Enable | Enlighten | Enrich (Deemed to be University) (Under Section 3 of UGC Act 1956)	**KARPAGAM ACADEMY OF HIGHER EDUCATION** **(*Established under section 3 of the UGC Act 1956*)** **COIMBATORE – 641021** **FACULTY OF ENGINEERING** **DEPARTMENT OF SCIENCE AND HUMANITIES** **SYLLABUS**

18BEEC301B, 18BEBME301B, 18BEEE301B

LINEAR ALGEBRA AND SPECIAL FUNCTIONS 3 2 0 4

OBJECTIVES:

1. To develop analytical skills for solving engineering problems
2. To make the students to study about linear algebra and some useful special functions.

INTENDED OUTCOMES:

1. Be able to acquire basic knowledge on vector spaces and linear transformations.
2. Be able to build and solve the special functions.

UNIT I VECTOR SPACES (12)

General vector spaces, real vector spaces, Euclidean n-space, subspaces, linear independence, basis and dimension, row space, column space and null space.

UNIT II LINEAR TRANSFORMATIONS (12)

Linear Transformations - The Null Space and Range - Isomorphisms - Matrix Representation of Linear Transformations –Eigen values and Eigen vectors - Similarity, Diagonalization.

UNIT III INNER PRODUCT SPACES (12)

The Dot Product on R^n and Inner Product Spaces - Orthonormal Bases - Orthogonal Complements - Application : Least Squares Approximation - Diagonalization of Symmetric M - Application: Quadratic Forms

UNIT IV HYPERBOLIC FUNCTIONS, BETA AND GAMMA FUNCTIONS (12)

Hyperbolic Functions: Hyperbolic functions and Inverse Hyperbolic functions – Identities – Real and imaginary parts – solving problems using hyperbolic functions.

Beta and Gamma Functions: Definitions – Properties – Relation between beta and gamma integrals – Evaluation of definite integrals in terms of beta and gamma functions.

UNIT V BESSEL FUNCTIONS (12)

Bessel Functions – Preliminaries – Definitions – Bessel Differential Equation – Differential recurrence relations – the pure recurrence relation – A generating function – Bessel's integral – Index half and odd integer.

Total : 60

TEXT BOOKS:

S. NO.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAR OF PUBLICATION
1	Kreyszig,E	Advanced Engineering Mathematics	John Wiley & Sons, New Delhi.	2014
2	ShahnazBathul	Text book of Engineering Mathematics(Special Functions and Complex Variables)	PHI Publications, New Delhi.	2009

REFERENCES:

S. NO.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAR OF PUBLICATION
1	Dr. Grewal B.S.	Higher Engineering Mathematics	Khanna Publishers, New Delhi.	2013
2	Anton and Rorres	Elementary Linear Algebra, Applications version	Wiley India Edition, New Delhi.	2012
3	Jim Defranza, Daniel Gagliardi	Introduction to Linear Algebra with Application	Tata McGraw-Hill, New Delhi.	2008

WEBSITES:

1. www.sosmath.com
2. www.nptel.ac.in
3. www.mathworld.wolfram.com

	KARPAGAM ACADEMY OF HIGHER EDUCATION **(*Established under section 3 of the UGC Act 1956*)** **COIMBATORE – 641021** **FACULTY OF ENGINEERING** **DEPARTMENT OF SCIENCE AND HUMANITIES**

LECTURE PLAN

Subject : LINEAR ALGEBRA AND SPECIAL FUNCTIONS

Code : **18BEEC301B/18BEBME301B**

Unit No.	List of Topics	No. of Hours
UNIT I	**VECTOR SPACES**	
	Introduction- Group, Field	**1**
	General vector spaces	**1**
	Real Vector Space - Problems	**1**
	Euclidean n-space, , and row space, column space and null space.	**1**
	Linear Algebra- Subspace of Linear Algebra	**1**
	Linear independence - Problems	**1**
	Basis- Problems	**1**
	Problems in Linear Independence and Basis	
	Dimension - Problems	**1**
	Row space, and.	**1**
	Column space	**1**
	Null space	**1**
	TOTAL	**12**
UNIT – II	**LINEAR TRANSFORMATIONS**	
	Introduction - Linear Transformations	1
	Linear Transformations	1
	The Null Space and Range	1
	Isomorphism	1
	Matrix Representation of Linear Transformations	1
	Problems in Matrix Representation of Linear Transformations	1
	Eigen values and Eigen vectors	1
	Eigen values and Eigen vectors - Problems	1
	Similarity of matrices - Problems	1
	Similarity of matrices - Problems	1
	Diagonalization of matrices - Problems	1
	Diagonalization of matrices - Problems	1
	TOTAL	**12**
UNIT – III	**INNER PRODUCT SPACES**	
	Introduction:	1
	The Dot Product on R^n	1
	Inner Product Spaces	1
	Orthonormal Bases	1
	Orthogonal Complements	1
	Least Squares Approximation- Problems	1
	Least Squares Approximation- Problems	1
	Least Squares Approximation- Problems	1
	Diagonalization of Symmetric M- Problems	1
	Diagonalization of Symmetric M- Problems	1
	Quadratic Forms- Problems	
	Quadratic Forms- Problems	1
	TOTAL	**12**

UNIT – IV	**HYPERBOLIC FUNCTIONS, BETA AND GAMMA FUNCTIONS**	
	Hyperbolic functions	1
	Inverse Hyperbolic functions	1
	Identities	1
	Real and imaginary parts	1
	Problems using hyperbolic functions.	1
	Problems using hyperbolic functions.	1
	Beta and Gamma Functions: Definitions	1
	Beta Functions: Properties	1
	Gamma Functions: Properties	1
	Relation between beta and gamma integrals	1
	Evaluation of definite integrals in terms of beta and gamma functions.	1
	Evaluation of definite integrals in terms of beta and gamma functions.	1
	TOTAL	**12**
UNIT – V	**BESSEL FUNCTIONS**	
	Bessel Functions – Preliminaries	1
	Bessel Functions – Preliminaries – Definitions	1
	Bessel Differential Equation	1
	Differential recurrence relations	1
	The pure recurrence relation	1
	The pure recurrence relation	1
	A generating function	1
	Bessel's integral	1
	Bessel's integral - Problems	1
	Bessel's integral - Problems	1
	Bessel's integral - Problems	1
	Index half and odd integer	1
	TOTAL	**12**
	TOTAL NO OF HOURS	**60**

STAFF IN-CHARGE **HOD**

Group:-

A nonempty set of elements G is said to form a group. If in G there is defined a binary operation, called the product and denoted by '.', such that

1. $a, b \in G \Rightarrow a.b \in G$
2. $a, b, c \in G \Rightarrow a.(b.c) = (a.b).c$
3. $\exists$ an element $e \in G \ni a.e = e.a = a \ \forall a \in G$
4. $\forall a \in G \ \exists a^{-1} \in G \ni a.a^{-1} = a^{-1}.a = e$.

Abelian Group:-

A Group G is said to be abelian (or Commutative) if for every $a, b \in G$.
$a.b = b.a$.

~~Field~~:-

Ring:-

A nonempty set R is said to b ring. if in R, there are defined two operations denoted by + and · respectively, such that for all a, b, c in R:

1. $a+b \in R$
2. $a+b = b+a$
3. $(a+b)+c = a+(b+c)$
4. $\exists\ 0 \in R \ni a+0 = a$
5. $\exists -a \in R \ni a+(-a) = 0$
6. $a.b \in R$
7. $a.(b.c) = (a.b).c$
8. $a.(b+c) = a.b + a.c$

Commutative ring:-

If the multiplication of R is such that $a \cdot b = b \cdot a \ \forall a, b \in R$ then we call R as a Commutative ring.

Division ring:-

A ring is said to be division ring if its nonzero elements form a group under multiplication.

field:-

A field is a Commutative division ring.

Vector space:-

A nonempty set V is said to be a vector space over a field F. If V is an abelian group under an operation which we denote by $+$ and if for every $\alpha \in F, v \in V$ there is defined an element, written αv in V subject to

1. $\alpha(v+w) = \alpha v + \alpha w$
2. $(\alpha+\beta)v = \alpha v + \beta v$
3. $\alpha(\beta v) = (\alpha\beta)v$ $\quad \forall \alpha, \beta \in F, v, w \in V$
4. $1 \cdot v = v$

Subspace:-

If V is a vector space over F and if $W \subset V$, then W is a subspace of V if under the operations of V, W itself forms a vector space over F.

or.

W is a subspace of V whenever $w_1, w_2 \in W$ $\alpha, \beta \in F$ implies that $\alpha w_1 + \beta w_2 \in W$.

Homomorphism:-

If U and V are vector spaces over F then the mapping T of U into V is said to be a homomorphism if

1. $(u_1+u_2)T = u_1T + u_2T$
2. $(\alpha u_1)T = \alpha(u_1T)$ $\forall u_1, u_2 \in U$ $\forall \alpha \in F$.

Lemma1:-

If V is a vetor space over F, then

1. $\alpha 0 = 0$ $\forall \alpha \in F$
2. $0 \cdot v = 0$ $\forall v \in V$
3. $(-\alpha)v = -(\alpha \cdot v)$ $\forall \alpha \in F, v \in V$
4. If $v \neq 0$, then $\alpha v = 0 \Rightarrow \alpha = 0$

Proof:-

(i) $\alpha \cdot 0 = 0$

$\alpha \cdot 0 = \alpha(0+0)$

$= \alpha 0 + \alpha \cdot 0$

$= 0 + 0$

$\alpha \cdot 0 = 0$

(ii) $0v = (0+0)v$

$= 0 \cdot v + 0 \cdot v$

$= 0 + 0$

$0 \cdot v = 0$

(iii) $0 = 0 \cdot v = (\alpha + (-\alpha)) \cdot v$

$= \alpha v + (-\alpha)v$

$-(\alpha \cdot v) = (-\alpha)v$.

Scanned by CamScanner

(iv) if $\alpha . v = 0$ and $\alpha \neq 0$

then $0 = \alpha^{-1} . 0$

$= \alpha^{-1}(\alpha . v)$

$= (\alpha^{-1}\alpha)\, v$

$= 1 . v$

$0 = v$

Prove that the set $V = \{(a,b) \,;\, a,b \in R\}$ is a vector space w.r.to composition of addition and scalar multiplication defined as

$$(a,b) + (c,d) = (a+c, b+d)$$
$$k(a,b) = (ka, kb)$$

Proof :-

let $u \in V \Leftrightarrow (a_1, b_1) = u \in V$ $a_1, a_2, a_3 \in R$

$v \in V \Rightarrow (a_2, b_2) = v \in V$ $b_1, b_2, b_3 \in R$.

$w \in V \Rightarrow (a_3, b_3) = w \in V$.

Closure.

(i) $a, b \in V \Rightarrow a + b \in V$.

$u, v \in V \Rightarrow u + v \in V$.

$(a_1, b_1) + (a_2 + b_2) = (a_1 + a_2, b_1 + b_2)$

$\in V$ $(\because a_1 + a_2 \in R$, $b_1 + b_2 \in R)$.

Associative

$u + (v + w) = (u + v) + w$

$u + (v + w) = (a_1, b_1) + [(a_2, b_2) + (a_3, b_3)]$

$= (a_1, b_1) + [a_2 + a_3, b_2 + b_3]$

$= (a_1, b_1) + (b_2, b_3)]$

$= (a_1 + (a_2 + a_3), b_1 + (b_2, b_3)]$

$= [(a_1 + a_2) + a_3, (b_1 + b_2) + b_3]$

$= (a_1 + a_2, b_1 + b_2) + w$

Scanned by CamScanner

Identity:-

$$a+e = e+a = a$$

$$u+e = e+u = u$$

$$(a,b) \cdot e = e(a,b)$$

$$e = (a,b) + (-(a,b))$$

$$= (a-a,\ b-b)$$

$$\boxed{e = (0,0)}$$

Inverse :-

$$aa^{-1} = a^{-1}a = e \Rightarrow uu^{-1} = u^{-1}u = e$$

$$(a,b) + u^{-1} = (0,0)$$

$$u^{-1} = (0,0) - (a,b)$$

$$= (0-a,\ 0-b)$$

$$u^{-1} = (-a,-b) \qquad -a \in R,\ -b \in R.$$

Commutative :-

$$u+v = v+u$$

$$(a_1,b_1) + (a_2,b_2) = (a_1+a_2,\ b_1+b_2)$$

$$= (a_2+a_1,\ b_2+b_1)$$

$$= (a_2,b_2) + (a_1,b_1)$$

$\Rightarrow$ It is an Abelian Group

To Prove vector space.

(i) $\alpha(u+v) = \alpha u + \alpha v$

$$\alpha((a_1,b_1) + (a_2,b_2)) = \alpha(a_1+a_2,\ b_1+b_2)$$

$$= (\alpha a_1 + \alpha a_2,\ \alpha b_1 + \alpha b_2)$$

Scanned by CamScanner

$= \alpha(a_1, b_1) + \alpha(a_2, b_2)$

$= \alpha u + \alpha v$

(ii) $(\alpha+\beta) u = \alpha u + \beta u$

$(\alpha+\beta)(a_1, b_1) = (\alpha a_1, \alpha b_1) + \cancel{\beta(a_1,}$ $(\beta a_1, \beta b_1)$

$= (\alpha(a_1, b_1) + \beta(a_1, b_1)$

$= \alpha u + \beta u$

(iii) $\alpha(\beta v) = (\alpha\beta)v$

$\alpha(\beta(a_1, b_1)) = \alpha(\beta a_1, \beta b_1)$

$= (\alpha\beta a_1, \alpha\beta b_1)$

$= \alpha\beta(a_1, b_1)$

$= \alpha\beta(u)$

(iv) $1 \cdot v = v$

$1 \cdot (a_1, b_1) = (a_1, b_1)$

$= v$

$\Rightarrow$ It is a vector space over $\mathbb{R}$.

Prove that $V = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ under multiplication.

Proof :-

let $u = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \in V$ $v = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} \in V$ $w = \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix} \in V$

$a_1, a_2, a_3 \in \mathbb{R}$, $b_1, b_2, b_3 \in \mathbb{R}$.

Closure

(i) if $u, v \in V$

$$u+v = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+a_2 & 0 \\ 0 & b_1+b_2 \end{pmatrix}$$

$u+v \in V$

(ii) Associative

$$(u+v)+w = u+(v+w)$$

$$\left(\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}\right) + \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+a_2 & 0 \\ 0 & b_1+b_2 \end{pmatrix} + \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+a_2+a_3 & 0 \\ 0 & b_1+b_2+b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+(a_2+a_3) & 0 \\ 0 & b_1+(b_2+b_3) \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2+a_3 & 0 \\ 0 & b_2+b_3 \end{pmatrix}$$

$$= u+(v+w).$$

(iii) Identity

$$u+e = e+u = u$$

$$\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + e = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$e = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} - \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1-a_1 & 0 \\ 0-0 & b_1-b_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & 0 \end{pmatrix}$

Scanned by CamScanner

(iv) Inverse :

$$u + u^{-1} = u^{-1} + u = e$$

$$\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + u^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$u^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$u^{-1} = \begin{pmatrix} -a_1 & 0 \\ 0 & -b_1 \end{pmatrix}$$

(v) Commutative :-

$$u + v = v + u$$

$$u + v = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 + a_1 & 0 \\ 0 & b_2 + b_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$u + v = v + u$$

$\Rightarrow$ V is an abelian group

~~(vi) $(\alpha+\beta) v$~~ =

$$\alpha(v + w) = \alpha v + \alpha w$$

(vi) $\alpha(v+w) = \alpha v + \alpha w$

$$\alpha(v+w) = \alpha\left(\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}\right)$$

$$= \alpha \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 + \alpha a_2 & 0 \\ 0 & \alpha b_1 + \alpha b_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 & 0 \\ 0 & \alpha b_1 \end{pmatrix} + \begin{pmatrix} \alpha a_2 & 0 \\ 0 & \alpha b_2 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \alpha \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

$$\alpha(v+w) = \alpha v + \alpha w$$

(vii) $(\alpha+\beta)u = \alpha u + \beta u$

$$(\alpha+\beta)u = \alpha+\beta \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}.$$

$$= \begin{pmatrix} (\alpha+\beta)a_1 & 0 \\ 0 & (\alpha+\beta)b_1 \end{pmatrix}.$$

$$= \begin{pmatrix} \alpha a_1 + \beta a_1 & 0 \\ 0 & \alpha b_1 + \beta b_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 & 0 \\ 0 & \alpha b_1 \end{pmatrix} + \begin{pmatrix} \beta a_1 & 0 \\ 0 & \beta b_1 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \beta \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$= \alpha u + \beta u$$

(viii) $\alpha(\beta v) = (\alpha\beta)u$

$$\alpha(\beta u) = \alpha\left(\beta\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}\right)$$

$$= \alpha \begin{pmatrix} \beta a_1 & 0 \\ 0 & \beta b_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\beta a_1 & 0 \\ 0 & \alpha\beta b_1 \end{pmatrix}$$

$$= \alpha\beta \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$\alpha(\beta u) = (\alpha\beta)u.$$

(ix) $1.u = u$

$$1.\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}.$$

$\Rightarrow$ V is a vector space over the field $\mathbb{R}$

Scanned by CamScanner

Linear Combination:-

Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n \in V$. Then an element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $\alpha_i \in F$ is called a Linear Combination. of the vectors $v_1, v_2, \ldots v_n$.

Linear Span:-

Let S be a non-empty subset of a vector space V. Then the set of all linear combinations of finite sets of elements of S is called the Linear Span of S and is denoted by $L(S)$.

Any ~~element~~ element of $L(S)$ is of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $\alpha_1, \alpha_2, \ldots \alpha_n \in F$.

Finite Dimensional:-

Let V be a vector space over a field F. V is said to be finite dimensional if there exists a finite subset S of V such that $L(S) = V$.

Linearly Independent:-

Let V be a vector space over a field F. A finite set of vectors $v_1, v_2, \ldots v_n$ in V is said to be linearly independent. if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Linearly Dependent:-

If $v_1, v_2, \ldots v_n$ are not linearly independent, then they are said to be linearly dependent.

If the vectors $(0,1,x)$ $(x,1,0)$, $(1,x,1)$ of the vector space $R^3(R)$ are linearly dependent then find the value of x.

Solution

Let $v_1 = (0,1,x)$

$v_2 = (x,1,0)$

$v_3 = (1,x,1)$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1 (0,1,x) + \alpha_2 (x,1,0) + \alpha_3 (1,x,1) = 0$

$0\alpha_1 + x\alpha_2 + 1\alpha_3 = 0$

$1\alpha_1 + 1\alpha_2 + x\alpha_3 = 0$

$x\alpha_1 + 0\alpha_2 + 1\alpha_3 = 0$

Given that the vector space is linearly dependent

$\Rightarrow \alpha_1, \alpha_2, \alpha_3 \neq 0$.

$\Rightarrow$ |coefficient matrix| $\neq 0$.

$$\Rightarrow \begin{vmatrix} 0 & x & 1 \\ 1 & 1 & x \\ x & 0 & 1 \end{vmatrix} = 0$$

$= -x(1-x^2) + 1(0-x) = 0$

$-\cancel{x} + x^3 - \cancel{x} = 0$

$x(-2+x^2) = 0 \Rightarrow x = 0,\ x^2 = 2$

$x = \pm\sqrt{2}$

$x = 0, +\sqrt{2}, -\sqrt{2}$.

Basis:-

A linearly independent subset S of a vector space V which spans the whole space V is called a basis of the vector space.

Dimension:-

Let V be a finite dimensional vector space over a field F. The number of elements in any basis of V is called the dimension of V and is denoted by $\dim V$.

Show that the set $\{(1,2,1), (2,1,0), (1,-1,2)\}$ forms a basis for $V_3(F)$.

Solution:-

$v_1 = (1,2,1)$ $v_2 = (2,1,0)$ $v_3 = (1,-1,2)$

let $\alpha_1, \alpha_2, \alpha_3 \in F$.

let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1(1,2,1) + \alpha_2(2,1,0) + \alpha_3(1,-1,2) = 0$

$1\alpha_1 + 2\alpha_2 + 1\alpha_3 = 0$ —(1)

$2\alpha_1 + 1\alpha_2 - 1\alpha_3 = 0$ —(2)

$1\alpha_1 + 0\alpha_2 + 2\alpha_3 = 0$ —(3)

$\alpha_1 \quad \alpha_2 \quad \alpha_3$

$2 \quad 1 \quad 1 \quad 2$

$1 \quad -1 \quad 2 \quad 1$

$2-1 \quad 2+1 \quad 1-4$

$-3 \quad 3 \quad -3 \quad \div 3$

$\frac{\alpha_1}{-1} = \frac{\alpha_2}{1} = \frac{\alpha_3}{-1} = k$

$\alpha_1 = -k, \ \alpha_2 = k \ \alpha_3 = -k$

Sub in (3)

$-k + 0 - 2k = 0$

$-3k = 0 \Rightarrow k = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow$ It is linearly independent.

$\dim V_3(F) = 3$

$\Rightarrow$ B is a basis for $V_3(F)$

Scanned by CamScanner

Determine whether or not the following vectors form a basis of R^3. $(1,1,2)$ $(1,2,5)$ $(5,3,4)$

Solution :-

let $v_1 = (1,1,2)$ $v_2 = (1,2,5)$ $v_3 = (5,3,4)$

let $\alpha_1, \alpha_2, \alpha_3 \in R$.

& $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1 (1,1,2) + \alpha_2 (1,2,5) + \alpha_3 (5,3,4) = 0$

$1\alpha_1 + 1\alpha_2 + 5\alpha_3 = 0$ — ①

$1\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$ — ②

$2\alpha_1 + 5\alpha_2 + 4\alpha_3 = 0$ — ③

α_1		α_2		α_3
1	5	1	1	
2	3	2	2	
$3-10$		$5-3$		$2-1$
-7		2		1

$$\frac{\alpha_1}{-7} = \frac{\alpha_2}{2} = \frac{\alpha_3}{1} = k.$$

$\alpha_1 = -7k$, $\alpha_2 = 2k$ $\alpha_3 = k$.

Sub in ③.

$2(-7k) + 5(2k) + 4(k) = 0$

$(-14 + 10 + 4)k = 0$

$0 . k = 0 \Rightarrow k$ takes any value in R.

let $k = 1$

$\Rightarrow \alpha_1 = -7, \alpha_2 = 2, \alpha_3 = 1$

$\Rightarrow \{v_1, v_2, v_3\}$ is linearly dependent

$\Rightarrow$ It is not a base.

Theorem 2:-

Let V be a finite-dimensional vector space over a field F. Let A and B be supspaces of V.

Then $\dim(A+B) = \dim A + \dim B - \dim(A\cap B)$.

Proof:-

Given A and B are supspaces of V.

Hence $A\cap B$ is supspace of V.

Let $\dim(A\cap B) = r$.

Let $S = \{v_1, v_2, \ldots, v_r\}$ be a basis for $A\cap B$.

Since $A\cap B$ is a subspace of A and B, S is a part of a basis for A and B.

Let $\{v_1, v_2, \ldots v_r, u_1, u_2, \ldots u_s\}$ be a basis for A.

$\Rightarrow \dim A = r+s$.

Let $\{v_1, v_2, \ldots v_r, w_1, w_2, \ldots w_t\}$ be a basis for B

$\Rightarrow \dim B = r+t$.

Claim:-

$S' = \{v_1, v_2, \ldots v_r, u_1, u_2, \ldots, u_s, w_1, w_2, \ldots w_t\}$ is a basis for $A+B$

Proof of claim:-

(i) S' is Linearly independent.

Let $\alpha_1 v_1 + \ldots + \alpha_r v_r + \beta_1 u_1 + \ldots + \beta_s u_s + \gamma_1 w_1 + \gamma_2 \ldots + \gamma_t w_t = 0$

$\Rightarrow \beta_1 u_1 + \ldots + \beta_s u_s = -\underbrace{(\gamma_1 w_1 + \ldots + \gamma_t w_t)}_{B} - \underbrace{(\alpha_1 v_1 + \ldots + \alpha_r v_r)}_{B} \in B$.

$\Rightarrow \beta_1 u_1 + \ldots + \beta_s u_s \in B$

Also $\beta_1 u_1 + \ldots + \beta_s u_s \in A$.

$\Rightarrow \beta_1 u_1 + \cdots + \beta_s u_s \in A \cap B.$

$\therefore \beta_1 u_1 + \cdots + \beta_s u_s = \delta_1 v_1 + \cdots + \delta_r v_r$

$\therefore \beta_1 u_1 + \cdots + \beta_s u_s - \delta_1 v_1 \cdots - \delta_r v_r = 0$

$\therefore \beta_1 = \cdots = \beta_s = \delta_1 = \cdots = \delta_r = 0$

Since $\{u_1, u_2, \ldots u_s, v_1, v_2, \ldots, v_r\}$ is L.I.

Similarly $\gamma_1 = \gamma_2 = \cdots = \gamma_t = 0$

$\therefore \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$

$\beta_1 = \beta_2 = \cdots = \beta_s = 0$

$\gamma_1 = \gamma_2 = \cdots = \gamma_t = 0$

$\therefore S'$ is a Linearly Independent set.

Clearly, ~~S~~ S' spans $A+B$

~~Since~~ $\therefore S'$ is a basis for $A+B$, ~~S' spans $A+B$~~.

$\therefore$ ~~S' is~~

$\dim(A+B) = r+s+t$

$\dim(A \cap B) = r.$

$\dim A + \dim B - \dim(A \cap B) = (r+s) + (r+t) - r$

$= r+s+t$

$= \dim(A+B)$

H.T.P.

Lemma 2:-

Let V be a vector space over F. A non-empty subset W of V is a subspace of V iff W is closed w.r.to vector addition and scalar multiplication in V.

Scanned by CamScanner

Proof:-

Let W be a subspace of V.

$\Rightarrow$ W is itself is a vector space.

$\Rightarrow$ W is closed w.r.to vector addition and scalar multiplication.

Conversely,

Let W be a non-empty subset of V $\ni$

$v, u \in W \Rightarrow u+v \in W$

$u \in W, \alpha \in F \Rightarrow \alpha u \in W$.

To Prove : W is a subspace of V.

Closure:-

Since W is non empty, $\exists$ an element $u \in W$.

identity.. $0 \cdot u = 0 \in W$

Inverse :- $u \in W \Rightarrow (-1) u \in W \Rightarrow -u \in W$

$\therefore$ W contains 0 (identity) and the additive inverse of each of its element.

$\therefore$ W is an additive subgroup of V.

Also, $u \in W$ & $\alpha \in F \Rightarrow \alpha u \in W$.

The elements of W are the elements of V, + the elements of W are the elements of vector space V.

$\Rightarrow$ It satisfies all the properties of vector space V.

Hence W is a subspace of V.

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
A square matrix A is said to be ----if the determinant value of A is zero.	singular	non singular	symmetric	non symmetric			singular
A square matrix A is said to be ----if the determinant value of A is not equal to zero.	singular	non singular	symmetric	non symmetric			non singular
A square matrix A is said to be singular if the determinant value of A is ----.	1	2	non zero	zero			zero
A square matrix A is said to be non singular if the determinant value of A is ----.	1	2	non zero	zero			non zero
A square matrix in which all the elements below the leading diagonal are zeros,it is called an -----matrix.	upper triangular	lower triangular	symmetric	non symmetric			upper triangular
A square matrix in which all the elements above the leading diagonal are zeros,it is called an -----matrix.	upper triangular	lower triangular	symmetric	non symmetric			lower triangular
A unit matrix is a ----matrix.	scalar	lower triangular	symmetric	non symmetric			scalar
A system of equation is said to be consistent if they have	one solution	one or more solution	no solution	infinite solution			one or more solution
If rank of A is equal to the rank of [AB] then the system of equations is ------	Consistent	inconsistent	symmetric	non symmetric			Consistent
If rank of A is not equal to the rank of [AB] then the system of equations is ------	Consistent	inconsistent	symmetric	non symmetric			inconsistent
The maximum value of the rank of a 4x5 matrix is	1	5	4	3			4
A square matrix A which satisfies the relation A^2 = A is called	nilpotient	idempotent	Hermitian	Skew - Hermitian			idempotent
A matrix is idempotent if	A^3 = A	A^2 = 0	A^1 =A	A^2 = A			A^2 = A
If the rank of A is 2, then the rank of A^ -1 is	3	2	4	1			2
The sum of the main diagonal elements of a matrix is called------	trace of a matrix	quadratic form	eigen value	canonical form			trace of a matrix
Every square matrix satisfies its own ----------	characteristic polynomial	characteristic equation	orthogonal transformation	canonical form			characteristic equation
The orthogonal transformation used to diagonalise the symmetric matrix A is----	$N^T AN$	$N^T A$	NAN^{-1}	NA			$N^T AN$
If $\lambda_1, \lambda_2, \lambda_3,\ldots\ldots\ \lambda_n$ are the eigen values of A ,then $k\lambda_1, k\lambda_2, k\lambda_3,\ldots\ldots, k\lambda_n$ are the eigen values of --------------	kA	kA^2	kA^{-1}	A^{-1}			kA
Diagonalisation of a matrix by orthogonal reduction is true only for a ----- matrix.	diagonal	triangular	real symmetric	scalar			real symmetric

In a modal matrix, the columns are the -----------	eigen vectors of A	eigen vectors of adj A	eigen vectors of inverse ofA	eigen values of A	eigen vectors of A
If atleast one of the eigen values of A is zero, then det A = -----	0	1	10	5	0
det (A- λI) represents------	characteristic polynomial	characteristic equation	quadratic form	canonical form	characteristic polynomial
If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots \lambda_n$ are the eigen values of A ,then $1/\lambda_1$,$1/\lambda_2$, $1/\lambda_3, \ldots\ldots\ldots$,$1/\lambda_n$ are the eigen values of --------------	A^(-1)	A	A^n	A^p	A^(-1)
If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots \lambda_n$ are the eigen values of A ,then λ_1^p , λ_2^p , ……… λ_n^p are the eigen values of	A^(-1)	A^2	A^(-p)	A^p	A^p
Cayley -Hamilton theorem is used to find ------------	inverse and higher powers of A	eigen values	eigen vectors	quadratic form	inverse and higher powers of A
The eigen values of a --------------------- matrix are its diagonal elements	diagonal	symmetric	skew-matrix	triangular	triangular
In an orthogonal transformation $N^T AN = D$, D refers to a ---------- matrix.	diagonal	orthogonal	symmetric	skew-symmetric	diagonal
In a modal matrix, the columns are the eigen vectors of----------	A^{-1}	A^2	A	adj A	A
If the sum of two eigen values and trace of a 3x3 matrix A are equal, then det A = ---------	$\lambda_1 \lambda_2 \lambda_3$	0	1	2	0
If 1,5 are the eigen values of a matrix A, then det A = -------	5	0	25	6	5
The eigen vector is also known as-------	latent value	latent vector	column value	orthogonal value	latent vector
If 1,3,7 are the eigen values of A, then the eigen values of 2A are ------------	1,3,7	1,9,21	2,6,14	1,9,49	2,6,14
If the eigen values of 2A are 2, 6, 8 then eigen values of A are _________	1,3,4	2,6,8	1,9,16	12,4,3	1,3,4
If all the eigenvalues of A are positive then it is called as_______	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite	Positive definite
A Square matrix A and its transpose have _______ eigen values.	different	Same	Inverse	Transpose	Same
The sum of the __________ of a matrix A is equal to the sum of the principal diagonal elements of A.	characteristic polynomial	characteristic equation	eigen values	eigen vectors	eigen values
The product of the eigenvalues of a matrix A is equal to_________	Sum of main diagonal	Determinant of A	Sum of minors of Main diagonal	Sum of the cofactors of A	Determinant of A
The eigenvectors of a real symmetric are _______	equal	unequal	real	symmetric	real

If a real symmetric matrix of order 2 has ---------then the matrix is a scalar matrix.	equal eigen vectors	different eigen vectors	equal eigen values	different eigen values		equal eigen values
A matrix is called symmetric if and only if ----------	A=A^T	A=A^-1	A=-A^T	A=A		A=A^T
If a matrix A is equal to A^T then A is a -------- matrix.	symmetric	non symmetric	skew-symmetric	singular		symmetric
A matrix is called skew-symmetric if and only if ---------	A=A^T	A=A^-1	A=-A^T	A=A		A=-A^T
If a matrix A is equal to -A^T then A is a -------- matrix.	symmetric	non symmetric	skew-symmetric	singular		skew-symmetric
A matrix is called orthogonal if and only if -----	A^T=A^-1	A^T=-A^-1	A^T=A^-2	A^T=-A^-2		A^T=A^-1
A matrix is called ------------if and only if A^T=A^-1.	orthogonal	square	non symmetric	triangular		orthogonal

Linear Transformation.

Homomorphism:-

Let V and W be vector spaces over a field F. A mapping $T: V \to W$ is called a homomorphism.

If

(a) $T(u+v) = T(u) + T(v)$

(b) $T(\alpha u) = \alpha T(u)$ where $\alpha \in F$ and $u, v \in V$.

A homomorphism T of vectorspace is also called a Linear Transformation.

Problem:-

Prove that $T: R^2 \to R^2$ defined by $T(a,b) = (2a-3b, a+4b)$ is a Linear transformation.

Proof:-

Let $u = (a,b)$ & $v = (c,d)$ and $\alpha \in \mathbb{R}$

$$T(u+v) = T((a,b)+(c,d))$$
$$= T(a+c, b+d)$$
$$= (2(a+c)-3(b+d), (a+c)+4(b+d))$$
$$= (2a+2c-3b-3d, a+c+4b+4d)$$
$$= (2a-3b+2c-3d, a+4b+c+4d)$$
$$= (2a-3b, a+4b) + (2c-3d, c+4d)$$
$$= \cancel{2a} = T[(a,b)+(c,d)]$$
$$= T(u)+T(v).$$

$$T(\alpha u) = T(\alpha(a,b))$$
$$= T(\alpha a, \alpha b)$$
$$= (2\alpha a - 3\alpha b, \alpha a + 4\alpha b)$$
$$= \alpha(2a-3b, a+4b)$$
$$= \alpha T(a,b)$$
$$= \alpha T(u).$$

$\therefore$ T is a linear Transformation.

Problem 2:-

Find a linear map $T: R^2 \to R^2$. Show that

(i) $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$.

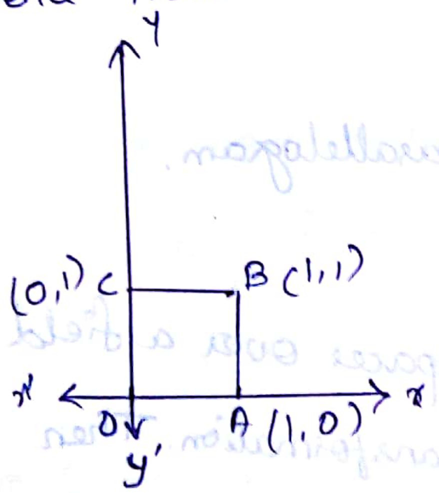
(ii) Prove that T maps square with vertices $(0,0)$ $(1,0)$ $(1,1)$ & $(0,1)$ into a parallelogram.

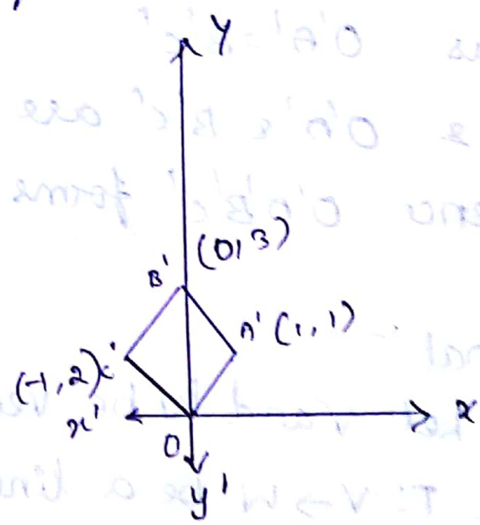
Solution:-

Let $e_1 = (1,0)$, $e_2 = (0,1)$, $u_1 = (1,1)$, $u_2 = (-1,2)$

Then $\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$.

Hence there exist a unique linear map.





Given $T: R^2 \to R^2$.

Prove that $T(e_1) = u_1$, $T(e_2) = u_2$

$$(x,y) = xe_1 + ye_2$$

$$T(x,y) = xTe_1 + yTe_2$$
$$= xu_1 + yu_2$$

$$T(x,y) = x(1,1) + y(-1,2)$$

This is a required map

$O(0,0)$, $A(1,0)$, $B(1,1)$, $C(0,1)$ respectively.

Let O', A', B', C' be their corresponding images

$O' = T(O) = T(0,0) = (0,0)$

$A' = T(A) = T(1,0) = (1,1)$

$B' = T(B) = T(1,1) = (0,3)$

$C' = T(C) = T(0,1) = (-1,2)$

$(O'A')^2 = [(1-0)^2 + (1-0)^2] = 2 \Rightarrow O'A' = \sqrt{2}$

$(B'C')^2 = (-1-0)^2 + (2-3)^2 = 2 \quad B'C' = \sqrt{2}$

~~B'C'~~ =

Slope of $O'A'$ is $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1-0}{1-0} = 1$

$(0,0)\ (1,1)$
$x_1\ y_1\ x_2\ y_2$

Slope of $B'C'$ is $m_2 = \dfrac{2-3}{-1-0} = \dfrac{-1}{-1} = 1$

Thus $O'A' = B'C'$

& $O'A'$ & $B'C'$ are $\parallel^{le}$.

Hence $O'A'B'C'$ forms a parallelogram.

Kernal :-

Let V and W be vector spaces over a field F and $T: V \to W$ be a linear transformation. Then the kernel of T is defined to be

$\{v \mid v \in V$ & $T(v) = 0\}$ and is denoted by $\ker T$

Isomorphism : T is 1-1 & onto

Monomorphism : T is 1-1

T is onto

Fundamental theorem of Homomorphism:-

Let V and W be vector spaces over a field F.

and $T: V \to W$ be an epimorphism

(i) $\ker T = V_1$ is a subspace of V

(ii) $\frac{V}{V_1} \cong W$

Proof:-

(i) Given $V_1 = \ker T$

$= \{ v / v \in V \text{ \& } T(v) = 0 \}$

Clearly $T(0) = 0$

Hence $0 \in \ker T = V_1$

$\therefore V_1$ is nonempty subset of V

Let $u, v \in \ker T$, and $\alpha, \beta \in F$

$\therefore T(u) = 0$ & $T(v) = 0$

$T(\alpha u + \beta v) = T(\alpha u) + T(\beta v)$

$= \alpha T(u) + \beta T(v)$

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0$

$\alpha u + \beta v \in \ker T$

$\therefore \ker T$ is a subspace of V.

(ii) We define a map $\varphi : \frac{V}{V_1} \to W$ by $\varphi(V_1 + v) = T(v)$

φ is well defined:-

Let $V_1 + v = V_1 + w$

$\therefore v \in V_1 + w$

$v = v_1 + w$ where $v_1 \in V$.

$T(v) = T(v_1 + w) = T(v_1) + T(w)$

$= 0 + T(w)$

$= T(w)$

$\varphi(V_1 + v) = \varphi(V_1 + w)$

φ is 1-1 :-

$\varphi(V_1+v) = \varphi(V_1+w)$

$T(v) = T(w)$

$T(v) - T(w) = 0$

$T(v) + T(-w) = 0$

$T(v-w) = 0$

$v-w \in \ker T = V_1$

$\Rightarrow v \in V_1 + w$

$\Rightarrow V_1 + v = V_1 + w.$

φ is onto :-

Let $w \in W$. Since T is onto $\exists\, v \in V \ni T(v) = w$

$\therefore \varphi(V_1+v) = w$

φ is homomorphism :-

$$\varphi[(V_1+v)+(V_1+w)] = \varphi[V_1+(v+w)]$$
$$= T(v+w)$$
$$= T(v) + T(w)$$
$$= \varphi(V_1+v) + \varphi(V_1+w)$$

$$\varphi[\alpha(V_1+v)] = \varphi(V_1+\alpha v)$$
$$= T(\alpha v)$$
$$= \alpha T(v)$$
$$= \alpha T(V_1+v)$$

$\therefore \varphi$ is an isomorphism from $\frac{V}{V_1}$ onto W.

$$\therefore \frac{V}{V_1} \cong W.$$

Scanned by CamScanner

Problem:-

1. Obtain the matrix representing the linear transformation $T: V_3(R) \rightarrow V_3(R)$ given by $T(a,b,c) = (3a, a-b, 2a+b+c)$ w.r.to the std. basis $\{e_1, e_2, e_3\}$.

Solution:-

$T(e_1) = T(1,0,0) = (3,1,2) = 3e_1 + e_2 + 2e_3$

$T(e_2) = T(0,1,0) = (0,-1,1) = -e_2 + e_3$

$T(e_3) = T(0,0,1) = (0,0,1) = e_3$

The matrix representing T is $\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

2. Find the linear transformation $T: V_3(R) \rightarrow V_3(R)$ determined by the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ w.r.to std. basis $\{e_1, e_2, e_3\}$.

Solution

$T(e_1) = e_1 + 2e_2 + e_3 = (1,2,1)$

$T(e_2) = 0 \cdot e_1 + e_2 + e_3 = (0,1,1)$

$T(e_3) = -e_1 + 3e_2 + 4e_3 = (-1,3,4)$

$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$(x,y,z) = xe_1 + ye_2 + ze_3$

$= ae_1 + be_2 + ce_3$

$T(a,b,c) = T(ae_1 + be_2 + ce_3)$

$= aT(e_1) + bT(e_2) + cT(e_3)$

$= a(1,2,1) + b(0,1,1) + c(-1,3,4)$

$= (a-c, 2a+b+3c, a+b+4c)$

This is an required Linear Transformation.

Scanned by CamScanner

Definition :-

Let $T: V \to W$ be a L.T

Range of T = dim Range $T = \rho(T)$

Nullity of T = dim ker $T = \nu(T)$

Sylvester's law :-

Let $T: V \to W$ be a L.T., then

Rank T + Nullity T = dim V

Proof :-

Let $\{x_1, x_2, \ldots, x_m\}$ be a basis of ker T

Then $\{x_1, x_2 \ldots x_m\}$ is L.I in ker T

$\Rightarrow \{x_1, x_2 \ldots x_m\}$ is L.I in V

It can be extended basis of V

dim ker T = nullity of $T = m$

dim $V = m+n$

To Prove: $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is a basis of Range T.

$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n) = 0$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = 0$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \in$ ker T

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_m x_m$

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n - \beta_1 x_1 - \beta_2 x_2 - \ldots - \beta_m x_m = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \beta_1 = \ldots = \beta_m = 0$

$\Rightarrow \alpha_i = 0 \ \forall i$

$\Rightarrow \{T(v_1), T(v_2) \ldots T(v_n)\}$ is L.I

Scanned by CamScanner

If $T(v) \in$ Range T be any element then as $v \in V$

$v = a_1 x_1 + \cdots + a_m x_n + b_1 v_1 + \cdots + b_n v_n$ $\quad a_i, b_j \in F$

$T(v) = a_1 T(x_1) + \cdots + a_m T(x_n) + b_1 T(v_1) + \cdots + b_n T(v_n)$

$= 0 + \cdots + 0 + b_1 T(v_1) + b_2 T(v_2) + \cdots + b_n T(v_n)$

(as $x_i \in \ker T$)

$= b_1 T(v_1) + b_2 T(v_2) + \cdots + b_n T(v_n)$

$\Rightarrow T(v)$ is a linear combination of $T(v_1), \ldots T(v_n)$

$\Rightarrow$ It forms a basis of Range T.

dim Range $T = n =$ rank T.

H.T.P

Find Range, rank, kernel, and nullity of the following Linear transformations

a) $T: R^2 \to R^3$ s.t $T(\underset{x,\,y}{a, b}) = (\underset{x,\; x+y,\; y}{a, a+b, b})$

b) $T: R^4 \to R^3$ s.t $T(u, v, w, z) = (u - x, v + w, w - x)$

Solution:- By the definition of T. elemenents of Type $(x, x+y, y)$ have Pre image in R^2 $\Rightarrow$ elemements of this type are in Range T.

$(x, x+y, y) = (x+0,\ x+y,\ 0+y)$

$= (x, x, 0) + (0, y, y)$

$= x(1,1,0) + y(0,1,1)$

$\Rightarrow$ Range T is spanned by $\{(1,1,0), (0,1,1)\}$

and since $\alpha_1(1,1,0) + \alpha_2(0,1,1) = (0,0,0)$

$\alpha_1 = 0 = \alpha_2$

$\Rightarrow$ It is Linearly independent

$\Rightarrow$ It forms a basis for Rang T

$\Rightarrow$ Rank T = dim Range $T = 2$.

$(x_1, y_2) \in \ker T$

$\Rightarrow T(x, y) = (0, 0, 0)$

$(x, x+y, y) = (0, 0, 0)$

$\Rightarrow x_1 = 0, y = 0$

$\Rightarrow \ker T = \{(0,0)\}$

$\Rightarrow$ Nullity $T = \dim \ker T = 0$

(ii) By the definition of T.

we find elements of the type $(u-x, v+w, w-x)$ have pre image in R^4

$(u-x, v+w, w-x) = (1u + 0v + 0w - 1x, \; 0u + 0v + 1w + 0x, \; 0u \cdot 0v + 1 \cdot w - 1x)$

$= u(1,0,0) + v(0,1,0) + w(0,1,1) + x(-1,0,-1)$

$\therefore$ Range of T spanned by

$\{(1,0,0), (0,1,0), (0,1,1) (-1,0,1)\}$.

Since Range T is a subspace of R^3 it is of dim 3

These 4 elements cannot form basis of Range T.

These are L.D elements as

$(-1,0,-1) + (1,0,0) + (0,1,0) + (0,1,1) = (0,0,0)$

Consider 3 members

$(1,0,0), (0,1,0), (0,1,1)$

$\alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,1,1) = (0,0,0)$

$\Rightarrow \alpha_i = 0 \; \forall i$

$\Rightarrow$ It is Linearly independent

$\Rightarrow$ It forms a basis of Range T.

Scanned by CamScanner

$\Rightarrow$ dim Range $T = 3 =$ rank of T.

$(-1,0,-1) = -1(1,0,0) + 1(0,1,0) - 1(0,1,1)$

The elements $(1,0,0), (0,1,0)\ (0,1,1)$ span Range T.

Range $T = \mathbb{R}^3$.

$(u, v, w, x) \in \ker T$ -

$\Rightarrow T(u,v,w,x) = (0,0,0)$

$\Rightarrow$ $u - x = 0$ — ① ① & ③ $\Rightarrow u = w = x$

$v + w = 0$ $v = -w$

$w - x = 0$ — ③

$\Rightarrow$ Elements of the type $(u, -u, u, u)$ are in $\ker T$.

$\Rightarrow \ker T$ is spanned by $(1,-1,1,1)$

$\Rightarrow$ $\ker T$. It is L.I forms a basis of $\ker T$.

$\Rightarrow$ dim $\ker T = 1$

nullity of $T = 1$

Problem:-

Let T be the linear Operator on $\mathbb{R}^3$, the matrix of which in the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Find a basis for the range of T and a basis for the null space of T.

Solution:-

Det $A = 1(4-3) - 2(1) + 1 = 0$

$\therefore A$ is not invertible and so T is not invertible.

Standard basis is $e_1 = (1,0,0)\ e_2 = (0,1,0)\ e_3 = (0,0,1)$

Scanned by CamScanner

$(x, y, z) \in \ker T$

$T(x, y, z) = 0$

$\Rightarrow T(x(1,0,0) + y(0,1,0) + z(0,0,1)) = 0$

$xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = 0$

$x(1,0,-1) + y(2,1,3) + z(1,1,4) = 0$

$\Rightarrow (x + 2y + z,\ y + z,\ -x + 3y + 4z) = 0$

$\Rightarrow \quad x + 2y + z = 0$

$0x + y + z = 0$

$-x + 3y + 4z = 0$

$$\begin{array}{cccc} & x & y & z \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ \hline 2-1 & 0-1 & 1-0 & \\ 1 & -1 & 1 & \\ \alpha & & & \end{array}$$

$\dfrac{\alpha_1}{1} = \dfrac{\alpha_2}{-1} = \dfrac{\alpha_3}{1} = k$

$\alpha_1 = k \quad \alpha_2 = -k \quad \alpha_3 = k$

$-k - 3k + 4k = 0$

$0 \cdot k = 0 \Rightarrow k \neq 0$

let $k = 1$, $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 1$

$\Rightarrow$ Every element in $\ker T$ is multiple of $(1,-1,1)$

$\Rightarrow$ $\ker T$ is spanned by $(-1,1,-1)$

Since $(-1,1,-1) \neq 0$ $\{(-1,1,-1)\}$ is a basis of $\ker T$

$\therefore \dim \ker T = 1$

$\Rightarrow \dim \text{Range } T = 2$

$\left.\begin{array}{l} Te_1 = (1,0,-1) \\ Te_2 = (2,1,3) \end{array}\right\} \in \text{Range } T$

$aTe_1 + bTe_2 = 0$

$a(1,0,-1) + b(2,1,3) = 0$

$\Rightarrow a = b = 0$

Scanned by CamScanner

$\Rightarrow \{Te_1, Te_2\}$ is Linearly independent set in Range T.

$\Rightarrow$ dim Range $T = 2$

$\Rightarrow \{(1,0,-1), (2,1,3)\}$ is a basis of Range T.

Two marks:-

1. Show that the following mapping is Linear.

$F: R^3 \to R^2$ given by $F(x,y,z) = (z, x+y)$

Solution:-

Let a map $F: R^3 \to R^2$ be given by

$F(x,y,z) = (z, x+y)$.

Let $u = (x,y,z)$, $v = (x_1, y_1, z_1)$

Then $u+v = (x+x_1, y+y_1, z+z_1)$

$ku = (kx, ky, kz)$

$F(u+v) = F(x+x_1, y+y_1, z+z_1)$

$= (z+z_1, x+x_1+y+y_1)$

$= (z+z_1, (x+y)+(x_1+y_1))$

$= (z, x+y) + (z_1, x_1+y_1)$

$= F(u) + F(v)$

$F(ku) = \cancel{k(z, x+y)}$

$= \cancel{(kz, k(x+y))}$

$= (kz, kx+ky)$

$= (kz, k(x+y))$

$= k(z, x+y)$

$= k\,F(u)$

$\therefore$ F is Linear.

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
What is the value of Gamma of one ?	0	1	2	3			1
Γ (n+1)=___________	(n+1)!	n Γ (n+1)	Γ (n-1)	n Γ (n)			n Γ (n)
what is the relation between Beta and Gamma functions?	β(m,n)=Γ(m)Γ(n)/Γ(m+n)	β(m,n)=Γ(m)Γ(m)/Γ(m+n)	β(m,m)=Γ(m)Γ(m)/Γ(m+n)	β(m,n)=Γ(n)Γ(n)/Γ(m+n)			β(m,n)=Γ(m)Γ(n)/Γ(m+n)
what is the value of Γ(1 /2)?	pi	0	1	root(pi)			root(pi)
Which one of the following statement is true?	Γ(2)=Γ(1)	Γ(1/2)= (√π	Γ (1/2)= 1	Γ(1/2)= 0			Γ(2)=Γ(1)
Which one of the following statement is false?	Γ(2)=Γ(1)	Γ(1)= 1	Γ (1/2)= √π	Γ (n+1)=n+1			Γ (n+1)=n+1
Γ(1/4) Γ(3/4)=_____	2π	π√2	√(2π)	1			π√2
The values of Γ(4)=______	1!	2!	3!	4!			3!
If C ′ is the evolute of the curve C then C is called the______ of the curve C ′	involute	curvature	radius of curvature	centre of curvature			involute
_______ of a curve is the envelope of the normals of that curve.	involute	curvature	radius of curvature	evolute			evolute
The parametric coordinates of the parabola x^2=4ay are_____.	x=at^2, y=2at	x=at, y=at	x=2at, y=at^2	x=a, y=t			x=2at, y=at^2
The parametric coordinates of the ellipse is given by _________.	x=acosθ, y=bsinθ	x=asinθ, y=bcosθ	x=atanθ, y=bsecθ	x=asecθ, y=btanθ			x=acosθ, y=bsinθ
The parametric coordinates of the hyperbola is given by _________.	x=acosθ, y=bsinθ	x=asinθ, y=bcosθ	x=atanθ, y=bsecθ	x=asecθ, y=btanθ			x=asecθ, y=btanθ
The parametric coordinates of the parabola y^2=4ax are_____.	x=at^2, y=2at	x=at, y=at	x=2at, y=at^2	x=a, y=t			x=at^2, y=2at
The locus of the centre of curvature for a curve is called its evolute and the curve is called an __________ of its evolute.	involute	evolute	envelope	curvature			involute
The locus of the centre of curvature for a curve is called its______.	involute	evolute	envelope	curvature			evolute
The parametric coordinates of the cycloid is given by _________.	x=a(θ+sinθ), y=a(1+cosθ)	x=a(θ-sinθ), y=a(1-cosθ)	x=(θ+sinθ), y=(1+cosθ)	x=(θ-sinθ), y=(1-cosθ)			x=a(θ-sinθ), y=a(1-cosθ)
If y=1/x, then y1=________	-1/x^2	1/x	ax	bx			-1/x^2
If y=x^2, then y1=________	x^2	1/x	2x	x			2x
If y=x^2, then y2=________	x^2	1/x	2x	2			2
If x=2at then dx/dt=	2at	2a	2t	0			2a
If x=at^2 then dx/dt=	2at	2a	2t	0			2at
If y=ax^2+2ax then dy/dx at (3,2) is ______	8a	4ax	2ax	6a			8a
If y=ax^2+2ax then dy/dx at (2,2) is ______	8a	4ax	2ax	6a			6a
If y=ax^2+2ax then dy/dx is ______	8ax+2a	4ax+2	2ax+2a	6a			2ax+2a
If y=ax^2+2ax then second derivative is ______	2a	4ax	6ax	6a			2a
The volume of the solid of revolution generated by revolving the plane area bounded by the circle x^2+y^2=a^2 about its diameter is..........	(4/3)πa^3	(2/3)πa^3	(1/3)πa^3	πa^3			(4/3)πa^3
The volume of the solid of revolution generated by revolving the plane area bounded by the circle x^2+y^2=2^2 about its diameter is..........	(32/3)π	(1/3)π	(2/3)π	π			(32/3)π
The volume of the solid of revolution generated by revolving the plane area bounded by the circle x^2+y^2=3^2 about its diameter is..........	16π	9π	36π	π			36π
The Volume of a sphere of radius 'a' is.................	2/3 π a^3	4/3 π a^3	1/3 π a^3	π a^3			4/3 π a^3
The surface are of the sphere of radius 'a' is.................	4πa^2	πa^2	3πa^2	2πa^2			4πa^2
The Volume of a sphere of radius '2' is.................	16/3 π	32/3 π	8/3 π	8 π			32/3 π
The surface area of the sphere of radius '3' is.................	36π	9π	27π	18π			36π
∫ dx=.............	x+C	1	0	x^2			x+C
∫cdx=…………….	cx+C	0	1	x+C			cx+C
∫ 5dx=.............	x+C	5x+C	x^2+C	5+C			5x+C
∫ x^n dx=.............	x^(n+1)/ (n	x^(n-1)/ (n-	nx^ (n-1)+	(n+1) x^ (n+1)+ C			x^(n+1)/ (n+1)+ C
∫xdx=…..	x^2+C	x^2/2+C	x^3/2+C	x^2/2+C			x^2/2+C
∫ x^ (2) dx=.............	(x^(2)/2)+C	(x^(3)/3)+C	x+C	2x+C			(x^(3)/3)+C
∫ 3x^(2) dx=.............	3x^(2)+C	x+C	x^2+C	x^(3) +C			x^(3) +C
∫ (1/x) dx=.............	1+ C	log x+C	(-1)+C	(-log x)+ C			log x+C
∫ e^(x) dx=.............	(-e^x)+ C	e^(-x) + C	(-e^(-x))+C	e^x + C			e^x + C
∫ e^(-x) dx=.............	(-e^x)+ C	e^(-x) + C	(-e^(-x))+C	e^x + C			(-e^(-x))+C
∫ e^(2x) dx=.............	(-e^2x)/2+	e^(-2x)/2 +	(-e^(-2x))/2	e^2x/2+ C			e^2x/2 + C
∫ e^(-2x) dx=.............	(-e^(-2x))/2	e^(-2x)/2 +	(-e^(-2x))/2	e^(-2x)/2+ C			e^2x/2 + C
∫ cosx dx=.............	sinx + C	cosx + C	(-cosx)+C	(-sinx)+C			sinx + C
∫ sinx dx=.............	sinx + C	cosx + C	(-cosx)+C	(-sinx)+C			(-cosx)+C
∫ cosmx dx=.............	(sinmx)/m	(cosmx)/m	(-cosmx)/m	(-sinmx)/m+C			(sinmx)/m+ C

Unit - III.

Inner Product Space :-

Let V be a vector space over F. An inner product on V is a function which assigns to each ordered pair of vectors u, v in V a scalar in F denoted by $\langle u, v\rangle$ satisfying the following conditions.

(i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(ii) $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$

(iii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ where $\overline{\langle v, u\rangle}$ is the complex conjucate of $\langle u, v\rangle$

(iv) $\langle u, u\rangle \geq 0$ & $\langle u, u\rangle = 0$ iff $u = 0$.

~~A~~ If is a vectorspace with an inner product ~~de~~ then it is called an inner product space.

An inner product space is called an Euclidean Space or Unitary Space according as F is field of real numbers or Complex numbers.

Example :-

1) $V_n(\mathbb{R})$ is a real inner product space with inner product defined by

$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$

This is called the standard inner product on $V_n(\mathbb{R})$

Proof :-

Let $x, y, z \in V_n(\mathbb{R})$ & $\alpha \in \mathbb{R}$

1) $\langle x+y, z\rangle = (x_1+y_1)z_1 + (x_2+y_2)z_2 + \dots + (x_n+y_n)z_n$

$= (x_1z_1 + x_2z_2 + \dots + x_nz_n)$
$+ (y_1z_1 + y_2z_2 + \dots + y_nz_n)$

$= \langle x, z\rangle + \langle y, z\rangle$

2) $\langle \alpha x, y\rangle = \alpha x_1y_1 + \alpha x_2y_2 + \dots + \alpha x_ny_n$

$= \alpha(x_1y_1 + x_2y_2 + \dots + x_ny_n)$

$= \alpha\langle x, y\rangle$

3) $\langle x, y\rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$

$= y_1x_1 + y_2x_2 + \dots + y_nx_n$

$= \langle y, x\rangle$

4) $\langle x, x\rangle = x_1x_1 + x_2x_2 + \dots + x_nx_n$

$= x_1^2 + x_2^2 + \dots + x_n^2$

$\langle x, x\rangle = 0$ iff $x_1 = x_2 = \dots = x_n = 0$

$\therefore \langle x, x\rangle = 0$ iff $x = 0$

2) Let V be the set of all continous real valued functions defined on the closed interval $[0,1]$. V is a real inner product space with inner product defined by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Scanned by CamScanner

i) $\langle f+g, h\rangle = \int_0^1 [f(t)+g(t)]\, h(t)\, dt$

$= \int_0^1 f(t)\, h(t)\, dt + \int_0^1 g(t)\, h(t)\, dt$

$= \langle f, h\rangle + \langle g, h\rangle$

ii) $\langle \alpha f, g\rangle = \int_0^1 \alpha f(t) g(t)\, dt$

$= \alpha \int_0^1 f(t) g(t)\, dt$

$= \alpha \langle f, g\rangle$

iii) $\langle f, g\rangle = \int_0^1 f(t) g(t)\, dt$

$= \int_0^1 g(t) f(t)\, dt$

$= \langle g, f\rangle$

(iv) $\langle f, f\rangle = \int_0^1 [f(t)]^2 dt \geq 0$

$\langle f, f\rangle = 0$ iff $f = 0$.

Norm:-

Let V be an inner product space and let $x \in V$. The norm or length of x, denoted by $\|x\|$, is defined by $\|x\| = \sqrt{\langle x, x\rangle}$.

x is called a unit vector if $\|x\| = 1$

Problem:-

1. Let V be the vector space of polynomials with inner product given by $\langle f, g\rangle = \int_0^1 f(t) g(t)\, dt$.

Let $f(t) = t + 2$ and $g(t) = t^2 - 2t - 3$.

Find (i) $\langle f, g\rangle$ (ii) $\|f\|$.

Solution:-

1) $$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$$

$$= \int_0^1 (t+2)(t^2-2t-3)\,dt$$

$$= \int_0^1 (t^3-7t-6)\,dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$= -\frac{37}{4}$$

2) $$\|f\|^2 = \langle f,f\rangle$$

$$= \int_0^1 f(t)f(t)\,dt$$

$$= \int_0^1 (t+2)^2\,dt$$

$$= \int_0^1 (t^2+4t+4)\,dt$$

$$= \left[\frac{t^3}{3} + 2t^3 + 4t\right]_0^1$$

$$= \frac{1}{3} + 2 + 4$$

$$= \frac{19}{3}$$

$$\|f\| = \sqrt{\frac{19}{3}}$$

Theorem 1:- The norm defined in an inner product space V has the following properties

i) $\|x\| \geq 0$ & $\|x\| = 0$ iff $x = 0$

ii) $\|\alpha x\| = |\alpha|\,\|x\|$

$\|x\|\;\|y\|$ (Schwartz's inequality)

Proof:-

i) $\|x\| = \sqrt{\langle x, x\rangle} \geq 0$ and $\|x\| = 0$ iff $x = 0$

(ii) $\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle$

$= \alpha \langle x, \alpha x\rangle$

$= \alpha \bar{\alpha} \langle x, x\rangle$

$= |\alpha|^2 \|x\|^2$

$\therefore \|\alpha x\|^2 = |\alpha|^2 \|x\|^2$

(iii) The inequality is trivially true when $x = 0$ or $y = 0$

Hence let $x \neq 0$ & $y \neq 0$

Consider $\quad z = y - \dfrac{\langle y, x\rangle}{\|x\|^2} \cdot x$

Then $\quad 0 \leq \langle z, z\rangle$

$$= \left\langle y - \frac{\langle y, x\rangle}{\|x\|^2} x \, , \, y - \frac{\langle y, x\rangle}{\|x\|^2} x \right\rangle$$

$$= \langle y, y\rangle - \frac{\overline{\langle y, x\rangle}}{\|x\|^2} \langle y, x\rangle - \frac{\langle y, x\rangle}{\|x\|^2} \langle x, y\rangle$$

$$+ \frac{\langle y, x\rangle \overline{\langle y, x\rangle}}{\|x\|^2 \, \|x\|^2} \langle x, x\rangle$$

$$= \|y^2\| - \frac{\overline{\langle y, x\rangle} \langle y, x\rangle}{\|x\|^2} - \frac{\langle y, x\rangle \langle x, y\rangle}{\|x\|^2}$$

$$+ \frac{\langle y, x\rangle \overline{\langle y, x\rangle}}{\|x\|^2}$$

$$= \|y\|^2 - \frac{\overline{\langle x, y\rangle} \langle x, y\rangle}{\|x\|^2}$$

$\therefore \quad 0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2$

Scanned by CamScanner

(iv) $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2$$

$$= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$\leq (\|x\| + \|y\|)^2$$

$\therefore \|x+y\| \leq \|x\| + \|y\|.$

Orthogonal :-

Let V be an inner product space and let $x, y \in V$. x is said to be orthogonal to y if $\langle x,y\rangle = 0$

Orthonormal :-

S is said to be an orthonormal set if S is orthogonal and $\|x\| = 1$ for all $x \in S$.

Theorem 1 :-

Let $S = \{v_1, v_2, \ldots v_n\}$ be an orthogonal set of non-zero vectors in V. Let $v \in V$ and $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ Then $\alpha_k = \dfrac{\langle v, v_k\rangle}{\|v_k\|^2}$

Proof :-

$$\langle v, v_k\rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n, v_k\rangle$$

$$= \alpha_1\langle v_1, v_k\rangle + \alpha_2\langle v_2, v_k\rangle + \ldots + \alpha_k\langle v_k, v_k\rangle + \ldots + \alpha_n\langle v_n, v_k\rangle$$

$$= \alpha_k\langle v_k, v_k\rangle$$

$$= \alpha_k \|v_k\|^2$$

$$\langle v, v_K \rangle = \alpha_K \, \|v_K\|^2$$

$$\alpha_K = \frac{\langle v, v_K \rangle}{\|v_K\|^2}.$$

Theorem 2:-

Every finite dimensional inner product space has an orthonormal basis

Proof:-

Let V be a finite dimensional inner product space. Let $\{v_1, v_2, \ldots v_n\}$ be a basis for V. From this basis we shall construct an orthonormal basis $\{w_1, w_2, \ldots w_n\}$ by means of a construction known as Gram - Schmidt orthogonalisation process.

First we take $w_1 = v_1$

Let $$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1.$$

we claim that $w_2 \neq 0$. For, if $w_2 = 0$ then v_2 is a scalar multiple of w_1 and hence of v_1 which is a contradiction. since v_1, v_2 are linearly independent.

$$\langle w_2, w_1 \rangle = \left\langle v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 , w_1 \right\rangle$$

$$= \left\langle v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1 , v_1 \right\rangle \quad (\because w_1 = v_1)$$

$$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_1 \rangle$$

$$= \langle v_2, v_1 \rangle - \langle v_2, v_1 \rangle$$

$$= 0.$$

suppose that we have constructed non-zero orthogonal vectors $w_1, w_2, \ldots w_k$.

Then $$w_{k+1} = v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} w_j$$

We claim that, $w_{k+1} \neq 0$
For if $w_{k+1} = 0$, then v_{k+1} is a linear combination of $w_1, w_2, \dots w_k$.
Hence It is a linear combination of $v_1, v_2, \dots v_k$
which is a contradiction.
Since $v_1, v_2 \dots v_{k+1}$ are linearly independent

Also

$$\langle w_{k+1}, w_i \rangle = \langle v_{k+1}, w_i \rangle - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} \langle w_j, w_i \rangle$$

$$= \langle v_{k+1}, w_i \rangle - \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} \langle w_i, w_i \rangle$$

$$= \langle v_{k+1}, w_i \rangle - \langle v_{k+1}, w_i \rangle$$

$$= 0$$

Thus, continuing in this way we ultimately obtain a non-zero orthogonal set $\{w_1, w_2, \dots w_n\}$

$\Rightarrow$ This set is linearly independent and hence a basis.

$\Rightarrow$ To obtain an orthonormal basis we replace each w_i by $\frac{w_i}{\|w_i\|}$.

—x—

Problem:

1. Apply Gram-Schmidth process to construct an orthonormal basis for $V_3(\mathbb{R})$ with standard inner product for the basis $\{v_1, v_2, v_3\}$ where $v_1 = (1,0,1)$; $v_2 = (1,3,1)$ & $v_3 = (3,2,1)$

<u>Solution:-</u>

Take $w_1 = v_1 = (1,0,1)$

Then

$$\|w_1\|^2 = \langle w_1, w_1\rangle = 1^2+0^2+1^2 = 2$$

$$\langle w_1, v_2\rangle = 1+0+1 = 2$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} \cdot w_1 = (1,3,1) - \frac{2}{2}(1,0,1) \quad w_1 = v_1$$

$$= (1,3,1) - (1,0,1)$$

$$= (0,3,0)$$

$$\|w_2\|^2 = 9$$

Also $\langle w_2, v_3\rangle = 0+6+0 = 6$

$$\langle w_1, v_3\rangle = 3+0+1 = 4$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2$$

$$= (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9}(0,3,0)$$

$$= (3,2,1) - 2(1,0,1) - \frac{2}{3}(0,3,0)$$

$$= (1,0,-1)$$

$$\|w_3\|^2 = 1^2 + 1^2 = 2$$

The orthogonal basis is $\{(1,0,1), (0,3,0), (1,0,-1)\}$

Hence the orthonormal basis is

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\right\}.$$

Problem 2:-

Let V be the set of all polynomials of degree ≤ 2 together with the zero polynomial. V is a real inner product space with inner product defined by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$. Starting with the basis $\{1, x, x^2\}$, obtain an orthonormal basis for V.

Scanned by CamScanner

Solution :-

$v_1 = 1$, $v_2 = x$ & $v_3 = x^2$.

Let $w_1 = v_1$

Then $\|w_1\|^2 = \langle w_1, w_1\rangle = \int_{-1}^{1} dx = 2$

$$\|w_1\| = \sqrt{2}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1$$

$$= x - \frac{1}{2}\int_{-1}^{1} x\,dx$$

$$= x$$

$$\|w_2\|^2 = \langle w_2, w_2\rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2$$

$$= x^2 - \frac{1}{2}\int_{-1}^{1} x^2 dx - \left(\frac{3x}{2}\right)\int_{-1}^{1} x^3 dx$$

$$= x^2 - \frac{1}{3}$$

$$\|w_3\|^2 = \langle w_3, w_3\rangle = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \frac{8}{45}$$

Hence the orthogonal basis is $\left\{1, x, x^2, -\frac{1}{3}\right\}$

The required orthonormal basis is

$$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}x, \frac{\sqrt{10}}{4}(3x^2-1)\right\}$$

Orthogonal Complement :

Let V be an inner product space. Let S be a subset of V. The orthogonal complement of S, denoted by $S^{\perp}$, is the set of all vectors in V ~~which~~ which are orthogonal to every vector of S.

Theorem :-

If S is any subset of V then $S^{\perp}$ is a subspace of V.

Proof :-

clearly $0 \in S^{\perp} \Rightarrow S^{\perp} \neq \phi$

let $x, y \in S^{\perp}$ & $\alpha, \beta \in F$.

Then $\langle x, u \rangle = \langle y, u \rangle = 0 \ \forall u \in S$

$\therefore \langle \alpha x + \beta y, u \rangle = \alpha \langle x, u \rangle + \beta \langle y, u \rangle$

$= 0 \quad \forall u \in S$

$\therefore \alpha x + \beta y \in S^{\perp}$

$\Rightarrow \quad S^{\perp}$ is a subspace of V.

Theorem :-

Let V be a finite dimensional inner product space. Let W be a subspace of V. Then V is the direct sum of W and $W^{\perp}$.

ie) $V = W \oplus W^{\perp}$

let $Z_n = \{0, 1, 2, \ldots n-1\}$

$a, b \in Z_n$

let $a + b = qn + r$

$0 \leq r < n$

$\Rightarrow a \oplus b = r$.

Proof :-

we shall prove that

(i) $W \cap W^{\perp} = \{0\}$

(ii) $W \oplus W^{\perp} = V$

(i) Let $v \in W \cap W^{\perp}$

$\Rightarrow \quad v \in W$ & $v \in W^{\perp}$

$\Rightarrow \quad v \in W^{\perp} \Rightarrow v$ is orthogonal to every vector in W.

In particular, v is orthogonal to itself.

$\langle v, v \rangle = 0$

$\Rightarrow \quad v = 0$

$\Rightarrow \quad W \cap W^{\perp} = \{0\}$.

(ii) Let $\{v_1, v_2, \ldots v_r\}$ be an orthonormal basis for W

Let $v \in V$.

consider.

$v_0 = v - \langle v, v_1 \rangle v_1 - \langle v, v_2 \rangle v_2 - \ldots - \langle v, v_r \rangle v_r$

$\langle v_0, v_i \rangle = \langle v, v_i \rangle - \langle v, v_i \rangle \langle v_i, v_i \rangle$

$= \langle v, v_i \rangle - \langle v, v_i \rangle$ $\left(\langle v_i, v_j \rangle = 0 \text{ if } \ldots, \ \langle v_i, v_i \rangle = 1 \right)$

$= 0$

$\therefore v_0$ is orthogonal to each of $v_1, v_2, \ldots v_r$ and hence is orthogonal to every vector in W.

Hence $v_0 \in W^{\perp}$

$v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \ldots + \langle v, v_r \rangle v_r + v_0$

$\in W + W^{\perp}$

$W = W \oplus W^{\perp}$

Hence the theorem.

Questions	opt1	opt2	opt3	opt4	opt5	opt6
The Taylor series of f(x) about the point 0 is ___________	Maclaurins	Taylor	power	binomial		
The expansion of f(x) by Taylor series is ___________	zero	unique	minimu	maximu		
The point at which function f(x) is either maximum or minimum is known as _______ point	Stationary	Saddle point	extremum	implicit		
A function f has ___ at 'c' if f(c) ≥ f(x) for all 'x' in D, where D is domain of 'f'.	an absolute maximu	an absolute minimu	local maximam	locam minimum		
If f(x) = x^2, then f(0) = 0 is the ___ value of f.	an absolute maximum	an absolute minimum	local maximam	an absolute and local minimu		
A function f has a ____ at 'c' if there is an open interval I containing 'c' such that f(c) ≥ f(x) for all 'x' in I.	an absolute maximu	an absolute minimu	local maximam	locam minimum		
A function f has a ____ at 'c' if there is an open interval I containing 'c' such that f(c) ≤ f(x) for all 'x' in I.	an absolute maximu	an absolute minimu	local maximam	local minimum		
If 'f' has a ___ at 'c' and if f'(c) exists then f'(c)=0.	critical number	stationary point	local extremum	an absolute maximu		
A function 'f' has ___ at 'c' if f(c) ≤ f(x) for all 'x' in D, where D is domain of 'f'.	an absolute maximu	an absolute minimu	local maximam	locam minimum		
If 'f' has a local extremum at 'c' and if f'(c) exists then f'(c)=___.	0	1	c	(-1)		
Evaluate: limit x tends to 0 (x / tan x) =	1	2	3	0		
Evaluate: limit x tends to infinity (x^2 / e^x) =	1	2	3	0		
L'Hopital's rule can be applied only to differentiable functions for which the limis are in the ______ form	real	indeterminate	complex	extremum		
L'Hopital's rule can be applied only to ___functions for which the limis are in the indeterminate form	differentiable	real	complex	extremum		
If f(x) = x^3, then the function has ___	eiher an absolute maximum or an absolute minimum	neiher an absolute maximum nor an absolute	local maximam	locam minimum		
A ___ of a function f is a number c in the domain of f such that either f'(c) =0 or f'(c) does not exist	critical number	stationary point	local extremum	an absolute maximu		
___ are critical numbers c in he domain of f, for which f'(c)=0	Critical number	Stationary points	Local extremum	An absolute maximu		
If f has a local extremum at c, then c is a ___ of f	critical number	stationary point	local extremum	an absolute maximu		
If f has a ____ at c, then c is a critical number of f	critical number	stationary point	local extremum	an absolute maximu		
If f(x)=x^2 - 4x+5 on [0,3] then the absolute maximum value is __	2	3	4	5		
Find the critical numbers, for the function f(x)=x^3 - 3x^2 +1.	(1,2)	(0,2)	(2,2)	(1,3)		
Find the critical numbers, for the function f(x)=x^3 - 3x +1.	(1,1)	(-1,1)	(0,1)	(-1,-1)		
Find the critical number, for the function f(x)=2x - 3x^2 .	(1/2)	(1/3)	(1/4)	1		
Find the critical number, for the function f(x)=x^2 - 2x +2.	0	1	2	3		
Find the critical number, for the function f(x)=1-2x-x^2.	0	1	2	3		
Find the critical numbers, for the function f(x)=x^3 - 12x +1.	(0,1)	(0,2)	(0,3)	(0,4)		
Find the stationary point of the function f(x)=2x - 3x^2	(1,1)	(1,2)	(1/3, 1/3)	(1/2,1)		
Find the stationary point of the function f(x)=x^3 - 3x +1	(1,-1) and (-1,3)	(1,-1)	(-1,3)	(1,1) and (1,3)		
Find the absolute maximum of the function f(x) = x^2-2x+2, [0,3	1	3	5	8		

Find the absolute minimum of the function f(x) = x^2-2x+2, [0,3]	1	3	5	8
Find the absolute maximum of the function f(x) = 1-2x-x^2 [-4,1]	1	2	7	8
Find the absolute minimum of the function f(x) = 1-2x-x^2 [-4,1]	1	2	(-7)	(-8)

Answer
Maclauri
ns
unique
Stationar
y
an
absolute
maximu
an
absolute
and
local
minimu

local
maximam

local
minimum

local
extremum
an
absolute
minimu

1
1
0
indetermi
nate
differentia
ble
neiher
an
absolute
maximu
m nor
an
absolute

critical
number

Stationar
y points

critical
number

local
extremum

5
(0,2)
(-1,1)
(1/3)
1
1
(0,4)
(1/3, 1/3)
(1,-1) and
(-1,3)
5

1

2

(-7)

Gamma Function.

The $\Gamma(n)$ is defined as the definite integral

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$$

Recurrence Formula :-

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx = -x^n e^{-x}\Big]_0^\infty + \int_0^\infty e^{-x} n x^{n-1} dx$$

$$= n\int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(1) = \int_0^\infty e^{-x} x^0 dx = -e^{-x}\Big]_0^\infty = 1$$

1) $\dfrac{\Gamma(n+1)}{n} = \Gamma(n).$

2) $\Gamma(1) = 1$

3) $\Gamma(2) = 1\,\Gamma(1) = 1 = 1!.$

4) $\Gamma(3) = 2\,\Gamma(2) = 1\cdot 2 = 2!.$

5) $\Gamma(4) = 3\,\Gamma(3) = 1\cdot 2\cdot 3 = 3!$

6) $\Gamma(n+1) = n\,\Gamma(n) = n(n-1)! = n!.$

when n is fraction :-

Let $n = \frac{1}{2}$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma(5/2) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$\vdots \qquad \vdots$

when n is a negative fraction:-

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-1/2} = -2\,\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{-2}{3}(-2)\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{4}{3}\,\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{-5}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}\right)}{\frac{-5}{2}} = \frac{-2^3}{3\cdot 5}\,\Gamma\left(\frac{1}{2}\right)$$

$\vdots \qquad \vdots$

Problem:-

1) Find the value of $\Gamma\left(\frac{1}{2}\right)$ and evaluate $\int_0^\infty e^{-x^2} dx$ using Γ function.

Solution:-

$$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt \qquad t = x^2$$
$$dt = 2x\,dx$$

$$\Gamma(n) = \int_0^\infty e^{-x^2}\, x^{2n-1} \cdot 2x\, dx$$

$$= 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx$$

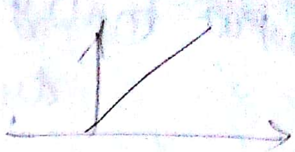
This is another

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dx \quad \text{——— (1)}$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\, dy \quad \text{——— (2)}$$

Multiply ① & ②

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$$



Transforming into polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$.

$x = 0 \Rightarrow \theta = \pi/2$

$x = \infty \Rightarrow \theta = 0$

$y = 0 \Rightarrow \theta$

$y = \infty \Rightarrow \theta$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^{1/2} = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr.d\theta$$

$$= 2\int_0^{\pi/2} d\theta \int_0^\infty e^{-r^2}\, 2r\, dr.$$

$$= -\pi . e^{-r^2}\Big]_0^\infty$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$= \pi.$$

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

$$\int_0^\infty e^{-x^2} dx = \frac{\Gamma\left(\frac{1}{2}\right)}{2} = \frac{\sqrt{\pi}}{2}.$$

Prove that $\int_0^1 x^m (\log x)^n dx = \dfrac{(-1)^n . n!}{(m+1)^{n+1}}$ where n is a positive integer. $m > -1$

Solution :-

Put $\log x = -y$

$e^{-y} = x$

$dx = -e^{-y} dy$

when $x = 0$, $y = \infty$

$x = 1$, $y = 0$

$$\Rightarrow \int_0^1 x^m (\log x)^n dx = \int_{+\infty}^0 e^{-my} (-y)^n (-e^{-y} dy)$$

$$= \int_0^\infty e^{-(m+1)y}\, y^n dy.$$

Put $(m+1)y = z$

$$dy = \frac{dz}{m+1}$$

$$\int_0^1 x^m (\log x)^n dx = (-1)^n \int_0^\infty e^{-z} \cdot \frac{z^n}{(m+1)^n} \cdot \frac{dz}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-z} z^n dz$$

$$= \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}}$$

$$= \frac{(-1)^n \cdot n!}{(m+1)^{n+1}}$$

i) Prove that $\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$ where $p>0, q>0$

Solution :-

$$\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy$$

Put $\log \frac{1}{y} = x$ $\quad\quad \log(1) = 0$

$e^{-x} = y$ $\quad\quad \log 0 = \infty$

$-e^{-x} dx = dy$

$$\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = -\int_\infty^0 e^{-qx+x} x^{p-1} e^{-x} dx$$

$$= \int_0^\infty e^{-qx} x^{p-1} dx$$

put $qx = z$, $dx = \frac{dz}{}$

Scanned by CamScanner

4) $\int_0^\infty \sqrt{x}\, e^{-x/3}\, dx$

Soln

Put $\frac{x}{3} = t$, $dx = 3dt$

$$\int_0^\infty \sqrt{x}\, e^{-x/3}\, dx = \int_0^\infty \sqrt{3}\sqrt{t}\, e^{-t}\, 3dt$$

$$= 3\sqrt{3}\int_0^\infty \sqrt{t}\, e^{-t}\, dt$$

$$= 3\sqrt{3}\,\Gamma\left(\frac{3}{2}\right)$$

$$= 3\sqrt{3}\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= 3\sqrt{3}\,\frac{1}{2}\sqrt{\pi}$$

$n-1 = \frac{1}{2}$

$n = \frac{3}{2}$

$$\int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma(n)$$

Relation Between β and Γ function :-

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

W.K.T $\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\, dt$

$t = x^2 \quad dt = 2x\,dx$

$$\Gamma(m) = \int_0^\infty e^{-x^2}\, x^{2m-2}\, 2x\,dx$$

$$= 2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\, dx \int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, x^{2m-1}\, y^{2n-1}\, dx\, dy$$

Scanned by CamScanner

Transforming to polar coordinates,

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\Rightarrow \Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2m-1}\cos^{2m-1}\theta \cdot r^{2n-1}\sin^{2n-1}\theta \cdot r\,dr\,d\theta$$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cdot \cos^{2m-1}\theta\,\sin^{2n-1}\theta \,.\, dr\,d\theta$$

$$= 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr \,.\, 2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\,\beta(m,n)$$

$$\Rightarrow \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

<u>Beta function</u>

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \,,\quad m>0, n>0.$$

2 mark)

Prove that $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$

<u>Solution</u> :-

put $x = \sin^{2/5}\theta$

$dx = \frac{2}{5}\sin^{-3/5}\theta\cos\theta\,d\theta$

$$\int_0^{\pi/2} \frac{\sin^{2/5}\theta \cdot \frac{2}{5}\sin^{-3/5}\theta \,,\, \cancel{\cos\theta}\,.d\theta}{\cancel{\cos\theta}}$$

$$= \frac{2}{5}\int_0^{\pi/2}\sin^{-1/5}\theta\,d\theta = \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$$

Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Proof:-

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\Gamma(m+1)\Gamma(n+1)\,(m+n)}{mn\,\Gamma(m+n+1)}$$

$$= \frac{\Gamma(m+1)\Gamma(n)\,m}{m\,\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)\,.\,n}{n\,\Gamma(m+n+1)}$$

$$= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \beta(m+1,n) + \beta(m,n+1)$$

Show that $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx.$

Proof:-

put $x = \frac{1}{1+y}$ when $x=0,\ y \to \infty$

$x = 1,\ y = 0$

$dx = \frac{-1}{(1+y)^2}\,dy$ $\qquad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$\beta(m,n) = \int_{-\infty}^{0} \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$\left[1-x = 1 - \frac{1}{1+y} = \frac{y}{1+y}\right]$$

$$= \int_0^{\infty} \frac{y^{n-1}\,dy}{(1+y)^{m+n}}$$

$$= \int_0^{1} \frac{y^{n-1}\,dy}{(1+y)^{m+n}} + \int^{\infty} \frac{y^{n-1}\,dy}{(1+y)^{m+n}}.$$

consider $\int_1^{\infty} \frac{y^{n-1}\,dy}{(1+y)^{m+n}}$

put $y = \frac{1}{x}$, $dy = \frac{-1}{x^2}dx$

$1+y = 1+\frac{1}{x} = \frac{x+1}{x}$.

$$\int_1^{\infty} \frac{y^{n-1}\,dy}{(1+y)^{m+n}} = -\int_1^0 \frac{x^{m+n}\,dx}{x^{n-1}(1+x)^{m+n} \cdot x^2}$$

$$= \int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+1}}$$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$= \int_0^1 \left(\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\right)dx.$$

Prove that $\beta(n,n) = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}$

Solution:-

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

$$\beta(m,n) = \int_0^1 (1-x)^{m-1} x^{n-1}\,dx$$

$\Gamma(n+1) = n\Gamma n.$

$\Gamma(n+\frac{1}{2}) =$

consider

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \frac{2n-1}{2}\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{2^n}\,\Gamma\left(\frac{1}{2}\right)$$

multiply & dividing by $2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n$.

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1.2.3\ldots\ 2n}{2^{2n}\, .\, n!}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1.2.3\ldots(2n-1).2n}{2^{2n}\ n!}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi} \quad \left(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$= \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)}$$

$$\left[\ \beta(n,n) = \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)}\right.$$

$$\left. \Rightarrow \frac{\Gamma(2n)}{\Gamma(n)} = \frac{\Gamma(n)\cancel{\Gamma}}{\beta(n,n)}\ \right]$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\Gamma(n)\sqrt{\pi}}{2^{2n-1}\beta(n,n)}$$

$$\Rightarrow \beta(n,n) = \frac{\Gamma(n)\sqrt{\pi}}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}.$$

Scanned by CamScanner

Hyperbolic Functions.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$\tan i\theta = i \tanh\theta$

$\sin i\theta = i \sinh\theta$

$\cos i\theta = \cosh\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\cosh^2\theta - \sinh^2\theta = 1$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

---

1) Prove that $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

Proof:-

$\sinh^{-1} x = y$

$x = \sinh y$

$$x = \frac{e^y - e^{-y}}{2}$$

$2x = e^y - 1/e^y$

$\Rightarrow 2xe^y = e^{2y} - 1$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow (e^y)^2 - 2x(e^y) - 1 = 0$

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - (4.1.(-1))}}{2(1)} = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$$

$e^y = x \pm \sqrt{x^2+1}$

taking log both sides we get,

$y = \log(x \pm \sqrt{x^2+1})$

2) $\tanh^{-1}x = \frac{1}{2}\log\frac{1+x}{1-x}$

Proof:- $\tanh^{-1}x = y$

$\therefore\ x = \tanh y$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^y - 1/e^y}{e^y + \frac{1}{e^y}}$$

$$x = \frac{e^{2y} - 1}{e^{2y} + 1}$$

$xe^{2y} + x = e^{2y} - 1$

$(x-1)(e^y)^2 + x + 1 = 0$

$$e^{2y} = \frac{-(x+1)}{x-1} = \frac{+x+1}{+(1-x)}$$

$$e^{2y} = \frac{x+1}{1-x}$$

Taking log

$$2y = \log\left(\frac{x+1}{1-x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{x+1}{1-x}\right)$$

$$\tanh^{-1}x = \frac{1}{2}\log\left(\frac{x+1}{1-x}\right)$$

3) If $\sin(A+iB) = x+iy$ then prove that

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1 \quad \& \quad \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

Proof

$\sin(A+iB) = \sin A \cos iB + \cos A \sin iB.$

$x + iy = \sin A \cosh B + i \cos A \sinh B$

$\Rightarrow x = \sin A \cosh B$

$\Rightarrow \sin A = \dfrac{x}{\cosh B}$

$\Rightarrow \cos A = \dfrac{y}{\sinh B}$

~~$\sin^2 A$~~ +

$\sin^2 A + \cos^2 A = 1$

$$\Rightarrow \boxed{\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1}$$

$\cosh B = \dfrac{x}{\sin A}$

$\sinh B = \dfrac{y}{\cos A}$

$\cosh^2 B - \sinh^2 B = 1$

$$\boxed{\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1}$$

Seperate the real and imaginary parts of $\tan^{-1}(a+ib)$.

Proof :- $\tan^{-1}(a+ib) = x+iy$

~~ta~~ $a+ib = \tan(x+iy)$ — ①

$a-ib = \tan(x-iy)$ — ②

① + ②

$\Rightarrow 2a = \tan(x+iy) + \tan(x-iy)$

$\tan((x+iy)+(x-iy)) = \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$\tan 2x = \dfrac{a+\cancel{ib}+a-\cancel{ib}}{1-(a+ib)(a-ib)} = \dfrac{2a}{1-(a^2+b^2)}$

$$\tan 2x = \frac{2a}{1-a^2-b^2}$$

$$2x = \tan^{-1}\left(\frac{2a}{1-a^2-b^2}\right)$$

$$x = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1-a^2-b^2}\right)$$

$$\tan\left[x+iy-x+iy\right] = \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{\cancel{a}+ib-\cancel{a}+ib}{1+a^2+b^2}$$

$$\tan 2iy = \frac{2ib}{1+a^2+b^2}$$

$$\cancel{i}\tanh 2y = \frac{\cancel{i}2b}{1+a^2+b^2}.$$

$$\tanh 2y = \frac{2b}{1+a^2+b^2}$$

$$2y = \tanh^{-1}\left(\frac{2b}{1+a^2+b^2}\right)$$

$$y = \frac{1}{2}\tanh^{-1}\left(\frac{2b}{1+a^2+b^2}\right)$$

$\therefore$

$$\tan^{-1}(a+ib) = x+iy$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2a}{1-a^2-b^2}\right) + \frac{i}{2}\tanh^{-1}\left(\frac{2b}{1+a^2+b^2}\right)$$

Seperate the real and imaginary parts of $\log \sin z$.

Solution:—

$\log \sin(x+iy)$

Scanned by CamScanner

$= \log(\sin x \cosh y + i \cos x \sinh y)$

$= \log re^{i\theta} = \log r + \log e^{i\theta}$

$= \log r + i\theta$ where $re^{i\theta} = \underbrace{\sin x \cosh y}_{x} + i \underbrace{\cos x \sinh y}_{y}$.

$r = \sqrt{x^2+y^2}$

$= \sqrt{\sin^2 x \cos^2 hx + \cos^2 x \sin^2 hy}$

$= \sqrt{\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cosh 2y}{2}\right) + \left(\frac{1+\cos 2x}{2}\right)\left(\frac{\cosh 2y - 1}{2}\right)}$

$= \frac{1}{2}\sqrt{\cancel{1} - \cos 2x + \cosh 2y - \cancel{\cos 2x \cosh 2y} + \cosh 2y + \cancel{\cos 2x \cosh 2y} - \cancel{1} - \cos 2x}$

$r = \frac{1}{2}\sqrt{2\cosh 2y - 2\cos 2x}.$

$=$

$\theta = \tan^{-1} \frac{\cos x \sinh y}{\sin x \cosh y}$

$= \tan^{-1}(\cot x \tanh y)$

$\operatorname{Log} \sin(x+iy) = \frac{1}{2}\sqrt{2\cosh 2y - 2\cos 2x} + i \tan^{-1}(\cot x \tanh y)$

If $\operatorname{cosec}\left(\frac{\pi}{4}+i\alpha\right)=u+iv$, Prove that $u^2+v^2=2(u^2-v^2)$

Solution

$$\boxed{\operatorname{cosec}(x+iy)=\frac{2(\sin x\cosh y-i\cos x\sinh y)}{\cosh 2y-\cos 2x}}$$

$$x=\frac{\pi}{4} \quad \& \quad y=\alpha$$

$$u=\frac{2\sin x\cosh y}{\cosh 2y-\cos 2x}=\frac{\frac{2}{\sqrt{2}}\cosh\alpha}{\cosh 2\alpha}=\sqrt{2}\,\frac{\cosh\alpha}{\cosh 2\alpha}$$

$$v=\frac{-2\cos x\sinh y}{\cosh 2y-\cos 2x}=-\frac{2}{\sqrt{2}}\,\frac{\sinh\alpha}{\cosh 2\alpha}$$

$$u^2+v^2=\frac{2\cosh^2\alpha}{\cosh^2 2\alpha}+\frac{2\sinh^2\alpha}{\cosh^2 2\alpha}$$

$$=\frac{2\cosh 2\alpha}{\cosh^2 2\alpha}$$

$$=2\left(\frac{\cosh^2\alpha}{\cosh^2 2\alpha}-\frac{\sinh^2\alpha}{\cosh^2 2\alpha}\right)$$

$$u^2+v^2=2(u^2-v^2)$$

If $\tanh\left(x+i\frac{\pi}{4}\right)=\alpha+i\beta$, then prove that $\alpha^2+\beta^2=1$

Solution:-

$$\tanh\left(x+i\frac{\pi}{4}\right)=\alpha+i\beta.$$

$$\tanh\left(x+i\frac{\pi}{4}\right)=\cancel{\alpha+i\beta}$$

$$=\frac{\tanh x+\tanh\left(i\frac{\pi}{4}\right)}{1+\tanh x\tanh i\frac{\pi}{4}}$$

$$=\frac{\tanh x+i}{1+i\tanh x}.$$

Scanned by CamScanner

$$= \frac{(\tanh x + i)(1 - i\tanh x)}{1+\tan^2 h\, x}$$

$$= \frac{\tanh x + \tanh x + i(1 - \tan^2 h\, x)}{1 + \tan^2 h\, x}$$

$$= \frac{2\tanh x + i(1 - \tan^2 h\, x)}{1+\tan^2 h\, x}$$

$$\alpha = \frac{2\tanh x}{1+\tan^2 h\, x} \qquad \beta = \frac{1-\tan^2 h x}{1+\tan^2 h x}$$

$$\alpha^2+\beta^2 = \frac{4\tan^2 h x + 1 + \tan^4 h x - 2\tan^2 h x}{(1+\tan^2 h x)^2}$$

$$= \frac{(1+\tan^2 h x)^2}{(1+\tan^2 h x)^2}$$

$$\alpha^2+\beta^2 = 1$$

If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$

Prove (i) $\cos 2x + \cos 2y = 2r^2$

(ii) $y = \frac{1}{2}\log \frac{\sin(x-\alpha)}{\sin(x+\alpha)}$.

Solution:-

Given $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$

$\cos x \cos iy - \sin x \sin(iy) = r\cos\alpha + i r\sin\alpha$

$\cos x \cosh y - i\sin x \sinh y = r\cos\alpha + i r \sin\alpha$

$\cos x \cosh y = r\cos\alpha$ — (1)

$-\sin x \sinh y = r\sin\alpha$ — (2)

$(1)^2 + (2)^2 = r^2\cos^2\alpha + r^2\sin^2\alpha = \cos^2 x\cosh^2 y + \sin^2 x \sinh^2 y$

$$= \cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y$$

$$= \cos^2 x + \sinh^2 y$$

$$\gamma^2 = \frac{1+\cos 2x}{2} + \frac{\cosh 2y - 1}{2}$$

$$2\gamma^2 = 1 + \cos 2x + \cosh 2y$$

$$\cos 2x + \cosh 2y = 2\gamma^2$$

② ÷ ①

$$\tan\alpha = \frac{-\sin x \sinh y}{\cos x \cosh y} = -\tan x \tanh y$$

$$\tanh y = -\frac{\tan\alpha}{\tan x}$$

$$y = \tanh^{-1}\left[-\frac{\tan\alpha}{\tan x}\right]$$

$$= \frac{1}{2}\log\left[\frac{1 - \frac{\tan\alpha}{\tan x}}{1 + \frac{\tan\alpha}{\tan x}}\right] \qquad \left[\because \tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right]$$

$$= \frac{1}{2}\log\left[\frac{\tan x - \tan\alpha}{\tan x + \tan\alpha}\right]$$

$$= \frac{1}{2}\log\left[\frac{\frac{\sin x}{\cos x} - \frac{\sin\alpha}{\cos\alpha}}{\frac{\sin x}{\cos x} + \frac{\sin\alpha}{\cos\alpha}}\right]$$

$$= \frac{1}{2}\log\left[\frac{\sin x\cos\alpha - \sin\alpha\cos x}{\sin x\cos\alpha + \sin\alpha\cos x}\right]$$

$$y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$$

$$2y = \log\left(\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right)$$

Scanned by CamScanner

Questions	opt1	opt2	opt3	opt4	opt5	opt6	Answer
The partial differentiation is a function of_________ or more variables .	two	zero	one	three			two
If z=f(x,y) where x and y are ______ function of another variable t	continuous	differential	two	one			continuous
If f(x,y)=0 then x and y are said to be an _________ function	implicit	extrem	explicit	differential			implicit
f(a,b) is said to be exturemumvalue of f(x,y) if it is either a______________	maximum or minimum	zero	minimum	maximum			maximum or minimum
The Lagrange multiplier is denoted by ________________	a	b	l	d			l
Every extremum value is a stationary value but a stationary value need not be an ___________ value.	infimum	minimum	maximum	extremum			extremum
If u1,u2……un are functions of n variables x1,x2……..xn then the Jacobian of the transformation from x1,x2….xn to u1,u2...un is defined by__________	2	0	1	-1			2
f(a,b) is a maximum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood ________	f(a,b)>f(a+h,b+k)	f(a,b)<f(a+h,b+k)	f(a,b)<0	f(a,b)>0			f(a,b)>f(a+h,b+k)
f(a,b) is a minimum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood ________	f(a,b)>f(a+h,b+k)	f(a,b)<f(a+h,b+k)	f(a,b)<0	f(a,b)>0			f(a,b)<f(a+h,b+k)
The necessary condition for maxima is __________	∂f/∂x (a,b)=0	∂f/∂x (a,b)= 1	∂f/∂y (a,b)=5	∂f/∂y (a,b)=1			∂f/∂x (a,b)=0
The necessary condition for minimum is ____________	∂f/∂x (a,b)=0	∂f/∂y (a,b)=0	∂f/∂x (a,b)=1	∂f/∂y (a,b)=1			∂f/∂y (a,b)=0
f(a,b) is said to be said to be a stationary value of f(x,y) if (x,y) is__________	∂f/∂x (a,b)=0 and ∂f/∂y (a,b)=0	∂f/∂x (a,b)=1	∂f/∂y (a,b)=0	∂f/∂y (a,b)=1			∂f/∂x (a,b)=0 and ∂f/∂y (a,b)=0
The expansion of f(x,y) by ________series is unique.	Maclaurins	Taylor	power	binomial			Taylor
If f(a,b) is said to be _________of f(x,y) if it is either maximum or minimum.	extremum value	boundary value	end	power			extremum value
The _______ differentiation is a function of two or more variables.	ODE	PDE	partial	total			partial
Any function of the type f(x,y)=	Implicit	Explicit	Constant	composite			Implicit

If u=f(x,y) ,where x=p(t),y=s(t) then u is a function of t and is called the ____ function	Implicit	Explicit	Constant	composite	composite
The point at which function f(x,y) is either maximum or minimum is known as ______ point	Stationary	Saddle poin	extremum	implicit	Stationary
If rt-s^2>0 and r<0 at (a,b) the f	Maximum	Minimum	maximum or minimum	zero	Maximum
If rt-s^2>0 and r>0 at (a,b) the f	Maximum	Minimum	maximum or minimum	zero	Minimum
If rt-s^2>0 at (a,b) the f(x,y) is	Stationary	Saddle poin	extremum	implicit	Saddle point
If Ñ.F=0 then F is	irrotational	solenoidal	rotational	curl	solenoidal
If Ñ×F=0 then F is	irrotational	solenoidal	rotational	curl	irrotational
Any motion in which the curl of the velocity vector is zero is said to be ___	irrotational	solenoidal	rotational	curl	irrotational
A function is said to be _______ if it associates a scalar with every point in space.	Scalar function	Vector function	Point function	vector point function	Scalar function
A variable quantity whose value at any point in a region of space depends upon the position of the point is called a ___	Scalar function	Vector function	Point function	vector point function	Point function
A function is said to be _______ if it associates with vector in every point in space.	Scalar function	Vector function	Point function	vector point function	Vector function
If the divergence of a flow is zero at all points then it is said to be _______	rotational	irrotational	solenoidal	conservative	solenoidal

______ gives the rate of outflow per unit volume at a point of the fluid.	curl V	div V	curl V=0	div V=0	div V
If div V=0 everywhere in some region R of space then V is called the _____ vector point function.	rotational	irrotational	solenoidal	conservative	solenoidal
_______ is a vector which measures the extent to which individual particles of the fluid are spnning or rotating.	curl V	div V	curl V=0	div V=0	curl V
div F is a ________ function.	point	vector	scalar	rotational	scalar
If curl V=0 then V is said to be an ___________.	rotational	irrotational	solenoidal	conservative	irrotational
If r=xI+yJ+zK then div r=________	0	1	2	3	3
If r=xI+yJ+zK then curl r=________	0	1	2	3	0
div (curl V)=	0	div V	curl V	V	0
curl (grad f)=	0	div V	curl V	f	0
Two surfaces are said to cut orthogonally at a point of intersection, if the respective normals at that point are ______.	parallel	perpendicular	equal	zero	perpendicular
Any integral which is to be evaluated over a surface is called a ___	Line integral	Volume integral	surface integral	closed integral	surface integral
When the circulation of F around every closed curve in a region vanishes, then F is said to be _______ in that region.	rotational	irrotational	solenoidal	conservative	irrotational
A force field F is said to be ____________ if it is derivable from a potential function f such that F = grad f.	rotational	irrotational	solenoidal	conservative	conservative
If F is ______ then cur F=0.	rotational	irrotational	solenoidal	conservative	conservative
If S has a unique normal at each of its points whose direction depends continuously on the point of S then the surface S is called a ______ surface.	Orientable	smooth	plane	twisted	smooth
If (3x-2y+z)I+(4x+ay-z)J+(x-y-2z)K is solenoidal then a=	0	1	-1	2	-1
If f=x+y+z-8 then grad f is ____	I+J+K	I+J-K	I-J+K	0	I+J+K

Unit - V

Jacobian Series.

The solution of the differential equation.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0.$$

is $$y = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}.$$

which is known as Bessel function.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \quad (\Gamma(n+1) = n!)$$

& $$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}.$$

(1) 1) Prove that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}$

Proof:-

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{x^n (-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(2n+2r)(-1)^r x^{2n+2r-1}}{\Gamma(r+1)\Gamma(n+r+1)\, 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^n \cdot x^{n+2r-1} (n+r)}{\Gamma(r+1)(n+r)!\, 2^{n+2r-1}} \quad \left[\because \Gamma(n+r+1) = (n+r)!\right]$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)(n+r-1)!} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n J_{n-1}(x)$$

(★)

(14) 2) Prove that $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Proof:-

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{x^{-n}(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot x^{2r}}{\Gamma(r+1)\Gamma(n+r+1)\, 2^{n+2r}}$$

$$\frac{d\left[x^{-n}J_n(x)\right]}{dx} = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2r\, x^{2r-1}}{\Gamma(r+1)\,\Gamma(n+r+1)\, 2^{n+2r}}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^{r-1} . 2r . x^{n+2r-1}}{\Gamma(r+1)\,\Gamma(n+r+1)\, 2^{n+2r}}$$

$$= -x^{-n} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}\, r\, x^{n+2r-1}}{r!\,\Gamma(n+r+1)\, 2^{n+2r-1}}$$

$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{n+2k+1}}{k!\,\Gamma(n+k+2)\, 2^{n+2k+1}}$$

Put $r-1=k$

$2r-1 = 2k+2-1 = 2k+1$

$$\frac{d\left[x^{-n}J_n(x)\right]}{dx} = -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\,\Gamma(n+k+2)} \left(\frac{x}{2}\right)^{n+2r+1}$$

$$= -x^{-n} J_{n+1}(x).$$

Two mark — 2, 3, 4 Ref with another 2 mark

1. $xJ_n' = xJ_{n-1} - nJ_n$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}$$

$$x^n J_n' + n x^{n-1} J_n = x^n J_{n-1}$$

Dividing by x^n.

$$J_n' + \frac{n}{x} J_n = J_{n-1}$$

$$xJ_n' = xJ_{n-1} - nJ_n.$$

(14)

(3). Prove that the coefficient of z^n in the expansion of $e^{\frac{x}{2}(z-\frac{1}{z})}$ is $J_n(x)$. (X)

Proof:-

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = 1+\frac{x}{2}\left(z-\frac{1}{z}\right)+\frac{x^2}{4.2!}\left(z-\frac{1}{z}\right)^2+\frac{x^3}{8.3!}\left(z-\frac{1}{z}\right)^3+\dots$$

$$+\frac{x^n}{2^n.n!}\left(z-\frac{1}{z}\right)^n$$

$$= 1+\frac{x}{2}\left(z-\frac{1}{z}\right)+\frac{x^2}{4.2!}\left(z^2-2+\frac{1}{z^2}\right)+\frac{x^3}{8.3!}\left(z^3-3z+\frac{3}{z}+\frac{1}{z^3}\right)$$

$$+\dots+\frac{x^n}{2^n.n!}\left(z^n-nz^{n-2}+\frac{(n)(n-1)}{2}.z^{n-2}.\frac{1}{z^2}+\dots-\frac{1}{z^n}\right)$$

$$= 1-\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^4\frac{1}{(2!)^2}-\left(\frac{x}{2}\right)^6.\frac{1}{(3!)^2}+\dots$$

$$+\left(\frac{x}{2}-\left(\frac{x}{2}\right)^3.\frac{1}{2!}+\left(\frac{x}{2}\right)^5.\frac{1}{2!}.\frac{1}{3!}+\dots\right)z$$

$$\vdots \quad \vdots$$

$$+z^n\left(\left(\frac{x}{2}\right)^n\frac{1}{n!}-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4}+\dots\right]+\dots$$

The coefficient of z^n is

$$= \left(\frac{x}{2}\right)^n\left[\frac{1}{n!}-\left(\frac{x}{2}\right)^2\frac{1}{(n+1)!}+\left(\frac{x}{2}\right)^4\frac{1}{2!\,(n+2)!}+\dots\right]$$

$$= \left(\frac{x}{2}\right)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{(n+r)!\,r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{0}^{\infty}\frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= J_n(x)$$

1) Prove that $J_n(-x) = (-1)^n J_n(x)$

soln

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(-\frac{x}{2}\right)^{n+2r}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= (-1)^n J_n(x).$$

2) Prove that $J_{-n}(x) = (-1)^n J_n(x)$

soln

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}.$$

$$\left[\because \ \Gamma(-n) = \infty \text{ if } n>0 \right.$$
$$\left. = 0 \text{ if } -n+r+1<1 \text{ (ie) } r<n \right]$$

Put $r = n+k$

$-n+2r = k+r$
$= n+2k.$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{\Gamma(n+k+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n J_n(x).$$

3. Find $J_0(x)$

Soln $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{n+2r}$

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(r+1)} \left(\frac{x}{2}\right)^{2r}$$

$$= 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{(2!)^2} - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots$$

4) Find $J_1(x)$

Soln $J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(r+2)} \left(\frac{x}{2}\right)^{1+2r}$

$$= \frac{x}{2} - \left(\frac{x}{2}\right)^3 \frac{1}{2!} + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^7 + \ldots$$

5) Express J_2 in terms of J_0 and J_1

Soln

The recurrence relation is

$$2n J_n = x\left[J_{n+1} + J_{n-1}\right]$$

$n=1$ $\quad 2J_1 = x\left[J_2 + J_0\right]$

$$x J_2 = 2J_1 - xJ_0$$

$$J_2 = \frac{2}{x} J_1 - J_0$$

6) Prove that $xJ_n' = nJ_n - xJ_{n+1}$

Soln

$$x^{-n} J_n' - n x^{-n-1} J_n = -x^{-n} J_{n+1}$$

$$J_n' - \frac{n}{x} J_n = -J_{n+1} \qquad \times x^n$$

$$x J_n' = nJ_n - xJ_{n+1}$$

w.k.t

$$x J_n' = x J_{n-1} - n J_n \quad \text{— (1)}$$

$$+ x J_n' = n J_n - x J_{n+1} \quad \text{— (2)}$$

$$\Rightarrow \quad x J_n' = \frac{1}{2}\left[x J_{n-1} - x J_{n+1}\right]$$

$$J_n' = \frac{1}{2}\left[J_{n-1} - J_{n+1}\right]$$

ii) Prove that $2n J_n = x\left[J_{n+1} + J_{n-1}\right]$

(1) - (2) $\Rightarrow$ $2n J_n - x J_{n+1} - x J_{n-1} = 0$

$$2n J_n = x J_{n+1} + x J_{n-1}$$

$$= x\left[J_{n+1} + J_{n-1}\right].$$

Prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

Soln

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Substitute $n = -1/2$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma\left(-\frac{1}{2}+r+1\right)} \left(\frac{x}{2}\right)^{-\frac{1}{2}+2r}$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{2}{x}} - \sqrt{\frac{2}{x}}\left(\frac{x}{2}\right)^2 \frac{1}{\Gamma\left(\frac{3}{2}\right)} + \sqrt{\frac{2}{x}}\left(\frac{x}{2}\right)^4 \frac{1}{2!\,\Gamma\left(\frac{5}{2}\right)}$$

$$- \sqrt{\frac{2}{x}}\left(\frac{x}{2}\right)^6 \frac{1}{3!\,\Gamma\left(\frac{7}{2}\right)} + \cdots$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{x^2}{4\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + \frac{x^4}{16\cdot 2!\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{x^6}{2^6\cdot 3!\,\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + \cdots\right]$$

Scanned by CamScanner

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cancel{+} \frac{x^6}{6!} + \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x \qquad \left[\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \cos x\right]$$

Note:- $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

Prove that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ X

Soln

w.k.t

$$2n J_n = x\left[J_{n+1} + J_{n-1}\right]$$

Substitute $n = 1/2$.

$$J_{1/2} = x\left[J_{3/2} + J_{-1/2}\right]$$

$$J_{3/2} = \frac{1}{x}\left[J_{1/2} - J_{-1/2}\right]$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x\Big)$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

Soln:-

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma\left(r+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2r}$$

$$= \left(\frac{x}{2}\right)^{1/2}\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \left(\frac{x}{2}\right)^{5/2}\frac{1}{\Gamma\left(\frac{5}{2}\right)} + \left(\frac{x}{2}\right)^{9/2}\frac{1}{2!\,\Gamma\left(\frac{7}{2}\right)} + \cdots$$

$$=$$

Scanned by CamScanner

$$= \frac{\sqrt{x}}{\sqrt{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{x^{5/2}}{2^2 \cdot \sqrt{2}\,\frac{3}{2}} + \frac{x^{9/2}}{2 \cdot 2^4 \cdot \sqrt{2}\,\frac{5}{2}\cdot\frac{3}{2}\frac{1}{2}\Gamma(}$$

$$- \frac{x^{11/2}}{3! \cdot 2^6 \sqrt{2}\,\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + \cdots \quad (\Gamma(n+1) = n\Gamma(n$$

Dividing & multiplying by $\sqrt{x}$.

$$= \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\ \sin x \quad \left(\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

Prove that $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1$

Soln w.k.t

$$\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right)$$

Substituting $n = 0, 1, 2, 3, \ldots$

$$\frac{d}{dx}\left(J_0^2 + J_1^2\right) = 2\left(0 - \frac{J_1^2}{x}\right)$$

$$\frac{d}{dx}\left(J_1^2 + J_2^2\right) = 2\left(\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right)$$

$$\frac{d}{dx}\left(J_2^2 + J_3^2\right) = 2\left(\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right)$$

$$\frac{d}{dx} \quad \vdots$$

Adding

$$\frac{d}{dx}\left[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)\right] = 0$$

Integrating,

$$J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = C$$

$$J_0(0) = 1, \quad J_1(0) = J_2(0) = \cdots = 0$$

Substituting $x = 0$

$$C = 1$$

$$J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1.$$

Scanned by CamScanner

Prove that $J_2 = J_0'' - \frac{J_0'}{x}$.

Soln

Recurrence formula

$$x J_n' = n J_n - x J_{n+1}$$

Substitute $n=0$

$$\cancel{x J_n' = n} \quad J_0' = -J_1 \quad — (1)$$

$n=1$

$$x J_1' = J_1 - x J_2 \quad — (2)$$

Diff (1) $\Rightarrow$ $J_0'' = J_1'$ — (3)

Sub from (3) & (1) in (2)

$$-x J_0'' = -J_0' - x J_2$$

$$J_2 = J_0'' - \frac{J_0'}{x}.$$

State & Prove Jacobi series.

1) Prove that

$$\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + 2J_4 \cos 2\theta + \cdots$$

$$\sin(x \sin\theta) = 2J_1 \sin\theta + 2J_3 \sin 3\theta + \cdots$$

Soln:-

The Generating function for $J_n(x)$ is

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0 + zJ_1 + z^2 J_2 + z^3 J_3 + \cdots + \frac{J_{-1}}{z} + \frac{J_{-2}}{z^2} + \frac{J_{-3}}{z^3}$$

Put $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$

$$e^{\frac{x}{2}\left(e^{i\theta} - e^{-i\theta}\right)} = J_0 + e^{i\theta} J_1 + e^{2i\theta} J_2 + e^{3i\theta} J_3 + \cdots$$

$e^{i\theta} = \cos\theta + \sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$.

Scanned by CamScanner

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$e^{x i\sin\theta} = J_0 + (\cos\theta + i\sin\theta)J_1 + (\cos 2\theta + i\sin 2\theta)J_2$$
$$+ (\cos 3\theta + i\sin 3\theta)J_3 + \cdots$$
$$+ J_{-1}(\cos\theta - i\sin\theta) + J_{-2}(\cos 2\theta - i\sin 2\theta)$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta)$$
$$= J_0 + J_1\cos\theta + J_2\cos 2\theta + \cdots$$
$$- J_1\cos\theta + J_2\cos 2\theta - J_3\cos 3\theta + \cdots$$
$$+ i(J_1\sin\theta + J_2\sin 2\theta + \cdots$$
$$+ J_1\sin\theta - J_2\sin 2\theta + J_3\sin 3\theta + \cdots)$$

$J_{-n} = (-1)^n J_n$.

Comparing real and imaginary parts.

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots$$
$$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + \cdots$$

These are called Jacobi series.

Prove that (i) $x\sin x = 2[2^2J_2 - 4^2J_4 + 6^2J_6 - \cdots]$

(ii) $x\cos x = 2[J_1 - 3^2J_3 + 5^2J_5 - \cdots]$

Soln :-

w.k.t

$$\cos x(\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots \quad (1)$$
$$\sin x(\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta + \cdots \quad (2)$$

Diff w.r.to θ

$$-\sin(x\sin\theta)\,x\cos\theta = -4J_2\sin 2\theta - 8J_4\sin 4\theta + \cdots$$

Again Diff w.r.to θ

$$-\cos(x\sin\theta)\,x^2\cos^2\theta + x\sin\theta.\sin(x\sin\theta)$$
$$= -8J_2\cos 2\theta - 32J_4\cos 4\theta + \cdots$$

Scanned by CamScanner

Put $\theta = \pi/2$

$$\sin x = 8J_2 - 32J_4 + \cdots$$

$$= 2[2^2 J_2 - 4^2 J_4 + \cdots]$$

Diff ② w.r.to θ

$$\cos(x\sin\theta)\, x\cos\theta = 2J_1\cos\theta + 2.3J_3\cos 3\theta + 2.5J_5\cos 5\theta$$

Again diff w.r.to θ

$$-\sin(x\sin\theta)x^2\cos^2\theta - x\sin\theta\cos(x\sin\theta)$$
$$= -2J_1\sin\theta - 2.3^2J_3\sin 3\theta - 2.5^2 J_5\sin 5\theta$$

Put $\theta = \pi/2$

$$-x\cos x = -2J_1 + 2.3^2J_3 - 2.5^2J_5$$
$$x\cos x = 2[J_1 - 3^2J_3 + 5^2J_5 - \cdots]$$

Using Jacobi series prove that

$$[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots = 1$$

Soln :-

Jacobi series is

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots \quad ①$$
$$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta + \cdots \quad ②$$

Squaring and integrating w.r.to θ between 0 and π on both sides of ① & ②

$$\int_0^\pi \cos^2(x\sin\theta)d\theta = \int_0^\pi J_0^2 d\theta + \int_0^\pi 4J_2^2\cos^2 2\theta\, d\theta$$
$$+ \int_0^\pi 4J_4^2\cos^2 4\theta\, d\theta$$
$$+ \cdots + 2\int_0^\pi J_0J_2\cos 2\theta\, d\theta + 4\int_0^\pi J_2J_4\cos 2\theta\cos 4\theta\, d\theta$$

$$\int_0^\pi \cos n\theta \cos m\theta \, d\theta = 0$$

$$\int_0^\pi \cos^2(x\sin\theta)\,d\theta = J_0^2\pi + 4J_2^2\frac{\pi}{2} + 4J_4^2\frac{\pi}{2} + \cdots$$

$$= \frac{\pi}{2}$$

$$= \pi\left[J_0^2 + 2J_2^2 + 2J_4^2 + \cdots\right] \quad (3)$$

$$\left(\because \int_0^\pi \cos^2 n\theta\,d\theta = \frac{\pi}{2}\right)$$

$$\int_0^\pi \sin^2(x\sin\theta)\,d\theta = 4\int_0^\pi J_1^2\sin^2\theta\,d\theta + 4\int_0^\pi J_3^2\sin^2 3\theta\,d\theta$$

$$+ \cdots + \int_0^\pi 4J_1J_3\sin\theta\sin 3\theta\,d\theta + \cdots$$

$$\frac{\pi}{2} = 4J_1^2\frac{\pi}{2} + 4J_3^2\frac{\pi}{2} + 4J_5^2\frac{\pi}{2} + \cdots \quad (4)$$

$$\int_0^\pi \sin^2 n\theta\,d\theta = \frac{\pi}{2}$$

$$\int_0^\pi \sin n\theta \sin m\theta\,d\theta = 0 \quad (\text{if } n \neq m).$$

From (3) & (4)

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi\left[J_0^2 + 2J_2^2 + 2J_4^2 + 2J_1^2 + 2J_3^2 + 2J_5^2 + \cdots\right]$$

$$\Rightarrow \boxed{J_0}[x]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots = 1,$$

Prove that $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$.

Soln :-

$\cos(x\sin\theta) = J_0(x) + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots$ (1)

$\sin(x\sin\theta) = 2J\sin\theta + 2J_3\sin 3\theta + \cdots$ (2)

Multiply (1) by $\cos n\theta$ and integrating w.r.to θ between 0 and π.

$$\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta$$

$$= \int_0^\pi J_0 \cos n\theta\, d\theta + \int_0^\pi 2J_2 \cos 2\theta \cos n\theta\, d\theta + \int_0^\pi 2J_4 \cos 4\theta \cos n\theta\, d\theta$$

$$= \int_0^\pi 2J_{2m} \cos^2 2m\theta\, d\theta \qquad (\because n = 2m)$$

$$\left(\int_0^\pi \cos n\theta \cos m\theta\, d\theta = 0, \ m \neq n\right)$$

$$= \cancel{2m}\, J_{2m}\pi$$

$= J_n \pi$ when n is even.

Multiply ② by $\sin n\theta$ and integrating w.r.to θ from 0 to π.

$$\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta$$

$$= 2\int_0^\pi J_1 \sin\theta \sin n\theta\, d\theta + \int_0^\pi 2J_3 \sin 3\theta \sin n\theta\, d\theta$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \pi J_n & \text{when } n \text{ is odd.} \end{cases}$$

$$\left(\because \int_0^\pi \sin n\theta \sin m\theta\, d\theta = 0 \text{ if } n \neq m\right)$$

$$\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta)\, d\theta = \pi J_n$$

$$\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta = \pi J_n(x)$$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi ((\cos n\theta - x\sin\theta)\, d\theta.$$

Prove that $\frac{d}{dx}\left[x J_n J_{n+1}(x)\right] = x\left[J_n^2 - J_{n+1}^2\right]$

Soln

$$\frac{d}{dx}(x J_n J_{n+1}) = J_n . J_{n+1} + x J_n' J_{n+1} + x J_{n+1}' J_n \quad \text{--- (1)}$$

W.K.T

$$x J_n' = x J_{n-1} - n J_n$$

Replacing n by $n+1$

$$x J_{n+1}' = x J_n - (n+1) J_{n+1} \quad \text{--- (2)}$$

$$x J_n' = n J_n - x J_{n+1} \quad \text{--- (3)}$$

Sub (2) & (3) in (1)

$$\frac{d}{dx}(x J_n J_{n+1}) = J_n . J_{n+1} + J_{n+1}(n J_n - x J_{n+1}) + J_n(x J_n - (n+1) J_{n+1})$$

$$= J_n J_{n+1} + n J_n J_{n+1} - x J_{n+1}^2 + x J_n^2 - n J_n J_{n+1} - J_n J_{n+1}$$

$$= x(J_n^2 - J_{n+1}^2).$$

Scanned by CamScanner

Questions	opt 1	opt 2	opt 3	opt 4	opt5	opt6	Answer
A sequence {2^n} is	Convergent	divergent	Oscillatory	unique			divergent
A sequence (-1)^n+2 is	Convergent	divergent	Oscillatory	unique			Oscillatory
A sequence {2n+1/3n-2} is	Convergent	divergent	Oscillatory	unique			Convergent
A sequence {2n^2+n/3n^2-3} is	Convergent	divergent	Oscillatory	unique			Convergent
The series ∑ cos(1/n) is	Convergent	divergent	Oscillatory	unique			Convergent
The series ∑ x^n/(n^3+1) at x=1 is	Convergent	divergent	divergent	Not unique			Convergent
The series 1-(1/2^2)+(1/3^2)-(1-4^2)+… is	Convergent	divergent	Oscillatory	Not unique			Convergent
The series 1-2x+3x^2-4x^3+... where 0<x<1 is	Convergent	divergent	divergent	unique			Convergent
The series whose nth term is ∑ sin (1/n) is	Convergent	divergent	divergent	Not unique			Convergent
An ordered set of real number a_1,a_2,…a_n is called a__________	Series	sequence	Montonic sequence	Montonic sequence			sequence
If a sequence has a _________,it is called a convergent sequence	Finite limit	Infinite limit	limit	Bounded			Finite limit
A sequence is said to be bounded above if there exists a number k, such that _____ for every n.	a_n>k	a_n≥k	a_n≤k	a_n<k			a_n≤k
Both increasing and decreasing sequence are called ______ sequence.	Convergent	Montonic	Bounded	divergent			Montonic
If limit n tends to ∞ a_n is equal to _______then the sequence is said to be Convergent	finite and unique	Infinite	unique	not unique			finite and unique
If $u_1, u_2, \ldots u_n, \ldots$ be an infinite sequence or real numbers,then u1+u2+….+un+…is called__________	infinite series	finite series	finite terms	infinite terms			infinite series
The series 1+2+3+ +n+…+...∞ is	Convergent	divergent	Oscillatory	not unique			divergent
Every absolutely convergent series is a ______ series	Convergent	divergent	Oscillatory	not unique			Convergent
Any convergent series of ________ terms is also absolutely convergent	negative	positive	zero	unique			positive
If limit n tends to infinite u_n/u_n+1 = m is a series of positive terms ∑ u_n is convergent if _____	m>0	m<1	m>1	m=1			m>1
If limit n tends to ∞ u_n/u_n+1 = m is a series of positive terms ∑ u_n is divergent if _____	m>0	m<1	m>1	m=1			m<1
If limit n tends to ∞ u_n/u_n+1 = m is a series of positive terms .when the ratio test fails	m>0	m<1	m>1	m=1			m=1
Which of the following functions has the period 2π?	cos x	sin nx	tan nx	tan x			cos x

If a function satisfies the condition f(-x) = f(x) then which is true?	$a_0 = 0$	$a_n = 0$	$a_0 = a_n = 0$	$b_n = 0$	$b_n = 0$
If a function satisfies the condition f(-x) = -f(x) then which is true?	$a_0 = 0$	$a_n = 0$	$a_0 = a_n = 0$	$b_n = 0$	$a_0 = a_n = 0$
Which of the following is an odd function?	sin x	cos x	$x^{\wedge 2}$	x^4	sin x
Which of the following is an even function?	$x^{\wedge 3}$	cos x	sin x	$\sin^{\wedge 2} x$	cos x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	f(-x) = f(-x)	f(-x) = - f(x)
The function f(x) is said to be an even function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	f(-x) = f(-x)	f(-x) = f(x)
∫f(x) dx = 2∫f(x) dx between the limits -a to a if f(x) is ------	even	continuous	odd	discontinuous	even
∫f(x) dx = 0 between the limits -a to a if f(x) is ------	even	continuous	odd	discontinuous	odd
If a periodic function f(x) is odd, it's Fourier expansion contains no ------ terms.	coefficient a_n	sine	coefficient a_0	cosine	cosine
If a periodic function f(x) is even, it's Fourier expansion contains no ------ terms.	cosine	sine	coefficient a_0	coefficient a_n	sine
In dirichlet condition, the function f(x) has only a ----- number of maxima and minima.	uncountable	continuous	infinite	finite	finite
In Fourier series, the function f(x) has only a finite number of maxima and minima. This condition is known as -------	Dirichlet	Kuhn Tucker	Laplace	Cauchy	Dirichlet
In dirichlet condition, the function f(x) has only a ----- number of discontinuities .	uncountable	continuous	infinite	finite	finite
The Fourier series of f(x) is given by ----	$a_0/2 + \sum (a_n \cos nx + bn \sin nx)$	$a_0/2 + \sum (a_n \cos nx - bn \sin nx)$	$a_n/2 + \sum (a_n \sin nx + bn \sin nx)$	$a_0/2 + \sum (a_0 \sin n\pi x/ l)$	$a_0/2 + \sum (a_n \cos nx + bn \sin nx)$

In Fourier series, the expansion $f(x) = a_0/2 + \sum (a_n \cos nx + b_n \sin nx)$ is possible only if in the interval $c_1 \leq x \leq c_2$ the function f(x) satisfies ---condition.	kuhn-Tucker	Laplace	Dirichlet	Cauchy	Dirichlet
If the periodic function f(x) is even, then the Fourier expansion is of the form ---	$a_0/2 + \sum a_n \sin(n\pi x/ l)$	$a_0/2 + \sum a_n \cos(n\pi x/ l)$	$a_n/2 + \sum a_n \cos(n\pi x/ l)$	$a_0/2 + \sum a_0 \sin(n\pi x/ l)$	$a_0/2 + \sum a_n \cos(n\pi x/ l)$
If the periodic function f(x) is even, then it's Fourier co-efficient an is of the form ---	$2/ l \int f(x) \sin(n\pi x/ l)\, dx$	$2/ l \int f(x) \cos(n\pi x/ l)\, dx$	$1/ l \int f(x) / l\, dx$	$\int f(x)\, dx$	$2/ l \int f(x) \cos(n\pi x/ l)\, dx$
If the periodic function f(x) is even, then it's Fourier co-efficient a_0 is of the form ---	$2/ l \int f(x)\, dx$	$1/ l \int f(x)\, dx$	$2/ l \int f(x)/ 1\, dx$	$\int f(x)\, dx$	$2/ l \int f(x)\, dx$
If the periodic function f(x) is odd, then it's Fourier co-efficient bn is of the form ---	$2/ l \int f(x) \cos(n\pi x/ l)\, dx$	$2/ l \int f(x) \sin(n\pi x/ l)\, dx$	$\int f(x)\, dx$	$1/ l \int f(x) / l\, dx$	$2/ l \int f(x) \sin(n\pi x/ l)\, dx$
If the periodic function f(x) is even, then it's Fourier co-efficient -------- is zero.	a_0	a_1	b_n	a_0 & a_n	b_n
If the periodic function f(x) is odd, then it's Fourier co-efficient -------- is zero.	a_0 & a_n	a_1	b_n	b_1	a_0 & a_n
If the periodic function f(x) is even, then the Fourier expansion is of the form ---	$\sum b_n \sin n\pi x/ l$	$\sum b_n \sin n\pi x/ l$	$\sum b_n \cos n\pi x/ l$	$a_0/2+\sum a_n \cos (n\pi x/ l)$	$a_0/2+\sum a_n \cos (n\pi x/ l)$
If the periodic function f(x) is odd, then the Fourier expansion is of the form ---	$\sum b_n \sin n\pi x/ l$	$\sum a_n \sin n\pi x/ l$	$\sum b_n \cos n\pi x/ l$	$\sum a_n \cos n\pi x/ l$	$\sum b_n \sin n\pi x/ l$
$1/\pi \int f(x) \cos nx\, dx$ gives the Fouier coefficient -----------	a_0	b_1	b_n	a_n	a_n
$1/\pi \int f(x)\, dx$ gives the Fourier coefficient	a_0	a_n	b_n	b_1	a_0
$1/\pi \int f(x) \sin nx\, dx$ gives the Fouier coefficient -----------	a_0	a_n	b_n	b_1	b_n