15BTAR701 **SPACE MECHANICS** **3 1 0 4**
100

OBJECTIVES:

To study the basic concepts of orbital Mechanics with particular emphasis on interplanetary trajectorie

UNIT - I BASIC CONCEPTS

The Solar System – References Frames and Coordinate Systems – The Celestial Sphere – The Ecliptic – Motion of Vernal Equinox – Sidereal Time – Solar Time – Standard Time – The Earth's Atmosphere.

UNIT - II THE GENERAL N-BODY PROBLEM

The many body Problem – Lagrange – Jacobin Identity –The Circular Restricted Three Body Problem – Libration Points- Relative Motion in the N-body Problem – Two –Body Problem – Satellite Orbits – Relations Between Position and Time – Orbital Elements.

UNIT - III SATELLITE INJECTION AND SATELLITE ORBIT PERTURBATIONS

General Aspects of satellite Injections – Satellite Orbit Transfer –Various Cases – Orbit Deviations Due to Injection Errors – Special and General Perturbations – Cowell's Method – Encke's Method – Method of vibrations of Orbital Elements – General Perturbations Approach.

UNIT - IV INTERPLANETARY TRAJECTORIES

Two Dimensional Interplanetary Trajectories –Fast Interplanetary Trajectories – Three Dimensional Interplanetary Trajectories – Launchif Interplanetary Spacecraft – Trajectory about the Target Planet.

UNIT - V BALLISTIC MISSILE TRAJECTORIES AND MATERIALS

The Boost Phase – The Ballistic Phase –Trajectory Geometry- Optimal Flights – Time of Flight – Re – entry Phase – The Position of the Impact Point – Influence Coefficients. Space Environment – Peculiarities – Effect of Space Environment on the Selection of Spacecraft Material.

TEXT BOOK:

S.NO.	Author(s)	Title of the Book	Publisher	Year of Publication
1.	William Tyrrell Thomson	Introduction to Space Dynamics	Courier Corporation, New York.	2012
2.	Cornelisse J.W.	Rocket Propulsion and Space Dynamics	W.H.Freeman& Co. New York.	1984
3.	Sutton G.P	Rocket Propulsion Elements	John Wiley, New York.	2011

REFERENCES BOOKS:

S.NO.	Author(s)	Title of the Book	Publisher	Year of Publication
1.	WiilliamV.Wieser	Space Craft Dynamics	McGraw Hill, New York.	2007
2.	Van de Kamp P	Elements of Astro Mechanics	Pitman, New York.	1979
3.	Parker E.R.	Materials for Missiles and Spacecraft	McGraw-Hill, New York.	1982

WEB REFERENCES:

www.sccs.swarthmore.edu/users/08/ajb/.../N-body_problem.html
www.math.uvic.ca/faculty/diacu/Sphere15-releq.pdf
www.philsrockets.org.uk/interplanetary.pdf
www.princeton.edu/sgs/publications/sgs/archive/15_2_Forden.pdf
see.msfc.nasa.gov/

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FACULTY OF ENGINERRING

DEPARTMENT OF MECHANICAL ENGINEERING (Aerospace)

COURSE PLAN

Subject Name : **SPACE MECHANICS**
Subject Code : **15BTAR701 (Credits - 3)**
Name of the Faculty : **B.AKILAN**
Designation : **ASSISTANT PROFESSOR**
Year/Semester/Section : **IV Year/VII SEM**
Branch : **B.Tech Aerospace Engineering**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – I : SPACE ENVIRONMENT			
1.	1	Introduction to Space Flight Mechanics.	
2.	1	Fundamentals of Space Mechanics	T [1] ,R [1] ,R [2]
3.	1	Space environment	T [1] ,R [1] ,R [2]
4.	1	Peculiarities of space environment and its description	T [1] ,R [1] ,R [2]
5.	1	Effect of space environment on materials of spacecraft structure	T [1] ,R [1] ,R [2]
6.	1	Effect of space environment on astronauts	T [1] ,R [1] ,R [2]
7.	1	Manned space missions	T [1] ,R [1] ,R [2]
8.	1	Unmanned space missions	T [1] ,T [2]] ,R [1]
9.	1	Effect on satellite life time	T [1] ,T [2]] ,R [1]
10.	1	Objective type Questions discussion	
Total No. of Hours Planned for Unit - I			**10**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – II : PARTICLE DYNAMICS			
11.	1	Dynamics of Particles: reference frames and rotations	T [2] ,R [1] ,R [2]
12.	1	Energy, angular momentum	T [2] ,R [1] ,R [2]
13.	1	Kepler's laws of planetary motion and proof of the laws	T [2] ,R [1] ,R [2]
14.	1	Newton's universal law of gravitation	
15.	1	The general ballistic missile problem	T [2] ,R [1] ,R [2]
16.	1	Geometry of the trajectory	T [2] ,R [1] ,R [2]

17.	1	Free flight range equations	T [2] ,R [1] ,R [2]
18.	1	Flight path angle equation	T [2] ,R [1] ,R [2]
19.	1	Maximum range trajectory	T [2] ,R [1] ,R [2]
20.	1	Time of free flight	T [2] ,R [1] ,R [2]
21.	1	Objective type Questions discussion	
Total No. of Hours Planned for Unit - II			**11**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – III : ORBIT TRANSFER			
22.	1	General Aspects of orbit transfer	T [3] ,R [1] ,R [2]
23.	1	Satellite Orbit Transfer	T [3] ,R [1] ,R [2]
24.	1	Coplanar Transfer Hohmann and Bielliptic transfer	T [3] ,R [1] ,R [2]
25.	1	Orbital Change due to Impulsive Thrust	T [3] ,R [1] ,R [2]
26.	1	Non-coplanar Transfer and Interception and Rendezvous	T [3] ,R [1] ,R [2]
27.	1	Continuous Thrust Transfer effects	T [3] ,R [1] ,R [2]
28.	1	Critical inclination	T [3] ,R [1] ,R [2]
29.	1	Sun-synchronous orbits and J3 effects and frozen orbits	T [3] ,R [1] ,R [2]
30.	1	Earth’s triaxiality effects and east- west station keeping.	T [3] ,R [1] ,R [2]
31.	1	Objective type Questions discussion	
Total No. of Hours Planned for Unit - III			**10**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – IV : ATTITUDE DYNAMICS AND REENTRY VEHICLE			
32.	1	Rigid Body Dynamics	T [3] ,R [1] ,R [2]
33.	1	Attitude Control	T [3] ,R [1] ,R [2]
34.	1	Gravity Gradient Satellite	T [3] ,R [1] ,R [2]
35.	1	Dual Spin Satellite	T [3] ,R [1] ,R [2]
36.	1	Reentry flight dynamics	T [3] ,R [1] ,R [2]
37.	1	Fundamentals of entry flight mechanics	T [3] ,R [1] ,R [2]
38.	1	Fundamentals of entry heating	T [3] ,R [1] ,R [2]
39.	1	Entry vehicle design	T [3] ,R [1] ,R [2]
40.	1	landing and recovery techniques	T [3] ,R [1] ,R [2]
41.	1	Objective type Questions discussion	
Total No. of Hours Planned for Unit - IV			**10**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – V : ROCKET MOTION			
42.	1	Principle of operation of rocket motor	T [3] ,R [1] ,R [2]
43.	1	Thrust equation	T [3] ,R [1] ,R [2]

44.	1	One dimensional and two dimensional rocket motions in free space	T [3] ,R [1] ,R [2]
45.	1	One dimensional and two dimensional rocket motions in homogeneous gravitational fields	T [3] ,R [1] ,R [2]
46.	1	Description of vertical turn trajectories	T [3], T [2], R [1]
47.	1	Description of inclined turn trajectories	T [3], T [2], R [1]
48.	1	Description of gravity turn trajectories	T [3], T [2], R [1]
49.	1	Determinations of range and altitude	T [3], T [2], R [1]
50.	1	Simple approximations to burnout velocity.	T [3], T [2], R [1]
51.	1	Objective type Questions discussion	
52.	1	Previous Year Question paper Discussion	
Total No. of Hours Planned for Unit - V			**10+1**

TOTAL PERIODS : 52

TEXT BOOKS

T [1] – Introduction to Space Dynamics - William Tyrrell Thomson.
T [2] - Rocket Propulsion and Space Dynamics - Cornelisse J.W.
T [3] - Rocket Propulsion Elements – Sutton G.P.

REFERENCES

R [1] - Space Craft Dynamics – Wiilliam V.Wieser
R [2] - Elements of Astro Mechanics – Van de Kamp P

JOURNALS

J [1] - Aerospace Science and Technology - Journal - Elsevier
J [2] –Journal of Aerospace Engineering | ASCE Library
J [3] –The Aeronautical Journal - Royal Aeronautical Society

UNIT	Total No. of Periods Planned	Lecture Periods	Tutorial Periods
I	10	09	1
II	11	10	1
III	10	09	1
IV	10	09	1
V	09+2	09	2
TOTAL	52	46	6

I. CONTINUOUS INTERNAL ASSESSMENT : 40 Marks

(Internal Assessment Tests: 30, Attendance: 5, Assignment/Seminar: 5)

II. END SEMESTER EXAMINATION : 60 Marks

TOTAL : 100 Marks

FACULTY **COORDINATOR/ AERO** **HOD / MECH** **DEAN / FOE**

UNIT I

Introduction

To develop an understanding and a basic description of any dynamical system, a physical model of that system must be constructed which is consistent with observations. The fundamentals of orbital mechanics, as we know them today, have evolved over centuries and have continued to require improvements in the dynamical models, coordinate systems and systems of time. The underlying theory for planetary motion has evolved from spheres rolling on spheres to precision numerical integration of the equations of motion based on general relativity. Time has evolved from using the motion of the Sun to describe the fundamental unit of time to the current use of atomic clocks to define the second. As observational accuracy has increased, models have generally increased in complexity to describe finer and finer detail.

To apply the laws of motion to a dynamical system or orbital mechanics problem, appropriate coordinate and time systems must first be selected. Most practical problems involve numerous reference frames and the transformations between them. For example, the equations of motion of a satellite of Mars are normally integrated in a system where the equator of the Earth at the beginning of year 2000 is the fundamental plane. But, o includ the Mars non-spheric gravitational forces (Section 5.4) requires the satellite position in the current Mars equatorial system. Planetary ephemerides (Section 2.5) are usually referred to the ecliptic, so inclusion of solar or Jovian gravitational forces require transformations between the ecliptic and the equator. The correct development of these transformations is tedious and a prime candidate for implementation errors.

Likewise, there are usually numerous time systems in a problem. Spacecraft events might be time tagged by an on board clock or tagged with the universal time that the telemetry is received at the tracking station. In the latter case, tracking station clocks must be synchronized and the time required for the telemetry signal to travel from the s/c to the tracking station must be calculated using the s/c orbit. Depending on the precision desired, this time difference might require special and general relativistic corrections. The independent variable for the equations of motion is called ephemeris time or dynamical time which is offset from universal time. By international agreement, atomic time is the basis of time and is obtained by averaging and correcting numerous atomic clocks around the world. Finally, the location of the zero or prime meridian and the equator are defined by averaging observations of specified Earth "fixed" stations. The understanding of these and other coordinate systems and time systems is fundamental to practicing orbital mechanics.

In this chapter only first order effects will be discussed. This book will also limit coverage to the classical mechanics approach, i.e. special and general relativistic effects might be mentioned but will not be included in any mathematical developments. Calculation for precise orbital mechanics and spacecraft tracking must however include many of these neglected effects. The definitive

reference for precise definitions of models and transformations is the Explanatory Supplement to
The first issue that must be addressed in any dynamics problem is to define the relevant coordinate systems. To specify the complete motion of a spacecraft, a coordinate system fixed in the spacecraft at the center of mass is usually selected to specify orientation and a coordinate system fixed in some celestial body is used to specify the trajectory of the center of mass of the spacecraft. The interest here is primarily in the latter system.

Coordinate systems are defined by specifying

1. the location of the *origin*,
2. the orientation of the *fundamental plane*, and
3. the orientation of the *fundamental direction or line* in the fundamental plane.

The **origin** is the (0,0,0) point in a rectangular coordinate system. The **fundamental plane** passes through the origin and is specified by the orientation of the positive normal vector, usually the z-axis. The fundamental direction is a directed line in the fundamental plane, usually specifying the +x-axis. The origin, fundamental plane and fundamental line are defined either relative to some previously defined coordinate system or in operational terms. The definitions are usually specified in a seemingly clear statement like: "The origin is the center of mass of the Earth, the fundamental plane (x-y) is the Earth equator and the x-axis points to the vernal equinox." Left as details are subtle issues like the fact that the center of mass of the Earth "moves" within the Earth, that the Earth is not a rigid body and the spin axis moves both in space and in the body, and that the vernal equinox is not a fixed direction. Some of these details are handled by specifying the epoch at which the orientation is defined, i.e. Earth mean equator of 2000.0 is frequently used. Further, it must be recognized that there is no fundamental inertial system to which all motion can be referred. Any system fixed in a planet, the Sun, or at the center of mass of the solar system is undergoing acceleration due to gravitational attraction from bodies inside and outside the solar system. The extent to which these accelerations are included in the dynamical model depends on accuracy requirements and is a decision left to the analyst.

Like many other fields, conventions and definitions are often abused in the literature and this

abuse will continue in this text. So "the equator" is jargon for the more precise statement "the plane through the center of mass with positive normal along the spin axis." Likewise, angles should always be defined as an angular rotation about a specified axis or as the angle between two vectors. The angle between a vector and a plane (e.g. latitude) is to be interpreted as the complement of the angle between the vector and the positive normal to the plane. The angle between two planes is defined as the angle between the positive normals to each plane. The more precise definitions often offer computational convenience. For example, after checking the orthogonality of the direction cosines of the positive unit normal (usually +z axis) and the direction cosines of the fundamental direction in the plane (usually +x), the direction cosines of the +y axis can be obtained by a vector cross product. Thus, the entire transformation or rotation matrix is defined by orthogonal x and z unit vectors.

Common origins for coordinate systems of interest in astrodynamics include:

1. **Topocentric**: at an observer fixed to the surface of a planet,
2. **Heliocentric, Geocentric, Areocentric, Selenocentric**, etc.: at the center of mass of the Sun, Earth, Mars, Moon, etc.
3. **Barycentric**: at the center of mass of a system of bodies, i.e. the solar system, Earth-Moon system, etc.

Astronomical observations were traditionally referred to topocentric coordinates since the local vertical and the direction of the spin axis could be readily measured at the site. For dynamics problems, topocentric coordinates might be used for calculating the trajectory of a baseball or a launch vehicle. For the former case, the rotation of the Earth and the variation in gravity with altitude can be ignored because these effects are small compared the errors introduced by the uncertainty in the aerodynamic forces acting on a spinning, rough sphere. For the latter case, these effects cannot be ignored; but, gravitational attraction of the Sun and Moon might be ignored for approximate launch trajectory calculations. The decision is left to the analyst and is usually base on "back of the envelope" calculations of the order of magnitude of the effect compared to the desired accuracy.

Heliocentric, areocentric, etc. coordinates are traditionally used for calculating and specifying the orbits of both natural and artificial satellites when the major gravitational attraction is due to the body at the origin. During calculation of lunar or interplanetary trajectories, the origin is shifted from one massive body to another as the relative gravitational importance changes; however, the fundamental plane is often kept as the Earth equator at some epoch. Often in what follows only Earth geocentric systems are discussed, but the definitions and descriptions generally apply to planets and moons. Geocentric systems are either terrestrial or celestial. **Terrestrial** systems are fixed to the rotating Earth and can be topocentric, geocentric, or geodetic. **Celestial** systems have either the equator or the ecliptic as the fundamental plane and the vernal equinox as the fundamental direction.

1.2.1 Spherical trigonometry

Transformations of position and velocity vectors between coordinate systems are represented in matrix notation and developed by vector outer and inner products as mentioned above. However, the understanding of the basic concepts of spherical trigonometry is also a necessity when dealing with orbital mechanics. It is convenient to introduce the concept of the celestial sphere . The **celestial sphere** is a spherical surface of infinite radius. The location of the center of the celestial sphere is therefore unimportant. For example, one can think of the center as being simultaneously apart, the great circle is unique.

The distance or length between two points on the surface is the central angle subtended by the points, which is also the shorter arc length on the great circle connecting the points. Three points, not on the same great circle, form the vertices of a spherical triangle. The three sides are the great circle arcs connecting each pair of vertices ($0<a,b,c<\pi$ in Figure 1-1). The length of a side of a spherical triangle is often referred to as simply the "side." With each vertex is associated an "angle" ($0 < \alpha, \beta, \gamma < \pi$) that is, the angle between the planes that form the adjacent sides. A spherical triangle has the following properties:

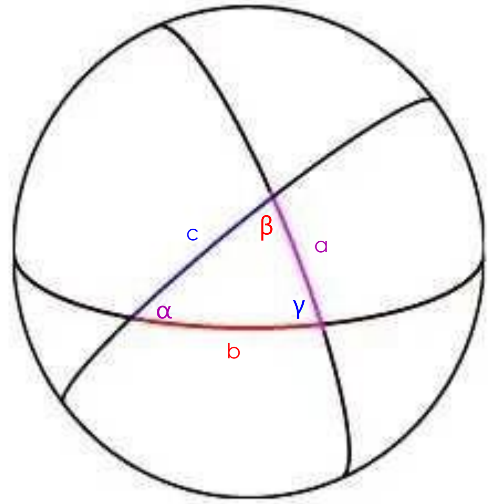


Figure 1-1. Spherical triangle

$$\pi < \alpha + \beta + \gamma < 3\pi$$
$$0 < a + b + c < 2\pi$$
$$a + b > c, \text{etc.}$$

Exercise 1-2. Draw a spherical triangle where both a+b+c is nearly zero and $\alpha+\beta+\gamma$ is nearly π. Draw a spherical triangle where both a+b+c is nearly 2π and $\alpha+\beta+\gamma$ is nearly 3π. Check equation (1-5) using the latter triangle.

Like plane trigonometry, spherical trigonometry relations involve four parts of the triangle. When three parts are known, the following four formulae are generally sufficient to obtain a solution for the fourth part (refer to Figure 1-1).

As in plane trigonometry there is the law of sines:

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma} \tag{1-1}$$

For spherical triangles there are two laws of cosines. The first is used when three sides and one angle are involved

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha \tag{1-2}$$

and the second is used when three angles and one side are involved

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a \tag{1-3}$$

[2] provides proofs of some spherical trigonometry formulae using vector analysis.

The solid angle subtended by the triangle is $\alpha+\beta+\gamma-\pi$ steradian, so if the sphere has radius R, the area of the spherical triangle is given by

$$\text{Area} = R^2(\alpha + \beta + \gamma - \pi) \tag{1-5}$$

A **right spherical triangle** has either a side or an angle of 90^o and equations (1-1) to (1-4) can be reduced to two rules and **Napier's Circle**. Consider the latter case and wolog assume $\gamma = 90^o$. Napier's Circle, shown in Figure 1-2, is created by putting the side opposite to the 90° angle at the top and proceeding around the triangle in the same direction to fill in the four remaining parts of the circle. The upper three parts are subtracted from 90°. Now consider any three parts of the triangle. The three parts will either be (1) "adjacent" parts, e.g. b, α and c in which case α would be called the "middle" part, or (2) two parts will be opposite the third part, e.g. b, α and β and β would be called the "opposite" part. **Napier's Rules of Circular Parts** are then:

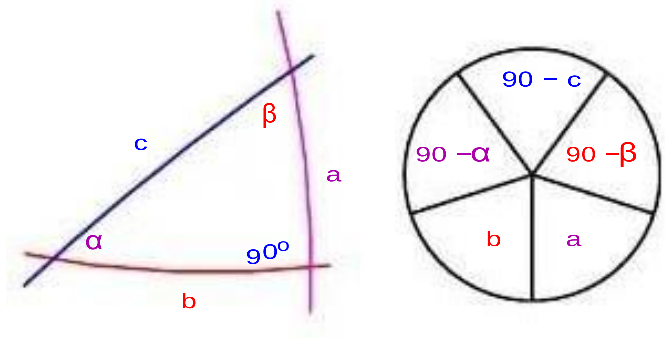


Figure 1-2. Napier's circle.

1. The sine of the middle part equals the product of the tangents of the adjacent parts.
2. The sine of the middle part equals the product of the cosines of the opposite parts.

As stated above, the first equation is used when the three parts of interest in the triangle are adjacent, e.g. a, β and c are related by $\cos(\beta)=\tan(a)\cot(c)$, which can be verified using equation (1-4). The second equation is used when one of the parts is opposite the other two, e.g. with b, α, and β: $\cos(\beta)=\cos(b)\sin(\alpha)$, which can be verified using equation (1-3). Note that the quadrant is not always determined from the basic equation. Since all parts are less than π, quadrant can not be determined from sine but can be determined from tangent or cosine. Therefore, care must be exercised in determining the quadrant.

Visit http://mathworld.wolfram.com/SphericalTrigonometry.html for additional spherical trigonometry relations.

1.2.2 Celestial coordinate systems

The two conventional **celestial coordinate** system [1,95], projected onto the celestial sphere, are shown in Figure 1-3. The two great circles or fundamental planes of interest are the equator of the the time of the vernal equinox. This convention is one of the few remaining concepts from Ptolemy. The angle between the equator and the ecliptic is known as the **obliquity** (ε). The obliquity for the Earth is approximately 23.45^o [1,171] and changes about 0.013^o per century. The two intersections of the ecliptic and the equator on the celestial sphere are known as the equinoctial points. When the Sun appears to move southward through the node, it is the **autumnal** equinox. The vernal equinox occurs within a day of March 21 and the autumnal occurs within a day of September 21. At either equinox, the length of the day and night are equal at all points on the Earth and the Sun rises (sets) due east (west). When the Sun has maximum northerly declination it is summer solstice in the northern hemisphere and winter solstice in the southern hemisphere, and conversely. At summer solstice in the northern hemisphere, the longest day occurs and the Sun rises and sets at maximum northerly azimuth. Nevertheless, due to the eccentricity of the orbit of the Earth, neither the earliest sunrise nor latest sunset occurs at summer solstice. A fact that, when properly phrased, has won small wagers from non-celestial mechanicians.

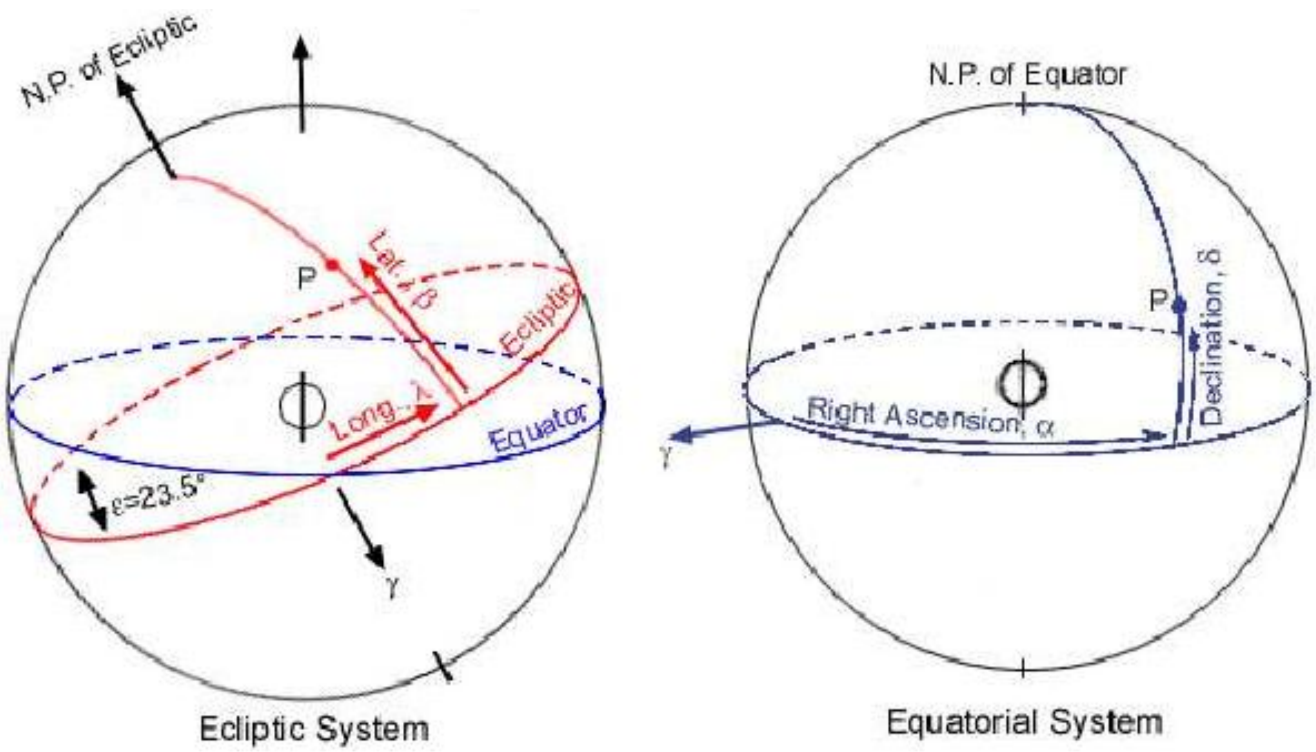


Figure 1-3. Celestial coordinate systems.

It must be recognized that neither the ecliptic nor the equator are fixed planes. Variations in the vernal equinox due to the motion of these planes are termed precession and nutation. Precession [1,99] is the secular component that produces a westward change in the direction of γ that is linear with time. Nutation [1,109] is the quasi-periodic residual that averages to zero over many years. The mean equator or ecliptic refers to the position that includes only precession. The true equator or ecliptic refers to the position that includes both precession and nutation. The Earth the Sun. If the equator was fixed, the planetary precession of the ecliptic would cause γ to move along the equator about 12" of arc per century and the obliquity would decrease by 47" per century. To eliminate the need to consider precession and nutation in dynamics problems, the coordinate system is usually specified at some epoch, i.e. mean equator and mean equinox of 2000.0, otherwise, known as J2000. In this case, Earth based observations must be corrected for precession and nutation. Transformations between the J2000 coordinates and the true or apparent systems are then required [1,145].

Another plane that is use in the celestial system is the invariant plane. The positive normal to the invariant plane is along the total angular momentum (i.e. rotational plus orbital) of the solar system. In Newtonian mechanics, only gravitational attraction from the distant stars and unmodeled masses can cause this plane to change orientation.

Consider some point P in the geocentric reference system of Figure 1-4. The position of point P is projected onto each fundamental plane.

In the equatorial system the angle from γ to this projection is call right ascension $(0 \le \alpha < 2\pi)$ and the angle between the point P and the equator is called the declination $(-\pi/2 \le \delta \le \pi/2)$. In the ecliptic system the corresponding angles are the celestial longitude $(0 \le \lambda < 2\pi)$ and celestial latitude $(-\pi/2 \le \beta \le \pi/2)$. The "celestial" qualifier is to assure no confusion with traditional terrestrial longitude and latitude. When the context is clear, the qualifier is often omitted. "Celestial" is also sometimes replaced with "ecliptic." The rotation matrix from the ecliptic system to the equatorial system is a single

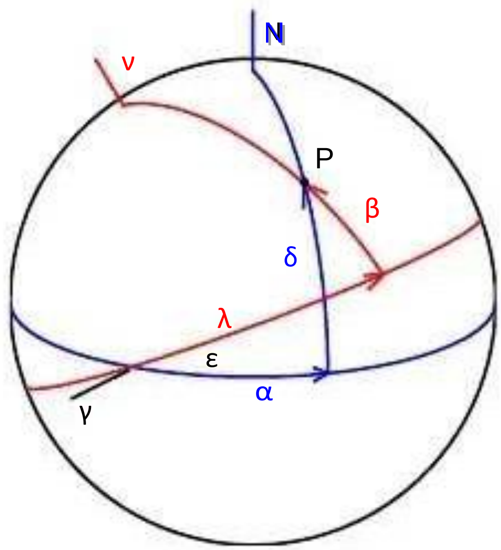


Figure 1-4. Transforming between celestial coordinate systems

rotation about the x axis by the obliquity ε. As the following example illustrates, solving spherical trigonometry problems often involves drawing numerous spherical triangle combinations until the proper combination of knowns and unknowns appears.

1.2.3 Terrestrial coordinate systems

Astrodynamics problems are generally framed in either the ecliptic or equatorial celestial coordinate system. The locations of observers, receivers, transmitters, and observation targets are usually specified in one of the terrestrial coordinate systems. A terrestrial coordinate system [1,199] is "fixed" in the rotating Earth and is either geocentric or topocentric. Transformations between terrestrial and celestial coordinates are an essential part of orbital mechanics problems involving Earth based observations. These transformations are defined by the physical

ephemeris (Section 1.4); that is, the definition of the pole location and the rotational orientation of the Earth. Precise definitions must include elastic deviations [1,237] in the solid Earth, plate tectonics [1,249], motion of the spin axis in the Earth [1,238], and numerous other effects. The largest of these effects is polar motion which produces deviations between the instantaneous and mean spin axis of order 10 meters. Pole location is determined by numerous observation stations and published by international agreement. Irregularities in the rotational rate of the Earth can change the length of the day by a few milli-seconds over time scales of interest for orbital

The fundamental terrestrial coordinate system has the origin at the center of mass and the equator as the fundamental plane. The intersection of the reference meridian with the equator is the fundamental direction. The origin, the equator, and reference meridian [1,223] are defined operationally by measurements made at a number of "fixed" stations on the surface. In the past, the prime meridian was the Greenwich meridian and was defined by the center of a plaque at Greenwich. The phrase "reference meridian" is used to clearly distinguish the fundamental difference in definitions. Nevertheless, the reference meridian is often referred to as the Greenwich meridian, and that practice will be used herein. For remote solid planets, prime meridians are still defined by easily observed sharp surface features. An observer's local meridian is defined by the plane through the observer that also contains the spin axis of the Earth. An observer's longitude (λ) is the angle between the reference meridian and the local meridian, more precisely referred to as "terrestrial longitude." Since the spin axis moves in the Earth, an observers true longitude deviates from the mean longitude.

specify the convention. For example, 75^{o}W = 285^{o}E longitude. Colatitude is the angle between the position vector and the normal to the equator and is unambiguous, but latitude is sometimes specified by using a sign convention e.g. -37.5°=37.5°S. Also note that geocentric latitude is often denoted by φ, i.e. the "prime" is omitted when the meaning is clear.

Geodetic coordinates are generally limited to points near the surface of the Earth. Geodetic latitude is the angle between the local vertical and the equator. The local vertical is determined by the local "gravity" force which is the combination of gravity and a centrifugal contribution due to rotation. An equipotential surface for the two terms is nearly an ellipsoid of revolution. Hence it is convenient to define a reference ellipsoid (spheroid) for the mean equipotential surface of the Earth which is approximately the mean sea level. This ellipsoid, which is symmetric about the equator and has rotational symmetry about the pole, is defined by the

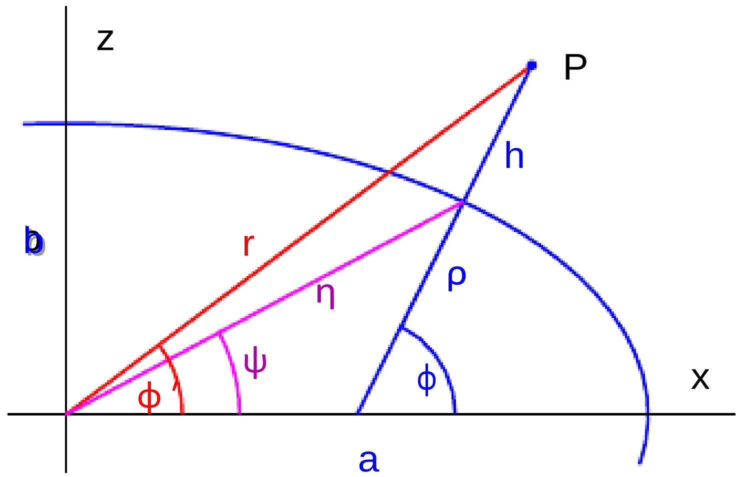


Figure 1-5. Geodetic and geocentric latitude

equatorial radius (a) and the flattening (f). The polar radius is given by b =a(1-f). Reference values [1,223] are a=6378137m and 1/f=298.25722. Figure 1-5 shows a cross section of the reference ellipsoid with greatly exaggerated flattening. For the figure, it is assumed that the cross section contains the x-axis, so the equation of the elliptical cross-section is

$$f(x,z) = \frac{x^2}{a^2} + \frac{z^2}{b^2} - 1 = 0 \tag{1-7}$$

Exercise 1-4. The gradient of f, ∇f, when evaluated at a point on the surface of the reference ellipsoid (h=0 in Figure 1-5) is a vector normal to the surface (B-2) and pointing outward. From this vector develop the following relationship between geodetic and geocentric latitude [3,78]

$$\tan\varphi = \frac{\tan\varphi'}{(1-f)^2} = \left(\frac{a}{b}\right)^2 \tan\varphi' \quad \text{(EE)} \tag{1-8}$$

Geodetic longitude and geocentric longitude are equal. For a point above the reference ellipsoid, the geodetic altitude (h in Figure 1-5) is defined as the closest distance to the reference ellipsoid and the geodetic latitude (φ in Figure 1-5) is defined as the angle between the normal to the ellipsoid and the equator at this closest point. Points with the same geodetic altitude are nearly on the same equipotential surface. Global atmospheric models, often used to calculate drag on a satellite Section 5.5.1, generally assume hydrostatic equilibrium and geodetic altitude is often an

$$
\begin{aligned}
x &= r\cos\varphi' = \eta\cos\psi + h\cos\varphi \\
z &= r\sin\varphi' = \eta\sin\psi + h\sin\varphi \\
\tan\psi &= (1-f)^2\tan\varphi
\end{aligned}
\tag{1-9}
$$

These equations can be solved by Newton-Raphson iteration or successive substitution for geodetic altitude and geodetic latitude and ψ, noting that η is a known function of ψ. Or ψ can be eliminated by noting that $\eta\cos\psi = aC\cos\varphi$ and $\eta\sin\psi = a(1-f)^2 C\sin\varphi = aS\sin\varphi$, where $C = \left[\cos^2\varphi + (1-f)^2\sin^2\varphi\right]^{-1/2}$ and $S=(1-f)^2 * C$. The transformation from geodetic to geocentric is obtained directly from equations (1-9).

Topocentric coordinate systems are also of interest. The origin of the system is fixed on the surface of the planet. For example, the location of a satellite relative to a ground based tracking system utilizes this frame. The fundamental plane for topocentric coordinates is either normal to the geocentric radius (geocentric topocentric) or tangent to the reference ellipsoid (geodetic topocentric). In both cases the fundamental plane is called the **horizon**. The points directly overhead and directly beneath the origin or observer are called the **zenith** and the **nadir**, respectively. The plane, formed by zenith and the north pole, is called the **meridian** and where it intersects the horizon is usually the fundamental line. Coordinates of points in the topocentric frame are specified by **range** (ρ), **azimuth** (A) and **elevation** (a). Range is the distance from the origin to the point. Azimuth is specified as

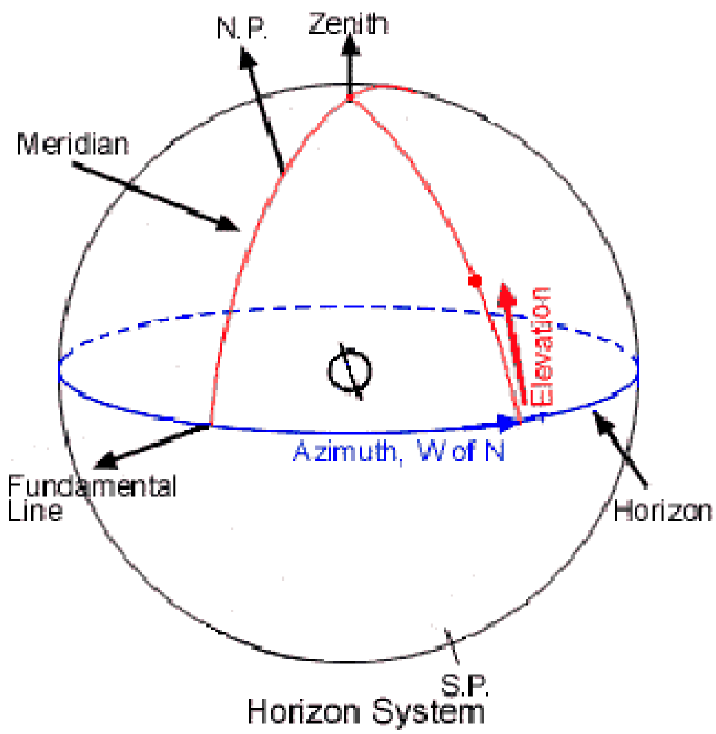


Figure 1-6. Topocentric coordinate system.

either east or west of North. Sometimes "east" is taken as the fundamental direction and azimuth is given as north or south of east. It is best to be explicit, e.g. 32.5° E of N. The elevation angle is zero for points on the horizon and 90° for points at zenith. The zenith angle is the complement of the elevation. Be aware that astronomers call elevation "altitude" and in Sonnet 116, Shakespeare calls it "height."

1.3 Time Systems

The above descriptions of the various spatial coordinate systems may initially leave the reader in a confused state of mind. The situation is only slightly better for time systems. St. Augustineand (2) the zero value or epoch i.e. like any well defined line, a slope and an intercept. In celestial
mechanics problems, three time systems are used. These are

Universal time or civil clock time which accounts for both the rotation and orbital motion of the Earth with respect to the Sun and is generally the independent variable for measurements.

Sidereal time which is a measure of the rotation of the Earth relative to the vernal equinox and locates the Earth based observer in the celestial coordinate system.

Ephemeris time or dynamical time which is the independent variable for orbit calculations and locates spacecraft, planets, etc. in the celestial coordinate system.

All of these times are related to the atomic time which is fundamental by international agreement. The following description of these four systems are short versions of and in some places approximations to the detailed descriptions given in Reference 1.

1.3.1 Atomic time

The fundamental unit of atomic time [1,40] is the Système International second or SI second. This is defined as the duration of 9,192,631,770 periods of the radiation from the transition between two levels of the ground state of the cesium-133 atom. This duration was adopted to be consistent with ephemeris time (Section 1.3.3). Within our current understanding of physics, the SI second is a fixed number. However, the definition is operational so measurements are required to determine atomic time. Further, relativistic corrections must be made to these Earth based measurements. The time standard that most closely follows the definition is the International Atomic Time or Temps Atomique International (TAI). TAI is supplied by the Bureau International des Poids et Measures in Sèveres, France. To obtain TAI an intermediate time scale is determined by combining data from a number of high-precision atomic standard clocks. This intermediate time scale is available in real time. After the fact, corrections are made for known effects to achieve a time as close as possible to atomic time. This adjusted time scale is published as the TAI.

1.3.2 Dynamical time

The independent variable in the equations of planetary motion [1,41] is dynamical time. Theories of relativity states that this value depends upon the reference coordinate system as well as the particular theory. To reduce periodic contributions and produce a nearly constant duration, the origin of the reference system is taken at the barycenter of the solar system and is called barycentric dynamical time (TDB). On the other hand, terrestrial dynamical time (TDT) is a theoretical time scale constructed from apparent geocentric ephemerides of bodies in the solar system. Dynamic time in other systems are then available by transformations and conversely.

s

1.3.3 Ephemeris time

Ephemeris time (ET) was developed as the independent variable for Newton's laws of motion and theory of gravitation. ET is a uniform time scale to depict observations of bodies in the solar system. There are three different forms of ET (ET0, ET1, and ET2), each based on more complex models of lunar motion. There is no detectable rate difference between ET and UTC (Section 1.3.6), but the epoch difference is updated with leap seconds. Although ephemeris time has been formally replaced by dynamical time the two are often used synonymously.

1.3.4 Julian date

The Julian date is simply a means of continuously counting the number of days from an epoch sufficiently far in the past to precede the historical record of astronomical observations. This continuous count is done with Julian day numbers. The first Julian day number (0) is defined as Greenwich mean noon on January 1, 4713 BC in the Julian proleptic calendar or Nov. 24, 4713 BC in the modern calendar. Note: JD starts at noon! Julian dates can be expressed in UT or dynamical time. For celestial reference coordinate systems, the epoch is defined in TDB. Thus, J2000.0 is 2000 January 1.5 TDB, which is JD 2451545.0 TDB. A Julian century is 36525 days. For convenience, the modified Julian date (MJD) was defined as the value of JD minus 2400000.5. MJD starts at midnight! There are a number of formula for converting from a Gregorian date to Julian date. The issue is of course how to handle the leap years. A year is a leap year if it not a century year and is divisible by 4. If the year is a century year, it is a leap year only if it is divisible by 400 (e.g. 1600 and 2000 are leap years while 1700, 1800, 1900 are not). The algorithm[3,61]

$$A = \langle Y / 100 \rangle \qquad B = 2 - A + \langle A / 4 \rangle$$
$$JD = \langle 365.25(Y + 4716) \rangle + \langle 30.6001(M + 1) \rangle + D + B - 1524.5 \tag{1-10}$$

is valid for all positive JD, Y is Gregorian year, M is month (3 to 14), D is day of the month including any fractional part. Note that month is increased by 12 for January and February and year is decreased by one. The 0.5 on the last term accounts for JD starting at noon and the 0.0001 addition is to assure largest integer operator performance. The symbol $\langle x \rangle$ is the largest integer operator which is the largest integer less than or equal to x. In MATLAB use the "floor" operator. The following inverse transformation is valid only for JD>2299161.

$$z = \langle JD + 0.5 \rangle \qquad f = JD + 0.5 - z$$
$$a = \langle (z - 1867216.25) / 36524.25 \rangle \qquad b = z + a - \langle a / 4 \rangle + 1525$$
$$c = \langle (b - 122.1) / 365.25 \rangle \qquad d = \langle 365.25c \rangle \qquad e = \langle (b - d) / 30.6001 \rangle$$
$$D = b - d - \langle 30.6001e \rangle + f \tag{1-11}$$
$$\text{If } e < 14, \quad M = e - 1 \quad \text{else} \quad M = e - 13$$
$$\text{If } M > 2, \quad Y = c - 4716 \quad \text{else} \quad Y = c - 4715$$

(DOY) can be calculated from

$$\begin{aligned} DOY &= \langle (275M)/9 \rangle - 2\langle (M+9)/12 \rangle + D - 30 \qquad \text{non – leap – year} \\ DOY &= \langle (275M)/9 \rangle - \langle (M+9)/12 \rangle + D - 30 \qquad \text{leap – year} \end{aligned} \tag{1-12}$$

Other useful algorithms on time transformations can be found in Reference 3 (Note that this reference is not without typographical error.)

1.3.5 Sidereal time

Sidereal time [1,48] is defined as the hour angle of the vernal equinox. An observers **hour angle** is the angle from the vernal equinox, measured eastward, to the observers meridian. As such, sidereal time is a measure of the diurnal rotation of the Earth. Apparent sidereal time is the hour angle of the true equinox as defined by the true equator and true ecliptic of date, i.e. apparent sidereal time includes the nutation in γ and therefore includes periodic inequalities. **Mean sidereal time** is the hour angle of the mean equinox and includes only the precession of γ and therefore only secular inequalities. Apparent sidereal time minus mean sidereal time is the equation of the equinoxes.

Sidereal time on the Greenwich meridian is called **Greenwich sidereal time** (Section 1.3.8). **Local sidereal time** is the Greenwich sidereal time added to the local east longitude. Sidereal time is traditionally stated in hours, minutes, and seconds with one hour corresponding to fifteen degrees of rotation relative to the vernal equinox. A **sidereal day** is defined as the period of consecutive passes of the equinox. Due to precession in γ the mean sidereal day is shorter than the period of rotation of the Earth by about 0.0084 seconds. The sidereal day begins with the first transit of the vernal equinox (sidereal noon) and ends with the second transit.

1.3.6 Universal time

The basis for all civil time-keeping [1,50] is known as Universal Time (UT). **Universal Time** is derived from the mean diurnal motion of the Sun and incorporates the rotational and orbital motion of the Earth with respect to the Sun. UT0 is determined directly from measurements of fixed stellar radio sources and depends on the observer location. UT0 accounts for variations in pole location and non-uniform rotation. These effects must be considered in precision orbital mechanics problems requiring tracking station location or any other geo-location to a few meters.

UT1 is obtained when UT0 is corrected for the shift in longitude caused by the motion of the pole relative to the surface of the Earth. UT1 is global because it is based on a mean pole location. UT1 is not a uniform time scale due to variations in the rotational rate of the Earth. The current definition of UT1 was created to fulfill the following conditions [1,51]:

1. UT1 is proportional to the angle of rotation of the Earth in space, reckoned around the true position of the rotation axis,

UT2 is UT1 corrected for variations in the Earth rotation rate and has a uniform rate but not the same as TAI. The final form of universal time, Coordinated Universal Time (UTC) is used by broadcast time services. UTC differs from TAI by an integer number of seconds and is kept within 0.9 seconds of UT1 by the use of leap seconds generally at the end of June or December.

By definition, UTC and TAI run at the same rate.

Figure 1-7 shows a graphical description of the relevant times scales. The reference is TAI on the x-axis. TAI vs. TAI has a slope of 1 and intercept at zero. TDT or ephemeris time has an observed slope of one but an offset of 32.184 seconds. UT1 on average has a slope less than 1 because of a rate difference between the rotation of the Earth and TAI. UTC is kept within 0.9 seconds of UT1 by introduction of a "leap second" (http://hpiers.obspm.fr/webiers/general/earthor/utc/UTC.html) as appropriate. There was a "leap second" at the beginning of 1999 and during the rest of the year there was a constant offset between TDT and UTC of 64.184 seconds. This is a critical number in celestial mechanics problems because the ephemerides are integrated in TDT (ephemeris time) and observations are time tagged in UTC.

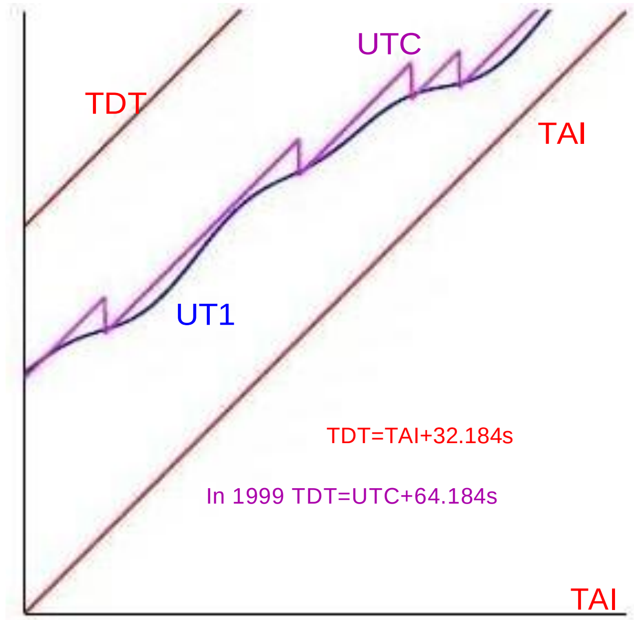


Figure 1-7. Summary of time scales.

1.3.7 UT1, UTC and Pole Location for 1998

Recall that UT1 is determined operationally by satellite tracking, lunar laser ranging and very long baseline radio interferometry (VLBI) data averaged ov r the globe. VLBI provides the most accurate measurements with an accuracy of about 0.00005 seconds when averaged over one day [1, 62]. Figure 1-8 shows the measured difference between UT1 and UTC during 1998. Since the difference between UT1 and UTC did not approach 0.9 seconds, there was no "leap second" in 1998. Recent leap seconds were Jan. 1, 1999, July 1, 1997, Jan. 1, 1996. There have been 22 leap seconds from 1972 through 2005 or about every 18 months. Leap seconds were included every year since 1972 except for 1984, 1986, 1987, 1989, 1995 and 1998-2005. A leap second will occur at the

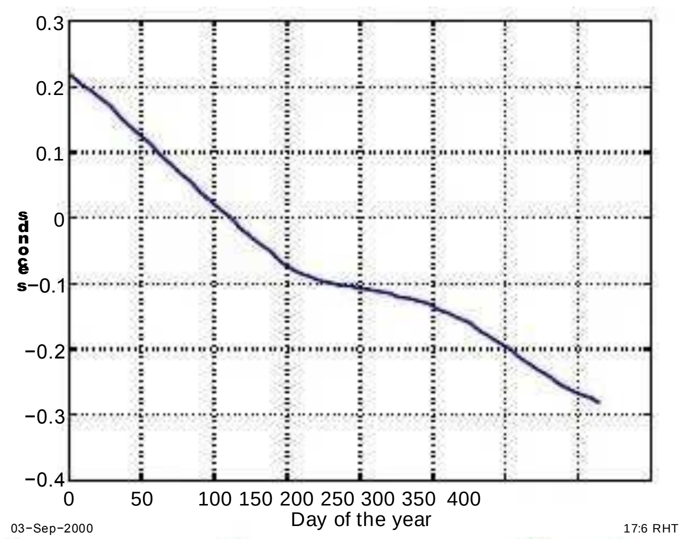


Figure 1-8. UT1 - UTC for 1998.

rily due to the location of the pole, but continental drift and other small effects contribute to the time difference. If the surface features of the Earth at some instant of time are considered fixed, then the instantaneous axis of rotation of this surface defines the pole of the Earth. Even if there were no external torques on the Earth, the pole would move with respect the surface due to natural precession of a torque free, rotationally symmetric, rigid body, as discussed in most dynamics book. For the Earth this motion is called the Chandler wobble after the person who provided an explanation in 1891 of the observed variation in latitude. If there were no external torques, the amplitude of this motion would be expected to damp to zero over long periods of time due to friction in the oceans and the elastic Earth. However continual excitation is provided by external torques due to lunar and solar gravity and internal motions of the Earth due to seasonal variations in atmospheric and ocean mass distributions, earthquakes, and any other phenomena that change planetary moments of inertia. The data were taken from http://hpiers.obspm.fr/webiers/general/earthor/polmot/PMOT.html.

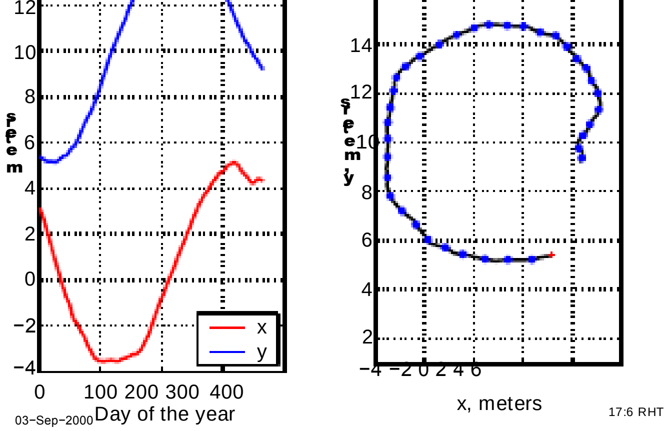


Figure 1-9. Pole location for 1998, x is along the Greenwich meridian, y is west.

Exercise 1-5. Visit http://maia.usno.navy.mil and http://hpiers.obspm.fr. Write a two page paper on what you discovered about UT1-UTC, pole location and anything else related to this chapter. E.g. make a plot of x vs. y pole location in meters and/or UT1-UTC for the last full year. Check the figures above.

1.3.8 Greenwich and local mean sidereal time

Greenwich mean sidereal time (GMST) is the angle between the Greenwich meridian and the mean vernal equinox and would be sensitive to the same variations in rotation as UT1. Due to the three conditions above, there is now a defined relationship between the two at midnight:

$$\begin{aligned} \text{GMST1-0hUT1} &= 24110.54841\text{sec}+8640184.812866T+0.093104T^2-6.2\text{x}10^{-6}T^3 \\ &= 100.4606184° + 36000.77005T + 0.00038793T^2 - 2.6\text{x}10^{-7}T^3 \end{aligned} \quad (1\text{-}13)$$

where in the first equation the coefficients are in seconds of time, T = d/36525 is the number of Julian centuries and d {±0.5, ±1.5...} is the number of days of UT elapsed since Julian Date 2451545.0 UT1 (Jan. 1, 12h, 2000). In the second form of the equation, GMST1 has been converted to degrees of rotation from γ to the Greenwich meridian by multiplying the first equation by 360/86400. The "1" at the end of GMST notates that the value is based on UT1 and correction to UT0 may be required for local observers.

Some consequences of these relationship are discussed in Reference 1. It is readily shown that the ratio of mean sidereal rate to UT1 rate is r'=1.002737909350795 plus secular terms that affect the

now in place. Refer to Figure 1-7 where "GM" represents the Greenwich meridian and "LM" represents the local meridian.

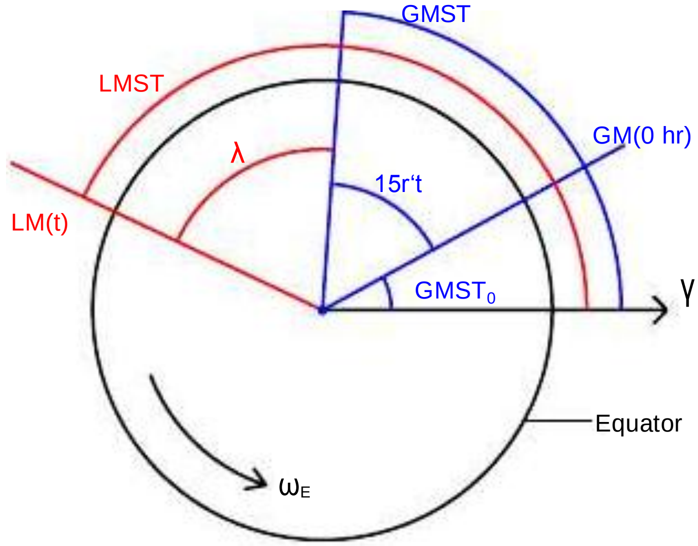


Figure 1-10. Local and Greenwich sidereal time

Given year, month, day and clock time or UTC in hours h, e.g. h=17.5678395 hrs:

1. Correct UTC to UT1 if necessary. This is only required if geographic locations to better than a few meters are required.
2. Use equation (1-11) to calculate the Julian date (JD) at 0 hours.
3. Calculate $T = \dfrac{JD - 2451545.0}{36525}$ Julian centuries from noon of Jan. 1, 2000.
4. Calculate the $GMST1_0°$ at 0 hours using equation (1-13).
5. Calculate the change in GMST1 since 0 hours using $\Delta GMST1° = 15r'h$.
6. Finally $GMST1° = GMST_0° + \Delta GMST1°$ gives the angle from the mean equinox to the Greenwich mean meridian.

The local mean sidereal time at east longitude λ is $MST(\lambda°) = GMST1° + \lambda°$; thereby providing the final information necessary to transform between the mean celestial and geocentric, Earth fixed coordinate systems. Transformations to J2000 [1,99] would have to include precession (50.290966"/year) and nutation.

UNIT II

Introduction

The description of the motion of a system of n bodies due to their mutual gravitational attraction is the fundamental problem in orbital mechanics. Applications range from the stability of the solar system to the formation of galaxies. No closed form solution exists for the general n-body problem when n is greater than two. However, it was shown by Lagrange that solutions do exist for special cases of the three-body problem, all of which require that the motion of the bodies takes place in the same plane. These special cases will be discussed in Chapter 4.

Before discussing the n-body problem, some of the fundamental principles of mechanics will be reviewed. Among these are: Newton’s laws of motion, the concepts of work and energy, and the concept of angular momentum. It is also useful to be aware of the theory of general relativity equations of motion for the n-body problem.

2.1 Newtonian Mechanics

Newton formalized the physical laws which determine the dynamics of massive bodies. Based on earlier work of Galileo, Kepler and others, he established three laws of mechanics and one for gravitational attraction. These laws were adequate to predict the dynamical motion of the planets and terrestrial objects for hundreds of years. Only after significant increases in observational precision was it necessary to seek modifications. The laws were formulated for particles and integration over the volume is required for application to finite bodies. The laws are only valid in an inertial frame. It is often said that such a frame is at “rest” or moving with constant velocity. Such a statement implies the existence of some absolute frame to which such motion can be referred. It might be said that an inertial system is at rest or moving with uniform velocity relative to the fixed stars. The problem has now been transformed to defining the “fixed” stars. An equally acceptable definition is to say a system is inertial if Newton's laws of motion are valid in that system. For practical applications, the analyst can pick a system moving through space with origin at the center of mass of the solar system or perhaps one whose origin coincides with the center of the Earth. It may even be reasonable to regard a system of coordinates attached to the Earth's surface as inertial, provided the accelerations resulting from the translation and rotation of the system are negligible compared with the acceleration of the body under consideration. The choice of coordinate systems is purely an issue of the accuracy desired in the prediction of the motion, there is no system that is exact and the choice is left to the analyst.

2.1.1 Laws of motion

Newton’s three classic laws can be stated as follows:

First Law: *If there are no forces acting on a particle, the particle will move in a straight line with* or can be considered to be a particle if the physical dimensions are small compared to the distance to other bodies. In Newtonian mechanics, force and position are also fundamental notions requiring no definition. Denote by f the force vector and by v the velocity (i.e. the time derivative of position) in an inertial space.

Second Law: *A particle acted upon by a force moves so that the force is equal to the mass times the time rate of change of the velocity.*

In equation form

$$f = m \frac{v}{dt} = ma \tag{2-1}$$

If "force" is fundamental, then "mass" is just the proportionally constant and conversely. Force and mass can not be defined independently. The first law, which Galileo discovered by rolling spheres down incline planes, is a special case of the second law.

Third Law: *When two particles exert forces upon one another, the forces are of equal magnitude and in opposite directions.*

This law is often called the law of action and reaction. Denoting by f_{ij} the force exerted by particle j upon particle i, then the law states $f_{ij} = -f_{ji}$.

2.1.2 Law of universal gravitation

Newton's law of universal gravitation was based on Kepler's laws of planetary motion [Section 3.2] and is the force model required to satisfy the condition that the orbital period is proportional to the 3/2 power of the semi major axis. The universal gravitation law is stated as: *two particles of mass* m_1 *and* m_2 *attract each other with a force along the line joining the two particles and with a magnitude proportional to the product of the masses and inversely proportional to the square of the distance between the particles.* Following the notation above, this is mathematically

$$f_{ij} = \frac{-Gm_i m_j \underline{e}_{ij}}{r_{ij}^2} = \frac{-Gm_i m_j \underline{r}_{ij}}{r_{ij}^3} \tag{2-2}$$

Where e_{ij} and r_{ij} are the unit vector and position vector from m_j to m_i, and G is the universal gravitational constant (6.672×10^{-11} $m^3/kg/s^2$). The law is known as the inverse square law. In practice G is almost never used because observations determine the product GM to much higher precision than G can be determined. For the Earth GM=398600.5 km^3/s^2.

As shown in Figure 2-1, the mass m_1 is at a distance R from the shell center. From equation (2-2), m_1 is attracted with a force

$$f = Gm_1 \int \frac{\mathbf{r}}{r^3} dm_2$$

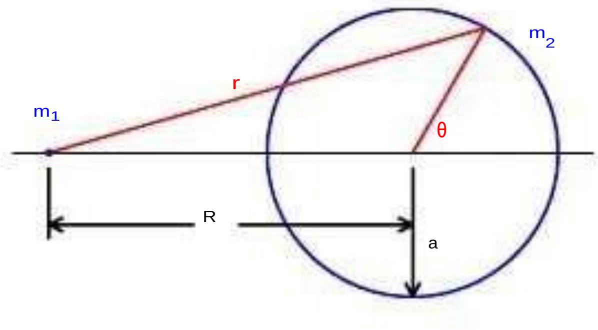


Figure 2-1. Attraction of a homogeneous spherical shell

Due to the symmetry of the problem, all components of f normal to the line between m_1 and the center of the shell will cancel, so the direction of the resulting force is along the line between m_1 and the center of m_2. By integration it can be shown that the magnitude is

$$f = \frac{Gm_1 m_2}{R^2} \tag{2-3}$$

Thus, the inverse square law holds for homogeneous spheres as well as for particles. This is one example where a finite mass can be considered a particle.

Exercise 2-1. Fill in the steps to verify equation (2-3).

2.1.3 Kinetic and potential energy

The concepts of work, kinetic energy and potential energy are also important in celestial mechanics. Work is a scalar quantity defined as the line integral of force along a particular path

$$W_{12} = \int_{r_1}^{r_2} \mathbf{f} \cdot d\mathbf{r} \tag{2-4}$$

between positions r_1 and r_2. Note that the definition has nothing to do with dynamics, particles or time, and implicit in the definition is the assumption that f depends only on position. The concept can be extended to work performed by the force that produces the motion of a particle by using Newton's second law to eliminate f in the integral and set $d\mathbf{r}$=vdt. In this case, it can be shown that the work done on particle m is just the change in kinetic energy between the end points W

$$= -\left(v^2 - v^2\right) \quad (2\text{-}5)$$

the work defined by equation (2-4) is independent of the path from r_1 and r_2. The work and change in kinetic energy are functions of the end points only.

Exercise 2-3. Using only the definition, show that for a conservative force the work performed in moving between two points is independent of the path taken to get from one point to the other.

The concept of potential energy at a point V(r) can now be introduced as the negative of the work done by a conservative force in going from a reference point r_o to an arbitrary point r

$$V(\mathbf{r}) = -\int_{r_0}^{r} f \cdot d\mathbf{r} + V(\mathbf{r}_0) \tag{2-6}$$

Within an additive constant, a scalar potential can therefore be uniquely associated with every point in space. So that the work done in going from r_1 to r_2 given by equation (2-5) can be expressed in terms of the potential as

$$W12 = V(\mathbf{r}1) - V(\mathbf{r}2)$$

This implies that

$$\int_{r_1}^{r_2} f \cdot d\mathbf{r} = -\int_{r_1}^{r_2} dV$$

or since r_1 and r_2 are arbitrary

$$f \cdot d\,r = -dV$$

which permits the force to be expressed as the negative gradient (B-2) of the potential

$$f = -\nabla V(\mathbf{r}) \tag{2-7}$$

The force is called a conservative force because total energy is conserved during the motion due to such a force. That is, if along the trajectory r_i and v_i are the position and velocity at time t_i and T_i and V_i are the corresponding kinetic and potential energies, then

$$T_1 + V_1 = T_2 + V_2$$

2.1.4 Linear and angular momentum

The linear momentum of a particle is the mass times the velocity

$$p=mv. \tag{2-8}$$

Newton's second law is often stated as the time rate of change of linear momentum equals the force. The moment of momentum or angular momentum is another important concept in mechanics. For a particle of mass m at position r and with linear momentum p=mv. The angular momentum about the origin from which r is measured is defined by

$$h = r \times mv \tag{2-9}$$

Often no distinction is make between angular momentum and specific angular momentum, i.e. angular momentum per unit mass. Even though two satellites of the Earth can have significantly different masses, if they are in the same orbit they will be said to have the same angular momentum. This is done because, as will be seen in Section 3.3, the orbital characteristics are determined by the sum of the masses of the Earth and the satellite and the latter is generally of negligible mass compared to the former.

2.2 Equations of Motion

While the two body problem discussed in Chapter 3 can be applied to many cases, and has the advantage of having a closed form solution, certain problems cannot be modeled with sufficient accurately using this assumption, and must be solved as a general system of n bodies. Consider a system of n bodies where each body is either spherical symmetry or sufficiently far from other bodies that each can be regarded as a point mass. It will be assumed that the only forces acting upon the system are due to the mutual Newtonian gravitational attraction. Let the mass and the position of each body in the system be denoted by mass m_i and r_i and the vector from mass m_j to mass m_i by $r_{ij}=r_i-r_j$. From Newton's second law and law of universal gravitation, for each mass

$j \neq i \quad r_{ij}$

where the notation j≠i means the sum over all values of j excluding i.

2.3 Integrals of the Motion

Equations (2-10) are a set of n second order, non-linear, coupled, ordinary differential equations and the solution will require 6n independent constants of integration. The constants of integration are usually determined from the n position vectors and the n velocity vectors at some epoch. Of the 6n required integrals of the motion only 10 are known. The relationships between these integrals and the physical assumptions are

1. No external forces and "action and reaction" assures conservation of total linear momentum,
2. Mutual force along the line between bodies and no external torques assures conservation of total angular momentum, and
3. Conservative force field and no external energy transfer assures conservation of total system mechanical energy.

Each of these conservation laws will now be demonstrated from the equations of motion (EOM).

2.3.1 Conservation of total linear momentum

The location of the center of mass (CM) of the system is given by

$$R = \frac{1}{M}\sum_{i=1}^{n} m_i r_i$$

where M is the total mass. Since this equation is true for any time, it can be differentiated with respect to time to get the EOM of the CM location. Performing this operation and eliminating the $\dot{r}_i$ using equation (2-10) yields $\ddot{R} = 0$. Integrating twice yields

$$R(t) = V_0(t - t_0) + R_0$$

Thus the CM of the system or barycenter of the system moves with constant linear velocity V_o. The vectors V_o and R_o represent six integrals of the equations of motion.

The total linear momentum, P is defined as the sum of all the individual linear momenta, i.e.

$P \equiv \sum_{i=1}^{n} m_i v_i = M\dot{R} = MV_0$. So that the total system linear momentum is conserved.

The total angular momentum, H is the sum of the individual angular momenta. As usual, to test for conservation of angular momentum, each EOM is pre-crossed with the corresponding position vector and the result is summed over all bodies. Since $\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r} \times \ddot{\mathbf{r}}$, $r_{ij} = -r_{ji}$ and $r_i \times r_{ij} = r_j \times r_i$, the result is

$$\frac{dH}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{n} r_i \times m_i \dot{r}_i\right] = 0 \qquad (2\text{-}11)$$

which represents the statement of the conservation of the total angular momentum. The three constant components of H constitute three additional integrals of the motion. Equation (2-11) implies that both the magnitude and direction of vector H are constant. The constant direction of H can be used to define a plane through the center of mass of the system. This plane was called the invariant plane by Laplace. For the solar system the invariant plane is inclined at about 1°35' with respect to the ecliptic, between the orbital planes of the two most massive planets Jupiter and Saturn. Except for the attraction of mass outside the solar system, the invariant plane is inertial in Newtonian mechanics.

Exercise 2-6. Fill in the steps to verify equation (2-11).

2.4.3 Conservation of energy

The total mechanical energy, E is the sum of the individual kinetic and potential energies. To test for conservation of energy, use the equations of motion to form an expression that looks like the rate at which the forces are doing work. This is accomplished by forming the dot product of v with each EOM and summing the results over all bodies to get the total rate at which work is being done. Since $r\frac{dr}{dt} = \frac{1}{2}\frac{d}{dt}(r^2) = \frac{1}{2}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r} \cdot \dot{\mathbf{r}}$, the sum can be written as

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{n} m_i(\dot{r}_i \cdot \dot{r}_i) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j \neq i}^{n} \frac{Gm_i m_j}{r_{ij}}\right) = \frac{d}{dt}(T + V) = 0 \qquad (2\text{-}12)$$

Where the first term is recognized as the total kinetic energy and the second term as the total potential energy. Thus total mechanical energy is conserved, i.e. T+V=E=constant.

2.4 Planetary Ephemerides

An ephemeris is a tabular representation of the motion of some body. A planetary ephemeris is a tabulation of the motion or trajectory of a planet and a satellite ephemeris is a tabulation of the motion or orbit of a satellite. Prior to modern computer technology and the GPS constellation, the planetary ephemerides were published annually as a listing of the position of the planets throughout the year. This information was very useful to astronomers and navigators. These tables were of sufficient accuracy for most optical observations. Early tables were recorded on paper and generally included the position and difference tables at uniform time intervals. Lagrange or other interpolation polynomials were used to determine intermediate positions. Current ephemerides are in a similar format but of course recorded on computer compatible media. Ephemerides can be defined with various levels of accuracy. The most accurate ephemerides are generated using the equations of motion from general relativity. Less accurate ephemerides are generated by omitting various terms from the equations of motion.

2.4.1 General relativity

The theory of general relativity is thought to completely describe the gravitational interaction of bodies [2]. However, the interaction is so complicated that even the one body problem, i.e. a particle with negligible mass being attracted by a body of finite mass, has not been solved. Approximations must be made to even write the equations of motion. For a body in the solar system the equation of motion is given to order $1/c^2$ as:

$$\ddot{r}_i = -\sum_{j \neq i} \frac{\mu_j r_{ij}}{r_{ij}^3}\left[1 - \frac{4}{c^2}\sum_{k \neq i}\frac{\mu_k}{r_{ij}} - \frac{1}{c^2}\sum_{k \neq j}\frac{\mu_k}{r_{jk}} + \frac{v_i^2}{c^2} + 2\frac{v_j^2}{c^2} - 4\frac{v_i \cdot v_j}{c^2} + \frac{3}{2c^2}\left(\frac{\mathbf{r}_{ij} \cdot v_j}{r_{ij}}\right)^2 - \frac{r_{ij} \cdot \dot{v}_j}{2c^2}\right] + \frac{1}{c^2}\sum_{j \neq i}\frac{\mu_j}{r_{ij}^3}\left[r_{ij} \cdot (4v_i - 3v_j)\right]v_{ij} + \frac{7}{2c^2}\sum_{j \neq i}\frac{\mu_j \dot{v}_j}{r_{ij}} - \sum_{m=1}^{5}\frac{\mu_m r_{im}}{r_{im}^3} \quad (2\text{-}13)$$

where the last term includes the Newtonian effects of the five largest asteroids. Note that the acceleration depends on the position, velocity and acceleration of the other bodies. Observe that most of the general relativistic terms are of the form $(v/c)^2$. Also note that the right hand side includes accelerations, a phenomena that can not occur in Newtonian mechanics. This equation is included just to demonstrate the complexity of calculating precision ephemerides.

elements. Over a century, errors in these simple models can be millions of kilometers for the outer planets and somewhat less for the terrestrial planets. Nevertheless, they are generally adequate for mission analysis studies and optical observations. Listed below are the orbital elements for Venus, Earth and Mars at J2000. The complete listing is given in [3,316].

Table 2-1. Planetary Orbital Elements

Planet	a	e	i	Ω	$\varpi = \omega + \Omega$	λ_o
Venus	0.72333199	0.00677323	3.39471	76.68069	131.53298	181.97973
Earth-Moon Barycenter	1.00000011	0.01671022	0.00005	-11.26064	102.94719	100.46435
Mars	1.52366231	0.09341233	1.85061	49.57854	336.04084	355.45332

The orbits elements are from left to right, semi-major axis in astronomical units [3,696] (1 AU= 149,597,870.66 km), eccentricity, inclination to the mean ecliptic (Section 1.2.2) of J2000, longitude of the ascending node relative to the mean equinox of J2000, longitude of perihelion, and mean longitude at J2000=JED 2451545.0 (Section 1.3.4). The argument of perihelion is ω and $\lambda_o = M_o + \varpi$ where M_o is the mean anomaly at J2000. The four angles are in degrees. Note

UNIT II

Introduction

The relative motion of two particles under their mutual gravitational attraction is the corner stone of the planetary ephemerides, lunar motion, the motions of planetary moons, and artificial satellite theories. Almost all interpretations of the effects of other forces, such as non-spherical gravity fields (Section 5.4.2), N-body gravitational attraction (Section 2.3), atmospheric drag (Section 5.5.1), and solar pressure (Section 5.4.5), are described in terms of perturbations, i.e. small or slowly varying changes to the two body solution [Chapter 5].

3.1 Kepler's Laws

Using the relatively precise measurements of his mentor, Tycho Brahe, the essentials of two body motion were determined empirically by Kepler and captured in the three simple laws:

1. Elliptic motion law: The heliocentric orbit of each planet is in a fixed plane and elliptical with the Sun at one focus (1609).
2. Equal area law: The line from the sun to the planet sweeps out equal area in equal time (1609).
3. Orbital period law: The square of a planetary period is proportional to the cube of the mean distance from the Sun (1619).

Kepler tried for a number of years to fit variations of moving circles and ovals to the observations of Mars, at that time the only planet with an observable eccentric orbit. It was on the verge of quitting that he tried an ellipse with the Sun at a focus [1,141].

3.2 Integrals of the Two Body Problem

The equations of motion for two particles are given by equations (2-10) with n=2

$$m_1\dot{r}_1 = \frac{-Gm_1m_2(\mathbf{r}_1-\mathbf{r}_2)}{r^3} \qquad m_2\dot{\mathbf{r}}_2 = \frac{-Gm_1m_2(r_2-r_1)}{r^3}$$

where $r = r_1 - r_2$ defines the relative position of m_1 with respect to m_2. With $M=m_1+m_2$, the equation of relative motion of m_1 with respect to m_2 is obtained by forming $\dot{r}$ and substituting from the equations above to obtain the fundamental equation of motion for the two body problem

3.2.1 Angular momentum.

From Section 2.4.2, the total system angular momentum is conserved. The specific relative angular momentum (simply referred to as angular momentum) $h = r \times v$ is also conserved. To test for conservation of angular momentum, it is natural to form the cross product of r with equation (3-1) to obtain $r \times \dot{v} = \dot{h} = 0$. Clearly angular momentum is conserved for any central force system. Since h is a vector, it represents three constants of integration and one immediate implication is that the relative position and velocity vectors must lie in the plane normal to h and through the center of mass of the reference body. This plane is called the orbit plane. Both bodies move in the same plane which contains the barycenter of the system. Kepler's observation that the planetary motion is planar is thus a result of the conservation of angular momentum. The second of Kepler's laws is also derived from this result as follows. Let θ be the angular position in the orbital plane measured from an arbitrary reference line. The magnitude of the angular momentum is the radial distance times the angular component of the velocity, i.e. $h = r^2\frac{d\theta}{dt}$. But, from elementary calculus the area sweep out in time dt is $\frac{1}{2}r^2 d\theta$. Thus, conservation of angular momentum implies that the orbital motion will sweep out equal area in equal time. This is a verification to Kepler's equal area law (Section 3.2.). Angular momentum can also be written as $h = rv\cos\gamma$ where γ is the flight path angle or angle between the velocity and the local horizontal.

3.2.2 Energy.

Total system energy is conserved as seen in Section 2.4.3. The test for relative conservation of mechanical energy is to form the rate at which the system forces are doing work i.e. $\dot{W} = v \cdot F$. Forming the dot product of the velocity with equation (3-1) yields

$$v \cdot \dot{v} = \frac{1}{2}\frac{d}{dt}(v \cdot v) = \frac{-\mu(r \cdot v)}{r^3} = \mu\frac{d}{dt}\left(\frac{1}{r}\right)$$

since $r\dot{r} = r \cdot v$ (B-2). This leads immediately to the energy integral

$$\frac{v^2}{2} - \frac{\mu}{r} = E \qquad (3\text{-}2)$$

be negative. Specifically, if initially $rv^2 < 2\mu$ then the energy is negative. In this case, the energy integral alone limits the motion to the bounded, spherical region $r < \mu/(-E)$. This spherical surface is called a zero velocity surface because it is a surface that can not be crossed once the energy is known. If, on the other hand, $E \geq 0$ there are no spatial limit to the motion provided by the energy integral. When E>0 equation (3-2) is often written as $v^2 = v_\infty^2 + \frac{2\mu}{r} = v_\infty^2 + v_e^2$, to show that there is a non-zero velocity as the particle approaches an infinite distance. The velocity at infinity is v_∞ which is zero if E=0 corresponding to an infinite radius for the surface of zero velocity. The escape velocity or parabolic velocity is v_e, which is the minimum velocity at distance r that will provide "escape" from the central body.

Exercise 3-1. Use the energy integral to show that if the initial conditions are such that $r_o v_o^2 = \mu$, then the maximum distance between the bodies is $2r_o$.

3.2.3 In-plane orbit geometry

Equation (3-1) describes the three dimensional motion; but, from above it is known that the motion is in the plane normal to h. If h=0 then r and v are co-linear and the motion is a straight line toward or away from the center of attraction. This case will be considered later. Otherwise, it is desirable to have a form of the EOM that only describes the motion in the plane. To this end, cross h($\neq$0) with equation (3-1) to get

$$\dot{v} \times h = \frac{-\mu}{r^3}(r \times h)$$

Recalling that h and r are orthogonal and using the magnitude of h from above gives

$$\frac{d}{dt}(v \times h) = \frac{\mu}{r^3} r (r^2 \dot{\theta} e_\theta) = \mu \dot{\theta} e_\theta = \mu \frac{d}{dt} e_r$$

since $\frac{d e_r}{dt} = \dot{\theta} e_\theta$ (B-2). Straight forward integration yields

$$v \times h = \mu(e_r + c) \tag{3-3}$$

where c is the vector constant of integration. It is seen that c lies in the orbit plane, is dimensionless and was obtained by the integration of a vector equation. Since c is in the orbital plane, it does not provide three new constants of integration. As shown below c provides the direction of the line of apsides of the conic orbit and thus c only provides the fifth of the necessary integrals. This can be seen by reducing equation (3-3) to a scalar equation by forming the dot

$$r = \frac{p}{1 + e\cos f} \qquad (3\text{-}4)$$

where the semi-latus rectum is $p=h^2/⍰$, the eccentricity $e = c \geq 0$ and the true anomaly f is the angle from the line defined by the minimum r (periapsis) to the current value of **r**. Note that p is completely determined by the angular momentum. For an ellipse $p=a(1-e^2)$ where a is the length of the semi-major axis of the ellipse. Equation (3-4) is called the equation of the orbit and is the mathematical statement of Kepler's elliptic motion law (Section 3.2).

Exercise 3-2. Starting with the equation of an ellipse with origin at the center, $\xi^2/a^2 + \eta^2/b^2 = 1$, show that equation (3-4) is the equation of the ellipse with origin at a focus and that $p=b^2/a$ where b is the length of the semi-minor axis.

If $p\neq 0$, the type of conic section is determined by the eccentricity. The minimum radius occurs when the denominator of equation (3-4) is a maximum i.e. when f=0. When e=0 the radius is a constant so the motion is in a circle. If e<1 there is a maximum radius at $f=\pi$, and the motion is elliptical. If $p\neq 0$ and e=1 the motion is parabolic and the radius is infinite at $f=\pi$. Finally, if e>1 the motion is hyperbolic and the asymptotes correspond to values of true anomaly (f_∞) that make the denominator zero, where $f_\infty = \cos^{-1}(-1/e)$. There is thus a range of forbidden values of true anomaly for hyperbolic motion. For hyperbolas, the geometric definition of p is $a(e^2-1)$ with a>0. In orbital mechanics it is convenient to set $p=a(1-e^2)$ and let a<0 for hyperbolic motion. It is generally clear by inspection which convention is being used. However, computers do not have such reasoning capability, so care must be exercised in computer programs to pick a convention and use it throughout all procedures.

Figure 3-1 shows the orbit geometry for the elliptical case. The periapsis distance is $r_p=a(1-e)$, apoapsis distance $r_a=a(1+e)$, semi-latus rectum $p=a(1-e^2)$ and semi-minor axis is $b = a\sqrt{1-e^2}$. The distance from the center to the focus is ae.

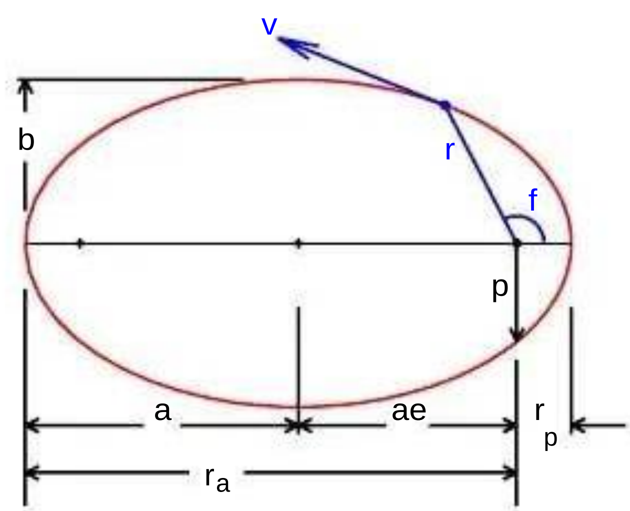


Figure 3-1. Elliptical orbit geometry

Regardless of the type of conic, at periapsis the velocity must be normal to the radius so $h^2=r_p^2 v_p^2=⍰p$. From above, $r_p=a(1-e)$ and $p=a(1-e^2)$ for elliptic and hyperbolic motion, so

$$v_p^2 = \frac{⍰}{a}\left(\frac{1+e}{1-e}\right) \qquad r_p > 0$$

r a

Exercise 3-3. Fill in the steps from equation (3-2) to equation (3-5).

By comparing equation (3-3) and the vis-viva integral at periapsis, it can be seen that $c = e\mathbf{e}_r$, i.e. c points toward periapsis and has magnitude equal to the eccentricity. So c is redesignated as e, the eccentricity vector and is given by

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \mathbf{e}_r \qquad (3\text{-}6)$$

Note that the eccentricity vector is NOT a unit vector. It is well defined for all cases with $e \neq 0$. Since r can never be zero, $\mathbf{e}_r$ is always defined. If $\mathbf{h} = \mathbf{r} \times \mathbf{v} = 0$ the conic is degenerate and the motion is rectilinear, i.e. a straight line either toward or away from the center of attraction. In either case $\mathbf{e} = -\mathbf{e}_r$, and as would be expected the periapsis is in the opposite direction of the position vector.

Exercise 3-4. Derive equation (3-6) starting with equation (3-3).

Exercise 3-5. Draw and annotate a sketch like Figure 3-1 for the parabolic and hyperbolic cases. Show the asymptotes for the latter case.

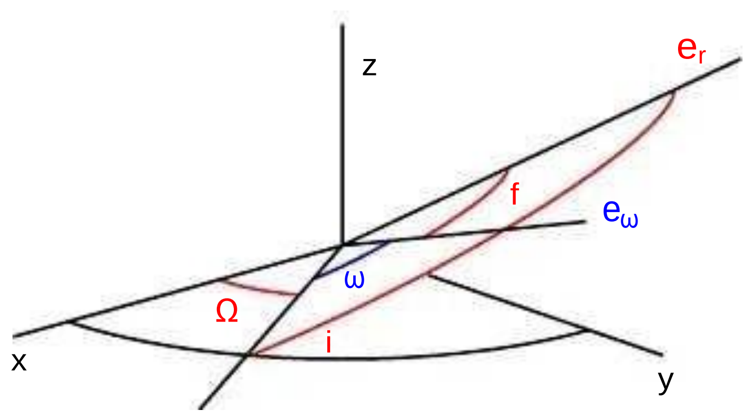
3.2.4 Orbital plane orientation

The orientation of the orbit plane in three dimensional space and the location of the line of apsides in the orbit plane are usually defined by the [3,1,3] Euler rotation angles [Ω, i, ω]. These angles are illustrated in Figure 3-2 and can be calculated from the angular momentum vector and eccentricity vector. Like all three parameter representations there is a singularity, i.e. a situation is which the angles are not unique. With latitude and longitude the singularity is at the pole where longitude is undefined. For the [3,1,3] rotation, the singularity is when $i=0$ or π and neither Ω nor ω are uniquely defined. However, the longitude of periapsis, $\varpi = \Omega + \omega$ may still be well defined if $e \neq 0$ even though it is not an angle in the usual sense.

If $h \neq 0$, the orbital inclination is given by

$$\cos i = \mathbf{e}_h \cdot \mathbf{e}_z = \frac{h_z}{h} \qquad 0 \leq i \leq \pi \qquad (3\text{-}7)$$

where the $h=0$ case is discussed in Section 3.11. An orbit with $i<\pi/2$ is said to be a direct orbit. An orbit with $i=\pi/2$ is called a polar orbit and if $i>\pi/2$ the orbit is said to be a



retrograde orbit.

by

$$e_\Omega = \frac{e_z \times e_h}{|e_z \times e_h|} \tag{3-8}$$

The cases where this definition does not provide a unique vector (i.e. $h=0$ or $h=he_z$) are discussed in Section 3.11. The longitude of the ascending node is given by

$$\sin\Omega = e_x \times e_\Omega \cdot e_z = \frac{h_x}{\sqrt{h_x^2 + h_y^2}} \qquad \cos\Omega = e_x \cdot e_\Omega = \frac{-h_y}{\sqrt{h_x^2 + h_y^2}} \qquad 0 \le \Omega < 2\pi \tag{3-9}$$

Which can also be written as

$$h_x = h\sin i\sin\Omega \qquad h_y = -h\sin i\cos\Omega \qquad h_z = h\cos i \tag{3-10}$$

and provide the tradition means of determining both inclination and longitude of the node.
If $e \neq 0$ define a unit vector toward periapsis $e_\omega = e/e$. The argument of periapsis is then given by

$$\sin\omega = e_\omega \cdot (e_h \times e_\Omega) \qquad \cos\omega = e_\omega \cdot e_\Omega \qquad 0 \le \omega < 2\pi \tag{3-11}$$

Equations (3-5) through (3-11) can be evaluated from the initial conditions $r(t_o)$ and $v(t_o)$ to determine the five Keplerian elements a, e, i, Ω, and ω as five constants of integration. The sixth orbital element, and last integration constant, is developed in the next section.

3.2.5 Motion in the orbital plane

None of the above five integrals of the motion explicitly involve time, i.e. given a time there is no relation above that will provide the position and velocity of the body. One approach to developing a relation between time and position in orbit can be derived from the vis-viva integral and conservation of angular momentum in the form

$$v^2 = \dot{r}^2 + r^2\dot{f}^2 = \dot{r}^2 + \frac{h^2}{r^2} = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

giving as the differential equation for r

$$\frac{dr}{dt} = \pm\frac{1}{r}\sqrt{-\frac{\mu r^2}{a} + 2\mu r - h^2} \tag{3-12}$$

This integral involves the square root of a quadratic polynomial and can therefore be integrated in terms of elementary functions to yield r as a function of time. The form of the solution depends on the sign of the coefficient of r^2, yielding regular trigonometric functions if the coefficient is negative (elliptical motion, a>0), hyperbolic functions if the sign is positive (hyperbolic motion, variable by a trigonometric variable. Without justification, new variables E and F are introduced for the elliptic and hyperbolic cases:

$$
\begin{array}{llll}
r = a(1 - e\cos E) & a > 0 & e \leq 1 & 0 \leq E < 2\pi \\
r = a(1 - e\cosh F) & a < 0 & e \geq 1 & -\infty < F < \infty
\end{array}
\tag{3-13}
$$

The elliptic eccentric anomaly, E, is seen to be a well defined variable permitting r to vary from a(1-e) to a(1+e) as required by the equation of the orbit, equation (3-4). Also it is seen that if 0<E<π then 0<f<π and likewise if π<E<2π then π<f<2π. Similarly for the hyperbolic eccentric anomaly F.

From this point, only the elliptical case will be developed in detail. Since E is well defined, the definition above can be differentiate with respect to time to yield $\dot{r} = ae\sin E\dot{E}$. After this expression and the definition are substituted into equation (3-12) a little algebra leads to $(1 - e\cos E)\dot{E} = \sqrt{\mu / a^3}$. This equation can be immediately integrated to yield Kepler's equation for the elliptical case

$$
M = n(t - \tau) = E - e\sin E \tag{3-14}
$$

which provides the sixth and final constant of integration τ, the time of periapsis passage. The mean motion is denoted $n = \sqrt{\mu / a^3}$. The time from one periapsis passage to the next is the period, P. Since E would change by 2π during this time, $P = \frac{2\pi}{n} = 2\pi\sqrt{a^3 / \mu}$, which is Kepler's orbital period law (Section 3.2). In practical orbital analysis it is not uncommon for t-τ to be larger than the orbital period. Thus the analyst must be prepared for |E| > 2π. The mean anomaly is defined by M=n(t-τ) and describes an angle that evolves linearly with time. The mean anomaly permeates orbital mechanics but is purely for notational convenience as a surrogate for time. From Kepler's equation, the difference between mean anomaly and eccentric anomaly is periodic and is never greater than the eccentricity.

Exercise 3-6. Make the substitution (3-13) in (3-12) to verify equation (3-14).

Following the same steps for the hyperbolic case and recalling that a<0 and e>1, Kepler's equation for hyperbolic motion can be shown to be

$$
M = n(t - \tau) = e\sinh F - F \tag{3-15}
$$

where $n = \sqrt{-\mu / a^3}$ and if t<τ then F<0. The concept of orbital period is of course meaningless for this case, nevertheless the notation M=n(t-τ) is still used. Most text do not associate Kepler with this equation since he did not derive it. Nevertheless, to shorten terminology, both elliptic and hyperbolic forms will be referred to as Kepler's equation.

1 + cosf 2 2 2 dt

can be integrated to yield Barker's equation

$$M = \sqrt{\frac{\mu}{p^3}}(t-\tau) = \frac{1}{6}\tan^3\frac{f}{2} + \frac{1}{2}\tan\frac{f}{2} \tag{3-16}$$

For notational continuity, M is also defined for parabolic motion, but the functional form is different than in Kepler's equation.

Exercise 3-7. Perform the elementary integration to derive equation (3-16).

In equations (3-14) through (3-16) the time of periapsis is an ephemeris time (Section 1.3.3) epoch often defined in either Julian day (Section 1.3.4) or YYMMDDHHMNSS.SS notation. Time must generally be carried to the microsecond level and is often represented by two numbers to maintain such accuracy. Typical representations are (1) modified Julian date (1.3.4) and seconds into the day, (2) year and seconds from beginning of the year, (3) YYYYMMDDHHMM and SS.SSS form and (4) year and day of the year.

This completes the development of the classical Keplerian orbital elements for the two body problem. For elliptical and hyperbolic motion a, e, i, Ω, ω and τ are utilized. For parabolic motion a and e are replaced by the single parameter p. Parabolic motion has only 5 independent parameters to define the orbit since it is known that e=1.

The trigonometric relationships between the true and eccentric anomalies can be derived directly from the equation of the orbit, equation (3-4), and (3-13), giving for f(E) and f(F)

$$\cos f = \frac{\cos E - e}{1 - e\cos E} \qquad \cos f = \frac{e - \cosh F}{e\cosh F - 1}$$

$$\sin f = \frac{\sqrt{1-e^2}\,\sin E}{1 - e\cos E} \qquad \sin f = \frac{\sqrt{e^2-1}\,\sinh F}{e\cosh F - 1} \tag{3-17}$$

Exercise 3-8. Invert equations (3-17) to obtain E(f) and F(f) as given in Table 3-1

3.3 Orbital Elements from Initial Position and Velocity

The calculation of the classical Keplerian orbital elements (a, e, i, Ω, ω, τ) given r and v at time t is relatively straight forward using the equations above. Modern tracking accuracies require that double precision calculations be performed for most orbits. The issues in calculating the orbital elements are (1) when to assume a non-degenerate orbit is parabolic ($e \approx 1$), and any special consideration for (2) the circular orbit case ($e \approx 0$), (3) the low inclination case ($i \approx 0, \pi$) and (4) the degenerate conic case ($h \approx 0$). One approach to these issues is given in Section 3.11. For the major axis. The reciprocal is used since it is well defined even for parabolic orbits.

2. Calculate angular momentum and related variables $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, h, and $p = h^2/\mu$.
3. Use equation (3-7) to determine inclination, i.
4. Use equations (3-9) or (3-10) to determine ascending node longitude, Ω.
5. Utilize equations (3-6) and (3-11) to determine argument of periapsis, ω.
6. Finally, τ, the time of periapsis is calculated using either equation (3-14), (3-15) or (3-16) depending on the sign of z. Quadrants are determined using r and $r\dot{r} = \mathbf{r} \cdot \mathbf{v}$ along with either:

   a. $\sin E = \frac{z^2 r\dot{r}}{ne}$ $\qquad \cos E = \frac{1}{e}(1 - zr)$ $\qquad$ for elliptical (z>0) motion.

   b. $\sin f = \dot{r}\sqrt{\frac{p}{\mu}}$ $\qquad \cos f = \frac{p}{r} - 1$ $\qquad$ for parabolic (z=0) motion.

   c. $\sinh F = \frac{z^2 r\dot{r}}{ne}$ $\qquad \cosh F = \frac{1}{e}(1 - zr)$ $\qquad$ for hyperbolic (z<0) motion.

Return z, p, i, Ω, ω and τ as the element set.

Exercise 3-9. From equations in Section 3.3, develop the expression for sinE in part a. and sinf in part b.

Numerous other sets of six elements have been developed. Some of these are combinations of Kepler elements utilized to eliminate a singularity for a particular problem. For examples, $P = e\sin\omega$ and $Q = e\cos\omega$ have been used for low eccentricity orbits, while $R = \sin i \sin\Omega$ and $S = \sin i \cos\Omega$ have been used for low inclination orbits. Various sets of "universal variables" have also been utilized. These are valid for all three types of orbits at the cost of introducing new functions defined in terms of infinite series [2,168] or in terms of continued fraction [1,187] or simply branching and using the equations above. However, the classical elements provide physical insight into orbit geometry and are adequate with careful handling of degenerate or nearly-degenerate cases as discussed in Section 3.11.

3.4 Solution of Kepler's and Barker's Equations

If the position and velocity are given at some time t, then either Kepler's or Barker's equation can be used to calculate the time of periapsis. On the other hand, these equations are transcendental functions of the anomalies. So if the orbital elements are given and position and velocity at time t

Referring to (3-16) let $B = 3\sqrt{\frac{\mu}{p^3}}(t-\tau) = 3M$, $A = \left(B + \sqrt{1 + B^2}\right)^{2/3}$ then the solution to Barker's equation is

$$\tan\frac{f}{2} = \frac{2AB}{1 + A + A^2} \tag{3-18}$$

Exercise 3-10. Verify that equation (3-18) is a solution by substitution into (3-16).

For the elliptical motion case there are two popular approaches to solving Kepler's equation. The first is successive substitution

$$E_{k+1} = M + e\sin E_k \tag{3-19}$$

and the second is Newton iteration

$$E_{k+1} = E_k + \frac{M - E_k + e\sin E_k}{1 - e\cos E_k} \tag{3-20}$$

Danby [2,149] and Meeus [3,181] provide excellent discussions of iteration method, valuations of various starting values, and the advantages of including higher order Taylor series'terms in the Newton iteration. Colwell [4] provides a history of solving this equation. Successive substitution is easiest to implement but can require 10 or more iterations even for e<0.1 and may not converge for e>0.8. The traditional starting value is $E_1 = M$, but Newton's method can become unstable due to the denominator being small when $|M|<\pi/6$ and 0.95<e<1. However, with the proper starting condition Newton's method will converge in less than five iteration for all M and any e<1 [3,181]. Danby [2,152] suggest an initial guess of $E_1 = M + 0.85\, e\, \mathrm{sign}(\sin(M))$ which will converge in six or less Newton iterations to eleven decimal places for $0 \le e < 1$.

Exercise 3-11. Implement both equation (3-19) and equation (3-20) with different starting conditions and evaluate the convergence properties for $0<M<2\pi$ and 0.05<e<0.95. Write a 2-3 page paper on the results. Use the starting values above and consider equation (3-23).

For the hyperbolic case e>1 so that successive substitution must take the form

$$F_{n+1} = \sinh^{-1}\left(\frac{n(t-\tau) + F_n}{e}\right)$$

to assure convergence. Recall that inverse hyperbolic functions can be written in terms of logarithms, for example, $\sinh^{-1}x = \log\left(x + \sqrt{1 + x^2}\right)$.

3.5 Position and Velocity from Orbital Elements

b.) Determine the eccentricity from p and z.

c.) Solve Barker's or Kepler's equation for the anomalies, $-\pi < f, E < \pi$ or F as appropriate.

d.) Calculate r, $\dot{r}$ and r $\dot{\theta}$

e.) Determine the position vector using the [3,1,3] rotation [Ω, i, ω+f] starting with (r,0,0)

f.) Determine the velocity vector using the [3,1,3] rotation [Ω, i, ω +f] starting with ($\dot{r}$,r$\dot{\theta}$,0)

The explicit transformations are

$$\mathbf{r} = r\,\mathbf{e}_r = r\begin{bmatrix} \cos\theta\cos\Omega - \sin\theta\sin\Omega\cos i \\ \cos\theta\sin\Omega + \sin\theta\cos\Omega\cos i \\ \sin\theta\sin i \end{bmatrix} \tag{3-21}$$

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\mathbf{e}_\theta = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\begin{bmatrix} -\sin\theta\cos\Omega - \cos\theta\sin\Omega\cos i \\ -\sin\theta\sin\Omega + \cos\theta\cos\Omega\cos i \\ \cos\theta\sin i \end{bmatrix} \tag{3-22}$$

where $\theta=\omega+f$ corresponds to the third rotation. Note that $\mathbf{e}_\theta = \dfrac{\partial \mathbf{e}_r}{\partial \theta}$.

3.6 Expansions for Elliptic Motion

In the era of analytic solutions it was often necessary to make approximations to arrive at any solutions at all. Taylor series expansions are a familiar tool. For periodic orbital motion the Fourier series representations are generally more useful and there are numerous such representations in the two body problem. These are developed in detail in a number of reference books [1,206], [5,33], [6, Chapter II] and will not be developed here. Such expansions are useful for making initial estimates for iterative solutions, for obtaining approximate solutions, for making order of magnitude estimates, and in orbit perturbation problems (Section 5.4.1). A few of these expansions are given below to terms through eccentricity cubed. The inversion of Kepler's equation yields

$$E = M + 2\sum_{k=1}^{\infty} J_k(ke)\,\frac{\sin kM}{k}$$

where J_k are Bessel functions of the first kind of order k. Bessel invented these functions for the

$$E = M + \left[e - \frac{e^3}{8}\right]\sin M + \left[\frac{e^2}{2}\right]\sin 2M + \left[\frac{3e^3}{8}\right]\sin 3M + O(e^4) \qquad (3\text{-}23)$$

In this equation and those below, any term in square brackets [] is a truncated infinite series. Within the radius of convergence, the complete expansion could be used to solve Kepler's equation without iteration. But, this is impractical since iteration is faster. Nevertheless, the first few terms can be used to obtain a first estimate for the iterative solution. It is to be noted that this series does not converge rapidly for large eccentricity. A similar expansion for r

$$\frac{r}{a} = 1 + \frac{e^2}{2} - \left[e - \frac{3e^3}{8}\right]\cos M - \left[\frac{e^2}{2}\right]\cos 2M - \left[\frac{3e^3}{8}\right]\cos 3M + O(e^4) \qquad (3\text{-}24)$$

can be used to directly estimate r(t) without solving Kepler's equation. Since integration of M over 2π is the same as integrating over an orbital period, note that the mean value of r over an orbit in not a. The expansion for true anomaly is also a double infinite sum [1,212]

$$f = M + \left[2e - \frac{e^3}{4}\right]\sin M + \left[\frac{5e^2}{4}\right]\sin 2M + \left[\frac{13e^3}{12}\right]\sin 3M + O(e^4) \qquad (3\text{-}25)$$

If $e < 0.01$, as is common for LEO and many other satellites, the last two equations can be used to calculate the position and velocity to six significant figures without solving Kepler's equation. Two additional series will be used in Chapter 5.

$$\frac{r}{a}\cos f = -\frac{3}{2}e + \left[1 - \frac{3}{8}e^2\right]\cos M + \left[\frac{e}{2}\right]\cos 2M + \left[\frac{3}{8}e^2\right]\cos 3M + O(e^3) \qquad (3\text{-}26)$$

$$\frac{r}{a}\sin f = \left[1 - \frac{5}{8}e^2\right]\sin M + \left[\frac{e}{2}\right]\sin 2M + \left[\frac{3}{8}e^2\right]\sin 3M + O(e^3) \qquad (3\text{-}27)$$

3.7 F and G Functions

The solution of equation (3-1) can be written as a Taylor series expanded about some time t_o with initial conditions $\mathbf{r}_o$ and $\mathbf{v}_o$ i.e. $\mathbf{r}(t) = \mathbf{r}(t_o) + \dot{\mathbf{r}}(t_o)(t - t_o) + \frac{1}{2}\ddot{\mathbf{r}}(t_o)(t - t_o) + \ldots$. Second and higher order derivatives can be eliminated using (3-1). Following Danby [2,163], let $\sigma = \mu / r_o^3$ and $\varepsilon = \dot{r}_o / r_o$, then $\ddot{\mathbf{r}}_o = -\sigma\mathbf{r}_o$ and it can be shown that $\left.\frac{d^3\mathbf{r}}{dt^3}\right|_o = 3\sigma\varepsilon\mathbf{r}_o - \sigma\mathbf{v}_o$ etc. So the series can be written [2,437] in terms of $\mathbf{r}_o$ and $\mathbf{v}_o$ and constants σ, ε and $\delta = (v_o / r_o)^2$.

$$\mathbf{r}(t) = \left[1 - \frac{\sigma}{2}(t - t_o)^2 + \frac{\sigma\varepsilon}{2}(t - t_o)^3 + \ldots\right]\mathbf{r}_o + \left[(t - t_o) - \frac{\sigma}{6}(t - t_o)^3 + \ldots\right]\mathbf{v}_o$$

of convergence. Thus the position and velocity can be written as

$$\mathbf{r}(t) = F(t,t_o)\mathbf{r}_o + G(t,t_o)\mathbf{v}_o \qquad \mathbf{v}(t) = \dot{F}(t,t_o)\mathbf{r}_o + \dot{G}(t,t_o)\mathbf{v}_o \tag{3-28}$$

Since $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{r}(t_o) \times \mathbf{v}(t_o) = \mathbf{h}$, $F\dot{G} - G\dot{F} = 1$ and also note that these equations are valid component wise, i.e. $x(t)=F(t,t_o)x_o + G(t,t_o)\dot{x}_o$. Because of slow convergence, this form has had limited utility except as a basis for analytic approximations over short times. A more useful form can be obtained by first introducing the orbital coordinate system (ξ, η, ζ). The origin of this system is the center of attraction and the fundamental plane is the orbit plane, i.e. the ζ axis is along $\mathbf{h}$. The ξ axis points to periapsis to define the fundamental direction and the η axis completes the right hand system pointing in the direction of the velocity at periapsis. The following relations can be developed from the above

$$\xi = r\cos f = a(\cos E - e) \qquad \frac{d\xi}{dt} = \frac{-na\sin f}{\sqrt{1-e^2}} = \frac{-na^2\sin E}{r}$$

$$\eta = r\sin f = a\sqrt{1-e^2}\sin E \qquad \frac{d\eta}{dt} = \frac{na(\cos f + e)}{\sqrt{1-e^2}} = \frac{na^2\sqrt{1-e^2}\cos E}{r} \tag{3-29}$$

Equations (3-28) are independent of the particular coordinate system chosen and so are equally applicable to the orbital system, i.e.

$$\xi_t = F(t,t_o)\xi_o + G(t,t_o)\dot{\xi}_o \qquad \eta_t = F(t,t_o)\eta_o + G(t,t_o)\dot{\eta}_o \tag{3-30}$$

If η_t and ξ_t are considered as being known, then these equations can be thought of as two equations in the two unknowns F and G so that

$$F(t,t_o) = \frac{1}{h}[\xi_t\dot{\eta}_o - \eta_t\dot{\xi}_o] \qquad G(t,t_o) = \frac{1}{h}[\eta_t\xi_o - \xi_t\eta_o] \tag{3-31}$$

where it is noted that the determinant of the coefficients of F and G is the angular momentum, h>0. Similar arguments can be made for the velocities. The states in the orbital coordinate system can be eliminated in favor of either the true or eccentric anomaly using equations (3-29) to yield

$$F(t,t_o) = 1 - \frac{r}{p}[1 - \cos(f - f_o)] = 1 + \frac{a}{r_o}[\cos(E - E_o) - 1]$$

$$G(t,t_o) = \frac{rr_o}{h}\sin(f - f_o) = (t - t_o) + \frac{1}{n}[\sin(E - E_o) - (E - E_o)]$$

$$\dot{F}(t,t_o) = \frac{-h}{p^2}[\sin(f - f_o) + e(\sin f - \sin f_o)] = \frac{-na^2}{rr_o}\sin(E - E_o) \tag{3-32}$$

at the respective times to obtain E and E_o. The orbit position can then be propagated from any time t_o to any time t using (3-28). The three Euler angles (Ω, ω, i) are not required in this approach.

Exercise 3-12. Utilize equations (3-29) through (3-31) to derive the first line of equations (3-32).

3.8 Coordinate System Rotation

The (3,1,3) rotation matrix Φ from the orbital coordinate system $\rho = (\xi, \eta, \varsigma)$ to the r=(x,y,z) system ($r = \Phi\rho$), where

$$\Phi = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix} \quad (3\text{-}33)$$

can be determined directly using the spherical trigonometry relations given in Section 1.2.1 or

from the multiplication of the three rotation matrices (B-1).

3.9 State Propagation

Mapping or propagating the state at time t_o to some other time t is one of the most common problems in orbital mechanics. For two body motion, two common approaches are

1. Transform the state at time t_o to orbital elements at t_o and then transform the orbital elements to the state at time t. This process would use the X2ORB and ORB2X procedures developed for the toolbox. This approach will also permit inclusion of secular and long period variations in the orbital elements due to perturbations to be discussed in Chapter 5.

2. Determine only a, e, τ and E^o from the state at time t^o. Solve Kepler's equation at time t for E, evaluate F, G, $\dot{F}$ and $\dot{G}$, then use equations (3-28) or the parabolic or hyperbolic equivalents to determine the mapped position and velocity. Unless orbital elements are specifically desired or orbital perturbations must be included, this approach is the preferred method and utilizes the X2X procedure developed for the toolbox.

3.10 Degenerate, Circular and Nearly Parabolic Orbits

Numerical calculations will generally not exactly satisfy the conditions for determining the orbital elements for degenerate, low inclination, zero eccentricity, or parabolic orbits. When the

ω. The zero inclination case has a similar problem in that the line of nodes is poorly defined by (3-9) because h_x and h_y are nearly zero. Finally, degenerate conics, i.e. h=0 can occur for elliptical, parabolic and hyperbolic orbits. As seen from (3-6), all degenerate orbits have unit eccentricity and the eccentricity vector is in the opposite direction of the position vector. For degenerate orbits, (3-4) is not valid and true anomaly is undefined. However equations (3-13) through (3-15) are still valid. Barker's equation must be derived for the degenerate parabolic motion (Problem 3-1). The following steps provide general direction for the calculation of the orbital elements in these cases. The tolerance parameter "tol" is analyst supplied and depends on the accuracy requirements of the problem and the computer. For double precision 1e-8< tol < 1e-10 might be considered.

- If |zr| < tol the semi-major axis is very large compared to the initial position so set z=0 to assure parabolic motion.
- If e < tol, put periapsis at the initial position i.e. set $e = e_r$ and τ=time of the initial conditions.

If p/r < tol, set $e = -e_r$, $f = -\pi$, and p = 0. For parabolic motion, use the results of Problem 3-1 to determine τ; otherwise, use E or F calculated from equation (3-13). Use $\dot{r}$ to remove ambiguities. There are a number of options for ascending node and inclination for rectilinear orbits. One option is to set $e_\Omega = e_x$ and select e_h to assure the orbit plane passes through e_r. Another option is to set $i=\pi/2$ and $\tan\Omega = y/x$.

- If 1-|cos(i)| < tol, set i=0 or π and $e_\Omega = e_x$ or $e_\Omega = e_r$.

Some of the following relations are not valid if p=0. For hyperbolic orbits a<0.

Table 3-1. Two-body Problem Relationships

Variable	Ellipse a>0, e<1	Parabola z=0, e=1	Hyperbola a<0, b<0, e>1
v^2	$\mu\left(\frac{2}{r}-\frac{1}{a}\right)$	$\frac{2\mu}{r}$	$\mu\left(\frac{2}{r}-\frac{1}{a}\right) = \frac{2\mu}{r}+v_\infty^2$
r	$\frac{p}{1+e\cos f} = a(1-e\cos E)$	$\frac{p}{1+\cos f} = \frac{p}{2}\sec^2\frac{f}{2}$	$\frac{p}{1+e\cos f} = a(1-e\cosh F)$
n	$\sqrt{\frac{\mu}{a^3}}$	$\sqrt{\frac{\mu}{p^3}}$	$\sqrt{-\frac{\mu}{a^3}}$
$M = n(t-\tau)$	$E - e\sin E$	$\frac{1}{2}\tan\frac{f}{2}+\frac{1}{6}\tan^3\frac{f}{2}$	$e\sinh F - F$
cos f, sin f	$\frac{\cos E - e}{1-e\cos E}, \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}$	cos f, sin f	$\frac{e-\cosh F}{e\cosh F - 1}, \frac{\sqrt{e^2-1}\sinh F}{e\cosh F-1}$
cos E, cosh F	$\frac{e+\cos f}{1+e\cos f}$	NA	$\frac{e+\cos f}{1+e\cos f}$
sin E, sinh F	$\frac{\sqrt{1-e^2}\sin f}{1+e\cos f}$	NA	$\frac{\sqrt{e^2-1}\sin f}{1+e\cos f}$
$\frac{dE}{dt}, \frac{dF}{dt}$	$\frac{na}{r}$	NA	$-\frac{na}{r}$
$\frac{dr}{dt}$	$\frac{na^2 e\sin f}{b} = \frac{na^2 e\sin E}{r}$	$np\sin f$	$-\frac{na^2 e\sin f}{b} = \frac{na^2 e\sinh F}{r}$
ξ	$r\cos f = a(\cos E - e)$	$r\cos f$	$r\cos f = a(\cosh F - e)$
η	$r\sin f = b\sin E$	$r\sin f$	$r\sin f = -b\sinh F$
$\frac{d\xi}{dt}$	$\frac{-na^2\sin f}{b} = \frac{-na^2\sin E}{r}$	$-np\sin f$	$\frac{na^2\sin f}{b} = -\frac{na^2\sinh F}{r}$
$\frac{d\eta}{dt}$	$\frac{na^2(\cos f+e)}{b} = \frac{nab\cos E}{r}$	$\frac{np^2}{r}$	$-\frac{na^2(\cos f+e)}{b} = \frac{nab\cosh F}{r}$
$F(t,t_0)$	$\frac{1}{h}(\xi_t\dot{\eta}_o - \eta_t\dot{\xi}_o)$ where $\xi_t = \xi(t)$, etc.		
$G(t,t_0)$	$\frac{1}{h}(\eta_t\xi_o - \xi_t\eta_o)$		
$\dot{F}(t,t_0)$	$\frac{1}{h}(\dot{\xi}_t\dot{\eta}_o - \dot{\eta}_t\dot{\xi}_o)$		

3-1. Starting with equation (3-12), derive a form of Barker's equation (3-16) for degenerate parabolic motion.

3-2. Show that for small $|t-t_o|$, equation (3-32) reduces to the expected limit.

3-3. Develop the equivalent of equations (3-32) for parabolic orbits.

3-4. Develop the equivalent of equations (3-32) for hyperbolic orbits.

3-5. Verify the $\Phi(2,2)$ term in equation (3-33) using spherical trigonometry relations.

3.14 Astronautics Toolbox

1. Write a procedure that returns the rotation matrix (3 by 3) for an arbitrary [3,1,3] set of rotations $[\alpha,\beta,\gamma]$, Φ=Rotate313(α,β,γ,ichk).
2. Write a procedure to solve Barker's equation (3-16), f=Barker(t,τ,p,□,ichk). Assume t is (n by 1).
3. Write a procedure to solve Kepler's equation (3-14) for elliptic motion using Newton-Raphson iteration, E=Kepler(M,e,tol,ichk). Assume M is (n by 1) and “tol” is the relative error in E for convergence.
4. Write a procedure to solve Kepler's equation (3-15) for hyperbolic motion using Newton-Raphson iteration, F=KeplerH(M,e,tol,ichk). Assume M is (n by 1) and “tol” is the relative error in F for convergence.
5. Write a procedure to transform from rectangular coordinates to orbital elements for any type of motion. [OE]=X2Orb(t,r,v,□,ichk), where r and v are given at a single time t and OE is the six vector $(z,p,i,\Omega,\omega,\tau)$.
6. Write a procedure to provide position and velocity at an array of times for any type of orbit. [r,v]=Orb2X(t,OE,□,tol,ichk) where t is (n by 1), OE is the six elements used above, and "tol" is the relative accuracy for convergence of Kepler’s equation. Output position and velocity are both (n by 3).
7. Write a procedure, using the F and G function approach, to transform from an initial state at time t_1 to states at an array of times t. [r,v]=X2X(t,t_1,r_1,v_1,□,tol,ichk), where, t (n by 1) and r and v are (n by 3).

UNIT IV

Introduction

The three body problem has been of considerable interest for centuries because the Earth-Moon-Sun system can be approximated as a three body problem. During the Apollo era, the problem received renewed interest since the two dominate forces action on the spacecraft were due to the Earth and the Moon. Even for this system, requiring 18 integrals, a closed-form solution of the general problem does not appear feasible and there are no known integrals beyond those discussed in Chapter 2. The three body problem was recognized by Poincare as being what we now call a chaotic system, i.e. the characteristics of the motion are very sensitive to the initial conditions. There exist, however, particular solutions of the three-body problem obtained by Lagrange in 1772, which will be discussed later.

4.1 Restricted Problem

Of interest in the problem of three bodies is the special case in which the mass m of one of the bodies is so small that its motion does not affect the motion of the two remaining bodies. The motion of the two massive bodies m_1 and m_2 is obtained from the solution of the two-body problem and can be assumed to be known. The problem is further restricted by considering the case in which m_1 and m_2 move in circular orbits about their barycenter with constant angular velocity ω whereas the infinitesimal mass m moves under the combined gravitational attraction of both m_1 and m_2. Under these circumstances, the problem is reduced to the investigation of a 3-degree of freedom (DOF) system. This problem is called the restricted problem of three bodies. If m is further restricted to the plane of motion of m_1 and m_2, there is a 2-DOF system.

The classical coordinate system has the origin at the barycenter and the fundamental plane is the plane of motion of the two finite bodies. From equations (2-10) the equations of motion are

$$\dot{\rho} = -Gm_1 \frac{\rho - \rho_1}{\rho_{01}} - Gm_2 \frac{\rho - \rho_2}{\rho_{02}} \qquad (4\text{-}1)$$

where ρ_{0i} is the distance from m to m_i and ρ_i is the position vector of m_i. Select as the unit of length the constant distance between m_1 and m_2 and the unit of mass so that $m_1+m_2=1$. It is readily shown (Section 3.3.5) that the mean motion of the two finite masses is unity, i.e. $n=\omega=1$.

$$r_1^3 \quad r_2^3 \quad y \quad y$$

Now transform to a rotating coordinate system so that the two finite masses remain on the x axis. Let r be the position vector in the rotating system, then the transformation is

$$\cos t \quad -\sin t \quad 0 \quad [\xi]$$

$$\ddot{x} - 2\dot{y} = x - (1-\mu)$$

$$\ddot{y} + 2\dot{x} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} \tag{4-2}$$

$$\ddot{z} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3}$$

where wolog $\mu \le 0.5$ is the normalized mass of m_2, $x_1=-\mu$ is the location of m_1 and $x_2=1-\mu$ is the location of m_2 on the x axis, and r_i is the distance from m to m_i. These equations can be written as

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \qquad \ddot{z} = \frac{\partial U}{\partial z} \tag{4-3}$$

where the pseudo-force function is defined by

$$U = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{4-4}$$

The latter two terms come from the gravity potential and the first term comes from the "centrifugal potential."

Exercise 4-1. Fill in the steps from equation (4-1) to equation (4-2) for the x-component

Exercise 4-2. Verify that equations (4-3) and (4-4) are equivalent to equation (4-2)

The only integral of this system is an energy type integral discovered by Jacobi. To seek an energy integral, multiply each of equations (4-2) by the corresponding velocity component and add the three equations. The sum is integrable and leads to Jacobi's integral

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2 = 2U - C = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C \qquad (4\text{-}5)$$

The constant C is called Jacobi's constant. Although this is the only known integral of the six required, it has proven very useful in studying orbital motion in systems that can be approximated by the restricted problem.

Exercise 4-3. Fill in the steps to develop equation (4-5) starting with equation (4-2).

4.1.1 Jacobi’s integral and Tisserand's criteria

It is believed that the Oart cloud is the source of observed comets. This cloud is well outside the subsequently escape the solar system. Observed periodic comets will therefore be in orbits with eccentricities near unity and large semi-major axes. Most of the orbital period is spent in the outer part of the solar system, otherwise the comet would have long ago been destroyed by the Sun's radiant energy.

If the elliptical orbit of a comet is unperturbed by one of the planets, subsequent appearances of the comet can be identified by the two body orbital elements about the Sun. On the other hand, if the comet is perturbed by a single planet then Jacobi's integral can be applied to the Sun, the perturbing planet, and the comet three body system so that the comet can be identified by Jacobi's constant. Tisserand found a simple way of relating Jacobi's constant to the Keplerian heliocentric orbital elements. Transform Jacobi's integral back to the non-rotating (ξ,η,ζ) system to get

$$\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2-2\left(\xi\frac{d\eta}{dt}-\eta\frac{d\xi}{dt}\right)=\frac{2(1-\mu)}{\rho_{01}}+\frac{2\mu}{\rho_{02}}-C \qquad (4\text{-}6)$$

where ρ_{0i} is defined above. For planets $\mu<<1$ so that the first three terms can be interpreted as the velocity relative to the Sun and the next two terms as the ζ component of angular momentum relative to the orbital plane of the planet about the Sun.

Exercise 4-4. Fill in the steps to develop equation (4-6) starting with equation (4-5)

Using the vis-viva integral equation (3-5) and $h^2 = a(1-e^2)$ reduces Jacobi's integral to

$$\frac{2}{\rho_{01}}-\frac{1}{a}-2\sqrt{a(1-e^2)}\cos i=\frac{2}{\rho_{01}}+\frac{2\mu}{\rho_{02}}-C$$

where 1-μ has been set to unity on both sides of the equation. If Jacobi's constant is evaluated when the comet is far from the perturbing planet $\left(\frac{2\mu}{\rho_{02}}\ll\frac{1}{a}\right)$ this equation becomes

$$\frac{1}{a}+2\sqrt{a(1-e^2)}\cos i=C$$

Evaluating this relation before and after the encounter with the perturbing planet yields Tisserand's criteria

$$\frac{1}{a_1} + 2\sqrt{a_1\left(1 - e^2\right)}\cos i = \frac{1}{a_2} + 2\sqrt{a_2\left(1 - e^2\right)}\cos i \qquad (4\text{-}7)$$

for the identification of comets that have been perturbed by a single planetary encounter. Based on the before and after heliocentric orbits, the planet at which the close encounter occurred can generally be identified. If not, since the semi-major axis and inclination in the equation areThe value of C in equation (4-5) can be determined from a set of initial conditions. If C>0 equation (4-5) places a constraint on the possible spatial locations of the trajectory. In particular, motion can only occur in regions where $v^2 \geq 0$. Recall that equation (3-2) limits the possible spatial locations for the two body problem. For a given C the surface defined by $v^2=0$ is called the zero velocity surface. Motion can only occur on one 'side' of the zero velocity surface.

From analytic geometry, a single equation relating the three spatial coordinates x, y, and z defines a two dimension subspace. For example, $x^2+y^2=R^2$ defines the surface of an infinite circular cylinder of radius R with the z-axis along the center of the cylinder, while $x^2+y^2<R^2$ defines the three dimensional space inside the cylinder. Similarly, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ defines the surface of an ellipsoid with principal axes a, b, and c along x, y and z.

Moulton [1,281] provides methods for calculating and an extensive discussion of the zero velocity surfaces. Figure 4-1 shows some of the zero velocity contours in the plane z=0 for μ=0.25 and

From equation (4-5) it is seen that motion can only occur in a region where

$$x^2+y^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}>C \tag{4-8}$$

For example, if the initial conditions have $z=\dot{z}=0$ and result in C=5, then motion is confined to the x-y plane in the nearly circular region about either mass or to the region outside the C=5 outer contour. If one wanted to design a trajectory that goes from a point near m_1 to m_2, then C must be less than about 3.87. Similarly, if the 2-d motion of m is initiated near either mass and C>3.56 then m can never escape from the figure-eight region defined by the C=3.56 contour. Finally, applications of Jacobi's integral like the

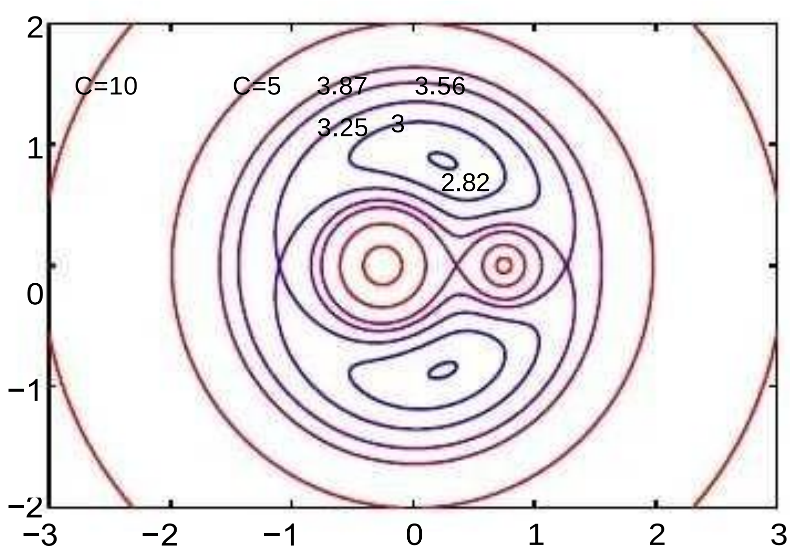


Figure 4-1 Zero velocity contours, μ=0.25

above are most useful for defining where motion cannot occur. There is no guarantee, that for a specified value of C, an actual trajectory exists between every two points in the region bounded by the C contour.

Lagrange discovered that there were five equilibrium points for the restricted three body problem. These positions correspond to solutions (x, y, z) to equations (4-2) when the velocity and acceleration terms are zero. In the inertial system, the particle at such an equilibrium point, would be in a circular orbit about the center of mass of m_1 and m_2. At an equilibrium point, equations (4-2) can be written as

$$x - (1-\mu)\frac{x-x_1}{r_1^3} - \mu\frac{x-x_2}{r_2^3} = -x\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} - 1\right) - (1-\mu)\mu\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) = 0$$

$$y\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} - 1\right) = 0 \qquad (4\text{-}9)$$

$$z\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right) = 0$$

The first equation is factored in two ways for later use. The third of these equations implies that all equilibria must be in the x-y plane. The second equation admits to two types of solutions, y=0 and $r_1=r_2=1$. Solutions with y=0 must therefore be on the x-axis and satisfy the condition

$$f(x) = x - (1-\mu)\frac{x-x_1}{|x-x_1|^3} - \mu\frac{x-x_2}{|x-x_2|^3} = 0 \qquad (4\text{-}10)$$

The zeros or roots of f(x) are the equilibrium points. Note that f(x)>0 for sufficiently large and positive x. As x becomes smaller and approaches x_2, the gravity potential term for m_2 dominates and $f(x_2^+)<0$. Thus there is exactly one equilibrium point with $x>x_2=1-\mu$ denoted L_2. When x is slightly less than x_2 this potential term still dominates but is now positive so $f`(x_2^-)>0$. But as x becomes smaller and approaches m_1, this potential dominates so that $f(x_2^+)<0$. Thus there is exactly one equilibrium point between the two masses called L_1. By the same arguments, there is exactly one equilibrium point with $x<x_1=-\mu$ called L_3. These three equilibria are called the straight line solutions since all three masses remain in a line. Moulton [1] provides power series expansions in $\sqrt[3]{\mu}$ for calculating the locations along the x axis for L1 and L2. Retaining only the first term in the series, the distance from m_2 to L1 and L2 is $\tilde{\rho}_2 = \sqrt[3]{\mu/3}$. Likewise, if $\mu \ll 1$, $\tilde{\rho}_1 = 1 - \tilde{\rho}_2$ is the distance from m_1 to L3. Newton-Raphson iteration works effectively for finding the roots of equation (4-10) if μ has a numerical value.

x-equation for equilibrium. These two points, $\left(-\frac{1}{2}\ \square, \pm\frac{\sqrt{3}}{2}, 0\right)$, are called the equilateral triangle solutions and denoted L_4 and L_5. In Figure 4-1, L_1 is where the C=3.87 contour crosses itself between the masses, L_2 is where the C=3.56 contour crosses near x=1.2, and L_3 is located where the C=3.25 contour crosses near x=-1. L_4 and L_5 are inside the C=2.82 contour.

The Lagrange solutions of the restricted three-body problem are of more than purely academic interest. If the Sun-Earth system is considered, satellites have been located at L_1 [2] to permit measurements of the solar wind before it arrives at the Earth and produces changes in the ionosphere and the geomagnetic field. The ionosphere is important for low frequency radio transmission and over-the-horizon radar. Disruptions in the ionosphere can be very dramatic during solar storms. Further, the electrical power distribution system, on numerous occasions, has had major black outs over large geographical areas when the geomagnetic field has changed drastically during a solar storm. One astrophysical phenomena which has been attributed to these solutions is the Gegenschein (counterglow). The Gegenschein is a faint glow observed at night in a position exactly opposite the sun and may result from reflection of sunlight off dust that is near the Earth-Sun equilibrium position L_3.

For the Sun and Jupiter system, there are a number of asteroids, called the Trojan asteroids, oscillating about L_4 or L_5. For the Earth-Moon system there have been numerous studies of placing a relay satellite near L_2 but sufficiently far away that the satellite could be seen from the Earth. Though not at the equilibrium point, the unbalanced forces acting on the satellite would be small and this position would require limited station keeping propulsion. There are "halo orbits" about these equilibria that have been exploited for various scientific purposes [2]. There was also a report in the early 1960's that clouds of dust were observed near L_4. This dust was attributed to a contemporary meteor impact on the back side of the Moon. Though these observation were never independently confirmed, numerous simulations were performed to study the possibility.

4.2.4 Stability of Lagrange points

After determining the existence of equilibrium points, the next issue is to determine the stability of each point. Only linearized stability will be considered here so that no conclusions will be drawn about global stability. Let (x_o, y_o, z_o) be an equilibrium point and (ξ, η, ζ) be small deviations from equilibrium, i.e. $x=\xi+x_o$, $y=\eta+y_o$ and $z=\zeta+z_o=\zeta$. Linearizing equations (4-3) about the equilibrium point yields

$$
\begin{aligned}
\dot{\xi} - 2\dot{\eta} &= U_{xx}\,\xi + U_{xy}\eta + U_{xz}\zeta \\
\dot{\eta} + 2\dot{\xi} &= U_{yx}\,\xi + U_{yy}\,\eta + U_{yz}\zeta
\end{aligned}
\tag{4-11}
$$

so the solution is a sum of exponentials. The characteristic equation, which completely defines the dynamics of this system, is obtained by substituting $\xi(t)=\xi_o e^{\lambda t}$, $\eta(t)=\eta_o e^{\lambda t}$, $\zeta(t)=\zeta_o e^{\lambda t}$ into equation (4-11) and seeking non-trivial solutions. Before performing this operation, note that U has continuous derivatives (except at two uninteresting points) and is symmetric in z, therefore U_{zx} and U_{zy} vanish in the x-y plane. In this case, equations (4-11) shows that the perturbed motion in the z direction is uncoupled from the motion in the x-y plane and reduces to

$$\ddot{\zeta} - U_{zz}|_{z=0}\,\zeta = \ddot{\zeta} + \omega_z^2 \zeta = 0 \tag{4-12}$$

where $\omega_z^2 = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} > 0$. <u>At all five equilibrium points, motion in the z direction is uncoupled and harmonic</u>. Also note that at L_4 and L_5 $\omega_z=1$, which is the same as the mean motion of m_1 and m_2, so that z-only linearized motion produces a closed orbit in 3-d inertial space as well as in the rotating system. In the inertial system, the motion describes a nearly circular orbit with a small inclination to the x-y plane.

Exercise 4-8. Starting with equations (4-3) develop the ξ component of equations (4-11)

The characteristic equation for perturbed motion in the x-y plane is obtained from the first two of equations (4-11) and reduces to the bi-quadratic

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx}U_{yy} - U_{xy}^2 = 0 \tag{4-13}$$

(4-14)

where all of the coefficients are evaluated at the particular equilibrium point under study. For stable solutions, λ must be pure imaginary so that λ^2 must be negative. Let $\nu=\lambda^2$ and write equation (4-13) as

$$\nu^2 + b\nu + c = 0$$

From the quadratic formula it is clear that the roots will be negative only if b>0, c>0 and $b^2>4c$. For any of the straight line solutions y=z=0 and by symmetry $U_{xy}=U_{yz}=U_{xz}=0$. Unfortunately, symmetry arguments are not applicable to the two remaining terms and analysis must be performed to show that $U_{xx} = 1+ 2\omega_z^2$ and $U_{yy} = 1- \omega_z^2$, so $b = 2 - \omega_z^2$ and $c = (1+2\omega_z^2)(1-\omega_z^2)$. The sign of c is determined from the sign of $1- \omega_z^2$, which from the second form of the first of

$$1-\omega_z^2 = \frac{(1-\mu)\mu}{x}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)$$

For L_1 and L_2, $x>0$ and both points are closer to m_2 than m_1 so c will be negative at these two points. At L_3 $x<0$ and L_3 is closer to m_1 than m_2 so c is also negative at this point. Thus all of the straight line solutions are unstable.

Exercise 4-9. Starting with equations (4-11) develop equation (4-13)

At L_4 and L_5, $\omega_z=1$ and the x-y characteristic equation (4-13) reduces to

$$\lambda^4+\lambda^2+\frac{27\mu(1-\mu)}{4} = \nu^2+\nu+\frac{27\mu(1-\mu)}{4} = 0$$

By Descartes rule of signs there are no positive roots and either 0 or 2 negative roots for ν. So if there are real roots they must be negative. Using the notation above, the condition for stable motion is $b^2>4c$ which reduces to

$$\mu < \frac{1}{2}-\sqrt{\frac{23}{108}} = 0.03852\ldots \approx \frac{1}{25.96}$$

Thus, the triangle equilibrium points are stable if the primary to secondary mass ratio is greater than about 24.96. All Sun/planet and planet/moon pairs in the solar system satisfy this criteria except for Pluto/Claron. Finally, it must be remembered that any particular Lagrange point may not be stable when other forces are included in the equations of motion or the motion of the primaries is not circular.

4.2 Finite Mass Particular Solutions

A number of particular solutions for the case of three finite masses have been found. These are generalizations of the straight line and triangle solution of Lagrange discussed above. Motion of all three bodies occurs in the same plane but the distance between all three of the bodies can vary with time. From equations (2-10) the equations of motion for three bodies are

$$\ddot{r}_i = -\sum_{j\neq i}^{3}\frac{\mu_j}{r_{ij}^3}r_{ij}, \quad i = 1, 2, 3 \tag{4-15}$$

where $\mu_i=Gm_i$. Wolog let the origin be at the barycenter so that

The generalization of the L_4 and L_5 equilateral triangle solution is to seek solutions where the three finite mass bodies form the vertices of an equilateral triangle at all times. However, the length of a side of the triangle is not necessarily constant. To determine if such a solution exist set $\rho = r_{ij}$ to be the equal distance between the bodies as shown in Figure 4-2. Further let $\mu = \mu_1 + \mu_2 + \mu_3$. Then the center of mass relation can be written as $\mu r_1 = \mu_2(r_1 - r_2) + \mu_3(r_1 - r_3)$. Using this expression in equation (4-15) and to determine r_1 (ρ) yields the equation of motion for m_1

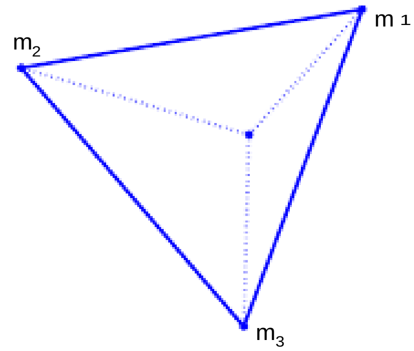


Figure 4-2. Equilateral triangle solution configuration.

$$\ddot{r}_1 = -M_1 \frac{r_1}{r_1^3} \qquad (4\text{-}16)$$

where $M_1 = \dfrac{\left(\mu_2^2 + \mu_2\mu_3 + \mu_3^2\right)^{3/2}}{\mu^2}$ - is a reduced gravitational mass. Equations of motion for the other two masses can be obtained by cyclic permutation. This is of course the equation of motion for a particle about a center with gravitational attraction M_1 and located at the origin. Thus the motion is a conic as studied in Section 3.3.3. The rest of the development assumes elliptical motion, but the conclusions are equally applicable for parabolic or hyperbolic motion for the three bodies.

Exercise 4-10. Verify equations (4-16)

For the three bodies to remain in an equilateral triangle configuration the initial conditions must be chosen so that the masses have the same period, i.e.

$$\frac{M_1}{a_1^3} = \frac{M_2}{a_2^3} = \frac{M_3}{a_3^3}$$

Hence, if the period of one mass is specified, the semi-major axis for each orbit can be calculated from the above. Also the angular rate must be the same so that the angles between the masses as measured at the center of mass remain constant, i.e.

$$\frac{h_1}{r_1^2} = \frac{h_2}{r_2^2} = \frac{h_3}{r_3^2}$$

where $h_i^2 = M_i p_i$. This condition requires each mass to have the same true anomaly at any specified time. Thus all masses are at the periapsis of their respective orbits at the same time. At where P is the orbital period common to all masses. The multiplier of P on the right is a monotone function of eccentricity on the interval (0,1), so the eccentricity of all three orbits is the same.

In summary, the dynamics of three finite bodies in an equilateral triangle configuration is completely defined by three mass values, an orbital period, an orbital eccentricity, a time of periapsis passage, the orientation of the orbit plane, and three arguments of periapsis that differ by 120°. More analytic detail can be found in Moulton [1,313] and Danby [3,266].

4.3.2 Straight line solution

To investigate the conditions for planar, straight line solutions, assume wolog that the plane of motion of the three masses is the x-y plane and the barycenter is at the origin. Let θ define the location of the line of centers in the plane, r_i define the location of each mass along the line of centers relative to the barycenter, and r_i^o be the location at the initial time t_o. For the barycenter to remain fixed, variations in position along the line must keep the ratio of distances from the center of mass constant. So introduce the time dependent variable ρ such that $r_i(t) = \rho(t) r_i^o$. With e_r being the unit vector along the line of centers, $r_i = \rho r_i^o e_r$. By direct differentiation the familiar expression

$$\dot{r}_i = (\dot{\rho} - \rho\dot{\theta}^2) r_i^o e_r + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) r_i^o e_\theta$$

is derived. Substituting into equations (4-15) gives for m_1

$$\dot{\rho} - \rho\dot{\theta}^2 = \left[\frac{\mu_2 (r_2^o - r_1^o)}{r_1^o |r_2^o - r_1^o|^3} + \frac{\mu_3 (r_3^o - r_1^o)}{r_1^o |r_3^o - r_1^o|^3}\right]\frac{1}{\rho^2} = -\frac{M_1}{\rho^2} \tag{4-17}$$

$$\frac{d}{dt}(\rho^2\dot{\theta}) = 0$$

These are of course the equation of motion for the two body problem. Since ρ is dimensionless, M_1 has dimensions of an angular rate squared, so denote $M_1 = \omega_1^2$. Angular momentum for each mass is again preserved. The equivalent mass or mean motion in the radial equation must be the same regardless of which mass is used to derive the equation of motion for ρ, i.e. $M_1=M_2=M_3$ and $\omega_i=\omega$, i=1,2,3. If three initial positions are specified consistent with the barycenter location, then the period of the motion and the initial velocity for m_1 can be obtained from

$$\omega^2 r_1^o - \mu_2 \frac{(r_2^o - r_1^o)}{|r_2^o - r_1^o|^3} - \mu_3 \frac{(r_3^o - r_1^o)}{|r_3^o - r_1^o|^3} = 0$$

and the other two initial velocities can be obtained by cyclic permutation. Multiplying each

$$\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 = 0$$

$$\omega^2 x_1 - \mu_2(x_1 - x_2) - \mu_3 \frac{x_1 - x_3}{r_{13}^3} = 0 \qquad (4\text{-}18)$$

$$\omega^2 x_2 - \mu_1(x_2 - x_1) - \mu_3 \frac{x_2 - x_3}{r_{23}^3} = 0$$

where for notational simplification $x_i \equiv \overset{o}{r}_i$. Wolog let $x_1 < x_2 < x_3$ and select the unit of length such that $r_{12}=1$ then equations (4-18) reduce to

$$\mu_1 x_1 + \mu_2(1 + x_1) + \mu_3 x_3 = 0$$

$$\mu_2 + \frac{\mu_3}{(x_3 - x_1)^2} + \omega^2 x_1 = 0$$

$$-\mu_1 + \frac{-\mu_3}{(x_3 - x_1 - 1)^2} + \omega^2(1 + x_1) = 0$$

Using the first equation to eliminate x_3 from the second and third equations and then eliminating ω^2 between the remaining two equations yields

$$\begin{aligned}&(\mu x_1 + \mu_2 + \mu_3)^2(\mu x_1 + \mu_2)^2[\mu_2 + (\mu_1 + \mu_2)x_1] \\ &+ \mu_3^3(1 + x_1)(\mu x_1 + \mu_2 + \mu_3)^2 - \mu_3^3 x_1(\mu x_1 + \mu_2)^2 = 0\end{aligned} \qquad (4\text{-}19)$$

where $\mu=\mu_1+\mu_2+\mu_3$ is the total gravitational constant. A little algebra will show that this quintic equation for x_1 has all positive coefficients. By Descartes rule of signs there are no positive roots and one, three or five negative roots. This result does not provide much new information since it is already known that x_1 must be negative. However, if x_1 is eliminated from equation (4-19) in favor of $x_{32} \equiv x_3 - x_2$ using

$$\mu x_1 + \mu_2 + \mu_3 + \mu_3 x_{32} = 0$$

then it can be shown that x_{32} must satisfy

$$\begin{aligned}&(\mu_1 + \mu_2)x_{32}^5 + (3\mu_1 + 2\mu_2)x_{32}^4 + (3\mu_1 + \mu_2)x_{32}^3 \\ &- (\mu_2 + 3\mu_3)x_{32}^2 - (2\mu_2 + 3\mu_3)x_{32} - (\mu_2 + \mu_3) = 0\end{aligned} \qquad (4\text{-}20)$$

value of ω can be obtained by either the second or third of equations (4-18). Three solutions to equation (4-20) can be obtained by cyclic permutation; but, these are the same as simply rearranging the mass values. After the x_i are found, the solution can be scaled to real dimensions. Initial velocities can be calculated from the equations above. Perhaps the simplest approach is to select $r_i^o = a_i$ so that $(1-e) \le \rho \le (1+e)$ and calculate the velocities at periapsis, i.e. ρ=1-e. The velocity at periapsis is given by $v_i = (\pm a_i)\omega\sqrt{1 \pm e}$, where the sign is selected depending on the location of m_i with respect to the center of mass.

4.3 Problems..

4-1. Use the Matlab meshdom, mesh and contour functions to generate zero velocity surfaces and contour plots in the (x-y), (x-z) and (y-z) planes for the Earth-Moon system. Same scale as Figure 4-1. Compare to Figure 4-1 and interpret results.

4-2. Write a Matlab procedure using ODE 45 to solve equations (4-2). Apply to motion near L_4 or L_5 in the Earth-Moon system. Provide five example plots of the trajectories in the x-y plane for five different initial conditions with $z=\dot{z}=0$. Vary initial conditions to show the transition from bound motion to unbound motion. At least one case should verify that your solution at the libration point is correct. Interpret results.

4-3. Provide a semi-log or log plot of the period of oscillation for the two solutions to equation (4-13) over the range of mass ratios from 0.001 to 0.5. Interpret results.

4.4 Astrodynamics Toolbox

1. Write a function x=XLn(,n,tol,ichk) that returns the x location of the n-th (n=1,2,3) Lagrange point for mass ratio to an accuracy specified by tol.
2. Write a function f=LnFreq(,n,ichk) that returns the two frequencies of oscillation at Lagrange point n (n=1:5) for mass ratio .

4.5 References

1. Moulton, F.R., An Introduction to Celestial Mechanics, The MacMillan Company, New York, 1914. Also available from Dover
2. Howell, K.C. and Pernicka, H.J., "Station Keeping Method for Libration Point Trajectories," *Journal of Guidance, Control and Dynamics*, Vol. 16, No. 1, January-February, 1993, pp. 151-159.

Chapter 5 - Orbital Perturbations

5.1 Introduction

The motion of planets and natural or artificial satellites can be approximated by modeling both bodies as point masses and assuming that the only forces acting between them are the mutual gravitational attraction. The relative motion is then described by the two body solution discussed in Chapter 3. There are numerous additional forces affecting the relative motion. Both the additional forces and the deviations from the two body motion are called perturbations. When these forces are small compared to the central gravitational attraction, they may cause only small and/or slow deviations from two body motion and might be addressed analytically. The analytic solutions can be used as computationally efficient approximations to the motion or perhaps more importantly, can provide insight into the effects of the perturbations. By adding a disturbing force onto equation (3-1), the equation for the motion of m_1 relative to m_2 can be written as

$$\ddot{r} + \mu \frac{r}{r^3} = f \tag{5-1}$$

where the perturbing force is f. Note the abuse of language here: f is actually the relative acceleration produced by whatever physical process is causing the deviation from two body motion. The relative acceleration is the acceleration produced on m_1 minus the acceleration produced on m_2. In addition to the gravitational attraction of other masses presented in Chapter 2, these perturbations can come from numerous sources and their effects on the orbit vary greatly. Perturbing forces include aerodynamic interaction with the atmosphere, electromagnetic interactions with the magnetic field and charged particle belts, and gravitational forces due to the non-spherical gravity field of the central body. Perturbations are produced by the momentum flux of electromagnetic energy from the Sun called radiation pressure and particle flux called the solar wind. Reflected and radiated flux from the Earth can also produce significant perturbations on low Earth orbit (LEO) satellites.

When a numerical solution is sought to (5-1) or an equivalent form of the EOM, the approach is called the method of special perturbations. Special perturbation methods will only be discussed briefly in Section 5.6. If an analytic solution is sought, the approach is called the method of general perturbations. General perturbation methods are usually based on a form of (5-1) that is derived using the variation of parameters method from the theory of ordinary differential equations.

5.2 Variation of Parameters

As a basis for developing Lagrange's planetary equations (5-18), the variation of parameters method for solving differential equations is reviewed by applying the approach to a harmonic

Letting $\rho = x, \nu = \dot{x}$ gives the first order form

$$\dot{\rho} = \nu \qquad \dot{\nu} = -\omega^2 \rho - \varepsilon\rho^3 \tag{5-3}$$

For ε=0 the solution is harmonic motion with period $2\pi/\omega$, amplitude A, and phase ø

$$\rho(t, A, \varphi) = A\sin(\omega t + \varphi) \qquad \nu(t, A, \varphi) = A\omega\cos(\omega t + \varphi) \tag{5-4}$$

where A and φ are determined from the initial conditions.

The variation of parameters method can be thought of as nothing more than a change in variables. In this case the dependent variables ρ and ν are replaced by A and ø. It is clear from (5-4) that the transformation is well defined in both directions, except for the trivial solution A=0. To derive the equations of motion for the new dependent variables, (5-4) are differentiated with respect to time to yield

$$\begin{aligned} \dot{\rho} &= A\omega\cos(\omega t + \varphi) + \sin(\omega t + \varphi)\dot{A} + A\cos(\omega t + \varphi)\dot{\varphi} \\ \dot{\nu} &= -A\omega^2\sin(\omega t + \varphi) + \omega\cos(\omega t + \varphi)\dot{A} - A\omega\sin(\omega t + \varphi)\dot{\varphi} \end{aligned} \tag{5-5}$$

Using (5-4) and (5-5) to eliminate ρ and ν from (5-3) yields

$$\begin{aligned} \sin(\omega t + \varphi)\frac{dA}{dt} + A\cos(\omega t + \varphi)\frac{d\varphi}{dt} &= 0 \\ \cos(\omega t + \varphi)\frac{dA}{dt} - A\sin(\omega t + \varphi)\frac{d\varphi}{dt} &= \frac{-\varepsilon A^3}{\omega}\sin^3(\omega t + \varphi) \end{aligned} \tag{5-6}$$

Using the fact that the coefficient matrix of the derivatives is non-singular along with a few trigonometry identities, (5-6) can be written as

$$\begin{aligned} \frac{dA}{dt} &= \frac{-\varepsilon A^3}{\omega}\cos(\omega t + \varphi)\sin^3(\omega t + \varphi) = \frac{-\varepsilon A^3}{8\omega}[2\sin 2(\omega t + \varphi) + \sin 4(\omega t + \varphi)] \\ \frac{d\varphi}{dt} &= \frac{\varepsilon A^2}{\omega}\sin^4(\omega t + \varphi) = \frac{3\varepsilon A^2}{8\omega} - \frac{\varepsilon A^2}{8\omega}[4\cos 2(\omega t + \varphi) - \cos 4(\omega t + \varphi)] \end{aligned} \tag{5-7}$$

These equations are exact. That is, let A(t) and ø(t) be solutions to (5-7). When these functions are substituted into (5-4) the results will be solutions to (5-3). Since (5-7) are much more complicated than (5-3), it might be said that nothing has been gained. Certainly, implementing a numerical solution to equation (5-2) would be less error prone than implementing a numerical solution to equation (5-7). But consider the case when ε<<1. In this case, from (5-7) both A and ø will change slowly with time. Assume that over one period of oscillation A and ø change so little that they can be considered constants on the right hand side of (5-7). The equations can then be integrated to show that over one period the net change in A is ΔA=0 and the net change in ø is Δø

2. produce **periodic** and **secular** variations in phase. That is, the phase continues to increase a small amount each period of oscillation and this secular drift increases with amplitude.

Note that the secular drift in phase is equivalent to an amplitude dependent frequency of oscillation.

If a new variable $\varphi' = \varphi - \Delta\varphi$ is defined, then φ' will only have periodic variation. This is the standard approach for dividing the perturbations into secular and periodic terms and is the approach for describing the motion of the vernal equinox as precession and nutation in Section 1.2.2.

Such insights into the motion are difficult to discern from (5-3). Second and higher order effects can be obtained by using the first order solutions as approximate solutions to equation (5-7) and performing another variation of parameter procedure. The next section applies the variation of parameters approach to the perturbed two body problem.

5.3 Lagrange's Planetary Equations

To study the effects of n-body perturbations on the motion of a planet, Lagrange applied the method of variation of parameters to equation (2-10) in the form of equation (5-1). It is for this reason that the results are given the name **Lagrange's planetary equations**. If the perturbing force $\mathbf{f}$ is derivable from a force function R then $\mathbf{f} = \nabla R$. This is of course the case for all gravitation perturbing forces. Both forms will be carried in the development. The equation numbers in brackets {} refer to the similar equation in Section 5.2. To begin, write equations (5-1) in the first order form with the non-perturbing force derivable from a potential V {(5-3)}

$$\frac{\mathbf{r}}{dt} = \mathbf{v} \qquad \frac{\mathbf{v}}{dt} + \nabla V(\mathbf{r}) = \mathbf{f}(\mathbf{r}, \mathbf{v}, t) \quad \text{or} \quad \nabla R(\mathbf{r}) \tag{5-8}$$

where $V = -\mu/r$ and the argument indicates the coordinate system to which the gradient operator applies.

The solution to these equations can be formally written as {(5-4)}

$$\mathbf{r}(t) = \mathbf{r}(t, \mathbf{c}) \qquad \mathbf{v}(t) = \mathbf{v}(t, \mathbf{c}) \tag{5-9}$$

where $\mathbf{c}$ is the six vector of Kepler orbital elements or any other six independent constants of integration. Because neither the position nor the velocity can be written explicitly as a function of time (recall equations (3-14), (3-15), and (3-16)) one must be careful with the explicit and implicit derivatives required below. The solution for $\mathbf{r}$ and $\mathbf{v}$ are given as implicit functions of time by equations (3-21) and (3-22) in which r and f are the terms that are functions of time.

Applying the variation of parameter approach to the first of equations (5-8) yields three equations

∂

where it is now assumed that r is a function of t and c {(5-5)}. But, $\frac{\partial \mathbf{r}}{\partial t} = v$ since the explicit dependence of r on time satisfies the equations of motion. Likewise, the second of equations (5-8) yields three more equations

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial c}\frac{dc}{dt} = -\nabla V(\mathbf{r}) + f \text{ or } -\nabla V(\mathbf{r}) + \nabla R(\mathbf{r})$$

Again the partial time derivative must satisfy the unperturbed equations of motion. So $\frac{\partial v}{\partial t} = -\nabla V$. The six differential equations of motion for c are therefore {(5-6)}

$$\frac{\partial \mathbf{r}}{\partial c}\frac{dc}{dt} = 0 \qquad \frac{\partial v}{\partial c}\frac{dc}{dt} = f \quad \text{or} \quad \nabla R(\mathbf{r}) \tag{5-10}$$

This form of the equations of perturbed motion are not convenient because the 3 by 6 coefficient matrices on the left are functions of time. Lagrange noted that this problem can be circumvented by multiplying the first equation by $\left[\frac{\partial v}{\partial c}\right]^T$ and the second by $\left[\frac{\partial r}{\partial c}\right]^T$ and then subtracting to get

$$L\frac{dc}{dt} = \left[\frac{\partial r}{\partial c}\right]^T f(c,t) \qquad \text{or} \qquad \nabla R(c) \tag{5-11}$$

where $L = \left[\frac{\partial \mathbf{r}}{\partial c}\right]^T\left[\frac{\partial v}{\partial c}\right] - \left[\frac{\partial v}{\partial c}\right]^T\left[\frac{\partial r}{\partial c}\right]$ is a 6 by 6 skew symmetric coefficient matrix, R(r) has been replaced by R(c) using equation (5-9), and the gradient operator on the right creates a six vector of partials of R wrt each component of c.

The most important property of L is that it is not an explicit function of time, that is, it can be evaluated at any point in the orbit and the same numerical values will result. To show this independence

$$\left[\frac{\partial L}{\partial t}\right] = \left[\frac{\partial}{\partial c}\left(\frac{\partial \mathbf{r}}{\partial t}\right)\right]^T\left[\frac{\partial v}{\partial c}\right] + \left[\frac{\partial \mathbf{r}}{\partial c}\right]^T\left[\frac{\partial}{\partial c}\left(\frac{\partial v}{\partial t}\right)\right] - \left[\frac{\partial}{\partial c}\left(\frac{\partial v}{\partial t}\right)\right]^T\left[\frac{\partial r}{\partial c}\right] - \left[\frac{\partial v}{\partial c}\right]^T\left[\frac{\partial}{\partial c}\left(\frac{\partial \mathbf{r}}{\partial t}\right)\right]$$

where the partials with respect to t and c have been commuted in each term. The first and fourth terms cancel since $\frac{\partial r}{\partial t} = v$ from above. The second term is the transpose of the third term and substituting $\frac{\partial v}{\partial t} = -\nabla V(\mathbf{r})$ into the third term yields which is a symmetric 6 by 6 matrix. Thus, when the force driving the unperturbed motion is derivable from a potential, all terms cancel and L is not an explicit function of time. To emphasize this result, equation (5-11) might be written as

$$L(c)\frac{dc}{dt} = \left[\frac{\partial r}{\partial c}\right]^T f(c,t) \qquad \text{or} \qquad \nabla R(c) \tag{5-13}$$

Exercise 5-1. Write explicitly every term in the first row of equation (5-13) for both the f and R cases.

5.3.1 Lagrange brackets

The individual components of L are called **Lagrange brackets** and denoted $[c_i, c_j]$, i.e.

$$L_{ij} \equiv [c_i, c_j] = \frac{\partial r}{\partial c_i}^T \frac{\partial v}{\partial c_j} - \frac{\partial v}{\partial c_i}^T \frac{\partial r}{\partial c_j} \tag{5-14}$$

As seen from this definition, the Lagrange brackets satisfy

$$[c_i, c_i] = 0 \qquad [c_i, c_j] = -[c_j, c_i] \qquad \frac{\partial}{\partial t}[c_i, c_j] = 0$$

which means there are no more than 30 non-zero brackets and only 15 have to be evaluated. Since all types of two body motion have a periapsis, L is traditionally evaluated at periapsis.

5.3.2 Rectangular coordinates

To compute the Lagrangian brackets, an appropriate c must be chosen. One set that results in the simplest forms of L is to chose the rectangular position and velocities at some epoch as the constants of integration. Though not usually thought of as constants of the motion, it is clear that they satisfy the necessary conditions of linear independence and are certainly sufficient to determine the state at any time. Wolog, let the epoch be t_o and specify the conditions at epoch as the six vector $c = [x_o, y_o, \ldots \dot{z}_o]^T$. From equation (5-14) the non-zero Lagrange brackets above the diagonal of L are

$$[x_o, \dot{x}_o] = [y_o, \dot{y}_o] = [z_o, \dot{z}_o] = 1$$

and the remaining 12 brackets are zero. Thus $L = \begin{bmatrix} 0_{3x3} & I_{3x3} \\ -I_{3x3} & 0_{3x3} \end{bmatrix}$ and the equations of motion (5-11) in terms of the force function reduce to the canonical form with similar expressions for the other two coordinates. There are numerous sets of choices of c that will lead to the canonical form; in particular, Lagrangian generalized coordinates and conjugate momenta lead to this form.

5.3.3 Keplerian orbital elements

A slight modification of the classical Keplerian elements leads to $c=(a, e, i, \Omega, \omega, \lambda)$, where $\lambda = -n\tau$ to simplify the partial derivatives. To evaluate the Lagrange brackets the position and velocity must be written in terms of these elements. Equations (3-21) and (3-22) provide the necessary relationships. These equations provide the explicit dependence of position and velocity on Ω, ω, and i. The dependence on a, e, and λ are implicit through r, f, $\dot{r}$ and $\dot{f}$. There are numerous methods for evaluating the brackets and the most extensive discussions are given in Battin[5], Fitzpatrick[3], and Moulton[7]. The development below is typical.

First equations (3-21) and (3-22) are written in the orbital coordinate system (Section 3.8) using the direction cosines from equation (3-33) and relations from Table 3-1

$$\begin{aligned} \mathbf{r} &= \Phi\rho = \Phi\left[a(\cos E - e),\ b\sin E,\ 0\right]^T \\ \mathbf{v} &= \Phi\dot{\rho} = \Phi\left[-na^2\sin E/r,\quad nab\cos E/r,\ 0\right]^T \end{aligned} \tag{5-15}$$

where Φ is a function only of the orientation angles (Ω, ω, i) and the position and velocity in the orbital system (Section 3.8) are functions only of a, e, and λ. Note that some of the partials of the Φ matrix yield terms in Φ, e.g.

$$\frac{\partial\Phi_{11}}{\partial\Omega} = -\Phi_{21} \qquad \frac{\partial\Phi_{11}}{\partial\omega} = \Phi_{12}$$

It is convenient to identify the columns of Φ as vectors

$$\Phi = \left[\vec{\varphi}^1\ \vec{\varphi}^2\ \vec{\varphi}^3\right]$$

The Lagrange brackets will be evaluated at periapsis, $t = \tau$. Since Φ is independent of time, for any orientation angle $\alpha = \Omega$, ω or i

$$\left.\frac{\partial \mathbf{r}}{\partial\alpha}\right|_\tau = \frac{\partial\Phi}{\partial\alpha}\rho(\tau) \qquad \left.\frac{\partial \mathbf{v}}{\partial\alpha}\right|_\tau = \frac{\partial\Phi}{\partial\alpha}\dot{\rho}(\tau)$$

But at periapsis, $\rho(\tau) = \left[r_p, 0, 0\right]^T$ and $\dot{\rho}(\tau) = \left[0, nab/r_p, 0\right]$, so it is straight forward to

$$\frac{\partial \mathbf{r}}{\partial \Omega} = r_p\begin{bmatrix}-\Phi_{21} & \Phi_{11} & 0\end{bmatrix}^T \qquad \frac{\partial \mathbf{v}}{\partial \Omega} = \frac{ab}{r_p}\begin{bmatrix}-\Phi_{22} & \Phi_{12} & 0\end{bmatrix}^T$$

$$\frac{\partial \mathbf{r}}{\partial \omega} = r_p \hat{\varphi}_2 \qquad \frac{\partial \mathbf{v}}{\partial \omega} = -\frac{ab}{r_p}\hat{\varphi}_1 \tag{5-16}$$

$$\frac{\partial \mathbf{r}}{\partial i} = r_p \sin\omega \hat{\varphi}_3 \qquad \frac{\partial \mathbf{v}}{\partial i} = \frac{nab\cos\omega}{r_p}\hat{\varphi}_3$$

Exercise 5-2. Starting with equation (5-15) verify the first line of equations (5-16) and (5-17)

It remains to take the partials with respect to a, e and λ. The rotation matrix Φ is not a function of these variables. So in equations (5-15), a and e appear explicitly and a, e and λ appear implicitly through E. The implicit relationship is defined purely by Kepler's equation (3-14)

$$nt + \lambda = E - e\sin E$$

where of course n is a function of a only. From which it is easy to show that when evaluated at periapsis

$$\frac{\partial E}{\partial a} = \frac{3\lambda}{2r_p} \qquad \frac{\partial E}{\partial e} = 0 \qquad \frac{\partial E}{\partial \lambda} = \frac{a}{r_p}$$

Combining the implicit and explicit derivatives leads to

$$\frac{\partial \mathbf{r}}{\partial a} = \frac{r_p}{a}\hat{\varphi}_1 + \frac{3b\lambda}{2r_p}\hat{\varphi}_2 \qquad \frac{\partial \mathbf{v}}{\partial a} = -\frac{3na^2\lambda}{2r_p^2}\hat{\varphi}_1 - \frac{bn}{2r_p}\hat{\varphi}_2$$

$$\frac{\partial \mathbf{r}}{\partial e} = -a\hat{\varphi}_1 \qquad \frac{\partial \mathbf{v}}{\partial e} = \frac{na^3}{br_p}\hat{\varphi}_2 \tag{5-17}$$

$$\frac{\partial \mathbf{r}}{\partial \lambda} = \frac{ab}{r_p}\hat{\varphi}_2 \qquad \frac{\partial \mathbf{v}}{\partial \lambda} = -\frac{na^3}{r_p^3}\hat{\varphi}_1$$

When the expression from equations (5-16) and (5-17) are substituted into equation (5-14) it is found that there are only 6 non-zero Lagrange brackets and these are [5,482]

$$[a, \Omega] = -\frac{nb}{2}\cos i \qquad [a, \omega] = -\frac{nb}{2} \qquad [a, \lambda] = -\frac{na}{2}$$

$$[e, \Omega] = \frac{nea^3}{\ }\cos i \qquad [e, \omega] = \frac{nea^3}{\ } \qquad [i, \Omega] = nab\sin i$$

Equations (5-13) in terms of the non-zero Lagrange brackets are

$$[i, \Omega]\frac{d\Omega}{dt} = \frac{\partial R}{\partial i}$$

$$[\lambda, a]\frac{da}{dt} = \frac{\partial R}{\partial \lambda}$$

$$[\omega, a]\frac{da}{dt} + [\omega, e]\frac{de}{dt} = \frac{\partial R}{\partial \omega}$$

$$[e, \Omega]\frac{d\Omega}{dt} + [e, \omega]\frac{d\omega}{dt} = \frac{\partial R}{\partial e}$$

$$[a, \Omega]\frac{d\Omega}{dt} + [a, \omega]\frac{d\omega}{dt} + [a, \lambda]\frac{d\lambda}{dt} = \frac{\partial R}{\partial a}$$

$$[\Omega, a]\frac{da}{dt} + [\Omega, e]\frac{de}{dt} + [\Omega, i]\frac{di}{dt} = \frac{\partial R}{\partial \Omega}$$

This is a set of linear algebraic equations with constant coefficients, so as long as the matrix of coefficients is not rank deficient, these equation can be easily solved to have only time derivatives on the left. The equations of motion for a and Ω can be found by division. Time derivatives for elements e and ω can then be obtained by elimination. Finally, the equation of motion for i and λ are also obtained by elimination to yield the Lagrange's planetary equations {(5-7)}[5,483]

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \lambda}$$

$$\frac{de}{dt} = \frac{-b}{na^3 e}\frac{\partial R}{\partial \omega} + \frac{b^2}{na^4 e}\frac{\partial R}{\partial \lambda}$$

$$\frac{di}{dt} = -\frac{1}{nab\sin i}\frac{\partial R}{\partial \Omega} + \frac{\cos i}{nab\sin i}\frac{\partial R}{\partial \omega}$$

$$\frac{d\Omega}{dt} = \frac{1}{nab\sin i}\frac{\partial R}{\partial i} \tag{5-18}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{nab\sin i}\frac{\partial R}{\partial i} + \frac{b}{na^3 e}\frac{\partial R}{\partial e}$$

$$\frac{d\lambda}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{b^2}{na^4 e}\frac{\partial R}{\partial e}$$

These are exact equations of motion and equivalent to equation (5-8). Even if other parameters are chosen for the orbital elements, the resulting EOM are called Lagrange's planetary equations. Other choices might depend on the particular orbit being analyzed. For example, to derive equations (5-18) division by e and sin(i) has been performed, hence application to orbits where either of these terms is zero or nearly zero must be done with care or the non-singular

5.4 Perturbations Derivable from a Potential

The two typical perturbations derivable from a potential function are the contributions due to other point masses as in the n-body problem (Chapter 2) and the contributions due to the gravity field of the primary being non-central. The former is discussed briefly in Section 5.5.2 and the latter is developed below.

5.4.1 Non-spherical gravity potential

The external gravity field of most bodies can not be represented as arising from a point mass. However sufficiently large, slowly rotating bodies will closely approximate a sphere because internal shear stresses due to self gravity cannot be supported. The external gravity field potential for any body satisfies Laplace's equation, $\nabla^2 V = 0$. For nearly spherical bodies, it is natural to represent the potential in spherical coordinates $V(r,\lambda,\varphi)$. Solution by separation of variables leads to the spherical harmonic representation, which is used here in the form

$$V(r,\varphi,\lambda) = -\frac{GM}{r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}(\sin\varphi)\right] \qquad (5\text{-}19)$$

where λ is longitude, φ is geocentric latitude, and R is a reference radius usually taken as the mean equatorial radius. P_{nm} is the associated Legendre polynomial of degree n and order m [1].

The terms $\cos m\lambda \; P_{nm}(\sin\varphi)$ and $\sin m\lambda \; P_{nm}(\sin\varphi)$ are called surface spherical harmonics of degree n and order m. The C_{nm} and S_{nm} are called the spherical harmonic coefficients and are the unknowns that would be selected to fit the boundary conditions to obtain the solution to Laplace's equation. The reference model in the Explanatory Supplement [2,226] is actually a force function and is the negative of equation (5-19). Some representations also change the sign between the '1' and the double sum. Thus care must be exercised by the analyst to check sign conventions. The expansion above is also “unnormalized” in that the relative importance of the terms on the orbit is not directly related to the numerical value. Various normalization approaches have been [2,226] used so that the numerical value is a direct measure of the “average” acceleration produced by the term. The complete set of surface spherical harmonics are divided into three sub classes:

1. zonal harmonics with m=0 are rotationally symmetric about the pole and have n zero crossings from pole to pole. Note that S_{n0}=0. The zonal coefficients are often represented by J's, i.e. $J_n = -C_{n0}$. Here the minus sign is used, but some authors will use a plus sign. J_2 is the “oblateness” and J_3 is the “pear shape” parameter. For planets with rotational rates sufficiently large to significantly affect the surface shape, J_2 is greater than zero and is the

$P_0(x)=1$ $\quad P_1(x)=x$ $\quad P_2(x)=(3x^2-1)/2$

where subsequent terms can be obtained from the recursion relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Note this is not a particularily useful recursion formula since errors in P_n produce errors that could be twice as large in P_{n+1}.

2. sectorial harmonics with n=m have no zero crossings from pole to pole since $P_{nn} \propto \cos^n \varphi$, but have 2m zeros in longitude due to the $\sin m\lambda$ and $\cos m\lambda$ terms.

3. tesseral harmonics with $n \neq m > 0$ have n-m zeros from pole to pole due to P_{nm} and 2m zeros in longitude due to the $\sin m\lambda$ and $\cos m\lambda$ terms. Recursion relations can be used [1] to increase the order so that all tesseral harmonics can be generated once the zonal harmonics are known from the recursion equation above.

In summary, all of the surface spherical harmonics have n-m zeros pole to pole and 2m zeros in longitude. Chobotov, Danby and other books presents graphical representation of some of these functions. In trajectory packages the gradient of the spherical harmonic functions are required. The gradient can be related to other harmonics and are also calculated recursively [1].

The spherical harmonic coefficients can be related to the inertia integrals of the body [3]. An inertia integral is a generalization of the traditional moments of inertia. The general inertial integral is defined by

$$I_{pqr} = \iiint \rho(x,y,z)\, x^p y^q z^r \, dx\,dy\,dz$$

where the integral is taken over the physical limits of the body and ρ is density. The moment of inertia about the x-axis is $I_{xx} = I_{020}+I_{002}$. Coefficients of degree n can be written as linear combinations of inertia integrals with p+q+r=n, for example $J_2 = \dfrac{I_{zz} - \frac{1}{2}(I_{xx} + I_{yy})}{MR^2}$ where M is the total mass and R is the mean radius of the body used in equation (5-19). From this form it is easier to see that J_2 is positive for oblate spheroids, i.e. most planets and other large, rotating bodies. Now $J_1 = \dfrac{I_{001}}{MR} = \dfrac{\bar{z}}{R}$, where $\bar{z}$ is the z component of the center of mass location. From this result, all the first degree coefficients are zero if the origin is taken at the center of mass. In precision Earth satellite orbit determination programs, the center of mass of the Earth is permitted is “pear shaped,” came from the orbital perturbation in eccentricity caused by C_{30} [Section 5.4.4]. The recommended [2,227] model of the gravity field is (36,36), i.e. n_{max}=36 and m_{max}=36. Larger models are used for precision orbit calculations. High order models for Mars and Venus have obtained from numerous orbiting missions to these planets. Generally, accuracy of the coefficients decrease with increase in degree and order. Exceptions correspond to coefficients of potential terms that produced a resonance with the orbital motion of a particular satellite.

5.4.2 Non-spherical gravity perturbations

To apply the planetary equations to the non-central part of the field, write the gravity potential function as $V(r,\varphi,\lambda) = -\frac{\mu}{r} - R(r,\varphi,\lambda)$. Substitution of the complete disturbing function R into equations (5-18) generally has little practical value. The general approach is to divide the perturbations to the orbital elements into secular perturbations, long period perturbations, and short period perturbations. The secular variations result from averaging the equations of motion over one orbital period by assuming constant, mean values of the elements over that time. Recall the variation of parameter results of Section 5.2. The result generally is that some of the angular variables (Ω, ω and λ) will change linearly with time. Inclusion of the slow change in these angular variables in the equations of motion produces the long period effects. When both the long period and secular effects are subtracted, only short period effects remain. These short period effects have periods no longer than the orbital period.

5.4.3 Oblateness Perturbations

As an example of this process, consider the J_2 term and the equation of motion for Ω for the elliptical orbit case

$$\frac{d\Omega}{dt} = -\frac{J_2\mu}{nab\sin i}\frac{\partial}{\partial i}\left[\frac{1}{r}\left(\frac{R}{r}\right)^2\left(\frac{3}{2}\sin^2\varphi - \frac{1}{2}\right)\right]$$

since $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$. To take the partial derivative r and φ must be replaced in favor of the orbital elements and time. Either the E or f forms could be used for r, so that r is only a function of time and the in-plane elements a, e and λ. For φ, write $\sin\varphi = \sin i \sin(\omega + f)$ from the law of sines, equation (1-1). Thus, the only dependence of this disturbing term on inclination is explicitly through sin(i). The final, exact equation of motion for Ω is

$$\frac{d\Omega}{dt} = -\frac{3\mu J_2}{nab}\frac{1}{r}\left(\frac{R}{r}\right)^2 \cos i \sin^2(\omega + f) \tag{5-20}$$

elements are constant on the right hand side. For this case, the independent variable in equation (5-20) is changed from time to f using $\frac{df}{dt} = \frac{h}{r^2}$ yielding

$$\frac{d\Omega}{df} = -\frac{3J_2R^2\cos i}{a^2(1-e^2)^2}(1+e\cos f)\sin^2(\omega+f)$$

The change in Ω in one orbit is obtained by integrating with respect to f from 0 to 2π to give

$$\Delta\Omega = \frac{-3\pi J_2\cos i}{(1-e^2)^2}\left(\frac{R}{a}\right)^2$$

Using the value of J_2 given above leads to about 0.5^o per orbit change for low inclination, LEO satellites. The secular rate per orbit is obtained by dividing by the period to give

$$\Delta\dot{\Omega} = \frac{-3nJ_2}{2(1-e^2)^2}\left(\frac{R}{a}\right)^2\cos i$$

yielding about 8^o per day change for low inclination, LEO satellites.

An alternate approach is to average the disturbing function over one orbital period before evaluating the partials on the right hand side of equations (5-18). The resulting disturbing function for the J_2 term is

$$\overline{R}_2 = -\frac{1}{P}\int_0^P \frac{\mu J_2}{r}\left(\frac{R}{r}\right)^2\left(\frac{3}{2}\sin^2\varphi - \frac{1}{2}\right)dt$$

Again switching to f as the independent variable yields after some algebra

$$\overline{R}_2 = -\frac{\mu J_2}{2a(1-e^2)^{3/2}}\left(\frac{R}{a}\right)^2\left(\frac{3}{2}\sin^2 i - 1\right) \tag{5-21}$$

Exercise 5-3. Perform the integration over one period to arrive at equation (5-21)

Thus the mean disturbing function for J_2 depends only on the orbital elements a, e and i. In view of equations (5-18) it is clear that there are no secular variations in a, e and i. Thus the mean orbit shape (a, e) is invariant and the mean inclination is constant. Physically this means that the average energy and z-component of angular momentum are preserved. The former should have been expected from the fact that the disturbing force is derivable from a potential function and the latter by the rotational symmetry of the potential due to J_2. The three secular variations due to J_2 are

$$\frac{d\Omega_s}{dt} = -\frac{3}{2} n J_2 \left(\frac{R}{p}\right)^2 \cos i$$

$$\frac{d\omega_s}{dt} = \frac{3}{4} n J_2 \left(\frac{R}{p}\right)^2 \left(5\cos^2 i - 1\right) \tag{5-22}$$

$$\frac{d\lambda_s}{dt} = \frac{3}{4} n J_2 \left(\frac{R}{p}\right)^2 \sqrt{1-e^2}\left(3\sin^2 i - 2\right)$$

Exercise 5-4. Begin with equation (5-18) and equation (5-21) and verify the second of equations (5-22)

Interpretation of these equations shows that $\dot{\omega}_s = 0$ if $\cos i = \pm\frac{1}{\sqrt{5}}$, or i=63.43° or 116.56°.

This angle is called the critical inclination. Since the argument of periapsis shows no secular variation at the critical inclination, the latitude of periapsis remains the same from orbit to orbit. The Molniya orbits in Section 7.3 are at the critical inclination so as to keep the periapsis at the latitude of the USSR. Below the critical inclination, periapsis regresses so that the time from one periapsis to the next is less than the "orbit period." The last equation suggest that the mean motion is biased by J_2 since $\dot{M} = n + \dot{\lambda}$ similar to the phase change for the non-linear spring example in Section 5.2.

The perturbations discussed above are typical of those caused by all even zonal harmonics e.g. J_4, J_6, etc. When calculating accurate values for critical inclination or secular variations in Ω, ω or λ, these additional terms must be considered.

Because of these types of perturbation to the orbit, a number of "periods" are in use. The nodal period is the time between successive ascending node passages. The anomalistic period is the time between successive periapsis passages based on the change in mean motion due to J_2 and other perturbations.

5.4.4 Odd-Zonal Perturbations

In a similar manner, the mean disturbing function for the third zonal harmonic is [4,349],

$$\bar{R}_3 = \frac{3\mu J_3 e}{8a}\left(\frac{R}{a}\right)^3\left(4\sin i - 5\sin^3 i\right)\left(1-e^2\right)^{-\frac{5}{2}}\sin\omega \tag{5-23}$$

Referring to equations (5-18) it is again seen that there is no change in mean energy due to J_3, but unlike J , there will be variations in inclination and eccentricity from the mean values. The

$$\frac{de}{dt} = \frac{3}{8} n J_3 \frac{R^3}{ap^2}\left(5\sin^3 i - 4\sin i\right)\cos\omega \tag{5-24}$$

If all the terms on the right side were considered to be constant, this equation would suggest a secular variation in eccentricity. However, due to the J_2 effects, ω is varying linearly with time unless the orbital inclination is critical, i.e. $\cos^2 i = 1/5$. Assuming that ω is the linear function of time given in equation (5-18) leads to the integral for the change in eccentricity from time t_o to time t

$$\Delta e_3(t, t_o) = \frac{-J_3}{2J_2}\left(\frac{R}{a}\right)\sin i\left[\sin\omega(t) - \sin\omega(t_o)\right] \tag{5-25}$$

The maximum amplitude for the variation in eccentricity occurs for polar orbits. Since $J_3 / J_2 \approx 0.002$ for the Earth, the maximum change in eccentricity for a LEO is about 0.001. This would produce a maximum variation in periapsis altitude of about 7 km with a period of $2\pi / \dot{\omega}$. For the Moon $J_2 = 2.03 \times 10^{-4}$, so the node and periapsis for a low altitude lunar orbiter (LLO) will precess 1/5 of the rate per orbit as a LEO satellite. Also the lunar $J_3 = 6 \times 10^{-6}$, yielding the ratio $J_3 / J_2 \approx 0.03$. Thus, the change in eccentricity due to J_3 is about 15 times larger. For LLO the effect of J_3 is a major consideration for orbit lifetimes.

Exercise 5-5. Derive equation (5-25) from equation (5-24).

There is also a long period variation in inclination which is of interest

$$\Delta i_3(t, t_o) = \frac{1}{2}\frac{J_3}{J_2}\left(\frac{R}{p}\right) e\cos i\left[\sin\omega - \sin(t_o)\right] \tag{5-26}$$

The perturbations discussed above are typical of those caused by all odd zonal harmonics e.g. J_5, J_7, etc. When calculating long term variations in i or e, these additional terms must be considered.

Additional long period variations due to J^3 as well as secular, long period and short period terms for J_2 through J_5 are given in Koelle[8,8-26] through terms of order J_2^2

5.4.5 Radiation pressure

Radiation pressure on an orbiting body occurs when photons strike the surface. These photons can be radiation directly from the sun (most of the energy is in the visible wavelengths), can be reflected from another body, or can be radiation, usually in the infrared, emitted from another body. Earth reflected radiation is particularly important for LEO satellites because of the high albedo of the Earth. The Earth has an albedo of about 0.3 at middle latitudes and 0.8 at the poles.

shadowing is ignored, solar pressure can be analyzed using Lagrange's planetary equations as will be demonstrated below.

Solar pressure has been proposed as a propulsion system for a satellite. Solar sails can be constructed to "catch" photons and reflect them in a manner to produce thrust. This method needs a very high effective area to produce a substantial thrust.

Radiation from a body is normally specified in energy flux. For example, the energy flux from the Sun at 1 AU is about 1340 watts/m^2. The momentum flux, from which pressure can be calculated, is the energy flux divided by the speed of light. Hence, the solar momentum flux P is about 4.5×10^{-6} N/m^2. The interactions of a photon with a surface ranges from passing through without any absorption (transparent material), having a probability greater than zero of being absorbed (translucent), completely absorbed (black body) and reflected (mirror). The solar pressure p_s is modeled as

$$p_s = \alpha P \tag{5-27}$$

where $0 \leq \alpha \leq 2$. Transparent materials have α=0 and mirrors have α=2, i.e. transparent materials absorb none of the momentum flux and mirrors reverse the direction of the momentum flux, effectively reacting to twice the incoming flux. The total force is obtained by integrating equation (5-27) over the exposed area of the body. Note that in addition to producing a net force on the satellite, solar pressure can also produce significant torques on unsymmetrical satellites which must be considered for attitude control.

Exercise 5-6. Show that a 100 kg satellite with a cross sectional area of 2 m ^{2} and albedo of 1 would experience about 1 micro-g of acceleration due to radiation pressure.

Now consider a spherical satellite of the Earth that is not passing through the shadow of the Earth. Assume a homogeneous reflecting surface so that the solar pressure is constant and away from the Sun. The radiation force is therefore $-\alpha P A e_s$, where e_s is the unit vector from the central body to the Sun and A is the effective cross sectional area. Assuming that the Sun does not move over one orbital period, this is a constant force and hence derivable from the disturbing function $R = -\alpha P A e_s \cdot \mathbf{r} = \beta e_s \cdot \mathbf{r}$ where $\mathbf{r}$ is the position vector and in terms of the orbital elements is given by equation (3-21). The average of R over an orbit can be obtained using equations (3-26) and (3-27) to yield the disturbing potential for long period and secular variations $\bar{R} = \kappa a e_s \cdot e$ where e is the eccentricity vector and $\kappa = -3\beta/2$. Hence, the eccentricity and the angle between the direction to the Sun and the major axis of the ellipse completely determine the perturbations.

Referring to the planetary equations (5-18), it is seen that there is no secular variation in energy because the work done by solar pressure as the satellite approaches the sun is equal to the work

Evaluating equations (5-18) for these two variables and then letting inclination become zero so that the Sun is in the orbit plane yields

$$\dot{e} = \frac{\kappa b}{na^2}\sin(\omega + \Omega) \qquad \dot{\omega} = \frac{\kappa b}{nea^2}\cos(\omega + \Omega)$$

These equations show that if solar pressure is the only perturbing force, the argument of periapsis will precess until $\cos(\omega+\Omega)=0$ and the precession is very rapid for small eccentricity. When the sun is along the semi-latus rectum, the precession stops, then e will either increase until the satellite hits the planet or decrease until a circular orbit is achieved. Though the equations here are not applicable when e=0, it can be shown that a circular orbit will become increasingly elliptical with the major axis at right angles to the sun line. For most satellites the J_2 secular variation in Ω and ω will dominate the solar pressure precession in ω, so the eccentricity will undergo a long period variation. Recall again that these results are for the no shadowing case.

Exercise 5-7. Use the toolbox to plot the 24 hour ground track of the LEO satellite a=7000, e=0.05, $i=55^o$, $\Omega=60^o$, $\omega=45^o$, τ=July 4, 2000, 13 hrs, 55 min, 34.56 sec. Compare tracks with and without J_2 precession.

5.5 Gauss' Form of the Perturbation Equations

Drag and some other forces can not be formulated as potential functions. It is therefore of interest to have the analog of the planetary equations in a form where the perturbing force or acceleration appear explicitly. There are numerous approaches to arrive at these equations. The direct method is to start with equation (5-10) and substitute the results from equations (5-13) and (5-14) directly into $\left[\frac{\partial r}{\partial c}\right]$. Another method is to consider the effect of an impulse or instantaneous change in v applied at some point in the orbit. Since the motion before and after the impulse is pure two body motion, the change in the elements across the impulse can be obtained by applying differential calculus to any two body equation. For example, from the vis-viva integral

$$2\mathbf{v}\cdot\delta\mathbf{v} = \frac{\mu\,\delta a}{a^2} \qquad \Rightarrow \qquad \delta a = \frac{2a^2}{\mu}\mathbf{v}\cdot\delta\mathbf{v}$$

where the fact that an impulse does not change r has been utilized. Interpretation of this equation shows that the most effective location in orbit to change the energy is to apply an impulse at periapsis (where v has the maximum value) along the velocity vector. Energy is not affected by an impulse normal to the velocity vector. If instead of an impulse, it is assumed that the δv occurred over a finite but small time δt, then taking the limit leads to

$$2va^2$$

The complete set, given below, is from [5,489] for the case where the perturbation acceleration is projected along orthogonal axes that are along the orbit tangent (a_t), normal to the orbit plane (a_h) and normal to the velocity vector in the orbit plane (a_n). In the Euler form of the perturbation equations, it is not convenient to utilize λ or τ as an orbital element for reasons discussed in [5]. Instead the last equation is given for M explicitly. Danby [4] and many other books present the equations of motions for perturbations that are projected in the radial, normal to the orbit, and circumferential directions.

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2va^2}{\mu} a_t \\
\frac{de}{dt} &= \frac{1}{v}\left(2(e+\cos f)\, a_t - \frac{r}{a}\sin f\, a_n\right) \\
\frac{di}{dt} &= \frac{r\cos\theta}{h} a_h \\
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{h\sin i} a_h \\
\frac{d\omega}{dt} &= \frac{1}{v}\left[2\sin f\, a_t + \left(2e + \frac{r}{a}\cos f\right) a_n\right] - \cos i\,\frac{d\Omega}{dt} \\
\frac{dM}{dt} &= n - \frac{b}{eav}\left[2\left(1 + \frac{re^2}{p}\right)\sin f\, a_t - \frac{r}{a}\cos f\, a_n\right]
\end{aligned}
\tag{5-28}
$$

Interpretation of equations (5-28) can lead to an understanding of where to apply impulses to achieve maximum change in an orbit parameter. Changing energy or semi-major has already been discussed. Changing orbital inclination is often a mission requirement. It is seen that only an impulse normal to the orbit plane will change inclination and that the most effective location is where $r\cos(\omega+f)$ is a maximum/minimum. The cosine reaches an extremum when the satellite is on the node line, so this is the most efficient location for a circular orbit. For the high eccentricity transfer orbits from LEO to GEO the optimal location for a single impulse would clearly be near apoapsis. Similar arguments can be made for Ω, but the other variables are not so obvious. Since there are three components to the impulse, at most three elements can be controlled with one maneuver. Of course, the remaining three elements may also change. For a particular orbit, the optimal location for performing an impulse can be formulated as a constrained optimization problem and the solution found by searching numerically around the orbit.

5.5.1 Drag

Any planetary atmosphere experienced by an orbiting body will cause drag and perhaps other forces and moments on the satellite. In the free molecular flow region that is usually associated

$$\mathbf{d} = -\frac{1}{2}\rho v\left(\frac{C_d A}{m}\right)\mathbf{v} \tag{5-29}$$

where A is the satellite reference area, m is the mass, C_d is the drag coefficient, $\mathbf{v}$ is the velocity vector, v is the speed, ρ is the atmospheric density, and $C_dA/m = \beta$ is the **ballistic coefficient.** Atmospheric density can vary with altitude, planet-sun distance, day/night, latitude, local solar time, solar activity, etc. and models of these variations are not precise. As a result the analyst must be careful in modeling drag phenomena. Letting A be the cross-section of the spacecraft exposed to free molecular flow, $C_d \geq 2$. $C_d = 2$ if the linear momentum of all incoming molecules is completely absorbed by the satellite. This situation occurs if the satellite surface has a momentum accommodation coefficient of unity. However, these gas-surface interactions are very complicated. Simple models assume that some fraction of the incoming molecules are not absorbed by the surface and that most absorbed molecules are quickly emitted from the surface after coming into thermal equilibrium with the surface. This generally leads to $C_d \approx 2.2$.

Drag is generally the dominate force that defines a satellite's orbital lifetime and requires propulsive capability for orbit maintenance. The definitive study of drag effects is given by King-Hele [6]. On the other hand, atmospheric drag has been used for aerobraking which is the process of reducing orbital energy to a desired level by dipping into an atmosphere. This process can significantly reduce propulsive requirements. In any case, if latitudinal and longitudinal variations in density are significant, the usual approach is to numerically integrate the equations of motion in rectangular coordinates using equation (5-29) for the perturbing force. If such variations are negligible or to gain insight into the effects of drag on orbit parameters, density can be assumed to only be a function of altitude.

From equations (5-28) it is seen that only a, e, ω and M are perturbed by a tangential drag force. For orbit lifetimes, the perturbations to a and e are of particular interest because $r_p = a(1-e)$. Referring to the first two of equations (5-27), change the independent variable from time to eccentric anomaly using Kepler's equation (3-14) and use the vis-viva integral (3-5) to write $v^2 = \frac{\mu}{a}\left(\frac{1 + e\cos E}{1 - e\cos E}\right)$ to finally yield

$$\frac{da}{dE} = -\beta a^2 \rho \frac{(1 + e\cos E)^{3/2}}{(1 - e\cos E)^{1/2}} \qquad \frac{de}{dE} = -\beta p \rho \cos E \sqrt{\frac{1 + e\cos E}{1 - e\cos E}} \tag{5-30}$$

If density is modeled as only a function of altitude, density can be written as a function of E. Further, if a and e can be assumed to be constant during a single pass through the atmosphere, these equations can be integrated numerically to yield the change in a and e during one orbit.

Exercise 5-8. Derive equations (5-30) following the directions above.

$$\cos E\sqrt{\frac{1+e\cos E}{1-e\cos E}} = \left(\frac{e}{2}+\dots\right) + \left(1+\frac{3}{8}e^2+\dots\right)\cos E + \left(\frac{e}{2}+\dots\right)\cos 2E + \dots$$

The coefficients are infinite power series in eccentricity. For analytic solutions, atmospheric density is modeled by an exponential in altitude, i.e.

$$\rho = \rho^o \exp\left(-\frac{h-h_o}{H_s}\right) \tag{5-31}$$

where ρ_o is the density at reference altitude h_o and H_s is the density scale height. For the Earth, H_s may range from 30 to 100 km depending on altitude, solar cycle, and other geophysical parameters. The reference altitude is usually taken as the periapsis altitude, h_o=h_p. To utilize this model for density in equations (5-30), note that $h-h_p = r-r_p = ae(1-e\cos E)$. With this substitution and the Fourier series expansions, the right hand sides can be integrated over one orbit. The results are infinite series of Bessel functions with coefficients that are infinite series in eccentricity[Reference 6, Chapter 4]. Reference 6 provides numerous approximations to equations (5-30) based on the values of e and α=ae/H_s. Only one of the expansions is given here as it applies to many LEO satellites and is applicable if $0.02<e<0.2$ and $\alpha>3$

$$\Delta a = -2\pi\beta a^2\rho_p \exp(-\alpha)\left[I_o + 2eI_1 + \frac{3}{4}e^2(I_o+I_2) + \frac{1}{4}e^3(3I_1+I_3) + O(e^4)\right] \tag{5-32}$$

where the argument of the modified Bessel functions is α, i.e. I_n=$I_n(\alpha)$.

5.5.2 N-Body Perturbations

To obtain the equation of relative motion of m_1 with respect to m_2 including the effects of the remaining n-2 bodies, subtract equation (2-10) with i=2 from the same equation with i=1,

$$\ddot{\mathbf{r}} = -\frac{G(m_1+m_2)\mathbf{r}}{r^3} - G\sum_{j=3}^{n}\left[\frac{\mathbf{r}_{1j}}{r_{1j}^3} - \frac{\mathbf{r}_{2j}}{r_{2j}^3}\right] \tag{5-33}$$

Even though the perturbation term on the right is derivable from a disturbing function, obtaining an average disturbing function, to study secular and long period variations, is difficult because the other bodies are in motion. Prior to the development of high speed computers, these equations were used as the basis for planetary theories. Generally, the approach is to consider a single disturbing body at a time. Double averaging is done over the two orbital periods to obtain the averaged disturbing function. The interested reader can consult Reference 9 for details. For LEO satellites, secular and long period variations due to the Moon and Sun are several orders of magnitude smaller than the J_2 secular terms, but can be substantial for high eccentricity orbits.

consideration is the selection of a method for numerical integration. The most efficient methods are second order, multi-step, constant step size which are very efficient for low to moderate eccentricity orbits. Reference 4 provides an introduction to the numerical procedures used in orbital mechanics including interpolation, extrapolation, differentiation and integration.

5.7 Problems

5-1. Apply the planetary equations to equation (5-23) to derive equation (5-25).

5-2. Apply the planetary equations to equation (5-23) to derive equation (5-26)

5-3. Derive an expression for the first order $d\omega/dt$ due to J_3 similar to equation (5-22).

5.8 Astronautics Toolbox

1. Develop a function [wdot, Wdot, Lamdot]=J2Precess(a,e,i,J2,mu,R) that will return the precession rates in radians per unit of time.

2. Modify the Orb2X routine (Section 3.14) so that $\dot{\Omega}$, $\dot{\omega}$ and $\dot{\lambda}$ due to J_2 are included to change Ω, ω and λ in the orbit propagation. Call the new routine Orb2XJ2 and add J2 and R to the input set. Use J2Precess.

3. Write a function that will plot ground tracks for a LEO satellite given the orbital elements in the J2000 equatorial system, (Section 1.2.2). Time of periapsis is in ymdhms format (Section 1.6) and time interval for the plot starts at periapsis and stops an input time in days later. Include an option for turning J_2 precession on or off. An option is to also limit applicability to years near 2000 so that precession of the vernal equinox does not have to be considered in the terrestrial longitude calculation using sidereal time (Section 1.3.5). Make maximum use of existing toolbox functions. It is recommended that a considerable design effort precede implementation of this function.

4. Develop a function that will integrate equations (5-30) over one orbital period. Assume an exponential atmosphere. [da,de]=dragI(a,e,Cd,A,m,rho0,h0,Hs,ichk).

5. Develop a function, [da,de]=dragKH(a,e,Cd,A,m,rho0,h0,Hs,ichk), that will evaluate equation (5-32)

6. Write a test program that will compare the relative error in da between dragKH and dragI over the applicable range of eccentricities using a=6800 km and Hs=40 km.

5.9 References

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Questions	opt1	opt2	opt3	opt4	Answer
The satellite of Ecuador which was damaged on collision with Russian debris in space?	Pegasus	lunar	aryabatta	suptnik	Pegasus
The Indian Space Research Organization (ISRO) Navigation centre was set up at _____________	karnataka	Byalalu	cuttack	jaipur	Byalalu
The first cosmonaut to spend about 17½ days in space endurance flight	Adrin Nikolayev	Neil amstrong	Samuel	Allen shepard	Adrin Nikolayev
In which year do the first indian satellite Aryabhatta was launched ?	1974	1975	1976	1977	1975
MIRV stands for	Multi dimensional independent reentry vehicle	Multi directional Independently Reoriented Vehicle	Multiple Independtly targetable Rentry Vehicle	Multi purpose Integrally targeted revolutionary vehicle	Multiple Independtly targetable Rentry Vehicle
Which one of the following is an Air to Air missile?	Astra	Akash	Becquerel	Prithivi	Astra
What was the name of the space shuttle that landed man on the moon?	Eagle	Columbia	Challenger	Apollo	Apollo
Who propounded the possibility of placing communications satellites in geosynchronous satelli	Edwin P.Hubble	William Herschel	Arthur C Clarke	Pierre ;Laplace	Arthur C Clarke
At what height geosynchronous orbit is located?	6KM	1000KM	3600KM	36000KM	36000KM
What is supernova?	a black hole	a dying star	an astroid	a comet	a dying star
Which of the following is the first missile which has been developed in India?	AKASH	PRITHIVI	AGNI	TRISHUL	PRITHIVI
What is INS Virat which serves the Indian Navy?	submarine	gunboat	aircraft carrier	fighter aircraft	aircraft carrier
The first person to enter into space was_______	valentina	Edward H.White	Yuri Gagarin	Allen shepard	Yuri Gagarin
An astronaut in outer space will observe sky as_______	black	white	blue	red	black
What is the name of the light combat aircraft developed by Indian indigenously?	BrahMos	Chetak	Astra	Tejas	Tejas
What is the range of AGNI III test fierd by India?	2250km	3500km	5000km	1000km	3500km
Nuclear explosive devices were tested in india at____	sriharikota	bangalore	pokharan	kanchipuram	pokharan
From where was india's multi purpose tele communication satellite INSTAT-IIE launched?	thumba	baikanour	kourou	sriharikota	kourou
Goestationary satellite revolves at___	any height	fixed height	height above the pole	height which depends upon its mass	fixed height
What is the name given to Met Sat launched in 2003?	vikram I	baskara I	kalpana I	aryabatta I	kalpana I
The first ever robot spacecraft to probe planet venus was named	challenger	newton	magellan	galileo	magellan
Which one of the following correctly describes AGNI?	a long range gun	a long range missile	a versatile tank	a fighter plane	a long range missile
The messenger satellite launched by NASA is to steady_____	jupiter	saturn	venus	mercury	mercury
Which launched the worlds first satellite dedicated to monitoring GGE in 2009?	USA	BRAZIL	JAPAN	INDIA	JAPAN
Which one of the following is the surface - air missile?	agni	brahmos	trishul	k15	trishul
Where is the first integrated solar combined cycle power project proposed to be setup?	patna	jaipur	cuttack	jodhpur	jodhpur
Which city recieves the highest cosmic radiation?	delhi	kolkata	mumbai	chennai	chennai
ISRO launched the worlds first satellite dedicated to education EDUSAT in the month of	Jun-04	Jul-04	Aug-04	Sep-04	Sep-04
Bhabha Atomic Research Center is situated in __________	Delhi	Mumbai	Chennai	Hyderabad	Mumbai
Saha Instituite of Nuclear Physics is situated at _________	New Delhi	Mumbai	Kolkata	chennai	Kolkata
Which of the following is a stealth aircraft virtually undetectable even by radar?	B-2 Spirit	B1-B Lancer	FA-18 Homets	B-52 Stratofortrees	B-52 Stratofortrees
The Name of India's research station at the North Pole is __________	Maitri	Himadri	Dakshin Gangotri	None of these	Himadri
A geostationary satellite revolves round the earth from ________	East to West	West to East	North to South	South to North	West to East
In which year was the ISRO founded?	1967	1969	1970	1972	1969
The first explosion of an atomic device in India was carried out in the State of _______	Rajasthan	Nagaland	Manipur	Jammu and Kashmir	Rajasthan
Name the country which launched the first Satellite into the space.	Japan	England	Soviet Union	USA	Soviet Union
Where is INS Ashwini anchored off?	Goa	Kochi	Mumbai	Vishakhapatnam	Mumbai
The Headquarters of MCF the nerve centre of the entire space craft operations in India is at ___	Hyderabad----Andhra Pradesh	Hassan ------Karnataka	Thumba ---- Kerala	Sriharikota ---- Andhra Pradesh	Hassan ------Karnataka
The rear side of the moon was photographed by _______	Ariner II	Luna II	Viking I	Viking II	Luna II
What is the india's first micro wave satellite?	RISAT I	GSAT 12	ROHINI	MEGHA TROPIQUIES	RISAT I
Which one the following is not a galilean satellite of jupiter?	deimos	calisto	europa	ganymede	deimos
The Vikram sarabhai space centre is located at ___	chennai	bangalore	trivandrum	sriharikota	trivandrum
What is the name given to india's lunar mission?	kalpana II	vikram I	chandrayan I	INSAT V	chandrayan I
Who developed ballistic missile ?	wernher Von Braun	Edward H.White	Samuel	J Robert	wernher Von Braun
ISRO 's master control facility is in______	orissa	karnataka	gujarat	andhara pradesh	karnataka
The largest circular storm in our solar system is on the surface of which of the following planet:	w) Jupiter	x)Venus	y)Uranus	z)Earth	Jupiter
The biggest asteroid known is:	w)Vesta	x)carus	y) Ceres	z) Eros	y) Ceres
Rounded to the nearest day, the Mercurian year is equal to:	w) 111 days	x)88 days	y)50 days	z)25 days	x)88 days
One of the largest volcanos in our solar system-if not the largest-is named Olympus Mons. This volcano is located on:	w) Jupiter's moon Callisto	x) Venus	y) Saturn's moon Titan	z) Mars	z) Mars
One Jupiter day is equal to which of the following?	w) 30 hrs 40 min	x)9 hrs 50 min	y)3 hrs 20 min	z)52 hrs 10 min	x)9 hrs 50 min
The time interval between two successive occurrences of a specific type of alignment of a planet (or the moon) with the sun and the earth is referred to as:	w) a conjunction	x) an opposition	y) a sidereal period	z) a synodic period.	z) a synodic period.
During the period between 1979 and 1998, what is the farthest planet from the sun?	Jupiter	Neptune	Uranus	Earth	Neptune
Of the following four times, which one best represents the time it takes energy generated in the core of the sun to reach the surface of the sun and be radiated?	w) Three minutes	x) Thirty days	y) One thousand years	z) One million years	z) One million years
The sunspot cycle is:	3 years	x) 11 years	y) 26 years	49 years	x) 11 years
The Hertzsprung-Russel Diagram of stars DIRECTLY compares what TWO of the following properties of stars?	w) size	x) temperature	y) luminosity	z) density	x) temperature
The andromeda Galaxy is which of the following types of galaxies?	w) elliptical	x) spiral	y) barred-spiral	z) irregular	x) spiral
About how many light years across is the Milky Way?	w) 1,000	x) 10,000	y) 100,000	1,000,000	y) 100,000
Which unlucky Apollo lunar landing was canceled after an oxygen tank exploded?	APOLLO 13	APOLLO 12	APOLLO 11	APOLLO 15	APOLLO 13
Heliocentric (pron: he-lee-o-sen-trik) means around:	w) Jupiter	x) the Moon	y) the Sun	z) Neptune	y) the Sun

Triton, Neptune's moon, has an ocean made of a liquid. What is this liquid?	w)nitrogen	x)hydrogen	y)oxygen	z) helium	x)NITROGEN
Who was the first man to classify stars according to their brightness?	w) Aristarchus	x) Pythagorus	y) Copernicus	z) Hipparchus	z) Hipparchus
For what reason was the Schmidt telescope specially built?	w) a sky camera	x) a radio telescope	y) an optical telescope	z) a solar telescope	w) a sky camera
What is the star nearest to the sun?	ALPHA CENTAURI	x) Icarus	y) Ceres	w) Vesta	ALPHA CENTAURI
The greatest distance of a planet from the sun is called what?	w) aphelion	x) perihelion	y) helix	z) eccentricity	w) aphelion
Multiple Choice: How is the atmospheric pressure of Mars as compared to the atmospheric pressure of the earth?	w) about the same as the earth's	x) about 100 times as great as the earth's	y) about 1/200th that of the earth's	z) half as much as that of the earth's	y) about 1/200th that of the earth's
What gas is the main component of the atmosphere of Mars?	O2	Co2	N	He	Co2
The planet Jupiter has a mass that is:	w) equal to the combined masses of the earth and Mars	x) equal to the combined masses of Saturn and Pluto	y) equal to the combined masses of Saturn, Neptune and Uranus	z) greater than the combined masses of all of the planets	z) greater than the combined masses of all of the planets
Which one of the following moon features is named Copernicus?	w) sea	x) crater	y) mountain range	z) rill	x) crater
On which day of the year does the summer solstice usually occur?	21-Jun	21-Jul	21-Sep	21-Oct	21-Jun
When the earth if farthest from the sun, what season is it in the Northern Hemisphere?	summer	winter	spring	autumn	summer
A typical galaxy, such as our Milky Way galaxy, contains how many billion stars?	w) 10 billion	x) 40 billion	y) 200 billion	z) 800 billion	y) 200 billion
Refracting telescopes always contain which one of the following?	w) mirrors	x) lenses	y) television systems	z) film	x) lenses
A comet's tail points in which direction?	w) toward the sun	x) toward the earth	y) behind the comet in its orbit	z) away from the sun	z) away from the sun
Which of the following statements is true for BOTH Saturn and Jupiter?	w) both rotate faster than the Earth	x) both rotate slower than the Earth	y) only one rotates rapidly while the other rotates very slowly	z) their periods of rotation are linked to their period of revolution	w) both rotate faster than the Earth
Which of the following is true for ORION?	w) the brightest star in the sky	x) a constellation	y) the name given to a NASA spacecraft	z) an asteroid	x) a constellation
Which of the following men wrote the book "On the Revolutions of the Heavenly Spheres"?	w) Kepler	x) Euclid	y) Copernicus	z) Newton	y) Copernicus
Beads of light visible around the rim of the moon at the beginning and end of a total solar eclips	w) Lunar Beads	x) Solar Beads	y) Baily's Beads	z) Rim Beads	y) Baily's Beads
Whereas latitude and longitude are the coordinates of places on earth, the coordinates used for star locations are two of the following, choose two.	w) ascension	x) right ascension	y) altitude	z) declination	x) right ascension
The 2.7 Kelvin cosmic background radiation is concentrated in the:	w) radio wavelengths	x) infrared	y) visible	z) ultraviolet	w) radio wavelengths
In which spectral region is it possible for astronomers to observe through clouds?	w) visual	x) radio	y) ultraviolet	z) x-ray	x) radio
The Magellanic Clouds are	w) irregular galaxies	x) spiral galaxies	y) elliptical galaxies	z) large clouds of gas and dust	w) irregular galaxies
The VISUAL aurora consists of luminous arcs, rays or bands in the night sky, usually confined to high latitudes and located in the:	w) troposphere	x) stratosphere	y) ozonosphere	z) ionosphere	z) ionosphere
When two heavenly bodies occupy the same longitude, the bodies are said to be in:	w) sympathy	x) conjunction	y) parallel	z) series	x) conjunction
The study of the origin and evolution of the universe is known as:	w) tomography	x) cystoscopy	y) cryology	z) cosmology	z) cosmology
What percentage of the Sun's mass has been converted to energy?	w) 50%	x) 1%	y) 2%	z) .001%	z) .001%
According to Kepler's Laws, all orbits of the planets are:	w) ellipses	x) parabolas	y) hyperbolas	z) square	w) ellipses
According to Kepler's Laws, the cube of the mean distance of a planet from the sun is proportio	w) area that is swept out	x) cube of the period	y) square of the period	z) fourth power of the mean distance	y) square of the period

1. The largest circular storm in our solar system is on the surface of which of the following planets?	Jupiter	Venus	Uranus	Earth	Jupiter
2. The biggest asteroid known is:	Vesta	Icarus	Ceres	Eros	Ceres
3. Rounded to the nearest day, the Mercurian year is equal to:	111 days	88 days	50 days	25 days	88 days
4. One of the largest volcanos in our solar system-if not the largest-is named Olympus Mons. This volcano is located on:	Jupiter's moon Callisto	Venus	Saturn's moon Titan	Mars	Mars
5. One Jupiter day is equal to which of the following?	30 hrs 40 min	9 hrs 50 min	3 hrs 20 min	52 hrs 10 min	9 hrs 50 min
6. The time interval between two successive occurrences of a specific type of alignment of a planet (or the moon) with the sun and the earth is referred to as:	a conjunction	an opposition	a sidereal period	a synodic perio	a synodic perio
7. Of the following four times, which one best represents the time it takes energy generated in the core of the sun to reach the surface of the sun and be radiated?	Three minutes	Thirty days	One thousand years	One million years	One million years
8. The sunspot cycle is:	3 years	11 years	26 years	49 years	11 years
9. The Hertzsprung-Russel Diagram of stars DIRECTLY compares what TWO of the following properties of stars?	size	temperature	luminosity	density	temperature
10. The andromeda Galaxy is which of the following types of galaxies?	elliptical	spiral	barred-spiral	irregular	spiral
11. About how many light years across is the Milky Way? Is it:	1,000	10,000	100,000	1,000,000	100,000
12. Heliocentric located around_____	Jupiter	the Moon	the Sun	Neptune	the Sun
13. Who was the first man to classify stars according to their brightness?	Aristarchus	Pythagorus	Copernicus	Hipparchus	Hipparchus
14. For what reason was the Schmidt telescope specially built?	a sky camera	a radio telescope	an optical telescope	a solar telescope	a sky camera
15. How is the atmospheric pressure of Mars as compared to the atmospheric pressure of the earth? Is it:	about the same as the earth's	about 100 times as great as the earth's	about 1/200th that of the earth's	half as much as that of the earth's	about 1/200th that of the earth's
16. The planet Jupiter has a mass that is:	equal to the combined masses of the earth and Mars	equal to the combined masses of Saturn and Pluto	equal to the combined masses of Saturn, Neptune and Uranus	greater than the combined masses of all of the planets	greater than the combined masses of all of the planets
17. Which one of the following moon features is named Copernicus? Is it a:	sea	crater	mountain range	rill	crater

18. A typical galaxy, such as our Milky Way galaxy, contains how many billion stars? Is it approximately:	10 billion	40 billion	200 billion	800 billion	200 billion
19. Refracting telescopes always contain which one of the following?	mirrors	lenses	television systems	film	lenses
20. A comet's tail points in which direction?	toward the sun	toward the earth	behind the comet in its orbit	away from the sun	away from the sun
21. Spectral line splitting due to the influence of magnetic fields is called:	Boltzmann Effect	Zeeman Effect	Planck Effect	Zanstra's Effect	Zeeman Effect
22. Which of the following statements is true for BOTH Saturn and Jupiter?	both rotate faster than the Earth	both rotate slower than the Earth	only one rotates rapidly while the other rotates very slowly	their periods of rotation are linked to their period of revolution	both rotate faster than the Earth
23. Which of the following is true for ORION? Orion is:	the brightest star in the sky	a constellation	the name given to a NASA spacecraft	an asteroid	a constellation
24. Which of the following men wrote the book "On the Revolutions of the Heavenly Spheres"?	Kepler	Euclid	Copernicus	Newton	Copernicus
25. Which of the following is TRUE for Retrograde motion? Retrograde motion is:	caused by epicycles	undergone only by superior planets	undergone only by inferior planets	an effect due to the projection of planet orbits onto the sky	an effect due to the projection of planet orbits onto the sky
26. Beads of light visible around the rim of the moon at the beginning and end of a total solar eclipse are called:	Lunar Beads	Solar Beads	Baily's Beads	Rim Beads	Baily's Beads
27. Whereas latitude and longitude are the coordinates of places on earth, the coordinates used for star locations are two of the following, choose two.	ascension	right ascension	altitude	declination	right ascension
28. The 2.7 Kelvin cosmic background radiation is concentrated in the:	radio wavelengths	infrared	visible	ultraviolet	radio wavelengths
29. If you were watching a star collapsing to form a black hole, the light would disappear because it:	is strongly redshifted	is strongly blueshifted	its color suddenly becomes black	none of the above	is strongly redshifted
30. In which spectral region is it possible for astronomers to observe through clouds?	visual	radio	ultraviolet	x-ray	radio
31. The Magellanic Clouds are	irregular galaxies	spiral galaxies	elliptical galaxies	large clouds of gas and dust	irregular galaxies
32. The VISUAL aurora consists of luminous arcs, rays or bands in the night sky, usually confined to high latitudes and located in the:	troposphere	stratosphere	ozonosphere	ionosphere	ionosphere
33. Which star ranks second in apparent brightness among the stars ?	Canopus	Alpha Centauri	Castor	Rigel	Canopus
34. When two heavenly bodies occupy the same longitude, the bodies are said to be in:	sympathy	conjunction	parallel	series	conjunction
35. The study of the origin and evolution of the universe is known as:	tomography	cystoscopy	cryology	cosmology	cosmology
36. What percentage of the Sun's mass has been converted to energy?	50%	1%	2%	0.00%	0.00%
37. According to Kepler's Laws, all orbits of the planets are:	ellipses	parabolas	hyperbolas	square	ellipses
38. According to Kepler's Laws, the cube of the mean distance of a planet from the sun is proportional to the:	area that is swept out	cube of the period	square of the period	fourth power of the mean distance	square of the period
39. What type of visible star is the coolest?	O	A	G	M	M
40. Which type of star is maintained by the pressure of an electron gas?	Main Sequence Star	White Dwarf	Neutron Star	Black Hole	White Dwarf
41. In our solar system, which planet has a moon with a mass closest to its own?	Earth	Mars	Jupiter	Pluto	Pluto
42. Io, Europa, Ganymede and Callisto are satellites of what planet?	Jupiter	Saturn	Neptune	Uranus	Jupiter
43. The universe is estimated to be between ten and twenty billion years ol This estimate is based on the value of which constant?	The mass of the Earth	The speed of light	The Hubble Constant	The mass of the electron	The Hubble Constant
44. Which of the following first hypothesized that the Earth orbited the sun?	Alexander the Great	Copernicus	Socrates	Tycho Brahe	Copernicus
45. The LAST manned moon flight was made in what year?	1971	1972	1973	1974	1972
46. The cosmic background radiation, a remnant of the Big Bang, is at what temperature?	100K	0K	5.3K	2.7K	2.7K
47. A planet is said to be at aphelion when it is:	closest to the sun	farthest from the sun	at its highest point above the ecliptic	at its lowest point below the ecliptic	farthest from the sun
48. At any time we may describe the position of an inferior planet by the angle it makes with the sun as seen from the earth. This angle is called the:	epicycle	elongation	proxima	ecliptic	elongation
49. Which of the following planets has the greatest eccentricity?	Pluto	Jupiter	Mars	Mercury	Pluto
50. The largest moon in our solar system has an atmosphere that is denser than the atmosphere of Mars. The name of this moon is:	Titan	Ganymede	Triton	Io	Titan
1) On which of the following planets would the sun rise in the west?	Saturn	Pluto	Mercury	Venus	Venus
2) Which planet seems to be turned on its side with an axis tilt of 98 degrees?	Uranus	Pluto	Neptune	Saturn	Uranus
3) The angle that the full moon takes up in the night sky is equal to which of the following values?	1/8 degree	1/2 degrees	1 degree	2 degrees	1/2 degrees
4) The period from one full moon to the next is:	30.3 days	30 days	29.5 days	28 days	29.5 days
5) When a superior planet is at opposition it is making an angle of how many degrees with the sun?	0 degrees	45 degrees	90 degrees	180 degrees	180 degrees
6) The word Albedo refers to which of the following?	The wobbling motion of a planet	The amount of light a planet reflects	The phase changes of a planet	The brightness of a star	The amount of light a planet reflects
7) Galileo discovered something about Venus with his telescope that shook the old theories. Which of the foll	Venus was covered in clouds	Venus had phases like the moon	Venus' surface was similar to the earth's	Venus had retrograde motion	Venus had phases like the moon
8) Cassini's division is described by which of the following?	A break in the rings of Saturn	A break in the clouds of Jupiter	The distance between the first two moons of Jupiter	The distance between the Van Allen belts	A break in the rings of Saturn
9) Name the phase that the moon is in for each type of eclipse, lunar and solar:	Full moon for both phases	New moon for both phases	Full moon for lunar and new moon for solar	New moon for lunar and full moon for solar	Full moon for lunar and new moon for solar
10) The orbital plane of the moon is how many degrees inclined from the ecliptic?	15 degrees	10 degrees	5 degrees	0 degrees	5 degrees
11) In the lowest level of the photosphere of the Sun, the temperature is:	1,000 degrees Kelvin	6,000 degrees Kelvin	10,000 degrees Kelvin	13,000 degrees Kelvin	6,000 degrees Kelvin
12) A Galactic year is the length of time that it takes our sun to orbit the galaxy. In Earth years, how long is a C	100 million years	230 million years	620 million years	940 million years	230 million years
13) A first magnitude star is how many times brighter than a second magnitude star?	2.5	7.3	10	12	2.5
14) Which of the following constellations has more bright stars than any other constellation?	Big Dipper	Cassiopeia	Orion	Scorpion	Orion

15) A line through the three stars in Orion's belt points toward which one of the following stars?	Mizor	Polaris	Sirius	Rigel	Sirius
16) A pulsar is actually a:	black hole	white dwarf	red giant	neutron star	neutron star
17) In the Milky Way there are approximately	2 million stars	100 million stars	400 million stars	200 billion stars.	200 billion stars.
18) Which of the following words best describes the shape of our galaxy?	spherical	elliptical	spiral	barred-spiral	spiral
19) On February 9, 1991, ROSAT, an orbiting observatory, finished the first ever all-sky survey of:	Radio waves.	Infrared radiation.	Ultraviolet radiation.	X-rays.	X-rays.
20) The ring nebula is an example of a planetary nebul It is:	a ring of planets circling a star.	an expanding ring of gas released from a dying star.	a contracting ring of gas and dust falling circling into a massive object.	a ring of stars in a circular orbit.	an expanding ring of gas released from a dying star.
21) A black hole with the mass of the earth would be the size of____	the Sun	the Moon	a bowling ball	a marble	a marble
22) Whose paradox asks why the sky is not ablaze with starlight if the universe is infinite in extent and unifor	Olber's	Greigheim's	Schuller's	Miller's	Olber's
23) Most stars are cooler than the sun. These stars, the planets, interstellar clouds and star-forming regions e	visible	x-ray region	ultraviolet	infrared	infrared
24) Astronomers use cepheids principally as measures of what? Is it:	size	speed	chemical composition	distance	distance
25) Where are most asteroids located? Is it between:	Jupiter and Saturn	Mars and Venus	Earth and Mars	Mars and Jupiter	Mars and Jupiter
26) Of the following phases of the moon, which is the one at which a spring tide occurs? Is it	new	new gibbous	first quarter	new crescent	new
27) A white dwarf has a mass of roughly one solar mass but a size of about:	a basketball	a car	Lake Michigan	the Earth.	the Earth.
28) Which of the following can be used to see through Venus's clouds?	refracting telescopes	radar	x-rays	ultraviolet	radar
29) Primary cosmic radiation is characterized by:	an intensity which is essentially constant in time	isotropic in space	very energetic particles	all of the above.	all of the above.
30) Globular clusters in our galaxy are primarily found:	in the spiral arms.	distributed throughout the disk including regions between the spiral arms.	in the bulge at the center of our galaxy.	in the halo of our galaxy.	in the halo of our galaxy.
31) The phenomenon that causes the Moon's rotation about it's own axis to be equal to the Moon's period of	Haner Effect	Landow Effect	Orbital Synchronization	Tidal Friction	Tidal Friction
32) The Sun rotates about its own axis approximately:	once every 60 minutes.	once every 24 hours.	once every 365 days.	varies with solar latitude.	varies with solar latitude.
33) PRESENTLY, what is the farthest planet from the sun?	Pluto	Neptune	Uranus	None of the above	Neptune
34) Andromeda, the nearest galaxy which is similar to the Milky Way, is how far from the Earth? Is it:	200,000 light years	2,000,000 light years	20,000,000 light years	200,000,000 light years	2,000,000 light years
35) The precession of the Earth refers to the:	change from night to day.	Earth's motion around the sun.	change in orientation of the Earth's axis.	effect of the moon on the Earth's orbit.	change in orientation of the Earth's axis.
36) The angular position of the sun at solar noon with respect to the plane of the equator is the definition of:	solar azimuth angle	latitude	solar declination angle	index of refraction	solar declination angle
37) Which of the following men was the first to make systematic use of a telescope?	Copernicus	Tycho Brahe	Kepler	Galileo	Galileo
38) The Magellanic cloud is a:	nebula	galaxy	super nova remnant	star cluster	galaxy
39) The comet known as Halley's Comet has an average period of:	56 years	66 years	76 years	86 years	76 years
40) Which one of the following planets has no moons?	Mars	Neptune	Venus	Jupiter	Venus
41) In kilometers, the earth's average distance from the sun is roughly which of the following distances?	250 million	91 million	150 million	350 million	150 million
42) The gravity on the moon is what fraction of the gravity on the earth?	1/3	2/3	1/6	1/10	1/6
What is the name given to india's lunar mission?	kalpana II	vikram I	chandrayan I	INSAT V	chandrayan I
Who developed ballistic missile ?	wernher Von Braun	Edward H.White	Samuel	J Robert	wernher Von Braun
ISRO 's master control facility is in______	orissa	karnataka	gujarat	andhara pradesh	karnataka
1) The Phythagoreans appear to have been the first to have taught that the Earth is:	at the center of the Universe.	spherical in shape.	orbits around the sun.	flat with sharp edges.	spherical in shape.
2) The light-gathering power of a reflecting telescope depends on which of the following?	the area of its objective mirror	focal length of its eyepiece	focal length of its objective mirror	the ratio of the focal lengths of objective and eyepiece	the area of its objective mirror
3) Who first used Tycho Brahe's observational data on the planet Mars to determine that Mars actually traversed an elliptical orbit, the sun being located at one of the foci?	Kepler	Galileo	Hipparchus	Aristarchus	Kepler
4) Approximately how many times could a beam of light travel around the earth in one second?	3 times	7.5 times	15 times	25 times	7.5 times
5) The diameter of the earth is approximately:	5,000 kilometers	13,000 kilometers	18,000 kilometers	25,000 kilometers	13,000 kilometers
6) The average density of the earth is approximately:	half the density of water	twice the density of water	five and a half times the density of water.	ten times the density of water	five and a half times the density of water.
7) Of the following colors, which is bent least in passing through a prism?	orange	violet	green	red	red
8) In a reflecting telescope where in the tube is the objective mirror placed?	the top to the tube	the middle of the tube	the bottom of the tube	the side of the tube	the bottom of the tube
9) What is the approximate age of our sun?	1 billion years	2 billion years	5 billion years	20 billion years	5 billion years
10) What does it mean when someone says that comets have eccentric orbits? Does it mean	they have open orbits	they have nearly circular orbits	their orbits are unpredictable	the sun is far from the foci of their orbits	the sun is far from the foci of their orbits
11) Which of the following planets has the lowest density?	Mercury	Venus	Saturn	Mars	Saturn
12) What causes the gas tail of a comet to always point away from the sun?	solar wind	air pressure	centrifugal force	gravity	solar wind
13) What are Saturn's rings composed of?	completely connected solid masses	billions of tiny solid particles	mixtures of gases	highly reflective cosmic clouds	billions of tiny solid particles
14) Of the following, which is the only planet which CANNOT be seen with the unaided eye?	Jupiter	Mars	Neptune	Saturn	Neptune
15) Accretion is:	the gradual accumulation of matter in one location usually due to gravity.	the process of moon formation for planets.	the process of matter accumulation due to centripetal force.	the disintegration of matter.	the gradual accumulation of matter in one location usually due to gravity.
16) If we look at a galaxy that is moving AWAY from us, we find that the wavelengths of light coming from this galaxy are:	lengthened	shortened	unchanged	narrowed	lengthened
17) The age of the Universe in billions of years is approximately	500	3.5	18	1000	18
18) The reason we call an astronomical body a black hole is that	it is a huge star which appears black at its center.	its gravity is so high that it absorbs its own photons.	it represents lack of matter in a portion of space.	it is a dead planet.	its gravity is so high that it absorbs its own photons.
19) A blue shift means a Doppler shift of light from a(an)	receding star.	blue star.	approaching star.	fixed star.	approaching star.

20) The first and largest asteroid discovered was:	Pallas.	Juno.	Ceres.	Trojan.	Ceres.
21) Which of the following Saturnian satellites is known to possess an atmosphere?	Tethys	Titan	Dione	Mimas	Titan
22) The Crab Nebula consists of the remnants of a supernova which was observed by:	Brahe in 1572.	Kepler and Galileo in 1604.	the Chinese in 1054	several ancient civilizations in 236	the Chinese in 1054
23) The atmosphere of Venus contains mostly	oxygen	carbon dioxide	nitrogen	water	carbon dioxide
24) What causes a planet to have a magnetic field?	the dynamo effect.	the Doppler effect.	the Photoelectric effect.	its rotation about its sun.	the dynamo effect.
25) On the celestial sphere, the annual path of the Sun is called	the eclipse path.	eclipti	diurnal.	solstice.	eclipti
26) The angular distance between a planet and the Sun, as viewed from the Earth, is called	angle of inclination.	elongation.	latitude.	opposition.	elongation.
27) Which of the following has the greatest density?	the sun	Venus	Mars	Jupiter	Venus
28) Mercury and Venus are said to be inferior planets because:	they are smaller than the other planets that circle the Sun.	their orbits are slower than the Earth's orbit.	their orbits are faster than the Earth's orbit.	their orbits are inside of the Earth's orbit.	their orbits are inside of the Earth's orbit.
29) Galileo made many astronomical discoveries. Which of the following was NOT one of his discoveries?	the phases of Venus	mountains on Venus	moons of Jupiter	sunspots	mountains on Venus
30) Identify the ripples in the overall geometry of space produced by the acceleration of moving objects.	Doppler effect.	granulation.	gravitational waves.	elongation.	gravitational waves.
31) Which one of the following planets has less mass than the Earth?	Jupiter	Saturn	Uranus	Pluto	Pluto
32) Which of the following planets is NOT a terrestrial planet?	Earth	Jupiter	Mars	Mercury	Jupiter
33) Why do we see lunar eclipses much more often than solar eclipses?	Lunar eclipses occur more often than solar eclipses.	Lunar eclipses last longer than solar eclipses.	The lunar eclipse is visible to much more of the Earth than a solar eclipse.	The moon is closer to the Earth than the sun.	The lunar eclipse is visible to much more of the Earth than a solar eclipse.
34) A starlike object with a very large redshift is a	neutron star.	nov	quasar.	supernov	quasar.
35) The apparent magnitude of an object in the sky describes its	size	magnification	brightness	distance	brightness
36) The Van Allen belts are:	caused by the refraction of sunlight like rainbows.	charged particles trapped in the Earth's magnetic fiel	caused by the reflection of polar snow.	caused by precession.	charged particles trapped in the Earth's magnetic fiel
37) A coordinate system based on the ecliptic system is especially useful for the studies of	planets	stars	the Milky Way	galaxies	planets
38) When originally discovered, how were planets such as Pluto distinguished from the multitude of stars in the sky?	The planets appear to be bigger than stars	The planets are brighter than most stars	The planets move relative to the stars	The planets are close to the ecliptic	The planets move relative to the stars
39) The moon was closer to earth in March 1993 than it has been for a dozen years. This near distance was about	200,000 miles	216,000 miles	231,000 miles	240,000 miles	216,000 miles
40) What is the name of the spacecraft that recently used Jupiter's gravitational field to redirect its course toward the Sun's polar regions?	Magellan	Ulysses	Galileo	Challenger	Ulysses