17BEEC305 ELECTRO MAGNETIC FIELDS L T P C
3 1 0 4

OBJECTIVE:

- To impart knowledge on the basics of static electric and magnetic field and the associated laws.
- To give insight into the propagation of EM waves and also to introduce the methods in computational electromagnetic
- To make students have depth understanding of antennas, electronic devices, Waveguides is possible

INTENDED OUTCOMES:

Upon completion of the course, the students would be able to

- Analyze field potentials due to static changes and static magnetic fields.
- Explain how materials affect electric and magnetic fields.
- Analyze the relation between the field sunder time varying situations.
- Discuss the principles of propagation uniform plane waves.

UNITI STATIC ELECTRICFIELD

Vector Algebra, Coordinate Systems, Vector differential operator, Gradient, Divergence, Curl, Divergence theorem, Stokes theorem, Coulombs law,Electric field intensity,Point,Line,Surface and Volume charged distributions, Electric flux density,Gauss law and its applications, Gauss divergence theorem, Absolute Electric potential,Potential difference,Calculation of potential differences for different configurations.Electric dipole,Electro static Energy and Energy density.

UNITII CONDUCTORS AND DIELECTRICS

Conductors and dielectrics in Static Electric Field,Current and current density,Continuity equation, Polarization, Boundary conditions, Methodofimages, Resistance of a conductor,Capacitance, Parallel plate,Coaxial and Spherical capacitors, Boundary conditionsfor perfect dielectric materials, Poisson'sequation,Laplace'sequation,Solution of Laplace equation,Application of Poisson's and Laplace's equations.

UNITIII STATIC MAGNETIC FIELDS

Biot-Savart Law,Magneticfield Intensity,Estimation of Magneticfield Intensity for straight and circular conductors, Ampere's Circuital Law,Point form of Ampere's Circuital Law,Stokes theorem, Magnetic flux and magnetic flux density, The Scalar and Vector Magnetic potentials,Derivation of Steady magneticfield Laws.

UNITIV MAGNETICFORCES AND MATERIALS

Forceona moving charge,Force on a differential current element,Force between current elements, Force and torque on a closed circuit, The nature magnetic materials,Magnetization and permeability,Magnetic boundary conditions in evolving magnetic fields,Themagnetic circuit, Potential energy and forceson magnetic materials,Inductance,Basic expressions fo rself and mutual inductances, Inductance evaluation for solenoid, toroid,coaxial cables and transmissionlines,Energy stored in Magneticfields.

UNITV TIME VARYING FIELDS AND MAXWELL'S EQUATIONS

Fundamental relations for Electro static and Magneto static fields,Faraday's law for Electromagnetic induction, Transformers, Motional Electromotive forces, Differential form of Maxwell's equations, Integral form of Maxwell's equations, Potential functions, Electromagnetic boundary conditions, Wave equations and their solutions,Poynting'stheorem, Time harmonic fields,Electro magnetic Spectrum.

TEXT BOOKS:

S.NO.	Author(s) Name	Title of the book	Publisher	Year of publication
1	WilliamH Haytand Jr.JohnA Buck	Engineering Electromagnetics	TataMcGraw-Hill PublishingCompanyLtd NewDelhi	2008
2	SadikuMH	PrinciplesofElectro magnetics	OxfordUniversityPressI nc, NewDelhi	2009

REFERENCES:

S.NO.	Author(s) Name	Title of the book	Publisher	Year of publication
1	DavidKCheng	FieldandWaveElect romagnetics	PearsonEducationInc,Delh i	2004
2	JohnDKrausandDaniel AFleisch,"	Electromagneticswi thApplications	McGrawHillBookCo	2005
3	KarlELongmanandSava VSavov	FundamentalsofEle ctromagnetics	PrenticeHallofIndiaNewD elhi	2006
4	AshutoshPramanic	Electromagnetism	PrenticeHall ofIndia,NewDelhi	2006

UNIT-1 STATIC ELECTRIC FIELD

INTRODUCTION:

Electromagnetic theory is a discipline concerned with the study of charges at rest and in motion. Electromagnetic principles are fundamental to the study of electrical engineering and physics. Electromagnetic theory is also indispensable to the understanding, analysis and design of various electrical, electromechanical and electronic systems. Some of the branches of study where electromagnetic principles find application are:

- RF communication,
- Microwave Engineering,
- Antennas,
- Electrical Machines,
- Satellite Communication,
- Atomic and nuclear research ,
- Radar Technology,
- Remote sensing
- Quantum Electronics,
- VLSI

Electromagnetic theory is a pre requisite for a wide spectrum of studies in the field of Electrical Sciences and Physics. Electromagnetic theory can be thought of as generalization of circuit theory. There are certain situations that can be handled exclusively in terms of field theory. In electromagnetic theory, the quantities involved can be categorized as **source quantities** and **field quantities**. Source of electromagnetic field is electric charges: either at rest or in motion. However an electromagnetic field may cause a redistribution of charges that in turn change the field and hence the separation of cause and effect is not always visible.

Sources of EMF:

Current carrying conductors.
Mobile phones.
Microwave oven.
Computer and Television screen.
High voltage Power lines.

Effects of Electromagnetic fields:

Plants and Animals.
Humans.
Electrical components.

Fields are classified as

Scalar field
Vector field.

Electric charge is a fundamental property of matter. Charge exist only in positive or negative integral multiple of **electronic charge**, -e, e= 1.60 × 10^{-19} coulombs. [It may be noted here that in 1962, Murray Gell-Mann hypothesized **Quarks** as the basic building blocks of matters. Quarks were predicted to carry a fraction ofelectronic charge and the existence of Quarks have been experimentally verified.] Principle of conservation of charge states that the total charge (algebraic sum of positive and negative charges) of an isolated system remains unchanged, though the charges may redistribute under the influence of electric field. Kirchhoff's Current Law (KCL) is an assertion of the conservative property of charges under the implicit assumption that there is no accumulation of charge at the junction.

Electromagnetic theory deals directly with the electric and magnetic field vectors where as circuit theory deals with the voltages and currents. Voltages and currents are integrated effects of electric and magnetic fields respectively. Electromagnetic field problems involve three space variables along with the time variable and hence the solution tends to become correspondingly complex. Vector analysis is a mathematical tool with which electromagnetic concepts are more conveniently expressed and best comprehended. Since use of vector analysis in the study of electromagnetic field theory results in real economy of time and thought, we first introduce the concept of vector analysis.

Vector Analysis:

The quantities that we deal in electromagnetic theory may be either **scalar** or **vectors** [There are other class of physical quantities called Tensors: where magnitude and direction vary with co ordinate axes]. Scalars are quantities characterized by magnitude only and algebraic sign. A quantity that has direction as well as magnitude is called a vector. Both scalar and vector quantities are function of time and position. A field is a function that specifies a particular quantity everywhere in a region. Depending upon the nature of the quantity under consideration, the field may be a vector or a scalar field. Example of scalar field is the electric potential in a region while electric or magnetic fields at any point is the example of vector field.

A vector $\vec{A}$ can be written as, $\vec{A} = \hat{a} A$, where, $A = |\vec{A}|$ is the magnitude and $\hat{a} = \frac{\vec{A}}{|A|}$ is the unit vector which has unit magnitude and same direction as that of $\vec{A}$.

Two vector $\vec{A}$ and $\vec{B}$ are added together to give another vector $\vec{C}$. We have

$$\vec{C} = \vec{A} + \vec{B} \quad \text{...}(1.1)$$

Let us see the animations in the next pages for the addition of two vectors, which has two rules: **1: Parallelogram law** and **2: Head & tail rule**

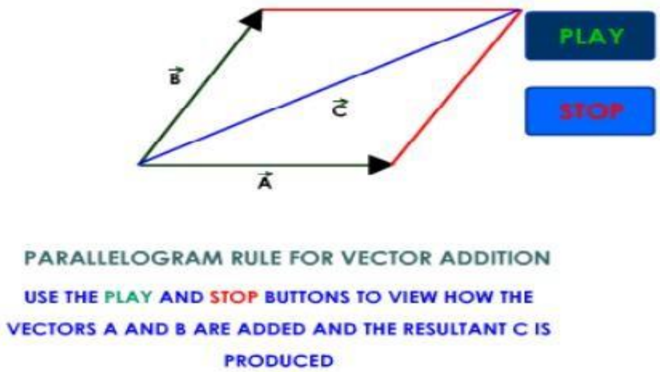
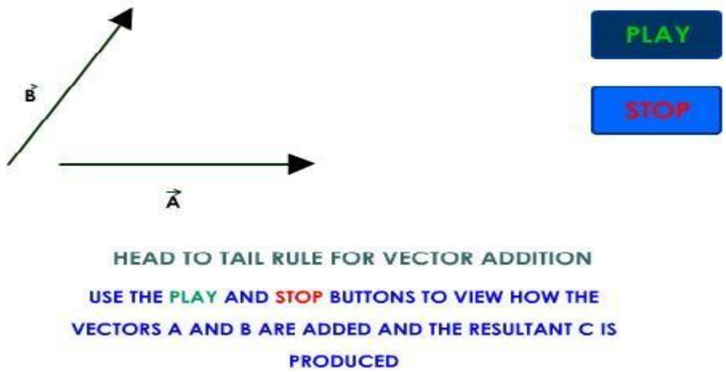


Fig 1.1(a):Vector Addition(Parallelogram Rule)



.2)

Fig 1.1(b): Vector Addition (Head & Tail Rule)

Vector Subtraction is similarly carried out: $\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ ....(1.2)

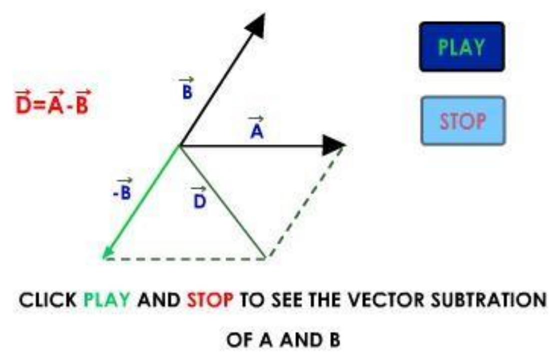


Fig 1.2: Vector subtraction

Scaling of a vector is defined as $\vec{C} = \alpha\vec{B}$, where $\vec{C}$ is scaled version of vector $\vec{B}$ and α is a scalar.

Some important laws of vector algebra are:

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ Commutative Law…..(1.3)

$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$ Associative Law...(1.4)

$\alpha(\vec{A}+\vec{B})=\alpha\vec{A}+\alpha\vec{B}$ Distributive Law ....(1.5)

The position vector $\vec{r_Q}$ of a point *P* is the directed distance from the origin (*O*) to *P*, i.e., $\vec{r_Q}=\overrightarrow{OP}$.

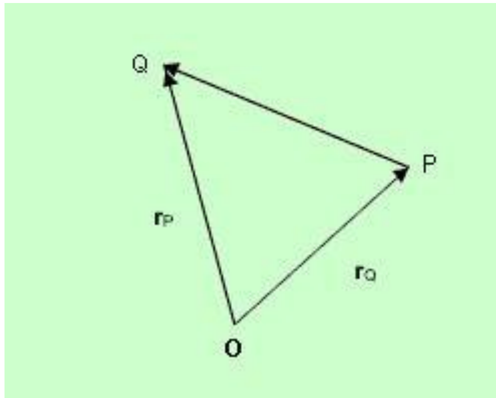


Fig 1.3: Distance Vector

If $\vec{r_Q}$ = OP and $\vec{r_P}$ = OQ are the position vectors of the points P and Q then the distance vector

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\vec{r_P}-\vec{r_Q}$$

Product of Vectors

When two vectors $\vec{A}$ and $\vec{B}$ are multiplied, the result is either a scalar or a vector depending how the two vectors were multiplied. The two types of vector multiplication are:

Scalar product (or dot product) $\vec{A}\cdot\vec{B}$ gives a scalar.
Vector product (or cross product) $\vec{A}\times\vec{B}$ gives a vector.
The dot product between two vectors is defined as $\vec{A}\cdot\vec{B}=|A||B|\cos\theta AB$ ....(1.6)
Vector product $\vec{A}\times\vec{B}=|A||B|\sin\theta_{AB}\cdot\vec{n}$
$\vec{n}$ is unit vector perpendicular to $\vec{A}$ and $\vec{B}$

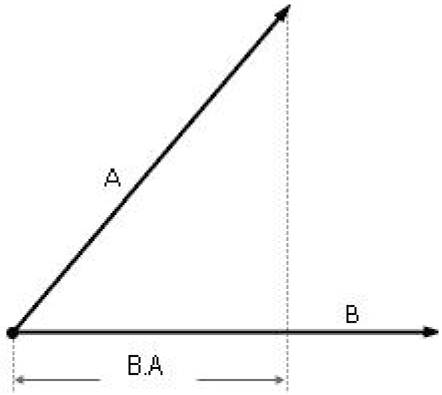


Fig 1.4: Vector dot product

The dot product is commutative i.e., $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ and distributive i.e., $\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$. Associative law does not apply to scalar product.

The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$. $\vec{A}\times\vec{B}$ is a vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, the magnitude is given by $|A||B|\sin\theta_{AB}$ and direction is given by right hand rule as explained in Figure 1.5.

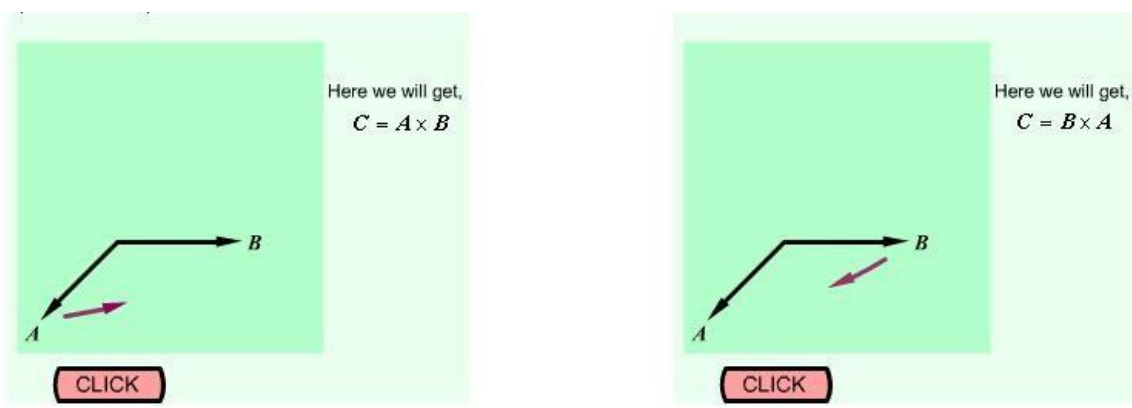


Fig 1.5 :Illustrating the left thumb rule for determining the vector cross product

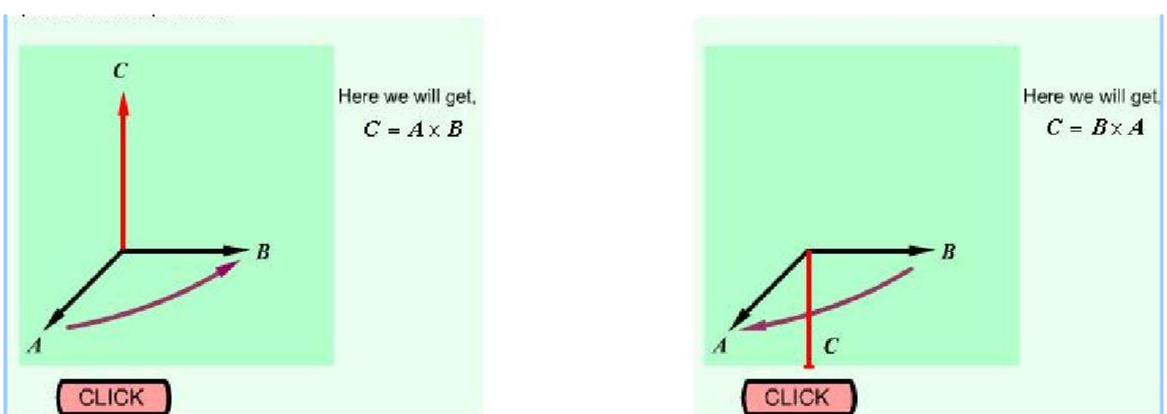


Fig 1.5 :Illustrating the left thumb rule for determining the vector cross product

$$\vec{A}\times\vec{B} = \hat{a}_n\, AB \sin\theta_{AB} \qquad \text{...(1.7)}$$

where $\hat{a}_n$ is the unit vector given by, $\hat{a}_n = \dfrac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}$.

The following relations hold for vector product.

$\vec{A}\times\vec{B} \quad -\vec{B}\times\vec{A}$ i.e., cross product is non commutative ....(1.8)

$\vec{A}\times\left(\vec{B} + \vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ i.e., cross product is distributive.......................(1.9)

$\vec{A}\times\left(\vec{B} \times \vec{C}\right) \neq \left(\vec{A}\times\vec{B}\right)\times\vec{C}$ i.e., cross product is non associative...........(1.10)

Scalar and vector triple product :

Scalar triple product $\vec{A}\cdot\left(\vec{B} \times \vec{C}\right) = \vec{B}\cdot\left(\vec{C}\times\vec{A}\right) = \vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$ ....(1.11)

Vector triple product $\vec{A}\times\left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$ ....(1.12)

Co-ordinate Systems

In order to describe the spatial variations of the quantities, we require using appropriate co-ordinate system. A point or vector can be represented in a **curvilinear** coordinate system that may be **orthogonal** or **non-orthogonal** .

An orthogonal system is one in which the co-ordinates are mutually perpendicular. Non- orthogonal co-ordinate systems are also possible, but their usage is very limited in practice .

Let *u* = constant, *v* = constant and *w* = constant represent surfaces in a coordinate system,

the surfaces may be curved surfaces in general. Furthur, let $\hat{a}_u$ $\hat{a}_v$ and $\hat{a}_w$ be the unit vectors in the three coordinate directions(base vectors). In a general right handed orthogonal curvilinear systems, the vectors satisfy the following relations :

$$\hat{a}_u \times \hat{a}_v = \hat{a}_w$$
$$\hat{a}_v \times \hat{a}_w = \hat{a}_u$$
$$\hat{a}_w \times \hat{a}_u = \hat{a}_v \quad \text{..} \quad (1.13)$$

These equations are not independent and specification of one will automatically imply the other two. Furthermore, the following relations hold

$$\hat{a}_u \cdot \hat{a}_v = \hat{a}_v \cdot \hat{a}_w = \hat{a}_w \cdot \hat{a}_u = 0$$
$$\hat{a}_u \cdot \hat{a}_u = \hat{a}_v \cdot \hat{a}_v = \hat{a}_w \cdot \hat{a}_w = 1 \quad \text{..} (1.14)$$

A vector can be represented as sum of its orthogonal components,

$$\vec{A} = A_u \hat{a}_u + A_v \hat{a}_v + A_w \hat{a}_w \quad \text{.................................} (1.15)$$

In general u, v and w may not represent length. We multiply u, v and w by conversion factors h_1,h_2 and h_3 respectively to convert differential changes du, dv and dw to corresponding changes in length dl_1, dl_2, and dl_3. Therefore

$$d\vec{l} = \hat{a}_u\, dl_1 + \hat{a}_v\, dl_2 + \hat{a}_w\, dl_3$$
$$= h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w \quad \text{...} (1.16)$$

In the same manner, differential volume dv can be written as $dv = h_1 h_2 h_3\, du\, dv\, dw$ and differential area ds_1 normal to $\hat{a}_u$ is given by, $ds_1 = h_2 h_3\, dv\, dw$. In the same manner, differential areas normal to unit vectors $\hat{a}_v$ and $\hat{a}_w$ can be defined.

In the following sections we discuss three most commonly used orthogonal co-ordinate systems, viz:

1. **Cartesian (or rectangular) co-ordinate system**
2. **Cylindrical co-ordinate system**
3. **Spherical polar co-ordinate system**

Cartesian Co-ordinate System :

In Cartesian co-ordinate system, we have, $(u,v,w) = (x,y,z)$. A point $P(x_0, y_0, z_0)$ in Cartesian co-ordinate system is represented as intersection of three planes $x = x_0$, $y = y_0$ and $z = z_0$. The unit vectors satisfies the following relation:

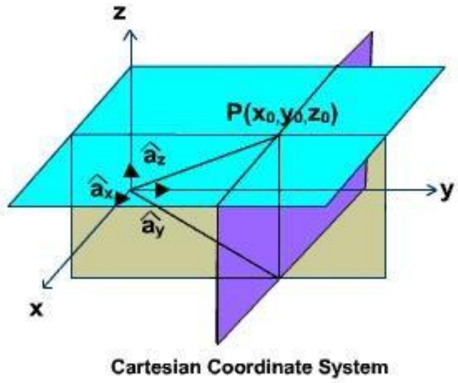


Fig 1.6 Intersection of three planes

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$

$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$

$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$

$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$$

$$\overrightarrow{OP} = \hat{a}_x x_0 + \hat{a}_y y_0 + \hat{a}_z z_0$$

In cartesian co-ordinate system, a vector $\vec{A}$ can be written as $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$. The dot and cross product of two vectors $\vec{A}$ and $\vec{B}$ can be written as follows:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \qquad \text{.................(1.19)}$$

$$\vec{A} \times \vec{B} = \hat{a}_x (A_y B_z - A_z B_y) + \hat{a}_y (A_z B_x - A_x B_z) + \hat{a}_z (A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

....................(1.20)

Since x, y and z all represent lengths, $h1 = h2 = h3 = 1$. The differential length, area

$$d\vec{l} = dx\, a_x + dy\, a_y + dz\, a_z$$

and volume are defined respectively as

$$d\vec{s}_x = dydz\, a_x$$

$$d\vec{s}_y = dxdz\, \hat{a}_y \qquad \text{................(1.21)}$$

$$d\vec{s}_z = dxdy\, \hat{a}_z$$

$$d\upsilon = dxdydz \qquad \text{.................................(1.22)}$$

Cylindrical Co-ordinate System :

For cylindrical coordinate systems we have $(u, v, w) = (r, \phi, z)$ a point $P(r_0, \phi_0, z_0)$ is determined as the point of intersection of a cylindrical surface *r = r0*, half plane containing the z-axis and making an angle $\phi = \phi_0$*;* with the xz plane and a plane parallel to *xy* plane located at *z=z0* as shown in figure.

In cylindrical coordinate system, the unit vectors satisfy the following relations

A vector $\vec{A}$ can be written as , $\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$ ....(1.24)

The differential length is defined as,

$$d\vec{l} = \hat{a}_\rho d\rho + \rho d\phi \hat{a}_\phi + dz \hat{a}_z \qquad h_1 = 1, h_2 = \rho, h_3 = 1$$ ....(1.25)

$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z$

$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$

$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$ ....(1.23)

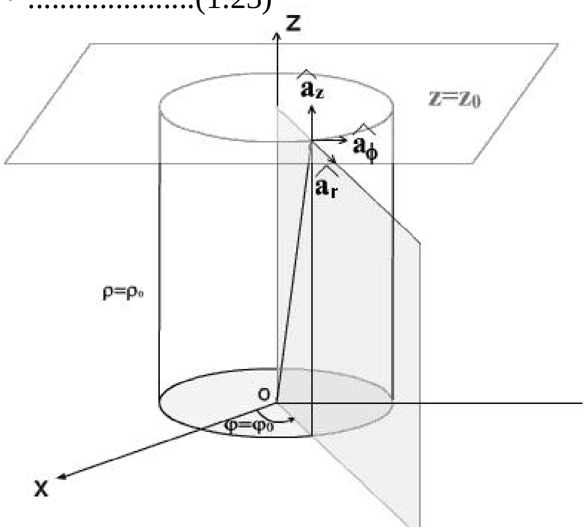
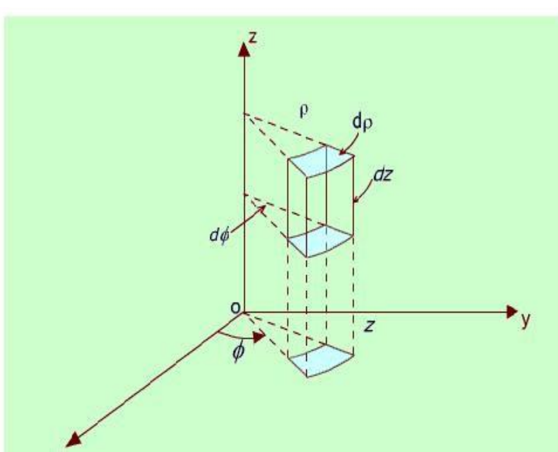


Fig 1.7 cylindrical co-ordinate system



Differential areas are:

$\overrightarrow{ds_\rho} = \rho d\phi dz \hat{a}_\rho$

$\overrightarrow{ds_\phi} = d\rho dz \hat{a}_\phi$ ....(1.26)

$\overrightarrow{ds_z} = \rho d\phi d\rho \hat{a}_z$

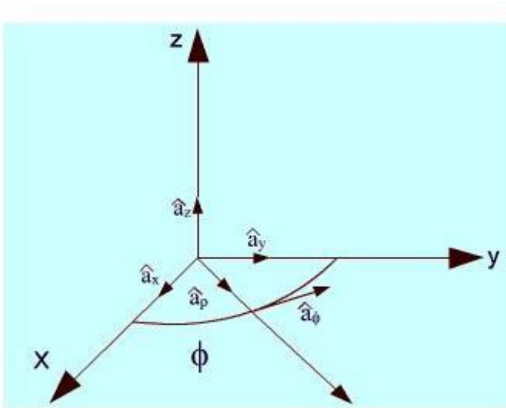
Differential volume,

$dv = \rho d\rho d\phi dz$ ....(1.27)

Fig1.8 cylindrical system surface

Transformation between Cartesian and Cylindrical coordinates:

Let us consider $\vec{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z$ is to be expressed in Cartesian co-ordinate as $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$. In doing so we note that $A_x = \vec{A}.\hat{a}_x = \left(\hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z\right).\hat{a}_x$ and it applies for other components as well.



$$\hat{a}_\rho.\hat{a}_x = \cos\phi$$
$$\hat{a}_\rho.\hat{a}_y = \sin\phi$$
$$\hat{a}_\phi.\hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi \quad \text{...............(1.28)}$$
$$\hat{a}_\phi.\hat{a}_y = \cos\phi$$

Therefore we can write,

$$A_x = \vec{A}.\hat{a}_x = A_\rho \cos\phi - A_\phi \sin\phi$$
$$A_y = \vec{A}.\hat{a}_y = A_\rho \sin\phi + A_\phi \cos\phi \quad \text{..........(1.29)}$$
$$A_z = \vec{A}.\hat{a}_z = A_z$$

Fig 1.9 : Unit Vectors in Cartesian and Cylindrical Coordinates

These relations can be put conveniently in the matrix form as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_y \\ A_z \end{bmatrix} \quad \text{.....................(1.30)}$$

A_ρ, A_ϕ and A_z themselves may be functions of ρ, ϕ and z as:

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z \quad \text{............................(1.31)}$$

The inverse relationships are:

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z \quad \text{........................(1.32)}$$

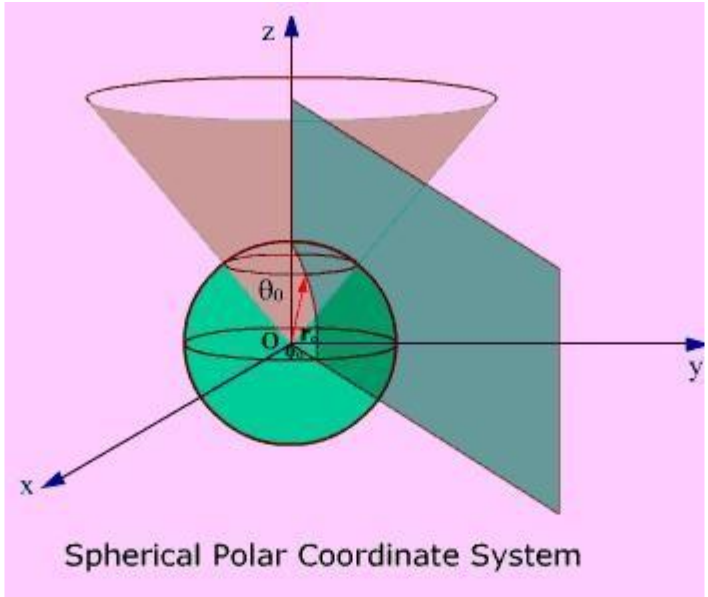


Fig 1.10: Spherical Polar Coordinate System

Thus we see that a vector in one coordinate system is transformed to another coordinate system through two-step process: Finding the component vectors and then variable transformation.

Spherical Polar Coordinates:

For spherical polar coordinate system, we have, $(u,v,w)=(r,\theta,\phi)$. A point $P(r_0,\theta_0,\phi_0)$ is represented as the intersection of

(i) Spherical surface *r=r0*

(ii) Conical surface $\theta=\theta_0$,and

(iii) half plane containing z-axis making angle $\phi=\phi_0$ with the *xz* plane as shown in the figure 1.10.

The unit vectors satisfy the following relationships:

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$$
$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$$
$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta \qquad ....(1.33)$$

The orientation of the unit vectors are shown in the figure 1.11.

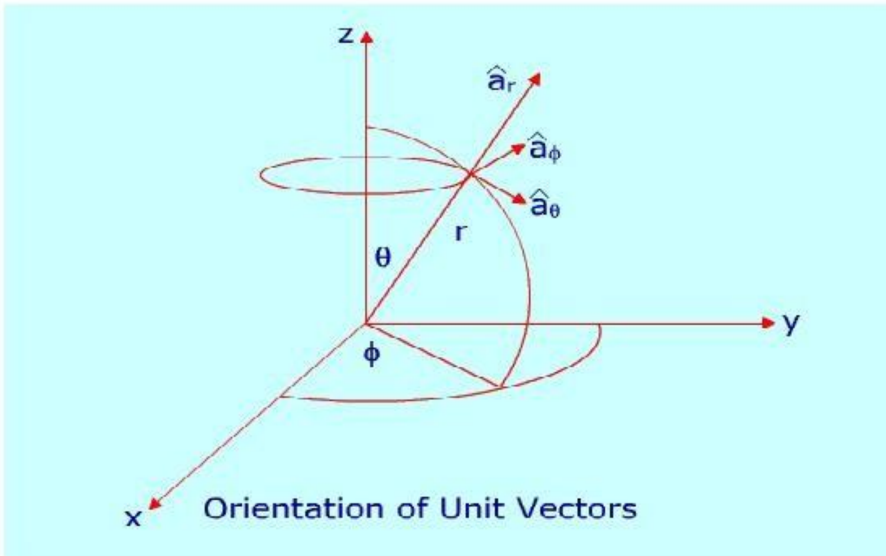


Fig 1.11: Orientation of Unit Vectors

A vector in spherical polar co-ordinates is written as : $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$ and

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r \sin\theta d\phi$$

For spherical polar coordinate system we have h_1=1, h_2= r and h_3= $r\sin\theta$.

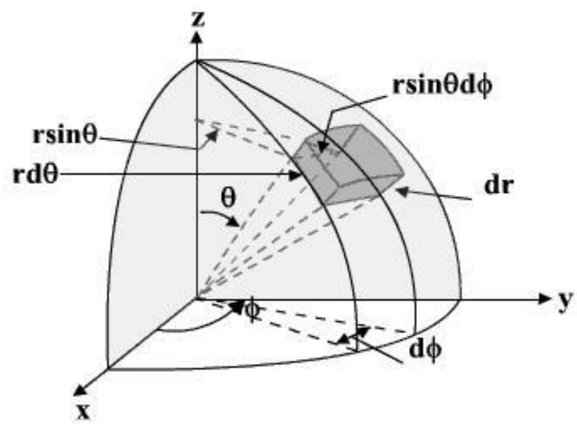


Fig 1.12(a) : Differential volume in s-p coordinates

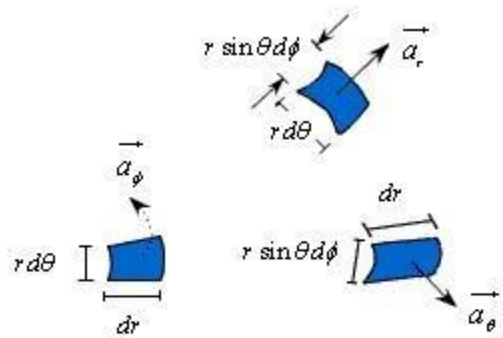


Fig 1.12(b) : Exploded view

With reference to the Figure 1.12, the elemental areas are:

$$ds_r = r^2 \sin\theta d\theta d\phi \hat{a}_r$$
$$ds_\theta = r \sin\theta dr d\phi \hat{a}_\theta$$
$$ds_\rho = r dr d\theta \hat{a}_\phi \qquad ....(1.34)$$

and elementary volume is given by

$$d\upsilon = r^2 \sin\theta dr d\theta d\phi \qquad ....(1.35)$$

Coordinate transformation between rectangular and spherical polar:

With reference to the figure 1.13 ,we can write the following equations:

$$\hat{a}_r \cdot \hat{a}_x = \sin\theta\cos\phi$$

$$\hat{a}_r \cdot \hat{a}_y = \sin\theta\sin\phi$$

$$\hat{a}_r \cdot \hat{a}_z = \cos\theta$$

$$\hat{a}_\theta \cdot \hat{a}_x = \cos\theta\cos\phi$$

$$\hat{a}_\theta \cdot \hat{a}_y = \cos\theta\sin\phi$$

$$\hat{a}_\theta \cdot \hat{a}_z = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$$

$$\hat{a}_\phi \cdot \hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi$$

$$\hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$

$$\hat{a}_\phi \cdot \hat{a}_z = 0$$

....(1.36)

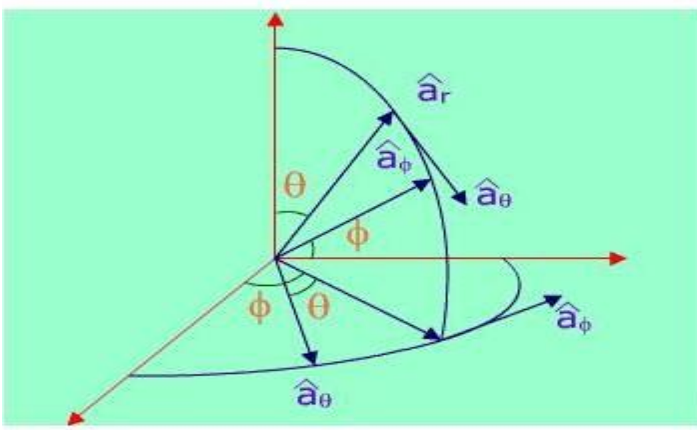


Fig 1.13: Coordinate transformation

Given a vector $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$ in the spherical polar coordinate system, its component in the cartesian coordinate system can be found out as follows:

$$A_x = \vec{A} . \hat{a}_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$$ ....(1.37)

Similarly,

$$A_y = \vec{A}\,\hat{a}_y = A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi \quad ....(1.38a)$$

$$A_z = \vec{A}\,\hat{a}_z = A_r \cos\theta - A_\theta \sin\theta \quad ....(1.38b)$$

The above equation can be put in a compact form:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \quad ....(1.39)$$

The components A_r, A_θ and A_ϕ themselves will be functions of r, θ and ϕ. r, θ and ϕ are related to *x*,*y* and *z* as:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta \quad ....(1.40)$$

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2} \quad ....(1.41a)$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad ....(1.41b)$$

$$\phi = \tan^{-1}\frac{y}{x} \quad ....(1.41c)$$

Using the variable transformation listed above, the vector components, which are functions of variables of one coordinate system, can be transformed to functions of variables of other coordinate system and a total transformation can be done.

Line, surface and volume integrals

In electromagnetic theory, we come across integrals, which contain vector functions. Some representative integrals are listed below:

$$\int_V \vec{F}\,dv \qquad \int_C \phi\, d\vec{l} \qquad \int_C \vec{F}.d\vec{l} \qquad \int_S \vec{F}.d\vec{s}$$

In the above integrals, $\vec{F}$ and ϕ respectively represent vector and scalar function of space coordinates. *C*,*S* and *V* represent path, surface and volume of integration. All these integrals are evaluated using extension of the usual one-dimensional integral as the limit of a sum, i.e., if a function *f*(*x*) is defined over arrange *a* to *b* of values of *x*, then the integral is given by

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f_i\,\delta x_i \quad \text{.................................(1.42)}$$

where the interval (*a*,*b*) is subdivided into n continuous interval of lengths $\delta x_1, \ldots\ldots\ldots, \delta x_n$.

Line Integral: Line integral $\int_C \vec{E}\cdot\vec{dl}$ is the dot product of a vector with a specified *C*; in other words it is the integral of the tangential component $\vec{E}$ along the curve *C*.

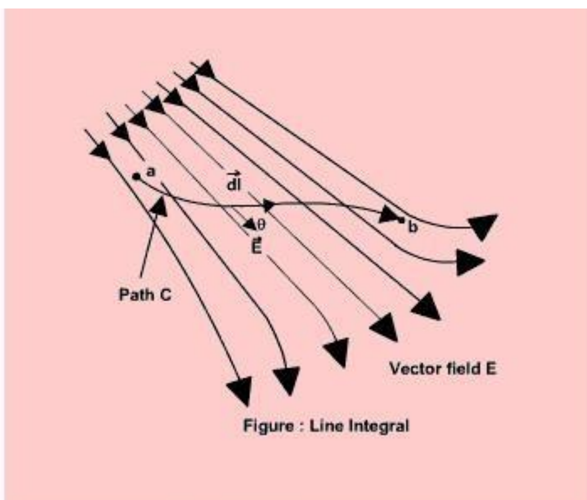


Fig 1.14: Line Integral

As shown in the figure 1.14, given a vector $\vec{E}$ around *C*, we define the integral

$$\int_C \vec{E}.\vec{dl} = \int_a^b E\cos\theta\,dl$$

as the line integral of E along the curve C.

If the path of integration is a closed path as shown in the figure the line integral becomes a closed line integral and is called the circulation of $\vec{E}$ around *C* and denoted as $\oint_C \vec{E}.\vec{dl}$ as shown in the figure 1.15.

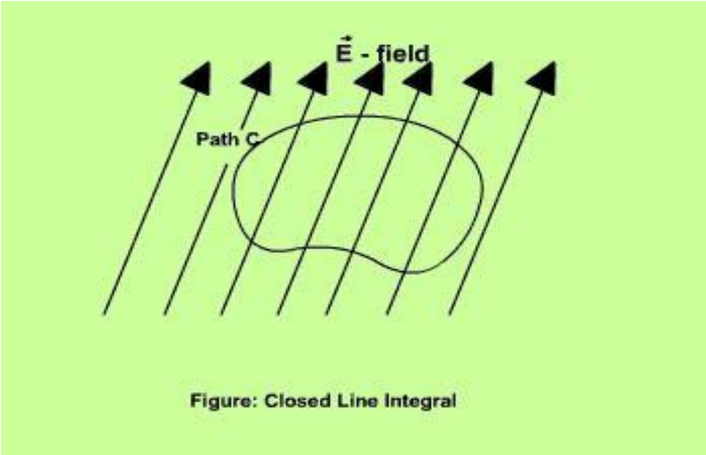


Fig 1.15: Closed Line Integral

Surface Integral :

Given a vector $\vec{A}$ field, continuous in a region containing the smooth surface *S*, we define the surface integral or the $\vec{A}$ flux of

$$\psi = \int_S A\cos\theta dS = \int_S \vec{A}.\hat{a}_n\, dS = \int_S \vec{A} d\vec{S}$$

as surface integral over surface S.

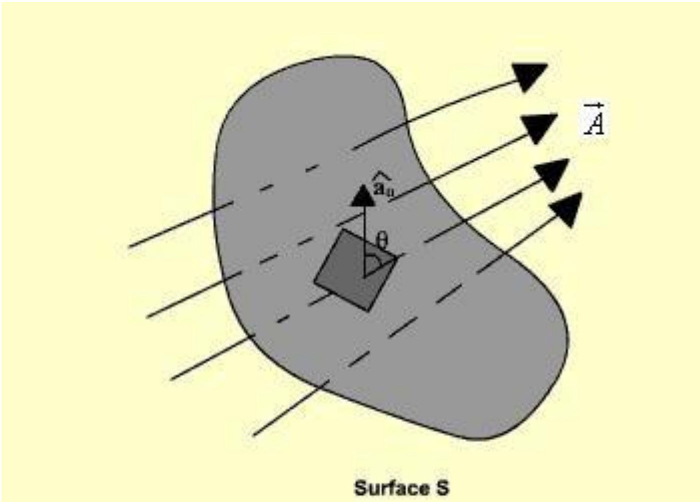


Fig 1.16 : Surface Integral

If the surface integral is carried out over a closed surface, then we write $\psi = \oint_S \vec{A} d\vec{S}$

Volume Integrals:

We define $\int_V f dV$ or $\iiint_V f dV$ as the volume integral of the scalar function f(function of spatial coordinates) over the volume V. Evaluation of integral of the form $\int_V \vec{F} dV$ can be carried out as a sum of three scalar volume integrals, where each scalar volume integral is a component of the vector $\vec{F}$

The Del Operator :

The vector differential operator ∇ was introduced by Sir W. R. Hamilton and later on developed by P. G. Tait.

Mathematically the vector differential operator can be written in the general form as:

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial}{\partial w}\hat{a}_w \quad \text{..................................(1.43)}$$

Gradient of a Scalar function:

In Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \quad \text{..(1.44)}$$

In cylindrical coordinates:

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z \quad \text{...(1.45)}$$

and in spherical polar coordinates:

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi \quad \text{.................................(1.46)}$$

Let us consider a scalar field $V(u,v,w)$, a function of space coordinates.

Gradient of the scalar field V is a vector that represents both the magnitude and direction of the maximum space rate of increase of this scalar field V.

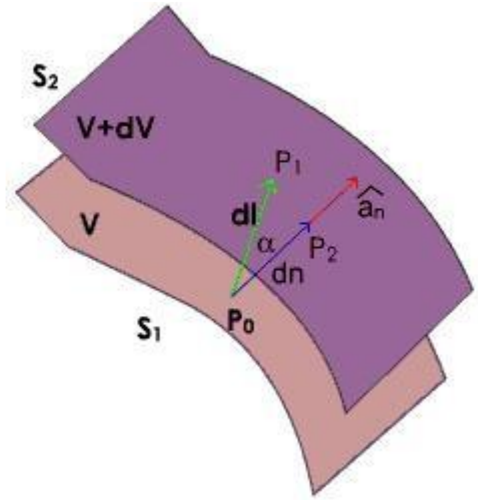


Fig 1.17 : Gradient of a scalar function

As shown in figure 1.17, let us consider two surfaces $S1$and $S2$ where the function V has constant magnitude and the magnitude differs by a small amount dV. Now as one moves from $S1$ to $S2$, the magnitude of spatial rate of change of V i.e. dV/dl depends on the direction of elementary path length dl, the maximum occurs when one traverses from $S1$to $S2$along a path normal to the surfaces as in this case the distance is minimum.

By our definition of gradient we can write:

$$\text{grad}V = \frac{dV}{dn}\hat{a}_n = \nabla V \quad \text{...(1.47)}$$

since $d\vec{n}$ which represents the distance along the normal is the shortest distance between the two surfaces.

For a general curvilinear coordinate system

$$d\vec{l} = \hat{a}_u\, dl_u + \hat{a}_v\, dl_v + \hat{a}_w\, dl_w = \left(h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w\right) \quad \text{..........(1.48)}$$

Further we can write

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \nabla V \cdot \hat{a}_l \quad \text{...(1.49)}$$

Hence,

$$dV = \nabla V.dl = \nabla V.(h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w) \quad \text{....................................(1.50)}$$

Also we can write,

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial l_u} dl_u + \frac{\partial V}{\partial l_v} dl_v + \frac{\partial V}{\partial l_w} dl_w \\
&= \left(\frac{\partial V}{\partial l_u} \hat{a}_u + \frac{\partial V}{\partial l_v} \hat{a}_v + \frac{\partial V}{\partial l_w} \hat{a}_w \right) \cdot \left(dl_u \hat{a}_u + dl_v \hat{a}_v + dl_w \hat{a}_w \right) \\
&= \left(\frac{\partial V}{h_1 \partial u} \hat{a}_u + \frac{\partial V}{h_2 \partial v} \hat{a}_v + \frac{\partial V}{h_3 \partial w} \hat{a}_w \right) \cdot \left(h_1 du \hat{a}_u + h_2 dv \hat{a}_v + h_3 dw \hat{a}_w \right)
\end{aligned}
$$
............................(1.51)

By comparison we can write,

$$\nabla V = \frac{1}{h_1} \frac{\partial V}{\partial u} \hat{a}_u + \frac{1}{h_2} \frac{\partial V}{\partial v} \hat{a}_v + \frac{1}{h_3} \frac{\partial V}{\partial w} \hat{a}_w$$
....(1.52)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for gradient can be written as:
In Cartesian coordinates:

$$\nabla V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z$$
....(1.53)

In cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z$$
....(1.54)

and in spherical polar coordinates:

$$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi$$
....(1.55)

The following relationships hold for gradient operator.

$$
\begin{aligned}
&\nabla (U + V) = \nabla U + \nabla V \\
&\nabla (UV) = V \nabla U + U \nabla V \\
&\nabla \left(\frac{U}{V}\right) = \frac{V \nabla U - U \nabla V}{V^2} \\
&\nabla V^n = n V^{n-1} \nabla V
\end{aligned}
$$
....(1.56)

where U and V are scalar functions and n is an integer.

It may further be noted that since magnitude of $\frac{dV}{dl}(= \Delta V . \hat{a}_1)$ depends on the direction of dl, it is called the **directional derivative**. If $A = \Delta V$, V is called the scalar potential function of the vector function $\vec{A}$.

Divergence of a Vector Field:

In study of vector fields, directed line segments, also called flux lines or streamlines, represent field variations graphically. The intensity of the field is proportional to the density of lines. For example, the number of flux lines passing through a unit surface S normal to the vector measures the vector field strength.

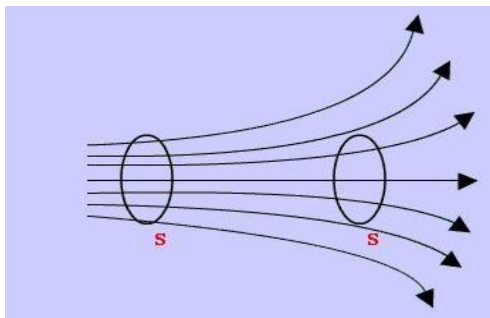


Fig 1.18: Flux Lines

We have already defined flux of a vector field as

$$\psi = \int_S A\cos\theta ds = \int_S \vec{A}\cdot\hat{a}_n ds = \int_S \vec{A}\cdot d\vec{s} \qquad \text{...(1.57)}$$

For a volume enclosed by a surface,

$$\psi = \oint_S \vec{A}\cdot d\vec{s} \qquad \text{...(1.58)}$$

We define the divergence of a vector field $\vec{A}$ at a point P as the net outward flux from a volume enclosing P, as the volume shrinks to zero.

$$div\,\vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v\to 0}\frac{\oint_s \vec{A}\cdot d\vec{s}}{\Delta v} \qquad \text{..(1.59)}$$

Here ΔV is the volume that encloses P and S is the corresponding closed surface.

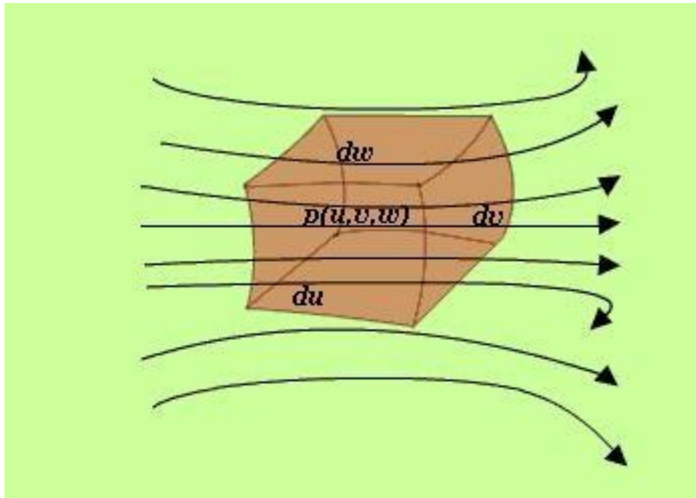


Fig 1.19: Evaluation of divergence in curvilinear coordinate

Let us consider a differential volume centered on point P(u,v,w) in a vector field $\vec{A}$. The flux through an elementary area normal to u is given by ,

$$\phi_u = \vec{A} \cdot \hat{a}_u h_2 h_3 dv dw \quad \text{.....................................(1.60)}$$

Net outward flux along u can be calculated considering the two elementary surfaces perpendicular to u .

$$\left[h_2 h_3 A_u \Big|_{\left(u+\frac{du}{2},v,w\right)} - h_2 h_3 A_u \Big|_{\left(u-\frac{du}{2},v,w\right)} \right] dv dw \cong \frac{\partial \left(h_2 h_3 A_u \right)}{\partial u} du dv dw$$

...............(1.61) Considering the contribution from all six surfaces that enclose the volume, we can write

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot \vec{ds}}{\Delta v} = \frac{du dv dw \frac{\partial \left(h_2 h_3 A_u \right)}{\partial u} + du dv dw \frac{\partial \left(h_1 h_3 A_v \right)}{\partial v} + du dv dw \frac{\partial \left(h_1 h_2 A_w \right)}{\partial w}}{h_1 h_2 h_3 du dv dw}$$

$$\therefore \nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial \left(h_2 h_3 A_u \right)}{\partial u} + \frac{\partial \left(h_1 h_3 A_v \right)}{\partial v} + \frac{\partial \left(h_1 h_2 A_w \right)}{\partial w} \right]$$

....(1.62)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for divergence ca written as:

In Cartesian coordinates:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{................................(1.63)}$$

In cylindrical coordinates:

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad \text{...(1.64)}$$

and in spherical polar coordinates:

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \quad \text{......................................(1.65)}$$

In connection with the divergence of a vector field, the following can be noted

- Divergence of a vector field gives a scalar.
- $$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$$
  $$\nabla \cdot (V\vec{A}) = V\nabla \cdot \vec{A} + \vec{A} \cdot \nabla V \quad \text{...(1.66)}$$

Divergence theorem :

Divergence theorem states that the volume integral of the divergence of vector field is equal to the net outward flux of the vector through the closed surface that bounds the volume. Mathematically, $\int_V \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot d\vec{s}$

Proof:

Let us consider a volume V enclosed by a surface S . Let us subdivide the volume in large number of cells. Let the k^{th} cell has a volume ΔV_K and the corresponding surface is denoted by S_k. Interior to the volume, cells have common surfaces. Outward flux through these common surfaces from one cell becomes the inward flux for the neighboring cells. Therefore when the total flux from these cells are considered, we actually get the net outward flux through the surface surrounding the volume. Hence we can write:

$$\oint_S \vec{A}\cdot d\vec{s} = \sum_k \oint_{S_k} \vec{A}\cdot d\vec{s} = \sum_k \frac{\oint_{S_k} \vec{A}\cdot d\vec{s}}{\Delta V_k}\Delta V_k \qquad \text{.....................................(1.67)}$$

In the limit, that is when $K \to \infty$ and $\Delta V_K \to 0$ the right hand of the expression can be written as $\int_V \nabla . A dV$.

Hence we get $\oint_S \vec{A}\cdot d\vec{S} = \int_V \nabla\cdot A dV$, which is the divergence theorem.

Curl of a vector field:

We have defined the circulation of a vector field *A* around a closed path as $\oint_l \vec{A}\cdot d\vec{l}$.

Curl of a vector field is a measure of the vector field's tendency to rotate about a point. Curl $\vec{A}$, also written as $\nabla\times\vec{A}$ is defined as a vector whose magnitude is maximum of the net circulation per unit area when the area tends to zero and its direction is the normal direction to the area when the area is oriented in such a way so as to make the circulation maximum.

Therefore, we can write:

$$Curl\ \vec{A} = \nabla\times\vec{A} = \lim_{\Delta S\to 0}\frac{\hat{a}_n}{\Delta S}\left[\oint_l \vec{A}\cdot dl\right]_{max} \qquad \text{......................................(1.68)}$$

To derive the expression for curl in generalized curvilinear coordinate system, we first compute $\nabla\times\vec{A}.\hat{a}_u$ and to do so let us consider the figure 1.20 :

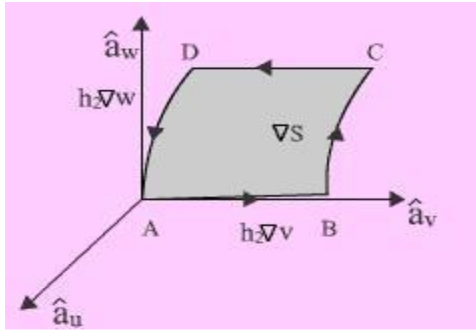


Fig 1.20: Curl of a Vector

$C1$ represents the boundary of ΔS, then we can write

$$\oint_{C_1} \vec{A}\cdot d\vec{l} = \int_{AB} \vec{A}\cdot d\vec{l} + \int_{BC} \vec{A}\cdot d\vec{l} + \int_{CD} \vec{A}\cdot d\vec{l} + \int_{DA} \vec{A}\cdot d\vec{l} \quad ....(1.69)$$

The integrals on the RHS can be evaluated as follows:

$$\int_{AB} \vec{A}\cdot d\vec{l} = \left(A_u\hat{a}_u + A_v\hat{a}_v + A_w\hat{a}_w\right)\cdot h_2\Delta v\hat{a}_v = A_v h_2 \Delta v \quad ....(1.70)$$

$$\int_{CD} \vec{A}\cdot d\vec{l} = -\left(A_v h_2 \Delta v + \frac{\partial}{\partial w}(A_v h_2 \Delta v)\Delta w\right) \quad ....(1.71)$$

The negative sign is because of the fact that the direction of traversal reverses. Similarly,

$$\int_{BC} \vec{A}\cdot d\vec{l} = \left(A_w h_3 \Delta w + \frac{\partial}{\partial v}(A_w h_3 \Delta w)\Delta v\right) \quad ....(1.72)$$

$$\int_{DA} \vec{A}\cdot d\vec{l} = -A_w h_3 \Delta w \quad ....(1.73)$$

Adding the contribution from all components, we can write:

$$\oint_{C_1} \vec{A}\cdot d\vec{l} = \left(\frac{\partial}{\partial v}(A_w h_3) - \frac{\partial}{\partial w}(A_v h_3)\right)\Delta v \Delta w$$

……………………………………………..............(1.74)Therefore,

$$\left(\nabla\times\vec{A}\right).\hat{a}_u = \frac{\oint_{C_1} \vec{A}\cdot d\vec{l}}{h_2 h_3 \Delta v \Delta w} = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 A_w)}{\partial v} - \frac{\partial(h_2 A_v)}{\partial w}\right) \quad ....(1.75)$$

In the same manner if we compute for $\left(\nabla\times\vec{A}\right).\hat{a}_v$ and $\left(\nabla\times\vec{A}\right).\hat{a}_w$ we can write,

$$\nabla\times\vec{A}=\frac{1}{h_2h_3}\left(\frac{\partial(h_3A_w)}{\partial v}-\frac{\partial(h_2A_v)}{\partial w}\right)\hat{a}_u+\frac{1}{h_1h_3}\left(\frac{\partial(h_1A_u)}{\partial w}-\frac{\partial(h_3A_w)}{\partial u}\right)\hat{a}_v+\frac{1}{h_1h_2}\left(\frac{\partial(h_2A_v)}{\partial u}-\frac{\partial(h_1A_u)}{\partial v}\right)\hat{a}_w$$
.......(1.76)

This can be written as,

$$\nabla\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1A_u & h_2A_v & h_3A_w \end{vmatrix}$$ ....(1.77)

In Cartesian coordinates:
$$\nabla\times\vec{A}=\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$ ....(1.78)

In Cylindrical coordinates,
$$\nabla\times\vec{A}=\frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$ ....(1.79)

In Spherical polar coordinates,
$$\nabla\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$ ....(1.80) Curl operation exhibits the following properties:

(i) *Curl of a vector field is another vector field.*

(ii) $\nabla\times(\vec{A}+\vec{B})=\nabla\times\vec{A}+\nabla\times\vec{B}$

(iii) $\nabla\times(V\vec{A})=\nabla V\times\vec{A}+V\nabla\times\vec{A}$

(iv) $\nabla.(\nabla\times\vec{A})=0$

(v) $\nabla\times\nabla V=0$

(vi) $\nabla\times(\vec{A}\times\vec{B})=\vec{A}\nabla.\vec{B}-\vec{B}\nabla.\vec{A}+(\vec{B}.\nabla)\vec{A}-(\vec{A}.\nabla)\vec{B}$ ....(1.81)

Stoke's theorem :

It states that the circulation of a vector $\vec{A}$ field around a closed path is equal to the integral of $\nabla\times\vec{A}$ over the surface bounded by this path. It may be noted that this equality holds provided $\vec{A}$ and $\nabla\times\vec{A}$ are continuous on the surface.

i.e,

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{s} \quad ....(1.82)$$

Proof: Let us consider an area S that is subdivided into large number of cells as shown in the figure 1.21.

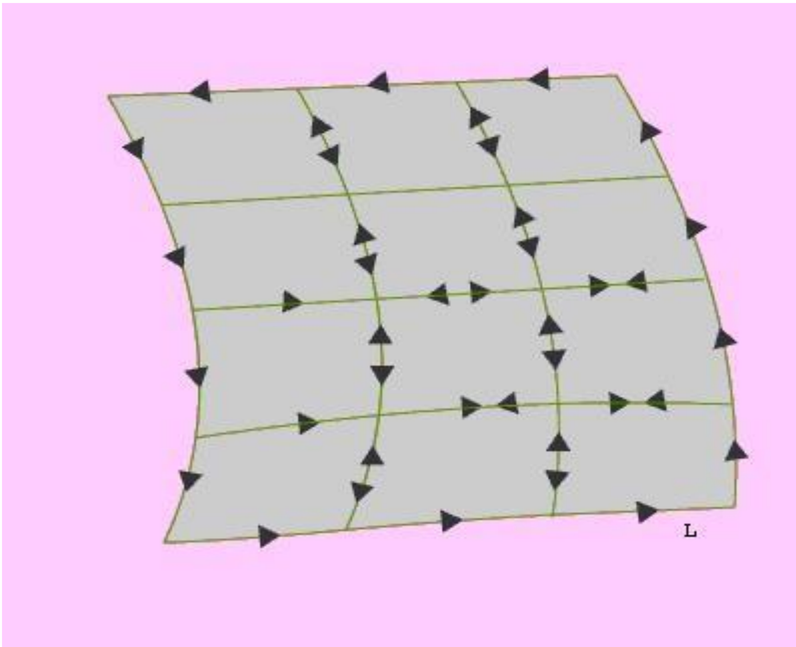


Fig 1.21: Stokes theorem

Let k^{th} cell has surface area ΔS_k and is bounded path L_k while the total area is bounded by path *L*. As seen from the figure that if we evaluate the sum of the line integrals around the elementary areas, there is cancellation along every interior path and we are left the line integral along path *L*. Therefore we can write,

$$\oint_L \vec{A}\cdot d\vec{l} = \sum_k \oint_{L_k} \vec{A}\cdot d\vec{l} = \sum_k \frac{\oint_{L_k} \vec{A}\cdot d\vec{l}}{\Delta S_k}\Delta S_k \quad ....(1.83)$$

As $\Delta S_k \rightarrow 0$

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{s} \quad ....(1.84)$$

which is the stoke's theorem.

Coulomb's Law

Coulomb's Law states that the force between two point charges $Q1$ and $Q2$ is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. Point charge is a hypothetical charge located at a single point in space. It is an idealised model of a particle having an electric charge.

$$F = \frac{kQ_1Q_2}{R^2}$$

Mathematically,where k is the proportionality constant.

In SI units, $Q1$ and $Q2$ are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $k = \frac{1}{4\pi\varepsilon_0}$, ε_0 is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0 \varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$ ....(2.1)

As shown in the Figure 2.1 let the position vectors of the point charges $Q1$ and $Q2$ are given by $\vec{r_1}$ and $\vec{r_2}$. Let $\vec{F_{12}}$ represent the force on $Q1$ due to charge $Q2$.

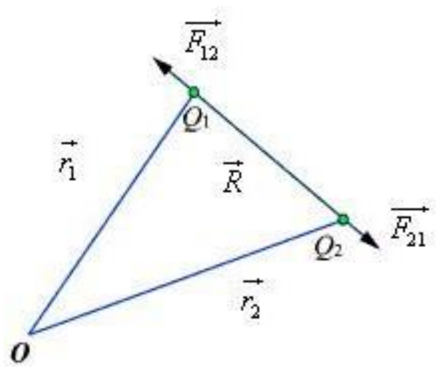


Fig 1.22: Coulomb's Law

The charges are separated by a distance of $R = \left|\vec{r_1} - \vec{r_2}\right| = \left|\vec{r_2} - \vec{r_1}\right|$. We define the unit vectors as

$$\widehat{a_{12}} = \frac{\left(\vec{r_2} - \vec{r_1}\right)}{R} \text{ and } \widehat{a_{21}} = \frac{\left(\vec{r_1} - \vec{r_2}\right)}{R}$$ ....(2.2)

$\vec{F_{12}}$ can be defined as $\vec{F_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\frac{(\vec{r_2} - \vec{r_1})}{\left|\vec{r_2} - \vec{r_1}\right|^3}$. Similarly the force on $Q1$

due to charge $Q2$ can be calculated and if $\overrightarrow{F_{21}}$ represents this force then we can write $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, *we apply principle of superposition*. If we have N number of charges

$Q1, Q2, \ldots\ldots\ldots QN$ located respectively at the points represented by the position vectors $\vec{r_1}$, $\vec{r_2}$, ...$\vec{r_N}$, the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i(\vec{r}-\vec{r_i})}{\left|\vec{r}-\vec{r_i}\right|^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

Electric Field

The electric field intensity or the electric field strength at a point is defined as the force per unit charge. That is

$$\vec{E} = \lim_{Q\to 0}\frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.4)$$

The electric field intensity E at a point r (observation point) due a point charge Q located at $\vec{r'}$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r}-\vec{r'})}{4\pi\varepsilon_0\left|\vec{r}-\vec{r'}\right|^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.5)$$

For a collection of N point charges $Q1$, $Q2$, ...QN located at $\vec{r_1}$, $\vec{r_2}$,......$\vec{r_N}$, the electric field intensity at point $\vec{r'}$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_k(\vec{r}-\vec{r_i})}{\left|\vec{r}-\vec{r_i}\right|^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.6)$$

The expression *(2.6)* can be modified suitably to compute the electric filed due to a continuous distribution of charges.
In figure 2.2 we consider a continuous volume distribution of charge d*(t)* in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r'})dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} = \frac{\rho(\vec{r}')dv'(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3}$$ ....*(2.7)*

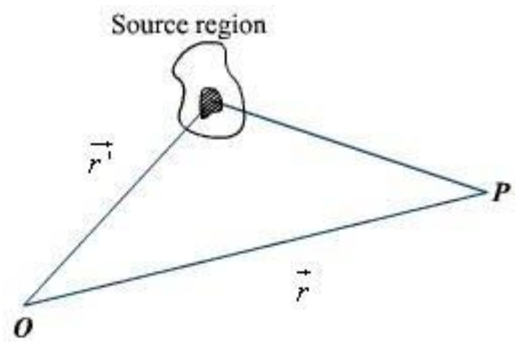


Fig 1.23: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overline{E(r)} = \int_V \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} dv'$$ .... (2.8)

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(r)} = \int_L \frac{\rho_L(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} dl'$$ .... (2.9)

$$\overline{E(r)} = \int_S \frac{\rho_s(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} ds'$$ ....(2.10)

Electric flux density:

As stated earlier electric field intensity or simply ‘Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon\vec{E}$$ ....*(2.11)*

We define the electric flux as

$$\psi = \int_S \vec{D}.d\vec{s} \qquad ....(2.12)$$

Gauss's Law: Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

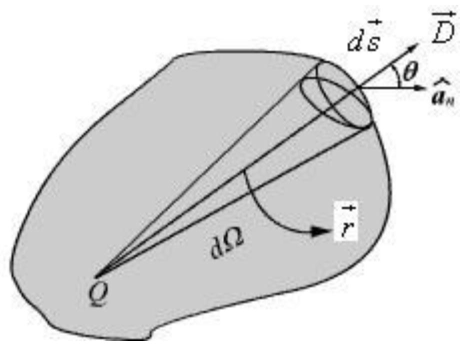


Fig 1.24: Gauss's Law

Let us consider a point charge *Q* located in an isotropic homogeneous medium of dielectric constant. The flux density at a distance *r* on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r \qquad ....(2.13)$$

If we consider an elementary area ***ds***, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2} ds\cos\theta \qquad ....(2.14)$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of *Q*. Therefore we can write $d\psi = \frac{Q}{4\pi} d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_S d\psi = \frac{Q}{4\pi}\oint_S d\Omega = Q$ which can seen to be same as what we have stated in the definition of Gauss's Law. Application of Gauss's Law

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1.15.1 An infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density LC/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 2.4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 2.4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_l l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s} \quad \text{.....................................(2.15)}$$

Considering the fact that the unit normal vector to areas $S1$ and $S3$ are perpendicular to

The electric field, the surface integrals for the top and bottom surfaces evaluates to zero.

Hence we can write, $\rho_l l = \varepsilon_0 E.2\pi\rho l$

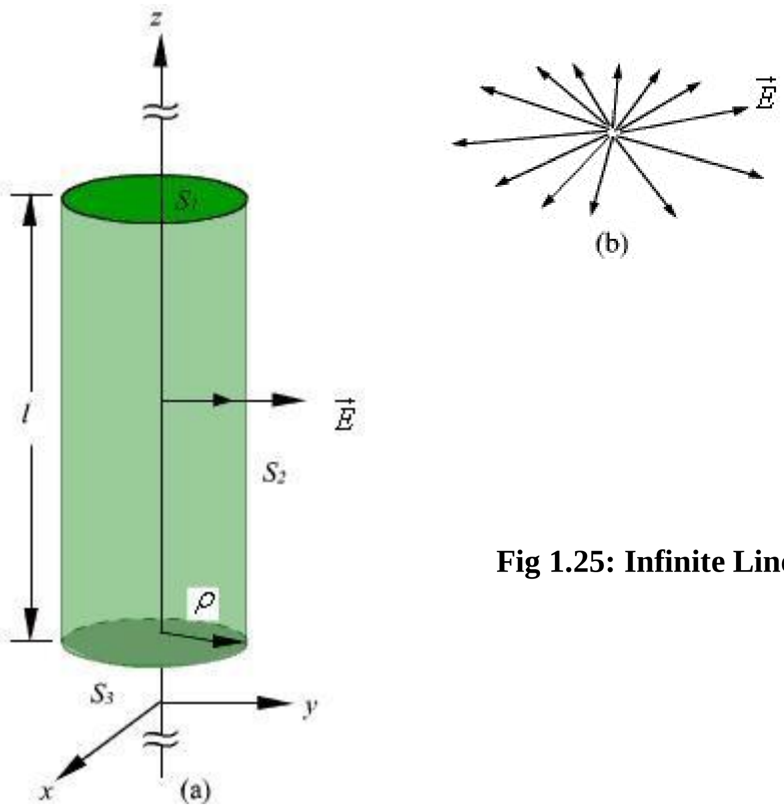


Fig 1.25: Infinite Line Charge

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0\rho}\hat{a}_\rho \quad \text{.....................................}(2.16)$$

Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x-z* plane as shown in figure *2*.5.

Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D} \cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y \quad \text{.............}(2.17)$$

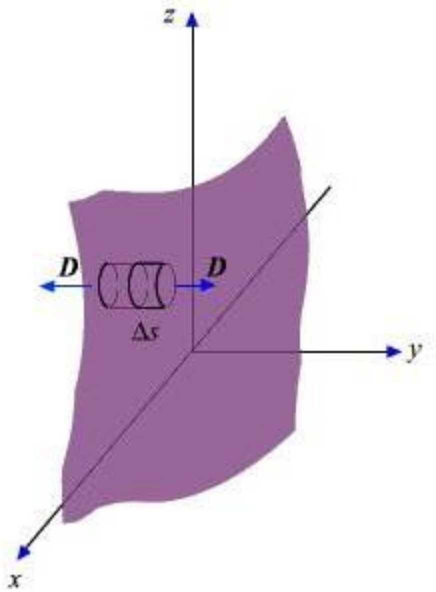


Fig 1.26: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

Uniformly Charged Sphere

Let us consider a sphere of radius r_0 having a uniform volume charge density of ρ_v C/m^3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius $r < r_0$ and $r > r_0$ as shown in Fig. *2*.6 (a) and Fig. *2*.6(b).

For the region $r < r_0$, the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3 \qquad \text{.........................}(2.18)$$

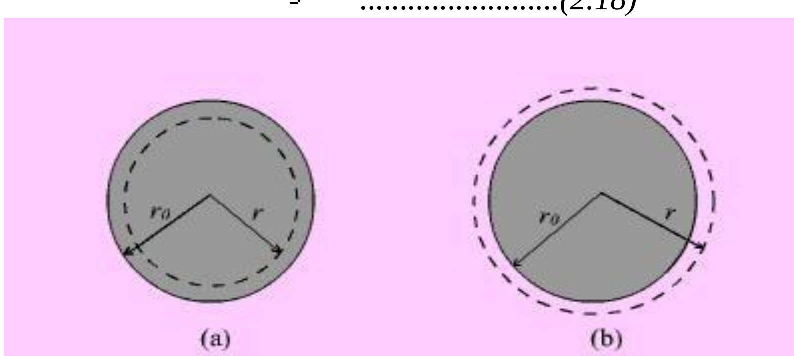


Fig 1.*27*:Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en}$$ ....*(2.19)*

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0$$ ....*(2.20)*

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3$$ ....(2.21

) By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0$$ ....(2.22)

Gauss divergence theorem:

The gauss law can be stated in the point form by the divergence of electric flux density is equal to the volume charge density.

Absolute Electric Potential and potential differences and its calculation.

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field.

Let us suppose that we wish to move a positive test charge Δq from a point *P* to another point *Q* as shown in the Fig. 2.8.

The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while Δq moves from *P* to *Q*. The work done by this external agent in moving the charge by a distance $d\vec{l}$ is given by:

$$dW = -\Delta q \vec{E} . d\vec{l} \quad ....(2.23)$$

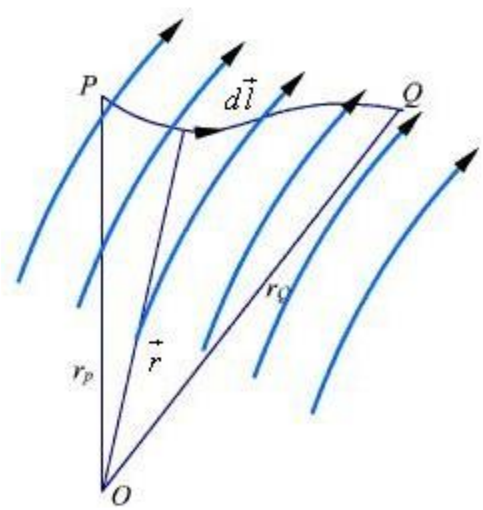


Fig 1.28: Movement of Test Charge in Electric Field

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l} \quad ....(2.24)$$

The potential difference between two points *P* and *Q* , *VPQ*, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_P^Q \vec{E} \cdot d\vec{l} \quad ....(2.25)$$

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in

Joules/Coulomb which is referred to as **Volts**.

Let us consider a point charge Q as shown in the Fig. *2*.9.

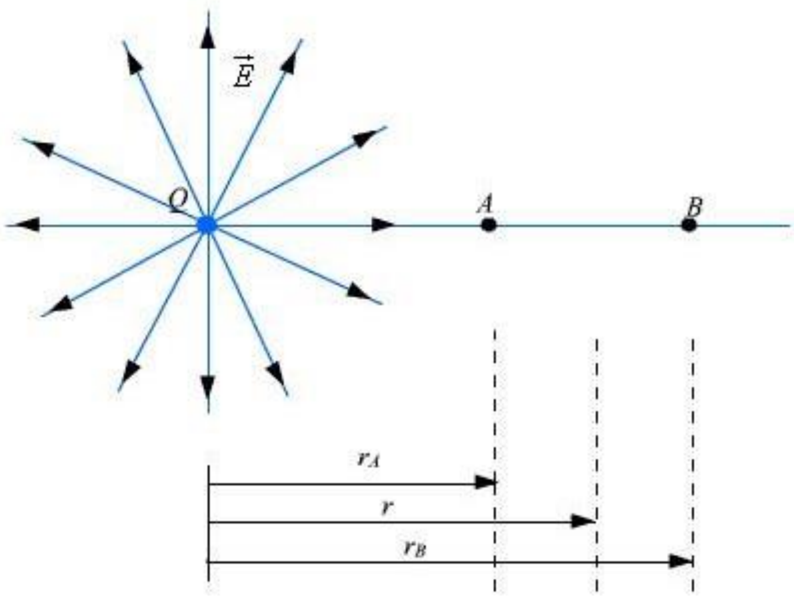


Fig 1.29: Electric Potential calculation for a point charge

Further consider the two points A and B as shown in the Fig. *2*.9. Considering the movement of a unit positive test charge from B to A , we can write an expression for the potential difference as:

$$V_{BA} = -\int_B^A \vec{E} \cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B$$ ....*(2.26)*

It is customary to choose the potential to be zero at infinity. Thus potential at any point ($r_A = r$) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. $r_B = 0$).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$ ....*(2.27)*

Or, in other words,

$$V = -\int_{\infty}^{r} E.dl$$ ....(2.28)

Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. *2*.10.

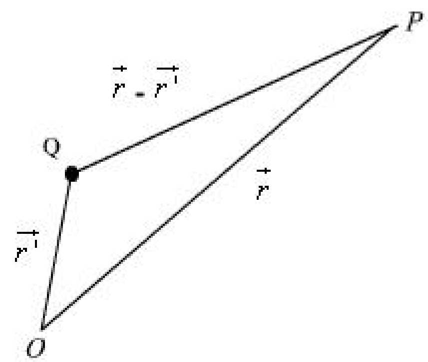


Fig *1.30*: Electric Potential due a Displaced Charge

The potential at a point *P* becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\left|\vec{r} - \vec{r}'\right|} \quad \text{..............................}(2.29)$$

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also.

Let us first consider *N* point charges *Q1, Q2,.....QN* located at points with position vectors $\vec{r_1}$, $\vec{r_2}$,...... $\vec{r_N}$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r} - \vec{r_1}\right|} + \frac{Q_2}{\left|\vec{r} - \vec{r_2}\right|} + \ldots\ldots \frac{Q_N}{\left|\vec{r} - \vec{r_N}\right|}\right) \quad \text{...............................}(2.30a)$$

or,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{\left|\vec{r} - \vec{r_n}\right|} \quad \text{..}(2.30b)$$

For continuous charge distribution, we replace point charges *Qn* by corresponding charge elements $\rho_L dl$ or $\rho_S ds$ or $\rho_V dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For line charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')dl'}{|\vec{r}-\vec{r}_n|}$$ ....*(2.31)*

For surface charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}')ds'}{|\vec{r}-\vec{r}_n|}$$ ....*(2.32)*

For volume charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}')dv'}{|\vec{r}-\vec{r}_n|}$$ ....*(2.33)*

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$ ....*(2.34)*

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$$V = -\int \vec{E}\cdot d\vec{l} + C$$ ....*(2.35)*

The potential difference is however independent of the choice of reference.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} = \frac{W}{Q}$$ ....*(2.36)*

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point P_1 to P_2 in one path and then from point P_2 back to P_1 over a different path.

If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position P_1. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable, the work must have to be independent of path and depends on the initial and final positions.

Since the potential difference is independent of the paths taken, $V_{AB} = -V_{BA}$, and over a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{l} = 0$$ ....(2.37)

Applying Stokes's theorem, we can write:

$$\oint \vec{E} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{E}\right) \cdot d\vec{s} = 0$$ ....(2.38)

from which it follows that for electrostatic field,

$$\nabla \times \vec{E} = 0$$ ....(2.39)

Any vector field $\vec{A}$ that satisfies $\nabla \times \vec{A} = 0$ is called an

irrotational field. From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial x} dz = -\vec{E} \cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right) \cdot \left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right) = -\vec{E} \cdot d\vec{l}$$

$$\nabla V \cdot d\vec{l} = -\vec{E} \cdot d\vec{l}$$ ....(2.40)

from which we obtain,

$$\vec{E} = -\nabla V$$ ....(2.41)

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that $\vec{E}$ is directed from higher to lower values of $\vec{V}$. This gives us another method of computing the electric field, i. e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function $\vec{V}$ contain all the information that three components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla \times \vec{E}$.

Electric Dipole

An electric dipole consists of two point charges of equal magnitude but of opposite sign and separated by a small distance.

As shown in figure *2*.11, the dipole is formed by the two point charges *Q* and -*Q*

separated by a distance d , the charges being placed symmetrically about the origin. Let us consider a
point P at a distance r, where we are interested to find the field.

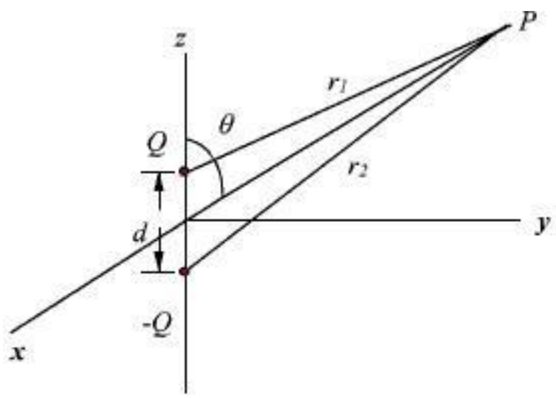


Fig *1*.31 : Electric Dipole

The potential at P due to the dipole can be written as:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$ ....*(2.42)*

When $r1$ and $r2 >> d$, we can write $r_2 - r_1 = 2\times\frac{d}{2}\cos\theta = d\cos\theta$ and . Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}$$ ....*(2.43)*

We can write,

$$Qd\cos\theta = Qd\hat{a}_z \cdot \hat{a}_r$$ ....(2.44)

The quantity $\vec{P} = Q\vec{d}$ is called the **dipole moment** of the electric dipole.

Hence the expression for the electric potential can now be written as:

$$V = \frac{\vec{P}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2}$$ ....*(2.45)*

It may be noted that while potential of an isolated charge varies with distance as $1/r$ that of an electric dipole varies as $1/r^2$ with distance. If the dipole is not centered at the origin, but the dipole center lies at $\vec{r}'$, the expression for the potential can be written as:

$$V = \frac{\vec{P}\cdot(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3}$$

......................*(2.46)*

The electric field for the dipole centered at the origin can be computed as

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta$$

$$= \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$\vec{E} = \frac{\vec{P}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

........................(2.47)

$\vec{P} = Q\vec{d}$ is the magnitude of the dipole moment. Once again we note that the electric field of electric dipole varies as $1/r^3$ where as that of a point charge varies as $1/r^2$.

Electrostatic Energy and Energy Density

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point. To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges Q_1, Q_2,......., Q_N are brought from infinity to their present position one by one. Since initially there is no field present, the amount of work done in bring Q_1 is zero. Q_2 is brought in the
presence of the field of Q_1, the work done $W_1 = Q_2V_{21}$ where V_{21} is the potential at the location
of Q_2 due to Q_1. Proceeding in this manner, we can write, the total work done

$$W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \dots\dots\dots\dots\dots + (V_{M1}Q_N + \dots\dots\dots + V_{N(N-1)}Q_N)$$

....(2.89)

Had the charges been brought in the reverse order,

Therefore,

Here V_{IJ} represent voltage at the I^{th} charge location due to J^{th} charge. Therefore,

$$2W = V_1Q_1 + \dots\dots\dots\dots + V_NQ_N = \sum_{I=1}^{N} V_IQ_I \quad .....(2.91)$$

Or,

$$W = \frac{1}{2}\sum_{I=1}^{N} V_IQ_I \quad ....(2.92)$$

If instead of discrete charges, we now have a distribution of charges over a volume *v* then we can write,

$$W = \frac{1}{2}\int_v V\rho_v dv \quad ....(2.93)$$

where ρ_v is the volume charge density and *V* represents the potential function.

$$\rho_v = \nabla\cdot\vec{D} \qquad W = \frac{1}{2}\int_v \left(\nabla\cdot\vec{D}\right)V dv \quad ....(2.94)$$

Using the vector identity,

$\nabla.(V\vec{D}) = \vec{D}\cdot\nabla V + V\nabla\cdot\vec{D}$, we can write

$$W = \frac{1}{2}\int_v \left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V\right) dv$$

$$= \frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s} - \frac{1}{2}\int_v \left(\vec{D}\cdot\nabla V\right) dv \quad ....(2.95)$$

In the expression $\frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s}$, for point charges, since *V* varies as $\frac{1}{r}$ and D varies as $\frac{1}{r^2}$, the term $V\vec{D}$ varies as $\frac{1}{r^3}$ while the area varies as r^2. Hence the integral term varies at least as $\frac{1}{r}$ and the as surface becomes large (i.e. $r \rightarrow \infty$) the integral term tends to zero.

Thus the equation for *W* reduces to

$$W = -\frac{1}{2}\int_v \left(\vec{D}.\nabla V\right)dv = \frac{1}{2}\int_v \left(\vec{D}.\vec{E}\right)dv = \frac{1}{2}\int_v \left(\varepsilon E^2\right)dv = \int_v w_e dv \quad ....(2.96)$$

$w_e = \frac{1}{2}\varepsilon E^2$, is called the energy density in the electrostatic field.

2-Marks and 16 Marks ELECTROMAGNETIC FIELDS

1. State stokes theorem.

The line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over any surface bounded by the path

H.dl = (ÑxH)ds

2. State coulombs law.

Coulombs law states that the force between any two point charges is directly proportional to the product of their magnitudes and inversely proportional to the square of the distance between them. It is directed along the line joining the two charges.

$F=Q_1Q_2 / 4\pi\varepsilon r^2$ ar

3. State Gauss law for electric fields

The total electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

4. Define electric flux.

The lines of electric force is electric flux.

5. Define electric flux density.

Electric flux density is defined as electric flux per unit area.

6. Define electric field intensity.

Electric field intensity is defined as the electric force per unit positive charge.

$E = F/ Q$

$= Q/4\pi\varepsilon r^2$ V/m

7. Name few applications of Gauss law in electrostatics.

Gauss law is applied to find the electric field intensity from a closed surface.e.g)Electric field can be determined for shell, two concentric shell or cylinders etc.

8. What is a point charge?

Point charge is one whose maximum dimension is very small in comparison with any other length.

9. Define linear charge density.

It is the charge per unit length.

10. Write poisson's and laplace 's equations.

Poisson 's eqn:

$$\tilde{N}^2V = -\rho_v / \varepsilon$$

Laplace' s eqn:

$$\tilde{N}^2V = 0$$

11. State the condition for the vector F to be solenoidal.

$$\tilde{N} \cdot F = 0$$

12. .State the condition for the vector F to be irrotational.

$$\tilde{N} x F = 0$$

13. Define potential difference.

Potential difference is defined as the work done in moving a unit positive charge from one point to another point in an electric field.

14. Define potential.

Potential at any point is defined as the work done in moving a unit positive charge from infinity to that point in an electric field.

$V=Q / 4\pi\varepsilon r$

15. Give the relation between electric field intensity and electric flux density.

$D=\varepsilon E$ C/m^2

16. Give the relationship between potential gradiant and electric field.

E= - ÑV

17. What is the physical significance of div D ?

Ñ·D=-ρv

The divergence of a vector flux density is electric flux per unit volume leaving a small volume. This is equal to the volume charge density.

18. Define current density.

Current density is defined as the current per unit area.

$J= I/A$ Amp/m^2

19. Write the point form of continuity equation and explain its significance.

Ñ·J= - ρv / t

20. Write the expression for energy density in electrostatic field.

$W=1 / 2\ \varepsilon E^2$

21. Write the boundary conditions at the interface between two perfect dielectrics.

i) The tangential component of electric field is continuous i.e)Et1=Et2

ii) The normal component of electric flux density is continuous I.e)Dn1=Dn2

22. Write down the expression for capacitance between two parallel plates.

$C = \varepsilon A / d$

23. What is meant by displacement current?

Displacement current is nothing but the current flowing through capacitor.

$J = D / t$

24. State point form of ohms law.

Point form of ohms law states that the field strength within a conductor is proportional to the current density.

$J = \sigma E$

25 Define surface charge density.

It is the charge per surface area.

26. State amperes circuital law.

Magnetic field intensity around a closed path is equal to the current enclosed by the path.

$H \cdot dl = I$

27. State Biot –Savarts law.

It states that the magnetic flux density at any point due to current element is proportional to the current element and sine of the angle between the elemental length and inversely proportional to the square of the distance between them

$dB = \mu_0 Idl \sin\theta / 4\pi r^2$

28. Define magnetic vector potential.

It is defined as that quantity whose curl gives the magnetic flux density.

B=Ñ x A

$= \mu / 4\pi \quad J/r \, dv \; web/m^2$

29. Write down the expression for magnetic field at the centre of the circular coil.

$H = I/2a$.

30. Give the relation between magnetic flux density and magnetic field intensity.

$B = \mu H$

31. Write down the magnetic boundary conditions.

i) The normal components of flux density B is continuous across the boundary.
ii) The tangential component of field intensity is continuous across the boundary.

32. Give the force on a current element.

$dF = BIdl\sin\theta$

33..Define magnetic moment.

Magnetic moment is defined as the maximum torque per magnetic induction of flux density.
$m=IA$

34. State Gauss law for magnetic field.

The total magnetic flux passing through any closed surface is equal to zero.
$B.ds = 0$

35. Define a wave.

If a physical phenomenon that occurs at one place at a given time is reproduced at other places at later times , the time delay being proportional to the space separation from the first location then the group of phenomena constitutes a wave.

36. Mention the properties of uniform plane wave.

i) At every point in space ,the electric field E and magnetic field H are perpendicular to each other.
ii)The fields vary harmonically with time and at the same frequency everywhere in space.

37. Write down the wave equation for E and H in free space.

$Ñ^2H - \mu 0\varepsilon 0 \quad ^2H / \quad t^2 \qquad =0.$

38. Define intrinsic impedance or characteristic impedance.

It is the ratio of electric field to magnetic field.or It is the ratio of square root of permeability to permittivity of medium.

39. Give the characteristic impedance of free space.

377ohms

40. Define propagation constant.

Propagation constant is a complex number

$\gamma = \alpha + j\beta$

where α is attenuation constant

β is phase constant

$\gamma = j\omega\mu \, (\sigma + j\omega\varepsilon)$

41. Define skin depth

It is defined as that depth in which the wave has been attenuated to 1/e or approximately 37% of its original value.

$\Delta = 1/\alpha = \quad 2 / j\omega\sigma$

42. Define Poynting vector.

The pointing vector is defined as rate of flow of energy of a wave as it propagates.

P =E X H

43. State Poyntings Theorem.

The net power flowing out of a given volume is equal to the time rate of decrease of the the energy stored within the volume- conduction losses.

44. Give significant physical difference between poisons and laplaces equations.

When the region contains charges poisons equation is used and when there is no charges laplaces equation is applied.

45. Give the difficulties in FDM.

FDM is difficult to apply for problems involving irregular boundaries and non homogenious material properties.

46. Explain the steps in finite element method.

i) Discretisation of the solution region into elements.
ii) Generation of equations for fields at each element
iii) Assembly of all elements
iv) Solution of the resulting system

47. State Maxwells fourth equation.

The net magnetic flux emerging through any closed surface is zero.

48. State Maxwells Third equation

The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

49. State the principle of superposition of fields.

The total electric field at a point is the algebraic sum of the individual electric field at that point.

50. Define ohms law at a point

Ohms law at appoint states that the field strength within a conductor is proportional to current density.

51. Define self inductance.

Self inductance is defined as the rate of total magnetic flux linkage to the current through the coil.

52. Define pointing vector.

The vector product of electric field intensity and magnetic field intensity at a point is a measure of the rate of energy flow per unit area at that point.

53. Give the formula to find potential at a point which is surrounded by four orthogonal points in FDM.

$V0= ¼(V1+V2+V3+V4)$

54. Give the formula to find potential at a point which is surrounded by six orthogonal points inFDM.

$V0= ¼(V1+V2+V3+V4 +V5+V6)$

55. State Lenz law.

Lenz's law states that the induced emf in a circuit produces a current which opposes the change in magnetic flux producing it.

56. What is the effect of permittivity on the force between two charges?

Increase in permittivity of the medium tends to decrease the force between two charges and decrease in permittivity of the medium tends to increase the force between two charges.

57. State electric displacement.

The electric flux or electric displacement through a closed surface is equal to the charge enclosed by the surface.

58. What is displacement flux density?

The electric displacement per unit area is known as electric displacement density or electric flux density.

59. What is the significance of displacement current?

The concept of displacement current was introduced to justify the production of magnetic field in empty space. It signifies that a changing electric field induces a magnetic field .In empty space the conduction current is zero and the magnetic fields are entirely due to displacement current.

60. Distinguish between conduction and displacement currents.

The current through a resistive element is termed as conduction current whereas the current through a capacitive element is termed as displacement current.

61. Define magnetic field strength.

The magnetic field strength (H) is a vector having the same direction as magnetic flux density.

$$H=B/\mu$$

62. Give the formula to find the force between two parallel current carrying conductors.

$$F=\mu\ I\ I_1\ /\ 2\pi R$$

63. Give the expression for torque experienced by a current carrying loop situated in a magnetic field.

$$T = IAB\sin\theta$$

64What is torque on a solenoid?

$$T = NIAB\sin\theta$$

65. Explain the conservative property of electric field.

The work done in moving a point charge around a closed path in a electric field is zero.

Such a field is said to be conservative.

$$\oint E.dl = 0$$

66. Write he expression for field intensity due to a toroid carrying a filamentary current I

$$H=NI / 2\pi R$$

67. What are equipotential surfaces?

An equipotential surface is a surface in which the potential energy at every point is of the same vale.

68. Define loss tangent.

Loss tangent is the ratio of the magnitude of conduction current density to displacement cuurrent density of the medium.

$$\text{Tan } \delta = \sigma / \omega\varepsilon$$

69. Defie reflection and transmission coefficients.

Reflection coefficient is defined as the ratio of the magnitude of the reflected field to that of the incident field.

70. Define transmission coefficients.

Transmission coefficient is defined as the ratio of the magnitude of the transmitted field to that of incident field.

71. What will happen when the wave is incident obliquely over dielectric –dielectric boundary?

When a plane wave is incident obliquely on the surface of a perfect dielectric part of the energy is transmitted and part of it is reflected .But in this case the transmitted wave will be refracted, that is the direction of propagation is altered.

72. What is the expression for energy stored in a magnetic field?

$$W = \frac{1}{2} LI^2$$

73. What is energy density in magnetic field?

$$W = \frac{1}{2} \mu H^2$$

74. Distinguish between solenoid and toroid.

Solenoid is a cylindrically shaped coil consisting of a large number of closely spaced turns of insulated wire wound usually on a non magnetic frame.

If a long slender solenoid is bent into the form of a ring and there by closed on itself it becomes a toroid.

75. Describe what are the sources of electric field and magnetic field?

Stationary charges produce electric field that are constant in time, hence the term electrostatics. Moving charges produce magnetic fields hence the term magnetostatics.

76. What are the significant physical differences between Poisson 's and laplace 's equations.

Poisson 's and laplace 's equations are useful for determining the electrostatic potential V in regions whose boundaries are known.

When the region of interest contains charges poissons equation can be used to find the potential.

When the region is free from charge laplace equation is used to find the potential.

77. State Divergence Theorem.

The integral of the divergence of a vector over a volume v is equal to the surface integral o f the normal component of the vector over the surface bounded by the volume.

78. Give the expression for electric field intensity due to a single shell of charge

$$E = Q / 4\pi\varepsilon r^2$$

79. Give the expression for potential between two spherical shells

$V = 1/ 4\pi (Q1/a – Q2/b)$

80. Define electric dipole.

Electric dipole is nothing but two equal and opposite point charges separated by a finite distance.

81. What is electrostatic force?

The force between any two particles due to existing charges is known as electrostatic force, repulsive for like and attractive for unlike.

82. Define divergence.

The divergence of a vector F at any point is defined as the limit of its surface integral per unit volume as the volume enclosed by the surface around the point shrinks to zero.

83. How is electric energy stored in a capacitor?

In a capacitor, the work done in charging a capacitor is stored in the form of electric energy.

84. What are dielectrics?

Dielectrics are materials that may not conduct electricity through it but on applying electric field induced charges are produced on its faces .The valence electron in atoms of a dielectric are tightly bound to their nucleus.

85. What is a capacitor?

A capacitor is an electrical device composed of two conductors which are separated through a dielectric medium and which can store equal and opposite charges ,independent of whether other conductors in the system are charged or not.

86. Define dielectric strength.

The dielectric strength of a dielectric is defined as the maximum value of electric field that can b applied to the dielectric without its electric breakdown.

87. What meaning would you give to the capacitance of a single conductor?

A single conductor also possess capacitance. It is a capacitor whose one plate is at infinity.

88. Why water has much greater dielectric constant than mica.?

Water has a much greater dielectric constant than mica .because water ha a permanent dipole moment, while mica does not have.

89. What is lorentz force?

Lorentz force is the force experienced by the test charge .It is maximum if the direction of movement of charge is perpendicular to the orientation of field lines.

90. Define magnetic moment.

Magnetic moment is defined as the maximum torque on the loop per unit magnetic induction.

91. Define inductance.

The inductance of a conductor is defined as the ratio of the linking magnetic flux to the current producing the flux.

$L = N\Phi / I$

92. What is main cause of eddy current?

The main cause of eddy current is that it produces ohmic power loss and causes local heating.

93. How can the eddy current losses be eliminated?

The eddy current losses can be eliminated by providing laminations. It can be proved that the total eddy current power loss decreases as the number of laminations increases.

94. What is the fundamental difference between static electric and magnetic fild lines?

There is a fundamental difference between static electric and magnetic field lines .The tubes of electric flux originate and terminates on charges, whereas magnetic flux tubes are continuous.

95. What are uniform plane waves?

Electromagnetic waves which consist of electric and magnetic fields that are perpendicular to each other and to the direction of propagation and are uniform in plane perpendicular to the direction of propagation are known as uniform plane waves.

96. Write short notes on imperfect dielectrics.

A material is classified as an imperfect dielectrics for $\sigma \ll \omega\varepsilon$, that is conduction current density is small in magnitude compared to the displacement current density.

97. What is the significant feature of wave propagation in an imperfect dielectric ?

The only significant feature of wave propagation in an imperfect dielectric compared to that in a perfect dielectric is the attenuation undergone by the wave.

98. What is the major drawback of finite difference method?

The major drawback of finite difference method is its inability to handle curved boundaries accurately.

99. What is method of images?

The replacement of the actual problem with boundaries by an enlarged region or with image charges but no boundaries is called the method of images.

100. When is method of images used?

Method of images is used in solving problems of one or more point charges in the presence of boundary surfaces.

Part-B

1. Find the electric field intensity of a straight uniformly charged wire of length 'L'm and having a linear charge density of +λ C/m at any point at a distance of 'h' m. Hence deduce the expression for infinitely long conductor.

Hints: Field due to charge element is given by:

$$dE = \lambda di / 4\pi\xi r^2$$

$$E_x = \lambda\,[\cos\alpha 1 + \cos\alpha 2]\,/4\pi\varepsilon h$$

$$E_y = \lambda\,[\sin\alpha 1 - \sin\alpha 2]\,/4\pi\varepsilon h$$

For infinitely long conductor

$$E = \lambda l\ /\ 4\pi\varepsilon h$$

2. Derive the boundary relations for electric fields.

Hints:

i) The tangential component of the electric field is continuous at the surface

$$.Et1 = Et2$$

ii) The normal component of the electric flux density is continuous if there is no surface charge density.

$$Dn1 = Dn2$$

3. Find the electric field intensity produced by a point charge distribution at P(1,1,1)caused by four identical 3nC point charges located at P1(1,1,0) P2(-1,1,0) P3(-1,-1,0) and P4(1,-1,0).

Hints:

Find the field intensity at P by using the formula

$$Ep = 1/4\varepsilon\pi[(Q1/r1p^2\ u1p) +(q2/r2p^2\ u2p) +(q3/r3p^2\ u3p)+(q4/r4p^2)u4p)]$$

4.A circular disc of radius ‘a’ m is charged with a charge density of $\sigma C/m^2$ .Find the electric field intensity at a point ‘h’m from the disc along its axis.

Hints:
Find the field due to the tangential and normal components
Total field is given by
E =ρ**s /2ε [1-cos α]**

5. Four positive charges of 10^{-9} C each are situated in the XY plane at points (0,0) (0,1) (1,0) and (1,1).Find the electric field intensity and potential at (1/2 ,1/2).

Hints:
Find the field intensity at point using the formula

$$\mathbf{E} = Q / 4\pi\varepsilon r^2 \ \mathbf{ur}$$

Find the potential at point using the formula

$$V = Q / 4\pi\varepsilon r$$

Find the field intensity at the point due to all four charges by using the superposition principle.

6. Given a electric field E = $(-6y/x^2)$ x + 6/x y + 5 z .Find the potential difference V_{AB} given A(-7,2,1) and B(4,1,2)

Hint:

Find the potential using the formula v=-/E.dl and substitute the points

7.Derive an expression for potential difference between two points in an electric field.

Hint:

The potential difference between two points r1 and r2 is

$V = V1 - V2$

$V = Q / 4\pi\varepsilon r1 \quad \_ \; Q / 4\pi\varepsilon r\,2$

8.Find the magnetic flux density at a point Z on the axis of a circular loop of radius ‘a’ that carries a direct current I.

Hints:
The magnetic flux density at a point due to the current element is given by

$dB = \mu\, Idl / \; 4\pi\, r^2$

$B = \mu\, Ia^2 / 2(a^2 + z^2)^{3/2}$

9.Determine the force per meter length between two long parallel wires A and B separated by 5cm in air and carrying currents of 40A in the same direction.

Hints:

Calculate the force per metre length using the formula
$F/L = \mu\, I_1 I_2 / 2\pi d$

In the same direction force is attractive.

10.Derive an expression for magnetic vector potential.

Hint:

magnetic vector potential is

$A = \mu / 4\pi$ ///J / r dv

11.Derive the magnetic boundary relations.

i) The tangential component of the magnetic field is continuous across the boundary

.$H_{t1} = H_{t2}$

ii) The normal component of the magnetic flux density is continuous across the boundary

$D_{n1} = D_{n2}$

12.Find the magnetic field intensity at a distance ‘h’m above an infinite straight wire carrying a steady current I.

Hints:
The magnetic flux density is calculated starting from Biot savarts law.
The magnetic flux density at any point due to aninfinite long conductor is given by

$B = \mu I / 2\pi d$

13.Two conducting concentric spherical shells with radii a and b are at potentials V_0 and 0 respectively. Determine the capacitance of the capacitor.

Hint:
Derive the capacitance between concentric spheres using the formula

$C = Q / V$

$= 4\pi\varepsilon [ab /(b-a)]$

14State and derive an expression for Poyntings theorem.

Hints:
The net power flowing out of a given volume v is equal to the time rate of decrease of the energy stored within the volume minus the conduction losses.

15.Find the forces /length between two long straight parallel conductors carrying a current of 10A in the same direction. A distance of 0.2m separates the conductors. Also find the force/length when the conductors carry currents in opposite directions.
Hints:

Calculate the force per metre length using the formula
$F/L = \mu I_1 I_2 / 2\pi d$

In opposite direction force is repulsive

16 Derive an expression for torque acting on a loop.

Hints
:When a current loop is placed parallel to a magnetic field forces act on the loop that tends to rotate the tangential force times the radial distance at which it acts is calledtorque or mechanicl moment of the loop.

T = m X B

17. Derive an expression for energy and energy density in a electric field.

Energy =$CV^2/2$

Energy density = $\varepsilon E^2/2$

18. .Derive an expression for energy and energy density in a magnetic field.

Energy =$LI^2/2$

Energy density = $\mu H^2/2$

19. Derive all the maxwells equations.

Hints:
i) Maxwells equation from electric Gauss law.
ii) Maxwells equation from magnetic Gauss law.
iii) Maxwells equation from Amperes law.
iv) Maxwells equation from Faradays law.

20. Derive an expression for displacement, conduction current densities. Also obtain an expression for continuity current relations

Hints:
Displacement current density Jd = $\varepsilon\delta E/\delta t$
Conduction current density Jcond = σE

21. Derive the general Electromagnetic wave equation.

Hint:

Starting from the maxwells equation from Faradays law and Amperes law derive the Equation

˘ ^{2} E - μ σ(δ E/ δt)-με (δ^2 E/δt^2)

22. Briefly explain reflection by a perfect dielectric when a wave is incident normally ona perfect dielectric and derive expression for reflection coefficient.

Hints:
When a plane electromagnetic wave is incident on the surface of aperfect dielectric part of the energy is transmitted and part of it is reflected.

Er / Ei = (2 – 1) /(2 + 1)

23. Briefly explain reflection by a perfect dielectric when a wave is incident normallyon a perfect conductor.

Hints
:When the plane wave is incident normally upon the surface of a perfect conductor the wave is entirely reflected. Since there can be no loss within a perfect conductor none of the energy is absorbed.

E (x,t) = 2Ei sinβx sinω t

24. Derive the relation between field theory and circuit theory for an RLC series circuit.

Hints :

Starting from field theory erquation for a series RLC circuit derive the

circuit equation V= IR + L dI/dt +(1 /C) / Idt

25. State and explain Faradays and Lenzs law of induction and derive maxwells equation.

Hints:
The total emf induced in a circuit is equal to the time rate of decrease of the total magnetic flux linking the circuit.

˘ X E = -δB/ δt

UNIT-2 CONDUCTORS AND DIELECTRICS

Boundary conditions for Electrostatic fields

In our discussions so far we have considered the existence of electric field in the homogeneous medium. Practical electromagnetic problems often involve media with different physical properties. Determination of electric field for such problems requires the knowledge of the relations of field quantities at an interface between two media. The conditions that the fields must satisfy at the interface of two different media are referred to as ***boundary conditions*** .

In order to discuss the boundary conditions, we first consider the field behavior in some common material media.

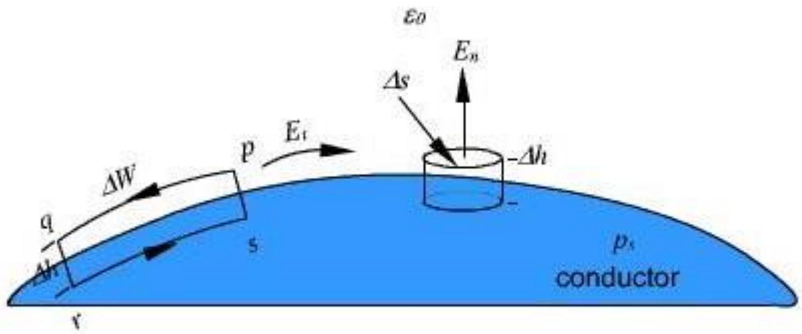


Fig *2*.1: Boundary Conditions for at the surface of a Conductor

In general, based on the electric properties, materials can be classified into three categories: conductors, semiconductors and insulators (dielectrics). In *conductor* , electrons in the outermost shells of the atoms are very loosely held and they migrate easily from one atom to the other. Most metals belong to this group. The electrons in the atoms of *insulators* or *dielectrics* remain confined to their orbits and under normal circumstances they are not liberated under the influence of an externally applied field. The electrical properties of *semiconductors* fall between those of conductors and insulators since semiconductors have very few numbers of free charges.

The parameter *conductivity* is used characterizes the macroscopic electrical property of a material medium. The notion of conductivity is more important in dealing with the current flow and hence the same will be considered in detail later on.

If some free charge is introduced inside a conductor, the charges will experience a force due
to mutual repulsion and owing to the fact that they are free to move, the charges will appear on the surface. The charges will redistribute themselves in such a manner that the field within the conductor is zero. Therefore, under steady condition, inside a conductor $\rho_v = 0$.

From Gauss's theorem it follows that

$\vec{E}=0$ ....*(2.51)*

The surface charge distribution on a conductor depends on the shape of the conductor. The charges on the surface of the conductor will not be in equilibrium if there is a tangential component of the electric field is present, which would produce movement of the charges. Hence under static field conditions, tangential component of the electric field on the conductor surface is zero. The electric field on the surface of the conductor is normal everywhere to the surface . Since the tangential component of electric field is zero, the conductor surface is an equipotential surface. As $\vec{E}$ = 0 inside the conductor, the conductor as a whole has the same potential. We may further note that charges require a finite time to redistribute in a conductor. However, this time is very small $\sim 10^{-19}$ sec for good conductor like copper.

Let us now consider an interface between a conductor and free space as shown in the figure *2*.1

Let us consider the closed path *pqrsp* for which we can write,

$$\oint \vec{E} \cdot d\vec{l} = 0$$ ....*(2.52)*

For $\Delta h \rightarrow 0$ and noting that $\vec{E}$ inside the conductor is zero, we can write

$$E_t \Delta w = 0$$ ....*(2.53)*

Et is the tangential component of the field. Therefore we find that

$E_t = 0$ ....*(2.54)*

In order to determine the normal component *En*, the normal component of $\vec{E}$, at the surface of the conductor, we consider a small cylindrical Gaussian surface as shown in the Fig.12. Let Δs represent the area of the top and bottom faces and Δh represents the height of the cylinder. Once again, as $\Delta h \rightarrow 0$, we approach the surface of the conductor. Since $\vec{E}$ = 0 inside the conductor is zero,

$$\varepsilon_0 \oint_s \vec{E} \cdot d\vec{s} = \varepsilon_0 E_n \Delta s = \rho_s \Delta s$$ ………………………………………..............*(2.55)*

$$E_n = \frac{\rho_s}{\varepsilon_0}$$ ……..(2.56)

Therefore, we can summarize the boundary conditions at the surface of a conductor as:

$E_t = 0$……………………………..…………….. (2.57)

$$E_n = \frac{\rho_s}{\varepsilon_0} \quad \text{.....................(2.58)}$$

Behavior of dielectrics in static electric field: Polarization of dielectric

Here briefly describe the behavior of dielectrics or insulators when placed in static electric field. Ideal dielectrics do not contain free charges. As we know, all material media are composed of atoms where a positively charged nucleus (diameter ~ 10^{-15}m) is surrounded by negatively charged electrons (electron cloud has radius ~ 10^{-10}m) moving around the nucleus. Molecules of dielectrics are neutral macroscopically; an externally applied field causes small displacement of the charge particles creating small electric dipoles. These induced dipole moments modify electric fields both inside and outside dielectric material.

Molecules of some dielectric materials posses permanent dipole moments even in the absence of an external applied field. Usually such molecules consist of two or more dissimilar atoms and are called *polar* molecules. A common example of such molecule is water molecule H_2O. In polar molecules the atoms do not arrange themselves to make the net dipole moment zero. However, in the absence of an external field, the molecules arrange themselves in a random manner so that net dipole moment over a volume becomes zero.

Under the influence of an applied electric field, these dipoles tend to align themselves along the field as shown in figure *2*.15. There are some materials that can exhibit net permanent dipole moment even in the absence of applied field. These materials are called *electrets* that made by heating certain waxes or plastics in the presence of electric field. The applied field aligns the polarized molecules when the material is in the heated state and they are frozen to their new position when after the temperature is brought down to its normal temperatures. Permanent polarization remains without an externally applied field.

As a measure of intensity of polarization, polarization vector $\vec{P}$ (in C/m^2) is defined as:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{i=1}^{n\Delta v} \vec{P_k}}{\Delta v} \quad \text{..}(2.59)$$

n being the number of molecules per unit volume i.e. $\vec{P}$ is the dipole moment per unit volume. Let us now consider a dielectric material having polarization $\vec{P}$ and compute the potential at an external point O due to an elementary dipole $\vec{P}$ *dv'*.

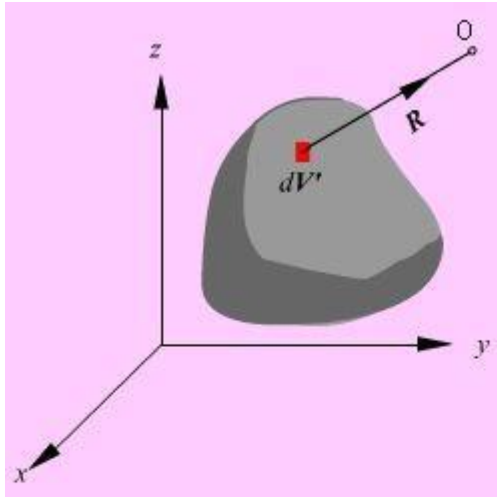


Fig 2.2: Potential at an External Point due to an Elementary Dipole $\vec{P}$ dv'.

With reference to the figure 2.16, we can write:
$$dV = \frac{\vec{P}dv' \cdot \hat{a}_R}{4\pi\varepsilon_0 R^2} \quad \text{...}(2.60)$$

Therefore,

$$R = \left\{ (x-x')^2 + (y-y')^2 + (z-z')^2 \right\}^{1/2} \quad \text{..................}(2.61)$$

where x,y,z represent the coordinates of the external point O and x',y',z' are the coordinates of the source point.

From the expression of R, we can verify that

$$\nabla'\left(\frac{1}{R}\right) = \frac{\hat{a}_R}{R^2} \quad \text{...}(2.63)$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{v'} \vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv' \quad \text{....................................}(2.64)$$

Using the vector identity, $\nabla'\cdot(f\vec{A}) = f\nabla'\cdot\vec{A} + \vec{A}\cdot\nabla' f$,where f is a scalar quantity , we have,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{v'}\nabla'\cdot\left(\frac{\vec{P}}{R}\right)dv' - \int_{v'}\frac{\nabla'\cdot\vec{P}}{R}dv'\right] \quad \text{.....................}(2.65)$$

Converting the first volume integral of the above expression to surface integral, we can write

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'} \frac{\vec{P}\cdot \hat{a}'_n}{R} ds' + \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\left(-\nabla\cdot\vec{P}\right)}{R} dv' \quad \text{.................(2.66)}$$

where $\hat{a}'_n$ is the outward normal from the surface element *ds'* of the dielectric. From the above expression we find that the electric potential of a polarized dielectric may be found from the contribution of volume and surface charge distributions having densities

$$\rho_{ps} = \vec{P}\cdot\hat{a}_n \quad \text{..(2.67)}$$

$$\rho_{pv} = -\nabla\cdot\vec{P} \quad \text{......................(2.68)}$$

These are referred to as polarisation or bound charge densities. Therefore we may replace a polarized dielectric by an equivalent polarization surface charge density and a polarization volume charge density. We recall that bound charges are those charges that are not free to move within the dielectric material, such charges are result of displacement that occurs on a molecular scale during polarization. The total bound charge on the surface is

$$\oint_S \rho_{ps} ds = \oint_S \vec{P}\cdot d\vec{s} \quad \text{..................(2.69)}$$

The charge that remains inside the surface is

$$\int_V \rho_{pv} dv = \int_V -\nabla\cdot\vec{P} dv \quad \text{....................(2.70)}$$

The total charge in the dielectric material is zero as

$$\oint_S \rho_{ps} ds + \int_V \rho_{pv} = \oint_S \vec{P}\cdot d\vec{s} + \int_V -\nabla\cdot\vec{P} dv = \int_V \nabla\cdot\vec{P} - \int_V \nabla\cdot\vec{P} = 0 \quad \text{....................(2.71)}$$

If we now consider that the dielectric region containing charge density ρ_v the total volume charge density becomes

$$\rho_t = \rho_v + \rho_{pv} \quad \text{.................(2.72)}$$

Since we have taken into account the effect of the bound charge density, we can write

$$\nabla \cdot \vec{E} = \frac{(\rho_v + \rho_{pv})}{\varepsilon_0} \quad \text{.....................................} \quad (2.73)$$

Using the definition of we have ρ_{pv}

$$\nabla.(\varepsilon_0 \vec{E} + \vec{P}) = \rho_v \quad \text{....................}(2.74)$$

Therefore the electric flux density $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

When the dielectric properties of the medium are linear and isotropic, polarisation is directly proportional to the applied field strength and

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \quad \text{....................}(2.75)$$

is the electric susceptibility of the dielectric. Therefore,

$$\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \quad \text{.......................}(2.76)$$

$\varepsilon_r = 1 + \chi_e$ is called relative permeability or the dielectric constant of the medium.
$\varepsilon_0 \varepsilon_r$ is called the absolute permittivity.

A dielectric medium is said to be linear when χ_e is independent of $\vec{E}$ and the medium is homogeneous if χ_e is also independent of space coordinates. A linear homogeneous and isotropic medium is called a **simple medium** and for such medium the relative permittivity is a constant.

Dielectric constant ε_r may be a function of space coordinates. For an isotropic materials, the dielectric constant is different in different directions of the electric field, D and E are related by a permittivity tensor which may be written as:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad \text{.......................}(2.77)$$

For crystals, the reference coordinates can be chosen along the principal axes, which make off diagonal elements of the permittivity matrix zero. Therefore, we have

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad \text{........................(2.78)}$$

Media exhibiting such characteristics are called **biaxial**. Further, if $\varepsilon_1 = \varepsilon_2$ then the medium is called **uniaxial**. It may be noted that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ for isotropic media, Lossy dielectric materials are represented by a complex dielectric constant, the imaginary part of which provides the power loss in the medium and this is in general dependant on frequency.

Another phenomenon is of importance is **dielectric breakdown**. We observed that the applied electric field causes small displacement of bound charges in a dielectric material that results into polarization. Strong field can pull electrons completely out of the molecules. These electrons being accelerated under influence of electric field will collide with molecular lattice structure causing damage or distortion of material. For very strong fields, avalanche breakdown may also occur. The dielectric under such condition will become conducting.

The maximum electric field intensity a dielectric can withstand without breakdown is referred to as the **dielectric strength** of the material.

Method Of Images:

The replacement of the actual problem with boundaries by an enlarged region or with image charges but no boundaries is called the method of images.
Method of images is used in solving problems of one or more point charges in the presence of boundary surfaces.

Continuity of equation:

The relation between density and the volume charge density at a point called continuity of equation

$\nabla \bullet J = - \rho / t_v$

Boundary Conditions for perfect Electric Fields:

Let us consider the relationship among the field components that exist at the interface between two dielectrics as shown in the figure 2.17. The permittivity of the medium 1 and medium 2 are ε_1 and ε_2 respectively and the interface may also have a net charge density ρ_s Coulomb/m.

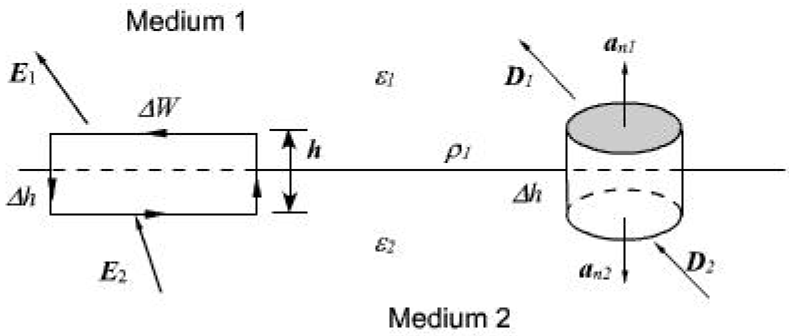


Fig *2.3*: Boundary Conditions at the interface between two dielectrics

We can express the electric field in terms of the tangential and

$$\vec{E_1} = \vec{E_{1t}} + \vec{E_{1n}}$$

normal $\vec{E_2} = \vec{E_{2t}} + \vec{E_{2n}}$ ....*(2.79)*

where E_t and E_n are the tangential and normal components of the electric field

respectively. Let us assume that the closed path is very small so that over the elemental

path length the
variation of E can be neglected. Moreover very near to the interface, $\Delta h \to 0$. Therefore

$$\oint \vec{E}.d\vec{l} = E_{1t}\Delta w - E_{2t}\Delta w + \frac{h}{2}\left(E_{1n} + E_{2n}\right) - \frac{h}{2}\left(E_{1n} + E_{2n}\right) = 0$$ ....(2.80)

Thus, we have,

$E_{1t} = E_{2t}$ or $\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$ i.e. the **tangential component of an electric field is continuous across the interface.**

For relating the flux density vectors on two sides of the interface we apply Gauss's law to a small pillbox volume as shown in the figure. Once again as $\Delta h \to 0$, we can write

$$\oint \vec{D} \cdot d\vec{s} = (\vec{D_1} . \hat{a}_{n2} + \vec{D_2} \cdot \hat{a}_{n1})\Delta s = \rho_s \Delta s$$ ....(2.81a)

i.e.,

$$D_{1n} - D_{2n} = \rho_s$$ ....(2.81b)

.e., $\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$ ....(2.81c)

Thus we find that the **normal component of the flux density vector *D* is discontinuous across an interface by an amount of discontinuity equal to the surface charge density at the interface.**

Example

Two further illustrate these points; let us consider an example, which involves the refraction of D or E at a charge free dielectric interface as shown in the figure 2.18.

Using the relationships we have just derived, we can write

$$E_{1t} = E_1 \sin\theta_1 = \frac{D_1}{\varepsilon_1}\sin\theta_1 = E_{2t} = E_2 \sin\theta_2 = \frac{D_2}{\varepsilon_2}\sin\theta_2 \quad ....(2.82a)$$

$$D_{1n} = D_1 \cos\theta_1 = D_{2n} = D_2 Cos\theta_2 \quad ....(2.82b)$$

In terms of flux density vectors,

$$\frac{D_1}{\varepsilon_1}\sin\theta_1 = \frac{D_2}{\varepsilon_2}\sin\theta_2 \quad ....(2.83a)$$

$$D_1 \cos\theta_1 = D_2 Cos\theta_2 \quad ....(2.83b)$$

Therefore,

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \quad ....(2.84)$$

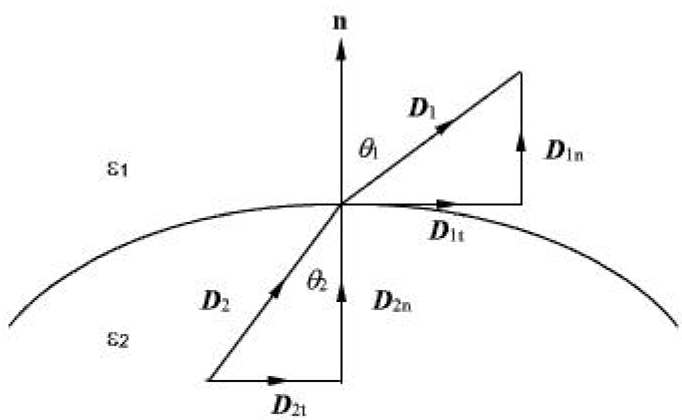


Fig 2.4: Refraction of D or E at a Charge Free Dielectric Interface

Capacitance and Capacitors

We have already stated that a conductor in an electrostatic field is an Equipotential body and any charge given to such conductor will distribute themselves in such a manner that electric field inside the conductor vanishes. If an additional amount of charge is supplied to an isolated conductor at a given potential, this additional charge will increase the surface charge density ρ_s. Since the potential of the conductor is given by $V = \frac{1}{4\pi\varepsilon_0}\int_{s'}\frac{\rho_s ds'}{r}$, the potential

of the conductor will also increase maintaining the ratio same $\frac{Q}{V}$ Thus we can write

$$C = \frac{Q}{V}$$

where the constant of proportionality *C* is called the capacitance of the isolated conductor. SI unit of capacitance is Coulomb/ Volt also called Farad denoted by *F*. It can It can be seen that if *V*=1, *C* = *Q*. Thus capacity of an isolated conductor can also be defined as the amount of charge in Coulomb required to raise the potential of the conductor by 1 Volt.

Of considerable interest in practice is a capacitor that consists of two (or more) conductors carrying equal and opposite charges and separated by some dielectric media or free space. The conductors may have arbitrary shapes. A two-conductor capacitor is shown in figure 2.5.

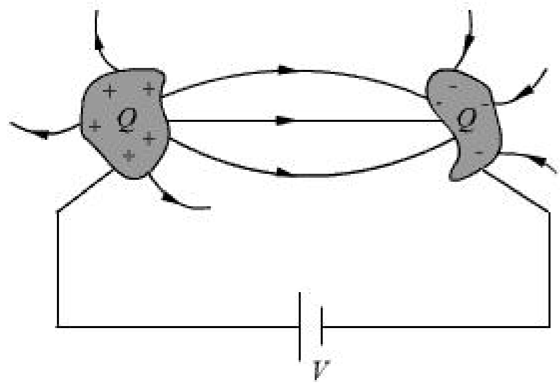


Fig ***2.5****:Capacitance and Capacitors*

When a d-c voltage source is connected between the conductors, a charge transfer occurs which results into a positive charge on one conductor and negative charge on the other conductor. The conductors are equipotential surfaces and the field lines are perpendicular to the conductor surface. If *V* is the mean potential difference between the conductors, the capacitance is given by $C = \frac{Q}{V}$. Capacitance of a capacitor depends on the geometry of the conductor and the permittivity of the medium between them and does not depend on the charge or potential difference between conductors. The capacitance can be computed by assuming *Q*(at the same time -*Q* on the other conductor), first determining $\vec{E}$ using Gauss's theorem and then determining $V = -\int \vec{E}.d\vec{l}$. We illustrate this procedure by taking the example of a parallel plate capacitor.

Parallel plate capacitor

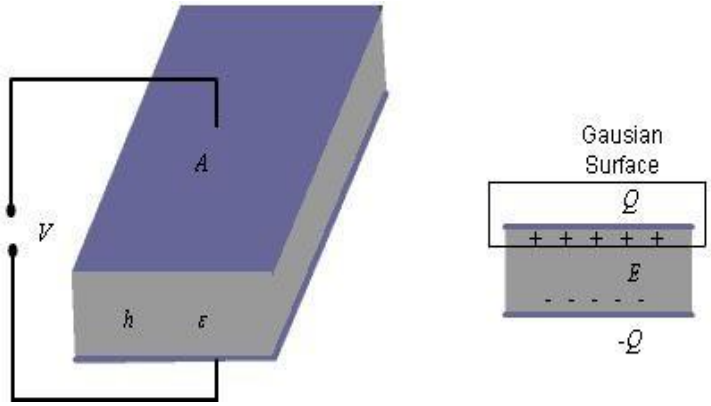


Fig 2.6: Parallel Plate Capacitor

For the parallel plate capacitor shown in the figure 2.20, let each plate has area A and a distance h separates the plates. A dielectric of permittivity ε fills the region between the plates. The electric field lines are confined between the plates. We ignore the flux fringing at the edges of the plates and charges are assumed to be uniformly distributed over the conducting plates with densities ρ_s and ρ_s, $\rho_s = \frac{Q}{A}$.

By Gauss's theorem we can write, $E = \frac{\rho_s}{\varepsilon} = \frac{Q}{A\varepsilon}$ ....*(2.85)*

As we have assumed ρ_s to be uniform and fringing of field is neglected, we see that E is constant in the region between the plates and therefore, we can write .

Thus, $V = Eh = \frac{hQ}{\varepsilon A}$

for a parallel plate capacitor we have,. $C = \frac{Q}{V} = \varepsilon \frac{A}{h}$.......................*(2.86)*

Series and parallel Connection of capacitors

Capacitors are connected in various manners in electrical circuits; series and parallel connections are the two basic ways of connecting capacitors. We compute the equivalent capacitance for such connections.

Series Case: Series connection of two capacitors is shown in the figure 2.21. For this case we can write,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{V}{Q} = \frac{1}{C_{eqs}} = \frac{1}{C_1} + \frac{1}{C_2} \quad ....(2.87)$$

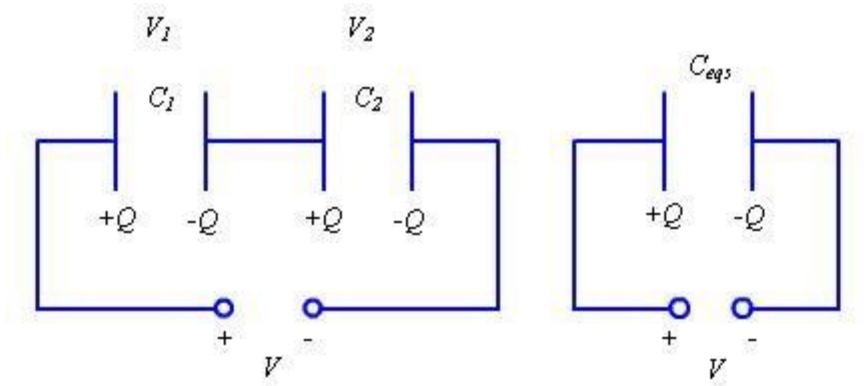
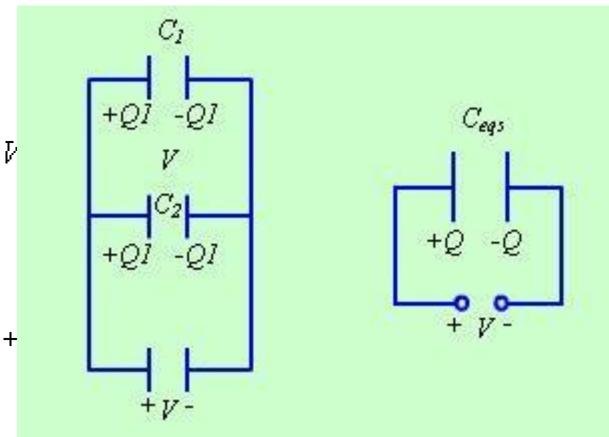


Fig 2.7: Series Connection of Capacitors

$W = (V_{1N}Q_1 + \ldots\ldots + V$ ... $_{-2)N}Q_{N-2}) + V_{(N-1)N}Q_{N-1}$



$2W = (V_{1N} + V_{1(N-1)} +$ ... $_{1}\ldots\ldots$

$\ldots\ldots V_{N2} + V_{N(N-1)})Q_N$

Fig2.7:Parallel Connection of Capacitors

The same approach may be extended to more than two capacitors connected in series. **Parallel Case:** For the parallel case, the voltages across the capacitors are the same. The total charge $Q = Q_1 + Q_2 = C_1V + C_2V$

Therefore,
$$C_{eqp} = \frac{Q}{V} = C_1 + C_2 \quad \text{.......................(2.88)}$$

Poisson's and Laplace's Equations

For electrostatic field, we have seen that

$$\nabla \cdot \vec{D} = \rho_v$$
$$\vec{E} = -\nabla V$$
....(2.97)

Form the above two equations we can write

$$\nabla \cdot (\varepsilon \vec{E}) = \nabla \cdot (-\varepsilon \nabla V) = \rho_v$$ ....(2.98)

Using vector identity we can write, $\varepsilon \nabla \cdot \nabla V + \nabla V \cdot \nabla \varepsilon = -\rho_v$ ....(2.99)

For a simple homogeneous medium, is constant and $\nabla \varepsilon = 0$. Therefore,

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{\rho_v}{\varepsilon}$$ ....(2.100)

This equation is known as **Poisson's equation**. Here we have introduced a new operator, ∇^2 (del square), called the Laplacian operator. In Cartesian coordinates,

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$ ....(2.101)

Therefore, in Cartesian coordinates, Poisson equation can be written as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\varepsilon}$$ ....(2.102)

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$ ....(2.103)

In spherical polar coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$ ....*(2.104)*

At points in simple media, where no free charge is present, Poisson's equation reduces to

$$\nabla^2 V = 0$$ ....*(2.105)*

which is known as Laplace's equation.

Application of poisons and Laplace's equations:

Laplace's and Poisson's equation are very useful for solving many practical electrostatic field problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found throughout the volume. We shall consider such applications in the section where we deal with boundary value problems.

ASSIGNMENT PROBLEMS

1. A charged ring of radius d carrying a charge of ρ_L C/m lies in the x-y plane with its centre at the origin and a charge Q C is placed at the point $(0,0,2d)$. Determine ρ_S in terms of Q and d so that a test charge placed at $(0,0,2d)$ does not experience any force.

2. A semicircular ring of radius a lies in the free space and carries a charge density ρ_S C/m. Find the electric field at the centre of the semicircle.

3. Consider a uniform sphere of charge with charge density ρ_0 and radius b, centered at the origin. Find the electric field at a distance r from the origin for the two cases: $r<b$ and $r>b$. Sketch the strength of the electric filed as function of r.

4. A spherical charge distribution is given by

$$\rho_v = \begin{cases} \rho_0 (a^2 - r^2), & r \le a \\ 0, & r > a \end{cases}$$

a is the radius of the sphere. Find the following:

i. The total charge.

ii. $\vec{E}$ for $r \le a$ and $r > a$.

iii The value of r where $\vec{E}$ becomes

5. With reference determine the potential and field at the point $P(0,0,h)$ if the shaded region contains uniform charge density ρ_s/m^2 .

6. A capacitor consists of two coaxial metallic cylinders of length L, radius a of the inner conductor and that of outer conductor b. A dielectric material having dielectric constant $\varepsilon_r = 3 + 2/\rho$, where ρ is the radius, fills the space between the conductors. Determine the capacitance of the capacitor.

7. Determine whether the functions given below satisfy Laplace 's equation

i) $V(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

ii) $V(\rho,\phi,z) = \rho z \sin\phi + \rho^2$

UNIT III MAGNETOSTATICS

Introduction

In previous chapters we have seen that an electrostatic field is produced by static or stationary charges. The relationship of the steady magnetic field to its sources is much more complicated.

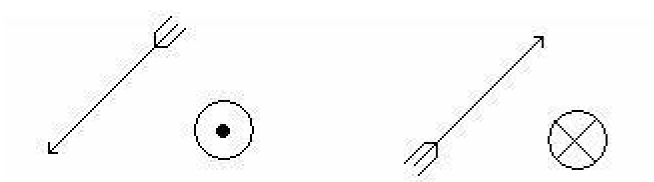
Laws governing magneto static fields

The source of steady magnetic field may be a permanent magnet, a direct current or an electric field changing with time. In this chapter we shall mainly consider the magnetic field produced by a direct current. The magnetic field produced due to time varying electric field will be discussed later. Historically, the link between the electric and magnetic field was established Oersted in 1820. Ampere and others extended the investigation of magnetic effect of electricity . There are two major laws governing the magnetostatic fields are:

1.Biot-Savart Law

2.Ampere's Law

Usually, the magnetic field intensity is represented by the vector $\vec{H}$. It is customary to represent the direction of the magnetic field intensity (or current) by a small circle with a dot or cross sign depending on whether the field (or current) is out of or into the page as shown in Fig. 3.1.



$\vec{H}$ (or l) out of the page $\vec{H}$ (or l) into the page

Fig. 3.1: Representation of magnetic field (or current) Biot- Savart Law

This law relates the magnetic field intensity *dH* produced at a point due to a differential current element $Id\vec{l}$ as shown in Fig. 3.2.

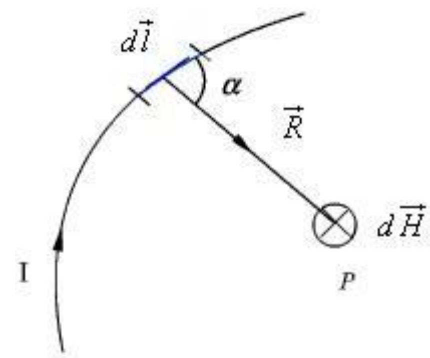


Fig. 3.2: Magnetic field intensity due to a current element

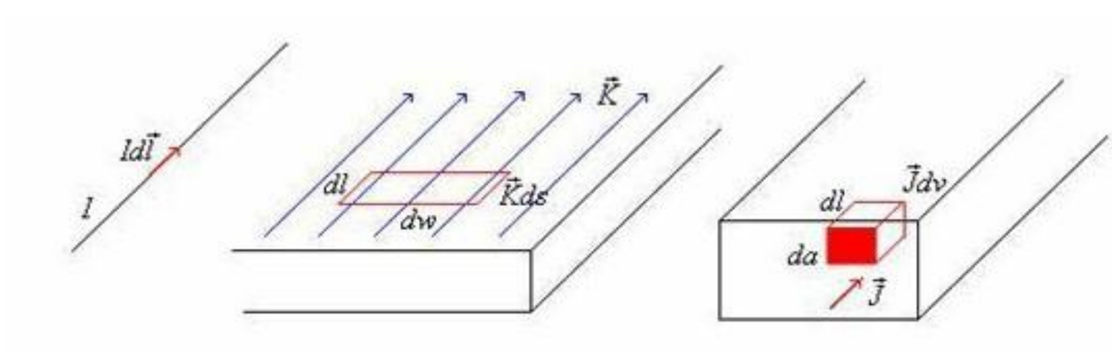
The magnetic field intensity at P can be written as, $d\vec{H}$

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3} \quad \text{.............................(3.1a)}$$

$$dH = \frac{IdlSin\alpha}{4\pi R^2} \quad \text{..(3.1b)}$$

where $R = |\vec{R}|$ is the distance of the current element from the point P.

Similar to different charge distributions, we can have different current distribution such as line current, surface current and volume current. These different types of current densities are shown in Fig. 3.3.



Line Current Surface Current Volume Current

Fig. 3.3: Different types of current distributions

By denoting the surface current density as K (in amp/m) and volume current density as J (in amp/m^2) we can write:

$$I d\vec{l} = \vec{K} ds = \vec{J} dv$$ ....(3.2)

(It may be noted that $I = Kdw = Jda$)

Employing Biot-Savart Law, we can now express the magnetic field intensity H. In terms of these current distributions.

$$\vec{H} = \int_l \frac{I d\vec{l} \times \vec{R}}{4\pi R^3}$$ ... **for line current.**...........................(3.3a)

$$\vec{H} = \int_s \frac{K d\vec{s} \times \vec{R}}{4\pi R^3}$$ **for surface current** ....(3.3b)

$$\vec{H} = \int_v \frac{\vec{J} dv \times \vec{R}}{4\pi R^3}$$ **for volume current**......................(3.3c)

To illustrate the application of Biot - Savart's Law, we consider the following example.

Example 3.1: We consider a finite length of a conductor carrying a current $\vec{r}$ placed along z- axis as shown in the Fig 3.4. We determine the magnetic field at point P due to this current carrying conductor.

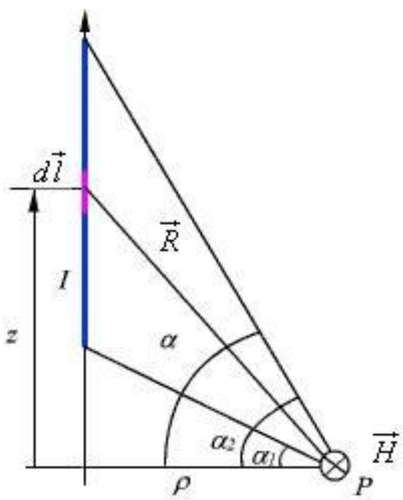


Fig. 3.4: Field at a point P due to a finite length current carrying conductor

With reference to Fig. 4.4, we find that

$$d\vec{l} = dz\,\hat{a}_z \quad and \quad \vec{R} = \rho\,\hat{a}_\rho - z\,\hat{a}_z$$

....(3.4)

Applying Biot -Savart's law for the current element $\vec{r}\ d\vec{l}$

we can write,

$$\overrightarrow{dH} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3} = \frac{\rho dz \hat{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}} \quad \text{.................(3.5)}$$

Substituting $\frac{z}{\rho} = \tan\alpha$ we can write,

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I}{4\pi} \frac{\rho^2 \sec^2\alpha d\alpha}{\rho^3 \sec^3\alpha} \hat{a}_\phi = \frac{I}{4\pi\rho}\left(\sin\alpha_2 - \sin\alpha_1\right)\hat{a}_\phi \quad \text{..........................(3.6)}$$

We find that, for an infinitely long conductor carrying a current I , $\alpha_2 = 90^0$ and $\alpha_1 = -90^0$

Therefore, $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$ ....(3.7)

Ampere's Circuital Law:

Ampere's circuital law states that the line integral of the magnetic field $\vec{H}$ (circulation of H) around a closed path is the net current enclosed by this path. Mathematically,

$$\oint \vec{H}.d\vec{l} = I_{enc} \quad \text{.....................................(3.8)}$$

The total current I enc can be written as,

$$I_{enc} = \int_S \vec{J}.d\vec{s} \quad \text{......................................(3.9)}$$

By applying Stoke's theorem, we can write

$$\oint \vec{H}.d\vec{l} = \int_S \nabla \times \vec{H}.d\vec{s}$$

$$\therefore \int_S \nabla \times \vec{H}.d\vec{s} = \int_S \vec{J}.d\vec{s}$$

$$\therefore \nabla \times \vec{H} = \vec{J} \quad \text{......................................(3.10)}$$

which is the Ampere's law in the point form.

3.3.1 Estimation of Magnetic field intensity for straight and circular conductors:

We illustrate the application of Ampere's Law with some examples.

We compute magnetic field due to an infinitely long thin current carrying conductor as shown in Fig. 4.5. Using Ampere's Law, we consider the close path to be a circle of radius ρ as shown in the Fig. 4.5.

If we consider a small current element $Id\vec{l}(= Idz\hat{a}_z)$, $d\vec{H}$ is perpendicular to the plane containing both $d\vec{l}$ and $\vec{R}(=\rho\hat{a}_\rho)$. Therefore only component $\vec{H}$ of that will be present is H_ϕ ,i.e., $\vec{H} = H_\phi \hat{a}_\phi$.

By applying Ampere's law we can write,

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho 2\pi - I$$ ....(4.11)

Therefore, $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$ which is same as equation (3.7)

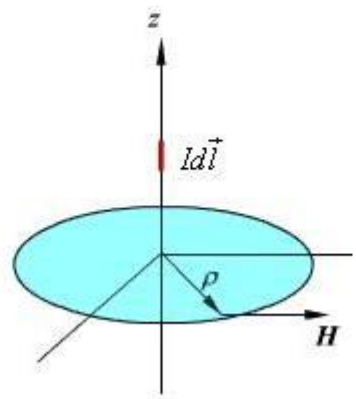


Fig. 3.5: Magnetic field due to an infinite thin current carrying conductor

We consider the cross section of an infinitely long coaxial conductor, the inner conductor carrying a current I and outer conductor carrying current - I as shown in figure 3.6. We compute the magnetic field as a function of ρ as follows:

In the region $0 \le \rho \le R_1$

$$I_{enc} = I\frac{\rho^2}{R_1^2}$$ ....(3.12)

$$H_\phi - \frac{I_{enc}}{2\pi\rho} - \frac{I\rho}{2\pi a^2}$$ ....(3.13)

In the region $R_1 < \rho < R_2$

$I_{enc} - I$

$$H_\varphi = \frac{I}{2\pi\rho} \qquad \text{......................................(3.14)}$$

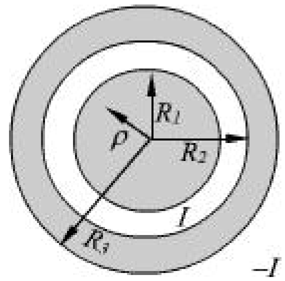


Fig. 3.6: Coaxial conductor carrying equal and opposite currents

In the region $R_2 < \rho < R_3$

$$I_{enc} = I - I\frac{\rho^2 - R_2^2}{R_3^2 - R_2^2} \qquad \text{......................................(3.15)}$$

$$H_\varphi = \frac{I}{2\pi\rho}\frac{R_3^2 - \rho^2}{R_3^2 - R_2^2} \qquad \text{......................................(3.16)}$$

In the region $\rho > R_3$ $\quad I_{enc} = 0 \qquad H_\varphi = 0$ ....(3.17)

Magnetic Flux and Density:
In simple matter, the magnetic flux density $\vec{B}$ related to the magnetic field intensity $\vec{H}$ as $\vec{B} = \mu\vec{H}$ where μ called the permeability. In particular when we consider the free space $\vec{B} = \mu_0\vec{H}$ where $\mu_0 - 4\pi\times10^{-7}$ H/m is the permeability of the free space. Magnetic flux density is measured in terms of Wb/m^2.

The magnetic flux density through a surface is given by:

$$\psi = \int_s \vec{B}.d\vec{s} \quad \text{Wb} \qquad ....(3.18)$$

In the case of electrostatic field, we have seen that if the surface is a closed surface, the net flux passing through the surface is equal to the charge enclosed by the surface. In case of magnetic field isolated magnetic charge (i. e. pole) does not exist. Magnetic poles always occur in pair (as N-S). For example, if we desire to have an isolated magnetic pole by dividing the magnetic bar successively into two, we end up with pieces each having north (N) and south (S) pole as shown in Fig. 3.7 (a). This process could be continued until the magnets are of atomic dimensions; still we will have N-S pair occurring together. This means that the magnetic poles cannot be isolated.

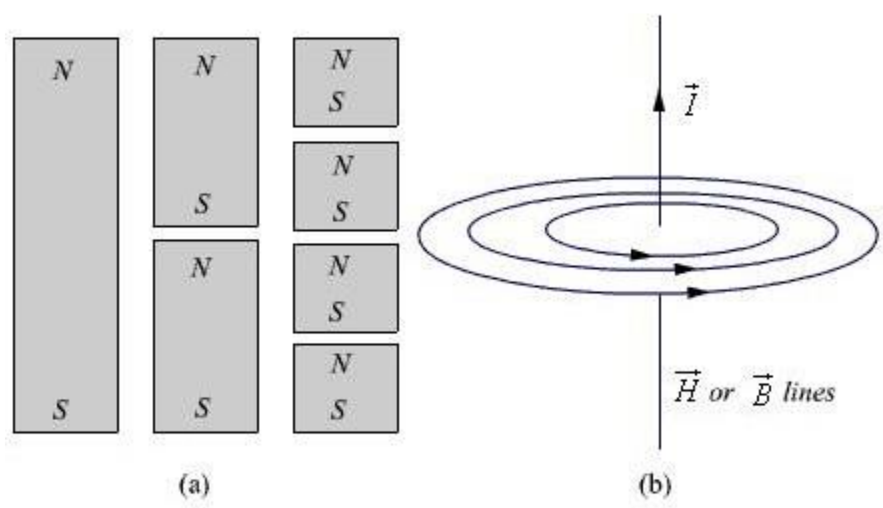


Fig. 3.7: (a) Subdivision of a magnet (b) Magnetic field/ flux lines of a straight current carrying conductor

Similarly if we consider the field/flux lines of a current carrying conductor as shown in Fig. 3.7 (b), we find that these lines are closed lines, that is, if we consider a closed surface, the number of flux lines that would leave the surface would be same as the number of flux lines that would enter the surface.

From our discussions above, it is evident that for magnetic field,

$$\oint_S \vec{B}.d\vec{s} = 0 \quad \text{......................................(3.19)}$$

which is the Gauss's law for the magnetic field. By applying divergence theorem, we can write:

$$\oint_S \vec{B}.d\vec{s} = \int_V \nabla.\vec{B}dv = 0$$

Hence, $$\nabla.\vec{B} = 0 \quad \text{......................................(3.20)}$$

which is the Gauss's law for the magnetic field in point form.

Magnetic Scalar and Vector Potentials:

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a **scalar magnetic potential** and write:

$$\vec{H} = -\nabla V_m \quad \text{...................................(3.21)}$$

From Ampere's law , we know that

$$\nabla \times \vec{H} = \vec{J} \quad \text{......................................(3.22)}$$

Therefore, $$\nabla \times (-\nabla V_m) = \vec{J} \quad \text{...........................(3.23)}$$

But using vector identity, $\nabla \times (\nabla V) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$. Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, *Vm* in general is not a single valued function of position.

This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 3.8.

In the region $a < \rho < b$, $\vec{J} = 0$ and $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$

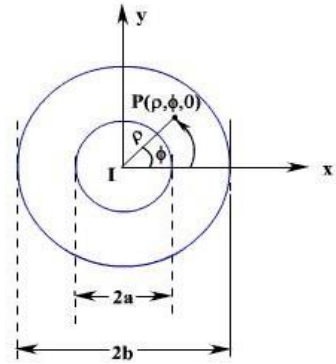


Fig. 3.8: Cross Section of a Coaxial Line

If V_m is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

$$= \frac{I}{2\pi\rho}$$

$$\therefore V_m = -\frac{I}{2\pi}\phi + c$$

If we set $V_m = 0$ at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi}\phi_0$$

We observe that as we make a complete lap around the current carrying conductor , we reach ϕ_0 again but V_m this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of V_m keeps changing as we complete additional laps to pass through the same point. We introduced V_m analogous to electostatic potential V. But for static electric fields, $\nabla\times\vec{E} = 0$ and $\oint \vec{E}\cdot d\vec{l} = 0$, whereas for steady magnetic field $\nabla\times\vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint \vec{H}\cdot d\vec{l} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the **vector magnetic potential** which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla.\vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla.\left(\nabla\times\vec{A}\right) = 0$, we can $\vec{B} = \nabla\times\vec{A}$ written .

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation.

We have introduced the vector function $\vec{A}$ and related its curl to $\vec{B}$. A vector function is defined fully in terms of its curl as well as divergence. The choice of $\nabla.\vec{A}$ is made as follows.

$$\nabla\times\nabla\times\vec{A} = \mu\nabla\times\vec{H} = \mu\vec{J} \quad \text{..(3.24)}$$

By using vector identity,

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A} \quad \text{..(3.25)}$$

$$\nabla(\nabla.\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} \quad \text{..(3.26)}$$

Great deal of simplification can be achieved if we choose $\nabla.\vec{A} = 0$.

Putting $\nabla.\vec{A} = 0$, we get $\nabla^2\vec{A} = -\mu\vec{J}$ which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

$$\nabla^2 A_x - -\mu J_x \quad \text{.....................................(3.27a)}$$

$$\nabla^2 A_y = \mu J_y \quad \text{.....................................(3.27b)}$$

$$\nabla^2 A_z - \mu J_z \quad \text{................................(3.27c)}$$

The form of all the above equation is same as that of

$$\nabla^2 V - \frac{\rho}{\varepsilon} \quad \text{..(3.28)}$$

for which the solution is

$$V = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho}{R}dv', \qquad R = \left|\vec{r} - \vec{r}'\right| \quad \text{..................(3.29)}$$

In case of time varying fields we shall see that $\nabla.\vec{A} = \mu\varepsilon \frac{\partial V}{\partial t}$, which is known as Lorentz condition, *V* being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi}\int_{v'} \frac{J_x}{R} dv' \quad \text{...................................(3.30)}$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi}\int_{v'} \frac{\vec{J}}{R} dv' \quad \text{.......................................(3.31)}$$

This equation enables us to find the vector potential at a given point because of a v olume current density $\vec{J}$. Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_{L'} \frac{I}{R} d\vec{l}' \quad \text{..(3.32)}$$

$$\vec{A} = \frac{\mu}{4\pi}\int_{S'} \frac{\vec{K}}{R} ds' \quad \text{respectively. ....(3.33)}$$

The magnetic flux ψ through a given area S is given by

$$\psi = \int_S \vec{B}.d\vec{s} \quad \text{...(3.34)}$$

Substitutig $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_S \nabla \times \vec{A}.d\vec{s} = \oint_C \vec{A}.d\vec{l} \quad \text{...(3.35)}$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

Boundary Condition for Magnetic Fields:

Similar to the boundary conditions in the electro static fields, here we will consider the behavior of $\vec{B}$ and $\vec{H}$ at the interface of two different media. In particular, we determine how the tangential and normal components of magnetic fields behave at the boundary of two regions having different permeabilities.

The figure 3.9 shows the interface between two media having permeabities μ_1 and μ_2, $\hat{a}_n$ being the normal vector from medium 2 to medium 1.

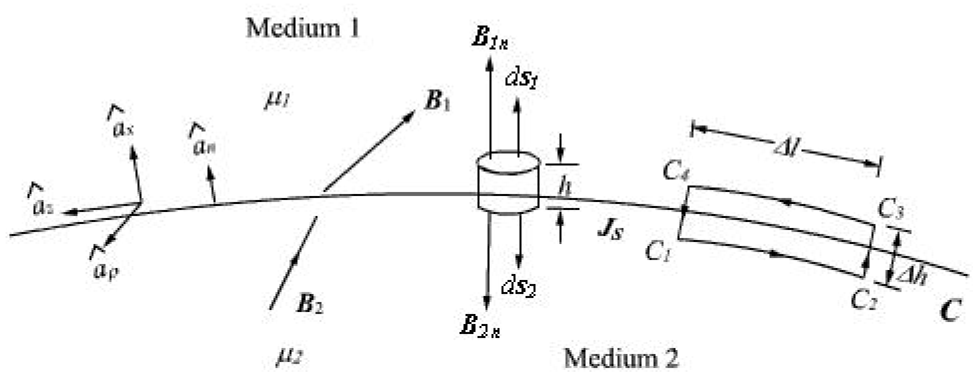


Figure 3.9: Interface between two magnetic media

To determine the condition for the normal component of the flux density vector $\vec{B}$, we consider a small pill box P with vanishingly small thickness *h* and having an elementary area ΔS for the faces. Over the pill box, we can write

$$\oint_S \vec{B}.d\vec{s} = 0 \quad ....(3.36)$$

Since h --> 0, we can neglect the flux through the sidewall of the pill box.

$$\therefore \int_{\Delta S} \vec{B}_1.d\vec{S}_1 + \int_{\Delta S} \vec{B}_2.d\vec{S}_2 = 0 \quad ....(3.37)$$

$$d\vec{S}_1 = dS\hat{a}_n \text{ and } d\vec{S}_2 = dS\left(-\hat{a}_n\right) \quad ....(3.38)$$

where

$$\int_{\Delta S} B_{1n} dS \quad \int_{\Delta S} B_{2n} dS = 0$$

$B_{1n} = \vec{B}_1 . \hat{a}_n$ and $B_{2n} = \vec{B}_2 . \hat{a}_n$ ....(3.39)

Since ΔS is small, we can write

$$(B_{1n} \quad B_{2n}) \Delta S \quad 0$$

or, $B_{1n} = B_{2n}$ ....(3.40)

That is, the normal component of the magnetic flux density vector is continuous across the interface.

In vector form,

$$\hat{a}_n . \left(\vec{B}_1 - \vec{B}_2\right) = 0$$ ....(3.41)

To determine the condition for the tangential component for the magnetic field, we consider a closed path C as shown in figure 3.8. By applying Ampere's law we can write

$$\oint \vec{H} . d\vec{l} = I$$ ....(3.42) Since

h -->0,

$$\int_{C_1 - C_2} \vec{H} . d\vec{l} + \int_{C_3 - C_4} \vec{H} . d\vec{l} = I$$ ....(3.43)

We have shown in figure 4.8, a set of three unit vectors $\hat{a}_n$, $\hat{a}_t$ $\hat{a}_P$ and such that they satisfy $\hat{a}_t \quad \hat{a}_P \times \hat{a}_n$ (R.H. rule). Here $\hat{a}_t$ is tangential to the interface and $\hat{a}_P$ is the vector perpendicular to the surface enclosed by C at the interface

The above equation can be written as

$$\vec{H}_1 . \Delta l \hat{a}_t - \vec{H}_2 . \Delta l \hat{a}_t = I = J_{Sn} \Delta l$$

or, $H_{1t} \quad H_{2t} \quad J_{Sn}$ ....(3.44)

., tangential component of magnetic field component is discontinuous across the interface where a free surface current exists.

If $J_S = 0$, the tangential magnetic field is also continuous. If one of the medium is a perfect conductor J_S exists on the surface of the perfect conductor.

In vector form we can write,

$$\left(\vec{H}_1 - \vec{H}_2\right).\hat{a}_t \Delta l$$

$$= \left(\vec{H}_1 - \vec{H}_2\right).\left(\hat{a}_\rho \times \hat{a}_n\right)\Delta l$$

$$= J_{Sn\Delta l} = \vec{J}_S.\hat{a}_\rho \Delta l \qquad \text{....................................(3.45)}$$

Therefore,

$$\hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S \qquad \text{....................................(3.46)}$$

ASSIGNMENT PROBLEMS

1. An infinitely long conductor carries a current *I A* is bent into an *L* shape and placed as shown in Fig. 3.10. Determine the magnetic field intensity at a point *P* (0,0, *a*).

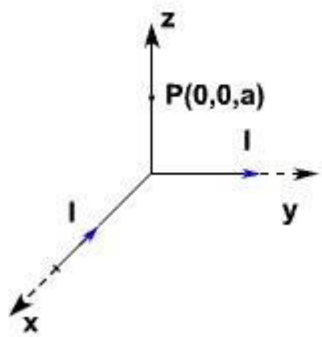


Figure 3.10

2. Consider a long filamentary carrying a current *IA* in the + *Z* direction. Calculate the magnetic field intensity at point *O* (- *a* , *a* ,0). Also determine the flux through this region described by $\rho_1 \le \rho \le \rho_2, \phi = 0, h \le z \le h$.
3. A very long air cored solenoid is to produce an inductance 0.1H/m. If the member of turns per unit length is 1000/m. Determine the diameter of this turns of the solenoid.
4. Determine the force per unit length between two infinitely long conductor each carrying current *IA* and the conductor are separated by a distance ? *d*'.

UNIT IV MAGNETIC FORCES AND MATERIALS

Force On A Moving Charge:

In electric field, force on a charged particle is

$$F=QE$$

Force is in the same direction as the electric field intensity
(positive charge)

A charged particle in motion in a magnetic field force magnitude is proportional to the product of magnitudes of the charge Q, its velocity V and the flux density B and to the sine of the angle between the vectors V and B.

The direction of force is perpendicular to both V and B and is given by a unit vector in the direction of V x B.

The force may therefore be expressed as

$$F=QV \times B$$

Force on a moving particle due to combined electric and magnetic fields is obtained by superposition.

$$F=Q (E + V \times B)$$

This equation is known as Lorentz force equation.

Force On A Differential Current Element:

The force on a charged particle moving through a steady magnetic field may be written as the differential; force exerted on a differential element of charge.

dF, dQ

Convection current density in terms of the velocity of the volume charge density

Differential element of charge may also be expressed in terms of volume charge density.

$$dQ={}_{V}\, dv$$

Thus,

$$dF={}_{V}\, dvVxB$$

JdV is the differential current element $dF = JxBdv$

$$Jdv, Kds, IdL$$

Lorentz force equation may be applied to surface current density.

$$dF = KxBds$$

Differential current element

$$dF = IdLxB$$

The magnitude of the force is given by the familiar equation

$$F = BIL\sin$$

Force on a current-carrying conductor

Charges confined to wires can also experience a force in a magnetic field. A current (I) in a magnetic field (**B**) experiences a force (**F**) given by the equation $\mathbf{F} = I\,\mathbf{l} \times \mathbf{B}$ or $F = IlB \sin \theta$, where **l** is the length of the wire, represented by a vector pointing in the direction of the current. The direction of the force may be found by a right-hand rule similar to the one shown in Figure . In this case, point your thumb in the direction of the current—the direction of motion of positive charges. The current will experience no force if it is parallel to the magnetic field.

Force and Torque on a current loop

A loop of current in a magnetic field can experience a torque if it is free to turn. Figure (a) depicts a square loop of wire in a magnetic field directed to the right. Imagine in Figure (b) that the axis of the wire is turned to an angle (θ) with the magnetic field and that the view is looking down on the top of the loop. The *x* in a circle depicts the current traveling into the page away from the viewer, and the dot in a circle depicts the current out of the page toward the viewer.

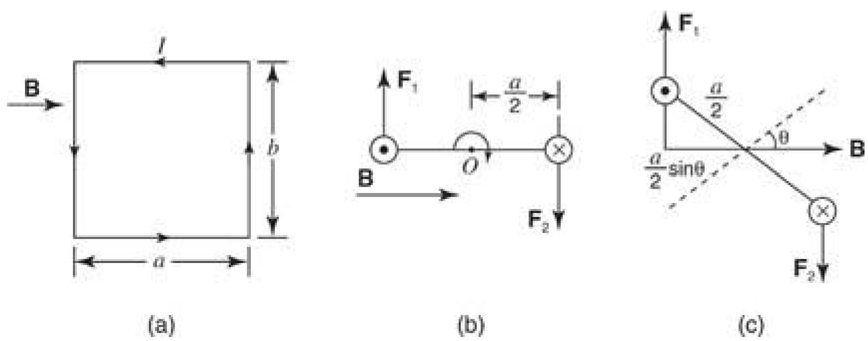


Figure 4.1
(a) Square current loop in a magnetic field **B**. (b) View from the top of the current loop. (c) If the loop is tilted with respect to **B**, a torque results.

MAGNETIC MATERIALS:

All material shows some magnetic effects. In many substances the effects are so weak that the materials are often considered to be non magnetic.

A vacuum is the truly nonmagnetic medium.

Material can be classified according to their magnetic behavior into
Diamagnetic
Paramagnetic
Ferromagnetic

DIAMAGNETIC:

In diamagnetic materials magnetic effects are weak. Atoms in which the small magnetic fields produced by the motion of the electrons in their orbit and those produced by the electron spin combine to produce a net field of zero.

The fields produced by the electron motion itself in the absence of any external magnetic field.

This material as one in which the permanent magnetic moment m_0 of each atom is zero. Such a material is termed diamagnetic.

PARAMAGNETIC:

In paramagnetic materials the magnetic moments of adjacent atoms align in opposite directions so that the net magnetic moment of a specimen is nil even in the presence of applied field.

FERROMAGNETIC:

In ferromagnetic substance the magnetic moments of adjacent atoms are also aligned opposite, but the moments are not equal, so there is a net magnetic moment.

It is less than in ferromagnetic materials.

The ferrites have a low electrical conductivity, which makes them useful in the cores of ac inductors and transformers.

Since induced currents are less and ohmic losses are reduced.

BOUNDARY CONDITIONS:

A boundary between two isotropic homogeneous linear materials with permeability 1 and 2.

The boundary condition on the normal components is determined by allowing the surface to cut a small cylindrical gaussian surface.

$$B_{N1} = B_{N2}$$

INDUCTANCE:

self inductance and mutual inductance:

Like capacitance, inductance *L* is a property of a physical arrangement of conductors. It is a measure of magnetic flux which links the circuit when a current *I* flows in the circuit. It is also a measure of how much energy is stored in the magnetic field of an inductor, such as a coil, solenoid, etc.

The definition of inductance rests on the concept of flux linkage. It is not a very precise concept unless one is willing to introduce a complicated topological description.

For our purposes it will be sufficient to define flux linkage A as the flux that links all the circuit, multiplied by the number of turns N. For example, in the case of the solenoid shown in Fig. , flux linkage will be given by

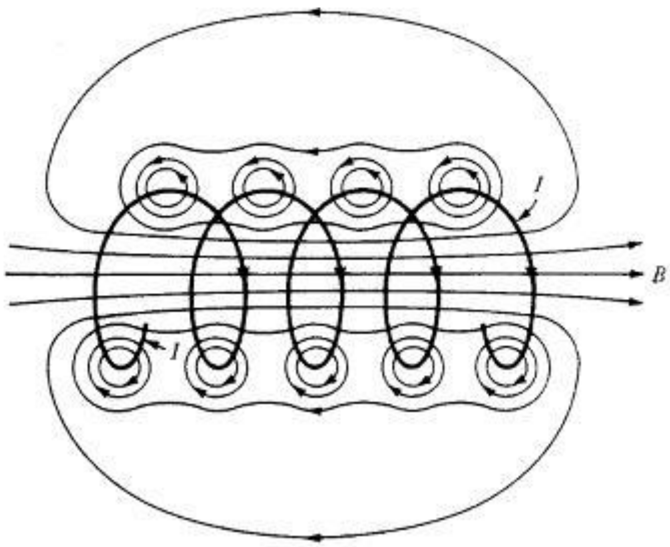


Fig 4.2 solenoid flux linkage

A = N¢ = N JJ B· **dA** ~ *NBA* Wb

that is, only the flux that goes through the inside of the solenoid and therefore links all turns is used. The small flux loops about each turn are ignored in a first-order analysis because they link only one or two turns and flow through a small area. The area *A* is that area through which the flux that links all turns flows. For the solenoid of Fig. 7.7, a good approximation to *A* is the cross section of the solenoid

The unit of inductance is the henry (H). Inductors for filter applications in power supplies are usually wire-wound solenoids on an iron core with inductances in the range from 1 to 10 H. Inductors found in high-frequency circuits are air-core solenoids with values in the millihenry (mH) range. The definition for inductance, (7.28), even though it is derived for steady currents, is valid up to very high frequencies.
Let us calculate *L* for some useful geometries.

Solenoid

A good approximation of the *B* field in a solenoid that links all turns is the *B* field at the center of the solenoid; that is, $B = J10\ Ni]l$ from (7.22) or (6.40). There is some leakage at the ends of the solenoid (recall that the value of the B field drops to one-half at the ends), which we will ignore because it occurs mainly at the ends. The inductance L of a solenoid is therefore

$$L = A = NBA = J1oN2A$$

where I is the length and A is the cross section of the solenoid.
If we have a short solenoid of N turns, that is, one where the length I is smaller

TOROID

For example, the inductance of a 2000-turn toroid having a cross-sectional area of 1 em? and mean radius of 5 em is

$$L = (4n \times 10^{-7}\ \text{H/m})(2000)2(10\text{-}4\ m2)/2n(O.05\ \text{m}) = 1.6\ \text{mH}$$

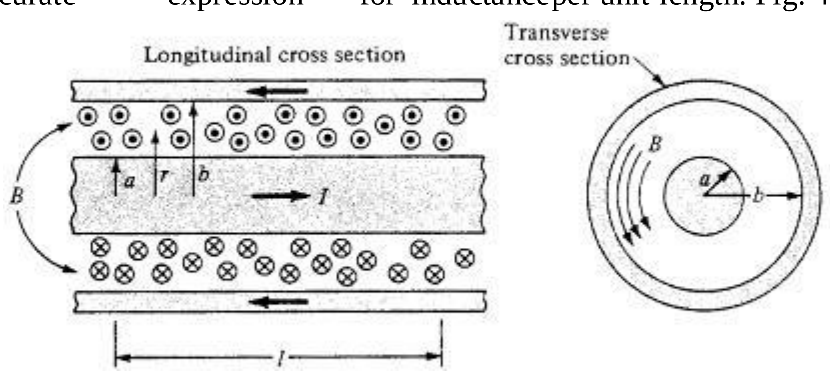
If the toroid were filled with iron instead of air, the inductance could be increased many thousand fold.

Note that we have neglected the variation of B across the cross section of the toroid. By using an average r, as, for example, $r = (a + b)/2$, we have in effect used an average value of B in the calculation of inductance. If this is not sufficiently accurate, the variation of B should be considered by integrating (7.25) between a and b.

Coaxial Transmission Line

The student usually does not have any difficulty in grasping the concept of inductance as long as the geometries involve windings (such as in coils and toroids). In the following examples flux linkage is used in a broader sense and should clarify that concept further. Figure 7.8 shows a longitudinal and transverse cross section of a coaxial line (already considered in Sec. 5.5 when the capacitance' per unit length was calculated). The current I flows in the center conductor and returns

We have ignored the contribution of the magnetic field inside the inner conductor for several reasons. t First, as shown in Fig. 7.4, the magnetic flux within the inner conductor (assuming the current I is distributed uniformly throughout the cross section of the inner conductor, which is a valid assumption for direct current and for current at low frequencies) links only a fraction of that conductor; that fraction is proportional to $(rlaf$ because $I_{atr} = (rla)2I$. Second, at the higher frequencies the current is effectively confined to a thin layer (skin depth) at $r = a$ for the inner conductor and at $r = b$ for the outer. Third, most practical transmission lines use a small inner conductor and a thin-walled outer conductor. Hence the flux linkages within the conductors can be neglected, and (7.34) is an accurate expression for inductanceper unit length. Fig: 4.3 coxial transmission-cross section



We have approximated the upper limit *d* - *a* by *d* because for practical transmis- sion lines *d* ~ *a*. This approximation also accounts for the flux from the lower conductor which partly links the current inside the upper wire. As a matter of fact it can be shown that the replacement of *d* - *a* by *a* gives an exact result for the flux linkages t The inductance per unit length *LII*. which is the desired result and gives the total stored magnetic energy in an inductance *L* carrying current *I*. For example, a solenoid with an inductance of 8 H and a current of -!- A has an energy stored of *W* = *-!-LI2* = 1 J.

Just as a capacitor stores energy in its electric field, so does an inductor in its magnetic field. A measure of the effectiveness of energy storage in the electric field is the capacitance ($W = \frac{1}{2}CV^2$); in the magnetic field, it is the inductance ($W = \frac{1}{2}LI^2$).

To derive the expression for the storage of energy in the magnetic field of an Inductor

$$V = RI + L\frac{dI}{dt} + \frac{1}{C}\int I\,dt$$

Since the instantaneous power is $P = VI = dW/dt$, we can obtain the energy in the inductor L by integrating power $P_L = V_L I = LI\,dI/dt$. We obtain

$$W = \int_0^t P_L\,dt = L\int_0^t I\frac{dI}{dt}dt = L\int_0^I I\,dI = \frac{1}{2}LI^2$$

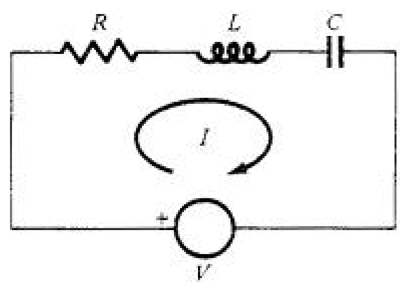


Fig:4.4 An RLC series circuit. Over a large range of current

which is the desired result and gives the total stored magnetic energy in an inductance L carrying current I. For example. a solenoid with an inductance of 8 H and a current of $\frac{1}{2}$ A has an energy stored of $W = \frac{1}{2}LI^2 = 1$ J.

4.8 ENERGY STORED INA MAGNETIC FIELD:

the charge on the capacitor plates or the electric field between the plates. We determined that the energy per unit volume in the electric E field is $w = \frac{1}{2}\varepsilon E^2$. Similarly, energy is stored in a magnetic field; the energy density will turn out to be $\frac{1}{2}\mu H^2$. To show this in the simplest way, we look for an inductor with a uniform field which is well confined. A long solenoid is suitable as is a toroid which has the magnetic field confined entirely to the region within the windings.

A toroid with a diameter that is large compared with that of its cross section

in the magnetic field is then

$$w = \frac{W}{v} = \frac{\frac{1}{2}LI^2}{A2\pi r} = \frac{1}{2}\mu_0 \left(\frac{NI}{2\pi r}\right)^2 = \frac{B^2}{2\mu_0} = \frac{1}{2}\mu_0 H^2$$

where A = cross section

$2\pi r$ = mean circumference

v = volume, $v = A2\pi r$

L = inductance, $L = \mu_0 N^2 A/2\pi r$

B = magnetic field of the toroid, $B = \mu_0 H = \mu_0 NI/2\pi r$

We state now without proof that (7.45) expresses the energy density in any magnetic field. The total magnetic energy stored in the field of an inductor can therefore be obtained by integrating the magnetic energy density:

$$W = \iiint w \, dv$$

The integration is over the entire volume in which the field exists and must, of course, be equal to $\frac{1}{2}LI^2$.

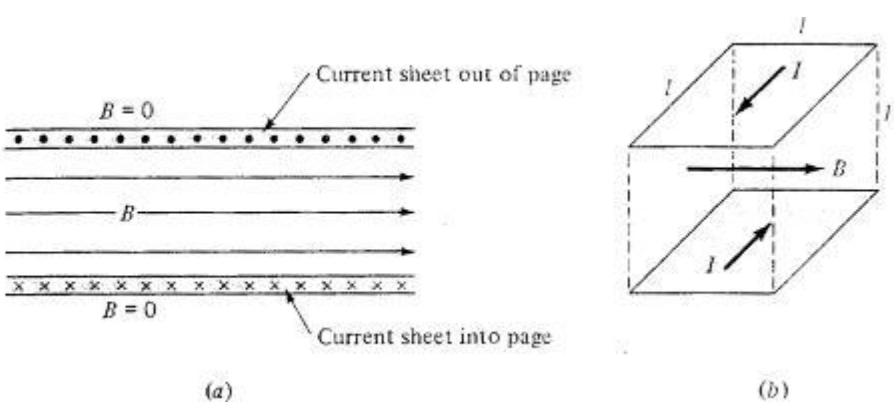


Fig:4.5 (*a*) The cross section of two infinite parallel current sheets; (*b*) a field cell is a cubical cut from the infinite sheets.

UNIT V TIME VARYING FIELDS AND MAXWELL'S EQUATIONS

Introduction:

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla \times \vec{E} = 0 \qquad (5.1a)$$

$$\nabla \cdot \vec{D} = \rho_v \qquad (5.1b)$$

For a linear and isotropic medium,

$$\vec{D} = \varepsilon \vec{E} \qquad (5.1c)$$

Similarly for the magnetostatic case

$$\nabla . \vec{B} = 0 \qquad (5.2a)$$

$$\nabla \times \vec{H} = \vec{J} \qquad (5.2b)$$

$$\vec{B} = \mu \vec{H} \qquad (5.2c)$$

It can be seen that for static case, the electric field vectors $\vec{E}$ and $\vec{D}$ and magnetic field vectors $\vec{B}$ and $\vec{H}$ form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

Faraday's Law of electromagnetic Induction

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law. Mathematically, the induced emf can be written as

$$\text{Emf} = \frac{d\phi}{dt} \quad \text{Volts} \qquad (5.3)$$

where ϕ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following: (a) time changing flux linkage a stationary closed path.

(a) relative motion between a steady flux a closed path.

(b) a combination of the above two cases.

The negative sign in equation (5.3) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

$$Emf = -N\frac{d\phi}{dt} \quad Volts \qquad (5.4)$$

By defining the total flux linkage as

$$\lambda = N\phi \qquad (5.5)$$

The emf can be written as

$$Emf = \frac{d\lambda}{dt} \qquad (5.6)$$

Continuing with equation (5.3), over a closed contour '*C*' we can write

$$Emf \oint_C \vec{E}.d\vec{l} \qquad (5.7)$$

where $\vec{E}$ is the induced electric field on the conductor to sustain the

current. Further, total flux enclosed by the contour '*C*' is given by

$$\phi = \int_s \vec{B}.d\vec{s} \tag{5.8}$$

Where *S* is the surface for which '*C*' is the contour.

From (5.7) and using (5.8) in (5.3) we can write

$$\oint_c \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_s \vec{B}.d\vec{s} \tag{5.9}$$

By applying stokes theorem

$$\int_s \nabla \times \vec{E}.d\vec{s} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{s} \tag{5.10}$$

Therefore, we can write

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.11}$$

which is the Faraday's law in the point form

We have said that non zero $\frac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a transformer emf **.**

Ideal transformers

As shown in figure 5.1, a transformer consists of two or more numbers of coils coupled magnetically through a common core. Let us consider an ideal transformer whose winding has zero resistance, the core having infinite permittivity and magnetic losses are zero.

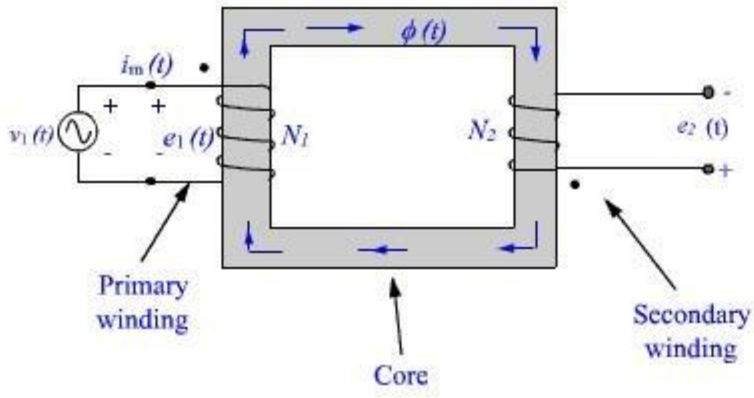


Fig 5.1: Transformer with secondary open

These assumptions ensure that the magnetization current under no load condition is vanishingly small and can be ignored. Further, all time varying flux produced by the primary winding will follow the magnetic path inside the core and link to the secondary coil without any leakage. If N1 and N2 are the number of turns in the primary and the secondary windings respectively, the induced emfs are

$$e_1 = N_1 \frac{d\phi}{dt} \qquad (5.12a)$$

$$e_2 = N_2 \frac{d\phi}{dt} \qquad (5.12b)$$

(The polarities are marked, hence negative sign is omitted. The induced emf is +ve at the dotted end of the winding.)

$$\frac{e_2}{\ } = \frac{N_2}{\ } \qquad (5.13)$$

i.e., the ratio of the induced emfs in primary and secondary is equal to the ratio of their turns. Under ideal condition, the induced emf in either winding is equal to their voltage rating.

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} = a \qquad (5.14)$$

where 'a' is the transformation ratio. When the secondary winding is connected to a load, the current flows in the secondary, which produces a flux opposing the original flux. The net flux in the core decreases and induced emf will tend to decrease from the no load value. This causes the primary current to increase to nullify the decrease in the flux and induced emf.

The current continues to increase till the flux in the core and the induced emfs are restored to the no load values. Thus the source supplies power to the primary winding and the secondary winding delivers the power to the load. Equating the powers

$$i_1 v_1 - i_2 v_2 \qquad (5.15)$$

$$\frac{i_2}{i_1} = \frac{v_1}{v_2} = \frac{e_1}{e_2} = \frac{N_1}{N_2} \qquad (5.16)$$

Further,

$$i_2 N_2 \quad i_1 N_1 \quad 0 \qquad (5.17)$$

i.e., the net magnetomotive force (mmf) needed to excite the transformer is zero under ideal condition.

Motional EMF:

Let us consider a conductor moving in a steady magnetic field as shown in the fig 5.2.

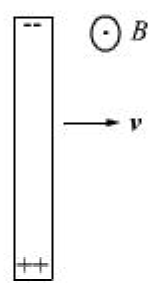


Fig 5.2

If a charge *Q* moves in a magnetic field $\vec{B}$, it experiences a force

$$\vec{F} = Q\vec{v} \times \vec{B} \qquad (5.18)$$

This force will cause the electrons in the conductor to drift towards one end and leave the other end positively charged, thus creating a field and charge separation continuous until electric and magnetic forces balance and an equilibrium is reached very quickly, the net force on the moving conductor is zero.

$$\frac{\vec{F}}{Q} = \vec{v} \times \vec{B}$$

field can be interpreted as an induced electric field which is called the motional electric

$$\vec{E}_m = \vec{v} \times \vec{B} \qquad (5.19)$$

If the moving conductor is a part of the closed circuit C, the generated emf around the circuit is $\oint_c \vec{v} \times \vec{B}.d\vec{l}$. This emf is called the motional emf.

Maxwell's Equation

Equation (5.1) and (5.2) gives the relationship among the field quantities in the static field. For time varying case, the relationship among the field vectors written as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (5.20a)$$

$$\nabla \times \vec{H} = \vec{J} \qquad (5.20b)$$

$$\nabla . \vec{D} = \rho \qquad (5.20c)$$

$$\nabla . \vec{B} = 0 \qquad (5.20d)$$

In addition, from the principle of conservation of charges we get the equation of continuity

$$\nabla . \vec{J} = -\frac{\partial \rho}{\partial t} \qquad (5.21)$$

The equation 5.20 (a) - (d) must be consistent with equation

(5.21). We observe that

$$\nabla . \nabla \times \vec{H} = 0 = \nabla . \vec{J} \qquad (5.22)$$

Since $\nabla . \nabla \times \vec{A}$ is zero for any vector $\vec{A}$.

Thus $\nabla \times \vec{H} = \vec{J}$ applies only for the static case i.e., for the scenario when . A classic example for this is given below .

$$\frac{\partial \rho}{\partial t} = 0$$

Suppose we are in the process of charging up a capacitor as shown in fig 5.3.

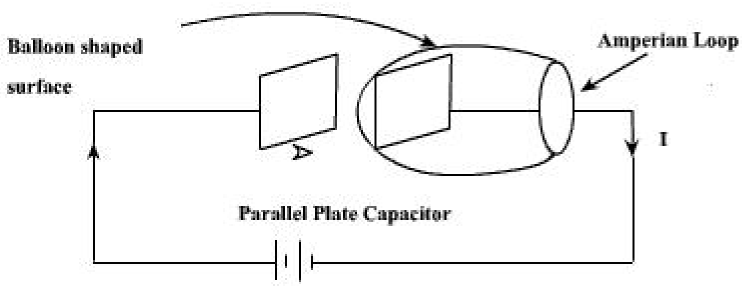


Fig 5.3 process of charging up a capacitor

Let us apply the Ampere's Law for the Amperian loop shown in fig 5.3. $I_{enc} = I$ is the total current passing through the loop. But if we draw a baloon shaped surface as in fig 5.3, no current passes through this surface and hence $I_{enc} = 0$. But for non steady currents such as this one, the concept of current enclosed by a loop is ill-defined since it depends on what surface you use. In fact Ampere's Law should also hold true for time varying case as well, then comes the idea of displacement current which will be introduced in the next few slides.

We can write for time varying case,

$$\nabla . \left(\nabla \times \vec{H} \right) = 0 = \nabla . \vec{J} + \frac{\partial \rho}{\partial t}$$

$$= \nabla . \vec{J} + \frac{\partial}{\partial t} \nabla . \vec{D}$$

$$= \nabla . \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad (5.23)$$

$$\therefore \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (5.24)$$

$\vec{J}$

The equation (5.24) is valid for static as well as for time varying case.

Equation (5.24) indicates that a time varying electric field will give rise to a magnetic field even in the absence of . The term $\frac{\partial \vec{D}}{\partial t}$ has a dimension of current densities (A/m^2) and is called the displacement current density.

Introduction of $\frac{\partial \vec{D}}{\partial t}$ in $\nabla \times \vec{H}$ equation is one of the major contributions of Jame's Clerk Maxwell. The modified set of equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (5.25b)$$

$$\nabla . \vec{D} = \rho \qquad (5.25c)$$

$$\nabla . \vec{B} = 0 \qquad (5.25d)$$

is known as the Maxwell's equation and this set of equations apply in the time varying scenario, static fields are being a particular case $\left(\frac{\partial}{\partial t} = 0\right)$.

In the integral form

$$\oint_c \vec{E}.d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{S} \qquad (5.26a)$$

$$\oint_c \vec{H}.d\vec{l} = \int_s \left(J + \frac{\partial D}{\partial t}\right).d\vec{S} = I + \int_s \frac{\partial \vec{D}}{\partial t}.d\vec{S} \qquad (5.26b)$$

$$\int_V \nabla . \vec{D} dv = \oint_s \vec{D}.d\vec{S} = \int_V \rho dv \qquad (5.26c)$$

$$\oint \vec{B}.d\vec{S} = 0 \qquad (5.26d)$$

The modification of Ampere's law by Maxwell has led to the development of a unified electromagnetic field theory. By introducing the displacement current term, Maxwell could predict the propagation of EM waves. Existence of EM waves was later demonstrated by Hertz experimentally which led to the new era of radio communication.

Boundary Conditions for Electromagnetic fields

The differential forms of Maxwell's equations are used to solve for the field vectors provided the field quantities are single valued, bounded and continuous. At the media boundaries, the field vectors are discontinuous and their behaviors across the boundaries are governed by boundary conditions.
The integral equations (eqn 5.26) are assumed to hold for regions containing discontinuous media. Boundary conditions can be derived by applying the Maxwell's equations in the integral form to small regions at the interface of the two media. The procedure is similar to those used for obtaining boundary conditions for static electric fields (chapter 2) and static magnetic fields (chapter 4). The boundary conditions are summarized as follows

With reference to fig 5.3

$$\widehat{a_n} \times \left(\overrightarrow{E_1} - \overrightarrow{E_2}\right) = 0 \qquad 5.27(a)$$

$$\widehat{a_n} \cdot \left(\overrightarrow{D_1} - \overrightarrow{D_2}\right) = \rho_S \qquad 5.27(b)$$

$$\widehat{a_n} \times \left(\overrightarrow{H_1} - \overrightarrow{H_2}\right) = \overrightarrow{J_S} \qquad 5.27(c)$$

$$\widehat{a_n} \cdot \left(\overrightarrow{B_1} - \overrightarrow{B_2}\right) = 0 \qquad 5.27(d)$$

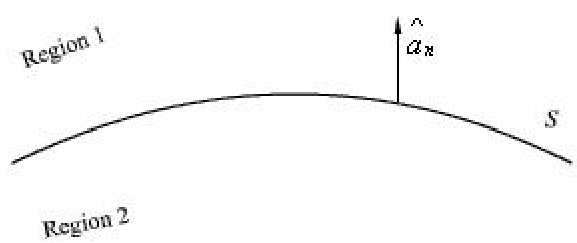


Fig 5.4

Equation 5.27 (a) says that tangential component of electric field is continuous across the interface while from 5.27 (c) we note that tangential component of the magnetic field is discontinuous by an amount equal to the surface current density. Similarly 5.27 (b) states that normal component of electric flux density vector $\overrightarrow{D}$ is discontinuous across the interface by an amount equal to the surface current density while normal component of the magnetic flux density is continuous.

If one side of the interface, as shown in fig 5.4, is a perfect electric conductor, say region 2, a surface current $\overrightarrow{J_S}$ can exist even though $\vec{E}$ is 0, $\sigma = \infty$
Thus eqn 5.27(a) and (c) reduces to

$$\widehat{a_n} \times \vec{H} = \overrightarrow{J_S} \qquad (5.28(a))$$
$$\widehat{a_n} \times \vec{E} = 0 \qquad (5.28(b))$$

WAVE EQUATION AND THEIR SOLUTION:

From equation 5.25 we can write the Maxwell's equations in the differential form as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{D} = \vec{\rho}$$
$$\nabla \cdot \vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ε, magnetic permeability μ and conductivity σ. The above set of equations can be written as

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (5.29(a))$$
$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (5.29(b))$$
$$\nabla \cdot \vec{E} = 0 \qquad (5.29(c))$$
$$\nabla \cdot \vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla \times \nabla \times \vec{A} = \nabla \cdot \left(\nabla \cdot \vec{A}\right) - \nabla^2 A$$

We can write from 5.29(b)

$$\nabla \times \nabla \times \vec{E} = \nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$
$$= -\nabla \times \left(\mu \frac{\partial \vec{H}}{\partial t}\right)$$

or $$\nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

Substituting $\nabla\times\vec{H}$ from 5.29(a)

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}\left(\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free medium $\nabla\cdot\vec{E}=0$ (eqn 5.29(c))

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad (5.30)$$

In the same manner for equation eqn 5.29(a)

$$\nabla\times\nabla\times\vec{H}=\nabla\cdot\left(\nabla\cdot\vec{H}\right)-\nabla^2\vec{H}$$

$$=\sigma\left(\nabla\times\vec{E}\right)+\varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

$$=\sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)+\varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H}=0$ from eqn 5.29(d), we can write

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right) \qquad (5.31)$$

These two equations

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

are known as wave equations.

It may be noted that the field components are functions of both space and time. For example, if we consider a Cartesian co ordinate system, $\vec{E}$ and $\vec{H}$ essentially represents $\vec{E}(x,y,z,t)$ and $\vec{H}(x,y,z,t)$. For simplicity, we consider propagation in free space, i.e. $\sigma = 0$, $\mu = \mu_0$, and $\varepsilon = \varepsilon_0$. The wave eqn in equations 5.30 and 5.31 reduces to

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \qquad (5.32(a))$$

$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right) \qquad (5.32(b))$$

Further simplifications can be made if we consider in Cartesian co ordinate system a special case where $\vec{E}$ and $\vec{H}$ are considered to be independent in two dimensions, say $\vec{E}$ and $\vec{H}$ are assumed to be independent of *y* and *z*. Such waves are called plane waves.

From eqn (5.32 (a)) we can write

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

The vector wave equation is equivalent to the three scalar equations

$$\frac{\partial^2 \vec{E_x}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_x}}{\partial t^2} \right) \qquad (5.33(a))$$

$$\frac{\partial^2 \vec{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_y}}{\partial t^2} \right) \qquad (5.33(b))$$

$$\frac{\partial^2 \vec{E_z}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_z}}{\partial t^2} \right) \qquad (5.33(c))$$

Since we have $\nabla \cdot \vec{E} = 0$,

$$\therefore \frac{\partial \overrightarrow{E_x}}{\partial x} + \frac{\partial \overrightarrow{E_y}}{\partial y} + \frac{\partial \overrightarrow{E_z}}{\partial z} = 0 \qquad (5.34)$$

As we have assumed that the field components are independent of y and z eqn (5.34)
reduces to

$$\frac{\partial E_x}{\partial x} = 0 \qquad (5.35)$$

i.e. there is no variation of E_X in the x direction.

Further, from 5.33(a), we find that $\frac{\partial E_x}{\partial x} = 0$ implies $\frac{\partial^2 E_x}{\partial t^2} = 0$ which requires any three of the conditions to be satisfied: (i) $E_X=0$, (ii)E_X = constant, (iii)E_X increasing uniformly with time.

A field component satisfying either of the last two conditions (i.e (ii) and (iii))is not a part of a plane wave motion and hence E_X is taken to be equal to zero. Therefore, a uniform plane wave propagating in x direction does not have a field component (*E* or *H*) acting along x.

Without loss of generality let us now consider a plane wave having E_Y component only (Identical results can be obtained for E_Z component) .

The equation involving such wave propagation is given by

$$\frac{\partial^2 \overrightarrow{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \overrightarrow{E_y}}{\partial t^2} \right) \qquad (5.36)$$

The above equation has a solution of the form

$$E_y = f_1(x - v_0 t) + f_2(x + v_0 t) \qquad (5.37)$$

where $v_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Thus equation (5.37) satisfies wave eqn (5.36) can be verified by substitution.
$f_1(x - v_0 t)$ corresponds to the wave traveling in the + x direction while $f_2(x + v_0 t)$ corresponds to a wave traveling in the -x direction. The general solution of the wave eqn thus consists of two waves, one traveling away from the source and other traveling back towards the source. In the absence of any reflection, the second form of the eqn (5.37) is zero and the solution can be written as

$$E_y = f_1(x - v_0 t) \qquad (5.38)$$

Such a wave motion is graphically shown in fig 5.5 at two instances of time t_1 and t_2.

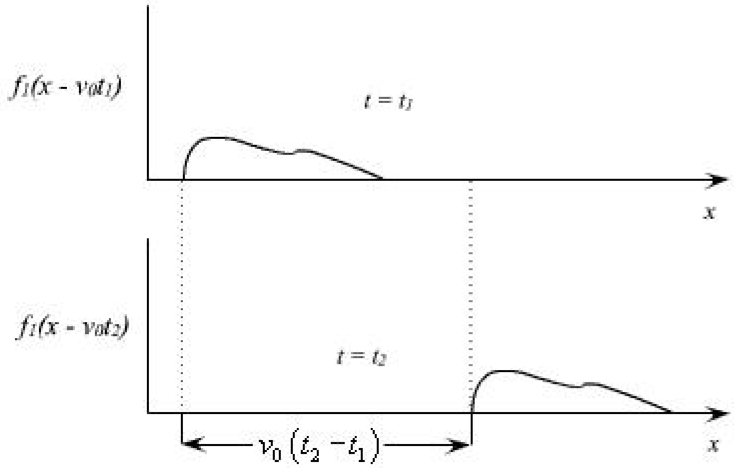


Fig 5.5 : Traveling wave in the + x direction

Let us now consider the relationship between E and H components for the forward traveling wave.

Since $\vec{E} = \widehat{a_y} E_y = \widehat{a_y} f_1(x - v_0 t)$ and there is no variation along y and z.

$$\nabla \times \vec{E} = \widehat{a_z} \frac{\partial E_y}{\partial x}$$

Since only z component of $\nabla \times \vec{E}$ exists, from (5.29(b))

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \qquad (5.39)$$

and from (5.29(a)) with $\sigma = 0$, only H_z component of magnetic field being present

$$\nabla \times \vec{H} = -\widehat{a_y} \frac{\partial H_z}{\partial x}$$

$$\therefore -\frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{\partial E_y}{\partial t} \qquad (5.40)$$

Substituting E_y from (5.38)

$$\frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t} = \varepsilon_0 v_0 f_1'(x - v_0 t)$$

$$\therefore \frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} f_1'(x - v_0 t)$$

$$\therefore H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \int f_1'\left(x - v_0 t\right)dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{\partial}{\partial x} f_1 dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} f_1 + c$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y + c$$

The constant of integration means that a field independent of x may also exist. However, this field will not be a part of the wave motion.

Hence $$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y \qquad (5.41)$$

which relates the *E* and *H* components of the traveling wave.

$$z_0 = \frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi \text{ or } 377\Omega$$

$z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the characteristic or intrinsic impedance of the free space

Harmonic fields

In the previous section we introduced the equations pertaining to wave propagation and discussed how the wave equations are modified for time harmonic case. In this section we discuss in detail a particular form of electromagnetic wave propagation called 'plane waves'. **The Helmhotz Equation:**

In source free linear isotropic medium, Maxwell equations in phasor form are,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \qquad \nabla \times \vec{H} = 0$$

$$\therefore \nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \times \vec{E}\right) - \nabla^2 \vec{E} = -j\omega\mu\nabla \times \vec{H}$$

or, $$-\nabla^2 \vec{E} = -j\omega\mu(j\omega\varepsilon\vec{E})$$

$$\text{or, } \nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0$$

$$\text{or, } \nabla^2 \vec{E} + k^2 \vec{E} = 0 \text{ where } k = \omega\sqrt{k\varepsilon}$$

An identical equation can be derived for $\vec{H}$.

$$\text{i.e., } \nabla^2 \vec{H} + k^2 \vec{H} = 0$$

These equations

$$\left.\begin{array}{l} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \ldots\ldots\ldots\ldots\ldots (a) \\ \& \quad \nabla^2 \vec{H} + k^2 \vec{H} = 0 \ldots\ldots\ldots\ldots\ldots (b) \end{array}\right\} \ldots\ldots (6.1)$$

are called homogeneous vector Helmholtz's equation.

$k = \omega\sqrt{\mu\varepsilon}$ is called the wave number or propagation constant of the medium.

Plane waves in Lossless medium:

In a lossless medium, ε and μ are real numbers, so k is real.

In Cartesian coordinates each of the equations 6.1(a) and 6.1(b) are equivalent to three scalar Helmholtz's equations, one each in the components E_X, E_y and E_Z or H_X, H_y, H_Z.

For example if we consider E_X component we can write

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.2)$$

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wavefront or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios.

Let us consider a plane wave which has only E_X component and propagating along z . Since the plane wave will have no variation along the plane perpendicular to z i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation (6.2) reduces to,

$$\frac{d^2E_x}{dz^2}+k^2E_x=0 \quad \text{...(6.3)}$$

The solution to this equation can be written as

$$E_x(z)=E_x^+(z)+E_x^-(z)$$
$$=E_0^+e^{-jkz}+E_0^-e^{jkz} \quad \text{...(6.4)}$$

E_0^+ & E_0^- are the amplitude constants (can be determined from boundary conditions). In the time domain, $\varepsilon_X(z,t)=\mathrm{Re}(E_x(z)e^{j\omega t})$

$$\varepsilon_X(z,t)=E_0^+\cos(\omega t-kz)+E_0^-\cos(\omega t+kz) \quad \text{............................(6.5)}$$

assuming E_0^+ & E_0^- are real constants.

Here, $\varepsilon_X^+(z,t)=E_0^+\cos(\omega t-\beta z)$ represents the forward traveling wave. The plot of $\varepsilon_X^+(z,t)$ for several values of t is shown in the Figure 6.1.

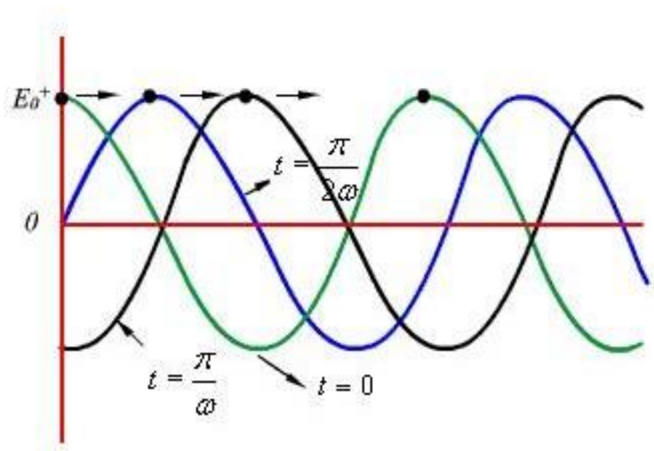


Figure 6.1: Plane wave traveling in the + *z* direction

As can be seen from the figure, at successive times, the wave travels in the +z direction.

If we fix our attention on a particular point or phase on the wave (as shown by the dot) i.e. , $\omega t-kz$ = constant

Then we see that as *t* is increased to $t+\Delta t$, z also should increase to $z+\Delta z$ so that

$$\omega(t+\Delta t)-k(z+\Delta z)=\text{constant}=\omega t-\beta z$$

Or, $\omega\Delta t = k\Delta z$

Or, $\dfrac{\Delta z}{\Delta t}=\dfrac{\omega}{k}$

When $\Delta t \to 0$,

we write $\lim_{\Delta t\to 0}\dfrac{\Delta z}{\Delta t}=\dfrac{dz}{dt}=$ phase velocity v_p.

$$\therefore v_p=\frac{\omega}{k} \quad \text{.....................................(6.6)}$$

If the medium in which the wave is propagating is free space i.e.,
$\varepsilon=\varepsilon_0$, $\mu=\mu_0$

Then $v_p=\dfrac{\omega}{\omega\sqrt{\mu_0\varepsilon_0}}=\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}=C$

Where '*C*' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength λ is defined as the distance between two successive maxima (or minima or any other reference points).

i.e., $(\omega t-kz)-\left[\omega t-k(z+\lambda)\right]=2\pi$

or, $k\lambda=2\pi$

or, $\lambda=\dfrac{2\pi}{k}$

Substituting $k=\dfrac{\omega}{v_p}$,

$$\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$$

or, $$\lambda f = v_p \qquad \text{................................(6.7)}$$

Thus wavelength λ also represents the distance covered in one oscillation of the wave. Similarly, $\varepsilon^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as follows:

From (6.4),

$$\vec{E}_x^{\,+}(z) = E_0^+ e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla \times \vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^+ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu}E_0^+ e^{-jkz}\hat{a}_y$$

$$= \frac{E_0^+}{\eta}e^{-jkz}\hat{a}_y = H_0^+ e^{-jkz}\hat{a}_y \qquad \text{............(6.8)}$$

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium. When the wave travels in free space

$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$ is the intrinsic impedance of the free space.

In the time domain,

$$\overrightarrow{H}^{+}(z,t) = \hat{a}_y \frac{E_0^{+}}{\eta} \cos\left(\omega t - \beta z\right) \quad \text{........... (6.9)}$$

Which represents the magnetic field of the wave traveling in the +z

direction. For the negative traveling wave,

$$\overrightarrow{H}^{-}(z,t) = -a_y \frac{E_0^{+}}{\eta} \cos\left(\omega t + \beta z\right) \quad \text{...........(6.10)}$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig 6.2.

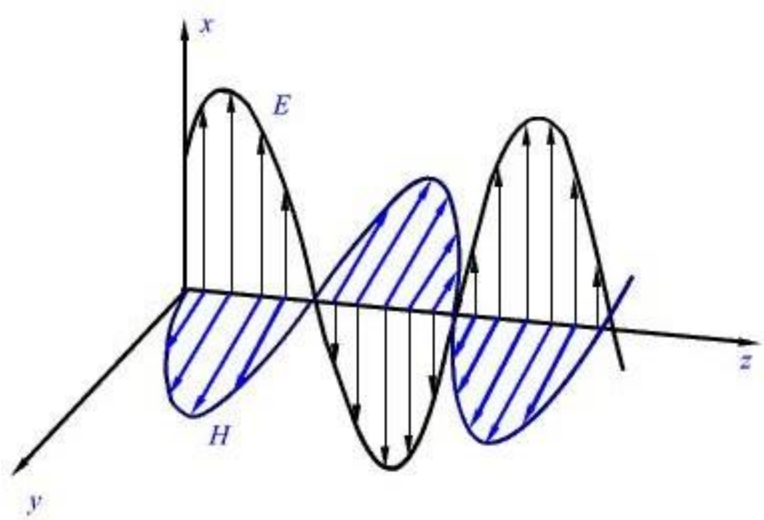


Figure 6.2 : E & H fields of a particular plane wave at time t.

TEM Waves:

So far we have considered a plane electromagnetic wave propagating in the z-direction. Let us now consider the propagation of a uniform plane wave in any arbitrary direction that doesn't necessarily coincides with an axis.

For a uniform plane wave propagating in z-direction

$\vec{E}(z) = E_0 e^{-jkz}$, E_0 is a constant vector..........................

(6.11) The more general form of the above equation is

$\vec{E}(x,y,z) = \vec{E}_0 e^{-jk_x x - jk_y y - jk_z z}$ ....

(6.12) This equation satisfies Helmholtz's equation $\nabla^2 \vec{E} + k^2 \vec{E} = 0$ provided,

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \varepsilon \quad \text{.......................... (6.13)}$$

We define wave number vector $\vec{k} = \hat{a}_x k_x + \hat{a}_y k_y + \hat{a}_z k_z = k \hat{a}_n$ ....

(6.14) And radius vector from the origin

$$\vec{r} = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z \quad \text{.......................... (6.15)}$$

Therefore we can write

$$\vec{E}(\vec{r}) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-jk\hat{a}_n\cdot\vec{r}} \quad \text{.......................... (6.16)}$$

Here $\hat{a}_n \cdot \vec{r}$ =constant is a plane of constant phase and uniform amplitude just in the case of $\vec{E}(z) = \vec{E}_0 e^{-jkz}$,

z =constant denotes a plane of constant phase and uniform

amplitude. If the region under consideration is charge free,

$$\nabla \cdot \vec{E} = 0$$

$$\therefore \nabla \cdot \left(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}\right) = 0$$

Using the vector identity $\nabla.\left(f\vec{A}\right) = \vec{A}\nabla f + f\nabla.\vec{A}$ and noting that $\vec{E}_0$ is constant we can write,

$$\vec{E}_0.\nabla\left(e^{-jk\hat{a}_n.\vec{r}}\right) = 0$$

$$or,\ \vec{E}_0.\left[\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)e^{-j(k_x x + k_y y + k_z z)}\right] = 0$$

$$or,\vec{E}_0.\left(-jk\,\hat{a}_n e^{-jk\hat{a}_n.\vec{r}}\right) = 0$$

$$\vec{E}_0.\hat{a}_n = 0 \quad ....(6.17)$$

i.e., $\vec{E}_0$ is transverse to the direction of the propagation.

The corresponding magnetic field can be computed as follows:

$$\vec{H}\left(\vec{r}\right) = -\frac{1}{j\omega\mu}\nabla\times\vec{E}\left(\vec{r}\right) = -\frac{1}{j\omega\mu}\nabla\times\left(\vec{E}_0 e^{-j\vec{k}.\vec{r}}\right)$$

Using the vector identity,

$$\nabla\times\left(\psi\vec{A}\right) = \psi\nabla\times\vec{A} + \nabla\psi\times\vec{A}$$

Since $\vec{E}_0$ is constant we can write,

$$\vec{H}(\vec{r}) = -\frac{1}{j\omega\mu}\nabla e^{-j\vec{k}.\vec{r}}\times\vec{E}_0$$

$$= -\frac{1}{j\omega\mu}\left[-jk\hat{a}_n\times\vec{E}_0 e^{-jk\hat{a}_n.\vec{r}}\right]$$

$$= \frac{k}{\omega\mu}\hat{a}_n\times\vec{E}\left(\vec{r}\right)$$

$$\vec{H}(\vec{r}) = \frac{1}{\eta}\hat{a}_n\times\vec{E}\left(\vec{r}\right) \quad ....(6.18)$$

Where η is the intrinsic impedance of the medium. We observe that $\vec{H}(\vec{r})$ is perpendicular to both $\hat{a}_n$ and $\vec{E}\left(\vec{r}\right)$. Thus the electromagnetic wave represented by $\vec{E}\left(\vec{r}\right)$ and $\vec{H}(\vec{r})$ is a TEM wave.

Plane waves in a lossy medium:

In a lossy medium, the EM wave looses power as it propagates. Such a medium is conducting with conductivity σ and we can write:

$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E} = (\sigma + j\omega\varepsilon)\vec{E}$$

$$= j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\vec{E}$$

$$= j\omega\varepsilon_c\vec{E} \qquad \text{.....................(6.19)}$$

Where $\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$ is called the complex permittivity.

We have already discussed how an external electric field can polarize a dielectric and give rise to bound charges. When the external electric field is time varying, the polarization vector will vary with the same frequency as that of the applied field. As the frequency of the applied filed increases, the inertia of the charge particles tend to prevent the particle displacement keeping pace with the applied field changes. This results in frictional damping mechanism causing power loss.

In addition, if the material has an appreciable amount of free charges, there will be ohmic losses. It is customary to include the effect of damping and ohmic losses in the imaginary part of ε_c. An equivalent conductivity $\sigma = \omega\varepsilon''$ represents all losses.

The ratio $\frac{\varepsilon''}{\varepsilon'}$ is called loss tangent as this quantity is a measure of the power loss.

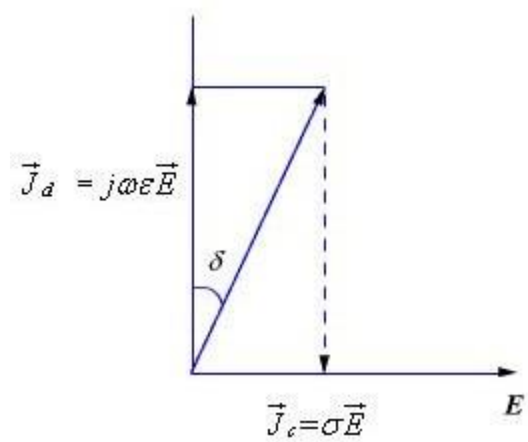


Fig 6.3 : Calculation of Loss Tangent

With reference to the Fig 6.3,

$$\tan\delta = \frac{|\vec{J}_c|}{|\vec{J}_d|} = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'} \qquad \text{.......................... (6.20)}$$

where $\vec{J}_c$ is the conduction current density and $\vec{J}_d$ is displacement current density. The loss tangent gives a measure of how much lossy is the medium under consideration. For a good dielectric medium $(\sigma << \omega\varepsilon)$, $\tan\delta$ is very small and the medium is a good conductor if $(\sigma \gg \omega\varepsilon)$. A material may be a good conductor at low frequencies but behave as lossy dielectric at higher frequencies.

For a source free lossy medium we can write

$$\left.\begin{aligned} \nabla\times\vec{H} &= (\sigma + j\omega\varepsilon)\vec{E} & \nabla.\vec{H} &= 0 \\ \nabla\times\vec{E} &= -j\omega\mu\vec{H} & \nabla.\vec{E} &= 0 \end{aligned}\right\} \qquad \text{...........................(6.21)}$$

$$\nabla\times\nabla\times\vec{E} = \nabla(\nabla.\vec{E}) - \nabla^2\vec{E} = -j\omega\mu\nabla\times\vec{H} = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}$$

$$or, \nabla^2\vec{E} - \gamma^2\vec{E} = 0 \qquad \text{.................... (6.22)}$$

Where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

Proceeding in the same manner we can write,

$$\nabla^2\vec{H} - \gamma^2\vec{H} = 0$$

$$\gamma = \alpha + i\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

is called the propagation constant.

The real and imaginary parts α and β of the propagation constant η can be computed as follows:

$$\gamma^2 = (\alpha + i\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$or, \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

And $\alpha\beta = \dfrac{\omega\mu\sigma}{2}$

$$\therefore \alpha^2 - \left(\frac{\omega\mu\sigma}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon = \omega^2\mu^2\sigma^2$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon + \omega^4\mu^2\varepsilon^2 = \omega^2\mu^2\sigma^2 + \omega^4\mu^2\varepsilon^2$$

$$or, \left(2\alpha^2 + \omega^2\mu\varepsilon\right)^2 = \omega^4\mu^2\varepsilon^2\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)$$

$$or, \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \quad \text{.................... (6.23a)}$$

Similarly $$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} \quad \text{.................. (6.23b)}$$

Let us now consider a plane wave that has only x -component of electric field and propagate along z .

$$\therefore \vec{E}_x(z) = \left(E_0^+ e^{-\gamma z} + E_0^- e^{-\gamma z}\right)\hat{a}_x \quad \text{................................... (6.24)}$$

Considering only the forward traveling wave

$$\vec{\varepsilon}(z,t) = \mathrm{Re}\left(E_0^+ e^{-\gamma z} e^{j\omega t}\right)\hat{a}_x$$

$$= E_0^+ e^{-\alpha z}\cos\left(\omega t - \beta z\right)\hat{a}_x \quad \text{................................... (6.25)}$$

Similarly, from $\vec{\mathcal{H}} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$, we can find

$$\vec{\mathcal{H}}(z,t) = \frac{E_0}{\eta} e^{-\alpha z}\cos\left(\omega t - \beta z\right)\hat{a}_y \quad \text{..................................... (6.26)}$$

Where $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |n| e^{j\theta_n}$

$$\therefore \vec{\mathcal{H}} = \frac{E_0}{|n|} e^{-\alpha z}\cos\left(\omega t - \beta z - \theta_n\right)\hat{a}_y \quad \text{..................................... (6.27)}$$

From (6.25) and (6.26) we find that as the wave propagates along z, it decreases in amplitude by a factor $e^{-\alpha z}$. Therefore α is known as attenuation constant. Further $\vec{E}$ and $\vec{H}$ are out of phase by an angle θ_{η}.

For low loss dielectric, $\frac{\sigma}{\omega\varepsilon} << 1$, i.e., $\varepsilon'' << \varepsilon'$.

Using the above condition approximate expression for α and β can be obtained as follows:

$$\gamma = \alpha + i\beta = j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{\varepsilon'}\right]^{1/2}$$

$$\cong j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{1}{2}\frac{\varepsilon''}{\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\left.\begin{aligned} \alpha &= \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} \\ \beta &= \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \end{aligned}\right\} \ldots\ldots\ldots(6.28)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right) \qquad (6.29)$$

& phase velocity

$$v_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \ldots\ldots\ldots\ldots\ldots (6.30)$$

For good conductors $\frac{\sigma}{\omega\varepsilon} >> 1$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \cong j\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{j\omega\varepsilon}}$$

$$= \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} \ldots\ldots\ldots\ldots\ldots (6.31)$$

We have used the relation

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)$$

From (6.31) we can write

$$\alpha + i\beta = \sqrt{\pi f \mu\sigma} + j\sqrt{\pi f \mu\sigma}$$

$$\therefore \alpha = \beta = \sqrt{\pi f \mu\sigma} \quad \text{............... (6.32)}$$

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(1+\frac{\sigma}{j\omega\varepsilon}\right)}}$$

$$\cong \sqrt{\frac{\mu}{\varepsilon}\frac{j\omega\varepsilon}{\sigma}} = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= (1+j)\frac{\alpha}{\sigma} \quad \text{..................... (6.33)}$$

And phase velocity

$$v_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{..................... (6.34)}$$

Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities asscociated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Using vector identity

$$\nabla.\left(\vec{E}\times\vec{H}\right) = \vec{H}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{H}$$

the above curl equations we can write

$$\nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

$$or,\ \nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\vec{J} - \vec{E}.\frac{\partial\vec{D}}{\partial t}$$ ....(6.35)

In simple medium where $\in, \mu$ and σ are constant, we can write

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right) \quad \text{and} \quad \vec{E}.\vec{J} = \sigma E^2$$

$$\therefore \nabla.\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV - \int_V \sigma E^2 dV$$ ....(6.36)

The term $\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2+\frac{1}{2}\mu H^2\right)dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (6.36) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S\left(\vec{E}\times\vec{H}\right).d\vec{S}=\oint_S \vec{P}.d\vec{S}$ where $\vec{P}=\vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction
losses.

Poynting vector for the time harmonic case:

For time harmonic case, the time variation is of the form $e^{j\omega t}$, and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z)=\hat{a}_x E_x(z)=\hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t)=\mathrm{Re}\left[\vec{E}(z)e^{j\omega t}\right]=E_0\cos(\omega t-\beta z)\hat{a}_x \quad \text{.................................(6.37)}$$

when *E0* is real.
Let us consider two instanteneous quantities A and B such that

$$A=\mathrm{Re}\left(Ae^{j\omega t}\right)=|A|\cos(\omega t+\alpha)$$

$$B=\mathrm{Re}\left(Be^{j\omega t}\right)=|B|\cos(\omega t+\beta)$$

where A and B are the phasor quantities.

i.e,

$$A=|A|e^{j\alpha}$$

$$B=|B|e^{j\beta}$$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right] \quad \text{..............................(6.39)}$$

Since A and B are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$$\overline{AB} = \frac{1}{2}|A||B|\cos(\alpha - \beta) \quad \text{.....................................(6.40)}$$

Further, considering the phasor quantities A and B, we find that

$$AB^* = |A|e^{j\alpha}\,|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}\left(AB^*\right) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}\left(AB^*\right) \quad \text{..(6.41)}$$

The poynting vector $\vec{P} = \vec{E}\times\vec{H}$ can be expressed as

$$\vec{P} = \hat{a}_x\left(E_yH_z - E_zH_y\right) + \hat{a}_y\left(E_zH_x - E_xH_z\right) + \hat{a}_z\left(E_xH_y - E_yH_x\right) \quad \text{...................................(6.42)}$$

If we consider a plane electromagnetic wave propagating in +z direction and has only E_x component, from (6.42) we can write:

$$\vec{P}_z = E_x(z,t)H_y(z,t)\hat{a}_3$$

Using (6.41)

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)H_y^*(z)\hat{a}_z\right)$$

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)\times H_y(z)\right) \quad \text{..(6.43)}$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration. For a general case, we can write

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right) \quad ....(6.44)$$

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}\left(\vec{S}\right)$.

5.10 Electromagnetic Spectrum:

The polarisation of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. For an electromagnetic wave, the specification of the orientation of the electric field is sufficent as the magnetic field components are related to electric field vector by the Maxwell's equations.

Let us consider a plane wave travelling in the +z direction. The wave has both E_x and E_y components.

$$\vec{E} = \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right)e^{-j\beta z} \quad ....(6.45)$$

The corresponding magnetic fields are given by,

$$\vec{H} = \frac{1}{\eta}\hat{a}_z\times\vec{E}$$

$$= \frac{1}{\eta}\hat{a}_z\times\left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right)e^{-j\beta z}$$

$$= \frac{1}{\eta}\left(-E_{oy}\hat{a}_x + E_{ox}\hat{a}_x\right)e^{-j\beta z}$$

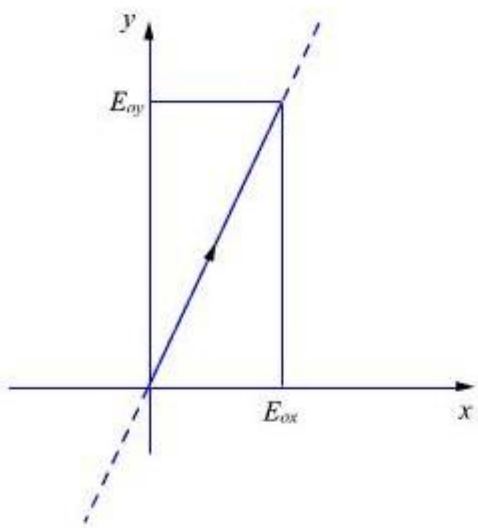
Depending upon the values of E_{ox} and E_{oy} we can have several possibilities:

1. If $E_{oy} = 0$, then the wave is linearly polarised in the *x*-direction.

2. If $E_{oy} = 0$, then the wave is linearly polarised in the *y*-direction.

3. If E_{ox} and E_{oy} are both real (or complex with equal phase), once again we get a

linearly polarised wave with the axis of polarisation inclined at an angle $\tan^{-1}\frac{E_{oy}}{E_{ox}}$, with respect to the x-axis. This is shown in fig 6.4.



emerging

Fig 6.4:Linear Polarisation

4. If *Eox* and *Eoy* are complex with different phase angles, $\vec{E}$ will not point to a single spatial direction. This is explained as follows:

$$\text{Let } E_{ox} = |E_{ox}|e^{ja}$$

$$E_{oy} = |E_{oy}|e^{jb}$$

Then,

$$E_x(z,t) = \text{Re}\left[|E_{ox}|e^{ja}e^{-j\beta z}e^{j\omega t}\right] = |E_{ox}|\cos(\omega t - \beta z + a)$$

and $E_y(z,t) = \text{Re}\left[|E_{oy}|e^{jb}e^{-j\beta z}e^{j\omega t}\right] = |E_{oy}|\cos(\omega t - \beta z + b)$ ....(6.46)

To keep the things simple, let us consider a =0 and $b = \frac{\pi}{2}$. Further, let us study the nature of the electric field on the z =0 plain.

From equation (6.46) we find that,

$$E_x(o,t) = |E_{ox}|\cos\omega t$$

$$E_y(o,t) = |E_{oy}|\cos\left(\omega t + \frac{\pi}{2}\right) = |E_{oy}|(-\sin \omega t)$$

$$\therefore \left(\frac{E_x(o,t)}{|E_{ox}|}\right)^2 + \left(\frac{E_y(o,t)}{|E_{oy}|}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1 \qquad \text{......................................(6.47)}$$

and the electric field vector at z = 0 can be written as

$$\vec{E}(o,t) = |E_{ox}|\cos(\omega t)\hat{a}_x - |E_{oy}|\sin(\omega t)\hat{a}_y \qquad \text{...(6.48)}$$

Assuming $|E_{ox}| > |E_{oy}|$, the plot of $\vec{E}(o,t)$ for various values of t is hown in figure 6.5.

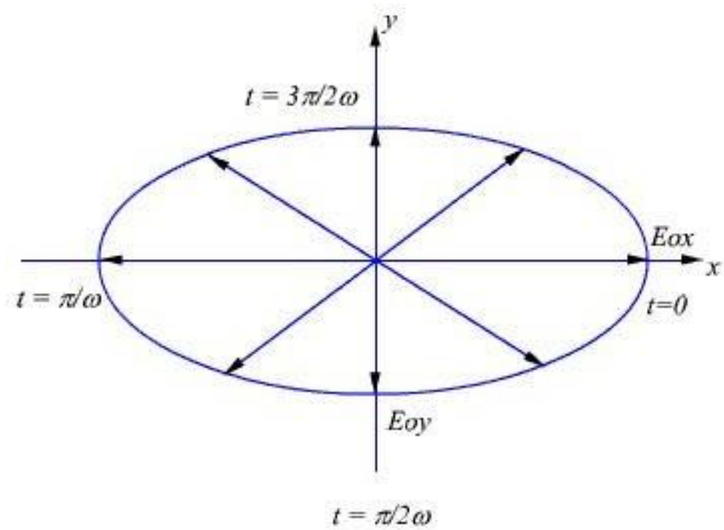


Figure 6.5:Plot of *E*(*o*,*t*)

From equation (6.47) and figure (6.5) we observe that the tip of the arrow representing electric field vector traces qn ellipse and the field is said to be elliptically polarised.

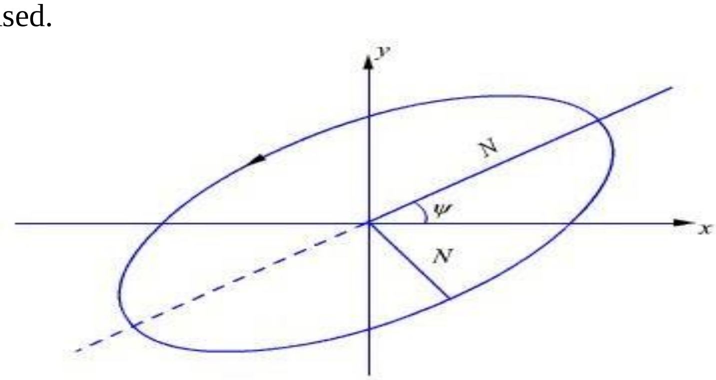


Figure 6.6: Polarisation ellipse

The polarisation ellipse shown in figure 6.6 is defined by its axial ratio(M/N, the ratio of semimajor to semiminor axis), tilt angle ψ (orientation with respect to xaxis) and sense of rotation(i.e., CW or CCW).

Linear polarisation can be treated as a special case of elliptical polarisation, for which the axial ratio is infinite.

In our example, if $|E_{ox}| = |E_{oy}|$, from equation (6.47), the tip of the arrow representing electric field vector traces out a circle. Such a case is referred to as Circular Polarisation. For circular polarisation the axial ratio is unity.

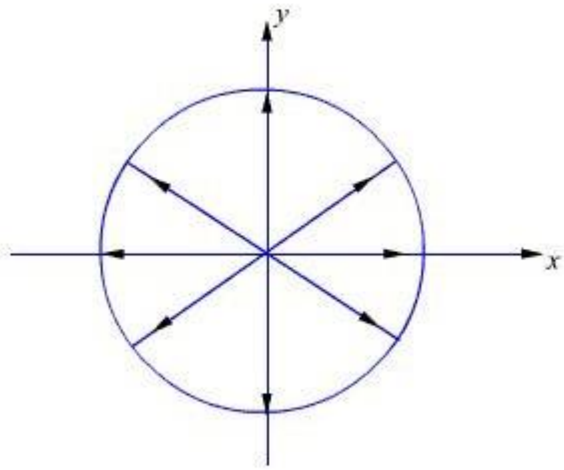


Figure 6.7:Circular Polarisation (RHCP)

Further, the circular polarisation is aside to be right handed circular polarisation (RHCP) if the electric field vector rotates in the direction of the fingers of the right hand when the thumb
points in the direction of propagation-(same and CCW). If the electric field vector rotates in the opposite direction, the polarisation is asid to be left hand circular polarisation (LHCP) (same as CW).

In AM radio broadcast, the radiated electromagnetic wave is linearly polarised with the $\vec{E}$ field vertical to the ground(vertical polarisation) where as TV signals are horizontally polarised waves. FM broadcast is usually carried out using circularly polarised waves.

In radio communication, different information signals can be transmitted at the same frequency at orthogonal polarisation (one signal as vertically polarised other horizontally polarised or one as RHCP while the other as LHCP) to increase capacity. Otherwise, same signal can be transmitted at orthogonal polarisation to obtain diversity gain to improve reliability of transmission.

Behaviour of Plane waves at the inteface of two media:

We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of ε, μ, σ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

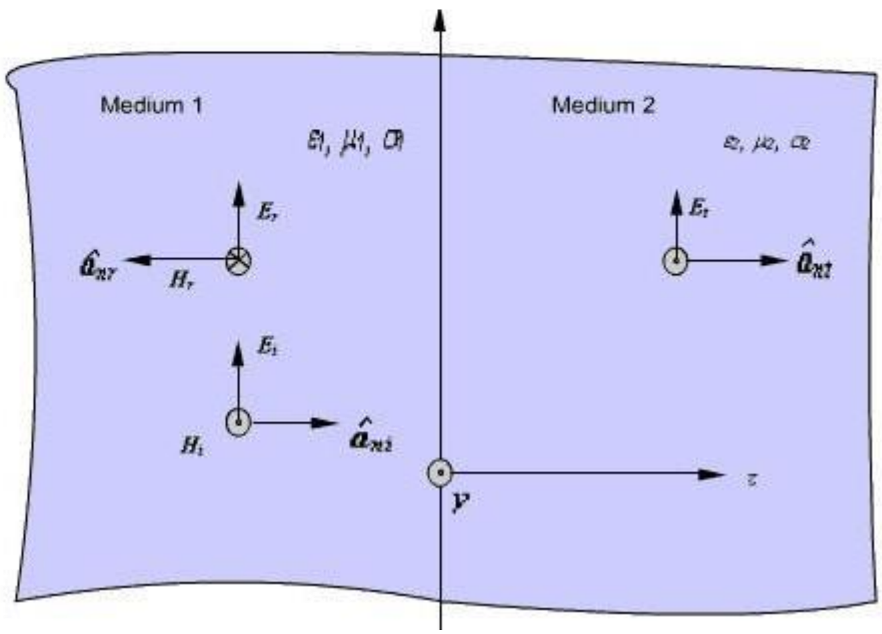


Fig 6.8 : Normal Incidence at a plane boundary

Case1: Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.

Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively.

The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium

1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x \quad \text{..................(6.49.a)}$$

$$\vec{H}_i(z) = \frac{1}{\eta_1} \hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_1} e^{-\gamma_1 z} \hat{a}_y \quad \text{......................(6.49.b)}$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\dfrac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at z =0, the incident wave will undergo partial reflection and partial transmission.
The reflected wave will travel along $-\hat{a}_z$ in medium

1. The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z} \hat{a}_x \text{ ....(6.50a)}$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z} \hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y \text{ ....(6.50b)}$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{a}_x \text{ ....(6.51a)}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y \text{ ....(6.51b)}$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\dfrac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1, $\vec{E}_1 = \vec{E}_i + \vec{E}_r$ and $\vec{H}_1 = \vec{H}_i + \vec{H}_r$

and in medium 2, $\vec{E}_2 = \vec{E}_t$ and $\vec{H}_2 = \vec{H}_t$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_t(0)$$

& $$\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$$

From equation 6.49 to 6.51 we get,

$$E_{io} + E_{ro} = E_{to} \quad \text{..(6.52a)}$$

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2} \quad \text{..(6.52b)}$$

Eliminating

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}\left(E_{io} + E_{ro}\right)$$

Eto ,

or, $$E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$$

or, $$E_{ro} = \tau E_{io}$$

...............(6.53)

is called the reflection coefficient. $\tau = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

From equation (6.52), we can

write

$$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$$

$$E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2} E_{io} = TE_{io}$$

or,

$$T = \frac{2\eta_2}{\eta_1 + \eta_2}$$

. .

....(6.54)

is called the transmission

coefficient. We observe that,

$$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau$$ ....(6.55)

The following may be noted

(i) both τ and T are dimensionless and may be complex

(ii) $0 \leq |\tau| \leq 1$

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\eta_2 = 0$$

$$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$$
$$= j\omega\sqrt{\mu_1\epsilon_1} = j\beta_1$$

From (6.53) and (6.54)

$$\tau = -1$$

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\,\hat{a}_x - E_{io}e^{j\beta_1 z}\,\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$$

&

$$\therefore \vec{E}_1(z,t) = \text{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x \quad \text{.................................(6.56)}$$

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t \quad \text{...(6.57)}$$

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given *'t'*, both $\vec{E}_1$ and $\vec{H}_1$ vary
sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

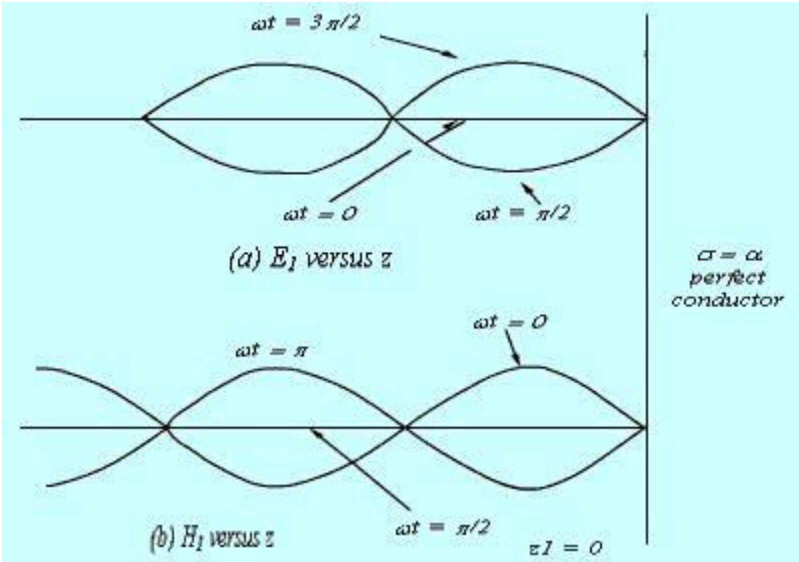


Figure 6.9: Generation of standing wave

Zeroes of $E_1(z,t)$ and Maxima of $H_1(z,t)$.

Maxima of $E_1(z,t)$ and zeroes of $H_1(z,t)$.

$$\left.\right\}\text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\frac{\lambda}{2}$$

$$\left.\right\}\text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \quad \text{or } z = -(2n+1)\frac{\lambda}{4},\quad n = 0,1,2,\ldots \qquad \text{.......(6.58)}$$

Case2: Normal incidence on a plane dielectric boundary

If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.
From equation (6.49(a)) and equation (6.53) we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x \quad \text{..................(6.59)}$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0, \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \quad \text{..................(6.60)}$$

In this case both η_1 and η_1 become real numbers.

$$\begin{aligned}\vec{E}_1 &= \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)\\ &= \hat{a}_x E_{io}\left((1+T)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right)\\ &= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma\left(2j\sin\beta_1 z\right)\right)\end{aligned} \quad \text{..................(6.61)}$$

From (6.61), we can see that, in medium 1 we have a traveling wave component with amplitude TE_{io} and a standing wave component with amplitude $2JE_{io}$.

The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows.

The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right) \quad \text{..................(6.62)}$$

If $\eta_2 > \eta_1$ i.e. Γ *>0*

The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{max} = E_{io}(1+T) \quad \text{..................(6.63)}$$

and this occurs when

$$2\beta_1 z_{max} = -2n\pi$$

or $z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{2\pi/\lambda_1} = -\frac{n}{2}\lambda_1$, n = 0, 1, 2, 3......................(6.64)

The minimum value of $\left|\vec{E}_1\right|$ is

$$\left|\vec{E}_1\right|_{min} = E_{io}\left(1-\Gamma\right)$$ ....(6.65)

And this occurs when

$$2\beta_1 z_{min} = -\left(2n+1\right)\pi$$

or $z_{min} = -\left(2n+1\right)\frac{\lambda_1}{4}$, n = 0, 1, 2,

3............................(6.66) For $\eta_2 < \eta_1$ i.e. $\Gamma < 0$

The maximum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1-\Gamma\right)$ which occurs at the z_{min} locations and the minimum

value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1+\Gamma\right)$ which occurs at z_{max} locations as given by the equations (6.64) and (6.66).

From our discussions so far we observe that $\frac{\left|E\right|_{max}}{\left|E\right|_{min}}$ can be written as

$$S = \frac{\left|E\right|_{max}}{\left|E\right|_{min}} = \frac{1+\left|\Gamma\right|}{1-\left|\Gamma\right|}$$ ....(6.67)

The quantity *S* is called as the standing wave ratio.

As $0 \le \left|\Gamma\right| \le 1$ the range of *S* is given by $1 \le S \le \infty$

From (6.62), we can write the expression for the magnetic field in medium 1 as

$$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1-\Gamma e^{j2\beta_1 z}\right)$$ ....(6.68)

From (6.68) we find that $|\vec{H}_1|$ will be maximum at locations where $|\vec{E}_1|$ is minimum and vice versa.
In medium 2, the transmitted wave propagates in the + z direction.

Oblique Incidence of EM wave at an interface

So far we have discuss the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

1. When the second medium is a perfect conductor.
2. When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal to the interface. We study two specific cases when the incident electric field $\vec{E}_i$ is perpendicular to the plane of incidence (perpendicular polarization) and $\vec{E}_i$ parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

Oblique Incidence at a plane conducting boundary i. Perpendicular Polarization
The situation is depicted in figure
6.10.

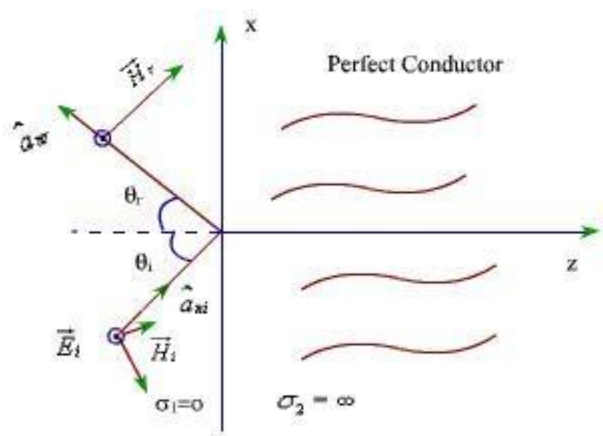


Figure 6.10: Perpendicular Polarization

As the EM field inside the perfect conductor is zero, the interface reflects the incident plane wave. $\hat{a}_{ni}$ and $\hat{a}_{nr}$ respectively represent the unit vector in the direction of propagation of the incident and reflected waves, θ_i is the angle of incidence and θ_r is the angle of reflection.

We find that

$$\hat{a}_{ni} = \hat{a}_z \cos\theta_i + \hat{a}_x \sin\theta_i$$

$$\hat{a}_{nr} = -\hat{a}_z \cos\theta_r + \hat{a}_x \sin\theta_r \quad \text{............................(6.69)}$$

Since the incident wave is considered to be perpendicular to the plane of incidence, which for the present case happens to be xz plane, the electric field has only y-component.

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1 \bar{a}_{ni}\cdot\bar{r}}$$

$$= \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

The corresponding magnetic field is given by

$$\vec{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{a}_n \times \vec{E}_i(x,z)\right]$$

$$= \frac{1}{\eta_1}\left[-\cos\theta_i \hat{a}_x + \sin\theta_i \hat{a}_z\right] E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

...........................(6.70)

Similarly, we can write the reflected waves as

$$\vec{E}_r(x,z) = \hat{a}_y E_{ro} e^{-j\beta_1 \bar{a}_{nr}\cdot\bar{r}}$$
$$= \hat{a}_y E_{ro} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad ....(6.71)$$

Since at the interface z=o, the tangential electric field is zero.

$$E_{io} e^{-j\beta_1 x\sin\theta_i} + E_{ro} e^{-j\beta_1 x\sin\theta_r} = 0 \quad ....(6.72)$$

Consider in equation (6.72) is satisfied if we have

$$E_{ro} = -E_{io}$$
$$\text{and } \theta_i = \theta_r \quad ....(6.73)$$

The condition $\theta_i = \theta_r$ is Snell's law of reflection.

$$\therefore \vec{E}_r(x,z) = -\hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)} \quad ....(6.74)$$

$$\text{and } \vec{H}_r(x,z) = \frac{1}{\eta_1}\left[\hat{a}_{nr} \times \vec{E}_r(x,z)\right]$$
$$= \frac{E_{io}}{\eta_1}\left[-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right] e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)} \quad ....(6.75)$$

The total electric field is given by

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$
$$= -\hat{a}_y 2jE_{io} \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} \quad ....(6.76)$$

Similarly, total magnetic field is given by

$$\vec{H}_1(x,z) = -2\frac{E_{io}}{\eta_1}\left[\hat{a}_x \cos\theta_i \cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} + \hat{a}_z j\sin\theta_i \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}\right] \quad ....(6.77)$$

From eqns (6.76) and (6.77) we observe that

1. Along z direction i.e. normal to the boundary
   y component of $\vec{E}$ and x component of $\vec{H}$ maintain standing wave patterns according to $\sin\beta_{1z} z$ and z where $\beta_{1z} = \beta_1 \cos\theta_i$. No average power propagates along z $\vec{E}$ $\vec{H}$

as y component of and x component of are out of phase.

2. Along x i.e. parallel to the interface

y component of $\vec{E}$ and z component of $\vec{H}$ are in phase (both time and space) and propagate with phase velocity

$$v_{p1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin\theta_i}$$

$$\text{and } \lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i} \qquad \text{.............................(6.78)}$$

The wave propagating along the x direction has its amplitude varying with z and hence constitutes a **non uniform** plane wave. Further, only electric field $\hat{E}_1$ is perpendicular to the direction of propagation (i.e. x), the magnetic field has component along the direction of propagation. Such waves are called transverse electric or TE waves.

ii. **Parallel Polarization**:

In this case also $\hat{a}_{ni}$ and $\hat{a}_{nr}$ are given by equations (6.69). Here $\vec{H}_1$ and $\vec{H}_r$ have only y component.

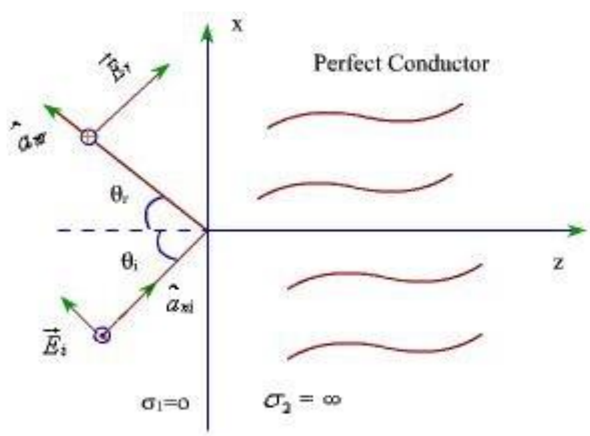


Figure 6.11: Parallel *Polarization*

With reference to fig (6.11), the field components can be written as: Incident field components:

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{n_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

............................(6.79)

Reflected field components:

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{n_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

............................(6.80)

Since the total tangential electric field component at the interface is zero.

$$E_i(x,0) + E(x,0) = 0$$

Which leads to $E_{io} = -E_{ro}$ and $\theta_i = \theta_r$ as before.

Substituting these quantities in (6.79) and adding the incident and reflected electric and magnetic field components the total electric and magnetic fields can be written as

$$\vec{E}_i(x,z) = -2E_{io}\left[\hat{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \hat{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)\right] e^{-j\beta_1 x\sin\theta_i}$$

and $$\vec{H}_i(x,z) = \hat{a}_y \frac{2E_{io}}{n_1} \cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}$$

...........................(6.81)

Once again, we find a standing wave pattern along z for the x and y components of $\vec{E}$ and $\vec{H}$, while a non uniform plane wave propagates along x with a phase velocity given by $v_{p1x} = \frac{vp_1}{\sin\theta_i}$ where $v_{p1} = \frac{\omega}{\beta_1}$. Since, for this propagating wave, magnetic field is in transverse direction, such waves are called transverse magnetic or TM waves.

Electromagnetic spectrum:

If the angle of incidence is larger than θ_c total internal reflection occurs. For such case an evanescent wave exists along the interface in the x direction (w.r.t. fig (6.12)) that attenuates exponentially in the normal i.e. z direction. Such waves are tightly bound to the interface and are called surface waves and waves spreading in the field of electric and magnetic together called electromagnetic spectrum.