KARPAGAM ACADEMY OF HIGHER EDUCATION

(Established Under Section 3 of UGC Act 1956)

COIMBATORE 641 021. INDIA

DEPARTMENT OF MECHANICAL ENGINEERING

SYLLABUS

15BEME702 FINITE ELEMENT METHODS 3 2 0 4 100

OBJECTIVES

1. To introduce the concepts of Mathematical Modeling of Engineering Problems.
2. To appreciate the use of FEM to a range of Engineering Problems.

UNIT I INTRODUCTION 9 + 3

Historical background – Matrix approach – Application to the continuum – Discretization – Matrix algebra – Governing equations for continuum – Classical Techniques in FEM – Weighted residual method – Ritz method

UNIT II ONE DIMENSIONAL PROBLEMS 9 + 3

Finite element modeling – Coordinates and shape functions– Potential energy approach – Galerkin approach – Assembly of stiffness matrix and load vector – Finite element equations – Quadratic shape functions – Applications to plane trusses

UNIT III TWO-DIMENSIONAL CONTINUUM 9 + 3

Introduction – Finite element modeling – Scalar valued problem – Poisson equation –Laplace equation – Triangular elements – Element stiffness matrix – Force vector – Galerkin approach – Stress calculation – Temperature effects

UNIT IV AXISYMMETRIC CONTINUUM 9 + 3

Axisymmetric formulation – Element stiffness matrix and force vector – Galerkin approach – Body forces and temperature effects – Stress calculations – Boundary conditions – Applications to cylinders under internal or external pressures

UNIT V ISOPARAMETRIC ELEMENTS FOR TWO-DIMENSIONAL CONTINUUM 9 + 3

The four-node quadrilateral – Shape functions – Element stiffness matrix and force vector – Numerical integration – Stiffness integration – Stress calculations – Four node quadrilateral element.

TOTAL 45 + 15 = 60 PERIODS

TEXT BOOKS

S. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
1	Rao S.S	The Finite Element Method in Engineering	Butter worth Heinemann imprint, USA	2011
2	Khanka S.S	A First course in the Finite Element Method	Cengage Learning, Stamford, USA	2006

REFERENCES

S. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
1	Chandrupatla T.R., and Belegundu A.D	Introduction to Finite Elements in Engineering	Pearson Education, Delhi	2011
2	David V Hutton	Fundamentals of Finite Element Analysis	McGraw–Hill Education	2005

WEB REFERENCES

1. http://nptel.iitm.ac.in/courses/Webcourse–contents/IIT–KANPUR/mathematics–2/node18.html
2. http://www.me.berkeley.edu/~lwlin/me128/FEMNotes.pdf
3. http://www.rose–hulman.edu/~fine/FE2004/Class2/Notes2.pdf
4. http://www.asiri.net/courses/meng412/m412sm04ex1sol.pdf
5. http://hyperphysics.phy–astr.gsu.edu/hbase/electric/laplace.html

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari Post, Coimbatore – 641 021. INDIA

FACULTY OF ENGINEERING

DEPARTMENT OF MECHANICAL ENGINEERING

LESSON PLAN

Subject Name : **Finite Element Method**

Subject Code : **15BEME702** **(Credits - 4)**

Name of the Faculty : **S. ARAVIND**

Designation : **Assistant Professor**

Year/Semester : **IV / VII**

Branch : **Mechanical Engineering**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – I: INTRODUCTION			
1.	1	Introduction to FEA, Basic concept of finite element method	T [1] 1-9, T [2] 1-28
2.	1	General procedure for FEA, Historical background, application of FEA	T [1] 1-9, T [2] 1-28
3.	1	Matrix approach, Application of the continuum, Discretization	R [1] 44-51
4.	1	Matrix algebra, Gaussian elimination method	R [1] 44-51, 51-61
5.	1	Solving Problems from Matrix algebra, Gaussian elimination method	R [1] 44-51, 51-61
6.	1	Tutorial 1: Problems from Gaussian elimination method	R [1] 44-51, 51-61
7.	1	Governing equations for continuum, Classical Techniques in FEM	T [1] 12-40
8.	1	Weighted residual method, Ritz method	T [1] 157-181, R [2] 131-152
9.	1	Solving Problems from Weighted residual method	T [1] 157-181
10.	1	Solving Problems from Ritz method	T [1] 157-181
11.	1	Tutorial 2: Problems from Weighted residual method and Ritz method	T [1] 157-181
12.	1	Discussion on University previous year questions	

Total No. of Hours Planned for Unit - I	12

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – II: ONE DIMENSIONAL PROBLEMS			
13.	1	Finite element modeling, Coordinates and shape functions	R [1] 67-74
14.	1	Potential energy approach, Galerkin approach	T [2] 31-66, R [1] 75-81
15.	1	Assembly of stiffness matrix and load vector	T [2] 31-66, R [1] 82-85
16.	1	Solving Problems from one dimensional bar elements	T [2] 31-66, R [1] 112-120
17.	1	Solving Problems from one dimensional beam elements	T [2] 31-66, R [1] 112-120
18.	1	Tutorial 3: Problems from one dimensional bar & beam elements	T [2] 31-66, R [1] 112-120
19.	1	Finite element equations, Quadratic shape functions	R [1] 82-107
20.	1	Applications to plane trusses	T [2] 72-91, R [1] 133-144
21.	1	Solving Problems from plane trusses	T [1] 312-322, R [1] 155-160
22.	1	Solving Problems from plane trusses	T [1] 312-322, R [1] 155-160
23.	1	Tutorial 4: Problems from plane trusses	T [1] 312-322, R [1] 155-160
24.	1	Discussion on University previous year questions	
Total No. of Hours Planned for Unit - II			**12**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – III: TWO-DIMENSIONAL CONTINUUM			
25.	1	Introduction, Finite element modeling, Scalar valued problem	T [1] 355-366, T [2] 328-348
26.	1	Poisson equation, Laplace equation, Triangular elements	T [1] 355-366
27.	1	Element stiffness matrix, Force vector	T [1] 355-366, R [1] 204-208
28.	1	Galerkin approach, Stress calculation, Temperature effects	T [2] 334-356, R [1] 214-227

29.	1	Tutorial 5: Derive element stiffness matrix for two-dimensional continuum	R [1] 228-250
30.	1	Solving Problems from two-dimensional constant strain triangular elements	T [2] 356-380, R [1] 241-253
31.	1	Solving Problems from two-dimensional constant strain triangular elements	T [2] 356-380, R [1] 241-253
32.	1	Solving Problems from two-dimensional constant strain triangular elements	T [2] 356-380, R [1] 241-253
33.	1	Solving Problems from two-dimensional constant strain triangular elements	T [2] 356-380, R [1] 241-253
34.	1	Solving Problems from two-dimensional constant strain triangular elements	T [2] 356-380, R [1] 241-253
35.	1	Tutorial 6: Problems from two-dimensional constant strain triangular elements	T [2] 356-380, R [1] 241-253
36.	1	Discussion on University previous year questions	
Total No. of Hours Planned for Unit - III			**12**

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – IV: AXISYMMETRIC CONTINUUM			
37.	1	Axisymmetric formulation – Element stiffness matrix and force vector	T [2] 452-474, R [1] 258-266
38.	1	Galerkin approach – Body forces and temperature effects	T [2] 452-474, R [1] 267-272
39.	1	Stress calculations – Boundary conditions	T [2] 452-474
40.	1	Applications to cylinders under internal or external pressures	T [2] 452-474, R [1] 272-278
41.	1	Tutorial 7: Derive element stiffness matrix for Axisymmetric element	T [2] 476-485
42.	1	Solving problems from Axisymmetric elements	T [2] 476-485, R [1] 279-286
43.	1	Solving problems from Axisymmetric elements	T [2] 476-485, R [1] 279-286
44.	1	Solving problems from Axisymmetric elements	T [2] 476-485, R [1] 279-286
45.	1	Solving problems from Axisymmetric elements	T [2] 476-485, R [2] 356-363
46.	1	Solving problems from Axisymmetric elements	T [2] 476-485, R [2] 356-363
47.	1	Tutorial 8: Solving problems from Axisymmetric elements	T [2] 476-485, R [2] 356-363
48.	1	Discussion on University previous year questions	

Total No. of Hours Planned for Unit - IV	12

Sl. No.	No. of Periods	Topics to be Covered	Support Materials
UNIT – V: ISOPARAMETRIC ELEMENTS FOR TWO-DIMENSIONAL CONTINUUM			
49.	1	The four-node quadrilateral, Shape functions	T [1] 401-428, T [2] 534-550
50.	1	Element stiffness matrixand force vector	T [1] 401-428, T [2] 534-550
51.	1	Numerical integration, Stiffness integration	T [2] 534-550, R [2] 206-213
52.	1	Solving Problems from Numerical integration	R [1] 295-302, R [2] 184-213
53.	1	Solving Problems from Numerical integration	R [1] 295-302, R [2] 184-213
54.	1	Tutorial 9: Problems from Numerical integration	R [1] 295-302, R [2] 184-213
55.	1	Stress calculations – Four node quadrilateral element.	T [2] 342-370, R [1] 310-320
56.	1	Solving problems from Four node quadrilateral element	T [2] 342-370, R [1] 310-320
57.	1	Solving problems from Four node quadrilateral element	T [2] 342-370, R [1] 310-320
58.	1	Solving problems from Four node quadrilateral element	T [2] 342-370, R [1] 310-320
59.	1	Tutorial 10: Problems from Four node quadrilateral element	T [2] 342-370, R [1] 310-320
60.	1	Discussion on University previous year questions	
Total No. of Hours Planned for Unit - V			**12**

TOTAL PERIODS : 60

TEXT BOOKS

TB. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
T [1]	Rao S. S	The Finite Element Method in Engineering	Butter worth Heinemann imprint, USA	2011
T [2]	Khanka S. S	A First course in the Finite Element Method	Cengage Learning, Stamford, USA	2006

REFERENCES

RB. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
R [1]	Chandrupatla T.R., and Belegundu A. D	Introduction to Finite Elements in Engineering	Pearson Education, Delhi	2011
R [2]	David V Hutton	Fundamentals of Finite Element Analysis	McGraw–Hill Education	2005

WEBSITES

W [1] - http://www.open.edu/openlearn/science-maths-technology/introduction-finite-element-analysis/content-section-0?intro=1
W [2] - http://nptel.ac.in/courses/112104193/
W [3] - http://nptel.ac.in/courses/112102101/17
W [4] - https://web.stanford.edu/class/me232a/1.1-1.2.pdf
W [5] - http://nptel.ac.in/courses/112104116/24
W [6] - http://users.metu.edu.tr/csert/me582/ME582%20Ch%2003.pdf
W [7] - http://www.ce.memphis.edu/7117/notes/presentations/chapter_09.pdf
W [8] - https://www.slideshare.net/ssuser69aeed/finite-element-axisymmetric-stress-and-strain
W [9] - https://www.youtube.com/watch?v=3He0rE5Arrs
W [10] - http://nptel.ac.in/courses/105105041/m3l14.pdf

JOURNALS

J [1] - Odeny D.de Matos Junior et al. Aeroelastic behavior of stiffened composite laminated panel with embedded SMA wire using the hierarchical Rayleigh–Ritz method. Composite Structures. Volume 181, 1 December 2017, Pages 26-45.
J [2] - IrwanKatili et al. Isogeometric Galerkin in rectangular plate bending problem based on UI approach. International European Journal of Mechanics - A/Solids. Volume 67, January 2018, Pages 92-107.
J [3] - Chien-YuanHou. Various remeshing arrangements for two-dimensional finite element crack closure analysis. Engineering Fracture Mechanics. Volume 170, 1 February 2017, Pages 59-76.
J [4] - X.CuiBen Q.Li. Discontinuous finite element solution of 2-D radiative transfer with and without axisymmetry. Journal of Quantitative Spectroscopy and Radiative Transfer. Volume 96, Issues 3–4, 15 December 2005, Pages 383-407.
J [5] - Gradimir V. et al. Generalized quadrature rules of Gaussian type for numerical evaluation of singular integrals. Journal of Computational and Applied Mathematics. Volume 278, 15 April 2015, Pages 306-325.

UNIT	Total No. of Periods Planned	Lecture Periods	Tutorial Periods
I	12	10	2
II	12	10	2
III	12	10	2
IV	12	10	2
V	12	10	2
TOTAL	60	50	10

I. CONTINUOUS INTERNAL ASSESSMENT : 40 Marks

(Internal Assessment Tests: 30, Attendance: 5, Assignment/Seminar: 5)

II. END SEMESTER EXAMINATION : 60 Marks

TOTAL : 100 Marks

UNIT 1

INTRODUCTION

1.1.2. Methods of Engineering Analysis

There are three different approaches to achieve the above mentioned objectives. They are:

1. Experimental methods.
2. Analytical methods.
3. Numerical methods or approximate methods.

1. Experimental Methods

In this methods, prototypes can be used. If we want to change the dimensions of the prototype, we have to disassemble the entire prototype and reassemble it and then testing should be carried out. It needs man power and materials. So, it is time consuming and costly process.

2. Analytical Methods or Theoretical Analysis

In these methods, problems are expressed by mathematical differential equations. It gives quick and closed form solutions. It is used only for simple geometries and idealized support and loading conditions.

3. Numerical Methods

Analytical solutions can be obtained only for certain simplified situations. For problems involving complex material properties and boundary conditions, the engineer prefers numerical methods that gives approximate but acceptable solutions. The following three methods are coming under numerical solutions.

(*i*) Functional Approximation.
(*ii*) Finite Difference Method (FDM).
(*iii*) Finite Element Method (FEM).

(i) Functional Approximation:

- ✓ The classical methods such as Rayleigh-Ritz methods (variational approach) and Galerkin methods (weighted residual methods) are based on functional approximation but vary in their procedure for evaluating the unknown parameters.
- ✓ Rayleigh-Ritz method is useful for solving complex structural problems, encountered in finite element analysis.
- ✓ Weighted residual method is useful for solving non-structural problems.

(ii) Finite Differential Method (FDM):

- ✓ Finite difference method is useful for solving heat transfer fluid mechanics and structural mechanics problems. It is a general method. It is applicable to any phenomenon for which differential equation along with the boundary conditions are available. It works well for two dimensional regions with boundaries parallel to the coordinate axes.
- ✓ The starting point in the finite difference method is that the differential equation must be known before solving. After that, the region is subdivided into a convenient number of divisions. The differential equation is applied successively at the various points of the subdivided region, a set of simultaneous equations are generated which upon solving lead to approximate solution to the problem. This is the essence of finite difference method.
- ✓ This method is difficult to use when regions have curved or irregular boundaries and it is difficult to write general computer programs.

(iii) Finite Element Method (FEM) or Finite Element Analysis (FEA):

- ✓ Finite element method is a numerical method for solving problems of Engineering and Mathematical Physics.
- ✓ In this method, a body or a structure in which the analysis to be carried out is subdivided into smaller elements of finite dimensions called finite elements. Then the body is considered as an assemblage of these elements connected at a finite number of joints called 'Nodes' or Nodal points. The properties of each type of finite element is obtained and assembled together and solved as whole to get solution.
- ✓ In other words, in the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and combine them to obtain the solution of the whole body.

- ✓ Finite element method is used to solve physical problems involving complicated geometrics, loading and material properties which cannot be solved by analytical method. This method is extensively used in the field of structural mechanics, fluid mechanics, heat transfer, mass transfer, electric and magnetic fields problems.
- ✓ Fig.1.1 shows the finite element discretization of spur gear teeth.

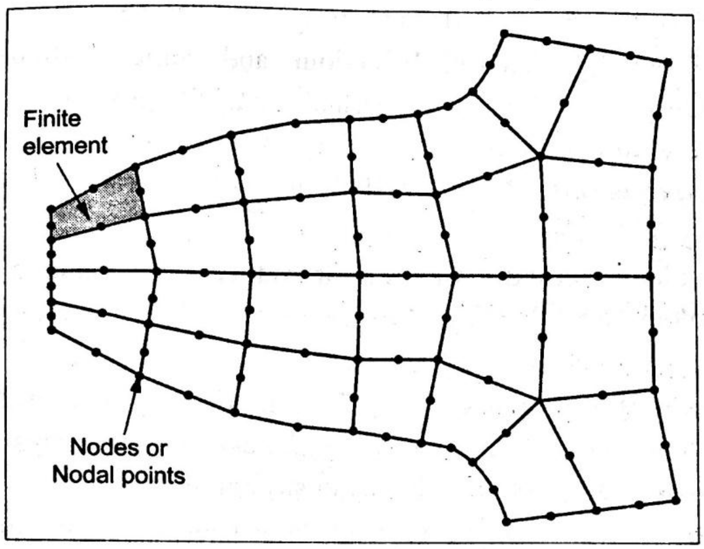


Fig. 1.1. Finite element discretization of spur gear teeth

- ✓ Based on application, the finite element problems are classified as follows:
  - (*i*) Structural problems.
  - (*ii*) Non-structural problems.

(i) Structural problems:

In structural problems, displacement at each nodal point is obtained. By using these displacement solutions, stress and strain in each element can be calculated.

(ii) Non-structural problems:

In non-structural problems, temperature or fluid pressure at each nodal point is obtained. By using these values, properties such as heat flow, fluid flow *etc.*, for each element can be calculated.

1.2. HISTORICAL BACKGROUND OF FEM

- ✓ Basic ideas of the finite element analysis were developed by aircraft engineers in the early 1940s. These were primarily the matrix methods of analysis.
- ✓ The modern development of the finite element method began in the year of 1945 in the field of structural engineering with the work by ***Hrennikoff.***
- ✓ In 1947 ***Levy*** introduced the flexibility or force method and in 1953 he suggested stiffness method which could be a promising alternative for use in analysing statically redundant aircraft structures.

- ✓ By using energy principles, ***Argyris and Kelsey*** developed matrix structural analysis methods in 1954. This development illustrated the important role that energy principles would play in the finite element method.
- ✓ The term finite element was first introduced by ***Clough*** in 1960 in the plane stress analysis and he used both triangular and rectangular elements in that analysis.
- ✓ Most of the finite element work upto early 1960s dealt with small strains and small displacements, elastic material behaviour and static loadings. In 1961, ***Turner*** considered large deflection and thermal analysis problems. In 1962, ***Gallagher*** introduced material non-linearities problems, whereas buckling problems were initially treated by ***Gallagher and Padlog*** in 1963. In 1968, ***Zinkiewicz*** extended the method to visco elasticity problems.
- ✓ Weighted residual methods was first introduced by ***Szabo*** and ***Lee*** in 1969 for structural analysis and then by ***Zinkiewicz*** and ***Parekh*** in 1970 for transient field problems.
- ✓ During the decades of the 1960s and 1970s, the finite element method was extended to applications in shell bending, plate bending, heat transfer analysis, fluid flow analysis and general three dimensional problems in structural analysis.
- ✓ From the early 1950s to present, enormous advances have been made in the application of finite element method to solve complicated engineering problems. It is curious to note that the present day finite element method does not have its root in one discipline. The mathematicians continue to put the finite element method on sound theoretical ground whereas the engineers continue to find interesting extensions in various branches of engineering. These concurrent developments have made the finite element method as one of the most powerful approximate methods.

1.3. GENERAL STEPS OF THE FINITE ELEMENT ANALYSIS

- ✓ This section presents the general procedure of finite element analysis. For simplicity's sake, we will consider only the structural problems.
- ✓ The following two general methods are associated with the finite element analysis. They are:
  - (*i*) Force method.
  - (*ii*) Displacement or stiffness method.
- ✓ In force method, internal forces are considered as the unknowns of the problem. In displacement or stiffness method, displacements of the nodes are considered as the unknowns of the problem.
- ✓ Among these two approaches, displacement method is more desirable because its formulation is simpler for most structural analysis problems. So, a vast majority of general purpose finite element programs have used the displacement formulation for solving structural problems.

✓ We now present the steps along with explanations used in the finite element method formulation.

Step 1: Discretization of Structure

The art of subdividing a structure into a convenient number of smaller elements is known as discretization.

Smaller elements are classified as follows:

(i) One dimensional elements. (ii) Two dimensional elements.
(iii) Three dimensional elements. (iv) Axisymmetric elements.

(i) One dimensional elements: A bar and beam elements are considered as one dimensional elements. The simplest line element also known as linear element has two nodes, one at each end as shown in Fig.1.2.

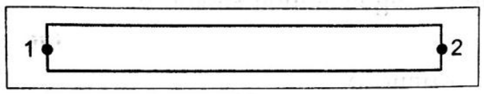


Fig. 1.2. Bar element

(ii) Two dimensional elements: Triangular and rectangular elements are considered as two dimensional elements. These elements are loaded by forces in their own plane. The simplest two dimensional elements have corner nodes as shown in Fig.1.3.

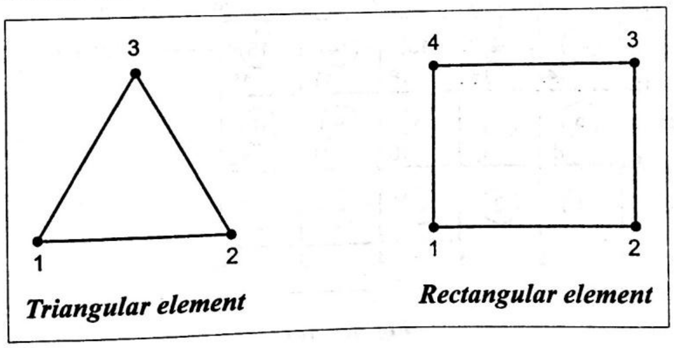


Fig. 1.3. Simple two dimensional elements

(iii) Three dimensional elements: The most common three dimensional elements are tetrahedral and hexahedral (Brick) elements. These elements are used for three dimensional stress analysis problems. The simplest three dimensional elements have corner nodes as shown in Fig.1.4.

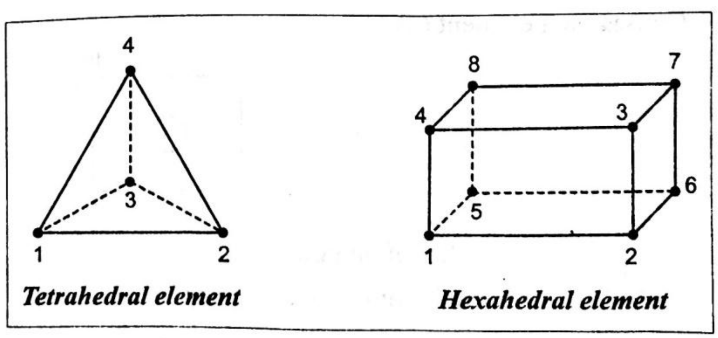


Fig. 1.4. Simple three dimensional elements

(iv) ***Axisymmetric elements:*** The axisymmetric element is developed by rotating a triangle or quadrilateral about a fixed axis located in the plane of the element through 360°. It is shown in Fig.1.5. When the geometry and loading of the problems are axisymmetric, these elements are used.

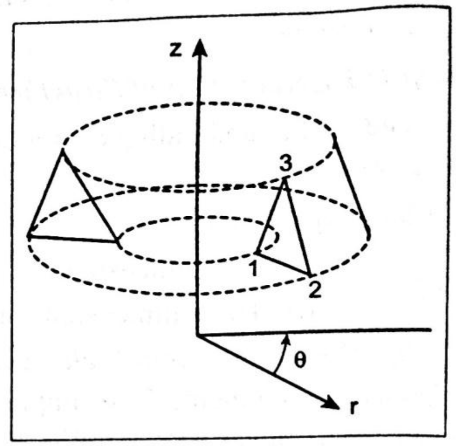


Fig. 1.5. Axisymmetric element

Step 2: Numbering of Nodes and Elements

The nodes and elements should be numbered after discretizaton process. The numbering process is most important since it decide the size of the stiffness matrix and it leads the reduction of memory requirement. While numbering the nodes, the following condition should be satisfied.

$$\begin{Bmatrix} \text{Maximum} \\ \text{node number} \end{Bmatrix} - \begin{Bmatrix} \text{Minimum} \\ \text{node number} \end{Bmatrix} = \text{Minimum}$$

It is explained in the Fig.1.6(a) and (b).

Longer Side Numbering Process:

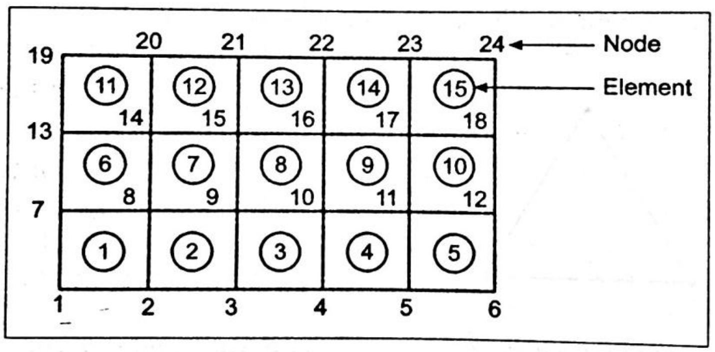
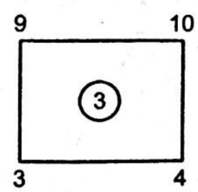


Fig. 1.6. (a)

[*Note:* Number with circle denotes element.
Number without circle denotes node]

Considering element (3),



Maximum node number = 10

Minimum node number = 3

Difference = 7 ... (1.1)

Shorter Side Numbering Process:

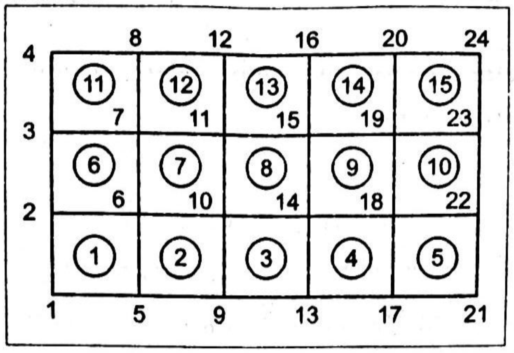
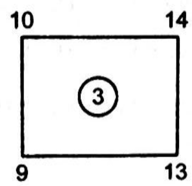


Fig. 1.6. (b)

Considering the same element (3).



Maximum node number = 14

Minimum node number = 9

Difference = 5 ... (1.2)

From equation (1.1) and (1.2), we came to know, shorter side numbering process is followed in the finite element analysis and it reduces the memory requirements.

Step 3: Selection of a Displacement Function or Interpolation Function

✓ It involves choosing a displacement function within each element. Polynomial of linear, quadratic and cubic form are frequently used as displacement functions because they are simple to work within finite element formulation.

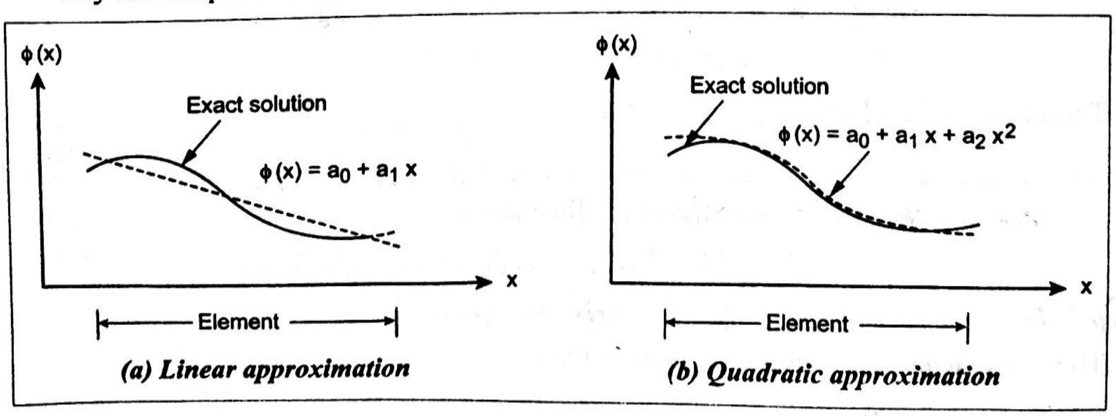


Fig. 1.7. Polynomial approximation in one dimension

✓ The polynomial type of interpolation functions are mostly used due to the following reasons.

1. It is easy to formulate and computerize the finite element equations.
2. It is easy to perform differentiation or integration.
3. The accuracy of the results can be improved by increasing the order of the polynomial.

Fig.1.7 shows the polynomial approximation in one dimension.

Let us consider $\phi(x)$ is a field variable.

Case (i): Linear Polynomial:

In compact matrix form as,

$$\{ F^e \} = [k^e] \{ u^e \}$$

where, e is a Element, { F } is the vector of element nodal forces, [k] is the element stiffness matrix and { u } is the element displacement vector.

This equation can be derived by any one of the following methods.

(i) Direct Equilibrium Method: This method is much easier to apply for line or one dimensional elements.

(ii) Variational Method: This method is most easily adaptable to the determination of element equations for complicated elements (*i.e.*, element having large number of degrees of freedom) like axisymmetric stress element, plate bending element and two or three dimensional solid stress element.

(iii) Weighted Residual Method: This method is (Galerkin's method) useful for developing the element equations in thermal analysis problems. They are especially useful when a functional such as potential energy is not readily available.

Step 6: Assemble the element equations to obtain the global or total equations:

The individual element equations obtained in step 5 are added together by using a method of superposition *i.e.*, direction stiffness method. The final assembled or global equation which is in the form of

$$\{ F \} = [K] \{ u \} \quad \ldots (1.5)$$

where, { F } → Global force vector.

[K] → Global stiffness matrix.

{ u } → Global displacement vector.

Step 7: Applying boundary conditions:

From equation (1.5), we know that, global stiffness matrix [K] is a singular matrix because its determinant is equal to zero. In order to remove this singularity problem, certain boundary conditions are applied so that the structure remains in place instead of moving as a rigid body. The global equation (1.5) to be modified to account for the boundary conditions of the problem.

Step 8: Solution for the unknown displacements:

A set of simultaneous algebraic equations formed in step 6 can be written in expanded matrix form as follows:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ k_{41} & k_{42} & k_{43} & \cdots & k_{4n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ u_n \end{Bmatrix}$$

These equation can be solved and unknown displacements $\{ u \}$ are calculated by using Gaussian elimination method or Gauss-Seidel method.

Step 9: Computation of the element strains and stresses from the nodal displacements, {u}:

In structural stress analysis problem, stress and strain are important factors. From the solution of displacement vector $\{ u \}$, stress and strain value can be calculated.

In case of one dimensional deformation, the strain-displacement relationship is given by,

$$\text{Strain, } e = \frac{du}{dx} \qquad \text{[From equation (1.3)]}$$

$$= \frac{u_2 - u_1}{x_2 - x_1}$$

where, u_1 and u_2 are displacement at node 1 and 2.

$x_2 - x_1$ = Actual length of the element.

From that, we can find the strain value.

By knowing the strain, stress value can be calculated by using the relation,

$$\text{Stress, } \sigma = E\,e$$

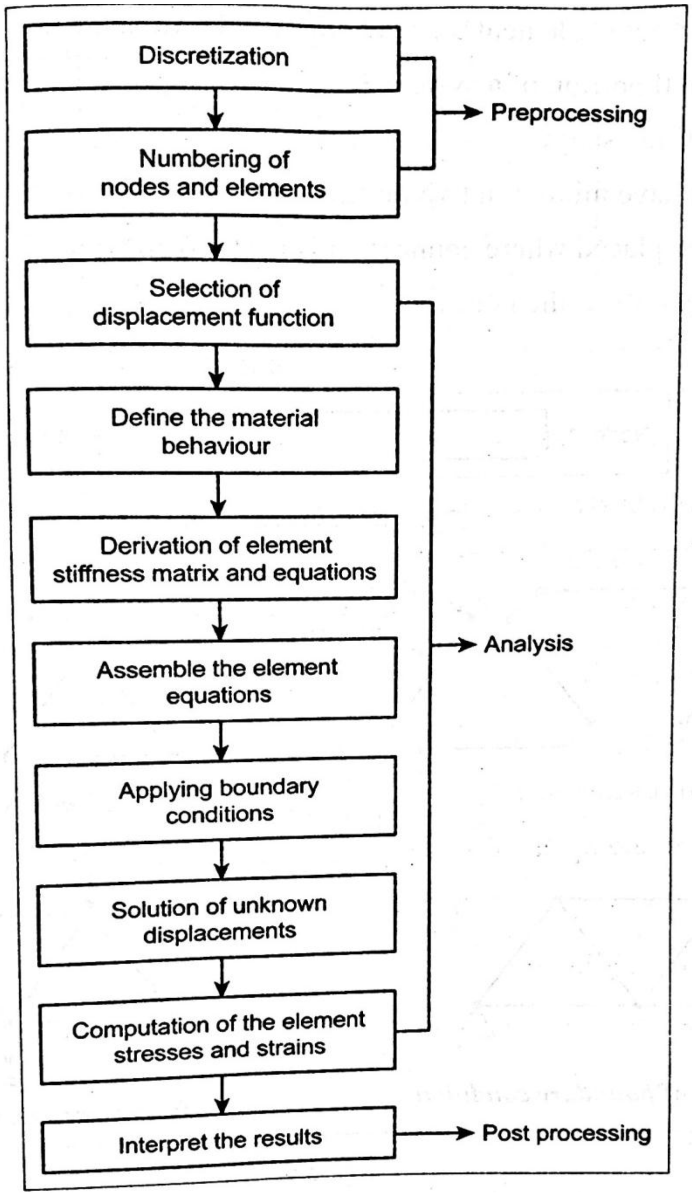
where, E → Young's Modulus.

e → Strain.

Step 10: Interpret the results (Post Processing):

Analysis and evaluation of the solution results is referred to as post-processing. Post processor computer programs help the user to interpret the results by displaying them in graphical form.

Steps 1 to 10 are summarized as follows:


Discretization
Numbering of nodes and elements
Preprocessing
Selection of displacement function
Define the material behaviour
Derivation of element stiffness matrix and equations
Assemble the element equations
Analysis
Applying boundary conditions
Solution of unknown displacements
Computation of the element stresses and strains
Interpret the results
Post processing

1.4.2. Discretization

The art of subdividing a structure into a convenient number of smaller components is known as Discretization. These smaller components are then put together. The process of uniting the various elements together is called Assemblage. The assemblage of such elements then represents the original body.

Discretization can be classified as follows:

(*i*) Natural.

(*ii*) Artificial (continuum).

1.4.3. Natural Discretization

In structural analysis, a truss is considered as a natural system. The various members of the truss constitute the elements. These elements are connected at various joints known as nodes.

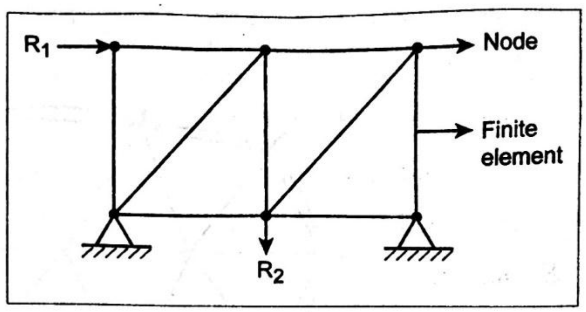


Fig. 1.10. Natural discretization of truss

Nodal Points: Each kind of finite element has a specific structural shape and is interconnected with the adjacent elements by nodal points or nodes.

Nodal forces: The forces that act at each nodal point are called nodal forces.

Degrees of freedom: When the force or reaction act at nodal point, node is subjected to deformation. This deformation includes displacements, rotations, and/or strains. These are collectively known as degrees of freedom or simply we can say nodal displacement is called degrees of freedom.

In Fig.1.10, the truss consists of 9 elements and 6 nodes. There are four freely moving and two extreme constrained nodes. The truss is a natural system as there is no possibility either to increase or decrease the number of elements and the nodes.

1.4.4. Artificial Discretization (Continuum)

Continuum is generally considered to be a single mass of material as found in a forging, concrete dam, deep beam, plate and so on.

Unlike the truss element which is physically present in the truss, in a continuum, the following three elements exist only in our imagination.

1. Triangular element.
2. Rectangular element.
3. Quadrilateral element.

They are shown in Fig.1.11.

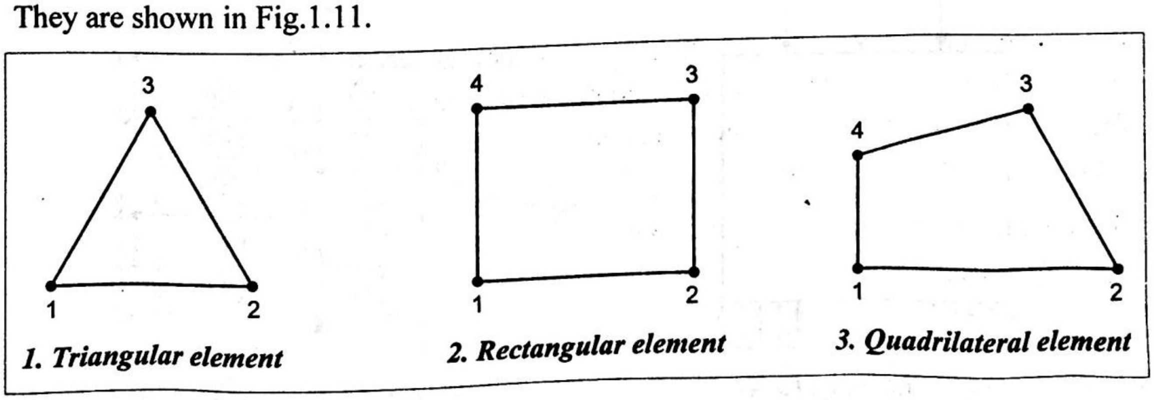


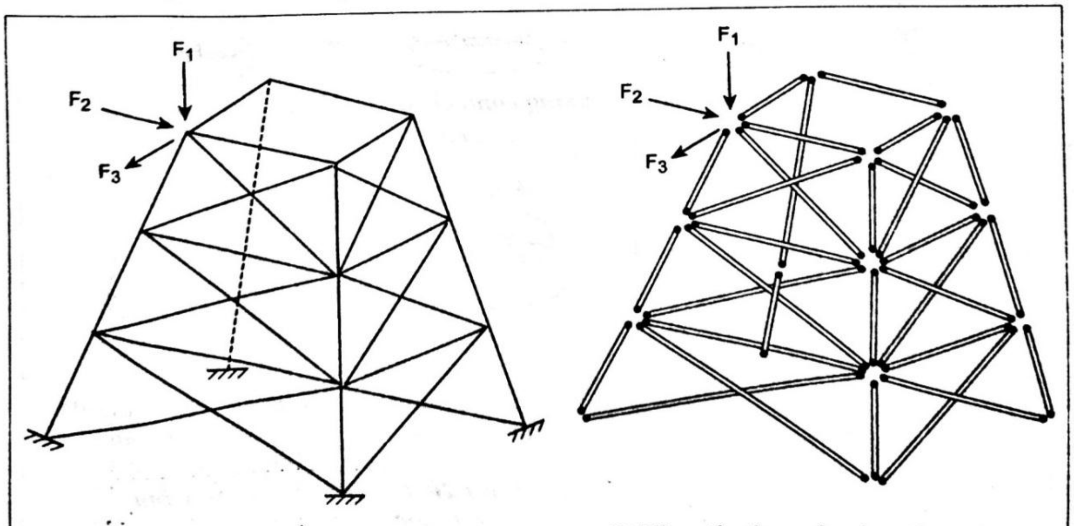
Fig. 1.11.

1.4.5. Discretization Process

The following points to be considered while analysing the discretization process.

(i) Type of elements:

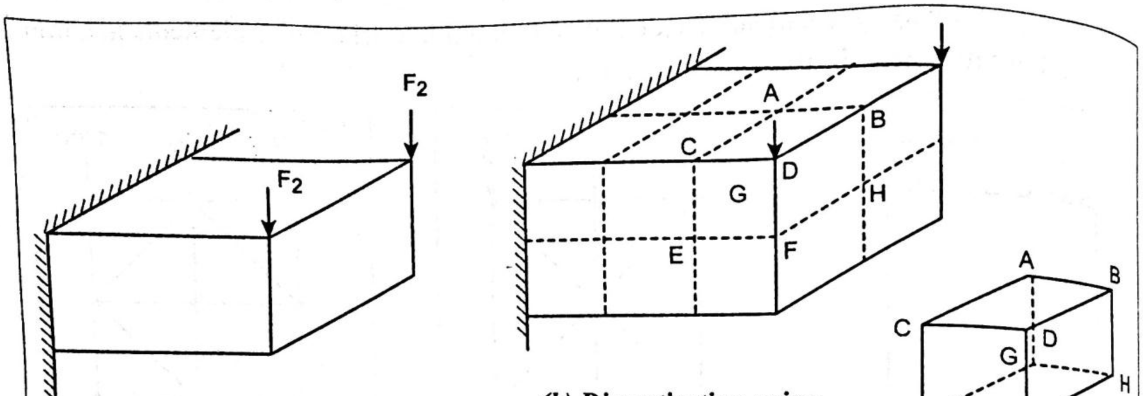
- ✓ The type of elements to be used will be evident from the physical problem.
- ✓ A structure, shown in Fig.1.18 is discretized by using line or bar elements.



(a) Original structure *(b) Discretization using bar elements*

Fig. 1.18.

- ✓ The finite element idealization can be done by using three dimensional rectangular element in stress analysis of short beam problem which is shown in Fig.1.19.



(a) Short beam *(b) Discretization using three-dimensional elements*

Fig. 1.19. (a) Short beam (b) Discretization using three-dimensional elements

- ✓ The choice of the element to be used for discretization depends upon the following factors.
  - (*i*) Number of degrees of freedom needed.
  - (*ii*) Expected accuracy.
  - (*iii*) Necessary equations required.
- ✓ However, in certain problems, the given structure cannot be discretized by using only one type of elements. In such cases, we can use two or more types of elements for discretization.

  Example: Air craft wing.

(ii) Size of elements:

- ✓ The size of elements influences the convergence of the solution of the problem directly. So, it should be chosen with more care.
- ✓ If the size of the element is small, the final solution is more accurate. But the computational time for the smaller size element is more when compared to larger size element.
- ✓ Another characteristic related to the size of elements that affects the finite element problem solution is the ***"Aspect ratio"*** of the elements.
- ✓ Aspect ratio is defined as the ratio of the largest dimension of the element to the smallest dimension. The conclusion of many researchers is that the aspect ratio should be close to unity as possible. For a two dimensional rectangular element, the aspect ratio is conveniently defined as length to breadth ratio. Aspect ratio closer to unity yields better results.

(iii) Location of nodes:

- ✓ If the structure has no abrupt changes in geometric, load, boundary conditions and material properties, the structure can be divided into equal subdivisions. So, the spacing of the nodes are uniform.
- ✓ If there are any discontinuities in geometric, load, boundary conditions and material properties of the structure, nodes should be introduced at these discontinuities as shown in the following figures.

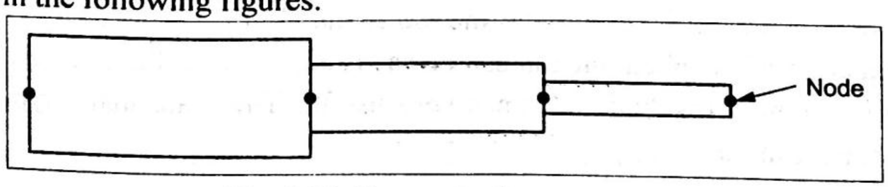


Fig. 1.21. Geometric discontinuities

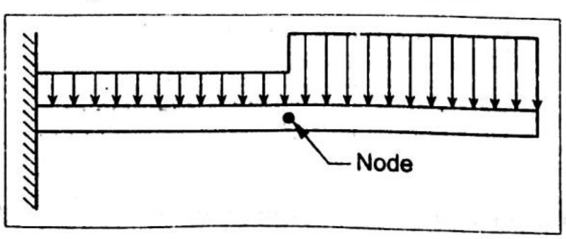


Fig. 1.22. Discontinuity in loading

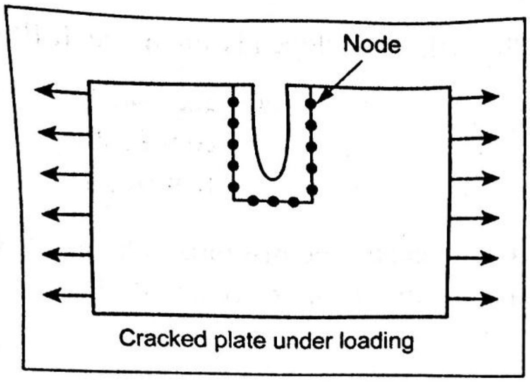


Fig. 1.23. Discontinuity of boundary conditions

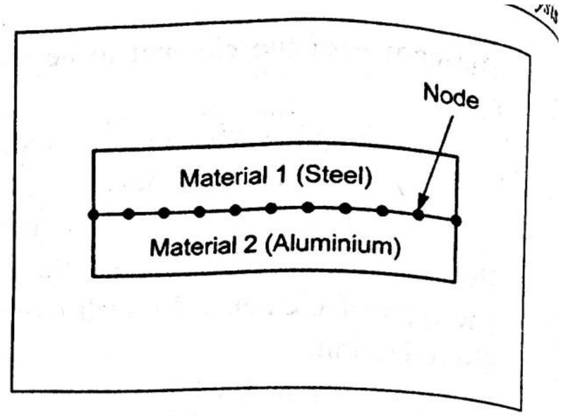


Fig. 1.24. Material discontinuity

(iv) Number of elements:

The number of elements to be selected for discretization depends upon the following factors:

1. Accuracy desired.
2. Size of the elements.
3. Number of degrees of freedom involved.

✓ If the number of element in the structure is increased, the final solution of the problem is expected to be more accurate. But the use of large number of elements involves a large number of degrees of freedom, it leads the storage problem in the available computer memory.

1.5. WEIGHTED RESIDUAL METHODS

1.5.1. Introduction

The method of weighted residuals is a powerful approximate procedure applicable to several problems. For structural problems, potential energy functional can be easily formed, so, Rayleigh-Ritz method is used. On the other hand for non-structural problems, the differential equation of the phenomenon can be easily formulated. For such type of problems, the method of weighted residuals becomes very useful. There are many types of weighted residuals. Of them four are very popular. They are:

(*i*) Point collocation method.

(*ii*) Subdomain collocation method.

(*iii*) Least squares method.

(*iv*) Galerkin's method.

Among these four methods, the Galerkin approach has the widest choice and is used in finite element analysis.

1.5.2. General Procedure

Our interest is to find y, which is the solution for the differential equation. If it is not possible to find a solution, we assume an approximate function for y. When we substitute the approximate solution in the differential equation, we can get residual and that residual can be expressed as,

$$R(x_i;\ a_1,\ a_2,\ a_3) = 0$$

where a_1, a_2 are unknown parameters present in assumed trial function.

The assumed trial function can be expressed as follows:

$$y = f(x; a_1, a_2, a_3, \ldots\ldots, a_n)$$

Trial function y must exactly satisfy the boundary conditions.

The method of weighted residuals needs the parameters a_1, a_2, a_3, ……, a_n, to be determined by satisfying the following equation.

$$\int_D w_i\, R(x; a_1, a_2, a_3, \ldots\ldots, a_n)\, dx = 0 \qquad \ldots (1.6)$$

where, w_i is a function of x and known as weighting function.

D is a domain; R is a residual.

1.5.3. Point Collocation Method

In the collocation method, also called point collocation, residuals are set to zero at n different locations X_i, and the weighting function w_i is denoted as $\delta(x - x_i)$.

$$\Rightarrow\ w_i = \delta(x - x_i)$$

Substituting w_i value in equation (1.6),

$$\Rightarrow \int_D \delta(x - x_i)\, R(x;\ a_1, a_2, a_3, \ldots\ldots, a_n)\, dx = 0 \qquad \ldots (1.7)$$

The x_i's are referred to as collocation points and are selected by the discretion of the analyst.

In equation (1.7), term $\int_D \delta\,(x - x_i) = 1$

So, $R(x; a_1, a_2, a_3, \ldots\ldots, a_n) = 0$

1.5.4. Subdomain Collocation Method

In this method, the weighting functions (w_i) are chosen to be unity over a portion of the domain and zero elsewhere. It is given as follows:

$$w_1 = \begin{cases} 1 & \text{for } x \text{ in } D_1 \\ 0 & \text{for } x \text{ not in } D_1 \end{cases}$$

$$w_2 = \begin{cases} 1 & \text{for } x \text{ in } D_2 \\ 0 & \text{for } x \text{ not in } D_2 \end{cases}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$w_n = \begin{cases} 1 & \text{for } x \text{ in } D_n \\ 0 & \text{for } x \text{ not in } D_n \end{cases}$$

where D is a domain.

1.5.5. Least Squares Method

In this method, the integral of the weighted square of the residual over the domain is required to be minimum.

$$i.e., \quad I = \int_D [R(x; a_1, a_2, a_3, \ldots\ldots a_n)]^2\, dx = \text{minimum}$$

$$\text{where,} \quad I = f(a_1, a_2, \ldots\ldots a_n)$$

$$\text{The requirement is } \frac{\partial I}{\partial a_i} = 0, \quad i = 1, 2, 3, \ldots\ldots n$$

1.5.6. Galerkin's Method

In this method, the trial function, $N_i(x)$, itself is considered as the weighting functions; that is,

$$w_i = N_i(x)$$

Substitute w_i value in equation (1.6),

$$\Rightarrow \int_D N_i(x)\, R(x; a_1, a_2, \ldots\ldots a_n)\, dx = 0 \qquad \ldots (1.8)$$

$$i = 1, 2, 3, \ldots\ldots n$$

Substitute the equation (6) in equation (5),

$$a_1 = \frac{16\, q_0}{\pi^2} \frac{12\,(1-\nu^2)}{\mathrm{E}\, h^3} \left[\frac{1}{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2}\right]^2$$

$$a_1 = \frac{16\, q_0}{\pi^2} \frac{12\,(1-\nu^2)}{\mathrm{E}\, h^3} \left[\frac{1}{4\left(\frac{\pi^4}{a^4}\right)}\right]$$

$$a_1 = \frac{16\, q_0}{\pi^2} \frac{12\,(1-\nu^2)}{\mathrm{E}\, h^3} \left[\frac{a^4}{4\,\pi^4}\right]$$

$$\boxed{a_1 = \frac{4\, q_0\, a^4}{\pi^6} \left[\frac{12\,(1-\nu^2)}{\mathrm{E}\, h^3}\right]} \quad \ldots (7)$$

Result: Parameter, by using Galerkin technique for rectangular plate,

$$\text{Parameter, } a_1 \text{ (for rectangular)} = \left(\frac{16\, q_0}{\pi^2}\right)\left(\frac{12\,(1-\nu^2)}{\mathrm{E}\, h^3}\right)\left[\frac{1}{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}\right]^2$$

$$\text{Parameter, } a_1 \text{ (for square)} = \frac{4\, q_0\, a^4}{\pi^6} \left[\frac{12\,(1-\nu^2)}{\mathrm{E}\, h^3}\right]$$

1.8. VARIATIONAL (WEAK) FORM OF THE WEIGHTED RESIDUAL STATEMENT

We know that the general weighted residual statement is,

$$\int w\, \mathrm{R}\, dx = 0 \quad \ldots (1.11)$$

In this variational method, integration is carried out by parts. It reduces the continuity requirement on the trial function assumed in the solution. So, it is referred to as the weak form. In this method, it is possible to have a wider choice of trial functions.

1.9. COMPARISON OF DIFFERENTIAL EQUATION, WEIGHTED RESIDUAL STATEMENT AND WEAK FORMULATION OF WEIGHTED RESIDUAL STATEMENT

1.9.1. Differential Equation

Consider a uniform rod subjected to uniform axial load q_0 as shown in Fig.1.25.

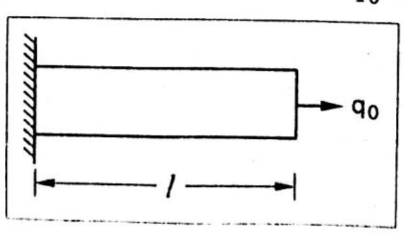


Fig. 1.25. Uniform rod

The deformation of the bar is governed by the differential equation,

$$A\,E\,\frac{d^2u}{dx^2} + q_0 = 0 \qquad \ldots (1.12)$$

With the boundary conditions,

$$u(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=l} = P_l \qquad \ldots (1.13)$$

1.9.2. Weighted Residual Statement

In order to find the solution for the above mentioned problem, the weighted residual statement can be developed as follows:

$$\int_0^l w(x)\left[A\,E\,\frac{d^2u}{dx^2} + q_0\right] dx = 0 \qquad \ldots (1.14)$$

With the boundary conditions,

$$u(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=l} = P_l \qquad \ldots (1.15)$$

1.9.3. Observations on the Weighted Residual Statement

- ✓ Weighted residual statement can be developed for any form of differential equations like linear, non-linear, ordinary, partial, *etc.*
- ✓ The weighted residual statement is developed only for differential equation and it is not suitable for boundary conditions.
- ✓ The trial solution should satisfy all the boundary conditions and it should be differentiable as many times as needed in the original differential equation.

1.9.4. Weak Form of Weighted Residual Statement

By performing integration by parts, the weak form of weighted residual statement of the above mentioned problem is obtained as follows:

$$\left[w(x)\,A\,E\,\frac{du}{dx}\right]_0^l - \int_0^l A\,E\,\frac{du}{dx}\cdot\frac{dw}{dx}\cdot dx + \int_0^l w(x)\,q\,dx = 0 \qquad \ldots (1.16)$$

With the boundary conditions,

$$u(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=l} = P_l$$

1.13. RAYLEIGH-RITZ METHOD (VARIATIONAL APPROACH)

1.13.1. Introduction

- ✓ Rayleigh-Ritz method is a integral approach method which is useful for solving complex structural problems, encountered in finite element analysis. This method is possible only if a suitable functional is available, otherwise Galerkin's method of weighted residual is used. By using this method stiffness matrices and consistent load vector can be assembled easily. This method is mostly used for solving solid mechanics problems.

- ✓ The phrase "Variational methods" refers to methods that make use of variational principles, such as the principles of virtual work and the principle of minimum potential energy in solid and structural mechanics, to determine the approximate solutions of the problems.

- ✓ In Rayleigh-Ritz method for continuous system we deal with the following functional.

$$\text{Potential energy, } \pi = \int_{x_1}^{x_2} f(y, y', y'')\, dx \qquad \dots (1.22)$$

- ✓ In our terminology, a functional is an integral expression that implicitly contains the governing differential equations for a particular problem.

- ✓ Total potential energy of the structure is given by,

$$\pi = \begin{Bmatrix} \text{Internal} \\ \text{potential} \\ \text{energy} \end{Bmatrix} - \begin{Bmatrix} \text{External} \\ \text{potential} \\ \text{energy} \end{Bmatrix}$$

$$= \text{Strain energy} - \text{Work done by external forces}$$

$$\pi = U - H$$

- ✓ In this method, the approximating functions must satisfy the boundary conditions and should be easy to use. Polynomials are generally used and sometimes sine and cosine terms are also used as approximating function.

- ✓ In general any exact function can be represented as a polynomial or trigonometric series with undetermined constants as shown below.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots$$

or

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} + \ldots\ldots$$

The constants a_0, a_1, a_2 are unknowns known as Ritz parameters of the curve. When the parameters are infinite, the particular polynomial tends to match the exact value. So, the accuracy depends upon the number of parameters chosen.

✓ The following two conditions must be fulfilled by the approximating function.

1. It should satisfy the geometric boundary conditions.
2. The function must have atleast one Ritz parameter.

✓ In general, a Rayleigh-Ritz solution is rarely exact except in some special simple cases, but it becomes more accurate with the use of more parameters.

✓ This method can be understood clearly by solving the following examples.

1.15. MATRIX ALGEBRA

1.15.1. Introduction

A matrix is an $m \times n$ array of numbers written in the form,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The a_{ij}'s are real or complex numbers. They are said to be the elements of the matrix. The above matrix have m rows and n columns. This matrix is said to be a $m \times n$ matrix or a rectangular matrix of the order $m \times n$. If $m = n$, then the matrix is called a square matrix of the order N.

Examples: $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 5 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ is a 3 × 3 matrix; $\begin{bmatrix} 2 & 3 \\ 7 & 9 \\ 4 & 2 \end{bmatrix}$ is a 3 × 2 matrix;

$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 3 & 5 \end{bmatrix}$ is a 2 × 3 matrix; $\begin{Bmatrix} 5 \\ 7 \\ 9 \end{Bmatrix}$ is a 3 × 1 matrix.

Equal Matrices: Two matrices A and B are called equal matrix, if they are of the same order and the corresponding elements are equal. In that case, we can write A = B.

Example: If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ and A = B,

we can write $p = 5, q = 2, r = 3, s = 4$.

Diagonal Matrix: A diagonal matrix is a square matrix in which all the elements other than the diagonal are zero.

Example: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Scalar matrix: A diagonal matrix in which all the diagonal elements are equal is known as scalar matrix.

Example: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Unit matrix: A square matrix in which all the diagonal elements are unity and other elements are zero is known as unit matrix. It is denoted by I.

Examples: 2 × 2 unit matrix. $I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 × 3 unit matrix. $I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Row matrix: A matrix containing only one row is known as row matrix.

$A = [a_1, a_2, a_3, \ldots, a_n]$ is a row matrix.

Column matrix: A matrix containing only one column is called a column matrix.

$B = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{Bmatrix}$ is a column matrix.

Partial differentiation of U with respect to x and y yields,

$$\frac{\partial U}{\partial x} = \frac{1}{2}[a_{11} 2x + 2a_{12} y + 0]$$

$$\boxed{\frac{\partial U}{\partial x} = a_{11} x + a_{12} y} \quad \text{... (1.47)}$$

$$\frac{\partial U}{\partial y} = \frac{1}{2}[0 + 2a_{12} x + 2a_{22} y]$$

$$\boxed{\frac{\partial U}{\partial y} = a_{12} x + a_{22} y} \quad \text{... (1.48)}$$

We can write equation (1.47) and (1.48) in the matrix form.

$$\begin{Bmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \quad \text{... (1.49)}$$

A general form of equation (1.49) is,

$$\frac{\partial u}{\partial x_i} = [a]\{X\} \quad \text{... (1.50)}$$

where x_i denotes x and y.

1.15.9. Matrix Integration

A matrix is integrated by integrating each element in the matrix by conventional manner.

Example:

$$A = \begin{bmatrix} 4x^3 & 3 & 4x \\ 5x^4 & 3x^2 & 3 \\ 1 & 5 & 4x \end{bmatrix}$$

$$\int A\,dx = \begin{bmatrix} \frac{4x^4}{4} & 3x & \frac{4x^2}{2} \\ \frac{5x^5}{5} & \frac{3x^3}{3} & 3x \\ x & 5x & \frac{4x^2}{2} \end{bmatrix} \Rightarrow \int A\,dx = \begin{bmatrix} x^4 & 3x & 2x^2 \\ x^5 & x^3 & 3x \\ x & 5x & 2x^2 \end{bmatrix}$$

In our finite element analysis, we often integrate an expression, which is in the form of

$$\iint \{X\}^T [A] \{X\}\,dx\,dy$$

The term $\{X\}^T [A] \{X\}$ is in quadratic form.

For example, $[A] = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ and $\{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$

1.15.2. Matrix Operation

(i) Scalar multiplication: In scalar multiplication, each element in the matrix to be multiplied by the scalar.

Example: $A = \begin{bmatrix} 3 & 5 & 2 \\ 4 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \Rightarrow 3A = \begin{bmatrix} 9 & 15 & 6 \\ 12 & 3 & 0 \\ 6 & 9 & 3 \end{bmatrix}$

(ii) Addition and subtraction of matrices: Matrices can be added and subtracted only if they are of the same order.

Example: $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 2 & 3 \\ 4 & 2 & 3 \end{bmatrix}$

A is 2×3 matrix and B is also 2×3 matrix. So, these matrices can be added or subtracted.

$$A + B = \begin{bmatrix} 2+5 & 4+2 & 1+3 \\ 3+4 & 1+2 & 2+3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 7 & 6 & 4 \\ 7 & 3 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-5 & 4-2 & 1-3 \\ 3-4 & 1-2 & 2-3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -3 & 2 & -2 \\ -1 & -1 & -1 \end{bmatrix}$$

(iii) Multiplication of two matrices: Two matrices A and B can be multiplied only if the number of columns in A is equal to the number of rows in B.

Example: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \\ 1 & 3 \end{bmatrix}$

A is 2×3 matrix; B is 3×2 matrix.

A	B		AB
2×3	3×2	=	2×2

Here, the number of columns in A matrix is equal to the number of rows in B matrix. So, the matrices A and B can be multiplied.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \\ 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \times 3 + 3 \times 2 + 2 \times 1 & 1 \times 5 + 3 \times 3 + 2 \times 3 \\ 2 \times 3 + 4 \times 2 + 3 \times 1 & 2 \times 5 + 4 \times 3 + 3 \times 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3+6+2 & 5+9+6 \\ 6+8+3 & 10+12+9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 11 & 20 \\ 17 & 31 \end{bmatrix}$$

In general, $AB \neq BA$

1.15.3. Transpose of a Matrix

Any matrix, whether a row, column, or rectangular matrix, can be transposed. This operation is frequently used in finite element equation formulations. The transpose of a matrix A is commonly denoted by A^T. If A is a $m \times n$ matrix, A^T is a $n \times m$ matrix.

Examples: $A = \begin{bmatrix} 3 & 5 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 3 \\ 5 & 2 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 3 & 2 \end{bmatrix}$$

Properties of Transpose: If A and B are two matrices of the same order then,

$$(A+B)^T = A^T + B^T$$

If A and B are conformed for multiplication, then $(AB)^T = B^T \cdot A^T$

Symmetric Matrices

If a square matrix is equal to its transpose, it is a symmetric matrix.

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

So, A is a symmetric matrix.

1.15.4. Determinant of a Matrix

Consider the 2 × 2 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

We can check that,

$$A A^{-1} = I$$

i.e.,

$$A A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A A^{-1} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{pmatrix} \frac{1}{17} \begin{pmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{pmatrix}$$

$$= \frac{1}{17} \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{pmatrix}$$

$$= \frac{1}{17} \begin{bmatrix} 36-3-16 & -9+5+4 & -24+2+22 \\ 12-12-0 & -3+20+0 & -8+8+0 \\ 24-0-24 & -6+0+6 & -16+0+33 \end{bmatrix}$$

$$= \frac{1}{17} \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, our answer is correct.

If the determinant of a matrix is zero, *i.e.,* $|A| = 0$, then the matrix is called singular. A singular matrix does not have an inverse. The stiffness matrices used in the finite element method are singular until sufficient boundary conditions are applied.

1.15.7. Row Reduction Method (Gauss Jordan Method) to Determine the Inverse of a Matrix

The inverse of a non-singular square matrix can be found by row reduction method (Gauss-Jordan Method). The following example illustrate the procedure for determining the inverse of a matrix.

Example:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

We know that, 3 × 3 unit matrix is,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We start by writing matrix A and unit matrix I side by side as follows:

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \ldots (1.43)$$

Step 1: Divide the first row of equation by 2. *i.e.*, $R_1 \rightarrow \frac{R_1}{2}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Step 2: Multiply the first row by –2 and add the result to the second row.

i.e., $R_2 \rightarrow -2R_1 + R_2$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Step 3: Subtract the first row from the third row. *i.e.*, $R_3 \rightarrow R_3 - R_1$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & 1 \end{array}\right]$$

Step 4: Multiply the second row by –1, *i.e.*, $R_2 \rightarrow -R_2$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & 1 \end{array}\right]$$

Step 5: Subtract the second row from the first row, *i.e.*, $R_1 \rightarrow R_1 - R_2$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{2} & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & 1 \end{array}\right] \quad \ldots (1.44)$$

From equations (1.43) and (1.44), we know that, the replacement of matrix A by the inverse matrix is completed.

The inverse of matrix A is the right side of equation (1.44).

$$A^{-1} = \begin{bmatrix} \frac{-1}{2} & 1 & 0 \\ 1 & -1 & 0 \\ \frac{-1}{2} & 0 & 1 \end{bmatrix}$$

1.15.8. Matrix Differentiation

A matrix is differentiated by differentiating each element in the matrix by conventional manner.

Examples:

$$A = \begin{bmatrix} 3x^3 & 2x & x^4 \\ x & x^2 & x^5 \\ 5x & x^3 & 2x \end{bmatrix}$$

$$\frac{dA}{dx} = \begin{bmatrix} 9x^2 & 2 & 4x^3 \\ 1 & 2x & 5x^4 \\ 5 & 3x^2 & 2 \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} x^2 & xy & xz \\ xz & yz & y^2 \\ xy & z^2 & xz \end{bmatrix}$$

Partial derivative of a matrix $\frac{\partial A}{\partial x}$ is given by,

$$\frac{\partial A}{\partial x} = \begin{bmatrix} 2x & y & z \\ z & 0 & 0 \\ y & 0 & z \end{bmatrix}$$

In structural analysis problem, we differentiate an expression, which is in the form of

$$U = \frac{1}{2}[x \;\; y]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$

where, U is a strain energy.

$[x \;\; y]$ is a row matrix.

$\begin{Bmatrix} x \\ y \end{Bmatrix}$ is column matrix.

By matrix multiplication of equation (1.45), we get

$$U = \frac{1}{2}[x \;\; y]\begin{bmatrix} a_{11}x + a_{12}y \\ a_{12}x + a_{22}y \end{bmatrix}$$

$$= \frac{1}{2}[a_{11}x^2 + a_{12}xy + a_{12}xy + a_{22}y^2]$$

$$U = \frac{1}{2}[a_{11}x^2 + 2a_{12}xy + a_{22}y^2]$$

MULTIPLE CHOICE QUESTIONS AND ANSWERS

S.No	Questions	Opt1	Opt2	Opt3	Opt4	Answer
1.	Finite element analysis deals with	Approximate numerical solution	Boundary value problem	Differential equations	All of these	**All of these**
2.	Indentify the wrong statement	In finite element method a continuous function can be approximated using a discrete model	A discrete model is composed of one or more interpolation polynominals	Continuous function is not divided in to finite elements	The end points of the finite element are called nodes.	**Continuous function is not divided in to finite elements**
3.	Shape function is usually	The coefficient that appears in the interpolation polynomial	Written for each individual node of finite element	Interchanged with the terminology interpolation polynomial	All of these	**All of these**
4.	The behavior of the element between the node points is described by	Stiffness matrix	Interpolation function	Displacement vector	Shape function	**Interpolation function**
5.	Essential boundary conditions are also called as	Mixed boundary condition	Dirichlet boundary condition	Natural boundary condition	None of these	**Dirichlet boundary condition**
6.	The boundary conditions usually specified on the surface at each end of the one dimensional domain is called	Geometric boundary condition	Essential boundary condition	Mixed boundary condition	None of these	**Essential boundary condition**
7.	The boundary condition specified for the first derivative of the problem is called as	Neuman boundary condition	Essential boundary condition	Mixed boundary condition	None of these	**Neuman boundary condition**
8.	Natural boundary condition is also called as	Dirichlet boundary condition	Neuman boundary condition	Displacement boundary condition	None of these	**Neuman boundary condition**
9.	A vector is defined as physical	Single magnitude	Multiple magnitude	Single magnitude	All	**Single magnitude**

	quantity that can be described by	and single direction	and single direction	and Multiple direction		**and single direction**
10.	Which one of the following is not an analysing software	ANSYS	ALGOR	NASTRAN	GAMBIT	**GAMBIT**
11.	Shape function is also called as	Blending function	Stiffness matrix	Polynomial equation	None of these	**Blending function**
12.	In FEA the continuous variation of the ---in terms of discrete values at the finite element nodes	Field variable	Shape function	Material property	Interpolation equation	**Field variable**
13.	The term used to describe the behavior of the node in an element is called as `	Shape function	Degree of freedom	Stiffness matrix	Differential equations	**Shape function**
14.	The weight of the body ,magnetic force are example for	Surface traction	Body force	Concentrated load	Uniformly distributed load	**Body force**
15.	The force acts entire volume of the element is called as	Body force	Surface traction	Concentrated load	Uniformly distributed load	**Body force**
16.	The force per unit area and acts over the outer surface of the body is called as	Body force	Surface traction	Concentrated load	Uniformly distributed load	**Surface traction**
17.	Pressure is the example for	Body force	Surface traction	Concentrated load	Uniformly distributed load	**Surface traction**
18.	The point load is the example for	Body force	Surface traction	Concentrated load	Uniformly distributed load	**Concentrated load**
19.	The art of sub dividing the structure into a convenient number of smaller elements is known as	Interpolation formulation	Isoparametri c formulation	Discretizatio n	Approximatio n	**Discretizatio n**
20.	In the force method internal	Constant value	Unknown value	Zero	None of these	**Unknown value**

	forces are considered as					
21.	In the stiffness method which one of the following is an unknown value	Displacement	Material property	Force	Element length	**Displacement**
22.	The processes involves preparation of data like type of loading ,material properties & boundaryconditions is called as	Post processing	Pre processing	Discritization	Polynomial process	**Pre processing**
23.	Post processing is a process of	Preparing the results	Initialization	Setting of boundary conditions	Setting of operating conditions	**Preparing the results**
24.	The processes involves the stiffness generation,stiffness modification and gives the solution of the equation is called as	Post processing	Pre processing	Processing	Polynomial processing	**Processing**
25.	Identify the wrong statement	The displacement method is often referred to as the stiffness method	The three moment equation and slope deflection method were fore runners of the displacement method	The displacement method is not used to solve for the displacements and slopes at each end of the beam	Use of computers is required for these methods	**The displacement method is not used to solve for the displacements and slopes at each end of the beam**
26.	Displacements and rotations at the end of the beam are accompanied by	Force reactions	Bending moment	Both force reactions & bending moment	Shear stress	**Both force reactions & bending moment**
27.	Any beam that is to be analysed is	Fixed at both ends	Subjected to an external applied load	Fixed at both ends & subjected to an external applied load	Not loaded at all	**Fixed at both ends & subjected to an external applied load**
28.	The truss element transmits	Axial force	Shearforce	Both a&b	Bending moment	**Axial force**
29.	Total degree of freedom of the truss element	1	2	3	4	**4**

30.	Beam,plate,shell elements transmits---------------------- type of forces	Transitional & axial	Transitional & rotational	Only transitional	Only rotational	**Transitional & rotational**
31.	The load is distributed accurately through the element is represented by	Consistent matrix	lumped matrix	Vector matrix	Stiffness matrix	**Consistent matrix**
32.	The load is distributed accurately through the Node is represented by	Consistent matrix	lumped matrix	Vector matrix	Stiffness matrix	**lumped matrix**
33.	Which plays a significant role in the isoparametric formulation	Discritization	Interpolation	Numerical integeration	All of these	**Numerical integeration**
34.	Identify the wrong statement	Numerical integeration is used extensively in finite element analysis	Elementary integration formulas,suc h as trapezoidal rule, often assume equally spaced data.	Gauss quadrature is not an accepted numerical integration scheme in finite element applications	The term quadrature means numerical integration	**Gauss quadrature is not an accepted numerical integration scheme in finite element applications**
35.	The gauss points in numerical intageration is called as	Null points	Sampling points	Weights	Sampling points &weights	**Sampling points &weights**
36.	Identify the wrong statement	Shape function for triangular finite elements of any order can be derived using the area coordinates.	Area coordinates depend on the number of nodes used to define the triangular element.	The number of nodes and placement of the nodes for higher order elements must satisfy certain requirements	All of these	**Area coordinates depend on the number of nodes used to define the triangular element.**
37.	The four node rectangular elements were derived	Using a local coordinate system that was identical	Not using local coordinate system	Not using global system	All of these	**Using a local coordinate system that was identical**

		to the global system				**to the global system**
38.	When fewer nodes are used to define the geometry than are used to define the shape function the element is called	Isoparametric formulation	Sub parametric element	Super parametric element	None of these	**Sub parametric element**
39.	Identify the wrong statement	The cubic element without interior nodes would probably be a more efficient choice.	Two dimensional lagrange cubic element will have 12 exterior nodes and 4 interior nodes.	The order of integration using Gaussian quadrature need not depends on order of finite element	A single integration point is satisfactory for linear quadrilateral and triangles	**The order of integration using Gaussian quadrature need not depends on order of finite element**
40.	Continuity in finite element analysis refers to	The continuity of the solution along element boundaries	Continuity of the derivatives of the shape functions being used to model the physical problem	The continuity of the solution along element boundaries &Continuity of the derivatives of the shape functions being used to model the physical problem	None of these	**The continuity of the solution along element boundaries &Continuity of the derivatives of the shape functions being used to model the physical problem**
41.	The element have a curved boundaries is called as	Bar	Triangle	Rectangle	Isoparametric	**Isoparametric**
42.	In ISO parametric element formulation concept of mapping means	Natural co-ordinate to global co-ordinate system	Global coordinate to natural coordinate system	Natural co-ordinate to spherical co-ordinate system	Natural co-ordinate to serendipity co-ordinate system	**Natural co-ordinate to global co-ordinate system**
43.	In isoparametric element shape function used for defining the geometry and displacement are	Equal	Not equal	Greater than or equal to	None of these	**Equal**

44.	When more nodes are used to define the geomentry than are used to define the shape function the element is called	Isoparametri c formulation	Sub parametric element	Super parametric element	None of these	**Super parametric element**
45.	Which one of the following is not to be represented as an isoparametric element	Beam	Rectangular	Triangle	All of these	**Beam**
46.	In higher order elements the field varible variation is	Linear	Non-linear	Curvilinear	Rectilinear	**linear**

UNIT 2
ONE DIMENSIONAL PROBLEMS

2.1. INTRODUCTION

Bar and beam elements are considered as one dimensional elements. These elements are often used to model trusses and frame structures.

A bar is a member which resist only axial loads, whereas a beam can resist axial, lateral and twisting loads. A truss is an assemblage of bars with pin joints and a frame is an assemblage of beam elements.

In this chapter, one dimensional elements and step-by-step procedure for the analysis of bars, trusses and beams are discussed. The total potential energy, stress-strain and strain-displacement relationships are used in developing the finite element method for a one dimensional problems. The basic one dimensional procedure is same for two and three dimensional problems.

2.2. STRESS, STRAIN, DISPLACEMENT AND LOADING

In one dimensional problems, stress (σ), strain (e), displacement (u) and loading depends only on the variable x. So, the vectors u, σ and e can be written as,

$$u = u(x)$$
$$\sigma = \sigma(x)$$
$$e = e(x)$$

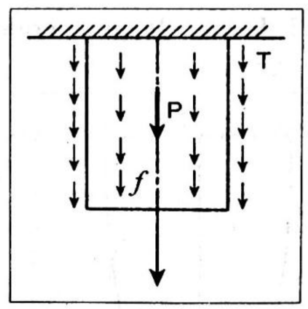


Fig. 2.1. A bar is subjected to loading

The stress-strain relationship is given by,

$$\sigma = E\,e$$

where, $\sigma \rightarrow$ Stress, N/mm^2.
$e \rightarrow$ Strain.
E $\rightarrow$ Young's modulus, N/mm^2.

The strain-displacement relationship is given by,

$$e = \frac{du}{dx}$$

The differential volume can be written as,

$$dV = A\,dx$$

There are three types of loading acts on the body. They are:

(*i*) Body force (f).
(*ii*) Traction force (T).
(*iii*) Point load (P).

Body Force (*f*)

A body force is a distributed force acting on every elemental volume of the body.

Unit: Force per unit volume.

Example: Self weight due to gravity.

Traction Force (T)

A traction force is a distributed force acting on the surface of the body.

Unit: Force per unit area but for one dimensional problems, unit is force per unit length.

Examples: Frictional resistance, viscous drag, surface shear, *etc.*

Point Load (P)

Point load is a force acting at a particular point which causes displacement.

2.3. FINITE ELEMENT MODELLING

Finite element modelling consists of the following:

(*i*) Discretization of structure.

(*ii*) Numbering of nodes.

(i) Discretization

The art of subdividing a structure into a convenient number of smaller components is known as discretization.

Consider a bar as shown in Fig.2.2. The first step is to model the bar as a stepped shaft. Let us model the bar using 5 finite elements, each having a uniform cross section as shown in Fig.2.3. Every elements connects two nodes. Five element, six node model element is shown in Fig.2.4.

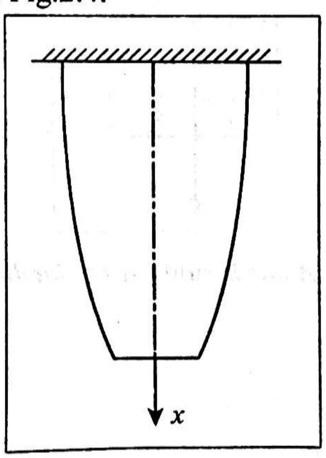


Fig. 2.2.

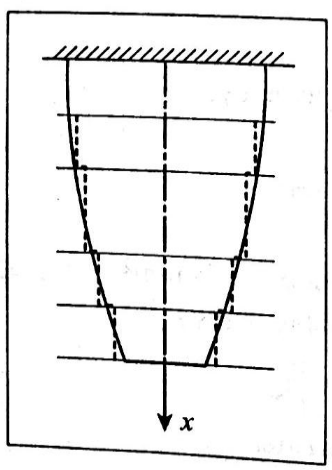


Fig. 2.3.

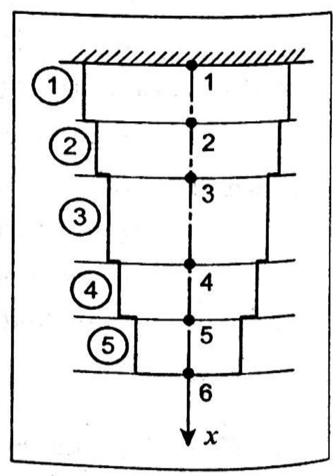


Fig. 2.4.

The element numbers are circled to distinguish them from node numbers. The cross-sectional area, traction forces and body forces are constant within each element. But, these are differ in magnitude from element to element. Better results are obtained by increasing the number of finite elements.

(ii) Numbering of nodes

In one dimensional problem, each node is allowed to move only in $\pm x$ direction. So, each node has one degrees of freedom. (Degrees of freedom is nothing but a nodal displacement).

A six node finite element model is shown in Fig.2.5. It has six degrees of freedom. Load is considered as positive if it is acting along the $+x$ direction.

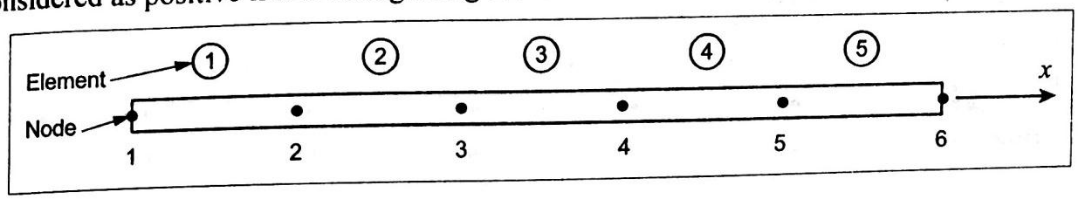
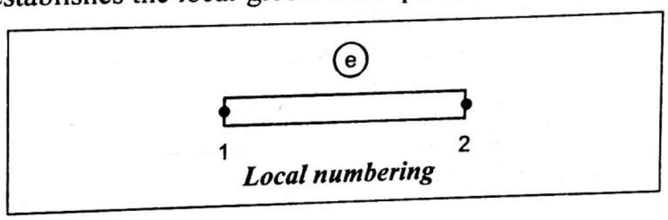


Fig. 2.5.

In the element connectivity table, the heading 1 and 2 refer to local node numbers of an element and the corresponding node numbers on the structure are called global numbers. Connectivity thus establishes the local-global correspondence.



Local numbering

Fig. 2.6. (a)

Element (e)	Nodes 1 2 (Local numbers)
(1)	1 2
(2)	2 3
(3)	3 4
(4)	4 5
(5)	5 6

(Node entries 1 2 … 5 6: Global numbers)

Fig. 2.6. (b) Connectivity table

2.4. CO-ORDINATES

The co-ordinates are generally classified as follows:

(*i*) Global co-ordinates.

(*ii*) Local co-ordinates.

(*iii*) Natural co-ordinates.

2.4.1. Global Co-ordinates

The points in the entire structure are defined using co-ordinate system is known as global co-ordinate system.

Example:

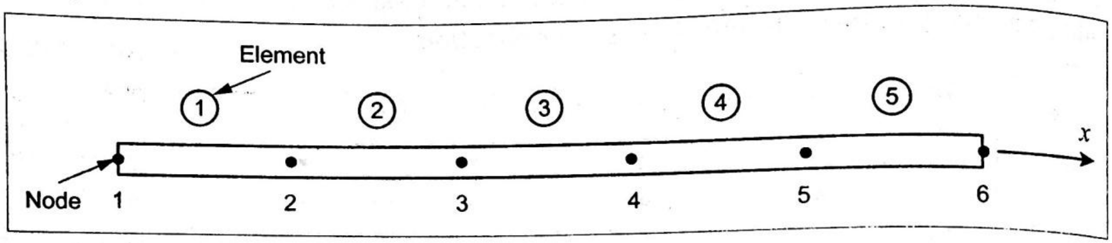


Fig. 2.7. One dimensional bar

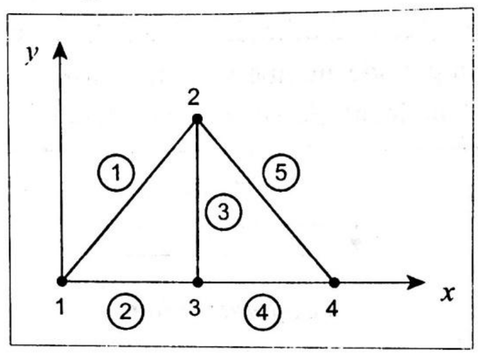
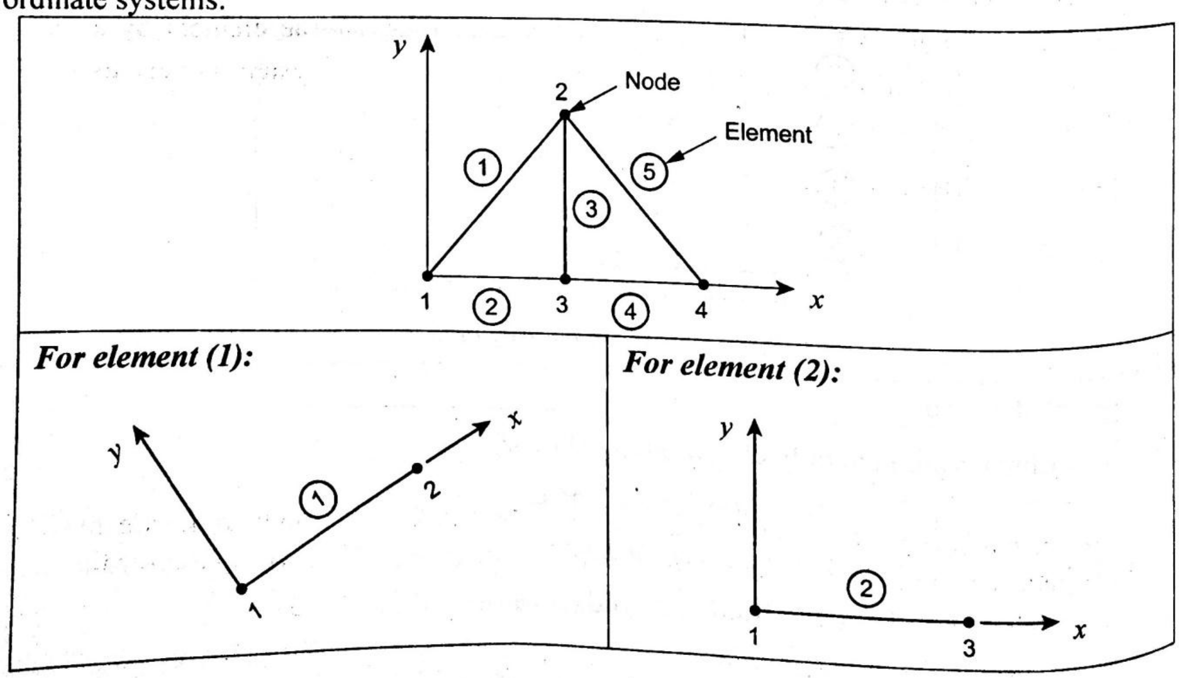


Fig. 2.8. Two dimensional triangular element

2.4.2. Local Co-ordinates

In finite element method, separate co-ordinate is used for each element. It is very useful for deriving element properties. But the final equations are to be formed only by global co-ordinate systems.



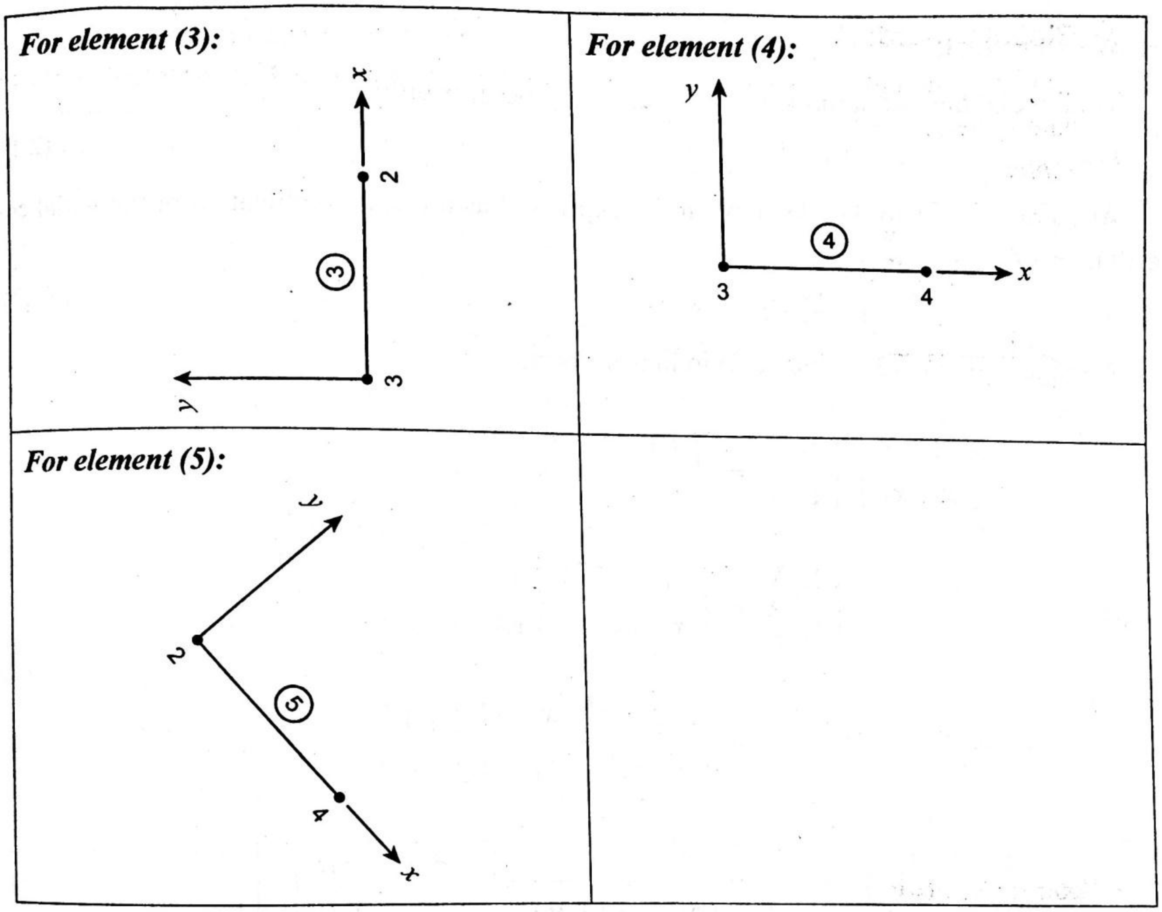


Fig. 2.9. Local co-ordinates system

2.4.3. Natural Co-ordinates

A natural co-ordinate system is used to define any point inside the element by a set of dimensionless numbers whose magnitude never exceeds unity. This system is very useful in assembling of stiffness matrices.

(1) Natural Co-ordinates in One Dimension

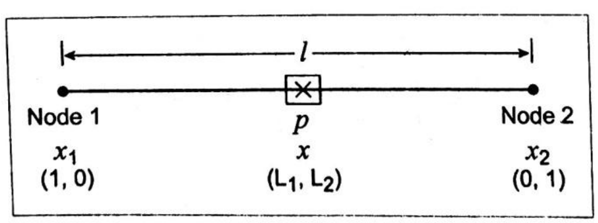


Fig. 2.10. Natural co-ordinates for a line element

Consider a two noded line element as shown in Fig.2.10. Any point p inside the line element is identified by two natural co-ordinates L_1 and L_2 and the cartesian co-ordinate x. Node 1 and node 2 have the cartesian co-ordinates x_1 and x_2 respectively.

We know that,

Total weightage of natural co-ordinates at any point is unity.

i.e., $$L_1 + L_2 = 1 \quad \ldots (2.1)$$

Any point x within the element can be expressed as a linear combination of the nodal co-ordinates of nodes 1 and 2 as,

$$L_1 x_1 + L_2 x_2 = x \quad \ldots (2.2)$$

Arrange equation (2.1) and (2.2) in matrix form,

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$= \frac{1}{(x_2 - x_1)} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\left[\textbf{Note:} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ -x_1 + x \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix}$$

$$= \frac{1}{l} \begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix} \qquad [\because x_2 - x_1 \text{ is the length of the element, } l]$$

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{x_2 - x}{l} \\ \dfrac{x - x_1}{l} \end{Bmatrix}$$

The variation of L_1 and L_2 is shown in Fig.2.12 and Fig.2.13. L_1 is one at node 1 and it is zero at node 2 whereas L_2 is one at node 2 and it is zero at node 1.

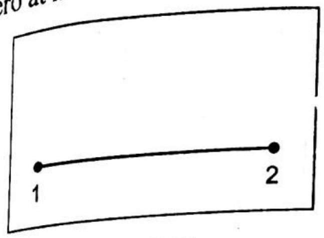


Fig. 2.11.

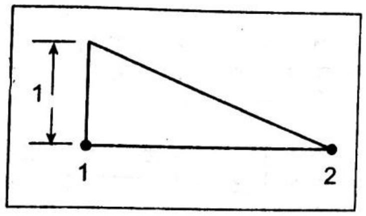


Fig. 2.12. Variation of L_1

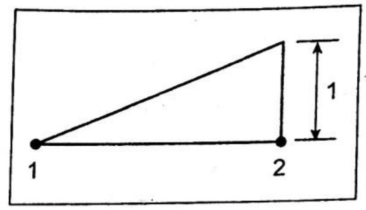


Fig. 2.13. Variation of L_2

Integration of polynomial terms in natural co-ordinates can be performed by using the simple formula,

$$\int_{x_1}^{x_2} (L_1)^{\alpha} (L_2)^{\beta}\, dx = \frac{\alpha!\, \beta!}{(\alpha+\beta+1)!} \times l_x \qquad \ldots (2.3)$$

where, $\alpha!$ is the factorial of α.

Natural Co-ordinate, ε

In one dimensional problem, the following type of natural co-ordinate is also used.

Consider a one dimensional element as shown in Fig.2.14.

In the local number scheme, the first node will be numbered 1 and the second node 2. c is the centre of nodes 1 and 2 and p is the point referred.

The natural co-ordinator ε for any point in the element is defined as,

$$\varepsilon = \frac{pc}{\left(\frac{x_2 - x_1}{2}\right)}$$

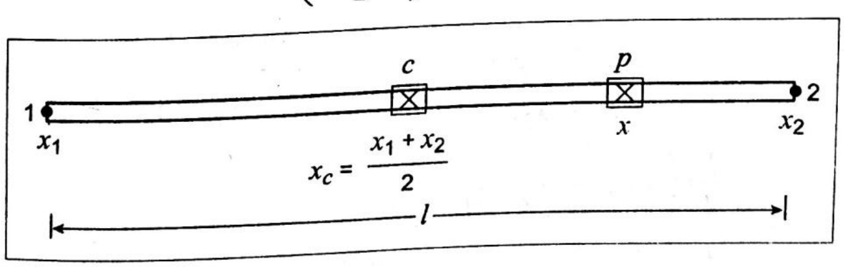


Fig. 2.14.

$$\Rightarrow \qquad \varepsilon = \frac{pc}{\frac{l}{2}} \qquad [\because\ x_2 - x_1 = l]$$

$$= \frac{2}{l} pc = \frac{2}{l}(x - x_c) \qquad [\because pc = x - x_c]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_1 + x_2}{2} \right) \right] \qquad \left[\because\ x_c = \frac{x_1 + x_2}{2} \right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_2 + x_1}{2} \right) \right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_2 - x_1 + 2x_1}{2} \right) \right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{l + 2x_1}{2} \right) \right]$$

$$\varepsilon = \frac{2}{l} \left[x - \left(\frac{l}{2} + x_1 \right) \right]$$

$$\Rightarrow \quad \frac{\varepsilon l}{2} = x - \frac{l}{2} - x_1$$

$$\Rightarrow \quad \frac{\varepsilon l}{2} + \frac{l}{2} = x - x_1$$

$$\Rightarrow \quad \boxed{\frac{l}{2}(\varepsilon + 1) = x - x_1} \qquad \ldots (2.4)$$

Applying boundary conditions,

At node 1, $x = x_1$

$$(2.4) \Rightarrow \quad \frac{l}{2}(1 + \varepsilon) = 0$$

$$\Rightarrow \quad 1 + \varepsilon = 0$$

$$\Rightarrow \quad \boxed{\varepsilon = -1}$$

At node 2, $x = x_2$

$$(2.4) \Rightarrow \quad \frac{l}{2}(1 + \varepsilon) = x_2 - x_1$$

$$\frac{l}{2}(1 + \varepsilon) = l$$

$$\Rightarrow \quad 1 + \varepsilon = 2$$

$$\Rightarrow \quad \boxed{\varepsilon = 1}$$

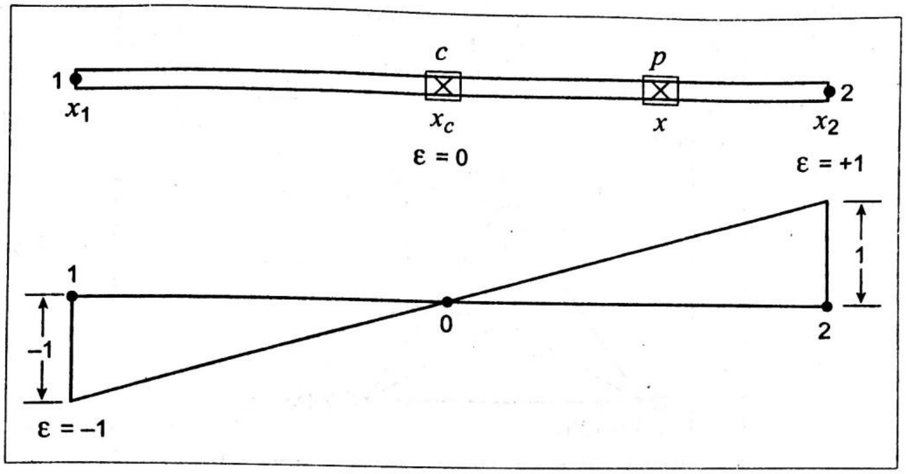


Fig. 2.15. Variation of natural co-ordinate, ε

Natural Co-ordinates in Two Dimensions

Consider a triangular element having 3 nodes as shown in Fig.2.16.

Let p is the point inside the element and it has 3 co-ordinates L_1, L_2 and L_3.

From the definition of natural co-ordinates, we know that,

$$L_1 + L_2 + L_3 = 1 \quad \text{... (2.5)}$$

$$L_1 x_1 + L_2 x_2 + L_3 x_3 = x \quad \text{... (2.6)}$$

$$L_1 y_1 + L_2 y_2 + L_3 y_3 = y \quad \text{... (2.7)}$$

Assemble the above equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad \text{... (2.8)}$$

$$\text{Let } D = \begin{bmatrix} \overset{+}{1} & \overset{-}{1} & \overset{+}{1} \\ \overset{-}{x_1} & \overset{+}{x_2} & \overset{-}{x_3} \\ \overset{+}{y_1} & \overset{-}{y_2} & \overset{+}{y_3} \end{bmatrix}$$

$$D^{-1} = \frac{C^T}{|D|} \qquad \dots (2$$

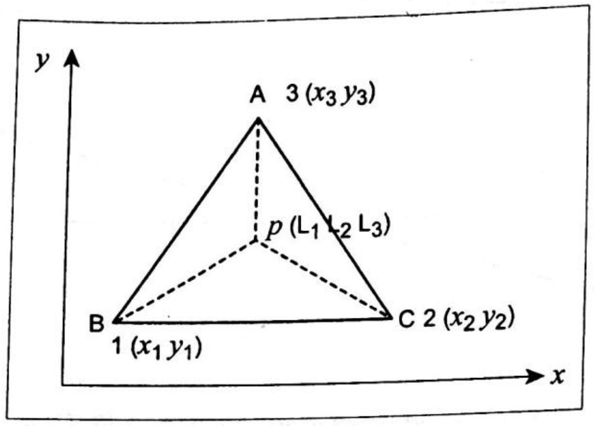


Fig. 2.16.

Coefficients of matrix D:

$$c_{11} = + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = x_2 y_3 - x_3 y_2$$

$$c_{12} = - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} = -(x_1 y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3$$

$$c_{13} = + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = (x_1 y_2 - x_2 y_1)$$

$$c_{21} = - \begin{vmatrix} 1 & 1 \\ y_2 & y_3 \end{vmatrix} = -(y_3 - y_2) = y_2 - y_3$$

$$c_{22} = + \begin{vmatrix} 1 & 1 \\ y_1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$c_{23} = - \begin{vmatrix} 1 & 1 \\ y_1 & y_2 \end{vmatrix} = -(y_2 - y_1) = y_1 - y_2$$

$$c_{31} = + \begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} = x_3 - x_2$$

$$c_{32} = - \begin{vmatrix} 1 & 1 \\ x_1 & x_3 \end{vmatrix} = -(x_3 - x_1) = x_1 - x_3$$

$$c_{33} = + \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$C = \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$$\Rightarrow \quad C^T = \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \quad \ldots (2.10)$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$|D| = 1\,(x_2 y_3 - x_3 y_2) - 1\,(x_1 y_3 - x_3 y_1) + 1\,(x_1 y_2 - x_2 y_1) \quad \ldots (2.11)$$

Substitute C^T and $|D|$ values in equation (2.9),

$$(2.9) \Rightarrow \quad D^{-1} = \frac{1}{(x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)} \times \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

Substitute D^{-1} value in equation (2.8),

$$\Rightarrow \quad \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{(x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)} \times \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad \ldots (2.12)$$

The area of the triangle ABC can be expressed as a function of the x, y co-ordinates of the nodes 1, 2 and 3.

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}[1(x_2 y_3 - x_3 y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)]$$

$$= \frac{1}{2}[x_2 y_3 - x_3 y_2 - x_1 y_3 + x_1 y_2 + x_3 y_1 - x_2 y_1]$$

$$A = \frac{1}{2}[x_2 y_3 - x_3 y_2 - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)]$$

$$\Rightarrow (x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1) = 2A \qquad \ldots (2.13)$$

Substitute (2.13) value in equation (2.12),

$$\Rightarrow \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

Integration of polynomial terms in natural co-ordinates for two dimensional elements can be performed by using the formula,

$$\oint_A (L_1)^\alpha (L_2)^\beta (L_3)^\gamma \, dA = \frac{\alpha!\,\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!} \times 2A \qquad \ldots (2.14)$$

2.6.1. Introduction

If the values of the field variable are computed only at nodes, how are values obtained at other nodal points within a finite element? This is a most important point of finite element analysis.

The values of the field variable computed at the nodes are used to approximate the values at non-nodal points by interpolation of the nodal values.

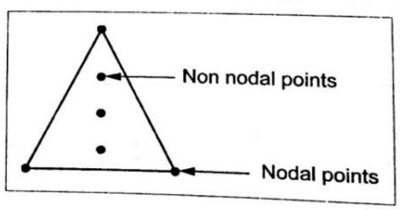


Fig. 2.17.

Consider the three noded triangular element as shown in Fig.2.17.

The nodes are exterior and at any point within the element the field variable is described by the following approximate relation.

$$\phi(x, y) = N_1(x, y)\,\phi_1 + N_2(x, y)\,\phi_2 + N_3(x, y)\,\phi_3$$

where ϕ_1, ϕ_2, ϕ_3 are the values of the field variable at the nodes, and N_1, N_2 and N_3 are the interpolation functions. N_1, N_2 and N_3 are also called as shape functions because they are used to express the geometry or shape of the element. Shape function has unit value at one nodal point and zero value at other nodal points.

In one dimensional problem, the basic field variable is displacement.

So, $u = \Sigma N_1 u_i$ where $u \rightarrow$ Displacement.

For two noded bar element, the displacement at any point within the element is given by,

$$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2$$

where, u_1 and u_2 are nodal displacements.

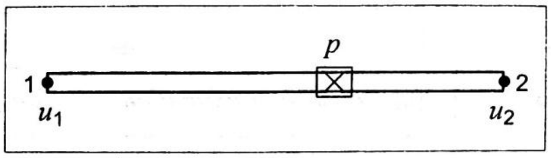


Fig. 2.18.

In two dimensional stress analysis problem, the basic field variable is displacement.

$$\text{So,} \quad u = \Sigma N_i u_i$$

$$v = \Sigma N_i v_i$$

For three noded triangular element, the displacement at any point within the element is given by,

$$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = \Sigma N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

where, u_1, u_2, u_3, v_1, v_2 and v_3 are nodal displacements.

In general, shape functions need to satisfy the following:

1. First derivatives should be finite within an element.
2. Displacement should be continuous across the element boundary.

The characteristics of shape function are:

1. The shape function has unit value at its own nodal point and zero value at other nodal points.
2. The sum of shape function is equal to one.
3. The shape functions for two dimensional elements are zero along each side that the node does not touch.
4. The shape functions are always polynomials of the same type as the original interpolation equations.

2.6.2. Polynomial Shape Functions

Polynomials are generally used as shape function due to the following reasons.

1. Differentiation and integration of polynomials are quite easy.
2. It is easy to formulate and computerize the finite element equations.
3. The accuracy of the results can be improved by increasing the order of the polynomial.

The approximation of a non-linear one dimensional function by using polynomials of different order is shown in Fig.2.19.

$$\Rightarrow u = [1 \quad x] \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [1 \quad x] \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [l - x \quad 0 + x] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

[∵ Matrix multiplication $(1 \times 2) \times (2 \times 2) = (1 \times 2)$]

$$u = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \ldots (2.20)$$

$$u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Displacement function, $u = N_1 u_1 + N_2 u_2$ ... (2.21)

where, Shape function, $N_1 = \frac{l-x}{l}$; Shape function, $N_2 = \frac{x}{l}$

We may note that N_1 and N_2 obey the definition of shape function, *i.e.*, the shape function will have a value equal to unity at the node to which it belongs and zero value at other nodes.

Checking: At node 1, $x = 0$.

$$\Rightarrow N_1 = \frac{l-x}{l} = \frac{l-0}{l}$$

$$\boxed{N_1 = 1}$$

$$\Rightarrow N_2 = \frac{x}{l} = \frac{0}{l}$$

$$\boxed{N_2 = 0}$$

At node 2, $x = l$

$$\Rightarrow N_1 = \frac{l-x}{l} = \frac{l-l}{l}$$

$$\boxed{N_1 = 0}$$

$$\Rightarrow N_2 = \frac{x}{l} = \frac{l}{l}$$

$$\boxed{N_2 = 1}$$

2.7. STIFFNESS MATRIX [K]

In order to get an expression for the stiffness matrix in finite element method, let us review the strain energy expression in structural mechanics.

Consider $\omega_1, \omega_2 \ldots\ldots \omega_n$ are nodal displacement parameters or otherwise known as degrees of freedom, $W_1, W_2, \ldots\ldots W_n$ are the corresponding nodal loads acting at degrees of freedom. $\{\omega\}$ and $\{W\}$ are column matrix.

$$\{W\} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_n \end{Bmatrix}$$

$$\{\omega\} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_n \end{Bmatrix}$$

$$\text{We know that,} \quad \{W\} = [K]\{\omega^*\} \qquad \ldots (2.22)$$

where, W = Nodal loads.
K = Stiffness matrix.
ω^* = Degrees of freedom.

From equation (2.22), we know that, nodal loads and the corresponding degrees of freedom are linked through stiffness matrix.

We know that,

Work done, P = Strain energy

$$\Rightarrow P = \frac{1}{2} W_1 \omega_1 + \frac{1}{2} W_2 \omega_2 + \frac{1}{2} W_3 \omega_3 + \ldots\ldots\ldots + \frac{1}{2} W_n \omega_n$$

We can write this equation in matrix form,

i.e.,
$$P = \frac{1}{2} [W_1 \; W_2 \; W_3 \ldots\ldots W_n] \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_n \end{Bmatrix}$$

$$P = \frac{1}{2} \{W\}^T \{\omega^*\} \qquad \ldots (2.23)$$

[*Note:* [ ] → Row matrix; { } → Column matrix]

Substitute equation (2.22) in equation (2.23),

$$\Rightarrow P = \frac{1}{2}\left\{[K]\{\omega^*\}\right\}^T \{\omega^*\}$$

$$= \frac{1}{2}[K]^T \{\omega^*\}^T \{\omega^*\}$$

$$\boxed{P = \frac{1}{2}\{\omega^*\}^T [K] \{\omega^*\}} \quad \dots (2.24)$$

[∵ K is a symmetric matrix. So, $[K]^T = [K]$]

Equation (2.24) is a strain energy equation for a structure.

Our aim is to find the expression for stiffness matrix [K]. Let us consider one dimensional element. $u_1, u_2, u_3 \dots\dots, u_n$ are the degrees of freedom of that element.

We know that,

$$\text{Strain, } \{e\} = [B]\{u^*\} \quad \dots (2.25)$$

$$\Rightarrow \boxed{\{e\}^T = [B]^T \{u^*\}^T} \quad \dots (2.26)$$

where, $\{e\}$ is a strain matrix [Column matrix].

$[B]$ is a strain-displacement matrix [Row matrix].

$\{u^*\}$ is a degree of freedom [Column matrix].

We know that,

$$\text{Stress } \{\sigma\} = [E]\{e\}$$

$$\boxed{\{\sigma\} = [D]\{e\}} \quad \dots (2.27)$$

where, $[E] = [D] =$ Young's modulus.

Strain energy expression is given by,

$$U = \int_v \frac{1}{2}\{e\}^T \{\sigma\}\, dv \quad \dots (2.28)$$

Substitute $\{e\}^T$ and $\{\sigma\}$ values,

$$\Rightarrow U = \int_v \frac{1}{2}[B]^T \{u^*\}^T [D]\{e\}\, dv$$

$$= \frac{1}{2}\{u^*\}^T \int_v [B]^T [D]\{e\}\, dv$$

Substitute { e } value,

$$\Rightarrow U = \frac{1}{2} \{u^*\}^T \int_v [B]^T [D] [B] \{u^*\} \, dv$$

$$U = \frac{1}{2} \{u^*\}^T \left[\int_v [B]^T [D] [B] \, dv \right] \{u^*\} \qquad \ldots (2.29)$$

From equation (2.24), we know that,

$$P = \frac{1}{2} \{\omega^*\}^T [K] \{\omega^*\} \qquad \ldots (2.24)$$

Comparing equation (2.29) and (2.24),

$$\Rightarrow \{\omega^*\}^T = \{u^*\}^T$$

$$\{\omega^*\} = \{u^*\}$$

$$[K] = \int_v [B]^T [D] [B] \, dv$$

So, Stiffness matrix, $[K] = \int_v [B]^T [D] [B] \, dv \qquad \ldots (2.30)$

where, [B] → Strain displacement relationship matrix.

[D] → Elasticity matrix or Stress-strain relationship matrix.

In one dimensional problem,

$$\text{Strain, } e = \frac{du}{dx}$$

where, u → Displacement function.

$$[D] = [E] = E = \text{Young's modulus.}$$

In Beam problem, Strain, e = Curvature $= \dfrac{d^2u}{dx^2}$

$$[D] = [EI] = \text{Flexural rigidity.}$$

2.7.1. Properties of Stiffness Matrix

1. It is a symmetric matrix.
2. The sum of elements in any column must be equal to zero.
3. It is an unstable element. So, the determinant is equal to zero.
4. The dimension of the global stiffness matrix [K] is N × N, where N is the number of nodes. This follows from the fact that each node has only one degree of freedom.
5. The diagonal coefficients are always positive and relatively large when compared to the off-diagonal values in the same row.

2.7.2. Derivation of Stiffness Matrix for One Dimensional Linear Bar Element

Consider a one dimensional bar element with nodes 1 and 2 as shown in Fig.2.21. Let u_1 and u_2 be the nodal displacement parameters or otherwise known as degrees of freedom.

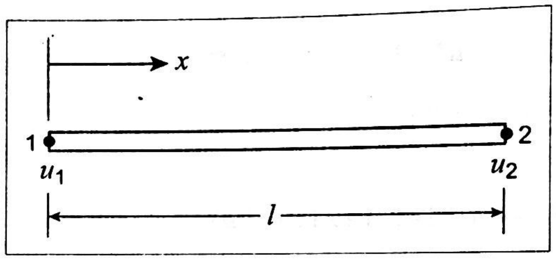


Fig. 2.21. A bar element with two nodes

We know that,

$$\text{Stiffness matrix } [K] = \int_v [B]^T [D] [B] \, dv \quad \text{[From equation no.(2.30)]}$$

In one dimensional bar element,

$$\text{Displacement function, } u = N_1 u_1 + N_2 u_2 \quad \text{[From equation no.(2.21)]}$$

$$\text{where, } N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

We know that,

$$\text{Strain-Displacement matrix, } [B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \quad \ldots (2.31)$$

$$\Rightarrow [B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \quad \ldots (2.32)$$

In one dimensional problems, $[D] = [E] = E$ = Young's modulus ... (2.33)

Substitute $[B]$, $[B]^T$ and $[D]$ values in stiffness matrix equation. [Limit is 0 to l].

$$\Rightarrow [K] = \int_0^l \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \times E \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv = \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} E\, dv$$

[∵ Matrix multiplication $(2 \times 1) \times (1 \times 2) = (2 \times 2)$]

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} E\, A\, dx \qquad [\because dv = A\, dx]$$

$$= AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^l dx = AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$= AE \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} (l - 0) = AEl \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix}$$

$$= \frac{AEl}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \qquad \dots (2.34)$$

The properties of a stiffness matrix are satisfied.

1. It is symmetric.
2. The sum of elements in any column is equal to zero. ?

2.8. DERIVATION OF FINITE ELEMENT EQUATION FOR ONE DIMENSIONAL LINEAR BAR ELEMENT

We know that, General force equation is,

$$\{F\} = [K]\{u\} \qquad \dots (2.35)$$

where, $\{F\}$ is a element force vector [Column matrix].

$[K]$ is a stiffness matrix [Row matrix].

$\{u\}$ is a nodal displacement [Column matrix].

For one dimensional bar element, stiffness matrix [K] is given by,

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ [From equation no.(2.34)]

For two noded bar element,

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute [K] { F } and { u } values in equation (2.35),

$$\Rightarrow \boxed{\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}} \quad \ldots (2.36)$$

This is a finite element equation for one dimensional two noded bar element.

2.9. ASSEMBLING THE STIFFNESS EQUATIONS OR GLOBAL EQUATIONS

Consider a bar as shown in Fig.2.22(a). This bar can be equally divided into 4 elements as shown in Fig.2.22(b).

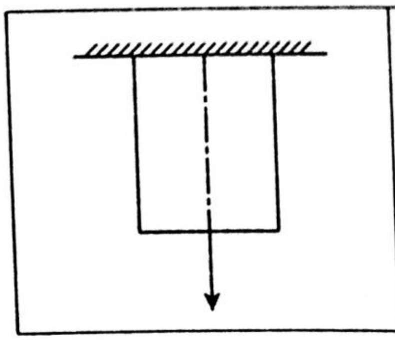


Fig. 2.22. (a)

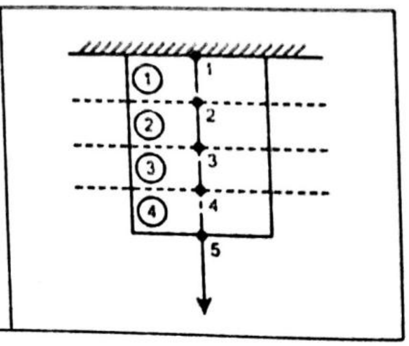


Fig. 2.22. (b)

Now the bar has 4 elements with 5 nodes.

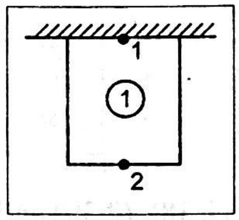
[*Note:* A number with circle denotes element and without circle denotes nodes]

We know that,

Finite element equation for two noded bar element is,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

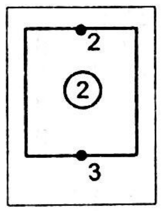
For element (1) (Nodes 1, 2):



Finite element equation is,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{-1} \\ \overset{a_{21}}{-1} & \overset{a_{22}}{1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \ldots (2.37)$$

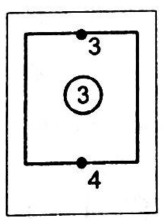
For element (2) (Nodes 2, 3):



Finite element equation is,

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} \overset{a_{22}}{1} & \overset{a_{23}}{-1} \\ \overset{a_{32}}{-1} & \overset{a_{33}}{1} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \quad \ldots (2.38)$$

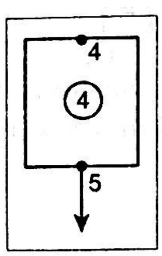
For element (3) (Nodes 3, 4):



Finite element equation is,

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} \overset{a_{33}}{1} & \overset{a_{34}}{-1} \\ \overset{a_{43}}{-1} & \overset{a_{44}}{1} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} \quad \ldots (2.39)$$

For element (4) (Nodes 4, 5):



Finite element equation is,

$$\begin{Bmatrix} F_4 \\ F_5 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} \overset{a_{44}}{1} & \overset{a_{45}}{-1} \\ \overset{a_{54}}{-1} & \overset{a_{55}}{1} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} \qquad \cdots (2.40)$$

Assembling the equations (2.37), (2.38), (2.39) and (2.40),

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{-1} & \overset{a_{13}}{0} & \overset{a_{14}}{0} & \overset{a_{15}}{0} \\ \overset{a_{21}}{-1} & \overset{a_{22}}{1+1} & \overset{a_{23}}{-1} & \overset{a_{24}}{0} & \overset{a_{25}}{0} \\ \overset{a_{31}}{0} & \overset{a_{32}}{-1} & \overset{a_{33}}{1+1} & \overset{a_{34}}{-1} & \overset{a_{35}}{0} \\ \overset{a_{41}}{0} & \overset{a_{42}}{0} & \overset{a_{43}}{-1} & \overset{a_{44}}{1+1} & \overset{a_{45}}{-1} \\ \overset{a_{51}}{0} & \overset{a_{52}}{0} & \overset{a_{53}}{0} & \overset{a_{54}}{-1} & \overset{a_{55}}{1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

[*Note:* The bar has 5 nodes and each node has one degree of freedom. So, the global stiffness matrix size is 5 × 5]

$$[K]_{global} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

2.10. THE LOAD OR FORCE VECTOR { F }

Consider a vertically hanging bar of length l, uniform cross-section A, density ρ and young's modulus E. This bar is subjected to self weight X_b.

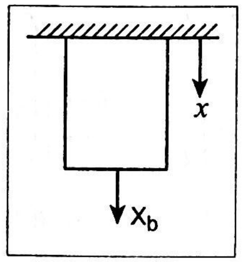


Fig. 2.23. Vertically hanging bar with self weight

The element nodal force vector is given by,

$$\{F\}_e = \int [N]^T X_b \qquad \ldots (2.41)$$

We know that,

Self weight due to loading force, $X_b = \rho A\, dx \qquad \ldots (2.42)$

For one dimensional bar element, the displacement function is given by,

$$u = N_1 u_1 + N_2 u_2 \qquad \text{[From equation no.(2.21)]}$$

where,
$$N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

$$\Rightarrow [N] = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

$$\Rightarrow [N]^T = \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} \qquad \ldots (2.43)$$

Substitute X_b and $[N]^T$ values in equation (2.41),

$$\Rightarrow \{F\}_e = \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} \rho A\, dx = \rho A \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= \rho A \int_0^l \begin{Bmatrix} 1 - \frac{x}{l} \\ \frac{x}{l} \end{Bmatrix} dx = \rho A \int_0^l \begin{Bmatrix} dx - \frac{x\,dx}{l} \\ \frac{x\,dx}{l} \end{Bmatrix}$$

Substitute the equation (2.49) in equation (2.50) and (2.51),

Equation (2.50) ⇒ $$u_2 = u_1 + a_1 l + a_2 l^2 \quad \ldots (2.52)$$

Equation (2.51) ⇒ $$u_3 = u_1 + \frac{a_1 l}{2} + \frac{a_2 l^2}{4} \quad \ldots (2.53)$$

Equation (2.52) ⇒ $$u_2 - u_1 = a_1 l + a_2 l^2 \quad \ldots (2.54)$$

Equation (2.53) ⇒ $$u_3 - u_1 = \frac{a_1 l}{2} + \frac{a_2 l^2}{4} \quad \ldots (2.55)$$

Arranging the equation (2.54) and (2.55) in matrix form,

$$\begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} l & l^2 \\ \frac{l}{2} & \frac{l^2}{4} \end{bmatrix}^{-1} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix}$$

$$= \frac{1}{\left(\frac{l^3}{4} - \frac{l^3}{2}\right)} \begin{bmatrix} \frac{l^2}{4} & -l^2 \\ \frac{-l}{2} & l \end{bmatrix} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix}$$

[Note: $\because \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11} a_{22} - a_{12} a_{21})} \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$]

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{\left(\frac{-l^3}{4}\right)} \begin{bmatrix} \frac{l^2}{4} & -l^2 \\ \frac{-l}{2} & l \end{bmatrix} \begin{Bmatrix} u_2 - u_1 \\ u_3 - u_1 \end{Bmatrix} \quad \ldots (2.56)$$

$$\Rightarrow a_1 = \frac{-4}{l^3} \left[\frac{l^2}{4} (u_2 - u_1) - l^2 (u_3 - u_1) \right] \quad \ldots (2.57)$$

$$\Rightarrow a_2 = \frac{-4}{l^3} \left[\frac{-l}{2} (u_2 - u_1) + l (u_3 - u_1) \right] \quad \ldots (2.58)$$

Equation (2.57) ⇒

$$a_1 = \frac{-4}{l^3} \left[\frac{l^2 u_2}{4} - \frac{l^2 u_1}{4} - l^2 u_3 + l^2 u_1 \right]$$

$$= \frac{-4\,l^2 u_2}{4\,l^3} + \frac{4\,l^2 u_1}{4\,l^3} + \frac{4\,l^2 u_3}{l^3} - \frac{4\,l^2 u_1}{l^3}$$

$$= \frac{-u_2}{l} + \frac{u_1}{l} + \frac{4\,u_3}{l} - \frac{4\,u_1}{l}$$

$$a_1 = \frac{-3\,u_1}{l} - \frac{u_2}{l} + \frac{4\,u_3}{l} \quad \ldots (2.59)$$

Equation (2.58) $\Rightarrow$

$$a_2 = \frac{-4}{l^3}\left[\frac{-l\,u_2}{2} + \frac{l}{2}\,u_1 + l\,u_3 - l\,u_1\right]$$

$$= \frac{4\,l\,u_2}{2\,l^3} - \frac{4\,l}{2\,l^3}\,u_1 - \frac{4\,l}{l^3}\,u_3 + \frac{4\,l}{l^3}\,u_1$$

$$= \frac{2\,u_2}{l^2} - \frac{2}{l^2}\,u_1 - \frac{4}{l^2}\,u_3 + \frac{4}{l^2}\,u_1$$

$$a_2 = \frac{2}{l^2}\,u_1 + \frac{2\,u_2}{l^2} - \frac{4}{l^2}\,u_3 \quad \ldots (2.60)$$

Arranging the equation (2.49), (2.59) and (2.60) in matrix form,

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (2.61)$$

Substitute the equation (2.61) in equation (2.48),

$$\{u\} = [1 \quad x \quad x^2] \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{l} & \frac{-1}{l} & \frac{4}{l} \\ \frac{2}{l^2} & \frac{2}{l^2} & \frac{-4}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (2.62)$$

$$\{u\} = \left[\left(1 - \frac{3}{l}\,x + \frac{2\,x^2}{l^2}\right)\left(\frac{-x}{l} + \frac{2\,x^2}{l^2}\right)\left(\frac{4\,x}{l} - \frac{4\,x^2}{l^2}\right)\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

2.20. THE POTENTIAL–ENERGY APPROACH

We know that,

The general expression for the potential energy is given by,

$$\pi = \frac{1}{2} \int_l \sigma^T e \, A \, dx - \int_l u^T f \, A \, dx - \int_l u^T \, T \, dx - \sum_i u_i P_i$$

where, σ → Stress, N/mm²

e → Strain

A → Area, mm²

u → Displacement, mm

f → Body force, N

T → Traction force, N

P → Point load, N

When the continuum has been discretized into finite elements, the expression for π becomes as follows:

$$\pi = \sum_e \frac{1}{2} \int_e \sigma^T e \, A \, dx - \sum_e \int_e u^T f \, A \, dx - \sum_e \int_e u^T \, T \, dx - \sum_i Q_i P_i$$

The above equation can be written as,

$$\pi = \sum_e U_e - \sum_e \int_e u^T f \, A \, dx - \sum_e \int_e u^T \, T \, dx - \Sigma \, Q_i \, P_i \qquad \dots (2.92)$$

where, Strain energy, $U_e = \frac{1}{2} \int \sigma^T e \, A \, dx$

Stiffness matrix for a bar element:

We know that,

$$\text{Strain energy, } U_e = \frac{1}{2} \int \sigma^T e \, A \, dx \qquad \dots (2.93)$$

From equation (2.25), we know that,

$$\boxed{\text{Strain, } e = B \, u}$$

$$\text{Stress, } \sigma = E \, e \qquad \left[\because \text{Young's modulus, } E = \frac{\text{Stress, } \sigma}{\text{Strain, } e}\right]$$

$$\Rightarrow \quad \boxed{\text{Stress, } \sigma = E \times B u}$$

Substitute σ and e values in equation (2.93),

$$\Rightarrow \quad U_e = \frac{1}{2} \int (E \, B \, u)^T \, (B \, u) \, A \, dx = \frac{1}{2} \int E \, B^T \, u^T \, B \, u \, A \, dx$$

Element stiffness matrix in global co-ordinates.

$$[K] = [L]^T [K'] [L] \quad \cdots (2.88)$$

Substitute [L] value from equation (2.81) and [K′] value from equation (2.85).

$$\Rightarrow [K] = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$\left[\because [L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}; \quad [L]^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \right]$$

$$\Rightarrow [K] = \frac{A_e E_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} l-0 & m-0 & 0-l & 0-m \\ -l+0 & -m+0 & 0+l & 0+m \end{bmatrix}$$

$$[\because (2 \times 2) \times (2 \times 4) = 2 \times 4]$$

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} l & m & -l & -m \\ -l & -m & l & m \end{bmatrix}$$

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} l^2-0 & lm-0 & -l^2+0 & -lm+0 \\ lm-0 & m^2-0 & -ml+0 & -m^2+0 \\ 0-l^2 & 0-lm & 0+l^2 & 0+lm \\ 0-lm & 0-m^2 & 0+lm & 0+m^2 \end{bmatrix}$$

$$[\because (4 \times 2) \times (2 \times 4) = 4 \times 4]$$

$$= \frac{1}{2} u^T \int E B^T B u A dx$$

$$= \frac{1}{2} u^T \left[A E B^T B u \int_0^l dx \right]$$

$$= \frac{1}{2} u^T \left[A E B^T B u [x]_0^l \right]$$

$$U_e = \frac{1}{2} u^T [A E B^T B u (l)] \quad \ldots (2.94)$$

From equation (2.31), we know that,

Strain–Displacement matrix, $[B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$

$$\Rightarrow [B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix}$$

Substitute B, B^T values in equation (2.94).

$$\Rightarrow \quad U_e = \frac{1}{2} u^T \cdot AE \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} u\, l$$

$$= \frac{1}{2} u^T \cdot AE \times \frac{1}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \times \frac{1}{l} [-1 \quad 1] u\, l$$

$$= \frac{1}{2} u^T \frac{AE}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} [-1 \quad 1] u$$

$$U_e = \frac{1}{2} u^T \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u \qquad [\because (2 \times 1) \times (1 \times 2) = 2 \times 2]$$

The above equation is in the form of

$$U_e = \frac{1}{2} u^T [K] u$$

where, Stiffness matrix, $[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \ldots (2.95)$

$$= \frac{1}{2} u^T \int E B^T B u A \, dx$$

$$= \frac{1}{2} u^T \left[A E B^T B u \int_0^l dx \right]$$

$$= \frac{1}{2} u^T \left[A E B^T B u [x]_0^l \right]$$

$$U_e = \frac{1}{2} u^T [A E B^T B u (l)] \quad \dots (2.94)$$

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Substitute B, B^T values in equation (2.94).

$$\Rightarrow \quad U_e = \frac{1}{2} u^T \cdot AE \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} u \, l$$

$$= \frac{1}{2} u^T \cdot AE \times \frac{1}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \times \frac{1}{l} [-1 \quad 1] \, u \, l$$

$$= \frac{1}{2} u^T \frac{AE}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} [-1 \quad 1] \, u$$

$$U_e = \frac{1}{2} u^T \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u \qquad [\because (2 \times 1) \times (1 \times 2) = 2 \times 2]$$

The above equation is in the form of

$$U_e = \frac{1}{2} u^T [K] u$$

where, Stiffness matrix, $[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \dots (2.95)$

$$= \frac{1}{2} u^T \int E B^T B u A dx$$

$$= \frac{1}{2} u^T \left[A E B^T B u \int_0^l dx \right]$$

$$= \frac{1}{2} u^T \left[A E B^T B u [x]_0^l \right]$$

$$U_e = \frac{1}{2} u^T [A E B^T B u (l)] \qquad \ldots (2.94)$$

From equation (2.31), we know that,

Strain–Displacement matrix, $[B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$

$$\Rightarrow [B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix}$$

Substitute B, B^T values in equation (2.94).

$$\Rightarrow \qquad U_e = \frac{1}{2} u^T \cdot AE \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} u\, l$$

$$= \frac{1}{2} u^T \cdot AE \times \frac{1}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \times \frac{1}{l} [-1 \quad 1] u\, l$$

$$= \frac{1}{2} u^T \frac{AE}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} [-1 \quad 1] u$$

$$U_e = \frac{1}{2} u^T \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u \qquad [\because (2 \times 1) \times (1 \times 2) = 2 \times 2]$$

The above equation is in the form of

$$U_e = \frac{1}{2} u^T [K] u$$

where, Stiffness matrix, $[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \ldots (2.95)$

Multiple Choice Questions and Answers

S.No	Questions	Opt1	Opt2	Opt3	Opt4	Answer
1	Secondary nodes are located at---------------------of the element	Mid side	Corner	Interior	Mid side &interior	**Mid side &interior**
2	In higher order elements number of nodes are equal to	Polynomial coefficients	Degree of freedom	Polynomial coefficients & degree of freedom	None of these	**Polynomial coefficients & degree of freedom**
3	In CST element the order of polynomial is	Linear	Quadratic	Cubic	Linear & Quadratic	**Linear**
4	In LST element the order of polynomial is	Linear	Quadratic	Cubic	Linear & Quadratic	**Quadratic**
5	The magnitude of the natural co-ordinate system is	1to -1	0.1to -0.1	0.01 to -.01	None of these	**1 to -1**
6	The relation between the derivatives of natural coordinate and global coordinate system is given by	Jacobian matrix	[B] matrix	Stiffness matrix	Triangular matrix	**Jacobian matrix**
7	Gauss quadrature method is suitable for variables specified in	Global coordinate system	Cartisiean coordinate system	Natural coordinate system	Spherical coordinate system	**Natural coordinate system**
8	Gauss quadrature method limits ranges to	1to -1	0.1to -0.1	0.01 to -.01	None of these	**1 to -1**
9	Number of nodes in CST element	1	2	3	4	**3**
10	Number of nodes in QST element	1	2	6	10	**10**
11	A graphical method of depicting complete two dimensional polynomial is called as	Pascals pyramid	Pascals triangle	Pascals rectangle	Pascals circle	**Pascals triangle**
12	The interior node is connected with	Interior node of another element	Interior node of same element	Not connected with any element	None of these	**Not connected with any element**
13	Which one of the following method used to derive the shape function for Triangular element	Rayleigh method	Ritz method	Area coordinates method	Weighted residual method	**Area coordinates method**
14	Total degree of freedom of the rectangular element is equal to	1	2	3	4	**4**

15	The choice of polynoimals for 8 noded rectangular element is	Quadratic	Cubic	Incompleted cubic	Quartic	**Incompleted cubic**
16	The extension of triangular elements in 3-dimensional elements is called as	Quadratic	Quadrilateral	Tetrahedrons	Brick	**Tetrahedrons**
17	The extension of rectangular elements in 3-dimensional elements is called as	Quadratic	Cubic	Tetrahedrons	Brick	**Brick**
18	The magnitude of jacobian is ------------------------------------ --area of the element	Equal to	Twice the	One third of the	One half	**Twice the**
19	In the triangular element N1+N2+N3=	1	2	3	0.1	**1**
20	[B] matrix is represents	Stress-strain relation ship	Strain-displacement relationship	Stress-displacement relationship	None of these	**Strain-displacement relationship**
21	The distributed load acting on the surface of the body is called as	Body force	Surface force	Traction force	Drag force	**Traction force**
22	The material property of the triangular element is representated by -------------matrix	A	B	B^T	D	**D**
23	The product of arbitary nodal displacement matrix*force vector is called as	Internal virtual work	External virtunal work	Potential work	None of these	**External virtunal work**
24	The product of DBq is called as	Stress	Strain	Displacement	None of these	**Stress**
25	When dividing the domain by triangular elements which one of the parameter have major preference	Length	Width	Aspect ratio	Height	**Aspect ratio**
26	At the notches and fillets the size of the element become	Increased	Decreased	Not varied	Equally spaced	**Decreased**
27	Coarse elements are recommended for	Initial trials	Filleted corners	Fluid flow analysis	For dynamic analysis	**Initial trials**
28	For the quadrilateral elements the numbering of nodes done in -----------------------direction	Clockwise	Anticlockwise	Opposite	None of these	**Anticlockwise**

29	The representation of quadrilateral elements in to triangular elements is called as	Convergent	Divergent	Degenerate	None of these	**Degenerate**
30	Degeneration will increases the	Accuracy	Error	Computing time	Computing cost	**Error**
31	Which one of the following element will gives the better results for analyzing the tubes	4 noded rectangular element	8 noded rectangular element	9 noded rectangular element	QST element	**9 noded rectangular element**
32	If the interior node is not properly located than the jacobian matrix becomes	1	0	It not effects the matrix	None of these	**0**
33	The order of stiffness matrix for CST element	6*6	7*7	4*4	8*8	**8*8**
34	The polynomial function for three noded isoparametric bar element is	Linear	Quadratic	Cubic	Quatric	**Quadratic**
35	The gauss quadrature method recommened for develop the stiffness of an 8 noded quadrilateral element is	One point method	Two point method	Three point method	Four point method	**Three point method**
36	The number of nodes at hexahedral elemrnt is equal to	4	6	8	10	**8**
37	The gauss quadrature method recommened for develop the stiffness of an hexahedral elemrnt is equal to	One point method	Two point method	Three point method	Four point method	**Two point method**
38	Which type of elements are used to analyze the arch dams ,forged parts	1-dimensional	2-dimesional	3-dimensional	Multi dimensional	**3-dimensional**
39	Which one of the following method of numerical integeration is more suitable analysis for FEA analysis	Newtons cotes qudrature	Simpsons rule	Trapezoid rule	Gauss quadrature	**Gauss quadrature**
40	The locatin of sample point taken for gauss quadrature one point method	0	-1	1	None of these	**0**
41	The locatin of sample point taken	±1	±.8888888888	±.577350	±.3339	**±.577350**

	for gauss quadrature two point method					
42	The number of sample point taken for gauss quadrature two point method	1	2	3	4	**2**
43	The number of sample point taken for gauss quadrature four point method	1	2	3	4	**4**
44	The order of body force vector for 4 nodedisoparametric quadrilateral element	2*2	8*1	4*4	3*4	**8*1**

UNIT 3

TWO-DIMENSIONAL CONTINUUM

3.1. INTRODUCTION

This chapter considers the two dimensional finite element. Two dimensional elements are defined by three or more nodes in a two dimensional plane (*i.e.*, x, y plane). The basic element useful for two dimensional analysis is the triangular element. The simplest two dimensional elements have corner nodes as shown in Fig.3.1. A quadrilateral (special forms of rectangle and parallelogram) element can be obtained by assembling two or four triangular elements, as shown in Fig.3.2. They are often used to model a wide range of Engineering problems.

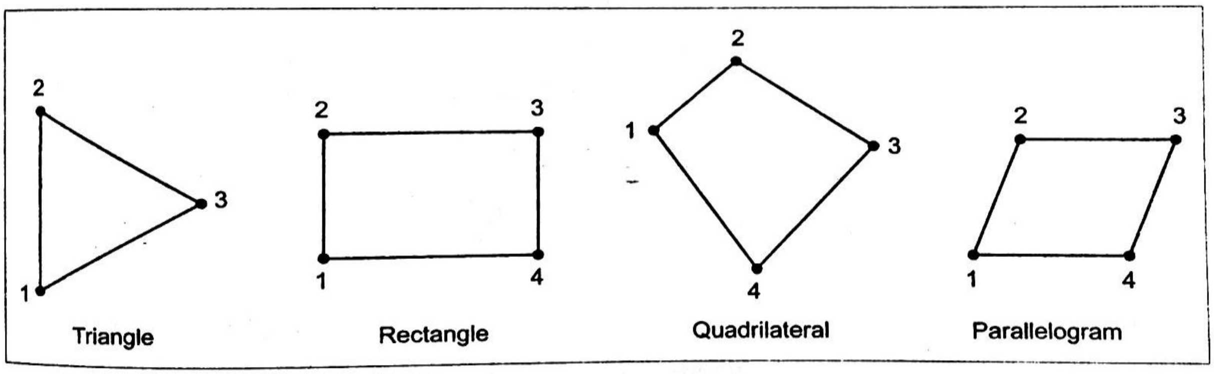


Fig. 3.1. Two dimensional elements

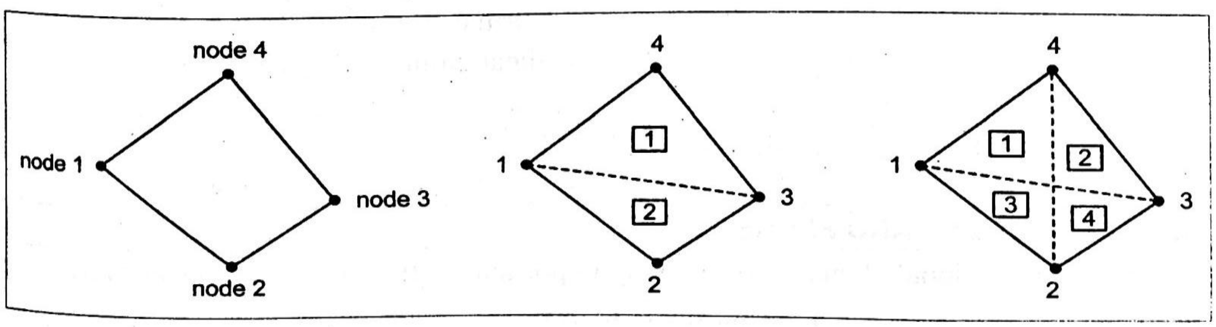


Fig. 3.2. A quadrilateral element as an assemblage of two or four triangular elements

The two dimensional analysis of hydraulic cylinder rod end with plane strain triangular elements is shown in Fig.3.3.

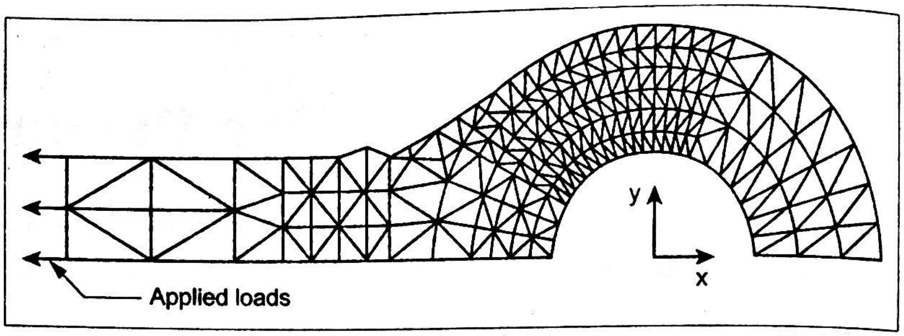


Fig. 3.3. Two dimensional analysis of hydraulic cylinder rod end

The two dimensional finite element formulation follows the same steps which is used in the one dimensional problems. The displacements and distributed body force values are functions of the position indicated by (x, y).

The displacement vector u is given by, $u = \begin{Bmatrix} u \\ v \end{Bmatrix}$

where u and v are the x and y components of u respectively.

The stresses and strains are given as,

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$e = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

where, σ → Normal stress
τ → Shear stress
e → Normal strain.
γ → Shear strain.

Body force is given by, $F = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$

3.2. PLANE STRESS AND PLANE STRAIN

The two dimensional element is extremely important for the following two analysis.

(*i*) Plane stress analysis.

(*ii*) Plane strain analysis.

(i) Plane Stress Analysis

Plane stress is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) directed perpendicular to the plane are assumed to be zero.

Generally, members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the $x-y$ plane can be considered to be under plane stress.

Plates with holes and plates with fillets are coming under plane stress analysis problems.

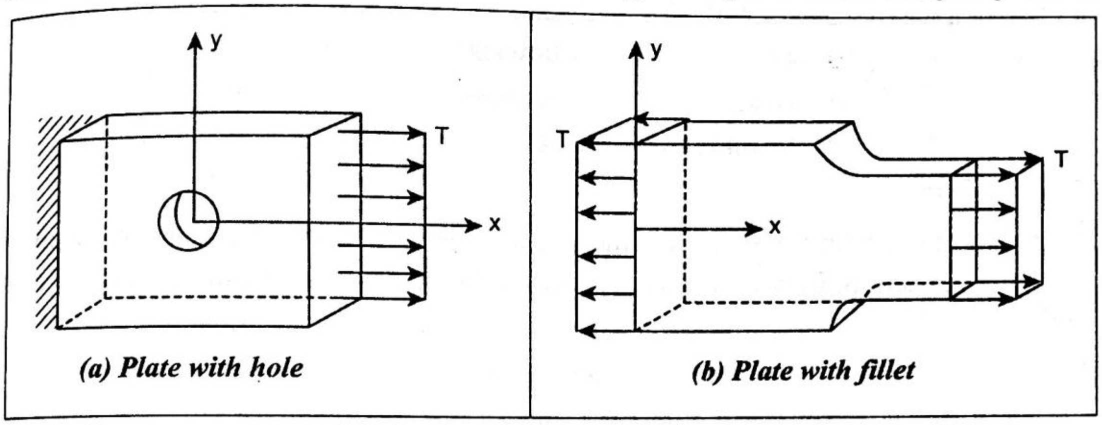


Fig. 3.4. Plane stress problems: (a) plate with hole; (b) plate with fillet

where, T → Surface tractions (*i.e.*, pressure acting on the surface edge or face of a member, unit → Force/Area → N/m^2)

Normal stress, $\sigma_z = 0$

Shear stresses τ_{xz} and $\tau_{yz} = 0$

(ii) Plane strain analysis

Plane strain is defined to be a state of strain in which the strain normal to the xy plane and the shear strains are assumed to be zero.

Dams and pipes subjected to loads that remain constant over their lengths are coming under plane strain analysis problems.

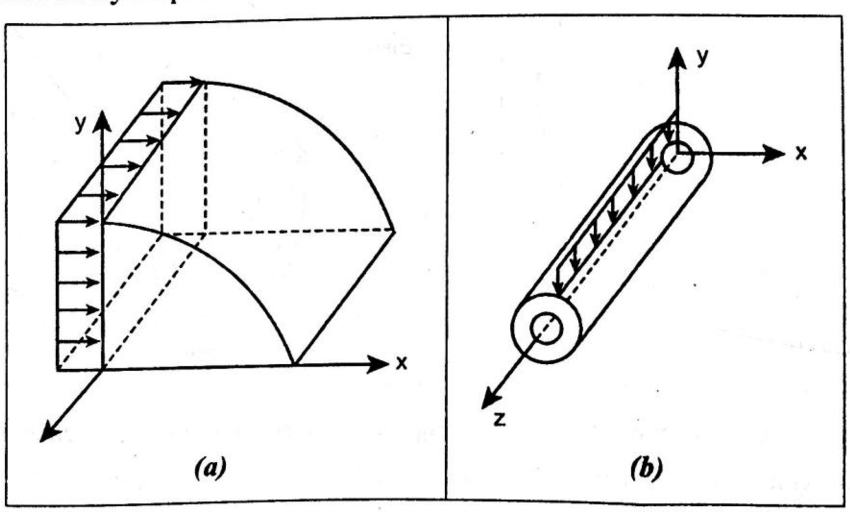


Fig. 3.5. Plane strain problems: (a) dam subjected to horizontal loading; (b) pipe subjected to a vertical load

Here, Normal strain, $e_z = 0$

Shear stresses γ_{xz} and $\gamma_{yz} = 0$

3.3. FINITE ELEMENT MODELLING

Finite element modelling consists of the following:

(*i*) Discretization of structure.

(*ii*) Numbering of nodes.

(i) Discretization

The art of subdividing a structure into a convenient number of smaller components is known as discretization. In two dimensional problems, three kinds of finite elements are used. They are:

(*i*) Triangular element.

(*ii*) Rectangular element.

(*iii*) Quadrilateral element.

In truss, the above three elements are physically present. But in a continuum, the above three elements exist only in our imagination.

The continuum shown in Fig.3.6 is discretized into eight triangular element as shown in Fig.3.7. The points where the corners of the triangles meet are called nodes. Each triangle formed by three nodes and three sides is called an element.

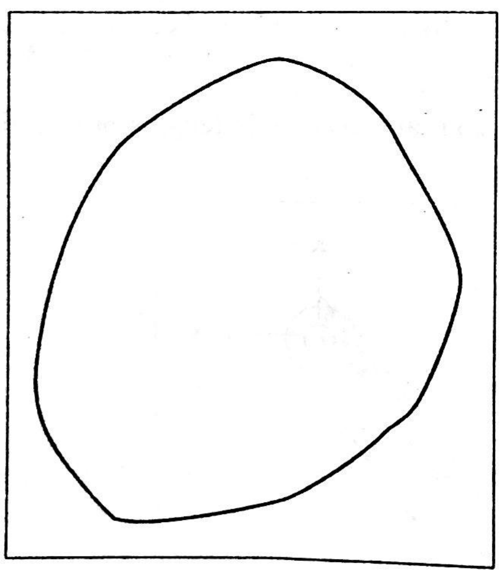


Fig. 3.6. Continuum

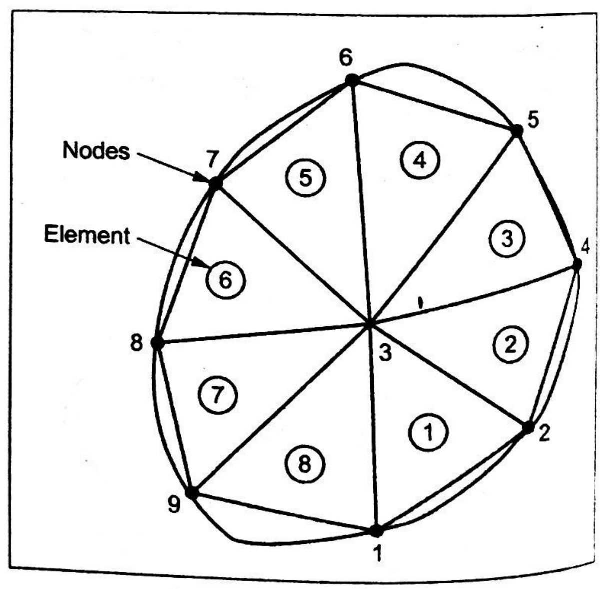


Fig. 3.7. Discretized into eight triangular elements

The element numbers are circled to distinguish from node numbers. The cross-section area, traction force and body force are constant within each element. But these are differ in magnitude from element to element. Better results are obtained by increasing the number of elements.

In Fig.3.7, the triangular elements fill the entire region except a small region at the boundary. This unfilled region can be eliminated by choosing smaller elements or elements with curved boundaries.

(ii) Numbering of Nodes

In one dimensional problem, each node is allowed to move only in $\pm x$ direction. But in two dimensional problem, each node is permitted to move in the two directions *i.e.*, x and y. Hence each node has two degrees of freedom (Nodal displacements). A three node finite element model is shown in Fig.3.8 has six degrees of freedom.

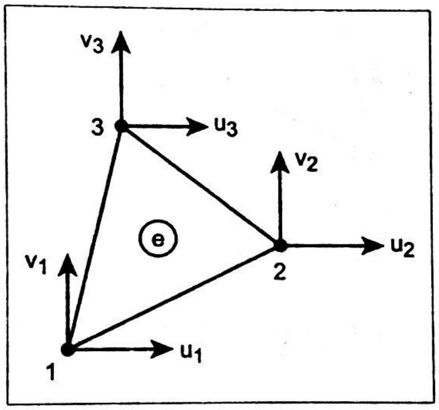


Fig. 3.8. Triangular element

The element connectivity table is given for Fig.3.7. The heading 1 and 2 refer to the local node numbers of an element and the corresponding node numbers on the body are called global numbers. Connectivity thus establishes the local-global correspondence.

Element (e)	Nodes 1 2 3
①	1 2 3
②	2 3 4
③	4 3 5
④	5 3 6
⑤	6 3 7
⑥	7 3 8
⑦	8 3 9
⑧	9 3 1

Local numbers (header row: Nodes 1 2 3)

Global numbers (rows ① to ⑧)

3.4. CONSTANT STRAIN TRIANGULAR (CST) ELEMENT

A three noded triangular element is known as constant strain triangular (CST) element which is shown in Fig.3.9. It has six unknown displacement degrees of freedom (u_1 v_1, u_2 v_2, u_3 v_3). The element is called CST because it has a constant strain throughout it.

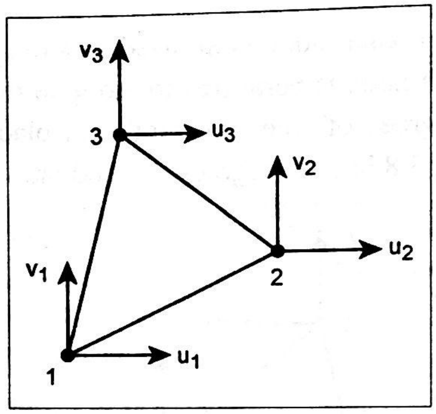


Fig. 3.9. Constant strain triangular element

3.5. SHAPE FUNCTION DERIVATION FOR THE CONSTANT STRAIN TRIANGULAR ELEMENT (CST)

We begin this section with the development of the shape function for a basic two dimensional finite element, called constant strain triangular element (CST).

We consider this CST element because its derivation is the simplest among the available two dimensional elements.

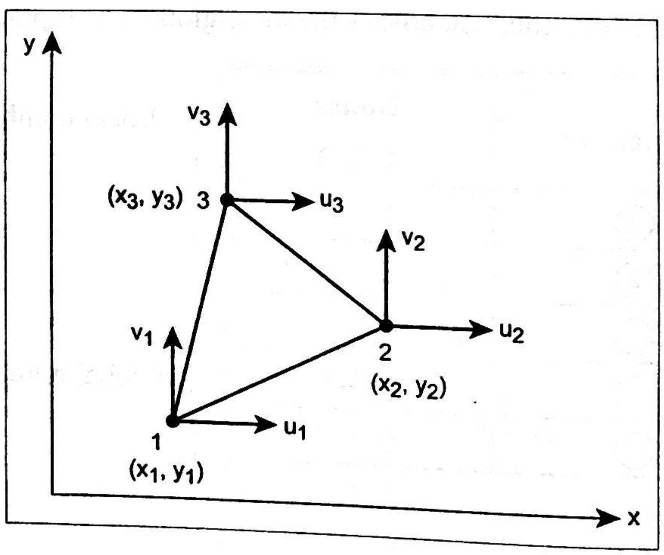


Fig. 3.10. Three noded CST element

Consider a typical CST element with nodes 1, 2 and 3 as shown in Fig.3.10 Let the nodal displacements be u_1, u_2, u_3, v_1, v_2 and v_3 .

$$\text{Displacement } \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Since the CST element has got two degrees of freedom at each node (u, v), the total degrees of freedom is 6. Hence it has 6 generalized coordinates.

Let, $u = a_1 + a_2 x + a_3 y$ ... (3.1)

$v = a_4 + a_5 x + a_6 y$ ... (3.2)

where, a_1, a_2, a_3, a_4, a_5 and a_6 are global or generalized co-ordinates.

$$\Rightarrow \quad u_1 = a_1 + a_2 x_1 + a_3 y_1$$

$$u_2 = a_1 + a_2 x_2 + a_3 y_2$$

$$u_3 = a_1 + a_2 x_3 + a_3 y_3$$

Write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{... (3.3)}$$

$$\text{Let } D = \begin{bmatrix} + & - & + \\ 1 & x_1 & y_1 \\ - & + & - \\ 1 & x_2 & y_2 \\ + & - & + \\ 1 & x_3 & y_3 \end{bmatrix}$$

We know, $D^{-1} = \dfrac{C^T}{|D|}$ ... (3.4)

Find the co-factors of matrix D.

$$c_{11} = + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2)$$

$$c_{12} = -\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} = -(y_3 - y_2) = y_2 - y_3$$

$$c_{13} = +\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = (x_3 - x_2)$$

$$c_{21} = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -(x_1 y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3$$

$$c_{22} = +\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$c_{23} = -\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} = -(x_3 - x_1) = x_1 - x_3$$

$$c_{31} = +\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$c_{32} = -\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} = -(y_2 - y_1) = y_1 - y_2$$

$$c_{33} = +\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$\Rightarrow \qquad C = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{vmatrix}$$

$$\Rightarrow \qquad C^T = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{vmatrix} \qquad \ldots (3.5)$$

We know that, $D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$|D| = 1\,(x_2 y_3 - x_3 y_2) - x_1\,(y_3 - y_2) + y_1\,(x_3 - x_2) \qquad \ldots (3.6)$$

Substitute C^T and D values in equation (3.4),

$$\Rightarrow \quad D^{-1} = \frac{1}{(x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)} \times \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

Substitute D^{-1} value in equation (3.3),

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{(x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)} \times \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (3.7)$$

The area of the triangle can be expressed as a function of the x, y co-ordinates of the nodes 1, 2 and 3.

$$\Rightarrow \quad A = \frac{1}{2} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$|A| = \frac{1}{2} \left[1 (x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)\right]$$

$$\Rightarrow \quad 2A = (x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2) \quad \ldots (3.8)$$

Substitute 2A values in equation (3.7),

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (3.9)$$

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \cdots (3.10)$$

where, $p_1 = x_2 y_3 - x_3 y_2$; $\quad p_2 = x_3 y_1 - x_1 y_3$; $\quad p_3 = x_1 y_2 - x_2 y_1$

$q_1 = y_2 - y_3$; $\quad q_2 = y_3 - y_1$; $\quad q_3 = y_1 - y_2$

$r_1 = x_3 - x_2$; $\quad r_2 = x_1 - x_3$; $\quad r_3 = x_2 - x_1$

From equation (3.1), we know that,

$$u = a_1 + a_2 x + a_3 y$$

We can write this equation in matrix form.

$$u = [1 \ x \ y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$ value, from equation no.(3.10)

$$\Rightarrow u = [1 \ x \ y] \times \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [1 \ x \ y] \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [p_1 + q_1 x + r_1 y \quad p_2 + q_2 x + r_2 y \quad p_3 + q_3 x + r_3 y] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$[\because (1 \times 3) \times (3 \times 3) = 1 \times 3]$

$$u = \begin{bmatrix} \frac{p_1 + q_1 x + r_1 y}{2A} & \frac{p_2 + q_2 x + r_2 y}{2A} & \frac{p_3 + q_3 x + r_3 y}{2A} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The above equation is in the form of

$$u = [N_1 \ N_2 \ N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \cdots (3.11)$$

$$\text{Similarly, } v = [N_1 \;\; N_2 \;\; N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \quad \ldots (3.12)$$

where, Shape function, $N_1 = \dfrac{p_1 + q_1 x + r_1 y}{2A}$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

Assembling the equations (3.11) and (3.12) in matrix form,

$$\text{Displacement function, } u = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \ldots (3.13)$$

3.6. STRAIN-DISPLACEMENT MATRIX [B] FOR CST ELEMENT

Displacement function for CST element is given by,

$$u(x, y) = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

or we can write

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

The strain components for CST element are,

Normal strain, $e_x = \dfrac{\partial u}{\partial x}$

$$\Rightarrow \boxed{e_x = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3}$$

Normal strain, $e_y = \frac{\partial v}{\partial y}$

$$\Rightarrow \boxed{e_y = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3}$$

Shear strain, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$$\Rightarrow \boxed{\gamma_{xy} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3}$$

Arranging the strains e_x, e_y and γ_{xy} in matrix form,

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{... (3.1}$$

From equation (3.11) or (3.12), we know that,

Shape function, $N_1 = \frac{p_1 + q_1 x + r_1 y}{2A}$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

Partial differentiation,

$$\frac{\partial N_1}{\partial x} = \frac{q_1}{2A}; \qquad \frac{\partial N_2}{\partial x} = \frac{q_2}{2A}; \qquad \frac{\partial N_3}{\partial x} = \frac{q_3}{2A}$$

$$\frac{\partial N_1}{\partial y} = \frac{r_1}{2A}; \qquad \frac{\partial N_2}{\partial y} = \frac{r_2}{2A}; \qquad \frac{\partial N_3}{\partial y} = \frac{r_3}{2A}$$

Substitute $\frac{\partial N_1}{\partial x}$, $\frac{\partial N_2}{\partial x}$, $\frac{\partial N_3}{\partial x}$, $\frac{\partial N_1}{\partial y}$, $\frac{\partial N_2}{\partial y}$ and $\frac{\partial N_3}{\partial y}$ values in equation (3.14),

$$(3.14) \Rightarrow \quad \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

The above equation is in the form of $\{ e \} = [B]\{ u \}$

where, $[B]$ = Strain-Displacement matrix $= \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$ ... (3.15)

where,
$$\left.\begin{aligned} q_1 &= y_2 - y_3 \\ q_2 &= y_3 - y_1 \\ q_3 &= y_1 - y_2 \\ r_1 &= x_3 - x_2 \\ r_2 &= x_1 - x_3 \\ r_3 &= x_2 - x_1 \end{aligned}\right\} \quad \text{[From equation no. (3.10)]}$$

3.7. STRESS-STRAIN RELATIONSHIP MATRIX OR CONSTITUTIVE MATRIX [D] FOR TWO DIMENSIONAL ELEMENT

Consider a three dimensional body which is subjected to the stresses σ_x, σ_y and σ_z independently as shown in Fig.3.11.

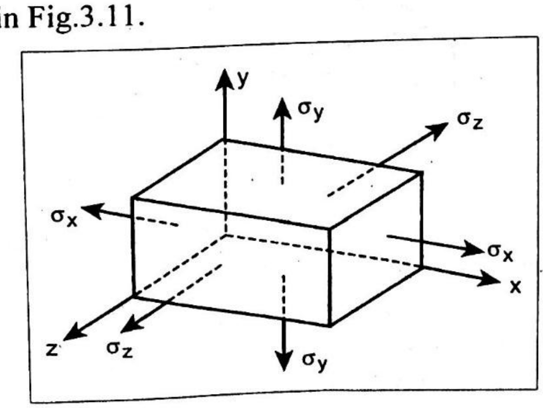


Fig. 3.11.

Hooke's law states that when a material is loaded within its elastic limit, the stress is directly proportional to the strain.

i.e.,

$$\text{Stress} \propto \text{Strain}$$

$$\sigma \propto e$$

$$\sigma = E\,e$$

$$e = \frac{\sigma}{E}$$

where. e = Strain

σ = Stress, N/mm^2

E = Young's modulus or Modulus of elasticity, N/mm^2

The stress in the x direction produces a positive strain in x direction as shown in Fig.3.12.

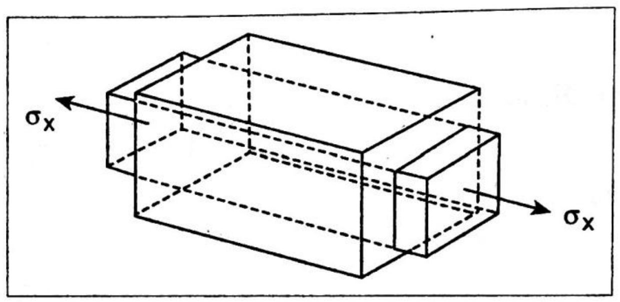


Fig. 3.12.

$$\text{Strain } e'_x = \frac{\sigma_x}{E} \quad \dots (3.16)$$

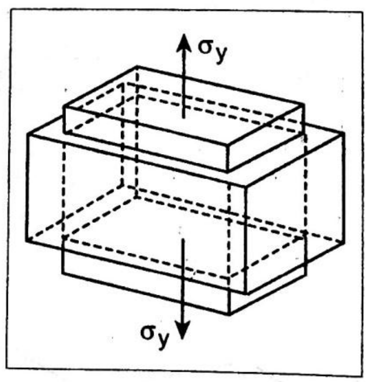


Fig. 3.13.

Fig.3.13 shows the positive stress in the y direction produces a negative strain in the x direction as a result of Poisson's effect which is given by,

$$-e''_x = \frac{\nu\sigma_y}{E}$$

$$\Rightarrow e''_x = \frac{-\nu\sigma_y}{E} \quad \dots (3.17)$$

where, $\nu \rightarrow$ Poisson's ratio.

Similarly, the stress in the z direction produces a negative strain in the x direction as shown in Fig.3.14.

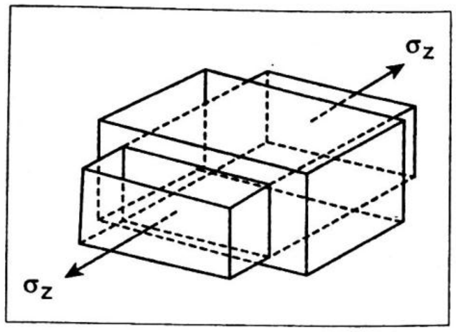


Fig. 3.14.

$$-e_x''' = \frac{\nu \sigma_z}{E}$$

$$e_x''' = \frac{-\nu \sigma_z}{E} \quad \dots (3.18)$$

By applying superposition principle to the equations (3.16), (3.17) and (3.18), we get

$$e_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \quad \dots (3.19)$$

This is a strain equation in x direction.

Similarly, the strains in y and z directions can be calculated as follows:

Strain in y direction, $e_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu \sigma_z}{E}$ ... (3.20)

Strain in z direction, $e_z = -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$ ... (3.21)

Solving equations (3.19), (3.20) and (3.21) for the normal stresses (σ_x, σ_y and σ_z), we get

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} [e_x(1-\nu) + \nu e_y + \nu e_z] \quad \dots (3.22)$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)} [\nu e_x + (1-\nu) e_y + \nu e_z] \quad \dots (3.23)$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)} [\nu e_x + \nu e_y + (1-\nu) e_z] \quad \dots (3.24)$$

The shear stress and shear strain relationship is given by,

$$\tau = G\gamma$$

where, $\tau \rightarrow$ Shear stress

$\gamma \rightarrow$ Shear strain

$G \rightarrow$ Modulus of rigidity or Shear modulus

The expressions for the three different sets of shear stresses are,

$$\tau_{xy} = G\gamma_{xy}$$
$$\tau_{yz} = G\gamma_{yz}$$
$$\tau_{zx} = G\gamma_{zx}$$

where, $G \rightarrow$ Modulus of rigidity $= \dfrac{E}{2(1+\nu)}$

$$\Rightarrow \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\tau_{xy} = \frac{E}{(1+\nu)(1-2\nu)} \times \left(\frac{1-2\nu}{2}\right) \times \gamma_{xy} \qquad \ldots (3.25)$$

$$\Rightarrow \tau_{yz} = \frac{E}{2(1+\nu)} \times \gamma_{yz}$$

$$\tau_{yz} = \frac{E}{(1+\nu)(1-2\nu)} \times \left(\frac{1-2\nu}{2}\right) \times \gamma_{yz} \qquad \ldots (3.26)$$

$$\Rightarrow \tau_{zx} = \frac{E}{2(1+\nu)} \times \gamma_{zx}$$

$$\tau_{zx} = \frac{E}{(1+\nu)(1-2\nu)} \times \left(\frac{1-2\nu}{2}\right) \times \gamma_{zx} \qquad \ldots (3.27)$$

Assembling the equations (3.22), (3.23), (3.24), (3.25), (3.26) and (3.27) in matrix form,

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \ldots (3.28)$$

The above equation is in the form of

$$\{\sigma\} = [D]\{e\}$$

The above equation (3.28) gives a three dimensional stress-strain relationship for an isotropic body,

where, [D] is a stress-strain relationship matrix or constitutive matrix.

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad \ldots (3.29)$$

where, E = Modulus of Elasticity or Young's modulus

ν = Poisson's ratio

3.7.1. Plane Stress

For two dimensional plane stress problems, the normal stress, σ_z and shear stresses τ_{xz}, τ_{yz} are zero.

i.e., $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

The shear strains γ_{xz}, γ_{yz} are zero, but $e_z \neq 0$.

i.e., $\gamma_{xz} = \gamma_{yz} = 0$

Substitute $\sigma_z = 0$ in equation (3.19),

$$\Rightarrow e_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \quad \ldots (3.30)$$

Substitute $\sigma_z = 0$ in equation (3.20),

$$\Rightarrow e_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} \quad \ldots (3.31)$$

Solving equation (3.30) and (3.31),

$$e_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E}$$

$$\nu e_y = -\nu^2 \frac{\sigma_x}{E} + \nu \frac{\sigma_y}{E} \quad \text{[Equation (3.31)} \times \nu]$$

$$e_x + \nu e_y = \frac{\sigma_x}{E} - \frac{\nu^2 \sigma_x}{E}$$

$$\Rightarrow e_x + \nu e_y = \frac{\sigma_x}{E}(1-\nu^2)$$

$$\Rightarrow \boxed{\sigma_x = \frac{E}{(1-\nu^2)}(e_x + \nu e_y)} \quad \ldots (3.32)$$

Solving equation (3.30) and (3.31),

$$\nu \; e_x = \nu \; \frac{\sigma_x}{E} - \nu^2 \; \frac{\sigma_y}{E} \qquad \text{[Equation (3.30)} \times \nu]$$

$$e_y = -\nu \; \frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$\nu \; e_x + e_y = -\nu^2 \; \frac{\sigma_y}{E} + \frac{\sigma_y}{E}$$

$$\nu \; e_x + e_y = \frac{\sigma_y}{E} \; (1 - \nu^2)$$

$$\Rightarrow \boxed{\sigma_y = \frac{E}{(1 - \nu^2)} \; (\nu \; e_x + e_y)} \qquad \dots (3.33)$$

We know that, Shear stress, $\tau_{xy} = G \; \gamma_{xy}$

where, $G \rightarrow$ Modulus of rigidity $= \dfrac{E}{2 \, (1 + \nu)}$

$\gamma_{xy} \rightarrow$ Shear strain.

$\nu \rightarrow$ Poisson's ratio

$$\Rightarrow \tau_{xy} = \frac{E}{2 \, (1 + \nu)} \; \gamma_{xy}$$

$$\Rightarrow \tau_{xy} = \frac{E}{(1 + \nu)(1 - \nu)} \times \frac{(1 - \nu)}{2} \times \gamma_{xy}$$

$$\boxed{\tau_{xy} = \frac{E}{(1 - \nu^2)} \times \left(\frac{1 - \nu}{2} \right) \times \gamma_{xy}} \qquad \dots (3.34)$$

Arranging equations (3.32), (3.33) and (3.34) in matrix form,

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} \qquad \dots (3.35)$$

The above equation is in the form of

$$\{ \sigma \} = [D] \{ e \}$$

The equation (3.35) gives the two dimensional stress-strain relationship for plane stress problems.

where, $[D]$ = Stress-Strain relationship matrix or Constitutive matrix

$$\Rightarrow [D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \ldots (3.36)$$

where, E = Modulus of Elasticity or Young's modulus

ν = Poisson's ratio

3.7.2. Plane Strain

For plane strain, we assume the following strains to be zero.

$$e_z = \gamma_{xz} = \gamma_{yz} = 0$$

The shear stresses $\tau_{xz} = \tau_{yz} = 0$, but $\sigma_z \neq 0$.

From equation (3.28), we know that,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

In the above equation, $e_z = 0$. So, delete third row and third column of [D] matrix. $\gamma_{yz} = 0$, so, delete fifth row and fifth column of [D] matrix. $\gamma_{xz} = 0$, hence, delete sixth row and sixth column of [D] matrix. The final reduced equation is,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} \quad \ldots (3.37)$$

The above equation is in the form of

$$\{\sigma\} = [D]\{e\}$$

The equation (3.37) gives the two dimensional stress-strain relationship for plane strain problems.

where, [D] = Stress-strain relationship or Constitutive matrix

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad \ldots (3.38)$$

where, E → Young's modulus.

ν → Poisson's ratio.

3.8. STIFFNESS MATRIX EQUATION FOR TWO DIMENSIONAL ELEMENT (CST ELEMENT)

We know that,

$$\text{Stiffness matrix, } [K] = \int_v [B]^T [D] [B]\, dV \qquad \text{[From Chapter 2]}$$

$$[K] = [B]^T [D] [B]\, V$$

$$\Rightarrow [K] = [B]^T [D] [B]\, A\, t \qquad [\because V = A \times t]$$

$$\boxed{\text{Stiffness matrix, } [K] = [B]^T [D] [B]\, A\, t} \qquad \ldots (3.39)$$

where, A → Area of the triangular element $= \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

t → Thickness of element

[B] = Strain – Displacement matrix

$$[B] = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \qquad \text{[From equation no.(3.15)]}$$

where, $q_1 = y_2 - y_3$; $q_2 = y_3 - y_1$; $q_3 = y_1 - y_2$

$r_1 = x_3 - x_2$; $r_2 = x_1 - x_3$; $r_3 = x_2 - x_1$

[D] = Stress-Strain relationship matrix

For plane stress problems,

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \text{[From equation no. (3.36)]}$$

For plane strain problems,

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

[From equation no. (3.38)]

where, E = Young's modulus or Modulus of elasticity

ν = Poisson's ratio

3.9. TEMPERATURE EFFECTS

When the distribution of the change in temperature (ΔT) is known, the strain due to this change in temperature can be considered as an initial strain e_0. For plane stress problem, the initial strain e_0 can be given by,

$$\{e_0\} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix} \quad \ldots (3.40)$$

where, $\alpha \rightarrow$ Coefficient of thermal expansion

$\Delta T \rightarrow$ Change in temperature

For plane strain problem,

$$\text{Initial strain } \{e_0\} = (1+\nu) \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix} \quad \ldots (3.41)$$

where, $\nu \rightarrow$ Poisson's ratio

The stresses and strains are related by,

$$\sigma = D(e - e_0) \qquad [\because e = B\,u]$$

$$\boxed{\sigma = D(B\,u - e_0)}$$

where, $[D] \rightarrow$ Stress-strain relationship matrix.

$[B] \rightarrow$ Strain-displacement relationship matrix

$\{u\} \rightarrow$ Displacement

The element temperature force can be represented by,

$$\{\theta\} \text{ or } \{F\} = [B]^T [D] \{e_0\}\, t\, A \quad \ldots (3.42)$$

where, $A \rightarrow$ Area

$t \rightarrow$ Thickness

3.10. GALERKIN APPROACH

A virtual displacement field is given by,

$$\phi = \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix}$$

and corresponding virtual strain,

$$e(\phi) = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}$$

Galerkin's variational form for the two dimensional elasticity problem is given by,

$$\int_A \sigma^T e(\phi)\, t\, dA - \left[\int_A \phi^T F\, t\, dA + \int_L \phi^T T\, t\, dl + \Sigma\, \phi^T P \right] = 0 \qquad \dots (3.43)$$

We know that, Stress, $\sigma = D\,e$

Substitute, σ value in the above equation,

$$\Rightarrow \int e^T D\, e(\phi)\, t\, dA - \left(\int \phi^T F\, t\, dA + \int \phi^T T\, t\, dl + \Sigma\, \phi^T P \right) = 0 \qquad \dots (3.44)$$

In the above equation, the first term represents the internal virtual work.

i.e., $\int e^T D\, e(\phi)\, t\, dA$ = Internal virtual work ... (3.45)

We know that, $\phi = N\psi$

$e(\phi) = B\psi$

where, ψ = Arbitrary nodal displacement of element.

Substitute $e(\phi)$ value in equation (3.45),

$$\Rightarrow \int e^T D\, e(\phi)\, t\, dA = \int e^T D B \psi\, t\, dA = e^T D B \psi\, t \int dA$$

$$= e^T D B \psi\, t\, A \qquad [\because \text{Strain, } e = uB]$$

$$= u^T \times B^T D B\, t\, A \times \psi$$

$$\Rightarrow \int_e e^T D\, e(\phi)\, t\, dA = u^T \times K_e \times \psi$$

where, K_e is element stiffness matrix, which is given by,

$$K_e = B^T D B\, t\, A$$

$$\text{Stiffness matrix } [K]_e = [B]^T [D] [B]\, t\, A \qquad \dots (3.46)$$

$$\text{Force vector is given by, } \{F\}_e = [K]_e \{u\} \qquad \dots (3.47)$$

3.11. POISSON EQUATION AND LAPLACE EQUATION

Consider a two dimensional plane region have volume V, boundary S in the xy plane and unit thickness in the z direction.

For steady state conditions, the governing heat conduction equation is given by,

$$\frac{\partial}{\partial x}(k_x \phi_x) + \frac{\partial}{\partial y}(k_y \phi_y) + Q = 0 \quad \dots (3.48)$$

If $k_x = k_y = k$, a constant, equation (3.48) becomes the Poisson equation.

$$i.e., \quad \boxed{k(\nabla^2 \phi) + Q = 0} \quad \dots (3.49)$$

[Poisson equation]

If $k_x = k_y = k$ = constant and Q = 0, equation (3.48) becomes Laplace's equation.

$$i.e., \quad k(\nabla^2 \phi) = 0$$

$$\Rightarrow \boxed{\nabla^2 \phi = 0} \quad \dots (3.50)$$

[Laplace equation]

3.12. LINEAR STRAIN TRIANGULAR (LST) ELEMENT

A six noded triangular element is known as Linear Strain Triangular (LST) element which is shown in Fig.3.15. It has twelve unknown displacement degrees of freedom. The displacement functions of the element are quadratic instead of linear as in the CST.

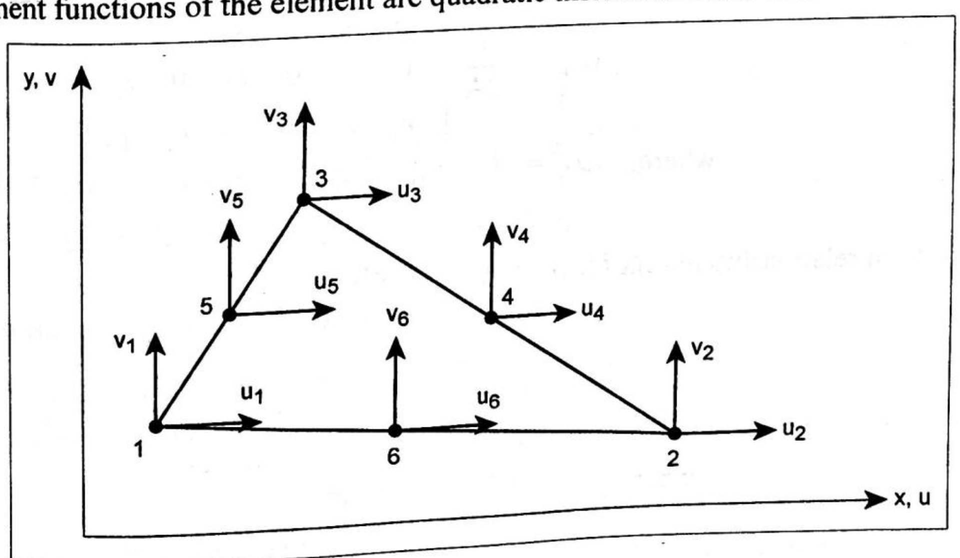


Fig. 3.15. Linear strain triangular element

The procedures for development of the stiffness matrix equations for the LST element follow the same steps as those used for the CST element. But the number of equations used for developing shift matrix equation is 12 instead of 6. It is a tedious process to solve those equations. Hence, we will use a computer to solve many of the mathematical equations.

LST element is preferred than the CST element for plane stress applications when relatively small numbers of nodes are used. LST element is not preferred when large numbers of nodes are used since the cost of formation of the element stiffnesses, equation bandwidth are high compared to CST element. Computer modelling for large number of nodes are also difficult for LST element.

3.13. FORMULAE USED

1. For constant strain triangle (CST) element,

$$\text{Shape function, } N_1 + N_2 + N_3 = 1$$

$$\text{Co-ordinate, } x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$\text{Co-ordinate, } y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

or

$$\text{Co-ordinate, } x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3$$

$$\text{Co-ordinate, } y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3$$

2. Area of the triangular element, $A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

3. Strain–Displacement matrix for CST element is,

$$[B] = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

where, $q_1 = y_2 - y_3$; $q_2 = y_3 - y_1$; $q_3 = y_1 - y_2$

$r_1 = x_3 - x_2$; $r_2 = x_1 - x_3$; $r_3 = x_2 - x_1$

4. Stress-Strain relationship matrix for plane stress problem,

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

where, $\nu \rightarrow$ Poisson's ratio

$E \rightarrow$ Young's modulus

5. Stress-Strain relationship matrix for plane strain problem,

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

6. Element stiffness matrix for CST element,

$$[K] = [B]^T [D] [B] A t$$

7. Element stress, $\{\sigma\} = [D] [B] \{u\}$

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D][B] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where, σ_x, σ_y $\rightarrow$ Normal stresses

τ_{xy} $\rightarrow$ Shear stress

u, v $\rightarrow$ Nodal displacements

8. Maximum normal stress, $\sigma_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Minimum normal stress, $\sigma_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

9. Principal angle, $\tan 2\theta_P = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}$

10. Element strain, $\{e\} = [B] \{u\} = [B] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$

11. Temperature effects

Initial strain, $\{e_0\}$ (For plane stress problems) $= \begin{Bmatrix} \alpha\,\Delta T \\ \alpha\,\Delta T \\ 0 \end{Bmatrix}$

Initial strain, $\{e_0\}$ (For plane strain problems) $= (1+\nu) \begin{Bmatrix} \alpha\,\Delta T \\ \alpha\,\Delta T \\ 0 \end{Bmatrix}$

where, α $\rightarrow$ Coefficient of thermal expansion

ν $\rightarrow$ Poisson's ratio

12. Element temperature force, $\{F\} = [B]^T [D] \{e_0\} t A$

Multiple Choice Questions and Answers:

S. No	Questions	Opt1	Opt2	Opt3	Opt4	Answer
1	The example for non structural problems in FEA is	Heat transfer analysis	Fluid flow analysis	Magneto statics	All of these	All of these
2	(-)kδT/δx is equation for	Conduction	Convection	Radiation	Heat flux	Conduction
3	the non linear heat transfer problems occurs due to	Conduction	Convection	Radiation	Heat flux	Radiation
4	The temperature at each point is independent of time and along one coordinate axis is called as	Steady state one dimensional analysis	Unsteady state one dimensional analysis	Single point state one dimensional analysis	None of these	Steady state one dimensional analysis
5	The values of the boundary conditions normally specified in the heat transfer analysis	Temperature	Heatflux	Convective heat transfer coefficent	All of these	All of these
6	The elements normaly used to solve the two dimensional heat transfer analysis is	Triangular	Linear	Rectangle	Triangular & rectangle	Triangular & rectangle
7	The example for one dimensional heat transfer problem is	Fins	Chimmney	Heatexchangers	All of these	Fins
8	In unsteady state problem which elemet matrix is addionaly added	Conduction matrix	Convection matrix	Capacitance matrix	Resistance matrix	Capacitance matrix
9	The order of polynomial used for one dimesional heattransfer analysis when the domain is discritisized by the triangular element	Linear	Quadratic	Algebric	Linear & quadratic	Linear & quadratic
10	Which one of the following is called Dirichlet boundary condition	Temperature	Heatflux	Thermal conductivity	All of these	Temperature
11	Which one of the following is called neuman boundary condition	Temperature	Heatflux	Thermal conductivity	All of these	Heatflux
12	The effect of heat flux wil appears on	Stiffness matrix	Load vector	Both a&b	None of these	Load vector
13	The effect of convective heat transfer effect wil appears on	Stiffness matrix	Load vector	Stiffness matrix & load vector	None of these	Stiffness matrix & load vector
14	The unsteady state problem is also called as-------------------problem	Transient	Propagation	Transient & propagation	None of these	Transient & propagation

15	The temperature of the hole body changes uniformly with time.This is the assumation for	Steady state one dimensional analysis	Two dimensional analysis	Lumped heat capacity system	Dynamic analysis	Lumped heat capacity system
16	The ratio between the conduction resistance with in the body to the convective resistance in the surface of the hot body is known as	Rayleigh number	Thermal conductivity	Reynolds number	Biot number	Biot number
17	The governing equation for fluid dynamics problem are	Fourier equation	FFT equation	Reynolds transport equation	Navier stroke equation	Navier stroke equation
18	Due to------------------------- variation the fluids are differensiated as compressible fluid and incompressible fluids	Viscosity	Density	Thermal conductivity	All of these	Density
19	Which method of describing the motion of fluid is suitable for fluiid flow analysis	Lagrangian	Euleraian	Euler	Laplace	Euleraian
20	The particle maintains the same orientation every where along the stream line with rotation is called-------------- flow	Rotational flow	Streamline flow	Laminar flow	Rayleigh flow	Rotational flow
21	Potential function will gives the ------------------------ value in the perticular direction	Velocity	Viscosity	Pressure	Density	Velocity
22	The fluid flow around the airfoil is example for -----------------flow	Inviscid	Incompressible	Inviscid& incompressible	None of these	Inviscid& incompressible
23	The choice of velocity potential and stream function formulation in FEA analysis is depends on	Reynolds number	Viscosity	Boundary conditions	Initial conditions	Boundary conditions
24	(-)kdØ/dx is equation based on	Fourier equation	Darcy's law	Continuity equation	None of these	Darcy's law
25	The flow with small velocity,inertia terms are negligible when comparing the viscous effects than the flow is called as	Viscous flow	Intertia flow	Laminar flow	Storkes flow	Storkes flow
26	Most of the fluid flow problems the unkown value tobe find is	Viscosity	Density	Temperature	Discharge	Discharge
27	The resistance to due to the pipe friction is reflected in --------------------------------	Stiffness matrix	Lumped mass matrix	Discharge vector	Stress-strain relationship	Stiffness matrix
28	Irrorational flow of ideal fluid is called as	Viscous flow	Intertia flow	Laminar flow	Pontential flow	Pontential flow
29	The stream function and potential function is governed by	Darcy's equation	Laplace equation	Continuity equation	None of these	Laplace equation
30	The flow perpencular to the streamline	Have maximum velocity	Zero velocity	Minimum pressure	None of these	Zero velocity

31	The reformulation of constraind problem by finding the stationary points of unconstrained functional is called--------------method	Penalty	Lagrangian	Potential energy	Rayleigh	Lagrangian
32	Interia force+damping force+elastic force=Applied force. This is governing equation for	Free vibration	Equation of motion	Rotation	None of these	Equation of motion
33	The damping methods are	Rayleigh damping	Modal damping	Rayleigh damping & Modal damping	None of these	Rayleigh damping & Modal damping
34	Multiple loding is possibe in -----------------------method of transient response analysis	Direct method	Modal damping	Discrete damping	Continous damping	Modal damping
35	Single loding is possibe in -----------------------method of transient response analysis	Direct method	Modal damping	Discrete damping	Continous damping	Direct method
36	Direct method of trasient response analysis allows-----------------type of loading	Shock	Periodic	Constant	Heavy	Shock
37	In the mass matrix which property of the material is constant	Viscosity	Density	Pressure	None of these	Density
38	The order of mass matrix for simple struss element	2*2	3*3	4*4	8*8	4*4
39	The order of mass matrix for simple bar element	2*2	3*3	4*4	8*8	2*2
40	The practical engineers use -----------------------------mass matrix	Consistent mass matrix	Lumped mass matrix	Stiffness	None of these	Lumped mass matrix
41	The accurate results are given by	Consistent mass matrix	Lumped mass matrix	Stiffness matrix	None of these	Consistent mass matrix
42	The eigen value will gives the ---------------------------value	Displacement	Frequency of vibration	Mass	Force	Frequency of vibration
43	The eigen value-eigen vector will gives by the -------------------------- method	Characterstic polynomial	Vector ietration	Tansformation	All of these	All of these

UNIT IV

AXISYMMETRIC CONTINUUM

4.3.1. Introduction

In previous chapters, we have been concerned with one dimensional elements and two dimensional elements. In this chapter, we consider a special two dimensional element called the axisymmetric element.

Many three dimensional problems in engineering exhibit symmetry about an axis of rotation. Such types of problems are known as axisymmetric problems. These problems can be solved by using two dimensional finite elements. These elements are most conveniently described in cylindrical (r, θ, z) co-ordinates. The required conditions for a problem to be axisymmetric are as follows:

1. The problem domain must be symmetric about the axis of revolution, which is conventionally taken as the z-axis.
2. All boundary conditions must be symmetric about the axis of revolution.
3. All loading conditions must be symmetric about the axis of revolution.

An axisymmetric solid is generated by revolving a plane figure about an axis in the plane.

Finite elements for axisymmetric solids are pictured as triangular element or quadrilateral element as shown in Fig.4.3 and 4.4. But these shapes are actually cross-sections of ring elements.

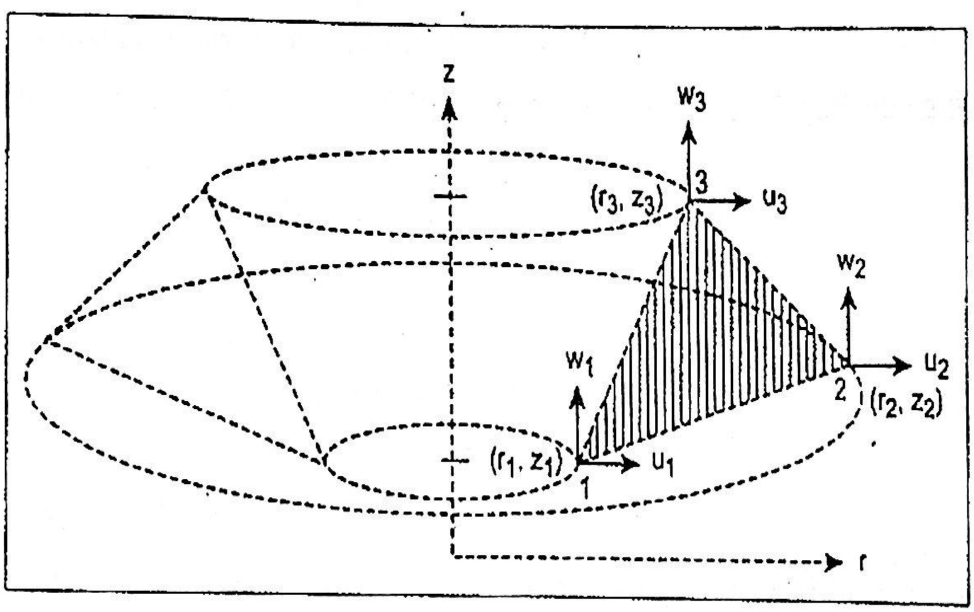


Fig. 4.3. Three-node axisymmetric triangular element

We begin with the development of the stiffness matrix for the simplest axisymmetric element, the triangular torus, whose vertical cross-section is a plane triangle.

4.3.2. Axisymmetric Formulation

Consider a typical axisymmetric triangular element with nodes 1, 2 and 3 as shown in Fig.4.5.

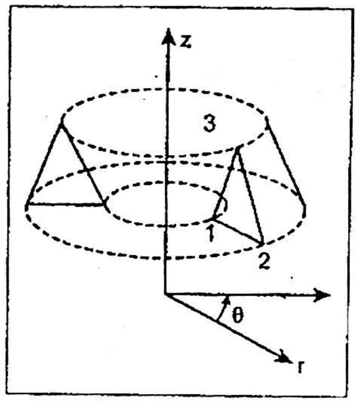


Fig. 4.5. Typical axisymmetric element

In two dimensional problems, the displacements and distributed body force values are indicated by x–y plane. But in case of axisymmetric problems, these values are indicated by r–z plane as shown in Fig.4.5.

For two dimensional problem, the displacement vector u is given by,

$$u(x, y) = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

where, u and v are the x and y components of u respectively.

In case of axisymmetric problems, the displacement vector u is given by,

$$u(r, z) = \begin{Bmatrix} u \\ w \end{Bmatrix}$$

where, u and w are r and z components of u respectively.

The stresses and strains for two dimensional element are given by,

$$\text{Stress, } \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\text{Strain, } \{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

In case of axisymmetric element, stresses and strains are given by.

$$\text{Stress, } \{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

where, $\sigma_r \rightarrow$ Radial stress
$\sigma_z \rightarrow$ Longitudinal stress
$\sigma_\theta \rightarrow$ Circumferential stress
$\tau_{rz} \rightarrow$ Shear stress

$$\text{Strain, } \{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

where, $e_r \rightarrow$ Radial strain
$e_z \rightarrow$ Longitudinal strain
$e_\theta \rightarrow$ Circumferential strain
$\gamma_{rz} \rightarrow$ Shear strain

For two dimensional problem, body force is given by,

$$F = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

In case of axisymmetric problem, $F = \begin{Bmatrix} F_r \\ F_z \end{Bmatrix}$

4.3.3. Derivation of Shape Function for Axisymmetric Element (Triangular Element)

Consider an axisymmetric triangular element with nodes 1, 2 and 3 as shown in Fig.4.3 Let the nodal displacements be u_1 w_1, u_2 w_2, and u_3 w_3.

$$\text{Displacement, } \{u\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

Since the triangular element has two degrees of freedom at each node, it has 6 generalized co-ordinates.

Displacement functions, $u = a_1 + a_2 r + a_3 z$ ... (4.14)

$w = a_4 + a_5 r + a_6 z$ ... (4.15)

where, a_1, a_2, a_3, a_4, a_5 and a_6 are global or generalized co-ordinates.

Let

$$u_1 = a_1 + a_2 r_1 + a_3 z_1$$
$$u_2 = a_1 + a_2 r_2 + a_3 z_2$$
$$u_3 = a_1 + a_2 r_3 + a_3 z_3$$

Write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{... (4.16)}$$

Let, $D = \begin{bmatrix} \overset{+}{1} & \overset{-}{r_1} & \overset{+}{z_1} \\ \overset{-}{1} & \overset{+}{r_2} & \overset{-}{z_2} \\ \overset{+}{1} & \overset{-}{r_3} & \overset{+}{z_3} \end{bmatrix}$

$$D^{-1} = \frac{C^T}{|D|} \quad \text{....(4.17)}$$

Find the co-factors of matrix D.

$$C_{11} = + \begin{vmatrix} r_2 & z_2 \\ r_3 & z_3 \end{vmatrix} = (r_2 z_3 - r_3 z_2)$$

$$C_{12} = - \begin{vmatrix} 1 & z_2 \\ 1 & z_3 \end{vmatrix} = -(z_3 - z_2) = (z_2 - z_3)$$

$$C_{13} = + \begin{vmatrix} 1 & r_2 \\ 1 & r_3 \end{vmatrix} = +(r_3 - r_2)$$

$$C_{21} = - \begin{vmatrix} r_1 & z_1 \\ r_3 & z_3 \end{vmatrix} = -(r_1 z_3 - r_3 z_1) = r_3 z_1 - r_1 z_3.$$

Since the triangular element has two degrees of freedom at each node, it has 6 generalized co-ordinates.

Displacement functions,
$$u = a_1 + a_2 r + a_3 z \quad \ldots (4.14)$$
$$w = a_4 + a_5 r + a_6 z \quad \ldots (4.15)$$

where, a_1, a_2, a_3, a_4, a_5 and a_6 are global or generalized co-ordinates.

Let
$$u_1 = a_1 + a_2 r_1 + a_3 z_1$$
$$u_2 = a_1 + a_2 r_2 + a_3 z_2$$
$$u_3 = a_1 + a_2 r_3 + a_3 z_3$$

Write the above equations in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (4.16)$$

$$\text{Let, } D = \begin{bmatrix} \overset{+}{1} & \overset{-}{r_1} & \overset{+}{z_1} \\ \overset{-}{1} & \overset{+}{r_2} & \overset{-}{z_2} \\ \overset{+}{1} & \overset{-}{r_3} & \overset{+}{z_3} \end{bmatrix}$$

$$D^{-1} = \frac{C^T}{|D|} \quad \ldots (4.17)$$

Find the co-factors of matrix D.

$$C_{11} = + \begin{vmatrix} r_2 & z_2 \\ r_3 & z_3 \end{vmatrix} = (r_2 z_3 - r_3 z_2)$$

$$C_{12} = - \begin{vmatrix} 1 & z_2 \\ 1 & z_3 \end{vmatrix} = -(z_3 - z_2) = (z_2 - z_3)$$

$$C_{13} = + \begin{vmatrix} 1 & r_2 \\ 1 & r_3 \end{vmatrix} = +(r_3 - r_2)$$

$$C_{21} = - \begin{vmatrix} r_1 & z_1 \\ r_3 & z_3 \end{vmatrix} = -(r_1 z_3 - r_3 z_1) = r_3 z_1 - r_1 z_3.$$

$$C_{22} = + \begin{vmatrix} 1 & z_1 \\ 1 & z_3 \end{vmatrix} = (z_3 - z_1)$$

$$C_{23} = - \begin{vmatrix} 1 & r_1 \\ 1 & r_3 \end{vmatrix} = -(r_3 - r_1) = (r_1 - r_3)$$

$$C_{31} = + \begin{vmatrix} r_1 & z_1 \\ r_2 & z_2 \end{vmatrix} = r_1 z_2 - r_2 z_1$$

$$C_{32} = - \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix} = -(z_2 - z_1) = (z_1 - z_2)$$

$$C_{33} = + \begin{vmatrix} 1 & r_1 \\ 1 & r_2 \end{vmatrix} = (r_2 - r_1)$$

$$\Rightarrow C = \begin{vmatrix} (r_2 z_3 - r_3 z_2) & (z_2 - z_3) & (r_3 - r_2) \\ (r_3 z_1 - r_1 z_3) & (z_3 - z_1) & (r_1 - r_3) \\ (r_1 z_2 - r_2 z_1) & (z_1 - z_2) & (r_2 - r_1) \end{vmatrix}$$

$$C^T = \begin{vmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{vmatrix} \quad \ldots (4.18)$$

We know that,

$$D = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}$$

$$|D| = \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$|D| = 1\,(r_2 z_3 - r_3 z_2) - r_1\,(z_3 - z_2) + z_1\,(r_3 - r_2) \quad \ldots (4.19)$$

Substitute C^T and D values in equation (4.17),

$$(4.17) \Rightarrow \quad D^{-1} = \frac{1}{(r_2 z_3 - r_3 z_2) - r_1\,(z_3 - z_2) + z_1\,(r_3 - r_2)} \times \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix}$$

Substitute D^{-1} value in equation (4.16),

$$(4.16) \Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{(r_2 z_3 - r_3 z_2) - r_1 (z_3 - z_2) + z_1 (r_3 - r_2)} \times$$

$$\begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (4.20)$$

The area of the triangle can be expressed as a function of the r, z co-ordinates of the nodes 1, 2 and 3.

$$A = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$A = \frac{1}{2} [1 (r_2 z_3 - r_3 z_2) - r_1 (z_3 - z_2) + z_1 (r_3 - r_2)]$$

$$\Rightarrow \boxed{2A = (r_2 z_3 - r_3 z_2) - r_1 (z_3 - z_2) + z_1 (r_3 - r_2)} \quad \ldots (4.21)$$

Substitute equation (4.21) in equation (4.20),

$$(4.20) \Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \ldots (4.22)$$

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \ldots (4.23)$$

where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\alpha_2 = r_3 z_1 - r_1 z_3$; $\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$; $\beta_2 = z_3 - z_1$; $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$; $\gamma_2 = r_1 - r_3$; $\gamma_3 = r_2 - r_1$

From equation (4.14), we know that,

$$u = a_1 + a_2 r + a_3 z$$

We can write this equation in matrix form,

$$u = [1 \ r \ z] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= [1 \ r \ z] \times \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$ [From equation (4.23)]

$$= \frac{1}{2A} [1 \ r \ z] \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} [\alpha_1 + \beta_1 r + \gamma_1 z \quad \alpha_2 + \beta_2 r + \gamma_2 z \quad \alpha_3 + \beta_3 r + \gamma_3 z] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

[*Note:* $(1 \times 3) \times (3 \times 3) = (1 \times 3)$]

$$\Rightarrow \quad u = \left[\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A} \quad \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A} \quad \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A} \right] \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The above equation is in the form of

$$u = [N_1 \ N_2 \ N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \ldots (4.24)$$

Similarly,

$$w = [N_1 \ N_2 \ N_3] \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} \qquad \ldots (4.25)$$

where, Shape function, $N_1 = \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}$

$$N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

We can write equations (4.24) and (4.25) as follows:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \quad \ldots (4.26)$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3 \quad \ldots (4.27)$$

Assembling the equations (4.26) and (4.27) in matrix form,

Displacement function,

$$u(r,z) = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \ldots (4.28)$$

4.4. Strain-Displacement Matrix [B] for Axisymmetric Triangular Element

Displacement function for axisymmetric triangular element is given by,

$$\text{Displacement function, } u(r,z) = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

or

We can write,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \quad \ldots (4.29)$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3 \quad \ldots (4.30)$$

The strain components are,

$$\text{Radial strain, } e_r = \frac{\partial u}{\partial r} = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3$$

$$\Rightarrow \boxed{e_r = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3} \quad \ldots (4.31)$$

$$\text{Circumferential strain, } e_\theta = \frac{u}{r}$$

$$\boxed{e_\theta = \frac{N_1}{r} u_1 + \frac{N_2}{r} u_2 + \frac{N_3}{r} u_3} \quad \ldots (4.32)$$

Longitudinal strain, $e_z = \frac{\partial w}{\partial z}$

$$\Rightarrow \boxed{e_z = \frac{\partial N_1}{\partial z} w_1 + \frac{\partial N_2}{\partial z} w_2 + \frac{\partial N_3}{\partial z} w_3} \quad \ldots (4.33)$$

Shear strain, $\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$

$$\boxed{\gamma_{rz} = \frac{\partial N_1}{\partial z} u_1 + \frac{\partial N_2}{\partial z} u_2 + \frac{\partial N_3}{\partial z} u_3 + \frac{\partial N_1}{\partial r} w_1 + \frac{\partial N_2}{\partial r} w_2 + \frac{\partial N_3}{\partial r} w_3} \quad \ldots (4.34)$$

Arranging equations (4.31), (4.32), (4.33) and (4.34) in matrix form,

$$\Rightarrow \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ 0 & \frac{\partial N_1}{dz} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \ldots (4.35)$$

From equation (4.24) or (4.25), we know that,

Shape function, $N_1 = \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}$

$$N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

Partial differentiation $\Rightarrow \frac{\partial N_1}{\partial r} = \frac{\beta_1}{2A}$

$$\frac{\partial N_2}{\partial r} = \frac{\beta_2}{2A}$$

$$\frac{\partial N_3}{\partial r} = \frac{\beta_3}{2A}$$

$$\frac{N_1}{r} = \frac{1}{2A}\left(\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}\right)$$

$$\frac{N_2}{r} = \frac{1}{2A}\left(\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}\right)$$

$$\frac{N_3}{r} = \frac{1}{2A}\left(\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}\right)$$

$$\frac{\partial N_1}{\partial z} = \frac{\gamma_1}{2A}$$

$$\frac{\partial N_2}{\partial z} = \frac{\gamma_2}{2A}$$

$$\frac{\partial N_3}{\partial z} = \frac{\gamma_3}{2A}$$

Substitute $\frac{\partial N_1}{\partial r}$, $\frac{\partial N_2}{\partial r}$, $\frac{\partial N_3}{\partial r}$, $\frac{N_1}{r}$, $\frac{N_2}{r}$, $\frac{N_3}{r}$, $\frac{\partial N_1}{\partial z}$, $\frac{\partial N_2}{\partial z}$ and $\frac{\partial N_3}{\partial z}$ values in equation (4.35).

$$(4.35) \Rightarrow \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

The above equation is in the form of,

$$\{ e \} = [B] \{ u \}$$

where,

$[B]$ = Strain-Displacement matrix

$$= \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad \ldots (4.36)$$

where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\quad \alpha_2 = r_3 z_1 - r_1 z_3$; $\quad \alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$; $\quad \beta_2 = z_3 - z_1$; $\quad \beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$; $\quad \gamma_2 = r_1 - r_3$; $\quad \gamma_3 = r_2 - r_1$

[From equation no. (4.23)]

4.3.5. Stress-Strain Relationship Matrix [D] for Axisymmetric Triangular Element

By using Hooke's law, we derived the following normal stresses equations. [Refer Chapter 3].

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[e_x(1-\nu)+\nu\, e_y+\nu\, e_z]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[\nu\, e_x+(1-\nu)\, e_y+\nu\, e_z]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\, e_x+\nu\, e_y+(1-\nu)\, e_z]$$

$$\tau_{xz} = \frac{E}{(1+\nu)(1-2\nu)} \times \left(\frac{1-2\nu}{2}\right) \times \gamma_{xz}$$

Substitute $x = r$ and $y = \theta$ in the above equations,

$\Rightarrow$ Radial stress, $$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}[e_r(1-\nu)+\nu\, e_\theta+\nu\, e_z] \quad \ldots (4.37)$$

Circumferential stress, $$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}[\nu\, e_r+(1-\nu)\, e_\theta+\nu\, e_z] \quad \ldots (4.38)$$

Longitudinal stress, $$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\, e_r+\nu\, e_\theta+(1-\nu)\, e_z] \quad \ldots (4.39)$$

Shear stress, $$\tau_{rz} = \frac{E}{(1+\nu)(1-2\nu)} \times \left(\frac{1-2\nu}{2}\right) \times \gamma_{rz} \quad \ldots (4.40)$$

Arranging the above equations, (4.37), (4.38), (4.39) and (4.40) in matrix form,

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} \quad \ldots (4.41)$$

The above equation is in the form of,

$$\{\sigma\} = [D]\{e\}$$

where, $[D]$ = Stress-Strain relationship matrix

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad \ldots (4.42)$$

4.3.6. Assemblage of the Element Stiffness Matrix [K]

We know that,

$$\text{Stiffness matrix, } [K] = \int_v [B]^T [D] [B]\, dV = [B]^T [D] [B] \int_v dV$$

$$= [B]^T [D] [B]\, V$$

$$\boxed{\text{Stiffness matrix, } [K] = 2\pi r A\, [B]^T [D] [B]} \quad \ldots (4.43)$$

$[\because V = 2\pi r A]$

where,

$$\text{Co-ordinate, } r = \frac{r_1 + r_2 + r_3}{3}$$

A = Area of the triangular element $= \frac{1}{2}(b \times h)$

[B] = Strain-Displacement matrix

$$= \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

where, $\alpha_1 = r_2 z_3 - r_3 z_2$; $\alpha_2 = r_3 z_1 - r_1 z_3$; $\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$; $\beta_2 = z_3 - z_1$; $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$; $\gamma_2 = r_1 - r_3$; $\gamma_3 = r_2 - r_1$

and $\quad$ Co-ordinate $z = \frac{z_1 + z_2 + z_3}{3}$

[D] = Stress-Strain relationship matrix

$$= \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

where, E → Young's modulus

ν → Poisson's ratio

4.3.7. Temperature Effects

When the free expansion is prevented in a axisymmetric element, the change in temperature causes stresses in the element.

Let ΔT be the rise in temperature and α be the coefficient of thermal expansion. The thermal force vector due to rise in temperature is given by,

$$\{F\}_t = [B]^T [D] \{e\}_t \times 2\pi r A \qquad \dots (4.44)$$

For axisymmetric triangular element,

$$\{F\}_t = \begin{Bmatrix} F_1 u \\ F_1 w \\ F_2 u \\ F_2 w \\ F_3 u \\ F_3 w \end{Bmatrix}$$

$$\text{Strain } \{e\}_t = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \\ \alpha \Delta T \end{Bmatrix}$$

4.3.8. Galerkin Approach

Virtual displacement field is given by,

$$\phi = \begin{Bmatrix} \phi_r \\ \phi_z \end{Bmatrix}$$

Virtual strain is given by,
$$e(\phi) = \begin{Bmatrix} \dfrac{\partial \phi_r}{\partial r} \\ \dfrac{\phi_r}{r} \\ \dfrac{\partial \phi_z}{\partial z} \\ \dfrac{\partial \phi_r}{\partial z} + \dfrac{\partial \phi_z}{\partial r} \end{Bmatrix}$$

For axisymmetric problems, Galerkin's variational form is given by,

$$2\pi \int_A \sigma^T e(\phi)\, r\, dA - \left[2\pi \int_A \phi^T F\, r\, dA + 2\pi \int_L \phi^T T\, r\, dl + \Sigma \phi_i^T P_i \right] = 0 \qquad \dots (4.45)$$

In the above equation, the first term representing the internal virtual work, Internal virtual work $= 2\pi \int_A \sigma^T \, e(\phi) \, r \, dA$

$= 2\pi \int_A \{u\}^T [B]^T [D]^T \times e(\phi) \times r \, dA$

$[\because \text{Stress } \{\sigma\} = \{u\} [B] [D]]$

$= 2\pi \int_A \{u\}^T [B]^T [D] \times e(\phi) \times r \, dA \qquad [\because [D]^T = [D]]$

$= 2\pi \int_A \{u\}^T [B]^T [D] \times [B] \times \{\psi\} \times r \, dA$

$[\because e(\phi) = [B] [\psi]$ where, ψ = Arbitrary nodal displacement$]$

$= 2\pi \times \{u\}^T [B]^T [D] [B] \{\psi\} r \int dA$

$= 2\pi \times \{u\}^T [B]^T [D] [B] \{\psi\} r \times A$

Internal virtual work, $W_e = \{u\}^T \times 2\pi r A \, [B]^T [D] [B] \times \{\psi\}$ ... (4.46)

The above equation is in the form of,

$W_e = \{u\}^T \times [K] \times \{\psi\}$ ... (4.47)

Comparing equations (4.46) and (4.47),

Stiffness matrix, $[K] = 2\pi r A \, [B]^T [D] [B]$

Multiple Choice Questions and Answers

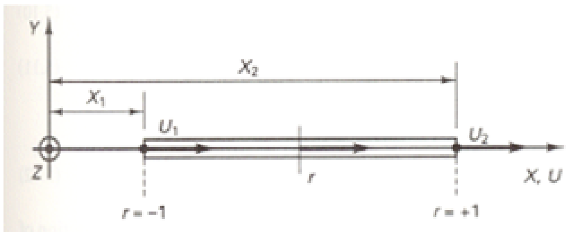
S.No	Questions	Opt1	Opt2	Opt3	Opt4	Answer
1	Large value of eigen value is evaluated by-------------------------------------method	power	Inverse	Subspace	All of these	Power
2	Subspace iteration method is suitable for ------------------------- type of problem	small scale	Large scale	Meadium scale	None of these	Large scale
3	Transformation of marix in dynamic analysis is done by	QR method	Jacobian method	Modal method	QR method & jacobian method	QR method & jacobian method
4	Which one of the following is the reduction method for DOF in dynamic analysis	guyan method	Jacobian method	Modal method	Both a&b	Guyan method
5	The governing equation for bar element in time dependent analysis is	Wave equation	Poissons equation	Euler equation	None of these	Wave equation
6	The free vibration decay expoentially to zero of the effect of	Small eigen value	Large eigen value	Damping	None of these	Damping
7	Damping is linearly propotional to	Nodal displacements	Nodal velocities	Nodal eigen value	Shape function	Nodal velocities
8	When there is a reduction in amplitude over every cycle of vibration, then the body is said to have	Free vibration	Forced vibration	Damped vibration	None of these	Damped vibration
9	Longitudinal vibrations are said to occur when the particles of a body moves	Perpendicular to its axis	Parallel to its axis	In a circle about its axis	Both perpendicular to its axis & parallel to its axis	Parallel to its axis
10	When a body is subjected to transverse vibrations, the stress induced in a body will be	Shear stress	Tensile stress	Compressive stress	Longitudinal stress	Tensile stress
11	The factor which affects the critical speed of a shaft is	Diameter of the disc	Span of the shaft	Eccentricity	All of these	All of these
12	The equation of motion for a vibrating system with viscous damping is $d^2x/dt^2+c/m(dx/dt)+sx/m=0$ if the roots of this equation are real, then the system will be	Over damped	Under damped	criticaly damped	Both over damped & under damped	Over damped
13	The ratio of the maximum displacement of the forced vibration to the deflection due to the static force is known as	Damping factor	Damping coefficient	Logarithmic decrement	Magnification factor	Magnification factor
14	In vibration isolation system, if W/Wn is less than √δ then for all values of the damping factor the transmissibility will be	Less than unity	Equal to unity	Greater than unity	Zero	greater than unity

15	At a modal point in a shaft the amplitude of torsional vibration is	Zero	Minimum	Maximum	None of these	Zero
16	A shaft carrying two rotors as its ends will have	No node	One node	Two node	Three nodes	Two node
17	If the periodic motion continuous after the cause of original distance is removed then the body is said to be under	Natural vibration	Forced vibration	Damped vibration	Undamped vibration	Natural vibration
18	When the body vibrates under the influence of external force then the body is said to be under	Natural vibration	Forced vibration	Damped vibration	Undamped vibration	Forced vibration
19	If no energy is lost or dissipated in friction or other resisting force during vibration such vibration is known as	Natural vibration	Forced vibration	Damped vibration	Undamped vibration	Undamped vibration
20	A motion which repeats itself after equal interval of time is kown as	Periodic motion	Time period	Cycle	None of these	Periodic motion
21	The resistance to the motion of a vibrating body is	Time period	Damping	Amplitude	None of these	Damping
22	When the particles of the shaft or disc moves parallel to the axis shaft then the vibration known as	Longitudinal	Transverse	Torsional	None of these	Longitudinal
23	If the load applied on the assembly is shared by two or more springs, then the springs are in	Parallel	Series	Both parallel & series	None of these	Parallel
24	Damping force per unit velocity is known as	Damping coefficient	Actual damping co efficient	Critical damping coefficient	None of these	Damping coefficient
25	The natural frequency of the free longitudinal vibration can be found by	Newton's method	Energy method	Ray leigh's method	All of these	All of these
26	A shaft fixed at one end and carrying a rotar and the free end is known as	Single rotar system	Two rotar system	Three rotar system	Four rotar system	Single rotar system
27	When the shaft is twisted & untwisted alternatively the torisional shear stresses are induced, this twisting & un twisting moments are known as	Longitudinal vibrations	Transverse vibrations	Torisional vibration	Damped vibration	Torisional vibration
28	The point at which the amplitude of vibration is maximum is known as	Node	No node	Anti node	Two node	Anti node
29	The volume of fluid flowing across the section per second is	Discharge	Velocity	Accelaration	All of these	Discharge
30	Continuity equation is	Q1=Q2	a1v1 = a2v2	q1/q2	Q1=Q2 & a1v1 = a2v2	Q1=Q2 & a1v1 = a2v2
31	It is a product of mass density and gravitational acceleration	Mass density	Specific weight	Specific volume	Specific gravity	Specific weight
32	When fluid mechanics is applied to fluid at rest is	Fluid statics	Fluid dynamics	Both(a)&(b)	None of these	Fluid statics
33	The volume of fluid flowing across the section per second is	Density	Velocity	Accelaration	None of the above	None of the above
34	Newton’s second law	F=m/a	m=f x a	F=m.a	None of these	F=m.a

35	Unit for power	Newton	Watt	Joule	None of these	Watt
36	One pascal is	N/m^2	N/mm^2	KN/m^2	KN/mm^2	N/m^2
37	An ideal fluid is defined as the fluid which	Is ompressible	Is compressible	Is incompressible and non-viscous (inviscid)	Has negligible surface tension.	Is incompressible and non-viscous (inviscid)
38	Newton's law of viscosity states that	Shear stress is directly proportional to the velocity	Shear stress is directly proportional to velocity gradient	Shear stress is directly proportional to shear strain	Shear stress is directly proportional to the viscosity.	Shear stress is directly proportional to velocity gradient
39	A Newtonian fluid is defined as the fluid which	Is incompressible and non-viscous	Obeys Newton's law of viscosity	Is highly viscous	Is compressible and non-viscous	Obeys Newton's Law of viscosity
40	Kinematic viscosity is defined as equal to	Dynamic viscosity x density	Dynamic velocity/density	Dynamic viscosity x pressure	Pressure x density	dynamic velocity/density
41	The expression weight per unit volume is	Mass density	Specific weight	Relative density	None of these	Specific weight
42	The symbol for viscosity	ρ	μ	ψ	Φ	μ
43	The expression inverse of mass density is	Mass density	Specific gravity	Specific volume	None of these	Specific volume
44	It is a product of mass density and gravitational acceleration	Mass density	Specific weight	Specific volume	Specific gravity	Specific gravity

UNIT V

Isoparametric Derivation of Bar(Truss) Element Stiffness Matrix



X, Y : global coord

r : natural coord

$$X=\sum_{i=1}^{2}h_iX_i,\ \text{where}\ h_1=\frac{1}{2}(1-r),\ \ h_2=\frac{1}{2}(1+r)\ \ \text{:shape functions}$$

$$\text{Also}\ U=\sum_{i=1}^{2}h_iU_i=\frac{1}{2}(1-r)U_1+\frac{1}{2}(1+r)U_2$$

$$\varepsilon=\frac{dU}{dX}=\frac{\frac{dU}{dr}}{\frac{dX}{dr}}=\frac{\frac{U_2-U_1}{2}}{\frac{X_2-X_1}{2}}=\frac{U_2-U_1}{L}=\frac{1}{L}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}U_1\\U_2\end{bmatrix}=B\hat{u}$$

$$K=\int_V B^T CBdV=\int_l B^T CBAdX=\int_l B^T CBA|J|dr=\frac{AE}{L}\int_{-1}^{1}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\frac{L}{2}dr=\frac{AE}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$(\text{Jacobian},\ J=\frac{dX}{dr})$$

Advantages of Isoparametric FE over General Coord FE

(1) Easily handle curved boundaries

(2) Easily construct element disp func since $\begin{cases}x=h_ix_i,\ y=h_iy_i,\ z=h_iz_i\\u=h_iu_i,\ v=h_iv_i,\ w=h_iw_i\end{cases}$

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\rightarrow\begin{matrix}\varepsilon_{xx}=\frac{\partial u}{\partial x},\ \varepsilon_{yy}=\frac{\partial v}{\partial y},\ \varepsilon_{zz}=\frac{\partial w}{\partial z}\\ \gamma_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\ \cdots\end{matrix}\rightarrow\begin{Bmatrix}\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}\\\frac{\partial}{\partial z}\end{Bmatrix}$$

$$x=f_1(r,s,t)\quad y=f_2(r,s,t)\quad z=f_3(r,s,t)$$

$$r=f_4(x,y,z)\quad s=f_5(x,y,z)\quad t=f_6(x,y,z)$$

Chain rule

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial}{\partial t}\frac{\partial t}{\partial x}\Leftrightarrow\frac{\partial}{\partial r}=\frac{\partial}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial}{\partial z}\frac{\partial z}{\partial r}$$

Implicit Function Theorem

Let $r_i \in R^n$ denotes a set of coord and $x_i \in R^n$ as Cartesian coord.

Let $x_i = f_i(\tilde{r}),\ i = 1, 2, \cdots, n$ where $f_i(\tilde{r})$ are differentiable.

It can be solved for $\tilde{r}$ as a differentiable func of $\tilde{x}$ if Jacobian matrix of $f_i(\tilde{r})$ is non-singular, i.e., $\left|\left[\frac{\partial f_i(\tilde{r})}{\partial r_i}\right]\right| \neq 0$

Theorem : If $f_i(\tilde{r})$ is continuously differentiable and Jacobian matrix is non-singular, $\iint_\Omega F(\tilde{x})d\tilde{x} = \iint_{\Omega'} F(f_i(\tilde{r}))|J|d\tilde{r}$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \rightarrow \quad \frac{\partial}{\partial \tilde{r}} = J\frac{\partial}{\partial \tilde{x}} \quad \rightarrow \quad \frac{\partial}{\partial \tilde{x}} = J^{-1}\frac{\partial}{\partial \tilde{r}}$$

J^{-1}, in general, exits except for the elements such as distorted or folded.

$$K = \int_V B^T CBdV = \iiint B^T CB|J|drdsdt = \iiint F\,drdsdt = \sum \alpha_{ijk} F_{ijk}$$

where F_{ijk} is the matrix F evaluated at the points (r_i, s_j, t_k)

α_{ijk} weighting factors

The more integration points the more accurate.

Once we know H, then M, R_B, R_s *and* R_I are the same as in chap.4.

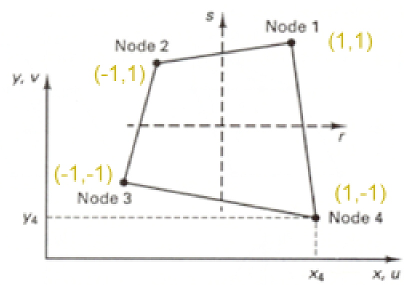
Example 5.5

$$x = \frac{1}{4}(1+r)(1+s)x_1 + \frac{1}{4}(1-r)(1+s)x_2 + \frac{1}{4}(1-r)(1-s)x_3 + \frac{1}{4}(1+r)(1-s)x_4$$

$$y = \frac{1}{4}(1+r)(1+s)y_1 + \frac{1}{4}(1-r)(1+s)y_2 + \frac{1}{4}(1-r)(1-s)y_3 + \frac{1}{4}(1+r)(1-s)y_4$$

$$u = \frac{1}{4}(1+r)(1+s)u_1 + \frac{1}{4}(1-r)(1+s)u_2 + \frac{1}{4}(1-r)(1-s)u_3 + \frac{1}{4}(1+r)(1-s)u_4$$

$$v = \frac{1}{4}(1+r)(1+s)v_1 + \frac{1}{4}(1-r)(1+s)v_2 + \frac{1}{4}(1-r)(1-s)v_3 + \frac{1}{4}(1+r)(1-s)v_4$$



$$\varepsilon^T = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{zz}]$$

$$where\ \varepsilon_{xx} = \frac{\partial u}{\partial x};\ \varepsilon_{yy} = \frac{\partial v}{\partial y};\ \gamma_{zz} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

$$or \qquad \frac{\partial}{\partial r} = J \frac{\partial}{\partial x}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix}$$

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1+s)x_1 - \frac{1}{4}(1+s)x_2 - \frac{1}{4}(1-s)x_3 + \frac{1}{4}(1-s)x_4$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}(1+r)x_1 + \frac{1}{4}(1-r)x_2 - \frac{1}{4}(1-r)x_3 - \frac{1}{4}(1+r)x_4$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}(1+s)y_1 - \frac{1}{4}(1+s)y_2 - \frac{1}{4}(1-s)y_3 + \frac{1}{4}(1-s)y_4$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(1+r)y_1 + \frac{1}{4}(1-r)y_2 - \frac{1}{4}(1-r)y_3 - \frac{1}{4}(1+r)y_4$$

$$\frac{\partial u}{\partial r} = \frac{1}{4}(1+s)u_1 - \frac{1}{4}(1+s)u_2 - \frac{1}{4}(1-s)u_3 + \frac{1}{4}(1-s)u_4$$

$$\frac{\partial u}{\partial s} = \frac{1}{4}(1+r)u_1 + \frac{1}{4}(1-r)u_2 - \frac{1}{4}(1-r)u_3 - \frac{1}{4}(1+r)u_4$$

$$\frac{\partial v}{\partial r} = \frac{1}{4}(1+s)v_1 - \frac{1}{4}(1+s)v_2 - \frac{1}{4}(1-s)v_3 + \frac{1}{4}(1-s)v_4$$

$$\frac{\partial v}{\partial s} = \frac{1}{4}(1+r)v_1 + \frac{1}{4}(1-r)v_2 - \frac{1}{4}(1-r)v_3 - \frac{1}{4}(1+r)v_4$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}_{\substack{at\ r=r_i \\ s=s_j}} = \frac{1}{4} J_{ij}^{-1} \begin{bmatrix} 1+s_j & 0 & -(1+s_j) & 0 & -(1-s_j) & 0 & 1-s_j & 0 \\ 1+r_j & 0 & 1-r_j & 0 & -(1-r_j) & 0 & -(1+r_j) & 0 \end{bmatrix} \hat{u}$$

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}_{\substack{at\ r=r_i \\ s=s_j}} = \frac{1}{4} J_{ij}^{-1} \begin{bmatrix} 0 & 1+s_j & 0 & -(1+s_j) & 0 & -(1-s_j) & 0 & 1-s_j \\ 0 & 1+r_j & 0 & 1-r_j & 0 & -(1-r_j) & 0 & -(1+r_j) \end{bmatrix} \hat{u}$$

where $\hat{u}^T = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{bmatrix}$

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \frac{1}{4} J_{ij}^{-1} \begin{bmatrix} 1+s_j & 0 & -(1+s_j) & 0 & -(1-s_j) & 0 & 1-s_j & 0 \\ 0 & 1+r_j & 0 & 1-r_j & 0 & -(1-r_j) & 0 & -(1+r_j) \\ 1+r_j & 1+s_j & 1-r_j & -(1+s_j) & -(1-r_j) & -(1-s_j) & -(1+r_j) & 1-s_j \end{bmatrix} \hat{u}$$

$$F_{ij} = B_{ij}^T C B_{ij} \det J_{ij} \Rightarrow K = \sum_{i,j} t_{ij} \alpha_{ij} F_{ij}$$

For 2-D element, we need to compute $\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$

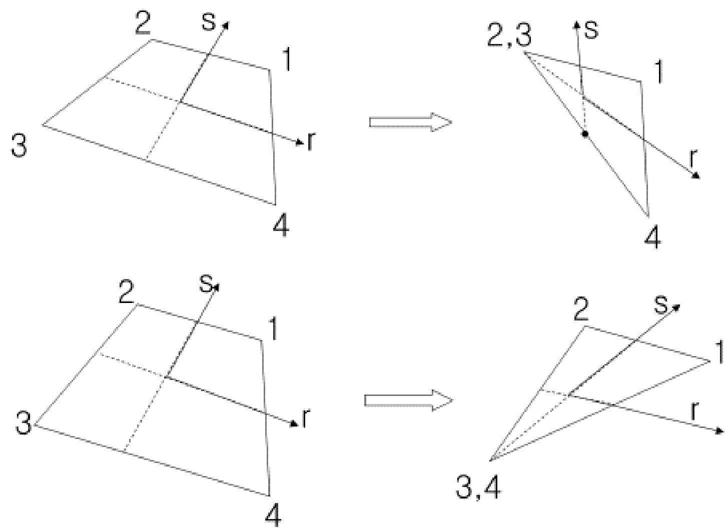
$$u = \sum h_i u_i, \qquad v = \sum h_i v_i$$

$$\frac{\partial u}{\partial x} = \sum \frac{\partial h_i}{\partial x} u_i, \qquad \frac{\partial u}{\partial y} = \sum \frac{\partial h_i}{\partial y} u_i, \qquad \frac{\partial v}{\partial x} = \sum \frac{\partial h_i}{\partial x} v_i, \qquad \frac{\partial v}{\partial y} = \sum \frac{\partial h_i}{\partial y} v_i$$

$$\tilde{\varepsilon} = B\tilde{u} \rightarrow \tilde{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 & \frac{\partial h_3}{\partial x} & 0 & \frac{\partial h_4}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} & 0 & \frac{\partial h_3}{\partial y} & 0 & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x} \end{bmatrix}$$

Triangular Elements

Degenerated triangle from quadrilateral



Example 5.15

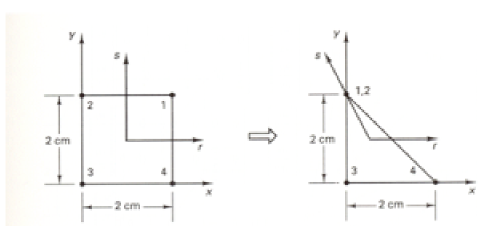
$$x=\frac{1}{4}(1+r)(1+s)x_1+\frac{1}{4}(1-r)(1+s)x_2+\frac{1}{4}(1-r)(1-s)x_3+\frac{1}{4}(1+r)(1-s)x_4$$

$$y=\frac{1}{4}(1+r)(1+s)y_1+\frac{1}{4}(1-r)(1+s)y_2+\frac{1}{4}(1-r)(1-s)y_3+\frac{1}{4}(1+r)(1-s)y_4$$

Using the conditions $x_1 = x_2$ *and* $y_1 = y_2$,

$$x=\frac{1}{2}(1+s)x_2+\frac{1}{4}(1-r)(1-s)x_3+\frac{1}{4}(1+r)(1-s)x_4$$

$$y=\frac{1}{2}(1+s)y_2+\frac{1}{4}(1-r)(1-s)y_3+\frac{1}{4}(1+r)(1-s)y_4$$



Example 5.15 (continued)

$$x = \frac{1}{2}(1+r)(1-s)$$

$$y = 1+s$$

$$\begin{aligned}\frac{\partial x}{\partial r} &= \frac{1}{2}(1-s) \quad \frac{\partial y}{\partial r} = 0 \\ \frac{\partial x}{\partial s} &= -\frac{1}{2}(1+r) \quad \frac{\partial y}{\partial s} = 1\end{aligned}; \qquad J = \frac{1}{2}\begin{bmatrix} (1-s) & 0 \\ -(1+r) & 2 \end{bmatrix}; \quad J^{-1} = \begin{bmatrix} \frac{2}{1-s} & 0 \\ \frac{1+r}{1-s} & 1 \end{bmatrix}$$

$$u = \frac{1}{2}(1+s)u_2 + \frac{1}{4}(1-r)(1-s)u_3 + \frac{1}{4}(1+r)(1-s)u_4$$

$$v = \frac{1}{2}(1+s)v_2 + \frac{1}{4}(1-r)(1-s)v_3 + \frac{1}{4}(1+r)(1-s)v_4$$

$$\frac{\partial u}{\partial r} = -\frac{1}{4}(1-s)u_3 + \frac{1}{4}(1-s)u_4; \quad \frac{\partial v}{\partial r} = -\frac{1}{4}(1-s)v_3 + \frac{1}{4}(1-s)v_4$$

$$\frac{\partial u}{\partial s} = \frac{1}{2}u_2 - \frac{1}{4}(1-r)u_3 - \frac{1}{4}(1+r)u_4; \quad \frac{\partial v}{\partial s} = \frac{1}{2}v_2 - \frac{1}{4}(1-r)v_3 - \frac{1}{4}(1+r)v_4$$

Example 5.15 (continued)

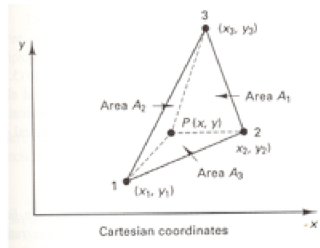
$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = J^{-1}\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} \Longrightarrow \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{2}{1-s} & 0 \\ \frac{1+r}{1-s} & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -\frac{1}{4}(1-s) & 0 & \frac{1}{4}(1-s) & 0 \\ \frac{1}{2} & 0 & -\frac{1}{4}(1-r) & 0 & -\frac{1}{4}(1+r) & 0 \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_2 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} u_2 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}$$

so we obtain

$$\varepsilon = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} u_2 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}$$

Triangular Elements by Area Coord.



$$\left.\begin{aligned} L_1 = \frac{A_1}{A},\ L_2 = \frac{A_2}{A},\ L_3 = \frac{A_3}{A} \quad &, L_1 + L_2 + L_3 = 1 \\ x = L_1 x_1 + L_2 x_2 + L_3 x_3 & \\ y = L_1 y_1 + L_2 y_2 + L_3 y_3 & \end{aligned}\right\} \rightarrow \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

Basic convergence requirements for Isoparameteric elements

Monotonic convergence { compatible : disp continuous with neighbor; complete : rigid body disp & constant strain }

Compatibility is satisfied if the elements have the same nodes on the common face and coords and disps on the common face are defined by the same interpolation functions

Completeness requires rigid body displacements and constant strain.

$$u = a_1 + b_1 x + c_1 y + d_1 z \qquad u_i = a_1 + b_1 x_i + c_1 y_i + d_1 z_i$$
$$v = a_2 + b_2 x + c_2 y + d_2 z \quad \Longrightarrow \quad v_i = a_2 + b_2 x_i + c_2 y_i + d_2 z_i$$
$$w = a_3 + b_3 x + c_3 y + d_3 z \qquad w_i = a_3 + b_3 x_i + c_3 y_i + d_3 z_i$$

Since $u = \sum h_i u_i \quad v = \sum h_i v_i \quad w = \sum h_i w_i$

$$u = a_1 \sum h_i + b_1 \sum h_i x_i + c_1 \sum h_i y_i + d_1 \sum h_i z_i$$
$$v = a_2 \sum h_i + b_2 \sum h_i x_i + c_2 \sum h_i y_i + d_2 \sum h_i z_i$$
$$w = a_3 \sum h_i + b_3 \sum h_i x_i + c_3 \sum h_i y_i + d_3 \sum h_i z_i$$

$$u = a_1 \sum h_i + b_1 x + c_1 y + d_1 z$$
$$v = a_2 \sum h_i + b_2 x + c_2 y + d_2 z \qquad => \sum h_i = 1 \ : Req't\ for\ completeness$$
$$w = a_3 \sum h_i + b_3 x + c_3 y + d_3 z$$

For general geometric shape, isoparametric elements always have the capability to represent the rigid body modes and constant strain stress. Therefore convergence is guaranteed.

If dimension of $\underline{r}$ and $\underline{x}$ are the same in $\frac{\partial}{\partial \underline{r}} = J \frac{\partial}{\partial \underline{x}}$ then J is the square matrix and can be inverted.

$$\frac{\partial}{\partial \underline{x}} = J^{-1} \frac{\partial}{\partial \underline{r}}$$

in this case, the element matrix correspond directly to the global disp.

If the order of global coordination system is higher than the other of natural coord system such as truss or plane element, transformation to the global coordinate should be included in the formulation.

Ex 5.22

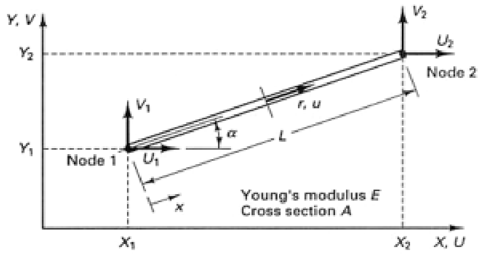
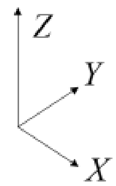
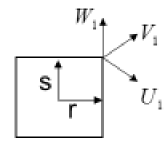
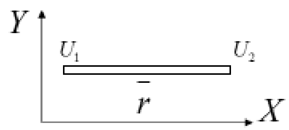


Figure E5.22 Truss element in global coordinate system



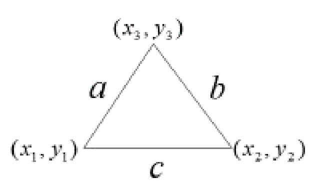




$$u = \begin{bmatrix} \frac{1}{2}(1-r) & \frac{1}{2}(1+r) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\Downarrow$$

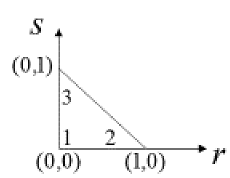
$$u = \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-r)U_1 + \frac{1}{2}(1+r)U_2 \\ \frac{1}{2}(1-r)V_1 + \frac{1}{2}(1+r)V_2 \end{bmatrix}$$



$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad : s = \frac{1}{2}(a+b+c)$$

$$A = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$u = \sum h_i u_i,\ v = \sum h_i v_i \Leftrightarrow x = \sum h_i x_i,\ y = \sum h_i y_i \ : h_i = L_i$$



$$h_1 = 1 - r - s$$

$$h_2 = r$$

$$h_3 = s$$

Ex 5.23 Shear Stress Energy

$$u = \int_A \frac{1}{2}\tau_a \gamma dA = \int_A \frac{1}{2G}\tau_a^2 dA \qquad \begin{pmatrix} \tau_a : \text{actual stress } (V/A_s) \\ V = \int_A \tau_a dA \end{pmatrix}$$

$$= \int_{A_s} \frac{1}{2G}\left(\frac{V}{A_s}\right)^2 dA_s \qquad A_s = kA$$

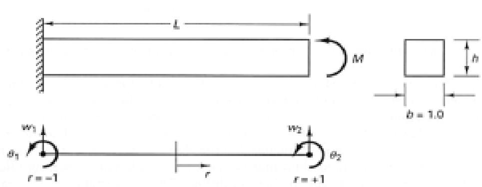
$$k = \frac{V^2}{A\int_A \tau_a^2 dA} \qquad \begin{pmatrix} \text{From Elemetary Beam Theory} \\ \tau_a = \frac{3}{2}\frac{V}{A}\left[\frac{(h/2)^2 - y^2}{(h/2)^2}\right] \Rightarrow k = \frac{5}{6} \end{pmatrix}$$

$$\Pi = \frac{EI}{2}\int_0^L \underbrace{\left(\frac{d\beta}{dx}\right)^2}_{\beta=\theta} dx + \frac{GAk}{2}\int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx - \int_0^L pwdx - \int_0^L m\beta dx$$

$$\tilde{\Pi} = \int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \underbrace{\frac{GAk}{EI}}_{\alpha}\int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx$$

$$(\alpha \to \infty \ \& \ h \to 0) \Rightarrow \text{zero shear deformation } \left(\frac{dw}{dx} = \beta\right)$$

Ex 5.25



$$\begin{cases} x = \frac{L}{2}(1+r) + x_1 \\ \beta = \frac{1+r}{2}\theta_2 \\ w = \frac{1+r}{2}w_2 \end{cases}$$

$$\gamma = \frac{dw}{dx} - \beta = \frac{dw/dr}{dx/dr} = \frac{w_2}{L} - \frac{1+r}{2}\theta_2$$

For applied moment

$$0 = \frac{w_2}{L} - \frac{1+r}{2}\theta_2$$

$$2\,\text{Gauss pts} \quad \begin{bmatrix} \frac{1}{L} & -\frac{1+1/\sqrt{3}}{2} \\ \frac{1}{L} & -\frac{1-1/\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to w_2 = \theta_2 = 0 \Rightarrow \begin{pmatrix} \text{shear locking} \\ \text{stiff behavior} \end{pmatrix}$$

1 Gauss pts (reduced integration)

$$\gamma = \frac{w_2}{2} - \frac{\theta_2}{2} \quad \text{(constant shear)}$$

Reissner & Mindlin Plate

$$u = -z\beta_x,\ v = -z\beta_y,\ w = w$$

$$\underline{\varepsilon} = B\underline{u}$$

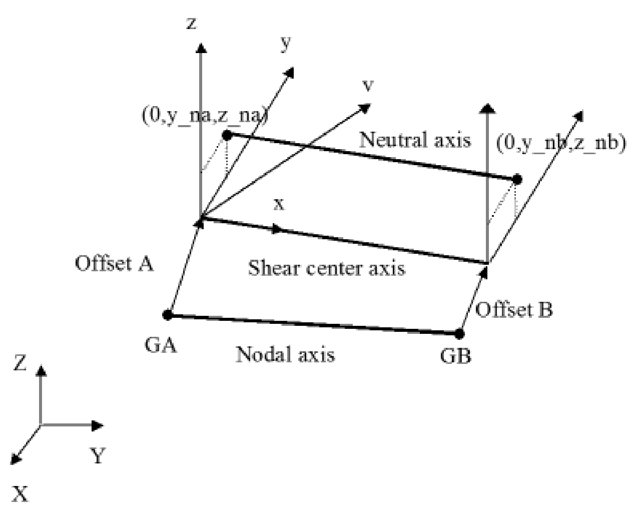
$$:\ \underline{\varepsilon} = \Big[\overbrace{\varepsilon_{xx}\ \ \varepsilon_{yy}\ \ \gamma_{xy}}^{bend}\ \ \ \overbrace{\gamma_{xz}\ \ \gamma_{yz}}^{transeverse\ \ shear}\Big]^T,\quad \underline{u} = \left[w\ \ \beta_x\ \ \beta_y\right]\quad B = \begin{bmatrix} 0 & -z\frac{\partial}{\partial x} & 0 \\ 0 & 0 & -z\frac{\partial}{\partial y} \\ 0 & z\frac{\partial}{\partial y} & z\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -1 & 0 \\ \frac{\partial}{\partial y} & 0 & -1 \end{bmatrix}\ :\ \underline{\tau} = C\underline{\varepsilon}$$

PVW

$$\iint_A \int_{-h/2}^{h/2} \left[\bar{\varepsilon}_{xx}\ \ \bar{\varepsilon}_{yy}\ \ \bar{\gamma}_{xy}\right] \begin{bmatrix} \tau_{xx} \\ -\tau_{yy} \\ \tau_{xy} \end{bmatrix} dzdA + k\int_A \int_{-h/2}^{h/2} \left[\bar{\gamma}_{xz}\ \ \bar{\gamma}_{yz}\right] \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} dzdA = \int_A \bar{w}pdA$$

$$\Pi = \frac{1}{2}\int_A \int_{-h/2}^{h/2} \left[\varepsilon_{xx}\ \ \varepsilon_{yy}\ \ \gamma_{xy}\right] C \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} dzdA + \frac{kG}{2}\int_A \int_{-h/2}^{h/2} \left[\gamma_{xz}\ \ \gamma_{yz}\right] \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} dzdA - \int_A \bar{w}pdA$$

General Cross-Section Isoparametric Beam with Offset



General Cross-Section Isoparametric Beam with Offset

$$a_{up}(\mathbf{z}_p,\bar{\mathbf{z}}_p)=\int_0^L [EI_{yp}\chi_{yp}(\mathbf{z}_p)\bar{\chi}^*_{yp}(\bar{\mathbf{z}}^*_p)+EI_{zp}\chi_{zp}(\mathbf{z}_p)\bar{\chi}^*_{zp}(\bar{\mathbf{z}}^*_p)$$

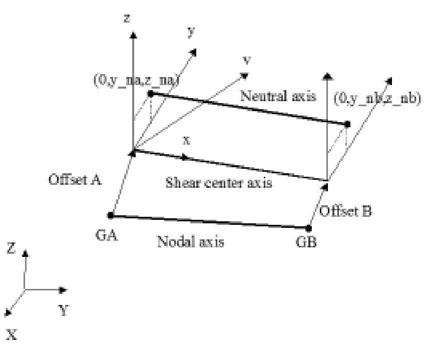
$$+GA^*_{yp}\gamma_{yp}(\mathbf{z}_p)\bar{\gamma}^*_{yp}(\bar{\mathbf{z}}^*_p)+GA^*_{zp}\gamma_{zp}(\mathbf{z}_p)\bar{\gamma}^*_{zp}(\bar{\mathbf{z}}^*_p)$$

$$+EA\mathbf{z}_{1,1p}\bar{\mathbf{z}}^*_{1,1p}+GJ\mathbf{z}_{4,1p}\bar{\mathbf{z}}^*_{4,1p}]dx$$

$$A^*_{zp}=\left(\frac{1}{k_{yp}A}+\frac{GL^2}{12EI_{zp}}\right)^{-1} \qquad A^*_{yp}=\left(\frac{1}{k_{zp}A}+\frac{GL^2}{12EI_{yp}}\right)^{-1}$$

$$\gamma_{yp}=\frac{\partial w_p}{\partial x}+\theta_{yp} \qquad \gamma_{zp}=\frac{\partial v_p}{\partial x}-\theta_{zp}$$

General Cross-Section Isoparametric Beam with Offset



$$\mathbf{T}=\mathbf{T}_2\mathbf{T}_3\mathbf{T}_1$$

$$=\begin{bmatrix}\mathbf{R}_{prin} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_{prin}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{T}_A\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}$$

$\mathbf{T}_1$: the global to local transformation

$\mathbf{T}_2$: the nonprincipal to principal transformation

$\mathbf{T}_3$: the translational transformation due to offset

$$\mathbf{R}_{prin}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_p & \sin\theta_p\\ 0 & -\sin\theta_p & \cos\theta_p\end{bmatrix}$$

General Cross-Section Isoparametric Beam with Offset

The translational matrix for displacement in calculating strain energy bilinear form

$$\mathbf{T}_A = \begin{bmatrix} 0 & \Delta\bar{z}_{A\_NA} & -\Delta\bar{y}_{A\_NA} \\ -\Delta\bar{z}_{A\_SC} & 0 & \Delta\bar{x}_{A\_NA} \\ \Delta\bar{y}_{A\_SC} & -\Delta\bar{x}_{A\_NA} & 0 \end{bmatrix}$$

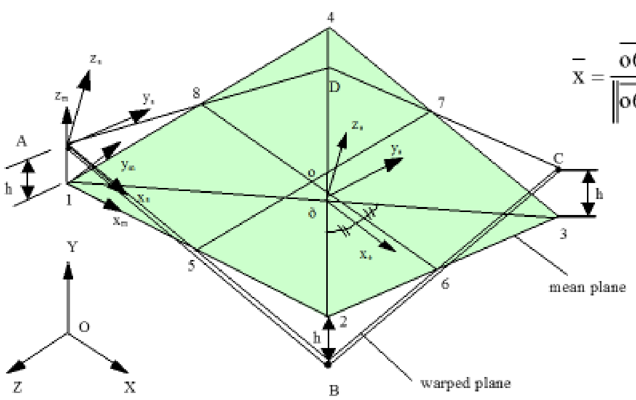
$$\mathbf{T}_B = \begin{bmatrix} 0 & \Delta\bar{z}_{B\_NA} & -\Delta\bar{y}_{B\_NA} \\ -\Delta\bar{z}_{B\_SC} & 0 & \Delta\bar{x}_{B\_NA} \\ \Delta\bar{y}_{B\_SC} & -\Delta\bar{x}_{B\_NA} & 0 \end{bmatrix}$$

The translational matrix for displacement in calculating kinetic energy bilinear form

$$\mathbf{T}_A = \begin{bmatrix} 0 & \Delta\bar{z}_{A\_NA} & -\Delta\bar{y}_{A\_NA} \\ -\Delta\bar{z}_{A\_NA} & 0 & \Delta\bar{x}_{A\_NA} \\ \Delta\bar{y}_{A\_NA} & -\Delta\bar{x}_{A\_NA} & 0 \end{bmatrix}$$

$$\mathbf{T}_B = \begin{bmatrix} 0 & \Delta\bar{z}_{B\_NA} & -\Delta\bar{y}_{B\_NA} \\ -\Delta\bar{z}_{B\_NA} & 0 & \Delta\bar{x}_{B\_NA} \\ \Delta\bar{y}_{B\_NA} & -\Delta\bar{x}_{B\_NA} & 0 \end{bmatrix}$$

Bilinear Isoparametric Plate with Warping



$$\bar{x} = \frac{\overline{o6}}{\left\|\overline{o6}\right\|} \qquad \bar{z} = \frac{\bar{x} \times \overline{o7}}{\left\|\bar{x} \times \overline{o7}\right\|} \qquad \bar{y} = \bar{z} \times \bar{x}$$

$$\bar{h} = \overline{oA} \bullet \bar{z}$$

$$\mathbf{f}_a = \mathbf{B}\mathbf{f}_e$$

$$\mathbf{u}_e = \mathbf{B}^T \mathbf{u}_a$$

$\mathbf{f}_a$: general forces (i.e. forces and moments) applied in the actual plane

$\mathbf{f}_e$: general forces applied in the mean plane

$\mathbf{u}_a$: general displacements occurred in the actual plane

$\mathbf{u}_e$: general displacements occurred in the mean plane

Plate Checking

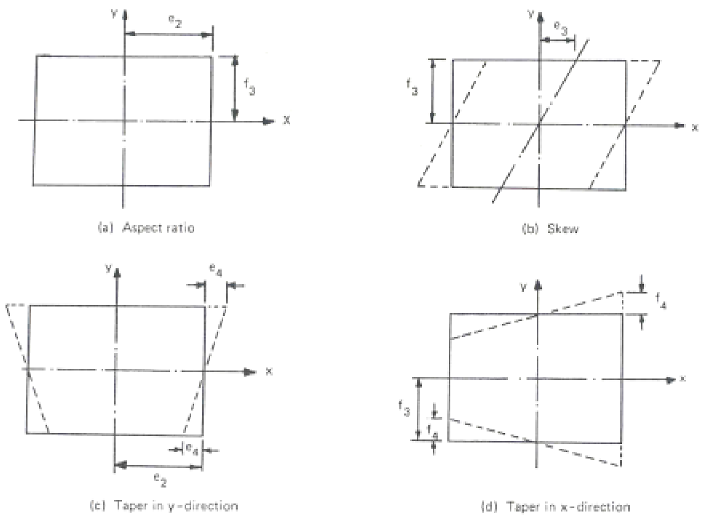


Fig. 5. Shape parameters.

Numerical Integration

$$K = \int F(r,s,t)drdsdt \qquad : when \;\; F = B^T CB \; \det J$$

$$= \sum_{i,j,k} \alpha_{ijk} F(r_i, s_j, t_k) + R_n \qquad : \begin{cases} \alpha_{ijk} : \text{weighting factor} \\ R_n : \text{Error matrix (practically not evaluated)} \end{cases}$$

- Polynomina ls $\rightarrow$ Lagrange interpolation
- Newton - Cotes $\rightarrow \begin{cases} \text{Trapezoidal rule} \\ \text{Simpson's rule} \end{cases} \rightarrow$ equal space
- Gauss Quadrature $\rightarrow$ nonequal space

Gauss Quadrature

Gauss Quadrature(Internal is not known)$\rightarrow$Optimize

$$\int_a^b F(r)dr = \sum_i \alpha_i F(r_i) + R_n \qquad \text{(position of sampling pts weights)}$$

$$\int_a^b F(r)dr = \sum_{j=1}^{n} F_j\left[\int_a^b l_j(r)dr\right] + \sum_{j=0}^{\infty}\beta_j\left[\int_a^b r^j p(r)dr\right]$$

$$r \rightarrow \int_a^b p(r)r^k dr = 0 \quad : k = 0 \sim n-1$$

$$\alpha \rightarrow \alpha_j = \int_{-1}^{1} l_j(r)dr \quad : j = 1 \sim n$$

tables for Gauss-Legendre(T5.6)

Integrations in 2-D and 3-D

$$\int_{-1}^{1}\int_{-1}^{1} F(r,s)drds = \sum_{i,j}\alpha_i\alpha_j F(r_i,s_j)$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F(r,s,t)drdsdt = \sum_{i,j,k}\alpha_i\alpha_j\alpha_k F(r_i,s_j,t_k)$$

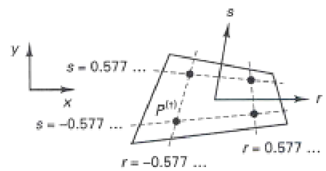
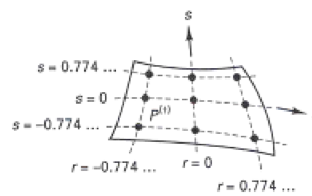
Gauss Quadrature

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

n	r_i			α_i		
1	0.	(15 zeros)		2.	(15 zeros)	
2	±0.57735	02691	89626	1.00000	00000	00000
3	±0.77459	66692	41483	0.55555	55555	55556
	0.00000	00000	00000	0.88888	88888	88889
4	±0.86113	63115	94053	0.34785	48451	37454
	±0.33998	10435	84856	0.65214	51548	62546
5	±0.90617	98459	38664	0.23692	68850	56189
	0.53846	93101	05683	0.47862	86704	99366
	0.00000	00000	00000	0.56888	88888	88889
6	±0.93246	95142	03152	0.17132	44923	79170
	±0.66120	93864	66265	0.36076	15730	48139
	±0.23861	91860	83197	0.46791	39345	72691

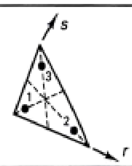
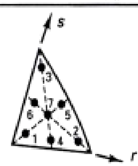
Gauss Quadrature

TABLE 5.7 *Gauss numerical integrations over quadrilateral domains*

Integration order	Degree of precision	Location of integration points
2 × 2	3	
3 × 3	5	

Gauss Quadrature

TABLE 5.8 *Gauss numerical integrations over triangular domains* $[\iint F\,dr\,ds = \frac{1}{2}\Sigma w_i F(r_i, s_i)]$

Integration order	Degree of precision	Integration points	r-coordinates	s-coordinates	Weights
3-point	2		$r_1 = 0.16666\ 66666\ 667$ $r_2 = 0.66666\ 66666\ 667$ $r_3 = r_1$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$	$w_1 = 0.33333\ 33333\ 333$ $w_2 = w_1$ $w_3 = w_1$
7-point	5		$r_1 = 0.10128\ 65073\ 235$ $r_2 = 0.79742\ 69853\ 531$ $r_3 = r_1$ $r_4 = 0.47014\ 20641\ 051$ $r_5 = r_4$ $r_6 = 0.05971\ 58717\ 898$ $r_7 = 0.33333\ 33333\ 333$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$ $s_4 = r_6$ $s_5 = r_4$ $s_6 = r_4$ $s_7 = r_7$	$w_1 = 0.12593\ 91805\ 448$ $w_2 = w_1$ $w_3 = w_1$ $w_4 = 0.13239\ 41527\ 885$ $w_5 = w_4$ $w_6 = w_4$ $w_7 = 0.225$

Polynomial

$$\psi(r) = a_0 + a_1 r + \cdots + a_n r^n$$

$$\underline{F} = V\underline{a} \rightarrow \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_n \end{bmatrix} \begin{bmatrix} 1 & r_0 & r_0^2 & \cdots & r_0^n \\ 1 & r_1 & r_1^2 & \cdots & r_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & r_n & r_n^2 & \cdots & r_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} : \left(\begin{array}{l} \text{unique solution} \\ \text{since } |V| \neq 0 \end{array} \right.$$

Lagrangian Interpolation

$$l_j(r) = \frac{(r-r_0)(r-r_1)\cdots(r-r_{j-1})(r-r_{j+1})\cdots(r-r_n)}{(r_j-r_0)(r_j-r_0)\cdots(r_j-r_{j-1})(r_j-r_{j+1})\cdots(r_j-r_n)} \quad : l_j(r_i) = \delta_{ij}$$

$$\psi(r) = F_0 l_0(r) + F_1 l_1(r) + \cdots + F_n l_n(r)$$

Newton-Cotes

Newton - Cotes $\left(\int_b^a F(r)dr\right)$

Sampling pts are spaced at equal distance $h = \frac{b-a}{n}$

$$\int_b^a F(r)dr = \sum_{i=0}^{n} \left[\int_b^a l_i(r)dr\right] F_i + R_n$$

$$= (b-a)\sum_{i=0}^{n} C_i^n F_i + R_n \quad , \; C_i^n : \text{N} - \text{C constants}$$

	C_0^n	C_1^n	C_2^n
1 :	$\frac{1}{2}$	$\frac{1}{2}$	
2 :	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$

$$\therefore \int_b^a F(r)dr = \frac{b-a}{6}(F_0 + F_1 + F_2)$$

To improve accuracy
- 1. higher - order N - C formula
- 2. lower - order formula in a repeated manner (composite formula)

Multiple Choice Questions and Answers

S.No	Questions	Opt1	Opt2	Opt3	Opt4	Answer
1	The ratio of specific weight of liquid to specific weight of water is	Specific gravity	Specific weight	Specific volume	All of these	Specific gravity
2	Kinematic viscosity is defined as equal to	Dynamic viscosity x density	Dynamic velocity/pressure	Dynamic viscosity x pressure	None of these	None of these
3	Specific gravity of water is	1000	1	9810	9.81	1
4	Relative density of mercury is	13.6	13600	1	9.8	13.6
5	In CGS system unit of kinematic viscosity is	Poise	Stokes	Mach number	All of these	Stokes
6	Standard atmospheric pressure in terms of mercury is ______________	13.6	760mm	10.3mm	None of these	760mm
7	The unit of bulk modulus in SI unit is	N/m^2	pa-s	kg/m-s	All of these	N/m^2
8	The unit of mass density in SI unit is	N/m^2	pa-s	kg/m^3	All of these	kg/m^3
9	In CGS system unit of viscosity is	Poise	Stokes	Mach number	All of these	Poise
10	The ratio of volume to mass of the fluid is termed as	Compressibility	Specific volume	Specific weight	None of these	Specific volume
11	The ratio of mass to volume of the fluid is termed as	Compressibility	Specific volume	Specific weight	Mass density	Mass density
12	The compressibility of the fluid is the reciprocal of	Density	Viscosity	Bulk modulus	None of these	Bulk modulus
13	The bulk modulus of the fluid is the reciprocal of	Compressibility	Viscosity	Pressure	None of these	Compressibility
14	It is a product of mass density and volume of the fluid	Mass	Specific weight	Specific volume	Specific gravity	Mass
15	The ratio of density of liquid to density of water is	Specific gravity	Specific weight	Specific volume	All of these	Specific gravity
16	___________ is one of the causes of the upward flow of water in the soil and in plants	Surface tension	Viscosity	Vapour pressure	None of these	None of these
17	When the pressure measured above atmospheric pressure it is called	Absolute pressure	Static pressure	Vacuum pressure	None of these	None of these

18	In capillary rise the angle of contact between mercury and glass tube is	0	128	60	None of these	128
19	An ideal fluid is defined as the fluid which	Is compressible	Is incompressible	Is incompressible and non-viscous (inviscid)	Has negligible surface tension.	Is incompressible and non-viscous (inviscid)
20	Newton's law of viscosity states that	shear stress is directly proportional to the velocity	Shear stress is directly proportional to velocity gradient	Shear stress is directly proportional to shear strain	Shear stress is directly proportional to the viscosity.	Shear stress is directly Proportional to velocity gradient
21	Kinematic viscosity is defined as equal to	dynamic viscosity x density	Dynamic velocity/density	Dynamic viscosity x pressure	Pressure x density	Dynamic velocity/density
22	If two fluids at different temperatures are mixed together, heat transfer occurs by______	Conduction	Convection	Radiation	Mass transfer	Convection
23	Fluid motion occurs by density difference is known as _____________	Conduction	Convection	Radiation	Mass transfer	Convection
24	Density difference in fluid may occur due to _________________	Temperature difference	Pressure difference	Viscosity difference	Flow variation	Temperature difference
25	Fluid motion may occur by density differences caused by temperature differences is known as	Natural convection	Forced convection	Boiling	Condensation	Natural convection
26	The unit of heat transfer coefficient is	W/ m^2	W/mK	W	W/m^2K	W/m^2K
27	The unit of thermal conductivity is	W/ m^2	W/mK	W	W/m^2K	W/mK
28	Heat transfer coefficient, 'h' is	A property of the surface material	A property of the fluid	Not a property of the surface material and of the fluid	Depends on flow condition	Not a property of the surface material and of the fluid
29	The property of the system is constant with respect to time is known as	Steady state	Unsteady state	Solid state	Liquid state	Steady state
30	The property of the system is varying with respect to time is known as	Steady state	Unsteady state	Solid state	Liquid state	Unsteady state
31	Fourier's law is defined for	Steady state, one dimensional heat flow	Steady state	One dimensional	Unsteady state	Steady state, one dimensional heat flow
32	Fourier's law is applicable to	Solid	Liquid	Gaseous	All of these	Solid

	___________states of matter.					
33	Thermal conductivity of materials depends upon	Material structure & density of material	Moisture content and pressure	Temperature	All of these	All of these
34	Heat treatment of pure metals _____the value of thermal conductivity	Increases	Decreases	Does not have any effect	All of these	Decreases
35	Thermal conductivity of alloy generally___as temperature increases	Increases	Decreases	Does not have any effect	All of these	Increases
36	Thermal diffusivity indicates	The ability of the material to conduct heat	The ability of the material to store heat	The ability of the material to withstand heat	The ability of the material to reject heat	The ability of the material to store heat
37	A wall made up of different thermal conductivity material is known as	Composite wall	Brick wall	Insulation wall	Guard wall of these	Composite wall of these
38	Temperature distribution is used to find out	Temperature of the atmosphere	Temperature of material	Temperature at any location in the material	Temperature of medium	Temperature at any location in the material
39	The unit of overall heat transfer coefficient is	W/m2K	W/mk	W/m	W/k	W/m2K
40	If the insulation material radius on a pipe is less than critical radius then the heat	loss is more	Loss is less	Loss is constant	Will be generated	Loss is more
41	Extended surface is used to	Increase the heat transfer surface area	Decrease the heat transfer surface area	Generate heat	Absorb heat	Increase the heat transfer surface area
42	The ratio of energy transferred by convection to that by conduction is called	Stanton number	Nusselt number	Biot number	Preclet number	Biot number
43	Free convection floe depends on all of the following except	Density	Coefficient of viscosity	Gravitational force	Velocity	Density