, 1$

The roots are real & equal:

$C.F = f_1(y + m_1 x) + x f_2(y + m_2 x)$

$C.F = f_1(y+x) + x f_2(y+x)$.

To find P.I

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y).$$

Replace $a = 1$, $b = -3$.

$D^2 \to -a^2 = -(1)^2 = -1$; $DD' \to -ab = -(1)(-3) = 3$

$D'^2 \to -b^2 = -(3)^2 = -9$.

$$= \frac{1}{-1 - 2(3) + (-9)} \cos(x-3y)$$

$$= \frac{1}{-1-6-9} \cos(x-3y)$$

$$P.I = \frac{1}{-16} \cos(x-3y).$$

$\therefore$ The complete solution is $z = C.F + P.I$

$$z = f_1(y+x) + x f_2(y+x) + \left(\frac{1}{-16} \cos(x-3y)\right)$$

$$z = f_1(y+x) + x f_2(y+x) - \frac{1}{16} \cos(x-3y)$$

—x—

Scanned with CamScanner

v.Imp 14m

(8) $[D^2 - DD' - 20D'^2]z = e^{5x+y} + \sin(4x-y)$

(8)

(9) soln:

The auxiliary equation is

$D \to m$

$D' \to 1$

$m^2 - m - 20 = 0.$

$-5 \quad 4$

$(m-5)(m+4) = 0$

$m = 5, -4.$

The roots are real and distinct.

$C.F = f_1(y+m_1x) + f_2(y+m_2x)$

$C.F = f_1(y+5x) + f_2(y-4x).$

To find P.I1

$$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$$

$D \to a = 5;\ D' \to b = 1$

$$= \frac{1}{5^2 - 5(1) - 20(1)} e^{5x+y}$$

$$= \frac{1}{25-5-20} e^{5x+y}$$

$$= \frac{1}{25-25} e^{5x+y}$$

$$= \frac{x}{2D - D'} e^{5x+y}$$

$$= \frac{x}{2(5)-1} e^{5x+y}$$

Scanned with CamScanner

$$= \frac{x}{10-1} e^{5x+y}$$

$$P.I_1 = \frac{x}{9} e^{5x+y}.$$

$$P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$D^2 \to -a^2 = -(4)^2 = -16$; $DD' \to -ab = -(4)(-1)$

$= 4$

$D'^2 \to -b^2 = -(-1)^2 = -1$

$$= \frac{1}{-16-4-20(-1)} \sin(4x-y)$$

$$= \frac{1}{-16-4+20} \sin(4x-y)$$

$$= \frac{1}{-20+20} \sin(4x-y)$$

$$= \frac{x}{2D-D'} \sin(4x-y)$$

$$= \frac{x}{2D-D'} \cdot \frac{(2D+D')}{(2D+D')} \cdot \sin(4x-y)$$

$$= \frac{x(2D+D')\sin(4x-y)}{4(D^2)-(D')^2}$$

$$= \frac{x[2D(\sin(4x-y)) + D'(\sin(4x-y))]}{4(-16)-(-1)}$$

Scanned with CamScanner

$$= \frac{x\left[2\cos(4x-y)\cdot 4 + \cos(4x-y)(-1)\right]}{-64+1}$$

$$= \frac{x\left[8\cos(4x-y) - \cos(4x-y)\right]}{-63}$$

$$P.I_2 = \frac{x\cos(4x-y)(8-1)}{-63}$$

$$= \frac{7x\cdot\cos(4x-y)}{-\cancel{63}\,9} = \frac{-x}{9}\cos(4x-y)$$

The complete solution is

$$z = C.F + P.I$$

$$= f_1(y+5x) + f_2(y-4x) + \frac{x}{9}e^{5x+y} - \frac{x}{9}\cos(4x-y)$$

—x—

Solve $(D+D'-2)z = 0$

Given

$$(D+D'-2)z = 0$$

$$(D-(-1)D'-2)z = 0$$

$$(D-mD'-a)z = 0$$

If $(D-mD'-a)z = 0$, then $z = e^{ax} f_1(y+mx)$

$m = -1, \; a = 2$

$$\therefore z = e^{2x} f_1(y-x).$$

Solve: $(D+3D'+4)^2 z = 0$

$$[D-(-3)D'-(-4)]^2 z = 0$$
$$(D-mD'-a)^2$$

$$m=-3, \quad a=-4$$

If $(D-mD'-a)^2$ then $z = e^{ax} f_1(y+mx) + x e^{ax} f_2(y+mx)$

$$z = e^{-4x} f_1(y-3x) + x e^{-4x} f_2(y-3x).$$

— x —.

~~Unit - 4~~

~~Applicati~~ ~~Partial Differential equation~~.

Solve

$$[2DD' + D'^2 - 3D']z = 3\cos(3x-2y)$$

Soln:

$$(2DD' + D'^2 - 3D')z = 0$$

$$D'(2D + D' - 3)z = 0$$

$$D'(D' + 2D - 3)z = 0$$

$$(D' - 0D - 0)(D' - (-2D - 3))z = 0.$$

$$m_1 = 0 \quad ; \quad a_1 = 0$$

$$m_1 = -2 \quad ; \quad a_2 = 3$$

$$\therefore \quad C.F = e^{0y} f_1(x+0y) + e^{3y} f_2(x-2y).$$

To find P.I

$$P.I = \frac{1}{2DD' - D'^2 - 3D'} 3\cos(3x-2y)$$

$$= 3\cos(3x-2y)$$

$$= \frac{3\cos(3x-2y)}{12-4-3D'}$$

$$= \frac{3\cos(3x-2y)}{8-3D'}$$

$$= 3\,\frac{(8+3D')}{(8-3D')(8+3D')}\cos(3x-2y)$$

$D'^2 = -1$

$$= 3\,\frac{[8\cos(3x-2y) + 3D'\cos(3x-2y)]}{64-9D'^2}$$

$$= 3\,\frac{[8\cos(3x-2y) - 3\sin(3x-2y)(-2)]}{64-9(-4)}$$

$$= \frac{3}{100}[8\cos(3x-2y) + 6\sin(3x-2y)]$$

$$P.I = \frac{3}{50}[4\cos(3x-2y) + 3\sin(3x-2y)]$$

The complete solution is

$$z = C.F + P.I$$

$$z = e^{0y} f_1(x+0y) + e^{3y} f_2(x-2y) + \frac{3}{50}[4\cos(3x-2y) + 3\sin(3x-2y)].$$

— x —

UNIT III PDE

Questions	opt1	opt2	opt3	opt4	Answer
In a PDE, there will be one dependent variable and ____ independent variables	only one	two or more	no	infinite number of	two or more
The ______ of a PDE is that of the highest order derivative occurring in it	degree	power	order	ratio	order
The degree of the a PDE is ______of the higest order derivative	power	ratio	degree	order	power
Afirst order PDE is obtained if______	Number of arbitrary constants is equal Number of independent variables	Number of arbitrary constants is lessthan Number of independent variables	Number of arbitrary constants is greater than Number of independent variables	Number of arbitrary constants is not equal to Number of independen t variables	Number of arbitrary constants= Number of independent variables
In the form of PDE, f(x,y,z,a,b)=0. What is the order?	1	2	3	4	1
What is form of the z=ax+by+ab by eliminating the arbitrary constants?	z=qx+py+pq	z=px+qy+pq	z=px+qy+p	z=py+qy+q	z=px+qy+pq
General solution of PDE F(x,y,z,p,q)=0 is any arbitray function F of specific functions u,v is____satisfying given PDE	F(u,v)=0	F(x,y,z)=0	F(x,y)=0	F(p,q)=0	F(u,v)=0
The PDE of the first order can be written as--------------	F(x,y,s,t)	F(x,y,z,p,q)=0	F(x,y,z,1,3,2)=0	F(x,y)=0	F(x,y,z,p,q)=0
The complete solution of clairaut's equation is _____	z=bx+ay+f(a,b)	z=ax+by+f(a,b)	z=ax+by	z=f(a,b)	z=ax+by+f(a,b)
The Clairaut’s equation can be written in the form --------	z=px+qy+f(p,q)	z=(p-1)x+qy+f(x,y)	z=Pp+Qq	Pq+Qp=r	z=px+qy+f(p,q)
Which of the following is the type f(z,p,q)=0 ?	p(1+q)=qx	p(1+q)=qz	p(1+q)=qy	p=2x f(y+2x)	p(1+q)=qz

The equation (D^2 z+2xy(Dz)^2+D'=5 is of order ____and degree____	2 and 2	2 and 1	1 and 1	0 and 1	2 and 1
The complementry function of (D^2 -4DD'+4D'^2)z=x+y is	f(y+2x)+xg(y+2x)	f(y+x)+xg(y+2x)	f(y+x)+xg(y+x)	f(y+4x)+xg(y+4x)	f(y+2x)+xg(y+2x)
The solution of xp+yq=z is _____	f(x^2,y^2)=0	f(xy,yz)	f(x,y)=0	f(x/y ,y/z)=0	f(x/y ,y/z)=0
A solution which contains the maximum possible number of arbitrary functions is called------------integral.	singular	complete	general	particular	general
The lagrange's linear equation can be written in the form -------------	Pq+Qp=r	Pq+Qp=R	Pp+Qq=R	F(x,y)=0	Pp+Qq=R
The complete solution of the PDE pq=1 is ------------------	z=ax+(1/a)y+b	z=ax+y+b	z=ax+ay/b+c	z=ax+b	z=ax+(1/a)y+b
The solution got by giving particular values to the arbitrary constants in a complete integral is called a --------	general	singular	particular	complete	particular
The general solution of Lagrange's equation is denoted as-------	f(u,v)=0	zx	f (x,y)	F(x,y,s,t)=0	f(u,v)=0
The subsidiary equations are px+qy=z is --------------	dx/y=dy/z=dz/x	dx/x=dy/y=dz/z	xdx=ydy=zdz	dz/z=dx/y=dy/x	dx/x=dy/y=dz/z
The general solution of equation p+q=1 is -------------	f(xyz,0)	f(x-y,y-z)	f(x-y,y+z)	F(x,y,s,t)=0	f(x-y,y-z)
The separable equation of the first order PDE can be written in the form of ------------	f(x,y)=g(x,y)	f(a,b)=g(x,y)	f(x,p)=g(y,q)	f(x)=g(a)	f(x,p)=g(y,q)
Complementary function is the solution of ------------------------	f(a,b)	f(1,0)=0	f(D,D')z=0	f(a,b)=F(x,y)	f(D,D')z=0
C.F+P.I is called ------------------- solution	singular	complete	general	particular	general
Particular integral is the solution of -------------	f(a,b)=F(x,y)	f(1,0)=0	[1/f(D,D')]F(x,y)	f(a,b)=F(u,v)	[1/f(D,D')]F(x,y)
Which is independent varible in the equation z= 10x+5y	x&y	z	x,y,z	x alone	x&y
Which is dependent varible in the equation z=2x+3y	x	z	y	x&y	z

Which of the following is the type f(z,p,q)=0	p(1+q)=qx	p(1+q)=qz	p(1+q)=qy	p=2xf'(x^2)-(y^2))	p(1+q)=qz
Which is complete integral of z=px+qy+(p^2)(q^2)	z=ax+by+(a^2)(b^2)	z=a+b+ab	z=ax+by+ab	z=a+f(a)x	z=ax+by+(a^2)(b^2)
The complete integral of PDE of the form F(p,q)=0 is	z=ax+f(a)y+c	z=ax+f(a)+b	z=a+f(a)x	z=ax+f(a)	z=ax+f(a)y+c
The relation between the independent and the dependent variables which satisfies the PDE is called-------	solution	complet solution	general solution	singular solution	solution
A solution which contains the maximum possible number of arbitrary constant is called---------	general	complete	solution	singular	complete
The equations which do not contain x & y explicitly can be written in the form---------------	f(z,p,q)=0	f(p,q)=0	(p,q)=0	f(x,p,q)=0	f(z,p,q)=0
The subsidiary equations of the lagranges equation 2y(z-3)p + (2x-z)q = y(2x-3)	dx/2y(z-3) = dy/(2x-z) =dz/y(2x-3)	dx/(2x-z) =dy/2y(z-3) =dz/y(2x-3)	dx/2y=dz/(z-3)	dx/2y=dz/(z-3)=dy/2x	dx/2y(z-3) =dy/(2x-z) =dz/y(2x-3)
A PDE ., the partial derivatives occuring in which are of the first degree is said to be --------------	linear	non-linear	order	degree	linear
A PDE., the partial derivatives occuring in which are of the 2 or more than 2 degree is said to be------	linear	non-linear	order	degree	non-linear
If z=(x^2+a)(y^2+b) then differentiating z partially with respect to x is -----	2x	3x(y^2+b)	2x(y^2+b)	3x+y	2x(y^2+b)
If z=ax+by+ab then differentiating z partially with respect to y is -----	a	a+b	0	b	b
The solution of differentiating z partially with respect to x twice gives -----	ax	ax+by+c	ax+b	ax=p	ax+b
The auxiliary equation of (D^2-4DD'+4 D'^2)z=0 is	m^2-4m+4=0	m^2+4m+4=0	m^2-4m-4=0	m^2+4m-4=0	m^2-4m+4=0
The auxiliary equation of (D^3-7DD'^2-6D'^3)z=0 is	m^3+7m+6=0	m^3-7m-6=0	m^3-7m+6=0	m^3+7m-6=0	m^3-7m-6=0
The auxiliary equation of (D^2-4DD'+4 D'^2)z=e^x is	m^2+4m+4=0	m^2-4m-4=0	m^2+4m-4=0	none	none

The roots of the partial differential equation (D^2-4DD'+4 D'^2)z=0 are	2,1	2,2	2,-2	2,-2	2,2
The roots of the partial differential equation (D^2-2DD'+D'^2)z=0 are	0,1	i,-1	1,2	1,1	1,1

UNIT – III Application of partial differential equation

PART – A

. State the three possible solutions of the One-dimensional Wave equation

[N/D 2016]

Ans:
The solution of One dimensional Wave equation are

(i) $y(x,t) = (Ae^{-px} + Be^{px})(Ce^{-pat} + De^{pat})$

(ii) $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

(iii) $y(x,t) = (Ax+B)(Ct+D)$

. State the assumptions in deriving One-dimensional Wave equation

[N/D 2016]

Ans: To derive the One dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, we make the following assumptions.

(i) The string is homogeneous and perfectly elastic so that it does not offer resistance to bending.

(ii) The tension T caused by stretching the string before fixing it at the ends is so large that the action of the gravitational force on the string can be neglected

(iii) The string performs small transverse motion in a vertical plane so that the deflection y and the slope $\frac{dy}{dx}$ are small in absolute value. Hence their higher powers can be neglected.

Classify the partial differential equation $U_{xx} + U_{yy} = f(x,y)$ [M|J 2016]

Soln: Given: $U_{xx} + U_{yy} = f(x,y)$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \rightarrow (1)$$

But the general form of second-order partial differential equation in two independent variables x and y

$$\Rightarrow A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0 \rightarrow (2)$$

Comparing (1) & (2) $\Rightarrow A = 1, \ B = 0, \ C = 1$

$\therefore B^2 - 4AC \Rightarrow 0 - 4 \times 1 \times 1 = -4 < 0$

$\therefore$ The given equation represent elliptic

Write all possible solutions of two dimensional heat equation. [N|D 2015]

Ans: (i) $U(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda x} + De^{-\lambda y})$

(ii) $U(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y)$

(iii) $U(x,y) = (Ax+B)(Cy+D)$ where A, B, C, D are arbitrary constants.

Classify the partial differential equation

$$(1-x^2) z_{xx} - 2xy z_{xy} + (1-y^2) z_{yy} + x z_x + 3x^2 y z_y - 2z = 0$$

[A/M 2015, N|D 2014]

Solution:

Given: $(1-x^2) z_{xx} - 2xy z_{xy} + (1-y^2) z_{yy} + x z_x + 3x^2 y z_y - 2z = 0$

$$A(x,y) \frac{\partial^2 z}{\partial x^2} + B(x,y) \frac{\partial^2 z}{\partial x \partial y} + C(x,y) \frac{\partial^2 z}{\partial y^2} + F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

$A =$ co-efficient of $\frac{\partial^2 z}{\partial x^2} = 1 - x^2$

$C =$ co-efficient of $\frac{\partial^2 z}{\partial y^2} = 1 - y^2$

$B =$ co-efficient of $\frac{\partial^2 z}{\partial x \partial y} = -2xy$

$$B^2 - 4AC = (-2xy)^2 - 4(1-x^2)(1-y^2)$$

$$= 4x^2y^2 - 4\{1 - y^2 - x^2 + x^2y^2\}$$

$$= \cancel{4x^2y^2} - 4 + 4y^2 + 4x^2 - \cancel{4x^2y^2}$$

$$= 4(x^2+y^2-1)$$

If $x^2+y^2<1$ then $B^2-4AC<0$

$\therefore$ The equation represent elliptic

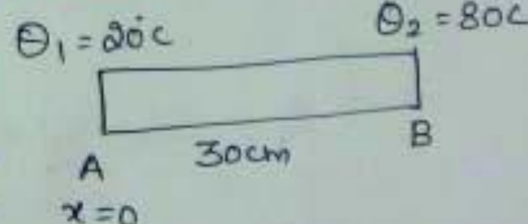
if $x^2+y^2>1$ then $B^2-4AC>0$

$\therefore$ The equation represent hyperbolic

if $x^2+y^2=1$ then $B^2-4AC=0$

$\therefore$ The equation represent parabolic

A rod 30 cm long has its ends A and B kept 20°c and 80°c respectively until steady state condition prevails. Find the steady state temperature in the rod. [A/M 2015]



Ans: The steady state temperature at anytime 't'

$$u(x) = \frac{\theta_2 - \theta_1}{l} x_1 + \theta_1 = \frac{80-20}{30} x + 20$$

$$= 2x + 20$$

7. Write down all the possible solution of One dimensional heat equation [MIJ 2016] [NID 2014] [NID 2017]

Ans: One dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The possible solution are

(i) $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\lambda^2\alpha^2 t}$

(ii) $u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})e^{\lambda^2\alpha^2 t}$

(iii) $u(x,t) = Ax+B$

Where A, B are arbitrary constants, λ is also constants.

Solve $3x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$; by method of separation of variables. [NID 2015]

Solns: Given: $3x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$

$$3x\frac{\partial u}{\partial x} = 2y\frac{\partial u}{\partial y}$$

$$\Rightarrow 3x \times \frac{1}{\partial x} = 2y\frac{1}{\partial y} \Rightarrow \frac{\partial y}{2y} = \frac{\partial x}{3x}$$

$$\Rightarrow \int\frac{1}{2y}\partial y = \int\frac{1}{3x}dx = \frac{1}{2}\log y = \frac{1}{3}\log x + \log c$$

$$\Rightarrow \log y^{1/2} = \log x^{1/3} + \log c \Rightarrow \log c = \log y^{1/2} - \log x^{1/3}$$

$$\Rightarrow \log c = \log\left(\frac{y^{1/2}}{x^{1/3}}\right)$$

$$\Rightarrow \boxed{C = y^{1/2}x^{-1/3}}$$

UNIT-III Application of partial differential equation

PART-B

Template-1 [With No velocity]

The one diamonsional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\frac{\partial y(x,0)}{\partial t} = 0$

(iv) $y(x,0) = f(x)$

The solution of the wave equation is.

$$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow ①$$

Apply (i) in equ ①

$$y(0,t) = [c_1][c_3 \cos pat + c_4 \sin pat] = 0$$

$$c_1 = 0 \text{ and } c_3 \cos pat + c_4 \sin pat \neq 0$$

sub $c_1 = 0$ in equ ①

$$y(x,t) = [c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \longrightarrow ②$$

Apply (ii) in equation ②

$$y(l,t) = [c_2 \sin pl][c_3 \cos pat + c_4 \sin pat] = 0$$

$$c_2 \sin pl = 0$$

either $c_2 = 0$ (or) $\sin pl = 0$

If $c_2 = 0$ We get trivial solution

$\sin pl = 0$

$pl = \sin^{-1}(0) = n\pi$

$pl = n\pi$

$p = \left(\frac{n\pi}{l}\right)$

sub p value in equ ②

$$y(x,t) = \left[c_2 \sin\left(\frac{n\pi x}{l}\right)\right]\left[c_3 \cos\left(\frac{n\pi at}{l}\right) + c_4 \sin\left(\frac{n\pi at}{l}\right)\right] \rightarrow ③$$

Diff w.r.t "t"

$$\frac{\partial y(x,t)}{\partial t} = \left[c_2 \sin\left(\frac{n\pi x}{l}\right)\right]\left[-c_3\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi at}{l}\right) + c_4\left(\frac{n\pi a}{l}\right)\cos\left(\frac{n\pi at}{l}\right)\right] \rightarrow ④$$

Apply (iv) in equ ④

$$\frac{\partial y(x,0)}{\partial t} = \left[c_2 \sin\left(\frac{n\pi x}{l}\right)\right]\left[c_4\left(\frac{n\pi a}{l}\right)\right] = 0$$

$c_2 \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$, $\left(\frac{n\pi a}{l}\right) \neq 0$, $\boxed{c_4 = 0}$

sub $c_4 = 0$ in equ ③

$$y(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right).$$

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \rightarrow ⑤$$

(1) A string is stretched and fastened to points at a distance l apart. Motion is started by displacing the string in the form $y = a\sin\left(\frac{\pi x}{l}\right)$, $0 < x < l$ from which it is released at time $t = 0$. Find the displacement at any time t [M/J 2014]

Solution:

The one diamensional wave equation is.

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i) $y(0, t) = 0$

(ii) $y(l, t) = 0$

(iii) $\frac{\partial y(x, 0)}{\partial t} = 0$

(iv) $y(x, 0) = a\sin\left(\frac{\pi x}{l}\right)$

~~Write~~ The solution of the wave equation is.

$y(x, t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (1)$

Write Template 1

$$y(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \longrightarrow (5)$$

Apply (iv) in equ (5)

$$y(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = a\sin\left(\frac{\pi x}{l}\right)$$

$$c_1 \sin\left(\frac{\pi x}{l}\right) + c_2 \sin\left(\frac{2\pi x}{l}\right) + c_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots = a\sin\left(\frac{\pi x}{l}\right)$$

Sub c_1 value in eqn (5)

$$y(x,t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi a t}{l}\right).$$

(2) A uniform string is stretched and fastened t two points l apart. Motion is started by displacing the string into the form of the curve $y = kx(l-x)$ and then released from this position at time $t=0$. Drive the expression for the displacement of any point of the string at a distance x from one end at time t

[N/D 2013] [A/M 2015]

Solution:

The one diamentional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\frac{\partial y(x,0)}{\partial t} = 0$

(iv) $y(x,0) = kx(l-x)$

The solution of the equation is

$$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (1)$$

Write templeat 1

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \longrightarrow (5)$$

Apply (iv) in eqn (5)

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = kx(l-x)$$

$$f(x) = kx(l-x), \quad c_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2k}{l}\left[\cancelto{0}{(lx-x^2)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right)} - \cancelto{0}{(l-2x)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)} + (-2)\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2k}{l}\left\{\left[(-2)\left(\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^3}\right)\right] - \left[(-2)\left(\frac{1}{\left(\frac{n\pi}{l}\right)^3}\right)\right]\right\}$$

$$= \frac{2k}{l} \times \frac{(-2)}{\left(\frac{n\pi}{l}\right)^3}\left[(-1)^n - 1\right]$$

$$= \frac{2k}{\cancel{l}} \times \frac{(-2)\, l^{\cancel{3}}}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$c_n = b_n = \begin{cases} 0 & \text{if } n = \text{even} \\ \dfrac{8kl^2}{n^3\pi^3} & \text{if } n = \text{odd} \end{cases}$$

Sub $c_n = b_n$ value in equ (5)

$$y(x,t) = \sum_{n=\text{odd}}^{\infty}\left[\frac{8kl^2}{n^3\pi^3}\right]\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi a t}{l}\right)$$ Ans.

Template - 2 [With velocity]

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $y(x,0) = 0$

(iv) $\frac{\partial y(x,0)}{\partial t} = f(x)$ [Given]

The solution of the equation is.

$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (1)$

Apply (i) in equ (1)

$y(0,t) = [c_1][c_3 \cos pat + c_4 \sin pat] = 0$

$c_1 = 0, \quad c_3 \cos pat + c_4 \sin pat \neq 0$

$y(x,t) = [c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (2)$

Apply (ii) in equ (2)

$y(x,t) = [c_2 \sin pl][c_3 \cos pat + c_4 \sin pat] = 0$

$c_2 \sin pl = 0$

either $c_2 = 0$ (or) $\sin pl = 0$

$pl = \sin^{-1}(0) = n\pi$

$p = \left(\frac{n\pi}{l}\right)$

Sub p value in equ (2)

$y(x,t) = \left[c_2 \sin\left(\frac{n\pi x}{l}\right)\right]\left[c_3 \cos\left(\frac{n\pi at}{l}\right) + c_4 \sin\left(\frac{n\pi at}{l}\right)\right] \rightarrow (3)$

Apply (iii) in equ (3)

$y(x,0) = \left[c_2 \sin\left(\frac{n\pi x}{l}\right)\right][c_3] = 0$

$c_2 \neq 0, \ \sin\left(\frac{n\pi x}{l}\right) \neq 0, \ \boxed{c_3 = 0}$

sub $c_3 = 0$ in equ (3)

$$y(x,t) = \left[c_2 c_4 \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right) \right]$$

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right) \longrightarrow (4)$$

Diff w.r.t "x" in equ (4)

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) \to (5)$$

(3) A tightly stretched string of length l initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial v}{\partial t}\right)_{t=0} = V_0 \sin^3\left(\frac{\pi x}{l}\right)$. Find the displacement $y(x,t)$ [N/D 2014]

solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0,t) = 0$
(ii) $y(l,t) = 0$
(iii) $y(x,0) = 0$
(iv) $\frac{\partial y(x,0)}{\partial t} = V_0 \sin^3\left(\frac{\pi x}{l}\right)$.

The solution of the equation is.

$$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \to (1)$$

write template 2

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) \to (5)$$

Apply (iv) in equ (5)

$$\boxed{\sin^3\theta = \tfrac{1}{4}\left[3\sin\theta - \sin 3\theta\right]}$$

$$\sum_{n=1}^{\infty}\left(\frac{n\pi a}{l}\right)c_n \sin\left(\frac{n\pi x}{l}\right) = V_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$= \frac{V_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right]$$

Equating co-efficient's

$$\frac{\pi a}{l}c_1 = \frac{3V_0}{4}, \Rightarrow c_1 = \frac{3lV_0}{4\pi a}. \quad ; \boxed{c_2 = 0}$$

$$\frac{3\pi a}{l}c_3 = \frac{-V_0}{4} \Rightarrow c_3 = \frac{-V_0 l}{12\pi a}.$$

$$c_4 = c_5 = \cdots = 0$$

sub c_1, c_3 value in equ ②

$$y(x,t) = \left(\frac{3V_0 l}{4\pi a}\right)\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi a t}{l}\right) + \left(\frac{-V_0 l}{12\pi a}\right)\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi a t}{l}\right).$$

③ A tightly stretched string between the fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If each of its points is given a velocity $kx(l-x)$ Find the displacement. [M/J 2013]

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $y(x,0) = 0$

(iv) $\frac{\partial y(x,0)}{\partial t} = kx(l-x)$

the solution of the equation is. (5)

$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (1)$

Write template - 2.

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) \rightarrow (5)$$

The half Range sine series is.

$$\cancel{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = f(x)} \quad \cancel{\rightarrow (6)}$$

Apply (iv) in equ (5)

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) = kx(l-x) \rightarrow (6)$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = f(x) \rightarrow (7)$$

$$c_n\left(\frac{n\pi a}{l}\right) = b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l kx(l-x) \sin\left(\frac{n\pi x}{l}\right)dx$$

$$c_n\left(\frac{n\pi a}{l}\right) = b_n = \begin{cases} 0 & \text{if } n = \text{even} \\ \dfrac{8kl^2}{n^3\pi^3} & \text{if } n = \text{odd} \end{cases}$$

$$c_n = \left(\frac{8kl^2}{n^3\pi^3}\right)\left(\frac{l}{n\pi a}\right) = \frac{8kl^3}{n^4\pi^4 a}. \text{ if } n = \text{odd}$$

Sub c_n value in equ (4)

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \left[\frac{8kl^3}{n^4\pi^4 a}\right] \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right),$$

(6)

(5) A tightly stretched string of length l with fixed end points is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity $V_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$, $0 < x < l$. Find the displacement of string

[N/D 2016]

Solution:

The wave equation is.

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $y(x,0) = 0$

(iv) $\frac{\partial y(x,0)}{\partial t} = V_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$

$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (1)$

Write template 2

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) \rightarrow (5)$$

Apply (iv) in equ (5)

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = V_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$$

$$\sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = \frac{V_0}{2}\left[\sin\left(\frac{4\pi x}{l}\right) + \sin\left(\frac{2\pi x}{l}\right)\right]$$

Equating co-efficients.

$$\frac{2\pi a}{l} c_2 = \frac{V_0}{2} \Rightarrow c_2 = \frac{V_0 l}{4\pi a}$$

$$\frac{4\pi a}{l} c_4 = \frac{V_0}{2} \Rightarrow c_4 = \frac{V_0 l}{8\pi a}.$$

$$c_1 = c_3 = c_5 = c_6 = \cdots = 0$$

$$y(x,t) = \frac{v_0 l}{4\pi a} \sin\left(\frac{2\pi x}{l}\right) \sin\left(\frac{2\pi a t}{l}\right)$$

$$+ \frac{l v_0}{8\pi a} \sin\left(\frac{4\pi x}{l}\right) \sin\left(\frac{4\pi a t}{l}\right)$$

(4) Find the displacement of a string stretched between two fixed points at a distance of $2l$ apart when the string is initially at rest in equilibrium position and points of the string are given initial velocity

$$f(x) = V = \begin{cases} \frac{x}{l}, & (0, l) \\ \frac{2l - x}{l}, & (l, 2l) \end{cases}$$, x being the distance measured from one end [M/J 2016] [A/M 2017]

solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $y(x,0) = 0$

(iv) $\frac{\partial y(x,0)}{\partial t} = f(x)$.

The solution of the equation is.

$$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (1)$$

Write template 2

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \rightarrow (5)$$

Apply (iv) in equ (5).

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = f(x) \rightarrow (6)$$

The half Range sine series is.

(7)

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2l}\right) = f(x) \longrightarrow (7) \left[\begin{matrix}\text{since}\\ \text{length } 2l\end{matrix}\right]$$

$$C_n\left(\frac{n\pi a}{2l}\right) = b_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{2l}\right)dx$$

$$= \frac{1}{l}\left\{\int_0^{l}\left(\frac{x}{l}\right)\sin\left(\frac{n\pi x}{2l}\right)dx + \int_l^{2l}\frac{(2l-x)}{l}\sin\left(\frac{n\pi x}{2l}\right)dx\right\}$$

$$= \frac{1}{l^2}\left[\left[(x)\left(\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n^2\pi^2}{4l^2}\right)}\right)\right]_0^{l}\right.$$

$$\left. + \left[(2l-x)\left(\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l}\right]$$

$$= \frac{1}{l^2}\left[\left(-\frac{2l^2}{n\pi}\cos n\pi/2 + \frac{4l^2}{n^2\pi^2}\sin n\pi/2\right) - (0+0)\right]$$

$$+ \frac{1}{l^2}\left[(0-0) - \left(\frac{-2l^2}{n\pi}\cos n\pi/2 - \frac{4l^2}{n^2\pi^2}\sin n\pi/2\right)\right]$$

$$= \frac{1}{\cancel{l^2}} \times \frac{8\cancel{l^2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\left(\frac{n\pi a}{2l}\right)C_n = \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$C_n = \frac{16}{n^4\pi^4 a}\sin\left(n\pi/2\right)$$

Sub C_n value in equ (6)

$$y(x,t) = \sum_{n=1}^{\infty}\left[\frac{16}{n^4\pi^4 a}\right]\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2l}\right)\sin\left(\frac{n\pi at}{2l}\right)$$

//Ans.

One diamensional Heat equation with Both ends are change to zero temperature

(1) Find the solution to the equation.

$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions.

(i) $u(0,t) = 0$ $u(l,t) = 0,$ $t > 0$ and

$u(x,0) = \begin{cases} x, & 0 \le x \le l/2 \\ l-x, & l/2 < x < l \end{cases}$ [AM 2015]

Solution:

The one diamensional heat flow equation is.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

(i) $u(0,t) = 0$

(ii) $u(l,t) = 0$

(iii) $u(x,0) = \begin{cases} x, & 0 \le x \le l/2 \\ l-x, & l/2 \le x \le l \end{cases}$

The solution of the equation is.

$$u(x,t) = [A\cos px + B\sin px] e^{-\alpha^2 p^2 t} \longrightarrow (1)$$

Apply (i) in equ (1)

$$u(0,t) = [A\cos(0) + B\sin p0] e^{-\alpha^2 p^2 t} = 0$$

$$A = 0 \text{ and } e^{-\alpha^2 p^2 t} \ne 0$$

Sub $A = 0$ in equ (1)

$$u(x,t) = [B\sin px] e^{-\alpha^2 p^2 t} \longrightarrow (2)$$

Apply (ii) in equation (2)

$$u(l,t) = [B\sin pl] e^{-\alpha^2 p^2 t} = 0$$

$$B\sin pl = 0$$

If $B=0$ we get trivial solution.

$$\sin pl = 0$$

$$pl = \sin^{-1}(0) = n\pi$$

$$\boxed{p = \left(\frac{n\pi}{l}\right)}$$

Sub p value in equ ②

$$u(x,t) = \left[B \sin\left(\frac{n\pi x}{l}\right)\right] e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t} \longrightarrow ③$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t} \longrightarrow ④$$

Apply (iii) in equ ④

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)] = f(x) = \begin{cases} x, & 0 \le x \le l/2 \\ l-x, & l/2 \le x \le l \end{cases}$$

The half Range sine series is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = f(x).$$

$$B_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left\{\int_0^{l/2} (x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} (l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{2}{l}\left\{\left[(x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2}{l}\left[\left\{\left[\cancel{(l/2)\left(\frac{-\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)}\right)} + \left(\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right] - [0]\right\} + \left\{[0] - \left[\cancel{(l/2)\left(\frac{-\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)}\right)} - \left(\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]\right\}\right]$$

$$= \frac{2}{l} \times \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \frac{2l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

sub bn value in equ ④

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}$$

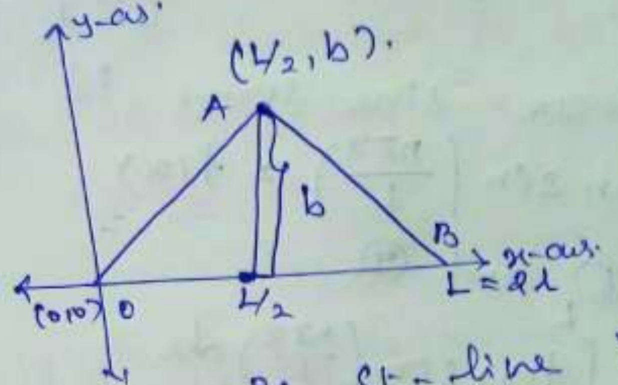
// Ans.

(7) A tightly streched string of length $2l$ is fastened at $x=0$ and $x=2l$. The mid point of the string is taken to hight b transversely and then relased from rest in that position. Find the lateral displacement of the string

Solution

let $2l = L$

The wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$



The equation is st-line is $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

The equation of OA $[O\underset{x_1\ y_1}{(0,0)}, A\underset{x_2\ y_2}{(L/2, b)}]$

$$\frac{y-0}{b-0} = \frac{x-0}{L/2 - 0} \Rightarrow \frac{y}{b} = \frac{2x}{L} \Rightarrow y = \frac{2bx}{L}, 0 \le x \le L/2$$

The equation of AB $[A\underset{x_1\ y_1}{(L/2, b)}\ B\underset{x_2\ y_2}{(L, 0)}]$

$$\frac{y-b}{0-b} = \frac{x - L/2}{L - L/2} \Rightarrow -\frac{y}{b} + 1 = \frac{x - L/2}{L/2}$$

$$-\frac{y}{b} + 1 = \frac{(2x-L)/\cancel{2}}{L/\cancel{2}} \Rightarrow -\frac{y}{b} = \frac{2x-L}{L} - 1$$

$$-\frac{y}{b} = \frac{2x - 2L}{L} \Rightarrow y = \frac{2b(L-x)}{L}, L/2 \le x \le L$$

The solution is the eqution is.

$$y(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pat + c_4 \sin pat] \rightarrow (1)$$

Write template - 1

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) \longrightarrow (5)$$

The boundary conditions are

(i) $y(0,t)=0$

(ii) $y(L,t)=0$

(iii) $\dfrac{\partial y(x,0)}{\partial t}=0$

(iv) $y(x,0)=f(x)$.

Apply (iv) in equation ⑤

$$y(x,0)=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)=f(x)=\begin{cases}\dfrac{2bx}{L}, & 0\le x\le L/2\\ \dfrac{2b(L-x)}{L}, & L/2\le x\le L\end{cases} \longrightarrow ⑥$$

The half range sine series is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)=f(x) \longrightarrow ⑦$$

from ⑥ and ⑦

$$C_n=b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{2}{L}\left\{\int_0^{L/2}\frac{2bx}{L}\sin\left(\frac{n\pi x}{L}\right)dx+\int_{L/2}^{L}\frac{2b}{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$=\frac{2}{L}\left\{\frac{2b}{L}\left[(x)\left(\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)}\right)-(1)\left(\frac{-\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}\right)\right]_0^{L/2}\right.$$
$$\left.+\frac{2b}{L}\left[(L-x)\left(\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)}\right)-(-1)\left(\frac{-\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}\right)\right]_{L/2}^{L}\right\}$$

$$=\frac{2}{L}\times\frac{2b}{L}\left\{\left\{\left[\cancel{(L/2)\left(\frac{-\cos(n\pi/2)}{(n\pi/L)}\right)}+\left(\frac{\sin(n\pi/2)}{\left(\frac{n\pi}{L}\right)^2}\right)\right]-[0]\right\}\right.$$
$$\left.+\left\{[0]-\left[\cancel{(L/2)\left(\frac{-\cos(n\pi/2)}{\left(\frac{n\pi}{L}\right)}\right)}-\left(\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}\right)\right]\right\}\right\}$$

$$=\frac{4b}{L^2}\times\frac{2\sin(n\pi/2)}{\left(\frac{n\pi}{L}\right)^2}=\frac{8b}{\cancel{L^2}}\times\frac{\cancel{L^2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)=\frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

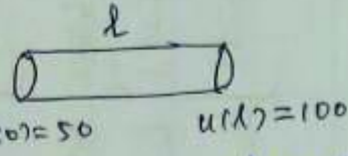
Sub $b_n=C_n$ value in equation ⑤

$$y(x,t)=\sum_{n=1}^{\infty}\left[\frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$ Ans.

one Dimensional Heat Equation with Both ends are change to Non-zero temperature

① A bar of 10 cm long with insulated sides has its ends A and B maintained at temperatures 50°c and 100°c resp until steady state conditions prevail The temperature at A is suddenly raised to 90°c and at is lowered to 60°c. Find the temperature distribution in the bar thereafter

Solution:



The study state temperature distribution on the rod is $u(x) = \left(\frac{b-a}{l}\right)x + a \longrightarrow$ ①

where a = temperature at the end $x=0$

b = Temperature at the end $x=l$

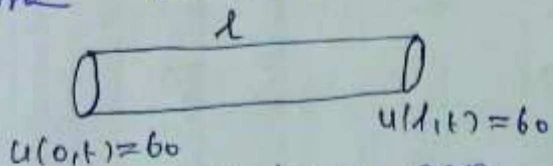
l = length of the rod

$a=50, \; b=100$ sub in ①

$$u(x) = \frac{50x}{l} + 50$$

which is temperature distribution of the rod

Now the temperature at A is raised to 90°c and at B is lowered to 60°c i) the study state is changed to unsteady state. Here the initial temperature distribution is $u(x) = \frac{50x}{l} + 50$



The boundary conditions are

(i) $u(0,t)=90$, (ii) $u(l,t)=60$ (iii) $u(x,0) = \frac{50x}{l} + 50$

We cannot find $u(x,t)$ for the non-zero boundary conditions

∴ We split the solution $u(x,t)$ into two parts

$$u(x,t) = u_s(x) + u_t(x,t) \longrightarrow \text{Ⓑ}$$

Where $u_s(x)$ is a solution of $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ ⑤
involving x only and satisfies the boundary conditions (i) and (ii) i) $u_s(0) = 90$ and $u_s(l) = 60$
ii) $u_s(x)$ is a steady state solution
$u_t(x,t)$ is a transient solution satisfies (B) which decreases as t increases.

To find $u_s(x)$
under the steady state condition

$$u_s(x) = a'x + b' \longrightarrow (C)$$

i) $x = 0 \Rightarrow u_s(0) = 0 + b' \Rightarrow \boxed{b' = 90}$

sub condition $u_s(l) = 60$ in (C)

$$u_s(l) = a'l + b' \Rightarrow 60 = a'l + 90 \Rightarrow a' = -\frac{30}{l}$$

sub in (C) we get $u_s(x) = -\frac{30x}{l} + 90$

To find $u_t(x,t)$

$$(B) \Rightarrow u_t(x,t) = u(x,t) - u_s(x) \longrightarrow (D)$$

We have to find the boundary condition for $u_t(x,t)$

put $x = 0$ in (D)

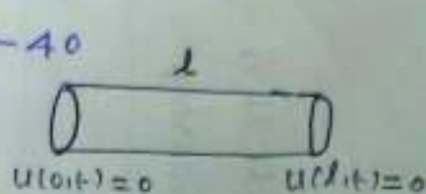
$$u_t(0,t) = u(0,t) - u_s(0) = 90 - 90 = 0$$

put $x = l$ in (D)

$$u_t(l,t) = u(l,t) - u_s(l) = 60 - 60 = 0$$

put $t = 0$ in (D)

$$u_t(x,0) = u(x,0) - u_s(x) = \frac{80x}{l} - 40$$

the new boundary conditions are 

(i) $u_t(0,t) = 0$
(ii) $u_t(l,t) = 0$
(iii) $u_t(x,0) = \frac{80x}{l} - 40$

The solution of the equation is.

$$u_t(x,t) = [A\cos px + B\sin px]\, e^{-\alpha^2 p^2 t} \longrightarrow (1)$$

Apply (i) in equ (1)

$$u_t(0,t) = [A]\, e^{-\alpha^2 p^2 t} = 0$$

$$e^{-\alpha^2 p^2 t} \neq 0, \quad \boxed{A = 0}$$

Sub $A=0$ in equ (1)

$$u_t(x,t) = [B\sin px]\, e^{-\alpha^2 p^2 t} \longrightarrow (2)$$

Apply (ii) in equ (2)

$$u_t(l,t) = [B\sin pl]\, e^{-\alpha^2 p^2 t} = 0$$

$B\sin pl = 0$ either $B=0$ (or) $\sin pl = 0$

If $B=0$ we get trivial solution.

$\sin pl = 0 \Rightarrow pl = \sin^{-1}(0) = n\pi \Rightarrow p = \left(\frac{n\pi}{l}\right)$

Sub p value in equ (2)

$$u_t(x,t) = \left[B\sin\left(\frac{n\pi x}{l}\right)\right] e^{-\left(\frac{n\pi}{l}\right)^2 t} \longrightarrow (3)$$

The most General equation is,

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t} \longrightarrow (4)$$

Apply (iii) in equ (4)

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = \frac{80x}{l} + 40.$$

The half Range sine series is.

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

$$B_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l \left[\frac{80x}{l} + 40\right]\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{80x}{l}-40\right)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right)-\left(\frac{80}{l}\right)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l \quad (4)$$

$$= \frac{2}{l}\left\{\left[\frac{-40l(-1)^n}{n\pi}\right]-\left[\frac{40l}{n\pi}\right]\right\}$$

$$= -\frac{80}{\pi n}\left[(-1)^n+1\right]$$

$$B_n = \begin{cases} 0 & \text{if } n=\text{odd} \\ \frac{-160}{n\pi} & \text{if } n=\text{even} \end{cases}$$

sub B_n value in equ (2)

$$u_t(x,t) = \sum_{n=\text{even}}^{\infty}\left[\frac{-160}{n\pi}\right]\sin\left(\frac{n\pi x}{l}\right)e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}.$$

$$u(x,t) = \left(-\frac{30x}{l}+90\right)+\sum_{n=\text{even}}^{\infty}\frac{-160}{n\pi}\sin\left(\frac{n\pi x}{l}\right)e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}.$$

put $l=10$.

Ⓧ Ⓧ Ⓧ 14m Ⓧ

D' Alembert's Solution of wave equation.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow ①$$

Let us introduce the new indepent variable $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right).$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right),$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right).$$

$$= \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \longrightarrow ②$$

Similarly

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \longrightarrow ③$$

Sub ② & ③ in ①

Scanned with CamScanner

$$c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

$$\cancel{\frac{\partial^2 y}{\partial u^2}} - 2\frac{\partial^2 y}{\partial u \partial v} + \cancel{\frac{\partial^2 y}{\partial v^2}} - \cancel{\frac{\partial^2 y}{\partial u^2}} - 2\frac{\partial^2 y}{\partial u \partial v} - \cancel{\frac{\partial^2 y}{\partial v^2}} = 0$$

$$-4\frac{\partial^2 y}{\partial u \partial v} = 0$$

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \rightarrow (4)$$

Integrating (4) with respect to v.

$$\int \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u}\right) = 0$$

$$\frac{\partial y}{\partial u} = f(u) \rightarrow (5)$$

where $f(u)$ is an arbitrary function of u.

Integrating (5) with respect to u,

$$\int \frac{\partial y}{\partial u} = f(u)$$

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an ~~integrating~~ arbitrary function of v

Since the integral is a function of u alone,

we may denote it by $\phi(u)$.

$$y = \phi(u) + \psi(v).$$

$$y(x,t) = \phi(x+ct) + \psi(x-ct) \longrightarrow (6)$$

This is the general solution of wave equation (1). Now to determine ϕ and ψ.

$\because u = x+ct$
$v = x-ct$

Suppose initially $y(x,0) = f(x)$ and $\frac{\partial y}{\partial t}(x,0) = 0$

Diff (6) with respect to t,

$$\frac{\partial y(x,t)}{\partial t} = \phi'(x+ct)\,c + \psi'(x-ct)(-c)$$

$$\frac{\partial y}{\partial t}(x,t) = c\,\phi'(x+ct) - c\,\psi'(x-ct)$$

~~At~~ $t=0$,

$$\frac{\partial y}{\partial t}(x,0) = c\,\phi'(x) - c\,\psi'(x)$$

$$0 = c\,\phi'(x) - c\,\psi'(x).$$

$$\cancel{c}\,\phi'(x) = \cancel{c}\,\psi'(x)$$

$$\phi'(x) = \psi'(x) \longrightarrow (7)$$

$t=0$ in (6)

$$y(x,0) = \phi(x+0) + \psi(x-0)$$

$$y(x,0) = \phi(x) + \psi(x) = f(x) \longrightarrow (8)$$

$$\Rightarrow \phi(x) + \psi(x) = f(x)$$

(7) gives $\quad \phi(x) = \psi(x) + k$

(7) Integrating,

(8) becomes, $\psi(x) + K + \psi(x) = f(x)$

sub $\phi(x)$ in (8),

$$2\psi(x) + K = f(x)$$

$$2\psi(x) = f(x) - K$$

$$\boxed{\psi(x) = \frac{1}{2}[f(x) - K]}$$

sub

$\psi(x) = \phi(x) - K$ in (8)

$\phi(x) + \psi(x) = K$
$\phi(x) - K = \psi(x)$

$$\phi(x) + \phi(x) - K = f(x)$$

$$2\phi(x) - K = f(x)$$

$$2\phi(x) = f(x) + K$$

$$\phi(x) = \frac{1}{2}[f(x) + K].$$

Hence the solution of (6) is

$$y(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}[f(x+ct) + K] + \frac{1}{2}[f(x-ct) - K]$$

$$= \frac{1}{2}[f(x+ct) + \cancel{K} + f(x-ct) - K]$$

$$\boxed{\therefore y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$$

which is the d'Alembert's solution of wave equation (1)

UNIT -IV

PARTIAL DIFFERENTIAL EQUATIONS

Questions	opt 1	opt 2	opt 3	opt 4	Answer
Partial differential equation of second order is said to Elliptic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	B^2-4AC<0
Partial differential equation of second order is said to Parabolic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	B^2-4AC=0
Partial differential equation of second order is said to Hyperbolic at a point (x,y) in the plane if ---------	B^2-4AC<0	B^2-4AC=0	B^2-4AC>0	B^2=4AC	B^2-4AC>0
Two dimensional Laplace Equation is ----------	u_xx+u_yy=1	u_xx+u_yy=0	u_x=u_y	u_x+u_y=0	u_xx+u_yy=0
One dimensional heat Equation is ----------	u_xx=(1/α^2)u_t	u_xx=[(1/α^2)u_t]+10	u_xx=u_tt	u_xx+u_tt=0	u_xx=(1/α^2)u_t
One dimensional wave Equation is ----------	u_xx=(1/α^2)u_t	u_xx+u_yy=0	u_xx=(1/α^2)u_t^2	u_xx=u_t	u_xx=(1/α^2)u_t^2
The D'Alembert's solution of the One dimensional wave Equation is-----	y(x,t)=φ(x-αt)+ψ(x+αt)	y(x,t)=0	u_xx=(1/α^2)u_t	u_xx=(1/α^2)u_t^2	y(x,t)=φ(x-αt)+ψ(x+αt)
The Possion equation is of the form -----	y(x,t)=φ(x-αt)+ψ(x+αt)	u_xx=(1/α^2)u_t	u_xx=(1/α^2)u_tt	u_xx+u_yy=f(x,y)	u_xx+u_yy=f(x,y)
The steady state temperature of a rod of length l whose ends are kept at 30 and 40 is	u(x)= 10x/l + 30	u(x)= 40x/l	u(x)= 30x/l	None	u(x)= 10x/l + 30
The temperature distribution of the plate in the steady state is ---------	u_xx=(1/α^2)u_t	u_xx+u_yy=0	u_xx=(1/α^2)u_t^2	u_xx=u_t	u_xx+u_yy=0
Two dimensional heat Equation is known as ----------equation.	partial	Radio	laplace	Poisson	laplace
In one dimensional heat flow equation ,if the temperature function u is independent of time, then the solution is------	u(x)= ax + b	u(x,t)= a(x,t)	u(t) = at + b	u(t) = at - b	u(x)= ax + b
f_xx+2f_xy+4f_yy=0 is a ______	Elliptic	Hyperbolic	Parabolic	circle	Elliptic
f_xx=2f_yy is a ---------	Elliptic	Hyperbolic	Parabolic	circle	Hyperbolic
f_xx-2f_xy+f_yy=0 is a -----------	Hyperbolic	Elliptic	Parabolic	circle	Parabolic
The diffusivity of substance is------	k/pc	pc	k	pc/k	k/pc
Heat flows from a ------- temperature	higher to lower	lower to higher	normal	high	higher to lower
The Amount of heat required to produce a given temperature change in a bodies propostional to the ------- of the body and to the temperature change.	temperature	heat	mass	wave	mass
The rate at which heat flows through an area is------ to the area and to the temperature gradient normal to the area.	equal	not equal	lessthan	proportional	proportional
In steady state conditions the temperature at any particular point does not vary with ---	Time	temperature	mass	none	Time
The wave equation is a linear and ------ equation	non homogeneous	homogeneous	quadratic	none	homogeneous
In method of separation of variables we assume the solution in the form of -----	u(x,y)=X(x)	u(x,t)=X(x)T(t)	u(x,0)=u(x,y)	u(x,y)=X(y)Y(x)	u(x,t)=X(x)T(t)
u(x,t)=(Acosλx+Bsinλx)Ce^(-(α^2))(λ^2)t) is the possible solution of ------ equation	heat	wave	laplace	none	heat

y=(Ax+B)(Ct+D) is the possible solution of ------ equation	heat	wave	laplace	none	wave
If the heat flow is one dimensional ,then the -------- is a function x and t only	heat	light	temperature	wave	temperature
The stream lines are parallel to the X-axis ,then the rate of change of the temperature in the direction of the Y-axis will be ---------.	one	two	zero	five	zero
To solve y_tt=(α^2)yxx, we need -------- boundary conditions.	y(0,t)=0 if t>=0; y(l,t)=0 if t>=0	y(x,t)=0 if t>0; y(t)=0 if t=0	y(x,t)=0 if t>0	none	y(0,t)=0 if t>=0; y(l,t)=0 if t>=0
The boundary condition with non zero value on the R.H.S of the wave equation should be taken as the -------------- boundary condition.	First	Second	Last	none	Last
In one dimensional heat equation u_t= (α^2)u_xx, What does α^2 stands for?	k/pc	pc	k	pc/k	k/pc
The possible solution of wave equation is ----------	y=(Ax+B)(Ct+D)	u(x,t)=(Acosλx+Bsinλx)(Ce^(λy)+De^(-λy))	u(x,t)=Acosλx+Bsinλx	u(x,t)=Acosλx-Bsinλx	y=(Ax+B)(Ct+D)
The possible solution of heat equation is ----------	u(x,t)=(Acosλx+Bsinλx)Ce^(-(α^2))(λ^2)t)	u(x,t)=(Acosλx+Bsinλx)(Ce^(λy)+De^(-λy))	u(x,t)=Acosλx+Bsinλx	u(x,t)=Acosλx-Bsinλx	u(x,t)=(Acosλx+Bsinλx)Ce^(-(α^2))(λ^2)t)
If B^2-4AC = 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced	parabolic
If B^2-4AC > 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced	hyperbolic
If B^2-4AC < 0, then the differential equation is said to be______	parabolic	elliptic	hyperbolic	equally spaced	elliptic
The laplace equation in the polar coordinates is of the form-----	u_r+u_θ=0	u_xx=(1/α^2)u_t^2	u_xx=(1/α^2)u_t	(r^2)u_rr+ru_r+u_θθ=0	(r^2)u_rr+ru_r+u_θθ=0
The flow is two dimensional the temperature at any point of the plane is ------- of Z-coordinates.	linear	independent	dependent	none	independent
u(x,y)=(Acosλx+Bsinλx)(Ce^(λy)De^(-λy)) is the possible solution of the _______equation.	heat	wave	laplace	none	laplace
U(r,θ)=(A log r+B)(Cθ+D) is the possible solution of ------ equation	heat	wave	laplace	none	laplace

1/3/19

UNIT - 4

Complex Integration.

Cauchy's Integral theorem or Cauchy's fundamental theorem:-

If a function of $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$.

Extension of Cauchy's integral theorem to multiply connected regions:-

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

Cauchy's Integral formula:-

If $f(z)$ is analytic, within and on a closed curve, C and if a is any point within C, then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\,dz$.

Cauchy's Integral formula for the derivative of an analytic function

If a function $f(z)$ is analytic in region R, then its derivatives at any point $z=a$, of R, is also analytic in R and is given by $f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$.

Similarly, $f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$

$f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-a)^4} dz \dots$

1. Use Cauchy's Integral formula, evaluate

$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$ where C is the circle $|z|=4$

$|z|=4$

$\Rightarrow x^2+y^2 = 4^2$

Consider $\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$1 = A(z-3) + B(z-2)$

$z=2$; $A=-1$

$z=3$; $B=1$

$$\frac{1}{(z-2)(z-3)} = \frac{-1}{(z-2)} + \frac{1}{z-3}$$

The poles are $z=2$ and $z=3$

Both the points lie inside $|z|=4$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} = -\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3} dz$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

By cauchy Integral theorem,

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$= \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz = -2\pi i f(2) + 2\pi i f(3)$$

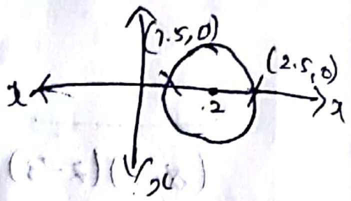
where $f(z) = \sin \pi z^2 + \cos \pi z^2$

$\Rightarrow 2\pi i [\sin 4\pi + \cos 4\pi] + 2\pi i [\sin 9\pi + \cos 9\pi]$

$\Rightarrow 2\pi i [0+1] + 2\pi i [0+(-1)]$

$\Rightarrow -2\pi i - 2\pi i \Rightarrow -4\pi i$

2 evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)}$ where C is the circle $|z-2| = 1/2$



The point $z=2$ lies inside the region $|z-2| = 1/2$ and $z=1$ lies outside the region $|z-2| = 1/2$

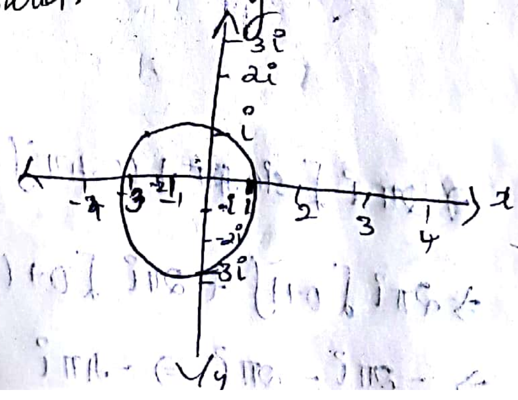
$$\int_C \frac{z\,dz}{(z-1)(z-2)} = \int_C \frac{\left(\frac{z}{z-1}\right)dz}{z-2}$$

$= 2\pi i \times f(2)$ where $f(z) = \frac{z}{z-1}$

$= 2\pi i \times 2$

$= 4\pi i$

3 Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where c is the circle $|z+1+i| = 2$ using Cauchy's integral formula.



$z = -1 - i$

$z^2 + 2z + 5$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)}$$

$$\Rightarrow \frac{-2 \pm \sqrt{4 - 20}}{2} \Rightarrow \frac{-2 \pm \sqrt{-16}}{2}$$

$$\Rightarrow \frac{-2 \pm 4i}{2} \Rightarrow -1 \pm 2i$$

$z = -1 \pm 2i$

$(z+1+2i)(z+1-2i) = 0$

$$\int_C \frac{z+4}{(z+1+2i)(z+1-2i)} dz$$

$$\int_C \frac{\left(\frac{z+4}{z+1-2i}\right)}{z+1+2i} dz \qquad f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$2\pi i f(a) = \int_C \frac{f(z)}{z-a}.$$

$\Rightarrow 2\pi i \, f(-1-2i)$ where $f(z) = \dfrac{z+4}{z+1-2i}$

$$\Rightarrow \cancel{2}\pi \cancel{i} \times \frac{3-2i}{-\cancel{4i}_{2}}$$

$$\Rightarrow \frac{(3-2i)\pi}{-2}$$

Scanned by CamScanner

4 Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the

$|z| = 2$

$z = -1$ lies inside $|z| = 2$

$$f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-a)^4} dz$$

By cauchy Integral formula,

$$\int_C \frac{f(z)}{(z-a)^4} dz = \frac{2\pi i}{3!} \times f'''(a).$$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f'''(-1) = 8e^{-2} = \frac{8}{e^2}$

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} \times f'''(-1)$$

$$= \frac{2\pi i}{6} \times \frac{8}{e^2}$$

$$= \frac{\cancel{16}^{8}\pi i}{e^2 \times \cancel{6}_3} \Rightarrow \frac{8\pi i}{3e^2}$$

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\, dz$$ where c is the circle $|z| = 3$

$z = +1, +2$ lies inside $|z| = 3$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)}\, dz$$

$$2\pi i \times f(a) = \int_C \frac{f(z)}{z-a}\, dz$$

consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$= \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

$B(1) = 1$

$A(-1) = 1$

$A = -1$

when $(z=2) \Rightarrow B = 1$

when $(z=1) \Rightarrow A = -1$

$$1 = \frac{-1}{z-1} + \frac{1}{z-2}$$

The poles are $z=1$ and $z=2$

Both the points lies inside $|z|=3$

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_C \frac{e^{2z}}{z-1}\,dz + \int_C \frac{e^{2z}}{z-2}\,dz$$

By Cauchy integral theorem,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$2\pi i\, f(a) = \int_C \frac{f(z)}{z-a}\,dz$$

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = 2\pi i\, f(1) + 2\pi i f(2)$$

$$= 2\pi i\,(e^{2z}) + 2\pi i\,(e^{2z})$$

$$= 2\pi i\,(e^{2}) + 2\pi i\,(e^{4})$$

$$= 2\pi i\,(e^{2}+e^{4})$$

$$= 2\pi i\,(e^{6})$$

Cauchy's Residue theorem:

If $f(z)$ is analytic within and on a simple closed curve C except that at a finite number of poles within C then $\oint_C f(z)\,dz = 2\pi i \times$ [sum of residues at the poles within C] calculation of residue at simple pole :

If $f(z)$ has a simple pole at $z=a$

then $\underset{z=a}{\text{Res}} f(z) = \underset{z \to a}{\text{Lt}} (z-a) f(z)$

calculation of residue at a multiple pole :

If $f(z)$ has a pole of order n at

$z=a$, then $\underset{z=a}{\text{Res}} f(z) = \frac{1}{(n-1)!} \underset{z \to a}{\text{Lt}} \left[\frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \right]$

1) Find the poles of $f(z) = \frac{2z}{(z-1)(z-2)^2(z-3)^3}$

$f(z) = \frac{2z}{(z-1)(z-2)^2(z-3)^3}$

poles of $f(z) = 1, 2, -3$

$z=1$ is a pole of order 1 or simple pol

$z=2$ is a pole of order 2

$z=-3$ is a pole of order 3

Scanned by CamScanner

Q. Find the residue of $f(z) = \dfrac{z^3}{(z-2)(z-3)^2}$

$z = 2$ is a simple pole

$z = 3$ is a pole of order 2.

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z)$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2) f(z)$$

$$= \lim_{z\to 2} \left[\frac{z^3}{(z-3)^2} \right]$$

$$= \frac{2^3}{(2-3)^2}$$

$$\Rightarrow \frac{8}{1} = 8$$

$$\operatorname*{Res}_{z=3} f(z) = \frac{1}{(2-1)!} \lim_{z\to 3} \left(\frac{d}{dz} (z-3)^2 f(z) \right)$$

$$= \lim_{z\to 3} \left[\frac{d}{dz} \left(\frac{z^3}{z-2} \right) \right]$$

$$= \lim_{z\to 3} \left[\frac{(z-2)(3z^2) - z^3}{(z-2)^2} \right]$$

$$= \lim \left[\frac{(3-2)(3(9)-27)}{(3-2)^2} \right] \Rightarrow \lim \left[\frac{(1)(27)-27}{(1)} \right]$$

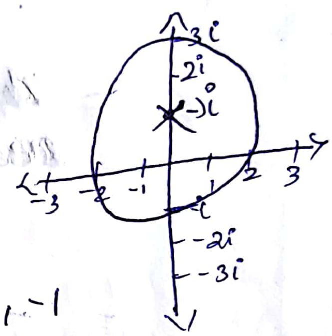
$$= \lim \left[\frac{(0)(12)-8}{0} \right) \rightarrow \frac{0}{0}$$

$$= 0$$

$$\operatorname*{Res}_{z=3} f(z) = 0.$$

Evaluate $\int_C \frac{z-1}{(z-2)(z+1)^2} dz$ where C is the circle $|z-i|=2$

$|z-i|=2$



The poles are $z = 2, -1$

By Cauchy's Residues theorem,

$$\int_C f(z)\,dz = 2\pi i \times [\text{sum of res at poles}]$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z \to 2} [(z-2)(f(z))]$$

$$= \lim_{z \to 2} \left[\frac{z-1}{(z+1)^2} \right] = \frac{1}{9}$$

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{(2-1)!} \lim \frac{d}{dz} ((z+1)^2 f(z)]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \lim_{z \to -1}\left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right]$$

$$= \lim_{z \to -1}\left[\frac{-1}{(z-2)^2}\right]$$

$$= \lim_{z \to -1}\left[\frac{-1}{(-1-2)^2}\right]$$

$$= \left(\frac{-1}{(9)}\right)$$

$$= \frac{-1}{9}$$

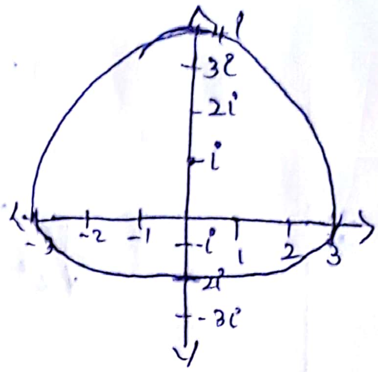
$$\operatorname*{Res}_{z=-1} f(z) = -1/9$$

$$\int_C \frac{z-1}{(z-2)(z+1)^2}\, dz = 2\pi i\left[\frac{1}{9}-\frac{1}{9}\right] = 0.$$

4 Evaluate $\int_C \frac{dz}{(z^2+9)^3}$ where C is $|z-i| = 3$ by using cauchy's residue theorem.

$|z-i| = 3$

$$f(z) = \frac{1}{(z^2+9)^3}$$



$$z^2+9 = 0$$

$$z = \pm 3i \Rightarrow +3i, -3i$$

$z = +3$ lies inside c and $z = -3$ lies outside c

Res $f(z)$

$$z = 3i \quad = \frac{1}{(3-1)!} \lim_{z \to 3i} \left[\frac{d^2}{dz^2} (z-3i)^3 f(z) \right]$$

$$f(z) = \frac{1}{[(z+3i)(z-3i)]^3}$$

$$f(z) = \frac{1}{(z+3i)^3 (z-3i)^3}$$

$$\frac{d}{dz} = \frac{d}{dz} \left(\frac{1}{(z+3i)^3} \right)$$

$$= \frac{d}{dz} (z+3i)^{-3}$$

$$= -3(z+3i)^{-3-1} (1)$$

$$\frac{d}{dz} = -3(z+3i)^{-4}$$

$$\frac{d^2}{dz^2} = \left(\frac{1}{(z+3i)^3} \right]$$

$$= 12(z+3i)^{-5}\ (1)$$

$$\frac{d^2}{dz^2} = \frac{12}{(z+3i)^5}$$

$$\underset{z=3i}{\text{Res}}\, f(z) = \frac{1}{2!} \lim_{z \to 3i} \left(\frac{12}{(z+3i)^5} \right)$$

$$= \frac{1}{2} \left(\frac{12}{(6i)^5} \right)$$

$$= \frac{1}{2} \left(\frac{12}{(6)^5 (i)^5} \right) \Rightarrow \frac{1}{\cancel{2}} \left(\frac{\overset{6}{\cancel{12}}}{(6)^5 (i)} \right)$$

$$\underset{z=3i}{\text{Res}}\, f(z) = \frac{6}{7776i}$$

$$\int_C \frac{dz}{(z^2+9)^3} = 2\pi i \left(\frac{6}{7776i} + 0 \right)$$

$$= 2\pi \cancel{i} \times \frac{6}{7776 \cancel{i}}$$

$$\int_C \frac{dz}{(z^2+9)^3} = \frac{12\pi}{7776}$$

a

Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, $a>b>0$

(I)

Let $z = e^{i\theta}$

$$\frac{dz}{d\theta} = ie^{i\theta}$$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{dz/iz}{a+b\left(\frac{1}{2}\left(\frac{z^2+1}{z}\right)\right)}$$

$$= \int_C \frac{dz/iz}{\frac{2az+bz^2+b}{2z}}$$

$$= \int_C \frac{dz}{iz} \times \frac{2z}{2az+bz^2+b}$$

$$= \frac{2}{i}\int_C \frac{dz}{2az+bz^2+b}$$

Consider $2az+bz^2+b=0$

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

Scanned by CamScanner

$$z = \frac{-\cancel{2}a \pm 2\sqrt{a^2-b^2}}{\cancel{2}b}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

(α) Let $z = \dfrac{-a + \sqrt{a^2-b^2}}{b}$ & (β) $z = \dfrac{-a - \sqrt{a^2-b^2}}{b}$

Let the root be α & β

$z = \alpha$ and $z = \beta$

$z = \alpha$ & $z = \beta$

Product of Roots $= |\alpha\beta| = 1$

Given $a > b > 0 \Rightarrow |\beta| > 1$; $|\alpha| < 1$

Since $z = \alpha$ and $z = \beta$ are the roots

and $|\alpha\beta| = 1$, $|\beta| > 1 \Rightarrow |\alpha| < 1$ Therefore

α is the only pole lies inside the C

(ie) $|z| = 1$. The roots of $bz^2 + 2az + b$

can be written as $b(z-\alpha)(z-\beta)$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\int_C \frac{1}{b(z-\alpha)(z-\beta)}\,dz = \frac{2}{i}\int_C f(z)$$

Let $f(z) = \dfrac{1}{b(z-\alpha)(z-\beta)}$

consider $\int_C f(z)dz$

$= 2\pi i \times$ sum of Residues

$\operatorname{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} [(z-\alpha) f(z)]$

$\Rightarrow \lim_{z\to\alpha} \left[\cancel{(z-\alpha)} \dfrac{1}{b\cancel{(z-\alpha)}(z-\beta)} \right]$

$\Rightarrow \dfrac{1}{b(\alpha-\beta)}$

$= \dfrac{1}{b} \left[\dfrac{1}{\dfrac{-\alpha+\sqrt{a^2+b^2}}{b} \quad \dfrac{+\alpha+\sqrt{a^2+b^2}}{b}} \right]$

$= \dfrac{1}{b} \left[\dfrac{1}{\dfrac{2\sqrt{a^2-b^2}}{b}} \right]$

$= \dfrac{1}{\cancel{b}} \left(\dfrac{\cancel{b}}{2\sqrt{a^2-b^2}} \right)$

$\operatorname{Res}_{z=\alpha} f(z) = \dfrac{1}{2\sqrt{a^2-b^2}}$

$$\int_C f(z)\,dz = \cancel{2}\pi i \times \frac{1}{\cancel{2}\sqrt{a^2-b^2}}$$

$$= \frac{\pi i}{\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{\cancel{i}} \times \frac{\pi \cancel{i}}{\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Hence proved.

Q (7) Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

Let $z = e^{i\theta}$ $\qquad \sin\theta = \frac{1}{2}\left(\frac{z+1}{z}\right)$

$\frac{dz}{d\theta} = ie^{i\theta}$ $\qquad \sin\theta = \frac{1}{2}\left(\frac{z^2-1}{z}\right)$

$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_C \frac{dz/iz}{13+5\left(\frac{1}{2}\left(\frac{z^2-1}{z}\right)\right)}$$

$$= \int_C \frac{dz/iz}{13+5\left(\frac{z^2+1}{2z}\right)} \Rightarrow \int_C \frac{dz/iz}{13+\frac{5z^2+5}{2z}}$$

$$= \int_C \frac{dz/iz}{\frac{26z+5z^2+5}{2z}}$$

$$= \int_C \frac{dz}{iz} \times \frac{2z}{5z^2+26z+5}$$

$$= \frac{2}{i}\int_C \frac{dz}{5z^2+26z+5}$$

Consider $5z^2+26z+5=0$.

$a=5$

$b=26$

$c=5$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-26 \pm \sqrt{(26)^2-4(5)(5)}}{10}$$

$$= \frac{-26 \pm \sqrt{(26)^2+100}}{10}$$

$$= \frac{-26 \pm \sqrt{676+100}}{10}$$

$$= \frac{-26 \pm \sqrt{776}}{10}$$

$\frac{\cancel{26}^{13}}{\cancel{10}_5}$ $\frac{\cancel{24}^{12}}{\cancel{10}_5}$

$\Rightarrow$ [illegible] [illegible]

Let $z = \frac{-13+12}{5}$ (or) $z = \frac{-13-12}{5}$

Let the roots be α and β.

$z=\alpha$, $z=\beta$.

Product of roots $= |\alpha\beta| = 1$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \frac{2}{i}\int_C \frac{1}{5(z-\alpha)(z-\beta)}\,dz$$

$$= \frac{2}{i}\int_C f(z)\,dz.$$

Let $f(z) = \frac{1}{5(z-\alpha)(z-\beta)}$

Consider $\int_C f(z)\,dz$,

$= 2\pi i \times$ sum of the residues.

$$\underset{z=\alpha}{\text{Res}}\, f(z) = \underset{z\to\alpha}{\text{Lt}} \left((z-\cancel{\alpha}) \frac{1}{5(z-\cancel{\alpha})(z-\beta)} \right)$$

$$= \frac{1}{5(z-\beta)} = \frac{1}{5(\alpha-\beta)}$$

$$= \frac{1}{5}\left(\frac{1}{\frac{-13+\cancel{12}-13-\cancel{12}}{5}} \right)$$

$$= \frac{1}{5}\left(\frac{1}{-26/5} \right)$$

$$= \frac{1}{\cancel{5}}\left(-\frac{\cancel{5}}{26}\right)$$

$$\underset{z=\alpha}{\text{Res}}\ f(z) = \frac{-1}{26}$$

$$\int_C f(z)\,dz = \cancel{2}\pi i \times -\frac{1}{\cancel{26}\ 13}$$

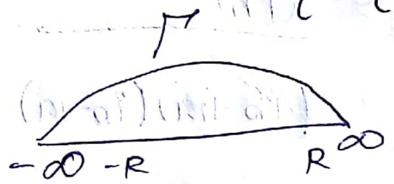
$$\Rightarrow \frac{-\pi i}{13}$$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \frac{2}{\cancel{i}} \times \frac{\pi \cancel{i}}{13}$$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \frac{-2\pi}{13}$$

3. $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx, \quad a>b>0.$$

$$\int_C \phi(z)\,dz = \int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$



where C consists the semicircle Γ and the bounding diameter

$$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz$$

$$\phi(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

$z = \pm ia$, $z = \pm ib$ are the poles

$z = +ia$ & $z = +ib$ lies inside c.

$$\operatorname*{Res}_{z=ia} \phi(z) = \lim_{z \to ia} (z-ia) \therefore \phi z$$

$$= \frac{(ia)^2}{(ia+ia)(ia+ib)(ia-ib)}$$

$$= \frac{+a^{\cancel{2}}}{2i\cancel{a}(a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

$$\operatorname*{Res}_{z=ib} \phi(z) = \lim_{z \to ib} (z-ib)\, \phi z$$

$$= \frac{(ib)^2}{(ib+ia)(ib-ia)(ib+ib)(ib-ib)}$$

$$= \frac{(ib)^2}{2ib\,(a^2-b^2)} = \frac{-b^{\cancel{2}}}{(a^2-b^2)}$$

$$= \frac{-b}{2i(a^2-b^2)} \Rightarrow \frac{-b}{2i(a^2-b^2)}$$

In ① let $R \to \infty$, $|z| \to \infty$, $\phi(z) = 0$

$$\int_C \phi(z)\,dz = \int_{-\infty}^{\infty} \phi(x)\,dx$$

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of Residue} \quad (\text{CRT})$$

$$\int_C \phi(z)\,dz = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= 2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{b}{2i(b^2-a^2)} \right]$$

$$= \pi \left[\frac{a(a^2-b^2) + b(a^2-b^2)}{(a^2-b^2)(a^2-b^2)} \right]$$

$$= \pi \left[\frac{\cancel{(a^2-b^2)}(a-b)}{(a^2-b^2)\cancel{(a^2-b^2)}} \right]$$

$$= \pi \left(\frac{a-b}{a^2-b^2} \right)$$

$$= \frac{\pi\,\cancel{(a-b)}}{(a+b)\cancel{(a-b)}}$$

$$\int_C \phi(z)\,dz = \frac{\pi}{a+b}$$

3/3/19

Taylor's and Laurent's series :-

Taylor's series :

If a function $f(z)$ is analytic at all points inside a circle C with its centre at the point a and radius R then at each point z inside C,

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} \dots + f^{(n)}(a)\frac{(z-a)^n}{n!} + \dots$$

The taylor's series at the point $a = 0$ is given by

$$f(z) = f(0) + f'(0)\frac{z}{1!} + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!} + \dots$$

This series is called ~~Maclaurin's~~ Mclaurin's series.

Laurent's series :-

If $f(z)$ is analytic on C_1 and C_2 and the annular region bounded by the two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_2 < r_1$) and with center at a, then for all in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n$$

where $a_n = \frac{1}{2\pi i} \int_{C} \frac{f(w)}{(w-a)^{n+1}} dw, \; n = 0, 1, 2, 3.$

$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, \; n = 1, 2, 3, \ldots$

1) Expand $f(x) = e^x$ in taylor's series, about $a = 0$.

2M

$$f(x) = f(0) + f''(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$$

$f(x) = e^x$	$f(0) = e^0 = 1$
$f''(x) = e^x$	$f'(0) = 1$
$f'(x) = e^x$	$f''(0) = 1$
$f'''(x) = e^x$	$f'''(0) = 1$

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

2. Obtain taylor's series expansion to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the region. $|z| < 2$

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

$$= 1 + \frac{-5z-7}{(z+2)(z+3)}$$

Consider $\frac{-5x-7}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{x+3}$

$$\frac{-5x-7}{(x+2)(x+3)} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

$-3+2$

when $x=-3$

$-5(-3)-7 = B(-1)$

$15-7 = -B$

$8 = -B \Rightarrow \boxed{B=-8}$

when $x=-2$

$-5(-2)-7 = A(1)+B(0)$

$10-7 = A$

$\boxed{3=A}$

$$= \frac{3}{(x+2)} + \frac{(-8)}{(x+3)}$$

$$f(x) = 1 + \frac{3}{x+2} + \frac{-8}{x+3}$$

Given $|x|<2$

$\left|\frac{x}{2}\right|<1 \quad ; \quad \left|\frac{x}{3}\right|<1$

$$f(x) = 1 + \frac{3}{2\left(\frac{x}{2}+1\right)} - \frac{8}{3\left(\frac{x}{3}+1\right)}$$

$\Rightarrow 1 + \frac{3}{2}(1+z/2)^{-1} - \frac{8}{3}(1+z/3)^{-1}$

$\Rightarrow 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right]$

$- \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right]$

$\Rightarrow 1 + \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n} - \frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n}$

3. Expand $\frac{1}{(z-1)(z-2)}$ in laurent's series valid for $|z|<1$ and $1<|z|<2$

$|z|<1 \Rightarrow \frac{|z|}{2}<1 \Rightarrow \left|\frac{z}{2}\right|<1$

$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$= \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$

$z=2$	$z=1$
$1 = B(1)$	$1 = A(-1)$
$B=1$	$A=-1$

$\Rightarrow \frac{-1}{(z-1)} + \frac{1}{(z-2)}$

$$\frac{1}{(z-2)(z-1)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{1}{1-z} + \frac{1}{z-2}$$

$$= \frac{1}{1-z} + \frac{1}{2\left(\frac{z}{2}-1\right)}$$

$$= (1-z)^{-1} + \frac{1}{2}(z/2-1)^{-1}$$

$$= (1-z)^{-1} - 1/2\,(1-z/2)^{-1}$$

$$\Rightarrow (1+z+z^2+z^3+\dots) - \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{3^3}+\dots\right]$$

$$\Rightarrow \frac{1}{2} + \frac{3z}{4} + \frac{7z^2}{8} + \dots$$

$$1 < |z| < 2$$

$$1 < |z| \Rightarrow \frac{1}{|z|} < 1 \Rightarrow \left|\frac{1}{z}\right| < 1$$

$$|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z(1-1/z)} + \frac{1}{2\left(\frac{z}{2}-1\right)}$$

$$\Rightarrow \frac{-1}{2}\left(1-1/z\right)^{-1} - \frac{1}{2\left(1-z/2\right)}$$

$$\Rightarrow -\frac{1}{2}\left(1-1/z\right)^{-1} - \frac{1}{2}\left(1-z/2\right)^{-1}$$

$$\Rightarrow -\frac{1}{2}\left(1+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right)$$

4 Find the laurent's series $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$

$1 < |z+1| < 3$

let $z+1 = u$
$z = u-1$

$$f(z) = \frac{7(u-1)-2}{(u-1+1)(u-1)(u-1-2)}$$

$$= \frac{7u-9}{(u)(u-1)(u-3)} \Rightarrow \frac{7u-9}{(u^2-u)(u-3)}$$

$$= \frac{7u-9}{u^3-3u^2-u^2+3u} \Rightarrow \frac{7u-9}{u^3-4u^2-3u}$$

$$\frac{7u-9}{(u)(u-1)(u-3)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u-3}$$

$$= \frac{A(u-1)(u-3) + B(u)(u-3) + C(u)(u-1)}{u(u-1)(u-3)}$$

$u = 1$

$7 - 9 = A(1-1)(1-3) + B(1)(1-3) + C(1)(1-1)$

$-2 \Rightarrow 0 + B(-2) + C(0)$

$\frac{-2}{-2} = B$

$\boxed{B = 1}$

$u = 3$

$7(3) - 9 = A(3-1)(3-3) + B(3)(3-3) + C(3)(3-1)$

$21 - 9 = A(0) + B(0) + C(6)$

$\frac{12}{6} = C$

$\boxed{C = 2}$

$u = 0$

$-9 = A(-1)(-3) + B(0) + C(0)$

$-9 = A(3)$

$\frac{-9}{3} = A$

$\boxed{A = -3}$

$$= \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

$1 < |u|$ | $|u| < 3$

$|1/u| < 1$ | $|u/3| < 1$

$$= \frac{-3}{u} + \frac{1}{u(1-1/u)} + \frac{2}{3(u/3-1)}$$

$$= \frac{-3}{u} + \frac{1}{u}(1-1/u)^{-1} + \frac{2}{3}(u/3-1)^{-1}$$

$$\Rightarrow \frac{-3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} - \frac{1}{u^2} + \frac{1}{u^3} + \dots\right) - \frac{2}{3}\left(1 + \frac{u}{3} - \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots\right)$$

UNIT 5 COMPLEX INTEGRATION

Questions	opt1	opt2	opt3	opt4	Answer
A curve is called a____________ if it does not intersect itself	Simple closed curve	multiple curve	simply connected region	multiple connected region	Simple closed curve
A curve is called ________ if it is not a simple closed curve	connected region	multiple curve	simply connected region	multiple connected region	multiple curve
If f(z) is analytic in a simply connected domain D and C is any simple closed path then ∫(from c)f(z)dz =	1	2πi	0	πi	0
If f(z) is analytic inside on a simple closed curve C and a be any point inside C then ∫(from c)f(z)dz /(z-a)=	2πi f(a)	2πi	0	πi	2πi f(a)
The value of ∫(from c) [(3z^2+7z+1)/(z+1)] dz where C is |z| = 1/2 is	2πi	-6πi	πi	πi/2	-6πi
The value of ∫(from c) (cos πz/z-1) dz if C is |z| =2	2πi	-2πi	πi	πi/3	-2πi
The value of ∫(from c) (1/z-1) dz if C is |z| =2	2πi	3πi	πi	πi/4	2πi
The value of ∫(from c) (1/z-3) dz if C is |z| =1	3πi	πi	πi/4	0	0
The value of ∫(from c) (1/(z-3)^3) dz if C is |z| =2	3πi	πi	πi/5	0	0
The Taylor's series of f(z) about the point z=0 is called ___________series	Maclaurin's	Laurent's	Geometric	Arithmetic	Maclaurin's
The value of ∫(from c) (1/z+4) dz if C is |z| =3	3πi	πi	πi/4	0	0
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____ part	regular	principal	real	imaginary	regular
In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____ part	regular	principal	real	imaginary	principal
The poles of the function f(z) = z/((z-1)(z-2)) are at z = ______	1, 2	2,3	1,-1	3,4	1, 2
The poles of cotz are______	2nπ	nπ	3nπ	4nπ	nπ
The poles of the function f(z) = cos z/((z+3)(z-4)) are at z = _____	- 3, 4	2,3	1,-1	3,4	- 3, 4
The isolated singular point of f(z) = z/((z-4)(z-5))	1,2	2,3	0,2	4,5	4,5
The isolated singular point of f(z) = z/((z(z-3))	1,3	2,4	0,3	4,5	0,3
A simple pole is a pole of order ______	1	2	3	4	1
The order of the pole z= 2 for f(z) = z/((z+1)(z-2)^2)	1	2	3	4	2
Residue of (cosz / z) at z = 0 is	0	1	2	4	1
The residue at z = 0 of ((1 + e^z) / (zcosz+sinz)) is	0	1	2	4	1
The residue of f(z) = cot z at z= 0 is_____	0	1	2	4	1
The singularity of f(z) = z / ((z-3)^3) is______	0	1	2	3	3
A point z=a is said to be a _______point of f(z), if f(z) is not analytic at z=a	Singular	isolated singular	removable	essential singular	Singular
A point z=a is said to be a _______point of f(z), if f(z) is analytic except at z=a	Singular	isolated singular	removable	essential singular	isolated singular

In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____point	Singular	isolated singular	removable singular	essential singular	essential singular
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____point	Singular	isolated singular	removable singular	essential singular	removable singular
In contour integration, cos θ=_______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz	(z^2+1)/2z
In contour integration, sin θ=_______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz	(z^2-1)/2iz