**18BEAE201/18BEME201 MATHEMATICS II 3 1 0 4**

**(Calculus, Ordinary DifferentialEquations and Complex Variable for Mechanical and Automobile Engineering)**

**COURSE OBJECTIVES**

The objective of this course is to familiarize the prospective engineers with techniques in Multivariate integration, ordinary and partial differential equations and complex variables. It aims to equip the students to deal with advanced level of mathematics and applications thatwould be essential for their disciplines.

**INTENDED OUTCOME**

The students will learn:
- The mathematical tools needed in evaluating multiple integrals and their usage.
- The effective mathematical tools for the solutions of differential equations that model physical processes.
- The tools of differentiation and integration of functions of a complex variable that are used in various techniques dealing engineering Problems.

**UNIT I: Multivariable Calculus (Integration) 12**

Multiple Integration: double and triple integrals (Cartesian and polar), change of order of integration in double integrals, Applications: areas and volumes, Center of mass and Gravity (constant and variable densities). Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds.

**UNIT II: First order ordinary differential equations 12**

Exact, linear and Bernoulli's equations, Euler's equations, Equations not of first degree :equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.

**UNIT III: Ordinary differential equations of higher orders 12**

Second order linear differential equations with variable coefficients, method of variation of parameters, Cauchy-Euler equation; Power series solutions; Legendre polynomials, Bessel functions of the first kind and their properties.

**UNIT IV: Analytic Functions 12**

Cauchy-Riemann equations, analytic functions, harmonic functions, finding harmonic conjugate; elementary analytic functions (exponential, trigonometric, logarithm)and their properties; Conformal mappings, Mobius transformations.

**UNIT V: Complex Integration 12**

Contour integrals, Cauchy- Goursat theorem (without proof), Cauchy Integral formula(without proof), zeros of analytic functions, singularities, Taylor's series, Laurent's series, Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral involving sine and cosine. **Total: 60**

**TEXT/REFERENCE BOOKS**

| S. NO. | AUTHOR(S) NAME | TITLE OF THE BOOK | PUBLISHER | YEAROF PUBLICATION |
|---|---|---|---|---|
| 1 | Hemamalini. P.T | Engineering Mathematics | McGraw Hill Education (India) Private Limited, New Delhi. | 2014 |
| 2 | G.B. Thomas and R.L. Finney | Calculus and Analytic geometry, 9th Edition | Pearson | 2002 |
| 3 | Erwin kreyszig | Advanced Engineering Mathematics, 9th Edition | John Wiley & Sons | 2006 |
| 4 | W. E. Boyce and R. C. DiPrima | Elementary Differential Equations and Boundary Value Problems9th Edn. | Wiley India | 2009 |
| 5 | S. L. Ross | Differential Equations, 3rd Ed. | Wiley India | 1984 |
| 6 | E. A. Coddington | An Introduction to Ordinary Differential Equations | Prentice Hall, India | 1995 |
| 7 | E. L. Ince | Ordinary Differential Equations | Dover Publications | 1958 |
| 8 | J. W. Brown and R. V. Churchill | Complex Variables and Applications, 7th Ed. | Mc-Graw Hill | 2004 |
| 9 | N.P. Bali and Manish Goyal | A text book of Engineering Mathematics | Laxmi Publications | 2008 |
| 10 | B.S. Grewal | Higher Engineering Mathematics, 36th Edition | Khanna Publishers | 2010 |

**Faculty Incharge** **HoD**

# KARPAGAM ACADEMY OF HIGHER EDUCATION

*(Deemed to be University Established Under Section 3 of UGC Act 1956)*

**COIMBATORE-641 021**

## DEPARTMENT OF SCIENCE AND HUMANITIES
## FACULTY OF ENGINEERING

## I B.E MECHANICAL / AUTOMOBILE ENGINEERING
## LECTURE PLAN

**Subject** : **MATHEMATICS – II**
**(Calculus, Ordinary Differential Equations and Complex Variable)**

**Code** : **18BEME201/18BEAE201**

| S.NO | Topics covered | No. of hours |
|---|---|---|
| | **UNIT I First order ordinary differential equations** | |
| 1 | Introduction of Multiple Integration: double and triple integrals | 1 |
| 2 | Multiple Integration: double integral | 1 |
| 3 | Multiple Integration: double and triple integrals (Cartesian and polar), | 1 |
| 4 | Multiple Integration: Triple integrals | 1 |
| 5 | change of order of integration in double integrals | 1 |
| 6 | change of order of integration in double integrals | 1 |
| 7 | Tutorial 1 - Problems based on change of order of integration in double integrals | 1 |
| 8 | Applications: areas and volumes | 1 |
| 9 | Applications: areas and volumes | 1 |
| 10 | Center of mass and Gravity (constant and variable densities). | 1 |
| 11 | Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds. | 1 |
| 12 | Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds. | 1 |
| 13 | Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds. | 1 |
| 14 | Tutorial 2 - Problems based on Theorems of Green, Gauss and Stokes | 1 |
| | **TOTAL** | **14** |
| | **UNIT II First order ordinary differential equations** | |
| 15 | Introduction of first order differential equations | 1 |
| 16 | Exact, linear and Bernoulli’s equations | 1 |
| 17 | Exact, linear and Bernoulli’s equations | **1** |
| 18 | Euler’s equations | 1 |
| 19 | Tutorial 3 - Problems based on Exact, linear and Bernoulli’s equations | 1 |
| 20 | Equations not of first degree:Equations solvable for p | 1 |
| 21 | Equations not of first degree:Equations solvable for p | 1 |
| 22 | Equations solvable for y | 1 |
| 23 | Equations solvable for y | 1 |
| 24 | Equations solvable for x | 1 |
| 25 | Equations solvable for x | 1 |
| 26 | Clairaut’s type | 1 |
| 27 | Clairaut’s type | 1 |
| 28 | Tutorial 4 - Problems based on Clairaut’s type, Equations solving for x and y, p | 1 |

| | | |
|---|---|---|
| | **TOTAL** | **14** |
| | **UNIT III Ordinary differential equations of higher orders** | |
| 29 | Introduction of ordinary differential equations | 1 |
| 30 | Second order linear differential equations with variable coefficients | 1 |
| 31 | Second order linear differential equations with variable coefficients | 1 |
| 32 | Second order linear differential equations with variable coefficients | 1 |
| 33 | Second order linear differential equations with variable coefficients | 1 |
| 34 | Second order linear differential equations with variable coefficients | 1 |
| 35 | Tutorial 5 - Problems based on second order differential equations with variable coefficients | 1 |
| 36 | Method of variation of parameters | 1 |
| 37 | Cauchy-Euler equation | 1 |
| 38 | Power series solutions; Legendre polynomials | 1 |
| 39 | Power series solutions; Legendre polynomials | 1 |
| 40 | Bessel functions of the first kind and their properties | 1 |
| 41 | Bessel functions of the first kind and their properties | 1 |
| 42 | Tutorial 6 - Problems based on Bessel functions and Legendre polynomials | 1 |
| | **TOTAL** | **14** |
| | **UNIT IV: Analytic Functions** | |
| 43 | Introduction – Analytic Function | 1 |
| 44 | Necessary and Sufficient conditions for an analytic function- Cauchy-Riemann equations –Cartesian form | 1 |
| 45 | Necessary and Sufficient conditions for an analytic function- Cauchy-Riemann equations –Cartesian form | 1 |
| 46 | Cauchy-Riemann equations – Polar form | 1 |
| 47 | Harmonic functions and its conjugate | 1 |
| 48 | Tutorial 7-Cauchy-Riemann equations Harmonic functions | 1 |
| 49 | Properties of analytic functions | 1 |
| 50 | Construction of an Analytic Function Milne-Thomson method | 1 |
| 51 | Construction of an Analytic Function Milne-Thomson method | 1 |
| 52 | Conformal mapping: The transformations $w = z+a, az$ | 1 |
| 53 | Conformal mapping: The transformations $w = 1/z, Z^2$ | 1 |
| 54 | Bilinear transformation | 1 |
| 55 | Mobius transformations | 1 |
| 56 | Tutorial 8 - Conformal mapping, Mobius transformations | 1 |
| | **TOTAL** | **14** |
| | **UNIT V Complex Integration** | |
| 57 | Introduction - Complex Integration, Line integral | 1 |
| 58 | Problems solving using Cauchy's integral theorem | 1 |
| 59 | Problems solving using Cauchy's integral formula | 1 |
| 60 | Taylor's Series Problems | 1 |
| 61 | Taylor's Series Problems | 1 |
| 62 | Laurent series problems | 1 |
| 63 | Laurent series problems | 1 |
| 64 | Tutorial 9 - Taylor's and Laurent's series problems | 1 |
| 65 | Theory of Residues | 1 |
| 66 | Cauchy Residue theorem (without proof) | 1 |
| 67 | Cauchy Residue theorem- Problems | 1 |
| 68 | Evaluation of definite integral involving sine and cosine. | 1 |

| | | |
|---|---|---|
| 69 | Evaluation of definite integral involving sine and cosine. | 1 |
| 70 | Tutorial 10 - Cauchy’s residue theorem, Applications | 1 |
| | **TOTAL** | **14** |
| **GRAND TOTAL** | | **70** |

**Staff- Incharge** **HoD**

# UNIT - II

## VECTOR CALCULUS

**Scalar point function**

At each pt P of a region R we may associate a Scalar denoted by $\phi(P)$. then we say that $\phi$ is a Scalar point-fn over the Region R.

**Vector point function.**

If to each pt P of a region R, there is associated a vector $\vec{F}(P)$ then the fn $\vec{F}$ is called a vector-point-fn.

The vector differential operator is defined as.

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors. along the three rectangular axis.

**Gradient**

Let $\phi(x,y,z)$ be a Scalar point fn and is continuously differentiable, then the vector.

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$
$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$ is called the gradient.

of the Scalar fn $\phi$ and it is written as grad $\phi$

$$\text{grad}\,\phi = \nabla\phi.$$

**Properties of gradient:-**

If f and g are two Scalar point fn then $\nabla(f+g) = \nabla f + \nabla g$.

**Proof:-**

$$\nabla(f+g) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(f+g)$$
$$= \vec{i}\frac{\partial}{\partial x}(f+g) + \vec{j}\frac{\partial}{\partial y}(f+g) + \vec{k}\frac{\partial}{\partial z}(f+g)$$
$$= \vec{i}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + \vec{j}\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) + \vec{k}\left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)f + \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)g = \underline{\nabla f + \nabla g}$$

② IIIly its true for $\nabla(f-g) = \nabla f - \nabla g$.

③. If f and g are any two Scalar point fn then.

$$\nabla(fg) = f\nabla g + g\nabla f.$$

<u>proof</u>

$$\nabla(fg) = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)fg$$

$$= \vec{i}\frac{\partial}{\partial x}(fg)+\vec{j}\frac{\partial}{\partial y}(fg)+\vec{k}\frac{\partial}{\partial z}(fg)$$

$$= f\vec{i}\frac{\partial g}{\partial x}+g\vec{i}\frac{\partial f}{\partial x}+f\vec{j}\frac{\partial g}{\partial y}+g\vec{j}\frac{\partial f}{\partial y}+\vec{k}\cdot f\frac{\partial g}{\partial z}+\vec{k}$$

$$= f\left[\vec{i}\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right](g)+g\left[\vec{i}\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right.$$

$$= f\nabla g + g\nabla f.$$

<u>hence proved</u>

④ gradient of a Constant is zero.

Let $\phi(x,y,z)$ be a constant. then $\frac{\partial\phi}{\partial x}=0$; $\frac{\partial\phi}{\partial y}=0$; $\frac{\partial\phi}{\partial z}=0$.

$$\therefore \nabla\phi = \left(i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial y}\right)=0.$$

hence the proof.

<u>Pbs</u>:-

1. If $\phi = \log(x^2+y^2+z^2)$ find $\nabla\phi$.

<u>Sol</u>:-

Given $\phi = \log(x^2+y^2+z^2)$.

W.K.T $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\log(x^2+y^2+z^2).$$

$$= \vec{i}\frac{\partial}{\partial x}\log(x^2+y^2+z^2)+\vec{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2)+\vec{k}\frac{\partial}{\partial z}\log(x^2+y^2+z$$

$$= \vec{i}\left[\frac{2x}{x^2+y^2+z^2}\right]+\vec{j}\left[\frac{2y}{x^2+y^2+z^2}\right]+\vec{k}\left[\frac{2z}{x^2+y^2+z^2}\right]$$

$$= \frac{2}{x^2+y^2+z^2}\left[x\vec{i}+y\vec{j}+z\vec{k}\right]. \quad \text{—①}$$

$$\left.\begin{array}{l} \therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \\ r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \\ \text{hence } r^2 = x^2+y^2+z^2 \end{array}\right\} \rightarrow ②$$

Sub ② in ① $\boxed{\nabla\phi = \frac{2}{r^2} \cdot \vec{r}}$

② If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ p.T (i) $\nabla r = \frac{\vec{r}}{r}$, (ii) $\nabla r^n = n r^{n-2} \vec{r}$, where $r = |\vec{r}|$.

Sol:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = |r| = \sqrt{x^2+y^2+z^2}$ ; $r^2 = x^2+y^2+z^2$ — ①

(i) $\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$ — ②

$$\left.\begin{array}{l} \text{from ① } 2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \\ 2r \cdot \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \\ 2r \cdot \frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \end{array}\right\} \rightarrow ③$$

$\frac{\partial r}{\partial x} = \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot 2x$

Sub ③ in ②

$\nabla r = \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$

$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})$

$\boxed{\nabla r = \frac{\vec{r}}{r}}$.

$\nabla r = \vec{i}\frac{x}{\sqrt{x^2+y^2+z^2}}$

$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2}}$

$= \frac{\vec{r}}{r}$

(ii)

$\nabla r^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r^n$

$= \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$

$= \vec{i} \cdot n r^{n-1} \cdot \frac{\partial r}{\partial x} + \vec{j}\, n \cdot r^{n-1} \cdot \frac{\partial r}{\partial y} + \vec{k}\, n \cdot r^{n-1} \frac{\partial r}{\partial z}$

$= n r^{n-1}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$

$= n r^{n-1} \cdot \nabla r = n r^{n-1} \cdot \frac{\vec{r}}{r} = \boxed{n r^{n-2}\vec{r} = \nabla r^n}$

## Directimal Derivative.

Directimal derivative of a Scalar fn $\phi$ at a pt is defined by the dot product of grad $\phi$ and the unit vector through that pt.

ie, Directimal Derivative = grad $\phi \cdot \hat{n}$.

$\because$ $\hat{n}$ is unit vector.

$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$\hat{n} \cdot \text{grad}\,\phi = |\hat{n}|\,|\text{grad}\,\phi| \cos\theta.$$

$$= |\text{grad}\,\phi| \cos\theta.$$

① Find the diretimal derivative of $\phi = xy + yz + zx$ in the directim of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ at $(1,2,0)$

Sol:-

directimal Derivative = grad $\phi \cdot \hat{n}$.

where grad $\phi = \nabla\phi$ & $\hat{n} = \frac{\phi}{|\phi|}$.

$\hat{n}$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$ is $\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$.

$$\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy + yz + zk)$$

$$= \vec{i}(y+z) + \vec{j}(x+z) + \vec{k}(y+x)$$

$$(\text{grad}\,\phi)_{1,2,0} = 2\vec{i} + \vec{j} + 3\vec{k}.$$

Hence Directimal derivative $= \frac{(2\vec{i} + \vec{j} + 3\vec{k})(\vec{i} + 2\vec{j} + 2\vec{k})}{3}$.

$$= \frac{2+2+6}{3}$$

$$= \frac{10}{3}$$

② Find the directional Derivative of $\phi = 3x^2+2y-3z$ at (1,1,1) in the direction of $2\vec{i}+2\vec{j}-\vec{k}$.

**Sol:-**

$$D.D = \text{grad}\,\phi \cdot \vec{n}$$

where $\text{grad}\,\phi = \nabla\phi = \left[\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right]\left[3x^2+2y-3z\right]$

$$= \vec{i}(6x) + \vec{j}(2) + \vec{k}(-3)$$

$$\boxed{\nabla\phi = 6x\vec{i}+2\vec{j}-3\vec{k}} \quad \boxed{(\nabla\phi)_{(1,1,1)} = 6\vec{i}+2\vec{j}-3\vec{k}}$$

$$\hat{n} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3}$$

hence $D.D = \dfrac{(6\vec{i}+2\vec{j}-3\vec{k})(2\vec{i}+2\vec{j}-\vec{k})}{3}$

$$= \frac{12+4+3}{3} = \frac{22}{3}$$

$$= \frac{19}{3}$$

**Def!-**

1) Normal Derivative of $\phi$ at a point $= |\text{grad}\,\phi| = |\nabla\phi$

2) unit normal vector to the surface $\phi = \dfrac{\nabla\phi}{|\nabla\phi|}$.

① Find a unit vector normal to the surface $x^2-y^2+z=2$. at the pt (1,-1,2). and also find normal derivative.

**Sol:-**

Given $\phi = x^2-y^2+z-2$.

$$\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(x^2-y^2+z-2)$$

$$= \vec{i}\cdot 2x + \vec{j}(-2y) + \vec{k}(1)$$

$$= 2x\vec{i} - 2y\vec{j} + \vec{k}$$

$$(\nabla\phi)_{(1,-1,2)} = 2\vec{i}+2\vec{j}+\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}.$$

hence unit normal vector $\boxed{\hat{n} = \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}}$.

normal derivative $\boxed{|\nabla\phi| = 3.}$

HW

② Find a unit normal vector to the surface $x^2+xy+z^2=$
at the pt $(1,-1,2)$.

~~Sol~~:

② Find a unit normal vector to the surfaces $x^2y+2xz^2=8$ at $(1,0,2)$. and normal derivative.

Sol: Given $\phi = x^2y+2xz^2=8$.

unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

To find $\nabla\phi = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\left[x^2y+2xz^2-8\right]$.

$$\nabla\phi = 2xy\vec{i} + 2z^2\vec{i} + x^2\vec{j} + 4xz\vec{k}$$

$$(\nabla\phi)_{1,0,2} = \cancel{4+1+8 = 13}\ 8\vec{i}+\vec{j}+8\vec{k}.$$

$$|\nabla\phi| = \sqrt{8^2+1^2+8^2} \Rightarrow \sqrt{129},$$

hence $\hat{n} = \frac{8\vec{i}+\vec{j}+8\vec{k}}{\sqrt{129}}$.

normal derivative $= |\nabla\phi| = \sqrt{129}$.

Green's Thm. ( In Cartesian Form)

If $u, v, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are continuous and one-valued fn in the region $R$ enclosed by the curve $C$, then.

$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$$

Note: 1. If $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, the value of the Integral $\int_C (u\,dx + v\,du)$ is Independent of the path of Integration.

2. If $R$ is a region bounded by a simple closed curve $C$, then the area of $R$ is $\frac{1}{2}\int_C x\,dy - y\,dx$.

In vector Notation.

① Verify Green's Thm in a plane for the Integral $\int_C \{(x-2y)\,dx + x\,dy\}$ taken around the circle $x^2+y^2=1$.

Sol:-

Green's Thm is $\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$ —①

Given $\int_C \{(x-2y)\,dx + x\,dy\}$ —②.

Comparing L.H.S of eqn ① with ② $u = x-2y$ $v = x$.

~~then $\frac{\partial u}{\partial y} = 0$ ; $\frac{\partial v}{\partial x} = 1$~~ $\frac{\partial u}{\partial y} = -2$ ; $\frac{\partial v}{\partial x} = 1$

The curve is the circle $x^2+y^2=1$.

The parametric eqn of the circle is $x = \cos\theta$ $y = \sin\theta$.

hence $dx = -\sin\theta\,d\theta$ ; $dy = \cos\theta\,d\theta$.

& ~~$u = \cos\theta - 2\sin\theta$~~

~~$\frac{\partial u}{\partial}$~~

To find $\int_C (udx + vdy)$

$$\int_C (udx + vdy) = \int_C \{(x-2y)dx + xdy\}.$$

$$= \int_0^{2\pi} \{(\cos\theta - 2\sin\theta)(-\sin\theta d\theta) + (\cos\theta)(\cos\theta d\theta)\}$$

$$= \int_0^{2\pi} \{ \cos\theta \sin\theta + 2\sin^2\theta + \cos^2\theta \} d\theta .$$

$$= \int_0^{2\pi} \{ -\cos\theta\sin\theta + \sin^2\theta + 1 \} d\theta .$$

$$= \int_0^{2\pi} \left\{\left\{ -\frac{\sin 2\theta}{2} \right\} + \left\{ \frac{1-\cos 2\theta}{2} \right\} + 1 \right\} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} -\sin 2\theta \, d\theta + \frac{1}{2}\int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} \cos 2\theta + \int_0^{2\pi} d\theta .$$

$$= \frac{1}{2}\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} + \frac{1}{2}[\theta]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} + (\theta)_0^{2\pi} .$$

$$= 0 + \pi + 2\pi = 3\pi .$$

hence $\int_C (udx + vdy) = 3\pi$ —— Ⓐ.

To find $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$.

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x-2y)\right] dx\,dy .$$

$$= \iint_R (1+2)\, dx\,dy .$$

$= 3 \iint_R dx\,dy$. $= 3$ [ Area of the region enclosed by

$= 3$ [ ~~radius~~ Area of the circle }

$= 3\pi$ [∵ radius = 1]

Ⓑ.

from Ⓐ & Ⓑ Green's Thm verified.

② verify Green's Thm for $\int_C [(x^2-y^2)dx + 2xy\,dy]$, where C is the boundary of the rectangle in the xoy plane bounded by the lines $x=0$, $x=a$, $y=0$ & $y=b$.

Sol:-

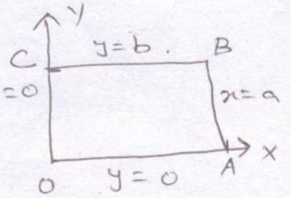
Green's Thm $\int_C [u\,dx + v\,dy] = \iint_R \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$. ①

Given $\int_C (x^2-y^2)dx + 2xy\,dy$ ← ②

Comparing L.H.S of eqn ① with ②.

$u = x^2 - y^2$ ; $v = 2xy$.

$\frac{\partial u}{\partial y} = -2y$. ; $\frac{\partial v}{\partial x} = 2y$.



To evaluate $\int_C (u\,dx + v\,dy)$.

$$\int_C (u\,dx + v\,dy) = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}.$$

(i) $\int_{OA} \{(x^2-y^2)\,dx + 2xy\,dy\} = \int_0^a x^2 dx \Rightarrow x^3/3 = a^3/3.$ $\left[\because \begin{matrix} y=0 \\ dy=0 \end{matrix}\right]$

(ii) $\int_{AB} \{ \quad \} = \int_0^b 2ay\,dy \Rightarrow \left[\frac{2ay^2}{2}\right]_0^b = ab^2$ $\left[\because \begin{matrix} x=a \\ dx=0 \end{matrix}\right]$

(iii) $\int_{BC} \{ \quad \} = -\int_a^a (x^2-b^2)dx = \left[\frac{x^3}{3} - b^2x\right]_0^a = -\left[\frac{a^3}{3} - ab^2\right].$

(iv) $\int_{CO} \{ \quad \} = 0.$

hence $\int_C (u\,dx + v\,dy) = \cancel{\frac{a^3}{3}} + ab^2 - \cancel{\frac{a^3}{3}} + ab^2 = 2ab^2$ — Ⓐ

To evaluate $\iint_R \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$.

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint [2y - (-2y)]\,dy\,dx$$

$$= \int_0^a\int_0^b 4y\,dy\,dx = \int_0^a 4\left[\frac{y^2}{2}\right]_0^b dx \Rightarrow \frac{4b^2}{2}(x)_0^a \Rightarrow \frac{4ab^2}{2} \Rightarrow 2ab^2$$ — Ⓑ

hence proved.

③ using Green's Thm evaluate $\int_C \{\{y-\sin x\}\,dx + \cos x\,dy\}$ where C is the triangle bounded by $y=0$ $x=\pi/2$, $y=\frac{2x}{\pi}$.

Sol :-

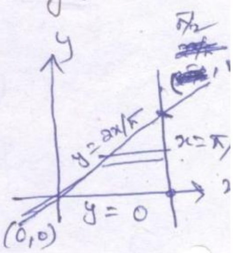
Green's Thm $\int_C (u\,dx + v\,dy) = \iint_R \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$.

here $u = y - \sin x$ $v = \cos x$.

$\frac{\partial u}{\partial y} = 1$ $\frac{\partial v}{\partial x} = -\sin x$.

hence $\int_C u\,dx + v\,dy = \iint_R (-\sin x - 1)\,dx\,dy$.

$= -\int_0^1 \int_{\frac{\pi y}{2}}^{\pi/2} (1+\sin x)\,dx\,dy = -\left[\frac{\pi^2+8}{4\pi}\right]$



~~Stoke's Thm~~.

④ verify Green's thm is the xy plane for $\int_C \{(xy+y^2)\,dx + x^2\,dy\}$ where C is the closed curve of the region bounded by $y=x$ & $y=x$

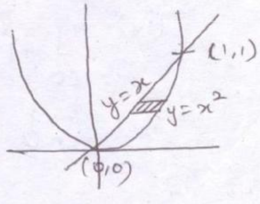
Sol! :-

Green's Thm is $\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$ —①.

Given $\int_C \{(xy+y^2)\,dx + x^2\,dy\}$ —②.

comparing L.H. segn ① & ②. $u = xy + y^2$ ; $V = x^2$.

hence $\frac{\partial u}{\partial y} = x + 2y$ ; $\frac{\partial v}{\partial x} = 2x$.



$\iint_R \left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} dx\,dy = \int_0^1 \int_x^{x^2} (x-2y)\,dy\,dx$.

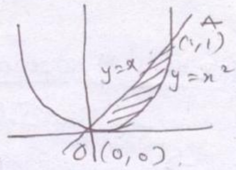
$= \int_0^1 \left[xy - \frac{2y^2}{2}\right]_x^{x^2} dx \Rightarrow \int_0^1 \left[x^3 - x^4 - \cancel{x^2} + \cancel{x^2}\right] dx$

$= \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 \Rightarrow \frac{1}{4} - \frac{1}{5} = -\frac{1}{20}$ $\left[xy - y^2\right]_{x^2}^{x}$

$\iint_R \left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} dx\,dy = -\frac{1}{20}$ —Ⓐ

$= x^2 - x^2 - \left[x^3 - x^4\right]$

$\Rightarrow -\left[\frac{1}{3} - \frac{1}{5}\right] \Rightarrow -\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$



To find $\int_C \{u\,dx + v\,dy\}$ take the curve $c$ into two different paths. i.e.,

(i) along OA ($y = x^2$) (ii) along AO ($y = x$). $\int_C = \int_{OA} + \int_{AO}$.

$$\int_C \{(xy + y^2)\,dx + x^2\,dy\} = \int_{OA} \left[ (x \cdot x^2 + x^4)\,dx + x^2 \cdot 2x\,dx \right] \quad ( \because y = x^2 \quad dy = 2x .$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx \Rightarrow \int_0^1 (x^4 + 3x^3)\,dx$$

$$\Rightarrow \left[ x^5/5 + 3x^4/4 \right]_0^1 \Rightarrow 1/5 + 3/4 = \frac{19}{20} \; //$$

$$\int_{AO} \{xy + y^2\}\,dx + x^2\,dy = \int_1^0 (x^2 + x^2)\,dx + x^2\,dx \Rightarrow \int_1^0 3x^2\,dx \Rightarrow \frac{3x^3}{3} = -$$

hence $\int_C = \int_{OA} + \int_{AO} = \frac{19}{20} - 1 = -1/20 \; //$ — Ⓑ.

from Ⓐ & Ⓑ Green's Thm verified.

<u>HW</u>.

① Evaluate the Integral using Green's Thm $\int_C \{(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy\}$. where $c$ is the boundary in the $xy$ plane of the area enclosed by the $x$ axis and the semi-circle $x^2 + y^2 = 1$ in the upper half $xy$ plane.

(ii) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) - \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(4x)$

$= 4x - 2x$

$\boxed{\nabla \cdot \vec{F} = 2x}$

$\iiint_V \nabla \cdot \vec{F} \, dV = \iiint_V 2x \, dx \, dy \, dz$

$= 8/3$ //

## Gauss Divergence Thm.

If $\vec{F}$ is a vector point fn, finite and diff in a region R bounded by a closed surface S, then the surface Integral of the normal component of $\vec{F}$ taken over S is equal to the Integral of divergence of $\vec{F}$ taken over V.

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dV$$

where $\hat{n}$ is the unit vector in the positive normal to

① verify G.D.T for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

Sol:-

Gauss Div Thm is $\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$.

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial z} yz$.

$= \cancel{\frac{4x^2z}{2}} - \cancel{\frac{y^2}{2}} + \cancel{\frac{yz^2}{2}}$

$= \frac{1}{2}) \, 4z - 2y + y$

$\nabla \cdot \vec{F} = 4z - y$.

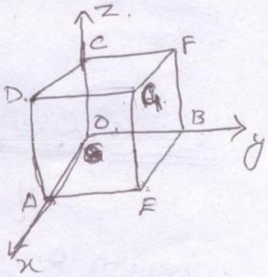
Hence $\iiint_V \nabla \cdot \vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z-y)\, dz\, dy\, dx.$

$= \int_0^1\int_0^1 \left[\frac{4z^2}{2} - yz\right]_0^1 dy\, dx.$

$= \int_0^1\int_0^1 (2-y)\, dy\, dx \Rightarrow \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \frac{3}{2}$

$\iiint_V \nabla \cdot \vec{F} dv = 3/2.$ —— Ⓐ.

$L.H.S = \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}.$



| Surface | Face | unit outward normal vector. |
|---|---|---|
| $S_1$ | AEGD | $\vec{i}$ |
| $S_2$ | OBFC | $-\vec{i}$ |
| $S_3$ | EBFG | $\vec{j}$ |
| $S_4$ | OADC | $-\vec{j}$ |
| $S_5$ | AEBO. | $\vec{k}$ |
| $S_6$ | DGFC | $-\vec{k}$. |

Evaluation of $\iint_{S_1} \vec{F}\cdot\hat{n}\, ds.$

$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = \iint_{AEGD} (4x\cancel{y}z\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{i}\ dy\, dz.$

$= \int_0^1\int_0^1 4xz\, dy\, dz. = \int_0^1\int_0^1 4z\, dy\, dz \quad (\because x=1)$

$= \cancel{4\int_0^1\int_0^1}\ 2$

Evaluation of $S_2$ $\iint_{S_2} \vec{F}\cdot\hat{n}\, ds.$

$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \iint_{OBFC} \vec{F}(-\vec{i})\, dy\, dz$

$= 0 \qquad (\because x = 0 .$

Evaluation of $\iint_{S_3} \vec{F}\cdot\hat{n}\,ds$.

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{EBFG} \vec{F}\cdot\vec{j}\,dx\,dz$$

$$= \int_0^1\int_0^1 -y^2\,dx\,dz = \int_0^1\int_0^1 -dx\,dz \quad (\because y=1).$$

$$= \underline{-1}.$$

Evaluation of $\iint_{S_4} \vec{F}\cdot\hat{n}\,ds$.

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{OADC} \vec{F}\cdot-\vec{j}\,dx\,dz = \iint_{OADC} y^2\,dx\,dz = 0 \quad (\because y=0).$$

Evaluation of $\iint_{S_5} \vec{F}\cdot\hat{n}\,ds$.

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{DGFC} yz\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy \quad (\because z=1)$$

$$= \underline{1/2}.$$

Evaluation of $\iint_{S_6} \vec{F}\cdot\hat{n}\,ds$.

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OAEB} -yz\,dx\,dy = 0 \quad (\because z=0)$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 2+0-1+0+1/2+0$$

$$= 3/2 \text{ —— Ⓑ}.$$

from Ⓐ & Ⓑ ~~Gauss~~ Div Thm proved.

③ Verify Divergence Thm. for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ taken over the rectangular parallelopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$. (or)

~~verify~~

**Def:-**

The angle '$\theta$' b/w the two surfaces $\phi_1$ & $\phi_2$ is .

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

**Pbs:-**

1. Find the angle of Intersection at the pt $(2,-1,2)$ of the surface $x^2+y^2+z^2=9$ & $x^2+y^2-z-3$ .

**Sol:-**

Let $\phi_1 = x^2+y^2+z^2-9$ $\phi_2 = x^2+y^2-z-3$

$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$ ; $\nabla\phi_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$.

$(\nabla\phi_1)_{2,-1,2} = 4\vec{i} \pm 2\vec{j} + 4\vec{k}$ $\nabla\phi_{2\,(2,-1,2)} = 4\vec{i} - 2\vec{j} - \vec{k}$ .

Angle $\theta$ b/w the surfaces $\phi_1$ & $\phi_2$ is.

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{(4\vec{i}-2\vec{j}+4\vec{k})(4\vec{i}-2\vec{j}-\vec{k})}{\sqrt{2^2+4^2+4^2}\ \sqrt{4^2+2^2+1^2}}.$$

$$= \frac{16+4-4}{\sqrt{36}\ \sqrt{21}} \Rightarrow \frac{16}{6\sqrt{21}} \Rightarrow \frac{8}{3\sqrt{21}}.$$

hence $\boxed{\cos\theta = \frac{8}{3\sqrt{21}}}$

② Find the values of a & b so that the surfaces $ax^3-by^2z=(a+3)x^2$ & $4x^2y-z^3=11$ may cut orthoganally at $($

**Sol:-**

Let $\phi_1 = ax^3-by^2z-(a+3)x^2$. ; $\phi_2 = 4x^2y-z^2-11$.

$$\nabla\phi_1 = 3x^2a\vec{i} - 2byz\vec{j} - 2(a+3)x\vec{i} - by^2\vec{k}$$

$$= [3x^2a - 2(a+3)x]\vec{i} - 2byz\vec{j} - by^2\vec{k}$$

$$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} - 2z\vec{k}$$

$$(\nabla\phi_1)_{2,-1,-3} = [12a - 2(a+3)2]\vec{i} - 6b\vec{j} - b\vec{k}$$

$$= [12a - 4(a+3)]\vec{i} - 6b\vec{j} - b\vec{k}$$

$\nabla\phi_2 = -16\vec{i} + 16\vec{j} - 27\vec{k}$.

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}.$$

When $\theta = 90°$ $\quad \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \cos 90° \Rightarrow \nabla\phi_1 \cdot \nabla\phi_2 = 0$.

$$[(8a-12)\vec{i} - 6b\vec{j} - b\vec{k}] \cdot [-16\vec{i} + 16\vec{j} - 27\vec{k}] = 0.$$

$$-16(8a-12) - 6b(16) - b(-27) = 0.$$

$$128a + 69b = 192. \quad — ①$$

Since $(2,-1,-3)$ lies on the surface $\phi_1(x,y,z) = 0$.

we get $\quad 8a + 3b - 4(a+3) = 0.$

$$4a + 3b = 12 \quad — ②.$$

Solving eqn ① & ②. $\quad 108a = -252.$

hence $\boxed{a = -2.33}$ — ③.

Sub ③ in ②. $\boxed{b = 7.11}$

HW

① Find a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

① If $\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$ find the scalar potential $\phi$.

Soln:-

$$\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}.$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}.$$

equating the R.H.S with L.H.S.

$$\frac{\partial\phi}{\partial x} = 2xyz. \quad — ①$$

$$\frac{\partial\phi}{\partial y} = x^2z, \quad — ②$$

$$\frac{\partial\phi}{\partial z} = x^2y. \quad — ③$$

from ①. $\partial\phi = 2xyz \cdot \partial x$.

Partially Integrate on both sides.

$$\phi = x^2yz.$$

from ② . $\partial\phi = x^2 z\, \partial y$

P.I on bothsides $\phi = x^2yz + f(x,z)$ — ⑤

from ③ $\partial\phi = x^2y\, \partial z$

P.I on bothsides $\phi = x^2yz + f(x,y)$ — ⑥

from ④, ⑤, ⑥ $\boxed{\phi = x^2yz + c}$

H.W.

① If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x,y,z)$.

Divergence and curl.

The divergence of the vector function $\vec{F}$ is defined as.

$$\nabla\cdot\vec{F} = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\cdot(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Where $F_1, F_2, F_3$ are the components along the coordinates axes.

Divergence is denoted by $\nabla\cdot\vec{F}$ or div $\vec{F}$

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$ (Note: div $\vec{F}$ is a Scalar quanti

Curl.

The curl or rotation of $\vec{F}(x,y,z)$ is denoted by.

$$\nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\therefore \nabla\times\vec{F} = \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \vec{j}\left[\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

Note 1: Curl $\vec{F}$ is a vector quantity

Note : 2 $\quad \text{curl } \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$.

Note : 3. If $\phi$ is a scalar fn of $x, y, z$, then.

$$\nabla^2 \phi = (\nabla \cdot \nabla) \phi$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2},$$

i.e., $\text{div}(\text{grad}\,\phi) = \nabla^2 \phi$.

The operator $\nabla^2 \equiv \nabla \cdot \nabla$ is called the Laplacian.

<u>Def</u> :- <u>Laplace Eqn</u>.

If $\nabla^2 \phi = 0$ is called Laplace eqn.

<u>Harmonic Function</u>

Any scalar fn $\phi$ which satisfies the partial differential eqn $\nabla^2 \phi = 0$ is called harmonic fn.

<u>Solenoidal</u> If $\nabla \cdot \vec{F} = 0$ then $\vec{F}$ is said to solenoidal.

~~If $\nabla \times \vec{F} = 0$~~

<u>Irrotational</u> : If $\nabla \times \vec{F} = 0$ then $\vec{F}$ is said to be irrotational.

① P.T $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

(or)

$$\text{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \text{curl } \vec{F} - \vec{F} \cdot \text{curl } \vec{G}.$$

<u>Proof</u> :

$$\nabla \cdot \vec{F} = \vec{i} \frac{\partial F}{\partial x} + \vec{j} \frac{\partial F}{\partial y} + \vec{k} \cdot \frac{\partial F}{\partial z}.$$

$$\nabla \cdot (\vec{F} \times \vec{G}) = \vec{i} \frac{\partial}{\partial x}(\vec{F} \times \vec{G}) + \vec{j} \frac{\partial}{\partial y}(\vec{F} \times \vec{G}) + \vec{k} \frac{\partial}{\partial z}(\vec{F} \times \vec{G})$$

$$= \vec{i} \cdot \left[\vec{F} \times \frac{\partial \vec{G}}{\partial x} + \frac{\partial \vec{F}}{\partial x} \times \vec{G}\right] + \vec{j} \cdot \left[\vec{F} \times \frac{\partial \vec{G}}{\partial y} + \frac{\partial \vec{F}}{\partial y} \times \vec{G}\right].$$

$$+ \vec{k} \cdot \left[\vec{F} \times \frac{\partial \vec{G}}{\partial z} + \frac{\partial \vec{F}}{\partial z} \times \vec{G}\right].$$

$$= \vec{i} \cdot \vec{F} \times \frac{\partial \vec{G}}{\partial x} + \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} \times \vec{G} + \vec{j} \cdot \vec{F} \times \frac{\partial \vec{G}}{\partial y} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} \times \vec{G}$$

$$+ \vec{k} \cdot \vec{F} \times \frac{\partial \vec{G}}{\partial z} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z} \times \vec{G}$$

$$= -\vec{i} \ast \frac{\partial \vec{u}}{\partial x} \times \vec{F} + \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} \times \vec{u} - \vec{j} \cdot \frac{\partial \vec{u}}{\partial y} \times \vec{F} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} \times \vec{u}$$
$$- \vec{k} \cdot \frac{\partial \vec{u}}{\partial z} \ast \vec{F} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z} \times \vec{u}.$$

By Interchanging the dot and cross products we get.

$$\nabla \cdot (\vec{F} \times \vec{u}) = -\vec{i} \times \frac{\partial \vec{u}}{\partial x} \vec{F} + \vec{i} \times \frac{\partial \vec{F}}{\partial x} \cdot \vec{u} - \vec{j} \times \frac{\partial \vec{u}}{\partial y} \cdot \vec{F} \ast$$
$$\vec{j} \times \frac{\partial \vec{F}}{\partial y} \cdot \vec{u} - \vec{k} \times \frac{\partial \vec{u}}{\partial z} \cdot \vec{F} + \vec{k} \times \frac{\partial \vec{F}}{\partial z} \cdot \vec{u}.$$

$$= \left[ \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \frac{\partial \vec{F}}{\partial y} + \vec{k} \frac{\partial \vec{F}}{\partial z} \right] \cdot \vec{u} + \left[ \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z} \right] \cdot \vec{F}$$

$$= \text{curl } \vec{F} \cdot \vec{u} - \text{curl } \vec{G} \cdot \vec{F}.$$

(or)

$$\text{div}(\vec{F} \times \vec{u}) = \vec{G} \cdot \text{curl } \vec{F} - \vec{F} \cdot \text{curl } \vec{G}.$$

⑦. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ of the vector pt fn $\vec{F} = xz^3 \vec{i} - 2x^2yz \vec{j} + 2yz^4 \vec{k}$ at the pt $(1,-1,1)$.

$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$ then

Sol:-

$$\nabla \cdot \vec{F} = \left( \vec{i} \cdot \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \left( xz^3 \vec{i} - 2x^2yz \vec{j} + 2yz^4 \vec{k} \right).$$

$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(2yz^4).$$

$$= z^3 - 2x^2 + 8yz.$$

$$(\nabla \cdot \vec{F})_{1,-1,1} = 1 - 2 - 8 = \underline{\underline{-9}}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}.$$

$$= \vec{i}(2z^4 + 2x^2y) - \vec{j}(0 - 3xy^2) + \vec{k}(-4xyz - 0).$$

$$(\nabla \times \vec{F})_{1,-1,1} = \underline{\underline{3\vec{j} + 4\vec{k}}}$$

② p.T curl(grad $\phi$) = 0 .

proof:-

$$\text{Curl}(\text{grad}\,\phi) = \text{Curl}(\nabla\phi).$$

$$= \nabla\times\nabla\phi.$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\right.$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \vec{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \right.$$

$$= 0.$$

③ HW. Prove that div(grad $\phi$) = $\nabla^2\phi$.

④. Find the value of the constant a, b, c so that the vector $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$. is irrotational.

Sol:-

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\times\left[\vec{F}\right].$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}. \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2).$$

Given $\vec{F}$ is irrotational.

i.e., $\nabla\times\vec{F} = 0$.

i.e., $(c+1)\vec{i} - (4-a)\vec{j} + (b-2)\vec{k} = 0$.

⑦

each component should be zero.

hence, $c+1=0$ ; $a-4=0$ ; $b-2=0$.

$\therefore \boxed{a=4 \; ; \; b=2 \; ; \; c=-1}$.

② Find the value of $a$ so that the vector $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal.

Sol:-

To prove $\vec{F}$ is solenoidal. To prove $\nabla \cdot \vec{F} = 0$.

$$\nabla \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(\vec{F})$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az)$$

$$= 1+0$$

Given $\vec{F}$ is solenoidal.

$$\nabla \cdot \vec{F} = 0.$$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{F}) = 0.$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0.$$

$$1+1+a=0.$$

$$\boxed{a=-2}$$

③. Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(z\vec{i} + x\vec{j} + y\vec{k}).$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$\nabla \cdot \vec{F} = 0$$

If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ find div(curl $\vec{F}$)

## Stoke's Theorem

In line Integral of the tangential component of a vector fn $\vec{F}$, around a simple closed curve C is equal to the surface Integral of the normal component of Curl $\vec{F}$ over any surface S having as its boundary.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds \Rightarrow \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds.$$

① verify Stoke's thm for a vector field defined by $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by $x=0, x=a, y=0$ To $y=a$.

**Sol:-**

By stoke's Thm $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds.$

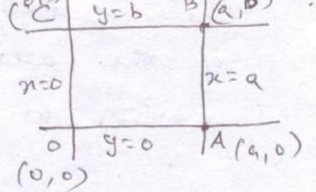
To find $\int_C \vec{F}\cdot d\vec{r}$

$$\vec{F} = (x^2-y^2)\vec{i} + 2xy\,\vec{j} \qquad d\vec{r} = dx\vec{i} + dy\,\vec{j}.$$

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)dx + (2xy)\,dy.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C [(x^2-y^2)\,dx + 2xy\,dy].$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}.$$



**Along OA** $(y=0)$.

$$\int_{OA}[(x^2-y^2)\,dx + 2xy\,dy] = \int_0^a x^2\,dx = \frac{a^3}{3}. \qquad [\because y=0, dy=0]$$

$$\int_{AB}[\qquad] = \int_0^b 2ay\,dy = ab^2 \qquad [\because x=a, dx=0]$$

$$\int_{BC}[\qquad] = \int_a^0 (x^2-b^2)\,dx = -\frac{a^3}{3} + ab^2 \quad [\because y=b; dy=0]$$

Along CO $(x=0)$.

$$\int_{CO} (x^2-y^2)\,dx + 2xy\,dy = 0 \qquad (\because x=0;\ dx=0.$$

hence $$\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2$$

$$\int_C \vec{F}\cdot d\vec{r} = 2ab^2 \quad \text{—} Ⓐ.$$

To find $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$.

Given $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$

hence $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = 4y\vec{k}$$

here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$. i.e., $\hat{n}=\vec{k}$.

$\therefore$ curl $\vec{F}\cdot\hat{n}\,ds = 4y\vec{k}\cdot\vec{k}\,dx\,dy$. $\quad$ ($\because$ area in the xy plane).

$= 4y\,dy\,dx$.

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 4\int_0^a\int_0^b y\,dy\,dx \Rightarrow 4\int_0^a b^2/2\,dx = \frac{4b^2}{2}(x)_0^a \Rightarrow 2a$$

from Ⓐ & Ⓑ proved.

② Verify stroke's Thm for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ & C is its bound

Soln:-

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds \quad \text{—} ①$$

To find $\int_C \vec{F}\cdot d\vec{r}$.

$$\vec{F}\cdot d\vec{r} = (y\vec{i}+z\vec{j}+x\vec{k})\cdot(dx\vec{i}+dy\vec{j}+dz\vec{k})$$

$$\vec{F}\cdot d\vec{r} = y\,dx + z\,dy + x\,dz \quad \text{—} ②$$

$\iint \vec{F}\cdot\hat{n}\,ds.$

Evaluation of $S_3$.

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{EBFG} \vec{F}\cdot\vec{j}\,dx\,dz$$

$$= \int_0^1\int_0^1 -y^2\,dx\,dz = \int_0^1\int_0^1 -dx\,dz \quad (\because y=1).$$

$$= \underline{-1}.$$

Evaluation of $\iint_{S_4} \vec{F}\cdot\hat{n}\,ds$.

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{OADC} \vec{F}\cdot -\vec{j}\,dx\,dz = \iint_{OADC} y^2\,dx\,dz = 0 \quad (\because y=0).$$

Evaluation of $\iint_{S5} \vec{F}\cdot\hat{n}\,ds$.

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{DGFC} yz\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy \quad (\because z=1)$$

$$= \underline{\tfrac{1}{2}}.$$

Evaluation of $\iint_{S_6} \vec{F}\cdot\hat{n}\,ds$.

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OAEB} -yz\,dx\,dy = 0 \quad (\because z=0)$$

$$\therefore \iint_{S} \vec{F}\cdot\hat{n}\,ds = 2+0-1+0+\tfrac{1}{2}+0$$

$$= \tfrac{3}{2} \quad \text{—— Ⓑ}.$$

from Ⓐ & Ⓑ ~~Gauss~~ Div Thm proved.

③ Verify Divergence Thm. for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ taken over the rectangular parallelopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$. (or)

~~verify~~

Q5 :-

Gauss Div. thm. $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\vec{F}\,dv$.

To find $\iiint_V \nabla.\vec{F}\,dv$.

Given $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$.

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x + 2y + 2z.$$

& also Given. the limits are $x=0$ to $x=a$, $y=0$ to $y=b$, $z=0$ to $z$

hence $\iiint_V \nabla.\vec{F}\,dv = 2\int_0^a\int_0^b\int_0^c (x+y+z)\,dz\,dy\,dx$

$$= 2\int_0^a\int_0^b \left(xz+yz+\frac{z^2}{2}\right)_0^c dy\,dx = 2\int_0^a\int_0^b\left[xc+yc+\frac{c^2}{2}\right]$$

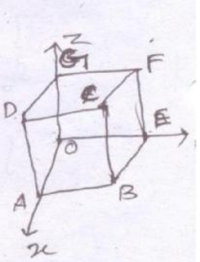
$$= 2\int_0^a\left[xyc+\frac{y^2}{2}c+\frac{c^2}{2}y\right]_0^b dx = 2\int_0^a\left[xbc+\frac{b^2}{2}.c+\frac{bc^2}{2}\right]dx.$$

$$= 2\left[\frac{x^2}{2}bc + \frac{xb^2}{2}.c + \frac{xbc^2}{2}\right]_0^a \Rightarrow 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right].$$

$$= a^2bc + ab^2c + abc^2 = abc(a+b+c) \text{ — Ⓐ}.$$

To prove $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$.

| Surface | Face | unit outward. Normal vectr. |
|---|---|---|
| $S_1$ | ABCD | $\vec{i}$ |
| $S_2$ | OEFG | $-\vec{i}$ |
| $S_3$ | CFEB | $\vec{j}$ |
| $S_4$ | AODG | $-\vec{j}$ |
| $S_5$ | GDCF | $\vec{k}$ |
| $S_6$ | OABE | $-\vec{k}$ |



Evaluation of $S_1$.

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{ABCD} \left((x^2-yz)\vec{i} + (y^2-2x)\vec{j} + (z^2-xy)\vec{k}\right)\cdot\vec{i}\;dy\,dz.$$

$$= \int_0^b\int_0^c (x^2-yz)\,dy\,dz \Rightarrow \int_0^b \left[x^2z - y\frac{z^2}{2}\right]_0^c dy \Rightarrow \int_0^b \left[x^2c - \frac{yc^2}{2}\right] dy.$$

$$\Rightarrow \left[x^2yc - \frac{y^2c^2}{4}\right]_0^b \Rightarrow a^2bc - \frac{b^2c^2}{4}.\;//$$

Evaluation of $S_2$.

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFG} -(x^2-yz)\,dy\,dz = \int_0^b\int_0^c yz\,dy\,dz = \int_0^b y\left(\frac{z^2}{2}\right)_0^c dy.$$

$$= \int_0^b y\frac{c^2}{2}\cdot dy \Rightarrow \frac{b^2c^2}{4}.\;/$$

Evaluation of $S_3$.

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{CFEB} (y^2-2x)\,dx\,dz \Rightarrow \int_0^a\int_0^c (b^2-zx)\,dz\,dx.$$

$$\Rightarrow \int_0^a \left(b^2z - \frac{xz^2}{2}\right)_0^c dx \Rightarrow \int_0^a \left[b^2c - \frac{xc^2}{2}\right] dx \Rightarrow \left[b^2cx - \frac{c^2x^2}{4}\right]_0^a$$

$$\Rightarrow ab^2c - \frac{a^2c^2}{4}.\;/$$

Evaluation of $S_4$.

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{AODG} -(y^2-zx)\,dx\,dz \Rightarrow \int_0^a\int_0^c zx\,dz\,dx \Rightarrow \int_0^a \left[\frac{z^2}{2}\right]_0^c x\,d$$

$$\Rightarrow \int_0^a \frac{c^2}{2}\,x\,dx \Rightarrow \frac{c^2}{2}\left(\frac{a^2}{2}\right)_0^a - \frac{a^2c^2}{4}\;//.$$

Evaluation of $S_5$.

$$\iint_{S_5} \vec{F}\;\hat{n}\,ds = \iint_{GDCF} (z^2-xy)\,dx\,dy. \Rightarrow \int_0^a\int_0^b (c^2-xy)\,dy\,dx.$$

$$\Rightarrow \int_0^a \left[c^2y - \frac{xy^2}{2}\right]_0^b dx \Rightarrow \int_0^a \left[c^2b - \frac{xb^2}{2}\right) dx \Rightarrow \left[c^2bx - \frac{b^2x^2}{4}\right]_0^a$$

$$\Rightarrow c^2ba - \frac{b^2a^2}{4}$$

Evaluation of $S_6$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \frac{1}{4}a^2b^2.$$

$\iint_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c)$ ← Ⓑ.

from Ⓐ & Ⓑ $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\cdot dv$.

③. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x=0$; $x=1$; $y=0$; $y=1$; $z=0$; $z=1$.

<u>Sol</u>:- By Div Thm $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$.

$\nabla\cdot\vec{F} = 4z - y$;

hence $\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1\int_0^1 (4z-y)\,dz\,dy\,dx = 3/2$.

④. using Div Thm to evaluate $\vec{F} = 4x\vec{i}^2 - 2y^2\vec{j} + z^2\vec{k}$ and S is the surface bounding the region $x^2+y^2=4$; $z=0$ and $z=3$.

Alog co $(x=0)$.

$$\int_{CO} (x^2-y^2)\,dx + 2xy\,dy = 0. \qquad (\because x=0\,;\ dx=0.$$

hence $\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2$

$$\int_C \vec{F}\cdot d\vec{r} = 2ab^2 \quad \text{—} Ⓐ.$$

To find $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$.

Given $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$

hence $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = 4y\vec{k}$

here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$. i.e., $\hat{n} = \vec{k}$.

$\therefore$ curl $\vec{F}\cdot\hat{n}\,ds = 4y\vec{k}\cdot\vec{k}\,dx\,dy$. ($\because$ area in the xy plane).

$= 4y\,dy\,dx$.

$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 4\int_0^a\int_0^b y\,dy\,dx \Rightarrow 4\int_0^a b^2/2\,dx = \frac{4b^2}{2}(x)_0^a \Rightarrow 2a$

from Ⓐ & Ⓑ proved.

② Verify stroke's Thm for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ & C is its bound

Soln:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds. \quad \text{—}①$$

to find $\int_C \vec{F}\cdot d\vec{r}$.

$\vec{F}\cdot d\vec{r} = (y\vec{i} + z\vec{j} + x\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$\vec{F}\cdot d\vec{r} = y\,dx + z\,dy + x\,dz$ —②.

Given $C$ is the boundary of the upper half of the Given Sphere $x^2+y^2+z^2=1$ which is clearly a circle $x^2+y^2=1$ & $z=0$.

hence eqn ② $\Rightarrow$ ~~$d\vec{r} = dx\vec{i} + dy\vec{j}$~~ $\cdot\ \vec{F}\cdot d\vec{r} = y\,dx$.

The parametric eqn of the Given circle is $x = \cos\theta$ $y = \sin\theta$

$dx = -\sin\theta\, d\theta$ ; $dy = \cos\theta \cdot d\theta$

$$\text{hence } \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin\theta \cdot -\sin\theta\, d\theta = -\int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= -\int_0^{2\pi} \left[\frac{1-\cos 2\theta}{2}\right] d\theta = -\tfrac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= -\tfrac{1}{2}(2\pi) = -\pi \quad \text{—— Ⓐ}$$

To find $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds$.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \vec{i}\left[\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(\right.$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y)\right]$$

$$= \vec{i}\,[0-1] - \vec{j}\,[1-0] + \vec{k}\,[0-1]$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

outward unit normal vector is $-\vec{k}$

hence $(\nabla\times\vec{F})\cdot\hat{n} = -(\vec{i}+\vec{j}+\vec{k})\cdot -\vec{k} = +1$.

$$\text{hence } \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\, ds = \iint_S ds$$

$$= \iint_S \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \iint_R \frac{dx\,dy}{-1}$$

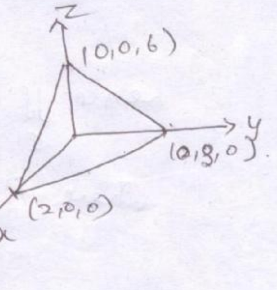
where $R$ is the region which is the projection of the surface on the xy plane. which is a circle whose radius is $\pi$

$$\text{hence } \iint_C \operatorname{curl}\vec{F}\cdot\hat{n}\, ds = -\iint_R dx\,dy$$

$= -$Area of the circle

$= -\pi r^2 = -\pi$ —— Ⓑ.

③ Evaluate The Integral $\int_C \{(x+y)dx + (2x-z)dy + (y+z)dz\}$ where C is the boundary of the triangle with vertices (2,0,0) (0,3,0) & (0,0,6) using stoke's Thm.



Sol:-

**Stoke's Thm.**

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$

To find $\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F} \cdot d\vec{r} = (x+y)\,dy + (2x-z)\,dy + (y+z)\,dz \quad \text{—①}$$

From eqn ① $\vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

and $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

To find $\int_C \vec{F} \cdot d\vec{r}$ we can find $\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$.

hence
$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}\left[1+1\right] - \vec{j}\left[0-0\right] + \vec{k}\left[2-1\right] = \vec{k} + 2\vec{i}$$

$$\text{curl}\,\vec{F} = 2\vec{i} + \vec{k}$$

To find the unit normal vector first we have to find the surface eqn. i.e., $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$$3x + 2y + z = 6 \quad \text{i.e.,} \quad \phi = 3x + 2y + z - 6 = 0$$

unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\therefore \nabla\phi = 3\vec{i} + 2\vec{j} + \vec{k}$$

$$|\nabla\phi| = \sqrt{3^2 + 2^2 + 1^2}$$

hence
$$\hat{n} = \frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}}$$

hence $\text{curl}\,\vec{F}\cdot\hat{n} = (2\vec{i}+\vec{k})\cdot\left(\dfrac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}\right)$

$$= \frac{6+1}{\sqrt{14}} = 7/\sqrt{14}.$$

hence $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S 7/\sqrt{14}\,ds \Rightarrow 7/\sqrt{14}\iint_R \dfrac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$

where R is the projection of the surface ABC on the xoy plane

$$= \frac{7}{\sqrt{14}}\iint_R \frac{dx\,dy}{1/\sqrt{14}} = 7\iint_R dx\,dy$$

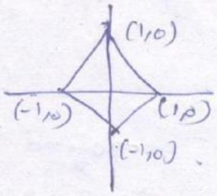
= 7 × Area of the triangle ABC.

$$= 7(3) \quad \left(\tfrac{1}{2}bh = \tfrac{1}{2}(3)(2) = 3\right)$$

$$= 21$$

hence $\boxed{\int_C \vec{F}\cdot d\vec{r} = 21}$.

④ Evaluate $\int_C (xy\,dx + xy^2\,dy)$ by Stoke's Thm where C is the square in the xy plane with vertices (1,0) (−1,0) (0,1) & (0,−1).



**Sol:-** $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = (y^2 - x)\vec{k}$$

unit normal vector is $\vec{k} = \hat{n}$.

hence $\text{curl}\,\vec{F}\cdot\hat{n} = y^2 - x.$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (y^2 - x)\,dx\,dy \Rightarrow \int_{-1}^{1}\int_{-1}^{1} (y^2 - x)\,dx\,dy.$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = 4/3}$$

| Objective type questions | Opt 1 |
|---|---|
| The triple integral ∫∫∫ dv gives the ___________ over the region v | area |
| The value of ∫∫ dx dy , inner integral limt varies from 1 to 2 and the outer integral limit varies from 0 to 1 | 0 |
| ∫∫∫ dx dy dz, the inner integral limit varies from 0 to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1 | 2 |
| When the limits are not given, the integral is named as ___________ | Definite integral |
| The Double integral ∫∫ dx dy gives _________ of the region R | area |
| The value of ∫∫ (x+y) dx dy , inner integral limt varies from 0 to 1 and the outer integral limit varies from 0 to 1 | 0 |
| The value of ∫∫∫ x^2 yz dx dy dz, the inner integral limit varies from 1 to 2, the central integral limit varios from 0 to 2 and outer integral limit various from 0 to 1 | 7/3 |
| Evaluate ∫∫ 4xy dx dy, the inner integral limit varies from 0 to 1 and outer integral limit varies from 0 to 2 | 10 |
| The value of ∫∫ d xdy ⁄xy, the inner integral limit varies from 0 to b and the outer limit varies from 0 to a | 0 |
| If the limits are given in the integral , then the integral is name as_____________ | Definite integral |
| The value of ∫∫(x^2+3y^2) dy dx , the inner integral limit varies from 0 to 1, the outer integral limit varies from 0 to 3 | 10 |
| The value od ∫∫∫ dxdy d, the inner integral limit varies from 0to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1 | 6 |
| If the limits are not given in the integral , the the integral is name as_____________ | Definite integral |
| The value of ∫∫(x^2+y^2) dy dx ,the inner integral limit varies from 0 to x, the outer integral limit varies from 0 to 1 | 1 |
| The value of ∫∫dy dx, the inner integral limit various from 0 to x , the outer integral limit varies from -a to a | 0 |
| The Double integral ∫∫ dx dy gives _________ of the region R | area |
| The value of ∫∫∫ dx dy dz, the inner integral limit varies from 0 to a , the central integral limit varies from 0 to a and the outer integral limit varies from 0 to a | 0 |
| The value of ∫∫(x+y) dx dy , the inner integral limit varies from 0 to 1 and the outer integral limit varies from 0 to 1 | 0 |
| The concept of line integral as a generalization of the concept of ________ integral | Single |
| The extension of double integral is nothing but ________ integral | Single |
| The concept of _________ integral as a generalization of the concept of double integral | Single |
| Evaluate ∫x^2⁄2 dx, the limit varies from 0 to 1 | 2 |
| Evaluate ∫42y dy, the limit varies from 0 to 10 | 10 |
| The value of ∫∫2 xy dy dx, the inner integral limit varies from 0 to x and the outer integral limit varies from 1 to 2 | 15⁄4 |

| Opt2 | Opt3 | Opt4 | Opt5 | Opt6 | Answer |
|---|---|---|---|---|---|
| volume | Direction | weight | | | volume |
| 1 | 2 | 3 | | | 1 |
| 4 | 6 | 8 | | | 6 |
| Infinite integral | volume integral | Surface integral | | | Infinite integral |
| modulus | Direction | weight | | | |
| 1 | 2 | 3 | | | 1 |
| 1/3 | 2/3 | 3 | | | 7/3 |
| 4 | 5 | 1 | | | 4 |
| 1 | ab | loga log b | | | loga log b |
| Infinite integral | volume integral | Surface integral | | | Definite integral |
| 15 | 12 | 30 | | | 12 |
| 1 | 16 | 12 | | | 6 |
| Infinite integral | volume integral | Surface integral | | | Infinite integral |
| 1⁄3 | 2⁄3 | 3⁄2 | | | 1⁄3 |
| 1 | 2 | 3 | | | 0 |
| modulus | Direction | weight | | | area |
| a^3 | a^2 | a^4 | | | a^3 |
| 1 | 2 | 3 | | | 1 |
| Double | change of order | Triple | | | Double |
| Line | volume integral | Triple | | | Triple |
| Surface | Line | Triple | | | Line |
| 1⁄6 | 1⁄10 | 34 | | | 1⁄6 |
| 2100 | 2000 | 100 | | | 2100 |
| 9⁄2 | 3⁄2 | 4⁄3 | | | 15⁄4 |

p

| | |
|---|---|
| The value of ∫∫dy dx, the inner integral limit varies from 2 to 4 ,the outer integral limit varies from  1 to 5 | 8 |
| The value of ∫∫xy dy dx, the inner integral limit varies from 0 to 3 , the outer integral limit varies from 0 to 4 | 12 |
| The value of ∫∫dy dx, the inner integral limit varies from 0 to 2 , the outer integral limit varies from 0 to 1 | 2 |
| The value of ∫∫dx dy, the inner integral limit varies from  y to 2 , the outer integral limit varies from 0 to 1 | 1⁄2 |
| The value of ∫∫dx dy, the inner integral limit varies from 2 to 4 , the outer integral limit varies from 1 to 2 | 2 |
| When a function f(x) is integrated with respect to x between the limits a and b, we get _________ | Definite  integral |
| In two dimensions the x and y axes divide the entire xy- plane into ___________ quadrants | 1 |
| In three dimensions the xy and yz and zx planes divide the entire space into ___________ parts called octants | 3 |
| Evaluate ∫(2x+3) dx, the integral limitvaries from 0 to 2 | 10 |
| _________ provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R. | Cauchy's Theorem |
| ________ is also called the first fundamental theorem of integral vector calculus. | Cauchy's Theorem |
| _________ transforms line integrals into surface integrals. | Cauchy's Theorem |
| _______ transforms surface integrals into a volume integrals. | Cauchy's Theorem |
| _________ is stated as surface integral of the component of curl F along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function F taken along the closed curve C. | Cauchy's Theorem |
| ________ is stated as the surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S. | Cauchy's Theorem |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 4 | 5 | | | 8 |
| 36 | 1⁄2 | 4 | | | 12 |
| 1 | 3⁄2 | 4 | | | 2 |
| 1 | 3⁄2 | 4 | | | 3⁄2 |
| 6 | 3 | 1 | | | 2 |
| infinite integralv | volume integral | Surface integral | | | Definite  integral |
| 2 | 3 | 4 | | | 2 |
| 2 | 8 | 4 | | | 8 |
| 42 | 51 | 1 | | | 10 |
| Green's Theorem | Stoke's Theorem | Gauss Theorem | | | Stoke's Theorem |
| Green's Theorem | Stoke's Theorem | Gauss Theorem | | | Green's Theorem |
| Green's Theorem | Stoke's Theorem | Gauss Theorem | | | Green's Theorem |
| Green's Theorem | Stoke's Theorem | Gauss Theorem | | | Gauss Theorem |
| Green's Theorem | Stoke's Theorem | Gauss Theorem | | | Stoke's Theorem |
| Green's Theorem | Stoke's Theorem | Gauss Theorem | | | Gauss Theorem |
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# [Differential equations].

UNIT: I

## [First order Ordinary differential Equations]

**Differential equation:**

* A differential equation is an equation which involves differential co-efficients.

**Ordinary differential equations: (O.D.E).**

* An ordinary differential equation is that in which all the differential co-efficients has a single independent variable.

Ex:

$$\frac{dy}{dx} = 2x.$$

**Partial differential equations: (P.D.E)**

* A Partial differential equation is that in which there are two or more independent variable.

Ex:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

## Exact differential equation:

* A differential equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if its left hand number is the exact differential of some function $u(x,y)$.

ie) $du = M\,dx + N\,dy = 0$.

$\therefore$ The solution is $u(x,y) = c$

## Theorem :

* The Necessary and Sufficient condition for that differential equation $M\,dx + N\,dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

### Necessary condition:

* The equation $M\,dx + N\,dy = 0$ will be exact if $M\,dx + N\,dy = du$. where 'u' is the some function of x and y.

* $\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition for exactness.

### Sufficient condition:

* If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = 0$ is exact

## Methods of solution :

* The equation $M dx + N dy = 0$ becomes

$d[u + \int f(y)\, dy] = 0$, Integrating $\int d[u + \int f(y)\, dy] = 0$,

∵ ~~The solution~~ $u + \int f(y)\, dy = 0$.

$$u = \int_{y \text{ constant}} M\, dx$$

$f(y)$ = terms of N not containing $x$.

∴ The solution of $M dx + N dy = 0$ is $\int_{(y \text{ constant})} M\, dx + \int \begin{pmatrix} \text{term of } N \\ \text{not containing } x \end{pmatrix} dy =$

Provided $\frac{du}{dy} = \frac{dN}{dx}$.

### Example : 1

Solve $\underset{M\ dx}{\left(y^2 e^{xy^2} + 4x^3\right) dx} + \underset{N\ dy}{\left(2xy e^{xy^2} - 3y^2\right) dy} = 0$.

Here,

$M = y^2 e^{xy^2} + 4x^2$ ; $N = 2xy e^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = 2y e^{xy^2} + y^2 . e^{xy^2} . 2xy$$

$$= 2y e^{xy^2} + 2xy^3 . e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2y\left(e^{xy^2} + x . e^{xy^2} . y^2\right)$$

$$= 2y e^{xy^2} + 2xy^3 e^{xy^2}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ constant})} M\, dx + \int \begin{pmatrix} \text{terms of } N \text{ not} \\ \text{containing } x \end{pmatrix} dy = c$$

$$\int (y^2 e^{xy^2} + 4x^3)\, dx + \int (-3y^2)\, dy = c$$

$$\int y^2 e^{xy^2} dx + \int 4x^3 dx - \int 3y^2 dy = c$$

$$\cancel{y^2} \cdot \frac{e^{xy^2}}{\cancel{y^2}} + \cancel{4}\, \frac{x^4}{\cancel{4}} - \cancel{3}\, \frac{y^3}{\cancel{3}} = c$$

$$\boxed{e^{xy^2} + x^4 - y^3 = c.}$$

—x—

Solve $\underbrace{\left[y\left(1+\frac{1}{x}\right) + \cos y\right] dx}_{M\,dx} + \underbrace{[x + \log x - x \sin y]\, dy}_{N\,dy} = 0.$

Soln:

$M = y\left(1+\frac{1}{x}\right) + \cos y \quad ; \quad N = x + \log x - x \sin y.$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y \quad ; \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

The equation is exact and its solution is

$$\underset{(y \text{ constant})}{\int M\, dx} + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int \left(y\left(1+\frac{1}{x}\right) + \cos y\right) dx + \int (0)\, dy = c$$

$$y\left[\int dx + \int \frac{1}{x} dx\right) + \int \cos y\, dx\right]$$

$$\boxed{y\,[x + \log x] + \cos y\, x = c}$$

—x—.

solve $\underset{M\,dx}{(1+2xy\cos x^2 - 2xy)\,dx} + \underset{N\,dy}{(\sin x^2 - x^2)\,dy} = 0.$

④ soln:

$$M = 1+2xy\cos x^2 - 2xy \quad ; \quad N = \sin x^2 - x^2$$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x \quad ; \quad \frac{\partial N}{\partial x} = \cos x^2 . 2x - 2x.$$

$$= 2x\cos x^2 - 2x.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is exact and its solution is

$$\underset{(y \text{ constant})}{\int M\,dx} + \int \left(\begin{array}{c}\text{terms of } N \text{ not}\\ \text{containing } x\end{array}\right) dy = c$$

$$\int (1+2xy\cos x^2 - 2xy)\,dx + \int (0)\,dy = c$$

$$\int dx + y\int \cos x^2\, 2x\,dx - \int 2xy\,dx = c$$

$$x + y\int d(\sin x^2) - \cancel{2}y\frac{x^2}{\cancel{2}} = c$$

$$\boxed{x + y\sin x^2 - yx^2 = c}$$

—x—

solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$$\frac{(\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx}{(\sin x + x\cos y + x)\,dx} = 0$$

$$\underset{M\,dx}{(y\cos x + \sin y + y)\,dx} + \underset{N\,dy}{(\sin x + x\cos y + x)\,dy} = 0.$$

$M = y\cos x + \sin y + y$ $\qquad$ $N = \sin x + x\cos y + x$

$\partial M/\partial y = \cos x + \cos y + 1$ $\qquad$ $\partial N/\partial x = \cos x + \cos y + 1$

$$\frac{\partial H}{\partial y} = \frac{\partial N}{\partial x}$$

$$\underset{(y\ \text{constant})}{\int H\,dx} + \int \left(\begin{matrix}\text{terms of N not} \\ \text{containing } x\end{matrix}\right) dy = \cancel{x}\ C$$

$$\int (y \cos x + \sin y + y)\,dx + \int (0)\,dy = C$$

$$y \sin x + (\sin y + y)\,x = C$$

$$\boxed{y \sin x + x \sin y + xy = C}$$

—x—

Linear equation:

* A differential equation is ~~saind~~ said to be linear if it dependent variable and its differential co-efficient aquire only in the first degree

* The Standard form of a Linear equation of the first order, commonly known as Leibnitz's linear equation.

$$\frac{dy}{dx} + Py = Q$$

where, P, Q are the functions of x

$$y\,e^{\int Pdx} = \int Q\,e^{\int Pdx}\,dx + C$$

$$I.F = e^{\int e p dx}$$

$$\boxed{y\,(I.F) = \int Q\,(I.F)\,dx + C}$$

solve the linear equation $(x+1)\frac{dy}{dx} - y = e \quad (x+1)$.

<u>soln</u>:-

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2.$$

$\div$ by $(x+1)$

$$\frac{dy}{dx} - \frac{y}{(x+1)} = e^{3x}(x+1)$$

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{-1}{x+1} \quad ; \quad Q = e^{3x}(x+1)$$

$$I.F = e^{\int P dx} = e^{\int\left(-\frac{1}{x+1}\right)dx}$$

$$= e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)}$$

$$= e^{\log(x+1)^{-1}}$$

$$= (x+1)^{-1} = \frac{1}{x+1}$$

$$\boxed{I.F = \frac{1}{x+1}}$$

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x}\cancel{(x+1)}\frac{1}{\cancel{(x+1)}}\,dx + c$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x}\,dx + c$$

$$y\left(\frac{1}{x+1}\right) = \frac{e^{3x}}{3} + c$$

$$\boxed{y = \left(\frac{e^{3x}}{3} + c\right)(x+1)}$$

——— x ———

Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

$$\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

which is Leibnitz's line
( equation

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{\sqrt{x}} \;;\; Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$= \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}} dx + c$$

$$= \int \frac{1}{\sqrt{x}} e^{0} dx + c$$

$$= \int \frac{dx}{\sqrt{x}} \cdot 1 + c$$

$$\boxed{y e^{2\sqrt{x}} = 2\sqrt{x} + c.}$$

$$\int x^{n} dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2} dx$$

$$= \frac{x^{-1/2+1}}{-1/2+1}$$

$$= \frac{x^{1/2}}{1/2}$$

$$= 2x^{1/2}$$

$$= 2\sqrt{x}.$$

—x—

Solve

$$\frac{dx}{dy} + Px = Q$$

$P, Q \rightarrow$ functions of $y$.

$$I.F = e^{\int P dy}$$

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + c.$$

Solve $(y\log y)\,dx + (x-\log y)\,dy = 0$

soln: $y\log y\,dx = -(x-\log y)dy$

$$\frac{dx}{dy} = \frac{\log y - x}{y\log y} = \frac{1}{y}\frac{(\log y - x)}{\log y}$$

$$= \frac{1}{y}\left[1-\frac{x}{\log y}\right].$$

$$\frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y}$$

$$\frac{dx}{dy}+Px=Q$$

$$P=\frac{1}{y\log y} \quad ; \quad Q=\frac{1}{y}.$$

$$\text{I.F} = e^{\int P dy} = e^{\int \frac{1}{y\log y}dy}$$

$$= e^{\int \frac{1/y\,dy}{\log y}} = e^{\log(\log y)}$$

$$= \log y.$$

$$x e^{\int Pdy} = \int Q\cdot e^{\int Pdy}\, dy + c.$$

$$x\log y = \int \frac{1}{y}\log y\,dy + c$$

$\log y = t$

$\frac{1}{y}dy = dt.$

$$= \int t'\, dt + c$$

$$x\log y = \frac{t^2}{2}+c$$

$$x\log y = \frac{1}{2}(\log y)^2 + c$$

$$\boxed{x = \frac{1}{2}\log y + c(\log y)^{-1}.}$$

——x——

Solve: $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ .

soln: $$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\frac{dx}{dy} + Px = Q$$

$$P = \frac{1}{1+y^2} \quad ; \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

$$x\, e^{\int P dy} = \int Q . e^{\int P dy} dy + c$$

$$x . e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} . e^{\tan^{-1}y} dy + c$$

$$= \int t\, e^{t} . dt + c$$

$$= t e^{t} - \int e^{t} dt + c$$

$$= t e^{t} - e^{t} + c$$

$$= e^{t}(t-1) + c$$

$\int u dv = uv - \int v du$

$u = t$ , $dv = e^t$

$du = dt$ $\quad v = \int e^t dt =$

u - D

v - I

$$\boxed{x\, e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c}$$

——x——

Put $t = \tan^{-1}y$

$dt = \frac{1}{1+y^2} dy$

$x = (\tan^{-1}y - 1) + c e^{\tan^{-1}}$

**Bernoulli's** Equation:

$$\frac{dy}{dx} + Py = Qy^n \rightarrow ①$$

To solve ①

$(\div)$ both sides by $y^n$

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Put $y^{1-n} = z$.

$$(1-n)\, y^{\cancel{1}-n-\cancel{1}}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

(or) $$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

which is Leibnitz's linear in z & can be solved easily.

— x —

Solve $$x\frac{dy}{dx} + y = x^3y^6.$$

Soln:

$\div$ by $x$

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^6$$

$\div$ by $y^6$ $$y^{-6}\frac{dy}{dx} + \frac{y}{x \cdot y^6} = x^2$$

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \rightarrow ①$$

Put $z = y^{-5}$

$$\frac{dz}{dx} = -5y^{-6}\frac{dy}{dx} \Rightarrow y^{-6}\frac{dy}{dx} = \frac{1}{-5}\frac{dz}{dx}$$

Sub $\frac{dy}{dx}$ in ①

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$$

$$\div \text{ by } \left(-\frac{1}{5}\right) \quad \frac{dz}{dx} + \frac{z/x}{-1/5} = \cancel{x^2/_{-5}} \frac{x^2}{-1/5}$$

$$\frac{dz}{dx} - \frac{5}{x} z = -5x^2$$

which is Leibnitz's linear equation in z

$$\frac{dz}{dx} + Pz = Q$$

$$P = -5/x \; ; \; Q = -5x^2.$$

$$I.F = e^{\int P dx} = e^{\int -5/x \, dx} = e^{-5\int \frac{dx}{x}} = e^{-5 \log x}$$

$$= e^{\log x^{-5}} = x^{-5}$$

$$z e^{\int P dx} = \int Q . e^{\int P dx} dx + c$$

$$z x^{-5} = \int -5x^2 . x^{-5} dx + c$$

$$z x^{-5} = -5 \int x^{-3} dx + c$$

$$= -5\left(\frac{x^{-3+1}}{-3+1}\right) + c$$

$$z x^{-5} = -5 \frac{x^{-2}}{-2} + c$$

$$y^{-5} x^{-5} = \frac{5}{2} x^{-2} + c$$

$$(\div) \text{by } y^{-5} x^{-5}$$

$$1 = \frac{\frac{5}{2} x^{-2} + c}{x^{-5} y^{-5}}$$

$$1 = \left(\frac{5}{2} + c x^2\right) x^3 y^5.$$

— x —.

Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

Soln:

$$xy(1+xy^2) = \frac{dx}{dy}$$

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\frac{dx}{dy} - xy = x^2y^3.$$

÷ by $x^2$

$$x^{-2}\frac{dx}{dy} - \frac{\cancel{x}y}{x^{\cancel{2}}} = \frac{\cancel{x^2}y^3}{\cancel{x^2}}$$

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \longrightarrow ①$$

put $x^{-1} = z$

$$-1x^{-1-1}\frac{dx}{dy} = \frac{dz}{dy}$$

$$-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$$

① becomes,

$$-\frac{dz}{dy} - yz = y^3$$

$$\frac{dz}{dy} + yz = -y^3 \longrightarrow ②$$

which is Leibnitz's linear equation in z

$$\frac{dz}{dy} + Pz = Q\cancel{y}.$$

Here $P = y$

$$I.F = e^{\int P dy} = e^{\int y dy} = e^{y^2/2}.$$

∴ The solution is $z(I.F) = \int Q(I.F)\,dy + c$

$$z e^{y^2/2} = \int (-y^3) e^{y^2/2} dy + c$$

$$= -\int y^2 e^{y^2/2} y \, dy + c$$

$$= -\int 2t e^t dt + c$$

$$= -2\int t e^t dt + c$$

$$= -2 [t e^t - \int e^t dt] + c$$

$$= -2 [t e^t - e^t] + c$$

$$= -2 (t-1) e^t + c$$

$$= -2 [\frac{y^2}{2} - 1] e^{y^2/2} + c$$

$$z = (-y^2+2) e^{y^2/2} + c e^{-y^2/2}$$

$$\frac{1}{x} = (2-y^2) e^{y^2/2} + c e^{-y^2/2}$$

$\frac{y^2}{2} = t$

$y^2 = 2t$

$\int u dv = uv - \int$

$u = t$ ; $dv=$

$du = 1$ ; $v$

Put $\frac{y^2}{2} = t$

$\frac{\cancel{2} y dy}{\cancel{2}} = dt$

—×— .

solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

÷ by $\cos^2 y$

$$\frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{2x \sin y \cancel{\cos y}}{\cos^{\cancel{2}} y} = \frac{x^3 \cancel{\cos^2 y}}{\cancel{\cos^2 y}}$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \rightarrow ①$$

Put $\tan y = z$

$$\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$$

∴ ① becomes

$$\frac{dz}{dx} + 2xz = x^3 \rightarrow ②$$

$$\frac{dz}{dx} + Pz = Q$$

which is Leibnitz's linear equation in z

Put $P = 2x$ ; $Q = x^3$

$$I.F = e^{\int P dx} = e^{\int 2x dx} = e^{\cancel{2}x^2/\cancel{2}} = e^{x^2}$$

The solution is

$$z(I.F) = \int Q(I.F)\, dx + c$$

$$z e^{x^2} = \int x^3 e^{x^2} dx + c$$

$$z e^{x^2} = \int x^2 x e^{x^2} dx + c$$

$u = t$ ; $dv = e^t$

$du = 1$ ; $v = e^t$

$uv - \int v du$.

$$= \int \cancel{x} \underset{u}{t}\, \underset{dv}{e^t}\, dt + c$$

$$= \frac{1}{2}\left[t e^t - \int e^t\right] + c$$

$$= \frac{1}{2}\left[t e^t - e^t\right] + c$$

Put $x^2 = t$

$2x dx = dt$

$x dx = \frac{dt}{2}$.

$$= \frac{1}{2}(t-1)e^t + c$$

$$z e^{x^2} = \frac{1}{2}\left[x^2 - 1\right] e^{x^2} + c$$

$\div$ by $e^{x^2}$

$$z = \frac{1}{2}(x^2 - 1) + c e^{-x^2}$$

— x —

## Equations of first order and higher degree.

* The general form of the differential equation of the first ~~degree~~ order and $n^{th}$ degree.

$$\left(\frac{dy}{dx}\right)^n + f_1(x,y)\left(\frac{dy}{dx}\right)^{n-1} + f_2(x,y)\left(\frac{dy}{dx}\right)^{n-2} + \cdots\cdots + f_{n-1}(x,y)\left(\frac{dy}{dx}\right) + f_n(x,y) = 0$$

If $\frac{dy}{dx} = P$

$$p^n + f_1(x,y)\,p^{n-1} + f_2(x,y)\,p^{n-2} + \dots + f_{n-1}(x,y)P + f_n(x,y)$$

Since equation ① is the first order its general solution will contain only one orbitary constant To solve ① is to be identified as an equation any one of the types

- * Solvable for P
- * Solvable for y
- * Solvable for x
- * ~~Solvable~~ clairaut's form.

* A differential equation of the first order but of nth degree is of the form

$$p^n + f_1(x,y)\,p^{n-1} + f_2(x,y)\,p^{n-2} + \dots + f_{n-1}(x,y)P + f_n(x,y) = 0$$

L.H.S of ① can be resolved in n linear factors

then ① becomes

$$(P-F_1)(P-F_2)\dots\dots(P-F_n) = 0$$

$$P = F_1, \quad P = F_2, \quad P = F_n.$$

$$\phi_1(x,y,c) = 0 \; ; \; \phi_2 = (x,y,c) = 0 \; \dots\dots \phi_n(x,y,c$$

The general solution is obtained.

$\phi_1(x,y,c)\ \phi_2(x,y,c) \cdots \phi_n(x,y,c) = 0.$

— x —

solve $\left(\frac{dy}{dx}\right)^2 - 6\left(\frac{dy}{dx}\right) + 8 = 0.$

soln:

Put $\frac{dy}{dx} = P$

The given equation is $P^2 - 6P + 8 = 0.$

$(P-4)(P-2) = 0$

$\begin{matrix} 8 \\ \wedge \\ -4 \quad -2 \end{matrix}$

$P = 4$ (or) $P = 2$

$\frac{dy}{dx} = 4$ (or) $\frac{dy}{dx} = 2$

Integrating

$\int dy = \int 4dx$

$y = 4x + c$

$(y - 4x - c) = 0$

Integrating

$\int dy = \int 2dx$

$y = 2x + c$

$(y - 2x - c) = 0$

$(y - 4x - c)\ (y - 2x - c) = 0$

— x — .

solve

$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Put $\frac{dy}{dx} = P$

$P - \frac{1}{P} = \frac{x}{y} - \frac{y}{x}$

$\frac{P^2 - 1}{P} = \frac{x}{y} - \frac{y}{x}.$

$P^2 - 1 = P\left(\frac{x}{y} - \frac{y}{x}\right).$

~~$$p^2 - p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$~~

~~$$\left(p + \frac{x}{y}\right)\left(p - y/x\right) = 0$$~~

~~$p + x/y = 0$ (or) $p - y/x = 0$~~

~~$p = -\frac{x}{y}$ (or) $p = y/x$~~

~~$-1$~~

~~$\frac{x}{y}$ $-y/x$~~

$$p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$$

$$\left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$$

$-1$

$y/x$ $-x/y$

$p + \frac{y}{x} = 0$ (or) $p - \frac{x}{y} = 0$

$p = -\frac{y}{x}$ (or) $p = \frac{x}{y}$

$\frac{dy}{dx} = \frac{-y}{x}$ (or) $\frac{dy}{dx} = \frac{x}{y}$

$x\,dy = -y\,dx$ (or) $y\,dy = x\,dx$

$x\,dy + y\,dx = 0$ $\qquad$ $x\,dx - y\,dy = 0$

Integrating $\qquad$ Integrating

$\int d(x,y) = 0$ $\qquad$ $\int(x\,dx - y\,dy) = 0$

$xy = c$ $\qquad$ $\frac{x^2}{2} - \frac{y^2}{2} = c$

$(xy - c) = 0$ ; $\qquad$ $x^2 - y^2 - c = 0$.

$$(xy - c)(x^2 - y^2 - c) = 0.$$

— x — .

Solve $p^2 + 2py\cot x = y^2$

V.Imp

$$p^2 + 2py\cot x - y^2 = 0$$

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$ ; $b = 2y\cot x$ ; $c = -y^2$

$$P = \frac{-2y\cot x \pm \sqrt{(2y\cot x)^2 - 4(1)(-y^2)}}{2(1)}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm 2\sqrt{y^2\cot^2 x + y^2}}{2}$$

$$= \frac{\cancel{2}(-y\cot x \pm \sqrt{y^2\cot^2 x + y^2})}{\cancel{2}}$$

$$= -y\cot x \pm \sqrt{y^2\cot^2 x + y^2}$$

$$P = -y\cot x \pm \sqrt{y^2(1 + \cot^2 x)}$$

$$= -y\cot x \pm y\sqrt{\operatorname{cosec}^2 x}$$

$$\frac{dy}{dx} = -y\cot x \pm y\operatorname{cosec} x$$

$$= y\operatorname{cosec} x - y\cot x$$

$$\frac{dy}{dx} = y(\operatorname{cosec} x - \cot x)$$

$$= y\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$\sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\frac{dy}{dx} = y\left(\frac{1 - \cos x}{\sin x}\right) = y\left(\frac{\cancel{2}\sin^{\cancel{2}} x/2}{\cancel{2}\sin\cancel{\frac{x}{2}}\cos\frac{x}{2}}\right)$$

$\int \sin ax$..

$$\frac{dy}{dx} = y \tan\frac{x}{2}$$

$\int \tan x = \log$

$$\frac{dy}{y} = \tan\frac{x}{2}\,dx$$

$$\int dy/y = \int \tan\frac{x}{2}\,dx$$

$$\log y = \frac{\log \sec\left(\frac{x}{2}\right)}{\frac{1}{2}} + \log c$$

$$\log y = 2\log \sec\left(\frac{x}{2}\right) + \log c$$

$$= \log \sec^2\frac{x}{2} + \log c$$

$$\cancel{\log} y = \cancel{\log}\left(c \sec^2\frac{x}{2}\right)$$

$$y = c\sec^2\frac{x}{2} = c\,\frac{1}{1+\cos x}\cdot 2$$

$$y(1+\cos x) = 2c$$

$\int \tan x = \log \sec x$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\log x = \int \frac{1}{x}\,dx$$

$$1+\cos x = 2\cos^2\frac{x}{2}$$

$\log ab = \log a + \log$

— x —

Type 1:

Integrating faction

equation reduciable to exact equation

Differential

* ~~Diffrential~~ equation which are not exact

k can sometime we made exact after Multiplying by a suitable ~~k~~ $[f(x,y)]$ called the integrating factor.

* Integrating factor found by Inspection.

Example:

solve $ydx - xdy = 0$

$ydx - xdy \to$ ①

$Mdx - Ndy = 0$

$M = y$ ; $N = -x$.

$\frac{dM}{\partial y} = 1$ ; $\frac{\partial N}{\partial x} = -1$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Multiplying ① by $\frac{1}{y^2}$

$$\frac{ydx - xdy}{y^2} = 0$$

$d\left(\frac{x}{y}\right) = 0.$

which is exact

① × $\frac{1}{x^2}$ , $\frac{ydx - xdy}{x^2} = 0$

$d\left(y/x\right) = 0$

Multiply ① by $1/xy$.

$$\frac{ydx - xdy}{xy} = 0$$

$$\frac{\cancel{y}dx}{x\cancel{y}} - \frac{\cancel{x}dy}{\cancel{x}y} = 0$$

$$\int \frac{dx}{x} - \int \frac{dy}{y} = 0$$

$\log x - \log y = \log c$

~~$\log(x-y)$ = $\log c$~~ $\log(x/y) = \log c$

$\cancel{x}$

$\frac{x}{y} = c$

$x = cy$

∴ $\frac{1}{y^2}$, $\frac{1}{x^2}$, $\frac{1}{xy}$ are integrating factor of ①

—x—

## * Integrating factor of a homogeneous equation.

solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

$$M\,dx + N\,dy = 0$$

$$M = x^2y - 2xy^2$$

$$N = -(x^3 - 3x^2y)$$

This equation is homogeneous in x and y.

$$\text{Integrating factor} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x + (3x^2y - x^3)y}$$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 + 3x^2y^2 - \cancel{x^3y}}$$

$$I.F = \frac{1}{x^2y^2}$$

Multiplying by $\frac{1}{x^2y^2}$

$$\frac{1}{x^2y^2}[x^2y - 2xy^2]\,dx - \frac{1}{x^2y^2}[x^3 - 3x^2y]\,dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

$$M = \frac{1}{y} - \frac{2}{x} \quad ; \quad N = -\left(\frac{x}{y^2} - \frac{3}{y}\right)$$

$$\frac{\partial M}{\partial y} = -1/y^2 \quad ; \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

∴ which is exact

The solution is $\underset{(y \text{ constant})}{\int M\,dx} + \int \left( \begin{matrix} \text{term of } N \\ \text{not containing } x \end{matrix} \right) dy =$

$$M = \frac{1}{y} - \frac{2}{x} \; ; \; N = -\left( \frac{x}{y^2} - \frac{3}{y} \right)$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \; ; \; \frac{\partial N}{\partial x} = \frac{-1}{y^2} .$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int \left( \frac{1}{y} - \frac{2}{x} \right) dx + \int \frac{3}{y}\, dy = c$$

$$\frac{1}{y} \int dx - 2 \int \frac{1}{x}\, dx + 3 \int \frac{1}{y}\, dy = c$$

$$\frac{x}{y} - 2 \log x + 3 \log y = c .$$

—x—.

Type 3:

I.F for an equation of the type

$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$

If the equation $M dx + N dy = 0.$

be of this form, $\frac{1}{Mx - Ny}$ is an I.F $(Mx - Ny) \neq 0$

—x—.

Ⓔ

Solve $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0.$

This is of the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0 .$$

$$M = (1+xy)\,y \; ; \; N = (1-xy)\,x .$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{(1+xy)yx - (1-xy)xy}$$

$$= \frac{1}{\cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2}$$

$$= \frac{1}{2x^2y^2}$$

Multiplying by $\frac{1}{2x^2y^2}$

$$\frac{1}{2x^2y^2}(1+xy)\cancel{y}\,dx + \frac{1}{2x^2y^2}(1-xy)\cancel{x}\,dy = 0$$

$$\frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

$$M = \frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right) \quad ; \quad N = \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2x^2y^2} \quad ; \quad \frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact

$\int M\,dx + \int$ (terms of N not containing x) dy = c

(y constant)

$$\frac{1}{2}\int\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \int \frac{1}{2}\left(-\frac{1}{y}\right)dy = c$$

$$\frac{1}{2}\left(\frac{-1}{xy}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$x^n = \frac{x^{n+1}}{n+1}$$

$$\int x^{-2}dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x}$$

÷ by $\frac{1}{2}$

$$-\frac{1}{xy} + \log x - \log y = c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$$

— × —

Scanned with CamScanner

In the equation

$$M dx + N dy = 0$$

a) if $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ be a function of $x$ only $= f(x)$,

then $e^{\int f(x) dx}$ is an I.F

b) if $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ be a function of $y$ only $= F(y)$ then $e^{\int F(y) dy}$ is an I.F.

—x—

solve

$$(xy^2 - e^{1/x^3}) dx - x^2 y dy = 0.$$

$M = xy^2 - e^{1/x^3}$ ; $N = -x^2 y$

$\frac{\partial M}{\partial y} = 2xy$ ; $\frac{\partial N}{\partial x} = -2xy$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2 y} = \frac{4\cancel{xy}}{-x^{\cancel{2}}\cancel{y}} = \frac{-4}{x}$$

which is a function of $x$ only.

$$I.F = e^{\int -4/x \, dx} = e^{-4 \log x} = e^{\log x^{-4}} = x^{-4}$$

Multiple by $x^{-4}$

$$x^{-4}(xy^2 - e^{1/x^3}) dx - x^{-4}(x^2 y dy) = 0$$

$$(x^{-3}y^2 - x^{-4}e^{1/x^3}) dx + x^{-2} y dy = 0$$

$M = x^{-3}y^2 - x^{-4}e^{1/x^3}$ ; $N = x^{-2}y$

$\frac{\partial M}{\partial y} = 2yx^{-3}$ ; $\frac{\partial N}{\partial x} = 2x^{-3}y$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact

The solution is

$\int M\,dx + \int$ (terms of N not containing $x$) $dy = c$
($y$ constant)

$$\int \left(x^{-3}y^{2} - x^{-4}\, e^{1/x^3}\right) dx + \int 0 = c$$

$$-y^{2}\frac{x^{-2}}{2} + \frac{1}{3}\int e^{-x^{3}}\left(-3x^{-4}\right) dx = c$$

$$\frac{1}{3} e^{-x^{3}} - \frac{1}{2}\, y^{2}/x^{2} = c$$

—x—.

Solve $(xy^{3}+y)\,dx + 2(x^{2}y^{2}+x+y^{4})\,dy = 0.$

$M = xy^{3}+y \qquad N = 2(x^{2}y^{2}+x+y^{4})$

$$\frac{\partial M}{\partial y} = x3y^{2}+1 \;;\; \frac{\partial N}{\partial x} = 2\left[2xy^{2}+1\right]$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is not exact

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{2(2xy^{2}+1)-(3xy^{2}+1)}{xy^{3}+y} = \frac{4xy^{2}+2-3xy^{2}-1}{xy^{3}+y}$$

$$= \frac{xy^{2}+1}{xy^{3}+y} = \frac{\cancel{xy^{2}+1}}{y(\cancel{xy^{2}+1})} = \frac{1}{y}$$

∴ which is function of $y$ alone.

$$I.F = e^{\int 1/y\,dy} = e^{\log y} = y$$

Multiple by $y$

$$y(xy^{3}+y)\,dx + 2y(x^{2}y^{2}+x+y^{4})\,dy = 0$$

$$(xy^{4}+y^{2})\,dx + (2x^{2}y^{3} + 2xy + 2y^{5})\,dy = 0$$

$M = xy^4 + y^2$  $N = 2x^2y^3 + 2xy + 2y^5$

$\frac{\partial M}{\partial y} = 4y^3x + 2y$  $\frac{\partial N}{\partial x} = 4xy^3 + 2y$,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is exact.

The solution is

$$\underset{(y\ \text{constant})}{\int M\,dx} + \int \left(\text{terms of } N \text{ not containing } x\right) dy = c$$

$$\int (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$$

$$\frac{x^2}{2}y^4 + y^2x + \frac{\cancel{2}y^6}{\cancel{6}\,3} = c$$

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

—x—.

**<u>Equation</u> <u>not</u> <u>of</u> <u>first</u> <u>degree</u>**

<u>Solvable</u> <u>for</u> <u>y</u>.

$$y = f(x, p)$$

$$P = \frac{dy}{dx} = \phi\left(x, P, \frac{dp}{dx}\right)$$

Let its should be $F(x, p, c) = 0$

$$x = F_1(P, c)\ ;\ y = F_2(P, c)$$

$$y - 2Px = \tan^{-1}(xp^2)$$

$$y = 2px + \tan^{-1}(xp^2). \rightarrow ①$$

Diff ① with respect to $x$ on both sides.

$$\frac{dy}{dx} = 2\left(P\cdot 1 + \frac{dp}{dx}x\right) + \frac{1}{1+(xp^2)^2}\left(x\cdot 2p\frac{dp}{dx} + p^2\right)$$

$$p = 2\left(p + x\frac{dp}{dx}\right) + \frac{1}{1+x^2p4}\left(2px\frac{dp}{dx} + p^2\right)$$

$$p = \left[\left(2p + 2x\frac{dp}{dx}\right) + \frac{1}{1+x^2p4}p\left(2x\frac{dp}{dx} + p\right)\right].$$

$$p = \left(p + 2x\frac{dp}{dx}\right)\left(p + \frac{p}{1+x^2p4}\right)$$

$$\cancel{p} = \left(p + 2x\frac{dp}{dx}\right)\cancel{p}\left(1 + \frac{1}{1+x^2p4}\right)$$

$$\left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{1}{x^2p4}\right) = 0$$

$$p + 2x\frac{dp}{dx} = 0$$

$$2x\frac{dp}{dx} = -p$$

$$2\frac{dp}{p} = -\frac{dx}{x}$$

$$2\frac{dp}{p} + \frac{dx}{x} = 0$$

$$2\int\frac{dp}{p} + \int\frac{dx}{x} = 0$$

$$2\log p + \log x = \log c$$

$$\log p^2 + \log x = \log C$$

$$\log(xp^2) = c$$

$$\boxed{xp^2 = c}$$

$$p^2 = c/x$$

$$P = \sqrt{c/x} \rightarrow ②$$

Eliminate P from ①. & ②

$$y = 2\sqrt{\frac{c}{x}}\, x + \tan^{-1} c$$

$$= 2\sqrt{c}\,\frac{\sqrt{x}\,\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} + \tan^{-1} c$$

$$y = 2\sqrt{cx} + \tan^{-1} c.$$

— x —

Solve $y = 2px - p^2$

$$\frac{dy}{dx} = P = 2\left(P \cdot 1 + x\frac{dp}{dx}\right) - 2p\frac{dp}{dx}$$

$$P = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$2P - P + 2(x-P)\frac{dp}{dx} = 0$$

$$P + 2(x-P)\frac{dp}{dx} = 0$$

$$P = -2(x-P)\,dp/dx$$

$$P\frac{dx}{dp} = -2x + 2P$$

$$P\,dx/dp + 2x = 2p$$

$$\frac{dx}{dp} + \frac{2x}{P} = \frac{2\cancel{p}}{\cancel{p}}$$

$$\frac{dx}{dp} + \underset{\substack{\downarrow \\ P}}{\frac{2x}{P}} = \underset{\substack{\downarrow \\ Q}}{2}$$

$$I.F = e^{\int P dp} = e^{\int 2/p\, dp} = e^{\log P} = e^{\log P^2} = P^2 \quad \boxed{I.F = P^2}$$

$$x(I.F) = \int Q\,(I.F)\,dp + c$$

$$\frac{dy}{dx} + Py = Q$$
$$I.F = e^{\int P dx}$$

$$xp^2 = \int 2p^2\,dp + c$$

$$xp^2 = \frac{2p^3}{3} + c \Rightarrow \div p^2; \quad x = \frac{2p^3}{3p^2} + \frac{c}{p^2}$$

$$\boxed{x = \frac{2P}{3} + cP^{-2}.}$$

— x —

solve $y + px = x^4p^2$

soln: $y = -px + x^4p^2$.

diff with respect to $x$

$$\frac{dy}{dx} = P = -\left(P.1 + x\frac{dp}{dx}\right) + \left(4x^3p^2 + x^4\,2p\frac{dp}{dx}\right)$$

$$P + P + x\frac{dP}{dx} - 4x^3p^2 - 2x^4p\frac{dp}{dx} = 0$$

$$2P - 4x^3p^2 + x\frac{dp}{dx} - 2x^4p\frac{dp}{dx} = 0$$

$$2p(1-2x^3p) + (1-2x^3p)\,x\,dp/dx = 0.$$

$$(1-2x^3p)(2p + x\,dp/dx) = 0$$

Discarding the factor $(1-2x^3p)$

$$2p + x\frac{dp}{dx} = 0$$

$$x\frac{dp}{dx} = -2p \Rightarrow \cancel{\frac{dp}{dx}}\ \frac{dp}{p} = -2\frac{dx}{x}$$

Integrating $\int dp/p = -2\int dx/x$

$$\log p = -2\log x + \log c$$

$$\log p + 2\log x = \log c$$

$$\log p + \log x^2 = \log c \Rightarrow Px^2 = c \Rightarrow P = c/x^2$$

Sub $P$ in ① $y = -\frac{c}{x^2}.x + x^4\left(\frac{c}{x^2}\right)^2$

$$= -c/x + x^4\frac{c^2}{x^4}$$

$$\boxed{y = \frac{-c}{x} + c^2}$$

Equation solvable of $x$

* The equation of this type $x = f(y, p) \longrightarrow ①$

Differentiating ① with respect to y.

$\frac{dy}{dx} = p$

$$x = f(y, p) \longrightarrow ①$$

Diff ① with respect to y,

$$\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \longrightarrow ②$$

② is the differential equation of first order in p and y

Solution of ② is $\phi(y, p, c) = 0 \longrightarrow ③$

* Eliminate p from equation ① & ③ gives the required equation.

— x —

⊗ v. Imp

Solve

$$y = 2px + y^2p^3 \longrightarrow ⊛$$

$$y - y^2p^3 = 2px$$

$$\Rightarrow x = \frac{y - y^2p^3}{2p}$$

Solving for $x$, $x = \frac{1}{2}\left[\frac{y}{p} - y^2p^2\right] \longrightarrow ①$

Diff ① with respect to y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} + y\left(\frac{-1}{p^2}\right)\frac{dp}{dy} - \left(2y \cdot p^2 + y^2 2p\frac{dp}{dy}\right)\right]$$

$$\frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - y^2\, 2p\frac{dp}{dy}\right].$$

$$\frac{1}{p} = \frac{1}{2} \cdot \frac{1}{p} \left( 1 - \frac{y}{p} \frac{dp}{dy} - 2p^2 py - y^2 \, 2p \cdot p \frac{dp}{dy} \right]$$

$$2p = p \left( 1 - y/p \; dp/dy - 2yp^3 - 2y^2p^2 \frac{dp}{dy} \right).$$

$$2p = p - y \frac{dp}{dy} - 2yp^4 - y^2 2p^3 \frac{dp}{dy}$$

$$\boxed{\begin{array}{l} 2p - p \\ = p \end{array}}$$

$$p + y \frac{dp}{dy} + 2yp^4 + 2y^2p^3 \frac{dp}{dy} = 0.$$

$$(p + 2yp^4) + (y + 2y^2p^3) \frac{dp}{dy} = 0$$

$$p(1 + 2yp^3) + (1 + 2yp^3) y \frac{dp}{dy} = 0$$

$$(1 + 2yp^3)\left(p + y \frac{dp}{dy}\right) = 0$$

Dis carding the factor $(1+2yp^3)$, we get

$$p + y \frac{dp}{dy} = 0$$

$$y \frac{dp}{dy} = -p$$

$$\frac{dp}{\cancel{dy}} \cancel{+} + \frac{\cancel{dy}}{\cancel{y}} \qquad \frac{dp}{p} = -\frac{dy}{y}$$

$$\frac{dp}{p} + \frac{dy}{y} = 0,$$

Integrating,

$$\int \frac{dp}{p} + \int \frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = c/y$$

[illegible] Sub p in ⊛

$$y = 2px + y^2p3$$

$$y = \frac{2cx}{y} + y^2\left(\frac{c}{y}\right)^3$$

$$y = \frac{2cx}{y} + y^2 \frac{c^3}{y^3}$$

$$y = \frac{2cx + c^3}{y}$$

$$\boxed{y^2 = 2cx + c^3.}$$

— x —

([illegible])

solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$

$$\tan^{-1}p = x - \frac{p}{1+p^2}$$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu'-u}{v^2}$

$$x = \tan^{-1}p + \frac{p}{1+p^2}$$

$\tan^{-1}x = \frac{1}{1+x^2}$

Diff with respect to y

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p\left(0+2p\frac{dp}{dy}\right)}{(1+p^2)^2}$$

$$\frac{1}{p} = \left[\frac{1}{(1+p^2)} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$= \left[\frac{(1+p^2) + (1+p^2) - 2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{2 + \cancel{2p^2} - \cancel{2p^2}}{(1+p^2)^2}\frac{dp}{dy}$$

$$1/p = \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$dy = \frac{2p}{(1+p^2)^2} dp$$

Integrating $$\int dy = \int \frac{2p}{(1+p^2)^2} dp$$ | $1+p^2 = t$

$2pdp = dt$

$$y = \int \frac{dt}{t^2}$$

$$= \int t^{-2} dt$$

$$= \frac{t^{-2+1}}{-2+1} = \frac{t^{-1}}{-1}$$

$$= \frac{-1}{t}$$

$$y = \frac{-1}{1+p^2} + c$$ | $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\boxed{y = c - \frac{1}{1+p^2}}$$

— x —

⊛ 2m

Clairaut's type equation:

* An equation of the form $y = Px + f(P) \rightarrow ①$ is known as clairaut's equation.

Differentiating ① with respect to $x$ we get

$$\frac{dy}{dx} = P + x\frac{dp}{dx} + f'(P)\frac{dp}{dx}$$

$$P = P + [x + f'(P)]\frac{dp}{dx}$$

$$[x + f'(P)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(P)]$

$$\frac{dp}{dx} = 0$$

Integrate , $P = C$

Puting $P = c$ in ①

$$\boxed{y = cx + f(c)}$$

* Thus the solution clairaut's equation is obtain by writing c for p.

—x—

Ⓧ 2m

Solve $(y - px)(p-1) = p$

The given equation is

$$(y - px)(p-1) = p$$

$$y - px = \frac{p}{p-1}$$

$$y = px + \frac{p}{p-1}$$

$$y = px + f(p)$$

which is clairaut's equation.

Putting $P = c$ We get the solution is

$$\boxed{y = cx + \frac{c}{c-1}}$$

*Thus the solution clairaut's equation is obtain by writing c for P

—x—

Ⓧ Solve $e^{4x}(p-1) + e^{2y}p^2 = 0.$

The given equation is $e^{4x}(p-1) + e^{2y}p^2 = 0$

$$y = px + f(p)$$

$$e^{lx} \qquad e^{my}$$

$$X = e^{kx} \qquad Y = e^{ky}$$

$K \to$ H.C.F of $l$ & $m$.

Putting $X = e^{2x}$ $\quad Y = e^{2y}$

$$dX = 2e^{2x}\,dx \quad ; \; dY = 2e^{2y}\,dy.$$

$$P = \frac{dy}{dx} = \frac{dY/2e^{2y}}{dX/2e^{2x}} = \frac{dY}{2e^{2y}} \times \frac{2e^{2x}}{dX}$$

$$= \frac{X}{Y}\frac{dY}{dX} = \frac{X}{Y}P$$

$$\boxed{P = \frac{X}{Y}P}$$

The given equation is

$$X^2\left(\frac{X}{Y}P - 1\right) + Y\left(\frac{XP}{Y}\right)^2 = 0.$$

$$X^2\left(\frac{XP - Y}{Y}\right) + \frac{\cancel{Y}X^2P^2}{Y^2} = 0$$

$$\frac{X^2}{Y}\left[XP - Y + P^2\right] = 0$$

$$XP - Y + P^2 = 0$$

$$PX + P^2 = Y$$

$$\boxed{Y = PX + P^2}$$

which is of clairaut's equation

$$Y = cX + c^2$$

$$e^{2y} = ce^{2x} + c^2.$$

—x—.

Solve $(Px-y)(Py+x) = 2P$

⊛⊛

V.Imp The given equation is

$$(Px-y)(Py+x) = 2P \longrightarrow ①$$

Putting $X = x^2 ; \ Y = y^2$

$$dX = 2x\,dx ; \quad dY = 2y\,dy$$

$$\frac{dX}{2x} = dx ; \quad \frac{dY}{2y} = dy.$$

$$P = \frac{dy}{dx} = \frac{dY/2y}{dX/2x} = \frac{dY}{2y} \times \frac{2x}{dX}$$

$$= \frac{x}{y}\frac{dY}{dX}$$

$$= \frac{\sqrt{X}}{\sqrt{Y}}\frac{dY}{dX}$$

$\frac{dY}{dX} = P$

$x^2 = X$

$x = X^{1/2}$

$x = \sqrt{x}$

$$\boxed{P = \frac{\sqrt{X}}{\sqrt{Y}}P}$$

where $\frac{dY}{dX} = P$

∴ The equation ① is

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\left(\frac{\sqrt{X}P\sqrt{X} - \sqrt{Y}\sqrt{Y}}{\sqrt{Y}}\right)(\sqrt{X}P + \sqrt{X}) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\left(\frac{XP-Y}{\sqrt{Y}}\right)\sqrt{X}(P+1) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\frac{\sqrt{X}}{\sqrt{Y}}(XP-Y)(P+1) = 2\frac{\sqrt{X}}{\ }P$$

$$(XP-Y)(P+1) = 2\frac{\sqrt{\cancel{x}}}{\sqrt{\cancel{y}}}P\frac{\sqrt{\cancel{y}}}{\sqrt{\cancel{x}}}$$

$$(XP-Y)(P+1) = 2P$$

$$(PX-Y) = \frac{2P}{P+1}$$

$$PX-\frac{2P}{P+1} = Y$$

$$Y = PX-\frac{2P}{P+1}$$

$$\boxed{y = Px + f(p)}$$

which is a clairaut's equation

By Putting $P=c$, we get the solution is

$$Y = cX - \frac{2c}{c+1}$$

$$y^2 = cx^2 - \frac{2c}{c+1},$$

—x—

Solve

$$(Px-y)(Py+x) = a^2P.$$

The given equation is

$$(Px-y)(Py+x) = a^2P \longrightarrow ①$$

Putting $X = x^2$ ; $Y = y^2$

$$dX = 2x\,dx \; ; \; dY = 2y\,dy.$$

$$\frac{dX}{2x} = dx \; ; \; \frac{dY}{2y} = dy.$$

$$P = \frac{dy}{dx} = \frac{dy/\cancel{dy}\,2y}{dx/\cancel{dx}\,2x} = \frac{dy}{\cancel{2}y} \times \frac{\cancel{2}x}{dx}$$

$$= \frac{x}{y}\,\frac{dy}{dx}$$

$$= \frac{\sqrt{x}}{\sqrt{y}}\,\frac{dy}{dx}$$

$$P = \frac{\sqrt{x}}{\sqrt{y}}\,p.$$

The equation ① is

$$\left(\frac{\sqrt{x}}{\sqrt{y}}\,P\sqrt{x} - \sqrt{y}\right)\left(\frac{\sqrt{x}}{\sqrt{y}}\,P\sqrt{y} + \sqrt{x}\right) = \alpha^2\frac{\sqrt{x}}{\sqrt{y}}\,P$$

$$\left(\frac{\sqrt{x}\;P\sqrt{x} - \sqrt{y}\;\sqrt{y}}{\sqrt{y}}\right)\left(\sqrt{x}\,P + \sqrt{x}\right) = \alpha^2\frac{\sqrt{x}}{\sqrt{y}}\,P$$

$$\left(\frac{xP - y}{\sqrt{y}}\right)\sqrt{x}\,(P+1) = \alpha^2\frac{\sqrt{x}}{\sqrt{y}}\,P$$

$$(xP - y)\,(P+1) = \alpha^2\frac{\sqrt{x}}{\cancel{\sqrt{x}}}\,P \times \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}$$

$$(xP - y)\,(P+1) = \alpha^2 P$$

$$(xP - y) = \frac{\alpha^2 P}{P+1}$$

$$Px - \frac{\alpha^2 P}{P+1} = y$$

$$y = Px - \frac{\alpha^2 P}{P+1}$$

$$\boxed{y = P(x) + f(P)}$$

| Objective type questions |
|---|
| The necessary and sufficient condition for the differential equation to be exact is |
| The equation is known is $dy/dx+Py=Q y^2$ |
| The integrating factor of $dy/dx+y/x=x^2$ |
| The solution of $Mdx+Ndy=0$ is posses an |
| A differential equation is said to be ______ if the dependent variable and its derivative occur only in the first degree and are not multiplied together |
| The order of $d^2y/dx^2+y=x^2-2$ is |
| The integrating factor of $dy/dx+y\sin x = 0$ is |
| The integrating factor of $dy/dx-y\cot x = \sin x$ is |
| The solution of $y=(x-a)p-p^2$ |
| An equation of the form $y=px+f(p)$ is known as |
| The order of $d^2y/dx^2+y=0$ is |
| The clairaut's form of $p=\tan(px-y)$ |
| An equation involving one dependent variable and its derivatives with respect to one independent variable is called __________ |
| The _______ is differentiation of a function of two or more variables |
| A differential equation is said to be linear if the dependent variable and its derivative occur only in the ______degree and are not multiplied together |
| The highest derivative of the differential equation is________ |
| The powerof the hightest derivative of the differential equation is called____ |
| The order of $y''-y'+7=x^2+4$ is |
| The order of $y'''+xy'+7x=0$ is |
| The degree of the $(d^2y/dx^2)^2+(dy/dx)^3+3y=0$ |
| The degree of the $(d^2y/dx^2)^3+(dy/dx)^3+7y=0$ |
| The order and degree of $(d^3/dx^3)^2+dy/dx+9y=0$ |
| The standard form of a linear equation of the first order |
| The integrating factor of linear equation of the form $dx/dy+Px=Q$ is |
| The integrating factor of linear equation of the form $dy/dx+Py=Q$ is |
| The integrating factor of $dy/dx+y\sin x=0$ is |
| The integrating factor of $dy/dx-y\cot x=0$ is |
| If the given equation$Mdx+Ndy=0$ is homogenous and $Mx+Ny\neq0$ then the integrating factor is ___ |
| The solution of $Mdx+Ndy$ is |
| If $Mdx+Ndy=0$ be a homogeneous equation in x and y, then______is an integrating factor$(Mx+Ny\neq0)$ |

| Opt 1 | Opt2 | Opt3 |
|---|---|---|
| M_x = N_y | M_y = N_x | M_x = N_x |
| Euler equation | Bernoulli’s Equation | Legendre equation |
| x | y | logx |
| e no of integrating factor | finite no of integrating factor | e |
| Linear | nonlinear | quadratic |
| 0 | 1 | 2 |
| e^-cosx | ye^-cosx | logx |
| sinx | - sinx | cosx |
| y = (x-a)c-c^2 | y = (x-a)c+c^2 | 0 |
| linear | Bernoulli's Equation | exact |
| 2 | 1 | 0 |
| y=cx+tan^-1 c | y=cx-tan^-1 c | c=tan(cx-y) |
| ODE | PDE | Partial |
| ODE | PDE | Partial |
| first | second | third |
| Order | Degree | Power |
| Order | Degree | Power |
| 0 | 1 | 2 |
| 0 | 1 | 2 |
| 0 | 1 | 2 |
| 0 | 1 | 2 |
| 3,2 | 2,3 | 1,2 |
| dy/dx+Py=Q | dy/dx+py=Q | dy/dx+Py=q |
| e^integral Qdx | e^integral Pdy | e^integral Qdx |
| e^integral Qdy | e^integral Pdx | e^integral Qdx |
| e^(-cosx) | e^(-cosx)y | logx |
| cos x | (-cos x) | cosec x |
| 1/(Nx-My) | 1/(Mx+Ny) | 1/(Mx-Ny) |
| intergral y constant Mdx+integral of terms of N not containing x dy | intergral y constant Mdx+integral of terms not containing x dx | intergral y constant Ndx+integral of terms not containing x dx |
| 1/(Mx+Ny) | 1/(Mx-Ny) | Mdy+Ndx |

| Opt4 | Opt5 | Opt6 | Answer |
|---|---|---|---|
| M_y = N_y | | | M_y = N_x |
| Homogeneous | | | Bernoulli’s Equation |
| 0 | | | x |
| one integrating factor | | | Infinite no of integrating factor |
| PDE | | | Linear |
| 3 | | | 2 |
| e^sinx | | | e^-cosx |
| - cosx | | | - sinx |
| -1 | | | y = (x-a)c-c^2 |
| lairaut’s equation | | | Clairaut’s equation |
| -1 | | | 2 |
| c=tan(px+y) | | | y=cx-tan^-1 c |
| Total | | | ODE |
| Total | | | PDE |
| first and second | | | first |
| second degree | | | Order |
| second degree | | | Degree |
| 3 | | | 2 |
| 3 | | | 3 |
| 3 | | | 2 |
| 3 | | | 3 |
| 2,1 | | | 3,2 |
| 5dy/dx+Py=Q | | | dy/dx+Py=Q |
| e^Qdx | | | e^integral Pdy |
| e^Qdx | | | e^integral Pdx |
| e^(sinx) | | | e^(-cosx) |
| sin x | | | cosec x |
| 1/(Nx+My) | | | 1/(Mx+Ny) |
| intergral y constant Mdx+integral of terms not containing y dx | | | intergral y constant Mdx+integral of terms of N not containing x dy |
| Mdy-Ndx | | | 1/(Mx+Ny) |

If Mdx+Ndy=0 be a homogeneous equation in x and y, then______is an integrating factor(Mx-Ny≠0)

| 1/(Mx+Ny) | 1/(Mx-Ny) | Mdy+Ndx |
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| Mdy-Ndx | | | 1/(Mx-Ny) |
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DEPARTMENT OF MATHEMATICS

# MATHEMATICS II (MA6251)

FOR

SECOND SEMESTER ENGINEERING STUDENTS
ANNA UNIVERSITY SYLLABUS

This text contains some of the most important short-answer (Part A) and long-answer questions (Part B) and their answers. Each unit contains 30 university questions. Thus, a total of 150 questions and their solutions are given. A student who studies these model problems will be able to get pass mark (hopefully!!).

Prepared by the faculty of Department of Mathematics

## UNIT I ORDINARY DIFFERENTIAL EQUATIONS

## Part – A

Problem 1 Solve the equation $\left(D^2 - D + 1\right) y = 0$

Solution:

The A.E is $m^2 - m + 1 = 0 \Rightarrow m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$.

$$m = \frac{1 \pm \sqrt{3}i}{2} \text{ and } \alpha = \frac{1}{2};\ \beta = \frac{\sqrt{3}}{2}$$

$$G.S : y = e^{\alpha x}\left(A\cos \beta x + B\sin \beta x\right)$$

$$G.S : y = e^{\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right)$$ where A, B are arbitrary constants.

Problem 2 Find the particular integral of $\left(D^2 + a^2\right) y = b\cos ax + c\sin ax$.

Solution:

Given $\left(D^2 + a^2\right) y = b\cos ax + c\sin ax$.

$$\text{P.I} = b\frac{1}{D^2 + a^2}\cos ax + c.\frac{1}{D^2 + a^2}\sin ax.$$

$$= \frac{bx\sin ax}{2a} - \frac{cx\cos ax}{2a}$$

$$= \frac{x}{2a}\left[b\sin ax - c\cos ax\right].$$

Problem 3 Find the particular integral of $(D+1)^2\ y = e^{-x}\cos x$.

Solution:

$$\text{P.I} = \frac{1}{(D+1)^2}e^{-x}\cos x$$

$$= \frac{e^{-x}}{(D-1+1)^2}\cos x$$

$$= e^{-x}\frac{1}{D^2}\cos x$$

$$= e^{-x}\frac{1}{D}\sin x$$

$$= -e^{-x}\cos x.$$

Problem 4 Find the particular integral of $\left(D^2+4\right)y=x^4$.

Solution:

$$\text{P.I}=\frac{1}{D^2+4}x^4$$

$$=\frac{1}{4\left(1+\frac{D^2}{4}\right)}x^4$$

$$=\frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^4$$

$$=\frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}\right)x^4$$

$$=\frac{1}{4}\left(x^4-\frac{4.3x^2}{4}+\frac{4.3.2.1}{16}\right)$$

$$=\frac{1}{4}\left(x^4-3x^2+\frac{3}{2}\right).$$

Problem 5 Solve $\left(D^2+6D+9\right)y=e^{-2x}x^3$.

Solution:

The A.E is $m^2+6m+9=0$

$\Rightarrow (m+3)^2=0$

$m=-3,-3$

C.F: $(A+Bx)e^{-3x}$

$$=\frac{1}{(D+3)^2}e^{-2x}x^3$$

$$=\frac{e^{-2x}}{(D-2+3)^2}x^3$$

$$=\frac{e^{-2x}}{(1+D)^2}x^3=e^{-2x}(1+D)^{-2}x^3$$

$$\text{P.I}=e^{-2x}\left(1-2D+3D^2-4D^3\right)x^3$$

$$=e^{-2x}\left(x^3-2\left(3x^2\right)+3(3.2x)-4(3.2.1)\right)$$

$$=e^{-2x}\left(x^3-6x^2+18x-24\right)$$

G.S. $y=(A+Bx)e^{-3x}+\left(x^3-6x^2+18x-24\right)e^{-2x}$.

### Problem 6 Solve $\left(D^2+2D-1\right)y=x$

### Solution:

The A.E is $m^2+2m-1=0 \Rightarrow m=\dfrac{-2\pm\sqrt{4+4}}{2}$ y

$\Rightarrow m=-1\pm\sqrt{2}$

C.F: $Ae^{\left(-1+\sqrt{2}\right)x}+Be^{\left(-1-\sqrt{2}\right)x}=Ae^{-x}e^{\sqrt{2}x}+Be^{-x}e^{-\sqrt{2}x}$

$$\text{P.I}=\frac{1}{\left(D^2+2D-1\right)}x$$

$$=\frac{1}{-\left(1-2D-D^2\right)}x$$

$$=-\left[1-\left(2D+D^2\right)\right]^{-1}x$$

$$\text{P.I}=-\left[1+2D+D^2\right]x=-x-2$$

G.S: $y=e^{-x}\left(Ae^{\sqrt{2}x}+Be^{-\sqrt{2}x}\right)-\left(x+2\right).$

### Problem 7 Find the particular integral $\left(D^2+4D+5\right)y=e^{-2x}\cos x$

### Solution:

$$\text{P.I}=\frac{1}{D^2+4D+5}e^{-2x}\cos x$$

$$=\frac{1}{\left(D+2\right)^2+1}\left(e^{-2x}\cos x\right)$$

$$=e^{-2x}\frac{1}{\left(D-2+2\right)^2+1}\cos x$$

$$=e^{-2x}\frac{1}{D^2+1}\cos x$$

$$\text{P.I}=\frac{xe^{-2x}}{2}\sin x.$$

### Problem 8 Solve for $x$ from the equations $x'-y=t$ and $x+y'=1$.

### Solution:

$x'-y=t\rightarrow(1)\Rightarrow x''-y'=1\Rightarrow x''-1=y'$

$x+y'=1\rightarrow(2)\Rightarrow x+x''-1=1$

Thus $x''+x=2\,(\text{or})\left(D^2+1\right)x=2$

$m^2+1=0\Rightarrow m=\pm\, i$

C.F. $A\cos t+B\sin t$

$$\text{P.I} = \frac{1}{\left(D^2+1\right)}(2) = \left(D^2+1\right)^{-1}(2) = 2$$

G.S: $x = A\cos t + B\sin t + 2.$

## Problem 9 Solve $\left[D^3 - 3D^2 - 6D + 8\right]y = x$.

## Solution:

The A.E is $m^3 - 3m^2 - 6m + 8 = 0$

$(m-1)(m+2)(m-4) = 0$

$m = 1, -2, 4$

$\therefore C.F$ is $C_1e^x + C_2e^{-2x} + C_3e^{4x}$

$$\text{P.I } = \frac{1}{D^3-3D^2-6D+8}x$$

$$= \frac{1}{8\left[1+\dfrac{D^3-3D^2-6D}{8}\right]}x$$

$$= \frac{1}{8}\left[1+\frac{D^3-3D^2-6D}{8}\right]^{-1}x$$

$$= \frac{1}{8}\left[1-\left(\frac{D^3-3D^2-6D}{8}\right)+..\right]x$$

$$= \frac{1}{8}\left[x+\frac{6}{8}\right] = \frac{1}{8}\left[x+\frac{3}{4}\right].$$

Complete solution is $y = C.F + P.I$

$$y = C_1\, e^x + C_2\, e^{-2x} + C_3e^{4x} + \frac{1}{8}\left[x+\frac{3}{4}\right].$$

## Problem 10 Solve the equation $\left[D^2 - 4D + 13\right]y = e^{2x}$

## Solution:

Given $\left[D^2 - 4D + 13\right]y = e^{2x}$

The A.E is $m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

C.F $y = e^{2x}\left(A\cos 3x + B\sin 3x\right)$

$$\text{P.I.} = \frac{1}{D^2-4D+13}e^{2x}$$
$$= \frac{1}{4-8+13}e^{2x} = \frac{1}{9}e^{2x}$$

G.S: $y = C.F + P.I$

$$y = e^{2x}\left(A\cos 3x + B\sin 3x\right) + \frac{e^{2x}}{9}.$$

## Problem 11 Solve the equation $\left(D^5 - D\right)y = 12e^x$

## Solution:

Given $\left(D^5 - D\right)y = 12e^x$

The A.E is $m^5 - m = 0$

$m\left(m^4 - 1\right) = 0$

$m^4 - 1 = 0$

$m = 0(or)m^4 - 1 = 0$

$\left(m^2 - 1\right)\left(m^2 + 1\right) = 0$

$m = 0,\ m = \pm 1,\ m = \pm i$

C.F $= C_1 e^{0x} + C_2 e^x + C_3 e^{-x} + \left[C_4\cos x + C_5\sin x\right]$

$$\text{P.I} = \frac{1}{D^5 - D}12e^x$$
$$= \frac{1}{1-1}12e^x \quad (\text{Replacing D by 1})$$
$$= \frac{x}{5D^4-1}12e^x \quad (\text{Replacing D by 1})$$
$$= \frac{x}{5-1}12e^x = \frac{x}{4}12e^x = 3xe^x$$

G.S. $y = C.F + P.I$

$$= C_1 + C_2 e^x + C_3 e^{-x} + \left[C_4\cos x + C_5\sin x\right] + 3xe^x.$$

## Problem 12 Solve the equation $\left(D^2 + 5D + 6\right)y = e^{-7x}\sinh 3x$

## Solution:

The A.E is $m^5 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$

C.F. is $C_1 e^{-2x} + C_2 e^{-3x}$

$$\text{P.I} = \frac{1}{D^2 + 5D + 6}e^{-7x}\sinh 3x$$

$$=\frac{1}{D^2+5D+6}e^{-7x}\left(\frac{e^{3x}-e^{-3x}}{2}\right)$$

$$=\frac{1}{2}\left[\frac{1}{D^2+5D+6}e^{-4x}-\frac{1}{D^2+5D+6}e^{-10x}\right]$$

$$=\frac{1}{2}\left[\frac{e^{-4x}}{16-20+6}-\frac{e^{-10x}}{10-50+6}\right]$$

$$=\frac{1}{2}\left[\frac{e^{-4x}}{2}+\frac{e^{-10x}}{34}\right]$$

$\therefore$ G.S. $y=C_1e^{-2x}+C_2e^{-3x}+\frac{e^{-4x}}{4}+\frac{e^{-10x}}{68}$.

## Problem 13 Solve the equation $\left(D^3-3D^2+4D-2\right)y=e^x$

## Solution:

Given $m^3-3m^2-4m-2=0$

$(m-1)\left(m^2-2m+2\right)=0$

$m=1\ (or)\ m=1\pm i$

Complementary function $=Ae^x+e^x\left(B\cos x+C\sin x\right)$

P.I $=\frac{1}{D^3-3D^2+4D-2}e^x$

$$=\frac{1}{(1)^3-3(1)^2+4(1)-2}e^x \quad \text{(Replacing D by 1)}$$

$$=\frac{1}{1-3+4-2}e^x=\frac{1}{0}e^x$$

$$=\frac{x}{3D^2-6D+4}e^x$$

$$=\frac{1}{3-6+4}e^x \quad \text{(Replacing D by 1)}$$

$$=xe^x$$

G.S: $y=C.F.+P.I$

$$=Ae^x+e^x\left(B\cos x+C\sin x\right)+xe^x.$$

## Problem 14 Solve the equation $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=e^{-2x}$

Solution:

Given $\left(D^2+4D+4\right)y=e^{-2x}$

The A.E is $m^2+4m+4=0$

$\left(m^2+2\right)\left(m+2\right)=0$

$m=-2,-2$

C.F.: $(Ax+B)e^{-2x}$

$$\text{P.I}=\frac{1}{D^2+4D+4}e^{-2x}$$

$$=\frac{1}{(-2)^2+4(-2)+4}e^{-2x}\quad (\text{Replacing D by - 2})$$

$$=\frac{1}{4-8+4}e^{-2x}\quad (\therefore \text{Dr is } 0)$$

$$=\frac{x}{2D+4}e^{-2x}\quad (\text{Replacing D by -2})$$

$$=\frac{x}{2(-2)+4}e^{-2x}$$

$$=\frac{x^2}{2}e^{-2x}\quad (\therefore \text{Dr is } 0)$$

G.S is $y=(Ax+B)e^{-2x}+\frac{x^2}{2}e^{-2x}.$

## Problem 15 Solve the equation $(D^2+2D+1)y=e^{-x}+3$

## Solution:

Given $(D^2+2D+1)y=e^{-x}+3$

The A.E is $m^2+2m+1=0$

$(m+1)(m+1)=0$

$m=-1,-1$

C.F.: $(Ax+B)e^{-x}$

$\text{P.I}=P.I_1+P.I_2$

$$P.I_1=\frac{1}{D^2+2D+1}e^{-x}$$

$$=\frac{1}{(-1)^2+2(-1)+1}e^{-x}\quad (\text{Replacing D by -1})$$

$$=\frac{1}{1-2+1}e^{-x}$$

$$=\frac{x}{2D+2}e^{-x}\quad (\because \text{Dr is } 0)$$

$$=\frac{x}{2(-1)+2}e^{-x}\quad (\text{Replacing D by -1})$$

$$PI_2=\frac{1}{D^2+2D+1}3e^{0x}$$

$$=\frac{1}{(0)^2+2(0)+1}3e^{0x}\quad (\text{Replacing D by } 0)$$

G.S is $y=(Ax+B)e^{-x}+\frac{x^2}{2}e^{-x}+3.$

## Part-B

**Problem 1** Solve $(D^2-2D-8)y=-4\cosh x\sinh 3x+(e^{2x}+e^x)^2+1.$

**Solution:**

The A.E. is $(m^2-2m-8)=0$

$\Rightarrow (m-4)(m+2)=0$

$\Rightarrow m=-2,4$

C.F.: $Ae^{-2x}+Be^{4x}$

R.H.S $=-4\cosh x\sin 3x+(e^{2x}+e^x)^2+1$

$$=-4\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^{3x}-e^{-3x}}{2}\right)+(e^{2x}+e^x)^2+1$$

$$=-(e^{4x}-e^{-2x}+e^{2x}-e^{-4x})+e^{4x}+2e^{3x}+e^{2x}+1$$

$$=e^{-2x}+e^{-4x}+2e^{3x}+1e^{0x}$$

$$\text{P.I.}=\frac{1}{(D-4)(D+2)}(e^{-2x})+\frac{1}{(D-4)(D+2)}(-e^{-4x}+2e^{3x}+e^{0x})$$

$$=\frac{-1}{(-2-4)(D+2)}e^{-2x}-\frac{e^{-4x}}{(-8)(-2)}-\frac{2e^{3x}}{(-1)(5)}+\frac{1}{(-4)(2)}$$

$$=\frac{-xe^{-2x}}{6}-\frac{e^{-4x}}{16}-\frac{2e^{3x}}{5}-\frac{1}{8}$$

G.S is $y=Ae^{-2x}+Be^{4x}-\frac{xe^{-2x}}{6}-\frac{e^{-4x}}{16}-\frac{2e^{3x}}{5}-\frac{1}{8}.$

**Problem 2** Solve $y''+y=\sin^2 x+\cos x\cos 2x\cos 3x$

**Solution:**

The A.E is $m^2+1=0\Rightarrow m^2=-1\Rightarrow m=\pm i$

C.F.: $A\cos x+B\sin x$

$$\text{R.H.S}=\frac{\cos x(2\cos 2x\cos 3x)}{2}=\frac{\cos x}{2}[\cos 5x+\cos x]$$

$$=\frac{1}{4}\left[2\cos x\cos 5x+2\cos^2 x\right]$$

$$=\frac{1}{4}\left[\cos 6x+\cos 4x+1+\cos 2x\right]$$

$\left[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)\right]\quad A=3x,\ \ B=2x$

P.I. $=\dfrac{1}{\left(D^2+1\right)}\left[\sin^2 x+\cos x\cos 2x\cos 3x\right]$

$$=\frac{1}{D^2+1}\left[\frac{e^{0x}}{2}-\frac{\cos 2x}{2}+\frac{\cos 6x}{4}+\frac{\cos 4x}{4}+\frac{\cos 2x}{4}+\frac{e^{0x}}{4}\right]$$

$$=\frac{1}{2}-\frac{\cos 2x}{(-4+1)^2}+\frac{\cos 6x}{4(-36+1)}+\frac{\cos 4x}{4(-16+1)}+\frac{\cos 2x}{4(-4+1)}+\frac{1}{4}$$

$$=\frac{3}{4}+\frac{\cos 2x}{6}-\frac{\cos 6x}{140}-\frac{\cos 4x}{60}-\frac{\cos 6x}{140}+\frac{3}{4}.$$

G.S. is $y=A\cos x+B\sin x+\dfrac{\cos 2x}{12}-\dfrac{\cos 4x}{60}-\dfrac{\cos 6x}{140}+\dfrac{3}{4}.$

## Problem 3 Solve $\dfrac{d^2x}{dy^2}+10x=\cos 8y.$

## Solution:

Here $y$ is independent and $x$ is dependent variable

Let $D=\dfrac{d}{dy}$.

The A.E is $m^2+10=0$

$\Rightarrow m^2=-10$

$\Rightarrow m=\pm\sqrt{10}i$

C.F.: $A\cos\sqrt{10}y+B\sin\sqrt{10}y$

P.I $=\dfrac{1}{\left(D^2+10\right)}\cos 8y=\dfrac{\cos 8y}{-64+10}=\dfrac{-\cos 8y}{54}$

G.S. is $x=A\cos\sqrt{10}y+B\sin\sqrt{10}y-\dfrac{\cos 8y}{54}$

## Problem 4 Solve $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+9y=\sin x\cos 2x$.

## Solution:

The A.E is $m^2+6m+9=0$

$(m+3)^2=0$

$m=-3,\,-3.$

C.F.: $(A+Bx)e^{-3x}$

R.H.S $=\dfrac{2\sin x\cos 2x}{2}=\dfrac{1}{2}\left[\sin 3x+\sin(-x)\right]$

$$= \frac{1}{2}\left[\sin 3x - \sin x\right]$$

$$\left[2\sin A\cos B = \sin\left(A+B\right)+\sin\left(A-B\right)\right] \quad A = x,\ B = 2x$$

$$\text{P.I.} = \frac{1}{2}\frac{1}{\left(D+3\right)^2}\sin 3x - \frac{1}{2}\frac{1}{\left(D+3\right)^2}\sin x$$

$$= \frac{1}{2}\frac{1}{D^2+6D+9}\sin 3x - \frac{1}{2}\frac{1}{D^2+6D+9}\sin x$$

$$\text{P.I.} = \frac{1}{2}\times\frac{1}{-9+6D+9}\sin 3x - \frac{1}{2}\times\frac{1}{-1+6D+9}\sin x$$

$$= \frac{1}{12}\times\frac{1}{D}\sin 3x - \frac{1}{2}\times\frac{1}{8+6D}\sin x$$

$$= \frac{-\cos 3x}{12\left(3\right)} - \frac{\left(4-3D\right)\sin x}{4\left(4+3D\right)\left(4-3D\right)}$$

$$= \frac{-1}{36}\cos 3x - \frac{1}{4}\ \frac{1}{16-9D^2}\left(4\sin x - 3\cos x\right)$$

$$= \frac{-1}{36}\cos 3x - \frac{1}{4}\times\frac{4\sin x - 3\cos x}{16+9}$$

$$= \frac{-\cos 3x}{36} - \frac{\sin x}{25} + \frac{3\cos x}{100}$$

Problem 5 Solve $\left(D^2+4\right)y = x^4 + \cos^2 x$

Solution:

The A.E. is $m^2+4=0$

$m = \pm 2i$

C.F.: $A\cos 2x + B\sin 2x$

$$\text{P.I} = \frac{1}{D^2+4}x^4 + \frac{1}{D^2+4}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{4}\frac{1}{\left(1+\frac{D^2}{4}\right)}x^4 + \frac{1}{2}\frac{1}{\left(D^2+4\right)}e^{0x} + \frac{1}{2}\frac{1}{D^2+4}\cos 2x$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^4 + \frac{1}{2\left(4\right)} + \frac{\left(x\sin 2x\right)}{2\left(2\right)\left(2\right)}$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}\right)x^4 + \frac{1}{8} + \frac{x\sin 2x}{8}$$

$$= \frac{x^4}{4} - \frac{12x^2}{16} + \frac{4.3.2.1}{64} + \frac{1}{8} + \frac{x\sin 2x}{8}$$

G.S. is $y = A\cos 2x + B\sin 2x + \frac{4}{8} - \frac{3x^2}{4} + \frac{x^4}{4} + \frac{x\sin 2x}{8}$

## Problem 6 Solve $\left(D^2+2D-1\right)y = \left(x+e^x\right)^2 + \cos 2x \cosh x$ .

## Solution:

The A.E is $m^2 + 2m - 1 = 0$

$$m = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

C.F.: $Ae^{\left(-1+\sqrt{2}\right)x} + Be^{\left(-1-\sqrt{2}\right)x}$

$$\text{P.I.} = \frac{1}{\left(D^2+2D-1\right)}\left(x^2+2xe^x+e^x\right) + \frac{1}{\left(D^2+2D-1\right)}\cos 2x \frac{\left(e^x+e^{-x}\right)}{2}$$

$$\frac{1}{\left(D^2+2D-1\right)}x^2 = -\frac{1}{\left[1-\left(2D+D^2\right)\right]}x^2$$

$$= -\left[1+\left(2D+D^2\right)+\left(2D+D^2\right)^2+....\right]x^2$$

$$= -\left[1+2D+D^2+4D^2\right]x^2 = -x^2$$

$$= -x^2+4x+(5)(2)$$

$$\frac{1}{D^2+2D-1}x^2 = -x^2+4x+10$$

$$\frac{2}{D^2+2D-1}xe^x = \left(\frac{2e^x}{(D+1)^2+2(D+1)-1}\right)x$$

$$= 2e^x \frac{1}{D^2+2D+1+2D+2-1}x$$

$$= \frac{2e^x}{D^2+4D+2}(x)$$

$$= \frac{2e^x}{2}\frac{1}{\left[1+\left(2D+\frac{D^2}{2}\right)\right]}x$$

$$= e^x\left[1+\left(2D+\frac{D^2}{2}\right)\right]^{-1} x$$

$$= e^x\left[1+\left(2D+\frac{D^2}{2}\right)+...\right]x$$

$$= e^x\left[1+2D\right]x$$

$$\frac{2}{\left(D^2+2D-1\right)}xe^x = e^x\left[x+2\right] = (x+2)e^x$$

$$\frac{1}{D^2+2D-1}e^{2x}=\frac{1}{(4+4-1)}e^{2x}=\frac{e^{2x}}{7}$$

$$\frac{1}{D^2+2D-1}\frac{e^x\cos 2x}{2}+\frac{1}{D^2+2D-1}\frac{e^{-x}\cos 2x}{2}$$

$$=\frac{e^x}{2}\frac{1}{(D+1)^2+2(D+1)-1}\cos 2x+\frac{e^{-x}}{2}\frac{1}{(D-1)^2+2(D-1)-1}\cos 2x$$

$$=\frac{e^x}{2}\frac{1}{D^2+2D+1+2D+2-1}\cos 2x+\frac{e^{-x}}{2}\frac{1}{-4-2}\cos 2x$$

$$=\frac{e^x}{2}\frac{(2D+1)\cos 2x}{2(2D-1)(2D+1)}-\frac{e^{-x}}{12}\cos 2x$$

$$=\frac{e^x}{2}\frac{1}{2(4D^2-1)}(-2.2\sin 2x+\cos 2x)-\frac{e^{-x}\cos 2x}{12}$$

$$=\frac{e^x}{4}\frac{(-4\sin 2x+\cos 2x)}{(-16-1)}-\frac{e^{-x}\cos 2x}{12}$$

$$=-\frac{e^x(\cos 2x-4\sin 2x)}{17}-\frac{e^{-x}\cos 2x}{12}$$

The General Solution is

$$y=Ae^{(-1+\sqrt{2})x}+Be^{-(1+\sqrt{2})x}+10+4x-x^2+\frac{e^{2x}}{7}+(x+2)e^x$$

$$-\frac{e^x}{17}(\cos 2x-4\sin 2x)-\frac{e^{-x}}{12}\cos 2x$$

### Problem 7 Solve $(D^2+4)y=x^2\cos 2x$

### Solution:

The A.E is $m^2+4=0$

$\Rightarrow m^2=-4$

$\Rightarrow m=\pm 2i$

C.F.: $A\cos 2x+B\sin 2x$

$$\text{P.I}=\frac{1}{D^2+4}\left(x^2\cos 2x\right)$$

$$=R.P\ of\ \frac{1}{D^2+4}x^2e^{i2x}=R.P.\ of\ \frac{e^{2ix}}{(D+2i)^2+4}x^2$$

$$\text{P.I.}=R.P\ of\ e^{2ix}\frac{1}{D^2+4iD-4+4}x^2$$

$$=R.P\ of\ e^{2ix}\frac{1}{D^2+4iD}x^2=R.P\ \ of\ e^{2ix}\frac{1}{D(D+4i)}x^2$$

$$= R.P\ of\ \ e^{2ix}\frac{1}{D}\frac{1}{4i\left(1+\frac{D}{4i}\right)}x^{2}$$

$$= R.P\ \ of\ \ \frac{e^{2ix}}{4i}\frac{1}{D}\left(1+\frac{D}{4i}\right)^{-1}x^{2}$$

$$= R.P\ \ of\ \ \frac{e^{2ix}}{4i}\frac{1}{D}\left(1-\frac{D}{4i}-\frac{D^{2}}{16}\right)x^{2}$$

$$= R.P\ \ of\ \ \left(\frac{-ie^{2ix}}{4}\right)\left(\frac{x^{3}}{3}-\frac{x^{2}}{4i}-\frac{x}{8}\right)$$

$$= R.P\ \ of\ \ \left(\frac{-ie^{2ix}}{4}\right)\left(\frac{x^{3}}{3}+\frac{ix^{2}}{4}-\frac{x}{8}\right)$$

$$= R.P\ of\ \left(\frac{e^{2ix}}{4}\right)\left(\frac{-x^{3}i}{3}+\frac{x^{2}}{4}+\frac{ix}{8}\right)$$

$$= R.P\ of\ \ \frac{(\cos 2x+\sin 2x)}{4}\left(\frac{-x^{3}i}{3}+\frac{x^{2}}{4}+\frac{ix}{8}\right)$$

$$=\frac{1}{4}\left[\frac{x^{2}\cos 2x}{4}+\frac{x^{3}\sin 2x}{3}-\frac{x\sin 2x}{8}\right]$$

$$\text{P.I.}\ =\frac{1}{4}\left[\frac{x^{2}\cos 2x}{4}+\frac{x^{3}\sin 2x}{3}-\frac{x\sin 2x}{8}\right]$$

$$\text{G.S.:}\ \ y = A\cos 2x + B\sin 2x + \frac{x^{2}\cos 2x}{16}+\frac{x^{3}\sin 2x}{12}-\frac{x\sin 2x}{32}.$$

## Problem 8 Solve $\left(D^{2}+a^{2}\right)y=\sec a\,x$.

## Solution:

The A.E. is $m^{2}+a^{2}=0$

$\Rightarrow m^{2}=-a^{2}$

$\Rightarrow m=\pm\, a\,i$

C.F.: $A\cos ax + B\sin a\,x$

$$\text{P.I}\ =\frac{1}{(D+ai)(D-ai)}\sec a\,x \to (1)$$

Using partial fractions

$$\frac{1}{D^{2}+a^{2}}=\left[\frac{C_{1}}{D+ai}+\frac{C_{2}}{D-ai}\right]$$

$$1=C_{1}\left(D-ai\right)+C_{2}\left(D+ai\right)$$

$$C_{1}=-\frac{1}{2ia},\ \ C_{2}=\frac{1}{2ia}$$

$$\text{P.I.} = -\frac{1}{2ia}\frac{1}{(D+ai)}\sec ax + \frac{1}{2ia}\frac{1}{(D-ai)}\sec a\,x$$

$$= -\frac{1}{2ia}\frac{1}{D-(-ai)}\sec ax + \frac{e^{aix}}{2ia}\int e^{-aix}\sec a\,x\,dx$$

$$= -\frac{e^{-aix}}{2ia}\int e^{aix}\sec a\,x\,dx + \frac{e^{aix}}{2ia}\int e^{-aix}\sec a\,x\,dx$$

$$\text{P.I.} = -\frac{e^{-aix}}{2ia}\int\frac{(\cos ax + i\sin ax)}{\cos ax}dx + \frac{e^{aix}}{2ia}\int\frac{(\cos ax - i\sin ax)}{\cos ax}dx$$

$$\text{P.I.} = -\frac{e^{-aix}}{2ia}\int(1+i\tan ax)\,dx + \frac{e^{aix}}{2ia}\int(1-i\tan ax)\,dx$$

$$= -\frac{e^{-aix}}{2ia}\left[x+\frac{i}{a}\log\sec ax\right] + \frac{e^{aix}}{2ia}\left[x-\frac{i}{a}\log\sec ax\right]$$

$$= \frac{2x}{2a}\left[\frac{e^{aix}-e^{-aix}}{2i}\right] - \frac{2i}{2ia^2}[\log\sec ax]\left[\frac{e^{aix}+e^{-aix}}{2}\right]$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}(\log\sec ax)(\cos ax)$$

$$= \frac{1}{a^2}[ax\sin ax + \cos ax\log\cos ax]$$

G.S. is $y = C.F + P.I.$

Problem 9 Solve $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$.

Solution:

The A.E is $m^2+4m+3=0$

$(m+1)(m+3)=0$

$m=-1,-3$

C.F.: $Ae^{-x}+Be^{-3x}$

$$\text{P.I} = \frac{1}{(D+3)(D+1)}e^{-x}\sin x + \frac{1}{(D+1)(D+3)}xe^{3x}$$

$$= \frac{e^{-x}}{(D-1+3)(D-1+1)}(\sin x) + \frac{e^{3x}}{(D+3+1)(D+3+3)}(x)$$

$$= e^{-x}\frac{1}{(D+2)D}\sin x + e^{3x}\frac{1}{(D+4)(D+6)}x$$

$$= -e^{-x}\frac{D-2}{(D+2)(D-2)}\cos x + e^{3x}\frac{1}{D^2+10D+24}x$$

$$= -e^{-x}\frac{1}{(D^2-4)}(-\sin x - 2\cos x) + \frac{e^{3x}}{24}\frac{1}{1+\frac{10D}{24}+\frac{D^2}{24}}x$$

$$\text{P.I.} = e^{-x}\frac{1}{(D^2-4)}(\sin x + 2\cos x) + \frac{e^{3x}}{24}\left[1+\frac{5D}{12}+\frac{D^2}{24}\right]^{-1} x$$

$$= \frac{e^{-x}(\sin x + 2\cos x)}{(-1-4)} + \frac{e^{3x}}{24}\left[1-\frac{5D}{12}\right]x$$

$$= -\frac{e^{-x}}{5}(\sin x + 2\cos x) + \frac{e^{3x}}{24}\left(x-\frac{5}{12}\right)$$

G.S. is $y = Ae^{-x} + Be^{-3x} - \frac{e^{-x}}{5}(\sin x + 2\cos x) + \frac{e^{3x}}{24}\left(x-\frac{5}{12}\right).$

Problem 10 Solve the Legendre's linear equation

$$\left[(3x+2)^2 D^2 + 3(3x+2)D - 36\right]y = 3x^2+4x+1$$

Solution:

Let $\left[(3x+2)^2 + D^2 + 3(3x+2)D - 36\right]y = 3x^2+4x+1$

Let $3x+2 = e^t$ or $t = \log(3x+2)$

$$\Rightarrow \frac{dt}{dx} = -\frac{3}{3x+2}$$

$$3x = e^z - 2$$

$$x = \frac{1}{3}e^z - \frac{2}{3}$$

$$Let(3x+2)D = 3D'$$

$$(3x+2)^2 D^2 = 9D'(D'-1)$$

$$\left[9D'(D'-1)+3(3D')-36\right]y = 3\left[\frac{1}{3}e^z-\frac{2}{3}\right]+4\left[\frac{1}{3}e^z-\frac{2}{3}\right]+1$$

$$\left[9D'^2-9D'+9D'-36\right]y = 3\left[\frac{1}{9}e^{2z}+\frac{4}{9}-\frac{4}{9}e^z\right]+\frac{4}{3}e^z-\frac{8}{3}+1$$

$$\left[9D'^2-36\right]y = \frac{1}{3}e^{2z}+\frac{4}{3}-\frac{4}{3}e^z+\frac{4}{3}e^z-\frac{8}{3}+1$$

$$= \frac{1}{3}e^{2z}-\frac{1}{3}$$

A.E is $9m^2 - 36 = 0$

$9m^2 = 36$

$m^2 = 4$

$m = \pm 2$

$$C.F = Ae^{2z} + Be^{-2z}$$

$$= A(3x+2)^2 + B(3x+2)^{-2}$$

$$P.I_1 = \frac{1}{9D'^2-36}\frac{e^{2z}}{3}$$

$$= \frac{1}{3}.\frac{1}{36-36}e^{2z}$$

$$= \frac{1}{3}z\frac{1}{18D'}e^{2z}$$

$$= \frac{1}{54}z\frac{e^{2z}}{2}$$

$$= \frac{1}{108}ze^{2z}$$

$$= \frac{1}{108}[\log 3x+2](3x+2)^2$$

$$P.I_2 = \frac{1}{9D^{12}-36}\frac{e^{0z}}{3}$$

$$= \frac{1}{3}.\frac{1}{-36}e^{0z} = -\frac{1}{108}$$

$$y = C.F + P.I_1 - PI_2$$

$$= A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}(3x+2)^2\log(3x+2) + \frac{1}{108}$$

$$= A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right].$$

Problem 11 Solve $(D^2+5D+4)=e^{-x}\sin 2x + x^2+1$ where $D=\frac{d}{dx}$.

Solution:

The $A.E\ m^2+5m+4=0$

$m=-4$ or $m=-1$

$C.F = Ae^{-4x}+Be^{-x}$

$$P.I = \frac{1}{D^2+5D+4}\left(e^{-x}\sin 2x + 2x^2+1\right)$$

$$= e^{3x}\frac{1}{(D-1)^2+5(D-1)+4}\sin 2x + \frac{1}{4\left(1+\frac{D^2+5D}{4}\right)}\left(x^2+1\right)$$

$$= e^{3x}\frac{1}{D^2+3D}\sin 2x + \frac{1}{4}\left(1-\frac{5D}{4}+\frac{5D^2}{16}\right)\left(x^2+1\right)$$

$$= \frac{-e^{-x}}{26}[2\sin 2x + 3cos2x] + \frac{1}{4}\left(x^2-\frac{5}{2}x+\frac{13}{8}\right)$$

$G.S: y = C.F.+P.I$

$$y = Ae^{-4x} + Be^{-x} - \frac{e^{-x}}{26}(2\sin 2x + 3\cos 2x) + \frac{1}{4}\left(x^2 - \frac{5}{2}x + \frac{13}{8}\right).$$

Problem 12 Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = \sin(\log x)$.

Solution:

Given equation is $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = \sin(\log x)$.

$\left(x^2D^2 - 4xD + 6\right)y = \sin(\log x) \to (1)$

Put $x = e^z$ $(or)$ $z = \log x$

$xD = D' \to (2)$

$x^2D^2 = D'(D'-1) \to (3)$ Where $D'$ denotes $\frac{d}{dz}$

Sub (2) & (3) in (1) we get

$\left(D'(D'-1) + 4D' + 2\right)y = \sin z$

$(i.e)\left(D'^2 - D' + 4D' + 2\right)y = \sin z$

$\left(D'^2 + 3D' + 2\right)y = \sin z \to (4)$

The $A.E$ is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

C.F.: $Ae^{-z} + Be^{-2z}$

$$\text{P.I.} = \frac{1}{D'^2 + 3D' + 2}\sin z$$

$$= \frac{1}{-1 + 3D' + 2}\sin z$$

$$= \frac{1}{3D' + 1}\sin z$$

$$= \frac{3D' - 1}{9D'^2 - 1}\sin z$$

$$= \frac{(3D'-1)\sin z}{9(-1) - 1} \quad \left[\text{Replace } D'^2 \text{ by } -1\right]$$

$$= \frac{3D'(\sin z) - \sin z}{-10}$$

$$= \frac{3\cos z - \sin z}{-10}$$

∴ The solution of (4) is

$$y = Ae^{-z} + Be^{-2z} + \frac{3\cos z - \sin z}{-10}$$

Sub $z=\log x \; or \; x=e^{z}$, we get

$$y=Ae^{-\log x}+Be^{-2\log x}-\frac{3\cos(\log x)-\sin(\log x)}{10}$$

$$y=Ax^{-1}+Bx^{-2}-\frac{3\cos(\log x)-\sin(\log x)}{10}$$

$$y=\frac{A}{x}+\frac{B}{x^{2}}-\frac{3\cos(\log x)-\sin(\log x)}{10}$$

This gives the solution of the given differential equation.

## Problem 13 Solve the simultaneous ordinary differential equation

$(D+4)x+3y=t,\; 2x+(D+5)y=e^{2t}$

### Solution:

Given $(D+4)x+3y=t \rightarrow (1)$

$2x+(D+5)y=e^{2t} \rightarrow (2)$

$2\times(1)-(D+4)\times(2)$

$6y-(D+4)(D+5)y=2t-(D+4)e^{2t}$

$\left[6-D^{2}-9D-20\right]y=2t-2e^{2t}-4e^{zt}$

$\left(D^{2}+9D+14\right)y=6e^{2t}-2t$

The A.E. is $m^{2}+9m+14=0$

$(m+7)(m+2)=0$

$m=-2,-7$

C.F.: $Ae^{-2t}+Be^{-7t}$

$$\text{P.I.}=\frac{6}{\left(D^{2}+9D+14\right)}e^{2t}-\frac{2}{\left(D^{2}+9D+14\right)}t$$

$$=\frac{6e^{2t}}{4+18+14}-\frac{2}{14}\frac{1}{1+\frac{9D}{14}+\frac{D^{2}}{14}}(t)$$

$$=\frac{6e^{2t}}{36}-\frac{1}{7}\left(1+\frac{9D}{14}+\frac{D^{2}}{14}\right)^{-1}(t)$$

$$=\frac{e^{2t}}{6}-\frac{1}{7}\left(1-\frac{9D}{14}\right)(t)=\frac{e^{2t}}{6}-\frac{1}{7}\left(t-\frac{9}{14}\right)$$

G.S. is $y=Ae^{-2t}+Be^{-7t}+\frac{e^{2t}}{6}-\frac{t}{7}+\frac{9}{98}$

To Calculate $x$

$$Dy=-2Ae^{-2t}-7Be^{-7t}+\frac{2e^{2t}}{6}-\frac{1}{7}$$

$$5y = 5Ae^{-2t} + 5Be^{-7t} + \frac{5e^{2t}}{6} - \frac{5t}{7} + \frac{45}{98}$$

$$(D+5)y = 3Ae^{-2t} - 2Be^{-7t} + \frac{7e^{2t}}{6} - \frac{5t}{7} - \frac{1}{7} + \frac{45}{98}$$

$$(2) \Rightarrow 2x = -(D+5)y + e^{2t}$$

$$= -3Ae^{-2t} + 2Be^{-7t} - \frac{7e^{2t}}{6} + \frac{5t}{7} - \frac{31}{98} + e^{2t}$$

$$x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{7}{72}e^{2t} + \frac{5t}{14} - \frac{31}{196}$$

The General solution is

$$x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{e^{2t}}{12} + \frac{5t}{14} - \frac{31}{196}$$

$$y = Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}.$$

## Problem 14 Solve: $\frac{d^2y}{dx^2} + y = \tan x$ by method of variation of parameters

## Solution:

A.E is $m^2 + 1 = 0$

$m = \pm i$

C.F = $c_1 \cos x + c_2 \sin x$

$P.I = PI_1 + PI_2$

$f_1 = \cos x$; $f_2 = \sin x$

$f_1' = -\sin x$; $f_2' = \cos x$

$f_2' f_1' - f_1' f_2 = 1$

Now, $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$

$$= -\int \sin x \tan x dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{(-1+\cos^2 x)}{\cos x} dx$$

$$= -\int \sec x dx + \int \cos x dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \cos x \tan x dx$$

$$= -\cos x$$

∴ y = C.F + Pf1 + Qf2

$$= c_1\cos x + c_2\sin x + [-\log(\sec x + \tan x) + \sin x]\cos x - \cos x \sin x$$

$$= c_1\cos x + c_2\sin x - \log(\sec x + \tan x)\cos\cos x.$$

Problem 15 Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + 4y = \sec 2x$

Solution:

The A.E is $m^2 + 4 = 0$

$m = \pm 2i$

C.F = $c_1\cos 2x + c_2\sin 2x$

P.I = P$f_1$ + Q$f_2$

$f_1$ = cos 2x; $f_2$ = sin 2x

$f_1' = -2\sin 2x$; $f_2' = 2\cos 2x$

$f_2' f_1 - f_1' f_2 = 2$

Now, $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}dx$

$$= -\int \frac{\sin 2x}{2}\sec 2x dx$$

$$= -\frac{1}{2}\int \tan 2x dx = \frac{1}{4}\log(\cos 2x)$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}dx$$

$$= \frac{1}{2}\int \cos 2x \sec 2x dx = \frac{1}{2}x$$

∴ y = C.F + P$f_1$ + Q$f_2$

$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\log(\cos 2x)\cos 2x + \frac{1}{2}x\sin 2x.$$

# UNIT II VECTOR CALCULUS

## Part-A

**Problem 1** Prove that $div(grad\,\phi)=\nabla^2\phi$

**Solution:**

$$
\begin{aligned}
div(grad\,\phi) &= \nabla.\nabla\phi \\
&= \nabla\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right) \\
&= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right) \\
&= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} \\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi \\
&= \nabla^2\phi .
\end{aligned}
$$

**Problem 2** Find a, b, c, if $\vec{F}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$ is irrotational.

**Solution:**

$\vec{F}$ is irrotational if $\nabla\times\vec{F}=\vec{0}$

$$
\nabla\times\vec{F}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}
$$

$$
=\vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z)-\frac{\partial}{\partial z}(bx-3y-z)\right]-\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right]
$$

$$
+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y+2z)-\frac{\partial}{\partial y}(x+2y+az)\right]
$$

$$
=\vec{i}[c+1]+\vec{j}[a-4]+\vec{k}[b-2]
$$

$\because \nabla\times\vec{F}=0 \Rightarrow 0\vec{i}+0\vec{j}+0\vec{k} \;=\vec{i}[c+1]+\vec{j}[a-4]+\vec{k}[b-2]$

$\therefore\ c+1=0,\ a-4=0,\ b-2=0$

$\Rightarrow c=-1,\ a=4,\ b=2.$

**Problem 3** If $S$ is any closed surface enclosing a volume $V$ and $\vec{r}$ is the position vector of a point, prove $\iint\limits_S(\vec{r}.\hat{n})\,ds=3V$

## Solution:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

By Gauss divergence theorem

$$\int\int_S \overline{F}\cdot\hat{n}\,ds = \int\int\int_V \nabla\cdot\overline{F}dV \qquad \text{Here } \overline{F} = \nabla.\vec{r}$$

$$\int\int_S \vec{r}\cdot\hat{n}\,ds = \int\int\int_V \nabla\cdot\vec{r}dV$$

$$= \iiint_V \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(x\vec{i} + y\vec{j} + z\vec{k}\right)dV$$

$$= \int\int\int(1+1+1)dV$$

$$\int\int_S \vec{r}\cdot\hat{n}\,ds = 3V.$$

## Problem 4 If $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$, where $\vec{a}, \vec{b}, n$ are constants show that $\vec{r}\times\frac{d\vec{r}}{dt} = n\left(\vec{a}\times\vec{b}\right)$

## Solution:

Given $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$

$$\frac{d\vec{r}}{dt} = -n\vec{a}\sin nt + n\vec{b}\cos nt$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \left(\vec{a}\cos nt + \vec{b}\sin nt\right)\times\left(-n\vec{a}\sin nt + n\vec{b}\cos nt\right)$$

$$= n\left(\vec{a}\times\vec{b}\right)\cos^2 nt - \left(\vec{b}\times\vec{a}\right)\sin^2 nt$$

$$= n\left(\vec{a}\times\vec{b}\right)\cos^2 nt + \left(\vec{a}\times\vec{b}\right)\sin^2 nt \qquad \left(\because \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}\right)$$

$$= n\left(\vec{a}\times\vec{b}\right)(1) = n\left(\vec{a}\times\vec{b}\right)$$

## Problem 5 Prove that $div\left(curl\vec{A}\right) = 0$

## Solution:

$$\nabla.(\nabla\times \mathrm{A}) = \nabla\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla\cdot\left[\vec{i}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z}\right) - \vec{j}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z}\right)\right] + \vec{k}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y}\right)$$

$$=\left(\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}\right)+\left(\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}\right)$$

$$\therefore div\left(curl\vec{A}\right)=0$$

**Problem 6** Find the unit normal to surface $xy^3z^2=4$ at $(-1,-1,2)$

**Solution:**

Let $\phi = xy^3z^2-4$

$$\nabla\phi = y^3z^2\vec{i}+3xy^2z^2\vec{j}+2xy^3z\vec{k}$$

$$\nabla\phi_{(-1,-1,2)}=(-1)^3(2)^2\vec{i}+3(-1)(-1)^2(2)^2\vec{j}+2(-1)(-1)^3(2)\vec{k}$$

$$=-4\vec{i}-12\vec{j}+4\vec{k}$$

Unit normal to the surface is $\hat{n}=\dfrac{\nabla\phi}{|\nabla\phi|}$

$$=\frac{-4\vec{i}-12\vec{j}+4\vec{k}}{\sqrt{16+144+16}}$$

$$=-\frac{-4\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{176}}$$

$$=\frac{-4\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{16\times 11}}=\frac{-\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{11}}.$$

**Problem 7** Applying Green's theorem in plane show that area enclosed by a simple closed curve $C$ is $\frac{1}{2}\int(xdy-ydx)$

**Solution:**

$$\int_C Pdx+Qdy=\int\int_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$P=-y,\ Q=x$

$$\frac{\partial P}{\partial y}=-1,\ \frac{\partial Q}{\partial x}=1$$

$$\therefore \int(x\,dy-ydx)=\int\int_R(1+1)\,dxdy=2\int\int_R dx\,dy$$

= 2 Area enclosed by C

$\therefore$ Area enclosed by C $=\frac{1}{2}\int(xdy-ydx)$.

**Problem 8** If $\vec{A}$ and $\vec{B}$ are irrotational show that $\vec{A}\times\vec{B}$ is solenoidal

**Solution:**

Given $\vec{A}$ is irrotational i.e., $\nabla\times\vec{A}=\vec{0}$

$\vec{B}$ is irrotational i.e., $\nabla\times\vec{B}=\vec{0}$

$$\nabla\left(\vec{A}\times\vec{B}\right)=\vec{B}.\left(\nabla\times\vec{A}\right)-\vec{A}.\left(\nabla\times\vec{B}\right)$$

$$=\vec{B}\cdot\vec{0}-\vec{A}\cdot\vec{0}=\vec{0}$$

$\therefore\ \vec{A}\times\vec{B}$ is solenoidal.

Problem 9 If $\vec{F}=grad\left(x^3+y^3+z^3-3xyz\right)$ find curl $\vec{F}$

Solution:

$$\vec{F}=\nabla\left(x^3+y^3+z^3-3xyz\right)$$

$$=\left(3x^2-3yz\right)\vec{l}+\left(3y^2-3xz\right)\vec{j}+\left(3z^2-3xy\right)\vec{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy\end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial}{\partial y}\left(3z^2-3xy\right)-\frac{\partial}{\partial z}\left(3y^2-3xz\right)\right]-\vec{j}\left[\frac{\partial}{\partial x}\left(3z^2-3xy\right)-\frac{\partial}{\partial z}\left(3x^2-3yz\right)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}\left(3y^2-3xz\right)-\frac{\partial}{\partial y}\left(3x^2-3yz\right)\right]$$

$$=\vec{i}\left[-3x+3x\right]-\vec{j}\left[-3y+3y\right]+\vec{k}\left[-3z+3z\right]$$

$$=\vec{i}\,0+\vec{j}0+\vec{k}\,0=0.$$

Problem 10 If $\vec{F}=x^2\vec{i}+y^2\vec{j}$, evaluate $\int\vec{F}\cdot d\vec{r}$ along the straight line $y=x$ from $(0,0)$ to $(1,1)$.

Solution:

$$\vec{F}.d\vec{r}=\left(x^2\vec{i}+y^2\vec{j}\right)\cdot\left(dx\vec{i}+dy\right)$$

$$=x^2dx+y^2dy$$

Given $y=x$

$dy=dx$

$$\therefore\int_C\vec{F}\cdot d\vec{r}=\int_0^1\left(x^2dx+y^2dy\right)$$

$$=\int_0^1 x^2dx+x^2dx=2\int_0^1 x^2dx=2\left[\frac{x^3}{3}\right]_0^1=\frac{2}{3}$$

Problem 11 What is the unit normal to the surface $\phi(x,y,z)=C$ at the point $(x,y,z)$?

Solution:

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

Problem 12 State the condition for a vector $\vec{F}$ to be solenoidal

Solution:

$$\nabla.\vec{F} = div\vec{F} = 0$$

Problem 13 If $\vec{a}$ is a constant vector what is $\nabla\times\vec{a}$?

Solution:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\nabla\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{0}$$

Problem 14 Find grad $\phi$ at $(2,2,2)$ when $\phi = x^2 + y^2 + z^2 + 2$

Solution:

$$grad\phi = \nabla\phi$$

$$= \vec{i}\frac{\partial}{\partial x}\left(x^2 + y^2x^2 + 2\right) + \vec{j}\frac{\partial}{\partial y}\left(x^2 + y^2 + z^2 + 2\right) + \vec{k}\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2 + 2\right)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{(2,2,2)} = 4\vec{i} + 4\vec{j} + 4\vec{k}$$

Problem 15 State Gauss Divergence Theorem

Solution:

The surface integral of the normal component of a vector function $F$ over a closed surface $S$ enclosing volume $V$ is equal to the volume integral of the divergence of $\vec{F}$ taken over V. i.e., $\int\int_S \vec{F}.\vec{n}ds = \int\int\int_V \nabla.\vec{F}dV$

## Part –B

Problem 1 Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \left(2xyz+4z^2\right)\vec{i} + x^2z\,\vec{j} + \left(x^2y+8xz\right)\vec{k}$$

$$\nabla\phi_{(1,-2,-1)} = \left[2(1)(-2)(-1)+4(-1)^2\right]\vec{i} + (1)^2(-1)\,\vec{j} + \left[(1)^2(-2)+8(1)(-1)\right]\vec{k}$$

$$= (4+4)\vec{i} - \vec{j} + (-2-8)\vec{k}$$

$$= 8\vec{i} - \vec{j} - 10\vec{k}$$

Directional derivative $\vec{a}$ is $= \dfrac{\nabla\phi.\vec{a}}{|\nabla\phi|}$

$$= \frac{\left(8\vec{i}-\vec{j}-10\vec{k}\right).\left(2\vec{i}-\vec{j}-2\vec{k}\right)}{\sqrt{4+1+4}}$$

$$= \frac{16+1+20}{3} = \frac{37}{3}.$$

Problem 2 Find the maximum directional derivative of $\phi = xyz^2$ at $(1,0,3)$.

Solution:

Given $\phi = xyz^2$

$$\nabla\phi = yz^2\vec{i} + xz^2\,\vec{j} + 2xyz\vec{k}$$

$$\nabla\phi_{(1,0,3)} = 0(3)^2\,\vec{i} + (1)(3)^2\,\vec{j} + 2(1)(0)(3)\vec{k} = 9\,\vec{j}$$

Maximum directional directive of $\phi$ is $\nabla\phi = 9\,\vec{j}$

Magnitude of maximum directional directive is $|\nabla\phi| = \sqrt{9^2} = 9.$

Problem 3 Find the angle between the surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at the point $(2,-1,2)$.

Solution:

Let $\phi_1 = x^2+y^2+z^2-9$

$$\nabla\phi_1 = 2x\vec{i} + 2y\,\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{1(2,-1,2)} = 2(2)\vec{i} + 2(-1)\,\vec{j} + 2(2)\vec{k} = 4\vec{i} - 2\,\vec{j} + 4\vec{k}$$

$$\phi_2 = x^2+y^2-z-3$$

$$\nabla\phi_2 = 2x\vec{i} + 2y\,\vec{j} - \vec{k}$$

$$\nabla\phi_{2(2,-1,2)} = 4\vec{i} - 2\,\vec{j} - 2\vec{k}$$

If $\theta$ is the angle between the surfaces then

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{\left(4\vec{i}-2\,\vec{j}+4\vec{k}\right).\left(4\vec{i}-2\,\vec{j}-2\vec{k}\right)}{\sqrt{16+4+16}\sqrt{16+4+4}}$$

$$= \frac{16+4-8}{\sqrt{36}\sqrt{24}}$$

$$= \frac{12}{6\times 2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right).$$

Problem 4 Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to the point $(1,1)$ along $y^2 = x$.

Solution:

Given $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}.d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Given $y^2 = x$

$2ydy = dx$

$\therefore \vec{F}.d\vec{r} = (x^2 - x + x)dx - (2y^3 + y)dy$

$= x^2 dx - (2y^3 + y)dy$

$$\int_C \vec{F}d\vec{r} = \int_0^1 x^2 dx - \int_0^1 (2y^3 + y)dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2}\right]_0^1$$

$$= \left(\frac{1}{3} - 0\right) - \left[\left(\frac{2}{4} + \frac{1}{2}\right) - (0+0)\right]$$

$$= \frac{1}{3} - \left[\frac{1}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{3} - 1 = \frac{-2}{3}$$

$$\therefore \text{Work done} = \int_C \vec{F}.d\vec{r} = \frac{2}{3}$$

Problem 5 Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential.

Solution:

Given $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2\end{vmatrix}$$

$$=\vec{i}[0-0]-\vec{j}\left[3z^2-3z^2\right]+\vec{k}[2y\cos x-2y\cos]$$

$$=0\vec{i}-0\vec{j}+0\vec{k}=0$$

$\nabla\times\vec{F}=0$

Hence $\vec{F}$ is irrotational

$\vec{F}=\nabla\phi$

$$\left(y^2\cos x+z^3\right)\vec{i}\left(2y\sin x-4\right)\vec{j}+3xz^2\vec{k}=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{yz}$$

Equating the coefficient $\vec{i},\vec{j},\vec{k}$

$$\frac{\partial\phi}{\partial x}=y^2\cos x+z^3\Rightarrow\int\partial\phi=\int y^2\cos x+z^3dx$$

$$\phi_1=y^2\sin x+z^3x+C_1$$

$$\frac{\partial\phi}{\partial x}=2y\sin x-4\Rightarrow\int\partial\phi=\int\left(2y\sin x-4\right)dy$$

$$\phi_2=2\left(\sin x\right)\frac{y^2}{2}-4y+C_2$$

$$\frac{\partial\phi}{\partial x}=3xz^2\Rightarrow\int\partial\phi=\int 3xz^2dy$$

$$\phi_3=3x\frac{z^3}{3}+C_3$$

$$\therefore\phi=y^2\sin x+xz^3-4y+C$$

Problem 6 If $\vec{F}=3xy\vec{i}-y^2\vec{j}$ evaluate $\int\vec{F}.\vec{dr}$ when $C$ is curve in the $xy$ plane $y=2x^2$, from $(0,0)$ to $(1,2)$

Solution:

$\vec{F}=3xy\vec{i}-y^2\vec{j}$

$\vec{dr}=dx\vec{i}+dy\vec{j}+dz\vec{k}$

$\vec{F}.\vec{dr}=3xydx-y^2dy$

Given $y=2x^2$

$dy=4xdx$

$$\therefore\vec{F}.\vec{dr}=3x(2x^2)dx-\left(2x^2\right)^2 4x\,dx$$

$$=6x^3dx-4x^4(4x)dx$$

$$=6x^3dx-16x^5dx$$

$$\int_C \vec{F}d\vec{r} = \int_0^1 \left(6x^3 - 16x^5\right)dx$$

$$= \left[6\frac{x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = -\frac{-7}{6}.$$

Problem 7 Find $\int_C \vec{F}.d\vec{r}$ when $\vec{F} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$ where the cure $C$ is the rectangle in the $xy$ plane bounded by $x = 0,\ x = a,\ y = b,\ y = 0$.

Solution:

Given $\vec{F}\left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}d\vec{r} = (x^2 + y^2)dx - 2xy\,dy$

$C$ is the rectangle $OABC$ and $C$ consists of four different paths.

OA (y = 0)

AB (x = a)

BC (y = b)

CO (x = 0)

$$\therefore \int_C \vec{F}.d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along

$OA,\quad y = 0,\quad dy = 0$

$AB,\quad x = a,\quad dx = 0$

$BC,\quad y = b,\quad dy = 0$

$CO,\quad x = 0,\quad dx = 0$

$$\therefore C\int_C \vec{F}.d\vec{r} = \int_{OA} x^2dx \int_{AB} -2aydy + \int_{BC}\left(x^2 + b^2\right)dx + \int_{CO} 0$$

$$= \int_0^a x^2dx - 2a\int_0^b ydy + \int_a^0 \left(x^2 + b^2\right)dx$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_o^b + \left[\frac{x^3}{3} + b^2x\right]_a^o$$

$$= \left(\frac{a^3}{3} - 0\right) - 2a\left(\frac{b^2}{2} - 0\right) + \left((0+0) - \left(\frac{a^3}{3} + ab^2\right)\right) = -2ab^2.$$

Problem 8 If $\vec{F} = \left(4xy - 3x^2z^2\right)\vec{i} + 2x^2\vec{j} - 2x^3zk$ check whether the integral $\int_C \vec{F}.d\vec{r}$ is independent of the path $C$.

Solution:

Given

$$\vec{F}=\left(4xy-3x^2z^2\right)\vec{i}+2x^2\vec{j}-2x^3zk$$

$$d\vec{r}=dx\vec{i}+dy\vec{j}+dz\overleftarrow{k}$$

$$\int_C \vec{F}.d\vec{r}=\int_C\left(4xy-3x^2z^2\right)dx+\int_C 2x^2dy-\int_C 2x^3zdz$$

This integral is independent of path of integration if

$$\vec{F}=\nabla\phi\Rightarrow\nabla\times\vec{F}=0$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 4xy-3x^2z^2 & 2x^2 & -2x^3z\end{vmatrix}$$

$$=\vec{i}\left(0,0\right)-j\left(-6x^2z+6x^2z\right)+\vec{k}\left(4x-4x\right)$$

$$=0\vec{i}-0\vec{i}-0\vec{j}+0\vec{k}=0.$$

Hence the line integral is independent of path.

Problem 9 Verify Green's Theorem in a plane for $\int_C\left(x^2(1+y)dx+(y^3+x^3)dy\right)$ where $C$ is the square bounded $x=\pm a, y=\pm a$

Solution:

Let $P=x^2(1+y)$

$$\frac{\partial P}{\partial y}=x^2$$

$$Q=y^3+x^3$$

$$\frac{\partial Q}{\partial x}=3x^2$$

By green's theorem in a plane

$$\int_C\left(Pdx+Qdy\right)=\iint_C\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

Now $\iint_R\left(\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)dx\,dy$

$$=\int_{-a}^{a}\int_{-a}^{a}\left(3x^2-x^2\right)dx\,dy$$

$$=\int_{-a}^{a}dy\int_{-a}^{a}2x^2dx$$

$$=\left(y\right)_{-a}^{a}\left(\frac{2x^3}{3}\right)_{-a}^{a}$$

$$=\left(a+a\right)\frac{2}{3}\left(a^3+a^3\right)$$

$$=\frac{8a^4}{3}-(1)$$

Now $\int_C (Pdx+Qdy)=\int_{AB}+\int_{BC}+\int_{CD}+\int_{DA}$

Along $AB,\ y=-a,\ dy=0$

$X$ varies from $-a$ to $a$

$$\int_{AB}(Pdx+Qdy)=\int_{-a}^{a}\left(x^2(1+y)dx+\left(x^3+y^3\right)dy\right)$$

$$=\int_{-a}^{a}x^2(1-a)dx+0$$

$$=(1-a)\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$=\left(\frac{1-a}{3}\right)\left(a^3+a^3\right)=\frac{2a^3}{3}-\frac{2a^4}{3}$$

Along $BC$

$x=a,\ dx=0$

$Y$ varies from $=-a$ to $a$

$$\int_{BC}(Pdx+Qdy)=\int_{-a}^{a}\left(x^2(1+y)dx+\left(x^3+y^3\right)dy\right)$$

$$=\int_{-a}^{a}(a^3+y^3)dy$$

$$=\left[a^3y+\frac{y^4}{4}\right]_{-a}^{a}$$

$$=\left(a^4+\frac{a^4}{4}\right)-\left(-a^4+\frac{a^4}{4}\right)=2a^4$$

Along $CD$

$y=a,\ dy=0$

$X$ varies from $a$ to $-a$

$$\int_{CD}(Pdx+Qdy)=\int_{a}^{-a}\left(x^2(1+y)dx+\left(x^3+y^3\right)dy\right)$$

$$=\int_{a}^{-a}x^2(1+a)dx$$

$$=(1+a)\left(\frac{x^3}{3}\right)_{a}^{-a}dx$$

$$=(1+a)\left[\frac{-a^3-a^3}{3}\right]$$

$$= -\frac{2a^3}{3} - \frac{2a^4}{3}$$

Along $DA$,

$x = -a,\ dx = 0$

$Y$ Varies from $a$ to $-a$

$$\int_{DA} (Pdx + Qdy) = \int_{a}^{-a} \left( x^2(1+y)dx + \left(x^3 + y^3\right)dy \right)$$

$$= \int_{+a}^{-a} \left( a^2(1+y)dx + \left(y^3 - a^3\right)dy \right)$$

$$= \left[ \frac{y^4}{4} - a^3 y \right]_a^a$$

$$= \left( \frac{a^4}{4} + a^4 \right) - \left( \frac{a^4}{4} - a^4 \right) = 2a^4$$

$$\int_C (Pdx + Qdy) = \frac{2a^3}{3} - \frac{2a^4}{3} + 2a^4 - \frac{2a^3}{3} - \frac{2a^4}{3} + 2a^4$$

$$= 4a^4 - \frac{4}{3}a^4$$

$$= \frac{8a^4}{3} ......(2)$$

From (1) and (2)

$$\int_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \frac{8a^4}{3}.$$

Hence Green's theorem verified.

Problem 10 Verify Green's theorem in a plane for $\int_C \left(3x^2 - 8y^2\right)dx + (4y - 6xy)dy$ where $C$ is the boundary of the region defined by $x = y^2,\ y = x^2$.

Solution:

Green's theorem states that

$$\int_C udx + vdy = \iint_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

$$\int_C Pdx + Qdy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy$$

Given $\int_C \left(3x^2 - 8y^2\right)dx + (4y - 6xy)dy$

$P = 3x^2 - 8y^2$

$$\frac{\partial P}{\partial y} = -16y$$

$$Q = 4y - 6xy$$

$$\frac{\partial Q}{\partial x} = -6y$$

Evaluation of $\int_C Pdx + Qdy$

(i) Along $OA$

$$y = x^2 \Rightarrow dy = 2xdx$$

$$\int_{OA} Pdx + Qdy = \int_{OA} \left(3x^2 - 8x^4\right)dx + \left(4x^2 - 6x^3\right)2xdx$$

$$= \int_0^1 \left(3x^2 - 8x^4 + 8x^3 - 12x^4\right)dx$$

$$= \int_0^1 \left(-20x^4 + 8x^3 + 3x^2\right)dx$$

$$= \left[-20\frac{x^5}{5} + 8\frac{x^4}{4} + \frac{3x^3}{3}\right]_0^1$$

$$= \frac{-20}{5} + \frac{8}{5} + \frac{3}{3}$$

$$= -4 + 2 + 1 = -1$$

Along $AO$

$$y^2 = x \Rightarrow 2ydy = dx$$

$$\int_{Ao} Pdx + Qdy = \int_{Ao} \left(3y^4 - 8y^2\right)2y\,dy + \left(4y - 6y^3\right)dy$$

$$= \int_{AO} (6y^5 - 16y^3 + 4y - 6y^3)dy$$

$$= \int_1^0 \left(6y^5 - 22y^3 + 4y\right)dy$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + \frac{4y^2}{2}\right]_1^0$$

$$= \left[y^6 - \frac{11}{2}y^4 + 2y^2\right]_1^0 = \frac{5}{2}$$

$$\therefore \int_C Pdx + Qdy = \int_{OA} + \int_{AO} = -1 + \frac{5}{2} = \frac{3}{2} \rightarrow (1)$$

Evaluation of $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy = \iint_R \left(-6y + 16y\right)dxdy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\,dx\,dy = \int_0^1 [10\,xy]_{x=y}^{x=\sqrt{y}}\,dy$$

$$= \int_0^1 10y\left(\sqrt{y}-y^2\right)dy$$

$$= 10\int_0^1 \left(y^{\frac{3}{2}}-y^3\right)dy$$

$$= 10\left[\frac{y^{\frac{5}{2}}}{\frac{5}{2}}-\frac{y^4}{4}\right]_0^1$$

$$= 10\left[\frac{2}{5}-\frac{1}{4}\right]$$

$$= 10\left[\frac{8-5}{20}\right]$$

$$= \frac{30}{20} = \frac{3}{2} \rightarrow (2)$$

For (1) and (2)

Hence Green's theorem is verified.

Problem 11 Verify Gauss divergence theorem for $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9,\ Z = 0$ and $Z = 2$.

Solution:

Gauss divergence theorem is

$$\iint_S \vec{F}.\vec{n}\ \ ds = \iiint_V div\,\vec{F}\,dV$$

$$div\,\vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}\left(z^2\right) = 2z$$

$$\iiint_V div\,\vec{F}\ dV = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^2 2z\,dzdy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left(\frac{z^2}{2}\right)_0^2 dydx$$

$$= 4\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dydx$$

= 4 (Area of the circular region)

$$= 4\left(\pi\,(3)^2\right)$$

$$= 36\pi\ldots\ldots\ldots\ldots\ldots..(1)$$

$$\iint_S \vec{F}.\vec{n}\ ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$S_1$ is the bottom of the circular region, $S_2$ is the top of the circular region and $S_3$ is the cylindrical region

On $S_1, \vec{n} = -\vec{k}, \quad ds = dxdy, \quad z = 0$

$$\iint_{S_1} \vec{F}.\vec{n}\, ds = \iint -z^2 dxdy = 0$$

On $S_2$, $\vec{n} = \vec{k}$, $ds = dxdy$, $z = 2$

$$\iint_{s_2} \vec{F}.\vec{n}\ ds = \iint z^2 dx\, dy$$

$$= 4\iint dxdy$$

$= 4$ (Area of circular region)

$$= 4\left(\pi (3)^2\right) = 36\pi$$

On $S_3$, $\phi = x^2 + y^2 - 9$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4\left(x^2+y^2\right)}}$$

$$= \frac{x\vec{i} + y\vec{j}}{3}$$

$$\iint_{S_3} \vec{F}.\vec{n}\, ds = \iint \left(y\vec{i} + x\vec{j} + z^2\vec{k}\right)\left(\frac{x\vec{i} + y\vec{j}}{3}\right) ds$$

$$= \iint \frac{yx + yx}{3} ds = \frac{2}{3}\iint_S xy\, ds$$

Let $x = 3\cos\theta, \quad y = 3\sin\theta$

$ds = 3\ d\theta dy$

$\theta$ varies from 0 to $2\pi$

$z$ varies from 0 to $2\pi$

$$= \frac{2}{3}\int_0^2 \int_0^{2\pi} (9\sin\theta\cos\theta) 3\, d\theta\, dz$$

$$= \frac{18}{2}\int_0^2 \int_0^{2\pi} \sin 2\theta\, d\theta\, dz$$

$$= 9\int_0^2 \left(-\frac{\cos 2\theta}{2}\right)_0^{2\pi} dz$$

$$= -\frac{9}{2}\int_0^2 [1-1]\ dz = 0$$

$$\therefore \iint_S \vec{F}.\vec{n}\, ds = 0 + 36\pi + 0 = 36\pi \ldots\ldots\ldots\ldots..(2)$$

from (1) and (2)

$$\int_C \vec{F}.\vec{n}\ ds = \iiint_V div\ \vec{F} dV$$

Problem 12 Verify Stoke's theorem for the vector field defined by $\vec{F} = \left(x^2 - y^2\right)\vec{i} - 2xy\vec{j}$ in the rectangular region in the $xy$ plane bounded by the lines $x = 0,\ x = a,\ y = 0,\ y = b.$

Solution:

$$\vec{F} = \left(x^2 - y^2\right)\vec{i} - 2xy\vec{j}$$

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\vec{n}\,ds$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y] = -4y\vec{k}$$

As the region is in the $xy$ plane we can take $\vec{n} = \vec{k}$ and $ds = dxdy$

$$\iint_S curl\,\vec{F}.\vec{n}\,ds = \iint -4y\vec{k}.\vec{k}\,dx\,dy$$

$$= -4\int_0^b\int_0^a y\ dx\,dy$$

$$= -4\left(\frac{y^2}{2}\right)_0^b (x)_0^a$$

$$= -2ab^2..............(1)$$

$$\int_C \vec{F}.dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA

$y = 0 \Rightarrow dy = 0,$

$x$ varies from 0 to $a$

$$\therefore \int_{OA} = \int_0^a \left(x^2 + y^2\right)dx - 2xy\ dy$$

$$= \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along AB

$x = a \Rightarrow dx = o,\ \ y$ varies from 0 to $b$

$$\int_{AB} = \int_0^b \left(a^2+y^2\right).0 - 2ay\ dy$$

$$= -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2$$

Along BC

$y = b,\ \ dy = 0$

$x$ varies from $a$ to 0

$$\int_{BC} = \int_a^0 \left(x^2+b^2\right)dx - 0$$

$$= \left(\frac{x^3}{3}+b^2x\right)_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

Along CO

$x = 0,\ \ dx = 0,$

$y$ varies from $b$ to 0

$$\int_{CO} = \int_b^0 \left(0+y^2\right)0 + 0 = 0$$

$$\therefore \int_c \vec{F}.d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$= -2ab^2 .........(2)$$

For (1) and (2)

$$\int_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\vec{n}ds$$

Here Stoke’s theorem is verified.

Problem 13 Find $\iint_S \vec{F}.\vec{n}ds$ if $\vec{F} = \left(x+y^2\right)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ where $S$ is the surface in the plane $2x+y+2z=6$ in the first octant.

Solution:

Let $\phi = 2x+y+2z-6$ be the given surface

Then $\nabla\phi = 2\vec{i}+\vec{j}+2\vec{k}$

$$\left|\nabla\phi\right| = \frac{2\vec{i}+\vec{j}+2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{3}$$

$\therefore$ The unit outward normal $\vec{n}$ to the surface $S$ is $\hat{n} = \frac{1}{3}\left[2\vec{i}+\vec{j}+2\vec{k}\right]$

Let $R$ be the projection of $S$ on the $xy$ plane

$$\therefore \iint_S \vec{F}.\vec{n}\, ds = \iint_{R_1} \vec{F}.\vec{n}\frac{dx\,dy}{\left|\vec{n}.\vec{k}\right|}$$

$$\vec{n}.\vec{k} = \frac{1}{3}\left(2\vec{i}+\vec{j}+2\vec{k}\right).\vec{k} = \frac{2}{3}$$

$$\vec{F}.\vec{n} = \left[\left(x+y^2\right)\vec{i}-2x\vec{j}+2yz\vec{k}\right].\frac{1}{3}\left(2\vec{i}+\vec{j}+2\vec{k}\right)$$

$$= \frac{2}{3}\left(x+y^2\right)-\frac{2}{3}x+\frac{4}{3}yz$$

$$= \frac{2}{3}\left(y^2+2yz\right)$$

$$= \frac{2}{3}y\left(y+2z\right)$$

$$= \frac{2}{3}y\left[y+6-y-2x\right]$$

$$= \frac{2}{3}y\left[6-2x\right]$$

$$= \frac{4}{3}y\left(3-x\right)$$

$$\therefore \iint_S \vec{F}.\vec{n}\, ds = \iint_S \vec{F}.\vec{n}\frac{dx\,dy}{\left|\vec{n}.\vec{k}\right|}$$

$$= \iint \frac{4}{3}y\left(3-x\right)\frac{dxdy}{2/3}$$

$$= 2\iint_{R_1}\left(3-x\right)dx\,dy$$

$$= 2\int_0^3 \int_0^{6-2-x}\left(3-x\right)\, dx\ dy$$

$$= 2\int_0^3 \left(3-x\right)\left(\frac{y^2}{2}\right)_0^{6-2-x} dx$$

$$= 4\int_0^3 \left(3-x\right)^3 dx$$

$$= 4\left[\frac{(3-x)^4}{-4}\right]_0^3$$

$$= 81 \text{ units.}$$

Problem 14 Evaluate $\int_C \left[\left(x+y\right)dx+\left(2x-3xy\right)\right]$ where C is the boundary of the triangle with vertices $\left(2,0,0\right),\left(0,3,0\right)\&\left(0,0,6\right)$ using Stoke’s theorem.

Solution:

Stoke's theorem is $\int_C \vec{F}.\vec{dr} = \iint_S curl\vec{F}.\hat{n}ds$ where $S$ is the surface of the triangle and $\hat{n}$ is the unit vector normal to surface $S$.

Given $\vec{F}.\vec{dr} = (x+y)dx + (2x-z)dy + (y+z)dz$

$\therefore \vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

$\vec{dr} = \vec{i}\,dx + \vec{j}dy + \vec{k}\,dz$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}(1+1) - \vec{j}(0-0) + \vec{k}(2-1)$$

$curl\,\vec{F} = 2\vec{i} + \vec{k}$

Equation of the plane ABC is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$3x + 2y + z = 6$

Let $\phi = 3x + 2y + z - 6$

$\nabla\phi = 3\vec{i} + 2\vec{j} + \vec{k}$

Unit normal vector to the surface ABC $(or\,\phi)$ is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\vec{i} + 2\overleftarrow{j} + \vec{k}}{\sqrt{14}}$$

$$curl\,\vec{F}.\hat{n} = (2\vec{i} + \vec{k}).\left(\frac{3\vec{i} + 2\overleftarrow{j} + \vec{k}}{\sqrt{14}}\right) = \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$$

Hence $\iint_S curl\vec{F}.\hat{n}ds = \iint_S \frac{7}{\sqrt{14}} ds$

$= \frac{7}{\sqrt{14}} \iint_R \frac{dxdy}{|\hat{n}\vec{k}|}$ where $R$ is the projection of surface ABC on XOY plane

$= \frac{7}{\sqrt{14}} \iint_R \frac{dxdy}{\frac{1}{\sqrt{14}}}$ $\left(\because \vec{n}\,\vec{k} = \left(\frac{3i+2j+k}{\sqrt{14}}\right).k = \frac{1}{\sqrt{14}}\right)$

$= 7\iint_R dxdy$

$= 7 \times (Area\ of\ \Delta^{le}\ OAB)$

$= 7 \times 3 = 21.$

**Problem 15** Verify Stoke's theorem for $\vec{F}=(y-z)\vec{i}+yz\vec{j}-xz\vec{k}$ where $S$ is the surface bounded by the planes $x=0,\ x=1,\ y=0,\ y=1,\ z=0$ and $z=1$ above the XOY plane.

## Solution:

Stoke's theorem is $\int\limits_C \vec{F}.d\vec{r}=\iint\limits_S \nabla\times\vec{F}.\hat{n}ds$

$$\vec{F}=(y-z)\vec{i}+yz\vec{j}-xz\vec{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y-z & yz & -xz\end{vmatrix}$$

$$=-y\vec{i}+(z-1)\vec{j}-\vec{k}$$

$$\iint\limits_S \nabla\times\vec{F}.\hat{n}\ ds=\iint\limits_{S_1}+\iint\limits_{S_2}+\iint\limits_{S_3}+\iint\limits_{S_4}+\iint\limits_{S_5}$$

$\iint\limits_{S_6}$ is not applicable, since the given condition is above the XOY plane.

$$\iint\limits_{S_1}=\iint\limits_{AEGD}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right].\vec{i}dydz$$

$$=\iint\limits_{AEGD}-ydy\,dz$$

$$=\int_0^1\int_0^1-y\,dy\,dz=\int_0^1\left[-\frac{y^2}{2}\right]_0^1dz$$

$$=-\frac{1}{2}(z)_0^1=-\frac{1}{2}$$

$$\iint\limits_{S_2}=\iint\limits_{OBFC}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right].\left(-\vec{i}\right)dydz$$

$$=\int_0^1\int_0^1 y\,dy\,dz=\int_0^1\left[\frac{y^2}{2}\right]_0^1dz=\frac{1}{2}$$

$$\iint\limits_{S_3}=\iint\limits_{EBFG}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\vec{j}dxdz$$

$$=\int_0^1\int_0^1(z-1)\,dx\,dz=\int_0^1(xz-x)_0^1\,dz$$

$$=\left(\frac{z^2}{2}-z\right)_0^1=\frac{1}{2}-1=-\frac{1}{2}$$

$$\iint\limits_{S_4}=\iint\limits_{OADC}\left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\left(-\vec{j}\ \right)dxdz$$

$$=\int_0^1\int_0^1(-z+1)\,dx\,dz$$

$$=\int_0^1(-xz+x)_0^1=\int_0^1(-z+1)\,dz$$

$$=\left(\frac{-z^2}{2}+z\right)_0^1=\frac{-1}{2}+1=\frac{1}{2}$$

$$\iint_{S_5}=\iint_{DGFC}\left(-y\vec{i}+(z-1)\vec{j}-\vec{k}\right).\vec{k}dxdy$$

$$=\int_0^1\int_0^1(-1)dxdy=\int_0^1(-x)_0^1\,dy$$

$$=\int_0^1(-1)\,dy=(-y)_0^1=-1$$

$$\iint_S=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}+\iint_{S_4}+\iint_{S_5}$$

$$=-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-1=-1$$

$$L.H.S=\int_C\vec{F}.\vec{dr}=\int_{OA}+\int_{AE}+\int_{EB}+\int_{BO}$$

$$\int_{OA}=\int_{OA}(y-z)\,dx+yzdy-xzdz$$

$$=\int_{OA}0=0\qquad[\because y=0,\ z=0,\ dy=0,\ dz=0]$$

$$\int_{AE}=\int_{AE}(y-z)\,dx+yzdy-xzdz$$

$$=\int_{AE}0=0\qquad[\because x=1,\ z=0,\ dx=0,\ dz=0]$$

$$\int_{EB}=\int_{EB}(y-z)\,dx+yzdy-xzdz$$

$$=\int_1^0 1\,dx\qquad(y=1,\ z=0,)$$

$$=[x]_1^0=0-1=-1$$

$$\int_{BO}=\int_{BO}(y-z)\,dx+yzdy-xzdz$$

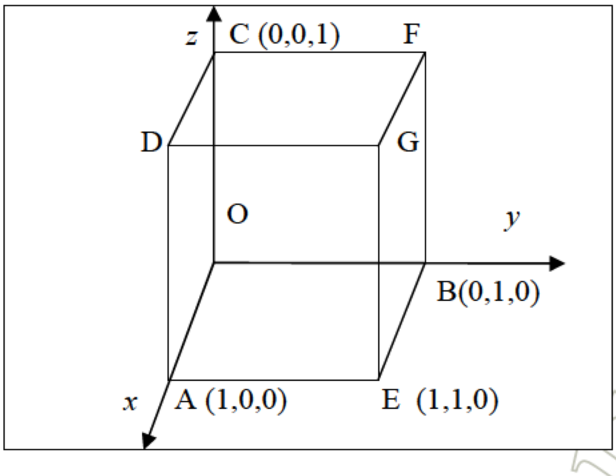
$$=\int_{BO}0=0\qquad[x=0,\ z=0]$$

$$=0$$

$$\therefore\int_C=\int_{OA}+\int_{AE}+\int_{EB}+\int_{BO}$$

$$=0+0-1+0=-1$$

∴ L.HS = R.HS.
Hence Stoke's theorem is verified.



# UNIT III ANALYTIC FUNCTIONS

## Part-A

**Problem 1** State Cauchy – Riemann equation in Cartesian and Polar coordinates.

**Solution:**

Cartesian form:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Polar form:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

**Problem 2** State the sufficient condition for the function f(z) to be analytic.

**Solution:**

The sufficient conditions for a function $f(z)=u+iv$ to be analytic at all the points in a region $R$ are

(1) $u_x = v_y,\quad u_y = -v_x$

(2) $u_x, u_y, v_x, v_y$ are continuous functions of $x$ and $y$ in region $R$.

**Problem 3** Show that $f(z)=e^z$ is an analytic Function.

**Solution:**

$$f(z)=u+iv=e^z$$
$$=e^{x+iy}$$
$$=e^x e^{iy}$$
$$=e^x\left[\cos y+i\sin y\right]$$

$u=e^x\cos y,\ v=e^x\sin y$

$u_x=e^x cosy,\ v_x=e^x siny$

$u_y=-e^x siny,\ v_y=e^x cosy$

i.e., $u_x=v_y,\ u_y=-v_x$

Hence C-R equations are satisfied.

$\therefore f(z)=e^z$ is analytic.

**Problem 4** Find whether $f(z)=\bar{z}$ is analytic or not.

**Solution:**

Given $f(z)=\bar{z}=x-iy$

i.e., $u=x,\ v=-y$

$$\frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial x}=0$$

$$\frac{\partial u}{\partial y}=0,\ \frac{\partial v}{\partial y}=-1$$

$\therefore u_x \neq v_y$

C-R equations are not satisfied anywhere.

Hence $f(z)=\bar{z}$ is not analytic.

## Problem 5 State any two properties of analytic functions

## Solution:

(i) Both real and imaginary parts of any analytic function satisfy Laplace equation.

i.e., $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ or $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$.

(ii) If $w=u+iv$ is an analytic function, then the curves of the family $u(x,y)=c$, cut orthogonally the curves of the family $v(x,y)=c$.

## Problem 6 Show that $f(z)=|z|^2$ is differentiable at $z=0$ but not analytic at $z=0$.

## Solution:

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{z\to 0}\frac{|z|^2}{z}=\lim_{z\to 0}\frac{z\bar{z}}{z}=\lim_{z\to 0}\bar{z}=0$$

$\therefore f(z)$ is differentiable at $z=0$.

Let $z=x+iy$

$\bar{z}=x-iy$

$|z|^2=z\bar{z}=(x+iy)(x-iy)=x^2+y^2$

$f(z)=x^2+y^2+i0$

$u=x^2+y^2,\ v=0$

$u_x=2x,\ v_x=0$

$u_y=2y,\ v_y=0$

The C-R equation $u_x=v_y$ and $u_y=-v_x$ are not satisfied at points other than z = 0.

Therefore $f(z)$ is not analytic at points other than $z=0$. But a function can not be analytic at a single point only. Therefore $f(z)$ is not analytic at z = 0 also.

## Problem 7 Determine whether the function $2xy+i(x^2-y^2)$ is analytic.

## Solution:

Given $f(z)=2xy+i(x^2-y^2)$

i.e., $u=2xy$ , $v=x^2-y^2$

$$\frac{\partial u}{\partial x} = 2y, \quad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2x, \frac{\partial v}{\partial y} = -2y$$

$\therefore u_x \neq v_y$ and $u_y \neq -v_x$

C-R equations are not satisfied.

Hence $f(z)$ is not analytic function.

## Problem 8 Show that $v = sinhx\, cosy$ is harmonic

## Solution:

$v = sinhxcosy$

$$\frac{\partial v}{\partial x} = coshxcosy, \frac{\partial v}{\partial y} = -sinhxsiny$$

$$\frac{\partial^2 v}{\partial x^2} = sinhxcosy, \frac{\partial^2 v}{\partial y^2} = -sinhycosy$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = sinhxcosy - sinhycosy = 0$$

Hence $v$ is a harmonic function.

## Problem 9 Construct the analytic function $f(z)$ for which the real part is $e^x cosy$.

## Solution:

$u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

Assume $\frac{\partial u}{\partial x}(x, y) = \phi_1(z, 0)$

$\therefore \phi_1(z, 0) = e^z$

$$\frac{\partial u}{\partial y} = -e^x siny$$

Assume $\frac{\partial u(x, y)}{\partial y} = \phi_2(z, 0)$

$\therefore \phi_2(z, 0) = 0$

$$f(z) = \int f^{'}(z) dz = \int \phi_1(z, 0)\, dz - i\int \phi_2(z, 0)\, dz$$

$$= \int e^z\, dz - i\int 0$$

$f(z) = e^z + C$.

## Problem 10 Prove that an analytic function whose real part is constant must itself be a constant.

Solution:

Let $f(z)=u+iv$ be an analytic function

$\Rightarrow u_x=v_y,\ u_y=-v_x$.............(1)

Given

$u=c(\text{a constant})$

$u_x=0,\ u_y=0$

$\Rightarrow v_y=0\ \&\ v_x=0\ by(1)$

We know that $f(z)=u+iv$

$f'(z)=u_x+iv_x$

$f'(z)=0+i0$

$f'(z)=0$

Integrating with respect to $z$, $f(z)=C$

Hence an analytic function with constant real part is constant.

## Problem 11 Define conformal mapping

Solution:

A transformation that preserves angle between every pair of curves through a point both in magnitude and sense is said to be conformal at that point.

## Problem 12 If $w=f(z)$ is analytic prove that $\frac{dw}{dz}=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}$ where $w=u+iv$ and prove that $\frac{\partial^2 w}{\partial z\partial \bar{z}}=0$

Solution:

$w=u(x,y)+iv(x,y)$ is an analytic function of $z$.

As $f(z)$ is analytic we have $u_x=v_y$ , $u_y=-v_x$

Now $\frac{dw}{dz}=f^{'}(z)=u_x+iv_x=v_y-iu_y=-i(u_y+iv_y)$

$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\left[\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right]$$

$$=\frac{\partial}{\partial x}(u+iv)=-i\frac{\partial}{\partial y}(u+iv)$$

$$=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}$$

W.K.T. $\frac{\partial w}{\partial \bar{z}} = 0$

$\therefore \frac{\partial^2 w}{\partial z \, \partial \bar{z}} = 0$

**Problem 13** Define bilinear transformation. What is the condition for this to be conformal?

**Solution:**

The transformation $w = \frac{az+b}{cz+d}, \; ad - bc \neq 0$ where $a, b, c, d$ are complex numbers is called a bilinear transformation.

The condition for the function to be conformal is $\frac{dw}{dz} \neq 0$.

**Problem 14** Find the invariant points or fixed points of the transformation $w = 2 - \frac{2}{z}$.

**Solution:**

The invariant points are given by $z = 2 - \frac{2}{z}$

i.e., $z = 2 - \frac{2}{z}$

$z^2 = 2z - 2$

$z^2 - 2z + 2 = 0$

$$z = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2}$$

$= 1 \pm i$

The invariant points are $z = 1 + i, \; 1 - i$

**Problem 15** Find the critical points of (i) $w = z + \frac{1}{z}$ (ii) $w = z^3$.

**Solution:**

(i). Given $w = z + \frac{1}{z}$

For critical point $\frac{dw}{dz} = 0$

$\frac{dw}{dz} = 1 - \frac{1}{z^2} = 0$

$z = \pm i$ are the critical points

(ii). Given $w = z^3$

$$\frac{dw}{dz} = 3z^2 = 0$$

$$z = 0$$

$\therefore\ z = 0$ is the critical point.

## Part-B

Problem 1 Determine the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\phi_1(z,0) = 3z^2 + 6z$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = 6xy - 6y$$

$$\phi_2(z,0) = 0$$

By Milne Thomason method

$$f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$$

$$= \int (3z^2 + 6z)\,dz - 0$$

$$= 3\frac{z^3}{3} + 6\frac{z^2}{2} + C = z^3 + 3z^2 + C$$

Problem 2 Find the regular function $f(z)$ whose imaginary part is $v = e^{-x}[x\cos y + y\sin y]$

Solution:

$$v = e^{-x}(x\cos y + y\sin y)$$

$$\phi_2(x, y) = \frac{\partial v}{\partial x} = e^{-x}[\cos y] + (x\cos y + y\sin y)(-e^{-x})$$

$$\phi_2(z,0) = e^{-z} + (z)(-e^{-z}) = e^{-z} - ze^{-z} = e^{-z}(1-z)$$

$$\phi_1(x, y) = \frac{\partial u}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y(1)]$$

$$\phi_1(z,0) = e^{-z}[0+0+0] = 0$$

By Milne's Thomson Method

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

$$= \int 0\,dz + i\int (1-z)e^{-z}dz$$
$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right]-(-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right]+C$$
$$= i\left[-(1-z)e^{-z}+e^{-z}\right]+C$$
$$= i\left[-e^{-z}+ze^{-z}+e^{-z}\right]+C = i\left[ze^{-z}\right]+C$$

## Problem 3

Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

## Solution:

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$
$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$
$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$
$$= \frac{2\cos 2z - 2(1+\cos 2z)}{1-\cos 2z} = \frac{2\cos 2z - 2 - 2\cos 2z}{1-\cos 2z}$$
$$= \frac{-2}{1-\cos 2z} = -\frac{1}{\left(\frac{1-\cos 2z}{2}\right)}$$
$$= -\frac{1}{\sin^2 z} = -\cos ec^2 z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x[2\sinh 2y]}{(\cosh 2y - \cos 2x)^2}$$
$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\varphi_2(z,0) = 0$$

By Milne's Thomson method

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$
$$= \int -cosec^2 z\,dz - 0 = cot\,z + C$$

Problem 4 Prove that the real and imaginary parts of an analytic function $w = u + iv$ satisfy Laplace equation in two dimensions viz $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

Solution:

Let $f(z) = w = u + iv$ be analytic

To Prove: $u$ and $v$ satisfy the Laplace equation.

i.e., To prove: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Given: $f(z)$ is analytic

$\therefore$ $u$ and $v$ satisfy C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ….. (1)

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ….. (2)

Diff. (1) p.w.r to $x$ we get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ….. (3)

Diff. (2) p.w.r. to $y$ we get $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$ ….. (4)

The second order mixed partial derivatives are equal

i.e., $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$

(3) + (4) $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$

$\therefore$ $u$ satisfies Laplace equation

Diff. (1) p.w.r to $y$ we get $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ … (5)

Diff. (2) p.w.r. to $x$ we get $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$ … (6)

(5) + (6) $\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0$

i.e., $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore$ $v$ Satisfies Laplace equation

Problem 5 If f(z) is analytic, prove that $\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)|f(z)|^2 = 4.|f'(z)|^2$

Solution:

Let f(z) = u + iv be analytic.

Then $u_x = v_y$ and $u_y = -v_x$ (1)

Also $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)

Now $|f(z)|^2 = u^2 + v^2$ and $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u.u_x + 2v.v_x$$

and $$\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right] \quad (3)$$

Similarly $$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right] \quad (4)$$

Adding (3) and (4)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) + v_x^2 + v_y^2 + v(v_{xx} + v_{yy})\right]$$

$$= 2\left[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

$$= 4.|f'(z)|^2$$

Problem 6 Prove that $\nabla^2 \left|\operatorname{Re} f(z)\right|^2 = 2\left|f'(z)\right|^2$

Solution:

Let $f(z) = u + iv$

$$\left|\operatorname{Re} f(z)\right|^2 = u^2$$

$$\frac{\partial}{\partial x}\left(u^2\right) = 2uu_x$$

$$\frac{\partial^2}{\partial x^2}\left(u^2\right) = \frac{\partial}{\partial x}\left(2uu_x\right)$$

$$= 2\left[uu_{xx} + u_x u_x\right]$$

$$= 2\left[uu_{xx} + u_x^{\,2}\right]$$

$$\frac{\partial^2}{\partial y^2}\left(u^2\right) = 2\left[uu_{yy} + u_y^2\right]$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(u^2\right) = 2\left[u\left(u_{xx} + u_{yy}\right) + u_x^2 + u_y^2\right]$$

$$= 2\left[u(0) + u_x^2 + u_y^2\right]$$

$$= 2\left|f'(z)\right|^2$$

**Problem 7** Find the analytic function $f(z)=u+iv$ given that $2u+v=e^{x}[\cos y-\sin y]$

**Solution:**

Given $2u+v=e^{x}[\cos y-\sin y]$

$f(z)=u+iv \ldots\ldots\ldots\ldots\ldots(1)$

$if(z)=iu-v \ldots\ldots\ldots\ldots\ldots(2)$

$(1)\times 2\Rightarrow 2f(z)=2u+i2v \ldots\ldots\ldots\ldots(3)$

$(3)-(2)\Rightarrow(2-i)f(z)=(2u+v)+i(2v-u) \ldots\ldots\ldots\ldots(4)$

$F(z)=U+iV$

$\therefore 2u+v=U=e^{x}[\cos y-\sin y]$

$$\phi_1(x,y)=\frac{\partial U}{\partial x}=e^{x}\cos y-e^{x}\sin y$$

$\phi_1(z,o)=e^{z}$

$$\phi_2(x,y)=\frac{\partial u}{\partial x}=-e^{x}\sin y-e^{x}\cos y$$

$\phi_2(z,o)=-e^{z}$

By Milne Thomson method

$F'(z)=\phi_1(z,o)-i\phi_2(z,o)$

$\int F'(z)\,dz=\int e^{z}dz-i\int -e^{z}dz$

$F(z)=(1+i)e^{z}+C------(5)$

From (4) & (5)

$(1+i)e^{z}+C=(2-i)f(z)$

$$f(z)=\frac{1+i}{2-i}e^{z}+\frac{C}{2-i}$$

$$f(z)=\frac{1+3i}{5}e^{z}+\frac{C}{2-i}$$

**Problem 8** Find the Bilinear transformation that maps the points $1+i,\ -i,\ 2-i$ of the z-plane into the points $0,\ 1,\ i$ of the w-plane.

Solution:

Given $z_1=1+i,\ w_1=0$

$z_2=-i,\ w_2=1$

$z_3=2-i,\ w_3=i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(1-i)}{(0-1)(i-w)} = \frac{[z-(1+i)][-i-(2-i)]}{[(1+i)-(-i)][(2-i)-z]}$$

$$\frac{w(1-i)}{(-1)(i-w)} = \frac{(z-1-i)(-i-2+i)}{(1+i+i)(2-i-z)}$$

$$\frac{w(1-i)}{(w-i)} = \frac{(z-1-i)(-2)}{(1+2i)(2-i-z)}$$

$$\frac{w(1-i)}{(w-i)} = \frac{(-2)(z-1-i)}{(1+2i)(2-i-z)}$$

$$\frac{w}{w-i} = \frac{(-2)}{(1+2i)(1-i)}\frac{(z-1-i)}{(2-i-z)}$$

$$\frac{w}{w-i} = \frac{(-2)}{(1-i+2i+2)}\frac{(z-1-i)}{(2-i-z)}$$

$$\frac{w}{w-i} = \frac{(-2)}{(3+i)}\frac{(z-1-i)}{(2-i-z)}$$

$$\frac{w-i}{w} = \frac{(3+i)(2-i-z)}{(-2)(z-1-i)}$$

$$1-\frac{i}{w} = \frac{(3+i)(2-i-z)}{(-2)(z-1-i)}$$

$$\frac{i}{w} = 1-\frac{3+i}{(-2)}\frac{(2-i-z)}{(z-1-i)}$$

$$\frac{i}{w} = 1+\frac{3+i}{2}\frac{(2-i-z)}{(z-1-i)}$$

$$\frac{i}{w} = \frac{2(z-1-i)+(3+i)(2-i-z)}{2(z-1-i)}$$

$$\frac{w}{i} = \frac{2(z-1-i)}{2(z-1-i)+(3+i)(2-i-z)}$$

$$w = \frac{2i(z-1-i)}{2(z-1-i)+(3+i)(2-i-z)}$$

$$w = \frac{2i(z-1-i)}{2z-2-2i+6-3i-3z+2i+1-zi}$$

$$w = \frac{2i(z-1-i)}{-z+5-3i-zi}.$$

Problem 9 Prove that an analytic function with constant modulus is constant.

Solution:

Let $f(z) = u+iv$ be analytic

By C.R equations satisfied

i.e., $u_x = v_y,\ u_y = -v_x$

$\therefore f(z) = u + iv$

$\Rightarrow |f(z)| = \sqrt{u^2 + v^2} = C$

$\Rightarrow |f(z)|^2 = u^2 + v^2 = C^2$

$u^2 + v^2 = C^2$...............(1)

Diff (1) with respect to $x$

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$uu_x + vv_x = 0$.............(2)

Diff (1) with respect to $y$

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$-uv_x + vu_x = 0$............(3)

$(2)\times u + (3)\times v \Rightarrow (u^2 + v^2)u_x = 0$

$\Rightarrow u_x = 0$

$(2)\times v - (3)\times u \Rightarrow (u^2 + v^2)v_x = 0$

$\Rightarrow v_x = 0$

W.K.T $f'(z) = u_x + iv_x = 0$

$f'(z) = 0$

Integrate w.r.to $z$

$f(z) = C$

**Problem 10** When the function $f(z) = u + iv$ is analytic show that $u(x, y) = C_1$ and $v(x, y) = C_2$ are Orthogonal.

**Solution:**

If $f(z) = u + iv$ is an analytic function of $z$, then it satisfies C-R equations

$u_x = v_y,\ u_y = -v_x$

Given $u(x, y) = C_1$.............(1)

$v(x, y) = C_2$.............(2)

By total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Differentiate equation (1) & (2) we get $du = 0$, $dv = 0$

$$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1(say)$$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2(say)$$

$$\therefore m_1\, m_2 = -\frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial v/\partial x}{\partial v/\partial y} \quad (\therefore u_x = v_y\, u_y = -v_x)$$

$$\therefore m_1\, m_2 = -1$$

The curves $u(x, y) = C_1$ and $v(x, y) = C_2$ cut orthogonally.

## Problem 11

Show that the function $u = \frac{1}{2}\log\left(x^2 + y^2\right)$ is harmonic and determine its conjugate.

## Solution:

Given $u = \frac{1}{2}\log\left(x^2 + y^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}.\frac{1}{x^2 + y^2}(2x) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}.\frac{1}{x^2 + y^2}(2y) = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2 + y^2\right) - x(2x)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2 + y^2\right)(1) - 2y^2}{\left(x^2 + y^2\right)^2} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2 + x^2 - y^2}{\left(x^2 + y^2\right)^2} = 0$$

Hence $u$ is harmonic function

To find conjugate of $u$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$$

$$\phi_1(z, o) = \frac{1}{z}$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\phi_2(z,o)=0$$

By Milne Thomson Methods

$$f'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

$$\int f'(z)\,dz=\int\frac{1}{z}dz+0$$

$$=\log z+c$$

$$f(z)=\log re^{i\theta}$$

$$f(z)=u+iv=\log r+i\theta$$

$u=\log r$ , $v=\theta$

$$u=\log\sqrt{x^2+y^2} \qquad \left[\therefore r^2=x^2+y^2,\ \ \theta=\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$v=\tan^{-1}\left(\frac{y}{x}\right)$ $\therefore$ Conjugate of $u$ is $\tan^{-1}\left(\frac{y}{x}\right)$.

Problem 12 Find the image of the infinite strips $\frac{1}{4}<y<\frac{1}{2}$ under the transformation $w=\frac{1}{z}$.

Solution: $w=\frac{1}{z}\Rightarrow z=\frac{1}{w}$

$$z=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x=\frac{u}{u^2+v^2}\text{.........(1)}$$

$$y=-\frac{v}{u^2+v^2}\text{.........(2)}$$

Given strip is $\frac{1}{4}<y<\frac{1}{2}$ when $y=\frac{1}{4}$

$$\frac{1}{4}=-\frac{v}{u^2+v^2}\quad (by\ 2)$$

$$u^2+(v+2)^2=4\text{..........(3)}$$

which is a circle whose centre is at $(0,-2)$ in the $w$-plane and radius 2.

When $y=\frac{1}{2}$

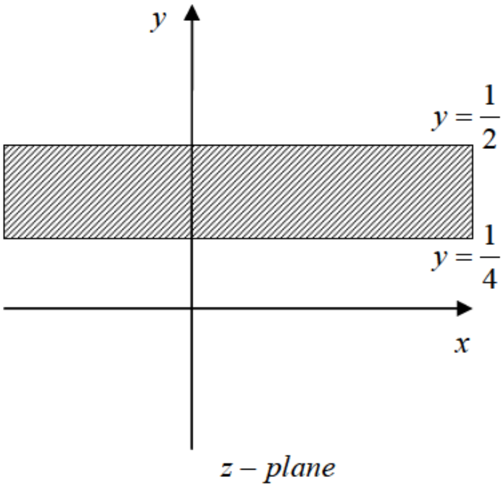
$$\frac{1}{2}=\frac{-v}{u^2+v^2}\ (by\ 2)$$

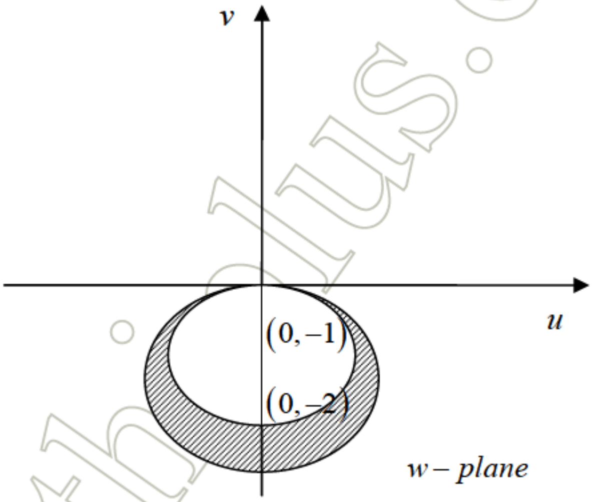
$$u^2+v^2+2v=0$$

$$u^2+(v+1)^2=1\text{..........(4)}$$

which is a circle whose centre is at $(0,-1)$ and radius is 1 in the $w$-plane.

Hence the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ is transformed into the region between circles $u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the $w$-plane.





**Problem 13** Obtain the bilinear transformation which maps the points $z=1,\ i,\ -1$ into the points $w=0,1,\infty$.

**Solution:** We know that

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(1-\infty)}{(0-1)(\infty-w)}=\frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\frac{w}{-1}(-1)=\frac{z-1}{1-i}\cdot\frac{i+1}{-(1+z)}$$

$$w=-\frac{z-1}{z+1}\cdot\frac{1+i}{1-i}$$

$$w=(-i)\frac{z-1}{z+1}$$

**Problem 14** Find the image of $|z-2i|=2$ under the transform $w=\frac{1}{z}$

**Solution:**

Given $w=\frac{1}{z}\Rightarrow z=\frac{1}{w}$

Now $w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

i.e., $x+iy=\dfrac{u-iv}{u^2+v^2}$

$\therefore\ x=\dfrac{u}{u^2+v^2}$..........(1)

$y=\dfrac{-v}{u^2+v^2}$..........(2)

Given $|z-2i|=2$

$|x+iy-2i|=2$

$|x+i(y-2)|=2$

$x^2+(y-2)^2=4$

$x^2+y^2-4y=0$..................(3)

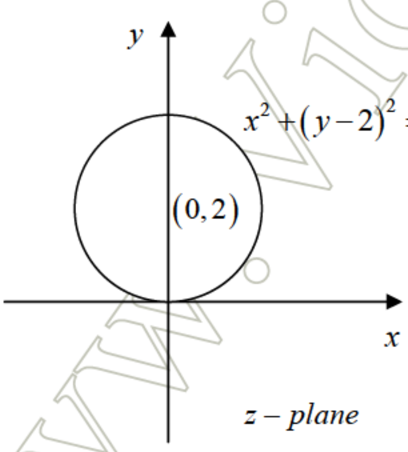
Sub (1) and (2) in (3)

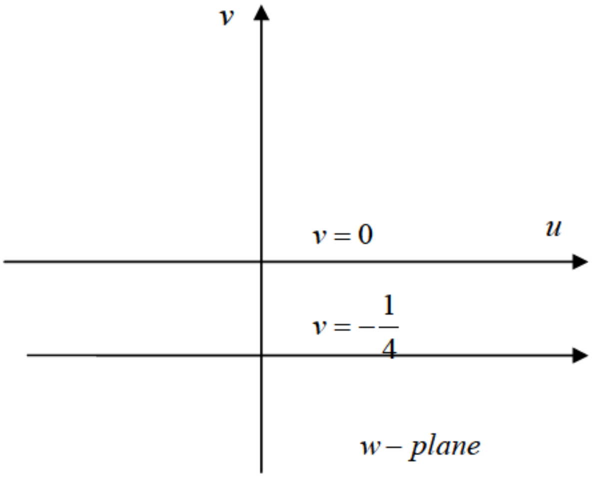
$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left[\frac{-v}{u^2+v^2}\right]=0$$

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+4\left[\frac{4v}{u^2+v^2}\right]=0$$

$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2}=0\qquad\frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2}=0$$

$1+4v=0\Rightarrow v=-\dfrac{1}{4}$ $(\therefore u^2+v^2\neq 0)$ This is straight line in $w$-plane.





**Problem 15** Prove that $w=\dfrac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane.

**Solution:**

$$w=\frac{z}{1-z}\Rightarrow w(1-z)=z$$

$$w-wz=z$$

$$w=\left(w+1\right)z$$

$$w=(w+1)z$$

$$z=\frac{w}{w+1}$$

Put $z=x+iy,\quad w=u+iv$

$$x+iy=\frac{u+iv}{u+iv+1}$$

$$=\frac{\left(u+iv\right)\left(u+1\right)-iv}{\left(u+iv+1\right)\left(u+1\right)-iv}$$

$$=\frac{u\left(u+1\right)-iuv+iv\left(u+1\right)+v^2}{\left(u+1\right)^2+v^2}$$

$$=\frac{\left(u^2+v^2+u\right)+iv}{\left(u+1\right)^2+v^2}$$

Equating real and imaginary parts

$$x=\frac{u^2+v^2+u}{\left(u+1\right)^2+v^2},\ y=\frac{v}{\left(u+1\right)^2+v^2}$$

$$y=0\Rightarrow\frac{v}{\left(u+1\right)^2+v^2}=0$$

$$y>0\Rightarrow\frac{v}{\left(u+1\right)^2+v^2}>0\Rightarrow v>0$$

Thus the upper half of the $z$ plane is mapped onto the upper half of the $w$ plane.

# UNIT IV COMPLEX INTEGRATION

## Part-A

**Problem 1** Evaluate $\int_C \frac{z}{(z-1)^3}\,dz$ where $C$ is $|z|=2$ using Cauchy's integral formula

**Solution:**

Given $\int_C \frac{z}{(z-1)^3}\,dz$

Here $f(z)=z$, $a=1$ lies inside $|z|=2$

$$\therefore \int_C \frac{zdz}{(z-1)^3}=\frac{2\pi i}{2!}f''(1)$$

$$=\pi i[0] \because f''(1)=0$$

$$\therefore \int_C \frac{zdz}{(z-1)^3}=0.$$

**Problem 2** State Cauchy's Integral formula

**Solution:**

If $f(z)$ is analytic inside and on a closed curve $C$ that encloses a simply connected region $R$ and if $'a'$ is any point in $R$, then $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$.

**Problem 3** Evaluate $\int_C e^{\frac{1}{z}}dz$ where $C$ is $|z-2|=1$.

**Solution:**

$e^{\frac{1}{z}}$ is analytic inside and on $C$.

Hence by Cauchy's integral theorem $\int_C e^{\frac{1}{z}}dz=0$

**Problem 4** Classify the singularities of $f(z)=\frac{e^{\frac{1}{z}}}{(z-a)^2}$.

**Solution:**

Poles of $f(z)$ are obtained by equating the denominator to zero.

i.e., $(z-a)^2=0$, $z=a$ is a pole of order 2

The principal part of the Laurent's expansion of $e^{1/z}$ about z = 0 contains infinite number terms. Therefore there is an essential singularity at z = 0.

**Problem 5** Calculate the residue of $f(z)=\frac{1-e^{2z}}{z^3}$ at the poles.

**Solution:**

Given $f(z)=\frac{1-e^{2z}}{z^3}$

Here $z=0$ is a pole of order 3

$$\therefore\left[\operatorname{Res} f(z)\right]_{z=0}=\underset{z\to 0}{Lt}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-0)^3\frac{1-e^{2z}}{z^3}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d^2}{dz^2}\left[1-e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d}{dz}\left[-2e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}-4e^{2z}$$

$$=\frac{1}{2}(-4)=-2\,.$$

**Problem 6** Evaluate $\int_C\frac{\cos\pi z}{z-1}dz$ if $C$ is $|z|=2$.

**Solution:**

We know that, Cauchy Integral formula is $\int_C\frac{f(z)}{z-a}dz=2\pi if(a)$ if $'a'$ lies inside $C$

$\int_C\frac{\cos\pi z}{z-1}dz$, Here $f(z)=\cos\pi z$

$\therefore z=1$ lies inside $C$

$\therefore f(1)=\cos\pi(1)=-1.$

$$\therefore\int_C\frac{\cos\pi z}{z-1}dz=2\pi i(-1)=-2\pi i\,.$$

**Problem 7** Define Removable singularity

**Solution:**

A singular point $z=z_0$ is called a removable singularity of $f(z)$ is $\underset{z\to z_0}{Lt}f(z)$ exists finitely

Example: For $f(z)=\frac{\sin z}{z}$, z = 0 is a removable singularity since $\underset{z\to 0}{Lt} f(z)=1$

## Problem 8 Test for singularity of $\frac{1}{z^2+1}$ and hence find corresponding residues.

Solution:

Let $f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$

Here $z=-i$ is a simple pole

$z=i$ is a simple pole

$$Res(z=i)=\underset{z\to i}{Lt}(z-i)\frac{1}{(z+i)(z-i)}$$

$$=\underset{z\to i}{Lt}\frac{1}{(z+i)}=\frac{1}{2i}$$

$$Res(z=-i)=\underset{z\to -i}{Lt}(z+i)\frac{1}{(z+i)(z-i)}=\frac{1}{-2i}.$$

## Problem 9 What is the value of $\int_C e^z dz$ where $C$ is $|z|=1$.

Solution:

Put $z=e^{i\theta}$

$dz=ie^{i\theta}d\theta$

$$\int_C e^z dz=\int_0^{2\pi} e^{e^{i\theta}} ie^{i\theta}d\theta..........(1)$$

Put $t=e^{i\theta}$ $\Rightarrow dt=e^{i\theta}d\theta$

When $\theta=0,\ t=1,\ \theta=2\pi,\ sssst=1$

$$\therefore (1)\Rightarrow \int_C e^z dz=\int_1^1 e^t dt=\left[e^t\right]_1^1=0$$

## Problem 10 Evaluate $\int_C \frac{3z^2+7z+1}{z+1}dz$, where $|z|=\frac{1}{2}$.

Solution:

Given $\int_C \frac{3z^2+7z+1}{z+1}dz$

Here $f(z)=3z^2+7z+1$

$z=-1$ lies outside $|z|=\frac{1}{2}$

Here $\int_C \frac{3z^2+72+1}{z+1}dz = 0$.(By Cauchy Theorem)

## Problem 11 State Cauchy's residue theorem

Solution:

If $f(z)$ be analytic at all points inside and on a simple closed curve $C$, except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ inside $C$ then $\int_C f(z)dz = 2\pi i \times [\text{sum of the residue of } f(z) \text{ at } z_1, z_2, \ldots, z_n]$.

## Problem 12 Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$

Here $z=-1$ is a pole of order 2

$$[Resf(z)]_{z=-1} = \underset{z\to -1}{Lt}\frac{1}{1!}\frac{d}{dz}(z+1)^2\frac{e^{2z}}{(z+1)^2}$$

$$= \underset{z\to -1}{Lt} 2e^{2z} = 2e^{-2}.$$

## Problem 13 Using Cauchy integral formula evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$, where $|z| = \frac{3}{2}$

Solution:

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz = \int_C \frac{-\cos \pi z^2}{z-1}dz + \int_C \frac{\cos \pi z^2}{(z-2)}dz$$

$$\left[\therefore \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{z-2}, \quad A=-1 \quad B=1\right]$$

Here $f(z) = \cos \pi z^2$

$z=1$ lies inside $|z| = \frac{3}{2}$

$z=2$ lies outside $|z| = \frac{3}{2}$

Hence by Cauchy integral formula

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i \, f(z)$$
$$= -2\pi i(-1)$$
$$= 2\pi i \qquad [\therefore f(z) = \cos \pi z, \; f(1) = \cos \pi = -1]$$

## Problem 14 State Laurent's series

Solution:

If $C_1$ and $C_2$ are two concentric circles with centres at $z = a$ and radii $r_1$ and $r_2$ $(r_1 < r_2)$ and if $f(z)$ is analytic on $C_1$ and $C_2$ and throughout the annular region $R$ between them, then at each point $z$ in $R$,

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n},$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^{n+1}}, \; n = 0,1,2,\ldots, b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^{-n+1}}, \; n = 1,2,3,\ldots$

## Problem 15 Find the zeros of $\frac{z^3-1}{z^3+1}$.

Solution:

The zeros of $f(z)$ are given by $f(z) = 0, \; \frac{z^3-1}{z^3+1} = 0$

i.e., $z^3 - 1 = 0, \; z = (1)^{\frac{1}{3}}$

$z = 1, \; w, w^2$ (Cubic roots of unity)

## Part-B

## Problem 1 Using Cauchy integral formula evaluate $\int_C \frac{dz}{(z+1)^2 (z-2)}$ where $C$ the circle $|z| = \frac{3}{2}$.

Solution:

Here $z = -1$ is a pole lies inside the circle

$z = 2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2 (z-2)} = \int \frac{\frac{1}{z-2}}{(z+1)^2} dz$$

Here $\mathrm{f}(\mathrm{z}) = \frac{1}{z-2}$

$$f'(z) = -\frac{1}{(z-2)^2}$$

Hence by Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2} dz$$

$$= \frac{2\pi i}{1!} f'(-1)$$

$$= 2\pi i\left[\frac{-1}{(-1-2)^2}\right] \quad \left(\therefore f'|z| = \frac{-1}{(z-2)^2}\right)$$

$$= 2\pi i\left[\frac{-1}{9}\right]$$

$$= \frac{-2}{9}\pi i.$$

## Problem 2

Evaluate $\int_C \frac{z-2}{z(z-1)} dz$ where $C$ is the circle $|z| = 3$.

### Solution:

W.K.T $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} dz$

Given $\int_C \frac{z-2}{z(z-1)} dz$ Here $z = 0$, $z = 1$lies inside the circle

Also $f(z) = z-2$

Now $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

Put $z = 0 \Rightarrow A = -1$

$z = 1 \Rightarrow B = 1$

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{z-2}{z(z-1)} dz = -\int_C \frac{z-2}{z} dz + \int_C \frac{z-2}{z-1} dz$$

$$= -2\pi i\ f(0) + 2\pi i\ f(1)$$

$$= 2\pi i[f(1) - f(0)]$$

$$= 2\pi i[-1-(-2)]$$

$$= 2\pi i[2-1] = 2\pi i.$$

**Problem 3** Find the Laurent's Series expansion of the function $\frac{z-1}{(z+2)(z+3)}$, valid in the region $2<|z|<3$.

**Solution:**

Let $f(z)=\frac{z-1}{(z+2)(z+3)}$

$$\frac{z-1}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$z-1=A(z+3)+B(z+2)$

Put $z=-2$

$-2-1=A(-2+3)+0$

$A=3$

Put $z=-3$

$-3-1=A(0)+B(-3+2)$

$-4=-B$

$B=4$

$\therefore f(z)=\frac{-3}{z+2}+\frac{4}{z+3}$

Given region is $2<|z|<3$

$2<|z|$ and $|z|<3$

$\left|\frac{2}{z}\right|<1$ and $\left|\frac{z}{3}\right|<1$

$$\therefore f(z)=\frac{-3}{z\left(1+\frac{2}{z}\right)}+\frac{4}{3\left(1+\frac{z}{3}\right)}$$

$$=\frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1}+\frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{-3}{z}\left[\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^{2}-....\right)\right]+\frac{4}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}-...\right]$$

**Problem 4** Expand $f(z)=\frac{7z-2}{z(z-2)(z+1)}$ valid in $1<|z+1|<3$

**Solution:**

Given $f(z)=\frac{7z-2}{z(z-2)(z+1)}$

$$f(z)=\frac{7z-2}{z(z-2)(z+1)}=\frac{A}{z}+\frac{B}{z-2}+\frac{C}{z+1}$$

$$7z-2=A(z-2)(z+1)+Bz(z+1)+Cz(z-2)$$

Put $z=2$

$B=2$

Put $z=0$

$A=1$

Put $z=-1$

$C=-3$

$$\therefore f(z)=\frac{1}{z}+\frac{2}{z-2}-\frac{3}{z+1}$$

Given region is $1<|z+1|<3$

Let $u=z+1 \Rightarrow 1<|u|<3$

$z=u-1 \Rightarrow 1<|u|$ & $|u|<3$

$$\Rightarrow \frac{1}{|u|}<1 \text{ \& } \left|\frac{u}{3}\right|<1$$

$$\therefore f(z)=\frac{1}{u-1}+\frac{2}{u-3}-\frac{3}{u}$$

$$=\frac{1}{u(1-\frac{1}{u})}+\frac{2}{-3(1-\frac{u}{3})}-\frac{3}{u}$$

$$=\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}-\frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}-\frac{3}{u}$$

$$=\frac{1}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^{2}+\dots\right]-\frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^{2}+\dots\right)-\frac{3}{u}$$

$$=\frac{1}{z+1}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^{2}+\dots\right]-\frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^{2}+\dots\right]-\frac{3}{z+1}$$

$$\therefore f(z)=-\frac{2}{z+1}+\sum_{n=2}^{\infty}\frac{1}{(z+1)^{n}}-\frac{2}{3}\sum_{n=0}^{\infty}\frac{(z+1)^{n}}{3^{n}}.$$

**Problem 5** Expand $f(z)=\dfrac{z^{2}-1}{(z+2)(z+3)}$ as a Taylor series valid in the region $|z|<2$.

**Solution:**

Given $f(z)=\dfrac{z^{2}-1}{(z+2)(z+3)}$

Now $(z+2)(z+3)=z^{2}+5z+6$

$$\therefore \frac{z^{2}-1}{(z+2)(z+3)}=1+\frac{-5z-7}{(z+2)(z+3)}$$

Now $\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$-5z-7 = A(z+3)+B(z+2)$

Put $z=-2$

$A=3$

Put $z=-3$

$B=-8$

$\therefore f(z) = 1+\frac{3}{z+2} - \frac{8}{z+3}$

Given $|z|<2$

$$f(z) = 1+\frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2+\ldots\right)$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$f(z) = 1+\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n .$$

Problem 6 Using Cauchy Integral formula Evaluate $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where $C$ is circle $|z|=1$.

Solution:

Here $f(z) = \sin^6 z$

$f'(z) = 6\sin^5 \cos z$

$f''(z) = 6\left[-\sin^6 z+\cos^2 z.5\sin^4 z\right]$

Here $a=\frac{\pi}{6}$, clearly $a=\frac{\pi}{6}$ lies inside the circle $|z|=1$

By Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^3} = \frac{2\pi i}{2!} f''(a)$$

$$\therefore \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$$

$$= \pi i 6\left[-\sin^6\left(\frac{\pi}{6}\right)+5\cos^2\left(\frac{\pi}{6}\right)\sin^4\frac{\pi}{6}\right]$$

$$=6\pi i\left[-\frac{1}{64}+\frac{5}{16}\times\frac{3}{4}\right]$$

$$=6\pi i\left[-\frac{1}{64}+\frac{15}{64}\right]$$

$$=6\pi i\left[\frac{15-1}{64}\right]=\frac{21\pi i}{16}$$

**Problem 7** Expand $f(z)=\sin z$ into a Taylor's series about $z=\frac{\pi}{4}$.

**Solution:**

Given $f(z)=\sin z$

$f'(z)=\cos z$

$f''(z)=-\sin z$

$f'''(z)=-\cos z$

Here $a=\frac{\pi}{4}$

$$\therefore f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f'\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f''\left(\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f'''\left(\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

W.K.T Taylor's series of $f(z)$ at $z=a$ is

$$f(z)=f(a)+\frac{z-a}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots$$

$$f(z)=f\left(\frac{\pi}{4}\right)+\frac{z-\frac{\pi}{4}}{1!}f'\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\ldots$$

$$=\frac{1}{\sqrt{2}}+\left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}-\left(\frac{z-\frac{\pi}{4}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)+\ldots$$

**Problem 8** Evaluate $\int_C \frac{z\sec z}{\left(1-z^2\right)}dz$ where $C$ is the ellipse $4x^2+9y^2=9$, using Cauchy's residue theorem.

**Solution:**

Equation of ellipse is

$$4x^2+9y^2=9$$

$$\frac{x^2}{9/4}+\frac{y^2}{1}=1$$

i.e., $\frac{x^2}{\left(3/2\right)^2}+\frac{y^2}{1}=1$

$\therefore$ Major axis is $\frac{3}{2}$, Minor axis is 1.

The ellipse meets the $x$ axis at $\pm\frac{3}{2}$ and the $y$ axis at $\pm 1$

Given $f(z)=\frac{z\sec z}{1-z^2}$

$$=\frac{z}{(1+z)(1-z)\cos z}$$

The poles are the solutions of $(1+z)(1-z)\cos z=0$

i.e., $z=-1,\ z=1$ are simple poles and $z=(2n+1)\frac{\pi}{2}$

Out of these poles $z\pm 1$ lies inside the ellipse

$z=\pm\frac{\pi}{4},\ \pm 3\frac{\pi}{4}$ lies outside the ellipse

$$\left[\operatorname{Re}s\ f(z)\right]_{z=1}=\underset{z\to 1}{Lt}(z-1)\frac{z}{(1+z)(1-z)\cos z}$$

$$=\underset{z\to 1}{Lt}\frac{-z}{(1+z)\cos z}=\frac{-1}{2\cos 1}$$

$$\left[\operatorname{Re}s\ f(z)\right]_{z=-1}=\underset{z\to -1}{Lt}(z+1)\frac{z}{(1+z)(1-z)\cos z}$$

$$=\underset{z\to -1}{Lt}\frac{z}{(1-z)\cos z}$$

$$= \frac{-1}{2\cos 1} = \frac{-1}{2\cos 1}$$

$$\therefore \int_C \frac{z\sec z}{1-z^2}\,dz = 2\pi i\left[\text{sum of the residues }\right]$$

$$= 2\pi i\left[\frac{-1}{2\cos 1} - \frac{1}{2\cos 1}\right]$$

$$= -2\pi i\left[\sec 1\right].$$

## Problem 9

Using Cauchy integral formula evaluate (i) $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where $C$ is the circle $|z+1-i| = 2$ (ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$, $C$ is the circle $|z| = \frac{3}{2}$.

### Solution:

(i) Given $|z+1-i| = 2$

$|z-(-1+i)| = 2$ is a circle whose centre is $-1+i$ and radius 2.

i.e., centre $(-1,1)$ and radius 2

$z^2+2z+5 = [z-(-1+2i)][z-(-1-2i]$

$-1+2i$ i.e., $(-1,2)$ lies inside the C

$-1-2i$ i.e., $(-1,-2)$ lies out side the C

$$\left[\therefore z^2+2z+5=0 \Rightarrow z=-2\pm\sqrt{\frac{4-20}{2}},\ z=-1\pm 2i\right]$$

$$\therefore \int_C \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}\,dz$$

$$= \int_C \frac{\frac{z+4}{[z-(-1-2i)]}}{z-(-1+2i)}\,dz$$

Hence $f(z) = \frac{z+4}{[z-(-1-2i)]}$

Here by Cauchy integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{z+4}{z^2+2z+5} = 2\pi i\, f(-1+2i)$$

$$= 2\pi i\left[\frac{-1+2i+4}{(-1+2i)-(-1-2i)}\right]$$

$$= 2\pi i\left[\frac{3+2i}{4i}\right] = \frac{\pi}{2}[3+2i].$$

(ii) $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$

$z=0,\ z=1$ lie inside the circle $|z| = \frac{3}{2}$

$z=2$ lies outside the circle

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$

$4-3z = A(z-1)(z-2) + B(z)(z-2) + C(z)(z-1)$

Put $z=0$

$4 = 4A$

$A = 1$

Put $z=1$

$B=-1$

Put $z=2$

$C=-1$

$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}$

$\int_C \frac{4-3z}{z(z-1)(z-2)}dz = \int_C \frac{2}{z}\,dz - \int_C \frac{1}{z-1}dz - \int_C \frac{1}{z-2}dz$

$\int_C \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{n}(0)$

$= 2\,[2\pi i\, f(0)] - 2\pi i\ f(1) - 0$

$= 4\pi i\ f(0) - 2\pi i\ f(1)$

$= 4\pi i\ (1) - 2\pi i\ (1)$

$= 2\pi i \qquad (\therefore f(0)=1\, f(1)=1)$

**Problem 10** Using Cauchy's integral formula evaluate $\int_C \frac{dz}{(z^2+4)^2}$ where $C$ is circle $|z-i|=2$

## Solution:

$\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

Given $|z-i|=2$, centre $(0,1)$, radius 2

$\therefore z=-2i$ lies outside the circle

$z=2i$ lies inside the circle

$$\therefore \int_C \frac{dz}{\left(z^2+4\right)^2} = \int_c \frac{\dfrac{1}{(z+2i)^2}}{(z-2i)^2}\,dz$$

Here $f(z) = \dfrac{1}{(z+2i)^2}$

$$f'(z) = \frac{-2}{(z+2i)^3}$$

$$f'(2i) = -\frac{2}{(2i+2i)^3} = -\frac{2}{(4i)^3}$$

$$= -\frac{2i}{64} = -\frac{i}{32}$$

Hence by Cauchy Integral Formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$

$$\int_C \frac{f(z)}{\left(z^2+4\right)^2} = \frac{2\pi i}{1!} f'(2i) = \frac{\pi}{16}.$$

## Problem 11

Find the Laurent's series which represents the function $\dfrac{z}{(z+1)(z+2)}$ in (i) $|z|>2$ (ii) $|z+1|<1$

### Solution:

(i). Let $f(z) = \dfrac{z}{(z+1)(z+2)}$

Now $\dfrac{z}{(z+1)(z+2)} = \dfrac{A}{z+1} + \dfrac{B}{z+2}$

$z = A(z+2) + B(z+1)$

Put $z=-1$

$A=-1$

Put $z=-2$

$B=1$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

Given $|z|>2$, $2<|z|$ i.e., $\left|\dfrac{2}{z}\right|<1 \Rightarrow \dfrac{1}{|z|}<1$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{z\left(1+\frac{1}{z}\right)} + \frac{2}{z\left(1+\frac{2}{z}\right)}$$

$$= \frac{-1}{z}\left(1+\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1+\frac{2}{z}\right)^{-1}$$

(ii). $|z+1| < 1$

Let $u = z+1$

i.e., $|u| < 1$

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{u} + \frac{2}{1+u}$$

$$= \frac{-1}{u} + 2(1+u)^{-1}$$

$$= \frac{-1}{u} + 2(1-u+u^2-...)$$

$$= \frac{-1}{1+z} + 2\left[1-(1+z)+(1+z)^2-...\right]$$

**Problem 12** Prove that $\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1} = \frac{2\pi}{1-a^2}$, given $a^2 < 1$.

**Solution:** Let $I = \int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1}$

Put $z = e^{i\theta}$

Then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\therefore\ I = \int_C \frac{dz/iz}{a^2 - a\left(z+\frac{1}{z}\right)+1} \text{ where C is } |z| = 1.$$

$$= \frac{1}{ai}\int_C \frac{dz}{\left(a+\frac{1}{a}\right)z - z^2 - 1}$$

$$= \frac{i}{a}\int_C \frac{dz}{z^2 - \left(a+\frac{1}{a}\right)z + 1}$$

$$= \int_C f(z)dz \text{ where } f(z) = \left(\frac{i}{a}\right)\frac{1}{z^2 - \left(a+\frac{1}{a}\right)z+1}$$

$$= \left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

The singularities of f(z) are simple poles at a and $\frac{1}{a}$. $a^2 < 1$ implies $|a| < 1$ and $\frac{1}{|a|} > 1$

$\therefore$ The pole that lies inside C is z = a.

$$\text{Res}[f(z);\ a] = \lim_{z\to a}(z-a).\left(\frac{i}{a}\right)\frac{1}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$= \left(\frac{i}{a}\right)\frac{1}{\left(a-\frac{1}{a}\right)}$$

$$= \frac{i}{a^2-1}$$

$$\text{Hence } I = 2\pi i.\frac{i}{a^2-1} = \frac{2\pi}{1-a^2}$$

## Problem 13 Show that $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta.d\theta}{5+4\cos\theta} = \frac{\pi}{6}$

Solution: Let $\displaystyle I = \int_0^{2\pi}\frac{\cos 2\theta.d\theta}{5+4\cos\theta}$

Put $z = e^{i\theta}$

Then $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$

$$I = \text{Real Part of } \int_0^{2\pi}\frac{e^{i2\theta}.d\theta}{5+4\cos\theta}$$

$$= \text{Real Part of } \int_C \frac{z^2.\,{}^{dz}\!/\!_{iz}}{5+2\left(z+\frac{1}{z}\right)} \text{ where C is } |z| = 1.$$

$$= \text{Real Part of } \frac{1}{2i}\int_C \frac{z^2.dz}{z^2+\frac{5}{2}z+1}$$

$$= \text{Real Part of } \frac{1}{2i}\int_C \frac{z^2.dz}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \text{Real Part of } \int_C f(z)dz \text{ where } f(z) = \frac{1}{2i}.\frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

z = -½ and z = -2 are simple poles of f(z).

z = -½ lies inside C.

$$\text{Res}\left[f(z); -\tfrac{1}{2}\right] = \lim_{z\to -\frac{1}{2}}\left(z+\tfrac{1}{2}\right).\frac{1}{2i}.\frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1}{2i}.\frac{{}^{1}\!/\!_{4}}{{}^{3}\!/\!_{2}} = \frac{1}{12i}$$

$$\therefore I = \text{Real Part of } 2\pi i.\frac{1}{12i}$$

$= \text{Real Part of } \pi/6$

$= \pi/6.$

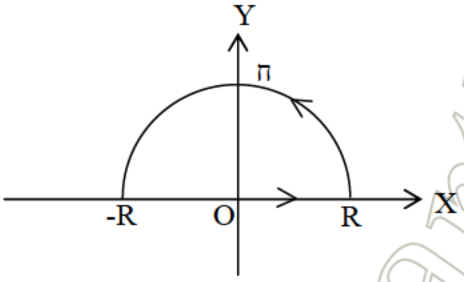
## Problem 14 Prove that $\int_0^\infty \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$

## Solution:

Let $\int_C \phi(z)\,dz = \int_C \frac{dz}{\left(z^2+1\right)^2}$

Where $\phi(z) = \frac{1}{\left(z^2+1\right)^2}$

Here $C$ is the semicircle $\Gamma$ bounded by the diameter $[-R, R]$



By Cauchy residue theorem,

$$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz ......(1)$$

To evaluate of $\int_C \phi(z)\,dz$

The poles of $\phi(z) = \frac{1}{\left(z^2+1\right)^2}$ is the solution of $\left(z^2+1\right)^2 = 0$

i.e., $(z+i)^2(z-i)^2 = 0$

i.e., the poles are $z = i, z = -i$

$z = i$ lies with inside the semi circle

$z = -i$ lies outside the semi circle

Now $\left[Res\,\phi(z)\right]_{z=i} = \underset{z \to i}{Lt} \frac{1}{1!}\frac{d}{dz}(z-i)^2 \phi(z)$

$$= \underset{z\to i}{Lt}\frac{1}{1!}\left[(z-i)^2\frac{1}{(z^2+1)^2}\right]$$

$$= \underset{z\to i}{Lt}\frac{1}{1!}\frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] \qquad \because z^2+1=(z+i)(z-i)$$

$$= \underset{z\to i}{Lt}\frac{-2}{(z+i)^3}$$

$$= \frac{-2}{i+i} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\therefore \int_C \phi(z)\,dz = 2\pi i\left[Sum\ of\ \ residues\ of\ \phi(z)\text{at}\ its\ \ poles\ \ which\ lies\, in\, C\right]$$

$$= 2\pi i\left[\frac{1}{4i}\right] = \frac{\pi}{2}..........(2)$$

Let $R\to\infty$, then $|z|\to\infty$ so that $\phi(z)=0$

$$\therefore \underset{|z|\to\infty}{Lt}\int_\Gamma \phi(z)dz = 0.........(3)$$

Sub (2) and (3) in (1)

$$\int_C \phi(z)dz = \int_{-\infty}^{\infty}\phi(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow 2\int_0^{\infty}\frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty}\frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

## Problem 15 Evaluate $\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx$

Solution:

$$2\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx = \int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}dx$$

$$\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{z\sin z}{z^2+a^2}dz$$

$$=\frac{1}{2}I........(1)$$

Now $z\sin z$ is the imaginary part of $ze^{iz}$

$$\therefore \text{I} = \int_{-\infty}^{\infty}\frac{z\sin z}{z^2+a^2}dz$$

$$=\text{I.P.}\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+a^2}dz$$

Let $\phi(z)=\frac{z\,e^{iz}}{z^2+a^2}=\frac{ze^{iz}}{(z+ia)(z-ia)}$

The poles are $z=-ia$ , $z=ia$

Now the poles $z=ia$ lies in the upper half – plane

But $z=-ia$ lies in the lower half – plane.

Hence

$$\left[Res\phi(z)\right]_{z=ia}=\underset{z\to ia}{Lt}(z-ia)\frac{ze^{iz}}{(z+ia)(z-ia)}$$

$$=\underset{z\to ia}{Lt}\frac{ze^{iz}}{(z+ia)}$$

$$=\frac{iae^{-a}}{2ia}=\frac{e^{-a}}{2}$$

$$\therefore \int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+a^2}dz=2\pi i\left[\text{Sum of the residues at each poles in the upper half plane }\right]$$

$$=2\pi i\left[\frac{e^{-a}}{2}\right]$$

$$=\pi ie^{-a}$$

I=I.P. of $\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+a^2}dz$

$$=\text{I.P. of }\left(\pi ie^{-a}\right)$$

$\text{I}=\pi e^{-a}$...........(2)

Sub (2) in (1)

$$\int_{0}^{\infty}\frac{x\sin x}{x^2+a^2}dx=\frac{1}{2}x=\frac{1}{2}\pi e^{-a}$$

## UNIT V LAPLACE TRANSFORM

### Part – A

Problem 1 State the conditions under which Laplace transform of $f(t)$ exists

Solution:

(i) $f(t)$ must be piecewise continuous in the given closed interval $[a,b]$ where $a>0$ and

(ii) $f(t)$ should be of exponential order.

Problem 2 Find (i) $L\left[t^{3/2}\right]$ (ii) $L\left[e^{-at}\cos bt\right]$

Solution:

(i) We know that

$$L\left[t^n\right]=\frac{\Gamma(n+1)}{s^{n+1}}$$

$$L\left[t^{n/3}\right]=\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} \quad \left[\because \Gamma(n+1)=n\Gamma(n)\right]$$

$$=\frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}}$$

$$=\frac{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{S^{5/2}}$$

$$=\frac{3\sqrt{\pi}}{4s^{5/2}} \quad \left[\therefore \Gamma(1/2)=\sqrt{\pi}\right]$$

ii)

$$L\left[e^{-at}\cos bt\right]=\left[L(\cos bt)\right]_{s\to s+a}$$

$$=\left[\frac{s}{s^2+b^2}\right]_{s\to s+a}$$

$$=\left[\frac{s+a}{(s+a)^2+b^2}\right]$$

Problem 3 Find $L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]$

Solution:

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=L\left[\sin 8t\cos 4t\right]+L\left[\cos^3 4t\right]+L[5]$$

$$L\left[\sin 8t+\cos 4t\right]=L\left[\frac{\sin 12t+\sin 4t}{2}\right] \quad \left[\because \sin A\cos B=\frac{\sin(A+B)+\sin(A-B)}{2}\right]$$

$$=\frac{1}{2}\left\{L\left[\sin 12t\right]+L\left(\sin 4t\right)\right\}$$

$$=\frac{1}{2}\left\{\frac{12}{s^2+144}+\frac{4}{s^2+16}\right\}$$

$$L\left[\cos^3 4t\right]=L\left[\frac{\cos 12t+3\cos 4t}{4}\right]\left[\because \cos^3\theta=\frac{\cos 3\theta+3\cos\theta}{4}\right]$$

$$=\frac{1}{4}\left\{L\left(\cos 12t\right)+3L\left(\cos 4t\right)\right\}$$

$$=\frac{1}{4}\left[\frac{s}{s^2+144}+\frac{3s}{s^2+16}\right]$$

$$L[5]=5L[1]=5\left[\frac{1}{s}\right]=\frac{5}{s}.$$

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=\frac{1}{2}\left\{\frac{12}{s^2+144}+\frac{4}{s^2+16}\right\}+\frac{1}{4}\left\{\frac{s}{s^2+144}+\frac{3s}{s^2+16}\right\}+\frac{5}{s}.$$

Problem 4 Find $L\{f(t)\}$ where $f(t)=\begin{cases}0 & ;when \quad 0<t<2\\ 3 & ;when \quad t>2\end{cases}$.

Solution:

W.K.T $L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt$

$$=\int_0^2 e^{-st}f(t)\,dt+\int_2^\infty e^{-st}f(t)\,dt$$

$$=\int_0^2 e^{-st}0.dt+\int_2^\infty e^{-st}3dt$$

$$=3\int_2^\infty e^{-st}dt=3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$=3\left[\frac{e^{-\infty}-e^{-2s}}{-s}\right]=\frac{3e^{-2s}}{s}.$$

Problem 5 If $L\left[f(t)\right]=F(s)$ show that $L\left\{f(\text{at})\right\}=\frac{1}{a}F\left(\frac{s}{a}\right)$.

(OR)

State and prove change of scale property.

Solution:

W.K.T $L\left\{f(t)\right\}=\int_0^\infty e^{-st}f(t)\,dt$

$$L\{f(\text{at})\}=\int_0^\infty e^{-st} f(\text{at})\,dt$$

Put $at=x$ when $t=0$, $x=0$

$adt=dx$ when $t=\infty$, $x=\infty$

$$L\{f(\text{at})\}=\int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\frac{dx}{a}$$

$$=\frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\,dt \quad [\because x \text{ is a dummy variable}]$$

$$=\frac{1}{a}F\left(\frac{s}{a}\right).$$

Problem 6 Does Laplace transform of $\frac{\cos \text{at}}{t}$ exist? Justify

Solution:

If $L\{f(t)\}=F(s)$ and $\frac{1}{t}f(t)$ has a limit as $t\to 0$ then $L\left\{\frac{f(t)}{t}\right\}=\int_s^\infty F(s)\,ds$.

Here $\underset{t\to 0}{Lim}\ \frac{\cos \text{at}}{t}=\frac{1}{0}=\infty$

$\therefore L\left\{\frac{\cos \text{at}}{t}\right\}$ does not exist.

Problem 7 Using Laplace transform evaluate $\int_0^\infty te^{-3t}\sin 2t\,dt$

Solution:

W.K.T $L\{f(t)\}=\int_0^\infty e^{st} f(t)\,dt$

$$=\int_0^\infty e^{-3t}t\sin 2t\,dt=L\left[(t\sin 2t)\right]_{s=3}$$

$$=\left[-\frac{d}{ds}L(\sin 2t)\right]_{s=3}=\left[-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right]_{s=3}$$

$$=\left[-\left(\frac{-4s}{(s^2+4)^2}\right)\right]_{s=3}$$

$$=\left[-\left(\frac{4s}{(s^2+4)^2}\right)\right]_{s=3}=\frac{12}{169}.$$

Problem 8 Find $L\left[\int_0^t \frac{\sin u}{u} du\right]$

Solution:

By Transform of integrals, $L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s} L\{f(t)\}$

$$L\left[\int_0^t \frac{\sin u}{u} du\right] = \frac{1}{s} L\left[\frac{\sin t}{t}\right] = \frac{1}{s}\int_s^\infty L[\sin t]\,ds = \frac{1}{s}\int_s^\infty \frac{1}{s^2+1} ds$$

$$= \frac{1}{s}\left[\tan^{-1} s\right]_s^\infty = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

$$= \frac{1}{s}\cot^{-1} s$$

Problem 9 Find the Laplace transform of the unit step function.

Solution:

The unit step function (Heaviside's) is defined as

$$U_a(t) = \begin{cases} 0 & ; \quad t < a \\ 1 & ; \quad t > a \end{cases}, \text{ where } a \geq 0$$

W.K.T $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{U_a(t)\} = \int_0^\infty e^{-st} U_a(t)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \left[\frac{e^{-\infty} - e^{-as}}{-s}\right] = \frac{e^{-as}}{s}$$

Thus $L\{U_a(t)\} = \frac{e^{-as}}{s}$

Problem 10 Find the inverse Laplace transform of $\frac{1}{(s+a)^n}$

Solution:

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L\left(t^{n-1}\right)=\frac{(n-1)!}{s^n}$$

$$L\left(e^{-at}t^{n-1}\right)=\left[\frac{(n-1)!}{s^n}\right]_{s\to s+a}=\frac{(n-1)!}{(s+a)^n}$$

$$e^{-at}t^{n-1}=L^{-1}\left(\frac{(n-1)!}{(s+a)^n}\right)$$

$$e^{-at}t^{n-1}=(n-1)!L^{-1}\left[\frac{1}{(s+a)^n}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)^n}\right]=\frac{1}{(n-1)!}e^{-at}t^{n-1}$$

Problem 11 Find the inverse Laplace Transform of $\frac{1}{s\left(s^2+a^2\right)}$

## Solution:

W.K.T $L^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t L^{-1}[F(s)]dt$

$$L^{-1}\left[\frac{1}{s\left(s^2+a^2\right)}\right]=\int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$$

$$=\int_0^t \frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]dt$$

$$=\frac{1}{a}\int_0^t \sin at\ dt$$

$$=\frac{1}{a}\left[\frac{\cos at}{a}\right]_0^t$$

$$=-\frac{1}{a^2}[\cos at-1]$$

$$=\frac{1}{a^2}[1-\cos at].$$

Problem 12 Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

## Solution:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=L^{-1}[sF(s)]=\frac{d}{dt}\quad L^{-1}[F(s)]$$

Where $F(s)=\frac{1}{(s+2)^2}$, $L\left[t^n\right]=\frac{n!}{s^{n+1}}$

$$L^{-1}\left[F(s)\right]=L^{-1}\left[\frac{1}{(s+2)^2}\right],\ L(t)=\frac{1}{s^2}$$

$$L^{-1}\left[F(s)\right]=e^{-2t}\,L^{-1}\left[\frac{1}{s^2}\right]=e^{-2t}\,t$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=\frac{d}{dt}\left[e^{-2t}\,t\right]=t\left(-2e^{-2t}\right)+e^{-2t}$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=e^{-2t}\ (1-2t)$$

Problem 13 Find $L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]$.

Solution:

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=L^{-1}\left[\frac{s+2}{\left((s+2)^2+1\right)^2}\right]$$

$$=e^{-2t}L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]\text{.......(1)}$$

$$L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]=t\ L^{-1}\int_s^\infty\frac{s}{\left(s^2+1\right)^2}\,ds$$

$$=t\,L^{-1}\int\frac{du}{2u^2}\qquad let\ u=s^2+1,\ du=2sds$$

$$=\frac{t}{2}L^{-1}\left(\frac{-1}{u}\right)$$

$$=\frac{t}{2}L^{-1}\left(\frac{-1}{s^2+1^2}\right)_s^\infty$$

$$=\frac{t}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$=\frac{t}{2}\sin t\text{........(2)}$$

Using (2) in (1)

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=e^{-2t}.\frac{t}{2}\sin t=\frac{1}{2}te^{-2t}\sin t.$$

**Problem 14** Find the inverse Laplace transform of $\frac{100}{s\left(s^2+100\right)}$

**Solution:**

Consider $\frac{100}{s\left(s^2+100\right)}=\frac{A}{s}+\frac{Bs}{s^2+100}$

$100=A\left(s^2+100\right)+\left(Bs+C\right)(s)$

Put $s=0,\ 100=A(100)$

$A=1$

$s=1,\quad 100=A(101)+B+C$

$B+C=-1$

Equating $s^2$ term

$0=A+B$

$\Rightarrow B=-1$

$\therefore B+C=-1$ i.e., $-1+C=-1$

$C=0$

$$\therefore L^{-1}\left[\frac{100}{s(s^2+100)}\right]=L^{-1}\left[\frac{1}{s}-\frac{s}{s^2+100}\right]$$

$$=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{s^2+100}\right]$$

$$=1-\cos 10t$$

**Problem 15** Solve $\frac{dx}{dy}-2y=\cos 2t$ and $\frac{dy}{dt}+2x=\sin 2t$ given $x(0)=1;\ y(0)=0$

**Solution:**

$x'-2y=cos2t$

$y'+2x=\sin 2t$ given $x(0)=1;\ y(0)=0$

Taking Laplace Transform we get

$$\left[sL(x)-x(0)\right]-2L\left[y\right]=L\left[\cos 2t\right]=\frac{s}{s^2+4}$$

$$\therefore sL\left[x\right]-2L\left[y\right]=\frac{s}{s^2+4}+1..........(1)$$

$$\left[sL(y)-y(0)\right]+2L\left[x\right]=L\left[\sin 2t\right]=\frac{2}{s^2+4}$$

$$2L\left[x\right]+sL\left[y\right]=\frac{2}{s^2+4}........(2)$$

$(1)\times 2-s\times(2)$ gives,

$$-\left(s^2+4\right)L(y)=\frac{-2}{s^2+4}$$

$$\therefore y = -L^{-1}\left[\frac{2}{s^2+4}\right]$$
$$= -\sin 2t$$
$$2x = \sin 2t - \frac{dy}{dt}$$
$$= \sin 2t + 2\cos 2t$$
$$\therefore x = \cos 2t + \frac{1}{2}\sin 2t$$

## Part-B

Problem 1 Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}dt$

Solution:

$$L\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{1}{s}L\left(\frac{\sin t}{t}\right)$$
$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty L(\sin t)\,ds$$
$$= \int_s^\infty \frac{1}{s^2+1}ds$$
$$= \left(\tan^{-1}(s)\right)_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1}(s)$$
$$= \cot^{-1}(s)$$
$$\therefore L\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{1}{s}\cot^{-1}(s)$$
$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}dt\right\} = \left[\frac{1}{s}\cot^{-1}(s)\right]_{s\to s+1}$$
$$= \frac{\cot^{-1}(s+1)}{s+1}.$$

Problem 2 Find $\int_0^\infty te^{-2t}\sin 3t\ dt$ using Laplace transforms.

Solution: $L(\sin 3t) = \frac{3}{s^2+9}$

$$L[t\sin 3t] = -\frac{d}{ds}\left(\frac{1}{s^2+9}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

$$\int_0^\infty e^{-st}(t\sin 3t)dt = L[t\sin 3t] \quad (\text{by definition})$$

$$= \frac{6s}{(s^2+9)^2}$$

i.e., $\int_0^\infty te^{-st}\sin 3t\,dt = \frac{6s}{(s^2+9)^2}$

Putting $s=2$ we get $\int_0^\infty te^{-2t}\sin 3t\,dt = \frac{12}{169}$

Problem 3 Find the Laplace transform of $t\int_0^t te^{-4t}\cos 3t\,dt + \frac{\sin 5t}{t}dt$

Solution:

$$L\left[t\int_0^t e^{-4t}\cos 3t\,dt\right] = -\frac{d}{ds}L\left[\int_0^t e^{-4t}\cos 3t\,dt\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left[L\left(e^{-4t}\cos 3t\right)\right]\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left[L\left(\cos 3t\right)\right]_{s\to s+4}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+9}\right)_{s\to s+4}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\frac{s+4}{(s+4)^2+9}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\frac{(s+4)}{(s^2+8s+25)}\right]$$

$$= -\frac{d}{ds}\left[\frac{s+4}{s^3+8s^2+25s}\right]$$

$$= -\left[\frac{\left(s^3+8s^2+25s\right)(1)-(s+4)\left(3s^3+16s+25\right)}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$= -\left[\frac{s^3+8s^2+25s-3s^3+16s^2-25s-12s^2-64s-100}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$= -\left[\frac{-2s^3-20s^2-64s-100}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$= 2\left[\frac{s^3+10s^2+32s+50}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$L\left[\frac{\sin 5t}{t}\right] = \int_s^\infty L\left(\sin 5t\right) ds$$

$$= \int_s^\infty \frac{5}{s^2+25} ds$$

$$= \left[5.\frac{1}{5}\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{5}\right)$$

$$= \cot^{-1}\left(\frac{s}{5}\right)$$

$$\therefore L\left[t\int_0^t e^{-4t}\cos 3t\,dt + \frac{\sin 5t}{t}\right] = L\left[t\int_0^t e^{-4t}\cos 3t\,dt\right] + L\left[\frac{\sin 5t}{t}\right]$$

$$= \frac{2\left(s^3+10s^2+32s+50\right)}{\left(s^3+8s^2+25s\right)^2} + \cot^{-1}\left(\frac{s}{5}\right)$$

Problem 4 Find $L\left[t^2e^{2t}\cos 2t\right]$.

Solution:

$$L\left[t^2e^{2t}\cos 2t\right] = (-1)^2\frac{d^2}{ds^2}L\left[e^{2t}\cos^{2t}\right]$$

$$= \frac{d^2}{ds^2}\left[\left(\frac{s}{s^2+4}\right)_{s\to s-2}\right]$$

$$= \frac{d^2}{ds^2}\left[\left(\frac{s-2}{(s-2)^2+4}\right)\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{s-2}{s^2-4s+8}\right]$$

$$= \frac{d}{ds}\left[\frac{\left(s^2-4s+8\right)(1)-(s-2)(2s-4)}{\left(s^2-4s+8\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{s^2-4s+8-2s^2+4s+4s-8}{\left(s^2-4s+8\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{-s^2+4s}{\left(s^2-4s+8\right)^2}\right]$$

$$=\frac{\left(s^2-4s+8\right)\left(-2s+4\right)-\left(s^2+4s\right)2\left(s^2-4s+8\right)\left(2s-4\right)}{\left(s^2-4s+8\right)^4}$$

$$=\frac{\left(s^2-4s+8\right)\left(-2s+4\right)-\left(s^2+4s\right)\left(2s-4\right)}{\left(s^2-4s+8\right)^3}$$

$$=\frac{-2s^3+8s^2-16s+4s^2-16s+32+4s^3-8s^2-16s^2+32s}{\left(s^2-4s+8\right)^3}$$

$$=\frac{2s^3-12s^2+32}{\left(s^2-4s+8\right)^3}.$$

Problem 5 Verify the initial and final value theorems for the function $f\left(t\right)=1+e^{-t}\left(\sin t+\cos t\right)$

Solution: Given $f\left(t\right)=1+e^{-t}\left(\sin t+\cos t\right)$

$$L\left\{f\left(t\right)\right\}=L\left\{1+e^{-t}\sin t+e^{-t}\cos t\right\}$$

$$=L\left(1\right)+L\left(e^{-t}\sin t\right)+L\left(e^{-t}\cos t\right)$$

$$=\frac{1}{s}+\frac{1}{\left(s+1\right)^2+1}+\frac{s+1}{\left(s+1\right)^2+1}$$

$$=\frac{1}{s}+\frac{s+2}{\left(s+1\right)^2+1}$$

Initial value theorem:

$$\underset{t\to 0}{Lt}\, f\left(t\right)=\underset{s\to\infty}{Lt}\, sF\left(s\right)$$

$$LHS=\underset{t\to 0}{Lt}\left[1+e^{-t}\left(\sin t+\cos t\right)\right]=1+1=2$$

$$RHS=\underset{s\to\infty}{Lt}\, sF\left(s\right)$$

$$=\underset{s\to\infty}{Lt}\, s\left[\frac{1}{s}+\frac{s+2}{\left(s+1\right)^2+1}\right]$$

$$=\underset{s\to\infty}{Lt}\left[1+\frac{s^2+2s}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt}\left[\frac{2s^2+4s+2}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt}\left[\frac{2+4/s+2/s^2}{1+2/s+2/s^2}\right]=2$$

$LHS = RHS$

Hence initial value theorem is verified.

Final value theorem:

$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$

$LHS = \underset{t\to\infty}{Lt}\ f(t)$

$$= \underset{t\to\infty}{Lt}\left(1+e^{-t}\sin t+e^{-t}\cos t\right)=1 \quad \left(\because e^{-\infty}=0\right)$$

$RHS = \underset{s\to 0}{Lt}\ sF(s)$

$$= \underset{s\to 0}{Lt}\ s\left[\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}\right]$$

$$= \underset{s\to 0}{Lt}\left[1+\frac{s^2+2s}{(s+1)^2+1}\right]=1$$

$LHS = RHS$

Hence final value theorem is verified.

Problem 6 Find $L^{-1}\left[\log\left(\frac{s^2+1}{s^2}\right)\right]$.

Solution:

$$L^{-1}\left[F(s)\right] = -\frac{1}{t}L^{-1}\left[F'(s)\right] \text{.......(1)}$$

$$F(s)=\log\left(\frac{s^2+1}{s^2}\right)$$

$$F'(s)=\frac{d}{ds}\log\left[\left(s^2+1\right)-\log\left(s^2\right)\right]$$

$$=\frac{2s}{s^2+1}-\frac{2s}{s^2}$$

$$L^{-1}\left[F'(s)\right]=L^{-1}\left[\frac{2s}{s^2+1}-\frac{2s}{s^2}\right]$$

$$=2L^{-1}\left[\frac{s}{s^2+1}-\frac{1}{s}\right]$$

$$=2\left[\cos t-1\right]$$

$$L^{-1}\left[\log\left(\frac{s^2+1}{s^2}\right)\right]=-\frac{1}{t}2\left[\cos t-1\right]$$

$$=\frac{2(1-\cos t)}{t}.$$

**Problem 7** Find the inverse Laplace transform of $\frac{s+3}{(s+1)(s^2+2s+3)}$

**Solution:** $\frac{s+3}{(s+1)(s^2+2s+3)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+2s+3}$....(1)

$$s+3=A(s^2+2s+3)+(Bs+c)(s+1)$$

Put $s=-1$

2=2A

A = 1

Equating the coefficients of $s^2$

0 = A+B $\Rightarrow$ B = -1

Put s = 0

3 = 3A+C

C = 0

$$(1)\Rightarrow\frac{s+3}{(s+1)(s^2+2s+3)}=\frac{1}{s+1}-\frac{s}{s^2+2s+3}$$

$$=\frac{1}{s+1}-\frac{s}{(s+1)^2+2}$$

$$=\frac{1}{s+1}-\frac{s+1}{(s+1)^2+2}+\frac{1}{(s+1)^2+2}$$

$$L^{-1}\left[\frac{s+3}{(s+1)(s^2+2s+3)}\right]=L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{s+1}{(s+1)^2+2}\right]+L^{-1}\left[\frac{1}{(s+1)^2+2}\right]$$

$$=e^{-t}-e^{-t}L^{-1}\left[\frac{s}{s^2+2}\right]+e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$=e^{-t}-e^{-t}\cos\sqrt{2}t+e^{-t}\sin\sqrt{2}t$$

$$=e^{-t}\left[1-\cos\sqrt{2}t+\sin\sqrt{2}t\right].$$

**Problem 8** Find $L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right)+2\right]$

**Solution:**

$$L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right)+2\right]=f(t)$$

$$\therefore L\left[f(t)\right]=s\log\left(\frac{s-1}{s+1}\right)+2$$

$$=s\log(s-1)-s\log(s+1)+2$$

$$L\{t\,f(t)\} = -\frac{d}{ds}\left[s\log(s-1) - s\log(s+1) + 2\right]$$

$$= -\left[\frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1)\right]$$

$$= -\left[\log\left(\frac{s-1}{s+1}\right) + \frac{s(s+1) - s(s-1)}{s^2-1}\right]$$

$$= \log\left(\frac{s-1}{s+1}\right) - \left(\frac{s^2+s-s^2+s}{s^2-1}\right)$$

$$= \log\left(\frac{s+1}{s-1}\right) - \frac{2s}{s^2-1}$$

$$tf(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2L^{-1}\left[\frac{s}{s^2-1}\right]$$

$$= L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2\cosh t \ldots.(1)$$

To find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Let $f(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

$$L[f(t)] = \log\left(\frac{s+1}{s-1}\right)$$

$$L\{t\,f(t)\} = -\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= \frac{1}{s-1} - \frac{1}{s+1} = \frac{2}{s^2-1}$$

$$\therefore t\ f(t) = 2L^{-1}\left[\frac{1}{s^2-1}\right] = 2\sinh t$$

$$f(t) = \frac{2\sinh t}{t} \ldots..(2)$$

Using (2) in (1)

$$tf(t) = \frac{2\sinh t}{t} - 2\cosh t$$

$$f(t) = \frac{2\sinh t}{t^2} - \frac{2\cosh t}{t}$$

$$= 2\left[\frac{\sinh t - t\cosh t}{t^2}\right].$$

Problem 9 Using convolution theorem find $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos \text{at}*\frac{1}{a}\sin \text{at}$$

$$=\frac{1}{a}\left[\cos \text{at}*\sin \text{at}\right]$$

$$=\frac{1}{a}\int_0^t \cos \text{a}\,u \sin a\left(t-u\right)du$$

$$=\frac{1}{a}\int_0^t \sin\left(\text{at}-au\right)\cos \text{a}\,u\,du$$

$$=\frac{1}{a}\int_0^t \frac{\sin\left(\text{at}-au+au\right)+\sin\left(\text{at}-au-au\right)}{2}du$$

$$=\frac{1}{2a}\int_0^t\left[\sin \text{at}+\sin a\left(t-2u\right)\right]du$$

$$=\frac{1}{2a}\left[\left(\sin \text{at}\right)u-\frac{\cos a\left(t-2u\right)}{-2a}\right]_0^t$$

$$=\frac{1}{2a}\left[t\sin \text{at}+\frac{\cos \text{at}}{2a}-\frac{\cos \text{at}}{2a}\right]$$

$$=\frac{t\sin \text{at}}{2a}.$$

Problem 10 Find the Laplace inverse of $\frac{1}{(s+1)\left(s^2+9\right)}$ using convolution theorem.

Solution:

$$L^{-1}\left[F(s).G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{1}{(s+1)\left(s^2+9\right)}\right]=L^{-1}\left[\frac{1}{(s+1)}.\frac{1}{\left(s^2+9\right)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)}\right] * L^{-1}\left[\frac{1}{(s^2+9)}\right]$$

$$= e^{t} * \frac{1}{3}\sin 3t$$

$$= \frac{1}{3}\int_0^t e^{-u}\sin\left[3(t-u)\right]du$$

$$= \frac{1}{3}\int_0^t e^{-u}\sin(3t-3u)\,du$$

$$= \frac{1}{3}\int_0^t e^{-u}\left[\sin 3t\cos 3u - \cos 3t\sin 3u\right]du$$

$$= \frac{1}{3}\sin 3t\int_0^t e^{-u}\cos 3u\ du - \frac{1}{3}\cos 3t\int_0^t e^{-u}\sin 3u\ du$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\cos 3u + 3\sin 3u)\right]_0^t - \frac{\cos 3t}{3}\left[\frac{e^{-u}}{10}(-\sin 3u - 3\cos 3u)\right]_0^t$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\cos 3t + 3\sin 3t) - \frac{1}{10}(-1)\right]$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\sin 3t - 3\cos 3t) - \frac{1}{10}(-3)\right]$$

Problem 11 Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem

Solution:

$$L^{-1}\left[F(s).G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[\frac{\sin\left[(a-b)u+bt\right]}{a-b} + \frac{\sin\left[(a+b)u-bt\right]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$=\frac{1}{2}\left[\frac{2a\sin at-2b\sin bt}{a^2-b^2}\right]$$

$$=\frac{a\sin at-b\sin bt}{a^2-b^2}.$$

Problem 12 Using convolution theorem find the inverse Laplace transform of $\frac{1}{\left(s^2+a^2\right)^2}$.

Solution:

$$L^{-1}\left[F(s).G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{1}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{1}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$=\frac{\sin \mathrm{at}}{a}*\frac{\sin \mathrm{at}}{a}$$

$$=\frac{1}{a^2}\int_0^t \sin au\sin a(t-u)\,du$$

$$=\frac{1}{2a^2}\int_0^t\left[\cos(2au-\mathrm{at})-\cos \mathrm{at}\right]du \quad \left[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)\right]$$

$$=\frac{1}{2a^2}\left[\frac{\sin(2au-\mathrm{at})}{2a}-(\cos \mathrm{at})u\right]_0^t$$

$$=\frac{1}{2a^2}\left[\frac{\sin \mathrm{at}}{2a}-t\cos \mathrm{at}-\left(\frac{-\sin \mathrm{at}}{2a}\right)\right]$$

$$=\frac{1}{2a^2}\left[\frac{2\sin \mathrm{at}}{2a}-t\cos \mathrm{at}\right]$$

$$=\frac{1}{2a^3}\left[\sin \mathrm{at}-at\cos \mathrm{at}\right]$$

Problem 13 Solve the equation $y''+9y=\cos 2t;\ y(0)=1 \text{and } y(\pi/2)=-1$

Solution:

Given $y''+9y=\cos 2t$

$$L\left[y''(t)+9y(t)\right]=L\left[\cos 2t\right]$$

$$L\left[y''(t)\right]+9L\left[y(t)\right]=L\left[\cos 2t\right]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

As $y'(0)$ is not given, it will be assumed as a constant, which will be evaluated at the end. $\therefore y'(0) = A.$

$$L[y(t)][s^2+9] - s - A = \frac{s}{s^2+4}$$

$$L[y(t)][s^2+9] = \frac{s}{s^2+4} + s + A$$

$$L[y(t)] = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9}.$$

Consider $\dfrac{s}{(s^2+4)(s^2+9)} = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+9}$

$$s = (As+B)(s^2+9) + (Cs+B)(s^2+4)$$

$$= As^3 + 9As + Bs^2 + 9B + Cs^3 + 4(s+1)s^2 + 4$$

Equating coefficient of $s^3$ A + C = 0 …….(1)

Equating coefficient of $s^2$ B + D = 0 …….(2)

Equating coefficient of $s$ 9A+ 4C = 1 …….(3)

Equating coefficient of constant 9B +4D = 0 …….(4)

Solving (1) & (3)

$$\begin{array}{r} 4A + 4C = 0 \\ \underline{-9A + 4C = -1} \\ -5A = -1 \end{array}$$

$$A = \frac{1}{5}$$

$$\frac{1}{5} + C = 0$$

$$C = -\frac{1}{5}$$

Solving (2) & (4)

$$\begin{array}{r} 9B + 9D = 0 \\ \underline{9B + AD = 0} \\ D = 0 \end{array}$$

$$\therefore B = 0 \,\&\, D = 0.$$

$$\therefore \frac{s}{(s^2+4)(s^2+9)} = \frac{1}{5}\frac{s}{s^2+4} - \frac{s}{5(s^2+9)}$$

$$\therefore L[y(t)] = \frac{1}{5}\left\{\frac{s}{s^2+4} - \frac{s}{s^2+9}\right\} + \frac{s}{s^2+9} + \frac{A}{s^2+4}$$

$$\therefore y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{A}{3}\sin 3t$$

$$= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

Given $y\left(\frac{\pi}{2}\right)=-1$

$$-1=-\frac{1}{5}-\frac{A}{5}$$

$$\therefore A=\frac{12}{5}$$

$$\therefore y(t)=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t$$

Problem 14 Using Laplace transform solve $\frac{d^2y}{dx^2}-\frac{3dy}{dx}+2y=4$ given that $y(0)=2,\ y'(0)=3$

Solution:

$$L[y''(t)]-3L[y'(t)]+2L[y(t)]=L[4]$$

$$s^2L[y(t)]-sy(0)-y'(0)-3sL[y(t)]+3y(0)+2L[y(t)]=\frac{4}{5}$$

$$(s^2-3s+2)L[y(t)]-2s-3+6=\frac{4}{s}$$

$$(s^2-3s+2)L[y(t)]=\frac{4}{s}+2s-3$$

$$L[f(t)]=\frac{2s^2-3s+4}{s(s^2-3s+2)}$$

$$L[f(t)]=\frac{2s^2-3s+4}{s(s-1)(s-2)}$$

$$\frac{2s^2-3s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}$$

$$2s^2-3s+4=A(s-1)(s-2)+Bs(s-2)+Cs(s-1)$$

Put s = 0 4 = 2A => A = 2

s = 1 $3=-B \Rightarrow B=-3$

$s=2$ $6=2c \Rightarrow C=3$

$$\therefore L[y(t)]=\frac{2}{s}-\frac{3}{s-1}+\frac{3}{s-2}$$

$$y(t)=2-3e^{t}+3e^{2t}$$

Problem 15 Solve $\frac{dx}{dy}+y=\sin t;\ x+\frac{dy}{dt}\cos t$ with $x=2$ and $y=0$ when $t=0$

Solution:

Given $x'(t)+y(t)=\sin t$

$x(t)+y'(t)=\cos t$

$L\left[x'(t)\right]+L\left[y(t)\right]=L[\sin t]$

$sL\left[x'(t)\right]-x(0)+L\left[y(t)\right]=\dfrac{1}{s^2+1}$

$sL\left[x(t)\right]+L\left[y(t)\right]=\dfrac{1}{s^2+1}+2.........(1)$

$L\left[x(t)\right]+L\left[y'(t)\right]=L[\cos 2t]$

$L\left[x(t)\right]+sL\left[y(t)\right]-=y(0)=\dfrac{1}{s^2+1}....(2)$

Solving (1) & (2)

$\left(1-s^2\right)L\left[y(t)\right]=2+\dfrac{1-s^2}{s^2+1}$

$\left(1-s^2\right)L\left[y(t)\right]=\dfrac{2s^2+2+1-s^2}{s^2+1}$

$L\left[y(t)\right]=\dfrac{2s^2+3}{\left(s^2+1\right)\left(1-s^2\right)}$

$\dfrac{s^2+3}{\left(s^2+1\right)\left(1-s^2\right)}=\dfrac{As+B}{s^2+1}+\dfrac{Cs+D}{1-s^2}$

$s^2+3=(As+B)\left(1-s^2\right)+(Cs+D)\left(s^2+1\right)$

Equating $s^3$ on both sides

$0=-A+C$ $\quad$ $put\, s=0$

$A=c$ $\quad$ $3=B+D$

$A=0\ \ C=0$

Equating $s^2$ on both sides

$1=-B+D$ $\quad$ $D=2$

$B=1$

Equation son both sides 0 = A + B

$\Rightarrow y(t)=L^{-1}\left[\dfrac{1}{s^2+1}\right]-2L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$=\sin t-2\sinh t$

To find $x(t)$ we have $x(t)+y'(t)=\cos t$, $x(t)=\cos t-y'(t)$, $y(t)=\sin t-2\sinh t$

$\dfrac{dy}{dt}=\cos t-2\cosh t$

$x(t)=\cos t-\cos t+2\cosh t$

$=2\cosh t$

Hence $x(t)=2\cosh t$

$y(t)=\sin t-2\sinh t$

| Objective type questions | opt1 | opt2 | opt3 |
|---|---|---|---|
| The solution of the differential equation (D^2 + 5D+6)y=0 is…………. | A e^ (-2x)+ B e^ (-3x) | A e^ (2x)+ B e^ (3x) | A e^ (-2x)+ B e^ (3x) |
| The solution of the differential equation (D^2 + 6D+9)y=0 is…………. | (A+Bx) e^ (3x) | (A+Bx) e^ (x) | (A+Bx) e^ (-2x) |
| The solution of the differential equation (D^2 -4 D+4)y=0 is…………. | (A+Bx) e^ (3x) | (A+Bx) e^ (-2x) | (A+Bx) e^ (-3x) |
| The particular integral of (D^2 -3D+2)y=12 is................. | (1/5) | (1/6) | (1/4) |
| The complementary function of (D^2 -2D+1)y=x sinx is............. | (A+Bx) e^ (-x) | (A+Bx) e^ (x) | (A+Bx) e^ (-2x) |
| If f(D)= D^ (2)- 2, 1/f(D) e^(-2x) is........... | 0.5 e^ (2x) | -0.5 e^ (2x) | 0.5 e^ (-2x) |
| The particular integral of (D^2+4) y= cos2x is ............... | (x cos2x)/2 | ( sin2x)/2 | (sin2x)/2 |
| If (D^2 +4)y=0 is a linear differential equation then general solution is | A cos2x+ B sin4x | Acos2x+Bsin2x | Asin2x+Bcos4x |
| If (D^2 − 6D+13) y = 0 is a linear differential equation then G.S. is ------- | e^(3x) (A cos2x+ B sin2x) | e^(3x) (A cos4x+ B sin4x) | e^(3x) (A cos2x+ B sin2x) |
| The solution of the differential equation (D^2 -4 D+3)y=0 is…………. | A e^ (x)+ B e^ (3x) | A e^ (-x)+ B e^ (3x) | A e^ (x)+ B e^ (-3 x) |
| The solution of the differential equation (D^2 +3D+2)y=0 is…………. | A e^ (x)+ B e^ (2x) | A e^ (-x)+ B e^ (2x) | A e^ (-x)+ B e^ (x) |
| The particular integral of (D^2 +3D+2)y= 2 e^(x) is…………. | e^(x)/3 | (-e^(x))/3 | e^(x)/6 |
| The particular integral of (D^2+4) y= e^(x) is ............... | 1/5* e^(x) | 1/5* e^(-x) | 1/6* e^(x) |
| If the roots of the auxilliary equation are real and distinct then the C.F is... | Ae^(m1x)+Be^ (m2x) | (A+Bx) e^ (m1x) | e^(αx) (Acosβx+Bsinβx) |
| If the roots of the auxilliary equation are real and equal then the C.F is... | Ae^(m1x)+Be^ (m2x) | e^(αx) (Acosβx+Bsinβx) | (A+Bx) e^ (mx) |
| If the roots of the auxilliary equation are complex then the C.F is... | Ae^(m1x)+Be^ (m2x) | e^(-αx) (Acosβx+Bsinβx) | (A+Bx) e^ (mx) |
| The particular integral of (D^2 +10D+24)y= e^(-x) is................. | (1/35) e^(-x) | (-1/35)e^(-x) | (-1/25)e^(-x) |
| The particular integral of (D^2+9) y= cos2x is ............... | cos2x/13 | (-cos2x)/13 | (-cos2x)/5 |
| The particular integral of (D^2+9) y= cos3x is ............... | x cos3x/2 | (-x cos3x)/2 | (xcos3x)/6 |
| The particular integral of (D^2 +12D+27)y= e^(-x) is.......... | (1/16) e^ (-x) | (-1/16) e^ (-x) | (1/16) e^ (x) |
| The solution of the differential equation (D^2 +19D+60)y=0 is…………. | A e^ (15x)+ B e^ (4x) | A e^ (-15x)+ B e^ (4x) | A e^ (15x)+ B e^ (-4x) |
| The solution of the differential equation (D^2 +13D+40)y=0 is…………. | A e^ (5x)+ B e^ (8x) | A e^ (5x)+ B e^ (-8x) | A e^ (-5x)+ B e^ (-8x) |
| The solution of the differential equation (D^2 -9 D+20)y=0 is…………. | A e^ (-5x)+ B e^ (4x) | A e^ (5x)+ B e^ (-4x) | A e^ (5x)+ B e^ (4x) |
| The solution of the differential equation (D^2 +D-72)y=0 is…………. | A e^ (-8x)+ B e^ (-9x) | A e^ (-8x)+ B e^ (9x) | A e^ (8x)+ B e^ (9x) |
| The solution of the differential equation (D^2-11D-42)y=0 is…………. | A e^ (14x)+ B e^ (-3x) | A e^ (-14x)+ B e^ (-3x) | A e^ (-14x)+ B e^ (3x) |
| The solution of the differential equation (D^2-12D-45)y=0 is…………. | A e^ (15x)+ B e^ (3x) | A e^ (-15x)+ B e^ (3x) | A e^ (15x)+ B e^ (-3x) |
| The solution of the differential equation (D^2-7D-30)y=0 is…………. | A e^ (-10x)+ B e^ (-3x) | A e^ (10x)+ B e^ (-3x) | A e^ (10x)+ B e^ (3x) |

| opt4 | opt5 | opt6 | Answer | |
|---|---|---|---|---|
| A e^ (2x)+ B e^ (-3 x) | | | A e^ (-2x)+ B e^ (-3x) | |
| (A+Bx) e^ (-3x) | | | (A+Bx) e^ (-3x) | |
| (A+Bx) e^ (2x) | | | (A+Bx) e^ (2x) | |
| (1/3) | | | (1/6) | |
| (A+Bx) e^ (2x) | | | (A+Bx) e^ (x) | |
| 0.5 e^ (3x) | | | 0.5 e^ (2x) | |
| (x sin2x)/4 | | | (x sin2x)/4 | |
| Asin4x+Bsin4x | | | Acos2x+Bsin2x | |
| e^(2x) (A cos2x+ B sin2x) | | | e^(3x) (A cos2x+ B sin2x) | |
| A e^ (2x)+ B e^ (-3 x) | | | A e^ (x)+ B e^ (3x) | |
| A e^ (-x)+ B e^ (-2 x) | | | A e^ (-x)+ B e^ (-2x) | |
| (-e^(x))/6 | | | e^(x)/3 | |
| 1/6* e^(x) | | | 1/5* e^(-x) | |
| (A+Bx) e^ (m2x) | | | Ae^(m1x)+Be^(m2x) | |
| (A+Bx) e^ (-mx) | | | (A+Bx) e^ (mx) | |
| e^(αx) (Acosβx+Bsinβx) | | | e^(αx) (Acosβx+Bsinβx) | |
| (1/25)e^(-x) | | | (1/25)e^(-x) | |
| cos2x/5 | | | cos2x/5 | |
| (-xcos3x)/6 | | | (xcos3x)/6 | |
| (-1/16) e^ (x) | | | (1/16) e^ (-x) | |
| A e^ (-15x)+ B e^ (-4x) | | | A e^ (-15x)+ B e^ (-4 x) | |
| A e^ (-5x)+ B e^ (8x) | | | A e^ (-5x)+ B e^ (-8x) | |
| A e^ (-5x)+ B e^ (-4x) | | | A e^ (5x)+ B e^ (4x) | |
| A e^ (8x)+ B e^ (-9 x) | | | A e^ (8x)+ B e^ (-9x) | |
| A e^ (14x)+ B e^ (3x) | | | A e^ (14x)+ B e^ (-3x) | |
| A e^ (-15x)+ B e^ (-3x) | | | A e^ (15x)+ B e^ (-3x) | |
| A e^ (-10x)+ B e^ (3x) | | | A e^ (10x)+ B e^ (-3x) | |

| | | | |
|---|---|---|---|
| The particular integral of (D^2 +19D+60)y= e^x is…………. | (-e^(-x))/80 | (e^(-x))/80 | (e^x)/80 |
| The particular integral of (D^2+25) y= cosx is ……………. | (cosx)/24 | (cosx)/25 | (-cosx)/24 |
| The particular integral of (D^2+1) y= sinx is ……………. | xcosx/2 | ( -xcosx)/2 | ( -xsinx)/2 |
| The particular integral of (D^2 -9D+20)y=e^(2x) is…………. | e^(2x) /6 | e^(2x) /(-6) | e^(2x) /12 |
| The particular integral of (D^2-1) y= sin2x is ……………. | (-sin2x)/5 | sin2x/5 | sin2x/3 |
| The particular integral of (D^2- 7D-30)y= 5 is…………. | (1/30) | (-1/30) | (1/6) |
| The solution of the differential equation (D^2-11D-42)y=21 is…………. | (-1/42) | (1/42) | (1/2) |
| Which one is Bessel's Equation of order n | x^2d^y/dx^2-xdy/dx+(x^2-n^2)y=0 | x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0 | x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0 |
| In this equation x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0 is called _______ | Legendre's Equation | Cauchy's equation | Partial Equation |
| Bessel's Equation is x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0 of order ______ | n+1 | n-1 | n |
| Which one is Bessel's Equation of order 0 | xd^2y/dx^2+dy/dx+xy=0 | xd^2y/dx^2+dy/dx-xy=0 | xd^2y/dx^2-dy/dx+xy=0 |
| Bessel's Equation is xd^y/dx^2+dy/dx+xy=0 of order ______ | 0 | 1 | 2 |
| d/dx[x^n J_n(x)] is equal to______ | x^n J_n+1(x) | x^n-1 J_n-1(x) | x^n J_n-1(x) |
| d/dx[x^(-n) J_n(x) ]is equal to______ | x^n J_n+1(x) | x^n-1 J_n-1(x) | x^n J_n-1(x) |
| J_n(x)= | (x/2n)[J_n+1 (x) +J_n+1 (x)] | (x/2n)[J_n-1 (x) +J_n+1 (x)] | (x/2n)[J_n-1 (x) +J_n-1 (x)] |
| x^-n J_n+1(x)= | _d/dx[x^(-n) J_n (x)] | _d/dx[x^(n) J_n (x)] | d/dx[x^(-n) J_n (x)] |
| J'_n (x)= | (1/2)[J_(n+1)(x) -J_(n+1)(x)] | (1/2)[J_(n-1)(x)-J_(n-1)(x)] | (1/2)[J_(n-1)(x) -J_(n+1)(x)] |
| J'_(n)(x)= | (n/x)[J_(n)(x)-J_(n+1)(x)] | (n/x^2)[J_(n)(x) -J_(n+1)(x)] | (n/x)[J_(n)(x)-J_(n-1)(x)] |
| J_n+1 (x)= | (n/x)[J_(n)(x)-J_(n+1)(x)] | (n/x^2)[J_(n)(x) -J_(n+1)(x)] | (n/x)[J_(n)(x)-J_(n-1)(x)] |
| J_-(1/2) (x)= | sqrt(2/pi) cosx | sqrt(4/pi) cosx | sqrt(2/pi) sinx |
| J_(1/2) (x)= | sqrt(2/pi) cosx | sqrt(4/pi) cosx | sqrt(2/pi) sinx |

| | | | | |
|---|---|---|---|---|
| (-e^x)/80 | | | (e^x)/80 | |
| (-cosx)/25 | | | cosx/24 | |
| xsinx/2 | | | ( -xcosx)/2 | |
| e^(2x) /(-12) | | | e ^ (2x) /6 | |
| (-sin2x)/3 | | | (-sin2x)/5 | |
| (-1/6) | | | (-1/6) | |
| (-1/2) | | | A e^ (14x)+ B e^ (-3x) | |
| x^2d^y/dx^2+xdy/dx+(x^2+n^2)y=0 | | | x^2d^y/dx^2+xdy/dx+(x^2-n^2)y=0 | |
| Bessel's Equation | | | Bessel's Equation | |
| 2n-1 | | | n | |
| x^2d^2y/dx^2+dy/dx+xy=0 | | | xd^2y/dx^2+dy/dx+xy=0 | |
| 3 | | | 0 | |
| _x^-n J_n+1(x) | | | x^n J_n-1(x) | |
| _x^-n J_n+1(x) | | | _x^-n J_n+1(x) | |
| (x/2n)[J_n-1 (x)-J_n+1 (x)] | | | (x/2n)[J_n-1 (x)+J_n+1 (x)] | |
| _d/dx[x^(-n) J_(-n)(x)] | | | _d/dx[x^(-n) J_n (x)] | |
| (1/4)[J_(n-1)(x)-J_(n+1)(x)] | | | (1/2)[J_(n-1)(x)-J_(n+1)(x)] | |
| (2n/x)[J_(n) (x)-J_(n-1) (x)] | | | (n/x)[J_(n)(x)-J_(n+1)(x)] | |
| | | | (2n/x)[J_(n) (x)-J_(n-1) (x)] | |
| sqrt(4/pi) sinx | | | sqrt(2/pi) cosx | |
| sqrt(4/pi) sinx | | | sqrt(2/pi) sinx | |

# Unit IX

# Analytic Functions

# 21 Complex Numbers

**Chapter Outline**

- Introduction
- Complex Numbers
- Complex Function
- Limit of a Function
- Derivative
- Analytic Function
- Cauchy–Riemann Equations
- Harmonic Function
- Properties of Analytic Functions
- Construction of Analytic Function (Milne–Thomson Method)

## 21.1 ❑ INTRODUCTION

Quite often, it is believed that complex numbers arose from the need to solve quadratic equations. In fact, contrary to this belief, these numbers arose from the need to solve cubic equations. In the sixteenth century, Cardano was possibly the first to introduce $a+\sqrt{-b}$, a complex number, in algebra. Later, in the eighteenth century, Euler introduced the notation $i$ for $\sqrt{-1}$ and visualized complex numbers as points with rectangular coordinates, but he did not give a satisfactory foundation for complex numbers. However, Euler defined the complex exponential and proved the identity $e^{i\varphi} = (\cos\varphi + i\sin\varphi)$, thereby establishing connection between trigonometric and exponential functions through complex analysis.

We know that there is no square root of negative numbers among real numbers.

However, algebra itself and its applications require such an extension of the concept of a number for which the extraction of the square root of a negative number would be possible.

We have repeatedly encountered the notion of extension of a number. Fractional numbers are introduced to make it possible to divide one integral number by another, negative numbers are introduced to make it possible to subtract a large number from a smaller one and irrational numbers become necessary in order to describe the result of measurement of the length of a segment in the case when the segment is incommensurable with the chosen unit of length.

The square root of the number –1 is usually denoted by the letter $i$ and numbers of the form $a + ib$ where $a$ and $b$ are ordinary real numbers known as **complex numbers**.

The necessity of considering complex numbers first arose in the sixteenth century when several Italian mathematicians discovered the possibility of algebraic solutions of third-degree equations.

The theoretical and applied values of complex numbers are far beyond the scope of algebra. The theory of functions of a complex variable, which was much advanced in the nineteenth century, proved to be a very valuable apparatus for the investigation of almost all the divisions of theoretical physics, such, for instance, as the theory of oscillations, hydrodynamics, the divisions of the theory of elementary particles, etc.

Many engineering problems may be treated and solved by methods involving complex numbers and complex functions. There are two kinds of such problems. The first of them consists of elementary problems for which some acquaintances with complex numbers are sufficient. This includes many applications to electric circuits or mechanical vibrating systems. The second kind consists of more advanced problems for which we must be familiar with the theory of complex analytic functions. Interesting problems in heat conduction, fluid flow and electrostatics belong to this category.

## 21.2 ❑ COMPLEX NUMBERS

A number of the form $x + iy$, where $x$ and $y$ are real numbers and $i = \sqrt{-1}$ ($i$ is pronounced as **iota**) is called a **complex number**. $x$ is called the **real part** of $x + iy$ and is written as $\text{Re}(x + iy)$ and $y$ is called the **imaginary part** and is written as $\text{Im}(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be **conjugates** of each other.

### *Properties*

(i) If $x_1 + iy_1 = x_2 + iy_2$ then $x_1 - iy_1 = x_2 - iy_2$

(ii) Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are said to be equal when $\text{Re}(x_1 + iy_1) = \text{Re}(x_2 + iy_2)$, i.e., $x_1 = x_2$ and $\text{Im}(x_1 + iy_1) = \text{Im}(x_2 + iy_2)$ i.e., $y_1 = y_2$

(iii) **Algebra of Complex Numbers**

The arithmetic operations on complex numbers follow the usual rules of elementary algebra of real numbers with the definition $i^2 = -1$. If $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$ are any two complex numbers then we define the following arithmetic operations.

**Addition**

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

**Subtraction**

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

**Multiplication**

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

**Division** Let $z_2 \neq 0$. Then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \left[\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right] + i\left[\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right]$$

i.e., sum, difference, product and quotient of any two complex numbers is itself a complex number.

(iv) Every complex number $x + iy$ can always be expressed in the form $r(\cos\theta + i\sin\theta)$.
i.e., $re^{i\theta}$ (Exponential form).

➢ **Note**

(i) The number $r = +\sqrt{x^2 + y^2}$ is called the **module** of $x + iy$ and is written as mod $(x + iy)$ or $|x + iy|$. The angle $\theta$ is called the **amplitude** or **argument** of $x + iy$ and is written as amp $(x + iy)$ or arg$(x + iy)$. Evidently, the amplitude $\theta$ has an infinite number of values. The value of $\theta$ which lies between $-\pi$ and $\pi$ is called the **principal value of the amplitude.**

(ii) $\cos\theta + i\sin\theta$ is briefly written as cis $\theta$ (pronounced as 'sis $\theta$')

(iii) If the conjugate of $z = x + iy$ be $\overline{z}$ then

(a) $\text{Re}(z) = \frac{1}{2}(z + \overline{z}), \text{Im}(z) = \frac{1}{2i}(z - \overline{z})$

(b) $|z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2} = |\overline{z}|$

(c) $z\overline{z} = |z|^2$

(d) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(e) $\overline{z_1z_2} = \overline{z_1} \cdot \overline{z_2}$

(f) $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}, z_2 \neq 0$

(iv) **De Moivre's Theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

## 21.3 ❑ COMPLEX FUNCTION

Recall from calculus that a real function $f$ defined on a set $S$ of real numbers is a rule that assigns to every $x$ in $S$ a real number $f(x)$, called the **value** of $f$ at $x$. Now in the complex region, $S$ is a set of complex numbers. A **function** $f$ defined on $S$ is a rule that assigns to every $z$ in $S$ a complex number $w$, called the value of $f$ at $z$.

We write $w = f(z)$. Here, $z$ varies in $S$ and is called a **complex variable**. The set $S$ is called the **domain** of $f$.

If to each value of $z$, there corresponds one and only one value of $w$ then $w$ is said to be a **single-valued function** of $z$; otherwise, it is a **multi-valued function**. For example, $w = \frac{1}{z}$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of $z$. The former is defined at all points of the $z$-plane except at $z = 0$ and the latter assumes two values for each value of $z$ except at $z = 0$.

➢ **Note**

(i) If $z = x + iy$ then $f(z) = u + iv$ (a complex number).

(ii) Since $e^{iy} = \cos y + i \sin y$, $e^{-iy} = \cos y - i \sin y$, the circular functions are

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2}, \text{ and so on}$$

∴ circular functions of the complex variable $z$ are given by $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\tan z = \dfrac{\sin z}{\cos z}$ with cosec $z$, sec $z$ and cot $z$ as their respective reciprocals.

(iii) **Euler's Theorem**

$$e^{iz} = \cos z + i \sin z$$

(iv) **Hyperbolic Functions**

If $x$ be real or complex, $\dfrac{e^{x} - e^{-x}}{2} = \sin hx$ (named hyperbolic sine of $x$)

$\dfrac{e^{x} + e^{-x}}{2} = \cos hx$ (named hyperbolic cosine of $x$)

Also, we define,

$$\tan hx = \frac{\sin hx}{\cos hx} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\cot hx = \frac{1}{\tan hx} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

$$\sec hx = \frac{1}{\cos hx} = \frac{2}{e^{x} + e^{-x}}$$

$$\operatorname{cosec} hx = \frac{1}{\sin hx} = \frac{2}{e^{x} - e^{-x}}$$

(v) **Relations between Hyperbolic and Circular Functions**

$\sin ix = i \sin hx$
$\cos ix = \cos hx$
$\tan ix = i \tan hx$

(vi) $\cos h^2x - \sin h^2x = 1$, $\sec h^2x + \tan h^2x = 1$
$\cot h^2x - \operatorname{cosec} h^2x = 1$

(vii) $\sin h(x \pm y) = \sin hx \cos hy \pm \cos hx \sin hy$
$\cos h(x \pm y) = \cos hx \cos hy \pm \sinh x \sinh y$

$$\tan h(x \pm y) = \frac{\tan hx \pm \tan hy}{1 + \tan hx \tan hy}$$

(viii) $\sin h2x = 2 \sin hx \cosh x$

$\cos h2x = \cos h^2x + \sin h^2x = 2 \cos h^2x - 1 = 1 + 2 \sin h^2x$

$\tan h2x = \dfrac{2 \tan hx}{1 + \tan h^2x}$

(ix) $\sin h3x = 3 \sin hx + 4 \sin h^3 x$

$\cos h3x = 4 \cos h^3 x - 3 \cos hx$

$$\tan h3x = \frac{3 \tan hx + \tan h^3 x}{1 + 3 \tan h^2 x}$$

(x)

$$\sin hx + \sin hy = 2 \sin h\frac{x+y}{2} \cos h\frac{x-y}{2}$$

$$\sin hx - \sin hy = 2 \cos h\frac{x+y}{2} \sin h\frac{x-y}{2}$$

$$\cos hx + \cos hy = 2 \cos h\frac{x+y}{2} \cos h\frac{x-y}{2}$$

$$\cos hx - \cos hy = 2 \sin h\frac{x+y}{2} \sin h\frac{x-y}{2}$$

(xi) $\cos h^2 x - \sin h^2 x = 1$

(xii) Complex trigonometric functions satisfy the same identities as real trigonometric functions.

$$\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z$$

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2 \sin z \cos z \quad \text{and} \quad \cos 2z = \cos^2 z - \sin^2 z$$

$$\sin \bar{z} = \overline{\sin z}$$

$\sin(z + 2n\pi) = \sin z$, $n$ is any integer

$\cos(z + 2n\pi) = \cos z$, $n$ is any integer

(xiii) **Inverse Trigonometric and Hyperbolic Functions**

Complex inverse trigonometric functions are defined by the following:

$$\cos^{-1} z = -i \log(z + \sqrt{z^2 + 1})$$

$$\sin^{-1} z = -i \log(iz + \sqrt{1 - z^2})$$

$$\tan^{-1} z = -\frac{i}{2} \log\left(\frac{1 + iz}{1 - iz}\right) = \frac{i}{2} \log \frac{i + z}{i - z}, z \neq \pm i$$

$$\operatorname{cosec}^{-1} z = \sin^{-1}\left(\frac{1}{z}\right) = -i \log\left(\frac{1 + \sqrt{z^2 - 1}}{z}\right), z \neq 0$$

$$\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right) = -i \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), z \neq 0$$

$$\cot^{-1} z = \tan^{-1}\left(\frac{1}{z}\right) = \frac{-i}{2} \log\left(\frac{z + i}{z - i}\right), z \neq \pm i$$

Complex inverse hyperbolic functions are defined by the following:

$$\cosh^{-1} z = \log(z + \sqrt{z^2 - 1}),\ \sinh^{-1} z = \log(z + \sqrt{z^2 + 1})$$

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), z \neq \pm 1$$

$$\operatorname{cosech}^{-1} z = \sinh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right), z \neq 0$$

$$\operatorname{sech}^{-1} z = \cosh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right), z \neq 0$$

$$\coth^{-1} z = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right), z \neq \pm 1$$

## 21.4 ❑ LIMIT OF A FUNCTION

A function $f(z)$ is said to have the **limit** '$b$' as $z$ approaches a point '$a$', written $\lim_{z \to a} f(z) = b$, if $f$ is defined in a neighborhood of '$a$' (except perhaps at '$a$' itself) and if the values of $f$ are close to '$b$' for all $z$ close to '$a$', i.e., the number $b$ is called the **limit** of the function $f(z)$ as $z \to a$, if the absolute value of the difference $f(z) - b$ remains less than any preassigned positive number $\in$ every time the absolute value of the difference $z - a$ for $z \neq a$, is less than some positive number $\delta$ (dependent on $\in$).

More briefly, the number $b$ is the limit of the function $f(z)$ as $z \to a$, if the absolute value $|f(z) - b|$ is arbitrarily small when $|z - a|$ is sufficiently small.

## 21.5 ❑ DERIVATIVE

A function $f(z)$ is said to be **differentiable** at a point $z = z_0$ if the limit $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists. This limit is then called the derivative of $f(z)$ at the point $z = z_0$ and is denoted by $f'(z_0)$.

If we write $z = z_0 + \Delta z$ then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

## 21.6 ❑ ANALYTIC FUNCTIONS

A function defined at a point $z_0$ is said to be **analytic** at $z_0$, if it has a derivative at $z_0$ and at every point in some neighborhood of $z_0$. It is said to be analytic in a region $R$, if it is analytic at every point of $R$. Analytic functions are otherwise named **holomorphic** or **regular** functions.

A point at which a function $f(z)$ is not analytic is called a **singular point** or **singularity** of $f(z)$.

## 21.7 ❑ CAUCHY–RIEMANN EQUATIONS

The necessary condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain $R$ is that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ must exist and satisfy the Cauchy–Riemann equations, namely,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The sufficient condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain $R$ is that the four partial derivatives $u_x$, $u_y$, $v_x$ and $v_y$ exist, are continuous and satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ at each point of $R$.

➢ **Note**

(i) The two partial differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called the **Cauchy–Riemann equations** and they may be written as $u_x = v_y$ and $u_y = -v_x$

(ii) The Cauchy–Riemann equations are referred as C-R equations

(iii) C-R equations in polar form are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

## 21.8 ❑ HARMONIC FUNCTION

A real function of two variables $x$ and $y$ that possesses continuous second-order partial derivatives and satisfies the Laplace equation is called a **harmonic function**.

If $u$ and $v$ are harmonic functions such that $u + iv$ is analytic then each is called the **conjugate harmonic function** of the other.

➢ **Note**

(i) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the **Laplacian operator** and is denoted by $\nabla^2$.

(ii) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$ is known as **Laplace equation** in two dimensions.

## 21.9 ❑ PROPERTIES OF ANALYTIC FUNCTIONS

### *Property 1*

The real and imaginary parts of an analytic function $f(z) = u + iv$ satisfy the Laplace equation in two dimensions.

- **Proof**

Since $f(z) = u + iv$ is an analytic function, it satisfies C-R equations,

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \tag{21.1}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{21.2}$$

Differentiating both sides of (21.1) partially with respect to *x*, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{21.3}$$

Differentiating both sides of (21.2) partially with respect to *y*, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \partial x} \tag{21.4}$$

By adding (21.3) and (21.4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\text{since } \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}, \text{ when they are continuous}\right)$$

$\Rightarrow$ *u* satisfies Laplace equation.

Now differentiating both sides of (21.1) partially with respect to *y*, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \tag{21.5}$$

Differentiating both sides of (21.2) partially with respect to *x* we get

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2} \tag{21.6}$$

Subtracting (21.5) and (21.6),

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

i.e.,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ *v* satisfies Laplace equation.

Hence, if *f*(*z*) is analytic then both real and imaginary parts satisfy Laplace's equation.

➢ **Note**

> If *f*(*z*) = *u* + *iv* is analytic then *u* and *v* are harmonic. Conversely, when *u* and *v* are any two harmonic functions then *f*(*z*) = *u* + *iv* need not be analytic.

### *Property 2*

If $f(z) = u + iv$ is an analytic function then the curves of the family $u(x, y) = C_1$ cut orthogonally the curves of the family $v(x, y) = C_2$ where $C_1$ and $C_2$ are constants.

- **Proof**

Given $u(x, y) = C_1$

Taking differentials on both sides, we get

$$du = 0$$

i.e.,

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$\therefore \; \dfrac{dy}{dx} = -\dfrac{\left(\dfrac{\partial u}{\partial x}\right)}{\left(\dfrac{\partial u}{\partial y}\right)} = m_1$ (say), where $m_1$ is the slope of the curve $u(x, y) = C_1$ at $(x, y)$

From the second curve $v(x, y) = C_2$, we get $\dfrac{dy}{dx} = -\dfrac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)} = m_2$, where $m_2$ is the slope of the curve $v(x, y) = C_2$ at $(x, y)$.

Now,
$$m_1 m_2 = \frac{\left(\dfrac{\partial u}{\partial x}\right)}{\left(\dfrac{\partial u}{\partial y}\right)} \cdot \frac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)}$$

$$= \frac{\left(\dfrac{\partial v}{\partial y}\right)}{-\left(\dfrac{\partial v}{\partial x}\right)} \cdot \frac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)} \quad \text{(as } f(z) \text{ is analytic, it satisfies C-R equation)}$$

$\Rightarrow m_1 m_2 = -1$

Hence, the curves cut each other orthogonally.

Here, the two families are said to be **orthogonal trajectories** of each other.

## 21.10 ❑ CONSTRUCTION OF ANALYTIC FUNCTIONS (MILNE–THOMSON METHOD)

### *To find f(z) when u is given*

We know that $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$.

i.e.,
$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(By C-R equations)} \tag{21.7}$$

Let
$$\frac{\partial u(x, y)}{\partial x} = \phi_1(x, y) \text{ and then calculate } \phi_1(z, 0) \tag{21.8}$$

and
$$\frac{\partial u(x, y)}{\partial y} = \phi_2(x, y) \text{ and then calculate } \phi_2(z, 0) \tag{21.9}$$

Substituting (21.8) and (21.9) in (21.7), we get

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

Integrating, we get $\int f^1(z)dz = \int \phi_1(z, 0)dz - i \int \phi_2(z, 0)dz$

i.e.,
$$f(z) = \int \phi_1(z, 0)dz - i\int \phi_2(z, 0)dz.$$

### *To find f(z) when v is given*

We know that $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \tag{21.10}$$

Let
$$\frac{\partial v(x, y)}{\partial y} = \phi_1(z, 0) \tag{21.11}$$

and
$$\frac{\partial v(x, y)}{\partial x} = \phi_2(z, 0) \tag{21.12}$$

Substituting (21.11) and (21.12) in (21.10), we get

$$f'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

Integrating, we get $\int f'(z)dz = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$

i.e.,
$$f(z) = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$$

## 21.11 ❑ APPLICATIONS

### *Irrotational Flows*

A flow in which the fluid particles do not rotate about their own axes while flowing is said to be irrotational.

Let there be an irrotational motion so that the velocity potential $\phi$ exists such that

$$u = \frac{-\partial \phi}{\partial x}, v = \frac{-\partial \phi}{\partial y} \tag{21.13}$$

In two-dimensional flow, the stream function $\psi$ always exists such that

$$u = \frac{-\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x} \tag{21.14}$$

From (21.13) and (21.14), we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x} \tag{21.15}$$

which are the well-known **Cauchy–Riemann equations**. Hence, $\phi + i\psi$ is an analytic function of $z = x + iy$. Moreover, $\phi$ and $\psi$ are known as conjugate functions.

On multiplying and rewriting, (21.15) gives

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0 \tag{21.16}$$

showing that the families of curves given by $\phi$ = constant and $\psi$ = constant intersect orthogonally. Thus, the curves of equi-velocity potential and the stream lines intersect orthogonally.

Differentiating the equation given in (21.15) with respect to $x$ and $y$ respectively, we get $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y}$ and $\frac{\partial^2 \phi}{\partial y^2} = \frac{-\partial^2 \psi}{\partial x \partial y}$. (21.17)

Since $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$, (21.17) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{21.18}$$

Again differentiating Eq. (21.15) with respect to $y$ and $x$ respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial y^2} \text{ and } \frac{\partial^2\phi}{\partial x\partial y}=\frac{-\partial^2\psi}{\partial x^2}$$

Subtracting these, $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \tag{21.19}$$

Equations (21.18) and (21.19) show that $\phi$ and $\psi$ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.

### *Complex Potential*

Let $w = \phi + i\psi$ be taken as a function of $x + iy$

Thus, suppose that $w = f(z)$

i.e., $$\phi + i\psi = f(x = iy) \tag{21.20}$$

Differentiating (21.20) with respect to $x$ and $y$ respectively, we get

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=f'(x+iy)$$

(21.21)

and $$\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}=i\,f'(x+iy)$$

or $$\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}=i\left(\frac{\partial\phi}{\partial x}+\frac{i\partial\psi}{\partial x}\right) \text{ by} \tag{21.22}$$

Equating real and imaginary parts, we get

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=\frac{-\partial\psi}{\partial x}$$

which are C-R equations. Then $w$ is an analytic function of $z$ and $w$ is known as the complex potential.

Conversely, if $w$ is an analytic function of $z$ then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion. The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called **equipotential lines** and **stream lines** respectively.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **equipotential lines** and **lines of force**.

In heat-flow problems, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **isothermals** and **heat-flow lines.**

## SOLVED EXAMPLES

**Example 1** Prove that the function $f(z) = |z|^2$ is differentiable only at the origin.

**Solution** Given $f(z) = |z|^2$

i.e., $$u + iv = |x + iy|^2 = [\sqrt{x^2+y^2}\,]^2 \qquad (\text{as } z = x + iy \text{ and } f(z) = u + iv)$$

$$= x^2 + y^2$$

$\Rightarrow$
$$u = x^2 + y^2$$
$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$
$$v = 0$$
$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

If $f(z)$ is differentiable then
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$\Rightarrow$ $2x = 0 \quad \Rightarrow \quad x = 0$

Also,
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$\Rightarrow$ $2y = 0 \quad \Rightarrow \quad y = 0$

$\therefore$ C-R equations are satisfied only when $x = 0$, $y = 0$

Hence, $f(z) = |z|^2$ is differentiable only at the origin (0, 0). **Proved.**

**Example 2** Prove that the function $f(z) = z\bar{z}$ is not analytic except at $z = 0$.

**Solution** Given $f(z) = z\bar{z}$

i.e., $u + iv = (x + iy)(x - iy)$
$$u + iv = x^2 + y^2$$

Equating real and imaginary parts.
$$u = x^2 + y^2$$
$\Rightarrow$
$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$
$$v = 0$$
$\Rightarrow$
$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C-R equations are not satisfied

$\therefore$ $f(z) = z\bar{z}$ is not analytic except at $z = 0$. **Proved.**

**Example 3** Show that (i) an analytic function with a constant real part is a constant, and (ii) an analytic function with a constant modulus is also a constant.

**[KU Nov. 2010, April 2012; AU Nov. 2010, Nov. 2011]**

**Solution** Let $f(z) = u + iv$ be an analytic function.

(i) Let $u = C_1$ (a constant)

Then $\frac{\partial u}{\partial x} = u_x = 0$ and $\frac{\partial u}{\partial y} = u_y = 0$.

Since $f(z)$ is an analytic function, by C-R equations $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ $v_y = 0$ and $v_x = 0$.

As $v_x = 0$ and $v_y = 0$, $v$ must be independent of $x$ and $y$ and must be a constant $C_2$.

$\therefore$ $f(z) = u + iv = C_1 + iC_2$ which is a constant.

(ii) Let $f(z) = u + iv$ be an analytic function.

Given $|f(z)| = \sqrt{u^2 + v^2} = k$ (a constant)

Differentiating partially with respect to $x$ and $y$, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and $$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Since $f(z)$ is an analytic function, it satisfies C-R equations.

∴ the above two equations may be written as,

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$

and $$v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

By solving, we get $\frac{\partial u}{\partial x} = u_x = 0$ and $\frac{\partial u}{\partial y} = u_y = 0$ .

By C-R equations, it implies that $\frac{\partial v}{\partial x} = v_x = 0$ and $\frac{\partial v}{\partial y} = v_y = 0$ .

Thus, $f(z) = u + iv$ is a constant. **Proved.**

**Example 4** If $f(z)$ is a regular function of $z$, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ .

**[AU May 2006, KU Nov. 2011, KU April 2013]**

**Solution** Let $f(z) = u(x, y) + iv(x, y)$

Then $|f(z)|^2 = u^2 + v^2$ and $|f'(z)|^2 = u_x^2 + v_x^2$

To prove $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2) = 4(u_x^2 + v_x^2)$

Now, $\frac{\partial}{\partial x}(u^2) = 2uu_x$ and $\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x)$

$$= 2[uu_{xx} + u_x u_x] = 2uu_{xx} + u_x^2 ]$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2[uu_{yy} + u_y^2]$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2u[u_{xx} + u_{yy}] + 2[u_x^2 + u_y^2]$$

$$= 2[u_x^2 + u_y^2] \qquad (\text{since } u_{xx} + u_{yy} = 0) \tag{1}$$

Again, $$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x^2]$$

and $$\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$$

$$\therefore \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(v^2)=2v(v_{xx}+v_{yy})+2(v_x^2+v_y^2)$$

$$=2(v_x^2+v_y^2) \qquad \text{(since } v_{zz}+v_{yy}=0) \qquad (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)=2[u_x^2+u_y^2+v_x^2+v_y^2]$$

$$=2[u_x^2+v_x^2+v_x^2+u_x^2] \qquad \text{(by using C-R equations)} \;=4[u_x^2+v_x^2].$$

Hence, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$ **Proved.**

**Example 5** Show that if $f(z)$ is a regular function of $z$ then $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|$ $=0$. **[AU May 2012]**

**Solution** $\text{Log}\,|f(z)|=\frac{1}{2}\log|f(z)|^2=\frac{1}{2}\log(u^2+v^2)$

$$\therefore \qquad \frac{\partial}{\partial x}\log|f(z)|=\frac{1}{2}\left[\frac{2uu_x+2v\cdot v_x}{u^2+v^2}\right]=\frac{uu_x+vv_x}{u^2+v^2}$$

$$\frac{\partial^2}{\partial x^2}\log|f(z)|=\frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_{xx}+v_x^2)-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$=\frac{1}{u^2+v^2}[uu_x+vv_{xx}+u_x^2+v_x^2]-\frac{2}{(u^2+v^2)^2}(uu_x+vv_x)^2 \qquad (1)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\log|f(z)|=\frac{1}{u^2+v^2}[uu_{yy}+vv_{yy}+u_y^2+v_y^2]-\frac{2}{(u^2+v^2)^2}(uu_y+vv_y)^2 \qquad (2)$$

Adding (1) and (2), we get $-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|$

$$=\frac{1}{u^2+v^2}[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+u_x^2+v_x^2+u_y^2+v_y^2]-\frac{2}{(u^2+v^2)^2}$$
$$[(uu_x+vv_x)^2+(uu_y+vv_y)^2]$$

$$=\frac{1}{(u^2+v^2)}[2(u_x^2+v_x^2)]-\frac{2}{(u^2+v^2)^2}[(uu_x+vv_x)^2+(-uv_x+vu_x)^2]$$

$$=\frac{2(u_x^2+v_x^2)}{u^2+v^2}-\frac{2}{(u^2+v^2)^2}[u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)]$$

$$=\frac{2(u_x^2+v_x^2)}{u^2+v^2}-\frac{2(u^2+v^2)(u_x^2+v_x^2)}{(u^2+v^2)^2}$$

$$=0$$ **Proved.**

**Example 6** Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and determine its conjugate. Also find $f(z)$. **[KU May 2010, KU April 2013]**

**Solution** Given $u = \frac{1}{2}\log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}; \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}; \frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$$

Hence, $u$ satisfies Laplace's equation.
$\therefore$ $u$ is harmonic.

**To find conjugate of *u***

We know that
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$= \frac{-\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$
$$= \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$
$$= \frac{x\,dy - y\,dx}{(x^2 + y^2)} = \frac{x\,dy - y\,dx}{x^2}\frac{1}{1 + \left(\frac{y}{x}\right)^2}$$
$$= \frac{1}{1 + \left(\frac{y}{x}\right)^2} d\left(\frac{y}{x}\right)$$
$$\int dv = \int \frac{d(y/x)}{1 + (y/x)^2}$$

i.e., $v = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore$ the required analytic function is $f(z) = u + iv$

$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

i.e., $f(z) = \log z$ **Ans.**

**Example 7** If $u(x, y) = e^x(x \cos y - y \sin y)$, find $f(z)$ so that $f(z)$ is analytic.

**Solution** Given $u = e^x(x \cos y - y \sin y)$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \cos y(xe^x + e^x) - y \sin y e^x$$

$$\therefore \quad \phi_1(z, 0) = ze^z + e^z \tag{1}$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = -xe^x \sin y - e^x(\sin y + y \cos y)$$

$\therefore$ $\phi_2(z, 0) = 0$ (2)

By Milne–Thomson method,

$$\begin{aligned} f'(z) &= \phi_1(z, 0) - i\phi_2(z, 0) \\ &= ze^z + e^z + 0 \\ &= e^z(z + 1) \end{aligned}$$

$\therefore$ $f(z) = \int e^z(z+1)dz$

$= ze^z - e^z + e^z + C$

i.e., $f(z) = ze^z + C$ **Ans.**

**Example 8** Find the analytic function $f(z) = u + iv$ given that $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

**[AU May 2006]**

**Solution** Given $u + iv = f(z)$ (1)

$\therefore$ $iu - v = i\,f(z)$ (2)

Adding (1) and (2), we get

$$(u - v) + i(u + v) = (1 + i)f(z)$$

Let $u - v = U$,

$u + v = V$ and $F(z) = (1 + i)\,f(z)$

$$\frac{\partial V}{\partial x} = \frac{(\cos h2y - \cos 2x)2\cos 2x - \sin 2x \cdot 2\sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(x, y) = \frac{\partial V}{\partial x}$$

$$= \frac{2\cos 2x \cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cos h2y - \cos 2x)^2}$$

$$\phi_1(x, y) = \frac{\partial V}{\partial y} = \frac{-\sin 2x(2\sin h2y)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{-2\sinh 2y \sin 2x}{(\cos h2y - \cos 2x)^2}$$

By Milne–Thomson method, we have

$$F'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

$$\phi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$\phi_1(z, 0) = 0$$

and $$F'(z) = i\frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= i\frac{-2}{1 - \cos 2z} = i\frac{-1}{\frac{1 - \cos 2z}{2}}$$

$$= i\frac{-1}{\sin^2 z} = -i\,\mathrm{cosec}^2 z$$

∴ $$f(z)=-\frac{i}{1+i}\int \text{cosec}^2 z\,dz$$

i.e., $$f(z)=\frac{i+1}{2}\cot z+C$$ **Ans.**

**Example 9** Find the analytic function $f(z) = u + iv$ if $u+v=\dfrac{x}{x^2+y^2}$ and $f(1) = 1$.

**[AU Nov. 2010]**

**Solution** Given $u + iv = f(z)$ (1)

$$iu - v = if(z) \quad (2)$$

Adding (1) and (2), we get

$$(u - v) + i(u + v) = (1 + i)f(z)$$

i.e., $$U + iV = F(z) \quad (3)$$

where $$U = u - v,\ V = u + v = \frac{x}{x^2+y^2},\ F(z) = (1 + i)\, f(z) \quad (4)$$

$$V=\frac{x}{x^2+y^2}$$

$$\phi_1(x,y)=\frac{\partial V}{\partial y}=\frac{-2xy}{(x^2+y^2)^2}$$

∴ $$\phi_1(z,0)=0 \quad (5)$$

$$\phi_2(x,y)=\frac{\partial V}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

∴ $$\phi_2(z,0)=\frac{-z^2}{z^4}=-\frac{1}{z^2} \quad (6)$$

By Milne's method, we have

$$F'(z)=\phi_1(z_1 0)+i\phi_2(z,0)$$

$$=0-i\frac{1}{z^2}$$

$$F(z)=-i\int\frac{1}{z^2}dz$$

∴ $$=-i\left(-\frac{1}{z}\right)+C$$

$$F(z)=\frac{i}{z}+C \quad (7)$$

But $F(z) = (1 + i)\, f(z)$ [from (4) and (8)]

From (7) and (8), we get

$$(1+i)\,f(z)=\frac{i}{z}+C$$

$$f(z)=\frac{i}{z(1+i)}+\frac{C}{1+i}$$

$$=\frac{i(1-i)}{(1+i)(1-i)z}+C_1,\text{ where } C_1=\frac{C}{1+i}$$

$$f(z)=\frac{1+i}{2z}+C_1$$

Given $f(1) = 1$

i.e., $$f(1)=\frac{1+i}{2}+C_1=1$$

$\Rightarrow$ $$C_1=1-\frac{(1+i)}{2}$$

$$=\frac{1-i}{2}$$

$\therefore$ $$f(z)=\frac{1+i}{2z}+\frac{1-i}{2}$$ **Ans.**

**Example 10** Show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\partial \bar{z}}$ . **[AU Nov. 2010]**

**Solution**

Let $z = x + iy$ (1)

$\therefore$ $\bar{z} = x - iy$ (2)

From (1) and (2), we get

$$x=\frac{z+\bar{z}}{2},\ y=\frac{z-\bar{z}}{2i}=\frac{-i}{2}(z-\bar{z})$$

Now, $$\frac{\partial x}{\partial z}=\frac{1}{2},\ \frac{\partial x}{\partial \bar{z}}=\frac{1}{2}$$

$$\frac{\partial y}{\partial z}=\frac{-i}{2},\ \frac{\partial y}{\partial \bar{z}}=\frac{i}{2}$$

Now, $$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} \quad (3)$$

$$=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}}+\frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

(4)

$$=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$\therefore$ $$\frac{\partial^2}{\partial z\partial \bar{z}}=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

$\Rightarrow$ $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\partial \bar{z}}$$ **Proved.**

**Example 11** If $f(z) = u + iv$ is analytic, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0.$ **[AU Nov. 2010]**

**Solution** We know that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\partial \bar{z}}$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\partial\bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\cdot\frac{1}{2}\log|f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\log[f'(z)\,f'(\bar{z})]$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}[\log f'(z)+\log f'(\bar{z})]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right] = 0$$

**Proved.**

**Example 12** If $u = x^2 - y^2$ and $v = -\frac{y}{x^2+y^2}$, prove that both $u$ and $v$ satisfy Laplace's equation but that $u + iv$ is not a regular function of $z$. **[KU Nov. 2011]**

**Solution** Given $u = x^2 - y^2$

Then $$\frac{\partial u}{\partial x} = u_x = 2x;\ \frac{\partial^2 u}{\partial x^2} = u_{xx} = 2;\ \frac{\partial u}{\partial y} = u_y = -2y;\ \frac{\partial^2 u}{\partial y^2} = u_{yy} = -2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

i.e., $u$ satisfies Laplace's equation.

$$v = -\frac{y}{x^2+y^2}$$

Then $$\frac{\partial v}{\partial x} = v_x = \frac{2xy}{(x^2+y^2)^2};\ v_{xx} = 2y\left[\frac{(x^2+y^2)\cdot - x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = v_y = -\left[\frac{(x^2+y^2)\cdot 1 - 2y^2}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = v_{yy} = \frac{(x^2+y^2)^2\,2y - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0$$

i.e., $v$ satisfies Laplace's equation.
Now, $u_x \neq v_y$ and $u_y \neq -v_x$
i.e., C-R equations are not satisfied by $u$ and $v$.
Hence, $u + iv$ is not an analytic (regular) function of $z$. **Ans.**

**Example 13** Show that the function $u(x, y) = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find a function $v(x, y)$ such that $u + iv$ is an analytic function.

**[AU June 2010]**

**Solution** Let $f(z) = u + iv$ be an analytic function with $u(x, y) = 3x^2y + x^2 - y^3 - y^2$

Then
$$\frac{\partial u}{\partial x} = u_x = 6xy + 2x; \frac{\partial^2 u}{\partial x^2} = u_{xx} = 6y + 2;$$
$$\frac{\partial u}{\partial y} = u_y = 3x^2 - 3y^2 - 2y; \frac{\partial^2 u}{\partial y^2} = u_{yy} = -6y - 2$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence, $u(x, y)$ is a harmonic function.

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy = \frac{-\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -u_y dx + u_x dy$$

$\therefore dv = (-3x^2 + 2y + 3y^2)dx + (6xy + 2x)dy$ where the RHS is a perfect differential equation.

$$dv = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$$
$$= -\int (3x^2 - 3y^2 - 2y)\, dx + \int (6xy + 2x)\, dy$$

$$\therefore \quad v = (3xy^2 + 2xy - x^3) + C$$
$$\therefore \quad f(z) = u + iv = 3x^2y + x^2 - y^3 - y^2 + i(3xy^2 + 2xy - x^3 + C)$$
$$= -i[x^3 + 3x^2(iy) + 3xi^2y^2 + i^3y^3] + [x^2 + 2xiy + i^2y^2] + iC$$
$$= -i[x + iy]^3 + [x + iy]^2 + iC$$
$$\therefore \quad f(z) = iz^3 + z^2 + iC$$

**Ans.**

## EXERCISE

### Part A

1. Define analytic function of a complex variable.
2. State any two properties of an analytic function.
3. Define a harmonic function with an example.
4. Verify whether the function $\phi(x, y) = e^x \sin y$ is harmonic or not.
5. Find the constant '$a$' so that $u(x, y) = ax^2 - y^2 + xy$ is harmonic.
6. Is $f(z) = z^3$ analytic? Justify.
7. What do you mean by a conjugate harmonic function? Find the conjugate harmonic of $x$.
8. Show that an analytic function with a constant real part is constant.
9. Write down the necessary condition for $w = f(z) = f(re^{i\theta})$ to be analytic.
10. Show that the function $u = \tan^{-1}\left(\frac{y}{x}\right)$ is harmonic.
11. Show that $xy^2$ cannot be the real part of an analytic function.
12. $f(z) = u + iv$ is such that $u$ and $v$ are harmonic. Is $f(z)$ analytic always? Justify.

13. State C-R equations in Cartesian coordinates.
14. Prove that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is a harmonic function.
15. Show that the function $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ satisfies Cauchy–Riemann equations.
16. Show that the real part $u$ of an analytic function satisfies the equation $\nabla^2 u = 0$.
17. Check whether the function $\frac{1}{z}$ is analytic or not.
18. Test the analyticity of the function $2xy + i(x^2 - y^2)$.
19. State the basic difference between the limit of a function of a real variable and that of a complex variable.
20. Find the analytic function $f(z) = u + iv$, given that (i) $u = y^2 - x^2$, (ii) $v = \sin hx \sin y$, and (iii) $u = \dfrac{x}{x^2+y^2}$.

## Part B

1. Prove that the following functions are not differentiable (and, hence, not analytic) at the origin.

   (i) $$f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

   (ii) $$f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

2. Prove that for the following function, C-R equations are satisfied at the origin but $f(z)$ is not analytic there.

   $$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

3. Show that $f(z) = \sin \bar{z}$ is not an analytic function of $z$.
4. Find whether the Cauchy–Riemann equations are satisfied for the following functions where $w = f(z)$.
   - (i) $w = 2xy + i(x^2 - y^2)$ **(Ans.** No**)**
   - (ii) $w = \dfrac{x - iy}{x^2 + y^2}$ **(Ans.** No**)**
   - (iii) $w = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$ **(Ans.** Yes**)**
   - (iv) $w = \cos x \sin hy$ **(Ans.** Yes**)**
   - (v) $w = z^3 - 2z^2$ **(Ans.** Yes**)**
5. Show that an analytic function with a constant imaginary part is constant.
6. Show that $u + iv = \dfrac{x - iy}{x - iy + a}$, where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

7. Find an analytic function $w = u + iv$ whose real part is given by
   - (i) $u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$ **[Ans.** $e^{-x}(x - iy)^2 (\cos y - i \sin y]$
   - (ii) $u = \dfrac{x}{x^2 + y^2}$ $\left(\textbf{Ans. } \dfrac{1}{z} + C\right)$
   - (iii) $u = e^x(x \cos y - y \sin y)$ **(Ans.** $ze^z + C$**)**
   - (iv) $u = x^4 - 6x^2y^2 + y^4$ **(Ans.** $z^4 + C$**)**
   - (v) $u = -\sin x \sin hy$ **(Ans.** $-i \cos z + iC$**)**
8. Find an analytic function $w = u + iv$ whose imaginary part is given by
   - (i) $v = e^x(x \cos y + y \sin y)$ **(Ans.** $ize^{-z} + C$**)**
   - (ii) $v = -2 \sin x(e^y - e^{-y})$ **(Ans.** $\log z + C$**)**
   - (iii) $v = \dfrac{\sin x \sin hy}{\cos 2x + \cos h 2y}$ $\left(\textbf{Ans. } \dfrac{1 + \sec z}{2}\right)$
   - (iv) $v = x^2 - y^2 + 2xy - 3x - 2y$ **[Ans.** $z^2 - 2z + i(z^2 - 3z)$**]**
   - (v) $v = x^3 - 3x^2y + 2x + 1 + y^3 - 3xy^2$ **[Ans.** $(i - 1)z^3 + 2z + C$**]**
9. Show that the following functions are harmonic and find their harmonic conjugates.
   - (i) $u = \cos x \cos hy$ **(Ans.** $-\sin x \sin hy + C$**)**
   - (ii) $u = e^x(\cos y - \sin y)$ **(Ans.** Not harmonic**)**
   - (iii) $u = e^{-x}(y \cos y - x \sin y)$ (**Ans.** $e^x(x \cos y + y \sin y) + C$)
   - (iv) $u = e^x \cos y$ (**Ans.** $e^x \sin y + C$)
   - (v) $u = 2xy + 3xy^2 - 2y^3$ **(Ans.** Not harmonic**)**
10. Find $f(z) = u + iv$, if $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, given that $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$.

    $\left[\textbf{Ans. } f(z) = \cot\left(\dfrac{z}{2}\right) + \left(\dfrac{1 - i}{2}\right)\right]$
11. Find $f(z) = u + iv$ if $2u - 3v = 3y^2 - 2xy - 3x^2 + 3y - x$ and $f(0) = 0$.

    (**Ans.** $f(z) = iz^2 - z$**)**
12. If $f(z) = u + iv$ is a regular function of $z$, then show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$.
13. If $u = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$, find $f(z)$ such that $f(z)$ is analytic.

    **(Ans.** $f(z) = \cot z + C$**)**
14. Show that $\phi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ can represent the velocity potential in an incompressible fluid flow. Also find the corresponding stream function and complex potential. $\left[\textbf{Ans. } \psi = 2xy - \dfrac{y}{x^2 + y^2} + C;\ f(z) = z^2 + \dfrac{1}{z} + iC\right]$

# 22 Conformal Mapping

## Chapter Outline

## 22.1 ❑ INTRODUCTION

Many physical problems involving ideal fluid flow, steady-state heat flow, electrostatics, magnetism, current flow etc., can be solved using conformal mapping techniques. These problems generally involve Laplacian in three-dimensional coordinates and also divergence and are of three-dimensional vector functions.

### *Geometrical Representation*

To draw the curve of a complex variable $(x, iy)$, we take two axes, i.e., the first one is the real axis and the other is the imaginary axis. A number of points $(x, y)$ are plotted on the $z$-plane, by taking different values of $z$ (different values of $x$ and $y$). The curve $C$ is drawn by joining the plotted points. The diagram obtained is called an **Argand diagram**.

Let $w = f(z) = f(x + iy) = u + iv$.

To draw a curve of $w$, we take the $u$-axis and $v$-axis. By plotting different points $(u, v)$ on the $w$-plane and joining them, we get a curve $C$ on the $w$-plane.

### *Transformation*

For every point $(x, y)$ in the $z$-plane, the relation $w = f(z)$ defines a corresponding point $(u, v)$ in the $w$-plane. We call this **transformation or mapping of $z$-plane into $w$-plane.** If a point $z_0$ maps into the point $w_0$, $w_0$ is also known as the image of $z_0$.

If the point $P(x, y)$ moves along the curve $C$ in the $z$-plane, the point $P'(u, v)$ will move along a corresponding curve $C_1$ in the $w$-plane. We then say that a curve $C$ in the $z$-plane is mapped into the corresponding curve $C_1$ in the $w$-plane by the relation $w = f(z)$.

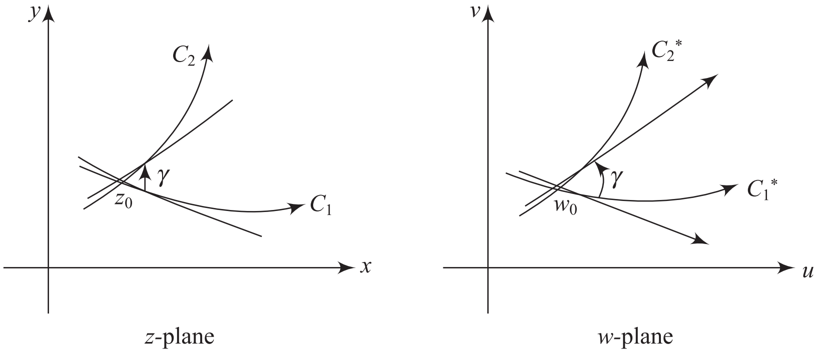


**Fig. 22.1**

## 22.2 ❑ CONFORMAL TRANSFORMATION (OR CONFORMAL MAPPING)

A mapping $w = f(z)$ is said to be **conformal** if the angle between any two smooth curves $C_1$, $C_2$ in the $z$-plane intersecting at the point $z_0$ is equal in magnitude and sense to the angle between their images $C_1^*, C_2^*$ in the $w$-plane at the point $w_0 = f(z_0)$.

Thus, **conformal mapping** preserves angles both in magnitude and sense (which is also known as conformal mapping of the first kind). If only the magnitude of the angle is preserved, then the mapping is known as **isogonal mapping** (or conformal mapping of the second kind).

Conformal mapping is used to map complicated regions conformally onto simpler, standard regions such as circular disks, half-planes and strips for which the boundary-value problems are easier.

Given two mutually orthogonal one-parameter family of curves, say $\phi(x, y) = C_1$ and $\phi(x, y) = C_2$. Their image curves in the $w$-plane $\phi(u, v) = C_3$ and $\phi(u, v) = C_4$ under a conformal mapping are also mutually orthogonal. Thus, conformal mapping preserves the property of mutual orthogonality of a system of curves in the plane.

➢ **Note**

(i) **Critical point** of a function $w = f(z)$ is a point $z_0$, where $f'(z_0) \neq 0$.

(ii) A mapping $w = f(z)$ is conformal at each point $z_0$ where $f(z)$ is analytic and $f'(z_0) \neq 0$

(iii) An analytic function $f(z)$ is conformal everywhere except at its critical points where $f'(z) \neq 0$.

(iv) Solutions of Laplace's equation are invariant under conformal transformation.

(v) Conjugate functions remain conjugate functions after conformal transformation. This is the main reason for the great importance of conformal transformations in applications.

## 22.3 ❑ CONFORMAL MAPPING BY ELEMENTARY TRANSFORMATIONS

General linear transformation, or simply transformation, is defined by the function

$$w = f(z) = az + b \tag{22.1}$$

where $a \neq 0$ and $b$ are arbitrary complex constants. The function maps conformally the extended complex $z$-plane onto the extended $w$-plane, since this function is analytic and $f'(z) = a \neq 0$ for any $z$. If $a = 0$ (22.1) reduces to a constant function.

## 22.4 ❑ SOME STANDARD TRANSFORMATIONS

### *Translation*

The transformation $w = z + c$, where $c$ is a complex constant, represents a translation. Consider the transformation $w = z + c$, where $c = a + ib$.

i.e., $u + iv = (x + iy) + (a + ib)$

$\Rightarrow$ $u = x + a$ and $v = y + b$

i.e., $x = u - a$ and $y = v - b$

On substituting the values of $x$ and $y$ in the equation of the curve to be transformed, we get the equation of the image in the $w$-plane.

The point $P(x, y)$ in the $z$-plane is mapped onto the point $P'(x + a, y + b)$ in the $w$-plane. Similarly, other points of the $z$-plane are mapped onto the $w$-plane. Thus, if the $w$-plane is superposed on the $z$-plane, the figure of the $w$-plane is shifted through a vector $c$.

In other words, the transformation is a mere **translation** of the axes.

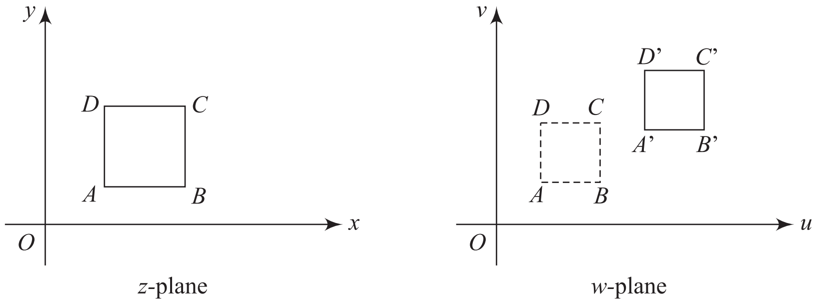


**Fig. 22.2**

### *Magnification and Rotation*

Consider the transformation $w = cz$ (22.2)

where $c$, $z$, $w$ are all complex numbers.

Let $z = re^{i\theta}$, $w = \text{R}e^{i\phi}$, $c = ae^{i\alpha}$

Substituting these values in (22.2), we have

$$\text{R}e^{i\phi} = (ae^{i\propto})(re^{i\theta}) = ar\, e^{i(\theta + \alpha)}$$

i.e., $R = ar$ and $\phi = \theta + \alpha$

Thus, we see that the transformation $w = cz$ corresponds to a rotation together with magnification.

Algebraically, $w = cz$ or $u + iv = (a + ib)(x + iy)$

$u + iv = ax - by + i(ay + bx)$

$\Rightarrow$ $u = ax - by$ and $v = ay + bx$.

On solving these equations, we can get the values of $x$ and $y$.

i.e., $$x = \frac{au + bv}{a^2 + b^2};\ y = \frac{-bu + av}{a^2 + b^2}$$

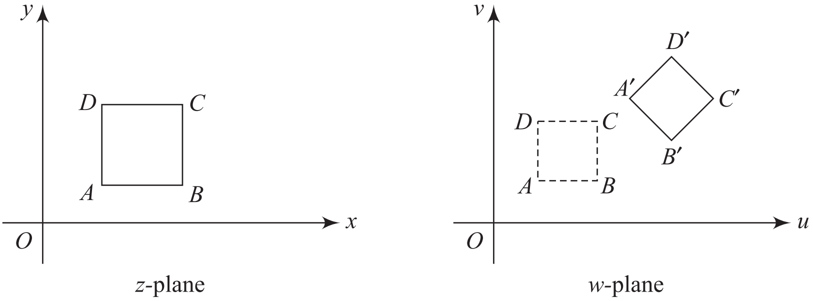


**Fig. 22.3**

On putting the values of $x$ and $y$ in the equation of the curve to be transformed, we get the equation of the image.

### *Inversion and Reflection* [KU April 2012]

Consider the transformation $w = \dfrac{1}{z}$ (22.3)

$$z = re^{i\theta} \quad \text{and} \quad w = \text{R}e^{i\phi}$$

Substituting these values in (22.3), we get

$$\text{R}e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\Rightarrow \qquad R = \frac{1}{r} \text{ and } \phi = -\theta$$

Thus, the point $P(r, \theta)$ in the $z$-plane is mapped onto the point $P'\left(\dfrac{1}{r}, -\theta\right)$ in the $w$-plane. Hence, the transformation is an inversion of $z$ followed by reflection into the real axis. The points inside the unit circle $|z| = 1$ map onto points outside it, and points outside the unit circle into points inside it.

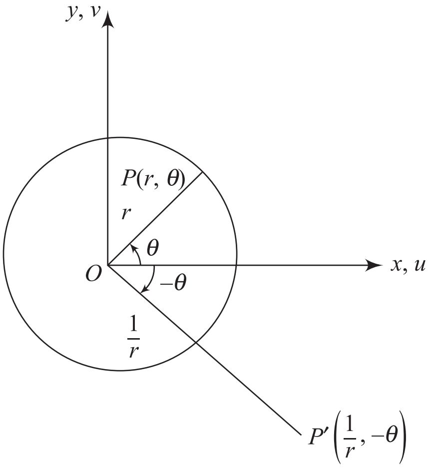


**Fig. 22.4**

Now consider the transformation $w = \dfrac{1}{z}$ or $z = \dfrac{1}{w}$.

i.e., $$x + iy = \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{(u + iv)(u - iv)} = \frac{u - iv}{u^2 + v^2}$$

$$\therefore \qquad x = \frac{u}{u^2 + v^2}, \; y = \frac{-v}{u^2 + v^2}$$

Let the circle $a(x^2 + y^2) + bx + cy + d = 0$ (22.4)
be in the $z$-plane.
If $a \neq 0$, (22.4) represents a circle and if $a = 0$, it represents a straight line.
On substituting the values of $x$ and $y$ in (22.4), we get

$$\frac{a}{u^2 + v^2} + \frac{bu}{u^2 + v^2} - \frac{cv}{u^2 + v^2} + d = 0$$

$$\Rightarrow \qquad d(u^2 + v^2) + bu - cv + a = 0 \tag{22.5}$$

If $d \neq 0$ Eq. (22.5) represents a circle and if $d = 0$ it represents a straight line.
The various cases are discussed as follows:

- **When $a \neq 0$, $d \neq 0$**

The transformation $w = \frac{1}{z}$ transforms circles not passing through the origin into circles not passing through the origin.

- **When $a \neq 0$, $d = 0$**

The transformation $w = \frac{1}{z}$ transforms circles passing through the origin in the $z$-plane and maps into the straight lines not passing through the origin in the $w$-plane.

- **When $a = 0$, $d \neq 0$**

The transformation $w = \frac{1}{z}$ transforms straight lines in the $z$-plane not passing through the origin into circles through the origin in the $w$-plane.

- **When $a = 0$, $d = 0$**

The transformation $w = \frac{1}{z}$ transforms straight lines through the origin in the $z$-plane into straight lines through the origin in the $w$-plane.

## 22.5 ❑ BILINEAR TRANSFORMATION (OR MÖBIUS TRANSFORMATION)

The transformation $w = f(z) = \frac{az + b}{cz + d}$ (22.8)

where $a$, $b$, $c$, $d$ are complex or real constants subject to $ad - bc \neq 0$ is known as bilinear transformation.
Differentiating (22.8), we get

$$\frac{dw}{dz} = \frac{(cz + d)a - (az + b)c}{(cz + d)^2}$$

$$= \frac{ad - bc}{(cz + d)^2}$$

If $ad - bc \neq 0$ then $\frac{dw}{dz} \neq 0$ for any $z$ and, therefore, bilinear transformation is conformal for all $z$, i.e., it maps the $z$-plane conformally onto the $w$-plane

If $ad - bc = 0$ then $\frac{dw}{dz} = 0$ for any $z$. Then every point of the $z$-plane is critical and the function is not conformal.

From (22.8), we get $w(cz + d) = az + b$,

i.e., $$cwz + dw - az - b = 0 \tag{22.9}$$

Equation (22.9) is linear in $z$ and linear in $w$ or bilinear in $z$ and $w$. Bilinear transformation is also known as **linear fractional transformation** or **Mobius transformation.**

For a choice of the constants $a, b, c, d$, we get special cases of bilinear transformation as

(i) $w = z + b$ (Translation)

(ii) $w = az$ (Rotation)

(iii) $w = az + b$ (Linear transformation)

(iv) $w = \frac{1}{z}$ (Inversion in the unit circle)

Thus, bilinear transformation can be considered as a combination of these transformations.

### *Fixed Points (or Invariant Points)*

Fixed (or invariant) points of a function $w = f(z)$ are points which are mapped onto themselves, i.e., $w = f(z) = z$.

- **Example**

$w = z$ has every point as a fixed point.

$w = \bar{z}$ infinitely many.

$w = \frac{1}{z}$ has two.

$w = z + b$ has no fixed point.

The fixed points of the bilinear transformation $w = \frac{az+b}{cz+d}$ are given by $\frac{az+b}{cz+d} = z$.

As this is quadratic in $z$, we will get two fixed points for the bilinear transformation.

### *Cross-ratio*

The **cross-ratio,** or **anharmonic ratio,** of four numbers $z_1, z_2, z_3, z_4$ is the linear function given by $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$.

➢ **Note**

(i) The cross-ratio of four points is invariant under a bilinear transformation, i.e., if $w_1, w_2, w_3, w_4$ are the images of $z_1, z_2, z_3, z_4$ respectively under a bilinear transformation, then $\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_1 - w_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$.

(ii) The bilinear transformation that maps three given points $z_2$, $z_3$, $z_4$ onto three given points $w_2$, $w_3$, $w_4$ is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w-w_3)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z-z_3)}$$

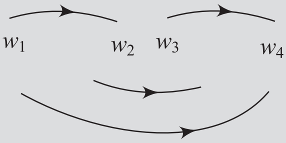


**Fig. 22.5**

## SOLVED EXAMPLES

**Example 1** Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$.

**Solution** Let $z = x + iy$; $w = u + iv$

Given $\quad w = z + 3 + 2i$

i.e., $\quad u + iv = (x + iy) + (3 + 2i)$

$\Rightarrow \quad u = x + 3$; $v = y + 2$

Given the circle $|z| = 2$

i.e., $\quad x^2 + y^2 = 4$

i.e., $\quad (u - 3)^2 + (v - 2)^2 = 4$

Hence, the circle $x^2 + y^2 = 4$ maps into $(u - 3)^2 + (v - 2)^2 = 4$ in the $w$-plane which is also a circle with centre at (3, 2) and radius of 2 units. **Ans.**

**Example 2** Find the image of the triangular region in the $z$-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ under the transformation $w = 2z$. **[KU May 2010]**

**Solution** Given $w = 2z$. i.e., $u + iv = 2(x + iy)$

$\therefore \quad u = 2x \quad$ and $\quad v = 2y$

When $x = 0$, $u = 0$, the line $x = 0$ is transformed into the line $u = 0$ in the $w$-plane.

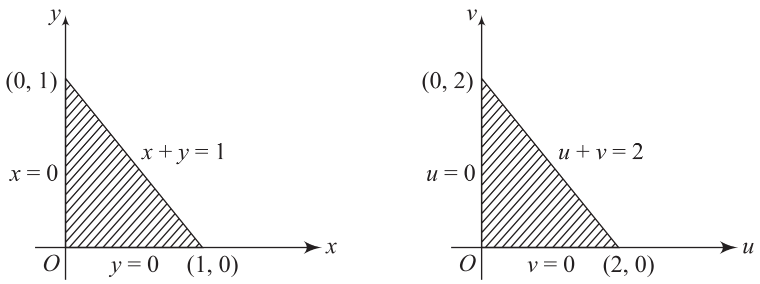
When $y = 0$, $v = 0$, the line $y = 0$ is transformed into the line $v = 0$ in the $w$-plane.

When $x + y = 1$, we get

$$\frac{u}{2}+\frac{v}{2}=1$$

$\Rightarrow \quad u + v = 2$

$\therefore$ the line $x + y = 1$ is transformed into the line $u + v = 2$ in the $w$-plane.



*z* Plane *w* Plane

**Fig. 22.6**

**Example 3** Find the image of the circle $|z-1| = 1$ in the complex plane under the mapping $w = \frac{1}{z}$.

**Solution** The given transformation is $w = \frac{1}{z}$

i.e., $$z = \frac{1}{w}$$

The equation of the circle is $|z-1| = 1$

i.e., $$|x+iy-1| = 1$$

$$(x-1)^2 + y^2 = 1 \quad \Rightarrow \quad x^2 - 2x + y^2 = 0 \qquad (1)$$

Now, $w = u + iv$

$$\therefore \quad z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \frac{u-iv}{u^2+v^2}$$

$$\therefore \quad x = \frac{u}{u^2+v^2} \qquad (2)$$

and $$y = \frac{-v}{u^2+v^2} \qquad (3)$$

Substituting (2) and (3) in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2 - 2\left(\frac{u}{u^2+v^2}\right) + \left(\frac{-v}{u^2+v^2}\right)^2 = 0$$

i.e., $$u^2 - 2u(u^2+v^2) + v^2 = 0$$

$$(u^2+v^2)(1-2u) = 0$$

$\Rightarrow$ $$1 - 2u = 0 \qquad (\text{since } u^2 + v^2 \neq 0)$$

i.e., $2u - 1 = 0$ which is a straight line in the $w$-plane. Hence, the circle $|z-1| = 1$ is mapped into a straight line under the transformation $w = \frac{1}{z}$. **Ans.**

**Example 4** Find the image of the infinite strips (i) $\frac{1}{4} < y < \frac{1}{2}$; and (ii) $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. **[KU April 2013]**

**Solution** Let $w = u + iv$, $z = x + iy$.

Given $$w = \frac{1}{z}$$

i.e., $$u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

i.e., $$u = \frac{x}{x^2+y^2} \qquad (1)$$

$$v = \frac{-y}{x^2+y^2} \qquad (2)$$

Now, $\frac{u}{v} = \frac{-x}{y}$.

i.e., $$x = \frac{-uy}{v} \tag{3}$$

Substituting (3) in (2), we get

$$v = \frac{-y}{\frac{u^2y^2}{v^2} + y^2} = \frac{-v^2}{(u^2+v^2)\cdot y}$$

or $$y = \frac{-v}{u^2+v^2} \tag{4}$$

(i) Consider a strip $\frac{1}{4} < y < \frac{1}{2}$.

When $y = \frac{1}{4}$,

From (4), $\frac{1}{4} = \frac{-v}{u^2+v^2}$

i.e., $u^2 + v^2 + 4v = 0$ or $u^2 + (v+2)^2 = 4$.

which is a circle whose centre is at (0, –2) in the $w$-plane and radius is 2 units.

When $y = \frac{1}{2}$,

From (4), $\frac{-v}{u^2+v^2} = \frac{1}{2}$

i.e., $u^2 + (v+1)^2 = 1$.

which is a circle whose centre is at (0, –1) in the $w$-plane and the radius is 1 unit.

Hence, the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region common to the circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the $w$-plane.

(ii) Consider a strip $0 < y < \frac{1}{2}$.

When $y = 0$,
from (4), we get $v = 0$.

When $y = \frac{1}{2}$,

from (4), we get $\frac{1}{2} = \frac{-v}{u^2+v^2}$.

i.e., $u^2 + v^2 + 2v = 0$

i.e., $u^2 + (v+1)^2 - 1 = 0$

which is a circle whose centre is at (0, –1) in the $w$-plane and radius is 1 unit.

$\therefore$ the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half-plane. **Ans.**

**Example 5** Find the invariant points of the transformation $w = -\dfrac{2z+4i}{iz+1}$.

**Solution** The invariant points of the transformation are given by $z = -\dfrac{2z+4i}{iz+1}$

$\Rightarrow$ $iz^2 + 3z + 4i = 0$

i.e., $z^2 - 3iz + 4 = 0$

i.e., $(z - 4i)(z + i) = 0$

i.e., $z = 4i, -i$ are the invariant points. **Ans.**

**Example 6** Find the image of $|z + 2i| = 2$ under the transformation $w = \dfrac{1}{z}$.

**[AU May 2010]**

**Solution** The given transformation is $w = \dfrac{1}{z}$

i.e., $z = \dfrac{1}{w}$

Given $|z + 2i| = 2$

$|x + iy + 2i| = 2$

i.e., $|x + i(y + 2)| = 2$

$\Rightarrow$ $x^2 + (y + 2)^2 = 4$

i.e., $x^2 + y^2 + 4y = 0$ (1)

Now, $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

i.e., $x + iy = \dfrac{u-iv}{u^2+v^2}$

$\Rightarrow$ $x = \dfrac{u}{u^2+v^2}$, (2)

and $y = \dfrac{-v}{u^2+v^2}$ (3)

Substituting (2) and (3) in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 + 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$u^2 + v^2 - 4v(u^2 + v^2) = 0$$

$$(u^2 + v^2)(1 - 4v) = 0$$

$\Rightarrow$ $1 - 4v = 0$ (as $u^2 + v^2 \neq 0$)

which is a straight line in the $w$-plane. **Ans.**

**Example 7** Find the bilinear transformation that maps the points $z_1 = -i$, $z_2 = 0$, $z_3 = i$ into the points $w_1 = -1$, $w_2 = i$, $w_3 = 1$ respectively. **[AU Oct. 2009, KU Nov. 2010]**

**Solution** Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \tag{1}$$

Given $z_1=-i, z_2=0, z_3=0; w_1=-1, w_2=i, w_3=1$ (2)

Substituting (2) in (1), we get

$$\frac{(w+1)(i-1)}{(-1-i)(1-w)}=\frac{(z+i)(0-i)}{(-i-0)(i-z)}$$

i.e., $$\frac{(w+1)}{(w-1)}\frac{(i-1)(i-1)}{(i+1)(i-1)}=\frac{-(z+i)}{(z-i)}$$

i.e., $$\frac{w+1}{w-1}\cdot\frac{-2i}{-2}=\frac{-(z+i)}{(z-i)}$$

$$\frac{w+1}{w-1}=\frac{i(z+i)}{z-i}$$

By componendo and dividendo,

$$\frac{(w+1)+(w-1)}{(w+1)-(w-1)}=\frac{i(z+i)+(z-i)}{i(z+i)-(z-i)}$$

$$\frac{2w}{2}=\frac{z(1+i)-(1+i)}{z(i-1)-(1-i)}$$

$$w=\frac{(1+i)(z-1)}{(i-1)(z+1)}$$

$$=\frac{(1+i)(-i-1)}{(i-1)(-i-1)}\cdot\frac{(z-1)}{(z+1)}$$

$\Rightarrow$ $$w=-\left(\frac{z-1}{z+1}\right)$$ **Ans.**

**Example 8** Find the bilinear transformation which maps the points $z_1=-1$, $z_2=0$, $z_3=1$ into the points $w_1=0$, $w_2=i$, $w_3=3i$ respectively.

**[AU Nov. 2010, KU April 2012]**

**Solution** Let the bilinear translation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \tag{1}$$

Given $z_1=-1, z_2=0, z_3=1; w_1=0, w_2=i, w_3=3i$ (2)

Substituting (2) in (1), we get

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)}=\frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{w(-2i)}{-i(3i-w)}=\frac{(z+1)}{1-z}$$

$$\frac{-2iw}{(w-3i)i}=-\left(\frac{z+1}{z-1}\right)$$

i.e., $$\frac{2w}{w-3i}=\frac{z+1}{z-1}$$

$$2w(z-1)=(z+1)(w-3i)$$

$$=zw-3iz+w-3i$$

⇒ $$w[2(z-1)-(z+1)]=-3i(z+1)$$

or $$w=-3i\frac{(z+1)}{z-3}$$ **Ans.**

**Example 9** Show that under the mapping $w=\frac{i-z}{i+z}$, the image of the circle $x^2 = y^2 < 1$ is the entire half of the $w$-plane to the right of the imaginary axis.

**[AU Nov. 2011]**

**Solution** Given $w=\frac{i-z}{i+z}$

i.e., $(i + z)w = i - z$

$iw + zw = i -z$

i.e., $z(w + 1) = i(1 - w)$

⇒ $$z=\frac{i(1-w)}{1+w}$$

Also given $x^2 + y^2 < 1$

i.e., $|z| < 1$, i.e., $\left|\frac{i(1-w)}{1+w}\right|<1$

i.e., $|i|\ |1 - w| < |1 + w|$, i.e., $|1 - u - iv| < |1 + u + iv|$ [as $|i| = 1$]

i.e., $(1 - u)^2 + v^2 < (1 + u)^2 + v^2$

i.e., $1 + u^2 - 2u + v^2 < 1 + u^2 + 2u + v^2$

⇒ $4u > 0$

or $u > 0$

Hence, the circle $x^2 + y^2 < 1$, i.e., $|z| < 1$ is mapped into the entire half of the $w$-plane to the right of the imaginary axis.

When $|z| = 1$ i.e., $x^2 + y^2 = 1$ which is the unit circle, we get $u = 0$ which is the imaginary axis of the $w$-plane. **Proved.**

## EXERCISE

### Part A

1. Define conformal mapping.
2. When is a transformation said to be isogonal? Prove that the mapping $w=\overline{z}$ is isogonal.
3. Define critical point of a transformation.

4. Find the images of the circle $|z| = a$ under the transformations (i) $w = z + 2 + 3i$, and (ii) $w = 2z$.
5. Under the transformation $w = iz + i$, show that the half-plane $x > 0$ maps into the half-plane $w > 1$.
6. Find the invariant point of the bilinear transformation $w = \dfrac{1+z}{1-z}$.
7. Find the fixed points of $w = \dfrac{3z-4}{z-1}$.
8. Define Mobius transformation.
9. Find the invariant point of the transformation $w = \dfrac{1}{z-2i}$.
10. Find the image of $x^2 + y^2 = 4$ under the transformation $w = 3z$.
11. Find the image of the circle $|z - \alpha| = r$ by the mapping $w = z + c$ where $c$ is a constant.
12. Find the fixed points of the transformation $w = \dfrac{1}{z+2i}$.
13. Find the invariant points of the transformation $w = \dfrac{1+z}{1-z}$.
14. Find the image of the circle $|z| = 3$ under the transformation $w = 2z$.
15. Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$.
16. Find the image of the real axis of the $z$-plane by the transformation $w = \dfrac{1}{z+i}$.
17. Define cross-ratio of four points in a complex plane.
18. Prove that a bilinear transformation has at most two fixed points.

## Part B

1. For the mapping $w = \dfrac{1}{z}$, find the image of the family of circles $x^2 + y^2 = ax$, where $a$ is real. $\left(\textbf{Ans. } u = \dfrac{1}{a}, \text{ is a straight line}\right)$
2. Determine the region of the $w$-plane into which the region bounded by $x = 1$, $y = 1$, $x + y = 1$ is mapped by the transformation $w = z^2$.
   **(Ans.** $4u + v^2 = 4$, $4u - v^2 = -4$, $u^2 = 2$, $v^2 = 1$**)**
3. Determine the images of the regions under $w = \dfrac{1}{z}$. (i) $x > 1, y > 0$ (ii) $0 < y < \dfrac{1}{2c}$.
   $$\left[\textbf{Ans. } \text{(i) } \left|w - \frac{1}{2}\right| < \frac{1}{2} \text{ (ii) } u^2 + (v+c)^2 > c^2\right]$$
4. Find an analytic function $w = f(z)$ which maps the half-plane $x \geq 0$ onto the region $u \geq 2$ such that $z = 0$ corresponds to $w = 2 + i$.
   **(***Hint*: $w_1 = z$, $w_2 = w_1 + 2$, $w = w_2 + \mathrm{i}$**)**
   **(Ans.** $w = z + 2 + i$**)**
5. Determine and plot the images of the regions under the transformation $w = z^2$.

   (i) $|z| = 2$ (ii) $|\arg z| \leq \dfrac{\pi}{2}$ (iii) $\dfrac{1}{2} < |z| < 2, \operatorname{Re} z \geq 0$
   $$\left[\textbf{Ans. } \text{(i) } 1w > 4 \text{ (ii) } |\arg w| \leq \pi \text{ (iii) } \frac{1}{4} < |w| < 4, -\pi \leq \phi \leq \pi\right]$$

6. Find the invariant (fixed) points of the transformation:

   (i) $w=\frac{z-1}{z+1}$ (ii) $w=z^2$ (iii) $w=\frac{2z-5}{z+4}$ (iv) $w=(z-i)^2$

   $$\left[\textbf{Ans.}\ \text{(i) } z=\pm i \ \text{ (ii) } z=0,1 \ \text{ (iii) } z=-1+2i \ \text{ (iv) } z=\frac{(1+2i)\pm\sqrt{1+4i}}{2}\right]$$

7. Find the bilinear transformation that maps $z_1, z_2, z_3$ onto $w_1, w_2, w_3$ respectively.
   (i) $z = -1, 0, 1$ onto $w = 0, i, 3i$
   (ii) $z = 0, -i, -1$ onto $w = i, 1, 0$
   (iii) $z = 1, i, -1$ onto $w = 2, i, -2$
   (iv) $z = \infty, i, 0$ onto $w = 0, i, \infty$
   (v) $z = 1, 0, -1$ onto $w = i, 1, \infty$

   $$\left[\begin{array}{l}\textbf{Ans.}\ \text{(i) } w=\frac{-3i(z+1)}{z-3},\ \text{(ii) } w=-i\left(\frac{z+1}{z-1}\right)\ \text{(iii) } w=\frac{-6z+2i}{iz-3}\\ \quad\text{(iv) } w=-\frac{1}{z}\ \text{(v) } w=\frac{(-1+2i)z+1}{z+1}\end{array}\right]$$

8. Verify that the equation $w=\frac{1+iz}{1+z}$ maps the exterior of the circle $|z| = 1$ into the upper half-plane $v > 0$.
9. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. **(Ans.** $i$, $2i$**)**
10. Show that the transformation $w=\frac{i(1-z)}{1+z}$ maps the circle $|z| = 1$ into the real axis of the $w$-plane and the interior of the circle $|z| < 1$ into the upper half of the $w$-plane.
11. Show that the transformation $w=\frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ onto the straight line $4u + 3 = 0$.
12. Show that transformation $w=\frac{i-z}{i+z}$ maps the circle $|z| = 1$ onto the imaginary axis of the $w$-plane. Find also the images of the interior and exterior of this circle.

| S.No | **Questions** |
|---|---|
| 1 | An example of single valued function of z is ________. |
| 2 | An example of multiple valued function of z is _______. |
| 3 | The distance between two points z and z0 is |
| 4 | A circle of radius 1 with centre at origin can be represented by _______. |
| 7 | If f(z) is differentiable at z0 then f(z) is ________ at z0. |
| 8 | A function is said to be _________ at a point if its derivative exists not only at point but also in some neighborhood of that point. |
| 9 | A function which is analytic everywhere in the finite plane is called ___________. |
| 11 | The necessary condition for f(z) to be analytic is_________ |
| 12 | A real function of two variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called __________. |
| 13 | If u and v are harmonic functions such that u+iv is analytic then each is called the _________ of the other. |
| 14 | A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is called _______ at that point. |
| 15 | A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be __________ at that point. |
| 16 | A mapping w = f(z) is said to be conformal at z = z0 if __________. |
| 17 | The point at which the mapping w = f(z) is not conformal, that is, f'(z) = 0 is called ________ of the mapping. |
| 18 | A _________ point of a mapping w = f(z) are points that are mapped onto themselves, are kept fixed under the mapping. |
| 19 | The transformation w = a+z where a is a complex constant, represents a __________. |
| 20 | The transformation _________ where a is a complex constant represents a translation. |
| 21 | The transformation _________ where a is a real constant represents magnification. |
| 22 | The transformation w = az where a is a real constant represents _________ |
| 23 | In general linear transformation, w = az+b where a and b are complex constants represents _______. |
| 24 | The transformation w=(az+b)/(cz+d), where a, b, c, d are complex numbers is called a ______. |
| 25 | A bilinear transformation is also called a _______. |
| 26 | The value of i = |
| 27 | _________ represents the interior of the circle excluding its circumference. |
| 28 | _________ represents the interior of the circle including its circumference. |
| 29 | _________ represents the exterior of the circle. |
| 30 | Cauchy-Riemann equations are necessary conditions for a function w = f (z) to be an _____. |
| 31 | Cauchy-Riemann equations are |
| 32 | The real and imaginary parts of an analytic function f(z) = u+iv satisfies the ______ equation in two dimensions. |
| 33 | An analytic function with a constant real part is __________. |

| Opt 1 | Opt 2 | Opt 3 | Opt 4 | |
|---|---|---|---|---|
| w = z^2 | w = z^(1/2) | w=SQRT(z) | w=z^-1 | |
| w = z^2 | w = z^(1/2) | w=SQRT(z) | w=z^-1 | |
| \|z-z0\| | \|z+z0\| | z | z0 | |
| \|z\|>1 | \|z\| < 1 | \|z\|=1 | \|z\| = 0 | |
| discontinuous | continuous | regular | irregular | |
| entire function | integral function | analytic | continuous | |
| analytic function | holomorphic functi | regular function | entire function | |
| u_x = v_y and v_x = -u | u_x = -v_y and v_x | u_x = v_y and v_x = | u_x = -v_y and v_x = -u_y | |
| analytic function | regular function | holomorphic function | harmonic function | |
| conjugate harmonic | analytic | entire function | not analytic | |
| Conformal | isogonal | entire function | unconformal | |
| Conformal | isogonal | entire function | unconformal | |
| f'(z0) = 0 | f'(z0) = f(z) | f'(z0) ≠ 0 | f'(z0) ≠ f(z) | |
| common | fixed | invariant | critical | |
| common | fixed | critical | variant | |
| translation | magnification | rotation | reflection | |
| w = az | w = az+b | w = a+z | w = 1/z | |
| w = a+z | w = 1/z | w = az+b | w = az | |
| translation | magnification | reflection | inversion | |
| magnification | rotation | translation | magnification, rotation and translation | |
| Linear transformation | bilinear transformation | fractional transformation | translation | |
| linear transformation | inversion | fractional transformation | linear fractional transformation | |
| SQRT(-1) | SQRT(1) | -1 | 1 | |
| \|z – z0\| > delta | \|z – z0\| < delta | \|z – z0\| ≥ delta | \|z – z0\| ≤ delta | |
| \|z – z0\| > delta | \|z – z0\| < delta | \|z – z0\| ≥ delta | \|z – z0\| ≤ delta | |
| \|z – z0\| > delta | \|z – z0\| < delta | \|z – z0\| ≥ delta | \|z – z0\| ≤ delta | |
| entire function | integral function | analytic function | continuous function | |
| u_x = v_y and v_x = -u | u_x = -v_y and v_x | u_x = v_y and v_x = | u_x = -v_y and v_x = -u_y | |
| Cauchy-Riemann | Homogeneous | Laplace | Euler | |
| a variable | a constant | an analytic function | an entire function | |

|  | Answer |
|---|---|
|  | w = z^2 |
|  | w = z^(1/2) |
|  | \|z-z0\| |
|  | \|z\| = 1 |
|  | continuous |
|  | analytic |
|  | entire function |
|  | u_x = v_y and v_x = |
|  | harmonic function |
|  | conjugate harmonic |
|  | Conformal |
|  | isogonal |
|  | f’(z0) ≠ 0 |
|  | critical |
|  | fixed |
|  | translation |
|  | w = a+z |
|  | w = az |
|  | magnification |
|  | magnification, rotation and translation |
|  | bilinear transformation |
|  | linear fractional transformation |
|  | SQRT(-1) |
|  | \|z – z0\| < delta |
|  | \|z – z0\| ≤ delta |
|  | \|z – z0\| > delta |
|  | analytic function |
|  | u_x = v_y and v_x = |
|  | Laplace |
|  | a constant |

| | |
|---|---|
| 34 | An analytic function with a constant modulus is __________. |
| 35 | A fixed point is also called as _________. |
| 36 | The fixed point of w=(5z+4)/(z+5) is |
| 37 | The critical point of z=(2z+1)/(z+2) is |
| 38 | Solutions of Laplace's equation are ________ under conformal transformation |
| 39 | If f(z) is analytic, and f'(z)=0 everywhere, then f(z) is _____ |
| 40 | An analytic function with a constant imaginary part is __________. |
| 41 | If u+iv is analytic, then v-iu is _______ |
| 44 | w=z has every point as a _______ point |
| 45 | w=1/z has _______ fixed points |
| 46 | w=z+b has ________ fixed points |

| | | | | |
|---|---|---|---|---|
| a variable | a constant | an analytic function | an entire function | |
| invariant points | critical points | common point | origin | |
| 2,1 | 1,-1 | -2, 2 | 0, 1 | |
| 1, 1 | 1, -1 | 1,2 | 0,1 | |
| common | fixed | invariant | critical | |
| a variable | a constant | an analytic function | an entire function | |
| a variable | a constant | an analytic function | an entire function | |
| entire function | integral function | analytic | continuous | |
| fixed | critical | invariant | common | |
| 1 | 2 | 3 | 4 | |
| 0 | 1 | 2 | 3 | |

| | |
|---|---|
| | a constant |
| | invariant points |
| | -2, 2 |
| | 1, -1 |
| | invariant |
| | a constant |
| | a constant |
| | analytic |
| | fixed |
| | 2 |
| | 0 |

# Unit X

# Complex Integration

# 23 Complex Integration

## Chapter Outline

- Introduction
- Line Integral in a Complex Plane
- Line Integral
- Basic Properties of Line Integrals
- Simply Connected Region and Multiply Connected Region
- Evaluation of Complex Integrals
- Cauchy's Integral Theorem
- Extension of Cauchy's Integral Theorem to Multiply Connected Regions
- Cauchy's Integral Formula
- Cauchy's Integral Formula for the Derivation of an Analytic Function

## 23.1 ❑ INTRODUCTION

Integration of functions of a complex variable plays a very important role in many areas of science and engineering. The advantage of complex integration is that certain complicated real integrals can be evaluated and properties of analytical functions can be established. Using integration, we shall prove a very important result in the theory of analytic functions:

**If a function $f(z)$ is analytic in a domain $D$ then it possesses derivatives of all orders in $D$, that is $f'(z)$, $f''(z)$ … are all analytic functions in $D$.**

Such a result does not exist in the real-variable theory. Also, the complex-integration approach can be used to evaluate many improper integrals of a real variable, which cannot be evaluated using real integral calculus. The concept of definite integral for functions of a real variable does not directly extend to the case of complex variables.

In the case of a real variable, the path of integration in the definite integral $\int_a^b f(x)dx$ is along a straight line. In complex integration, the path could be along any curve from $z = a$ to $z = b$.

## 23.2 ❑ LINE INTEGRAL IN COMPLEX PLANE

- **Continuous Arc**

The set of points $(x, y)$ defined by $x = \phi(t)$, $y = \psi(t)$, with parameter $t$ in the interval $(a, b)$, defines a continuous arc provided $\phi$ and $\psi$ are continuous functions.

- **Smooth Arc**

If $\phi$ and $\psi$ are differentiable, the arc is said to be smooth.

- **Simple Curve**

It is a curve having no self-intersections, i.e., no two distinct values of $t$ correspond to the same point $(x, y)$.

- **Closed Curve**

It is one in which end points coincide, i.e., $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$.

- **Simple Closed Curve**

It is a curve having no self-intersections and with coincident end points.

- **Contour**

It is a continuous chain of a finite number of smooth arcs.

- **Closed Contour**

It is a piecewise smooth closed curve without points of self-intersection.

## 23.3 ❑ LINE INTEGRAL

Definite integral or complex line integral or simply line integral of a complex function $f(z)$ from $z_1$ to $z_2$ along a curve $C$ is defined as

$$\begin{aligned}\int_C f(z)dz &= \int_C (u+iv)(dx+i\,dy)\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\end{aligned}$$

Here, $C$ is known as path of integration. If it is a closed curve, the line integral is denoted by $\oint_C$.

When the direction is in positive sense, it is indicated as $\int_{C+}$ or simply, $\int_C$ while negative direction is denoted by $\int_C$. Counter integral is an integral along a closed contour.

## 23.4 ❑ BASIC PROPERTIES OF LINE INTEGRALS

(i) Linearity: $\int_C (k_1 f(z) + k_2 g(z))\,dz = k_1 \int_C f(z)dz + k_2 \int_C g(z)dz$

(ii) Sense reversal: $\int_a^b f(z)dz = -\int_b^a f(z)dz$

(iii) Partitioning of path: $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$ where the curve $C$ consists of the curves $C_1$ and $C_2$.

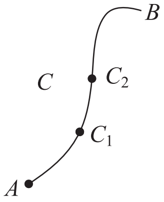


**Fig. 23.1**

➢ **Note**

> Although real definite integrals are interpreted as area, no such interpretation is possible for complex definite integrals.

## 23.5 ❑ SIMPLY CONNECTED REGION AND MULTIPLY CONNECTED REGION

A simply connected region $R$ is a domain such that every simple closed path in $R$ contains only points of $R$.

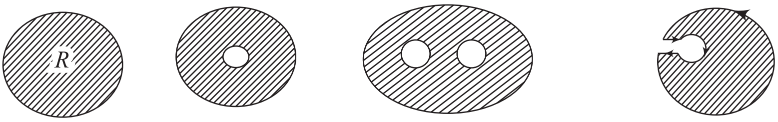
- **Example**

Interior of a circle, rectangle, triangle, ellipse, etc.

A multiply connected region is one that is not simply connected.

- **Example**

Annulus region, region with holes.



Simply connected region

Doubly connected region

Triply connected region

Simply connected region (or) Multiply connected region converted into simply connected region by cross-cuts.

**Fig. 23.2**

## 23.6 ❑ EVALUATION OF A COMPLEX INTEGRAL

To evaluate the integral $\int_C f(z)dz$, we have to express it in terms of real variables.

Let $\quad f(z) = u + iv$ where $z = x + iy$, $dz = dx + idy$

$$\therefore \quad \int_C f(z)dz = \int_C (u + iv)dz$$
$$= \int_C (u + iv)(dx + idy)$$
$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

## 23.7 ❑ CAUCHY'S INTEGRAL THEOREM OR CAUCHY'S FUNDAMENTAL THEOREM

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve $C$ then $\int_C f(z)dz = 0$.

- **Proof**

Let the region enclosed by a curve $C$ be $R$ and let

$$f(z) = u + iv,\ z = x + iy,\ dz = dx + i\,dy$$

$$\int_C f(z)dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{(by Green's theorem)}$$

Replacing $-\dfrac{\partial v}{\partial x}$ by $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ by $\dfrac{\partial u}{\partial x}$, we get

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$$= 0 + i0 = 0$$

or $\int_C f(z)dz = 0$

➢ **Note**

(i) Cauchy's integral theorem is also known as Cauchy's theorem.
(ii) Cauchy's theorem without the assumption that $f'$ is continuous is known as the **Cauchy–Goursat theorem.**
(iii) Simple connectedness is essential.

## 23.8 ❑ EXTENSION OF CAUCHY'S INTEGRAL THEOREM TO MULTIPLY CONNECTED REGIONS

If $f(z)$ is analytic in the region $R$ between two simple closed curves $C_1$ and $C_2$ then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

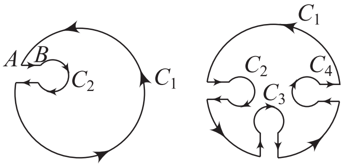


**Fig. 23.3**

- **Proof**

By Cauchy's integral theorem, we know that $\int_C f(z)dz = 0$ where the path of integration is along $AB$ and the curve $C_2$ in clockwise direction, and $BA$ and along $C_1$ in anticlockwise direction,

i.e., $$\int_{AB} f(z)dz + \int_{C_2} f(z)dz + \int_{BA} f(z)dz + \int_{C_1} f(z)dz = 0$$

or $$\int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0 \text{ (since } \int_{AB} f(z)dz = -\int_{BA} f(z)dz)$$

Reversing the direction of the integral around $C_2$, we get

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

➢ **Note**

By introducing as many cross-cuts as the number of inner boundaries, we can give the proof in a similar manner for the extension of Cauchy's integral theorem.

## 23.9 ❑ CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve $C$ and if $a$ is any point within $C$ then $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-a}dz$.

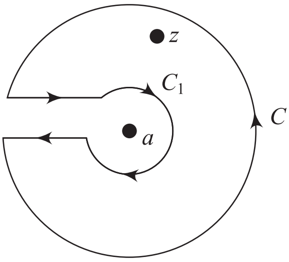


**Fig. 23.4**

- **Proof**

Consider the function $\dfrac{f(z)}{z-a}$, which is analytic at all points within $C$ except $z = a$.

With a point $a$ as centre and radius $r$, draw a small circle $C_1$ lying entirely within $C$. Now, $\dfrac{f(z)}{z-a}$ is analytic in the region between $C$ and $C_1$;

Hence, by Cauchy's integral theorem for a multiply connected region, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1}\frac{f(z)}{z-a}dz = \int_{C_1}\frac{f(z)-f(a)+f(a)}{z-a}dz$$

$$= \int_{C_1}\frac{f(z)-f(a)}{z-a}dz + f(a)\int_{C_1}\frac{dz}{z-a} \tag{23.1}$$

For any point on $C_1$

Now, $$\int_{C_1}\frac{f(z)-f(a)}{z-a}dz = \int_0^{2\pi}\frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}}ire^{i\theta}d\theta$$

[as $z - a = re^{i\theta}$ and $dz = ire^{i\theta}\,d\theta$]

$$= \int_0^{2\pi}[f(a+re^{i\theta})-f(a)]id\theta = 0$$

(where $r$ tends to zero]

$$\int_{C_1}\frac{dz}{z-a} = \int_0^{2\pi}\frac{ire^{i\theta}d\theta}{re^{i\theta}} = \int_0^{2\pi} id\theta = i[0]_0^{2\pi i} = 2\pi i$$

Putting the values of the integrals of RHS in (23.1), we have

$$\int_C \frac{f(z)}{z-a}dz = 0 + f(a)(2\pi i)$$

or $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

## 23.10 ❑ CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

If a function $f(z)$ is analytic in a region $R$ then its derivative at any point $z = a$ of $R$ is also analytic in $R$ and is given by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

where $C$ is any closed curve in $R$ surrounding the point $z = a$.

- **Proof**

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \tag{23.2}$$

Differentiating (23.2) with respect to $a$, we get

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right)\cdot dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

Similarly,

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz$$

$$f^n(a) = \frac{n!}{2\pi !}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

## SOLVED EXAMPLES

**Example 1** Use Cauchy's integral formula to evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)}dz$, where $C$ is the circle $|z| = 4$.

**[AU June 2009, April 2011; KU Nov. 2011]**

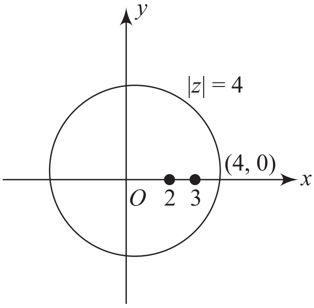


Fig. 23.5

**Solution**

$$\frac{1}{(z-2)(z-3)} = \frac{1}{(z-3)} - \frac{1}{(z-2)}$$

∴ given integral

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3}dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}dz$$

$$= \int_C \frac{f(z)}{(z-3)}dz - \int_C \frac{f(z)}{(z-2)}dz \tag{1}$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic on and inside $C$.

The points $z = 2$ and $z = 3$ lie inside $C$.

$\therefore$ by Cauchy's integral formula, from (1), we get,

$$\begin{aligned}\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz &= 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=3} - 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=2} \\ &= 2\pi i(\sin 9\pi + \cos 9\pi) - 2\pi i(\sin 4\pi + \cos 4\pi) \\ &= -2\pi i - 2\pi i = -4\pi i\end{aligned}$$

**Ans.**

**Example 2** Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$, where $C$ is the circle $|z-2| = \frac{1}{2}$, using Cauchy's integral formula. **[AU May 2012]**

**Solution** $|z-2| = \frac{1}{2}$ is the circle with centre at $z = 2$ and radius equal to $\frac{1}{2}$.

The point $z = 2$ lies inside the circle $|z-2| = \frac{1}{2}$

The given integral can be rewritten as

$$\int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2} dz = \int_C \frac{f(z)}{(z-2)^2} dz \text{ (say)}$$

$f(z) = \frac{z}{z-1}$ is analytic on and inside $C$ and the point $z = 2$ lies inside $C$.

$\therefore$ by Cauchy's integral formula,

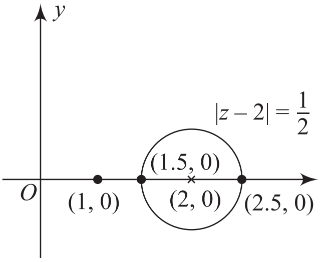


**Fig. 23.6**

$$\begin{aligned}\int_C \frac{z}{(z-1)(z-2)^2} dz &= \frac{2\pi i}{1!} f'(2) \\ &= 2\pi i \left\{ \frac{d}{dz}\left(\frac{z}{z-1}\right) \right\}_{z=2} \\ &= 2\pi i \left\{ \frac{-1}{(z-1)^2} \right\}_{z=2} = -2\pi i\end{aligned}$$

**Ans.**

**Example 3** Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where $C$ is the circle $|z+1+i| = 2$ using Cauchy's integral formula. **[AU Nov. 2011]**

**Solution** $|z+1+i| = 2$ is the circle whose centre is $-1-i$ and radius is 2 units.

Consider $\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$

$\therefore$ the integral is not analytic at $z = -1 - 2i$ and $-1 + 2i$.

The point $z = -1 - 2i$ lies inside $C$.

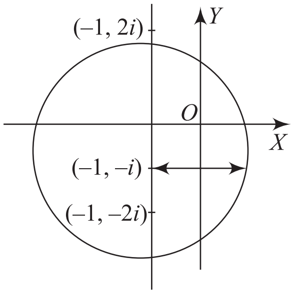


**Fig. 23.7**

We rewrite the given integral as

$$\int_C \frac{\left(\dfrac{z+4}{z+1-2i}\right)}{z+1+2i}dz = \int_C \frac{f(z)}{z-(-1-2i)}dz \text{ (say)}$$

*f*(*z*) is analytic on and inside *C* and the point (–1, –2*i*) lies inside *C*.
∴ by Cauchy's integral formula,

$$\int_C \frac{z+4}{z^2+2z+5}dz = 2\pi i\, f(-i-2i)$$

$$= 2\pi i\left\{\frac{-1-2i+4}{-1-2i+1-2i}\right\}$$

$$= \frac{-\pi}{2}(3-2i) \qquad \textbf{Ans.}$$

## EXERCISE

### Part A

1. The value of the integral $\int_C \frac{dz}{z^2-2z}$ where *C* is the circle $|z-2| = 1$, traversed in the counter-clockwise sense is
   (i) $-\pi i$ (ii) $2\pi i$ (iii) $\pi i$ (iv) 0
2. The value of the integral $\int_C \frac{z^2-z+1}{z-1}dz$, where *C* is the circle $|z| = \frac{1}{2}$ is
   (i) 0 (ii) $\pi i$ (iii) $-\pi i$ (iv) $-2\pi i$
3. What is the value of $\int_C e^z\,dz$ if $c : |z| = 1$?
4. State Cauchy's integral formula.
5. Evaluate $\int_C \frac{dz}{z-2}$ where *C* is the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).
6. Evaluate $\int_C \frac{3z^2+7z+1}{(z-3)}dz$ where $C : |z| = 2$.
7. Evaluate $\int_C \frac{dz}{z^2-5z+6}$ where *C* is the circle $|z-1| = \frac{1}{2}$.
8. State Cauchy's formula for the first derivative of an analytic function.
9. State Cauchy's fundamental theorem.
10. Evaluate $\int_C \frac{z\,dz}{z-2}$ where $C : |z| = 1$.
11. Evaluate $\int_C \frac{2}{(z+3)}dz$ where $C : |z| = 2$.
12. Evaluate $\int_C \frac{1}{2z-3}dz$ where $C : |z| = 1$.

13. Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{z^2+5}{z-3}dz$ where $C$ is $|z| = 4$ using Cauchy's integral formula.
14. Evaluate $\displaystyle\int_C \dfrac{dz}{(z-3)^2}$ where $C : |z| = 1$.
15. State the Cauchy–Goursat theorem.

## Part B

1. Evaluate $\displaystyle\int_C \dfrac{z-1}{(z+1)^2(z-2)}dz$ where $C$ is $|z-i| = 2$. $\left(\textbf{Ans.}\ \dfrac{-2\pi i}{9}\right)$
2. Evaluate $\displaystyle\int_C \dfrac{4-3z}{z(z-1)(z-2)}dz$ using Cauchy's integral formula. where $C$ is the circle $|z| = \dfrac{3}{2}$. (**Ans.** $2\pi i$)
3. Find the value of $\displaystyle\int_C \dfrac{2z^2+z}{z^2-1}dz$. (**Ans.** $3\pi i$)
4. Evaluate the following:
   (i) $\displaystyle\int_C \dfrac{dz}{(z^2+4)^2}$, where $C$ is $|z-i| = 2$
   (ii) $\displaystyle\int_C \dfrac{z^3+z+1}{z^2-7z+6}dz$ where $C$ is the ellipse $4x^2 + 9y^2 = 1$
   (iii) $\displaystyle\int_C \dfrac{z^3+1}{z^2-3iz}dz$ where $C$ is $|z| = 1$. $\left[\textbf{Ans.}\ \text{(i)}\ \dfrac{\pi}{16}, \text{(ii)}\ 0, \text{(iii)} -\dfrac{2\pi}{3}\right]$
5. Evaluate $\displaystyle\int_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}dz$ where $C$ is $|z| = 3$. (**Ans.** $-4\pi i$)
6. If $f(a) = \displaystyle\int_C \dfrac{4z^2+z+5}{z-a}dz$ where $C$ is $|z| = 2$, find the values of $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$. (**Ans.** $20\pi i$; $2\pi(i-1)$; $-14\pi i$; $16\pi i$)
7. Evaluate $\int_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1). (**Ans.** $-1 + i$)
8. Evaluate $\displaystyle\int_C \dfrac{z^2+1}{z^2-1}dz$ where (i) $C : |z-1| = 1$, (ii) $C : |z+1| = 1$, and (iii) $C : |z-i| = 1$. [**Ans.** (i) $2\pi i$ (ii) $-2\pi i$ (iii) 0]
9. Evaluate $\displaystyle\int_C \dfrac{\sin 2z}{(z+3)(z+1)^2}dz$ where $C$ is the rectangle with vertices at $3 + i$, $-2 + i$, $-2 - i$, $3 - i$. $\left[\textbf{Ans.}\ \pi i \dfrac{(4\cos 2 + \sin 2)}{2}\right]$
10. Evaluate $\displaystyle\int_C \dfrac{z^4-3z^2+6}{(z+i)^3}dz$ where $C : |z| = 2$. (**Ans.** $-18\pi i$)

# 24 Taylor and Laurent Series Expansions

**Chapter Outline**

- Introduction
- Taylor's Series
- Laurent's Series

## 24.1 ❑ INTRODUCTION

### *Power Series*

A power series in powers of $(z - z_0)$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0) + \cdots \qquad (24.1)$$

Here, $a_0$, $a_1$, $a_2$ ... are complex (or real) constants known as coefficients of the series. $z$ is a complex variable and $z_0$ is called the centre of the series. Equation (24.1) is also known as the power series about the point $z_0$.

Power series in powers of $z$ is

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

obtained as a particular case with $z_0 = 0$ in (24.1). The **region of convergence** of a series is the set of all points $z$ for which the series converges.

Three distinct possibilities exist regarding the region of convergence of a power series (24.1).

(i) The series converges only at the point $z = z_0$.

(ii) The series converges everywhere inside a circular disk $|z - z_0| < R$ and diverges everywhere outside the disk $|z - z_0| > R$. Here, $R$ is known as the **radius of convergence** and the circle $|z - z_0| = R$ as the **circle of convergence.**

➢ **Note**

(i) The series may converge or diverge at the points on the circle of convergence.

(ii) **Geometric Series:** $\sum_{m=0}^{\infty} z^m = 1 + z + z^2 + \cdots$ converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. (i.e., $R = 1$)

(iii) **Power series:** $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z$. (i.e., $R = \infty$)

Power series play an important role in complex analysis, since they represent analytic functions and conversely every analytic function has a power series representation called Taylor series similar to Taylor series in real calculus.

Analytic functions can also be represented by another type of series called **Laurent series,** which consist of positive and negative integral powers of the independent variable. They are useful for evaluating complex and real integrals.

## 24.2 ❑ TAYLOR'S SERIES (TAYLOR'S THEOREM)

If a function $f(z)$ is analytic at all points inside a circle $C$ with its centre at the point $a$ and radius $R$ then at each point $z$ inside $C$,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

- **Proof**

Take any point $z$ inside $C$. Draw a circle $C_1$ with centre $a$, enclosing the point $z$. Let $w$ be a point on the circle $C_1$.

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)}$$

$$= \frac{1}{(w-a)} \frac{1}{\left(1 - \frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

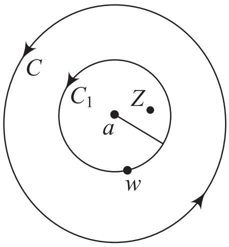


Fig. 24.1

Applying the binomial theorem,

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right]$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^{n+1}} + \cdots \quad (24.2)$$

As $|z-a| < |w-a|$ or $\frac{|z-a|}{|w-a|} < 1$,

so the series converges uniformly. Hence, the series is integrable.

Multiplying (24.2) by $f(w)$,

$$\frac{f(w)}{w-z}=\frac{f(w)}{w-a}+(z-a)\frac{f(w)}{(w-a)^2}+(z-a)^2\frac{f(w)}{(w-a)^3}+\cdots+(z-a)^n\frac{f(w)}{(w-a)^{n+1}}+\cdots$$

On integrating with respect to $w$, we get

$$\int_{C_1}\frac{f(w)}{w-z}dw=\int_{C_1}\frac{f(w)}{w-a}dw+(z-a)\int_{C_1}\frac{f(w)}{(w-a)^2}dw+\cdots$$
$$+(z-a)^n\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw+\cdots \tag{24.3}$$

We know that

$$\int_{C_1}\frac{f(w)}{(w-z)}dz=2\pi i\,f(z),\ \int_{C_1}\frac{f(w)}{w-a}dw=2\pi i\,f(a)$$

$$\int_{C_1}\frac{f(w)}{(w-a)^2}dw=2\pi i\,f'(a)\text{ , and so on.}$$

Substituting these values in (24.3), we get

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^n(a)}{n!}(z-a)^n+\cdots$$

➢ **Note**

(i) Putting $a=0$ in the Taylor's series, we get $f(z)=f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\cdots$

This series is called the **McLaurin's series** of $f(z)$.

(ii) **Standard McLaurin's Series**

(a) $e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$ for $|z|<\infty$

(b) $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$ for $|z|<\infty$

(c) $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}\cdots$ for $|z|<\infty$

(d) $\sin hz=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$ for $|z|<\infty$

(e) $\cos hz=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots$ for $|z|<\infty$

(f) $(1-z)^{-1}=1+z+z^2+z^3+\cdots$ for $|z|<1$

(g) $(1+z)^{-1}=1-z+z^2-z^3+\cdots$ for $|z|<1$

(h) $(1-z)^{-2}=1+2z+3z^2+\cdots$ for $|z|<1$

(iii) Expansion of a function $f(z)$ about a singular point $z=h$ means, expansion of $f(z)$ in powers of $(z-h)$.

## 24.3 ❑ LAURENT'S SERIES (LAURENT'S THEOREM)

If $f(z)$ is analytic on $C_1$ and $C_2$ and the annular region bounded by the two concentric circles $C_1$ and $C_2$ of radii $r_1$ and $r_2 (r_2 < r_1)$ and with centre at $a$ then for all in $R$,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots$$

where
$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, n = 0, 1, 2, 3...$$

$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, n = 1, 2, 3...$$

- **Proof**

By introducing a cross-cut $AB$, the multiply connected region $R$ is converted to a simply connected region. Now, $f(z)$ is analytic in this region.

Now by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}$$
$$\int_{AB} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_{BA} \frac{f(w)}{w-z} dw$$

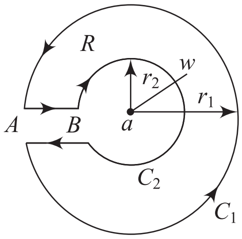


**Fig. 24.2**

Integral along $c_2$ is clockwise, so it is negative.

$$\therefore \qquad f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw \qquad (24.4)$$

For the first integral, $\frac{f(w)}{w-z}$ can be expended exactly as in Taylor's series since $w$ lies on $C_1$,

$$|z-a| \le |w-a| \text{ or } \frac{|z-a|}{|w-a|} \le 1$$

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-a} dw + \frac{(z-a)}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^2} dw$$
$$+ \frac{(z-a)^2}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^3} dw + \cdots$$
$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \qquad (24.5)$$

$$\left[\text{as } a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw\right]$$

In the second integral, $w$ lies on $C_2$

$$\therefore \qquad |w-a| < |z-a| \text{ or } \frac{|w-a|}{|z-a|} < 1$$

So here,
$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)} = \frac{-1}{(z-a)} \cdot \frac{1}{1-\frac{w-a}{z-a}}$$

$$= -\frac{1}{z-a}\left[1-\frac{w-a}{z-a}\right]^{-1}$$

$$= -\frac{1}{z-a}\left[1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\cdots+\left(\frac{w-a}{z-a}\right)^{n+1}+\cdots\right]$$

Multiplying by $\frac{-f(w)}{2\pi i}$, we get

$$-\frac{1}{2\pi i}\frac{f(w)}{w-z}=\frac{1}{2\pi i}\frac{f(w)}{z-a}+\frac{1}{2\pi i}\frac{(w-a)}{(z-a)^2}f(w)+\frac{1}{2\pi i}\frac{(w-a)^2}{(z-a)^3}f(w)+\cdots$$

$$=\left(\frac{1}{z-a}\right)\frac{1}{2\pi i}f(w)+\frac{1}{(z-a)^2}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-1}}+\frac{1}{(z-a)^3}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-2}}+\cdots$$

Integrating, we have

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw=\frac{1}{(z-a)}\frac{1}{2\pi i}\int_{C_2}f(w)dw+\frac{1}{(z-a)^2}\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-1}}dw$$

$$+\frac{1}{(z-a)^3}\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-2}}dw+\cdots$$

$$=\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+\cdots \tag{24.6}$$

$$\left[\text{as } b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw\right]$$

Substituting the values of both integrals from (24.5) and (24.6) in (24.4), we get

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+b_1(z-a)^{-1}+b_2(z-a)^{-2}+\cdots$$

or

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

➢ **Note**

(i) If $f(z)$ is analytic at all points inside $C_1$ (i.e., no singular points inside $C_2$) then by Cauchy's theorem, $b_n = 0$ for all $n - 1 \geq 0$. Hence, the Laurent series reduces to Taylor series. Thus, Laurent's series expansion about an analytic point $a$ is Taylor series expansion about $a$.

(ii) The region of convergence of Laurent's series is the annulus region $R_1 < |z - a| < R_2$.

(iii) If $f(z)$ has more than one singular point then several (more than one) Laurent series expansions can be obtained about the same singular point by appropriately considering analytic regions about (centred) at $a$.

(iv) The part $\sum_{n=0}^{\infty}a_n(z-a)^n$ consisting of positive integral powers of $(z - a)$ is called the **analytic part** of the Laurent's series, while $\sum_{n=1}^{\infty}b_n(z-a)^{-n}$ consisting of negative integral powers of $(z - a)$ is called the **principal part** of the Laurent's series.

## SOLVED EXAMPLES

**Example 1** Obtain Taylor's series expansion to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z| < 2$. **[KU Nov. 2010]**

**Solution** Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

$$= 1 + \frac{-5z-7}{(z+2)(z+3)} \quad (1)$$

Consider $$\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$-5z - 7 = A(z+3) + B(z+2)$$

Put $z = -3 \Rightarrow B = -8$

Put $z = -2 \Rightarrow A = 3$

$\therefore$ $$\frac{-5z-7}{(z+2)(z+3)} = \frac{3}{z+2} - \frac{8}{z+3}$$

$\therefore$ $$(1) \Rightarrow f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Given $|z| < 2$, i.e., $\frac{|z|}{2} < 1$, so clearly $\frac{|z|}{3} < 1$

i.e., $$\left|\frac{z}{2}\right| < 1 \text{ and } \left|\frac{z}{3}\right| < 1$$

$\therefore$ $$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

By using binomial theorem,

$$f(z) = 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n} - \frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n}$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n$$ **Ans.**

**Example 2** Expand $\frac{1}{(z-1)(z-2)}$ in Laurent's series valid for $|z| < 1$ and $1 < |z| < 2$. **[AU Nov. 2010]**

**Solution** Let $f(z)=\dfrac{1}{(z-1)(z-2)}=\dfrac{1}{z-2}-\dfrac{1}{z-1}$

(i) Given $|z|<1$ obviously $\dfrac{|z|}{2}<1$, i.e., $\left|\dfrac{z}{2}\right|<1$

$$\therefore \qquad \frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$$

$$=-\frac{1}{2\left(1-\dfrac{z}{2}\right)}+\frac{1}{1-z}$$

$$=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1-z)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right]+[1+z+z^2+\cdots]$$

i.e.,
$$f(z)=\frac{1}{2}+\frac{3z}{4}+\frac{7}{8}z^2+\cdots$$

(ii) Given $1<|z|<2$

$$1<|z| \;\Rightarrow\; \frac{1}{|z|}<1,\ \text{i.e.,}\ \left|\frac{1}{z}\right|<1$$

$$|z|<2 \;\Rightarrow\; \frac{|z|}{2}<1,\ \text{i.e.,}\ \left|\frac{z}{2}\right|<1$$

$$\therefore \qquad f(z)=\frac{1}{z-2}-\frac{1}{z-1}$$

$$=\frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$ **Ans.**

**Example 3** If $0<|z-1|<2$, express $f(z)=\dfrac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of $z-1$. **[AU April 2011]**

**Solution** Let $z-1=u$
$\therefore$ $0<|z-1|<2$ becomes $0<|u|<2$

Now,
$$\frac{z}{(z-1)(z-3)}=\frac{A}{z-1}+\frac{B}{z-3}$$

$$z=A(z-3)+B(z-1)$$

Put $\quad z=1, \;\Rightarrow\; A=-\dfrac{1}{2}$

Put $\quad z=3, \;\Rightarrow\; B=\dfrac{3}{2}$

$$\therefore \qquad \frac{z}{(z-1)(z-3)}=\frac{-\frac{1}{2}}{z-1}+\frac{\frac{3}{2}}{z-3}$$

(or) $$\frac{u+1}{u(u-2)}=-\frac{1}{2u}+\frac{3}{2(u-2)} \quad (\text{as } z-1=u \Rightarrow z=u+1)$$

So instead of expanding $\dfrac{z}{(z-1)(z-3)}$ in powers of $(z-1)$, it is enough to expand $\dfrac{u+1}{u(u-2)}$ in powers of $u$.

$$\frac{u+1}{u(u-2)}=-\frac{1}{2u}+\frac{3}{2(u-2)}$$

Since $|u|<2$, we have $\dfrac{|u|}{2}<1$. i.e., $\left|\dfrac{u}{2}\right|<1$.

$$\therefore \qquad \frac{u+1}{u(u-2)}=\frac{-1}{2u}-\frac{3}{4\left(1-\frac{u}{2}\right)}$$

$$=\frac{-1}{2u}-\frac{3}{4}\left(1-\frac{u}{2}\right)^{-1}$$

$$=\frac{-1}{2u}-\frac{3}{4}\left[1+\frac{u}{2}+\left(\frac{u}{2}\right)^2+\cdots\right]$$

$$=\frac{-1}{2u}-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n$$

$$\therefore \qquad \frac{z}{(z-1)(z-3)}=\frac{-1}{2(z-1)}-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$ **Ans.**

**Example 4** Obtain the Laurent's expansion for $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ which are valid in (i) $1<|z|<4$, and (ii) $|z|>4$. **[AU Nov. 2011]**

**Solution** Let $f(z)=\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$

$$\Rightarrow \qquad f(z)=1+\frac{-5z-8}{(z+1)(z+4)} \tag{1}$$

(since the degrees of $z$ in both numerator and in denominator are equal, divide it)

Consider $$\frac{-5z-8}{(z+1)(z+4)}=\frac{A}{(z+1)}+\frac{B}{(z+4)}$$

$$-5z-8 = A(z+4)+B(z+1)$$

Put $z=-1 \Rightarrow A=-1$

Put $z=-4 \Rightarrow B=-4$

$$\therefore \quad \frac{-5z-8}{(z+1)(z+4)} = \frac{-1}{(z+1)} - \frac{-4}{(z+4)} \tag{2}$$

Substituting (2) in (1), we get

$$f(z) = 1 - \frac{1}{(z+1)} - \frac{4}{(z+4)}$$

(i) Given $1 < |z| < 4$

$$1 < |z| \Rightarrow \frac{1}{|z|} < 1, \text{ i.e., } \left|\frac{1}{z}\right| < 1$$

$$|z| < 4 \Rightarrow \frac{|z|}{4} < 1, \text{ i.e., } \left|\frac{z}{4}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - 4\frac{1}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \left[-\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \left[-\frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{z^n} - \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{z}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z^n} - \left(\frac{z}{4}\right)^n\right]$$

(ii) Given $|z| > 4$

$$\frac{4}{|z|} < 1, \text{ i.e., } \left|\frac{4}{z}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{1}{1+z} - \frac{4}{z+4}$$

$$= 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{z\left(1+\frac{4}{z}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right] - \frac{4}{z}\left[1 - \frac{4}{z} + \left(\frac{4}{z}\right)^2 - \cdots\right]$$

$$= 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n} - \frac{4}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{z}\right)^n$$

$$= 1 - \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{z^{n+1}} + \left(\frac{4}{z}\right)^{n+1}\right]$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}}(1 + 4^{n+1})$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n(1 + 4^n)\cdot\frac{1}{z^n} \qquad \textbf{Ans.}$$

**Example 5** Find the Laurent's series of $f(z) = \dfrac{1}{z(1-z)}$ valid in the region (i) $|z+1| < 1$, (ii) $1 < |z+1| < 2$, and (iii) $|z+1| > 2$. **[KU May 2010, Nov. 2011]**

**Solution** Let $z + 1 = u$ or $z = u - 1$

$$\therefore \qquad f(z) = \frac{1}{z(1-z)} = \frac{1}{(u-1)(2-u)} = \frac{1}{u-1} + \frac{1}{2-u} \qquad (1)$$

(i) Given $|z+1| < 1 \quad \Rightarrow \quad |u| < 1$

$$\therefore \qquad f(z) = \frac{-1}{1-u} + \frac{1}{2\left(1 - \frac{u}{2}\right)}$$

$$= -(1-u)^{-1} + \frac{1}{2}\left(1 - \frac{u}{2}\right)^{-1}$$

$$= -[1 + u + u^2 + \cdots] + \frac{1}{2}\left[1 + \left(\frac{u}{2}\right) + \left(\frac{u}{2}\right)^2 + \cdots\right]$$

$$= -\sum_{n=0}^{\infty}u^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^{n+1}}\right)u^n$$

$$\text{i.e.,} \qquad f(z) = \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^{n+1}}\right)(z+1)^n$$

(ii) Given $1 < |z+1| < 2$. i.e., $1 < |u| < 2$

$$1 < |u| \quad \Rightarrow \quad \frac{1}{|u|} < 1, \text{ i.e., } \left|\frac{1}{u}\right| < 1$$

$$|u| < 2 \quad \Rightarrow \quad \frac{|u|}{2} < 1 \text{ i.e., } \left|\frac{u}{2}\right| < 1$$

Consider (1), $f(z) = \dfrac{1}{u-1} + \dfrac{1}{2-u}$

$$= \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{1}{2\left(1-\frac{u}{2}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right] + \frac{1}{2}\left[1+\left(\frac{u}{2}\right)+\left(\frac{u}{2}\right)^2+\cdots\right]$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{u^{n+1}} + \sum_{n=0}^{\infty}\frac{u^n}{2^{n+1}}$$

i.e., $$f(z) = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z+1)^n$$

(iii) $|z+1| > 2$, i.e., $|u| > 2 \Rightarrow \left|\dfrac{2}{u}\right| < 1$

$\therefore$ $$f(z) = \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{1}{u\left(1-\frac{2}{u}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1-\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right] - \frac{1}{u}\left[1+\frac{2}{u}+\left(\frac{2}{u}\right)^2+\cdots\right]$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} - \frac{1}{u}\sum_{n=0}^{\infty}\frac{2^n}{u^n}$$

$$= \sum_{n=0}^{\infty}(1-2^n)\frac{1}{u^{n+1}}$$

or $$f(z) = \sum_{n=0}^{\infty}(1-2^n)\frac{1}{(z+1)^{n+1}}$$ **Ans.**

## EXERCISE

### Part A

1. Define radius and circle of convergence of power series.
2. State Taylor's theorem and Laurent's theorem.
3. State McLaurin's series.
4. Give some standard McLaurin's series.
5. What do you mean by analytic part and principal part of Laurent's series of a function of $z$?
6. Expand $\dfrac{1}{z(z-1)}$ as Laurent's series about $z = 0$ in the annulus $0 < |z| < 1$.
7. Find the Laurent's series expansion of $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$.
8. Expand $f(z) = e^z$ in a Taylor's series about $z = 0$.
9. Expand $\cos z$ at $z = \dfrac{\pi}{4}$ in a Taylor's series.
10. In the power series $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$, $z_0$ is called the ______ of the series.

### Part B

1. Find the Taylor's series expansion of $f(z) = \dfrac{z}{z(z+1)(z+2)}$ about $z = i$.

   State also the region of convergence of the series.

   $$\left[\textbf{Ans.} \sum_{n=0}^{\infty}(-1)^n\left\{\frac{2}{(2+i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n\right]$$

2. Find the Laurent's series expansion of $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ valid in the region (i) $|z| < 2$, (ii) $2 < |z| < 3$, and (iii) $|z| > 3$ **[KU April 2013]**

   $$\left[\begin{array}{l}\textbf{Ans.}\ \text{(i)}\ 1 + \displaystyle\sum_{n=0}^{\infty}(-1)^n\left\{\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right\}z^n\ \text{(ii)}\ 1 + 3\sum(-1)^n\frac{2^n}{z^{n+1}} - 8\sum(-1)^n\frac{z^n}{3^{n+1}} \\ \text{(iii)}\ 1 + \displaystyle\sum(-1)^n\{3.2^n - 8.3^n\}1/z^{n+1}\end{array}\right]$$

3. Find the Laurent's series expansion of $f(z) = \dfrac{z}{(z-1)(z-2)}$, valid in the region (i) $|z+2| < 3$, (ii) $3 < |z+2| < 4$, and (iii) $|z+2| > 4$.

   $$\left[\begin{array}{l}\textbf{Ans.}\ \text{(i)}\ \displaystyle\sum_{n=0}^{\infty}\left[-\frac{1}{2.4^n} + \frac{1}{3^{n+1}}\right](z+2)^n\ \text{(ii)}\ -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z+2)^n}{4^n} - \sum_{n=0}^{\infty}\frac{3^n}{(z+2)^{n+1}} \\ \text{(iii)}\ \displaystyle\sum_{n=0}^{\infty}(2.4^n - 3^n)\cdot\frac{1}{(z+2)z^{n+1}}\end{array}\right]$$

4. Expand $\dfrac{z^2-6z-1}{(z-1)(z+2)(z-3)}$ in $3 < |z+2| < 5$.

$$\left[\textbf{Ans. } \frac{2}{z+2}+\frac{3}{(z+2)^2}+\frac{3^2}{(z+2)^3}+\cdots+\frac{1}{5}\left[1+\frac{z+2}{5}+\frac{(z+2)^2}{5^2}+\frac{(z+2)^3}{5^3}+\cdots\right]\right]$$

5. Find Laurent's series of $f(z)=\dfrac{e^z}{z(1-z)}$ about $z = 1$. Find the region of convergence.

$$\left[\begin{array}{c}\textbf{Ans. } f(z)=\dfrac{1}{e}\left[-\dfrac{1}{z-1}-\dfrac{3}{2}(z-1)+\dfrac{1}{3}(z-1)^2+\cdots\right] \\ \text{Region of convergence is } |z-1|<1\end{array}\right]$$

6. Obtain the Laurent's series expansion for $f(z)=\dfrac{1}{z(z-1)}$ for (i) $0 < |z| < 1$, and (ii) $0 < |z-1| < 1$.

$$\left[\textbf{Ans. (i) } -\frac{1}{z}(1+z+z^2+\cdots)\text{ (ii) }\frac{1}{z-1}(1-(z-1)+(z-1)^2\ldots)\right]$$

7. Find Laurent's series about the indicated singularity. (i) $\dfrac{e^{2z}}{(z-1)^3}, z=1$ (ii) $\dfrac{z}{(z+1)(z+2)}, z=-2$ (iii) $\dfrac{1}{z^2(z-3)^2}, z=3$

$$\left[\begin{array}{l}\textbf{Ans. (i) } \dfrac{e^2}{(z-1)^3}+\dfrac{2e^2}{(z-1)^2}+\dfrac{2e^2}{(z-1)}+\dfrac{4e^2}{3}+\dfrac{2e^2}{3}(z-1)+\cdots \\ \qquad\text{(ii) } \dfrac{2}{2+z}+1+(z+2)+(z+2)^2+\cdots \\ \qquad\text{(iii) } \dfrac{1}{9(z-3)^2}-\dfrac{2}{27(z-3)}+\dfrac{1}{27}-\dfrac{4(z-3)}{243}+\cdots\end{array}\right]$$

# 25

# Theory of Residues

## Chapter Outline

- Introduction
- Classification of Singularities
- Residues
- Cauchy's Residue Theorem
- Evaluation of Real Definite Integrals by Contour Integration

## 25.1 ❑ INTRODUCTION

The residue theorem is a very powerful and elegant theorem in complex integration. Using the residue theorem, many complicated real integrals can be evaluated. It is also used to sum a real convergent series and to find the inverse of a Laplace transform.

## 25.2 ❑ CLASSIFICATION OF SINGULARITIES

A point at which a function $f(z)$ is not analytic is known as a **singular point** or **singularity** of the function.

- **Example**

The function $f(z)=\dfrac{1}{z-5}$ has a singular point at $z-5=0$ or $z=5$.

If $z=a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z=a$ then $z=a$ is said to be an **isolated singularity** of the function $f(z)$. Otherwise, it is called **non-isolated**.

- **Example**

(i) The function $\dfrac{1}{(z-2)(z-7)}$ has two isolated singular points, namely, $z = 2$ and $z = 7$ [since $(z - 2)(z - 7) = 0$ or $z = 2, 7$].

(ii) The function $\dfrac{1}{\sin\frac{\pi}{z}}$ is not analytic at the points where $\sin\frac{\pi}{z}=0,$ i.e., at the points $\frac{\pi}{z}=n\pi$ .

i.e., at the points $z=\frac{1}{n}(n=1,2,3\ldots)$ .

Thus, $z=1,\frac{1}{2},\frac{1}{3}\ldots,z=0$ are the points of singularity. But $z = 0$ is the non-isolated singularity of the function $\dfrac{1}{\sin\frac{\pi}{z}}$ because in the neighbourhood $z = 0$, there are infinite number of other singularities $z=\frac{1}{n},$ where $n$ is very large.

Let a function $f(z)$ have an isolated singular point $z = a$. $f(z)$ can be expanded in a Laurent's series expansion around $z = a$ as

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\cdots+\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}$$

$$+\cdots+\frac{b_m}{(z-a)^m}+\frac{b_{m+1}}{(z-a)^{m+1}}+\cdots$$

In some cases, it may happen that the coefficients $b_{m+1} = b_{m+2} = \ldots = 0$,

Then the series reduces to

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\ldots+\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\ldots+\frac{b_m}{(z-a)^m}$$

Then $z = a$ is said to be **a pole of order *m*** of the function $f(z)$.

When $m = 1$, the pole is said to be a **simple pole**.

In this case, $f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + \dfrac{b_1}{(z-a)}$ .

If the number of terms of negative powers in the above expansion are infinite then $z = a$ is called an **essential singular point** of $f(z)$.

If a single-valued function $f(z)$ is not defined at $z = a$, but $\lim\limits_{z\to a} f(z)$ exists then $z = a$ is called a **removable singularity**.

- **Example**

$z = 0$ is a removable singularity of $f(z)=\dfrac{\sin z}{z},$ since $f(0)$ is not defined, but $\lim\limits_{z\to 0}\left(\dfrac{\sin z}{z}\right)=1$ .

## 25.3 ❑ RESIDUES

Residue of an analytic function *f*(*z*) at an isolated singular point $z = a$ is the coefficient say $b_1$ of $(z - a)^{-1}$ in the Laurent's series expansion of *f*(*z*) about *a*. Residue of *f*(*z*) at *a* is denoted by $\operatorname*{Res}_{z=a} f(z)$. From Laurent's series, we know that the coefficient $b_1$ is given by $b_1 = \dfrac{1}{2\pi i}\int_C f(z)dz.$

Thus, the residue of $f(z)$ at $z = a, = \operatorname*{Res}_{z=a} f(z) = b_1 = \dfrac{1}{2\pi i}\int_C f(z)dz.$

where *C* is any closed contour enclosing *a* (and such that *f* is analytic on and within *C*).

### Calculation of Residue at Simple Pole

(i) If *f*(*z*) has a simple pole at $z = a$, then $\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)f(z)$.

(ii) Suppose $f(z) = \dfrac{P(z)}{Q(z)}$ has a simple pole at *a* such that $P(a) \neq 0$.

Then $\operatorname*{Res}_{z=a} f(z) = \operatorname*{Res}_{z=a}\dfrac{P(z)}{Q'(z)} = \dfrac{P(a)}{Q'(a)}$

### Calculation of Residue at a Multiple Pole

If *f*(*z*) has a pole of order *n* at $z = a$, then

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(n-1)!}\lim_{z\to a}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}$$

## 25.4 ❑ CAUCHY'S RESIDUE THEOREM

If *f*(*z*) is analytic within and on a simple closed curve *C* except at a finite number of poles within *C* then $\oint_C f(z)dz = 2\pi i$ (sum of residues at the poles within *C*).

**Proof** Let $C_1, C_2, C_3 \ldots C_n$ be the non-intersecting circles with centre at $a_1, a_2 \ldots a_n$ respectively and radii so small that they lie entirely within the closed curve *C*. Then *f*(*z*) is analytic in the multiply connected legion lying between the curves *C* and $C_1, C_2 \ldots C_n$. Applying Cauchy's theorem,

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$
$$= 2\pi i \operatorname*{Res}_{z=a_1} f(z) + 2\pi i \operatorname*{Res}_{z=a_2} f(z) \cdots + 2\pi i \operatorname*{Res}_{z=a_n} f(z)$$
$$= 2\pi i\left[\operatorname*{Res}_{z=a_1} f(z) + \operatorname*{Res}_{z=a_2} f(z) \cdots + \operatorname*{Res}_{z=a_n} f(z)\right]$$

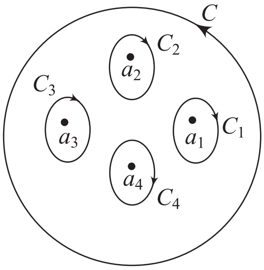


**Fig. 25.1**

$\therefore \oint_c f(z)dz = 2\pi i$ (sum of residues at the poles within *C*)

## 25.5 ❑ EVALUATION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated using Cauchy's theorem of residues. For finding the integrals, we take a closed curve $C$, find the poles of the function $f(z)$ and calculate residues at those poles only which lie within the curve $C$.

Then using Cauchy's theorem of residues, we have $\int_C f(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at the poles within $C$)

We call the curve a contour and the process of integration along a contour as contour integration.

### *Type 1*

**Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where $f$ is a rational function of $\cos\theta$ and $\sin\theta$**

In this type of integrals, put $z = e^{i\theta}$

Differentiating with respect to $\theta$, we get,

$$dz = ie^{i\theta}\, d\theta, \text{ i.e., } d\theta = \frac{dz}{iz}$$

We know that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

i.e., $$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and $$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\therefore \quad \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \int_C f\left(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i}\right)\frac{dz}{iz}$$

where $C$ is the unit circle $|z| = 1$

$$= \frac{1}{i}\int_C f\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\frac{dz}{z}$$

$$= \int_C \phi(z)\, dz \text{ (say)}$$

Clearly, $\phi(z)$ is a rational function of $z$.

Hence, by the residue theorem, $\int_C \phi(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at its poles inside $C$).

### *Type 2*

Consider the integral $\int_C \phi(z)dz$, where $C$ is the positively oriented semicircle $\Gamma$, $|z| = R$, $\text{Im } z \geq 0$ together with the line segment $L : [-R, R]$. Such integrals can be evaluated by integrating $f(z)$ round a contour $C$ consisting of a semicircle $\Gamma$ of radius $R$ large enough to include all the poles of $f(z)$

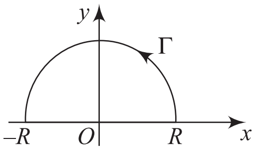


**Fig. 25.2**

and the part of the real axis from $x = -R$ to $x = R$. Here, the only singularities of $f(z)$ in the upper half-plane are poles.

When $\phi(z)$ has singularities on the real axis then $\int_C \phi(z)dz = \int_{-R}^{R} \phi(x)dx + \int_\Gamma \phi(z)dz$ .

By the residue theorem, we have $\int_C \phi(z)dz = 2\pi i$ (sum of the residues of the function $\phi(z)$ at its poles in the upper half-plane).

i.e., $\int_{-R}^{R} \phi(x)dx + \int_\Gamma \phi(z)dz = 2\pi i$ (sum of the residues of the function $\phi(z)$ at its poles within $C$).

Putting $R \to \infty$ we get, $\int_{-\infty}^{\infty} \phi(x)dx,$ provided $\int_\Gamma \phi(z)dz \to 0$.

### *Type 3*

Integrals of the form $\int_{-\infty}^{\infty} (\sin ax)\ f(x)dx$ or $\int_{-\infty}^{\infty} (\cos ax)\ f(x)dx$ . $a > 0$ where $f(z)$ is such that $f(z) \to 0$ as $z \to \infty$ and it does not have a pole on the real axis.

## SOLVED EXAMPLES

**Example 1** Find the residue of $f(z) = \dfrac{1}{(z^2+1)^2}$ about each singularity.

**Solution** Given $f(z) = \dfrac{1}{(z^2+1)^2} = \dfrac{1}{[(z-i)(z+i)]^2}$

$$= \frac{1}{(z-i)^2(z+i)^2}$$

Here, $z = i, -i$ are poles of order 2.

Now,
$$[\text{Res}\, f(z)]_{z=i} = \underset{z\to i}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}[(z-i)^2 f(z)]$$

$$= \underset{z\to i}{\text{Lt}} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{1}{(z-i)^2(z+i)^2}\right]$$

$$= \underset{z\to i}{\text{Lt}} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$$

$$= \underset{z\to i}{\text{Lt}} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$= \frac{-i}{4}$$

$$[\text{Res}\, f(z)]_{z=-i} = \underset{z\to -i}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}[(z+i)^2 f(z)]$$

$$= \underset{z\to -i}{\text{Lt}} \frac{d}{dz}\left[(z+i)^2 \cdot \frac{1}{(z-i)^2(z+i)^2}\right]$$

$$= \underset{z\to -i}{\text{Lt}} \frac{d}{dz}\left[\frac{1}{(z-i)^2}\right]$$

$$= \underset{z\to -i}{\text{Lt}} \frac{-2}{(z-i)^3} = \frac{-2}{8i} = \frac{i}{4}$$ **Ans.**

**Example 2** Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where $C$ is the circle $|z-i| = 2$.

**[AU June 2009, May 2012]**

**Solution** Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

Here, $z = -1$ is a pole of order 2.
And $z = 2$ is a simple pole.
Clearly, $z = 2$ lies outside the circle $|z-i| = 2$

$\therefore$ $[\text{Res}\, f(z)]_{z=2} = 0$

Now, $$[\text{Res}\, f(z)]_{z=-1} = \underset{z\to -1}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \underset{z\to -1}{\text{Lt}} \frac{d}{dz}\left[(z+1)^2 \cdot \frac{(z-1)}{(z+1)^2(z-2)}\right]$$

$$= \underset{z\to -1}{\text{Lt}} \frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \underset{z\to -1}{\text{Lt}} \left[\frac{(z-2)-(z-1)}{(z-2)^2}\right]$$

$$= \underset{z\to -1}{\text{Lt}} \left[\frac{-2+1}{(z-2)^2}\right] = \underset{z\to -1}{\text{Lt}} \left[-\frac{1}{(z-2)^2}\right]$$

$$= \frac{-1}{(-1-2)^2} = -\frac{1}{9}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i \;\; [\text{sum of the residues}]$$

$$= 2\pi i\left(-\frac{1}{9}\right) = \frac{-2\pi i}{9}$$ **Ans.**

**Example 3** Evaluate $\int_C \frac{dz}{c(z^2+9)^3}$, where $C$ is $|z-i| = 3$ by using Cauchy's residue theorem.

**[KU Nov. 2011]**

**Solution** Let $f(z) = \frac{1}{(z^2+9)^3}$

The singularities of $f(z)$ are obtained by $z^2 + 9 = 0$
$\Rightarrow z = \pm 3i$, of which $z = 3i$ lies inside the circle $|z - i| = 3$
$z = 3i$ is a triple pole of $f(z)$.

$$\therefore \qquad [\text{Res}\, f(z)]_{z=3i} = \frac{1}{2!}\left[\frac{d^2}{dz^2}\frac{1}{(z+3i)^3}\right]_{z=3i}$$

$$= \frac{1}{2!}\left[\frac{12}{(z+3i)^5}\right]_{z=3i}$$

$$= \frac{6}{6^5 i^5} = \frac{1}{1296i}$$

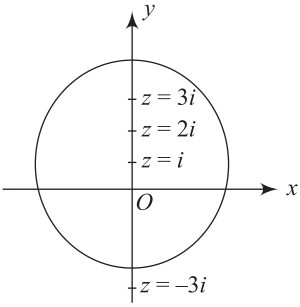


**Fig. 25.3**

By Cauchy's residue theorem,

$$\int_C \frac{dz}{(z^2+9)^3} = 2\pi i \times \frac{1}{1296i} = \frac{\pi}{648}$$ **Ans.**

**Example 4** Show that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, a > b > 0$ .

**[KU May 2010; AU Nov. 2010, Nov. 2011, April 2013]**

**Solution** Let $z = e^{i\theta}$

$$\Rightarrow \qquad d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\therefore \qquad \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{a + \frac{1}{2}b\left(z + \frac{1}{z}\right)} \text{ where } C \text{ is } |z| = 1$$

$$= \frac{1}{i}\int_C \frac{dz}{z\left[a + \frac{1}{2}b\left(z + \frac{1}{z}\right)\right]}$$

$$= \frac{1}{i}\int_C \frac{dz}{z\left[\frac{2az + bz^2 + b}{2z}\right]}$$

$$\text{i.e.,} \qquad \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{i}\int_C \frac{dz}{bz^2 + 2az + b}$$

$$= \frac{2}{i}\int_C f(z)dz \qquad (1)$$

The poles of $f(z)$ are given by the roots of $bz^2 + 2az + b = 0$

$$\therefore \qquad z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

i.e., $$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a-\sqrt{a^2-b^2}}{b}$$

Let $$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}; \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Since $a > b > 0$, $|\beta| > 1$

But the modulus of the product of the roots $|\alpha\beta| = 1$ (since if $az^2 + b + c = 0$, product of the roots $|\alpha\beta| = \frac{c}{a}$).

Since $|\beta| > 1$ and $|\alpha\beta| = 1$, we get $|\alpha| < 1$ so that $z = \alpha$ is the only simple pole inside $C$.

Since $z = \alpha$ and $z = \beta$ are the roots of $bz^2 + 2az + b = 0$, we can write $bz^2 + 2az + b = b(z - \alpha)(z - \beta)$

Hence, $$f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$$

Now, $$\begin{aligned}[\text{Res}\, f(z)]_{z=\alpha} &= \underset{z\to\alpha}{\text{Lt}}\,(z-\alpha)\cdot f(z) \\ &= \underset{z\to\alpha}{\text{Lt}}\,(z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)} \\ &= \underset{z\to\alpha}{\text{Lt}}\,\frac{1}{b(z-\beta)} = \frac{1}{b(\alpha-\beta)} \\ &= \frac{1}{b\left[\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right]} \\ &= \frac{1}{b\frac{2\sqrt{a^2-b^2}}{b}} \\ &= \frac{1}{2\sqrt{a^2-b^2}}\end{aligned}$$

From (1), since $|\beta| > 1$,

$\beta$ lies outside the circle $|z| = 1$

$\therefore$ $$[\text{Res}\, f(z)]_{z=\beta} = 0$$

Hence, (1) ⇒ $$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} &= \frac{2}{i}\int_C f(z)dz \\ &= \frac{2}{i}\,[2\pi i \times (\text{sum of the residues})] \\ &= \frac{2}{i}\cdot 2\pi i\left[\frac{1}{2\sqrt{a^2-b^2}}\right]\end{aligned}$$

$\therefore$ $$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$ **Ans.**

**Example 5** Evaluate $\int_0^{\pi} \frac{a d\theta}{a^2+\sin^2\theta}, a>0$. **[KU Nov. 2010]**

**Solution** Let

$$I = \int_0^{\pi} \frac{a d\theta}{a^2+\sin^2\theta}$$

$$= \int_0^{\pi} \frac{a d\theta}{a^2+\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$= \int_0^{\pi} \frac{2a d\theta}{2a^2+1-\cos 2\theta}$$

Put $2\theta = \phi \quad \Rightarrow \quad 2d\theta = d\phi$

When $\theta = 0$, $\phi = 0$ and when $\theta = \pi$, $\phi = 2\pi$

$$\therefore \qquad I = \int_0^{2\pi} \frac{2a\left(\frac{d\phi}{2}\right)}{2a^2+1-\cos\phi}$$

$$= \int_0^{2\pi} \frac{a d\phi}{2a^2+1-\cos\phi} \qquad (1)$$

Put $z = e^{i\phi}$, then $d\phi = \frac{dz}{iz}$

$$\cos\phi = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

Then

$$(1) \Rightarrow I = \int_C \frac{a\cdot\frac{dz}{iz}}{\left[2a^2+1-\frac{1}{2}\left(z+\frac{1}{z}\right)\right]}$$

where $C$ is the unit circle $|z| = 1$

$$= \frac{a}{i}\int_C \frac{dz}{\left[2a^2+1-\frac{1}{2}\left(\frac{z^2+1}{z}\right)\right]}$$

$$= \frac{a}{i}\int_C \frac{dz}{\left[\frac{4a^2z+2z-z^2-1}{2z}\right]}$$

$$= \frac{2a}{i}\int_C \frac{dz}{(4a^2+2)-z^2-1}$$

$$= -\frac{2a}{i}\int_C \frac{dz}{z^2-(4a^2+2)z+1}$$

$$= 2ai\int_C \frac{dz}{z^2-(4a^2+2)z+1}$$

$\therefore$ $I = \int_C f(z)dz$, where $f(z) = \frac{2ai}{z^2-(4a^2+2)z+1}$

The poles of $f(z)$ are the solutions of

$$z^2-(4a^2+2)z+1=0$$

$$z^2-(4a^2+2)z+1=0$$

$$\therefore \qquad z=\frac{(4a^2+2)\pm\sqrt{(4a^2+2)^2-4}}{2}$$

$$=\frac{2(2a^2+1)\pm 4a\sqrt{a^2+1}}{2}$$

$$=(2a^2+1)\pm 2a\sqrt{a^2+1}$$

$$\Rightarrow \qquad z=(2a^2+1)+2a\sqrt{a^2+1}\text{ or }(2a^2+1)-2a\sqrt{a^2+1}$$

Let $\alpha=(2a^2+1)+2a\sqrt{a^2+1}$ and $\beta=(2a^2+1)-2a\sqrt{a^2+1}$

Since $\alpha, \beta$ are the roots of $z^2-(4a^2+2)z+1=0$, the product of the roots $\alpha\beta=1$

Since $a>0$, $\alpha>1$ also, $\beta<1$.

$\therefore$ out of the two poles $\alpha$ and $\beta$, $z=\beta$ lies within the unit circle $|z|=1$ (since $|\beta|<1$)

Now,

$$[\operatorname{Re}s\, f(z)]_{z=\beta}=\underset{z\to\beta}{\text{Lt}}(z-\beta)\cdot f(z)$$

$$=\underset{z\to\beta}{\text{Lt}}(z-\beta)\cdot\frac{2ai}{(z-\alpha)(z-\beta)}$$

$$=\frac{2ai}{\beta-\alpha}$$

$$=\frac{2ai}{(2a^2+1-2a\sqrt{a^2+1})-(2a^2+1-2a\sqrt{a^2+1})}$$

$$=\frac{2ai}{-4a\sqrt{a^2+1}}=\frac{-i}{2\sqrt{a^2+1}}$$

$$\therefore \qquad I=\int_C f(z)dz$$

$$=2\pi i\,[\text{sum of the residues of } f(z) \text{ at its poles}]$$

$$=2\pi i\left[\frac{-i}{2\sqrt{a^2+1}}\right]$$

$$\therefore \qquad \int_0^{\pi}\frac{a\,d\theta}{a^2+\sin^2\theta}=\frac{\pi}{\sqrt{a^2+1}}$$

**Ans.**

**Example 6** Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx, a>0, b>0$.

**[KU May 2010, Nov. 2011]**

**Solution** Let $\displaystyle\int_C \phi(z)dz=\int_C\frac{z^2}{(z^2+a^2)(z^2+b^2)}dz$

where $C$ consists of the semicircle $\Gamma$ and the bounding diameter $[-R, R]$.

Now,

$$\int_C\phi(z)dz=\int_{-R}^{R}\phi(x)dx+\int_\Gamma\phi(z)dz \qquad (1)$$

Now, $$\phi(z)=\frac{z^2}{(z^2+a^2)(z^2+b^2)}$$
$$=\frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

Here, the poles are $z = ia, -ia, ib, -ib$

Here, $z = ia$ and $z = ib$ lie in the upper half-plane while $z = -ia$ and $z = -ib$ lie in the lower half-plane.

We have to find the residues of $\phi(z)$ at each of its poles which lies in the upper half-plane.

$\therefore$ $$[\text{Res}\, f(z)]_{z=ia}=\underset{z\to ia}{\text{Lt}}(z-ia)\cdot\phi(z)$$
$$=\underset{z\to ia}{\text{Lt}}(z-ia)\frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}$$
$$=\underset{z\to ia}{\text{Lt}}\frac{z^2}{(z-ia)(z^2+b^2)}$$
$$=\underset{z\to ia}{\text{Lt}}\frac{(ia)^2}{(ia+ia)((ia)^2+b^2)}$$
$$=\frac{-a^2}{2ia(-a^2+b^2)}$$
$$=\frac{a}{2i(a^2-b^2)}$$

$$[\text{Res}\, f(z)]_{z=ib}=\underset{z\to ib}{\text{Lt}}(z-ib)\phi(z)$$
$$=\underset{z\to ib}{\text{Lt}}(z-ib)\frac{z^2}{(z^2+a^2)(z+ib)(z-ib)}$$
$$=\underset{z\to ib}{\text{Lt}}\frac{z^2}{(z^2+a^2)(z+ib)}$$
$$=\frac{(ib)^2}{[(ib)^2+a^2][ib+ib]}$$
$$=\frac{-b^2}{(a^2-b^2)2ib}=\frac{-b}{2i(a^2-b^2)}$$

In (1), making $R\to\infty$, we get
$$\int_C\phi(z)dz=\int_{-\infty}^{\infty}\phi(x)dx+\int_\Gamma\phi(z)dz$$

When $R\to\infty$, $|z|\to\infty$ and $\phi(z)\to 0$

$\therefore$ $$\int_C\phi(z)dz=\int_{-\infty}^{\infty}\phi(x)dx\ \ [\text{from (1)}]$$

$\therefore$ $$\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}=\int_{-\infty}^{\infty}\frac{z^2dz}{(z^2+a^2)(z^2+b^2)}$$
$$=2\pi i$$

[sum of the residues of $\phi(z)$ at each pole in the upper half-plane]

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right]$$

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)}\right] = 2\pi i\left[\frac{a-b}{2i(a-b)(a+b)}\right]$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b} \qquad \textbf{Ans.}$$

**Example 7** Evaluate $\int_0^{\infty}\frac{dx}{x^4+1}$. **[KU Nov. 2010]**

**Solution** Consider $\int_0^{\infty}\frac{dx}{x^4+1}$

$$\int_0^{\infty}\frac{dx}{x^4+1} = \int_0^{\infty}\frac{dx}{z^4+1}$$

i.e.,
$$2\int_0^{\infty}\frac{dx}{x^4+1} = \int_{-\infty}^{\infty}\frac{dx}{z^4+1}$$

The poles are the roots of $z^4 + 1 = 0$

i.e., $z^4 = -1$

$\Rightarrow$ $z = (-1)^{\frac{1}{4}}$

$$= \left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right] \text{ where } n = 0, 1, 2, 3$$

When $n = 0$, $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

When $n = 1$, $z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = e^{\frac{i3\pi}{4}}$

When $n = 2$, $z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = e^{\frac{i5\pi}{4}}$

When $n = 3$, $z = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = e^{\frac{i7\pi}{4}}$

Hence, the poles are $z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$.

Out of these poles, $z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}$ lies in the upper half-plane.

$$\therefore \qquad [\text{Res}\,\phi(z)]_{z=e^{\frac{i\pi}{4}}} = \underset{z\to e^{\frac{i\pi}{4}}}{\text{Lt}}\frac{z-e^{\frac{i\pi}{4}}}{z^4+1}$$

$$= \underset{z\to e^{\frac{i\pi}{4}}}{\text{Lt}}\frac{1}{4z^3} = \frac{1}{4\left(e^{\frac{i\pi}{4}}\right)^3} \text{ (applying L'Hospital's rule)}$$

$$= \frac{1}{4e\dfrac{i3\pi}{4}}$$

$$[\text{Res}\,\phi(z)]_{z=e^{\frac{i3\pi}{4}}} = \underset{z\to e^{\frac{i3\pi}{4}}}{\text{Lt}} \frac{z-e^{\frac{i3\pi}{4}}}{z^4+1}$$

$$= \underset{z\to e^{\frac{i3\pi}{4}}}{\text{Lt}} \frac{1}{4z^3} = \frac{1}{4\left(e^{\frac{i3\pi}{4}}\right)^3}$$

$$= \frac{1}{4e^{i\frac{9\pi}{4}}}$$

∴ $$2\int_0^\infty \frac{dx}{x^4+1} = \int_{-\infty}^\infty \frac{dz}{z^4+1}$$

$= 2\pi i$ [sum of the residues at each pole in the upper half-plane]

$$= 2\pi i\left[\frac{1}{4e^{\frac{i3\pi}{4}}} + \frac{1}{4e^{\frac{i9\pi}{4}}}\right]$$

$$= \frac{\pi i}{2}\left[e^{-\frac{i3\pi}{4}} + e^{-\frac{i9\pi}{4}}\right]$$

$$= \frac{\pi i}{2}\left[\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + \left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right)\right]$$

$$= \frac{\pi i}{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = \frac{\pi i}{2}\left[\frac{-2i}{\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

∴ $$\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dz}{z^4+1} = \frac{1}{2}\frac{\pi}{\sqrt{2}}$$ **Ans.**

## EXERCISE

### Part A

1. Define essential singularity with an example.
2. Define removable singularity with an example.
3. Define simple pole and multiple pole of a function $f(z)$. Give one example for each.
4. Define the residue of a function at an isolated singularity.
5. State the formula for finding the residue of a function at a multiple pole.
6. Find the residues at the isolated singularities of each of the following:

   (i) $\frac{z}{(z+1)(z-2)}$ (ii) $\frac{ze^z}{(z-1)^2}$ (iii) $\frac{z\sin z}{(z-\pi)^3}$

7. Evaluate the following integrals using Cauchy's residue theorem:

   (i) $\int_C \frac{z+1}{z(z-1)}dz$ where $C : |z| = 2$

(ii) $\int_C \frac{e^{-z}}{z^2} dz$ where C : $|z| = 1$

8. Explain how to convert $\int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta$ into a contour integral, where *f* is a rational function.

9. Obtain the poles of $\frac{z+4}{z^2+2z+5}$.

10. By using residue theorem, find the value of $\int_C \frac{z-2}{z-1} dz$ where *C* is $|z| = 2$.

11. Find the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ at $z = -2$.

12. Find the singularities of $f(z) = \frac{z+4}{z^2+2z+2}$.

13. Find the residue of $f(z) = \frac{z}{z^2+1}$ about $z = i$.

14. Find the residue of $f(z) = \frac{1}{(z^2+a^2)^2}$ at $z = ai$

15. Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

16. Find the poles of $f(z) = \frac{1}{\sin\frac{1}{z-a}}$.

17. Find the singularities of the function $f(z) = \frac{\cot \pi z}{(z-a)^3}$.

18. Give the forms of the definite integrals which can be evaluated using the infinite semicircular contour above the real axis.

19. Define Cauchy's residue theorem.

20. Find the residue of $\frac{1}{(z^3-1)^2}$ at $z = 1$.

## Part B

1. Evaluate the following using Cauchy's residue theorem:

(i) $\int_C \frac{1-2z}{z(z-1)(z-2)} dz, C: |z| = \frac{3}{2}$

(ii) $\int_C \frac{2z-1}{z(z+2)(2z+1)} dz, C: |z| = 1$

(iii) $\int_C \frac{e^{-z}}{z^2} dz, C: |z| = 1$

(iv) $\int_C \frac{12z-7}{(z-1)^2(2z+3)} dz, C: |z+i| = \sqrt{3}$

$\left[\textbf{Ans.} \text{ (i) } 3\pi i \text{ (ii) } \frac{5\pi i}{3} \text{ (iii) } -2\pi i \text{ (iv) } 4\pi i\right]$

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

3. Evaluate $\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta}$. $\left(\textbf{Ans. } \frac{2\pi}{15}\right)$

4. Evaluate $\int_0^{\infty} \frac{dx}{x^4+a^4}$. $\left(\textbf{Ans. } \frac{\pi}{a^3\cdot\sqrt{2}}\right)$

5. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

6. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$. $\left(\textbf{Ans. } \frac{\pi}{4a^3}, a>0\right)$

7. Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx$. $\left(\textbf{Ans. } \frac{1}{2}\pi e^{-a}\right)$

8. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx$. $\left(\textbf{Ans. } \frac{\pi}{a}e^{-a}\right)$

9. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$.

10. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

11. Evaluate the integral $\int_0^{\infty} \frac{x^2}{x^4+1}dx$ using contour integration.

12. Evaluate $\int_0^{\infty} \frac{\cos x}{(1+x^2)^2}dx$. $\left(\textbf{Ans. } \frac{\pi}{2e}\right)$

| Questions | opt1 | opt2 | opt3 | opt4 | opt5 |
|---|---|---|---|---|---|
| A curve is called a_____________ if it does not intersect itself | Simple closed curve | multiple curve | simply connected region | multiple connected region | |
| A curve is called ________ if it is not a simple closed curve | connected region | multiple curve | simply connected region | multiple connected region | |
| If f(z) is analytic in a simply connected domain D and C is any simple closed path then ∫(from c)f(z) dz = | 1 | 2πi | 0 | πi | |
| If f(z) is analytic inside on a simple closed curve C and a be any point inside C then ∫(from c)f(z) dz /(z-a)= | 2πi f(a) | 2πi | 0 | πi | |
| The value of ∫(from c) [(3z^2+7z+1)/(z+1)] dz where C is \|z\| = 1/2 is | 2πi | -6πi | πi | πi/2 | |
| The value of ∫(from c) (cos πz/z-1) dz if C is \|z\| =2 | 2πi | -2πi | πi | πi/3 | |
| The value of ∫(from c) (1/z-1) dz if C is \|z\| =2 | 2πi | 3πi | πi | πi/4 | |
| The value of ∫(from c) (1/z-3) dz if C is \|z\| =1 | 3πi | πi | πi/4 | 0 | |
| The value of ∫(from c) (1/(z-3)^3) dz if C is \|z\| =2 | 3πi | πi | πi/5 | 0 | |
| The Taylor's series of f(z) about the point z=0 is called ___________series | Maclaurin's | Laurent's | Geometric | Arithmetic | |
| The value of ∫(from c) (1/z+4) dz if C is \|z\| =3 | 3πi | πi | πi/4 | 0 | |
| In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____ part | regular | principal | real | imaginary | |
| In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____ part | regular | principal | real | imaginary | |
| The poles of the function f(z) = z/((z-1)(z-2)) are at z = ______ | 1, 2 | 2,3 | 1,-1 | 3,4 | |
| The poles of cotz are______ | 2nπ | nπ | 3nπ | 4nπ | |
| The poles of the function f(z) = cos z/((z+3)(z-4)) are at z = _____ | - 3, 4 | 2,3 | 1,-1 | 3,4 | |
| The isolated singular point of f(z) = z/((z-4)(z-5)) | 1,2 | 2,3 | 0,2 | 4,5 | |
| The isolated singular point of f(z) = z/((z(z-3)) | 1,3 | 2,4 | 0,3 | 4,5 | |
| A simple pole is a pole of order ______ | 1 | 2 | 3 | 4 | |
| The order of the pole z= 2 for f(z) = z/((z+1)(z-2)^2) | 1 | 2 | 3 | 4 | |
| Residue of (cosz / z) at z = 0 is | 0 | 1 | 2 | 4 | |
| The residue at z = 0 of ((1 + e^z) / (zcosz+sinz)) is | 0 | 1 | 2 | 4 | |
| The residue of f(z) = cot z at z= 0 is_____ | 0 | 1 | 2 | 4 | |
| The singularity of f(z) = z / ((z-3)^3) is______ | 0 | 1 | 2 | 3 | |
| A point z=a is said to be a _______point of f(z), if f(z) is not analytic at z=a | Singular | isolated singular | removable | essential singular | |
| A point z=a is said to be a _______point of f(z), if f(z) is analytic except at z=a | Singular | isolated singular | removable | essential singular | |
| In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____point | Singular | isolated singular | removable singular | essential singular | |

| opt6 | Answer |
|---|---|
| | Simple closed curve |
| | multiple curve |
| | 0 |
| | 2πi f(a) |
| | -6πi |
| | -2πi |
| | 2πi |
| | 0 |
| | 0 |
| | Maclaurin's |
| | 0 |
| | regular |
| | principal |
| | 1, 2 |
| | nπ |
| | - 3, 4 |
| | 4,5 |
| | 0,3 |
| | 1 |
| | 2 |
| | 1 |
| | 1 |
| | 1 |
| | 3 |
| | Singular |
| | isolated singular |
| | essential singular |

| | | | | | |
|---|---|---|---|---|---|
| In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____point | Singular | isolated singular | removable singular | essential singular | |
| In contour integration, cos θ=_____ | (z^2+1) /2z | (z^2+1) /2iz | (z^2-1) /2z | (z^2-1) /2iz | |
| In contour integration,  sin θ=_____ | (z^2+1) /2z | (z^2+1) /2iz | (z^2-1) /2z | (z^2-1) /2iz | |

| | |
|---|---|
| | removable singular |
| | (z^2+1)/2z |
| | (z^2-1)/2iz |