
Semester-I

18BEEE101 **Mathematics –I** **4H-4C**

(Calculus and Differential Equations)

Instruction Hours/week: L:3 T:1 P:0 **Marks:**Internal:**40** External:**60** Total:**100**

End Semester Exam:3 Hours

Course Objectives

- To understand geometrical aspects of curvature and elegant application of differential calculus and improper integrals, Gamma, Beta and Error functions which are needed in engineering applications.
- The goal of this course is for students to gain proficiency in calculus computations. In calculus, we use three main tools for analyzing and describing the behavior of functions: limits, derivatives and vector calculus.
- To acquaint the student with mathematical tools needed in evaluating multiple integrals and their usage.
- To make the student acquire sound knowledge of techniques in solving ordinary differential equations and partial differential equations.
- To introduce sequence and series which is central to many applications in engineering.

Course Outcomes

The students will learn:

1. To apply differential and integral calculus to notions of curvature and to improper integrals. Apart from various applications, they will have a basic understanding of Beta and Gamma functions.
2. The tool of power series and Fourier series for learning advanced Engineering Mathematics.
3. To deal with functions of several variables that is essential in most branches of engineering.
4. To find an appropriate method for a given integral and use Green, Gauss and Stokes theorems to simplify calculations of integrals and prove simple results.
5. To understand the ideas of differential equations and facility in solving simple standard examples.
6. To improve facility in algebraic manipulation.

UNIT I - Calculus

Evolutes and involutes; Evaluation of definite and improper integrals; Beta and Gamma functions and their properties; Applications of definite integrals to evaluate surface areas and volumes of revolutions.

UNIT II - Multivariable Calculus: Differentiation

Limit, continuity and partial derivatives, directional derivatives, total derivative, Maxima, minima and saddle points; Method of Lagrange multipliers; Gradient, curl and divergence.

UNIT III - Multivariable Calculus: Integration

Multiple Integration: double and triple integrals (Cartesian and polar), change of order of integration in double integrals, Applications: areas and volumes, Center of mass and

Gravity (constant and variable densities). Theorems of Green, Gauss and Stokes, Simple applications involving cubes and rectangular parallelepipeds.

UNIT IV- Differential Equations

Introduction to Ordinary differential equations: Linear ordinary differential equations of second and higher order with constant coefficients. Introduction to Partial differential equations: Linear Partial differential equations of second and higher order with constant coefficients.

UNIT V - Sequences and Series

Convergence of sequence and series, tests for convergence, power series, Taylor's series. Series for exponential, trigonometric and logarithmic functions; Fourier series: Half range sine and cosine series, Parseval’s theorem.

SUGGESTED READINGS

1. B.S. Grewal, (2010), Higher Engineering Mathematics, 36th Edition, Khanna Publishers.
2. Veerarajan T, (2008),Engineering Mathematics for first year,Tata McGraw-Hill, New Delhi.
3. Ramana B.V, (2010),Higher Engineering Mathematics, 11th Reprint, Tata McGraw Hill New Delhi.
4. N.P. Bali and Manish Goyal, (2010), A text book of Engineering Mathematics, Laxmi Publications.
5. Hemamalini. P.T, (2014), Engineering Mathematics, McGraw Hill Education (India) Private Limited, New Delhi.
6. W. E. Boyce and R. C. DiPrima(2009), Elementary Differential Equations and Boundary Value Problems, 9th EditionWiley India.
7. S. L. Ross(1984)**,** Differential Equations, 3rd Ed.,Wiley India.
8. E. A. Coddington(1995), An Introduction to Ordinary Differential Equations, Prentice Hall India.
9. G.B. Thomas and R.L. Finney, (2002),Calculus and Analytic geometry, 9th Edition,Pearson.
10. E. L. Ince,(1958)**,**Ordinary Differential Equations**,** Dover Publications.
11. G.F. Simmons and S.G. Krantz,(2007), Differential Equations, Tata McGraw Hill.
12. Erwin kreyszig, (2006),Advanced Engineering Mathematics, 9th Edition,John Wiley & Sons.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

COIMBATORE-641 021

DEPARTMENT OF SCIENCE AND HUMANITIES

FACULTY OF ENGINEERING

I B.E ELECTRICAL AND ELECTRONICS ENGINEERING

LESSON PLAN

SUBJECT : MATHEMATICS – I (CALCULUS AND DIFFERENTIAL EQUATIONS)

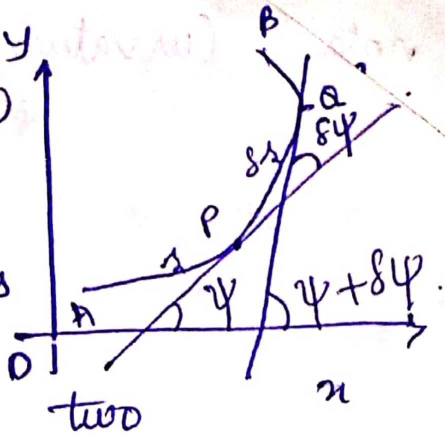
SUB.CODE : 18BEEE101

S.NO	Topics covered	No. of hours
	UNIT I CALCULUS	
1	Introduction to Calculus, Differentiation and Integration	1
2	Concept of Curvature, Evolutes and Involutes	1
3	Problems based on the concept of curvature	1
4	Problems based on Evolutes	1
5	Problems based on Involutes	1
6	Basic problems in integration	1
7	Evaluation of definite and improper integrals	1
8	Concept of Beta and Gamma functions and their properties	1
9	Problems based on Beta and Gamma functions	1
10	Tutorial 1 (Involutes, evolutes and Beta and Gamma functions)	1
11	Applications of definite integrals to evaluate surface areas	1
12	Problems based on Applications of definite integrals to evaluate surface areas	1
13	Problems based on Applications of definite integrals to evaluate surface areas	1
14	Applications of definite integrals to evaluate volumes of revolutions	1
15	Problems based on Applications of definite integrals to evaluate volumes of revolutions	1
16	Tutorial 2 (Applications of definite integrals to evaluate surface areas and volumes of revolutions)	1
	Total	16
	UNIT II MULTIVARIABLE CALCULUS: DIFFERENTIATION	
17	Introduction to Limits and Continuity	1
18	Continuity and partial derivatives	1
19	Problems based on Continuity and partial derivatives,	1
20	Directional derivatives	1
21	Definitions - Total derivative	1
22	Problems based on directional derivatives and total derivatives	1
23	Tutorial 3 (Problems based on limits, continuity and derivatives)	1
24	Maxima, minima and saddle points	1
25	Problems based on the concept of Maxima, minima and saddle points	1

26	Problems based on the concept of Maxima, minima and saddle points	1
27	Method of Lagrange multipliers.	1
28	Problems based on the method of Lagrange multipliers.	1
29	Gradient, curl and divergence	1
30	Problems based on Gradient, curl and divergence.	1
31	Problems based on Gradient, curl and divergence.	1
32	Tutorial 4 (Maxima, minima, and saddle points and Lagrange multipliers)	1
	Total	16
	UNIT III MULTIVARIABLE CALCULUS: INTEGRATION	
33	Introduction of Multiple Integration	1
34	Problems based on double	1
35	Problems based on triple integrals	1
36	Problems based on triple integrals	1
37	Change of order of integration in double integrals	1
38	Problems based on change of order of integration in double integrals	1
39	Tutorial 5 (Problems based on Multiple Integration)	1
40	Applications: Areas and Volumes	1
41	Applications: Center of mass	1
42	Applications: Gravity (constant and variable densities)	1
43	Theorems of Green, Gauss and Stokes	1
44	Problems based on Green	1
45	Problems based on Gauss	1
46	Problems based on Stokes	1
47	Simple applications involving cubes and rectangular parallelepipeds.	1
48	Tutorial 6 (Problems based on Green, Gauss and Stokes theorem)	1
		16
	UNIT IV DIFFERENTIAL EQUATIONS	
49	Introduction to Differential equations	1
50	Introduction to Ordinary differential equations	1
51	Linear ordinary differential equations of second order with constant coefficients	1
52	Problems based on ordinary differential equations of second order with constant coefficients	1
53	Problems based on ordinary differential equations of second order with constant coefficients	1
54	Linear ordinary differential equations of higher order with constant coefficients	1
55	Problems based on ordinary differential equations of higher order with constant coefficients	1
56	Tutorial 7 (Problems based on ODE of second and higher order with constant coefficients)	1
57	Introduction to Partial differential equations	1
58	Linear Partial differential equations of second order with constant coefficients	1
59	Problems based on partial differential equations of second order with constant coefficients	1
60	Problems based on partial differential equations of second order with constant coefficients	1

61	Linear Partial differential equations of higher order with constant coefficients	1
62	Problems based on partial differential equations of higher order with constant coefficients	1
63	Problems based on partial differential equations of higher order with constant coefficients	1
64	Tutorial 8 (Problems based on PDE of second and higher order with constant coefficients)	1
	Total	16
	UNIT V SEQUENCES AND SERIES	
65	Introduction of Convergence of sequence and series	1
66	Tests for convergence	1
67	Problems based on Convergence	1
68	Power series, Taylor's series	1
69	Problems based on Power series, Taylor's series	1
70	Series for exponential function	1
71	Tutorial 9 (Convergence of sequence and series)	1
72	Trigonometric and logarithm functions	1
73	Trigonometric and logarithm functions	1
74	Trigonometric and logarithm functions	1
75	Fourier series: Half range sine and cosine series	1
76	Half range sine and cosine series	1
77	Half range sine and cosine series	1
78	Parseval’s theorem	1
79	Problems based on Parseval’s theorem	1
80	Tutorial 10 (Fourier series)	1
	TOTAL	16
	GRAND TOTAL	**80**

STAFF INCHARGE **HOD**



Curvature of a curve :- (How fast a particle moves in the curve)

Let A be any fixed point on the curve from which arc lengths are measured. Let P & Q be two neighbouring points on the curve AB.

Let arc $AP = s$ & arc $AQ = s+\delta s$, then $PQ = \delta s$

Let ψ & $\psi + \delta\psi$ be the angles made by the tangents at P & Q respectively with the x-axis

Then the angle between the tangents at P&Q is $\delta\psi$.

For a length δs of the curve, change in the direction of the tangent $= \delta\psi$.

$\therefore$ The rate at which the angle has changed (or) average curvature (or) average bending of the ~~curve~~ arc PQ $\Big\} = \dfrac{\delta\psi}{\delta s}$.

Def : The rate of bending of a curve in any interval is called the curvature of the curve in that interval.

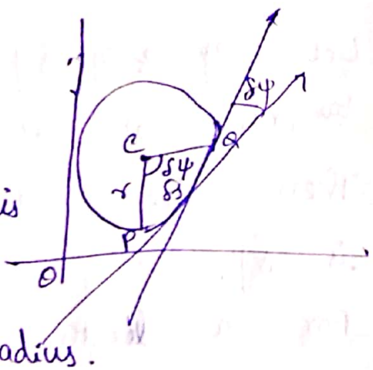
Curvature at the point P

$$= \lim_{Q \to P} \frac{\delta\psi}{\delta s}$$

$$= \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}.$$

$$= \frac{d\psi}{ds}$$

Curvature of a circle



The curvature of a circle at any point on it is the same and is equal to the reciprocal of its radius.

(i.e) Curvature at P $= \frac{d\psi}{ds}$

$$= \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}$$

$$= \frac{1}{r}$$

$\because \angle PCQ = \delta\psi$.
& δs = length of the arc PQ
$= r.\delta\psi$.
(sector of an arc formula)

Formula for the radius of curvature.

Let P be any point on a given curve & ρ be the radius of curvature of the curve at P.

Then $\rho = \frac{1}{\text{curvature}}$

$$= \frac{1}{\left(\frac{d\psi}{ds}\right)}$$

$$= \frac{ds}{d\psi}.$$

Cartesian formula for the radius of curvature

Let P be any point (x,y) on a given curve & ψ be the angle made by the tangent at P with the x-axis.

$\therefore$ radius of curvature at P is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

(or) $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ (or) $\rho = \frac{(1+y'^2)^{3/2}}{y''}$.

where $y_1 = y' = \frac{dy}{dx}$ & $y_2 = y'' = \frac{d^2y}{dx^2}$

note :

i) The def shows that its value depends only on the curve & not on the axes.

Hence, $$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

Radius of curvature in parametric coordinates.

Gn. $x = f(t)$, $y = \phi(t)$.

Then $$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

where $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$.

Radius of curvature in polar coordinates

Let $r = f(\theta)$ be the given curve in polar coordinates.

$x = r\cos\theta$, $y = r\sin\theta$.

$\dot{r} = f'(\theta)$, $\ddot{r} = f''(\theta)$.

$$\therefore \rho = \frac{(r^2+\dot{r}^2)^{3/2}}{r^2 + 2\dot{r}^2 - r\ddot{r}}$$

Centre of curvature:

Let C be the centre of curvature at the point P whose co-ordinates is $(\bar{x}, \bar{y})$.

where $$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

where $y_1 = \frac{dy}{dx}$ & $y_2 = \frac{d^2y}{dx^2}$

Circle of curvature

Let $(\bar{x}, \bar{y})$ be the centre of curvature and ρ be the radius of curvature corresponding to a point (x,y) of the given circle. So, the eqn to the circle of curvature at the point (x,y) is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$.

Problems.

1) What is the radius of curvature of the curve $x^4 + y^4 = 2$ at the point $(1,1)$?

Sol Diff w.r.t x,

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(y^4) = 0.$$

$\Rightarrow 4x^3 + \frac{d}{dy}(y^4) \cdot \frac{dy}{dx} = 0$ ($\because$ y depends on x)

$\Rightarrow 4x^3 + 4y^3 \frac{dy}{dx} = 0.$

$\Rightarrow x^3 + y^3 \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3}.$ —— (1)

$\left(\frac{dy}{dx}\right)_{(1,1)} = -1$

diff (1) w.r.t x,

$$\frac{d^2y}{dx^2} = -\frac{\left[y^3 3x^2 - 3x^3 y^2 \frac{dy}{dx}\right]}{y^6}$$

$$= \frac{3x^3 \frac{dy}{dx} - 3yx^2}{y^4}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1,1)} = \frac{-3-3}{1} = -6.$$

$$\therefore \rho_{(1,1)} = \frac{[1+1]^{3/2}}{-6} = \frac{2^{3/2}}{-6} = -\frac{\sqrt{2}}{3}$$

$\because$ radius of curvature cannot be -ive,

$\therefore \rho = \frac{\sqrt{2}}{3}.$

2) Find the radius of curvature at $(a,0)$ on the curve $xy^2 = a^3 - x^3$.

Sol. Gn $xy^2 = a^3 - x^3$. $\frac{d}{dx}(2x) =$ $2x)$.

diff w.r.t x,

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(a^3 - x^3)$$

$\Rightarrow x\frac{d(y^2)}{dx} + y^2\frac{dx}{dx} = \frac{d}{dx}(a^3) - \frac{d}{dx}(x^3)\,dx$

$\frac{d}{dy}(y^2) \cdot \frac{dy}{dx}$

$\Rightarrow 2xyy' + y^2 = 0 - 3x^2.$

$\Rightarrow 2xyy' = -3x^2 - y^2.$ 20 - 1m (4m)

$\Rightarrow y' = -\frac{(3x^2+y^2)}{2xy}$ 5 - 2m 5 - 14m

$$\left(\frac{dy}{dx}\right)_{(a,0)} = y'_{(a,0)} = \infty.$$

$\therefore \rho = \frac{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{3/2}}{\left(\frac{d^2x}{dy^2}\right)}$ should be used.

$\left(2yx + y^2\frac{dx}{dy}\right) = -3x^2 \frac{dx}{dy}$

$$\frac{dx}{dy} = -\frac{2xy}{(3x^2+y^2)}$$

$-2yx = \frac{dx}{dy}(y^2 + 3x^2)$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = +0.$$

$$\frac{d^2x}{dy^2} = -\left[\frac{(3x^2+y^2)2x - 2xy\left(3x\frac{dx}{dy}+2y\right)}{(3x^2+y^2)^2}\right]$$

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$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -\frac{\left[(3a^2+0)\,2a - 0\right]}{(3a^2)^2}$$

$$= -\frac{\overset{2}{\cancel{6}}a^3}{\underset{3}{\cancel{9}}a^4} = \frac{-2}{3a}.$$

$$\therefore \rho = \frac{[1+0]^{3/2}}{\left(-2/3a\right)} = -\frac{3a}{2}.$$

$$\therefore \rho = \frac{3a}{2} \quad (\because \text{radius cannot be -ve})$$

3) S.T the radius of curvature at the point θ on the curve $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$. ~~(Polar form)~~

Sol. Gn. $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$.

$$\left(\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}\right)$$

$$\frac{dx}{d\theta} = \dot{x} = -3a\sin\theta + 3a\sin 3\theta$$

$$\frac{dy}{d\theta} = \dot{y} = 3a\cos\theta - 3a\cos 3\theta.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{3a\cos\theta - 3a\cos 3\theta}{3a\sin 3\theta - 3a\sin\theta}$$

$$= \frac{\cancel{3a}\,(\cos\theta - \cos 3\theta)}{\cancel{3a}\,(\sin 3\theta - \sin\theta)}$$

$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

$$= -\frac{(\cos 3\theta - \cos\theta)}{(\sin 3\theta - \sin\theta)}$$

$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

$$= +\frac{\cancel{2}\sin 2\theta\,\cancel{\sin\theta}}{\cancel{2}\cos 2\theta\,\cancel{\sin\theta}}.$$

$$= \tan 2\theta.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}(\tan 2\theta)\cdot\frac{d\theta}{dx}.$$

$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$

$$= 2\cdot\sec^2 2\theta\cdot\frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$= 2\sec^2 2\theta\cdot\frac{1}{3a(\sin 3\theta - \sin\theta)}.$$

$\frac{1}{\cos\theta} = \sec\theta.$

$$= \frac{\cancel{2}\sec^2 2\theta}{\underset{3}{\cancel{6}}a\cos 2\theta\sin\theta}$$

$$= \frac{\sec^3 2\theta}{3a\sin\theta}.$$

$1 + \tan^2\theta = \sec^2\theta$

$$\therefore \rho = \frac{[1+\tan^2 2\theta]^{3/2}}{\sec^3 2\theta} \times 3a\sin\theta.$$

$$= \frac{(\sec^2 2\theta)^{3/2}}{\sec^3 2\theta} \times 3a\sin\theta = 3a\sin\theta.$$

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4) Find the radius of curvature of the curve $r = a(1+\cos\theta)$ at the point $\theta = \pi/2$.

Sol Given $r = a(1+\cos\theta)$.

$\dot{r} = -a\sin\theta$, $\ddot{r} = -a\cos\theta$.

$$\therefore \rho = \frac{(r^2+\dot{r}^2)^{3/2}}{r^2+2\dot{r}^2-r\ddot{r}}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1+\cos\theta)(-\cos\theta)}$$

$$= \cancel{\frac{a^3\left[1+2\cos\theta+\cos^2\theta+\sin^2\theta\right]^{3/2}}{a^2\left[(1+2\cos\theta+\cos^2\theta)+2\sin^2\theta-\sin\theta\cos\theta\right]}}$$

$\sin 2A = 2\sin A\cos A$

$$= \cancel{\frac{a\left[2+2\cos\theta\right]^{3/2}}{[}}$$

$$= \frac{a^3\left[1+2\cos\theta+\cos^2\theta+\sin^2\theta\right]^{3/2}}{a^2\left[1+2\cos\theta+\cancel{\cos^2\theta}+2\sin^2\theta+\cos\theta+\cancel{\cos^2\theta}\right]}$$

$$= \frac{a^{\cancel{3}}\left[2+2\cos\theta\right]^{3/2}}{\cancel{a^2}\left[3+3\cos\theta\right]}$$

$$= \frac{a\,2^{3/2}(1+\cos\theta)^{3/2}}{3(1+\cos\theta)}$$

$3/2 - 1$

$2^{1+1/2} = 2\cdot 2^{1/2}$

$$= \frac{a\,2\sqrt{2}}{3}\cdot(1+\cos\theta)^{1/2}$$

$1+\cos 2\theta = 2\cos^2\theta$

$1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$= \frac{a\,2\sqrt{2}}{3}\cdot\sqrt{2}\cdot\cos\frac{\theta}{2}$$

$$= \frac{4a}{3}\cos\frac{\theta}{2}$$

$\rho_{(\theta=\pi/2)} = \frac{4a}{3}\cdot\cos\left(\frac{\pi}{4}\right) = \frac{4a}{3}\cdot\frac{1}{\sqrt{2}}$

5) Find the eqn of the circle of curvature at (c, c) on $xy = c^2$. → rectangular hyperbolas

Sol. Equation of the circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

where $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$ & $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$.

Given:

$xy = c^2$.

$y = \frac{c^2}{x}$.

diff w.r.t x,

$$\frac{dy}{dx} = y_1 = -\frac{c^2}{x^2}.$$

$$(y_1)_{(c,c)} = -1.$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{2c^2}{x^3}.$$

$$(y_2)_{(c,c)} = \frac{2}{c}.$$

$$\therefore \quad \rho = \frac{\left(1+\frac{c^4}{x^4}\right)^{3/2}}{2c^2/x^3}$$

$$= \frac{(x^4+c^4)^{3/2}}{x^{\cancel{6}}\,x^3} \cdot \frac{x^3}{2c^2}.$$

$$\rho_{(c,c)} = \frac{(2c^4)^{3/2}}{c^3 \cdot 2c^2} = \frac{2^{3/2} \cdot c^{6}}{2\,\cancel{c^5}}$$

$$= \sqrt{2}\, c.$$

$$\bar{x}_{(c,c)} = x - \underbrace{\frac{y_1(1+y_1^2)}{y_2}} = \cancel{x} - \frac{(-1)(1+1)}{2/c}$$

$$= c + \frac{2}{2} \cdot c$$

$$= 2c$$

$$\bar{y}_{(c,c)} = y + \frac{(1+y_1^2)}{y_2}$$

$$= c + \frac{(1+1)}{2} \times c = 2c.$$

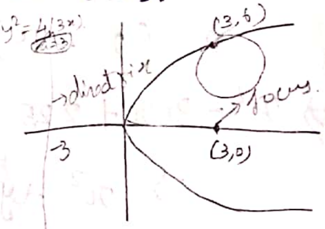
$\therefore$ The circle of curvature at (c,c) is

$$(x-2c)^2 + (y-2c)^2 = (\sqrt{2}c)^2 = 2c^2 \,//.$$

6) Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at $(3,6)$.

(7)

Sol. Given $y^2 = 12x$.



diff w.r.t x,

$$2y\frac{dy}{dx} = 12.$$

$$\Rightarrow \frac{dy}{dx} = 6/y.$$

$$\left(\frac{dy}{dx}\right)_{(3,6)} = \frac{6}{6} = 1. = y_1$$

diff w.r.t x,

$$\frac{d^2y}{dx^2} = 6\left(-\frac{1}{y^2}\right).$$

$$\left(\frac{d^2y}{dx^2}\right)_{(3,6)} = -\frac{1}{6}. = y_2$$

$$\therefore \quad \rho_{(3,6)} = \frac{\left[1+(1)^2\right]^{3/2}}{-1/6} = -2^{3/2} \cdot 6 = \cancel{2^{3/2} \times 2 \times 3} = -2\sqrt{2} \times 6 \cancel{\times 3}$$

$$= -12\sqrt{2}. \qquad \therefore |\rho| = 12\sqrt{2}.$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$\Rightarrow \bar{x}_{(3,6)} = 3 - \frac{1}{-(1/6)}(1+1) = 3+6(2) = 15.$

$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$

$\bar{y}_{(3,6)} = 6 - 6(1+1) = -6.$

$\therefore$ The equation of the circle of curvature is

$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2.$

$\Rightarrow (x-15)^2 + (y+6)^2 = 144 \times 2$ at $(3,6)$

$\Rightarrow x^2 - 30x + 225 + y^2 + 12y + 36 = 288$

$\Rightarrow x^2 + y^2 - 30x + 12y + 261 - 288 = 0.$

$\Rightarrow x^2 + y^2 - 30x + 12y - 27 = 0$ //.

7) S.T the circle of curvature of $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$ is $\left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \frac{a^2}{2}.$

Sol. Given is $\sqrt{x}+\sqrt{y}=\sqrt{a}$. $\Rightarrow x^{1/2}+y^{1/2}=a^{1/2}.$

Diff w.r.t x,

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\left(\frac{dy}{dx}\right) = 0.$

$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\left(\frac{dy}{dx}\right) = 0.$

$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = y_1$

$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)} = -\frac{\sqrt{a/4}}{\sqrt{a/4}} = -1.$

$u = y^{1/2}, v = x^{1/2}$

$u' = \frac{1}{2}y^{-1/2}y',$

$v' = \frac{1}{2\sqrt{x}}$

Diff w.r.t x,

$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\left(\frac{dy}{dx}\right) - \sqrt{y}\frac{1}{2\sqrt{x}}}{x}\right]$

$= \frac{\frac{\sqrt{y}}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{y}}\left(\frac{dy}{dx}\right)}{x}.$

$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)} = \frac{\frac{1}{2} - \frac{1}{2}(-1)}{a/4}$

$= \frac{1}{a} \times 4 = \frac{4}{a}.$

$\therefore \rho_{\left(\frac{a}{4},\frac{a}{4}\right)} = \frac{(1+1)^{3/2}}{4/a} = 2^{3/2} \cdot a \cdot 2^{-2}$

$= \frac{a}{\sqrt{2}}$

$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$\bar{x}_{(a/4,\, a/4)} = \frac{a}{4} + \frac{a}{4}(1+1) = \frac{a}{4}(1+2)$

$= \frac{3a}{4}.$

$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$.

$\bar{y}_{(\frac{a}{4},\frac{a}{4})} = \frac{a}{4} + \frac{a}{4}(1+1) = \frac{3a}{4}$.

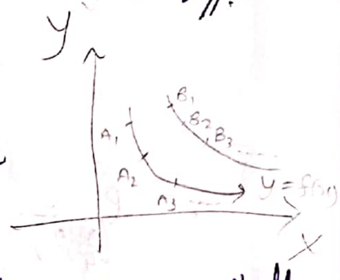
$\therefore$ The eqn of the circle of curvature at $(\frac{a}{4},\frac{a}{4})$ is

$$\left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \frac{a^2}{2} //$$

HW Find the eqn. of the circle of curvature of the curve $x^3+y^3=3axy$ at the point $\left(\frac{3a}{2},\frac{3a}{2}\right)$.

ans. $\left(x-\frac{21a}{16}\right)^2 + \left(y-\frac{21a}{16}\right)^2 = \left(\frac{9a^2}{128}\right) //$

Involutes & Evolutes.



Def :- The locus of the centre of curvature is called the evolute of the curve & the curve itself $(A_1, A_2, \dots)$ is called the involute of its evolute $(B_1, B_2, \dots)$

Family of curves :-

Consider the eqn $f(x,y,c) = 0$.
From this we can get different curves for different values of c. Hence this eqn represents a family of curves.

Envelope of a family of curves :-

The locus of the ultimate points of intersection of consecutive members of this family of curves is called the envelope of that family.

i) $f(x,y,c) = 0 \rightarrow$ family of curves

envelope can be got by eliminating c from $f(x,y,c)$ & $\frac{\partial f}{\partial c}(x,y,c) = 0$

ii) $f(x,y,\alpha,\beta)$ → constant parameters

envelope → write β in terms of α & eliminate c from $f=0$ & $\frac{\partial f}{\partial c} = 0$.

Evolute as the envelope of normals :-

★ The evolute of a curve is the ~~norma~~ envelope of the normals of that curve.

Problems.

To find evolute :-
i) write the parametric eqns $x=f(\theta), y=g(\theta)$
ii) find centre $(\bar{x},\bar{y})$
iii) eliminate θ
iv) write the evolute eqn by replacing $\bar{x}$ & $\bar{y}$ by x & y

1) Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Let $x = a\cos\theta$, $y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = b\cos\theta.$$

$$y_1 = y' = \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta.$$

$$y_2 = y'' = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}.$$

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\frac{1}{(-a\sin\theta)}.$$

$d(\cot\theta) = -\mathrm{cosec}^2\theta$

$$= -\frac{b}{a}(-\mathrm{Cosec}^2\theta)\cdot\frac{1}{-a\sin\theta}$$

$$= -\frac{b}{a^2\sin^3\theta}\;//.$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= a\cos\theta - \frac{\left[-\frac{b}{a}\cot\theta\right]}{\left[-\frac{b}{a^2\sin^3\theta}\right]}\left[1+\left(-\frac{b}{a}\cot\theta\right)^2\right]$$

$$= a\cos\theta - \left(\frac{b}{a}\frac{\cos\theta}{\sin\theta}\right)\times\left(\frac{a^2\sin^3\theta}{b}\right)\left[1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right]$$

$$= a\cos\theta - a\cos\theta\sin^2\theta\left[\frac{a^2\sin^2\theta+b^2\cos^2\theta}{a^2\sin^2\theta}\right]$$

$$= \frac{1}{a}\left[a^2\cos\theta - a^2\sin^2\theta\cos\theta - b^2\cos^3\theta\right]$$

$$= \frac{\cos\theta}{a}\left[a^2 - a^2\sin^2\theta - b^2\cos^2\theta\right]$$

$1+\cos2\theta = 2\cos^2\theta$

$1-\cos2\theta = 2\sin^2\theta$

$\sin^2\theta+\cos^2\theta = 1$

$$= \frac{\cos\theta}{a}\left[a^2(1-\sin^2\theta) - b^2\cos^2\theta\right]$$

$$= \frac{\cos\theta}{a}\left[a^2\cos^2\theta - b^2\cos^2\theta\right]$$

$$= \frac{\cos^3\theta}{a}(a^2-b^2). \quad \text{—— (1)}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\sin\theta + \frac{\left[1+\left(-\frac{b}{a}\cot\theta\right)^2\right]}{\left[-\frac{b}{a^2\sin^3\theta}\right]}$$

$$= b\sin\theta + \frac{\left[1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right]\cdot a^2\sin^3\theta}{-b}.$$

$$= b\sin\theta - \frac{\left[a^2\sin^2\theta+b^2\cos^2\theta\right]}{b\,a^2\sin^2\theta}\cdot a^2\sin^3\theta$$

$$= \frac{1}{b}\left[b^2\sin\theta - a^2\sin^3\theta - b^2\cos^2\theta\sin\theta\right]$$

$$= \frac{\sin\theta}{b}\left[b^2(1-\cos^2\theta) - a^2\sin^2\theta\right]$$

$$= \frac{\sin\theta}{b}\left[b^2\sin^2\theta - a^2\sin^2\theta\right] = \frac{\sin^3\theta}{b}\left[b^2-a^2\right]$$

$$= -\frac{\sin^3\theta}{b}(a^2-b^2) \quad \text{(2)}$$

From ①, $\cos^3\theta = \dfrac{a\bar{x}}{(a^2-b^2)}$

$\therefore \cos\theta = \left(\dfrac{a\bar{x}}{(a^2-b^2)}\right)^{1/3}$.

From ②,

$\sin^3\theta = -\dfrac{b\bar{y}}{(a^2-b^2)}$

$\therefore \sin\theta = \left[\dfrac{-b\bar{y}}{(a^2-b^2)}\right]^{1/3}$.

$\therefore \cos^2\theta + \sin^2\theta = 1 \Rightarrow \left(\dfrac{a\bar{x}}{(a^2-b^2)}\right)^{2/3} + \left(\dfrac{-b\bar{y}}{(a^2-b^2)}\right)^{2/3} = 1$

(ie) $(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$.

Locus of $(\bar{x},\bar{y})$ (ie) evolute is

$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$

2) S.T the evolute of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$.

$x = a\sec\theta$, $y = b\tan\theta$.

$\dot{x} = a\sec\theta\tan\theta$ $\dot{y} = b\sec^2\theta$.

$d(\operatorname{cosec}\theta) = (-\operatorname{cosec}\theta\cot\theta)$

3) Find the evolute of the parabola $y^2 = 4ax$

Sol.

Let $x = at^2$, $y = 2at$.

$\dfrac{dx}{dt} = 2at$, $\dfrac{dy}{dt} = 2a$.

$\therefore y_1 = y' = \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{1}{\left(\frac{dx}{dt}\right)}$

$= \dfrac{2a}{2at} = \dfrac{1}{t}$.

$y_2 = y'' = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$= \dfrac{d}{dt}\left(\dfrac{1}{t}\right) \cdot \dfrac{1}{\left(\frac{dx}{dt}\right)}$

$= -\dfrac{1}{t^2} \cdot \dfrac{1}{2at} = -\dfrac{1}{2at^3}$.

Let $(\bar{x},\bar{y})$ be the centre of curvature.

$\bar{x} = x - \dfrac{y_1}{y_2}(1+y_1^2)$

$= at^2 + \dfrac{1}{t} \cdot 2at^{3}\left(1 + \dfrac{1}{t^2}\right)$

$= at^2 + 2at^2\dfrac{(t^2+1)}{t^2}$

$= at^2 + 2at^2 + 2a = 3at^2 + 2a$. ①

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= 2at + \frac{\left(1+\frac{1}{t^2}\right)}{\left(-\frac{1}{2at^3}\right)}$$

$$= 2at - 2at^{\cancel{3}}\left(\frac{t^2+1}{\cancel{t^2}}\right)$$

$$= 2at - 2at^3 - 2at$$

$$= -2at^3. \quad \text{——— (2)}$$

eliminating 't' from (1) & (2).

(1) $\Rightarrow$ $\bar{x} = 3at^2 + 2a.$

$$\Rightarrow (\bar{x} - 2a) = 3at^2$$

$$\Rightarrow t^2 = \frac{(\bar{x}-2a)}{3a}$$

$$\Rightarrow t = \left(\frac{\bar{x}-2a}{3a}\right)^{1/2}$$

Sub t in (2),

$$\bar{y} = -2a\left(\frac{\bar{x}-2a}{3a}\right)^{3/2}$$

Squaring on both sides,

$$\bar{y}^2 = 4a^2 \frac{(\bar{x}-2a)^3}{27a^3}$$

$$\Rightarrow 27a^{\cancel{3}}\bar{y}^2 = 4\cancel{a^2}(\bar{x}-2a)^3$$

$$\Rightarrow 27a\bar{y}^2 = 4(\bar{x}-2a)^3$$

$\therefore$ The evolute is $27ay^2 = 4(x-2a)^3$

4) Find the evolute of the parabola $x^2 = 4ay$

$x = 2at, \; y = at^2$

ans: $27ax^2 = 4(y-2a)^3$

~~5) Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$ where a & b connected by the relation $a^2+b^2 = c^2$~~

~~Sol~~

5) Find the envelope of $y = mx + m^2$, m being the parameter.

Sol. Gn $y = mx + m^2$. ——— (1).

Diff partially w.r.t m, we get

$$0 = x + 2m$$

$$\Rightarrow m = \frac{-x}{2}. \quad \text{——— (2).}$$

Sub ② in ①

$$y = \left(-\frac{x}{2}\right)x + \left(\frac{-x}{2}\right)^2$$

$$\Rightarrow y = -\frac{x^2}{2} + \frac{x^2}{4} = -\frac{x^2}{4}.$$

$\cancel{\Rightarrow x^2 + 4y = 0}$.

$\Rightarrow x^2 + 4y = 0$ is the required envelope.

6) Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where a & b are connected by the relation $a^2+b^2 = c^2$.

Sol. Let a & b be ~~two~~ functions of some arbitrary parameter t.

Given $\frac{x}{a} + \frac{y}{b} = 1$ ——— ①.

Diff ① w.r.t t, we get (or) diff w.r.t b, $-\frac{x}{a^2}\frac{da}{db} - \frac{y}{b^2} = 0$. $\Rightarrow \frac{da}{db} = -\frac{y}{b^2}\cdot\frac{a^2}{x}$

$$-\frac{x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0.$$

$$\Rightarrow \frac{x}{a^2}\frac{da}{dt} = -\frac{y}{b^2}\frac{db}{dt}.$$

$$\Rightarrow \frac{\frac{da}{dt}}{\left(\frac{db}{dt}\right)} = \frac{-y}{b^2}\cdot\frac{a^2}{x}$$

$$\Rightarrow \frac{da}{db} = -\frac{a^2y}{xb^2}. \text{ ——— ②.}$$

diff $a^2+b^2 = c^2$ w.r.t t, we get.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0.$$

$$\Rightarrow a\frac{da}{dt} = -b\frac{db}{dt}$$

$$\Rightarrow \frac{\left(\frac{da}{dt}\right)}{(db/dt)} = -\frac{b}{a} \Rightarrow \frac{da}{db} = \frac{-b}{a} \text{ ——— ③.}$$

② = ③.

$$\Rightarrow -\frac{a^2y}{xb^2} = -\frac{b}{a}.$$

$$\Rightarrow \frac{y}{x} = \frac{b^3}{a^3} \quad \cancel{\Rightarrow a^3y = xb^3}$$

$$\Rightarrow \frac{y}{b^3} = \frac{x}{a^3} \Rightarrow \frac{\left(\frac{y}{b}\right)}{b^2} = \frac{\left(\frac{x}{a}\right)}{a^2}$$

$$= \frac{\left(\frac{x}{a}\right)+\left(\frac{y}{b}\right)}{(a^2+b^2)}$$

$$= \frac{1}{c^2}$$

$$\Rightarrow \frac{y}{b^3} = \frac{1}{c^2}, \quad \frac{x}{a^3} = \frac{1}{c^2} \Rightarrow yc^2 = b^3, \quad xc^2 = a^3$$

$$\Rightarrow b = (c^2y)^{1/3}, \quad a = (xc^2)^{1/3}.$$

Sub a & b in $a^2+b^2=c^2$

$\Rightarrow (xc^2)^{2/3} + (yc^2)^{2/3} = c^2$

$\Rightarrow c^{4/3}\left[x^{2/3}+y^{2/3}\right] = c^2$

$\Rightarrow x^{2/3}+y^{2/3} = c^{2/3}$ is the required envelope. $\frac{c^2}{c^{4/3}} = c^{2-4/3} = c^{2/3}$

7) Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the two (constant) parameters are connected by the relation $a+b=c$, where c is a constant.

Sol. Let a & b be fns of some 3rd arbitrary parameter t.

Gn: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. ——— ①.

diff ① w.r.t t, (or) assume $a=a(b)$ & diff w.r.t b

$-\frac{2x^2}{a^3}\frac{da}{dt} - \frac{2y^2}{b^3}\frac{db}{dt} = 0. \Rightarrow -\frac{x^2}{a^3}\frac{da}{dt} = \frac{y^2}{b^3}\frac{db}{dt}$

$\Rightarrow \frac{da}{db} = -\frac{y^2a^3}{x^2b^3}$ ——— ②.

~~And~~ Gn: $a+b=c$ ——— ③

diff ③ w.r.t t,

$\frac{da}{dt}+\frac{db}{dt}=0 \Rightarrow \frac{da}{db} = -1$ ——— ④.

② = ④

$\Rightarrow \frac{y^2a^3}{x^2b^3} = 1. \Rightarrow \frac{y^2}{b^3} = \frac{x^2}{a^3}$

$\Rightarrow \frac{\left(\frac{y^2}{b^2}\right)}{b} = \frac{\left(\frac{x^2}{a^2}\right)}{a} = \frac{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}{a+b} = \frac{1}{c}.$

$\Rightarrow \frac{y^2}{b^3} = \frac{1}{c}, \quad \frac{x^2}{a^3} = 1/c.$

$\Rightarrow b^3 = y^2c, \quad a^3 = x^2c.$

$\Rightarrow b = (y^2c)^{1/3}, \quad a = (x^2c)^{1/3}.$

Sub a & b in ③, we get.

$(x^2c)^{1/3} + (y^2c)^{1/3} = c.$

$\Rightarrow c^{1/3}\left[x^{2/3}+y^{2/3}\right] = c. \quad c^{1-1/3}$

$\Rightarrow x^{2/3}+y^{2/3} = c^{2/3}$ is the required envelope.

8) Find the envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$ where a & b are connected by the relation i) $a+b=c$ ii) $ab=c^2$ where c is constant.

i) $(x^{1/2}+y^{1/2}) = c^{1/2}$ → ans.

ii) $4xy = c^2$ is the envelope

Evaluation of definite and improper integrals.

Definite integral

If $\int f(x)\,dx = F(x)+c$ (indefinite) then $\int_a^b f(x)\,dx = F(b)-F(a)$ (constant)

where a & b are lower and upper limits

and $\int_a^b f(x)\,dx$ is called the definite integral of f(x). ~~Let~~

1) $\int x^3\,dx = \left[\frac{x^{3+1}}{4}\right]_1^2$ $\qquad \frac{x^{n+1}}{n+1}$

eg:

2) $\int_1^2 \left(x^2 - 3x^{1/2} + \frac{1}{x^2}\right)dx$

$= \left[\frac{x^3}{3} - \frac{3x^{3/2}}{3} \times 2 - \frac{1}{x}\right]_1^2$

$= \left[\frac{8}{3} - 2^{3/2}2 - \frac{1}{2}\right] - \left[\frac{1}{3} - 2 - 1\right]$ $\quad \frac{1}{3} - 3$

$= \left[\frac{8}{3} - 4\sqrt{2} - \frac{1}{2}\right] - \left[\frac{-8}{3}\right]$

$= \frac{16}{3} - \frac{1}{2} - 4\sqrt{2}$

$= \frac{29}{6} - 4\sqrt{2}$ //.

Improper integrals.

In an integral, if, either the range is infinite or the integrand has an infinite discontinuity in the range (i.e. the integrand tends to ∞ at some point of the range), then the integral is called an Infinite integral (or) Improper integral.

eg: 1) $\int_a^\infty x\,dx$ 2) $\int_{-\infty}^1 x\,dx$ 3) $\int_{-\infty}^\infty x\,dx$.

4) $\int_0^1 \frac{dx}{x} = [\log(x)]_0^1 = \log(1) - \log(0)$

$= 0 - (-\infty)$

$\boxed{\begin{aligned}\log 1 &= 0\\ \log 0 &= -\infty\\ \log\infty &= \infty\end{aligned}}$

Beta and Gamma functions

Def.

i) $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ for $m>0$, $n>0$ is known as Beta fn. (or) first Eulerian integral and is denoted by $\beta(m,n)$.

ii) $\int_0^\infty x^{n-1}e^{-x}\,dx$ for $n>0$ is known as Gamma function (or) Second Eulerian integral and is denoted by $\Gamma(n)$.

Recurrence formula of Gamma functions.

By def, $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$ ——①, $n>0$.

$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$, $n>-1$.

$e^\infty = \infty$
$e^{-\infty} = 0$
$e^0 = 1$

$= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx$ | $u = x^n$, $du = nx^{n-1}dx$, $dv = \int e^{-x} dx$, $v = -e^{-x}$

$= [0-0] + n\,\Gamma(n)$ (by ①).

$\therefore \boxed{\Gamma(n+1) = n\Gamma(n)}$, $n>0$.

Corollary:-

1) $\Gamma(n+1) = n\,\Gamma(n)$
$= n(n-1)\,\Gamma(n-1)$
$= n(n-1)\,(n-2)\,\Gamma(n-2)$
$= n(n-1)(n-2) \ldots\ldots 1.\Gamma(1)$ ——②

$\Gamma(1) = \int_0^\infty (1)\, e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty$
$= \left[-e^{-\infty} + e^0\right] = [0+1] = 1.$

$\therefore$ ② $\Rightarrow \Gamma(n+1) = n(n-1)(n-2) \ldots 1.$
$= n!$

2) $\Gamma(n+a) = (n+a-1)(n+a-2)\ldots a\,\Gamma(a)$, $n>0$

Properties of Beta functions.

i) $\beta(m,n) = \beta(n,m)$

Proof:- By def, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $1-x = y$. $\Rightarrow x = 1-y$
$-dx = dy$.
$\Rightarrow dx = -dy$.

x	0	1
y	1	0

$\therefore \beta(m,n) = \int_1^0 (1-y)^{m-1}\, y^{n-1}\,(-dy)$

$= \int_0^1 y^{n-1}(1-y)^{m-1}\, dy$ $\left(\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right)$

$= \beta(n,m)$

ii) $\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$

Proof.

By def, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $x = \frac{y}{1+y}$.

$dx = \frac{(1+y)dy - y\,dy}{(1+y)^2} = \frac{dy}{(1+y)^2}$

$$1-x = 1-\frac{y}{1+y} = \frac{1+y-y}{1+y} = \frac{1}{1+y}.$$

$x = \frac{y}{1+y}$

$x + xy - y = 0$

$x + y(x-1) = 0$

$\Rightarrow y(x-1) = -x$

$\Rightarrow y = \frac{x}{1-x}$

x	0	1
y	0	∞

$$\therefore \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}\,dy}{(1+y)^{m-1+n-1+2}}$$

$$= \int_0^\infty \frac{y^{m-1}\,dy}{(1+y)^{m+n}} \quad //$$

iii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x\,dx$

Proof. By def, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \sin^2\theta \Rightarrow \theta = \sin^{-1}(x^{1/2})$ $\quad x^{1/2} = \sin\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$dx = d\left(\frac{1-\cos 2\theta}{2}\right)$

$= 0 + \frac{2\sin 2\theta}{2}\,d\theta$

$= \sin 2\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\sin 2\theta\,d\theta.$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin 2\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \cos^{2n-2+1}\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta \quad //$$

Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$.

Proof. By def, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ — (1)

$\beta(m+1,n) = \int_0^1 x^{m+1-1}(1-x)^{n-1}\,dx = \int_0^1 x^m (1-x)^{n-1}\,dx$

$\beta(m,n+1) = \int_0^1 x^{m-1}(1-x)^n\,dx.$

$$\therefore \beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^m(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^n\,dx$$

$$= \int_0^1 \left[x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right]dx.$$

$$= \int_0^1 \left[x \cdot x^{m-1} (1-x)^{n-1} + x^{m-1} (1-x)(1-x)^{n-1} \right] dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} [\cancel{x} + (1-\cancel{x})] \, dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \beta(m,n) \quad (\text{by } ①)$$

write the result 1st here

⑤ Relation between Beta and Gamma functions.

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Proof:- By def, $\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} \, dx$

Put $x = t^2$,
$dx = 2t \, dt$

x	0	∞
t	0	∞

$$\therefore \Gamma(m) = \int_0^\infty t^{2m-2} e^{-t^2} (2t \, dt)$$

$$= 2 \int_0^\infty t^{2m-1} e^{-t^2} \, dt$$

$$= 2 \int_0^\infty x^{2m-1} e^{-x^2} \, dx \quad \text{(replacing dummy variable } t \text{ by } x)$$

Illy, $\Gamma(n) = 2 \int_0^\infty y^{2n-1} e^{-y^2} \, dy.$

$$\therefore \Gamma(m)\Gamma(n) = 4 \int_0^\infty x^{2m-1} e^{-x^2} dx \int_0^\infty y^{2n-1} e^{-y^2} dy$$

$$= 4 \int_0^\infty \int_0^\infty x^{2m-1} y^{2n-1} e^{-x^2-y^2} \, dx \, dy.$$

Put $x = r\cos\theta$, $y = r\sin\theta$. $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

~~$dx = -r\sin\theta \, d\theta + \cos\theta \, dr$, $dy = r\cos\theta \, d\theta + \sin\theta \, dr$~~

~~$dx\,dx = (-r\sin\theta \, d\theta + \cos\theta \, dr)(r\cos\theta \, d\theta + \sin\theta \, dr)$~~

~~$= r^2 \sin\theta \, d\theta$~~

$dx\,dy = r\,dr\,d\theta$. (by transformation from cartesian to polar)

Transformation from cartesian to polar coordinates.
Let the polar coordinates of point P whose cartesian coordinates are (x,y) be (r,θ)

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$
Jacobian

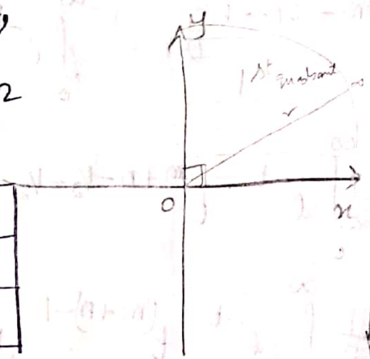
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$\Rightarrow dx\,dy = r\,dr\,d\theta$

$\frac{y}{x} = \tan\theta$,

& $r^2 = x^2 + y^2$



x	0	∞
y	0	∞
r	0	∞

Illy, limits of θ ranges from 0 to $\pi/2$ ($\because$ we are considering 1st quadrant)

$$\therefore \Gamma(m)\Gamma(n) = 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} e^{-r^2} \, r \, dr \, d\theta$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2m-1+2n-1+1} \sin^{2m-1}\theta \cos^{2n-1}\theta \, dr\, d\theta$$

$$= 4\int_0^\infty e^{-r^2} r^{2m+2n-1}\, dr \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \quad \text{—— (1)}$$

Consider, $\int_0^\infty e^{-r^2} r^{2m+2n-1}\, dr.$

Put $r^2 = t$, $2r\,dr = dt$

$\Rightarrow r = t^{1/2}$, $\Rightarrow dr = \frac{dt}{2r} = \frac{dt}{2\sqrt{t}}$

r	0	∞
t	0	∞

$$\therefore \int_0^\infty e^{-r^2} r^{2m-2n-1}\, dr = \int_0^\infty e^{-t}\, t^{1/2(2m+2n-1)} \frac{dt}{2t^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{m+n-1/2-1/2}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{(m+n)-1}\, dt.$$

$$= \frac{1}{2}\Gamma(m+n) \quad \text{(by def).} \quad \text{—— (2)}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n) \quad \text{—— (3)}$$

(by properties of beta fn)

Sub (2) & (3) in (1).

$$\Gamma(m)\Gamma(n) = \cancel{4} \times \frac{1}{\cancel{2}}\Gamma(m+n) \times \frac{1}{\cancel{2}}\beta(m,n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Cor (i) :-

$$\Gamma(1/2) = \sqrt{\pi}$$

Proof :- W.K.T, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Put $m = n = 1/2$.

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)}$$

$$2\int_0^{\pi/2} \sin^0\theta \cos^0\theta\, d\theta = \frac{[\Gamma(1/2)]^2}{\Gamma(1)}$$

$$\Rightarrow 2\int_0^{\pi/2} d\theta = \frac{[\Gamma(1/2)]^2}{1}$$

$$\therefore [\Gamma(1/2)]^2 = 2[\theta]_0^{\pi/2} = 2[\pi/2 - 0] = \pi.$$

$$\therefore \Gamma(1/2) = \sqrt{\pi} \quad //$$

(we can't solve normally by using def since it becomes repeated)

ii) ~~4a~~ $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

Put $m = 1-n$.

$\therefore \beta(1-n, n) = \dfrac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$

$\Rightarrow \Gamma(1-n)\,\Gamma(n) = \beta(1-n, n) = \beta(n, 1-n)$ ↳ by 1st beta property

$= \displaystyle\int_0^\infty \frac{y^{n-1}}{(1+y)^{n+1-n}}\,dy.$

$= \dfrac{\pi}{\sin n\pi}$ // $\left[\because \displaystyle\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}\right]$

if $n = 1/2$.

$\therefore \left[\Gamma(1/2)\right]^2 = \dfrac{\pi}{\sin \pi/2} = \dfrac{\pi}{1}$

$\Rightarrow \left[\Gamma(1/2)\right] = \sqrt{\pi}$ //

iii) W.K.T $\beta(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

Put $2m = p$ & $2n = q$. (here we won't diff. & find [ ] here we are replacing value)

$\therefore \displaystyle\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta\, d\theta = \frac{1}{2}\beta(p/2, q/2)$

$= \dfrac{1}{2} \dfrac{\Gamma(p/2)\,\Gamma(q/2)}{\Gamma\left(\frac{p+q}{2}\right)}$

Problems

1) HW. $\displaystyle\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$

Sol. Put $\log \dfrac{1}{x} = t$

Taking exponential on both sides,

$\dfrac{1}{x} = e^t \quad \Rightarrow x = \dfrac{1}{e^t} = e^{-t}$.

$\therefore dx = -e^{-t}\,dt$

x	0	1
t	∞	0

$\therefore \displaystyle\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \int_\infty^0 e^{-tm} (t)^n (-e^{-t})\,dt$

$= \displaystyle\int_0^\infty e^{-mt}\, t^n e^{-t}\, dt$

$= \displaystyle\int_0^\infty t^n e^{-(m+1)t}\, dt$

Put $(m+1)t = y$.

$(m+1)\,dt = dy \Rightarrow dt = \dfrac{dy}{(m+1)}$.

t	0	∞
y	0	∞

$$\therefore \int_0^1 x^m \left(\log\left(\frac{1}{x}\right)\right)^n dx = \int_0^\infty \frac{y^n}{(m+1)^n} e^{-y} \frac{dy}{(m+1)}$$

$$= \int_0^\infty \frac{y^n e^{-y}}{(m+1)^{n+1}} dy$$

$$= \frac{1}{(m+1)^{n+1}} \cdot \Gamma(n+1) \quad //$$

$$\left(\because \Gamma(n) = \int_0^\infty y^{n-1} e^{-y} dy\right)$$

2) Prove that $\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$ where $p>0, q>0$.

Sol. Put $\log\left(\frac{1}{y}\right) = t$

$\Rightarrow \frac{1}{y} = e^t \Rightarrow y = e^{-t}$

$dy = -e^{-t} \cdot dt$

y	0	1
t	∞	0

$$\therefore \int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = \int_\infty^0 (e^{-t})^{q-1} (t)^{p-1} (-e^{-t}) dt$$

$$= \int_0^\infty (e^{-t})^{q-\cancel{1}+\cancel{1}} t^{p-1} dt$$

$$= \int_0^\infty t^{p-1} e^{-qt} dt$$

$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx$.

Put $qt = x \Rightarrow t = \frac{x}{q}$

$q\,dt = dx \Rightarrow dt = \frac{dx}{q}$

t	0	∞
x	0	∞

$$\therefore \int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = \int_0^\infty \left(\frac{x}{q}\right)^{p-1} e^{-x} \left(\frac{dx}{q}\right)$$

$$= \int_0^\infty \frac{x^{p-1} e^{-x}}{q^{p-1+1}} dx$$

$$= \frac{1}{q^p} \int_0^\infty x^{p-1} e^{-x} dx$$

$$= \frac{1}{q^p} \Gamma(p) \quad // \quad (\text{by def})$$

3) Show that $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Sol We know that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad (\text{by properties of } \beta\text{-fn})$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \underbrace{\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx}_{①}$$

Consider, $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

Put $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2} dy$.

$\Rightarrow y = 1/x$

x	1	∞
y	1	0

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\,dy\right)$$

$$= \int_0^1 \frac{(1)^{m-1}}{y^{m-1}} \cdot \frac{1}{\left(\frac{y+1}{y}\right)^{m+n}} \left(\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m-1+2}} \cdot \frac{1}{(y+1)^{m+n}}\,dy.$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m+1}} \cdot \frac{dy}{(y+1)^{m+n}}$$

$$= \int_0^1 y^{\cancel{m}+n-\cancel{m}-1} \frac{dy}{(y+1)^{m+n}}$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad \text{(replace dummy variable } y \text{ by } x) \longrightarrow (2).$$

Sub (2) in (1).

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx \quad //.$$

4) evaluate $\int_0^\infty e^{-x^2}\,dx$ $\qquad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$

Sol Put $x^2 = t \Rightarrow x = \sqrt{t}$

(ie) $2x\,dx = dt$

$\Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

x	0	∞
t	0	∞

$$\therefore \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}}.$$

$$= \frac{1}{2}\int_0^\infty t^{-1/2} e^{-t}\,dt$$

$$= \frac{1}{2}\,\Gamma(1/2)$$

$$= \frac{1}{2}\sqrt{\pi} \quad //.$$

$n-1 = -1/2$
$n = -1/2 + 1$
$= 1/2.$

5) i) $\int_0^1 x^7 (1-x)^8\,dx.$

Sol. W.K.T, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$

here $m-1 = 7$, $n-1 = 8.$

$\Rightarrow m = 8$, $n = 9.$

$$\therefore \int_0^1 x^7 (1-x)^8\,dx = \beta(8,9)$$

$$= \frac{\Gamma(8)\,\Gamma(9)}{\Gamma(8+9)}$$

$$= \frac{7!\,8!}{16!} \quad //$$

$\left[\because \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$

$\left[\because \Gamma(n+1) = n!\right]$

ii) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$.

Sol. W.K.T $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

here $2m-1 = 7$, $2n-1 = 5$.

$\Rightarrow m = 4$, $n = 3$.

$$\therefore \int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{2}\beta(4,3)$$

$$= \frac{1}{2} \frac{\Gamma 4 \, \Gamma 3}{\Gamma 7}$$

$$= \frac{1}{2} \frac{3! \, 2!}{6!}$$

$$= \frac{3!}{6\times5\times4\times3!} = \frac{1}{120}$$

Result :

$\Gamma(3/2) = 1/2 \, \Gamma(1/2)$

$\Gamma(5/2) = 3/2 \, \Gamma(3/2) = 3/2 \cdot 1/2 \, \Gamma(1/2)$

$\Gamma(7/2) = 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma(1/2)$

$\Gamma(5/4) = \frac{1}{4}\Gamma(1/4)$

iii) $\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \int_0^{\pi/2} \sin^{10}\theta \cos^0\theta \, d\theta$.

Sol W.K.T $\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma(\frac{p}{2}) \Gamma(\frac{q}{2})}{\Gamma(\frac{p+q}{2})}$

where $p = 2m$ & $q = 2n$.

here $p-1 = 10 \Rightarrow p = 11$ & $q-1 = 0. \Rightarrow q = 1$.

$$\therefore \int_0^{\pi/2} \sin^{10}\theta \, d\theta = \frac{1}{2} \frac{\Gamma(\frac{11}{2}) \Gamma(1/2)}{\Gamma(6)} = \frac{1}{2} \cdot \frac{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot(\sqrt{\pi})^2}{5!}$$

$$= \frac{9.7.5.3 \, \pi}{\cancel{512} \, 5!} = \frac{63\pi}{512}$$

$\Gamma(n+1) = n!$

$n+1 = \frac{11}{2}$

$n = \frac{9}{2}$

iv) $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} \frac{\sin^{1/2}\theta}{\cos^{1/2}\theta} d\theta$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$

$\left[\text{W.K.T } 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(m,n)\right]$

here $2m-1 = 1/2$, $2n-1 = -1/2$.

$\Rightarrow 2m = 3/2$, $2n = 1/2$.

$\Rightarrow m = 3/4$, $n = 1/4$.

$$\therefore \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta = \frac{1}{2}\beta(3/4, 1/4)$$

$$= \frac{1}{2} \frac{\Gamma(3/4) \Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \Gamma(1-\tfrac{1}{4}) \Gamma(1/4)$$

$$= \frac{\pi}{2\sin\frac{\pi}{4}} \quad \left(\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right)$$

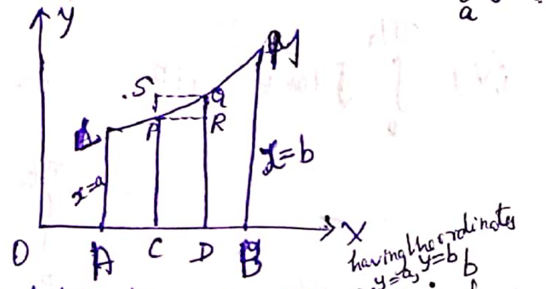
$\sin \pi/4 = \frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{2}\,\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Areas of cartesian curves.

i) Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a, x = b$ is $\int_a^b y\,dx$

iii) If $f(x) \le 0 \ \forall\ x$ in $a \le x \le b$ then Area $= \int_a^b (-y)\,dx$



ii) The area of the region bounded by the curve $x = g(y)$, having the ordinates $y = a, y = b$ is $\int_a^b x\,dy$.

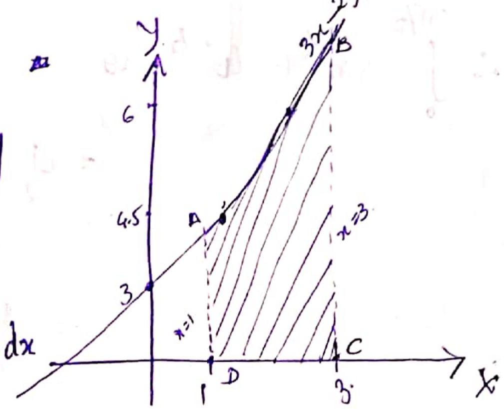
(X) Problems

1) Find the area of the region bounded by the line $3x - 2y + 6 = 0$, $x = 1$, $x = 3$ & x-axis.

Sol $3x - 2y + 6 = 0$.

$\Rightarrow y = \frac{3x+6}{2}$

x	0	1	2
y	3	9/2=4.5	6



$\therefore$ Area of the region ABCD $= \int_a^b y\,dx$

$= \int_1^3 \frac{(3x+6)}{2}\,dx = \frac{3}{2}\int_1^3 (x+2)\,dx = \frac{3}{2}\left[\frac{x^2}{2} + 2x\right]_1^3$

$= \frac{3}{2}\left[\left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} + 2\right)\right] = \frac{3}{2}\left[\frac{21}{2} - \frac{5}{2}\right] = \frac{3}{2} \times \frac{16}{2} = 12$ sq. units

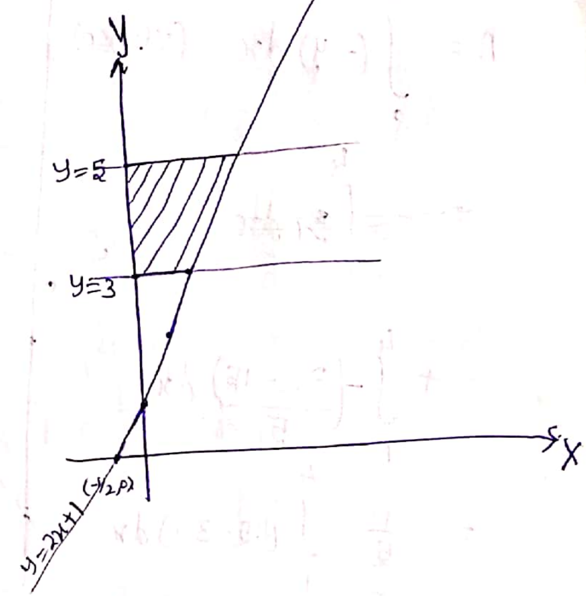
(X) 2) Find the area of the region bounded by $y = 2x+1$, $y = 3$, $y = 5$ & y-axis.

Sol. Given $y = 2x+1$.

$2x = y - 1$.

$x = \frac{y-1}{2}$.

y	0	1	2	3
x	-1/2	0	1/2	1



The required area

$A = \int_a^b x\,dy$

$= \int_3^5 \left(\frac{y-1}{2}\right) dy = \frac{1}{2}\left[\frac{y^2}{2} - y\right]_3^5$

$= \frac{1}{2}\left[\left(\frac{25}{2} - 5\right) - \left(\frac{9}{2} - 3\right)\right]$

$= \frac{1}{2}\left[\frac{15-3}{2}\right] = \frac{1}{2} \times \frac{12}{2} = 3$ sq. units.

(X) 3) Find the area of the region bounded by the line $3x - 5y - 15 = 0$, $x = 1$, $x = 4$ and x axis.

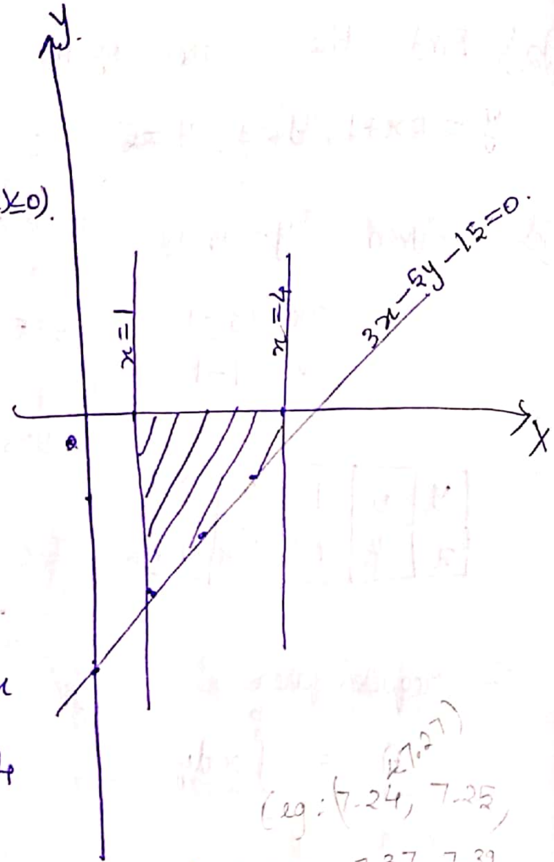
Sol. Given $3x - 5y - 15 = 0$

$y = \frac{3x-15}{5}$

x	0	1	2	3
y	-3	-2.4	-1.8	-1.2

The required area

$$A = \int_a^b (-y)\,dx \quad (\because f(x) \le 0).$$



$$\cancel{= - \int_1^4 3x\,dx}$$

$$= + \int_1^4 -\left(\frac{3x-15}{5}\right) dx$$

$$= \frac{1}{5} \int_1^4 (15-3x)\,dx$$

$$= \frac{1}{5}\left[15x - \frac{3x^2}{2}\right]_1^4$$

$$= \tfrac{1}{5}\left[\left(60 - \frac{48}{2}\right) - \left(15 - 3/2\right)\right]$$

$$= \frac{1}{5}\left[(60-24) - \left(\frac{27}{2}\right)\right] = \frac{1}{5}\left[36 - \frac{27}{2}\right]$$

$$= \frac{1}{5} \times \frac{45}{2} = 9/2 \text{ sq.units.}$$

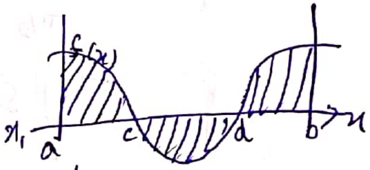
(4) Find the area of the region bounded $y = 2x+4$, $y=1$ & $y=3$ & y-axis.

Ans : 2 sq. units.

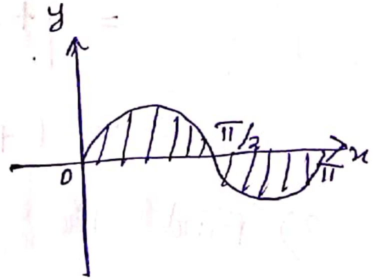
Remark:-

If the continuous curve $y=f(x)$ crosses the x-axis, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d (-f(x))\,dx + \int_d^b f(x)\,dx.$$



1) Find the area bounded by the curve $y = \sin 2x$ between the ordinates $x=0$, $x=\pi$ & x-axis.



Sol. Put $y=0$. inorder to obtain the point where the curve $y = \sin 2x$ meets the x-axis.

$$\Rightarrow \sin 2x = 0.$$

$$\Rightarrow 2x = \sin^{-1}(0) = \sin^{-1}(\sin n\pi) \quad \forall n \in \mathbb{Z}.$$

$$\Rightarrow 2x = n\pi \Rightarrow x = \frac{n\pi}{2} \quad \forall n \in \mathbb{Z}.$$

$\boxed{\sin n\pi = 0 \text{ always}}$

(ie) $x = \left\{0, \pm\frac{\pi}{2}, \pm\pi, \dots\right\}$.

$\therefore$ The values of x between $x=0$ and $x=\pi$ are $0, \pi/2, \pi$.

$\therefore$ The required area $A = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} (-f(x))\,dx$

$$= \int_0^{\pi/2} \sin 2x \, dx + \int_{\pi/2}^{\pi} (-\sin 2x)\, dx$$

$$= \left[\frac{-\cos 2x}{2}\right]_0^{\pi/2} + \left[\frac{\cos 2x}{2}\right]_{\pi/2}^{\pi}$$

$\boxed{\cos n\pi = (-1)^n}$

$$= \left[-\frac{\cos \pi}{2} + \frac{\cos 0}{2}\right] + \left[\frac{\cos 2\pi}{2} + \frac{\cos \pi}{2}\right]$$

$$= \left[+\frac{1}{2} + \frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{2}\right]$$

$= 1 + 1 = 2$ sq. units.

2) Find the area between the curves $y = x^2 - x - 2$, x-axis and the lines $x = -2$ and $x = 4$.

Sol. Given $y = x^2 - x - 2$.

x	0	1	2	3	4	-1
y	-2	-2	0	4	10	0.

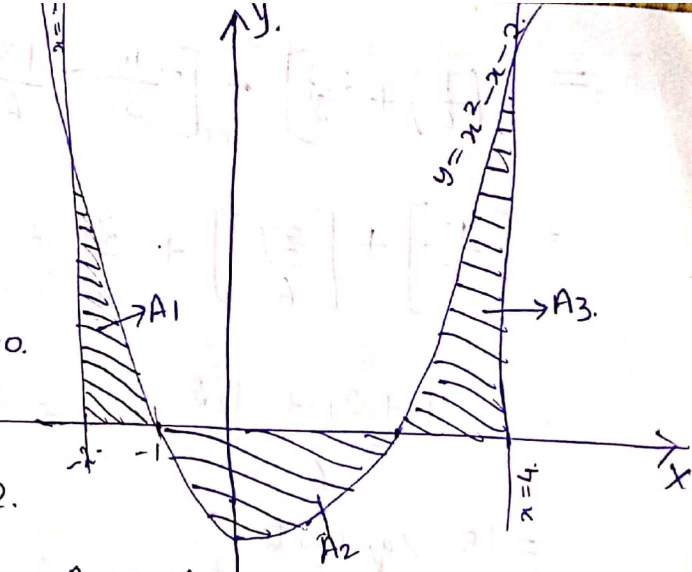
$\cancel{16-4-2}$
$9-3-2$
$1+1-2$
$4-2-2$

Now let us find where the curve intersects the x-axis.

(i.e) $y = x^2 - x - 2 = 0$.

$\Rightarrow (x+1)(x-2) = 0$

$\Rightarrow x = -1$ & $x = 2$.



$\therefore$ Required area $= A_1 + A_2 + A_3$

$$= \int_{-2}^{-1} y\,dx + \int_{-1}^{2} (-y)\,dx + \int_2^4 y\,dx$$

$$= \int_{-2}^{-1} (x^2 - x - 2)\,dx - \int_{-1}^{2} (x^2 - x - 2)\,dx + \int_2^4 (x^2 - x - 2)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} - \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2} + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_2^4$$

$$= \left[\left(\frac{-1}{3} - \frac{1}{2} + 2\right) - \left(\frac{-8}{3} - \frac{4}{2} + 4\right)\right]$$

doubt $- \left[\left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(\frac{-1}{3} + \frac{1}{2} + 2\right)\right] + \left[\left(\frac{64}{3} - \frac{16}{2} - 8\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)\right]$

$$= \left[\left(\frac{7}{6}\right) + \frac{2\times 2}{3\times 2}\right] - \underbrace{\left[-\frac{10\times 2}{3\times 2} - \frac{13}{6}\right]} + \left[\frac{16\times 2}{3\times 2} + \frac{20}{6}\right]$$

$$= \left[\frac{11}{6}\right] + \left[\frac{27}{6}\right] + \frac{26\times 2}{3\times 2}$$

$\frac{11+33+50}{6} = \frac{96}{6} = 16$

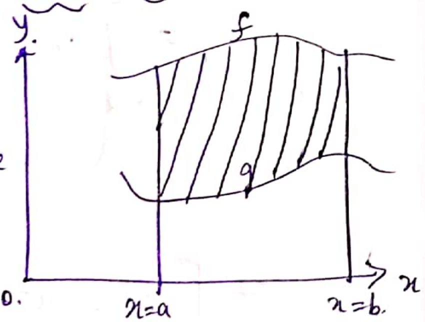
$$= \frac{11 + 27 + 52}{6} = \frac{90}{6}$$

$= 15$ sq. units.

General area principle.

(i.e) Area between two curves.

Let f & g be two continuous curves with the ordinates $x = a$, $x = b$.



Then the ~~radius~~ area is given by

$$A = \int_a^b f\,dx - \int_a^b g\,dx$$

$$= \int_a^b (f-g)\,dx. \quad \left[\text{where } f \text{ is above } g \text{ curve}\right].$$

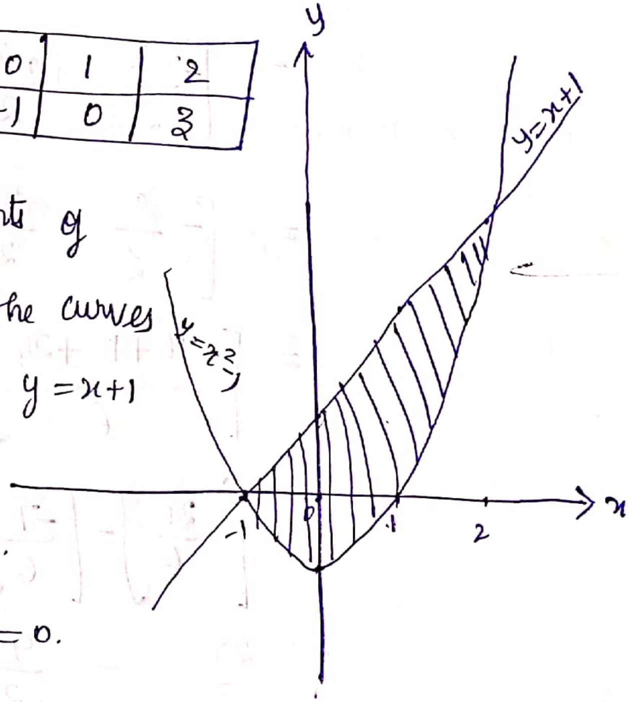
1) Find the area between the line $y = x+1$ & the curve $y = x^2 - 1$.

Sol. $y = x+1$

x	-1	0	1	2
y	0	1	2	3

$y = x^2 - 1$

x	-1	0	1	2
y	0	-1	0	3



To get the points of intersection of the curves we should solve $y = x+1$ & $y = x^2 - 1$.

$\Rightarrow x + 1 = x^2 - 1.$

$\Rightarrow x^2 - x - 2 = 0.$

$\Rightarrow (x+1)(x-2) = 0.$

$\Rightarrow x = -1$ & 2.

$\therefore$ The line intersects the curve at $x = -1$ & $x = 2$.

Required area $= \int_a^b [f(x)_{above} - f(x)_{below}]\, dx$

$= \int_{-1}^{2} [(x+1) - (x^2-1)]\, dx$

$= \left[\frac{x^2}{2} + x - \frac{x^3}{3} + x\right]_{-1}^{2}$

$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2}$

$= \left[-\frac{8}{3} + \frac{4}{2} + 4\right] - \left[\frac{1}{3} + \frac{1}{2} - 2\right]$

$= \left[\frac{-16+12+24}{6}\right] - \left[\frac{2+3-12}{6}\right]$

$= \left[\frac{20}{6}\right] - \left[\frac{-7}{6}\right] = \frac{20+7}{6}$

$= \frac{27}{6} = \frac{9}{2}$ sq. units.

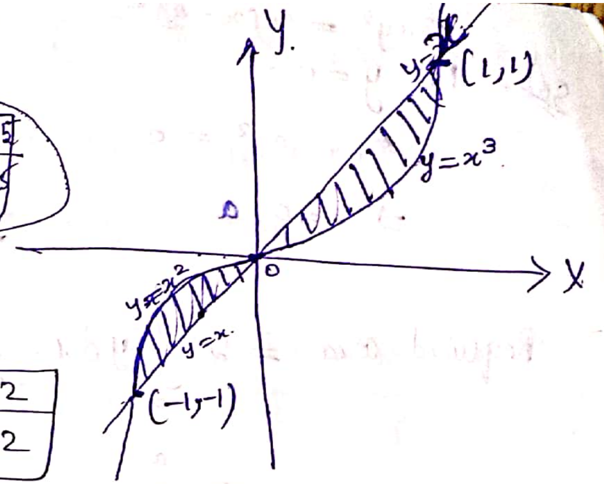
2) Find the area bounded by the curve $y = x^3$ and the line $y = x$.

sol. $y = x^3$.

x	-1	0	1	2	-2	0.5
y	-1	0	1	8	-8	0.125

$y = x$

x	-2	-1	0	1	2
y	-2	-1	0	1	2



$y = x^3$ & $y = x$ $\Rightarrow x^3 = x \Rightarrow x(x^2-1) = 0$.

$\Rightarrow x = 0, x = \pm 1$. Are the points of intersection.

$\therefore$ The required Area $= \int_{-1}^{0} (x^3 - x)\, dx + \int_0^1 (x - x^3)\, dx$

$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$

$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\frac{1}{2} - \frac{1}{4}\right]$

$= \left[\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{2} - \frac{1}{4}\right]$

$= 1 - \frac{1}{2} = \frac{1}{2}$ sq. units.

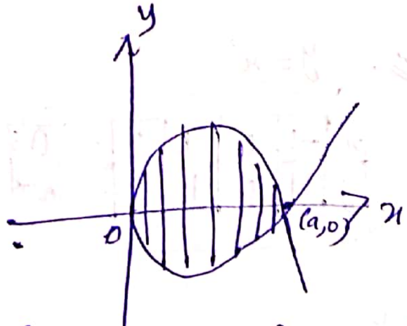
3) Find the area of the loop of the curve $3ay^2 = x(x-a)^2$.

$3ay^2 = x(x-a)^2$

Sol Put $y=0$.

$x\cdot(x-a)^2 = 0$

$\Rightarrow x = 0, a$



Required area $= 2\int_0^a y\,dx$ ($\because$ the curve is symmetrical about x-axis)

$$= 2\int_0^a \frac{\sqrt{x}\,(x-a)}{\sqrt{3a}}\,dx$$

$$= \frac{2}{\sqrt{3a}}\int_0^a \left[x^{3/2} - a x^{1/2}\right]dx$$

$$= \frac{2}{\sqrt{3a}}\left[\frac{x^{5/2}}{5/2} - \frac{a x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{2}{\sqrt{3a}}\left[\frac{2}{5}x^{5/2} - \frac{2a}{3}x^{3/2}\right]_0^a$$

$$= \frac{2}{\sqrt{3a}}\left\{\left[\frac{2}{5}a^{5/2} - \frac{2}{3}a\,a^{3/2}\right] - [0]\right\}$$

$$= \frac{2}{\sqrt{3a}}\left[\frac{2}{5}a^{5/2} - \frac{2}{3}a^{5/2}\right]$$

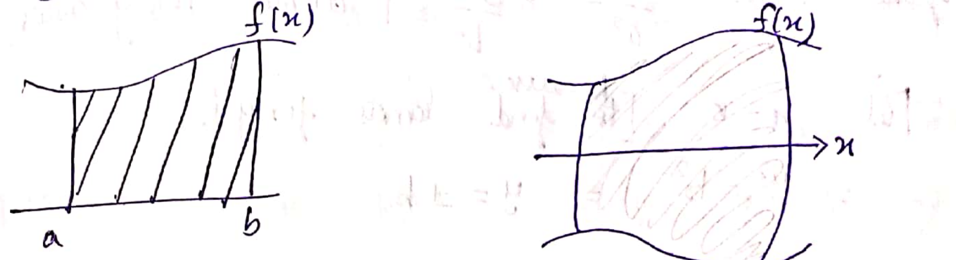
$$= \frac{2\,a^{5/2}}{\sqrt{3}\,a^{1/2}}\left[\frac{2\times 3}{5\times 3} - \frac{2\times 5}{3\times 5}\right] = -\frac{2}{\sqrt{3}}a^2\left[\frac{4}{15}\right]\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$A = -\frac{8\sqrt{3}\,a^2}{15\times 3} = -\frac{8\sqrt{3}\,a^2}{45}.$$

$\because$ Area cannot be negative.

$\therefore A = \frac{8\sqrt{3}a^2}{45}$ sq. units //.

Volumes of solids of revolution.



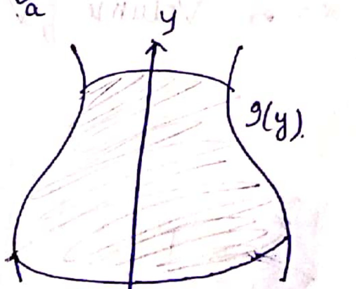
The cross-sectional area is $A(x) = \pi[f(x)]^2 = \pi y^2$

i) Volume of the solid is $V = \int_a^b \pi\,[f(x)]^2 dx$

$$= \int_a^b \pi y^2\,dx.$$

ii) $V = \int_a^b \pi\,[g(y)]^2\,dx$

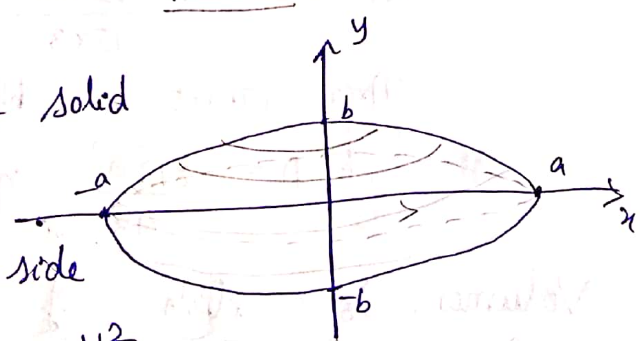
$$= \int_d^{b} \pi x^2\,dy.$$



1) Find the volume of the solid that results when the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a>b>0)$ is

revolved about the minor axis. (here it is y axis)

Sol. Volume of the solid is obtained by revolving the right side of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the y-axis.



Put $x=0$ [because to find limits for y].

$\Rightarrow y^2=b^2 \Rightarrow y=\pm b.$

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}=\frac{b^2-y^2}{b^2}$$

$$\Rightarrow x^2=\frac{a^2}{b^2}(b^2-y^2)$$

$\int_a^b x^n\,dx=\frac{x^{n+1}}{n+1}$

$\int y^2\,dy=\frac{y^{2+1}}{2+1}=\frac{y^3}{3}$

$$\therefore \text{Volume}, V=\int_{-b}^{b}\pi x^2\,dy.$$

$$=\int_{-b}^{b}\pi\frac{a^2}{b^2}(b^2-y^2)\,dy=\frac{\pi a^2}{b^2}\int_{-b}^{b}(b^2-y^2)\,dy$$

$$=\frac{\pi a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_{-b}^{b}$$

$$=\frac{\pi a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_{-b}^{b}$$

$$=\frac{\pi a^2}{b^2}\left[\left(b^3-\frac{b^3}{3}\right)-\left(-b^3+\frac{b^3}{3}\right)\right]$$

$$=\frac{\pi a^2}{b^2}\left[b^3-\frac{b^3}{3}+b^3-\frac{b^3}{3}\right]$$

$$=\frac{\pi a^2}{b^2}\left[\frac{2b^3}{3}+\frac{2b^3}{3}\right]$$

$$=\frac{\pi a^2}{\cancel{b^2}}\times\frac{4b^3}{3}=\frac{4\pi a^2 b}{3}\ \text{Cubic units.}$$

$2b^3-\frac{2b^3}{3}=\frac{6b^3-2b^3}{3}=\frac{4b^3}{3}$

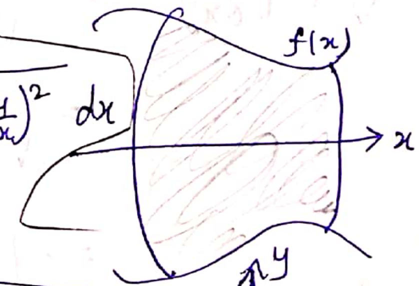
$=\frac{(6-2)b^3}{3}$

Length of the curve:-

1) If $f(x)$ & $f'(x)$ are continuous on $[a,b]$ then the arc length L of the curve $y=f(x)$ from $x=a$ to $x=b$ is $L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

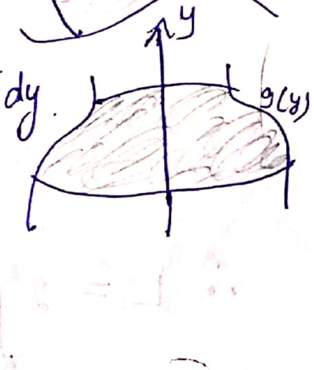
2) IIIrly if the curve is $x=g(y)$, then arc length is $L=\int_a^b\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$

Surface area of a solid.



1) Surface area $=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

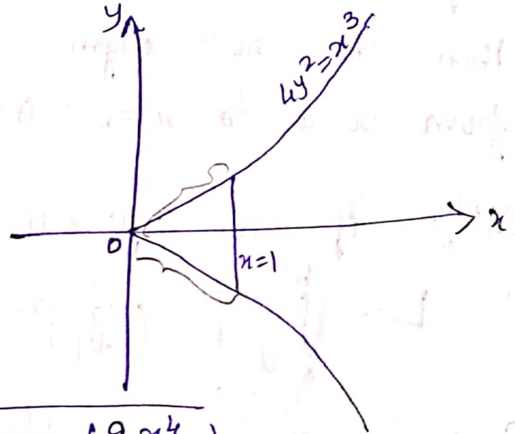
2) Surface area $=2\pi\int_a^b y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$



1) Find the length of the curve $4y^2 = x^3$ between $x=0$ and $x=1$

<u>Sol</u>. $\cancel{y = \frac{\sqrt{x^3}}{2}}$ $y^2 = \frac{x^3}{4} \Rightarrow y = \pm\frac{\sqrt{x^3}}{4}$.

x	0	1	2	3	4	5
y	0	± 0.5	± 0.70	± 1.29	± 2	± 2.795



Given $4y^2 = x^3$.

diff w.r.t x.

$8y \cdot dy = 3x^2 dx$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{8y}$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{9x^4}{64y^2}\right)}$$

$$= \sqrt{1+\frac{9x^4}{16 \times 4y^2}} \quad 4y^2 \to x^3 \text{ (gn)}$$

$$= \sqrt{1+\frac{9x^4}{16x^3}}$$

$$= \sqrt{1+\frac{9x}{16}}$$

$\therefore L = 2\int_0^1 \sqrt{1+\frac{9x}{16}}\, dx$ ($\because$ the curve is symmetrical about x-axis).

$$= 2\left[\frac{\left(1+\frac{9x}{16}\right)^{3/2}}{9/16 \times 3/2}\right]_0^1$$

$$= \frac{2\times 32}{27}\left[\left(1+\frac{9}{16}\right)^{3/2} - 1\right]$$

$$= \frac{64}{27}\left[\left(\frac{25}{16}\right)^{3/2} - 1\right]$$

$$= \frac{64}{27}\left[\frac{(5^2)^{3/2}}{(4^2)^{3/2}} - 1\right]$$

$$= \frac{64}{27}\left[\frac{125}{64} - 1\right] = \frac{64}{27}\left[\frac{61}{64}\right]$$

$$= \frac{61}{27}$$ //.

$\cancel{1+\frac{9x}{16} = y}$

$\cancel{\frac{9}{16}dx = dy}$

$\cancel{dx = \frac{16dy}{9}}$

2) Show that the surface area of the solid obtained by revolving the arc of the curve $y = \sin x$ from $x=0$ to $x=\pi$ about x-axis is $2\pi\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$.

<u>Sol</u>. Given $y = \sin x$.

diff w.r.t x,

$$\frac{dy}{dx} = \cos x.$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\cos^2 x}\ \cancel{= \sqrt{\sin^2 x} = \sin x}.$$

Surface area $= \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

when the area is rotated about the x-axis.

$\therefore\ S = \int_0^{\pi} 2\pi \sin x \sqrt{1+\cos^2 x}\ dx.$

Put $t = \cos x.$

$dt = -\sin x\, dx$

$\Rightarrow dx = -\frac{dt}{\sin x}.$

x	0	π
t	1	-1

$\therefore\ S = \int_1^{-1} 2\pi \cancel{\sin x} \sqrt{1+t^2} \cdot \left(\frac{-dt}{\cancel{\sin x}}\right)$

$= 2\pi \int_{-1}^{1} \sqrt{1+t^2}\ dt$ — f is even

$\int_{-a}^{a} f\,dx = 2\int_0^a f\,dx$ if f is even

$\int_{-a}^{a} f\,dx = 0$ if f is odd

$= 4\pi \int_0^1 \sqrt{1+t^2}\ dt$

$= 4\pi \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right]_0^1$

$\left[\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + c\right]$

$= 2\pi\left[\left\{\sqrt{1+1} + \log\left(1+\sqrt{1+1}\right)\right\} - \left\{0 + \frac{1}{2}\log(0+1)\right\}\right]$

$= 2\pi\left[\sqrt{2} + \log(1+\sqrt{2}) - 0\right]$

$= 2\pi\left[\sqrt{2} + \log(1+\sqrt{2})\right]$ //.

UNIT - 2.

Let $z = f(x, y)$ be a fn.

The variables x & y are called independent variables while z is called the dependent variable.

LIMIT:

The function $f(x,y)$ is said to tend to the limit l as $x \to a$ & $y \to b$ iff the limit l is independent of the path followed by the point (x,y) as $x \to a$ and $y \to b$. Then

$$\lim_{\substack{x\to a \\ y\to b}} f(x,y) = l.$$

1) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\left[\frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots}{\theta}\right]$

$= \lim_{\theta\to 0}\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots\right]$

$= 1$ //.

$$= 2\pi\left[\left\{\sqrt{1+1} + \log\left(1+\sqrt{1+1}\right)\right\} - \left\{0 + \frac{1}{2}\log(0+1)\right\}\right]$$

$$= 2\pi\left[\sqrt{2} + \log(1+\sqrt{2}) - 0\right]$$

$$= 2\pi\left[\sqrt{2} + \log(1+\sqrt{2})\right] //.$$

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1) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = \lim\limits_{\theta \to 0} \left[\dfrac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots}{\theta}\right]$

$$= \lim_{\theta \to 0}\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots\right]$$

$$= 1 //$$

[Working Rule to find the limit.

1) find $f(x,y)$ along $x \to a$ & $y \to b$.

2) " " " $y \to b$ & $x \to a$.

If the values of f in (1) & (2) are same, then the limit exists otherwise not.

3). If $a \to 0$ & $b \to 0$, find the limit along $y = mx$ or $y = mx^n$. If the value of the limit doesn't contain m then limit exists. If it contains m, the limit doesn't exist.

note: i) Put $x = 0$ & then $y = 0$ in f. Find the value f_1.

ii) Put $y = 0$ & then $x = 0$ in f. Find the value f_2.

if $f_1 \neq f_2$, limit doesn't exist.

If $f_1 = f_2$, then.

iii) Put $y = mx$ & find the limit f_3.

If $f_1 = f_2 \neq f_3$, then lim doesn't exist.

If $f_1 = f_2 = f_3$, then

iv) Put $y = mx^n$ & find the limit f_4.

If $f_1 = f_2 = f_3 \neq f_4$, then lim does not exist.

If $f_1 = f_2 = f_3 = f_4$, then limit exist]

$\lim\limits_{\substack{x \to 0 \\ y \to 0}}$ ...

Sol. i) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} (x^3 + y^3) = \lim\limits_{y \to 0} y^3 = 0 = f_1$ (say).

ii) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} (x^3 + y^3) = \lim\limits_{x \to 0} x^3 = 0 = f_2$ (say).

here $f_1 = f_2$, therefore.

iii) Put $y = mx$.

$$\lim\limits_{\substack{x \to 0 \\ y \to 0}} (x^3 + y^3) = \lim\limits_{x \to 0} \left[\lim\limits_{y \to mx} (x^3 + y^3)\right]$$

$$= \lim\limits_{x \to 0} \left[x^3 + m^3 x^3\right]$$

$$= 0 = f_3 \text{ (say)}.$$

here $f_1 = f_2 = f_3$, therefore.

iv) Put $y = mx^2$

$$\lim\limits_{\substack{x \to 0 \\ y \to 0}} (x^3 + y^3) = \lim\limits_{x \to 0} \left[\lim\limits_{y \to mx^2} (x^3 + y^3)\right]$$

$$= \lim\limits_{x \to 0} \left[x^3 + m^3 x^6\right]$$

$$= 0 = f_4 \text{ (say)}.$$

here $f_1 = f_2 = f_3 = f_4 = 0$.

Thus, limit exists with value 0.

Partial Derivatives.

$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Let $z = f(x,y)$ be function of two variables x & y.

The partial derivative of z w.r.t x keeping y as constant is

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}.$$

IIIrly, The partial derivative of z w.r.t y keeping x as constant is denoted by

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Notation:

$\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial y^2} = t$,

$\frac{\partial^2 z}{\partial x \partial y} = s$.

Function of function rule:

Let z be a fn. of u where u is a fn. of 2 independent variables x & y.

Then $\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x}$ & $\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$.

1) If $u = \frac{xy}{x+y}$, S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

Sol. $\frac{\partial u}{\partial x} \leftarrow$ (y constant) $= \frac{(x+y)y - xy}{(x+y)^2} = \frac{xy + y^2 - xy}{(x+y)^2}$

$= \frac{y^2}{(x+y)^2}$.

$\frac{\partial u}{\partial y} = \frac{(x+y)x - xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$.

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy^2 + y.x^2}{(x+y)^2}$

$= \frac{xy\,[y+x]}{(x+y)^2}$

$= \frac{xy}{(x+y)}$

$= u$ //.

Hence the proof.

4) If $u = \log(\tan x + \tan y + \tan z)$ S.T $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$.

2) (H.W) If $u = \sin^{-1}\left(\frac{x^2 - y^2}{x+y}\right)$ prove that the following

i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u$.

5) If $u = \log(x^2+y^2) + \tan^{-1}(\frac{y}{x})$ ... $u_{xx} + u_{yy} = 0$.

$$u = \log(x^2+y^2) + \tan^{-1}(y/x)$$

$$u_x = \frac{1}{x^2+y^2} 2x + \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{\cancel{x^2}} \frac{1}{x^2+y^2} \cancel{x^2}$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2}$$

$$= \frac{2x-y}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2)2 - (2x-y)2x}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2}$$

$$= \frac{2(y^2-x^2+xy)}{(x^2+y^2)^2} \quad \text{--- } ①$$

$$u_y = \frac{1}{x^2+y^2} \cdot 2y + \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$= \frac{1}{x^2+y^2} 2y + \frac{1}{x^2+y^2} x^2 \frac{1}{\cancel{x}}$$

$$= \frac{x+2y}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)2 - (x+2y)2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-2xy-4y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2} \quad \text{--- } ②$$

① + ② ⟹

$$u_{xx} + u_{yy} = \frac{1}{(x^2+y^2)^2}\left[\cancel{2y^2} - \cancel{2x^2} + \cancel{2xy} + \cancel{2x^2} - \cancel{2y^2} - \cancel{2xy}\right]$$

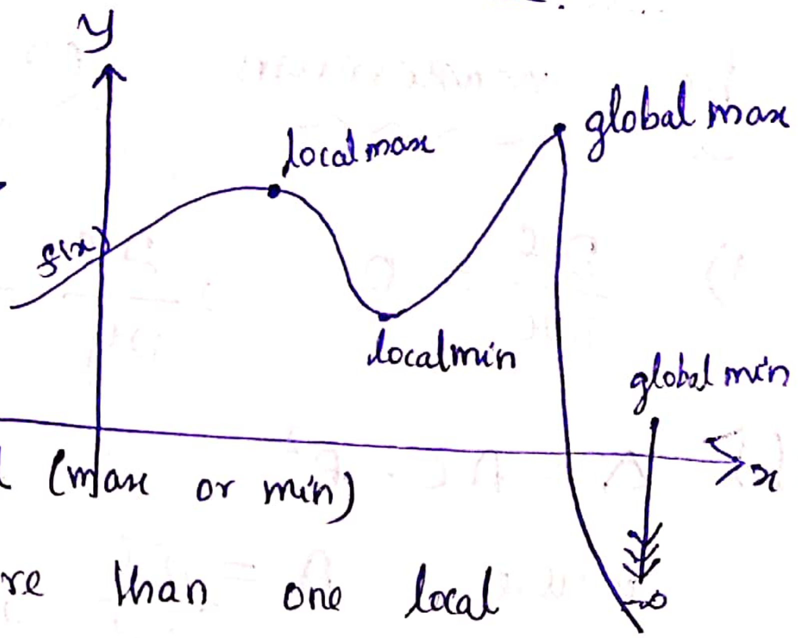
$$= 0$$

Scanned by CamScanner

Maxima, minima and saddle points.

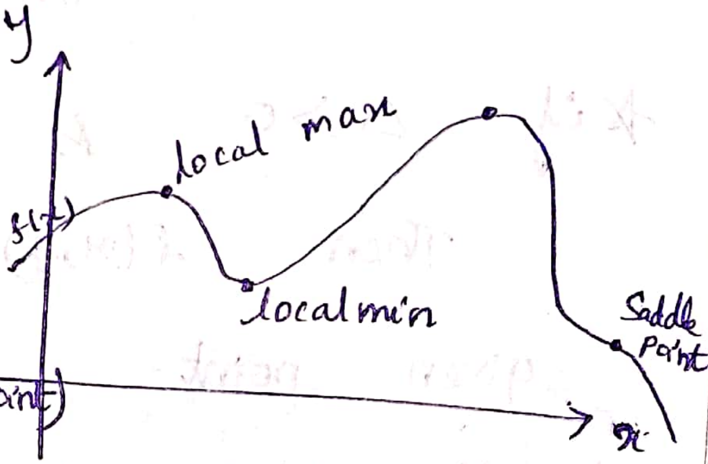
* Global (or) (absolute) maximum or minimum.

(ie) The max or min over the entire fn.

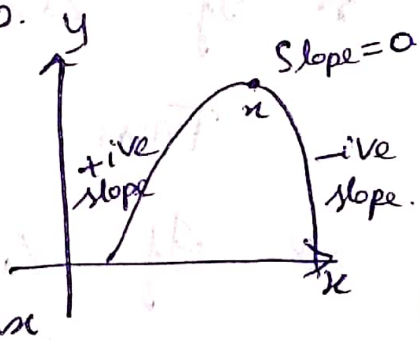


* There is only one global (max or min) but there can be more than one local max or min.

* In a smoothly changing fn. a max or min is always where the fn flattens out (except at the saddle point)



flattens out → where the $\underbrace{\text{slope}}_{1^{st}\text{ derivative}}$ is zero.



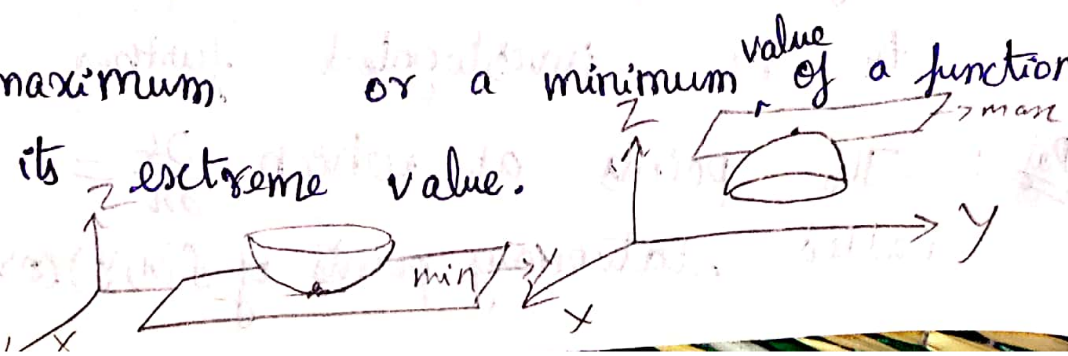
* 2^{nd} derivative at x:

<0 (slope), it is a local max

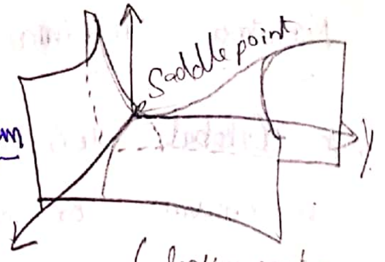
>0, it is local min

$=0$, then the test fails or saddle point.

* The maximum or a minimum value of a function is called its extreme value.



Sufficient Conditions for $f(x,y)$ to be maximum or minimum



1) $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

2) let $\Delta = AC - B^2 = rt - s^2$

where $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x \partial y}$ & $C = \frac{\partial^2 f}{\partial y^2}$.

★ if $\Delta > 0$ (necessary) & A (or) $C < 0$. ($r < 0$)

Then $f(x,y)$ is maximum at the given point.

★ If $\Delta > 0$ & A (or) $C > 0$ ($r > 0$)

Then $f(x,y)$ is minimum at that point.

★ If $\Delta < 0$, then $f(x,y)$ has a saddle point (neither max nor min).

★ If $\Delta = 0$, (no result) (or) failure. $f(x,y)$ has to be investigated further.

Def: The points at which $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ are called stationary points of $f(x,y)$ (or) Critical points.

1) Discuss the maximum and minimum of $x^2 + y^2 + 6x + 12$.

Sol. Let $f(x,y) = x^2 + y^2 + 6x + 12$.

$$\frac{\partial f}{\partial x} = 2x + 6.$$

$$\frac{\partial f}{\partial y} = 2y.$$

$$\frac{\partial^2 f}{\partial x^2} = 2 , \quad \frac{\partial^2 f}{\partial y^2} = 2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0.$$

For maxima & minima, $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$.

(ie) $2x + 6 = 0$ & $2y = 0$.

$\Rightarrow x = -3$ & $y = 0$.

(ie) $(x,y) = (-3,0)$ is the stationary point.

At $(-3,0)$, $\Delta = \left(\frac{\partial^2 f}{\partial x^2}\right) \cdot \left(\frac{\partial^2 f}{\partial y^2}\right) - \frac{\partial^2 f}{\partial x \partial y} = 4 - 0 = 4 > 0$

and. $A = 2 > 0$.

$\therefore f(x,y)$ is minimum at $(-3,0)$.

Minimum value $= f(-3,0) = 9 + 0 - 18 + 12 = 3$.

2) Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Sol. Let $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3$$

$$q = \frac{\partial f}{\partial y} = 3y^2 - 12$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x, \quad t = \frac{\partial^2 f}{\partial y^2} = 6y, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0.$$

$$\Delta = rt - s^2$$
$$= (6x)(6y) - 0 = 36xy$$

To find Stationary points: $\left(\frac{\partial f}{\partial x} = 0\right)$ & $\left(\frac{\partial f}{\partial y} = 0\right)$

$\Rightarrow 3x^2 - 3 = 0$ & $3y^2 - 12 = 0$.

$\Rightarrow x^2 = 1$ & $y^2 = 4$.

$\Rightarrow x = \pm 1$ & $y = \pm 2$.

$\therefore$ The stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$ //.

i) At $(1,2)$,

$r = 6 > 0$, $\Delta = 72 > 0$.

$\therefore f(x,y)$ is minimum at $(1,2)$.

ii) At $(1,-2)$.

$r = 6 > 0$, $\Delta = -72 < 0$.

$\therefore f(x,y)$ is neither maximum, nor minimum at $(1,-2)$. (i.e) $(1,-2)$ is a saddle point.

iii). At $(-1,2)$.

$r = -6 < 0$, $\Delta = -72 < 0$.

$\therefore (-1,2)$ is a saddle point.

iv) At $(-1,-2)$.

$r = -6 < 0$, $\Delta = 72 > 0$.

$\therefore f(x,y)$ is maximum at $(-1,-2)$.

The maximum value $= f(x,y)\big|_{(-1,-2)}$

$= f(-1,-2)$

$= -1 - 8 + 3 + 24 + 20$

$= 38$.

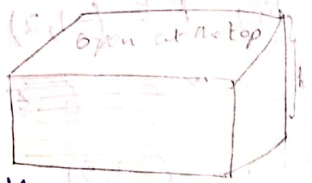
The minimum value $= f(1,2)$

$= 1 + 8 - 3 - 24 + 20$

$= 2$ //.

3) A rectangular box, open at the top is to have a volume of 32 cc. Find the dimensions of the box that requires the least material (surface area) for its construction.

Sol. Let l, b & h be the length, breadth and height of the box respectively & S be the surface area & v be the volume.



Given $v = 32$ cc. Volume of a rectangle $= lbh$

$\Rightarrow lbh = 32 \Rightarrow b = \frac{32}{lh}$ —— (1)

Surface area, $S = 2(l+b)h + lb$ —— (2). perimeter + area

Perimeter of a rectangle $= 2(a+b)$, area $= ab$

$S = 2\left(l + \frac{32}{lh}\right)h + l\left(\frac{32}{lh}\right)$. (by (1))

$= 2lh + \frac{64}{l} + \frac{32}{h}$ —— (3).

diff (3) partially w.r.t l,

$\frac{\partial S}{\partial l} = 2h - \frac{64}{l^2}$ —— (4).

diff (3) partially w.r.t 'h', we get

$\frac{\partial S}{\partial h} = 2l - \frac{32}{h^2}$ —— (5).

For maximum and minimum S,

$\frac{\partial S}{\partial l} = 0$ & $\frac{\partial S}{\partial h} = 0$.

$\Rightarrow 2h - \frac{64}{l^2} = 0$ & $2l - \frac{32}{h^2} = 0$.

$\Rightarrow h = \frac{32}{l^2}$ & $l = \frac{16}{h^2}$. —— (6).

$l = \frac{16}{32 \times 32} \times l^4 \Rightarrow \frac{32 \times 32}{16} = \frac{l^4}{l}$

$\Rightarrow l^3 = 64 = 4 \times 4 \times 4.$

$\Rightarrow \boxed{l = 4}$

$\therefore \boxed{h = \frac{32}{16} = 2.}$

from (1), $\boxed{b = \frac{32}{8} = 4.}$

(least material) → min or max At $(l, b, h) = (4, 4, 2)$.

$\frac{\partial^2 S}{\partial l^2} = +\frac{128}{l^3} = \frac{128}{64} = 2.$

$\frac{\partial^2 S}{\partial h^2} = \frac{64}{h^3} = \frac{64}{8} = 8.$

$\frac{\partial^2 S}{\partial l \partial h} = \frac{\partial}{\partial l}\left(\frac{\partial S}{\partial h}\right) = 2.$

$\therefore \Delta = \left(\frac{\partial^2 S}{\partial l^2}\right)\left(\frac{\partial^2 S}{\partial h^2}\right) - \left(\frac{\partial^2 S}{\partial l \partial h}\right)^2$

$= 16 - 4$

$= 12 > 0.$

& $\frac{\partial^2 S}{\partial l^2} > 0.$

$\therefore$ S attains its minimum at $(4,4,2)$. → S at (4,4,2)

And the minimum value $= S|_{(4,4,2)}$ — not necessary.

$= 2(4+4)2 + 16$

$= \cancel{64}\ 32 + 16 = 48$ sq m

Thus, the dimensions of the box are 4cm, 4cm and 2cm.

Lagrange's method of undetermined multipliers.

A.F: $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

Let $f(x,y,z)$ be a function of three variables x, y, z and the variables ~~are~~ be connected by the relation (constraint)

$\phi(x,y,z) = 0$ —— (1).

or $f(x,y,z)$ to have stationary values,

(i.e) $\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$

$\therefore$ The total differential,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0. \quad \text{—— (2)}$$

By total differentiation of (1),

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0. \quad \text{—— (3)}$$

(2) + λ(3), we get.

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0.$$

The above eqn holds if

$\Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$ —— (4)

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0.$ —— (5)

$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial y} = 0.$ —— (6)

necessary but not sufficient.

On solving (1), (4), (5) & (6), we can find the values of x, y, z & λ for which $f(x,y,z)$ has stationary values? (x,y,z)

And λ is the Lagrange multiplier.

Working rule:

1) form A.E.

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

2) Partially diff F w.r.t x, y, z respectively

3) Solve $F_x = 0$, $F_y = 0$ & $F_z = 0$. & $\phi(x,y,z) = 0$ for λ & stationary values x, y, z.

1) Find the maximum value of $x^m y^n z^p$ when $x+y+z=a$.

Sol. Let $f(x,y,z) = x^m y^n z^p$.

& $\phi(x,y,z) = x+y+z-a$.

The auxiliary Equation is

$$g(x,y,z) = f + \lambda\phi$$

$$= x^m y^n z^p + \lambda(x+y+z-a)$$

The stationary points are given by

$g_x = 0$, $g_y = 0$ & $g_z = 0$ & ~~$\phi = 0$~~. $g_\lambda = 0$

(i.e) $m x^{m-1} y^n z^p + \lambda = 0.$ —— (1).

$n x^m y^{n-1} z^p + \lambda = 0$ —— (2).

$p x^m y^n z^{p-1} + \lambda = 0$ —— (3)

$x + y + z - a = 0$ —— (4).

(1) $\Rightarrow$ $-\lambda = m x^{m-1} y^n z^p$

(2) $\Rightarrow$ $-\lambda = n x^m y^{n-1} z^p$

(3) $\Rightarrow$ $-\lambda = p x^m y^n z^{p-1}$.

$\because$ LHS are equal, so RHS are equal.

$\therefore -\lambda = m x^{m-1} y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$.

$\div x^m y^n z^p$,

$$\frac{m x^{m-1} y^n z^p}{x^m y^n z^p} = \frac{n x^m y^{n-1} z^p}{x^m y^n z^p} = \frac{p x^m y^n z^{p-1}}{x^m y^n z^p}$$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$$

$\therefore$ the maximum values of f occur when

$$x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p}$$

Thus, the maximum value of

$$f(x,y,z) = \frac{(am)^m}{(m+n+p)^m} \cdot \frac{(an)^n}{(m+n+p)^n} \frac{(ap)^p}{(m+n+p)^p}$$

$$= \frac{a^{m+n+p} \; m^m \; n^n \; p^p}{(m+n+p)^{m+n+p}} \quad //$$

3). A rectangular box, open at the top is to have a given quantity of 32cc. Find the dimensions of the box that requires the least material for its construction.

Hw ($S = xy + 2yz + 2zx$ with $\phi = xyz = 32$) [mostly use this method]

$x = 4, y = 4, z = 2$.

The Scalar and vector fields.

Gradient of a scalar function

If $\phi = \phi(x,y,z)$ be a scalar function then $\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$ is called the gradient of the scalar fn ϕ and is denoted by $\nabla\phi$ (or) grad ϕ

(ie) $\nabla\phi = \text{grad}\,\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

del or nabla

Divergence of a vector field: Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\text{div}\,\vec{f} = \nabla\cdot\vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{f}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

($\because \hat{i}\cdot\hat{i} = 1$, $\hat{i}\cdot\hat{j}$ (or) $\hat{i}\cdot\hat{k} = 0$)

$\Rightarrow$ div $\vec{f}$ is a scalar.

☆ If $\vec{f}$ is a vector such that div $\vec{f} = 0$ for all points in the region, then it is said to be solenoidal vector in that region.

Curl (or) rotation.

If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

then
$$\nabla\times\vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

☆ If $\vec{f}$ is a vector such that curl $\vec{f} = 0$ for all points in the region, then it is called an irrotational vector or lamellar vector in that region.

Properties of Gradient, curl and divergence.

1) If ϕ is a constant scalar point fn. then $\nabla\phi = \vec{0}$

2) If ϕ_1 & ϕ_2 are two scalar point functions then

i) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

ii) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1, c_2 are constants.

iii) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

iv) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}$, $\phi_2 \neq 0$.

3) div $\vec{F}$ is a scalar function and curl $\vec{F}$ is a vector quantity.

4) For a constant vector $\vec{a}$, div $\vec{a} = 0$, curl $\vec{a} = \vec{0}$

5) div $(\vec{a}+\vec{b})$ = div $\vec{a}$ + div $\vec{b}$

(ie) $\nabla\cdot(\vec{a}+\vec{b}) = \nabla\cdot\vec{a} + \nabla\cdot\vec{b}$

6) $\nabla\times(\vec{a}+\vec{b}) = \nabla\times\vec{a} + \nabla\times\vec{b}$.

7) If $\vec{a}$ is a vector fn & ϕ is a scalar fn then

div $(\phi\vec{a}) = \phi$ div $\vec{a}$ + (grad ϕ) $\cdot\vec{a}$

(or)

$\nabla\cdot(\phi\vec{a}) = \phi\,\nabla\cdot\vec{a} + (\nabla\phi)\cdot\vec{a}$.

8) curl $(\phi\vec{a})$ = (grad ϕ) $\times\vec{a}$ + ϕ curl $\vec{a}$

(or) $\nabla\times(\phi\vec{a}) = (\nabla\phi)\times\vec{a} + \phi(\nabla\times\vec{a})$

9) $\nabla(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a})$

10) $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$

11) $\nabla\times(\vec{a}\times\vec{b}) = (\nabla\cdot\vec{b})\vec{a} - (\nabla\cdot\vec{a})\vec{b} + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$

eg: If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Show that

i) div $\vec{r} = 3$ ii) curl $\vec{r} = \vec{0}$.

sol: i) div $\vec{r} = \nabla\cdot\vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})$

$= \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z$

$= 1+1+1$

$= 3$.

ii) curl $\vec{r} = \nabla\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$= \hat{i}\left[\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right] - \hat{j}\left[\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right] + \hat{k}\left[\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right]$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$

$= \vec{0}$.

Directional derivative of a Scalar point fn:

(Let P & Q be two neighbouring points whose position vectors w.r.t the origin O be $\vec{r}$ & $\vec{r}+\Delta\vec{r}$ respectively, so that $\vec{PQ} = \Delta\vec{r}$ & $PQ = \Delta r$. Let ϕ & $\phi+\Delta\phi$ be the values of a scalar point fn ϕ at the points P & Q respectively. Then $\frac{d\phi}{dr} = \lim_{\Delta r\to 0}\left(\frac{\Delta\phi}{\Delta r}\right)$ is called the directional derivative of ϕ in the direction OP.)

(or) The component $\nabla\phi$ in the direction of $\vec{a}$ is given by $\frac{\nabla\phi \cdot \vec{a}}{|\vec{a}|}$ and is called the directional derivative of ϕ in the direction of $\vec{a}$.

2) Find the divergence and curl of the vector $\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point $(2,-1,1)$.

Sol. $\text{div}\,\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[(xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}\right]$

$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$

$= yz + 3x^2 + 2xz - y^2$

$(\text{div}\,\vec{v})\big|_{(2,-1,1)} = (-1)(1) + 3(4) + 2(2)(1) - 1$

$= -1 + 12 + 4 - 1 = 14.$

$$\text{Curl}\,\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$= \hat{i}(-2yz - 0) - \hat{j}(z^2 - xy) + \hat{k}(6xy - xz)$

$\text{Curl}\,\vec{v}\big|_{(2,-1,1)} = \hat{i}(2) - \hat{j}(1+2) + \hat{k}(-12-2)$

$= 2\hat{i} - 3\hat{j} - 14\hat{k}.$

3) Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.

Then $\vec{F} = \text{grad}\,\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(\phi)$

$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}.$

$\therefore \text{div}\,\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[(3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}\right]$

$= 6x + 6y + 6z = 6(x+y+z).$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (3x^2 - 3yz) & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$= \hat{i}(-3x + 3x) - \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z)$

$= \vec{0}.$

4) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that

i) grad $r = \frac{\vec{r}}{r}$ ii) grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

iii) $\nabla r^n = n r^{n-2}\vec{r}$ iv) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector.

Sol. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. $\vec{r} \cdot \vec{r} = x^2 + y^2 + z^2$

$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2.$

Diff r partially w.r.t x, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$

Ill^{rly}, $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = z/r.$

i) grad $r = \nabla r = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) r$

$= \left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right)$

$= \left(\frac{x}{r}\right)\hat{i} + \left(\frac{y}{r}\right)\hat{j} + \left(\frac{z}{r}\right)\hat{k}$

$= \frac{1}{r}\left[x\hat{i} + y\hat{j} + z\hat{k}\right] = \frac{1}{r}\vec{r}$ //.

ii) grad $\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\underbrace{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)}_{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\cdot\frac{\partial r}{\partial x}}\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}\right)$

$= \left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\hat{i} - \frac{1}{r^2}\frac{\partial r}{\partial y}\hat{j} - \frac{1}{r^2}\frac{\partial r}{\partial z}\hat{k}\right)$

$= -\frac{1}{r^2}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$

$= -\frac{1}{r^3}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$

$= -\frac{\vec{r}}{r^3}$ //.

iii). $\nabla r^n = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) r^n$

(by fn. of fn. rule)

$= \left(n r^{n-1}\frac{\partial r}{\partial x}\hat{i} + n r^{n-1}\frac{\partial r}{\partial y}\hat{j} + n r^{n-1}\frac{\partial r}{\partial z}\hat{k}\right)$

$= \left(n r^{n-1}\frac{x}{r}\hat{i} + n r^{n-1}\frac{y}{r}\hat{j} + n r^{n-1}\frac{z}{r}\hat{k}\right)$

$= n r^{n-2}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$

$= n r^{n-2}\vec{r}$.

iv) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

6) Find the directional derivative of $\phi(x,y,z) = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the vector $(\hat{i} + 2\hat{j} + 2\hat{k})$

Sol: The directional derivative of ϕ is $\frac{\nabla\phi \cdot \vec{a}}{|\vec{a}|}$.

Let $\phi(x,y,z) = xy^2 + yz^3$

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy^2 + yz^3)$$

$$= \frac{\partial}{\partial x}(xy^2+yz^3)\hat{i} + \frac{\partial}{\partial y}(xy^2+yz^3)\hat{j} + \frac{\partial}{\partial z}(xy^2+yz^3)\hat{k}$$

$$= y^2\hat{i} + (2yx + z^3)\hat{j} + 3z^2y\hat{k}.$$

$$\nabla\phi\Big|_{(2,-1,1)} = \hat{i} + (-4+1)\hat{j} + 3(-1)\hat{k}$$

$$= \hat{i} - 3\hat{j} - 3\hat{k}.$$

$\therefore$ The directional derivative of ϕ in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$ is

$$\frac{(\hat{i} - 3\hat{j} - 3\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{1+4+4}}$$

$$= \frac{1-6-6}{3} = \frac{-11}{3} //.$$

* Unit vector normal to a gn. surface (ϕ) at a point $= \frac{\nabla\phi}{|\nabla\phi|}$, where $\phi(x,y,z) = c$ is the surface

H.W 7) Find the unit vector normal to the surface $x^2 + 2y^2 + z^2 = 7$ at $(1,-1,2)$.

Sol. Let $\phi = x^2 + 2y^2 + z^2 - 7$.

$$\nabla\phi = \frac{\partial}{\partial x}(x^2+2y^2+z^2-7)\hat{i} + \frac{\partial}{\partial y}(x^2+2y^2+z^2-7)\hat{j} + \frac{\partial}{\partial z}(x^2+2y^2+z^2-7)\hat{k}$$

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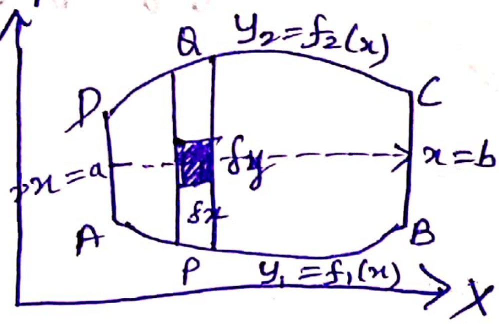
UNIT-3

Multiple Integrals

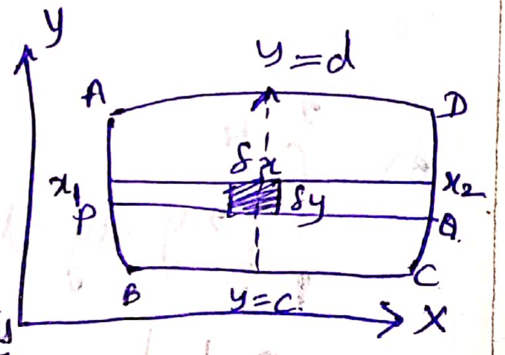
$$\text{Area} = \iint_A dx\,dy$$

Double integrals (cartesian form)

1) $$\iint_A f(x,y)\,dx\,dy = \int_a^b \left[\int_{y_1}^{y_2} f(x,y)\,dy \right] dx$$



2) $$\iint_A f(x,y)\,dx\,dy = \int_c^d \left[\int_{x_1}^{x_2} f(x,y)\,dx \right] dy.$$



Double integrals in polar co-ordinates

$$\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r f(r,\theta)\,dr\,d\theta$$

Problems: note: $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$

Evaluate

1) $$\int_0^1 \int_0^x dy\,dx.$$

Sol. $$\int_0^1 \left(\int_0^x dy \right) dx = \int_0^1 \left[y \right]_0^x dx$$

$$= \int_0^1 [x - 0]\,dx = \left[\frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2}$$

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2) evaluate $\int_0^1 \int_0^x xy \, dy \, dx$

Sol.

$$\int_0^1 \left(\int_0^x xy \, dy \right) dx = \int_0^1 \left[\frac{xy^2}{2} \right]_0^x dx$$

$$= \int_0^1 \frac{x^3}{2} \, dx$$

$$= \left[\frac{x^4}{8} \right]_0^1 = \frac{1}{8}$$

HW

3) $$\int_1^2 \left(\int_3^4 xy \, dy \right) dx = \int_1^2 \left[\frac{xy^2}{2} \right]_3^4 dx$$

$$= \int_1^2 \left[\frac{x16}{2} - \frac{x9}{2} \right] dx$$

$$= \int_1^2 \left[\frac{7x}{2} \right] dx = \left[\frac{7x^2}{4} \right]_1^2$$

$$= \frac{7(4)}{4} - \frac{\cancel{8}7}{4} = \frac{21}{4}$$

4) $$\int_0^3 \int_1^2 xy(x^2+y^2) \, dy \, dx = \int_0^3 \int_1^2 (x^3 y + xy^3) \, dy \, dx$$

$$= \int_0^3 \left[\frac{x^3 y^2}{2} + \frac{xy^4}{4} \right]_1^2 dx$$

$$= \int_0^3 \left\{ \left[2x^3 + 4x \right] - \left[\frac{x^3}{2} + \frac{x}{4} \right] \right\} dx.$$

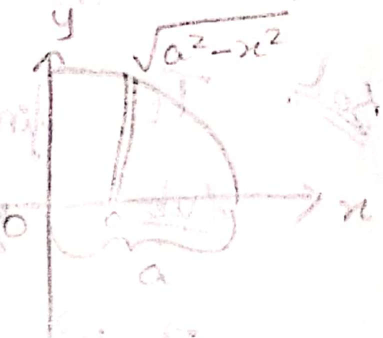
$$= \int_0^3 \left[\frac{3x^3}{2} + \frac{15x}{4}\right] dx$$

$$= \left[\frac{3x^4}{8} + \frac{15x^2}{8}\right]_0^3$$

$$= \left[\frac{3^5}{8} + \frac{15\times 9}{8}\right]$$

$$= \frac{243+135}{8} = \frac{378}{8} = 47.25$$

5). Evaluate $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$.



Sol Let us keep x as constant.

Then $y^2 = a^2 - x^2$

$\Rightarrow y = \sqrt{a^2-x^2}$.

Hence y varies from 0 to $\sqrt{a^2-x^2}$

And x varies from 0 to a.

$$\therefore \iint xy\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a \left[\frac{xy^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{xa^2 - x^3}{2}\right] dx$$

$$= \frac{1}{2}\left[\frac{x^2a^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} - 0\right]$$

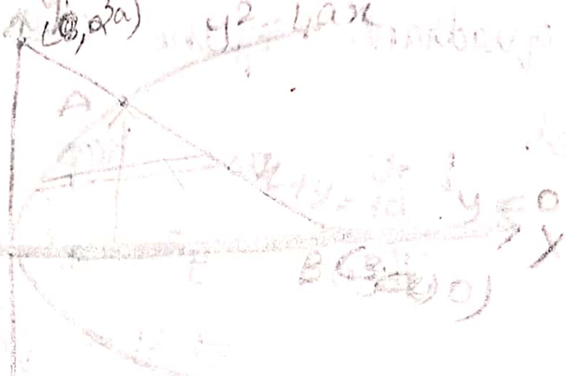
$$= \frac{1}{2}\,\frac{a^4}{4}$$

$$= \frac{a^4}{8} \;//.$$

x	$3a$	0	1
y	0	$3a$	$3a-1$

6) Find the area enclosed by the curve $y^2=4ax$ and the lines $x+y=3a$, $y=0$.

$y=3a-x$



Sol: To find the ~~range~~ limit of

Sub $x=3a-y$ in $y^2=4ax$.

$$\Rightarrow y^2 = 4a(3a-y).$$

$$= 12a^2 - 4ay.$$

$$\Rightarrow y^2+4ay-12a^2=0.$$

$$\Rightarrow (y+6a)(y-2a)=0.$$

$$\Rightarrow y=-6a \;(\text{or})\; y=2a.$$

$\therefore$ y varies from 0 to $2a$

and x varies from $x=\frac{y^2}{4a}$ to $x=3a-y$.

$$\therefore \text{Area} = \int_{y=0}^{2a}\int_{x=\frac{y^2}{4a}}^{x=3a-y} dx\,dy.$$

$$= \int_0^{2a} [x]_{\frac{y^2}{4a}}^{3a-y} \, dy$$

$$= \int_0^{2a} \left[3a - y - \frac{y^2}{4a}\right] dy$$

$$= \left[3ay - \frac{y^2}{2} - \frac{y^3}{12a}\right]_0^{2a}$$

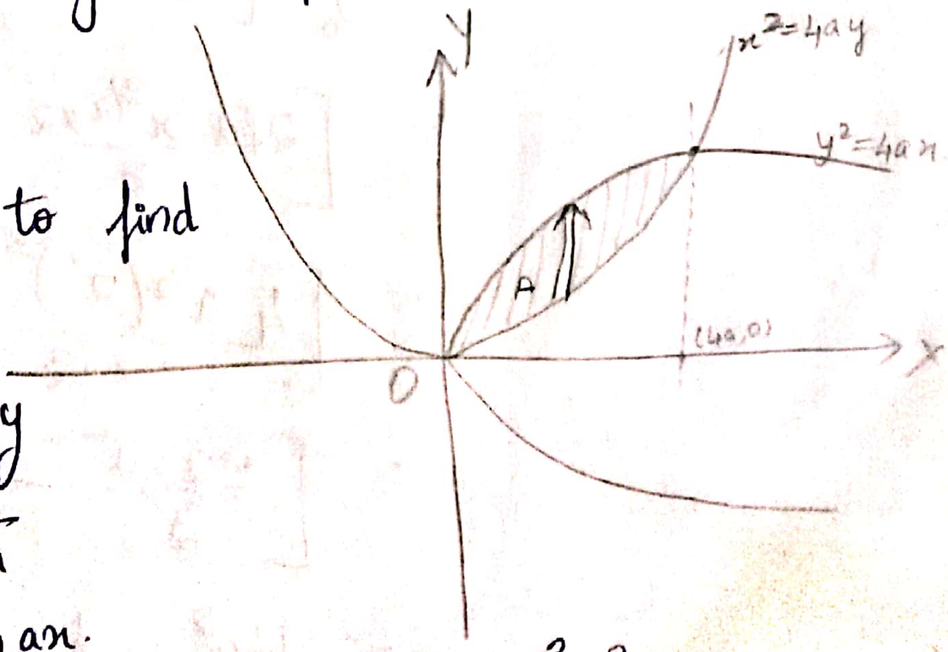
$$= \left[6a^2 - \frac{\cancel{4}^{2}a^2}{\cancel{2}_{1}} - \frac{{}^{2}\cancel{8}a^{\cancel{3}\,2}}{{}_{3}\cancel{12}\cancel{a}}\right] - [0]$$

$$= \left[4a^2 - \frac{2a^2}{3}\right]$$

$$= \frac{10a^2}{3}$$

7) By double integration, evaluate the area enclosed by the parabola $y^2 = 4ax$ & $x^2 = 4ay$.

First we have to find the limits.



Given $x^2 = 4ay$

$$y = \frac{x^2}{4a}$$

Sub 'y' in $y^2 = 4ax$.

$$\Rightarrow \frac{x^4}{4^2a^2} = 4ax \Rightarrow x^3 = 4^3a^3$$

$$\Rightarrow x = 4a.$$

$\therefore$ x varies from 0 to $4a$

And y varies from $y = \dfrac{x^2}{4a}$ to $y = \sqrt{4ax}$.

$\therefore$ Required area, $A = \iint dx\, dy$

$$= \int_{x=0}^{4a} \left[\int_{y=\frac{x^2}{4a}}^{y=\sqrt{4ax}} dy \right] dx$$

$$= \int_0^{4a} \left[y \right]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx$$

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx$$

$$= \int_0^{4a} \left[2\sqrt{a}\, x^{1/2} - \frac{x^2}{4a} \right] dx$$

$$= \left[2\sqrt{a}\, \frac{x^{3/2} \times 2}{3} - \frac{x^3}{12a} \right]_0^{4a} \cancel{dx}$$

$$= \left[\frac{4\, a^{1/2} (2^2)^{3/2} a^{3/2}}{3} - \frac{4^3 a^3}{12a} \right] - [0]$$

$$= \left[\frac{32\, a^2}{3} - \frac{16a^2}{3} \right]$$

$$= \frac{16a^2}{3} \;//$$

8) Find $\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$.

Sol
$$\int_0^{\pi}\left(\int_0^{\sin\theta} r\,dr\right)d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$
$$= \frac{1}{2}\int_0^{\pi} \sin^2\theta\, d\theta$$
$$= \frac{1}{2}\int_0^{\pi}\left[\frac{1-\cos 2\theta}{2}\right]d\theta$$
$$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$
$$= \frac{1}{4}\left\{\left[\pi - \frac{\sin 2\pi}{2}\right] - \left[0 - \frac{\sin 0}{2}\right]\right\}$$
$$= \frac{1}{4}\left[\pi - 0\right] - 0 \quad (\because \sin n\pi = 0 \ \forall n \in I)$$
$$= \frac{\pi}{4}$$

Triple integrals

$$I = \iiint_V f(x,y,z)\,dx\,dy\,dz.$$
$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz.$$
(or) $$\int_{z_1}^{z_2}\int_{f_1(z)}^{f_2(z)}\int_{\phi_1(y,z)}^{\phi_1(y,z)} f(x,y,z)\,dx\,dy\,dz.$$

1) Evaluate $\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx$.

Sol. $\int_0^2\int_1^3\left(\int_1^2 xy^2z\,dz\right)dy\,dx = \int_0^2\int_1^3\left[\frac{xy^2z^2}{2}\right]_1^2 dy\,dx$

$$= \frac{1}{2}\int_0^2\int_1^3 [4xy^2 - xy^2]\,dy\,dx$$

$$= \frac{1}{2}\int_0^2\left[\frac{4xy^3}{3} - \frac{xy^3}{3}\right]_1^3 dx$$

$$= \frac{1}{6}\int_0^2 \left\{[4\,3^3x - 3^3x] - [4x - x]\right\} dx$$

$3^3x(4-1) - 3x$

$$= \frac{1}{6}\int_0^2\left\{[3^4x - 3x]\right\}dx$$

$$= \frac{1}{6}\left[\frac{3^4x^2}{2} - \frac{3x^2}{2}\right]_0^2$$

$$= \frac{1}{6\times 2}\left\{[3^4 4 - 3\times 4] - [0]\right\}$$

$$= \frac{\cancel{12}}{\cancel{12}}[3^3 - 1]$$

$$= 27 - 1$$

$$= 26$$

HW

1) P.T: $\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dx\,dy = \frac{\pi a^3}{6}$.

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2) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$

Sol. $\int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \cdot dx\,dy\,dz$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{\underbrace{(1-x^2-y^2)}_{a^2}-z^2}}\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$\sin^{-1}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} [\sin^{-1}(1)-0]\,dy\,dx.$

$= \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx$

$= \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx = \frac{\pi}{2} \int_0^1 [\sqrt{1-x^2}]\,dx$

$= \frac{\pi}{2} \left[\frac{1}{2}\sin^{-1}(x) + \frac{x\sqrt{1-x^2}}{2}\right]_0^1$

$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2}$

$= \frac{\pi}{2}\left\{\left[\frac{1}{2}\sin^{-1}(1) + \frac{\sqrt{1-1}}{2}\right] - 0\right\}$

$= \frac{\pi}{2}\left\{\frac{1}{2}\cdot\frac{\pi}{2}\right\} = \frac{\pi^2}{8}$ //.

HW

2) $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \frac{\pi a^2}{4}$

3) Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$, where the region V is bounded by $x+y+z=a$ $(a>0)$, $x=0$, $y=0$, $z=0$.

In xyz space
$x+y+z=a$
$z=a-y-x.$
In xy plane
$y=a-x$
In x axis
$x=a$

Sol. $$\iiint_V (x+y+z)\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-y-x}(x+y+z)\,dz\,dy\,dx$$

$$= \int_0^a\int_0^{a-x}\left[xz+yz+\frac{z^2}{2}\right]_0^{a-y-x} dy\,dx$$

$$= \int_0^a\int_0^{a-x}\left[x(a-y-x)+y(a-y-x)+\frac{(a-y-x)^2}{2}\right]dy\,dx$$

$a-y-x=t$
$-dy=dt$

$$= \int_0^a\int_0^{a-x}\left[x(a-x)-xy+y(a-x)-y^2+\frac{(a-y-x)^2}{2}\right]dy\,dx$$

$-\frac{t^2}{2}dt$
$=-\frac{t^3}{3\times 2}$

$$= \int_0^a\left[xy(a-x)-\frac{xy^2}{2}+\frac{y^2(a-x)}{2}-\frac{y^3}{3}+\left(-\frac{1}{6}\right)(a-y-x)^3\right]_0^{a-x}dx$$

$$= \int_0^a\left\{\left[x(a-x)^2-\frac{x(a-x)^2}{2}+\frac{(a-x)^3}{2}-\frac{(a-x)^3}{3}-\frac{1}{6}(\cancel{a}-\cancel{a}+\cancel{x}-x)^3\right]-\left[0-0+0-0-\frac{1}{6}(a-x)^3\right]\right\}dx$$

$$= \int_0^a \left\{ \frac{x(a-x)^2}{2} + \frac{(a-x)^3}{6} - 0 + \frac{1}{6}(a-x)^3 \right] dx$$

$$= \int_0^a \left[\frac{x(a-x)^2}{2} + \frac{(a-x)^3}{3} \right] dx$$

$$= \int_0^a \left[\frac{x(a^2-2ax+x^2)}{2} + \frac{(a^3-3a^2x+3ax^2-x^3)}{3} \right] dx$$

$$= \int_0^a \left[\frac{xa^2-2ax^2+x^3}{2} + \frac{(a^3-3a^2x+3ax^2-x^3)}{3} \right] dx$$

$$= \cancel{\left[\frac{x^2a^2}{}\right.} = \frac{1}{2}\int_0^a (xa^2-2ax^2+x^3)\,dx$$

$$+ \frac{1}{3}\int_0^a (a^3-3a^2x+3ax^2-x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2a^2}{2} - \frac{2ax^3}{3} + \frac{x^4}{4}\right]_0^a + \frac{1}{3}\left[a^3x - \frac{3a^2x^2}{2} + \frac{3ax^3}{3} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2}{\times 6 \over \times 6} - \frac{2a^4}{3}{\times 4 \over \times 4} + \frac{a^4}{4}{\times 3 \over \times 3}\right] - [0] + \frac{1}{3}\left[\frac{a^4}{}{\times 4 \over 4} - \frac{3a^4}{2}{\times 2 \over \times 2} + \frac{a^4}{}{\times 4 \over 4} - \frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{6a^4-8a^4+3a^4}{12}\right] + \frac{1}{3}\left[\frac{8a^4-6a^4-a^4}{4}\right]$$

$$= \frac{1}{24}[a^4] + \frac{1}{12}[a^4] \times \frac{2}{2}$$

$$= \frac{3}{24}a^4 = \frac{a^4}{8}$$

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1) Volume $V = \iiint dx\,dy\,dz$.

2) Mass = Volume × density

$= \iiint \rho\,dx\,dy\,dz$ if ρ is the density.

3) In cylindrical co-ordinates,

$$V = \iiint_V r\,dr\,d\phi\,dz.$$

4) In spherical polar co-ordinates,

$$V = \iiint_V r^2 \sin\theta\,dr\,d\theta\,d\phi.$$

Ex) 1) Find the volume of the tetrahedron bounded by the plane $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

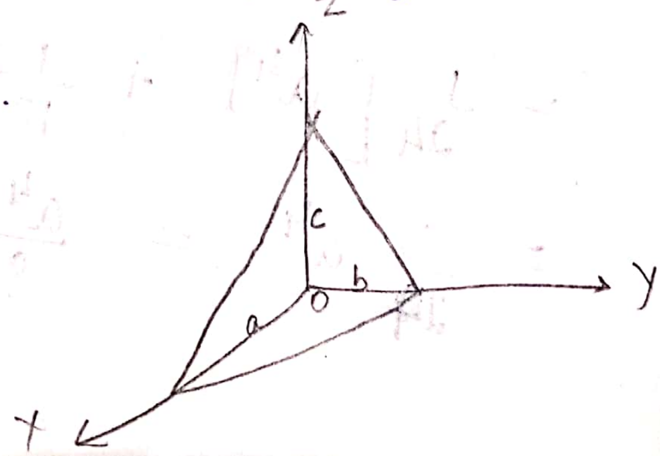
Sol. Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$$\frac{z}{c} = 1-\frac{y}{b}-\frac{x}{a}.$$

$$\Rightarrow z = c\left(1-\frac{y}{b}-\frac{x}{a}\right).$$

& $\frac{y}{b} = 1-\frac{x}{a} \Rightarrow y = b\left(1-\frac{x}{a}\right)$.

& $\frac{x}{a}=1 \Rightarrow x=a$.



Volume of the tetrahedron $= \iiint\limits_V dx\,dy\,dz$

$$= \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{z=0}^{c(1-y/b-x/a)} dz\,dy\,dx$$

$$= \int_0^a\int_0^{b(1-x/a)} \left[z\right]_0^{c(1-y/b-x/a)} dy\,dx$$

$$= c\int_0^a\int_0^{b(1-x/a)} \left(1-\frac{y}{b}-\frac{x}{a}\right) dy\,dx$$

$$= c\int_0^a \left[y-\frac{y^2}{2b}-\frac{xy}{a}\right]_0^{b(1-x/a)} dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)-\frac{b^{\cancel{2}}}{2\cancel{b}}\left(1-\frac{x}{a}\right)^2-\frac{xb}{a}\left(1-\frac{x}{a}\right)\right]dx$$

$$= bc\int_0^a\left[\left(1-\frac{x}{a}\right)-\underbrace{\frac{1}{2}\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)}_{\left(-\frac{1}{2}\right)+\frac{x}{a}-\frac{x^2}{2a^2}}-\frac{x}{a}+\frac{x^2}{a^2}\right]dx$$

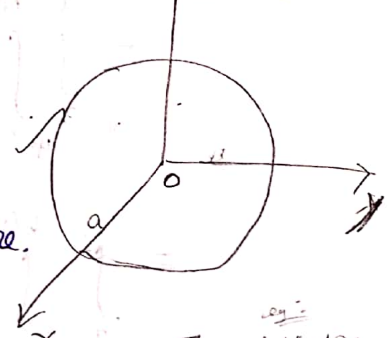
$$= bc\int_0^a\left[\frac{1}{2}-\frac{x}{a}+\frac{x^2}{2a^2}\right]dx.$$

$$= bc\left[\frac{x}{2}-\frac{x^2}{2a}+\frac{x^3}{6a^2}\right]_0^a$$

$$= bc\left\{\left[\frac{a}{2}-\frac{a^2}{2a}+\frac{a^3}{6a^2}\right]-[0]\right\}$$

$$= \frac{abc}{6}$$ //

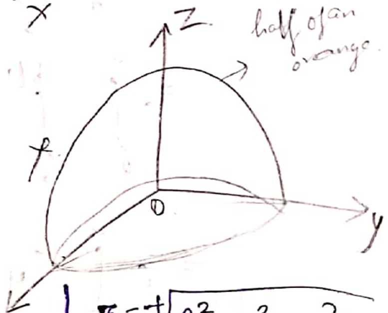
2) Find by triple integration, the volume of the sphere $x^2+y^2+z^2=a^2$.



Sol (Volume of the sphere

$= 2 \times$ Volume of the hemisphere.

$= 2 \iiint_V dz\,dy\,dx$)



$\cancel{=2\int_0}$

(or)

Volume of the sphere $= \iiint dz\,dy\,dx$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-y^2-x^2}}^{\sqrt{a^2-y^2-x^2}} dz\,dy\,dx$$

$z = \pm\sqrt{a^2-y^2-x^2}$

$y^2 = a^2-x^2$ (in xy plane)

$\Rightarrow y = \pm\sqrt{a^2-x^2}$

$x = \pm a$.

$$= 2\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\, dy\, dx.$$

$\int \sqrt{b^2-x^2}\,dx = \frac{b^2}{2}\sin^{-1}\left(\frac{x}{b}\right) + \frac{x\sqrt{b^2-x^2}}{2}$

$$= 2\int_{-a}^{a} \left[\frac{(a^2-x^2)}{2}^2 \sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right) + \frac{y\sqrt{a^2-x^2-y^2}}{2} \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left\{ \left[\left(\frac{a^2-x^2}{2}\right) \sin^{-1}\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right) + \frac{\sqrt{a^2-x^2}\sqrt{a^2-x^2-a^2+x^2}}{2} \right] - \left[\left(\frac{a^2-x^2}{2}\right) \sin^{-1}(-1) + 0 \right] \right\} dx.$$

$$= \cancel{2}\int_{-a}^{a} \left[\left(\frac{a^2-x^2}{\cancel{2}}\right) \frac{\pi}{2} + \left(\frac{a^2-x^2}{\cancel{2}}\right) \frac{\pi}{2} \right] dx$$

$$= \int_{-a}^{a} (a^2-x^2)\,\pi\, dx$$

$$= \pi \left[a^2x - \frac{x^3}{3} \right]_{-a}^{a}$$

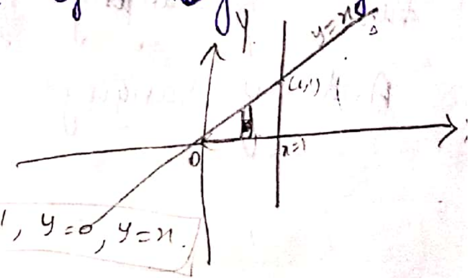
$$= \pi\left\{ \left[a^3 - \frac{a^3}{3} \right] - \left[-a^3 + \frac{a^3}{3} \right] \right\}$$

$$= \pi\left[\frac{2a^3}{3} + \frac{2a^3}{3} \right] = \frac{4\pi a^3}{3}$$

Change the order of Integration

1) Change the order of integration of

$$I = \int_{x=0}^{1} \int_{y=0}^{x} dy\, dx.$$



$x=0,\ x=1,\ y=0,\ y=x.$

Sol $I=\int_0^1\int_0^x dy\,dx = \int_{y=0}^{1}\int_{x=y}^{1} dx\,dy$

$= \int_0^1 [x]_y^1\, dy$

$= \int_0^1 [1-y]\, dy$

$= \left[y-\frac{y^2}{2}\right]_0^1$

$= 1-\frac{1}{2} = \frac{1}{2}.$

(Q.2) Change the order of integration in $\int_0^4\int_{x^2/4}^{2\sqrt{x}} dy\,dx$ and then evaluate it.

Sol Consider, $\int_{x=0}^{4}\int_{y=x^2/4}^{2\sqrt{x}} dy\,dx.$

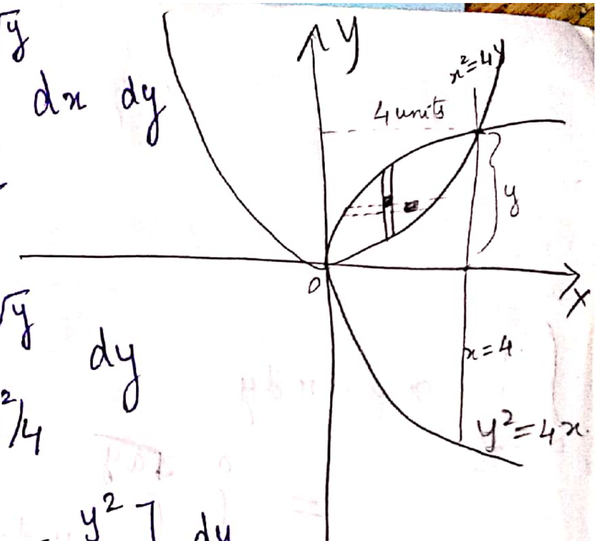
here, x ranging from $x=0$ to $x=4$.

And y ranging from $y=x^2/4$ to $y=2\sqrt{x}$.

$\Rightarrow x^2=4y$ & $y^2=4x$.

$\Rightarrow x=2\sqrt{y}$ & $\frac{y^2}{4}=x$

$\therefore \int_0^4\int_{x^2/4}^{2\sqrt{x}} dy\,dx = \int_{y=0}^{4}\int_{x=\frac{y^2}{4}}^{2\sqrt{y}} dx\,dy$



$= \int_0^4 [x]_{y^2/4}^{2\sqrt{y}}\, dy$

$= \int_0^4 \left[2\sqrt{y}-\frac{y^2}{4}\right] dy$

$= \left[\frac{4y^{3/2}}{3}-\frac{y^3}{12}\right]_0^4$

$= \frac{4\cdot 2^3}{3} - \frac{4(4)^2}{12_3} = \frac{32-16}{3} = \frac{16}{3}$

(Q.3) Change the order of integration in $\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$ and then evaluate it.

Sol Here x varies from $x=0$ to $x=a$.

And y varies from $y=\frac{x^2}{a}$ to $y=2a-x$.

$\Rightarrow x^2=ay$. $2a-y=x$

$x=(2a,0)$

$y=(0,2a)$

$\frac{x}{2a}+\frac{y}{2a}=1$

x	$2a$	1	0	a
y	0	$2a-1$	$2a$	a

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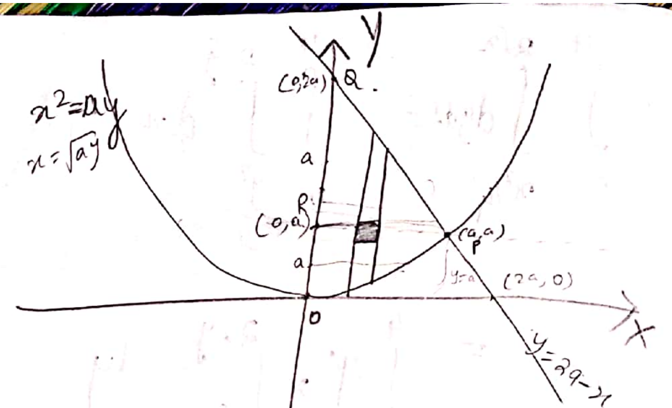


$$\int_{x=0}^{a}\int_{y=\frac{x^2}{a}}^{2a-x} x\,y\,dx\,dy$$

$$= \underbrace{\int_{y=0}^{a}\int_{x=0}^{\sqrt{ay}} xy\,dx\,dy}_{\text{region POR}} + \underbrace{\int_{y=a}^{2a}\int_{x=0}^{2a-y} xy\,dx\,dy}_{\text{PQR}}.$$

$$= \int_0^a \left[\frac{x^2y}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a}\left[\frac{x^2y}{2}\right]_0^{2a-y} dy$$

$(2a-y)^2$
$= 4a^2-4ay+y^2$

$$= \int_0^a \left[\frac{ay^2}{2}-0\right]dy + \int_a^{2a}\left[\frac{(2a-y)^2y}{2}-0\right]dy.$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\int_a^{2a}\left[4a^2y-4ay^2+y^3\right]dy$$

$$= \frac{a}{2}\left[\frac{a^3}{3}-0\right] + \frac{1}{2}\left[\frac{4a^2y^2}{2} - \frac{4ay^3}{3} + \frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left\{\left[\frac{2a^2 4a^2}{3} - \frac{4a\,8a^3}{3} + \frac{16a^4\times 3}{4\times 3}\right] - \left[\frac{2a^4\times 12}{12} - \frac{4a^4\times 4}{3\times 4} + \frac{a^4\times 3}{4\times 3}\right]\right\}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left\{\frac{1}{3}\left[24a^4-32a^4+12a^4\right] - \frac{1}{12}\left[24a^4-16a^4+3a^4\right]\right\}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left\{\frac{4a^4\times 4}{3\times 4} - \frac{11a^4}{12}\right\}$$

$$= \frac{a^4\times 4}{6\times 4} + \frac{1}{24}.5a^4.$$

$$= \frac{9a^4}{24}$$

$$= \frac{3a^4}{8}$$ //.

(Ex 4) H.W. Change the order of integration in

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$$ & evaluate it.

Centre of mass of an arc.

The point whose coordinates are $(\bar{x},\bar{y})$ given by the eqn. $\bar{x} = \frac{\Sigma mx}{\Sigma m}$, $\bar{y} = \frac{\Sigma my}{\Sigma m}$ is called the centre of mass of the system.

$$\bar{x} = \frac{\int \rho x \, dx \, dy}{\int \rho \, dx \, dy} \quad \& \quad \bar{y} = \frac{\int \rho y \, dx \, dy}{\int \rho \, dx \, dy} \quad \text{Illy for } (\bar{x}, \bar{y}, \bar{z})$$

1) Find the volume and the position of the centre of gravity of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ & the coordinate planes.

Sol. Refer to the sum (1) of triple integrals.

$$V = \frac{abc}{6}.$$

Let ρ be the density of the tetrahedron.

Then mass = volume × density

$$= \frac{abc}{6} \times \rho$$

$$= \frac{abc\rho}{6}.$$

To find: the position of the centre of gravity

W.k.T, $\int \rho \, ds \; \bar{y} = \int \rho \, ds \; y.$

$$\Rightarrow \frac{abc\rho \, \bar{y}}{6} = \rho \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} y \, dz \, dy \, dx$$

$$\frac{1}{6} abc\rho\bar{y} = \rho \int_0^a \int_0^{b(1-\frac{x}{a})} cy\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy \, dx.$$

$$= \rho c \int_0^a \left[\frac{y^2}{2} - \frac{xy^2}{2a} - \frac{y^3}{3b} \right]_0^{b(1-\frac{x}{a})} dx$$

$$= \rho c \int_0^a \left[\frac{b^2(1-\frac{x}{a})^2}{2} - \frac{xb^2(1-\frac{x}{a})^2}{2a} - \frac{b^3(1-\frac{x}{a})^3}{3b} \right] dx$$

$-3xb^2 + \frac{2ab^2x}{a}$

$$= \frac{\rho c}{6a} \int_0^a \left[\left(1 - \frac{x}{a}\right)^2 \left[3ab^2 - 3xb^2 - 2ab^2\left(1 - \frac{x}{a}\right)\right] \right] dx.$$

$$= \frac{\rho c}{6a} \int_0^a \left[\left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right)\left[ab^2 - xb^2\right] dx\right.$$

$$= \frac{\rho c}{6a} \int_0^a \left[ab^2 - xb^2 - 2b^2x + \frac{2x^2b^2}{a} + \frac{x^2b^2}{a} - \frac{x^3b^2}{a^2}\right] dx$$

$$= \frac{\rho c}{6a} \int_0^a \left[ab^2 - 3b^2x + \frac{3x^2b^2}{a} - \frac{x^3b^2}{a^2}\right] dx$$

$$= \frac{\rho c}{6a} \left[ab^2x - \frac{3b^2x^2}{2} + \frac{3x^3b^2}{3a} - \frac{x^4b^2}{4a^2}\right]_0^a$$

$$= \frac{\rho c}{6a} \left\{\left[a^2b^2 - \frac{3a^2b^2}{2} + a^2b^2 - \frac{a^2b^2}{4}\right] - [0]\right\}.$$

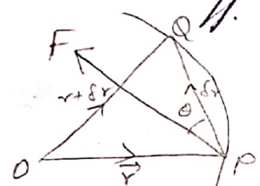
$$= \frac{\rho c}{6a} \cdot \frac{a^2b^2}{4} = \frac{\rho ab^2c}{24}.$$

$\therefore \frac{1}{\rho_1} abc\, \rho \bar{y} = \frac{\rho a b^2 c}{24 \cdot 4}$

$\Rightarrow \bar{y} = \frac{b}{4}$.

IIIly $\bar{x} = \frac{a}{4}$ & $\bar{z} = \frac{c}{4}$.

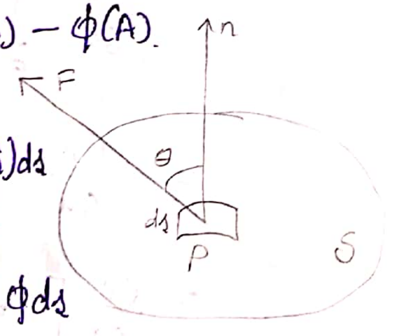
$\therefore$ the centre of gravity is $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$ //



Def

1) The integrals $\int_C \vec{F} \times d\vec{r}$ and $\int_C \phi\, d\vec{r}$ where ϕ is a scalar point function are also line integrals.

2) $\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B d\phi = \phi(B) - \phi(A)$.



3) Surface integral $= \iint_C (\vec{F} \cdot \hat{n})\, ds$
Total flux

4) $\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_S \vec{F} \cdot d\vec{s} = \iint_S \phi\, ds$

$= \iint_S \vec{F} \times ds = \iint_S \vec{u} \cdot \vec{v}\, ds$

5) Volume integral $= \iiint_V \phi\, dv = \iiint_V \vec{F}\, dv$.

Green's theorem in the plane.

If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x & y having continuous derivatives in R, then

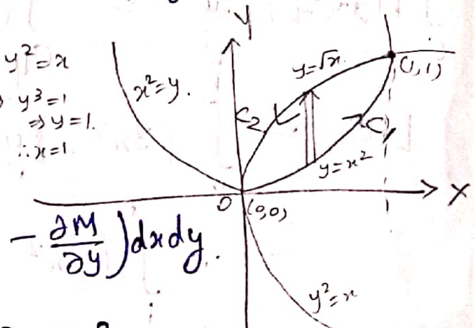
$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the anticlockwise direction (positive direction).

Problems:

1) Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region defined by $y = \sqrt{x}$ & $y = x^2$.

$x^2 = y$ & $y^2 = x$
$y^4 = y \Rightarrow y^3 = 1$
$\Rightarrow y = 1$.
$\therefore x = 1$



Sol. By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

From the given, $M = 3x^2 - 8y^2$, $N = 4y - 6xy$.

$\Rightarrow \frac{\partial M}{\partial y} = -16y$, $\frac{\partial N}{\partial x} = -6y$.

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$.

$\therefore \oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

RHS. $= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y\,dy\,dx$

$= \int_0^1 \left[\frac{10y^2}{2} \right]_{x^2}^{\sqrt{x}} dx.$

$= 5\int_0^1 \left[x - x^4 \right] dx$

$= 5\left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$

$= 5\left[\frac{1}{2} - \frac{1}{5} \right] - 5[0] = 5\left(\frac{3}{10} \right)$

$= \frac{3}{2}$ —— 1

$\oint_C M\,dx + N\,dy = \int_{C_1 \left[x^2 = y,\, x = \sqrt{y},\, 2x\,dx = dy \right]} (M\,dx + N\,dy) + \int_{C_2 \left[y^2 = x,\, y = \sqrt{x},\, 2y\,dy = dx \right]} (M\,dx + N\,dy)$

LHS. $= \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$+ \int_1^0 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy.$

$= \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)(2x\,dx)$

$+ \int_1^0 (3x^2 - 8y^2)(2y\,dy) + (4y - 6xy)\,dy.$

$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$

$+ \int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$

$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$

$+ \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy.$

$= \left[\frac{3x^3}{3} - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5} \right]_0^1$

$+ \left[\frac{6y^6}{6} - \frac{16y^4}{4} + \frac{4y^2}{2} - \frac{6y^4}{4} \right]_1^0$

$= \left[\frac{5 - 8 + 10 - 12}{5} \right] - [0] + [0] - \left[\frac{2 - 8 + 4 - 3}{2} \right]$

$= \frac{-5}{5} - \left[\frac{-5}{2} \right] = -1 + \frac{5}{2} = \frac{3}{2}$ —— 2

from 1 & 2, LHS = RHS.

Thus verified.

(xi) Verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the boundary of the region defined by $x=0$, $y=0$ & $x+y=1$.

Sol. By Green's theorem,

$y = 1-x$

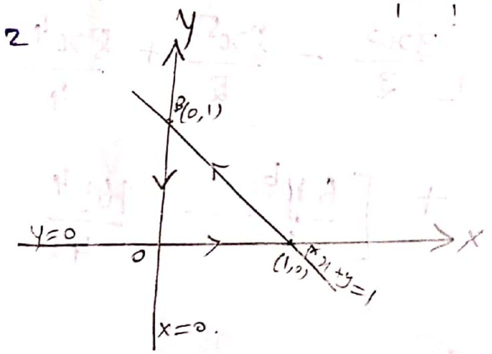
x	0	1	2	3
y	1	0	-1	-2

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\frac{x}{1} + \frac{y}{1} = 1$

here $M = 3x^2 - 8y^2$

$$\frac{\partial M}{\partial y} = -16y.$$



$N = 4y - 6xy$, $\frac{\partial N}{\partial x} = -6y$.

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = +10y.$$

$$\therefore \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \iint_R (10y)\,dx\,dy \quad \text{—— (1).}$$

$$\text{RHS} = \iint_R 10y\,dx\,dy = \int_{x=0}^{1}\left(\int_{y=0}^{1-x} 10y\,dy\right) dx$$

$$= \int_0^1 \left(\frac{\cancel{10}^{5} y^2}{\cancel{2}_1}\right)_0^{(1-x)} dx.$$

$$= 5\int_0^1 (1-x)^2\,dx$$

$$= 5\int_0^1 (1-2x+x^2)\,dx$$

$$= 5\left(x - \frac{\cancel{2}x^2}{\cancel{2}} + \frac{x^3}{3}\right)_0^1$$

$$= 5\left[\cancel{1} - \cancel{1} + \frac{1}{3}\right] - [0]5$$

$$= \frac{5}{3} \quad \text{—— (2).}$$

$$\text{LHS} = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \left(\int_{OA} + \int_{AB} + \int_{BO}\right)\{[3x^2-8y^2]dx + [4y-6xy]dy\}.$$

$\begin{bmatrix} y=0 \\ dy=0 \end{bmatrix}$ $\begin{bmatrix} x+y=1 \\ y=1-x \\ dy=-dx \end{bmatrix}$ $\begin{bmatrix} x=0 \\ dx=0 \end{bmatrix}$

$$= \int_0^1 3x^2\,dx + \int_1^0 \{[3x^2 - 8\underbrace{(1-x)^2}_{1-2x+x^2}]\,dx + [4(1-x) - 6x(1-x)](-dx)\}$$

$$+ \int_1^0 (0 + 4y)\,dy.$$

$$= \left[\frac{3x^3}{3}\right]_0^1 + \int_1^0 \left\{3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2\right\} dx + \left[\frac{4y^2}{2}\right]_1^0$$

$$= \{[1]-[0]\} + \left[\frac{3x^3}{3} - 8x + \frac{16x^2}{2} - \frac{8x^3}{3} - 4x + \frac{4x^2}{2} + \frac{6x^2}{2} - \frac{6x^3}{3}\right]_1^0 + [0-2]$$

$$= 1 + \left\{[0] - \left[1 - 8 + 8 - \frac{8}{3} - 4 + 2 + 3 - 2\right]\right\} - 2$$

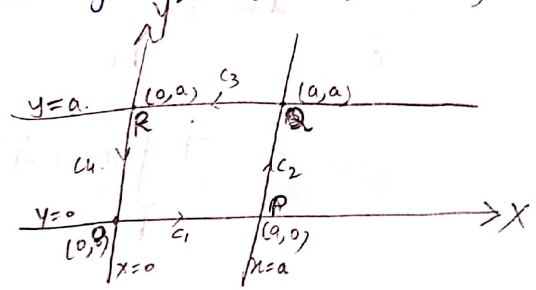
$$= -1 - \left[-\frac{8}{3}\right] = -1 + \frac{8}{3} = \frac{5}{3} \quad \text{———} (3)$$

from (2) & (3), LHS = RHS//.

~~Hence~~ Thus verified.

3) Verify Green's theorem in a plane with respect to $\int_C (x^2 dx + xy\, dy)$, where C is the boundary of the square formed by $x=0, y=0, x=a, y=a$, $a>0$

ans = $\frac{a^3}{2}$



Gauss's Divergence theorem:-

2M (X1) The surface integral of the normal component of a vector function $\vec{F}$ taken over a closed surface S enclosing a volume V is ~~given~~ equal to the volume integral of the divergence of $\vec{F}$ taken throughout the volume V.

In otherwords, if the vector function $\vec{F}$ has continuous first partial derivatives in V, then

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_V \operatorname{div} \vec{F}\, dv.$$

where dv is the volume element.

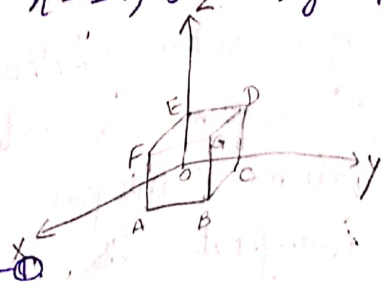
2M Stokes theorem :

(X1) The surface integral of the normal component of the curl of a vector point function $\vec{F}$ over an open surface S is equal to the line integral of the tangential component of $\vec{F}$ around the closed curve C bounding the open surface S.

In otherwords, if $\vec{F}$ is a vector function with continuous first partial derivatives and C is a simple closed curve bounding the open surface S, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{s} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, ds.$$

(※) 1) Verify divergence theorem for $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ over the cube formed by $x = \pm 1, y = \pm 1, z = \pm 1$.



Sol. Gauss's divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V (\text{div } \vec{F})\,dV \quad \text{— (1)}$$

$$\text{div } \vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x^2\hat{i} + z\hat{j} + yz\hat{k})$$

$$= 2x + 0 + y$$

$$= 2x + y.$$

$$\therefore \text{RHS} = \iiint_V (\text{div } \vec{F})\,dV$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x+y)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(\frac{\cancel{2}x^2}{\cancel{2}} + xy\right)_{-1}^{1} dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} [(1+y) - (1-y)]\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} (2y)\,dy\,dz = \int_{-1}^{1}\left[\frac{\cancel{2}y^2}{\cancel{2}}\right]_{-1}^{1} dz$$

$$= \int_{-1}^{1} [1-1]\,dz = 0. \quad \text{— (2)}$$

Now, the surface integral has to be evaluated over the six faces of the cube.

$$\text{LHS} = \iint_S \vec{F}\cdot\hat{n}\,dS = \left[\iint_{ABGF} + \iint_{OCDE} + \iint_{OAFE} + \iint_{CBGD} + \iint_{OABC} + \iint_{EFGD}\right](\vec{F}\cdot\hat{n})\,dS.$$

$$= \left[\iint_{\substack{x=1\\ \hat{n}=\hat{i}}} + \iint_{\substack{x=-1\\ \hat{n}=-\hat{i}}} + \iint_{\substack{y=-1\\ \hat{n}=-\hat{j}}} + \iint_{\substack{y=1\\ \hat{n}=\hat{j}}} + \iint_{\substack{z=-1\\ \hat{n}=-\hat{k}}} + \iint_{\substack{z=1\\ \hat{n}=\hat{k}}}\right][x^2\hat{i} + z\hat{j} + yz\hat{k}]\cdot\hat{n}\,dS.$$

$$= \iint_{x=1} x^2\,dS + \iint_{x=-1} -x^2\,dS + \iint_{y=-1} -z\,dS + \iint_{y=1} z\,dS + \iint_{z=-1} -yz\,dS + \iint_{z=1} yz\,dS.$$

$$= \int_{-1}^{1}\int_{-1}^{1} \cancel{dz\,dy} - \int_{-1}^{1}\int_{-1}^{1} \cancel{dz\,dy} - \int_{-1}^{1}\int_{-1}^{1} \cancel{z\,dx\,dz} + \int_{-1}^{1}\int_{-1}^{1} \cancel{z\,dx\,dz}$$

$$+ \int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy + \int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy.$$

$$= 2\int_{-1}^{1}\left[\frac{yx}{1}\right]_{-1}^{1} dy = 2\int_{-1}^{1}[y+y]\,dy = 4\int_{-1}^{1} y\,dy$$

$$= 4\left[\frac{y^2}{2}\right]_{-1}^{1} = \frac{4}{2}[1^2 - (-1)^2] = 0. \quad \text{— (3)}$$

from (2) & (3), LHS = RHS. Thus verified.

Example 21 Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$. **[KU Nov. 2010]**

Solution For verification of the divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

$$\text{Now div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2(x + y + z)$$

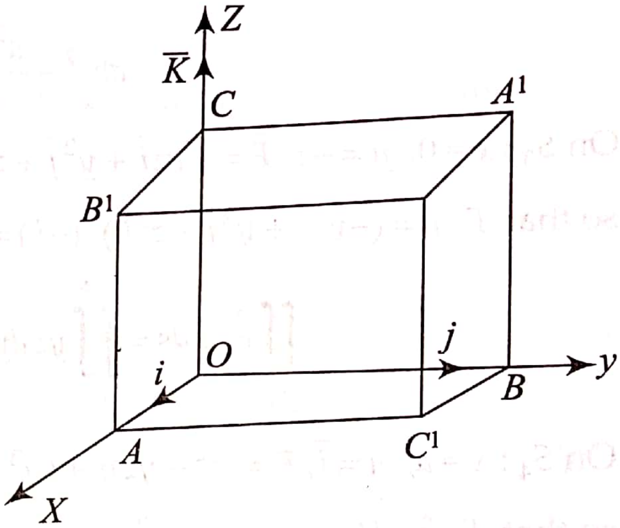


Fig. 25.16

$$\therefore \quad \iiint_V \text{div } \vec{F} dv$$

$$= \int_0^c \int_0^b \int_0^a 2(x + y + z)\, dx\, dy\, dz$$

$$= \int_0^c \int_0^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dy\, dz$$

$$= \int_0^c \int_0^b 2\left(\frac{a^2}{2} + ya + za\right) dy\, dz$$

$$= \int_0^c 2\left[\left(\frac{a^2}{2}y + \frac{y^2 a}{2} + azy\right)\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz = 2\left[\frac{a^2 b}{2}z + \frac{ab^2}{2}z + \frac{abz^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a + b + c) \quad (1)$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelepiped into 6 parts.

S_1: Face $OAC'B$

S_2: Face $CB'PA'$

S_3: Face $OBA'C$

S_4: Face $AC'PB'$

S_5: Face $OCB'A$

S_6: Face $BA'PC'$

Also,

$$\iint_S \vec{F} \cdot \hat{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n} ds + \iint_{S_2} \vec{F} \cdot \hat{n} ds + \iint_{S_3} \vec{F} \cdot \hat{n} ds$$

$$+ \iint_{S_4} \vec{F} \cdot \hat{n} ds + \iint_{S_5} \vec{F} \cdot \hat{n} ds + \iint_{S} \vec{F} \cdot \hat{n} ds \quad (2)$$

On $S_1: z=0,\ \hat{n}=-\vec{k},\ ds=dx\,dy$

so that $\vec{F}\cdot\hat{n}=(x^2\vec{i}+y^2\vec{j}-xy\vec{k})\cdot(-\vec{k})=xy$

$$\therefore \qquad \iint_{S_1}\vec{F}\cdot\hat{n}ds=\int_0^b\int_0^a xy\,dx\,dy=\int_0^b\left(y\frac{x^2}{2}\right)_0^a dy$$

$$=\frac{a^2}{2}\int_0^b y\,dy=\frac{a^2b^2}{4} \tag{3}$$

On $S_2: z=c,\ \hat{n}=\vec{k},\ ds=dx\,dy,\ \vec{F}=(x^2-cy)\vec{i}+(y^2-cx)\vec{j}+(c^2-xy)\vec{k}$.

so that $\vec{F}\cdot\hat{n}=[(x^2-cy)\vec{i}+(y^2-(x)\vec{j})+(c^2-xy)\vec{k}]\cdot\vec{k}=c^2-xy$.

$$\therefore \qquad \iint_{S_2}\vec{F}\cdot\hat{n}ds=\int_0^b\int_0^a(c^2-xy)\,dx\,dy=\int_0^b\left(c^2a-\frac{a^2}{2}y\right)dy$$

$$=abc^2-\frac{a^2b^2}{4} \tag{4}$$

On $S_3: x=0,\ \hat{n}=-\vec{i},\vec{F}=-yz\vec{i}+y^2\vec{j}+z^2\vec{k},dz=dy\,dz$

so that $\vec{F}\cdot\hat{n}=(-yz\vec{i}+y^2\vec{j}+z^2\vec{k})\cdot(-\vec{i})=yz,\ ds=dy\,dz$

$$\therefore \qquad \iint_{S_3}\vec{F}\cdot\hat{n}ds=\int_0^c\int_0^b yz\,dy\,dz=\int_0^c\frac{b^2}{2}z\,dz=\frac{b^2c^2}{4} \tag{5}$$

On $S_4: x=a,\ \hat{n}=\vec{i},\vec{F}=(a^2-yz)\vec{i}+(y^2-az)\vec{j}+(z^2-ay)\vec{k}$

so that $\vec{F}\cdot\hat{n}=[(a^2-yz)\vec{i}+(y^2-az)\vec{j}+(z^2-ay)\vec{k}]\cdot\vec{i}$

$=a^2-yz,\ ds=dy\,dz$

$$\therefore \qquad \iint_{S_4}\vec{F}\cdot\hat{n}ds=\int_0^c\int_0^b(a^2-yz)\,dy\,dz=\int_0^c\left(a^2b-\frac{b^2}{2}z\right)dz$$

$$=a^2bc-\frac{b^2c^2}{4} \tag{6}$$

On $S_5: y=0,\ \hat{n}=-\vec{j},\vec{F}=x^2\vec{i}-zx\vec{j}+z^2\vec{k},ds=dx\,dz$

so that $\vec{F}\cdot\hat{n}=(x^2\vec{i}-zx\vec{j}+z^2\vec{k})\cdot(-\vec{j})=zx$

$$\therefore \qquad \iint_{S_5}\vec{F}\cdot\hat{n}ds=\int_0^a\int_0^c zx\,dz\,dx=\int_0^a\frac{c^2}{2}x\,dx=\frac{a^2c^2}{4} \tag{7}$$

On $S_6: y=b,\ \hat{n}=\vec{j},\vec{F}=(x^2-bz)\vec{i}+(b^2-zx)\vec{j}+(z^2-bx)\vec{k}$

$ds=dx\,dz$

so that $\vec{F}\cdot\hat{n}=[(x^2-bz)\vec{i}+(b^2-zx)\vec{j}+(z^2-bx)\vec{k}]\cdot\vec{j}$

$=b^2-zx.$

$$\therefore \quad \iint_{S_6} \vec{F} \cdot \hat{n} = \int_0^a \int_0^c (b^2 - zx)\, dz\, dx$$

$$= \int_0^a \left(b^2 c - \frac{c^2}{2} x \right) \cdot dx = ab^2 c - \frac{a^2 c^2}{4} \tag{8}$$

By using (3), (4), (5), (6), (7) and (8), in (2), we get

$$\iint_S \vec{F} \cdot \hat{n} ds = \frac{a^2 b^2}{4} + abc^2 - \frac{a^2 b^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + ab^2 c - \frac{a^2 c^2}{4}$$

$$= abc(a + b + c) \tag{9}$$

The equalities (1) and (9) verify the divergence theorem. **Ans.**

3) Use the divergence theorem to evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ & $z=3$.

Sol. By Gauss's divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \text{div}\, \vec{F}\, dv.$$

$y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

$x^2 = 4$

$x = \pm 2$

$$= \iiint_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\, dV$$

$\left(\frac{\partial}{\partial x}4x - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}z^2\right)$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{z=3} (4 - 4y + 2z)\, dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + \frac{2z^2}{2}\right]_0^3 dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [12 - 12y + 9]\, dy\, dx$$

$$= \int_{-2}^{2} \left[12y - \frac{12y^2}{2} + 9y\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left\{\left[12(\sqrt{4-x^2}) - 6(4-x^2) + 9(\sqrt{4-x^2})\right] - \left[-12\sqrt{4-x^2} - 6(4-x^2) - 9(\sqrt{4-x^2})\right]\right\} dx$$

$$= \int_{-2}^{2} \left\{24\sqrt{4-x^2} + 18\sqrt{4-x^2}\right\} dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\, dx = 42\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 21\left\{\left[2\sqrt{4-4} + 4\sin^{-1}\left(\frac{2}{2}\right)\right] - \left[0 + 4\sin^{-1}\left(-\frac{2}{2}\right)\right]\right\}$$

$$= 21\left\{0 + \frac{4\pi}{2} + \frac{4\pi}{2}\right\}$$

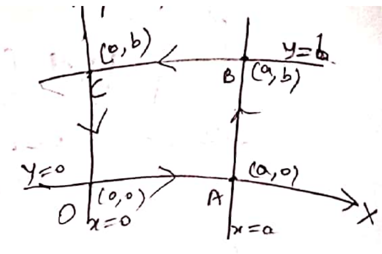
$$= 21(4\pi) = 84\pi \;//.$$

4) Verify Stoke's theorem for a vector field $\vec{A} = (x^2 - y^2)\hat{i} + (2xy)\hat{j}$ in the rectangular region of $z=0$ plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.

Sol By Stoke's theorem,

$$\iint_S \text{curl}\, \vec{A}\cdot\hat{n}\, dS = \int_C \vec{A}\cdot d\vec{r} \quad \text{——— (1)}$$

LHS:



$$\text{curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^2) & 2xy & 0 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial z}(2xy)\right) - \hat{j}\left(0 - \frac{\partial}{\partial z}(x^2-y^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right)$$

$$= 0 - 0 + \hat{k}(2y + 2y).$$

$$= 4y\hat{k}.$$

$$\text{curl } \vec{A} \cdot \hat{n} = (4y\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 4y. \quad \& \quad dS = dx\,dy \quad (\because z = 0 \text{ (gn)})$$

$$\therefore \iint_S \text{curl } \vec{A} \cdot \hat{n}\, dS = \int_{x=0}^{a}\int_{y=0}^{b} 4y\, dy\, dx$$

$$= \cancel{4}^{2} \int_0^a \left[\frac{y^2}{\cancel{2}}\right]_0^b dx$$

$$= 2\int_0^a [b^2 - 0]\, dx = 2b^2 \int_0^a dx$$

$$= 2b^2 [x]_0^a = 2b^2[a-0] = 2ab^2. \quad \text{—— (2)}$$

RHS: $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{A} \cdot d\vec{r} = \left((x^2-y^2)\hat{i} + 2xy\,\hat{j}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= (x^2-y^2)\,dx + 2xy\,dy$$

$$\int_C \vec{A} \cdot d\vec{r} = \Bigg(\int_{\substack{OA \\ [y=0,\ dy=0]}} + \int_{\substack{AB \\ [x=a,\ dx=0]}} + \int_{\substack{BC \\ [y=b,\ dy=0]}} + \int_{\substack{CO \\ [x=0,\ dx=0]}}\Bigg) \vec{A} \cdot d\vec{r}$$

$$= \int_{\substack{y=0 \\ dy=0}} (x^2-y^2)\,dx + 2xy\,dy + \int_{\substack{x=a \\ dx=0}} (x^2-y^2)\,dx + 2xy\,dy + \int_{\substack{y=b \\ dy=0}} (x^2-y^2)\,dx + 2xy\,dy$$

$$+ \int_{\substack{x=0 \\ dx=0}} (x^2-y^2)\,dx + 2xy\,dy.$$

$$= \int_{x=0}^{x=a} x^2\,dx + \int_{y=0}^{b} 2ay\,dy + \int_{x=a}^{0} (x^2-b^2)\,dx + \int_{y=b}^{0} (0+0)\,dy.$$

$$= \left[\frac{x^3}{3}\right]_0^a + 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} - b^2 x\right]_a^0 + 0$$

$$= \frac{a^3}{3} - 0 + \frac{\cancel{2}ab^2}{\cancel{2}} - 0 + 0 - \left[\frac{a^3}{3} - ab^2\right] + 0$$

$$= \cancel{\frac{a^3}{3}} + ab^2 - \cancel{\frac{a^3}{3}} + ab^2$$

$$= 2ab^2 \quad \text{—— (3)}$$

from (2) & (3), LHS = RHS//.
Thus verified.

Example 17 Verify Stokes' theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}$ over the surface of a cube $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 2$, $z = 2$ above the XOY plane (open at the bottom). **[KU May 2010]**

Solution Consider the surface of the cube as shown in the figure. Bounding path is $OABCO$ shown by arrows.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [(y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}] \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (y - z + 2)dx + (yz + 4)dy - xz\,dz$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r} \quad (1)$$

Along OA, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2dx = (2x)_0^2 = 4$$

Along AB, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4dy = 4(y)_0^2 = 8$$

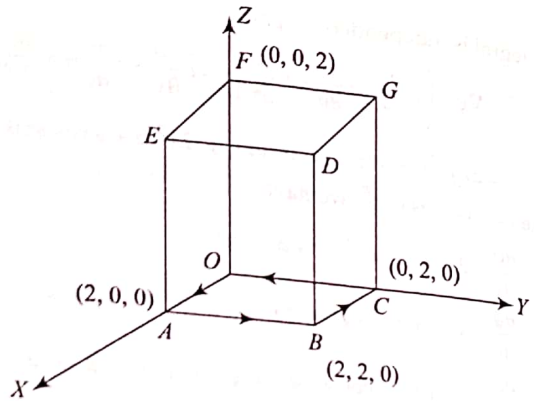


Fig. 25.12

Along BC, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^2 (2-0+2)dx = (4x)_0^2 = -8$$

Along CO, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{CO} \vec{F}\cdot d\vec{r} = \int (y-0+2)\times 0 + (0+4)dy - 0$$

$$= 4\int dy = 4(y)_2^0 = -8$$

On putting the values of these integrals in (1), we get

$$\int_C \vec{F}\cdot d\vec{r} = 4+8-8 = -4$$

To obtain surface integral

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= (0-y)\vec{i} - (-z+1)\vec{j} + (0-1)\vec{k} = -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

Here, we have to integrate over the five surfaces, $ABDE$, $OCGF$, $BCGD$, $OAEF$, $DEFG$.

Over the surface $ABDE$: $x = 2$, $\hat{n} = \vec{i}$, $ds = dydz$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}]\cdot\vec{i}\,dy\,dz$$

$$= \iint_S -y\,dydz = -\int_0^2 y\,dy\int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$$

Over the surface $OCGF$: $x = 0$, $\hat{n} = -\vec{i}$, $ds = dy\,dz$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}]\cdot(-\vec{i})dy\,dz$$

$$= \iint_S y\,dy\,dz = \int_0^2 y\,dy\int_0^2 dz = \left[\frac{y^2}{2}\right]_0^2 = 4$$

Over the surface $BCGD$: $y = 2$, $\hat{n} = \vec{j}$, $ds = dx\,dz$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}]\cdot\vec{j}\,dx\,dz$$

$$= \iint_S (z-1)dx\,dz$$

$$= \int_0^2 dx\int_0^2 (z-1)dz$$

$$= [x]_0^2\left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface $OAEF$: $y = 0$, $\hat{n} = -\vec{j}$, $ds = dx\,dz$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}]\cdot(-\vec{j})dx\,dz$$

$$= -\iint_S (z-1)dx\,dz$$

$$= -\int_0^2 dx\int_0^2 (z-1)dz$$

$$= -[x]_0^2\left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface $DEFG$: $z = 2$, $\hat{n} = \vec{k}$, $ds = dx\,dy$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}]\cdot\vec{k}\,dx\,dy$$

$$= -\iint dx\,dy = -\int_0^2 dx\int_0^2 dy$$

$$= -[x]_0^2[y]_0^2 = -4$$

Total surface integral $= -4+4+0+0-4 = -4$

Thus $\iint_S \text{curl}\ \vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot d\vec{r} = -4$

which verifies Stokes' theorem. **Verified.**

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Example 3 Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A}=(x+y^2)\vec{i}-2x\vec{j}+2yz\vec{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant. [KU May 2010]

Solution A vector normal to the surface S is given by

$$\nabla(2x+y+2z)=2\vec{i}+\vec{j}+2\vec{k}$$

$\therefore\ \hat{n}=a$ unit vector normal to the surface S

$$=\frac{2\vec{i}+\vec{j}+2\vec{k}}{\sqrt{2^2+1^2+2^2}}=\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}$$

$$\vec{k}\cdot\hat{n}=\vec{k}\cdot\left(\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}\right)=\frac{2}{3}$$

$$\therefore\qquad \iint_S \vec{A}\cdot\hat{n}\cdot ds=\iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$$

where R is the projection of S

Now, $$\vec{A}\cdot\hat{n}=[(x+y^2)\vec{i}-2x\vec{j}+2yz\vec{k}]\cdot\left(\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}\right)$$

$$=\frac{2}{3}(x+y^2)-\frac{2}{3}x+\frac{4}{3}yz=\frac{2}{3}y^2+\frac{4}{3}yz$$

$$=\frac{2}{3}y^2+\frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$\left(\text{since on the plane } 2x+y+2z=6,\ z=\dfrac{6-2x-y}{2}\right)$

$$=\frac{2}{3}y(y+6-2x-y)$$

$$=\frac{4}{3}y(3-x)$$

Hence, $$\iint_S \vec{A}\cdot\hat{n}\cdot ds=\iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}.$$

$$=\iint_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}dx\,dy$$

$$=\int_0^3\int_0^{6-2x} 2y(3-x)dy\,dx$$

$$= \int_0^3 2(3-x)\left(\frac{y^2}{2}\right)_0^{6-2x} dx$$

$$= \int_0^3 (3-x)(6-2x)^2\, dx$$

$$= 4\int_0^3 (3-x)^3\, dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3$$

$$= 81$$ **Ans.**

Ex. 2. If $\mathbf{F} = (2xy+z^3)\,\mathbf{i}+x^2\mathbf{j}+3xz^2\,\mathbf{k}$, find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is any path joining $(1, -2, 1)$ to $(3, 1, 4)$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = -\mathbf{j}\,(3z^2 - 3z^2) + \mathbf{k}\,(2x - 2x) = 0.$$

~~By Cor. (i) of § 3,~~ $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A $(1, -2, 1)$ to B $(3, 1, 4)$ and $\mathbf{F} = \nabla\phi$.

$$\therefore \quad \phi_x \mathbf{i} + \phi_y \mathbf{j} + \phi_z \mathbf{k} \equiv (2xy + z^3)\,\mathbf{i} + x^2\mathbf{j} = 3xz^2\,\mathbf{k}.$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2xy + z^3 \,;\, \frac{\partial \phi}{\partial y} = x^2 \,;\, \frac{\partial \phi}{\partial z} = 3xz^2.$$

Integrating partially these three equations,

$$\phi = x^2y + xz^3 + f(y, z)$$
$$\phi = x^2y + g(x, z)$$
$$\phi = xz^3 + h(x, y).$$

The above three values of ϕ agree if we choose $f(y, z) = 0$, $g(x, z) = xz^3$ and $h(x, y) = x^2y$.

Hence $\phi = x^2y + xz^3$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \nabla \phi \cdot d\mathbf{r} = \int d\phi = \big[\, \phi \,\big]_A^B = \big(\, x^2y + xz^3 \,\big)_{(1,\,-2,\,1)}^{(3,\,1,\,4)}$$

$$= 9 + 192 - (-2 + 1) = 202.$$

CHAPTER 1.5

LINEAR PARTIAL DIFFERENTIAL EQUAITONS OF SECOND AND HIGHER ORDER WITH CONSTNAT COEFFICENTS

A linear homogeneous partial differential equation of nth order with constant coefficients is represented as:

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots + a^n \frac{\partial^n z}{\partial y^n} = f(x, y) \quad \ldots(1)$$

NOTE

1. As the derivatives involved are of same order, the equation is called homogeneous.

2. If $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$

$$(D^n + a_1 D^{n-1} D' + \ldots + a_n D'^n) z = f(x, y)$$

$$F(D, D') z = f(x, y) \quad \ldots (2)$$

3. The complete solution of (2) consists of two parts, complementary function (C.F.) and the particular integral (P.I).

Complete solution is:

$$Z = C.F. + P.I. \quad \ldots (3)$$

Procedure to find the complementary function (C.F.)

Consider a third order P.D.E.

$$a_0 \frac{\partial^3 z}{\partial x^3} + a_1 \frac{\partial^3 z}{\partial x^2 \partial y} + a_2 \frac{\partial^3 z}{\partial x\, \partial y^2} + a_3 \frac{\partial^3 z}{\partial y^3} = f(x, y) \quad \ldots (1)$$

Put R.H.S = 0

$$a_0 \frac{\partial^3 z}{\partial x^3} + a_1 \frac{\partial^3 z}{\partial x^2 \partial y} + a_2 \frac{\partial^3 z}{\partial x\, \partial y^2} + a_3 \frac{\partial^3 z}{\partial y^3} = 0$$

Put $\frac{\partial}{\partial x} = D, \frac{\partial^2}{\partial x^2} = D^2 \ldots$

$\frac{\partial}{\partial y} = D', \frac{\partial^2}{\partial y^2} = D'^2 \ldots$

$(a_o D^3 + a_1 D^2 D' + a_2 D D'^2 + a_3 D'^3) z = 0 \quad \ldots (2)$

$f(D, D') z = 0 \quad \ldots (3)$

Put $D = m, D' = 1, f(m, 1) = 0$ or $f(D, D') = 0 \quad \ldots (4)$

(4) is called the auxiliary equation or A.E. m_1, m_2, m_3 are the roots of (4).

Case (i)

If $m_1 \neq m_2 \neq m_3$ (real or complex and different) Solution is:

$$z = f_1 (y + m_1 x) + f_2 (y + m_2 x) + f_3 (y + m_3 x) \quad \ldots (i)$$

Case (ii)

If $m_1 = m_2 = m_3$ (real and equal)

$$z = f_1 (y + m_1 x) + x f_2 (y + m_1 x) + x^2 f_3 (y + m_1 x) \quad \ldots (ii)$$

UNIT 1

SOLVED EXAMPLES

Example 1

$$\frac{\partial^2 z}{\partial x^2} - \frac{5\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$$

SOLUTION: $(D^2 - 5DD' + 6D'^2)\, z = 0$

$$f(D, D')\, z = 0$$

Put $D = m$, $D' = 1$, the A.E. is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2,\ m = 3$$

The complementary function is

C.F. is: $z = f_1(y + 2x) + f_2(y + 3x)$

Example 2

Solve $\frac{\partial^2 z}{\partial x^2} - 6\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$

SOLUTION: $(D^2 - 6DD' + 9D'^2)\, Z = 0$

$$D = m,\ D' = 1$$

A.E.: (Auxilary equation) $m^2 - 6m + 9 = 0$

$$m = 3, 3$$

Complementary function is:

$$Z = f_1(y + 3x) + x f_2(y + 3x)$$

Example 3

Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = 0$

SOLUTION: $(D^3 - 3D^2 D' + 4D'^3)\, z = 0$

$$f(D, D')\, z = 0$$

Put $D = m, D' = 1$

A.E.: $m^3 - 3m^2 + 4 = 0$

Put: $m = -1, -1 - 3 + 4 = 0$ $\therefore m = -1$ is a root

$$\begin{array}{r|l} & m^2 - 4m + 4 \\ \hline m = -1 & m^3 - 3m^2 + 4 \\ (m+1) & m^3 + m^2 \\ \hline & \quad -4m^2 + 4 \\ & \quad -4m^2 - 4m \\ \hline & \qquad 4m + 4 \\ & \qquad 4m + 4 \\ \hline & \qquad 0 \end{array}$$

Solve

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2, 2.$$

roots are $m = 2, 2, -1$

$\therefore$ Solution is $z = f_1(y + 2x) + x f_2(y + 2x) + f_3(y - x)$

Example 4 **(Anna Uni. Oct/Nov. 1996)**

Solve $(D^3 - 4D^2 D' + 4DD'^2]\, z = 0.$

SOLUTION: Given $(D^3 - 4D^2 D' + 4D D'^2)\, z = 0.$

A.E: $m^3 - 4m^2 + 4m = 0.$

$$m = 0,\ 2,\ 2$$

$\therefore$ Solutionis $z = f_1(y + 0 \cdot x) + f_2(y + 2x) + x f_3(y + 2x)$

Example 5 **(Anna Uni. April/May 2001)**

Solve $4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0.$

SOLUTION: $(4D^2 - D'^2)\, z = 0.$

A.E: $4m^2 - 1 = 0;\ m = \pm\frac{1}{2}$

Solution is: $z = f_1(y + 0.5x) + f_2(y - 0.5x)$

UNIT 1

Example 6 (Anna Uni. April/May 2003)

Solve $(D^3 - 3DD'^2 + 2D'^3)z = 0$

SOLUTION: $m^3 - 3m + 2 = 0.$

$$m = 1,\ 1,\ -2$$

Solution is $$z = f_1(y - 2x) + f_2(y + x) + xf_3(y + x)$$

Example 7 (Anna Uni. Nov/Dec. 2003)

Find the general solution of $4\dfrac{\partial^2 z}{\partial x^2} - 12\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{9\,\partial^2 z}{\partial y^2} = 0.$

SOLUTION: $4m^2 - 12m + 9 = 0$

$$m = \frac{3}{2},\ \frac{3}{2}$$

$$z = f_1(y + 1.5x) + xf_2(y + 1.5x).$$

Example 8 (Anna Uni. April/May 2005)

Solve $(D^3 + DD'^2 - D^2D' - D'^3)z = 0.$

SOLUTION: $m^3 - m^2 + m - 1 = 0.$

$$m = 1,\ m = \pm i$$

$$z = f_1(y + x) + f_2(y + ix) + f_3(y - ix)$$

Example 9 (Anna Uni. May 1996)

[Type: Non. homogeneous] ... (1)

Solve $(D^2 - DD' + D' - 1)z = 0.$

SOLUTION: Take the solution as:

$$z = ce^{hx + ky} \qquad \text{... (2)}$$

Replace D by h, D' by $k \Rightarrow$

$$h^3 - hk + k - 1 = 0.$$

$$h = 1,\ h = k - 1$$

complete solution is

$$z = \Sigma\, c_1 e^{x + ky} + \Sigma\, c_2 e^{(k-1)x + ky}.$$

$$z = e^x \phi_1(y) + e^{-x}\phi_2(y + x)$$

Type

R.H.S $= f(x, y)$, if $R.H.S. \neq 0$, we need to find the particular Integral (P.I.) Here there are three cases.

Case (i)

$$\text{R.H.S} = e^{ax + by}$$

$$\Rightarrow \quad P.I = \frac{e^{ax+by}}{f(D, D')} = \frac{e^{ax+by}}{f(a, b)}$$

Provided $f(a, b) \neq 0$, if $f(a, b) = 0$, it is a case of failure.

Case (ii)

(a) R.H.S $= \sin(ax + by)$ (or) $\cos(ax + by)$

If $f(D^2, DD', D'^2)\,z = \sin(ax + by)$, then

$$\text{P.I} = \frac{\sin(ax + by)}{f(-a^2, -ab, -b^2)},\ \text{provided } f(-a^2, -ab, -b^2) \neq 0$$

If R.H.S. $= \cos(ax + by)$

$$\text{P.I} = \frac{\cos(ax + by)}{f(-a^2, -ab, -b^2)},\ \text{provided } f(-a^2, -ab - b^2) \neq 0$$

Case (iii)

R.H.S. $= x^p y^q$ (p, q being positive integers), then

$$\text{P.I} = \frac{1}{f(D, D')}x^p y^q = [f(D, D')]^{-1} x^p y^q,$$

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Expand $f(D, D')$ in ascending powers of D, D' and then operate
$x^p y^q$

NOTE *If the above cases fails then we need to prefer the General method for finding P.I.*

Example 10

Solve $\dfrac{\partial^2 z}{\partial x^2} + 7\dfrac{\partial^2 z}{\partial x \partial y} + 12\dfrac{\partial^2 z}{\partial y^2} = e^{2x+5y}$

SOLUTION: Given $\dfrac{\partial^2 z}{\partial x^2} + 7\dfrac{\partial^2 z}{\partial x \partial y} + 12\dfrac{\partial^2 z}{\partial y^2} - e^{2x+5y}$

R.H.S = 0: $(D^2 + 7DD' + 12D'^2)\, z = 0$

$$f(D, D')\, z = 0$$

A.E. (put $D = m, D' = 1$)

$$m^2 + 7m + 12 = 0$$

$$m = -3, -4$$

C.F. $= f_1(y - 3x) + f_2(y - 4x)$ ... (1)

R.H.S $= e^{2x+5y} = e^{ax+by}$ (Using R.H.S find P.I)

$a = 2, b = 5$

$f(D, D') = D^2 + 7DD' + 12D'^2$

$f(a, b) = f(2, 5) = 4 + 70 + 300 = 374 \neq 0$

$$\text{P.I} = \frac{e^{ax+by}}{f(a, b)} = \frac{e^{2x+5y}}{374} \quad \text{... (2)}$$

Complete solution is: $z = CF + P.I$

$$z = f_1(y - 3x) + f_2(y - 4x) + \frac{e^{2x+5y}}{374}$$

Example 11

Solve $\dfrac{\partial^3 z}{\partial x^3} - 5\dfrac{\partial^3 z}{\partial x^2 \partial y} + 6\dfrac{\partial^3 z}{\partial y^3} = e^{4x+y}$

SOLUTION: $(D^3 - 5D^2 D' + 6D'^3)\, z = e^{4x+y}$ ... (1)

$$f(D, D') \cdot z = e^{4x+y}$$

R.H.S = 0, put $D = m, D' = 1$

$$m^3 - 5m^2 + 6 = 0$$

$m = -1: -1 - 5 + 6 = 0 \quad \therefore m = -1$ is a root

$$\begin{array}{l|l}
m+1 & m^2 - 6m + 6 \\
\hline
m=+1 & m^3 - 5m^2 + 6 \\
 & m^3 + m^2 \\
\hline
 & -6m^2 + 6 \\
 & -6m^2 - 6m \\
\hline
 & 6m + 6 \\
 & 6m + 6 \\
\hline
 & 0
\end{array}$$

$m^2 - 6m + 6 = 0$

$m = \dfrac{6 \pm \sqrt{36 - 24}}{2(1)}$

$m = \dfrac{6 \pm 2\sqrt{3}}{2}$

$m = \dfrac{6 \pm 2\sqrt{3}}{2}$

$m = 3 \pm \sqrt{3}$

roots are $m = -1, 3 + \sqrt{3}, 3 - \sqrt{3}$

C.F $= f_1(y - x) + f_2(y + (3 + \sqrt{3})\, x) + f_3(y + (3 - \sqrt{3})\, x)$ ... (2)

Using R.H.S. find the P.I

R.H.S $= e^{4x+y} = e^{ax+by} \Rightarrow a = 4, b = 1$

$f(D, D') = D^2 - 5D^2 D' + 6D'^3$

$f(a, b) = f(4, 1) = 64 - 80 + 6 = -10 \neq 0.$

$$\text{P.I} = \frac{e^{ax+by}}{f(a, b)} = \frac{e^{4x+y}}{-10} \quad \text{... (3)}$$

Solution is:

$$z = \text{C.F} + \text{P.I}$$

UNIT 1

$$z = f_1(y-x) + f_2\left(y + (3+\sqrt{3})x\right) + f_3\left(y + (3-\sqrt{3})x\right) - \frac{e^{4x+y}}{10}$$

This is the Complete solution.

Example 12

Solve $(D-2D')(D-D')^3 z = e^{3x+2y}$

☞ **SOLUTION:** Given $(D-2D')(D-D')^3 z = e^{3x+2y}$ ... (1)

Put **R.H.S = 0**

$$(D-2D')(D-D')^3 z = 0$$

$$f(D, D') \cdot z = 0$$

Put $D = m, D' = 1$

A.E: $(m-2)(m-1)^3 = 0$

$$m = 2, m = 1, 1, 1.$$

C.F. $= f_1(y+2x) + f_2(y+x) + xf_3(y+x) + x^2 f_4(y+x)$ ... (2)

R.H.S.$= e^{3x+2y} = e^{ax+by}$

$a = 3,\ b = 2$

$f(a, b) = (3-4)(3-2)^3 = -1$ ... (3)

$$\text{P.I} = \frac{e^{ax+by}}{f(a,b)} = \frac{e^{3x+2y}}{-1}$$

Complete solution is

$z = \text{C.F.} + \text{P.I}$

$$z = f_1(y+2x) + f_2(y+x) + xf_3(y+x) + x^2 f_4(y+x) - e^{3x+2y}$$

Example 13

Solve $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$

☞ **SOLUTION:** Given $(D^2 - 2DD' + D'^2) z = \sin(2x+3y)$ ... (1)

R.H.S. = 0

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A.E.: $m^2 - 2m + 1 = 0, m = 1, 1$

C.F. $= f_1(y+x) + x f_2(y+x)$ ... (2)

Using R.H.S find P.I

$\sin(ax+by) = \sin(2x+3y) \Rightarrow a = 2, b = 3$

Substitute: $D^2 = -a^2 = -4$

$DD' = -ab = -6$

$D'^2 = -b^2 = -9$

$$\text{P.I} = \frac{\sin(2x+3y)}{D^2 - 2DD' + D'^2} = \frac{\sin(2x+2y)}{-4+12-9}$$

$= -\sin(2x+3y)$ ... (3)

Complete solution is

$$z = CF + P.I$$

$$z = f_1(y+x) + x f_2(y+x) - \sin(2x+3y)$$

Example 14

Solve $(D^3 - 4D^2 D' + 4DD'^2) z = 12 \sin(2x+3y)$

☞ **SOLUTION:** Given $(D^3 - 4D^2 D' + 4 \quad ^2) z = 12 \sin(2x+3y)$ ... (1)

R.H.S. = 0: $(D^3 - 4D^2 D' + 4DD'^2) z = 0$

$f(D, D') \cdot z = 0$

A.E.: put $D = m, D' = 1 \Rightarrow m^3 - 4m^2 + 4m = 0$

$m(m^2 - 4m + 4) = 0$

$m = 0, 2, 2$

C.F. $= f_1(y + 0 \cdot x) + f_2(y+2x) + x f_3(y+2x)$ ... (2)

R.H.S $\Rightarrow \sin(ax+by) = \sin(2x+3y)$

$a = 2, b = 3$

$D^2 = -a^2 = -4 \mid DD' = -ab = -6 \mid D'^2 = -b^2 = -9$

$$\text{P.I} = 12 \cdot \frac{\sin(2x+3y)}{D^3 - 4D^2 D' + 4DD'^2} = 12 \frac{\sin(2x+3y)}{DD^2 - 4D^2 D' + 4DD'^2}$$

$$P.I = 12\frac{\sin(2x+3y)}{-4D+16D'-36D} = 12\cdot\frac{\sin(2x+3y)}{16D'-40D}$$

(multiply in numerator and Denominator by $40D+16D'$)

$$P.I = \frac{-12\sin(2x+3y)}{40D-16D'} = \frac{-12(40D+16D')\sin(2x+3y)}{1600D^2-256D'^2}$$

$$= -12\frac{[40D(\sin(2x+3y))+16D'(\sin(2x+3y))]}{-6400+2304}$$

$$= \frac{-12}{-4096}(40\cos(2x+3y)\cdot 2+16\cos(2x+3y)\cdot 3)$$

$$P.I. = \frac{1536}{4096}\cos(2x+3y) \quad \dots (3)$$

Complete solution is

$z = CF + P.I$

$$z = f_1(y)+f_2(y+2x)+xf_3(y+2x)+\frac{1536}{4096}\cos(2x+3y)$$

Example 15

Solve $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\partial y}-6\dfrac{\partial^2 z}{\partial y^2} = \cos(3x+2y)$

SOLUTION: $(D^2+DD'-6D'^2)z = \cos(3x+2y)$

$f(D, D')z = \cos(3x+2y)$

R.H.S. = 0, Put $D = m,\ D' = 1$

$m^2+m-6=0$

$m = -3, 2$

C.F. $= f_1(y-3x)+f_2(y+2x)$ ... (1)

R.H.S. $\Rightarrow \cos(ax+by) = \cos(3x+2y)$

$a = 3, b = 2$

Put $D^2 = -a^2 = -9,\ DD' - ab = -6,\ D'^2 = -b^2 = -4$

$$P.I = \frac{\cos(3x+2y)}{D^2+DD'-6D'^2} = \frac{\cos(3x+2y)}{-9-6+24} = \frac{\cos(3x+2y)}{9} \quad \dots (2)$$

Complete solution is $z = f_1(y-3x)+f_2(y+2x)+\dfrac{\cos(3x+2y)}{9}$

Example 16

Solve $\dfrac{\partial^2 z}{\partial x^2}-5\dfrac{\partial^2 z}{\partial x\partial y}+6\dfrac{\partial^2 z}{\partial y^2} = \sin 4x\cos 3y$

SOLUTION: Given $\dfrac{\partial^2 z}{\partial x^2}-5\dfrac{\partial^2 z}{\partial x\,\partial y}+6\dfrac{\partial^2 z}{\partial y^2} = \sin 4x\cos 3y$

RHS = 0

$(D^2-5DD'+6D'^2)z = 0$

A.E: $m^2-5m+6=0$

$m = 2, 3$

C.F. $= f_1(y+2x)+f_2(y+3x)$ ... (1)

(using RHS find the P.I)

RHS $= \sin 4x\cos 3y = \dfrac{1}{2}[\sin(4x+3y)+\sin(4x-3y)]$

$f(D, D') = D^2-5DD'^1+6D'^2$

(i) $\sin(4x+3y)$

$a = 4,\ b = 3$

$D^2 = -16,\ DD' = -12,\ D'^2 = -9$

$$P.I_1 = \frac{\sin(4x+2y)}{-16+60-54}$$

$$PI_1 = \frac{\sin(4x+2y)}{-10} \quad \dots(2)$$

(ii) $\sin(4x-3y)$

$a = 4, b = -3$

$D^2 = -16,\ DD' = 12,\ D'^2 = -9$

$$P.I_2 = \frac{\sin(4x-3y)}{-16-60-54}$$

$$PI_2 = \frac{\sin(4x-3y)}{-130} \quad \dots(3)$$

Solution is $z = CF + PI_1 + PI_2$

$$z = f_1(y+2x)+f_2(y+3x)-\frac{1}{20}\sin(4x+2y)-\frac{1}{260}\sin(4x-3y)$$

NOTE

Solve $(D^2-D'^2)z = \sin 2x\sin 3x$

(Anna Uni - April 1996)

Example 17

Solve $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{3x+4y} + \sin(4x-3y)$

SOLUTION: Given:

$$(D^2 - 3DD' + 2D'^2)\,z = e^{3x+4y} + \sin(4x-3y)$$

R.H.S $= 0 \Rightarrow (D^2 - 3DD' + 2D'^2)\,z = 0$

A.E: $m^2 - 3m + 2 = 0,\ m = 1, 2$

$\therefore$ C.F. $= f_1(y+x) + f_2(y+2x)$ ... (1)

$f(D, D') = D^2 - 3DD' + 2D'^2$

R.H.S.

$e^{3x+4y} = e^{ax+by}$	$\sin(4x-3x) = \sin(ax+by)$
$a = 3,\ b = 4$	$a = 4,\ b = -3$
$f(a, b) = 9 - 36 + 32$	$D^2 = -a^2 = -16$
	$DD' = -ab = 12$
$f(a, b) = 5$	$D'^2 = -b^2 = -9$
$PI_1 = \dfrac{e^{3x+4y}}{5}$ ...(2)	$PI_2 = \dfrac{\sin(4x-3y)}{-16-36-18} = \dfrac{\sin(4x-3y)}{-70}$ ...(3)

$\therefore$ Solution is

$z = CF + PI_1 + PI_2$

$$z = f_1(y+x) + f_2(y+2x) + \frac{e^{3x+4y}}{5} - \frac{\sin(4x-3y)}{70}$$

Example 18 (Anna Uni. April 2000, April/May 2004)

Solve $(D^3 - 7DD'^2 - 6D'^3)\,z = \sin(x+2y) + e^{2x+y}$

SOLUTION: R.H.S $= 0$

A.E: $m^3 - 7m - 6 = 0$

$m = -1, -2, 3$ ... (1)

$CF = f_1(y-x) + f_2(y-2x) + f_3(y+3x)$

R.H.S.

(i) $\sin(ax+by) = \sin(x+2y)$

$a = 1,\ b = 2$

$D^2 = -a^2 = -1,\ DD' = -ab = -2,\ D'^2 = -b^2 = -4$

$$\text{P.I}_1 = \frac{\sin(x+2y)}{D^3 - 7DD'^2 - 6D'^3} = \frac{\sin(x+2y)}{27D + 24D'}$$

$$= \frac{1}{3}\frac{\sin(x+2y)}{(9D+8D')} = \frac{1}{3}\frac{(9D-8D')\sin(x+2y)}{(81D^2 - 64D'^2)}$$

$$= \frac{(9D-8D')(\sin(x+2y))}{3\,(175)}$$

$$= \frac{1}{525}[\,9\cos(x+2y) - 16\cos(x+2y)\,]$$

$$\text{P.I}_1 = \frac{-7}{525}\cos(x+2y) = \frac{-1}{75}\cos(x+2y) \quad \text{... (2)}$$

(ii) $e^{ax+by} = e^{2x+y} \Rightarrow a = 2, b = 1$

$f(D, D') = D^3 - 7DD'^2 - 6D'^3$

$f(a, b) = f(2, 1) = -12$

$$\text{P.I}_2 = \frac{e^{2x+y}}{-12} \quad \text{... (3)}$$

Complete Solution is

$z = CF + PI_1 + PI_2$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) - \frac{e^{2x+y}}{12}$$

Example 19

Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = x + y$

SOLUTION: R.H.S. $= 0$

$(D^2 + DD' - 6D'^2)\,z = 0$

A.E: $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

C.F $= f_1(y-3x) + f_2(y+2x)$

$$\text{P.I} = \frac{x+y}{(D^2 + DD' - 6D'^2)} \quad \dots (1)$$

$$= \frac{(x+y)}{D^2\left[1+\left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right]}$$

$$= \frac{1}{D^2}\left[1+\left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}\left[x+y-\frac{1}{D}(0+1)\right]$$

$$PI = \frac{1}{D^2}[x+y-x] = \frac{1}{D^2}(y) = \frac{x^2y}{2}$$

$$PI = \frac{x^2y}{2} \quad \dots (2)$$

$D = \frac{\partial}{\partial x}$

$D' = \frac{\partial}{\partial y}$

$\frac{1}{D} = \int dx$

$\frac{1}{D'} = \int dy$

Complete solution is:

$$z = f_1(y-3x) + f_2(y+2x) + \frac{x^2y}{2}$$

Example 20

Solve $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = y^2 + x$

☞ **SOLUTION:** R.H.S= 0

$(D^2 + 4DD' - 5D'^2)z = 0$

$m^2 + 4m - 5 = 0$

$m = -5, 1$

C.F. $= f_1(y-5x) + f_2(y+x)$ ... (1)

$$\text{P.I} = \frac{(x+y^2)}{(D^2+4DD'-5D'^2)} = \frac{1}{D^2\left(1+4\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)}(x+y^2)$$

$$= \frac{1}{D^2}\left(1+\left(\frac{4D'}{D} - 5\frac{D'^2}{D^2}\right)\right)^{-1}(x+y^2)$$

$$= \frac{1}{D^2}\left(1-\left(\frac{4D'}{D} - 5\frac{D'^2}{D^2}\right) + \frac{16D'^2}{D^2}\right)(y^2+x)$$

$$PI = \frac{y^2x^2}{2} + \frac{x^3}{6} - 4\frac{x^3y}{3} + \frac{7x^4}{4}$$

Complete solution is $z = \text{CF} + \text{PI}$.

$$z = f_1(y-5x) + f_2(y+x) + \frac{x^2y^2}{2} + \frac{x^3}{6} - 4\frac{x^3y}{3} + \frac{7x^4}{4}$$

Example 21

Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$

☞ **SOLUTION:** $m^3 - 2m^2 = 0$

$m = 0, 0, 2$

C.F $= f_1(y) + xf_2(y) + f_3(y+2x)$ ... (1)

$$\text{P.I} = \frac{1}{D^3 - 2D^2D'}(2e^{2x}) + \frac{1}{D^3 - 2D^2D'}(3x^2y)$$

$$= \frac{2e^{2x}}{2^3 - 2.2^2 \cdot 0} + \frac{3}{D^3}\left(1 - 2\frac{D'}{D}\right)(x^2y)$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3}\left(x^2y + 2\frac{1}{D}x^2\right)$$

$$\text{P.I.} = \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60} \quad \dots (2)$$

Solution is

$$z = \text{C.F} + \text{P.I}$$

$$z = f_1(y) + x f_2(y) + f_3(y+2x) + \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Example 22

Solve $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$

☞ **SOLUTION:** R.H.S= 0

$$(D^2 - 6DD' + 9D'^2)\, z = 0$$

A.E: $m^2 - 6m + 9 = 0$

$$m = 3, 3$$

$$\text{C.F} = f_1(y+3x) + x f_2(y+3x) \quad \dots (2)$$

$$\text{P.I} = \frac{1}{(D^2 - 6DD' + 9D'^2)}(6x+2y)$$

$$= \frac{1}{D^2\left(1 - \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)\right)}(6x+2y)$$

$$= \frac{1}{D^2}\left(1 - \left(6\frac{D'}{D} - 9\frac{D'^2}{D^2}\right)\right)^{-1}(6x+2y)$$

(As the function is $6x + 2y$, go upto D, D' is enough, higher order in D and D' may be neglected)

$$= \frac{1}{D^2}\left[1 + 6\frac{D'}{D}\right](6x+2y) \qquad D = \frac{\partial}{\partial x}\; D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2}\left[(6x+2y) + 6\frac{1}{D}D'(6x+2y)\right] \qquad \frac{1}{D} = \int dx,\; \frac{1}{D_1} = \int dy$$

$$= \frac{1}{D^2}\left[6x + 2y + 6\frac{1}{D}\;(2)\right]$$

$$= \frac{1}{D^2}\left[6x + 2y + 12\frac{1}{D}(1)\right]$$

$$= \frac{1}{D^2}\,[6x + 2y + 12x].$$

$$= \frac{1}{D^2}\,[\,18x + 2y\,] = \frac{1}{D}\left(18\frac{x^2}{2} + 2xy\right)$$

$$= 9\frac{x^3}{3} + 2\frac{x^2}{2}y$$

$$\text{P.I} = 3x^3 + x^2 y = x^2(3x+y) \quad \dots (2)$$

Complete solution is

$$z = CF + P.I$$

$$z = f_1(y+3x) + x f_2(y+3x) + x^2(3x+y)$$

General Method for finding P.I.

> **NOTE**
>
> *If the above cases fails to find P.I. as well as if the R.H.S function is in different form, we need to use the general method to find the P.I.* $\frac{1}{D - mD'} f(x,y) = \int f(x, c - mx)\, dx$, *in which* C *is to be replaced by* $y + mx$ *after integration.*

Example 23

Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

☞ **SOLUTION:** Given $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

A.E: $m^2 - m - 2 = 0$

$$m = -1, 2$$

$$\text{C.F} = f_1(y-x) + f_2(y+2x) \quad \dots (1)$$

$$\text{P.I} = \frac{(y-1)e^x}{(D^2 - DD' - 2D'^2)} = \frac{1}{(D - 2D')(D + D')}\left((y-1)e^x\right)$$

$$(D = mD') = D + D'$$

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UNIT 1

$$m = -1, \quad f(x, y) = (y-1)\, e^x$$

replace y by $c - mx \Rightarrow$ put $y = c + x$

W.K.: $f(x, y) = (y-1)\, e^x$

$$f(x, c-x) = (c + x - 1)\, e^x$$

$$\text{P.I} = \frac{1}{D - 2D'} \int \underset{u}{(c + x - 2)} \; \underset{v}{e^x}\, dx$$

$$\text{P.I} = \frac{1}{(D - 2D')} \left[e^x (c + x - 2) \right] \qquad \ldots \text{(I)}$$

Substitute $c = y + mx = y - x$ *in* (I)

$$\text{P.I} = \frac{1}{D - 2D'} \left((y-2)\, e^x \right)$$

(Again use the same formula)

$$\text{P.I} = \int (c - 2x - 2)\, e^x\, dx$$

$$\text{P.I} = (c - 2x)\, e^x \quad \text{(II)} \quad (\text{ replace } c \text{ by } y + 2x \text{ in (II) })$$

$$\text{P.I} = y e^x \qquad \ldots (2)$$

Complete Solution is $z = f_1(y - x) + f_2(y + 2x) + e^x y$

Example 24

Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\, \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y \cos x$

☞ **SOLUTION:** R.H.S = 0, $(D^2 + DD' - 6D'^2)\, z = 0$

A.E: $m^2 + m - 6 = 0$

$$m = 2, -3$$

$$\text{CF} = f_1(y + 2x) + f_2(y - 3x) \qquad \ldots (1)$$

$$\text{P.I} = \frac{1}{(D^2 + DD' - 6D'^2)} (y \cos x)$$

$$= \frac{1}{(D + 3D')(D - 2D')} (y \cos x)$$

$$= \frac{1}{(D + 3D')} \frac{1}{(D - 2D')} (y \cos x)$$

$$= \frac{1}{(D + 3D')} \int (c - 2x) \cos x\, dx \qquad (\text{replacing } y \text{ by } c - 2x)$$

$$= \frac{1}{(D + 3D')} \left((c - 2x) \sin x - (2)(-\cos x) \right)$$

$$= \frac{1}{(D + 3D')} \left((c - 2x) \sin x - 2 \cos x \right)$$

$$= \frac{1}{(D + 3D')} (y \sin x - 2 \cos x) \qquad (\text{replacing } (c - 2x \text{ by } y)$$

$$= \int \left((c + 3x) \sin x - 2 \cos x \right) dx \qquad (\text{replacing } y \text{ by } (c + 3x)$$

$$= (c + 3x)(-\cos x) + \sin x$$

$$\text{P.I} = -y \cos x + \sin x \qquad (\text{replace } c + 3x \text{ by } y)$$

$\therefore$ complete solution is

$$z = f_1(y + 2x) + f_2(y - 3x) - y \cos x + \sin x$$

Example 25 **(Anna Uni. April/May 2003)**

Solve $(D^2 - 2DD' + D'^2)\, z = 8e^{x+2y}$

☞ **SOLUTION:** Given $(D^2 - 2DD' + D'^2)\, z = 8e^{x+2y} \qquad \ldots (1)$

$$\text{R.H.S} = 0$$

$$(D^2 - 2DD' + D'^2)\, z = 0.$$

$$\text{A.E: } m^2 - 2m + 1 = 0.$$

$$m = 1,\ 1$$

$$\text{C.F} = f_1(y + x) + x f_2(y + x) \qquad \ldots (2)$$

$$\text{P.I} = 8 \frac{e^{x+2y}}{D^2 - 2DD' + D'^2} = 8 \cdot \frac{e^{x+2y}}{1 - 4 + 4}$$

$$\text{P.I} = 8e^{x+2y} \qquad \ldots (3)$$

$\therefore$ complete solution is

$$z = f_1(y + x) + x f_2(y + x) + 8\, e^{x+2y}$$

Example 26 (Anna Uni. Oct/Nov. 1996)

Solve $(D^2+DD'-6D'^2)\,z=\cos(2x+y)+e^{x-y}$

SOLUTION: Given

$$(D^2+DD'-6D'^2)\,z=\cos(2x+y)+e^{x-y}$$

$$\text{R.H.S}=0 \Rightarrow (D^2+DD'-6D'^2)\,z=0.$$

$$\text{A.E}: m^2+m-6=0$$

$$m=-3,\ 2$$

$$\text{C.F}=f_1(y-3x)+f_2(y+2x) \quad \dots (2)$$

$$\text{P.I}_1=\frac{\cos 2x+y}{D^2+DD'-6D'^2}=\frac{x}{5}\sin(2x+y)$$

$$\text{P.I}_2=\frac{e^{x-y}}{D^2+DD'+6D'^2}=\frac{-e^{x-y}}{6} \quad \dots (3)$$

complete solution is:

$$z=\text{C.F.}+\text{P.I}_1+\text{P.I}_2$$

$$z=f_1(y-3x)+f_2(y+2x)+\frac{x}{5}\sin(2x+y)-\frac{1}{6}e^{x-y}$$

Example 27 (Anna Uni. March 1996)

Solve $(D^3-7DD'^2-6D'^3)\,z=\cos(x+2y)+x$

SOLUTION: Given $(D^3-7DD'^2-6D'^3)\,z=\cos(x+2y)+x \quad \dots (1)$

put R.H.S. = O

$$(D^3-7DD'^2-6D'^3)\,z=0$$

$$\text{A.E}: m^3-7m-6=0$$

$$m=-1,\ -2,\ 3.$$

$$\text{C.F}=f_1(y-x)+f_2(y-2x)+f_3(y+3x) \quad \dots (2)$$

$$\text{P.I}_1=\frac{\cos(x+2y)}{(D^3-7DD'^2-6D'^3)} \Rightarrow \text{Replace } D^2=-1,\ DD'=-2,\ D'^2=-4$$

$$=\frac{\cos(x+2y)}{(38D'-D)}=\frac{(38D'+D)(\cos x+2y)}{1444D'^2-D^2}$$

$$\text{P.I}_1=\frac{\sin(x+2y)}{75} \quad \dots (3)$$

$$\text{P.I}_2=\frac{1}{(D^3-7DD'^2-6D'^3)}(x)=\frac{1}{D^3\left[1-\left(\frac{7D'}{D^2}+\frac{6D'^3}{D^3}\right)\right]}(x)$$

$$=\frac{1}{D^3}\left[1-\left(\frac{7D'}{D^2}+\frac{6D'^3}{D^3}\right)\right]^{-1}(x)$$

$$\text{P.I}_2=\frac{1}{D^3}[x]=\frac{x^4}{24}$$

∴ complete solution

$$z=f_1(y-x)+f_2(y-2x)+f_3(y+3x)+\frac{\sin(x+2y)}{75}+\frac{x^4}{24}$$

Example 28 (Anna Uni. April/May 2003)

Solve $(D^2-DD'-20D'^2)\,z=e^{5x+y}+\sin(4x-y)$

SOLUTION: Given $(D^2-DD'-20D'^2)\,z=e^{5x+y}+\sin(4x-y) \quad \dots (1)$

$$\text{R.H.S}=0:$$

$$(D^2-DD'-20D'^2)\,z=0$$

$$\text{A.E}: m^2-m-20=0$$

$$m=5,\ -4.$$

$$\text{C.F}=f_1(y+5x)+f_2(y-4x) \quad \dots (2)$$

$$\text{P.I}_1=\frac{e^{5x+y}}{(D^2-DD'-20D'^2)}=\frac{xe^{5x+y}}{9} \qquad (\text{Replace } D=5,\ D'=1)$$

$$\text{P.I}_2=\frac{\sin(4x-y)}{(D^2-DD'-20D'^2}=\frac{-x\cos(4x-y)}{9}$$

(Replace $D^2=-16,\ DD'=-4,\ D'^2=-1$) ... (4)

U N I T 1

$\therefore$ complete solution is:

$$y = C.F + P.I_1 + P.I_2$$

$$y = f_1(y+5x) + f_2(y-4x) + \frac{xe^{5x+y}}{9} - \frac{x\cos(4x-y)}{9}$$

Example 29 (Anna Uni. April 2003, 2005)

Solve $(D^3 + D^2D' - DD'^2 - D'^3)\,z = e^{2x+y} + \cos(x+y)$

SOLUTION: Given:

$$(D^3 + D^2D' - DD'^2 - D'^3)\,z = e^{2x+y} + \cos(x+y) \quad \text{... (1)}$$

R.H.S: $(D^3 + D^2D' - DD'^2 - D'^3)\,z = 0.$

A.E: $m^3 + m^2 - m - 1 = 0.$

$m = 1, -1, -1$

$$C.F: = f_1(y+x) + f_2(y-x) + x f_3(y-x) \quad \text{... (2)}$$

$$P.I_1 = \frac{e^{2x+y}}{D^3 + D^2D' - DD'^2 - D'^3} = \frac{e^{2x+y}}{9} \quad \text{... (3)}$$

$$P.I_2 = \frac{\cos(x+y)}{(D-D')(D^2 + 2DD' + D'^2)}$$

$$= \frac{\cos(x+y)}{(D-D')(-1-2-1)}$$

$$\begin{bmatrix} \text{Put} \\ D^2 = -1 \\ DD' = -1 \\ D'^2 = -1 \end{bmatrix}$$

$$= \frac{-1}{4}\frac{1}{D-D'}\text{R.P}\left[e^{i(x+y)}(1)\right]$$

$$= \frac{-1}{4}\text{R.P}\,(e^{ix+iy})\frac{1}{D-D'}(e^{ox+oy})$$

$$P.I_2 = \frac{-x}{4}\cos(x+y) \quad \text{... (4)}$$

$\therefore$ solution is

$$z = CF + P.I_1 + P.I_2$$

$$z = f_1(y+x) + f_2(y-x) + x f_3(y-x) + x f_3(y-x) + \frac{e^{2x+y}}{9} - \frac{x}{4}\cos(x+y)$$

Example 30 (Anna Uni. Nov/Dec 2003)

Solve $(D^2 + 4DD' - 5D'^2)\,z = 3e^{2x-y} + \sin(x-2y)$

SOLUTION: $(D^2 + 4DD' - 5D'^2)\,z = 3e^{2x-y}$

R.H.S= 0

$(D^2 + 4DD' - 5D'^2)\,z = 0$

A.E : $m^2 + 4m - 5 = 0$

$m = 1, -5$

$$C.F = f_1(y+x) + f_2(y-5x) \quad \text{... (2)}$$

$$P.I_1 = \frac{1}{(D^2 + 4DD' - 5D'^2)}(3e^{2x-y}) = \frac{-e^{2x-y}}{3} \quad \text{... (3)}$$

$$P.I_2 = \frac{\sin(x-2y)}{D^2 + 4DD' - 5D'^2}$$

$$P.I_2 = \frac{\sin(x-2y)}{-1+8+20} = \frac{\sin(x-2y)}{27} \quad \text{... (4)}$$

$$\begin{bmatrix} \text{Put} \\ D^2 = -a^2 = -1 \\ DD' = -ab = 2 \\ D'^2 = -4 \end{bmatrix}$$

$\therefore$ complete solution is

$$z = C.F + P.I_1 + P.I_2$$

$$z = f_1(y-5x) + f_2(y+x) - \frac{e^{2x-y}}{3} + \frac{\sin(x-2y)}{27}$$

Example 31 (Anna Uni. April/May 2004)

Solve $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 8\sin(x+3y)$

SOLUTION: Given $(D^2 - 3DD' + 2D'^2)\,z = 8\sin(x+3y)$

R.H.S = 0.

$m^2 - 3m + 2 = 0,\ m = 2, 1$

$$C.F = f_1(y+2x) + f_2(y+x)$$

$$P.I = 8\,\frac{1}{(D^2 - 3DD' + 2D'^2)}\sin(x+3y)$$

Replace : $D^2 = -1$, $DD' = -3$, $DD' = -3$, $D'^2 = -9$

$$P.I = \frac{8\sin(x+3y)}{-10}$$

$\therefore$ solution is:

$$z = f_1(y+x) + f_2(y+2x) - \frac{4}{5}\sin(x+3y)$$

EXERCISE 1.5

Solve the following P.D.E

1. $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 Z}{\partial x \partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = e^{x+y}$
2. $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$
3. $(D^3 - 4D^2D' + 4DD'^2)\,Z = 6\sin(3x+2y)$
4. $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = x+y$
5. $(Dx + Dy)^2\,Z = e^{x-y}$
6. $(D^3 + D^2D' - DD'^2 - D'^3)\,z = e^{2x+y} + \cos(x+y)$
7. $(D^2 + D'^2)\,z = \dfrac{8}{x^2+y^2}$
8. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(3x+y)$
9. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$
10. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = x+y$

11. $\dfrac{\partial^2 z}{\partial x^2} - a^2\dfrac{\partial^2 z}{\partial y^2} = x^2$
12. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 3x^2y$
13. $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x-2y)$
14. $(D - 2D')(D - D')^2\,z = e^{x+y}$
15. $(D^2 - 2DD' + D'^2)\,z = \sin(x-2y) + e^x(x+2y)$
16. $(2D^2 - 2DD' - D'^2)\,z = 2e^{3y} + e^{x+y} + y^2$
17. $2\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^2 z}{\partial y^2} = 1$
18. $(D^2 - 3DD' + D'^2)\,z = \sin x \cos y$
19. $(D^3 - 7DD'^2 - 6D'^3)\,z = x^2y + \sin(x+2y)$
20. $(D^3 + D^2D' - DD'^2 - D'^3)\,z = e^x \cos 2y$

UNIT 1

NOTE

The notation for partial derivatives are

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2}\ s=\frac{\partial^2 z}{\partial x\,\partial y}\quad t=\frac{\partial^2 z}{\partial y^2}$$

Definition: *A partial differential equation is an equation, which involves partial derivatives such as* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$ ... etc. *Which can be simply denoted as (P.D.E). Which can also be denoted as* $F(x, y, t, z \ldots U_x, U_y, \ldots U_{xt} \ldots) = 0$

Example: $4\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}+z=x^2$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$$

(i) The order of the P.D.E is the order of the highest partial derivative occur in it.

(ii) The degree of the P.D.E is the degree of the highest order derivative occur in it.

Example: $\frac{\partial^3 z}{\partial x^3}+4\frac{\partial^2 z}{\partial x^2}+6\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=x^2+y^2$

order = 3; *degree* = 1

(iii) The solution of a PDE is a function of independent variables, which satisfies the P.D.E.

(iv) The general solution of the P.D.E contains arbitrary constants, or arbitrary functions or both.

(v) If the number of constants to be eliminated is equal to the number of independent variables, it will produce P.D.E of first order.

(vi) If the number of constants more than number of independent variables, if will produce P.D.E of higher order.

(vii) The order of the P.D.E is equal to number of arbitrary functions to be eliminated

INTRODUCTION

(i) Consider a Single variable function $y = f(x)$, for example: $y = 3x^2 + 2x + 11$, $y = \sin x$, $y = \cos x + x^2$, $y = \log x \dots etc.$ Then $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ... are said to be differential coefficients or differentials. An equation which involves the above differentials are said to be ordinary differential equations (O.D.E).

Example: $\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + y = \sin x$

$$(x^2 D^2 + 4xD + 3)\, y = e^x \dots etc.$$

(ii) Consider a multiple variable or several variable function ((function having two or more independent variable), $z = f(x, x, \dots x)$, and $u = f(x, y, z \dots)$) which are common occurrence in so many engineering application problems; particularly in Theory of Vibration, Heat transfer, Fluid mechanics, Thermodynamics....etc.

Example: $u = \sin(2x + 4y - 5z)$

$$z = e^{2x - 3y} \dots \text{etc}$$

Then $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\, \partial y}$, $\frac{\partial^2 z}{\partial y^2}$

are said to be partial derivatives (or) partial differential coefficients.

1.1.1 FORMATION OF P.D.E BY ELIMINATING ARBITRARY CONSTANTS

UNIT 1

SOLVED EXAMPLES

Example 1

Form the P.D.E from $z = ax + by + \sqrt{a^2 + b^2}$

☞ **SOLUTION:** Given $z = ax + by + \sqrt{a^2 + b^2}$ ... (1)

differentiate w.r.t x: $\frac{\partial z}{\partial x} = p = a + 0 + 0$

$$p = a$$ ... (3)

differentiate w.r.t y: $\frac{\partial z}{\partial y}$

$$q = b$$

substitute a and b in (1)

$$z = px + qy + \sqrt{p^2 + q^2}$$

This is the P.D.E of first order

Example 2 (Anna Uni. Nov/Dec 2004)

Form the PD from $z = (x^2 + a^2)(y^2 + b^2)$

☞ **SOLUTION:** Given $z = (x^2 + a^2)(y^2 + b^2)$ ... (1)

differentiate w.r.t x : $p = (2x)(y^2 + b^2)$ ... (2)

differentiate w.r.t y : $q = (2y)(x^2 + a^2)$ ... (3)

substitute (2) and (3) in (1)

$$z = \frac{p}{2x} \cdot \frac{q}{2y} \Rightarrow 4xyz = pq$$

This is the P.D.E. of first order.

Example 3 (Anna Uni. Nov/Dec. 2003)

From the P.D.E from $(x - a)^2 + (y - b)^2 + z^2 = 1$

☞ **SOLUTION:** Given $(x - a)^2 + (y - b)^2 + z^2 = 1$ ... (1)

UNIT-4 (Differential Equations)

Def. A equation which involves differential co-efficient is called a differential equation.

Differential equations

Ordinary Differential Equations (ODEs)	Partial Differential Equations (PDEs)
A differential equation involving derivatives with respect to a single independent variable is called an ODE.	A differential equation involving partial derivatives with respect to more than one independent variable is called a PDE.
eg: 1) $\frac{dy}{dx} = 2\sin x$, 2) $\frac{d^2y}{dx^2} + m^2 y = 0$	eg: 1) $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2}$, 2) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

Def. ★ The order of a diff eqn is the order of the highest differential coefficient present in the eqn.

★ The degree of a diff eqn is the degree of the highest derivative after removing the radicals and fractions.

eg: 1) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

here the order is 2 and degree is 2.

2) $\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} + \frac{y}{a} = \sin x$.

here the order is 2 and degree is 1.

Solution of a differential equation.

★ Any relation between dependent and independent variables which, when substituted in the diff eqn, reduce it to an identity [(or) satisfies] is called a solution (primitive) of the differential eqn. [free from derivative]

★ The solution, in which the number of arbitrary constants occurring is the same as the order of the eqn is called the general solution or complete primitive.

★ Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a Particular Integral.

★ General Solution = Complementary fn + Particular Integral
(ie) $(y = CF + PI)$

Linear differential Equations of Second and higher order with constant coefficients

★ The general form of linear eqn of 2^{nd} order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (1)}$$

where ~~P, Q, R~~ P, Q are constants and R is a fn of x or a constant.

Let D be the differential operator.

Then $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$.

$\therefore$ ① $\Rightarrow$ $D^2y + PDy + Qy = R$ (or) $(D^2+PD+Q)y = R$.

* The general form of a linear differential eqn of the n^{th} order with constant coefficients is

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = X \quad \text{②}$$

where $a_0 (\neq 0), a_1, a_2, \dots, a_n$ are constants and X is a function of x.

And by using the differential operator

② becomes,

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1}D + a_n)y = X$$

(or) $f(D)\, y = X$, where $f(D)$ is a polynomial in D.

* Homogenous eqn if $f(D)y = 0$. (Sol: $y = CF$).
* Non-homogenous eqn if $f(D)\, y = X$ where $X \neq 0$ (Sol: $y = CF + PI$)

To find Complementary function.

Let $f(D)\, y = 0$ ——③.

First find the auxiliary eqn (A.E).

(i.e) $f(m) = 0$ in ③ and later the roots are found from this eqn

Case (i): [The roots of A.E are real & distinct.]

C.F of ③ $= c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$.

where $m_1, m_2, \dots m_n$ are the roots of the A.E & $c_1, c_2, \dots c_n$ are ~~the~~ arbitrary constants.

Case (ii): [The A.E has got real roots, some of which are equal.]

(i.e) If $m_1 = m_2 = m_3$, then

C.F of ③ $= y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$.

Case (iii): [Two roots of the A.E are complex]

let $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$.

Then $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

Case (iv): [If $m_1 = m_3 = \alpha + i\beta$ & $m_2 = m_4 = \alpha - i\beta$]

$\therefore y = e^{\alpha x}\left[(c_1 x + c_2)\cos \beta x + (c_3 x + c_4)\sin \beta x\right] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$.

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To find Particular Integral (P.I)

Let $f(D)\,y = X$ ——— ④

$$P.I = \frac{1}{f(D)} X \quad \text{[(ie) P.I depends on X]}$$

Rule 1: when $X = e^{ax}$, where a is a constant.

* $P.I = \frac{1}{f(D)} e^{ax}$, Replace D by a if $f(a) \neq 0$.

* if $f(a) = 0$ (and) $(D-a)^r$ is a factor of

$f(D) = (D-a)^r \phi(D)$ where $\phi(a) \neq 0$.

Then $P.I = \frac{1}{\phi(a)} \cdot \frac{x^r}{r!} e^{ax}$.

Rule 2: when $X = \sin ax$ (or) $\cos ax$, where a is constant.

* Replace D^2 by $-a^2$ in P.I provided $\phi(-a^2) \neq 0$.

* if $\phi(-a^2) = 0$ then.

$\left[e^{iax} = \cos ax + i\sin ax\right]$

$$\frac{1}{D^2+a^2} \sin ax = \frac{1}{D^2+a^2} \times \text{Imaginary part of } e^{iax}$$

$$= \cdots\cdots = -\frac{x\cos ax}{2a}.$$

Rule 3: when $X = x^m$, $m > 0$.

$$P.I = \frac{1}{f(D)} x^m = \{f(D)\}^{-1} x^m.$$

And use binomial expression for $\{f(D)\}^{-1}$.

note:

1) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
2) $(1+x)^{-1} = 1 - x + x^2 - \cdots\cdots$
3) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
4) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots\cdots$
5) $(a+x)^n = a^n + n a^{n-1} x^{1} + \frac{n(n-1)}{2!} a^{n-2} x^2 + \cdots + x^n$.

Rule 4: If $X = e^{ax} V$, where V is any fn. of x.

$$P.I = \frac{1}{f(D)} . e^{ax} V = e^{ax} . \underbrace{\frac{1}{f(D+a)}}_{\text{evaluated by rule ③ or ④}} . V$$

Rule 5: If $X = x . V(x)$

$$P.I = x \frac{1}{f(D)} V(x) - \frac{f'(D)}{\{f(D)\}^2} V(x).$$

Rule 6: If X is any other fn of x.

$$P.I = \frac{1}{f(D)} X = \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} X$$

Then use partial fractions method and solve them by using the above rules.

Problems.

1) If $y = x^2$ find $\frac{d^2y}{dx^2}$.

Sol. $y = x^2$,

$\frac{dy}{dx} = 2x$.

$\frac{d^2y}{dx^2} = 2$.

2) Find the Solution of $(D^2 - 8D + 16)y = 0$.

Sol. The auxiliary equation is $m^2 - 8m + 16 = 0$.

$\Rightarrow m^2 - 4m - 4m + 16 = 0$.

$\Rightarrow m(m-4) - 4(m-4) = 0$.

$\Rightarrow (m-4)^2 = 0$.

$\Rightarrow m = 4, 4$.

$\therefore y = (A + Bx)e^{4x}$ //.

3). Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$ $\Rightarrow (D^2y - 5Dy + 4y) = 0$ $\Rightarrow (D^2 - 5D + 4)y = 0$.

Sol.: The A.E is $m^2 - 5m + 4 = 0$.

$\Rightarrow (m-4)(m-1) = 0$

$\Rightarrow m = 4, m = 1$.

$\therefore y = C.F = Ae^{x} + Be^{4x}$.

Ae^{0x}

4) Solve $(D^2 - 1)y = 0$.

Sol A.E is $m^2 - 1 = 0$. $\Rightarrow m^2 = 1$.

$\Rightarrow m = \pm 1$

$\therefore y = Ae^{-x} + Be^{x}$.

5) Solve $(D^2 + D + 1)y = 0$.

Sol. A.E is $m^2 + m + 1 = 0$.

$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{3}\, i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore y = e^{-\frac{x}{2}}\left[A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x\right]$

H.W 6) $(D^2 - 4D + 4)y = 0$.

7) $(D^2 + 6D + 8)y = 0$.

8) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$. $\rightarrow m = 5, 2$.

9) $\frac{d^4y}{dx^4} - y = 0$. $\Rightarrow (D^4 - 1)y = 0$.

Sol. A.E is $m^4 - 1^4 = 0$. ~~$\Rightarrow m^4 - 1$~~

$\Rightarrow (m^2)^2 - (1^2)^2 = 0$.

$a^2 - b^2 = (a+b)(a-b)$

$\Rightarrow (m^2 + 1^2)(m^2 - 1^2) = 0$.

$\Rightarrow (m^2 + 1)(m+1)(m-1) = 0$.

$\Rightarrow m = 1, -1, \pm i$

$m^2 = -1$

$m = \pm\sqrt{-1}$

$= \pm i$

$\therefore y = Ae^{x} + Be^{-x} + e^{0x}(C \cos x + D \sin x)$

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Solve $(D^2+1)y = e^{2x} \Rightarrow y = \frac{1}{(D^2+1)} \cdot e$

A.E is $m^2+1=0$.

$m^2=-1 \Rightarrow m = \pm i$.

$y = C.F = A\cos x + B\sin x$.

$a=2$

$P.I = \frac{1}{(D^2+1)} \cdot e^{2x}$

$= \frac{1}{(4+1)} e^{2x} = \frac{e^{2x}}{5}$.

$\therefore$ The general sol, $y = C.F + P.I$

$= A\cos x + B\sin x + \frac{e^{2x}}{5}$.

11) Solve $(D^2+4D+13)y = 2e^{-x}$.

Sol A.E is $m^2+4m+13=0$.

$m = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$

$= \frac{-4 \pm 6i}{2} = -2 \pm 3i$.

$[\alpha = -2 \ \& \ \beta = 3]$

$C.F = e^{-2x}[A\cos 3x + B\sin 3x]$.

$P.I = \frac{1}{(D^2+4D+13)} \cdot 2e^{-x}$

$= \frac{2}{(-1)^2+4(-1)+13} \cdot e^{-x}$

$= \frac{2}{10} e^{-x} = \frac{e^{-x}}{5}$.

$\therefore y = C.F + P.I = e^{-2x}[A\cos 3x + B\sin 3x] + \frac{e^{-x}}{5}$

12) Solve $(3D^2+D+14)y = 13e^{2x}$.

13) $(D^2-D-2)y = e^{2x}+e^x$.

Sol. A.E is $m^2-m-2=0$.

$\Rightarrow m^2-2m+m-2=0$

$\Rightarrow (m-2)(m+1)=0$.

$\Rightarrow m = 2, -1$.

$C.F = Ae^{-x} + Be^{2x}$.

$P.I = \frac{1}{(D^2-D-2)} \cdot (e^{2x}+e^x)$

$= \frac{1}{D^2-D-2} \cdot e^{2x} + \frac{1}{(D^2-D-2)} \cdot e^x$.

$= \frac{1}{(D-2)(D+1)} \cdot e^{2x} + \frac{1}{(1-1-2)} \cdot e^x$

$= \frac{1}{3} \cdot \frac{1}{(D-2)} \cdot e^{2x} + \frac{e^x}{(-2)}$

$= \frac{1}{3} x e^{2x} - \frac{e^x}{2}$.

$\therefore y = Ae^{-x} + Be^{2x} + \frac{1}{3} x e^{2x} - \frac{e^x}{2}$ //.

12) $(3D^2+D-14)y = 13e^{2x}$

Sol. $y = Ae^{2x} + Be^{-7x/3} + xe^{2x}$

14) $(D^2+4)y = \sin 3x.$

Sol A.E is $m^2+4=0. \Rightarrow m^2=-4 \Rightarrow m=\pm 2i$

$\sin ax, \cos ax$

C.F $= A\cos 2x + B\sin 2x.$

$$P.I = \frac{1}{(D^2+4)} \cdot \sin 3x$$

here $a=3$. replace D^2 by $-a^2$ (i.e $D^2=-9$)

$$= \frac{1}{(-9+4)} \cdot \sin 3x$$

$$= -\frac{\sin 3x}{5}.$$

$$\therefore y = A\cos 2x + B\sin 2x - \frac{\sin 3x}{5} //.$$

(x) 15) $(D^2+4D+4)y = 4\sin 2x.$

Sol. A.E is $m^2+4m+4=0 \Rightarrow (m+2)^2=0$

$\Rightarrow m=-2,-2.$

$\therefore$ C.F $= (A+Bx)e^{-2x}.$

$$P.I = \frac{1}{(D^2+4D+4)} \cdot 4\sin 2x$$

$D^2=-4.$

$\frac{1}{D} = \int.$

$$= 4 \cdot \frac{1}{(-4+4D+4)} \cdot \sin 2x$$

$$= \frac{4}{4}\int \sin 2x \, dx$$

$$= -\frac{\cos 2x}{2}$$

$$\therefore y = (A+Bx)e^{-2x} - \frac{\cos 2x}{2}.$$

16) Solve $(D^2+16)y = e^{-3x} + \cos 4x.$

Sol. A.E is $m^2+16=0 \Rightarrow m^2=-16 \Rightarrow m=\pm 4i.$

C.F $= A\cos 4x + B\sin 4x.$

$$P.I = \frac{1}{(D^2+16)}(e^{-3x}+\cos 4x)$$

$$= \frac{1}{(D^2+16)}e^{-3x} + \frac{1}{(D^2+16)}\cos 4x.$$

$$= \frac{e^{-3x}}{25} + \frac{1}{(D^2+16)} \text{Realpart of } e^{i4x}$$

here $D^2=-16.$

$e^{iax} = \cos ax + i\sin bx$

$$= \frac{e^{-3x}}{25} + \text{Real part of } \frac{1}{(D+4i)(D-4i)} \cdot e^{i4x}$$

$(D+4i)(D-4i) = D^2-4iD+4iD+16$

$i^2=-1$

$$= \frac{e^{-3x}}{25} + \text{Realpart of } \frac{1}{8i} \times \frac{i}{i}\, x e^{i4x}$$

$$= \frac{e^{-3x}}{25} + \text{Realpart of } \left(-\frac{i}{8}\right) x\,[\cos 4x + i\sin 4x]$$

$$= \frac{e^{-3x}}{25} + \text{Real part of} \left[-\frac{ix}{8}\cos 4x + \frac{x}{8}\sin 4x\right].$$

$$= \frac{e^{-3x}}{25} + \frac{x}{8}\sin 4x.$$

$$\therefore y = A\cos 4x + B\sin 4x + \frac{e^{-3x}}{25} + \frac{x}{8}\sin 4x //.$$

[x form]

17) Solve $(D^2-1)y = x.$

Sol A.E is $m^2-1=0 \Rightarrow m=\pm 1.$

$$C.F = Ae^{-x} + Be^{x}.$$

$$P.I = \frac{1}{(D^2-1)}x.$$

$$= \frac{1}{(-1)(1-D^2)}x.$$

$$= -(1-D^2)^{-1}x$$

$$= -[1+D^2+D^4+\cdots]x$$

$$= -[x + D^2x + D^4x + \cdots]$$

$$= -x$$

$(1-x)^{-1} = 1+x+x^2+\cdots$

x

$Dx = 1.$

$D^2x = 0.$

$$\therefore y = Ae^{-x} + Be^{x} - x //.$$

18) Solve. $(D^3+8)y = x^4+2x+1.$

Sol. A.E is $m^3+8=0.$

	m^3	m^2	m	c
$m=-2$	1	0	0	8
	0	-2	4	-8
	1	-2	4	$\boxed{0}$

$$\therefore (m+2)(m^2-2m+4) = 0.$$

$\Rightarrow m=-2$ & $m^2-2m+4=0.$

(ie) $m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2\pm\sqrt{-12}}{2}$

$= 1 \pm \sqrt{3}\,i.$

$\therefore m = -2,\ \overset{\alpha}{1} \pm \overset{\beta}{\sqrt{3}}\,i.$

$$C.F = Ae^{-2x} + e^{x}[B\cos\sqrt{3}x + C\sin\sqrt{3}x].$$

$$P.I = \frac{1}{(D^3+8)}(x^4+2x+1).$$

$$= \frac{1}{8\left(1+\frac{D^3}{8}\right)}(x^4+2x+1)$$

$$= \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+2x+1).$$

$$= \frac{1}{8}\left[1 - \frac{D^3}{8} + \frac{D^6}{8^2} - \cdots\right](x^4+2x+1)$$

$(1+x)^{-1} = 1-x+x^2-x^3\cdots$

x^4+2x+1

$D(x^4+2x+1) = 4x^3+2$

$D^2(x^4+2x+1) = 12x^2$

$D^3(x^4+2x+1) = 24x.$

$$= \frac{1}{8}\left[(x^4+2x+1) - \frac{D^3}{8}(x^4+2x+1) + \cdots\right]$$

$$= \frac{1}{8}\left[x^4+2x+1 - \frac{\overset{3}{\cancel{24}}x}{\cancel{8}}\right]$$

$$= \frac{1}{8}[x^4 - x + 1]$$

$$\therefore y = C.F + P.I$$

$$= Ae^{-2x} + e^{x}[B\cos\sqrt{3}x + C\sin\sqrt{3}x] + \frac{1}{8}[x^4-x+1].$$

19) Solve $[D^4 - 2D^3 + D^2]y = x^3.$

Sol. A.E is $m^4-2m^3+m^2 = 0.$

$\Rightarrow m^2(m^2-2m+1) = 0.$

$\Rightarrow m^2(m-1)^2 = 0.$

$$\begin{array}{ccc} & 1 & \\ -1 & & -1 \end{array}$$

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$\Rightarrow m = 0, 0, 1, 1.$

$$C.F = (A+Bx)e^{0x} + (C+Dx)e^{x}$$

$$= A + Bx + (C+Dx)e^{x}.$$

$$P.I = \frac{1}{(D^4 - 2D^3 + D^2)} \cdot x^3$$

$$= \frac{1}{D^2[1 + (D^2 - 2D)]} x^3$$

$$= \frac{1}{D^2} \cdot [1 + (D^2-2D)]^{-1} x^3$$

$$= \frac{1}{D^2}\left[1 - (D^2-2D) + (D^2-2D)^2 - (D^2-2D)^3 \ldots\right] x^3$$

$$= \frac{1}{D^2}\left[1 - D^2 + 2D + (D^4 - 4D^3 + 4D^2) - (D^6 - 6D^5 + 6D^4 - 8D^3) + \ldots\right] x^3$$

$$= \frac{1}{D^2}\left[x^3 - D^2(x^3) + 2D(x^3) + D^4(x^3) - 4D^3(x^3) + 4D^2(x^3) - D^6(x^3) + 6D^5(x^3) - 6D^4(x^3) + 8D^3(x^3) + \ldots\right]$$

$$= \frac{1}{D^2}\left[x^3 - 6x + 6x^2 + 0 - 24 + 24x - 0 + 0 - 0 + 48\right]$$

$$= \frac{1}{D^2}\left[x^3 + 18x + 6x^2 + 24\right].$$

x^3

$D(x^3) = 3x^2$

$D^2(x^3) = 3D(x^2) = 6x$

$D^3(x^3) = 6.$

$D^4(x^3) = 0.$

$$= \frac{1}{D}\int (x^3 + 18x + 6x^2 + 24)\,dx$$

$$= \frac{1}{D}\left[\frac{x^4}{4} + \frac{\cancel{18}^{9}x^2}{\cancel{2}_{1}} + \frac{\cancel{6}^{2}x^3}{\cancel{3}_{1}} + 24x\right].$$

$$= \int\left[\frac{x^4}{4} + 9x^2 + 2x^3 + 24x\right]dx.$$

$$= \frac{x^5}{20} + \frac{\cancel{9}^{3}x^3}{\cancel{3}_{1}} + \frac{\cancel{2}x^4}{\cancel{4}_{2}} + \frac{\cancel{24}^{12}x^2}{\cancel{2}}$$

$$= \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2.$$

$$\therefore y = A + Bx + (C+Dx)e^{x} + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2.$$

HW

20) $(D^2 - 2D + 1)y = (e^x + 1)^2.$

21) $(D^2 - 2D + 1)y = 2e^{-x} + x^2 + 3.$

[$e^{ax} \times$ form]. $PI = e^{ax} \frac{1}{f(D+a)} \cdot X$

22). Find the P.I of $(D^2+1)\,y = xe^{x}.$

Sol

$$P.I = \frac{1}{(D^2+1)} \cdot xe^{x}.$$

$$= e^{x} \cdot \frac{1}{((D+1)^2+1)} \cdot x$$

$$= e^{x} \cdot \frac{1}{[1 + D^2 + 2D + 1]} x = e^{x} \frac{1}{[2 + D^2 + 2D]} x$$

$$= \frac{e^{x}}{2} \cdot \frac{1}{\left[1 + \left(\frac{D^2+2D}{2}\right)\right]} x.$$

x

$D(x) = 1$

$D^2(x) = 0$

$$= \frac{e^x}{2}\left[1 + \left(\frac{D^2+2D}{2}\right)\right]^{-1} x$$

$$= \frac{e^x}{2}\left[1 - \left(\frac{D^2+2D}{2}\right) + \left(\frac{D^2+2D}{2}\right)^2 - \dots\right]x$$

$$= \frac{e^x}{2}\left[x - \frac{D^2(x)}{2} - D(x)\right]$$

$$= \frac{e^x}{2}\left[x - 1\right] //$$

23) Solve $(D+2)^2 y = e^{-2x}\sin x$.

Sol. A.E is $(m+2)^2 = 0 \Rightarrow m = -2, -2$.

$\therefore C.F = (A+Bx)e^{-2x}$.

$$P.I = \frac{1}{(D+2)^2} \cdot e^{-2x}\sin x$$

$$= e^{-2x}\frac{1}{((D-2)+2)^2}\sin x$$

$$= e^{-2x} \cdot \frac{1}{D^2}\sin x \qquad \begin{array}{|l} a = 1 \\ D^2 = -a^2 \\ \quad = -1. \end{array}$$

$$= -e^{-2x}\sin x.$$

$$y = (A+Bx)e^{-2x} - e^{-2x}\sin x //$$

Applications.

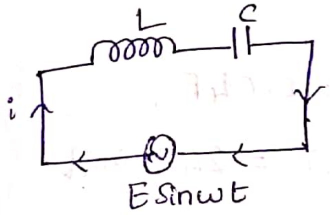
$\Rightarrow$ ~~LC circuit~~. $\quad i = \frac{dq}{dt}$ (or $i = q'$) (or) $q = \int i\, dt$

Elements in an RLC circuit.

Name	Symbol	notation	unit	Voltage Drop.
ohm's resistor	-WW-	R ohm's resistance.	ohms (Ω)	$RI \to Ri$
Inductor	-0000-	L Inductance	henrys (H)	$L\frac{dI}{dt}$
Capacitor	-||-	C capacitance	farads (F)	Q/C (or q/c)

Q (or) q $\to$ quantity of electricity (coulombs). (or capacitor charge).

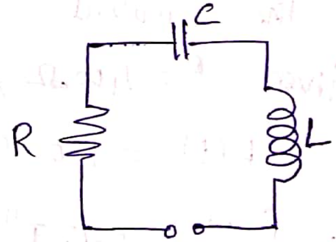
LC-circuit $V_R + V_L + V_C = V_S$.



$$L\frac{d^2I}{dt^2} + \frac{I}{c} = E'$$

$$L\frac{d^2q}{dt^2} + \frac{q}{c} = E_0\sin\omega t \quad (= E)$$

RLC-circuit

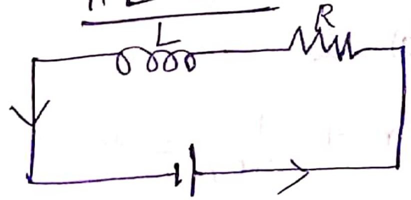


$E(t) = E_0\sin\omega t$.

$$LI'' + RI' + \frac{I}{c} = E'(t)$$

(or) $LQ'' + R.Q' + \frac{Q}{C} = E$ where $I = Q'$.

RL-circuit.



$$Ri + L\frac{di}{dt} = E \quad \text{(Kirchhoff's voltage law)}$$

$$\Rightarrow \frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

CR-circuit

By voltage law,

$$Ri + \frac{q}{c} = E$$

$$\Rightarrow R\frac{dq}{dt} + \frac{q}{c} = E.$$

at $q(0)=0$

(3) $\Rightarrow \quad 0 = -\dfrac{A}{1500.626} - \dfrac{B}{1832.707} - 0 - \dfrac{0.0092}{2}$.

$\Rightarrow 0.0007A + 0.0005B + 0.0046 = 0$.

$\Rightarrow 0.0007A + 0.0005B = -0.0046$ — (5).

(4) $\Rightarrow \quad A + B = 0.2997$

(4) $\times 0.0007 \Rightarrow 0.0007A + 0.0007B = 0.0002$ — (6).

(5) − (6).

$$\begin{aligned} \Rightarrow \quad & \cancel{0.0007A} + 0.0005B = -0.0046 \\ (-)\ & \cancel{0.0007A} \overset{(-)}{+} 0.0007B \overset{(-)}{=} 0.0002 \\ \hline & -0.0002B = -0.0048 \end{aligned}$$

$\Rightarrow B = \dfrac{0.0048}{0.0002} = 24.$

$\therefore$ (4) $\Rightarrow \quad A = 0.2997 - 24. = -23.7$

$\therefore$ (3) $\Rightarrow \quad I(t) = -23.7e^{-1500.626t} + 24e^{-1832.707t} - 0.2997\cos 2t + 0.0092\sin 2t$

) Find the current in the RLC circuit, assuming zero initial current and capacitor charge with the following data $R = 450\Omega$, $L = 0.95H$, $C = 0.07F$, $E(t) = e^{-t}\sin^2 3t$ V.

Sol. Given : $R = 450\Omega$, $L = 0.95H$, $C = 0.07F$,

$E(t) = e^{-t}\sin^2 3t\,V. = e^{-t}\left(\dfrac{1-\cos 6t}{2}\right)$

$= \frac{1}{2}(e^{-t} - e^{-t}\cos 6t)$

$E'(t) = \dfrac{-e^{-t}}{2} - \dfrac{1}{2}\left[6e^{-t}\sin 6t + (-1)e^{-t}\cos 6t\right]$

$= \dfrac{-e^{-t}}{2} + 3e^{-t}\sin 6t + \dfrac{1}{2}e^{-t}\cos 6t$.

$1 - \cos 2t = 2\sin^2 t$

$\Rightarrow \sin^2 t = \dfrac{1-\cos 2t}{2}$

$\Rightarrow \sin^2 3t = \dfrac{1-\cos 6t}{2}$

The differential eqn for the given problem is

$0.95I'' + 450I' + \dfrac{I}{0.07} = \dfrac{-e^{-t}}{2} + 3e^{-t}\sin 6t + \frac{1}{2}e^{-t}\cos 6t$

with $I(0)=0$ & $q(0)=0$ as the initial conditions — (1)

A.E of (1) is $\left(0.95m^2 + 450m + \dfrac{1}{0.07}\right) = 0$.

$m = \dfrac{-450 \pm \sqrt{202500 - 54.2857}}{1.9}$

$= -0.03175, \ -473.652$

C.F $= Ae^{-0.03175t} + Be^{-473.652t}$.

P.I $= \dfrac{1}{\left(0.95D^2 + 450D + \frac{1}{0.07}\right)}\left(\dfrac{-e^{-t}}{2}\right) + \dfrac{3}{\left(0.95D^2 + 450D + \frac{1}{0.07}\right)}e^{-t}\sin 6t$

$+ \dfrac{1}{2\left(0.95D^2 + 450D + \frac{1}{0.07}\right)}e^{-t}\cos 6t$

$$P.I = 0.00115e^{-t} - 0.000189e^{-t}\sin 6t - 0.00108e^{-t}\cos 6t + 0.00018050e^{-t}\sin 6t - 0.0000315e^{-t}\cos 6t$$

$$= 0.00115e^{-t} - 0.0000085e^{-t}\sin 6t - 0.001111e^{-t}\cos 6t$$

$$y = C.F + P.I$$

Find the current in the RLC circuit, assuming zero initial current and capacitor charge with the following data, $R = 400\,\Omega$, $L = 0.12\,H$, $C = 0.04\,F$, $E(t) = 120 \sin 2t\ V$.

Sol. The differential equation for an RLC circuit is $LI'' + RI' + \frac{I}{C} = E'(t)$

Given $R = 400\,\Omega$, $L = 0.12\,H$, $C = 0.04\,F$,

$E(t) = 120 \sin 2t\ V$,

$$E'(t) = 120 \cos 2t \times 2 = 240 \cos 2t$$

∴ The diff eqn becomes

$$0.12\, I'' + 400\, I' + \frac{I}{0.04} = 240 \cos 2t \quad \text{①}$$

with $I(0) = 0$ & $Q(0) = 0$ as initial conditions.

$$\text{①} \Rightarrow \quad 0.12\, I'' + 400\, I' + 25 I = 240 \cos 2t.$$

$$\Rightarrow \left(0.12 D^2 + 400 D + 25\right) I = 240 \cos 2t$$

A.E is $\quad 0.12 m^2 + 400 m + 25 = 0.$

$$\Rightarrow m = \frac{-400 \pm \sqrt{400^2 - 4(0.12)25}}{2(0.12)}$$

$$m = \frac{-400 \pm \sqrt{160000 - 12}}{0.24}$$

$$= \frac{-400 \pm 399.985}{0.24}$$

$$= -0.0625 \,,\; -3333.271$$

$$C.F = A e^{-0.0625t} + B e^{-3333.271t}$$

$$P.I = \frac{240}{(0.12D^2 + 400D + 25)} \cos 2t$$

here. $a = 2$.
Replace $D^2 = -a^2 = -4$.

$$= \frac{240}{((0.12)(-4) + 400D + 25)} \cos 2t$$

$$= \frac{240}{400D + 24.52} \cos 2t.$$

$$= \frac{240\,(400D - 24.52)}{400(400)D^2 - (24.52)^2} \cos 2t$$

$$= \frac{96000D(\cos 2t) - 5884.8 \cos 2t}{160000(-4) - 601.2304}$$

$$= \frac{-96000 \sin 2t \times 2 - 5884.8 \cos 2t}{-640601.2304}$$

$$P.I. = 0.299718 \sin 2t + 0.009186 \cos 2t$$

$$\therefore I(t) = C.F + P.I$$

$$= A e^{-0.0625t} + B e^{-3333.271t} + 0.299718 \sin 2t + 0.009186 \cos 2t \quad \text{—— (2)}$$

Gn that $I(0) = 0$.

$$\therefore (2) \Rightarrow 0 = A + B + 0.009186 \qquad (\because \sin 0 = 0,\ \cos 0 = 1\ \&\ e^0 = 1)$$

$$\Rightarrow A + B = -0.009186 \quad \text{—— (3)}$$

W.K.T, $Q(t) = \int I(t)\,dt$.

$$\therefore Q(t) = \int \left[A e^{-0.0625t} + B e^{-3333.271t} + 0.299718 \sin 2t + 0.009186 \cos 2t\right] dt \quad \text{—— } \cancel{(4)}$$

~~at Q(0) = 0~~, ~~(4) ⇒~~

$$\cancel{0 = A}$$

$$\cancel{Q(t)} = \cancel{\frac{A e^{-0.0625t}}{-0.0625}} - \cancel{\frac{B e^{-3333.271t}}{3333.271}}$$

$$\cancel{+ 0.599436 \cos 2t}$$

$$Q(t) = \frac{A e^{-0.0625t}}{-0.0625} - \frac{B e^{-3333.271t}}{3333.271}$$

$$+\frac{0.299718 \cos 2t}{2} + \frac{0.009186 \sin 2t}{2}$$

at $Q(0) = 0$, the above eqn becomes

$$0 = -\frac{A}{0.0625} - \frac{B}{3333.271} - \frac{0.299718}{2}$$

$$\Rightarrow 16A + 0.0003B = -0.149859 \quad (4)$$

(3) $\times 16$ $-$ (4)

$$\Rightarrow \cancel{16A} + 16B = -0.146976$$

$$(-)\ \cancel{16A} \overset{(-)}{+} 0.0003B = \overset{(+)}{-}0.149859$$

$$15.9997B = 0.002883$$

$$\therefore B = 0.00018$$

$\therefore$ (3) $\Rightarrow$ $A = -0.009186 - 0.00018$

$$= -0.0094$$

$\therefore$ (2) $\Rightarrow$ $I(t) = -0.0094\, e^{-0.0625t} + 0.00018\, e^{-3333.2t}$

$$+ 0.299718 \sin 2t + 0.009186 \cos 2t$$

CHAPTER

39

INFINITE SERIES

39.1 SEQUENCE

A *sequence* is a succession of numbers or terms formed according to some definite rule. The nth term in a sequence is denoted by u_n.

For example, if $u_n = 2n + 1$. $\{3, 5, 7, \ldots, 2n+1\}$

~~By giving different values of n in u_n, we get different terms of the sequence.~~

~~Thus, $u_1 = 3$, $u_2 = 5$, $u_3 = 7$, ...~~

A sequence having unlimited number of terms is known as an *infinite sequence*.

39.2 LIMIT

~~If a sequence tends to a limit l, then we write~~ $\lim\limits_{n \to \infty} (u_n) = l$

39.3 CONVERGENT SEQUENCE

If the limit of a sequence is finite, the sequence is convergent. If the limit of a sequence does not tend to a finite number, the sequence is said to be *divergent*.

e.g., $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2} + \ldots$. is a convergent sequence. $u_n = \frac{1}{n^2}$ $\lim\limits_{n \to \infty} \frac{1}{n^2} = \frac{1}{\infty} = 0$

$3, 5, 7, \ldots, (2n + 1), \ldots$ is a divergent sequence. $u_n = 2n+1$ lt $u_n = \infty$.

39.4 BOUNDED SEQUENCE

$u_1, u_2, u_3, \ldots, u_n \ldots$ is a bounded sequence if $u_n < k$ for every n.

39.5 MONOTONIC SEQUENCE

The sequence is either increasing or decreasing, such sequences are called *monotonic*.

e.g., $1, 4, 7, 10, \ldots$ is a monotonic sequence.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is also a monotonic sequenece.

$1, -1, 1, -1, 1, \ldots$ is not a monotonic sequence.

A sequence which is monotonic and bounded is a convergent sequence.

EXERCISE 39.1

Determine the general term of each of the following sequence. Prove that the following sequences are convergent.

1. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ **Ans.** $\frac{1}{2^n}$

2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **Ans.** $\frac{n}{n+1}$

3. $1, -1, 1, -1, \ldots$ **Ans.** $(-1)^{n-1}$

4. $\frac{1^2}{1!}, \frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \frac{5^2}{5!}, \ldots$ **Ans.** $\frac{n^2}{n!}$

Which of the following sequences are convergent ?

5. $u_n = \dfrac{n+1}{n}$ **Ans.** Convergent

6. $u_n = 3n$ **Ans.** Divergent

7. $u_n = n^2$ **Ans.** Divergent

8. $u_n = \dfrac{1}{n}$ **Ans.** Convergent

39.6 REMEMBER THE FOLLOWING LIMITS

(i) $\lim\limits_{n \to \infty} x^n = 0$ if $x < 1$ and $\lim\limits_{n \to \infty} x^n = \infty$ if $x > 1$

(ii) $\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = 0$ for all values of x

(iii) $\lim\limits_{n \to \infty} \dfrac{\log n}{n} = 0$

(iv) $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n = e$

(v) $\lim\limits_{n \to \infty} (n)^{1/n} = 1$

(vi) $\lim\limits_{n \to \infty} [n!]^{1/n} = \infty$

(vii) $\lim\limits_{n \to \infty} \left[\dfrac{(n!)}{n}\right]^{1/n} = \dfrac{1}{e}$

(viii) $\lim\limits_{n \to \infty} n\, x^n = 0$ if $x < 1$

(ix) $\lim\limits_{n \to \infty} n^n = \infty$

(x) $\lim\limits_{n \to \infty} \dfrac{1}{n^k} = 0$

(xi) $\lim\limits_{x \to \infty} \left[\dfrac{a^x - 1}{x}\right] = \log a$ or $\lim\limits_{n \to \infty} \dfrac{a^{1/n} - 1}{1/n} = \log a$

(xii) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

(xiii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

39.7 SERIES

A *series* is the sum of a sequence.

Let $u_1, u_2, u_3, \ldots\ldots\ldots, u_n, \ldots\ldots\ldots$ be a given sequence. Then, the expression $u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n + \ldots\ldots\ldots$ is called the series associated with the given sequence.

For example, $1 + 3 + 5 + 7 + \ldots$ is a series.

If the number of terms of a series is limited, the series is called *finite*. When the number of terms of a series are unlimited, it is called an *infinite series*.

$$u_1 + u_2 + u_3 + u_4 + \ldots + u_n + \ldots \infty$$

is called an infinite series and it is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or $\Sigma\, u_n$. The sum of the first n terms of a series is denoted by S_n.

39.8 CONVERGENT, DIVERGENT AND OSCILLATORY SERIES

Consider the infinite series $\Sigma\, u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Three cases arise:

(i) If S_n tends to a finite number as $n \to \infty$, the series $\Sigma\, u_n$ is said to be *convergent*.

(ii) If S_n tends to infinity as $n \to \infty$, the series $\Sigma\, u_n$ is said to be *divergent*.

(iii) If S_n does not tend to a unique limit, finite or infinite, the series $\Sigma\, u_n$ is called *oscillatory*.

39.9 PROPERTIES OF INFINITE SERIES

1. The nature of an infinite series does not change:
   (i) by multiplication of all terms by a constant k.
   (ii) by addition or deletion of a finite number of terms.
2. If two series Σu_n and Σv_n are convergent, then $\Sigma (u_n + v_n)$ is also convergent.

Example 1. *Examine the nature of the series* $1 + 2 + 3 + 4 + \ldots + n + \ldots \infty$.

Solution. Let $$S_n = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2} \qquad \text{[Series in A.P.]}$$

Since $$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n(n+1)}{2} \Rightarrow \infty$$

Hence, this series is divergent. **Ans.**

3

$2+\left(\frac{1}{2}\right)$

Example 2. *Test the convergence of the series* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$

$1+\frac{x}{2}+\left(\frac{1}{2}\right)$

Solution. Let $$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty \qquad \text{[Series in G.P.]}$$

$$= \frac{1}{1 - \frac{1}{2}} = 2 \qquad \left(S_n = \frac{a}{1-r}\right)$$

$$\lim_{n \to \infty} S_n = 2$$

Hence, the series is convergent. **Ans.**

Example 3. *Prove that the following series:*

$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots$ *is convergent and find its sum.* *(M.U. 2008)*

Solution. Here, $$u_n = \frac{n+1}{(n+2)!} = \frac{n+2-1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{1}{(n+2)!}$$

$$= \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

$$S_n = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots$$

$$+ \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right) = \frac{1}{2!} - \frac{1}{(n+2)!}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{1}{2!} - \frac{1}{(n+2)!}\right] = \frac{1}{2}$$

$\therefore$ Σu_n converges and its limit is $\frac{1}{2}$. **Ans.**

Example 4. *Discuss the nature of the series* $2 - 2 + 2 - 2 + 2 - \ldots \infty$.

Solution. Let $$S_n = 2 - 2 + 2 - 2 + 2 - \ldots \infty$$
$= 0$ if n is even
$= 2$ if n is odd.

Hence, S_n does not tend to a unique limit, and, therefore, the given series is oscillatory. **Ans.**

Properties of geometric series.

The series $1 + r + r^2 + r^3 + \ldots \infty$ is
i) Convergent if $|r| < 1$ ii) divergent if $r \geq 1$
iii) Oscillatory if $r \leq -1$.

necessary conditions for convergent series.

For every convergent series Σu_n. $\lim_{n\to\infty} u_n = 0$

Cauchy's fundamental test for divergence

If $\lim_{n\to\infty} u_n \neq 0$, the series is divergent.

eg: Test the convergence of the series $1+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\dots+\frac{n}{n+1}+\dots\infty$.

ans. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$

$= 1 \neq 0$.

hence by the Cauchy's fundamental test for divergence, the series is divergent.

EXERCISE 39.4

Examine for convergence:

1. $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}} + \dots + \frac{2^n}{\sqrt{4^n+1}} + \dots\infty$ **Ans.** Divergent

2. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ **Ans.** Divergent

3. $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ **Ans.** Divergent

4. $\sum \cos\frac{1}{n}$ **Ans.** Divergent

5. $1+\frac{1}{2}+2+\frac{1}{3}+3+\frac{1}{4}+4+\dots$ **Ans.** Divergent

6. $\Sigma\,(6-n^2)$ **Ans.** Divergent

7. $\Sigma\,(-2^n)$ **Ans.** Divergent

8. $\Sigma\,3^{n+1}$ **Ans.** Divergent

39.14 *p*-SERIES

The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots\infty$ *is* (*i*) *convergent if* $p > 1$ (*ii*) *Divergent if* $p \le 1$.

(*MDU, Dec. 2010*)

Solution. Case 1: $(p > 1)$

The given series can be grouped as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+$$
$$\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\frac{1}{11^p}+\frac{1}{12^p}+\frac{1}{13^p}+\frac{1}{14^p}+\frac{1}{15^p}\right)+\dots$$

Now
$$\frac{1}{1^p} = 1 \qquad \dots(1)$$
$$\frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p} = \frac{2}{2^p} \qquad \dots(2)$$
$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} < \frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} = \frac{4}{4^p} \qquad \dots(3)$$
$$\frac{1}{8^p}+\frac{1}{9^p}+\dots+\frac{1}{15^p} < \frac{1}{8^p}+\frac{1}{8^p}+\dots+\frac{1}{8^p} = \frac{8}{8^p} \qquad \dots(4)$$

On adding (1), (2), (3) and (4), we get:

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\dots+\frac{1}{15^p}\right)+\dots$$
$$< \frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\dots$$
$$< 1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{2}\right)^{2p-2}+\left(\frac{1}{2}\right)^{3p-3}+\dots$$
$$< \frac{1}{1-\left(\frac{1}{2}\right)^{p-1}} \qquad \left[\text{G.P., } r = \left(\frac{1}{2}\right)^{p-1},\ S = \frac{1}{1-r}\right]$$

< Finite number if $p > 1$

$2^{p+1}-1 = (2^{p-1}).2$ 2^{p-1}

Hence, the given series is convergent when $p > 1$.

Case 2: $p = 1$

When $p = 1$, the given series becomes

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \dots$$

$$1 + \frac{1}{2} = 1 + \frac{1}{2} \qquad \dots(1)$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \qquad \dots(2)$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2} \qquad \dots(3)$$

$$\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16} = \frac{8}{16} = \frac{1}{2} \qquad \dots(4)$$

On adding (1), (2), (3) and (4), we get

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \dots$$

$$> 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots}_{\text{times}}$$

$$> 1 + \frac{n}{2} \quad (n \to \infty)$$

$$> \infty$$

Hence, the given series is divergent when $p = 1$.

Case 3: $p < 1$

$$\frac{1}{2^p} > \frac{1}{2}, \qquad \frac{1}{3^p} > \frac{1}{3}, \qquad \frac{1}{4^p} > \frac{1}{4} \text{ and so on}$$

Therefore, $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

$>$ divergent series ($p = 1$) [From Case 2]

$$\left[\text{As the series on R.H.S. } \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) \text{ is divergent}\right]$$

Hence, the given series is divergent when $p < 1$.

39.15 COMPARISON TEST

If two positive terms $\Sigma\, u_n$ and $\Sigma\, v_n$ be such that

$\lim\limits_{n \to \infty} \frac{u_n}{v_n} = k$ **(finite number), then both series converge or diverge together.**

Proof. By definition of limit there exists a positive number ε, however small, such that

$$\left|\frac{u_n}{v_n} - k\right| < \varepsilon \text{ for } n > m \qquad \textit{i.e.}, \; -\varepsilon < \frac{u_n}{v_n} - k < +\varepsilon$$

$$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \text{ for } n > m$$

Ignoring the first m terms of both series, we have

$$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \text{ for all } n. \qquad ...(1)$$

Case 1. Σv_n is convergent, then

$$\lim_{n \to \infty} (v_1 + v_2 + ... v_n) = h \text{ (say)} \qquad \text{where } h \text{ is a finite number.}$$

From (1), $u_n < (k + \varepsilon) v_n$ for all n.

$$\lim_{n \to \infty} (u_1 + u_2 + ... + u_n) < (k + \varepsilon) \lim_{n \to \infty} (v_1 + v_2 + ... + v_n) = (k + \varepsilon)h$$

Hence, Σu_n is also convergent.

Case 2. Σv_n is divergent, then

$$\lim_{n \to \infty} (v_1 + v_2 + ... + v_n) \to \infty \qquad ...(2)$$

Now from (1) $\qquad k - \varepsilon < \dfrac{u_n}{v_n}$

$$u_n > (k - \varepsilon)v_n \text{ for all } n$$

$$\lim_{n \to \infty} (u_1 + u_2 + ... + u_n) > (k - \varepsilon) \lim_{n \to \infty} (v_1 + v_2 + ... + v_n)$$

From (2), $\displaystyle\lim_{n \to \infty} (u_1 + u_2 + ... + u_n) \to \infty$

Hence, Σu_n is also divergent.

Note. For testing the convergence of a series, this Comparison Test is very useful. We choose Σv_n (p-series) in such a way that

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \text{finite number.}$$

Then the nature of both the series is the same. The nature of Σv_n (p-series) is already known, so the nature of Σu_n is also known.

Example 8. *Test the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 10}$ *for convergence or divergence.*

Solution. Here, $\qquad u_n = \dfrac{1}{n + 10}$

Let $\qquad v_n = \dfrac{1}{n}$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n}{n + 10} = \lim_{n \to \infty} \frac{1}{1 + \frac{10}{n}} = 1 = \text{finite number.}$$

According to Comparison Test both series converge or diverge together, but Σv_n is divergent as $p = 1$.

$\therefore \quad \Sigma u_n$ is also divergent. **Ans.**

Example 9. *Test the convergence of the following series:*

$$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots \qquad \textit{(M.D.U., 2000)}$$

Solution. Here, we have

$$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{\sqrt{1+0}-0}{(1-0)^3-0} = 1$$

Which is finite and non-zero.

$\therefore \sum u_n$ and $\sum v_n$, converge or diverge together since $\sum v_n = \sum \frac{1}{n^{\frac{5}{2}}}$ is of the form

$\sum \frac{1}{n^p}$. where $P = \frac{5}{2} > 1$.

$\therefore \sum v_n$ is convergent $\Rightarrow \sum u_n$ is convergent. **Ans.**

Example 13. *Test the convergence and divergence of the following series.*

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{5+n^5}$$ *(Gujarat, I Semester, Jan. 2009)*

Solution. Here, $$u_n = \frac{2n^2+3n}{5+n^5} = \frac{n^2\left(2+\frac{3}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\,\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1}$$

Let $$v_n = \frac{1}{n^3}$$

By Comparison Test

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^3\left(2+\frac{3}{n}\right)}{n^3\left(\frac{5}{n^5}+1\right)} = \lim_{n\to\infty}\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1} = 2 = \text{Finite number.}$$

According to comparison test both series converge or diverge together but Σv_n is convergent as $p = 2$.

Hence, the given series is convergent. **Ans.**

Example 14. *Test the following series for convergence* $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

Solution. Given series is $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

$\frac{1}{3^P}+\frac{1}{5^P}+\frac{1}{7^P}+\ldots$

Here $$u_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$$

$u_n = \frac{1}{(2n+1)^P} = \frac{1}{n^P\left(2+\frac{1}{n}\right)^P}$

Let $$v_n = \frac{1}{n^{p-1}} \quad \therefore \quad \frac{u_n}{v_n} = \frac{1+\frac{1}{n}}{n^{p-1}} \times \frac{n^{p-1}}{1} = 1+\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = 1$$

Therefore, both the series are either convergent or divergent.

But Σv_n is convergent if $p-1>1$, *i.e.*, if $p>2$

and is divergent if $p-1 \le 1$, *i.e.*, if $p \le 2$ (P series)

$\therefore$ The given series is convergent if $p > 2$ and divergent if $p \le 2$. **Ans.**

ΣV_n is convergent if $P>1$.
& divergent if $P \le 1$.

$\therefore$ The gn series is convergent if $P>1$ & divergent if $P\le 1$

$V_n = \frac{1}{n^P}$

$\frac{u_n}{v_n} = \frac{1}{n^P\left(2+\frac{1}{n}\right)^P}\cdot n^P = \frac{1}{2^P}$

$\lim_{n\to\infty} \frac{u_n}{v_n}$

EXERCISE 39.5

Examine the convergence or divergence of the following series:

1. $2 + \frac{3}{2}\cdot\frac{1}{4} + \frac{4}{3}\cdot\frac{1}{4^2} + \frac{5}{4}\cdot\frac{1}{4^3} + \ldots \infty$ **Ans.** Convergent
2. $1 + \frac{1.2}{1.3} + \frac{1.2.3}{1.3.5} + \frac{1.2.3.4}{1.3.5.7} + \ldots \infty$ **Ans.** Convergent
3. $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \ldots \infty$ **Ans.** Divergent
4. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots\ldots \infty$ **Ans.** Convergent (*M.D. University, Dec. 2004*)
5. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots \infty$ **Ans.** Convergent
6. $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \ldots$ **Ans.** Convergent (*M.D. University, 2001*)
7. $\frac{1}{3} + \frac{2!}{3^2} + \frac{3!}{3^3} + \ldots \infty$ **Ans.** Convergent
8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$ **Ans.** Divergent
9. $\sum_{n=1}^{\infty} \frac{2n^3 + 5}{4n^5 + 1}$ **Ans.** Convergent
10. $\sum_{n=1}^{\infty} \frac{a^n}{x^n + n^a}$ **Ans.** If $x > a$, convergent; if $x \le a$, Divergent
11. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ **Ans.** Convergent
12. $\sum_{n=1}^{\infty} \sqrt{(n^2+1)} - n$ **Ans.** Divergent
13. $\sum_{n=1}^{\infty} \left[\sqrt{(n^4+1)} - \sqrt{(n^4-1)}\right]$ **Ans.** Convergent
14. $\sum_{n=1}^{\infty} \frac{2^n + 1}{3^n + n}$ **Ans.** Convergent
15. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **Ans.** Convergent
16. $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ **Ans.** Convergent

39.16 D'ALEMBERT'S RATIO TEST

Statement. *If Σu_n is a positive term series such that $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = k$ then*

(i) the series is convergent if $k < 1$ *(ii) the series is divergent if $k > 1$*

Solution. $K = 1$, ratio test fails.

Case I. When $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = k < 1$

By definition of a limit, we can find a number r (< 1) such that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n \ge m \qquad \left[\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r \ldots\right]$$

Omitting the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \ldots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots\right) = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots \infty\right)$$

$$< u_1\,(1 + r + r^2 + r^3 + \ldots \infty) \qquad (r < 1)$$

$$= \frac{u_1}{1-r}, \text{ which is a finite quantity.}$$

Hence, $\Sigma\, u_n$ is convergent.

Case 2. When $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = k > 1$

By definition of limit, we can find a number m such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$

$$\frac{u_2}{u_1} > 1, \qquad \frac{u_3}{u_2} > 1, \qquad \frac{u_4}{u_3} > 1$$

Ignoring the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \ldots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots\right) = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots \infty\right)$$

$$\geq u_1\,(1 + 1 + 1 + 1 \ldots \text{ to } n \text{ terms}) = nu_1$$

$$\left[\because \lim_{n\to\infty} (u_1 + u_2 + \ldots + u_n) = nu_1\right]$$

$$\lim_{n\to\infty} S_n \geq \lim_{n\to\infty} nu_1 = \infty$$

Hence, $\Sigma\, u_n$ is divergent.

Note. When $\frac{u_{n+1}}{u_n} = 1 \quad (k = 1)$

The ratio test fails.

For Example. Consider the series whose n^{th} term $= \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \qquad \ldots(1)$$

Consider the second series whose n^{th} term is $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1 \qquad \ldots(2)$$

Thus, from (1) and (2) in both cases $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

But we know that the first series is divergent as $p = 1$.
The second series is convergent as $p = 2$.

Hence, when $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, the series may be convergent or divergent.

Thus, ratio test fails when $k = 1$.

Example 15. *Test for convergence of the series whose n^{th} term is* $\frac{n^2}{2^n}$.

Solution. Here, we have $u_n = \frac{n^2}{2^n}, \qquad u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

By D'Alembert's Test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

Hence, the series is convergent by D'Alembert's Ratio Test. **Ans.**

Example 16. *Test for convergence the series whose n^{th} term is $\frac{2^n}{n^3}$.*

Solution. Here, we have $u_n = \frac{2^n}{n^3}$, $\quad u_{n+1} = \frac{2^{n+1}}{(n+1)^3}$

By D'Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} = \frac{2}{\left(1+\frac{1}{n}\right)^3} \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^3} = 2 > 1$$

Hence, the series is divergent. **Ans.**

Example 17. *Discuss the convergence of the series:*

$$\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n. \qquad (x > 0)$$

(M.D.University, Dec., 2001)

Solution. Here, we have

$$u_n = \sqrt{\frac{n}{n^2+1}} x^n$$

$$\therefore \quad u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}} . x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1}} . \sqrt{\frac{n^2+2n+2}{n^2+1}} \frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)}} \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)}} \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$, *i.e.* $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, the Ratio Test fails.

$$\text{When } x = 1,\ u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} . \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$v_n = \frac{1}{\sqrt{n}},$$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{n}} . \frac{1}{\sqrt{1+\frac{1}{n^2}}} . \frac{\sqrt{n}}{1} = \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

Which is finite and non-zero.

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$.

$\sum v_n$ diverges $\Rightarrow \sum u_n$ diverges.

Hence, the given series $\sum u_n$ converges is $x < 1$ and diverges if $x \geq 1$. **Ans.**

EXERCISE 39.6

Test the convergence for series:

1. $\sum_{n=1}^{n} \frac{n^2}{3^n}$ **Ans.** Convergent
2. $\sum_{n=1}^{n} \frac{n!}{n^n}$ **Ans.** Convergent
3. $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \ldots \infty$ **Ans.** Convergent
4. $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots \infty$ **Ans.** Convergent
5. $\sum_{n=1}^{n} \frac{n!.2^n}{n^n}$ **Ans.** Convergent
6. $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n.3^n}$ **Ans.** Convergent if $x > 3$, Divergent if $x < 3$
7. Prove that, if $u_{n+1} = \frac{k}{1+u_n}$, where $k > 0$, $u_1 > 0$, then the series Σu_n converges to the positive root of the equation $x^2 + x = k$.

39.17 RAABE'S TEST (HIGHER RATIO TEST)

If Σu_n is a positive term series such that $\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, then
(i) the series is convergent if $k > 1$ (ii) the series is divergent if $k < 1$.

Proof. Case I. $k > 1$

Let p be such that $k > p > 1$ and compare the given series Σu_n with $\sum \frac{1}{n^p}$ which is convergent as $p > 1$.

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \quad \text{or} \quad \left(\frac{u_n}{u_{n+1}}\right) > \left(1 + \frac{1}{n}\right)^p > 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

(Binomial Theorem)

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots.$$

If $$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

and $k > p$ which is true as $k > p > 1$; Σu_n is convergent when $k > 1$.

Case II. $k < 1$ Same steps as in Case I.

Notes:

1. Raabe's Test fails if $k = 1$
2. Raabe's Test is applied only when D'Alembert's Ratio Test fails.

CHAPTER 6

FOURIER SERIES

Definition : A function $f(x)$ is said to be periodic if and only if $f(x+p)=f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x is called *the period of the function.*

For example, for any integer n, $\sin(x+2n\pi)=\sin x$ for all x. Therefore, $\sin x$ is periodic. For $n=1$, $\sin(x+2\pi)=\sin x$. There is no positive number 'a' which is less than 2π such that $\sin(a+x)=\sin x$ for all x. Therefore, 2π is the period of $\sin x$. Similarly, 2π is the period for $\cos x$. But $\tan(\pi+x)=\tan x$ and π is the least positive value such that $\tan(\pi+x)=\tan x$.

So, $\tan x$ is periodic of period π.

Sin nx, cos nx are periodic functions of period $\dfrac{2\pi}{n}$.

Standard Results in integrals. If m, n are integers,

1. If $n \neq 0$, $\int_c^{c+2\pi} \sin nx\,dx = 0 \;\therefore\; \int_0^{2\pi} \sin nx\,dx = 0$

If $n = 0$ $\int_c^{c+2\pi} \sin nx\,dx = \int_c^{c+2\pi} 0\,dx = 0$

2. If $n \neq 0$, $\int_c^{c+2\pi} \cos nx\,dx = 0 \;\therefore\; \int_0^{2\pi} \cos nx\,dx = 0$

3. $\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0 \;\therefore\; \int_0^{2\pi} \sin mx \cos nx\,dx = 0$

4. If $m \neq n$. $\int_c^{c+2\pi} \sin mx \sin nx\,dx = 0 \;\therefore\; \int_0^{2\pi} \sin mx \sin nx\,dx = 0$ if $m \neq n$

5. If $n \neq 0$, $\int_c^{c+2\pi} \sin^2 nx\,dx = \pi \;\therefore\; \int_0^{2\pi} \sin^2 nx\,dx = \pi$

6. If $n \neq 0$, $\int_c^{c+2\pi} \cos^2 nx\,dx = \pi \;\therefore\; \int_0^{2\pi} \cos^2 nx\,dx = \pi$

7. If $n \neq m$, $\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0$

8. $\int e^{ax} \sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx] + k$

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx] + k$$

9. Bernoulli's generalised formula of integration by parts.

Integration by parts : $\int u\,dv = uv - \int v\,du$.

We extend this result.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$$

where suffix denotes the integration and primes denote the differentiation.

Example. *Evaluate (i)* $\int (x^2+7x+5)\cos 3x\,dx$ *(ii)* $\int x^2 e^{2x}\,dx$

Here take u = polynomial $= x^2+7x+5$

and $v = \cos 3x$

(i) $$\int uv\,dx = \int (x^2+7x+5)\cos 3x\,dx$$

$$= (x^2+7x+5)\left(\frac{\sin 3x}{3}\right) - (2x+7)\left(-\frac{\cos 3x}{9}\right) + (2)\left(-\frac{\sin 3x}{27}\right) + c.$$

(ii) $$\int x^2 e^{2x}\,dx = (x^2)\left(\frac{e^{2x}}{2}\right) - (2x)\left(\frac{e^{2x}}{4}\right) + (2)\left(\frac{e^{2x}}{8}\right) + c.$$

Some results. If n is any integer, $\sin n\pi = 0$, $\cos n\pi = (-1)^n$.

Fourier series of $f(x)$:

If $f(x)$ is defined in $(0, 2\pi)$ and if $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then the R.H.S. series of sines and cosines is called the Fourier series of $f(x)$ in the interval $(0, 2\pi)$.

Theorem. If $f(x)$ is defined in $(0, 2\pi)$ and if $f(x)$ can be represented by the trigonometric series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \ldots(i)$$

then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

then
$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Taking $c = 0$, we get previous results.

Taking $c = -\pi$, we get that in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Even and odd functions : In $(-\infty, \infty)$,

(*i*) If $f(-x) = f(x)$ for all x then $f(x)$ is even.

(*ii*) If $f(-x) = -f(x)$ for all x, then $f(x)$ is odd.

Also,

(*i*) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even.

(*ii*) $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd.

(*iii*) $\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx$

Example 1. *Find the Fourier series of periodicity* 2π *for* $f(x) = x^2$ *in* $(0, 2\pi)$.

Deduce $\sum_1^{\infty} \frac{1}{n^2} = \pi^2/6$.

Sol. Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots(1)$$

where
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$\therefore$
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$
$$= \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi}$$
$$= \frac{1}{3\pi}\left[8\pi^3 - 0\right] = \frac{8}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$
$$= \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{2(2\pi)}{n^2}\cos 2n\pi\right] \quad \text{since other terms vanish}$$
$$= \frac{4}{n^2} \quad \text{since } \cos 2n\pi = 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$
$$= \frac{1}{\pi}\left[(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3}(\cos 2n\pi - 1)\right]$$
$$= -\frac{4\pi}{n}$$

Substituting a_0, a_n, b_n values in (1)

$$\therefore \quad f(x) = x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$x = 0$ is an end point of the range.

Value of Fourier series at x = 0 is $\frac{f(0)+f(2\pi)}{2}$

$$\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{0^2+4\pi^2}{2}=2\pi^2$$

$$\sum\frac{4}{n^2} = 2\pi^2-\frac{4\pi^2}{3}=\frac{2\pi^2}{3}$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e., $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \pi^2/6.$

Example 2. *Find Fourier series of* $f(x) = x$ *in* $(0, 2\pi)$ *of periodicity*

Sol. Let $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}x\,dx=\frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi}=2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(1-1)\right]=0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\right]=-\frac{2}{n}$$

Substituting a_0, a_n, b_n in (1), we get

$$f(x) = x=\pi-2\sum_1^{\infty}\frac{1}{n}\sin nx.$$

Example 3. *Find Fourier series of* $f(x) = \frac{(\pi-x)^2}{4}$ *in* $(0, 2\pi)$ *of periodicity* 2π.

Sol. Let $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{(-3)}\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi}\left[-\pi^3-\pi^3\right]$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right)+2(\pi-x)\left(-\frac{\cos nx}{n^2}\right)+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{2}{n^2}\{+\pi+\pi\}\right]=\frac{1}{n^2}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)+2(\pi-x)\left(-\frac{\sin nx}{n^2}\right)+(2)\left(-\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^2}{n}+\frac{\pi^2}{n}\right]=0$$

$$\therefore \quad \frac{(\pi-x)^2}{4} = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

Example 4. *Expand* $f(x) = \frac{1}{2}(\pi-x)$ in $(0, 2\pi)$ *as a Fourier series of periodicity* 2π.

Deduce $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots=\frac{\pi}{4}$

Sol. Let $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ ...(1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{2\pi}$$

$$= \frac{1}{\pi}[0] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2}(1-1)\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] = \frac{1}{n}$$

$$\therefore \quad f(x) = \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

Put $x = \pi/2$. $f(x)$ is continuous at $x = \pi/2$

$$\therefore \quad \frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin\frac{4\pi}{2} + \ldots = \frac{\pi}{4}$$

i.e., $1 - \frac{1}{3} + \frac{1}{5}$ .... to $\infty = \frac{\pi}{4}$.

Example 5. *Obtain the Fourier series of periodicity* 2π *for* $f(x) = e^{-x}$ *in the interval* $0 < x < 2\pi$. *Hence deduce the value of* $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$.

Sol. Let $\quad f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad$ ...(1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left(-e^{-x}\right)_0^{2\pi}$$

$$= \frac{1}{\pi}(-e^{-2\pi} + 1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-1) - (-1)\right]$$

$$= \frac{1 - e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{-2\pi} + n\right]$$

$$= \frac{n}{\pi(1+n^2)}(1 - e^{-2\pi})$$

Substituting a_0, a_n, b_n in (1), we get

$$e^{-x} = \frac{(1 - e^{-2\pi})}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)\right] \quad \ldots(2)$$

In $(0, 2\pi)$, e^{-x} is continuous. Therefore, at $x = \pi$, the value of the Fourier series equals the value of the function. Hence replacing x by π in (2),

$$e^{-\pi} = \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\right]$$

$$\frac{\pi e^{-\pi}}{1 - e^{-2\pi}} = \frac{1}{2} + \left(-\frac{1}{2}\right) + \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore \quad \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{e^{\pi} - e^{-\pi}} = \frac{\pi}{2}\cdot\frac{2}{e^{\pi} - e^{-\pi}} = \frac{\pi}{2}\operatorname{cosech}\pi$$

Find the half range cosine series for $f(x) = (\pi - x)^2$ in $(0, \pi)$. Hence deduce $\frac{1}{1^2} + \frac{1}{2^2} + \cdots$

sol Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx.$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x)^2 dx = \frac{2}{\pi} \left(\frac{(\pi - x)^3}{-3} \right)_0^{\pi}$$

$$= -\frac{2}{3\pi} \left[(\pi - \pi)^3 - (\pi - 0)^3 \right] = \frac{2}{3\pi} \pi^3 = \frac{2}{3} \pi^2$$

$$= -\frac{2}{3\pi} \left[0 - \pi^3 \right]$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{2}{\pi} \left[\left((\pi - x)^2 \frac{\sin nx}{n} \right)_0^{\pi} + \int_0^{\pi} \frac{\sin nx}{n} 2(\pi - x) dx \right]$$

put $(\pi - x)^2 = u$

$2(\pi - x)(-dx) = du$

$\cos nx\, dx = dv$

$\frac{\sin nx}{n} = v$

$$= \frac{2}{\pi} \left[0 - 0 + \frac{2}{n} \int_0^{\pi} (\pi - x) \sin nx\, dx \right]$$

$$= \frac{4}{n\pi} \int_0^{\pi} (\pi - x) \sin nx\, dx$$

$$= \frac{4}{n\pi}\left[\left((\pi - x)\left(-\frac{\cos nx}{n}\right)\right)_0^{\pi} - \frac{1}{n}\int_0^{\pi} \cos nx \, dx\right]$$

$u = \pi - x$

$du = -dx$

$dv = \sin nx \, dx$

$v = -\frac{\cos nx}{n}$

$$= \frac{4}{n\pi}\left[\left(0 + \frac{\pi \cos 0}{n}\right) - \frac{1}{n}\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{4\pi}{n^2\pi} - \frac{4}{n^3\pi}(\sin n\pi - \sin 0)$$

$$= \frac{4\pi}{n^2\pi} = \frac{4}{n^2}$$

$$f(x) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

$$(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

Put $x = 0$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$4\sum \frac{1}{n^2} = \pi^2 - \pi^2/3 = \frac{2}{3}\pi^2$$

$$4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{2}{3}\pi^2$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\cancel{2}}{3 \times \cancel{4}2}\pi^2 = \frac{\pi^2}{6}$$

3. Find the half range sine series for $f(x) = k(l-x)$ in $(0, l)$.

Sol. The half range sine series of

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \int_0^l k(l-x) \sin \frac{n\pi x}{l} dx$$

put

$l - x = u$

$-dx = du$

$$dv = \sin \frac{n\pi x}{l} dx$$

$$v = \frac{-\cos \frac{n\pi x}{l}}{\left(\frac{n\pi x}{l}\right)}$$

$$= k \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right)_0^l \right.$$

$$\left. + \frac{l}{n\pi x} \int_0^l \cos \frac{n\pi x}{l} (-dx) \right]$$

$$= k \left[0 + \frac{l^2}{n\pi} \cos 0 \right]$$

$$= \frac{kl^2}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{kl^2}{n\pi} \sin \frac{n\pi x}{l}$$

Questions	opt1	opt2
What is the value of Gamma of one ?	0	1
Γ (n+1)=___________	(n+1)!	n Γ (n+1)
what is the relation between Beta and Gamma functions?	$\beta(m,n)=\Gamma(m)\Gamma(n)/\Gamma(m+n)$	$\beta(m,n)=\Gamma(m)\Gamma(m)/\Gamma(m+n)$
The value of $\beta(1/2,1/2)$ is _________.	$\sqrt{\pi}$	$\sqrt{\pi}/2$
The value of $\Gamma(1)$=_________.	1	n
what is the value of $\Gamma(1/2)$?	pi	0
Which one of the following statement is true?	$\Gamma(2)=\Gamma(1)$	$\Gamma(1/2)=(\sqrt{\pi})^2$
Which one of the following statement is false?	$\Gamma(2)=\Gamma(1)$	$\Gamma(1)=1$
$\Gamma(1/4)\,\Gamma(3/4)$=_____	2π	$\pi\sqrt{2}$
The values of $\Gamma(4)$=______	1!	2!
If C ′ is the evolute of the curve C then C is called the______ of the curve C ′	involute	curvature
Curvature of the circle is the _________	its radius	the reciprocal of its radius
_______ of a curve is the envelope of the normals of that curve.	involute	curvature
The parametric coordinates of the parabola x^2=4ay are_____.	x=at^2, y=2at	x=at, y=at
The parametric coordinates of the ellipse is given by _________.	x=acosθ, y=bsinθ	x=asinθ, y=bcosθ
The parametric coordinates of the hyperbola is given by _________.	x=acosθ, y=bsinθ	x=asinθ, y=bcosθ
The parametric coordinates of the parabola y^2=4ax are_____.	x=at^2, y=2at	x=at, y=at
The locus of the centre of curvature for a curve is called its evolute and the curve is called an __________ of its evolute.	involute	evolute
The locus of the centre of curvature for a curve is called its______.	involute	evolute
The parametric coordinates of the cycloid is given by _________.	x=a(θ+sinθ), y=a(1+cosθ)	x=a(θ-sinθ), y=a(1-cosθ)
If y=1/x, then y1=________	-1/x^2	1/x
If y=x^2, then y1=________	x^2	1/x
If y=x^2, then y2=________	x^2	1/x
If x=2at then dx/dt=	2at	2a
If x=at^2 then dx/dt=	2at	2a
If y=ax^2+2ax then dy/dx at (3,2) is _______	8a	4ax
If y=ax^2+2ax then dy/dx at (2,2) is _______	8a	4ax
If y=ax^2+2ax then dy/dx is _______	8ax+2a	4ax+2
If y=ax^2+2ax then second derivative is _______	2a	4ax
The volume of the solid of revolution generated by revolving the plane area bounded by the circle x^2+y^2=a^2 about its diameter is..........	(4/3)πa^3	(2/3)πa^3
The volume of the solid of revolution generated by revolving the plane area bounded by the circle x^2+y^2=2^2 about its diameter is..........	(32/3)π	(1/3)π
The volume of the solid of revolution generated by revolving the plane area bounded by the circle x^2+y^2=3^2 about its diameter is..........	16π	9π

opt3	opt4	opt5	opt6	Answer
2	3			1
Γ (n-1)	n Γ (n)			n Γ (n)
β(m,m)=Γ(m)Γ(m)/Γ(m+n)	β(m,n)=Γ(n)Γ(n)/Γ(m+n)			β(m,n)=Γ(m)Γ(n)/Γ(m+n)
π	π/2			π
n!	0			1
1	root(pi)			root(pi)
Γ (1/2)= 1	Γ(1/2)= 0			Γ(2)=Γ(1)
Γ (1/2)= √π	Γ (n+1)=n+1			Γ (n+1)=n+1
√(2π)	1			π√2
3!	4!			3!
radius of curvature	centre of curvature			involute
its centre	the reciprocal of its centre			the reciprocal of its radius
radius of curvature	evolute			evolute
x=2at, y=at^2	x=a, y=t			x=2at, y=at^2
x=atanθ, y=bsecθ	x=asecθ, y=btanθ			x=acosθ, y=bsinθ
x=atanθ, y=bsecθ	x=asecθ, y=btanθ			x=asecθ, y=btanθ
x=2at, y=at^2	x=a, y=t			x=at^2, y=2at
envelope	curvature			involute
envelope	curvature			evolute
x=(θ+sinθ), y=(1+cosθ)	x=(θ-sinθ), y=(1-cosθ)			x=a(θ-sinθ), y=a(1-cosθ)
ax	bx			-1/x^2
2x	x			2x
2x	2			2
2t	0			2a
2t	0			2at
2ax	6a			8a
2ax	6a			6a
2ax+2a	6a			2ax+2a
6ax	6a			2a
(1/3)πa^3	πa^3			(4/3)πa^3
(2/3)π	π			(32/3)π
36π	π			36π

The Volume of a sphere of radius 'a' is..................	2/3 π a^3	4/3 π a^3
The surface are of the sphere of radius 'a' is..................	4πa^2	πa^2
The Volume of a sphere of radius '2' is..................	16/3 π	32/3 π
The surface area of the sphere of radius '3' is..................	36π	9π
∫ dx=.............	x+C	1
∫cdx=…………….	cx+C	0
∫ 5dx=.............	x+C	5x+C
∫ x^n dx=.............	x^(n+1)/ (n+1)+ C	x^(n-1)/ (n-1)+ C
∫xdx=…..	x^2+C	x^2/2+C
∫ x^ (2) dx=.............	(x^(2)/2)+C	(x^(3)/3)+C
∫ 3x^(2) dx=.............	3x^(2)+C	x+C
∫ (1/x) dx=.............	1+ C	log x+C
∫ e^(x) dx=.............	(-e^x)+ C	e^(-x) + C
∫ e^(-x) dx=.............	(-e^x)+ C	e^(-x) + C
∫ e^(2x) dx=.............	(-e^2x)/2+ C	e^(-2x)/2 + C
∫ e^(-2x) dx=.............	(-e^(-2x))/2+ C	e^(-2x)/2 + C
∫ cosx dx=.............	sinx + C	cosx + C
∫ sinx dx=.............	sinx + C	cosx + C
∫ cosmx dx=.............	(sinmx)/m + C	(cosmx)/m + C

1/3 π a^3	π a^3			4/3 π a^3
3πa^2	2πa^2			4πa^2
8/3 π	8 π			32/3 π
27π	18π			36π
0	x^2			x+C
1	x+C			cx+C
x^2+C	5+C			5x+C
nx^ (n-1)+ C	(n+1) x^ (n+1)+ C			x^(n+1)/ (n+1)+ C
x^3/2+C	x^2/2+C			x^2/2+C
x+C	2x+C			(x^(3)/3)+C
x^2+C	x^(3) +C			x^(3) +C
(-1)+C	(-log x)+ C			log x+C
(-e^(-x))+C	e^x + C			e^x + C
(-e^(-x))+C	e^x + C			(-e^(-x))+C
(-e^(-2x))/2+C	e^2x/2+ C			e^2x/2 + C
(-e^(-2x))/2+C	e^(-2x)/2+ C			e^2x/2 + C
(-cosx)+C	(-sinx)+C			sinx + C
(-cosx)+C	(-sinx)+C			(-cosx)+C
(-cosmx)/m+C	(-sinmx)/m+C			(sinmx)/m+ C

Questions	opt1	opt2	opt3
The partial differentiation is a function of_________ or more variables .	two	zero	one
If z=f(x,y) where x and y are ______ function of another variable t	continuous	differential	two
If f(x,y)=0 then x and y are said to be an _________ function	implicit	extremum	explicit
f(a,b) is said to be exturemumvalue of f(x,y) if it is either a_______________	maximum or minimum	zero	minimum
The Lagrange multiplier is denoted by _________________	a	b	l
Every extremum value is a stationary value but a stationary value need not be an ____________ value.	infimum	minimum	maximum
If u1,u2……un are functions of n variables x1, x2……..xn then the Jacobian of the transformation from x1,x2….xn to u1,u2...un is defined by___________	2	0	1
f(a,b) is a maximum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood _________	f(a,b)>f (a+h,b+k)	f(a,b)<f(a+h, b+k)	f(a,b)<0
f(a,b) is a minimum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood _________	f(a,b)>f (a+h,b+k)	f(a,b)<f(a+h, b+k)	f(a,b)<0
The necessary condition for maxima is ___________	$\partial f/\partial x$ (a,b) =0	$\partial f/\partial x$ (a,b)= 1	$\partial f/\partial y$ (a,b)=5
The necessary condition for minimum is _____________	$\partial f/\partial x$ (a,b) =0	$\partial f/\partial y$ (a,b)=0	$\partial f/\partial x$ (a,b)=1
f(a,b) is said to be said to be a stationary value of f(x,y) if (x,y) is___________	$\partial f/\partial x$ (a,b) =0 and $\partial f/\partial y$ (a,b) =0	$\partial f/\partial x$ (a,b)=1	$\partial f/\partial y$ (a,b)=0

opt4	opt5	opt6	Answers
three			two
one			continuous
differential			implicit
maximum			maximum or minimum
d			1
extremum			extremum
-1			2
f(a,b)>0			f(a,b)>f (a+h,b+k)
f(a,b)>0			f(a,b)<f(a+h,b+
∂f/∂y (a,b)=1			∂f/∂x (a,b) =0
∂f/∂y (a,b)=1			∂f/∂y (a,b) =0
∂f/∂y (a,b)=1			∂f/∂x (a,b) =0 and ∂f/∂y (a,b) =0

The expansion of f(x,y) by ________series is unique.	Maclaurins	Taylor	power
If f(a,b) is said to be _________of f(x,y) if it is either maximum or minimum.	extremum value	boundary value	end
The _______ differentiation is a function of two or more variables.	ODE	PDE	partial
Any function of the type f(x,y)=c is called an __	Implicit	Explicit	Constant
If u=f(x,y) ,where x = φ(t),y =Ψ(t) then u is a function of t and is called the ____ function	Implicit	Explicit	Constant
The point at which function f(x,y) is either maximum or minimum is known as ______ point	Stationary	Saddle point	extremum
If rt-s^2>0 and r<0 at (a,b) the f(x,y) is maximur	Maximum	Minimum	maximum or minimum
If rt-s^2>0 and r>0 at (a,b) the f(x,y) is minimun	Maximum	Minimum	maximum or minimum
If rt-s^2>0 at (a,b) the f(x,y) is neither maximun	Stationary	Saddle point	extremum
If Ñ.F=0 then F is	irrotational	solenoidal	rotational
If Ñ×F=0 then F is	irrotational	solenoidal	rotational
Any motion in which the curl of the velocity vector is zero is said to be ___	irrotational	solenoidal	rotational
A function is said to be _______ if it associates a scalar with every point in space.	Scalar function	Vector function	Point function
A variable quantity whose value at any point in a region of space depends upon the position of the point is called a ___	Scalar function	Vector function	Point function
A function is said to be _______ if it associates with vector in every point in space.	Scalar function	Vector function	Point function
If the divergence of a flow is zero at all points then it is said to be _______	rotational	irrotational	solenoidal

binomial			Taylor
power			extremum value
total			partial
composite			Implicit
composite			composite
implicit			Stationary
zero			Maximum
zero			Minimum
implicit			Saddle point
curl			solenoidal
curl			irrotational
curl			irrotational
vector point function			Scalar function
vector point function			Point function
vector point function			Vector function
conservative			solenoidal

______ gives the rate of outflow per unit volume at a point of the fluid.	curl V	div V	curl V=0
If div V=0 everywhere in some region R of space then V is called the _____ vector point function.	rotational	irrotational	solenoidal
_______ is a vector which measures the extent to which individual particles of the fluid are spnning or rotating.	curl V	div V	curl V=0
div F is a ________ function.	point	vector	scalar
If curl V=0 then V is said to be an ___________.	rotational	irrotational	solenoidal
If r=xI+yJ+zK then div r=________	0	1	2
If r=xI+yJ+zK then curl r=________	0	1	2
div (curl V)=	0	div V	curl V
curl (grad f)=	0	div V	curl V
Two surfaces are said to cut orthogonally at a point of intersection, if the respective normals at that point are ______.	parallel	perpendicular	equal
Any integral which is to be evaluated over a surface is called a ___	Line integral	Volume integral	surface integral
When the circulation of F around every closed curve in a region vanishes, then F is said to be _______ in that region.	rotational	irrotational	solenoidal
A force field F is said to be ____________ if it is derivable from a potential function f such that F = grad f.	rotational	irrotational	solenoidal
If F is ______ then curl F=0.	rotational	irrotational	solenoidal
If S has a unique normal at each of its points whose direction depends continuously on the point of S then the surface S is called a ______ surface.	Orientable	smooth	plane

div V=0			div V
conservative			solenoidal
div V=0			curl V
rotational			scalar
conservative			irrotational
3			3
3			0
V			0
f			0
zero			perpendicular
closed integral			surface integral
conservative			irrotational
conservative			conservative
conservative			conservative
twisted			smooth

If (3x-2y+z)I+(4x+ay-z)J+(x-y-2z)K is solenoidal then a=	0	1	-1
If f=x+y+z-8 then grad f is ____	I+J+K	I+J-K	I-J+K
The function f(x,y)=2x^2+2xy-y^3 has __________.	only one stationary point at (0,0)	two stationary points at (0,0)and (1/6, 1/3)	two stationary point at (0,0) and (1,-1)
The function f(x)=10+x6	is a decreasing function of x	has a minimum at x=0	saddle point
curl (gradϕ)=	0	div V	curl V
if **F=**xy**i-**yz**j-**xz**k** then at (1,1,1), div F=	i+j+k	i-j-k	i-j-k
div (curl **V**)=	0	div **V**	curl **V**

2			-1
0			I+J+K
not stationary points			two stationary points at (0,0)and (1/6,1/3)
has neither a maximum nor a minimum at x = 0			has neither a maximum nor a minimum at x = 0
			0
i+j-k			i-j-k
V			0

Questions	opt1	opt2	opt3	opt4	opt5
$\int_0^1\int_0^2\int_0^3 dx\,dy\,dz =$	2	4	6	8	
$\int_1^2\int_2^4 dx\,dy =$	2	6	3	1	
The value of ∫∫ dx dy , inner integral limt varies from 1 to 2 and the outer integral limit varies from 0 to 1	0	1	2	3	
∫∫∫ dx dy dz, the inner integral limit varies from 0 to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1	2	4	6	8	
When the limits are not given, the integral is named as ___________	Definite integral	Infinite integral	volume integral	Surface integral	
The triple integral ∫∫∫ dx dy dz gives __________ over the region v.	area	volume	Direction	weight	
The value of ∫∫ (x+y) dx dy , inner integral limt varies from 0 to 1 and the outer integral limit varies from 0 to 1	0	1	2	3	
The value of ∫∫∫ x^2 yz dx dy dz, the inner integral limit varies from 1 to 2, the central integral limit varios from 0 to 2 and outer integral limit various from 0 to 1	7/3	1/3	2/3	3	
Evaluate ∫∫ 4xy dx dy, the inner integral limit varies from 0 to 1 and outer integral limit varies from 0 to 2	10	4	5	1	

opt6	Answer
	6
	2
	1
	6
	Infinite integral
	volume
	1
	7/3
	4

The value of ∫∫ dxdy/xy, the inner integral limit varies from 0 to b and the outer limit varies from 0 to a	0	1	ab	loga log b	
The value of ∫∫ dxdy/xy, the inner integral limit varies from 0 to x and the outer limit varies from -a to a	0	1	2	3	
If the limits are given in the integral , the the integral is name as____________	Definite integral	Infinite integral	volume integral	Surface integral	
The value of ∫∫(x^2+3y^2) dy dx , the inner integral limit varies from 0 to 1, the outer integral limit varies from 0 to 3	10	15	12	30	
The value od ∫∫∫ dxdy d, the inner integral limit varies from 0to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0 to 1	6	1	16	12	
If the limits are not given in the integral , the the integral is name as____________	Definite integral	Infinite integral	volume integral	Surface integral	
The value of ∫∫(x^2+y^2) dy dx ,the inner integral limit varies from 0 to x, the outer integral limit varies from 0 to 1	1	1⁄3	2⁄3	3⁄2	
The value of ∫∫dy dx, the inner integral limit various from 0 to x ,the outer integral limit varies from -a to a	0	1	2	3	
The Double integral ∫∫ dx dy gives _________ of the region R	area	modulus	Directi on	weight	

	loga log b
	0
	Definite integral
	12
	6
	Infinite integral
	1⁄3
	0
	area

The value of ∭ dx dy dz, the inner integral limit varies from 0 to a , the central integral limit varies from 0 to a and the outer integral limit varies from 0 to a	0	a^3	a^2	a^4	
The value of ∬(x+y) dx dy , the inner integral limit varies from 0 to 1 and the outer integral limit varies from 0 to 1	0	1	2	3	
The concept of line integral as a generalization of the concept of ________ integral	Single	Double	change of order	Triple	
The extension of double integral is nothing but ________ integral	Single	Line	volume integral	Triple	
The concept of _________ integral as a generalization of the concept of double integral	Single	Surface	Line	Triple	
Evaluate $\int x^2/2$ dx, the limit varies from 0 to 1	2	1⁄6	1⁄10	34	
Evaluate ∫42y dy, the limit varies from 0 to 10	10	2100	2000	100	
The value of ∬2 xy dy dx, the inner integral limit varies from 0 to x and the outer integral limit varies from 1 to 2	15⁄4	9⁄2	3⁄2	4⁄3	
The value of ∬dy dx, the inner integral limit varies from 2 to 4 ,the outer integral limit varies from 1 to 5	8	2	4	5	

	a^3
	1
	Double
	Triple
	Line
	1⁄6
	2100
	15⁄4
	8

The value of ∫∫xy dy dx, the inner integral limit varies from 0 to 3 , the outer integral limit varies from 0 to 4	12	36	1⁄2	4	
The value of ∫∫dy dx, the inner integral limit varies from 0 to 2 , the outer integral limit varies from 0 to 1	2	1	3⁄2	4	
The value of ∫∫dx dy, the inner integral limit varies from y to 2 , the outer integral limit varies from 0 to 1	1⁄2	1	3⁄2	4	
The value of ∫∫dx dy, the inner integral limit varies from 2 to 4 , the outer integral limit varies from 1 to 2	2	6	3	1	
When a function f(x) is integrated with respect to x between the limits a and b, we get __________	Definite integral	infinite integralv	volume integral	Surface integral	
In two dimensions the x and y axes divide the entire xy- plane into ____________ quadrants	1	2	3	4	
In three dimensions the xy and yz and zx planes divide the entire space into ____________ parts called octants	3	2	8	4	
Evaluate ∫(2x+3) dx, the integral limitvaries from 0 to 2	10	42	51	1	
__________ provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R.	Cauchy's Theorem	Green's Theore m	Stoke's Theore m	Gauss Theorem	

	12
	2
	3⁄2
	2
	Definite integral
	2
	8
	10
	Stoke's Theorem

________ is also called the first fundamental theorem of integral vector calculus.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	
_________ transforms line integrals into surface integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	
_______ transforms surface integrals into a volume integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	
_________ is stated as surface integral of the component of curl F along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function F taken along the closed curve C.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	
________ is stated as the surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem	

	Green's Theorem
	Green's Theorem
	Gauss Theorem
	Stoke's Theorem
	Gauss Theorem

Questions	opt1	opt2
An equation involving one dependent variable and its derivatives with respect to independent variable is called ___________	Ordinary Differential Equation	Partial Differential Equation
The ODE of the first order can be written as	F(x,y,s,t)	F(x,y,z,p,q)
C.F+P.I is called_____solution	Singular	Complete
The roots of the A.E of D.E, (D^2-2D+1)y=0 are	(0 1)	(3 2)
The quadratic equation of roots are real and distinct. What is the Complementary function?	$C.F = Ae^{m_1x} + Be^{m_2x}$	
The order of the (D^2+D)y=0 is	2	1
The roots of the A.E of D.E, (D^4-1)y=0 are	(1 ,1, 1, 1)	(1, 1, -1, 1)
The roots of the A.E of D.E, (D^3-D^2+D-1)y=0 are	(1,-i, i)	(i, i, -i)
The roots of the A.E of D.E, (D^3-7D-6)y=0 are	(1, 2, 3)	(1, -2, 3)

opt3	opt4	opt5	opt6	Answer	
Difference Equation	Integral Equation			Ordinary Differential Equation	
F(x,y,z)	F(x,y,y')=0			F(x,y,y')=0	
General	particular			General	
(1 2)	(1 1)			(1 1)	
				$C.F = Ae^{m_1 x} + Be^{m_2 x}$	
0	-1			2	
(1 ,-1, 1, -1)	(1, -1, i, -i)			(1, -1, i, -i)	
(1, i, -i)	(1, 1, 1)			(1, -i, i)	
(3, 2, -1)	(-1, -2, 3)			(-1, -2, 3)	

The degree of the (D^2+2D+2) y=0 is	1	3
The particular integral of (D^2-2D+1)y=e^x is__________	((x^2)/2) e^x	(x/2) e^x
The roots of the A.E of D.E, (D^2-4D+4)y=0 are	(2, 1)	(2, 2)
If y=ax+b then differentiating with respect to x=________	a	a+b
A Differential Equation is said to be _____________ if the dependent variable and its differential co-efficient occur only in the first degree.	Linear equation	Non-Linear equation
The P.I of the Differential equation (D^2 -3D+2) y=12 is________	1 / 2	1 / 7
If f(D)=D^2 -2, (1/f(D))e^2x=______	(1 / 2) e^x	(1 / 4) e^2x
If f(D)=D^2 +5, (1/f(D)) sin 2x =______	sin x	cos x
To transform (xD^2+D+7)y=1/x into a linear differential equation with constant coefficient. Put x=______	e^(-t)	e^(2t)

0	2			1	
((x^2)/4) e^x	((x^3)/3) e^x			((x^2)/2) e^x	
(2, -2)	(-2, 2)			(2, 2)	
b	ab			a	
Homogeneous equation	Non-Homogeneous equation			Linear equation	
6	10			6	
(1 / 2) e^(-2x)	(1 / 2) e^2x			(1 / 2) e^2x	
sin 2x	-sin 2x			sin 2x	
e^(t)	e^(-2t)			e^(t)	

The particular integral of (D^2 +19D+60)y= e^x is_______	(-e^(-x))/80	(e^(-x))/80
The particular integral of (D^2+25) y= cosx is __________	(cosx)/24	(cosx)/25
The particular integral of (D^2+25) y= sin4x is ___________	(-sin4x)/9	(sin4x)/9
The particular integral of (D^2+1) y= sinx is ____________	xcosx/2	(-xcosx)/2
The particular integral of (D^2 -9D+20)y=e^(2x) is____________	e^(2x) /6	e^(2x) /(-6)
The particular integral of (D^2 +D-72)y= e^(7x) is ___________	e^(7x)/16	e^(-7x)/16
The particular integral of (D^2-1) y= sin2x is __________	(-sin2x)/5	sin2x/5
The particular integral of (D^2+2) y= cosx is __________	(-cosx)	(-sinx)
In a PDE, there will be one dependent variable and ____ independent variables	only one	two or more

(e^x)/80	(-e^x)/80			(e^x)/80	
(-cosx)/24	(-cosx)/25			cosx/24	
(sin4x)/41	(-sin4x)/41			(sin4x)/9	
(-xsinx)/2	xsinx/2			(-xcosx)/2	
e^(2x) /12	e^(2x) /(-12)			e ^ (2x) /6	
e^(7x)/(-16)	e^(-7x)/(-16)			e^(7x)/(-16)	
sin2x/3	(-sin2x)/3			(-sin2x)/5	
cosx	sinx			cosx	
no	infinite number of			two or more	

The ______ of a PDE is that of the highest order derivative occurring in it	degree	power
The degree of the a PDE is ______of the higest order derivative	power	ratio
Afirst order PDE is obtained if______	Number of arbitrary constants is equal Number of independent variables	Number of arbitrary constants is lessthan Number of independent variables
In the form of PDE, f(x,y,z,a,b)=0. What is the order?	1	2
What is form of the z=ax+by+ab by eliminating the arbitrary constants?	z=qx+py+pq	z=px+qy+pq
A solution obtained from the complete integral by giving paticular values to the arbitrary constant is called a _____ solution.	complete	general
The solution f(x,y,z,a,b)= 0 of the first order PDE, Which contains two arbitrary constants is called a ____ solution.	complete	general
General solution of PDE F(x,y,z,p,q)=0 is any arbitray function F of specific functions u,v is____satisfying given PDE	F(u,v)=0	F(x,y,z)=0
The Lagrange's linear PDE is of the form _______	Pp+Qq=r	Pp+Qq= R

order	ratio			order	
degree	order			power	
Number of arbitrary constants is greater than Number of independent variables	Number of arbitrary constants is not equal to Number of independent variables			Number of arbitrary constants= Number of independent variables	
3	4			1	
z=px+qy+p	z=py+qy+q			z=px+qy+pq	
particular	singular			particular	
particular	singular			complete	
F(x,y)=0	F(p,q)=0			F(u,v)=0	
Pp+Qp= R	Pq+Qq= R			Pp+Qq= R	

_________ is of the form of the Lagrange's auxiliary equation	dx/P=dy/Q=dz/R	dx/Q=dy/P=dz/R
The complete solution of the PDE, pq=1 is____	z=ax+(1/a)y+b	z=ax+y+b
The order and degree of the solution of the PDE is y=f(y+x)+g(y+x)+e^2x_______	1 and 2	2 and 1
The complete solution of clairaut's equation is _____	z=bx+ay+f(a,b)	z=ax+by+f(a,b)
The clairaut's equation can be written in the form	z=px+qy+f(p,q)	z= py+qx
From the PDE by eliminating the arbitrary function from z=f(x^2 -y^2) is	xp+yq=0	p=-(x/y)
Which of the following is the type f(z,p,q)=0 ?	p(1+q)=qx	p(1+q)=qz
The equation (D^2 z+2xy(Dz)^2+D'=5 is of order ____and degree____	2 and 2	2 and 1
The complementry function of (D^2 -4 DD'+4D'^2)z=x+y is	f(y+2x)+xg(y+2x)	f(y+x)+xg(y+2x)

dx/R=dy/Q=dz/P	dx/P=dy/R=dz/ Q			dx/P=dy/Q=dz/R	
z= ax+(1-2x)/y+c	z=ax+b			z=ax+(1/a)y+b	
0 and 1	1 and 1			2 and 1	
z=ax+by	z=f(a,b)			z=ax+by+f(a,b)	
z=px+f(a,b)	z=py+qy+f(p,q)			z=px+qy+f(p,q)	
q=yp/x	yp+xq=0			yp+xq=0	
p(1+q)=qy	p=2x f(y+2x)			p(1+q)=qz	
1 and 1	0 and 1			2 and 1	
f(y+x)+xg(y+x)	f(y+4x)+xg (y+4x)			f(y+2x)+xg(y+2x)	

The solution of xp+yq=z is ______	f(x^2,y^2)=0	f(xy,yz)
The solution of p+q=z is _____	f(xy,ylogz)=0	f(x+y, y+logz)=0
The roots of the PDE(D^2-2DD'+D' ^2)z=0 are	0,1	i,-i
The particular integral of e^(ax+by)/ (D-(aD' /b))^2 is ------	e^(ax+by)	(x2/2) e^(ax+by)
The particular integral of e^(ax+by)/ (D-(aD' /b)) is ---------	ax-by+c	e^(ax+by)
The subsidiary equations of the Lagrange's equation $(z-y)p+(x-z)q=y-x$ is____	$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}$	

f(x,y)=0	f(x/y ,y/z)=0			f(x/y ,y/z)=0	
f(x-y, y-logz)=0	f(x-y,y+logz)=0			f(x-y, y-logz)=0	
1,2	1,1			1,1	
ax-by+c	ax+by			(x2/2)e^(ax+by)	
ax+by	xe^(ax+by)			xe^(ax+by)	
$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{z-x}$	$\frac{dx}{x-y}=\frac{dy}{z-y}=\frac{dz}{y-x}$			$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}$	

Questions	opt1	opt2	opt3
The Taylor,s series of f(x,y) at the point (0,0) is ___________ series.	Maclaurins	Taylor	power
The expansion of f(x,y) by Taylor series is __________	zero	unique	minimum
The period of *cos nx*, where n is the positive integer is ________.	2π/n	n/2π	2π
f(x,y) =e^x siny at (1,π/2)then ___________	f=0	f=1	f=2
f(x,y) = e^xy at(1,1) then __________	f=1	f=e	f=0
Which of the following functions has the period 2π?	cos x	sin nx	tan nx
1/π ∫ f(x) sinnx dx between the limits c to c+2π gives the Fourier coefficient_____	a_0	a_n	b_n
If f(x) = -x for -π< x< 0 then its Fourier coefficient a0 is_______-	(π^2)/2	π/2	π/3
If a function satisfies the condition f(-x) = f (x) then which is true?	a_0 = 0	a_n = 0	a_0 = a_n = 0

opt4	opt5	opt6	Answer
binomial			Maclaurins
maximum			unique
nπ			2π/n
f=e			f=e
f=2			f=2
tan x			cos x
b_1			b_n
π			π
b_n = 0			b_n = 0

If a function satisfies the condition f(-x) = -f (x) then which is true?	a0 = 0	an = 0	a_0 = a_n = 0
Which of the following is an odd function?	sin x	cos x	x^2
Which of the following is an even function?	x^3	cos x	sin x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)
The function f(x) is said to be an even function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)
∫f(x) dx = 2∫f(x) dx between the limits -a to a if f(x) is ------	even	continuous	odd
∫f(x) dx = 0 between the limits -a to a if f(x) is ------	even	continuous	odd
If a periodic function f(x) is odd, it's Fourier expansion contains no ------ terms.	coefficient an	sine	coefficient a0
If a periodic function f(x) is even, it's Fourier expansion contains no ------ terms.	cosine	sine	coefficient a_0

b_n = 0			a_0 = a_n = 0
x^4			sin x
sin^2x			cos x
f(-x) = f(-x)			f(-x) = - f(x)
f(-x) = f(-x)			f(-x) = f(x)
discontinues			even
discontinues			odd
cosine			cosine
coefficient a_n			sine

In dirichlet condition, the function f(x) has only a ----- number of maxima and minima.	uncountable	continuous	infinite
In Fourier series, the function f(x) has only a finite number of maxima and minima. This condition is known as -------	Dirichlet	Kuhn Tucker	Laplace
In dirichlet condition, the function f(x) has only a ----- number of discontinuities .	uncountable	continuous	infinite
A sequence {2^n} is	Convergent	divergent	Oscillatory
A sequence (-1)^n+2 is	Convergent	divergent	Oscillatory
A sequence {2n+1/3n-2} is	Convergent	divergent	Oscillatory
A sequence {2n^2+n/3n^2-3} is	Convergent	divergent	Oscillatory
A sequence 5+(-1)^n is	Convergent	divergent	Oscillatory
The series ∑ cos(1/n) is	Convergent	divergent	Oscillatory

finite			finite
Cauchy			Dirichlet
finite			finite
unique			divergent
unique			Oscillatory
unique			Convergent
unique			Convergent
unique			Oscillatory
unique			Convergent

The series $\sum x^n/(n^3+1)$ at $x=1$ is	Convergent	divergent	Oscillatory
The series $1-(1/2^2)+(1/3^2)-(1-4^2)+\ldots$ is	Convergent	divergent	Oscillatory
The series $2-(3/2)+(4/3)-(5/4)+\ldots$ is	Convergent but not absolutely	divergent	absolutely Convergent
The series $1+(1/\sqrt{2})+(1/\sqrt{3})+\ldots$ is	Convergent but not absolutely	Oscillatory	divergent
In a series positive terms $\sum u_n$ if limit n tends to ∞ u_n/u_n+1 is not equal to zero then the series $\sum u_n$ is	Convergent	divergent	not Convergent
The series $1-(1/2)+1-(3/4)+1-(7/8)+\ldots$ is	Convergent	conditionally Convergent	absolutely Convergent
The series $(1/(a+1) -(1/(a+2) +(1/a+3)-(1/a+4)+\ldots$ convergent if	$a>0$	$a<0$	$a<-1$
The series $1-2x+3x^2-4x^3+\ldots$ where $0<x<1$ is	Convergent	divergent	Oscillatory
The series $1/(1+2^{(-1)}) +1/(1+2^{(-2)})+1/(1+2^{(-3)})\ldots$ is	Convergent	divergent	Oscillatory

Not unique			Convergent
Not unique			Convergent
Oscillates finitely			Oscillates finitely
absolutely Convergent			divergent
Oscillatory			not Convergent
Oscillatory			Oscillatory
a≤0			a>0
unique			Convergent
unique			divergent

The series whose nth term is ∑ sin (1/n) is	Convergent	divergent	Oscillatory
The series 2+(3/4)+(4/9)+(5/16)+…+(n+1)/n^2 +… is	Convergent	divergent	Oscillatory
If p and q are positive real number, then the series 2^p/1^q+3^p/2^q+4^p/3^q+… converges	p<q-1	p<q+1	p≥q-1
An ordered set of real number a_1,a_2,…a_n is called a_____________	Series	sequence	Montonic sequence
If a sequence has a ________,it is called a convergent sequence	Finite limit	Infinite limit	limit
A sequence is said to be bounded above if there exists a number k, such that _____ for every n.	a_n>k	a_n≥k	a_n≤k
Both increasing and decreasing sequence are called ______ sequence.	Convergent	Montonic	Bounded
If limit n tends to ∞ a_n is equal to _______then the sequence is said to be Convergent	finite and unique	Infinite	unique
If u1,u2,….un,…be an infinite sequence or real numbers,then u1+u2+….+un+…is called__________	infinite series	finite series	finite terms

Not unique			Convergent
Not unique			divergent
p≥q+1			p<q-1
Montonic sequence			sequence
Bounded			Finite limit
a_n<k			a_n≤k
divergent			Montonic
not unique			finite and unique
infinite terms			infinite series

The series 1+2+3+ +n+…+...∞ is	Convergent	divergent	Oscillatory
Every absolutely convergent series is a ______ series	Convergent	divergent	Oscillatory
Any convergent series of ________ terms is also absolutely convergent	negative	positive	zero
If limit n tends to infinite u_n/u_n+1 = m is a series of positive terms ∑ u_n is convergent if _____	m>0	m<1	m>1
If limit n tends to ∞ u_n/u_n+1 = m is a series of positive terms ∑ u_n is divergent if _____	m>0	m<1	m>1
If limit n tends to ∞ u_n/u_n+1 = m is a series of positive terms .when the ratio test fails	m>0	m<1	m>1

not unique			divergent
not unique			Convergent
unique			positive
m=1			m>1
m=1			m<1
m=1			m=1