17BECC202, 17BTAR202, 17BTCE202

ENGINEERING MATHEMATICS II **3 2 0 4**

OBJECTIVES:

1. To have knowledge in integral calculus and Vector calculus
2. To expose the concept of Analytical function and Complex integration.

INTENDED OUTCOMES:

The student will be able to

1. Solve problems in Fluid Dynamics, Theory of Elasticity, Heat and Mass Transfer etc.
2. Find the areas and volumes using Multiple Integrals
3. Improve their ability in Vector calculus
4. Expose to the concept of Analytical function.
5. Apply Complex integration in their Engineering problems

UNIT I INTEGRAL CALCULUS (12)

Definite and indefinite integrals – Techniques of integration – Substitution rule, Trigonometric integrals, Integration by parts , Integration of rational functions by partial fraction, Integration of irrational functions – Improper Integrals.

UNIT II MULTIPLE INTEGRALS (12)

Double integral – Cartesian coordinates – Polar coordinates – Area as double integrals- Change the order of integration – Triple integration in Cartesian co-ordinates.

UNIT III VECTOR INTEGRATION (12)

Integration of vectors – line integral- surface integral- volume integral- Green's theorem - Gauss divergence theorem and Stoke's theorems (Statement Only), hemisphere and rectangular parallelopipeds problems.

UNIT IV ANALYTIC FUNCTIONS (12)

Analytic functions - Cauchy-Riemann equations in Cartesian and polar forms – Sufficient condition for an analytic function (Statement Only) - Properties of analytic functions – Constructions of an analytic function - Conformal mapping: w = z+a, az, 1/z and bilinear transformation.

UNIT V COMPLEX INTEGRATION (12)

Complex Integration - Cauchy's integral theorem and integral formula (Statement Only) – Taylor series and Laurent series - Residues – Cauchy's residue theorem (Statement Only) - Applications of Residue theorem to evaluate real integrals around unit circle and semi-circle (excluding poles on the real axis).

Total: 60

TEXT BOOKS:

S.NO.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAR OF PUBLICATION
1	Hemamalini. P.T	Engineering Mathematics I & II	McGraw-Hill Education Pvt.Ltd, New Delhi	2017
2	Grewal, B.S.	Higher Engineering Mathematics	Khanna Publishers, Delhi.	2014

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3	Narayanan. S, Manicavachagam pillay.T.K and Ramaniah.G	Advanced Mathematics for Engineering Students.	Viswanathan S.(Printers and Publishers) Pvt. Ltd. Chennai.	2002
4	Michael D. Greenberg	Advanced Engineering Mathematics	Pearson Education, India	2009

WEBSITES:

1.www.efunda.com
2. www.mathcentre.ac.uk
3. www.sosmath.com/diffeq/laplace/basic/basic.html
4. www.mathworld.wolframe.com

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established under Section 3 of UGC Act 1956)
Eachanari, Coimbatore-641 021. INDIA

First Year B.E - Second Semester

Engineering Mathematics – II
(17BECC202, 17BTCE202, 17BTAR202)
Lecture Plan

S.No	Topic covered	No. of hours	Supporting Material
	UNIT I : INTEGRAL CALCULUS		
1	Basics - Integration	1	T1:16.5-16.8
2	Definite and indefinite integrals	1	T1:16.9-16.13
3	Basic Problems	1	T1:16.13-16.15
4	Techniques of integration	1	T1:17.1-17.2
5	Substitution rule	1	T1:17.2-17.4
6	Tutorial 1- Definite and indefinite integrals, Substitution rule	1	
7	Trigonometric integrals	1	T1:17.10-17.14
8	Integration by parts	1	T1:17.5-17.10
9	Integration of rational functions by partial fraction	1	T1:17.14-17.18
10	Integration of irrational functions	1	T1:17.18-17.24
11	Improper Integrals	1	T1:17.24-17.27
12	Tutorial 2 - Techniques of integration	1	
	Total	12	
	UNIT II : MULTIPLE INTEGRALS		
13	Integration – Basic Problems	1	T1:18.3-18.4
14	Double integral	1	T2:294-295
15	Double integral - Problems	1	T1:18.3-18.4
16	Problems in Cartesian coordinates	1	T1:18.5-18.8
17	Problems in Polar coordinates	1	T1:18.5-18.8
18	Area as double integrals	1	T1:18.7-18.8
19	Tutorial 3 - Double integral, Area as double integrals	1	
20	Change the order of integration	1	T1:19.1-19.4
21	Change the order of integration	1	T2:297-302
22	Triple integration in Cartesian co-ordinates	1	T1:19.4-19.7
23	Triple integration in Cartesian co-ordinates	1	T2:305-310
24	Tutorial 4 - Change the order of integration, Triple integration in Cartesian co-ordinates	1	
		12	
	UNIT III : VECTOR INTEGRATION		
25	Integration of vectors	1	T1:20.3-20.4
26	Line integral problems	1	T1:20.5-20.8
27	Surface integral problems	1	T1:20.3-20.4
28	Volume integral problems	1	T1:20.3-20.4
29	Green's theorem problems	1	T1:20.9-20.28
30	Green's theorem problems	1	R:485-490
31	Tutorial 5 – Line, Surface and Volume integral problems	1	
32	Gauss divergence theorem problems	1	T1:20.9-20.28
33	Gauss divergence theorem problems	1	T2:376-381
34	Stoke's theorems problems	1	T1:20.9-20.28
35	Stoke's theorems problems	1	T2:372-375
36	Tutorial 6 – Gauss Divergence theorem and Stoke's theorem problems	1	
	Total	12	

	UNIT IV : ANALYTIC FUNCTIONS		
37	Introduction – Analytic Function	1	T1:21.3-21.8
38	Necessary and Sufficient conditions for an analytic function	1	T1:21.9
39	Cauchy-Riemann equations –Cartesian form	1	R:740-744
40	Cauchy-Riemann equations – Polar form	1	R:740-744
41	Cauchy-Riemann equations – Properties	1	T1:21.9-21.12
42	Cauchy-Riemann equations – Problems based on Properties	1	T1:21.13-21.22
43	Construction of an Analytic Function - Problems	1	T2:745-747
44	Tutorial 7 - Cauchy-Riemann equations , Construction of an Analytic Function	1	
45	Conformal mapping: w = z+a, az	1	T1:22.1-22.12
46	Conformal mapping: w = 1/z	1	T1:22.1-22.12
47	Bilinear transformation – Problems	1	T2:756-762
48	Tutorial 8 - Conformal mapping, Bilinear transformation	1	
	Total	12	
	UNIT V: COMPLEX INTEGRATION		
49	Introduction - Complex Integration	1	T1:23.1-23.5
50	Problems solving using Cauchy's integral theorem	1	T1:23.6-23.10
51	Problems solving using Cauchy's integral formula	1	T2:765-769
52	Tutorial 10 - Problems solving using Cauchy's integral theorem and integral formula	1	
53	Taylor and Laurent expansions	1	T1:24.1-24.11
54	Taylor Series and Laurent Series Problems	1	T2:771-776
55	Tutorial 11 - Taylor and Laurent expansions	1	
56	Theory of Residues	1	T1:25.1-25.3
57	Cauchy's residue theorem	1	T1:25.3-25.13
58	Applications of Residue theorem to evaluate Unit circle	1	T1:25.3-25.13
59	Applications of Residue theorem to evaluate semi – circle.	1	T2:776-723
60	Tutorial 12 - Cauchy's residue theorem, Applications	1	
	Total	12	
	TOTAL	**50+10=60**	

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SEMESTER II

17BECC202, 17BTAR202, 17BTCE202

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Unit IV

Pre Requisite

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$.

2. $\int e^{ax} dx = \frac{e^{ax}}{a}$

3. $\int \cos x \, dx = \sin x$.

4. $\int \sin x \, dx = -\cos x$.

5. $\int \cos ax \, dx = \frac{\sin ax}{a}$.

6. $\int \sin ax \, dx = \frac{-\cos ax}{a}$.

7. $\int \frac{dx}{x} = \log x$.

Problems:

1. Evaluate $\int_c x^2 dy + y^2 dx$ where c is the path $y=x$ from $(0,0)$ to $(1,1)$.

Sol. Given $y=x$.

$dy = dx$.

x varies from 0 to 1.

$$\int_c x^2 dy + y^2 dx = \int_0^1 x^2 dx + x^2 dx.$$

$$= \int_0^1 2x^2 dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{3} - 0\right)$$

$$= \frac{2}{3}$$

2. Evaluate $\int_C (3xy^2+y^3)\,dx + (x^3+3xy^2)\,dy$ where C is the parabola $y^2 = 4ax$ from $(0,0)$ to $(a,2a)$.

Sol: Given $y^2 = 4ax \Rightarrow x = y^2/4a$.

$$dx = \frac{2y}{4a}\,dy = \frac{y}{2a}\,dy.$$

y varies from 0 to $2a$.

$$\int_C (3xy^2+y^3)\,dx + (x^3+3xy^2)\,dy.$$

$$= \int_0^{2a} \left(3\frac{y^2}{4a}y^2 + y^3\right)\frac{y}{2a}\,dy + \left(\frac{y^2}{4a}\right)^3 + 3\left(\frac{y}{4a}\right)^2 y^2\,dy.$$

$$= \int_0^{2a} \left(\frac{3}{8a^2}y^5 + \frac{y^4}{2a} + \frac{y^6}{64a^3} + \frac{3y^4}{4a}\right) dy$$

$$= \left[\frac{3}{8a^2}\times\frac{y^6}{6} + \frac{y^5}{10a} + \frac{y^7}{7}\frac{1}{64a^3} + \frac{3}{4a}\times\frac{y^5}{5}\right]_0^{2a}$$

$$= \frac{3}{8a^2}\cdot\frac{64a^6}{6} + \frac{1}{2a}\cdot\frac{32a^5}{5} + \frac{1}{64a^3}\cdot\frac{128a^7}{7} + \frac{3}{4a}\cdot\frac{32a^5}{5}$$

$$= 4a^4 + \frac{16a^4}{5} + \frac{2}{7}a^4 + \frac{24}{5}a^4.$$

$$= a^4\left[\frac{140+112+10+168}{35}\right]$$

$$= a^4\left[\frac{430}{35}\right]$$

$$= \frac{86}{7}a^4.$$

(3). Evaluate $\int_0^5\int_0^2 (x^2+y^2)\,dx\,dy$.

Sol: $\int_0^5\int_0^2 (x^2+y^2)\,dx\,dy = \int_0^5 \left[\frac{x^3}{3}+y^2x\right]_0^2 dy$.

$= \frac{x^3}{3}+y^2x$

$= \int_0^5 \left[\frac{8}{3}+2y^2\right] dy$

$= \left[\frac{8y}{3}+\frac{2y^3}{3}\right]_0^5$

$= \frac{40}{3}+\frac{250}{3} = \frac{290}{3}$

(4) Evaluate $\int_1^b\int_1^a \frac{dx\,dy}{xy}$.

$\int_1^b\int_1^a \frac{dx\,dy}{xy} = \left(\int_1^b \frac{dy}{y}\right)\left(\int_1^a \frac{dx}{x}\right)$

$= [\log y]_1^b\,[\log x]_1^a$

$= (\log b - \log 1)(\log a - \log 1)$

$= \log b \cdot \log a$.

(5). Evaluate $\int_0^1\int_x^{\sqrt{x}} xy(x+y)\,dx\,dy$.

$\int_0^1\int_x^{\sqrt{x}} xy(x+y)\,dx\,dy = \int_0^1\int_x^{\sqrt{x}} x^2y+xy^2\,dx\,dy$

$= \int_0^1 \left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_x^{\sqrt{x}} dx$.

$= \int_0^1 \left(\frac{x^3}{2}+\frac{x^{5/2}}{3}-\frac{x^4}{2}-\frac{x^4}{3}\right) dx$.

$$= \left[\frac{1}{2} \cdot \frac{x^4}{4} + \frac{1}{3} \cdot \frac{x^{7/2}}{7/2} - \frac{x^5}{10} - \frac{x^5}{15} \right]_0^1$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{10} - \frac{1}{15}$$

$$= 3/56.$$

(6). Evaluate $\int_0^1 \int_0^x e^{y/x}\, dy\, dx$.

$$\int_0^1 \int_0^x e^{y/x}\, dy\, dx = \int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^x dx.$$

$$\int_0^1 \int_0^x e^{y/x}\, dy\, dx = \int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^x dx.$$

$$= \int_0^1 \left[\frac{e}{1/x} - \frac{1}{1/x} \right] dx.$$

$$= \int_0^1 x(e-1)\, dx$$

$$= (e-1) \left[\frac{x^2}{2} \right]_0^1$$

$$= (e-1) \left[\frac{1}{2} - 0 \right]$$

$$= \frac{e-1}{2}.$$

(7) Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta$.

Sol. $\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{\sin\theta} d\theta$

$$= \int_0^{\pi} \frac{\sin^2\theta}{2}\, d\theta = \frac{1}{2} \int_0^{\pi} \frac{(1-\cos 2\theta)}{2}\, d\theta.$$

$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$

$= \frac{1}{4}\left[\pi - \frac{\sin 2\pi}{2} - (0+0)\right]$

$= \pi/4.$

8\. Evaluate $\int_0^2 \int_0^{\pi} r \sin^2\theta \, d\theta \, dr.$

Sol.

$\int_0^2 \int_0^{\pi} r \sin^2\theta \, d\theta \, dr = \int_0^2 \int_0^{\pi} r \left[\frac{1-\cos 2\theta}{2}\right] d\theta \, dr.$

$= \int_0^2 \frac{r}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} dr.$

$= \int_0^2 \frac{r}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right] dr.$

$= \int_0^2 \frac{r}{2} \pi \, dr.$

$= \frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^2$

$= \frac{\pi}{2}\left(\frac{4}{2} - 0\right)$

$= \frac{\pi}{4} \times 4$

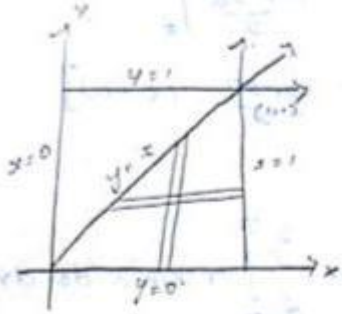
$= \pi.$

Change of order of Integration.

① change the order of integration for the double in

$\int_0^1 \int_0^x f(x,y)\,dx\,dy.$ dydx

Sol. $\int_0^1 \int_0^x f(x,y)\, dy\, dx$.



$y = x$

x	0	1	2	3
y	0	1	2	3

Given $y=0 \quad y=x$

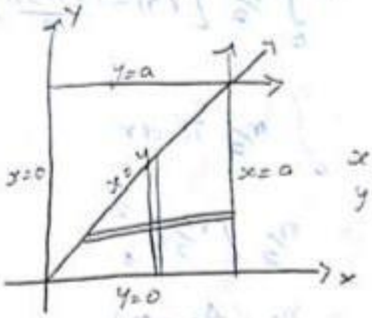
$x=0 \quad x=1$

$\int_0^1 \int_0^x f(x,y)\, dy\, dx$

$= \int_0^1 \int_y^1 f(x,y)\, dx\, dy$.

2. Change the order of integration for the double integral $\int_0^a \int_y^a f(x,y)\, dx\, dy$.

Sol. Given $\int_0^a \int_a^y f(x,y)\, dx\, dy$



x	0	1	2	3
y	0	1	2	3

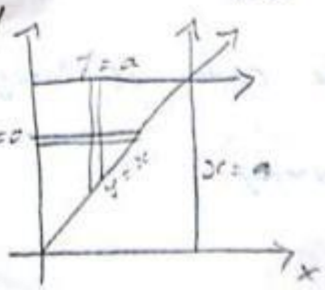
$x = y \quad x = a$

$y = 0 \quad y = a$.

$\int_0^a \int_y^a f(x,y)\, dx\, dy = \int_0^a \int_0^x f(x,y)\, dy\, dx$

3 change of order of integration in $\int_0^a \int_x^a f(x,y)\, dy\, dx$.

Sol. Given $\int_0^a \int_x^a f(x,y)\, dy\, dx$



$y = x \quad y = a$

$x = 0 \quad x = a$.

$\int_0^a \int_0^y f(x,y)\, dx\, dy$

4. Change the order of Integration in

$$I = \int_0^1 \int_{x^2}^{2-x} f(x,y)\, dx\, dy.$$

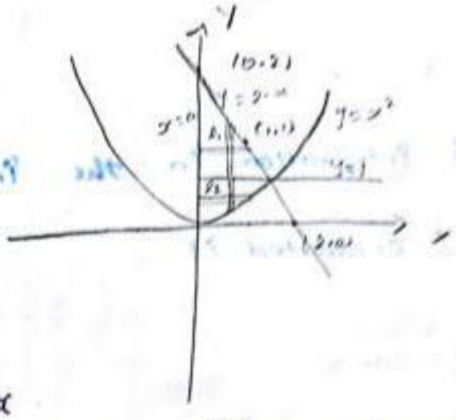
Sol.

Given: $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)\, dy\, dx$.

$y = x^2$, $y = 2-x$, $x = 0$, $x = 1$.

$x = \sqrt{y}$ $y =$

x	0	1	1	9	x	0	1	2
y	0	1	0	3	y	2	1	0



$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \iint_{R_1} xy\, dx\, dy + \iint_{R_2} xy\, dx\, dy$$

$$= \int_0^2 \int_0^{2-y} xy\, dx\, dy + \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy$$

$$= \int_1^2 \int_0^{2-y} xy\, dx\, dy + \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy$$

$$= \int^2 \left[\frac{x^2}{2} y\right]^{2-y} dy + \int^1 \left[\frac{x^2}{2} y\right]^{\sqrt{y}} dy$$

$$= \frac{1}{2}\int_1^{2} (4y + y^3 - 4y^2)\,dy + \frac{1}{2}\int_0^{1} y^2\,dy.$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2 + \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{4(2)^2}{2} + \frac{(2)^4}{4} - \frac{4(2)^3}{3}\right) - \left(\frac{1}{2} + \frac{1}{4} - \frac{1}{3}\right)\right] + \frac{1}{2}\left[\frac{1}{3} - 0\right]$$

$$= \frac{1}{2}\left[\frac{16}{2} + \frac{32}{4} - \frac{32}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} + \frac{1}{3}\right]$$

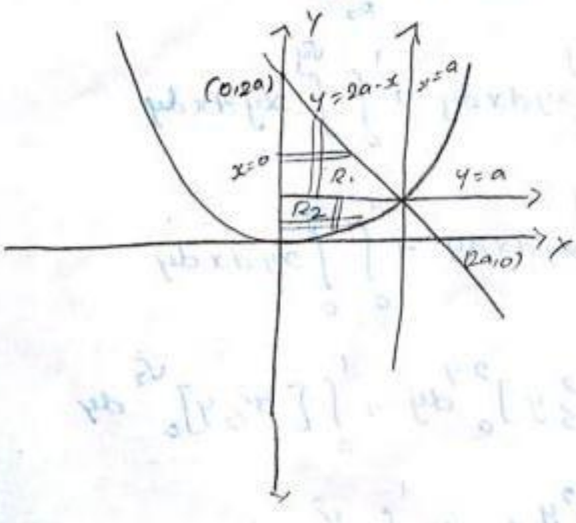
$$= \frac{1}{2} \times \frac{3}{4}$$

$$= \frac{3}{8}.$$

5. Change the order of integration in the interval $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ and hence evaluated it.

Sol. Given $y = x^2/a$, $y = 2a - x$

$x = 0$ & $x = a$.



$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy.$$

$$= \int_0^{2a}\int_0^{2a-y} xy\,dx\,dy + \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy.$$

$$= \int_0^{2a} \left[\frac{x^2}{2}\right]_0^{2a-y} y\,dy + \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y\,dy.$$

$$= \int_0^{2a} \frac{(2a-y)^2}{2} y\,dy + \int_0^a \left(\frac{ay}{2}\right) y\,dy.$$

$$= \frac{1}{2}\left[\int_0^{2a} (4a^2+y^2-4ay)\,y\,dy + \int_0^a ay^2\,dy.\right].$$

$$= \frac{1}{2}\left[\left(\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right)_0^{2a} + \left(\frac{ay^3}{3}\right)_0^3\right]$$

$$= \frac{1}{2}\left[\frac{16a^4}{2} + \frac{16a^4}{4} - \frac{32a^4}{3} - \frac{4a^4}{2} - \frac{a^4}{4} - \frac{4a^4}{3} + \frac{a^4}{3}\right]$$

$$= \frac{3a^4}{8}.$$

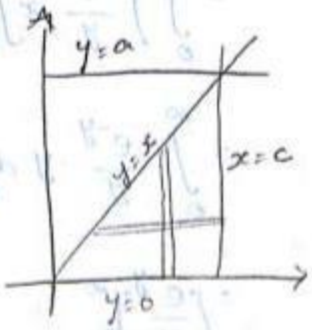
6. Change the order of in $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ and hence evaluate it.

$\int \frac{1}{x^2+y^2}dy = \frac{1}{x}\tan^{-1}$

Sol. Given $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy.$

$x = y$ & $x = a$

$y = 0$ & $y = a.$



$$\int_0^a \int_0^a \frac{x}{x^2+y^2}\,dx\,dy = \int_0^a \int_0^x \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a \left[\tan^{-1}(y/x) \right]_0^x dx$$

$$= \int_0^a \left[\tan^{-1}(1) - \tan^{-1}(0) \right] dx$$

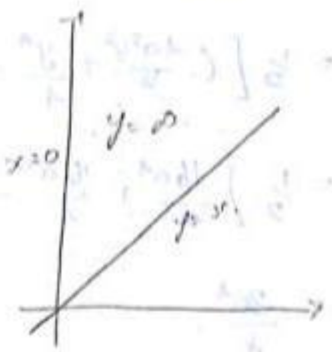
$$= \int_0^a \pi/4 \, dx$$

$$= \pi/4 \left[x \right]_0^a = \pi/4 (a-0) = \pi/4 \, a.$$

7. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx\,dy$ by changing the order of integration.

Sol. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx\,dy.$

$$= \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy\,dx.$$



$y = x \qquad y = \infty.$

$x = 0 \qquad x = \infty.$

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx\,dy.$$

$$= \int_0^\infty \left[\frac{e^{-y}}{y} x \right]_0^y dy.$$

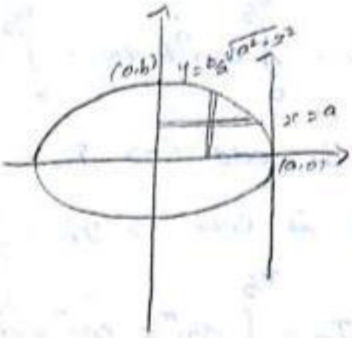
$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y \, dy.$$

$$= \left[\frac{e^{-y}}{-1} \right]_0^\infty = -(e^{-\infty} - e^{0}) = 0+1 = 1.$$

8. Change the order of integration in $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\,dy\,dx$ and then evaluate it.

Sol. Given $y=0$, $y=\frac{b}{a}\sqrt{a^2-x^2}$.

$x=0$, $x=a$.



$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\,dy\,dx$$

$$= \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x^2\,dx\,dy.$$

$$= \int_0^b \left[\frac{x^3}{3}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}}$$

$$= \int_0^b \frac{a^3}{3b^3}(b^2-y^2)^{3/2}\,dy.$$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}.$

$x^2=\frac{a^2}{b^2}(b^2-y^2)$

$x=\frac{a}{b}\sqrt{b^2-y^2}.$

Put $y=b\sin\theta$ $\quad dy=b\cos\theta\,d\theta$.

$$= \frac{a^3}{3b^3}\int_0^{\pi/2}(b^2-b^2\sin^2\theta)^{3/2}\,b\cos\theta\,d\theta.$$

$$= \frac{a^3}{3b^3}\int_0^{\pi/2}(b^2)^{3/2}(1-\sin^2\theta)^{3/2}\,b\cos\theta\,d\theta.$$

$$= \frac{a^3}{3b^3}\times b^4\int_0^{\pi/2}\cos^3\theta\,\cos\theta\,d\theta. \qquad \left(I_n=\int_0^{\pi/2}\cos^n\theta\,d\theta\right)$$

$$= \frac{a^3b}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta.$$

$$= \frac{a^3b}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi a^3 b}{16}.$$

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\, dy\, dx = \frac{\pi a^3 b}{16}.$$

$$I_n = \int_0^{\pi/2} \cos^n\theta\, d\theta = \int_0^{\pi/2} \sin^n\theta\, d\theta.$$

n is odd $\Rightarrow I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1$

n is even $\Rightarrow I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}.$

$$I_n = \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}.$$

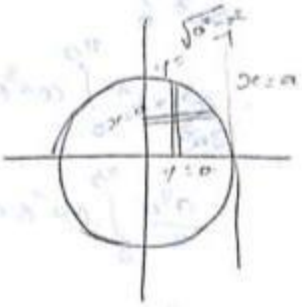
Changing cartesian to polar co-ordinates.

1. Transform the integral into polar co-ordinates and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

Sol Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r.$

y varies



$y = 0$ & $y = \sqrt{a^2-x^2}$

$r\sin\theta = 0$ $\quad y^2 = a^2 - x^2$

$r = 0$ $\quad \sin\theta = 0$ $\quad y^2 + x^2 = a^2$

$\theta = \sin^{-1} 0$ $\quad r^2 = a^2$

$\theta = 0$ $\quad r = a.$

x varies

$x = 0$ & $x = a.$

$r\cos\theta = 0.$

$r = 0$ | $\cos\theta = 0 \Rightarrow \theta = \cos^{-1} 0 \Rightarrow \theta = \pi/2.$

r varies from 0 to a.

θ varies from 0 to $\pi/2$.

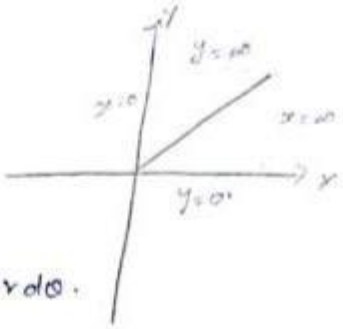
$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^{\pi/2}\int_0^a r \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \int_0^{\pi/2} \frac{a^3}{3} d\theta = \frac{a^3}{3}[\theta]_0^{\pi/2}$$

$$= \frac{a^3}{3}\cdot\frac{\pi}{2} = \frac{\pi a^3}{6}.$$

2. Transform the integral into polar co-ordinates & hence evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx\, dy = \pi/4$. and deduce the value of $\int_0^\infty e^{-x^2} dx$.



Sol. let $I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx\, dy.$

Put $x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta.$

x varies from 0 to ∞.

$r\cos\theta = 0 \qquad\qquad r\cos\theta = \infty$

$r = 0,\ \cos\theta = 0 \qquad\qquad r = \infty.$

$\theta = \pi/2$

y varies from 0 to ∞.

$r\sin\theta = 0$

$r = 0,\ \sin\theta = 0 \Rightarrow \theta = 0$

θ varies from 0 to $\pi/2$.

r varies from 0 to ∞.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot r \cdot dr\,d\theta.$$

Take $t = r^2$ $\quad dt = 2r\,dr$.

$$\frac{dt}{2} = r\,dr.$$

$r = 0 \Rightarrow t = 0$

$r = \infty \Rightarrow t = \infty$

$$= \int_0^{\pi/2} \int_0^\infty e^{-t} \frac{dt}{2} d\theta. = \frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{-t} dt\,d\theta.$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{e^{-t}}{-1} \right]_0^\infty d\theta.$$

$$= \frac{1}{2} \int_0^{\pi/2} -[e^{-\infty} - e^0]\,d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta.$$

$$= \frac{1}{2} [\theta]_0^{\pi/2}$$

$$= \pi/4.$$

Let $I = \int_0^\infty e^{-x^2} dx.$

$$I = \int_0^\infty e^{-y^2} dy.$$

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \pi/4.$$

$I^2 = \pi/4. \quad \Rightarrow I = \frac{\sqrt{\pi}}{2}.$

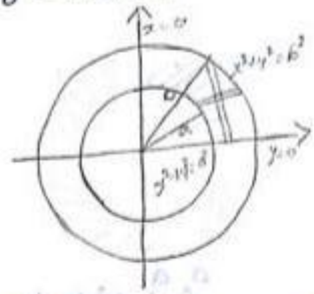
$I = \sqrt{\pi}/2$

$$\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$$

3. Evaluate $\iint \frac{x^2y^2}{x^2+y^2} dxdy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$ by transforming into polar co-ordinates.

Sol. $I = \iint \frac{x^2y^2}{x^2+y^2} dxdy$.

put $x = r\cos\theta$, $y = r\sin\theta$ $\quad dxdy = r\,dr\,d\theta$.



θ varies from 0 to 2π

r varies from a to b

$$x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$
$$= r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2.$$

$$x^2y^2 = (r^2\cos^2\theta)(r^2\sin^2\theta)$$
$$= r^4\cos^2\theta.\sin^2\theta.$$

$$I = \int_0^{2\pi}\int_a^b \frac{r^4\cos^2\theta\,\sin^2\theta}{r^2}\, r\,dr.d\theta.$$

$$= \int_0^{2\pi}\int_a^b r^3\cos^2\theta\,\sin^2\theta\,dr\,d\theta.$$

$$= \int_0^{2\pi} \cos^2\theta\,\sin^2\theta\left[\frac{r^4}{4}\right]_a^b d\theta.$$

$$= \frac{b^4-a^4}{4}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta \quad = \frac{b^4-a^4}{4}\int_0^{2\pi}\sin^2\theta\,(1-\sin^2\theta)\,d\theta.$$

$$= \frac{b^4-a^4}{4}\left[4\int_0^{\pi/2}\sin^2\theta\,d\theta - \int_0^{\pi/2}\sin^4\theta\,d\theta\right]$$

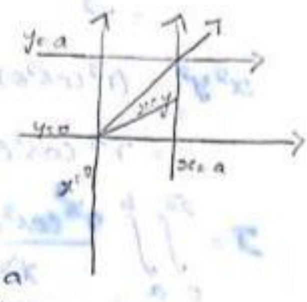
$$= \frac{b^4-a^4}{4}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{b^4-a^4}{}\left(\frac{\pi}{4}\right)\left(1-\frac{3}{4}\right) =$$

$$= b^4-a^4\left(\frac{\pi}{16}\right)$$

4. Express $\int_0^a\int_y^a \frac{x^2\,dx\,dy}{(x^2+y^2)^{3/2}}$ in polar Co-ordinates and then evaluate it.

Sol.



Given $I = \int_0^a\int_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy.$

$x = y$	$x = a$	$y = 0$	$y = a$
$r\cos\theta = r\sin\theta$	$r\cos\theta = a$	$r\sin\theta = 0$	$r\sin\theta = a$
$\cos\theta = \sin\theta$	$r = \frac{a}{\cos\theta}$	$\sin\theta = 0$	$r = \frac{a}{\sin\theta}$, $r = a\,\text{cosec}\,\theta$.
$\theta = \pi/4.$	$r = a\sec\theta$	$\theta = 0$, $r = 0$	

$$I = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{(r^2\cos^2\theta + r^2\sin^2\theta)^{3/2}}\,r\,dr\,d\theta.$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{(r^2)^{3/2}(\cos^2\theta+\sin^2\theta)^{3/2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{r^3}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \cos^2\theta\, dr\,d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta\,[r]_0^{a\sec\theta}\, d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta\,(a\sec\theta)\, d\theta = \int_0^{\pi/4} \cos^2\theta \times \frac{a}{\cos\theta}\, d\theta$$

$$= \int_0^{\pi/4} a\cos\theta\, d\theta$$

$$= a\,[\sin\theta]_0^{\pi/4} = a\,[\sin\pi/4 - \sin 0]$$

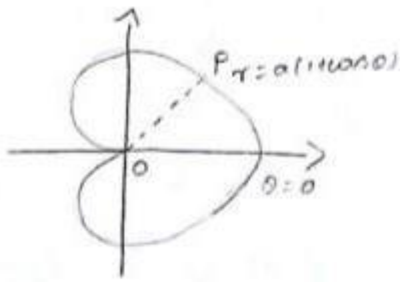
$$= a\left(\frac{1}{\sqrt{2}}\right).$$

$$I = a/\sqrt{2}.$$

Area as double integral (polar co-ordinates)

$$\text{Area} = \iint r\,dr\,d\theta.$$

①. Find using a double integral, the area of the cardiod. $r = a(1+\cos\theta)$.



Sol.

$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta.$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi} 1+2\cos\theta+\left(\frac{1+\cos2\theta}{2}\right)d\theta.$$

$$= \frac{a^2}{2}\int_0^{\pi} 2+4\cos\theta+1+\cos2\theta\,d\theta$$

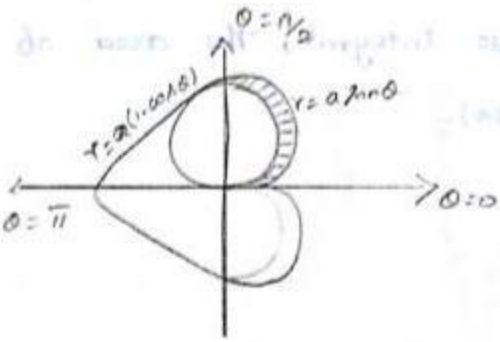
$$= \frac{a^2}{2}\left[3\theta+4\sin\theta+\frac{\sin2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{2}[3\pi]$$

$$= \frac{3}{2}a^2\pi.$$

(2). Find the area inside the circle $r = a\sin\theta$ and outside the cardiod $r = a(1-\cos\theta)$.

Sol.



∴ θ varies from 0 to $\pi/2$

r varies from $a(1-\cos\theta)$ to $a\sin\theta$.

$$\text{Area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} (a^2\sin^2\theta - a^2(1-\cos\theta)^2\, d\theta.$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\, d\theta.$$

$$\neq \frac{a^2}{2}\int_0^{\pi/2} -2\cos^2\theta + 2\cos\theta\, d\theta.$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) - 1 - \left(\frac{1+\cos 2\theta}{2}\right) + 2\cos\theta\, d\theta.$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{1}{2} - \frac{\cos 2\theta}{2} - 1 - \frac{1}{2} - \frac{\cos 2\theta}{2} + 2\cos\theta\, d\theta.$$

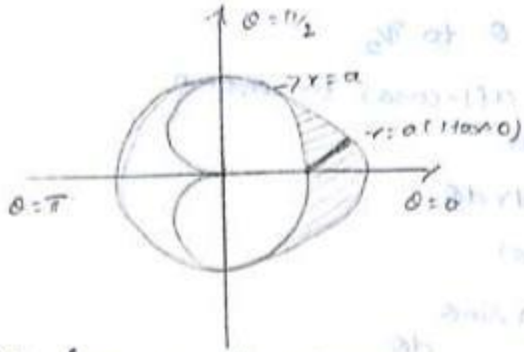
$$= \frac{a^2}{2}\left[-\frac{\sin 2\theta}{4} - \theta - \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\right]$$

$$= \frac{a^2}{2}\left(\frac{4-\pi}{2}\right)$$

$$\text{Area} = \frac{a^2}{4}(4-\pi) \text{ sq units.}$$

(3) Find the area that lies inside the cardiod $r = a(1+\cos\theta)$ and outside the circle $r = a$ by double integration.



r varies from a to $a(1+\cos\theta)$.

θ varies from 0 to $\pi/2$.

$$\text{Area} = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2}\frac{1}{2}\left(a^2(1+\cos\theta)^2 - a^2\right)d\theta$$

$$= \int_0^{\pi/2} a^2 + a^2\cos^2\theta + 2a^2\cos\theta - a^2\,d\theta$$

$$= \int_0^{\pi/2} a^2\left(\frac{1+\cos 2\theta}{2}\right) + 2a^2\cos\theta\,d\theta$$

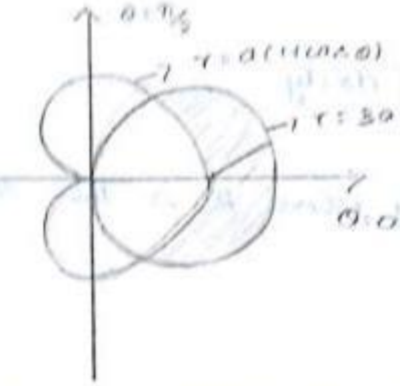
$$= a^2\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} + 2a^2\left[\sin\theta\right]_0^{\pi/2}$$

$$= a^2\left[\frac{\pi}{4}\right] + \left[2a^2\right]$$

$$= \frac{a^2}{4}(\pi) + 2a^2$$

$$= \frac{a^2}{4}(\pi + 8) \text{ sq units.}$$

④ Find the area which is inside the circle $r = 3a\cos\theta$ and outside the cardiod $r = a(1+\cos\theta)$.



r varies from $a(1+\cos\theta)$ to $3a\cos\theta$.

θ varies from 0 to $\pi/2$

$$\text{Area} = 2\int_0^{\pi/2}\int_{a(1+\cos\theta)}^{3a\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1+\cos\theta)}^{3a\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{9a^2\cos^2\theta}{2} - \frac{a^2(1+\cos\theta)^2}{2}\right) d\theta$$

$$= \frac{2}{2}\int_0^{\pi/2}(a^2 + 2a^2\cos\theta - 8a^2\cos^2\theta)\,d\theta$$

$$= \left[a^2\theta + 2a^2\sin\theta - 4a^2\theta - \frac{4a^2\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2\pi}{2} + 2a^2\sin\pi/2 - 4a^2\pi/2 - 2a^2\sin 2\pi/2$$

$$= \frac{\pi a^2}{2} + 2a^2 - 2a^2\pi$$

$$= a^2(\pi/2 + 2 - 2\pi)$$

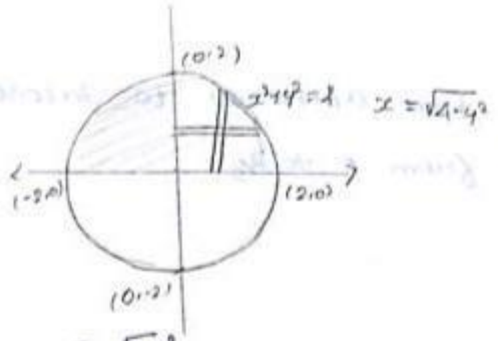
$$= \frac{a^2}{2}(-3\pi + 4).$$

$$\text{Area} = \frac{a^2}{2}(4 - 3\pi).$$

Area :

$$\text{Area} = \iint_R dx\,dy .$$

① Evaluate $\iint_R dx\,dy$ where R is the shaped region in the figure



$$\iint_R dx\,dy = 2\int_0^2\int_0^{\sqrt{4-y^2}} dx\,dy$$

$$= 2\int_0^2 [x]_0^{\sqrt{4-y^2}} dy$$

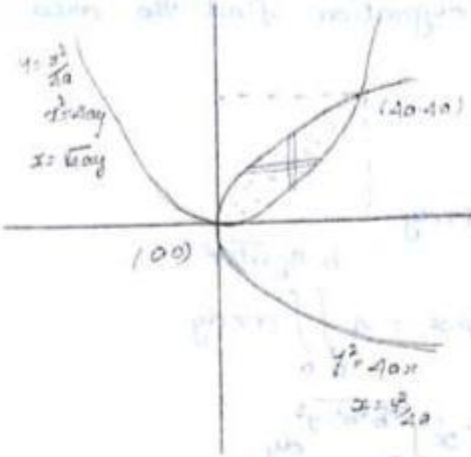
$$= 2\int_0^2 \sqrt{4-y^2}\,dy . = 2\left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}(y/2)\right]_0^2$$

$$= 2\left[2\sin^{-1}1\right]$$

$$= 2\times 2\times \pi/2$$

$$= 2\pi .$$

② Find by double integration, the area enclosed by the curve $y^2 = 4ax$ & $x^2 = 4ay$.



Given $y^2 = 4ax$ & $x^2 = 4ay$.

$$\text{Area} = \iint_R dx\,dy.$$

$$= \int_0^{4a} \int_{y^2/4a}^{\sqrt{4ay}} dx\,dy.$$

$$= \int_0^{4a} [x]_{y^2/4a}^{\sqrt{4ay}} dy = \int_0^{4a} \left(\sqrt{4ay} - \frac{y^2}{4a}\right) dy.$$

$$= \int_0^{4a} 2\sqrt{a}\sqrt{y} - \frac{1}{4a} y^2\, dy.$$

$$= 2\sqrt{a}\left[\frac{y^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a} \times \frac{2}{3} \times (4a)^{3/2} - \frac{1}{12a}(4a)^3$$

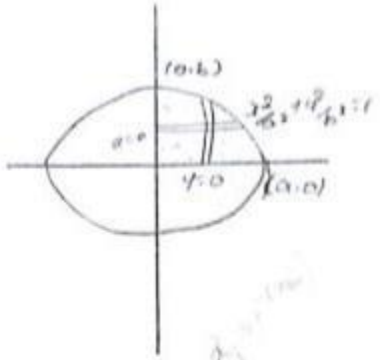
$$= \frac{4\sqrt{a}}{3} \times 8a\sqrt{a} - \frac{1}{12a} \times 64a^{3\,2}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\text{Area} = \frac{16a^2}{3} \text{ sq units.}$$

③ using double integration find the area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol.



$$\text{Area} = \iint_R dx\,dy.$$

$$\text{Area of the ellipse} = 4\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$$

$$= 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy.$$

$$= 4\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\,dy$$

$$= \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right)\right]_0^b.$$

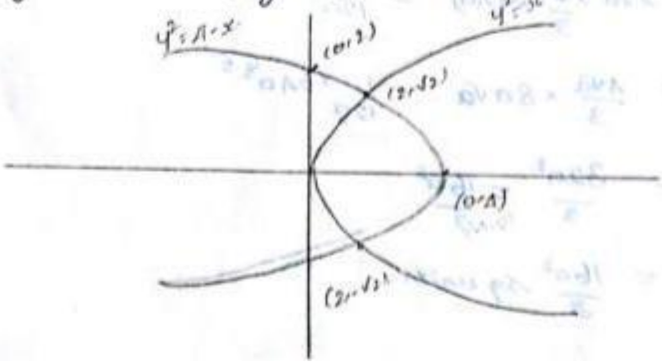
$$= \frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}1\right]$$

$$= \frac{4a}{b}\left[\frac{b^2}{2}\cdot\frac{\pi}{2}\right]$$

$$= 4\,\frac{\pi ab}{4}$$

$$\text{Area} = \pi ab \text{ sq units}$$

④ Find the area bounded by the parabola $y^2=4-x$ & $y^2=x$ by double integration



Sol.

Given $y^2 = 1 - x$ & $y^2 = x$

$y^2 = 1 - x$

x	0	1	2
y	2	0	$\sqrt{2}$.

$$\text{Area} = \iint_R dx\, dy$$

$$= 2\int_0^{\sqrt{2}} \int_{y^2}^{1-y^2} dx\, dy.$$

$$= 2\int_0^{\sqrt{2}} [x]_{y^2}^{1-y^2} dy = 2\int_0^{\sqrt{2}} (1 - y^2 - y^2)\, dy$$

$$= 2\int_0^{\sqrt{2}} 1 - 2y^2\, dy.$$

$$= 2\left[1y - \frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

$$= 2\left[1\sqrt{2} - \frac{1\sqrt{2}}{3}\right]$$

$$= 2 \times \frac{8\sqrt{2}}{3}$$

$$\text{Area} = \frac{16\sqrt{2}}{3} \text{ Sq units.}$$

(5) Find by double integration the area between the two parabolas $3y^2 = 25x$ & $5x^2 = 9y$.

Sol.

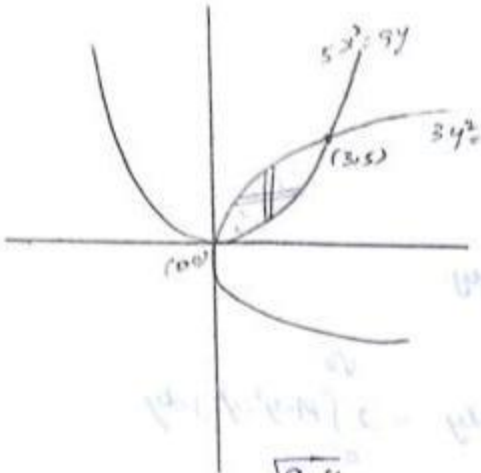
Given $3y^2 = 25x$ $\qquad$ $5x^2 = 9y$.

$y^2 = 25/3\, x$ $\qquad$ $x^2 = 9y/5$

$y = \pm \frac{5\sqrt{x}}{\ }$ $\qquad$ ~~$x = \pm 3\sqrt{y}$~~ $\quad y = \frac{5x^2}{9}$

x	0	1	2	3
y	0	$\frac{5}{\sqrt{3}}$	$\frac{5\sqrt{2}}{\sqrt{3}}$	5

x	0	1	2	3
y	0	$5/9$	$20/9$	5



$$\text{Area} = \iint_R dx\,dy = \int_0^5 \int_{\frac{3}{25}y^2}^{\sqrt{\frac{9}{5}y}} dx\,dy.$$

$$= \int_0^5 [x]_{\frac{3}{25}y^2}^{\sqrt{\frac{9}{5}y}}\, dy.$$

$$= \int_0^5 \sqrt{\frac{9y}{5}} - \frac{3}{25}y^2\, dy.$$

$$= \int_0^5 \frac{3}{\sqrt{5}} y^{1/2} - \frac{3}{25}y^2\, dy = \left[\frac{3}{\sqrt{5}}\left(\frac{y^{3/2}}{3/2}\right) - \frac{3}{25}\left(\frac{y^3}{3}\right)\right]_0^5$$

$$= \frac{3}{\sqrt{5}} \times \frac{2}{3} \times (5)^{3/2} - \frac{3}{25} \times \frac{5^3}{3}$$

$$= 2 \times 5 - 5$$

$$= 10 - 5$$

Area $= 5$ sq units.

Find the smaller of the areas bounded by the ellipse $4x^2+9y^2=36$ and the straight line $2x+3y=6$

Sol.

Given $4x^2+9y^2=36$. $\qquad 2x+3y=6$

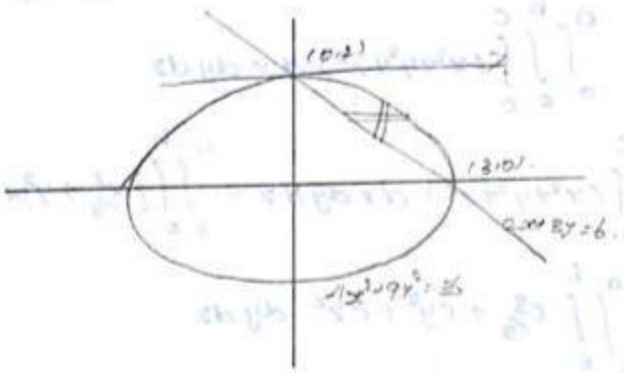
$\frac{x^2}{9}+\frac{y^2}{4}=1$. $\qquad 3y=6-2x$

$y^2 = 4\left(1-\frac{x^2}{9}\right)$ $\qquad y = 2-\frac{2x}{3}$.

$y^2 = \frac{36-x^2}{9}$

x :	0	1	2	3
y :	± 2	$\frac{\sqrt{32}}{3}$	$\frac{\sqrt{20}}{3}$	0

x	0	1	2	3
y	2	$\frac{4}{3}$	$\frac{2}{3}$	0



$$\text{Area} = \iint_R dx\,dy$$

$$= \int_0^2 \int_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{4-y^2}} dx\,dy.$$

$$= \int_0^2 [x]_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{4-y^2}}\,dy$$

$$= \int_0^2 \left[\frac{3}{2}\sqrt{4-y^2} - \frac{3}{2}(2-y)\right] dy.$$

$$= \frac{3}{2}\int_0^2 \left(\sqrt{1-y^2} - (2-y)\right) dy$$

$$= \frac{3}{2}\left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}\left(\frac{y}{2}\right) - 2y + \frac{y^2}{2}\right]_0^2$$

$$= \frac{3}{2}\left[0 + 2\sin^{-1}1 - 2(2) + \frac{1}{2}\right]$$

$$= \frac{3}{2}\left[2\cdot\frac{\pi}{2} - 1 + \frac{1}{2}\right].$$

$$= \frac{3}{2}(\pi - 2).$$

Area $= \frac{3}{2}(\pi-2)$ sq units

Triple integrals

① Evaluate $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$.

Sol. $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^a\int_0^b \left[\frac{x^3}{3} + y^2x + z^2x\right]_0^c dy\,dz$

$$= \int_0^a\int_0^b \frac{c^3}{3} + cy^2 + cz^2 \; dy\,dz.$$

$$= \int_0^a \left(\frac{c^3}{3}y + \frac{cy^3}{3} + cz^2y\right)_0^b dz.$$

$$= \int_0^a \frac{bc^3}{3} + \frac{cb^3}{3} + cbz^2\; dz.$$

$$= \left[\frac{bc^3}{3}z + \frac{cb^3}{3}z + cb\frac{z^3}{3}\right]_0^a$$

$$= \frac{abc^3}{3} + \frac{ab^3c}{3} + \frac{a^3bc}{3}$$

② Evaluate $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

Sol. $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$\int_0^{\log a}\int_0^{x}\left[e^{x+y+z}\right]_0^{x+y} dy\,dx = \int_0^{\log a}\int_0^{x} e^{2x+2y} - e^{x+y}\,dy\,dx$

$= \int_0^{\log a}\left(\frac{e^{2(x+y)}}{2} - e^{x+y}\right)_0^{x} dx$

$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right) dx$

$= \left(\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x}\right)_0^{\log a}$

$= \frac{a^4}{8} - \frac{a^2}{2} - \frac{a^2}{4} + a - \frac{1}{8} + \frac{1}{2} + \frac{1}{4} - 1$

$= \frac{a^4 - 4a^2 - 2a^2 + 8a - 1 + 4 + 2 - 8}{8}$

$= \frac{1}{8}(a^4 - 6a^2 + 8a - 3)$

③ Evaluate $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$

Sol. $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$ $\left(\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)\right)$

$\sqrt{(\sqrt{a^2-x^2-y^2})^2 - z^2}$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} [\sin^{-1}1 - \sin^{-1}0]\, dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} [\pi/2]\, dy\,dx$$

$$= \pi/2 \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx = \pi/2 \int_0^a \sqrt{a^2-x^2}\, dx.$$

$$= \pi/2 \left[x/2 \sqrt{a^2-x^2} + a^2/2 \sin^{-1} x/a \right]_0^a$$

$$= \pi/2 \left[(0 + a^2/2\, \pi/2) - (0+0) \right]$$

$$= \frac{\pi^2 a^2}{8}.$$

(4). Evaluate $\iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$, over the region of integration bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$.

Sol. The given region is a tetrahedron.

$x=0 \quad y=0 \quad z=0 \quad x+y+z=1.$

x varies from $x=0$ to $x=1$.

y varies from $y=0$ to $y=1-x$.

z varies from $z=0$ to $z=1-x-y$.

$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx.$$

$$= \int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x} -\tfrac{1}{2}\left[(x+y+1-x-y+1)^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\tfrac{1}{2}\int_0^1\int_0^{1-x}\left[(2^{-2}) - (x+y+1)^{-2}\right] dy\,dx.$$

$$= -\tfrac{1}{2}\int_0^1\left[\tfrac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx.$$

$$= -\tfrac{1}{2}\int_0^1\left[\tfrac{1}{4}y + (x+y+1)^{-1}\right]_0^{1-x} dx.$$

$$= -\tfrac{1}{2}\int_0^1\left[\left(\tfrac{1}{4}(1-x) + (2)^{-1}\right) - \left(0 + (x+1)^{-1}\right)\right] dx$$

$$= -\tfrac{1}{2}\int_0^1\left[\tfrac{1}{4} - \tfrac{x}{4} + \tfrac{1}{2} - (x+1)^{-1}\right] dx$$

$$= -\tfrac{1}{2}\int_0^1\left(\tfrac{3}{4} - \tfrac{x}{4} - \tfrac{1}{1+x}\right) dx$$

$$= -\tfrac{1}{2}\left[\tfrac{3}{4}x - \tfrac{x^2}{8} - \log(1+x)\right]_0^1$$

$$= -\tfrac{1}{2}\left[\left(\tfrac{3}{4} - \tfrac{1}{8} - \log 2\right) - (0-0-0)\right]$$

$$= -\tfrac{1}{2}\left(\tfrac{5}{8} - \log 2\right).$$

$$= \tfrac{1}{2}\log 2 - \tfrac{5}{16}$$

Volume as Triple integral

$$\text{Volume} = \iiint_V dz\,dy\,dx$$

① Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol.

Volume of ellipsoid $= 8\iiint dz\,dy\,dx$

$z=0$ to $z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$y=0$ to $y = b\sqrt{1-\frac{x^2}{a^2}}$

$x=0$ to $x=a$.

$$\text{volume} = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\,dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{\frac{b^2(1-x^2/a^2)-y^2}{b^2}}\,dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \frac{c}{b}\sqrt{b^2(1-x^2/a^2)-y^2}\,dy\,dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{b^2(1-x^2/a^2)-y^2} + \frac{b^2(1-x^2/a^2)}{2}\sin^{-1}\left(\frac{y}{\sqrt{b^2(1-x^2/a^2)}}\right)\right]_0^{b\sqrt{1-x^2/a^2}} dx$$

$$= \frac{8c}{b}\int_0^a \frac{b^2(1-x^2/a^2)}{2}\sin^{-1}1\,dx$$

$= \frac{8c}{2b} \times b^2 \int_0^a (1 - x^2/a^2) \cdot \pi/2 \, dx$

$= \frac{4cb\pi}{2} \left[x - \frac{x^3}{3a^2} \right]_0^a$

$= 2cb\pi \left[a - a^3/3a^2 \right]$

$= 2cb\pi (2a/3)$

$I = \frac{4\pi abc}{3}$

(2). Express the volume of the Sphere $x^2+y^2+z^2 = a^2$ as a volume integral and hence evaluate it.

Sol.

$$\text{Volume} = \iiint_V dz\,dy\,dx.$$

$$\text{volume of the sphere} = 8 \iiint_V dz\,dy\,dx.$$

$z = 0$ to $z = \sqrt{a^2 - x^2 - y^2}$

$y = 0$ to $y = \sqrt{a^2 - x^2}$

$x = 0$ to $x = a$.

$$\text{Volume of the Sphere} = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} \, dy\,dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)^{1/2} \, dy\,dx.$$

$$= 8 \int_0^a \left[\frac{y}{2} \sqrt{(a^2-x^2) - y^2} + \frac{(a^2-x^2)}{2} \sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \frac{a^2-x^2}{2} \sin^{-1} 1 \, dx.$$

$$= \frac{4}{2}\frac{8\pi}{2}\int_0^a \frac{a^2-x^2}{2}\,dx = \frac{2}{1}\frac{4\pi}{2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[a^3 - \frac{a^3}{3}\right] = 2\pi \times \frac{2a^3}{3}$$

$$I = \frac{4\pi a^3}{3}.$$

Find the volume of the tetrahedron bounded by the Coordinate planes and the plane $x/a + y/b + z/c = 1$.

Sol. $\text{volume} = \iiint_V dz\,dy\,dx$

$z = 0$ to $z = c(1 - x/a - y/b)$

$y = 0$ to $y = b(1 - x/a)$.

$x = 0$ to $x = a$.

$$\text{volume} = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-x/a)} [z]_0^{c(1-x/a-y/b)}\,dy\,dx.$$

$$= \int_0^a \int_0^{b(1-x/a)} c(1 - x/a - y/b)\,dy\,dx$$

$$= \int_0^a \left[cy - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-x/a)} dx$$

$$= c\int_0^a \left[b(1-x/a) - \frac{xb}{a}(1-x/a) - \frac{1}{2b}b^2(1-x/a)^2\right]dx$$

$$= c\int_0^a (1-x/a)\left[b - \frac{xb}{a} - b/2\,(1-x/a)\right]dx.$$

$$= c\int_0^a (1-x/a)\left[b(1-x/a) - b/2\,(1-x/a)\right]dx$$

I

$$= c\int_0^a b(1-x/a)^2\left[1-1/2\right]dx.$$

$$= c\int_0^a \frac{b(1-x/a)^2}{2}dx$$

$$= \frac{bc}{2}\int_0^a 1 + x^2/a^2 - 2x/a \; dx$$

$$= \frac{bc}{2}\left[x + \frac{x^3}{3a^2} - \frac{2x^2}{2a}\right]_0^a$$

$$= \frac{bc}{2}\left[a + \frac{a^3}{3a^2} - a^2/a\right]$$

$$= \frac{bc}{2}\left[a + a/3 - a\right]$$

$$I = \frac{abc}{6}$$

Objective type questions	Opt 1
The triple integral ∫∫∫ dv gives the __________ over the region v	area
The value of ∫∫ dx dy , inner integral limt varies from 1 to 2 and the outer integral limit varies from 0 to 1	0
∫∫∫ dx dy dz, the inner integral limit varies from 0 to 3, the central integral limit varies from 0 to 2 and outer integral limit varies from 0	2
When the limits are not given, the integral is named as __________	Definite integral
The Double integral ∫∫ dx dy gives _________ of the region R	area
The value of ∫∫ (x+y) dx dy , inner integral limt varies from 0 to 1 and the outer integral limit varies from 0 to 1	0
to 2, the central integral limit varios from 0 to 2 and outer integral limit various from 0 to 1	7/3
Evaluate ∫∫ 4xy dx dy, the inner integral limit varies from 0 to 1 and outer integral limit varies from 0 to 2	10
The value of ∫∫ d xdy ⁄xy, the inner integral limit varies from 0 to b and the outer limit varies from 0 to a	0
If the limits are given in the integral , then the integral is name as____________	Definite integral
The value of ∫∫(x^2+3y^2) dy dx , the inner integral limit varies from 0 to 1, the outer integral limit varies from 0 to 3	10
The value od ∫∫∫ dxdy d, the inner integral limit varies from 0to 3, the central integral limit varies from 0 to 2 and outer integral limit	6
If the limits are not given in the integral , the the integral is name as____________	Definite integral
The value of ∫∫(x^2+y^2) dy dx ,the inner integral limit varies from 0 to x, the outer integral limit varies from 0 to 1	1
The value of ∫∫dy dx, the inner integral limit various from 0 to x ,the outer integral limit varies from -a to a	0
The Double integral ∫∫ dx dy gives _________ of the region R	area
the central integral limit varies from 0 to a and the outer integral limit varies from 0 to a	0
The value of ∫∫(x+y) dx dy , the inner integral limit varies from 0 to 1 and the outer integral limit varies from 0 to 1	0
The concept of line integral as a generalization of the concept of ________ integral	Single
The extension of double integral is nothing but ________ integral	Single
The concept of _________ integral as a generalization of the concept of double integral	Single
Evaluate ∫x^2⁄2 dx, the limit varies from 0 to 1	2
Evaluate ∫42y dy, the limit varies from 0 to 10	10
The value of ∫∫2 xy dy dx, the inner integral limit varies from 0 to x and the outer integral limit varies from 1 to 2	15⁄4

The value of ∫∫dy dx, the inner integral limit varies from 2 to 4 ,the outer integral limit varies from 1 to 5	8
The value of ∫∫xy dy dx, the inner integral limit varies from 0 to 3 , the outer integral limit varies from 0 to 4	12
The value of ∫∫dy dx, the inner integral limit varies from 0 to 2 , the outer integral limit varies from 0 to 1	2
The value of ∫∫dx dy, the inner integral limit varies from y to 2 , the outer integral limit varies from 0 to 1	1⁄2
The value of ∫∫dx dy, the inner integral limit varies from 2 to 4 , the outer integral limit varies from 1 to 2	2
When a function f(x) is integrated with respect to x between the limits a and b, we get __________	Definite integral
In two dimensions the x and y axes divide the entire xy- plane into ____________ quadrants	1
In three dimensions the xy and yz and zx planes divide the entire space into ____________ parts called octants	3
Evaluate ∫(2x+3) dx, the integral limitvaries from 0 to 2	10
__________ provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R.	Cauchy's Theorem
_________ is also called the first fundamental theorem of integral vector calculus.	Cauchy's Theorem
__________ transforms line integrals into surface integrals.	Cauchy's Theorem
________ transforms surface integrals into a volume integrals.	Cauchy's Theorem
__________ is stated as surface integral of the component of curl F along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function F taken along the closed curve C.	Cauchy's Theorem
_________ is stated as the surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.	Cauchy's Theorem

Opt2	Opt3	Opt4	Opt5	Opt6	Answer
volume	Direction	weight			volume
1	2	3			1
4	6	8			6
Infinite integral	volume integral	Surface integral			Infinite integral
modulus	Direction	weight			
1	2	3			1
1/3	2/3	3			7/3
4	5	1			4
1	ab	loga log b			loga log b
Infinite integral	volume integral	Surface integral			Definite integral
15	12	30			12
1	16	12			6
Infinite integral	volume integral	Surface integral			Infinite integral
1/3	2/3	3/2			1/3
1	2	3			0
modulus	Direction	weight			area
a^3	a^2	a^4			a^3
1	2	3			1
Double	change of order	Triple			Double
Line	volume integral	Triple			Triple
Surface	Line	Triple			Line
1/6	1/10	34			1/6
2100	2000	100			2100
9/2	3/2	4/3			15/4

2	4	5	8
36	1⁄2	4	12
1	3⁄2	4	2
1	3⁄2	4	3⁄2
6	3	1	2
infinite integralv	volume integral	Surface integral	Definite integral
2	3	4	2
2	8	4	8
42	51	1	10
Green's Theorem	Stoke's Theorem	Gauss Theorem	Stoke's Theorem
Green's Theorem	Stoke's Theorem	Gauss Theorem	Green's Theorem
Green's Theorem	Stoke's Theorem	Gauss Theorem	Green's Theorem
Green's Theorem	Stoke's Theorem	Gauss Theorem	Gauss Theorem
Green's Theorem	Stoke's Theorem	Gauss Theorem	Stoke's Theorem
Green's Theorem	Stoke's Theorem	Gauss Theorem	Gauss Theorem

p

Unit VIII

Vector Integration

Chapter 20: Line Integral, Surface Integral and Integral Theorems

20 Line Integral, Surface Integral and Integral Theorems

Chapter Outline

- Introduction
- Integration of Vectors
- Line Integral
- Circulation
- Application of Line Integrals
- Surfaces
- Surface Integrals
- Volume Integrals
- Integral Theorems

20.1 ❑ INTRODUCTION

In multiple integrals, we generalized integration from one variable to several variables. Our goal in this chapter is to generalize integration still further to include integration over curves or paths and surfaces. We will define integration not just of functions but also of vector fields. Integrals of vector fields are particularly important in applications involving the "field theories" of physics, such as the theory of electromagnetism, heat transfer, fluid dynamics and aerodynamics.

In this chapter, we shall define line integrals and surface integrals. We shall see that a line integral is a natural generalization of a define integral and a surface integral is a generalization of a double integral. Line integrals can be transformed into double integrals or into surface integrals and conversely. Triple integrals can be transformed into surface integrals and vice versa. These transformations are of great practical importance. Theorems of Green, Gauss and Stokes serve as powerful tools in many applications as well as in theoretical problems.

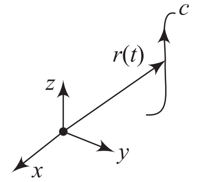


Fig. 20.1

In this chapter, we study the three main theorems of Vector Analysis: Green's Theorem, Stokes' Theorem and the Divergence Theorem. This is a fitting conclusion to the text because each of these theorems is a vector generalization of the Fundamental Theorem of calculus. This chapter is thus the culmination of efforts to extend the concepts and methods of single-variable calculus to the multivariable setting. However, far from being a terminal point, vector analysis the gateway to the field theories of mathematics physics and engineering. This includes, first and foremost, the theory of electricity and magnetism as expressed by the famous *Maxwell's equations*. It also includes fluid dynamics, aerodynamics, analysis of continuous matter, and at a more advanced level, fundamental physical theories such as general relativity and the theory of elementary particles.

Curves

Curves in space are important in calculus and in physics (for instance, as paths of moving bodies).

A curve C in space can be represented by a vector function

$$\vec{r}(t) = [x(t), y(t), z(t)] \qquad (20.1)$$
$$= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

where x, y, z are Cartesian coordinates. This is called a **parametric representation** of the curve (Fig. 20.1), t is called the **parameter** of the representation. To each value t_0 of t, there corresponds a point of C with position vector $\vec{r}(t_0)$, that is with coordinates $x(t_0)$, $y(t_0)$ and $z(t_0)$.

The parameter t may be time or something else. Equation (20.1) gives the **orientation** of C, a direction of travelling along C, so that t increasing is called the **positive sense** on C given by (20.1) and that of decreasing t is the **negative sense**.

- **Examples**

Straight line, ellipse, circle, etc.

The concept of a line integral is a simple and natural generalization of a definite

integral $$\int_a^b f(x)dx \qquad (20.2)$$

In (20.2), we integrate the **integrand** $f(x)$ from $x = a$ to $x = b$ along the x**-**axis. In a line integral, we integrate a given function, called the integrand, along a curve C in space (or in the plane).

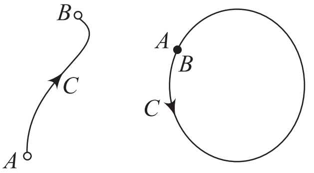
Hence, curve integral would be a better turn, but line integral is standard.

We represent a curve C by a parametric representation

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, (a \le t \le b)$$

We call C the **path of integration**, $A{:}\vec{r}(a)$ its **initial point** and $B{:}\vec{r}(b)$, its **terminal point**. The curve C is now oriented. The direction from A to B, in which t increases, is called the positive direction on C. We can indicate the direction by an arrow [Fig. 20.2(a)].

The points A and B may coincide [Fig. 20.2(b)]. Then C is called a **closed path**.



(a) **(b)**

Fig. 20.2

➢ **Note**

(i) A **plane curve** is a curve that lies in a plane in space.
(ii) A curve that is not plane is called a **twisted curve**.

20.2 ❑ INTEGRATION OF VECTORS

If two vector functions $\vec{F}(t)$ and $\vec{G}(t)$ be such that $\frac{d\vec{G}(t)}{dt} = \vec{F}(t)$, then $\vec{G}(t)$ is called an integral of $\vec{F}(t)$ with respect to the scalar variable t and we write $\int \vec{F}(t)dt = \vec{G}(t)$. If $\vec{C}$ be an arbitrary constant vector, we have $\vec{F}(t) = \frac{d\vec{G}(t)}{dt} = \frac{d}{dt}[\vec{G}(t) + \vec{C}]$, then $\int \vec{F}(t)dt = \vec{G}(t) + \vec{C}$. This is called the indefinite integral of $\vec{F}(t)$ and its definite integral is $\int_a^b \vec{F}(t)dt = [\vec{G}(t) + \vec{C}]_a^b = \vec{G}(b) - \vec{G}(a)$.

20.3 ❑ LINE INTEGRAL

Any integral which is to be evaluated along a curve is called a **line integral**. Consider a continuous vector point function $\vec{F}(R)$ which is defined at each point of the curve C in space. Divide C into n parts at the points $A = p_0, p_1 \ldots p_{i-1}, p_i \ldots p_n = B$

Let their position vectors be $\vec{R}_0, \vec{R}_1 \ldots \vec{R}_{i-1}, \vec{R}_i \ldots \vec{R}_n$

Let $\vec{v}_i$ be the position vector of any point on the arc $P_{i-1} P_i$

Now consider the sum $S = \sum_{i=0}^{n} \vec{F}(\vec{v}_i) \cdot \delta\vec{R}_i$ where $\delta\overrightarrow{R_i} = \overrightarrow{R_i} - \overrightarrow{R_{i-1}}$.

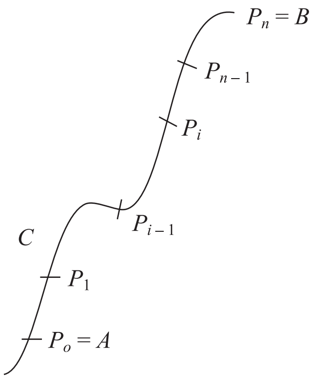


Fig. 20.3

The limit of this sum as $n - \infty$ in such a way that $|\delta\overrightarrow{R_i}| \to 0$, provided it exists, is called the **tangential line integral** of $\vec{F}(\vec{R})$ along C which is a scalar and is symbolically written as

$$\int_C \vec{F}(\vec{R}) \cdot \overrightarrow{dR} \text{ or } \int_C \vec{F} \cdot \frac{d\vec{R}}{dt} \cdot dt$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\vec{F}(\vec{R}) = f(x,y,z)\vec{i} + \phi(x,y,z)\vec{j} + \psi(x,y,z)\vec{k}$ and $d\vec{R} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

then $\int_C \vec{F}(\vec{R}) \cdot d\vec{R} = \int_C (f dx + \phi dy + \psi dz)$.

Two other types of line integrals are $\int_C \vec{F} \times d\vec{R}$ and $\int_C f d\vec{R}$ which are both vectors.

20.4 ❑ CIRCULATION

In fluid dynamics, if $\vec{F}$ represents the velocity of a fluid particle then the line integral $\int_C \vec{F}\cdot d\vec{r}$ is called the circulation of $\vec{F}$ around the curve. When the circulation of $\vec{F}$ around every closed curve in a region E vanishes, $\vec{F}$ is said to be **irrotational** in E.

Conservative Vector

If the value of $\int_A^B \vec{F}\cdot\overrightarrow{dr}$ does not depend on the curve C, but only on the terminal points A and B, $\vec{F}$ is called a **conservative vector**.

A force field $\vec{F}$ is said to be **conservative** if it is derivable from a potential function ϕ, i.e., $\vec{F} =$ grad ϕ. Then curl $(\vec{F})$ = curl $(\nabla\phi) = 0$.
$\therefore$ if $\vec{F}$ is **conservative** then $\text{curl}(\vec{F}) = 0$ and there exists a scalar potential function ϕ such that $\vec{F} = \nabla\phi$.

20.5 ❑ APPLICATIONS OF LINE INTEGRALS

Work Done by a Force

Let $\vec{v}\,(x, y, z) = v_1(x, y, z)\,\vec{i} + v_2(x, y, z)\,\vec{j} + v_3(x, y, z)\,\vec{k}$ be a vector function defined and continuous at every point on C. Then, the integral of the tangential component of $\vec{v}$ along the curve C from a point P on to the point Q is given by

$$\int_P^Q \vec{v}\cdot\overrightarrow{dr} = \int_{C_1} \vec{v}\cdot d\vec{r} = \int_{C_1} v_1dx + v_2dy + v_3dz$$

where C_1 is the part of C, whose initial and terminal points are P and Q.

Let $\vec{v} = \vec{F}$, variable force acting on a particle which moves along a curve C. Then the work done W by the force $\vec{F}$ in displacing the particle from the point P to the point Q along the curve C is given by

$$W = \int_P^Q \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}$$

where C_1 is the part of C whose initial and terminal points are P and Q.

Suppose $\vec{F}$ is a conservative vector field; then $\vec{F}$ can be written as $\vec{F}$ = grad ϕ, where ϕ is a scalar potential.
Then, the work done

$$W = \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (\text{grad}\,\phi)\cdot d\vec{r}$$

$$= \int_{C_1}\left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right] = \int_P^Q d\phi = \left[\phi(x, y, z)\right]_P^Q$$

$\therefore$ work done depends only on the initial and terminal points of the curve C_1, i.e., the work done is independent of the path of integration. The units of work depend on the units of $|\vec{F}|$ and on the units of distance.

➢ **Note**

(i) **Condition for $\vec{F}$ to be conservative**

If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = 0$.

It is possible only when $\vec{F} = \nabla\phi$. which $\Rightarrow$ $\vec{F}$ is conservative.

$\therefore$ if $\vec{F}$ is an irrotational vector, it is conservative.

(ii) If $\vec{F}$ is irrotational (and, hence, conservative) and C is a closed curve then $\oint_C \vec{F}\cdot d\vec{r} = 0$. [$\because$ $\phi(A) = \phi(B)$, as A and B coincide].

20.6 ❑ SURFACES

A surface S may be represented by $F(x, y, z) = 0$.

The parametric representation of S is of the form

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

and the continuous functions $u = \phi(t)$ and $v = \phi(t)$ of a real parameter t represent a curve C on this surface S.

If S has a unique normal at each of its points whose direction depends continuously on the points of S then the surface S is called a **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions then it is called a **piecewise smooth surface**. For example, the surface of a sphere is smooth while the surface of a cube is piecewise smooth.

If a surface S is smooth from any of its points P, we may choose a unit normal vector $\vec{n}$ of S at P. The direction of $\vec{n}$ is then called the **positive normal direction of S at *P***. A surface S is said to be **orientable** or **two-sided**, if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface. If the positive direction of the normal is reversed as we move around a curve on S passing through P then the surface is **non-orientable** (i.e., one-sided) (Fig. 20.4).

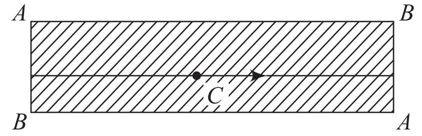


Fig. 20.4

- **Example**

A sufficiently small portion of a smooth surface is always orientable (Fig. 20.5).

A Mobius strip is an example of a non-orientable surface. A model of a Mobius strip can be made by taking a long rectangular piece of paper, making a half-twist and sticking the shorter sides together so that the two points A and the two points B coincide; then the surface generated is non-orientable.

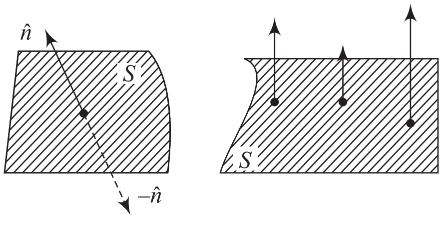


Fig. 20.5

20.7 ❑ SURFACE INTEGRALS

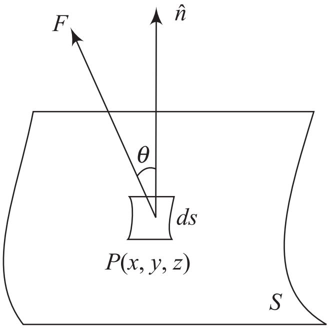


Fig. 20.6

Any integral which is to be evaluated over a surface is called a **surface integral**.

Let S be a two-sided surface, one side of which is considered arbitrarily as the positive side.

Let $\vec{F}$ be a vector point function defined at all points of S. Let ds be the typical elemental surface area in S surrounding the point $P(x, y, z)$.

Let $\hat{n}$ be the unit vector normal to the surface S at $P(x, y, z)$, drawn in the positive side (or outward direction).

Let θ be the angle between $\vec{F}$ and $\hat{n}$.

$\therefore$ the normal component of $\vec{F} = \vec{F} \cdot \hat{n} = F \cos\theta$.

The integral of this normal component through the elemental surface area ds over the surface S is called the **surface integral** of $\vec{F}$ over S and denoted as $\int_S F \cos\theta \cdot ds$ or $\int_S \vec{F} \cdot \hat{n} ds$.

If $d\vec{s}$ is a vector whose magnitude is ds and whose direction is that of $\hat{n}$, then $\overrightarrow{ds} = \hat{n} \cdot ds$. $\therefore \int_S \vec{F} \cdot \hat{n} ds$ can also be written as $\int_S \vec{F} \cdot \overrightarrow{ds}$.

➢ **Note**

(i) If S in a closed surface, the outer surface is usually chosen as the positive side.

(ii) $\int_S \phi d\vec{s}$ and $\int_S \vec{F} \times d\vec{s}$ where ϕ is a scalar point function are also surface integrals.

(iii) The surface integral $\int_S \vec{F} \cdot d\vec{s}$ is also denoted as $\iint_S \vec{F} \cdot d\vec{s}$.

(iv) If $\vec{F}$ represents the velocity of a fluid particle then the total outward flux of $\vec{F}$ across a closed surface S is the surface integral $\int_S \vec{F} \cdot d\vec{s}$.

(v) When the flux of $\vec{F}$ across every closed surface S in a region E vanishes, $\vec{F}$ is said to be a **solenoidal vector point function** in E.

(vi) It may be noted that $\vec{F}$ could equally well be taken as any other physical quantity such as gravitational force, electric force, magnetic force, etc.

20.8 ❑ VOLUME INTEGRALS

Any integral which is to be evaluated over a volume is called a **volume integral**.

If V is a volume bounded by a surface S then the triple integrals $\iiint_V \phi dv$ and $\iiint_V \vec{F} dv$ are called volume integrals. The first of these is a scalar and the second is a vector.

20.9 ❑ INTEGRAL THEOREMS

The following three theorems in vector calculus are of importance from theoretical and practical considerations:

(i) Green's theorem in a plane
(ii) Stokes' theorem
(iii) Gauss' divergence theorem

Green's theorem provides a relationship between a double integral over a region *R* and the line integral over the closed curve *C* bounding *R*. Green's theorem is also called the **first fundamental theorem** of integral vector calculus.

Stokes' theorem transforms line integrals into surface integrals and conversely. This theorem is a generalization of Green's theorem. It involves the curl.

Gauss' divergence theorem transforms surface integrals into a volume integral. It is named Gauss' divergence theorem because it involves the divergence of a vector function.

We shall give the statements of the above theorems (without proof) and apply them to solve problems.

Green's Theorem in a Plane

If *C* is a simple closed curve enclosing a region *R* in the *xy*-plane and $P(x, y)$, $Q(x, y)$ and its first-order partial derivatives are continuous in *R* then $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ where *C* is described in the anticlockwise direction.

Stokes' Theorem (Relation between Line Integral and Surface Integral)

Surface integral of the component of curl $\vec{F}$ along the normal to the surface *S*, taken over the surface *S* bounded by curve *C* is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve *C*.

Mathematically, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} \cdot ds$

Gauss' Divergence Theorem or Gauss' Theorem of Divergence (Relation between Surface Integral and Volume Integral)

The surface integral of the normal component of a vector function $\vec{F}$ taken around a closed surface *S* is equal to the integral of the divergence of $\vec{F}$ taken over the volume *V* enclosed by the surface *S*.

Mathematically, $\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V \text{div}\, \vec{F} \cdot dv$.

SOLVED EXAMPLES

Example 1 If $\vec{F} = 3xy\vec{i}, - y^2\vec{j}$, evaluate $\int_C \vec{F} \cdot dr$, where *C* is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution Let $x = t$, then the parametric equations of the parabola $y = 2x^2$ are $x = t$, $y = 2t^2$.

At the point (0, 0), $x = 0$ and so $t = 0$.

At the point (1, 2), $x = 1$ and so $t = 1$.

If $\vec{r}$ is the position vector of any point (x, y) in C, then

$$\vec{r} = x\vec{i} + y\vec{j}$$
$$= t\vec{i} + 2t^2\vec{j}$$

Also in terms of t, $\vec{F} = 3t(2t^2)\vec{i} - (2t^2)^2\vec{j}$

$$= 6t^3\vec{i} - 4t^4\vec{j}$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$
$$= \int_0^1 (6t^3\vec{i} - 4t^4\vec{j})\cdot(\vec{i} + 4t\vec{j})dt$$
$$= \int_0^1 (6t^3 - 16t^5)dt$$
$$= \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1$$
$$= \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$

Ans.

Example 2 Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant. **[KU May 2010]**

Solution A vector normal to the surface S is given by

$$\nabla(2x + y + 2z) = 2\vec{i} + \vec{j} + 2\vec{k}$$

$\therefore$ $\hat{n} = a$ unit vector normal to the surface S

$$= \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\vec{k}\cdot\hat{n} = \vec{k}\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = \frac{2}{3}$$

$$\therefore \quad \iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$$

where R is the projection of S

Now, $$\vec{A}\cdot\hat{n} = [(x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}]\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$$
$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$
$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$\left(\text{since on the plane } 2x + y + 2z = 6,\ z = \frac{6 - 2x - y}{2}\right)$$

$$= \frac{2}{3}y(y + 6 - 2x - y)$$

$$= \frac{4}{3}y(3 - x)$$

Hence, $\iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$.

$$= \iint_R \frac{4}{3}y(3 - x)\cdot\frac{3}{2}dx\,dy$$

$$= \int_0^3\int_0^{6-2x} 2y(3 - x)dy\,dx$$

$$= \int_0^3 2(3 - x)\left(\frac{y^2}{2}\right)_0^{6-2x} dx$$

$$= \int_0^3 (3 - x)(6 - 2x)^2\,dx$$

$$= 4\int_0^3 (3 - x)^3\,dx$$

$$= 4\left[\frac{(3 - x)^4}{4(-1)}\right]_0^3$$

$$= 81$$ **Ans.**

Example 3 If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$ then evaluate $\iiint \nabla\cdot\vec{F}\cdot dV$, where V is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$

$$= 4x - 2x = 2x$$

$\therefore$ $$\iiint_V \nabla\cdot F\cdot dv = \iiint_V 2x\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x[z]_0^{4-2x-2y}\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2x(4 - 2x - 2y)\,dy\,dx = \int_0^2 \int_0^{2-x} [4x(2 - x) - 4xy]\,dy\,dx$$

$$= \int_0^2 [4x(2 - x)y - 2xy^2]_0^{2-x} \cdot dx$$

$$= \int_0^2 [4x(2 - x)^2 - 2x(2 - x)^2]dx$$

$$= \int_0^2 2x(2 - x)^2 dx$$

$$= 2\int_0^2 (4x - 4x^2 + x^3) \cdot dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left[8 - \frac{32}{3} + 4\right] = \frac{8}{3}$$ **Ans.**

Example 4 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ and the curve C is the rectangle in the xy-plane bounded by $y = 0$, $y = b$, $x = 0$, $x = a$.

Solution In the xy-plane, $z = 0$

$$\vec{r} = x\vec{i} + y\vec{j},\ d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (x^2 + y^2)dx - 2xydy \quad (1)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r} \quad (2)$$

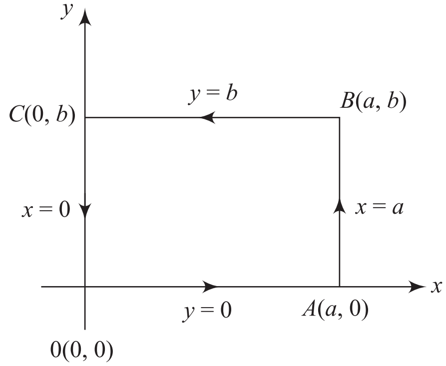


Fig. 20.7

Along OA, $y = 0$; $dy = 0$ and x varies from 0 to a
Along AB, $x = a$; $dx = 0$ and y varies from 0 to b

Along BC, $y = b$; $dy = 0$ and x varies from a to 0
Along CO, $x = 0$; $dx = 0$ and y varies from b to 0
Hence, from (1) and (2),

$$\int_C \vec{F}\cdot\overrightarrow{dr} = \int_{x=0}^{a} x^2\,dx - \int_{y=0}^{b} 2ay\,dy + \int_{x=a}^{0}(x^2+b^2)\,dx + \int_b^0 0\cdot dy$$

$$= \left(\frac{x^3}{3}\right)_0^a - (ay^2)_0^b + \left(\frac{x^3}{3}+b^2x\right)_a^0 + 0$$

$$= \left(\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2\right) = -2ab^2$$ **Ans.**

Example 5 Find the work done by the force $\vec{F} = (2xy+z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ when it moves a particle from (1, –2, 1) to (3, 1, 4) along any path. **[AU Dec. 2011]**

Solution Since the equation of the path is not given, the work done by the force $\vec{F}$ depends only on the terminal points.

Consider $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy+z^3) & x^2 & 3xz^2 \end{vmatrix}$

$$= \vec{i}[0-0] - \vec{j}[3z^2-3z^2] + \vec{k}[2x-2x] = 0$$

$\Rightarrow$ $\vec{F}$ is irrotational

Hence, $\vec{F}$ is conservative

Since $\vec{F}$ is irrotational, we have $\vec{F} = \nabla\phi$

It is easy to see that $\phi = x^2y + xz^3 + C$

$\therefore$ work done by $\vec{F} = \displaystyle\int_{(1,-2,1)}^{(3,1,4)} \vec{F}\cdot d\vec{r}$

$$= \int_{(1,-2,1)}^{(3,1,4)} \nabla\phi\cdot\overrightarrow{dr} = \int_{(1,-2,1)}^{(3,1,4)} d\phi \;\; [\text{as } \nabla\phi\cdot\overrightarrow{dr} = d\phi]$$

$$= [\phi]_{(1,-2,1)}^{(3,1,4)}$$

$$= [x^2y + xz^3 + C]_{(1,-2,1)}^{(3,1,4)}$$

$$= (201 + C) - (-1 + C) = 202$$ **Ans.**

Example 6 Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = e^x \sin y\vec{i}$ $+ e^x \cos y\vec{j}$; and C is the rectangle whose vertices are $(0,0), (1,0), \left(1, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.

Solution

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\therefore \quad \vec{F} \cdot d\vec{r} = e^x \sin y \cdot dx + e^x \cos y \cdot dy$$

Now along OA, $y = 0$; $dy = 0$
along AB, $x = 1$; $dx = 0$

along $BC, y = \dfrac{\pi}{2}; dy = 0$

along CO, $x = 0$; $dx = 0$
$\therefore$ circulation round the rectangle $OABC$ is

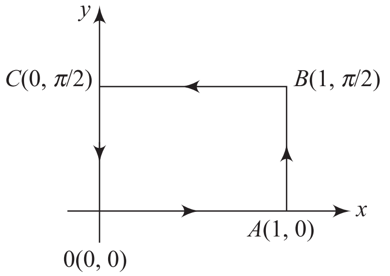


Fig. 20.7

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (e^x \sin y\, dx + e^x \cos y\, dy)$$

$$= \int_{OA} o + \int_{AB} e^1 \cos y\, dy + \int_{BC} e^x \sin\frac{\pi}{2} dx + \int_{CO} \cos y\, dy$$

$$= 0 + \int_0^{\frac{\pi}{2}} e \cos y \cdot dy + \int_1^0 e^x \sin\frac{\pi}{2} dx + \int_{\frac{\pi}{2}}^0 \cos y\, dy$$

$$= [e \sin y]_0^{\frac{\pi}{2}} + [e^x]_1^0 + [\sin y]_{\frac{\pi}{2}}^0 = e + (1 - e) - 1 + 0 = 0 \qquad \textbf{Ans.}$$

Example 7 Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Solution Total work done

$$= \int_C \vec{F} \cdot d\vec{r} = \int_C (3xy\vec{i} - 5z\vec{j} + 10x\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C [3xydx - 5zdy + 10xdz]$$

$$= \int_{t=1}^{2} [3(t^2+1)(2t^2)d(t^2+1) - 5t^3 d(2t^2) + 10(t^2+1)d(t^3)]$$

$$= \int_{t=1}^{2} [6t^2(t^2+1)(2tdt) - 20t^4 dt + 30t^2(t^2+1)dt]$$

$$= \int_{t=1}^{2} [12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2]dt$$

$$= \int_{t=1}^{2} [12t^5 + 10t^4 + 12t^3 + 30t^2]dt$$

$$= 12\left[\frac{t^6}{6}\right]_1^2 + 10\left[\frac{t^5}{5}\right]_1^2 + 12\left[\frac{t^4}{4}\right]_1^2 + 30\left[\frac{t^3}{3}\right]_1^2$$

$$= 12\left[\frac{2^6}{6} - \frac{1}{6}\right] + 10\left[\frac{2^5}{5} - \frac{1}{5}\right] + 12\left[\frac{2^4}{4} - \frac{1^4}{4}\right] + 30\left[\frac{2^3}{3} - \frac{1^3}{3}\right]$$

$$= 12 \cdot \frac{63}{6} + 10 \cdot \frac{31}{5} + 12 \cdot \frac{15}{4} + 30 \cdot \frac{7}{3}$$

$$= 126 + 62 + 45 + 70$$

$$= 303$$

Ans.

Example 8 If $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$, evaluate $\iint_S \vec{F} \cdot \hat{n} ds$ where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. **[AU Dec. 2009]**

Solution The surface of the cube consists of the following six faces:

(a) Face *LMND*
(b) Face *TQPO*
(c) Face *QPNM*
(d) Face *TODL*
(e) Face *TQMl*
(f) Face *ODNP*

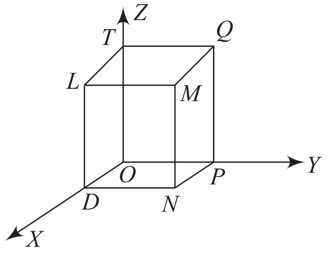


Fig. 20.8

Now, for the face *LMND*:

$$\hat{n} = \vec{i}, x = OD = 1$$

Hence, $$\iint_S F \cdot \hat{n} ds = \iint_{LMND} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{i}\, dy\, dz$$

$$= \iint_{LMND} 4xz\, dy\, dz = 4 \int_{LMND} z\, dy\, dz \quad (\because x = 1)$$

$$= 4 \int_{z=0}^{1} \int_{y=0}^{1} z\, dy\, dz = 4\left[\left(\frac{z^2}{2}\right)_0^1 (y)_0^2\right] = 2 \quad (1)$$

For the face *TQPO*: $\hat{n} = -\vec{i}, x = 0$

Hence, $$\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iint_{TQPO} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{i}) dy\, dz$$

$$= \iint_{TQPO} (-4xz) dy\, dz = 0 \quad (\because x = 0) \quad (2)$$

For the face *OPNM*: $\hat{n} = \vec{j}, y = 1$

Hence, $$\iint_S \vec{F} \cdot \hat{n} ds = \iint_{QPNM} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{j}\, dx\, dz$$

$$= \iint_{QPNM} (-y^2 dx\, dz) = \iint_{QPNM} -dx\, dz \quad (\because y = 1)$$

$$= -\int_{z=0}^{1}\int_{x=0}^{1} dx\,dz = -[x]_0^1 [z]_0^1 = -1 \quad (3)$$

For the face *TODL*: $\hat{n} = -\vec{j}, y = 0$

Hence, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{TODL} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j})dx\,dz$

$$= \iint_{TODL} (y^2 dxdz) = 0 \quad (\because y = 0) \quad (4)$$

For the face *TQML*: $\hat{n} = \vec{k}, z = 1$

Hence, $\iint_{TQML} \vec{F}\cdot\hat{n}\,ds = \iint_{TQML} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\,dx\,dy$.

$$= \iint_{TQML} yz\,dx\,dy = \iint_{TQML} y\,dx\,dy \quad (\because z = 1)$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} y\,dx\,dy = [x]_0^1\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} \quad (5)$$

For the face *ODNP*: $\hat{n} = -\vec{k}, z = 0$

Hence, $\iint_{ODNP} \vec{F}\cdot\hat{n}\,ds = \iint_{ODNP} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{k})\cdot dx\,dy$

$$= \iint_{ODNP} (-yz)dx\,dy = 0, \quad (\because z = 0) \quad (6)$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$$ **Ans.**

Example 9 Verify Stokes' theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}$ over the surface of a cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the *XOY* plane (open at the bottom). **[KU May 2010]**

Solution Consider the surface of the cube as shown in the figure. Bounding path is *OABCO* shown by arrows.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}]\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (y - z + 2)dx + (yz + 4)dy - xz\,dz$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \quad (1)$$

Along *OA*, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2dx = (2x)_0^2 = 4$$

Along *AB*, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4dy = 4(y)_0^2 = 8$$

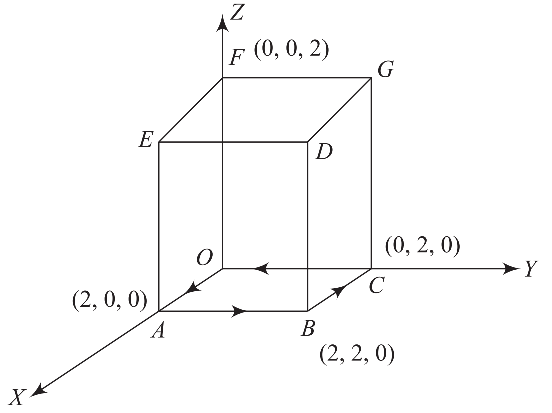


Fig. 20.9

Along *BC*, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_0^2 (2 - 0 + 2)dx = (4x)_2^0 = -8$$

Along *CO*, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int (y - 0 + 2) \times 0 + (0 + 4)dy - 0$$

$$= 4\int dy = 4(y)_2^0 = -8$$

On putting the values of these integrals in (1), we get

$$\int_C \vec{F} \cdot d\vec{r} = 4 + 8 - 8 = -4$$

To obtain surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= (0 - y)\vec{i} - (-z + 1)\vec{j} + (0 - 1)\vec{k} = -y\vec{i} + (z - 1)\vec{j} - \vec{k}$$

Here, we have to integrate over the five surfaces, *ABDE*, *OCGF*, *BCGD*, *OAEF*, *DEFG*.
Over the surface *ABDE*: $x = 2,\ \hat{n} = \vec{i}\ ,\ ds = dydz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{i}\, dy\, dz$$

$$= \iint_S -y\, dydz = -\int_0^2 y\, dy \int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$$

Over the surface *OCGF*: $x = 0,\ \hat{n} = -\vec{i},\ ds = dy\ dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{i}) dy\, dz$$

$$= \iint_S y\, dy\, dz = \int_0^2 y\, dy \int_0^2 dz = \left[\frac{y^2}{2}\right]_0^2 = 4$$

Over the surface *BCGD*: $y = 2,\ \hat{n} = \vec{j}\ ,\ ds = dx\ dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{j}\, dx\, dz$$

$$= \iint_S (z-1) dx\, dz$$

$$= \int_0^2 dx \int_0^2 (z-1)\, dz$$

$$= [x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *OAEF*: $y = 0,\ \hat{n} = -\vec{j}\ ,\ ds = dx\ dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{j}) dx\, dz$$

$$= -\iint_S (z-1) dx\, dz$$

$$= -\int_0^2 dx \int_0^2 (z-1) dz$$

$$= -[x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *DEFG*: $z = 2$, $\hat{n} = \vec{k}$, $ds = dx\,dy$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{k}\,dx\,dy$$

$$= -\iint dx\,dy = -\int_0^2 dx \int_0^2 dy$$

$$= -[x]_0^2 [y]_0^2 = -4$$

Total surface integral $= -4 + 4 + 0 + 0 - 4 = -4$

Thus $\iint_S \text{curl } \vec{F} \cdot \hat{n}\,ds = \int_C \vec{F} \cdot d\vec{r} = -4$

which verifies Stokes' theorem. **Verified.**

Example 10 Verify Green's theorem in the plane for $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where *C* is a square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

Solution Given integrand is of the form $Mdx + Ndy$, where $M = x^2 - xy^3$, $N = y^2 - 2xy$.

Now to verify Green's theorem, we have to verify that

$$\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \iint_R (-2y + 3xy^2)dx\,dy \qquad (1)$$

Consider $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where the curve *C* is divided into four parts, hence the line integral along *C* is nothing but the sum of four line integrals along four lines *OA*, *AB*, *BC* and *CO*.

Along *OA* : $y = 0$, $dy = 0$ and x varies from 0 to 2.

Hence, $$\int_{OA} [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \int_{x=0}^{2} x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

Along *AB* : $x = 2$, $dx = 0$, and y varies from 0 to 2.

Hence, $$\int_{AB} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_0^2 (y^2 - 4y)dy = \left(\frac{y^3}{3} - 4\frac{y^2}{2}\right)_0^2$$

$$= \left(\frac{8}{3}\right) - 8 = -\frac{16}{3}$$

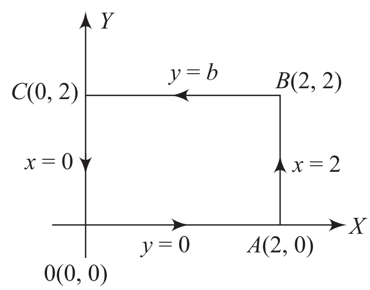


Fig. 20.10

Along *BC*: $y = 2$, $dy = 0$ and x varies from 2 to 0.

Hence, $$\int_{BC} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_{x=2}^{0} (x^2 - 8x)dx = \left(\frac{x^3}{3} - 8\frac{x^2}{2}\right)_2^0$$

$$= 0 - 0 - \frac{8}{3} + 16 = \frac{40}{3}$$

Along $CO : x = 0$, $dx = 0$ and y varies from 2 to 0

Hence, $\int_{CO} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$

$$= \int_{y=2}^{0} y^2 dy = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$$

$$\therefore \qquad \int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8 \qquad (2)$$

Now consider

$$\iint_R (-2y + 3xy^2)dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2} (-2y + 3xy^2)dy\,dx$$

$$= \int_{x=0}^{2} \left(-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_{x=0}^{2} \left[-4 + 3x\left(\frac{8}{3}\right)\right] dx = \left(-4x + 8\frac{x^2}{2}\right)_0^2$$

$$= -8 + 16 + 0 = 8 \qquad (3)$$

From (2) and (3), we observe that the relation (1) is true.
Hence, Green's theorem is verified. **Ans.**

Example 11 Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. **[KU Nov. 2010]**

Solution For verification of the divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$

$$\therefore \qquad \iiint_V \text{div}\,\vec{F}\,dv$$

$$= \int_o^c \int_o^b \int_o^a 2(x + y + z)\,dx\,dy\,dz$$

$$= \int_o^c \int_o^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dy\,dz$$

$$= \int_0^c \int_o^b 2\left(\frac{a^2}{2} + ya + za\right) dy\,dz$$

$$= \int_o^c 2\left[\left(\frac{a^2}{2}y + \frac{y^2 a}{2} + azy\right)\right]_o^b dz$$

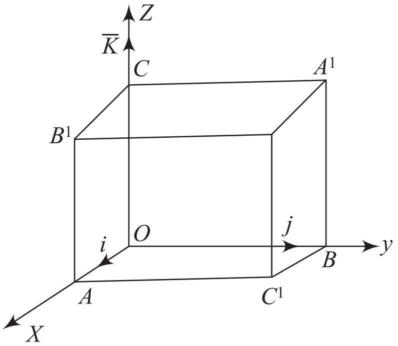


Fig. 20.11

$$= 2\int_o^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right)dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + \frac{abz^2}{2}\right]_o^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a + b + c) \quad (1)$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelepiped into 6 parts.

S_1 : Face $OAC'B$

S_2 : Face $CB'PA'$

S_3 : Face $OBA'C$

S_4 : Face $AC'PB'$

S_5 : Face $OCB'A$

S_6 : Face $BA'PC'$

Also,
$$\iint_S \vec{F}\cdot\hat{n}ds = \iint_{S_1} \vec{F}\cdot\hat{n}ds + \iint_{S_2} \vec{F}\cdot\hat{n}ds + \iint_{S_3} \vec{F}\cdot\hat{n}ds + \iint_{S_4} \vec{F}\cdot\hat{n}ds + \iint_{S_5} \vec{F}\cdot\hat{n}ds + \iint_{S} \vec{F}\cdot\hat{n}ds \quad (2)$$

On $S_1 : z = 0,\ \hat{n} = -\vec{k}\ , ds = dx\,dy$

so that $\vec{F}\cdot\hat{n} = (x^2\vec{i} + y^2\vec{j} - xy\vec{k})\cdot(-\vec{k}) = xy$

$$\therefore \quad \iint_{S_1} \vec{F}\cdot\hat{n}ds = \int_o^b\int_o^a xy\,dx\,dy = \int_o^b \left(y\frac{x^2}{2}\right)_o^a dy$$

$$= \frac{a^2}{2}\int_o^b y\,dy = \frac{a^2b^2}{4} \quad (3)$$

On $S_2 : z = c,\ \hat{n} = \vec{k}\ , ds = dx\,dy,\ \vec{F} = (x^2 - cy)\,\vec{i} + (y^2 - cx)\,\vec{j} + (c^2 - xy)\,\vec{k}$.

so that $\vec{F}\cdot\hat{n} = [(x^2 - cy)\vec{i} + (y^2 - (x)\vec{j}) + (c^2 - xy)\vec{k}]\cdot\vec{k} = c^2 - xy$.

$$\therefore \quad \iint_{S_2} \vec{F}\cdot\hat{n}ds = \int_o^b\int_o^a (c^2 - xy)\,dx\,dy = \int_o^b \left(c^2a - \frac{a^2}{2}y\right)dy$$

$$= abc^2 - \frac{a^2b^2}{4} \quad (4)$$

On $S_3 : x = 0,\ \hat{n} = -\vec{i}\,, \vec{F} = -yz\vec{i} + y^2\vec{j} + z^2\vec{k}, dz = dy\,dz$

so that $\vec{F}\cdot\hat{n} = (-yz\vec{i} + y^2\vec{j} + z^2\vec{k})\cdot(-\vec{i}) = yz, ds = dy\,dz$

$$\therefore \quad \iint_{S_3} \vec{F}\cdot\hat{n}ds = \int_o^c\int_o^b yz\,dy\,dz = \int_o^c \frac{b^2}{2}z\,dz = \frac{b^2c^2}{4} \quad (5)$$

On $S_4 : x = a,\ \hat{n} = \vec{i}\,, \vec{F} = (a^2 - yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}$

so that $\vec{F}\cdot\hat{n}=[(a^2-yz)\vec{i}+(y^2-az)\vec{j}+(z^2-ay)\vec{k}]\cdot\vec{i}$

$=a^2-yz,\ ds=dy\,dz$

$$\therefore \qquad \iint_{S_4}\vec{F}\cdot\hat{n}ds=\int_o^c\int_o^b(a^2-yz)dy\,dz=\int_o^c\left(a^2b-\frac{b^2}{2}z\right)dz$$

$$=a^2bc-\frac{b^2c^2}{4} \qquad (6)$$

On $S_5: y=0,\ \hat{n}=-\vec{j},\vec{F}=x^2\vec{i}-zx\vec{j}+z^2\vec{k}, ds=dxdz$

so that $\vec{F}\cdot\hat{n}=(x^2\vec{i}-zx\vec{j}+z^2\vec{k})\cdot(-\vec{j})=zx$

$$\therefore \qquad \iint_{S_5}\vec{F}\cdot\hat{n}ds=\int_o^a\int_o^c zx\,dz\,dx=\int_o^a\frac{c^2}{2}x\,dx=\frac{a^2c^2}{4} \qquad (7)$$

On $S_6: y=b,\ \hat{n}=\vec{j},\vec{F}=(x^2-bz)\vec{i}+(b^2-zx)\vec{j}+(z^2-bx)\vec{k}$

$ds=dxdz$

so that $\vec{F}\cdot\hat{n}=[(x^2-bz)\vec{i}+(b^2-zx)\vec{j}+(z^2-bx)\vec{k}]\cdot\vec{j}$

$=b^2-zx.$

$$\therefore \qquad \iint_{S_6}\vec{F}\cdot\hat{n}=\int_o^a\int_o^c(b^2-zx)dz\,dx$$

$$=\int_o^a\left(b^2c-\frac{c^2}{2}x\right)\cdot dx=ab^2c-\frac{a^2c^2}{4} \qquad (8)$$

By using (3), (4), (5), (6), (7) and (8), in (2), we get

$$\iint_S\vec{F}\cdot\hat{n}ds=\frac{a^2b^2}{4}+abc^2-\frac{a^2b^2}{4}+\frac{b^2c^2}{4}+a^2bc-\frac{b^2c^2}{4}+\frac{a^2c^2}{4}+ab^2c-\frac{a^2c^2}{4}$$

$$=abc(a+b+c) \qquad (9)$$

The equalities (1) and (9) verify the divergence theorem. **Ans.**

Example 12 Verify Green's theorem in the plane for $\int_C(3x^2-8y^2)dx+(4y-6xy)dy$ where C is the boundary of the region defined by (i) $y=\sqrt{x}$, $y=x^2$ and (ii) $x=0, y=0, x+y=1$. **[AU July 2010, June 2012 ; KU Nov. 2011, KU April 2013]**

Solution

(i) $y=\sqrt{x}$, i.e., $y^2=x$ and $y=x^2$ are two parabolas intersecting at 0(0, 0) and $A(1, 1)$. Here, $P=3x^2-8y^2$, $Q=4y-6xy$

$$\frac{\partial P}{\partial y}=-16y,\frac{\partial Q}{\partial x}=-6y$$

$$\therefore \qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=10y$$

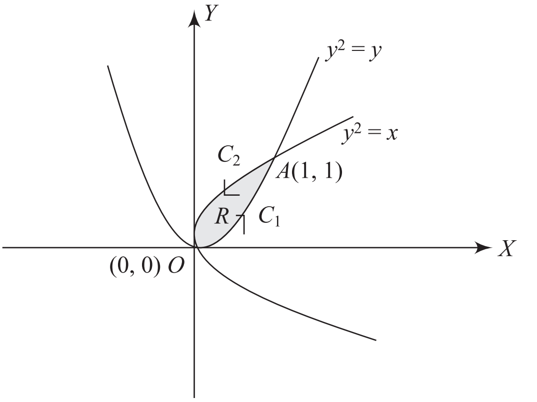


Fig. 20.12

If R is the region bounded by C then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx = \int_0^1 10\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\left[\frac{3}{10}\right] = \frac{3}{2} \qquad (1)$$

Also, $\int_C P\,dx + Q\,dy = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy)$

Along C_1, $x^2 = y$. $\therefore$ $2x\, dx = dy$ and the limits of x are from 0 to 1.

$$\therefore \qquad \int_{C_1} (P\,dx + Q\,dy)$$

$$= \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x \cdot x^2) \cdot 2x\,dx \text{ (since } x^2 = y)$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= [x^3 + 2x^4 - 4x^5]_0^1 = -1$$

Along C_2, $y^2 = x$. $\therefore$ $2y\, dy = dx$ and the limits of y are from 1 to 0.

$$\int_{C_2}(P\,dx + Q\,dy)$$

$$= \int_1^0 (3y^4 - 8y^2)2y\,dy + (4y - 6y^2 \cdot y) \cdot dy$$

$$\therefore \quad = \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$$\therefore \quad \int_C (P\,dx + Q\,dy) = -1 + \frac{5}{2} = \frac{3}{2} \qquad (2)$$

The equalities of (1) and (2) verify Green's theorem in the plane. **Ans.**

(ii) Here, $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx$$

$$= \int_0^1 5[y^2]_0^{1-x}\,dx$$

$$= 5\int_0^1 (1-x)^2\,dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= \frac{-5}{3}(0-1) = \frac{5}{3} \qquad (1)$$

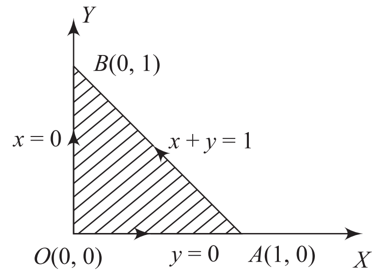


Fig. 20.13

Along OA, $y = 0$ $\therefore$ $dy = 0$ and the limits of x are from 0 to 1.

$$\therefore \quad \int_{OA} P\,dx + Q\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$$

Along AB, $y = 1 - x$. $\therefore$ $dy = -dx$ and the limits of x are from 1 to 0.

$$\therefore \quad \int_{AB} P\,dx + Q\,dy = \int_1^0 [3x^2 - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)](-dx)$$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)dx$$

$$= \int_1^0 (-12 + 26x - 11x^2) \cdot dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

Along BO, $x = 0$. $\therefore$ $dx = 0$ and the limits of y are from 1 to 0

$$\therefore \quad \int_{BO} P\,dx + Q\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$\therefore$ line integral along C (i.e., along $OABO$) $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$

i.e., $$\int_C (P\,dx + Q\,dy) = \frac{5}{3} \qquad (2)$$

The equality of (1) and (2) verifies Green's theorem in the plane. **Verified.**

Example 13 Evaluate $\int_C (e^x dx + 2y\,dy - dz)$ by using Stokes' theorem, where C is the curve $x^2 + y^2 = 4$, $z = 2$. **[AU May 2010]**

Solution

$$\begin{aligned}
&\int_C (e^x\,dx + 2y\,dy - dz)\\
&\quad = \int_C (e^x\vec{i} + 2y\vec{j} - \vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})\\
&\quad = \int_C \vec{F}\cdot\overrightarrow{dr} \text{ where } \vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}
\end{aligned}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

$\therefore$ by Stokes' theorem, $$\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\cdot ds$$

$$= 0 \text{ (since curl } \vec{F} = 0)$$ **Ans.**

Example 14 Find the work done by the force $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, when it moves a particle along the arc of the curve $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ from $t = 0$ to $t = 2n$. **[AU Dec. 2007]**

Solution From the vector equation of the curve C, we get the parametric equations of the curve as $x = \cos t$, $y = \sin t$, $z = t$.

Work done by the force $\vec{F} = \int_C \vec{F}\cdot\overrightarrow{dr}$

$$\begin{aligned}
&= \int_C (z\vec{i} + x\vec{j} + y\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})\\
&= \int_C (z\,dx + x\,dy + y\,dz)\\
&= \int_0^{2\pi} [t(-\sin t) + \cos^2 t + \sin t]dt\\
&= \left[t\cos t - \sin t + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) - \cos t\right]_0^{2\pi}\\
&= (2\pi + \pi - 1) - (-1)\\
&= 3\pi
\end{aligned}$$ **Ans.**

Example 15 Verify Stokes' theorem for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\vec{k}$ where S is the open surface of the rectangular parallelepiped formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ above the XOY-plane. **[AU Dec. 2007]**

Solution Stokes' theorem is given by

$$\int_C \vec{F} \cdot \overrightarrow{dr} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds$$

$$\text{Here, curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -zx \end{vmatrix}$$

$$= 2y\vec{i} + z\vec{j} + x\vec{k} \quad \therefore \int_C (xy\,dx - 2yz\,dy - zx\,dz) - \iint_S (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\, ds \qquad (1)$$

The open cuboid S is made up of the five faces $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ and is bounded by the rectangle $OAC'B$ lying on the XOY plane. LHS of (1) is

$$= \int_{OAC'B} (xy\,dx - 2yzdy - zxdz)$$

$$= \int_{OAC'B} xy\,dx$$

(since the boundary C lies on the XOY plane, $z = 0$)

$$= \int_{OA} xy\,dx + \int_{AC'} xy\,dx + \int_{C'B} xy\,dx + \int_{BO} xy\,dx$$

Along OA, $y = 0$, $dy = 0$
Along AC', $x = 1$, $dx = 0$
Along $C'B$, $y = 2$, $dy = 0$
Along BO, $x = 0$, $dx = 0$

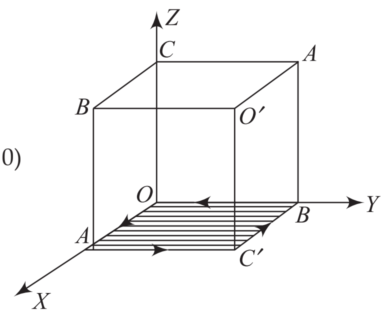


Fig. 20.14

$$\therefore \qquad \int_{OAC'B} xy\,dx = 0 + 0 + \int_{C'B} xy\,dx + 0 = \int_1^0 2x\,dx$$

(as along $C'B$, x varies from 1 to 0).

$$= -1 \qquad (2)$$

RHS of (1) is

$$\iint_S (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\, ds = \iint_{O'C'AB'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\, ds + \iint_{A'BOC} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\, ds$$

$$+ \iint_{A'BC'O'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\, ds + \iint_{COAB'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\, ds$$

$$+ \iint_{A'O'B'C} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n}\, ds$$

$$= \int_0^3\int_0^2 2y\,dy\,dz - \int_0^3\int_0^2 2y\,dy\,dz - \int_0^1\int_0^3 z\,dz\,dx$$
$$- \int_0^1\int_0^3 z\,dz\,dx - \int_0^2\int_0^1 x\,dx\,dy$$
$$= -\int_0^2\int_0^1 x\,dx\,dy = -\int_0^2 \left(\frac{x^2}{2}\right)_0^1 dy = -1 \qquad (3)$$

From (2) and (3), Stokes' theorem is verified. **Verified.**

Example 16 Verify the divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. **[AU Dec. 2007, KU Nov. 2011]**

Solution Gauss' divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}ds = \iiint_V (\text{div}\,\vec{F})dv \qquad (1)$$

$$\text{LHS of } (1) = \iint_{x=1} x^2\,ds + \iint_{x=-1} -x^2\,ds + \iint_{y=1} z\,ds + \iint_{y=-1} -z\,ds + \iint_{z=1} yz\,ds + \iint_{z=-1} -yz\,ds = 0 \qquad (2)$$

$$\text{RHS of } (1) = \iiint_V (\text{div}\,\vec{F})\cdot dv$$
$$= \iiint_V (2x+y)\,dx\,dy\,dz$$
$$= \int_{-1}^1\int_{-1}^1\int_{-1}^1 (2x+y)\,dx\,dy\,dz$$
$$= \int_{-1}^1\int_{-1}^1 2y\,dy\,dz = 0 \qquad (3)$$

From (2) and (3), Gauss' divergence theorem is verified. **Verified.**

Example 17 Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot\vec{dr}$, where $\vec{F} = (\sin x - y)\vec{i} - \cos x\vec{j}$ and C is the boundary of the triangle whose vertices are $(0,0), \left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

[KU Nov. 2011]

Solution By Stokes' theorem, we have $\int_C \vec{F}\cdot\vec{dr} = \iint_S \text{ curl } \vec{F}\cdot\hat{n}\cdot ds$.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x - y & -\cos x & 0 \end{vmatrix}$$
$$= (\sin x + 1)\vec{k}$$

$\therefore$ the given line integral

$$= \iint_R (1+\sin x)\,dx\,dy$$

$$= \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (1+\sin x)\,dx\,dy$$

$$= \int_0^1 \left[x - \cos x\right]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \left[\frac{\pi}{2} - \frac{\pi y}{2} + \cos\frac{\pi y}{2}\right] dy$$

$$= \left[\frac{\pi}{2}y - \frac{\pi y^2}{4} + \frac{2}{\pi}\sin\frac{\pi y}{2}\right]_0^1$$

$$= \frac{\pi}{4} + \frac{2}{\pi}$$

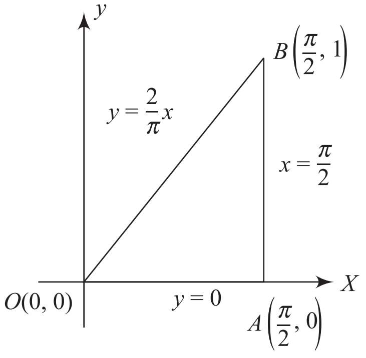


Fig. 20.15

$\therefore \quad \int_C \vec{F}\cdot\overrightarrow{dr} = \frac{\pi}{4} + \frac{2}{\pi}$ **Ans.**

EXERCISE

Part A

1. State Green's theorem in a plane.
2. Give the relation between a line integral and a surface integral.
3. State Gauss' divergence theorem.
4. Deduce Green's theorem in a plane from Stokes' theorem.
5. In Gauss' divergence theorem, surface integral is equal to _______ integral.
6. The integral of $\vec{F}\cdot d\vec{r}$ is
   (i) line integral (ii) zero
   (iii) surface integral (iv) one
7. Using Green's theorem, prove that the area enclosed by a simple closed curve C is $\frac{1}{2}\int (x\,dy - y\,dx)$.
8. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$ where C is the part of the curve $y = x^3$ between $x = 1$ and $x = 2$.
9. If $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$ along the straight line $y = x$ from (0, 0) to (1, 1).
10. If C is a simple closed curve and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, prove that $\int_C \vec{r}\cdot\overrightarrow{dr} = 0$.
11. Evaluate $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ where C is the circle given by $x^2 + y^2 + z^2 = 1$ and $z = 0$.
12. Use the integral theorems to prove $\nabla\cdot(\nabla\times\vec{F}) = 0$.

13. Evaluate $\int_C (x\,dy - y\,dx)$, where C is the circle $x^2 + y^2 = a^2$.

14. Evaluate $\int_C \vec{F} \cdot \overrightarrow{dr}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and C is the curve $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, t varying from –1 to 1.

Part B

1. If a force $\vec{F} = 2x^2y\vec{i} + 3xy\vec{j}$ displaces a particle in the xy plane from (0, 0) to (1, 4) along a curve $y = 4x^2$, find the work done. $\left(\textbf{Ans. } \frac{104}{5}\right)$

2. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to (1, 1) along a parabola $y^2 = x$. $\left(\textbf{Ans. } \frac{2}{3}\right)$

3. Verify Green's theorem in a plane with respect to $\int_C (x^2dx + xy\,dy)$, where C is the boundary of the square formed by $x = 0, y = 0, x = a, y = a$. **[AU Dec. 2009]** $\left(\textbf{Ans. } \frac{a^3}{2}\right)$

4. Use Green's theorem to evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is the square formed by the lines $y = \pm 1, x = \pm 1$. **(Ans.** 0**)**

5. Use divergence theorem to evaluate $\iint_S (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k}) \cdot \hat{n}ds$ where S is the closed surface bounded by the XOY-plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane. **(Ans.** πa^4**)**

6. Verify Stokes' theorem for $\vec{F} = (x^2 + y - 4)\vec{i} + 3xy\vec{j} + (2xz + z^2)\vec{k}$ over the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above the XOY plane. **(Ans.** -16π**)**

7. Use the divergence theorem to evaluate $\int_S \vec{A} \cdot \overrightarrow{ds}$ where $\vec{A} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2 = y^2 + z^2 = a^2$. $\left(\textbf{Ans. } \frac{12\pi a^5}{5}\right)$

8. Use the divergence theorem to evaluate $\iint_S x^3dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where S is the surface of the region bounded by the closed cylinder $x^2 + y^2 = a^2$, $(0 \le z \le b)$ $z = 0$ and $z = b$. $\left(\textbf{Ans. } \frac{5\pi a^4 b}{4}\right)$

9. Using Green's theorem, evaluate $\int_C [(y - \sin x)dx + \cos x\,dy]$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$. $\left[\textbf{Ans. } -\left(\frac{\pi^2 + 8}{4\pi}\right)\right]$

10. Evaluate $\int_C [(x^2 + y^2)dx - 2xy\,dy]$ where C is the rectangle bounded by $y = 0$, $x = 0, y = b, x = a$ using Green's theorem. **(Ans.** $-2ab^2$**)**

11. Verify Stokes' theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **(Ans.** $-\pi$**)**

12. Verify Stokes' theorem for $\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is the boundary. **(Ans.** 9π**)**

13. Find the area of $x^{2/3} + y^{2/3} = a^{2/3}$ using Green's theorem. $\left(\textbf{Ans. } \frac{3\pi a^2}{8}\right)$

14. Using Stokes' theorem, evaluate $\int_C (xy\,dx + xy^2 dy)$ taking C to be a square with vertices (1, 1), (–1, 1), (–1, –1) and (1, –1). $\left(\textbf{Ans. } \frac{4}{3}\right)$

15. Verify Gauss' divergence theorem for $\vec{F} = y\vec{i} + x\vec{j} + z^3\vec{k}$ over the cylindrical region $x^2 + y^2 = 9$, $z = 0$, $z = 6$. **(Ans.** 1944π**)**

Unit VIII

Vector Integration

Chapter 20: Line Integral, Surface Integral and Integral Theorems

20 Line Integral, Surface Integral and Integral Theorems

Chapter Outline

- Introduction
- Integration of Vectors
- Line Integral
- Circulation
- Application of Line Integrals
- Surfaces
- Surface Integrals
- Volume Integrals
- Integral Theorems

20.1 ❑ INTRODUCTION

In multiple integrals, we generalized integration from one variable to several variables. Our goal in this chapter is to generalize integration still further to include integration over curves or paths and surfaces. We will define integration not just of functions but also of vector fields. Integrals of vector fields are particularly important in applications involving the "field theories" of physics, such as the theory of electromagnetism, heat transfer, fluid dynamics and aerodynamics.

In this chapter, we shall define line integrals and surface integrals. We shall see that a line integral is a natural generalization of a define integral and a surface integral is a generalization of a double integral. Line integrals can be transformed into double integrals or into surface integrals and conversely. Triple integrals can be transformed into surface integrals and vice versa. These transformations are of great practical importance. Theorems of Green, Gauss and Stokes serve as powerful tools in many applications as well as in theoretical problems.

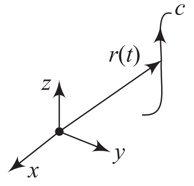


Fig. 20.1

In this chapter, we study the three main theorems of Vector Analysis: Green's Theorem, Stokes' Theorem and the Divergence Theorem. This is a fitting conclusion to the text because each of these theorems is a vector generalization of the Fundamental Theorem of calculus. This chapter is thus the culmination of efforts to extend the concepts and methods of single-variable calculus to the multivariable setting. However, far from being a terminal point, vector analysis the gateway to the field theories of mathematics physics and engineering. This includes, first and foremost, the theory of electricity and magnetism as expressed by the famous *Maxwell's equations*. It also includes fluid dynamics, aerodynamics, analysis of continuous matter, and at a more advanced level, fundamental physical theories such as general relativity and the theory of elementary particles.

Curves

Curves in space are important in calculus and in physics (for instance, as paths of moving bodies).

A curve C in space can be represented by a vector function

$$\vec{r}(t) = [x(t), y(t), z(t)] \qquad (20.1)$$
$$= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

where x, y, z are Cartesian coordinates. This is called a **parametric representation** of the curve (Fig. 20.1), t is called the **parameter** of the representation. To each value t_0 of t, there corresponds a point of C with position vector $\vec{r}(t_0)$, that is with coordinates $x(t_0)$, $y(t_0)$ and $z(t_0)$.

The parameter t may be time or something else. Equation (20.1) gives the **orientation** of C, a direction of travelling along C, so that t increasing is called the **positive sense** on C given by (20.1) and that of decreasing t is the **negative sense**.

- **Examples**

Straight line, ellipse, circle, etc.

The concept of a line integral is a simple and natural generalization of a definite integral $\int_a^b f(x)dx$ (20.2)

In (20.2), we integrate the **integrand** $f(x)$ from $x = a$ to $x = b$ along the x**-**axis. In a line integral, we integrate a given function, called the integrand, along a curve C in space (or in the plane).

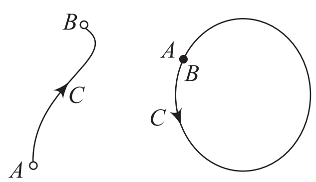
Hence, curve integral would be a better turn, but line integral is standard.

We represent a curve C by a parametric representation

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, (a \le t \le b)$$

We call C the **path of integration**, $A{:}\vec{r}(a)$ its **initial point** and $B{:}\vec{r}(b)$, its **terminal point**. The curve C is now oriented. The direction from A to B, in which t increases, is called the positive direction on C. We can indicate the direction by an arrow [Fig. 20.2(a)].

The points A and B may coincide [Fig. 20.2(b)]. Then C is called a **closed path**.



(a) **(b)**

Fig. 20.2

➢ **Note**

(i) A **plane curve** is a curve that lies in a plane in space.
(ii) A curve that is not plane is called a **twisted curve**.

20.2 ❑ INTEGRATION OF VECTORS

If two vector functions $\vec{F}(t)$ and $\vec{G}(t)$ be such that $\frac{d\vec{G}(t)}{dt} = \vec{F}(t)$, then $\vec{G}(t)$ is called an integral of $\vec{F}(t)$ with respect to the scalar variable t and we write $\int \vec{F}(t)dt = \vec{G}(t)$. If $\vec{C}$ be an arbitrary constant vector, we have $\vec{F}(t) = \frac{d\vec{G}(t)}{dt} = \frac{d}{dt}[\vec{G}(t)+\vec{C}]$, then $\int \vec{F}(t)dt = \vec{G}(t)+\vec{C}$. This is called the indefinite integral of $\vec{F}(t)$ and its definite integral is $\int_a^b \vec{F}(t)dt = [\vec{G}(t)+\vec{C}]_a^b = \vec{G}(b) - \vec{G}(a)$.

20.3 ❑ LINE INTEGRAL

Any integral which is to be evaluated along a curve is called a **line integral**. Consider a continuous vector point function $\vec{F}(R)$ which is defined at each point of the curve C in space. Divide C into n parts at the points $A = p_0, p_1 \dots p_{i-1}, p_i \dots p_n = B$

Let their position vectors be $\vec{R}_0, \vec{R}_1 \dots \vec{R}_{i-1}, \vec{R}_i \dots \vec{R}_n$

Let $\vec{v}_i$ be the position vector of any point on the arc $P_{i-1} P_i$

Now consider the sum $S = \sum_{i=0}^{n} \vec{F}(\vec{v}_i)\cdot \delta\vec{R}_i$ where $\delta\overrightarrow{R_i} = \overrightarrow{R_i} - \overrightarrow{R_{i-1}}$.

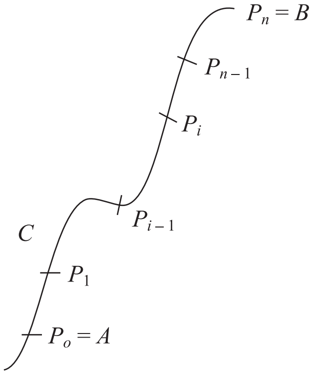


Fig. 20.3

The limit of this sum as $n - \infty$ in such a way that $|\delta\overrightarrow{R_i}| \to 0$, provided it exists, is called the **tangential line integral** of $\vec{F}(\vec{R})$ along C which is a scalar and is symbolically written as

$$\int_C \vec{F}(\vec{R})\cdot \overrightarrow{dR} \text{ or } \int_C \vec{F}\cdot \frac{d\vec{R}}{dt}\cdot dt$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\vec{F}(\vec{R}) = f(x,y,z)\vec{i} + \phi(x,y,z)\vec{j} + \psi(x,y,z)\vec{k}$ and $d\vec{R} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

then $\int_C \vec{F}(\vec{R})\cdot d\vec{R} = \int_C (fdx + \phi dy + \psi dz)$.

Two other types of line integrals are $\int_C \vec{F}\times d\vec{R}$ and $\int_C fd\vec{R}$ which are both vectors.

20.4 ❑ CIRCULATION

In fluid dynamics, if $\vec{F}$ represents the velocity of a fluid particle then the line integral $\int_C \vec{F}\cdot d\vec{r}$ is called the circulation of $\vec{F}$ around the curve. When the circulation of $\vec{F}$ around every closed curve in a region E vanishes, $\vec{F}$ is said to be **irrotational** in E.

Conservative Vector

If the value of $\int_A^B \vec{F}\cdot\overrightarrow{dr}$ does not depend on the curve C, but only on the terminal points A and B, $\vec{F}$ is called a **conservative vector**.

A force field $\vec{F}$ is said to be **conservative** if it is derivable from a potential function ϕ, i.e., $\vec{F} =$ grad ϕ. Then curl $(\vec{F})$ = curl $(\nabla\phi) = 0$.

$\therefore$ if $\vec{F}$ is **conservative** then $\text{curl}(\vec{F}) = 0$ and there exists a scalar potential function ϕ such that $\vec{F} = \nabla\phi$.

20.5 ❑ APPLICATIONS OF LINE INTEGRALS

Work Done by a Force

Let $\vec{v}\,(x, y, z) = v_1(x, y, z)\,\vec{i} + v_2(x, y, z)\,\vec{j} + v_3(x, y, z)\,\vec{k}$ be a vector function defined and continuous at every point on C. Then, the integral of the tangential component of $\vec{v}$ along the curve C from a point P on to the point Q is given by

$$\int_P^Q \vec{v}\cdot\overrightarrow{dr} = \int_{C_1} \vec{v}\cdot d\vec{r} = \int_{C_1} v_1 dx + v_2 dy + v_3 dz$$

where C_1 is the part of C, whose initial and terminal points are P and Q.

Let $\vec{v} = \vec{F}$, variable force acting on a particle which moves along a curve C. Then the work done W by the force $\vec{F}$ in displacing the particle from the point P to the point Q along the curve C is given by

$$W = \int_P^Q \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}$$

where C_1 is the part of C whose initial and terminal points are P and Q.

Suppose $\vec{F}$ is a conservative vector field; then $\vec{F}$ can be written as $\vec{F}$ = grad ϕ, where ϕ is a scalar potential.

Then, the work done

$$W = \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (\text{grad}\,\phi)\cdot d\vec{r}$$

$$= \int_{C_1} \left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right] = \int_P^Q d\phi = \left[\phi(x, y, z)\right]_P^Q$$

$\therefore$ work done depends only on the initial and terminal points of the curve C_1, i.e., the work done is independent of the path of integration. The units of work depend on the units of $|\vec{F}|$ and on the units of distance.

➢ **Note**

(i) **Condition for $\vec{F}$ to be conservative**

If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = 0$.

It is possible only when $\vec{F} = \nabla\phi$. which $\Rightarrow$ $\vec{F}$ is conservative.

$\therefore$ if $\vec{F}$ is an irrotational vector, it is conservative.

(ii) If $\vec{F}$ is irrotational (and, hence, conservative) and C is a closed curve then $\oint_C \vec{F} \cdot d\vec{r} = 0$. [$\because$ $\phi(A) = \phi(B)$, as A and B coincide].

20.6 ❑ SURFACES

A surface S may be represented by $F(x, y, z) = 0$.

The parametric representation of S is of the form

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

and the continuous functions $u = \phi(t)$ and $v = \phi(t)$ of a real parameter t represent a curve C on this surface S.

If S has a unique normal at each of its points whose direction depends continuously on the points of S then the surface S is called a **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions then it is called a **piecewise smooth surface**. For example, the surface of a sphere is smooth while the surface of a cube is piecewise smooth.

If a surface S is smooth from any of its points P, we may choose a unit normal vector $\vec{n}$ of S at P. The direction of $\vec{n}$ is then called the **positive normal direction of S at *P***. A surface S is said to be **orientable** or **two-sided**, if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface. If the positive direction of the normal is reversed as we move around a curve on S passing through P then the surface is **non-orientable** (i.e., one-sided) (Fig. 20.4).

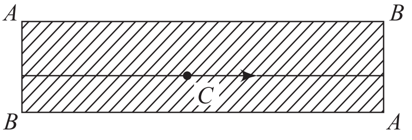


Fig. 20.4

● **Example**

A sufficiently small portion of a smooth surface is always orientable (Fig. 20.5).

A Mobius strip is an example of a non-orientable surface. A model of a Mobius strip can be made by taking a long rectangular piece of paper, making a half-twist and sticking the shorter sides together so that the two points A and the two points B coincide; then the surface generated is non-orientable.

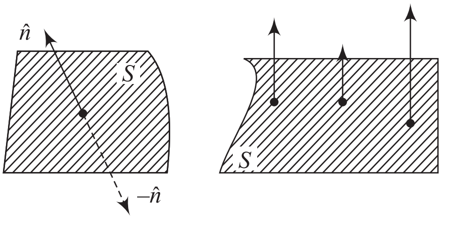


Fig. 20.5

20.7 ❑ SURFACE INTEGRALS

Any integral which is to be evaluated over a surface is called a **surface integral**.

Let S be a two-sided surface, one side of which is considered arbitrarily as the positive side.

Let $\vec{F}$ be a vector point function defined at all points of S. Let ds be the typical elemental surface area in S surrounding the point $P(x, y, z)$.

Let $\hat{n}$ be the unit vector normal to the surface S at $P(x, y, z)$, drawn in the positive side (or outward direction).

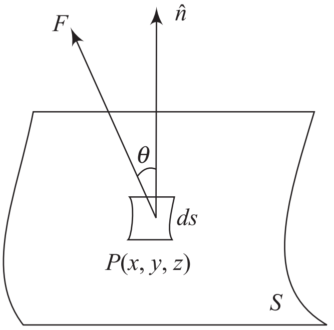


Fig. 20.6

Let θ be the angle between $\vec{F}$ and $\hat{n}$.

∴ the normal component of $\vec{F} = \vec{F} \cdot \hat{n} = F \cos\theta$.

The integral of this normal component through the elemental surface area ds over the surface S is called the **surface integral** of $\vec{F}$ over S and denoted as $\int_S F\cos\theta \cdot ds$ or $\int_S \vec{F} \cdot \hat{n}\, ds$.

If $d\vec{s}$ is a vector whose magnitude is ds and whose direction is that of $\hat{n}$, then $\overrightarrow{ds} = \hat{n} \cdot ds$. ∴ $\int_S \vec{F} \cdot \hat{n}\, ds$ can also be written as $\int_S \vec{F} \cdot \overrightarrow{ds}$.

➢ **Note**

(i) If S in a closed surface, the outer surface is usually chosen as the positive side.

(ii) $\int_S \phi\, d\vec{s}$ and $\int_S \vec{F} \times d\vec{s}$ where ϕ is a scalar point function are also surface integrals.

(iii) The surface integral $\int_S \vec{F} \cdot d\vec{s}$ is also denoted as $\iint_S \vec{F} \cdot d\vec{s}$.

(iv) If $\vec{F}$ represents the velocity of a fluid particle then the total outward flux of $\vec{F}$ across a closed surface S is the surface integral $\int_S \vec{F} \cdot d\vec{s}$.

(v) When the flux of $\vec{F}$ across every closed surface S in a region E vanishes, $\vec{F}$ is said to be a **solenoidal vector point function** in E.

(vi) It may be noted that $\vec{F}$ could equally well be taken as any other physical quantity such as gravitational force, electric force, magnetic force, etc.

20.8 ❑ VOLUME INTEGRALS

Any integral which is to be evaluated over a volume is called a **volume integral**.

If V is a volume bounded by a surface S then the triple integrals $\iiint_V \phi\, dv$ and $\iiint_V \vec{F}\, dv$ are called volume integrals. The first of these is a scalar and the second is a vector.

20.9 ❑ INTEGRAL THEOREMS

The following three theorems in vector calculus are of importance from theoretical and practical considerations:

(i) Green's theorem in a plane
(ii) Stokes' theorem
(iii) Gauss' divergence theorem

Green's theorem provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R. Green's theorem is also called the **first fundamental theorem** of integral vector calculus.

Stokes' theorem transforms line integrals into surface integrals and conversely. This theorem is a generalization of Green's theorem. It involves the curl.

Gauss' divergence theorem transforms surface integrals into a volume integral. It is named Gauss' divergence theorem because it involves the divergence of a vector function.

We shall give the statements of the above theorems (without proof) and apply them to solve problems.

Green's Theorem in a Plane

If C is a simple closed curve enclosing a region R in the xy-plane and $P(x, y)$, $Q(x, y)$ and its first-order partial derivatives are continuous in R then $\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ where C is described in the anticlockwise direction.

Stokes' Theorem (Relation between Line Integral and Surface Integral)

Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \cdot ds$

Gauss' Divergence Theorem or Gauss' Theorem of Divergence (Relation between Surface Integral and Volume Integral)

The surface integral of the normal component of a vector function $\vec{F}$ taken around a closed surface S is equal to the integral of the divergence of $\vec{F}$ taken over the volume V enclosed by the surface S.

Mathematically, $\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V \text{div } \vec{F} \cdot dv$.

SOLVED EXAMPLES

Example 1 If $\vec{F} = 3xy\vec{i}, - y^2\vec{j}$, evaluate $\int_C \vec{F} \cdot dr$, where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution Let $x = t$, then the parametric equations of the parabola $y = 2x^2$ are $x = t$, $y = 2t^2$.

At the point (0, 0), $x = 0$ and so $t = 0$.

At the point (1, 2), $x = 1$ and so $t = 1$.

If $\vec{r}$ is the position vector of any point (x, y) in C, then

$$\vec{r} = x\vec{i} + y\vec{j}$$
$$= t\vec{i} + 2t^2\vec{j}$$

Also in terms of t, $\vec{F} = 3t(2t^2)\vec{i} - (2t^2)^2\vec{j}$

$$= 6t^3\vec{i} - 4t^4\vec{j}$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$
$$= \int_0^1 (6t^3\vec{i} - 4t^4\vec{j})\cdot(\vec{i} + 4t\vec{j})dt$$
$$= \int_0^1 (6t^3 - 16t^5)dt$$
$$= \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1$$
$$= \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$

Ans.

Example 2 Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = (x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant. **[KU May 2010]**

Solution A vector normal to the surface S is given by

$$\nabla(2x + y + 2z) = 2\vec{i} + \vec{j} + 2\vec{k}$$

$\therefore\ \hat{n} = a$ unit vector normal to the surface S

$$= \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\vec{k}\cdot\hat{n} = \vec{k}\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = \frac{2}{3}$$

$$\therefore \quad \iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\cdot\frac{dxdy}{|\vec{k}\cdot\hat{n}|}$$

where R is the projection of S

Now, $$\vec{A}\cdot\hat{n} = [(x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}]\cdot\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$$
$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$
$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right)$$

$\left(\text{since on the plane } 2x + y + 2z = 6,\ z = \dfrac{6 - 2x - y}{2}\right)$

$$= \frac{2}{3}y(y + 6 - 2x - y)$$

$$= \frac{4}{3}y(3 - x)$$

Hence, $\displaystyle\iint_S \vec{A} \cdot \hat{n} \cdot ds = \iint_R \vec{A} \cdot \hat{n} \cdot \frac{dxdy}{|\vec{k} \cdot \hat{n}|}.$

$$= \iint_R \frac{4}{3}y(3 - x) \cdot \frac{3}{2}dx\,dy$$

$$= \int_0^3 \int_0^{6-2x} 2y(3 - x)dy\,dx$$

$$= \int_0^3 2(3 - x)\left(\frac{y^2}{2}\right)_0^{6-2x} dx$$

$$= \int_0^3 (3 - x)(6 - 2x)^2 dx$$

$$= 4\int_0^3 (3 - x)^3 dx$$

$$= 4\left[\frac{(3 - x)^4}{4(-1)}\right]_0^3$$

$$= 81$$ **Ans.**

Example 3 If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$ then evaluate $\iiint \nabla \cdot \vec{F} \cdot dV$, where V is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x = 2x$$

$$\therefore \quad \iiint_V \nabla \cdot F \cdot dv = \iiint_V 2x\,dx\,dy\,dz$$

$$= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2x[z]_0^{4-2x-2y} dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2x(4-2x-2y)\,dy\,dx = \int_0^2 \int_0^{2-x} [4x(2-x)-4xy]\,dy\,dx$$

$$= \int_0^2 [4x(2-x)y - 2xy^2]_0^{2-x} \cdot dx$$

$$= \int_0^2 [4x(2-x)^2 - 2x(2-x)^2]\,dx$$

$$= \int_0^2 2x(2-x)^2\,dx$$

$$= 2\int_0^2 (4x - 4x^2 + x^3)\cdot dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left[8 - \frac{32}{3} + 4\right] = \frac{8}{3}$$ **Ans.**

Example 4 Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ and the curve C is the rectangle in the xy-plane bounded by $y = 0$, $y = b$, $x = 0$, $x = a$.

Solution In the xy-plane, $z = 0$

$$\vec{r} = x\vec{i} + y\vec{j},\ d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+y^2)dx - 2xydy \qquad (1)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \qquad (2)$$

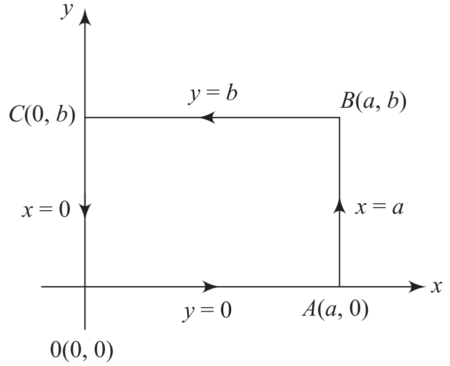


Fig. 20.7

Along OA, $y = 0$; $dy = 0$ and x varies from 0 to a
Along AB, $x = a$; $dx = 0$ and y varies from 0 to b

Along BC, $y = b$; $dy = 0$ and x varies from a to 0
Along CO, $x = 0$; $dx = 0$ and y varies from b to 0
Hence, from (1) and (2),

$$\int_C \vec{F}\cdot\overrightarrow{dr} = \int_{x=0}^{a} x^2\,dx - \int_{y=0}^{b} 2ay\,dy + \int_{x=a}^{0}(x^2+b^2)\,dx + \int_b^0 0\cdot dy$$

$$= \left(\frac{x^3}{3}\right)_0^a - (ay^2)_0^b + \left(\frac{x^3}{3}+b^2x\right)_a^0 + 0$$

$$= \left(\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2\right) = -2ab^2$$ **Ans.**

Example 5 Find the work done by the force $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ when it moves a particle from (1, –2, 1) to (3, 1, 4) along any path. **[AU Dec. 2011]**

Solution Since the equation of the path is not given, the work done by the force $\vec{F}$ depends only on the terminal points.

Consider $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy+z^3) & x^2 & 3xz^2 \end{vmatrix}$

$$= \vec{i}[0-0] - \vec{j}[3z^2 - 3z^2] + \vec{k}[2x - 2x] = 0$$

$\Rightarrow$ $\vec{F}$ is irrotational

Hence, $\vec{F}$ is conservative

Since $\vec{F}$ is irrotational, we have $\vec{F} = \nabla\phi$

It is easy to see that $\phi = x^2y + xz^3 + C$

$\therefore$ work done by $\vec{F} = \displaystyle\int_{(1,-2,1)}^{(3,1,4)} \vec{F}\cdot d\vec{r}$

$$= \int_{(1,-2,1)}^{(3,1,4)} \nabla\phi\cdot\overrightarrow{dr} = \int_{(1,-2,1)}^{(3,1,4)} d\phi \ \ [\text{as } \nabla\phi\cdot\overrightarrow{dr} = d\phi]$$

$$= [\phi]_{(1,-2,1)}^{(3,1,4)}$$

$$= [x^2y + xz^3 + C]_{(1,-2,1)}^{(3,1,4)}$$

$$= (201 + C) - (-1 + C) = 202$$ **Ans.**

Example 6 Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = e^x \sin y\vec{i} + e^x \cos y\vec{j}$; and C is the rectangle whose vertices are $(0,0), (1,0), \left(1, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.

Solution

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\therefore \quad \vec{F} \cdot d\vec{r} = e^x \sin y \cdot dx + e^x \cos y \cdot dy$$

Now along OA, $y = 0$; $dy = 0$
along AB, $x = 1$; $dx = 0$

along BC, $y = \dfrac{\pi}{2}$; $dy = 0$

along CO, $x = 0$; $dx = 0$
$\therefore$ circulation round the rectangle $OABC$ is

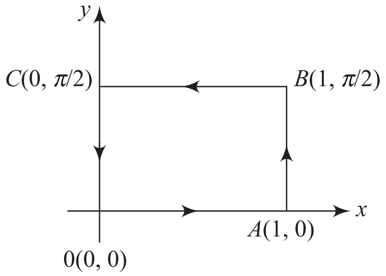


Fig. 20.7

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (e^x \sin y\, dx + e^x \cos y\, dy)$$

$$= \int_{OA} o + \int_{AB} e^1 \cos y\, dy + \int_{BC} e^x \sin\frac{\pi}{2} dx + \int_{CO} \cos y\, dy$$

$$= 0 + \int_0^{\frac{\pi}{2}} e \cos y \cdot dy + \int_1^0 e^x \sin\frac{\pi}{2} dx + \int_{\frac{\pi}{2}}^0 \cos y\, dy$$

$$= [e \sin y]_0^{\frac{\pi}{2}} + [e^x]_1^0 + [\sin y]_{\frac{\pi}{2}}^0 = e + (1 - e) - 1 + 0 = 0 \qquad \textbf{Ans.}$$

Example 7 Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Solution Total work done

$$= \int_C \vec{F} \cdot d\vec{r} = \int_C (3xy\vec{i} - 5z\vec{j} + 10x\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C [3xydx - 5zdy + 10xdz]$$

$$= \int_{t=1}^{2} [3(t^2+1)(2t^2)d(t^2+1) - 5t^3 d(2t^2) + 10(t^2+1)d(t^3)]$$

$$= \int_{t=1}^{2} [6t^2(t^2+1)(2tdt) - 20t^4 dt + 30t^2(t^2+1)dt]$$

$$= \int_{t=1}^{2} [12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2]dt$$

$$= \int_{t=1}^{2} [12t^5 + 10t^4 + 12t^3 + 30t^2]dt$$

$$= 12\left[\frac{t^6}{6}\right]_1^2 + 10\left[\frac{t^5}{5}\right]_1^2 + 12\left[\frac{t^4}{4}\right]_1^2 + 30\left[\frac{t^3}{3}\right]_1^2$$

$$= 12\left[\frac{2^6}{6} - \frac{1}{6}\right] + 10\left[\frac{2^5}{5} - \frac{1}{5}\right] + 12\left[\frac{2^4}{4} - \frac{1^4}{4}\right] + 30\left[\frac{2^3}{3} - \frac{1^3}{3}\right]$$

$$= 12 \cdot \frac{63}{6} + 10 \cdot \frac{31}{5} + 12 \cdot \frac{15}{4} + 30 \cdot \frac{7}{3}$$

$$= 126 + 62 + 45 + 70$$

$$= 303$$

Ans.

Example 8 If $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$, evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. **[AU Dec. 2009]**

Solution The surface of the cube consists of the following six faces:

(a) Face *LMND*
(b) Face *TQPO*
(c) Face *QPNM*
(d) Face *TODL*
(e) Face *TQMl*
(f) Face *ODNP*

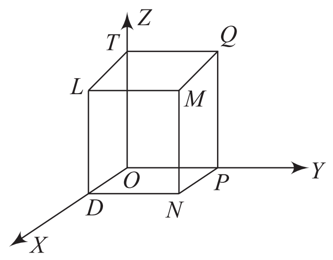


Fig. 20.8

Now, for the face *LMND*:

$$\hat{n} = \vec{i},\ x = OD = 1$$

Hence, $$\iint_S F \cdot \hat{n}\, ds = \iint_{LMND} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{i}\, dy\, dz$$

$$= \iint_{LMND} 4xz\, dy\, dz = 4 \int_{LMND} z\, dy\, dz \qquad (\because x = 1)$$

$$= 4 \int_{z=0}^{1} \int_{y=0}^{1} z\, dy\, dz = 4\left[\left(\frac{z^2}{2}\right)_0^1 (y)_0^2\right] = 2 \qquad (1)$$

For the face *TQPO*: $\hat{n} = -\vec{i}$, $x = 0$

Hence, $$\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iint_{TQPO} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{i}) dy\, dz$$

$$= \iint_{TQPO} (-4xz) dy\, dz = 0 \qquad (\because x = 0) \qquad (2)$$

For the face *OPNM*: $\hat{n} = \vec{j}$, $y = 1$

Hence, $$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{QPNM} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{j}\, dx\, dz$$

$$= \iint_{QPNM} (-y^2\, dx\, dz) = \iint_{QPNM} -dx\, dz \qquad (\because y = 1)$$

$$= -\int_{z=0}^{1}\int_{x=0}^{1} dx\,dz = -[x]_0^1[z]_0^1 = -1 \quad (3)$$

For the face *TODL*: $\hat{n} = -\vec{j}, y = 0$

Hence, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{TODL} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j})dx\,dz$

$$= \iint_{TODL} (y^2 dxdz) = 0 \quad (\because y = 0) \quad (4)$$

For the face *TQML*: $\hat{n} = \vec{k}, z = 1$

Hence, $\iint_{TQML} \vec{F}\cdot\hat{n}\,ds = \iint_{TQML} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\,dx\,dy$.

$$= \iint_{TQML} yz\,dx\,dy = \iint_{TQML} y\,dx\,dy \quad (\because z = 1)$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} y\,dx\,dy = [x]_0^1\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} \quad (5)$$

For the face *ODNP*: $\hat{n} = -\vec{k}, z = 0$

Hence, $\iint_{ODNP} \vec{F}\cdot\hat{n}\,ds = \iint_{ODNP} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{k})\cdot dx\,dy$

$$= \iint_{ODNP} (-yz)dx\,dy = 0, \quad (\because z = 0) \quad (6)$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$$ **Ans.**

Example 9 Verify Stokes' theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}$ over the surface of a cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the *XOY* plane (open at the bottom). **[KU May 2010]**

Solution Consider the surface of the cube as shown in the figure. Bounding path is *OABCO* shown by arrows.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(y - z + 2)\vec{i} + (yz + 4)\vec{j} - (xz)\vec{k}]\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (y - z + 2)dx + (yz + 4)dy - xz\,dz$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \quad (1)$$

Along OA, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2dx = (2x)_0^2 = 4$$

Along AB, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4dy = 4(y)_0^2 = 8$$

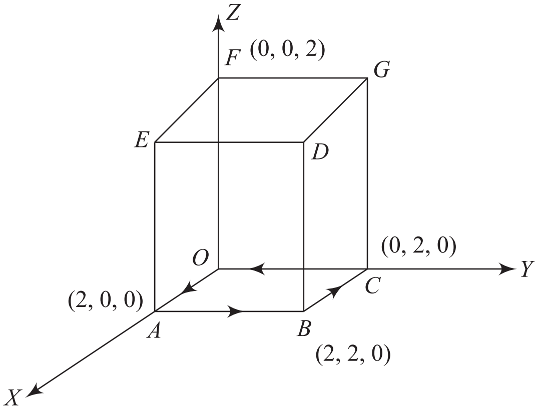


Fig. 20.9

Along BC, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_0^2 (2 - 0 + 2)dx = (4x)_2^0 = -8$$

Along CO, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int (y - 0 + 2) \times 0 + (0 + 4)dy - 0$$

$$= 4\int dy = 4(y)_2^0 = -8$$

On putting the values of these integrals in (1), we get

$$\int_C \vec{F} \cdot d\vec{r} = 4 + 8 - 8 = -4$$

To obtain surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= (0 - y)\vec{i} - (-z + 1)\vec{j} + (0 - 1)\vec{k} = -y\vec{i} + (z - 1)\vec{j} - \vec{k}$$

Here, we have to integrate over the five surfaces, *ABDE*, *OCGF*, *BCGD*, *OAEF*, *DEFG*.
Over the surface *ABDE*: $x = 2,\ \hat{n} = \vec{i}\ ,\ ds = dydz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{i}\, dy\, dz$$

$$= \iint_S -y\, dydz = -\int_0^2 y\, dy \int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$$

Over the surface *OCGF*: $x = 0,\ \hat{n} = -\vec{i},\ ds = dy\ dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{i}) dy\, dz$$

$$= \iint_S y\, dy\, dz = \int_0^2 y\, dy \int_0^2 dz = \left[\frac{y^2}{2}\right]_0^2 = 4$$

Over the surface *BCGD*: $y = 2,\ \hat{n} = \vec{j}\ ,\ ds = dx\ dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{j}\, dx\, dz$$

$$= \iint_S (z-1) dx\, dz$$

$$= \int_0^2 dx \int_0^2 (z-1)\, dz$$

$$= [x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *OAEF*: $y = 0,\ \hat{n} = -\vec{j}\ ,\ ds = dx\ dz$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot (-\vec{j}) dx\, dz$$

$$= -\iint_S (z-1) dx\, dz$$

$$= -\int_0^2 dx \int_0^2 (z-1) dz$$

$$= -[x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2$$

$$= 0$$

Over the surface *DEFG*: $z = 2$, $\hat{n} = \vec{k}$, $ds = dx\,dy$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint [-y\vec{i} + (z-1)\vec{j} - \vec{k}] \cdot \vec{k}\, dx\, dy$$

$$= -\iint dx\, dy = -\int_0^2 dx \int_0^2 dy$$

$$= -[x]_0^2 [y]_0^2 = -4$$

Total surface integral = –4 + 4 + 0 + 0 – 4 = –4

Thus $\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \int_C \vec{F} \cdot d\vec{r} = -4$

which verifies Stokes' theorem. **Verified.**

Example 10 Verify Green's theorem in the plane for $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where C is a square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

Solution Given integrand is of the form $Mdx + Ndy$, where $M = x^2 - xy^3$, $N = y^2 - 2xy$.

Now to verify Green's theorem, we have to verify that

$$\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \iint_R (-2y + 3xy^2)dx\, dy \quad (1)$$

Consider $\int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where the curve C is divided into four parts, hence the line integral along C is nothing but the sum of four line integrals along four lines *OA*, *AB*, *BC* and *CO*.

Along *OA* : $y = 0$, $dy = 0$ and x varies from 0 to 2.

Hence, $$\int_{OA} [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \int_{x=0}^{2} x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

Along *AB* : $x = 2$, $dx = 0$, and y varies from 0 to 2.

Hence, $$\int_{AB} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_0^2 (y^2 - 4y)dy = \left(\frac{y^3}{3} - 4\frac{y^2}{2}\right)_0^2$$

$$= \left(\frac{8}{3}\right) - 8 = -\frac{16}{3}$$

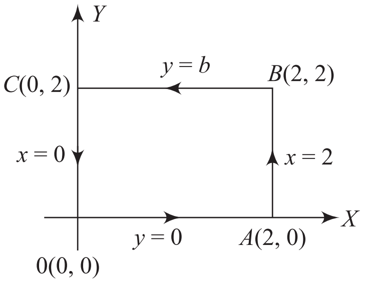


Fig. 20.10

Along *BC*: $y = 2$, $dy = 0$ and x varies from 2 to 0.

Hence, $$\int_{BC} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$= \int_{x=2}^{0} (x^2 - 8x)dx = \left(\frac{x^3}{3} - 8\frac{x^2}{2}\right)_2^0$$

$$= 0 - 0 - \frac{8}{3} + 16 = \frac{40}{3}$$

Along CO : $x = 0$, $dx = 0$ and y varies from 2 to 0

Hence, $\int_{CO} [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$

$$= \int_{y=2}^{0} y^2 dy = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$$

$$\therefore \quad \int_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy] = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8 \qquad (2)$$

Now consider

$$\iint_R (-2y + 3xy^2)dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2} (-2y + 3xy^2)dy\,dx$$

$$= \int_{x=0}^{2} \left(-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_{x=0}^{2} \left[-4 + 3x\left(\frac{8}{3}\right)\right]dx = \left(-4x + 8\frac{x^2}{2}\right)_0^2$$

$$= -8 + 16 + 0 = 8 \qquad (3)$$

From (2) and (3), we observe that the relation (1) is true.
Hence, Green's theorem is verified. **Ans.**

Example 11 Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. **[KU Nov. 2010]**

Solution For verification of the divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$

$$\therefore \quad \iiint_V \text{div}\,\vec{F}dv$$

$$= \int_o^c \int_o^b \int_o^a 2(x + y + z)\,dx\,dy\,dz$$

$$= \int_o^c \int_o^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dy\,dz$$

$$= \int_0^c \int_o^b 2\left(\frac{a^2}{2} + ya + za\right)dy\,dz$$

$$= \int_o^c 2\left[\left(\frac{a^2}{2}y + \frac{y^2 a}{2} + azy\right)\right]_o^b dz$$

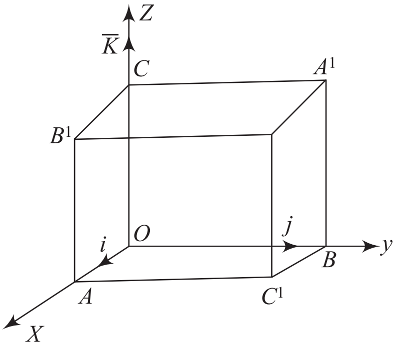


Fig. 20.11

$$= 2\int_o^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz \right) dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + \frac{abz^2}{2} \right]_o^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a + b + c) \quad (1)$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelepiped into 6 parts.

S_1 : Face $OAC'B$

S_2 : Face $CB'PA'$

S_3 : Face $OBA'C$

S_4 : Face $AC'PB'$

S_5 : Face $OCB'A$

S_6 : Face $BA'\,PC'$

Also,
$$\iint_S \vec{F} \cdot \hat{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n} ds + \iint_{S_2} \vec{F} \cdot \hat{n} ds + \iint_{S_3} \vec{F} \cdot \hat{n} ds + \iint_{S_4} \vec{F} \cdot \hat{n} ds + \iint_{S_5} \vec{F} \cdot \hat{n} ds + \iint_{S} \vec{F} \cdot \hat{n} ds \quad (2)$$

On $S_1 : z = 0,\ \hat{n} = -\vec{k}$, $ds = dx\, dy$

so that $\vec{F} \cdot \hat{n} = (x^2\vec{i} + y^2\vec{j} - xy\vec{k}) \cdot (-\vec{k}) = xy$

$$\therefore \quad \iint_{S_1} \vec{F} \cdot \hat{n} ds = \int_o^b \int_o^a xy\, dx\, dy = \int_o^b \left(y\frac{x^2}{2} \right)_o^a dy$$

$$= \frac{a^2}{2} \int_o^b y\, dy = \frac{a^2b^2}{4} \quad (3)$$

On $S_2 : z = c,\ \hat{n} = \vec{k}$, $ds = dx\, dy,\ \vec{F} = (x^2 - cy)\,\vec{i} + (y^2 - cx)\,\vec{j} + (c^2 - xy)\,\vec{k}$.

so that $\vec{F} \cdot \hat{n} = [(x^2 - cy)\vec{i} + (y^2 - (x)\vec{j}) + (c^2 - xy)\vec{k}] \cdot \vec{k} = c^2 - xy$.

$$\therefore \quad \iint_{S_2} \vec{F} \cdot \hat{n} ds = \int_o^b \int_o^a (c^2 - xy)\, dx\, dy = \int_o^b \left(c^2a - \frac{a^2}{2}y \right) dy$$

$$= abc^2 - \frac{a^2b^2}{4} \quad (4)$$

On $S_3 : x = 0,\ \hat{n} = -\vec{i}, \vec{F} = -yz\vec{i} + y^2\vec{j} + z^2\vec{k}, dz = dy\, dz$

so that $\vec{F} \cdot \hat{n} = (-yz\vec{i} + y^2\vec{j} + z^2\vec{k}) \cdot (-\vec{i}) = yz, ds = dy\, dz$

$$\therefore \quad \iint_{S_3} \vec{F} \cdot \hat{n} ds = \int_o^c \int_o^b yz\, dy\, dz = \int_o^c \frac{b^2}{2} z\, dz = \frac{b^2c^2}{4} \quad (5)$$

On $S_4 : x = a,\ \hat{n} = \vec{i}, \vec{F} = (a^2 - yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}$

so that $\vec{F}\cdot\hat{n} = [(a^2 - yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}]\cdot\vec{i}$

$= a^2 - yz,\ ds = dy\,dz$

$$\therefore \quad \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_o^c\int_o^b (a^2 - yz)\,dy\,dz = \int_o^c \left(a^2 b - \frac{b^2}{2}z\right)dz$$

$$= a^2bc - \frac{b^2c^2}{4} \qquad (6)$$

On $S_5 : y = 0,\ \hat{n} = -\vec{j},\ \vec{F} = x^2\vec{i} - zx\vec{j} + z^2\vec{k},\ ds = dxdz$

so that $\vec{F}\cdot\hat{n} = (x^2\vec{i} - zx\vec{j} + z^2\vec{k})\cdot(-\vec{j}) = zx$

$$\therefore \quad \iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_o^a\int_o^c zx\,dz\,dx = \int_o^a \frac{c^2}{2}x\,dx = \frac{a^2c^2}{4} \qquad (7)$$

On $S_6 : y = b,\ \hat{n} = \vec{j},\ \vec{F} = (x^2 - bz)\vec{i} + (b^2 - zx)\vec{j} + (z^2 - bx)\vec{k}$

$ds = dxdz$

so that $\vec{F}\cdot\hat{n} = [(x^2 - bz)\vec{i} + (b^2 - zx)\vec{j} + (z^2 - bx)\vec{k}]\cdot\vec{j}$

$= b^2 - zx.$

$$\therefore \quad \iint_{S_6} \vec{F}\cdot\hat{n} = \int_o^a\int_o^c (b^2 - zx)\,dz\,dx$$

$$= \int_o^a \left(b^2c - \frac{c^2}{2}x\right)\cdot dx = ab^2c - \frac{a^2c^2}{4} \qquad (8)$$

By using (3), (4), (5), (6), (7) and (8), in (2), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

$$= abc(a + b + c) \qquad (9)$$

The equalities (1) and (9) verify the divergence theorem. **Ans.**

Example 12 Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by (i) $y = \sqrt{x}$, $y = x^2$ and (ii) $x = 0, y = 0, x + y = 1.$ **[AU July 2010, June 2012 ; KU Nov. 2011, KU April 2013]**

Solution

(i) $y = \sqrt{x}$, i.e., $y^2 = x$ and $y = x^2$ are two parabolas intersecting at $0(0, 0)$ and $A(1, 1)$.

Here, $P = 3x^2 - 8y^2,\ Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y,\ \frac{\partial Q}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$$

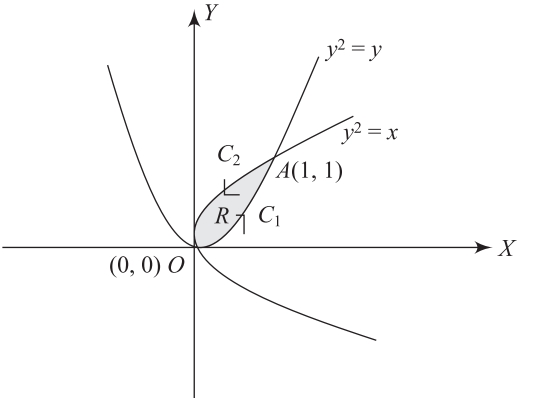


Fig. 20.12

If R is the region bounded by C then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx = \int_0^1 10\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\left[\frac{3}{10}\right] = \frac{3}{2} \qquad (1)$$

Also, $\int_C P\,dx + Q\,dy = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy)$

Along C_1, $x^2 = y$. $\therefore$ $2x\,dx = dy$ and the limits of x are from 0 to 1.

$\therefore$ $\int_{C_1} (P\,dx + Q\,dy)$

$$= \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x \cdot x^2) \cdot 2x\,dx \text{ (since } x^2 = y)$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= [x^3 + 2x^4 - 4x^5]_0^1 = -1$$

Along C_2, $y^2 = x$. $\therefore$ $2y\,dy = dx$ and the limits of y are from 1 to 0.

$$\int_{C_2} (P\,dx + Q\,dy)$$

$$= \int_1^0 (3y^4 - 8y^2)2y\,dy + (4y - 6y^2 \cdot y) \cdot dy$$

$$\therefore \quad = \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$$\therefore \quad \int_C (P\,dx + Q\,dy) = -1 + \frac{5}{2} = \frac{3}{2} \qquad (2)$$

The equalities of (1) and (2) verify Green's theorem in the plane. **Ans.**

(ii) Here, $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx$$

$$= \int_0^1 5[y^2]_0^{1-x}\,dx$$

$$= 5\int_0^1 (1-x)^2\,dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= \frac{-5}{3}(0-1) = \frac{5}{3} \qquad (1)$$

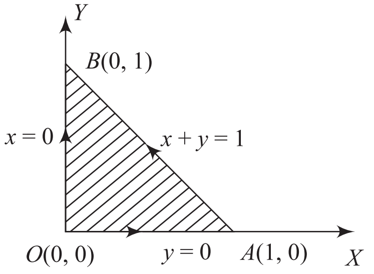


Fig. 20.13

Along OA, $y = 0 \;\therefore\; dy = 0$ and the limits of x are from 0 to 1.

$$\therefore \quad \int_{OA} P\,dx + Q\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$$

Along AB, $y = 1 - x$. $\therefore\; dy = -dx$ and the limits of x are from 1 to 0.

$$\therefore \quad \int_{AB} P\,dx + Q\,dy = \int_1^0 [3x^2 - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)](-dx)$$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)dx$$

$$= \int_1^0 (-12 + 26x - 11x^2) \cdot dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

Along BO, $x = 0$. $\therefore\; dx = 0$ and the limits of y are from 1 to 0

$$\therefore \quad \int_{BO} P\,dx + Q\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$\therefore$ line integral along C (i.e., along $OABO$) $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$

i.e., $$\int_C (P\,dx + Q\,dy) = \frac{5}{3} \tag{2}$$

The equality of (1) and (2) verifies Green's theorem in the plane. **Verified.**

Example 13 Evaluate $\int_C (e^x dx + 2y\,dy - dz)$ by using Stokes' theorem, where C is the curve $x^2 + y^2 = 4$, $z = 2$. **[AU May 2010]**

Solution

$$\begin{aligned}
&\int_C (e^x\,dx + 2y\,dy - dz)\\
&\quad = \int_C (e^x\vec{i} + 2y\vec{j} - \vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})\\
&\quad = \int_C \vec{F}\cdot\overrightarrow{dr} \text{ where } \vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}
\end{aligned}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

$\therefore$ by Stokes' theorem, $\displaystyle\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\cdot ds$

$$= 0 \text{ (since curl } \vec{F} = 0)$$ **Ans.**

Example 14 Find the work done by the force $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, when it moves a particle along the arc of the curve $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ from $t = 0$ to $t = 2n$. **[AU Dec. 2007]**

Solution From the vector equation of the curve C, we get the parametric equations of the curve as $x = \cos t$, $y = \sin t$, $z = t$.

Work done by the force $\vec{F} = \int_C \vec{F}\cdot\overrightarrow{dr}$

$$\begin{aligned}
&= \int_C (z\vec{i} + x\vec{j} + y\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})\\
&= \int_C (z\,dx + x\,dy + y\,dz)\\
&= \int_0^{2\pi} [t(-\sin t) + \cos^2 t + \sin t]dt\\
&= \left[t\cos t - \sin t + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) - \cos t\right]_0^{2\pi}\\
&= (2\pi + \pi - 1) - (-1)\\
&= 3\pi
\end{aligned}$$ **Ans.**

Example 15 Verify Stokes' theorem for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\vec{k}$ where S is the open surface of the rectangular parallelepiped formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ above the XOY-plane. **[AU Dec. 2007]**

Solution Stokes' theorem is given by

$$\int_C \vec{F} \cdot \overrightarrow{dr} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} ds$$

$$\text{Here, curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -zx \end{vmatrix}$$

$$= 2y\vec{i} + z\vec{j} + x\vec{k} \quad \therefore \int_C (xy\,dx - 2yz\,dy - zx\,dz) - \iint_S (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n} ds \qquad (1)$$

The open cuboid S is made up of the five faces $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ and is bounded by the rectangle $OAC'B$ lying on the XOY plane. LHS of (1) is

$$= \int_{OAC'B} (xy\,dx - 2yzdy - zxdz)$$

$$= \int_{OAC'B} xy\,dx$$

(since the boundary C lies on the XOY plane, $z = 0$)

$$= \int_{OA} xy\,dx + \int_{AC'} xy\,dx + \int_{C'B} xy\,dx + \int_{BO} xy\,dx$$

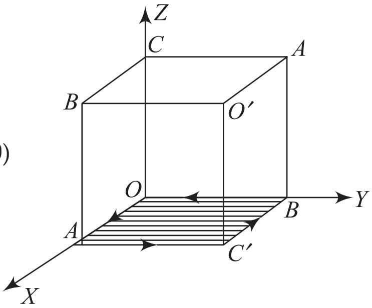


Fig. 20.14

Along OA, $y = 0$, $dy = 0$
Along AC', $x = 1$, $dx = 0$
Along $C'B$, $y = 2$, $dy = 0$
Along BO, $x = 0$, $dx = 0$

$$\therefore \quad \int_{OAC'B} xy\,dx = 0 + 0 + \int_{C'B} xy\,dx + 0 = \int_1^0 2x\,dx$$

(as along $C'B$, x varies from 1 to 0).

$$= -1 \qquad (2)$$

RHS of (1) is

$$\iint_S (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n} ds = \iint_{O'C'AB'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n} ds + \iint_{A'BOC} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n} ds$$

$$+ \iint_{A'BC'O'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n} ds + \iint_{COAB'} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n} ds$$

$$+ \iint_{A'O'B'C} (2y\vec{i} + z\vec{j} + x\vec{k}) \cdot \hat{n} ds$$

$$= \int_0^3\int_0^2 2y\,dy\,dz - \int_0^3\int_0^2 2y\,dy\,dz - \int_0^1\int_0^3 z\,dz\,dx$$

$$- \int_0^1\int_0^3 z\,dz\,dx - \int_0^2\int_0^1 x\,dx\,dy$$

$$= -\int_0^2\int_0^1 x\,dx\,dy = -\int_0^2 \left(\frac{x^2}{2}\right)_0^1 dy = -1 \quad (3)$$

From (2) and (3), Stokes' theorem is verified. **Verified.**

Example 16 Verify the divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. **[AU Dec. 2007, KU Nov. 2011]**

Solution Gauss' divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\text{div } \vec{F})\,dv \quad (1)$$

$$\text{LHS of } (1) = \iint_{x=1} x^2\,ds + \iint_{x=-1} -x^2\,ds + \iint_{y=1} z\,ds + \iint_{y=-1} -z\,ds + \iint_{z=1} yz\,ds + \iint_{z=-1} -yz\,ds = 0 \quad (2)$$

$$\text{RHS of } (1) = \iiint_V (\text{div } \vec{F})\cdot dv$$

$$= \iiint_V (2x + y)\,dx\,dy\,dz$$

$$= \int_{-1}^1\int_{-1}^1\int_{-1}^1 (2x + y)\,dx\,dy\,dz$$

$$= \int_{-1}^1\int_{-1}^1 2y\,dy\,dz = 0 \quad (3)$$

From (2) and (3), Gauss' divergence theorem is verified. **Verified.**

Example 17 Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$, where $\vec{F} = (\sin x - y)\vec{i} - \cos x\vec{j}$ and C is the boundary of the triangle whose vertices are $(0, 0), \left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

[KU Nov. 2011]

Solution By Stokes' theorem, we have $\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_S \text{ curl } \vec{F}\cdot\hat{n}\cdot ds$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x - y & -\cos x & 0 \end{vmatrix}$$

$$= (\sin x + 1)\vec{k}$$

$\therefore$ the given line integral

$$= \iint_R (1 + \sin x)\,dx\,dy$$

$$= \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (1 + \sin x)\,dx\,dy$$

$$= \int_0^1 \left[x - \cos x\right]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \left[\frac{\pi}{2} - \frac{\pi y}{2} + \cos\frac{\pi y}{2}\right] dy$$

$$= \left[\frac{\pi}{2}y - \frac{\pi y^2}{4} + \frac{2}{\pi}\sin\frac{\pi y}{2}\right]_0^1$$

$$= \frac{\pi}{4} + \frac{2}{\pi}$$

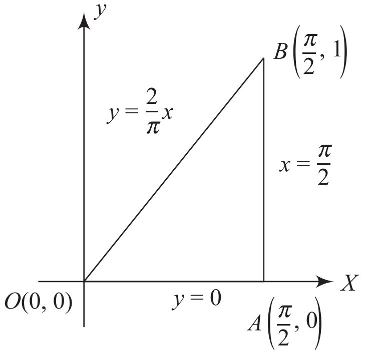


Fig. 20.15

$\therefore$ $$\int_C \vec{F}\cdot\vec{dr} = \frac{\pi}{4} + \frac{2}{\pi}$$ **Ans.**

EXERCISE

Part A

1. State Green's theorem in a plane.
2. Give the relation between a line integral and a surface integral.
3. State Gauss' divergence theorem.
4. Deduce Green's theorem in a plane from Stokes' theorem.
5. In Gauss' divergence theorem, surface integral is equal to _______ integral.
6. The integral of $\vec{F}\cdot d\vec{r}$ is
   (i) line integral (ii) zero
   (iii) surface integral (iv) one
7. Using Green's theorem, prove that the area enclosed by a simple closed curve C is $\frac{1}{2}\int(x\,dy - y\,dx)$.
8. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F}\cdot\vec{dr}$ where C is the part of the curve $y = x^3$ between $x = 1$ and $x = 2$.
9. If $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F}\cdot\vec{dr}$ along the straight line $y = x$ from (0, 0) to (1, 1).
10. If C is a simple closed curve and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, prove that $\int_C \vec{r}\cdot\vec{dr} = 0$.
11. Evaluate $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ where C is the circle given by $x^2 + y^2 + z^2 = 1$ and $z = 0$.
12. Use the integral theorems to prove $\nabla\cdot(\nabla\times\vec{F}) = 0$.

13. Evaluate $\int_C (x\,dy - y\,dx)$, where C is the circle $x^2 + y^2 = a^2$.

14. Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and C is the curve $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, t varying from –1 to 1.

Part B

1. If a force $\vec{F} = 2x^2y\vec{i} + 3xy\vec{j}$ displaces a particle in the xy plane from (0, 0) to (1, 4) along a curve $y = 4x^2$, find the work done. $\left(\textbf{Ans. } \frac{104}{5}\right)$

2. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to (1, 1) along a parabola $y^2 = x$. $\left(\textbf{Ans. } \frac{2}{3}\right)$

3. Verify Green's theorem in a plane with respect to $\int_C (x^2dx + xy\,dy)$, where C is the boundary of the square formed by $x = 0, y = 0, x = a, y = a$. **[AU Dec. 2009]** $\left(\textbf{Ans. } \frac{a^3}{2}\right)$

4. Use Green's theorem to evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is the square formed by the lines $y = \pm1, x = \pm1$. **(Ans.** 0**)**

5. Use divergence theorem to evaluate $\iint_S (yz^2\vec{i} + zx^2\vec{j} + 2z^2\vec{k})\cdot\hat{n}ds$ where S is the closed surface bounded by the XOY-plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane. **(Ans.** πa^4**)**

6. Verify Stokes' theorem for $\vec{F} = (x^2 + y - 4)\vec{i} + 3xy\vec{j} + (2xz + z^2)\vec{k}$ over the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above the XOY plane. **(Ans.** -16π**)**

7. Use the divergence theorem to evaluate $\int_S \vec{A}\cdot\vec{ds}$ where $\vec{A} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2 = y^2 + z^2 = a^2$. $\left(\textbf{Ans. } \frac{12\pi a^5}{5}\right)$

8. Use the divergence theorem to evaluate $\iint_S x^3dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where S is the surface of the region bounded by the closed cylinder $x^2 + y^2 = a^2$, $(0 \le z \le b)$ $z = 0$ and $z = b$. $\left(\textbf{Ans. } \frac{5\pi a^4 b}{4}\right)$

9. Using Green's theorem, evaluate $\int_C [(y - \sin x)dx + \cos x\,dy]$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$. $\left[\textbf{Ans. } -\left(\frac{\pi^2+8}{4\pi}\right)\right]$

10. Evaluate $\int_C [(x^2 + y^2)dx - 2xy\,dy]$ where C is the rectangle bounded by $y = 0$, $x = 0, y = b, x = a$ using Green's theorem. **(Ans.** $-2ab^2$**)**

11. Verify Stokes' theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **(Ans.** $-\pi$**)**

12. Verify Stokes' theorem for $\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is the boundary. **(Ans.** 9π**)**

13. Find the area of $x^{2/3} + y^{2/3} = a^{2/3}$ using Green's theorem. $\left(\textbf{Ans.}\ \dfrac{3\pi a^2}{8}\right)$

14. Using Stokes' theorem, evaluate $\int_C (xy\,dx + xy^2 dy)$ taking C to be a square with vertices (1, 1), (–1, 1), (–1, –1) and (1, –1). $\left(\textbf{Ans.}\ \dfrac{4}{3}\right)$

15. Verify Gauss' divergence theorem for $\vec{F} = y\vec{i} + x\vec{j} + z^3\vec{k}$ over the cylindrical region $x^2 + y^2 = 9$, $z = 0$, $z = 6$. **(Ans.** 1944π**)**

Questions	opt1	opt2	opt3	opt4
If ∇.**F**=0 then **F** is	irrotational	solenoidal	rotational	curl
If ∇×**F**=0 then **F** is	irrotational	solenoidal	rotational	curl
Any motion in which the curl of the velocity vector is zero is said to be ___	irrotational	solenoidal	rotational	curl
A function is said to be _______ if it associates a scalar with every point in space.	Scalar function	Vector function	Point function	vector point function
A variable quantity whose value at any point in a region of space depends upon the position of the point is called a ___	Scalar function	Vector function	Point function	vector point function
A function is said to be _______ if it associates with vector in every point in space.	Scalar function	Vector function	Point function	vector point function
If the divergence of a flow is zero at all points then it is said to be ________	rotational	irrotational	solenoidal	conservative
_______ gives the rate of outflow per unit volume at a point of the fluid.	curl **V**	div **V**	curl **V=0**	div **V=0**
If div **V**=0 everywhere in some region R of space then **V** is called the _____ vector point function.	rotational	irrotational	solenoidal	conservative
________ is a vector which measures the extent to which individual particles of the fluid are spnning or rotating.	curl **V**	div **V**	curl **V=0**	div **V=0**
div **F** is a _________ function.	point	vector	scalar	rotational
If curl **V**=0 then **V** is said to be an ____________.	rotational	irrotational	solenoidal	conservative
If **r**=x**I**+y**J**+z**K** then div **r**=_________	0	1	2	3
If **r**=x**I**+y**J**+z**K** then curl **r**=_________	0	1	2	3
div (curl **V**)=	0	div **V**	curl **V**	**V**
curl (grad ϕ)=	**0**	div **V**	curl **V**	ϕ
Two surfaces are said to cut orthogonally at a point of intersection, if the respective normals at that point are _______.	parallel	perpendicular	equal	zero
A sufficiently small portion of a smooth surface is always ________	plane	smooth	twisted	orientable
A curve that is not plane is called a ________ curve.	plane	point	twisted	closed
Any integral which is to be evaluated over a surface is called a ___	Line integral	Volume integral	surface integral	closed integral
When the circulation of **F** around every closed curve in a region vanishes, then **F** is said to be ________ in that region.	rotational	irrotational	solenoidal	conservative

A force field **F** is said to be _____________ if it is derivable from a potential function ϕ such that **F =** grad ϕ.	rotational	irrotational	solenoidal	conservative
If **F** is ______ then cur **F**=0.	rotational	irrotational	solenoidal	conservative
If S has a unique normal at each of its points whose direction depends continuously on the point of S then the surface S is called a ______ surface.	Orientable	smooth	plane	twisted
_________ provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem
________ is also called the first fundamental theorem of integral vector calculus.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem
_________ transforms line integrals into surface integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem
_______ transforms surface integrals into a volume integrals.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem
_________ is stated as surface integral of the component of curl **F** along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function **F** taken along the closed curve C.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem
________ is stated as the surface integral of the normal component of a vector function **F** taken around a closed surface S is equal to the integral of the divergence of **F** taken over the volume V enclosed by the surface S.	Cauchy's Theorem	Green's Theorem	Stoke's Theorem	Gauss Theorem
If ∇ϕ is solenoidal, then ∇^2(ϕ)=	ϕ	1	0	-1
If (3x-2y+z)**I**+(4x+ay-z)**J**+(x-y-2z)**K** is solenoidal then a=	0	1	-1	2
If ϕ=x+y+z-8 then grad ϕ is ____	**I**+**J**+**K**	**I**+**J**-**K**	**I**-**J**+**K**	0
If ϕ=x^2+y^2+z^2-8 then grad ϕ at(2,2,2) is ____	4**I**+4**J**+4**K**	4**I**+4**J**-4**K**	4**I**-4**J**+4**K**	0
If ϕ =x^2+y^2+z^2-8 then grad ϕ at(2,0,2) is ____	4**I**+4**K**	4**J**+4**K**	4**I**+4**J**	0
If **F** = (x+2y+az)**I**+(bx-3y-z)**J**+(4x+cy+2z)**K** is irrotational, then the values of a,b and c are _______	a=2, b=4, c=-1	a=-1, b=2, c=4	a=4, b=2, c=1	a=4, b=2, c=-1
If **F**= xy**I**-yz**J**-zx**K** then curl **F** =	x**I**+y**J**+z**K**	x**I**-y**J**-z**K**	y**I**+z**J**+x**K**	y**I**+z**J**-x**K**
If **F**= xy**I**-yz**J**-zx**K** then div **F** =	x**I**+y**J**+z**K**	x**I**-y**J**-z**K**	y**I**-z**J**-x**K**	y**I**+z**J**-x**K**

If **F**= xy**I**-yz**J**-zx**K** then at (1,1,1), div **F** =	**I**+**J**+**K**	**I**-**J**+**K**	**I**-**J**-**K**	**I**+**J**-**K**
If **F**= x^2-y^2+2z^2 then at (1,2,3), div **F** =	**2I**+4**J**+12**K**	**2I**-4**J**+12**K**	**2I-4J-6K**	**2I**+4**J**-12**K**
div F is a ________ function.	point	vector	scalar	rotational
If curl V=0 then V is said to be an ____________.	rotational	irrotational	solenoidal	conservative
If **F**= x^2+y^2+2z^2 then grad F at (2,0,2)is-----	4i+4k	4j+4k	4i+4j	0
If **F** is an irrotational vector, it is ______	rotational	irrotational	solenoidal	conservative
A ______ _curve that lies in a plane in space.	plane	point	twisted	closed
If F is conservative then cur F=0 and there exists a scalar potential function ϕ such that ____	rotational	irrotational	solenoidal	conservative
Any integral which is to be evaluated along a curve is called a ___	Line integral	Volume integral	surface integral	closed integral
Any integral which is to be evaluated over a volume is called a ___	Line integral	Volume integral	surface integral	closed integral
If F is conservative then cur F=0 and there exists a scalar potential function f such that ____	rotational	irrotational	solenoidal	conservative
The integral of vector F.dr is -----.	line integral	zero	surface integral	one
The integral of vector F.dr is is conservative if the terminal points A and B ______	Coinside	split	different	deviate
Greens theorem is called the _____theorem of integral vector calculus.	second fundamental	first fundamental	third fundamental	fourth fundamental
If del x F then vector F is _______	conservative	non conservative	curl	solenoidal
If a force moves a particle from one place to another place in any curve then integral of vector F.dr is called ---------- by that force.	work done	rest taken	conservative	displacement
If a force--------a particle from one place to another place in any curve then integral of vector F.dr is called work done by that force.	moves	still	constant	idle
If S is not smooth but can be divided into finitely many smooth portions then it is called a ______ surface.	Orientable	smooth	piecewise smooth	twisted
If F is an irrotational vector, it is ______	rotational	irrotational	solenoidal	conservative
A force field F is said to be ____________ if it is derivable from a potential function f such that F = grad f.	rotational	irrotational	solenoidal	conservative

opt5	opt6	Answer
		solenoidal
		irrotational
		irrotational
		Scalar function
		Point function
		Vector function
		solenoidal
		div **V**
		solenoidal
		curl **V**
		scalar
		irrotational
		3
		0
		0
		0
		perpendicular
		orientable
		twisted
		surface integral
		irrotational

conservative

conservative

smooth

Green's Theorem

Green's Theorem

Stoke's Theorem

Gauss Theorem

Stoke's Theorem

Gauss Theorem

0

-1

I+**J**+**K**

4I+4**J**+4**K**

4I+4**K**

a=4, b=2, c=-1

y**I**+z**J**-x**K**

y**I-**z**J-**x**K**

I-J-K

2I-4**J**+12**K**

scalar

irrotational

4i+4k

conservative

plane

F = grad ϕ.

Line integral

Volume integral

F = grad f.

line integral

Coinside

first fundamental

conservative

work done

moves

piecewise smooth

conservative

conservative

Unit IX

Analytic Functions

21 Complex Numbers

Chapter Outline

- Introduction
- Complex Numbers
- Complex Function
- Limit of a Function
- Derivative
- Analytic Function
- Cauchy–Riemann Equations
- Harmonic Function
- Properties of Analytic Functions
- Construction of Analytic Function (Milne–Thomson Method)

21.1 ❑ INTRODUCTION

Quite often, it is believed that complex numbers arose from the need to solve quadratic equations. In fact, contrary to this belief, these numbers arose from the need to solve cubic equations. In the sixteenth century, Cardano was possibly the first to introduce $a+\sqrt{-b}$, a complex number, in algebra. Later, in the eighteenth century, Euler introduced the notation i for $\sqrt{-1}$ and visualized complex numbers as points with rectangular coordinates, but he did not give a satisfactory foundation for complex numbers. However, Euler defined the complex exponential and proved the identity $e^{i\varphi} = (\cos\varphi + i\sin\varphi)$, thereby establishing connection between trigonometric and exponential functions through complex analysis.

We know that there is no square root of negative numbers among real numbers.

However, algebra itself and its applications require such an extension of the concept of a number for which the extraction of the square root of a negative number would be possible.

We have repeatedly encountered the notion of extension of a number. Fractional numbers are introduced to make it possible to divide one integral number by another, negative numbers are introduced to make it possible to subtract a large number from a smaller one and irrational numbers become necessary in order to describe the result of measurement of the length of a segment in the case when the segment is incommensurable with the chosen unit of length.

The square root of the number –1 is usually denoted by the letter i and numbers of the form $a + ib$ where a and b are ordinary real numbers known as **complex numbers**.

The necessity of considering complex numbers first arose in the sixteenth century when several Italian mathematicians discovered the possibility of algebraic solutions of third-degree equations.

The theoretical and applied values of complex numbers are far beyond the scope of algebra. The theory of functions of a complex variable, which was much advanced in the nineteenth century, proved to be a very valuable apparatus for the investigation of almost all the divisions of theoretical physics, such, for instance, as the theory of oscillations, hydrodynamics, the divisions of the theory of elementary particles, etc.

Many engineering problems may be treated and solved by methods involving complex numbers and complex functions. There are two kinds of such problems. The first of them consists of elementary problems for which some acquaintances with complex numbers are sufficient. This includes many applications to electric circuits or mechanical vibrating systems. The second kind consists of more advanced problems for which we must be familiar with the theory of complex analytic functions. Interesting problems in heat conduction, fluid flow and electrostatics belong to this category.

21.2 ❑ COMPLEX NUMBERS

A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ (i is pronounced as **iota**) is called a **complex number**. x is called the **real part** of $x + iy$ and is written as $\text{Re}(x + iy)$ and y is called the **imaginary part** and is written as $\text{Im}(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be **conjugates** of each other.

Properties

(i) If $x_1 + iy_1 = x_2 + iy_2$ then $x_1 - iy_1 = x_2 - iy_2$

(ii) Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are said to be equal when $\text{Re}(x_1 + iy_1) = \text{Re}(x_2 + iy_2)$, i.e., $x_1 = x_2$ and $\text{Im}(x_1 + iy_1) = \text{Im}(x_2 + iy_2)$ i.e., $y_1 = y_2$

(iii) **Algebra of Complex Numbers**

The arithmetic operations on complex numbers follow the usual rules of elementary algebra of real numbers with the definition $i^2 = -1$. If $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$ are any two complex numbers then we define the following arithmetic operations.

Addition

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Division Let $z_2 \neq 0$. Then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \left[\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right] + i\left[\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right]$$

i.e., sum, difference, product and quotient of any two complex numbers is itself a complex number.

(iv) Every complex number $x + iy$ can always be expressed in the form $r(\cos\theta + i\sin\theta)$.
i.e., $re^{i\theta}$ (Exponential form).

➢ **Note**

(i) The number $r = +\sqrt{x^2 + y^2}$ is called the **module** of $x + iy$ and is written as mod $(x + iy)$ or $|x + iy|$. The angle θ is called the **amplitude** or **argument** of $x + iy$ and is written as amp $(x + iy)$ or $\arg(x + iy)$. Evidently, the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the **principal value of the amplitude.**

(ii) $\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')

(iii) If the conjugate of $z = x + iy$ be $\bar{z}$ then

(a) $\text{Re}(z) = \frac{1}{2}(z + \bar{z}), \text{Im}(z) = \frac{1}{2i}(z - \bar{z})$

(b) $|z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2} = |\bar{z}|$

(c) $z\bar{z} = |z|^2$

(d) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(e) $\overline{z_1z_2} = \overline{z_1} \cdot \overline{z_2}$

(f) $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}, \overline{z_2} \neq 0$

(iv) **De Moivre's Theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

21.3 ❑ COMPLEX FUNCTION

Recall from calculus that a real function f defined on a set S of real numbers is a rule that assigns to every x in S a real number $f(x)$, called the **value** of f at x. Now in the complex region, S is a set of complex numbers. A **function** f defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z.

We write $w = f(z)$. Here, z varies in S and is called a **complex variable**. The set S is called the **domain** of f.

If to each value of z, there corresponds one and only one value of w then w is said to be a **single-valued function** of z; otherwise, it is a **multi-valued function**. For example, $w = \frac{1}{z}$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

➢ **Note**

(i) If $z = x + iy$ then $f(z) = u + iv$ (a complex number).

(ii) Since $e^{iy} = \cos y + i \sin y$, $e^{-iy} = \cos y - i \sin y$, the circular functions are

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2}, \text{ and so on}$$

$\therefore$ circular functions of the complex variable z are given by $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$,

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\tan z = \dfrac{\sin z}{\cos z}$ with cosec z, sec z and cot z as their respective reciprocals.

(iii) **Euler's Theorem**

$$e^{iz} = \cos z + i \sin z$$

(iv) **Hyperbolic Functions**

If x be real or complex, $\dfrac{e^{x} - e^{-x}}{2} = \sin hx$ (named hyperbolic sine of x)

$\dfrac{e^{x} + e^{-x}}{2} = \cos hx$ (named hyperbolic cosine of x)

Also, we define,

$$\tan hx = \frac{\sin hx}{\cos hx} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\cot hx = \frac{1}{\tan hx} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

$$\sec hx = \frac{1}{\cos hx} = \frac{2}{e^{x} + e^{-x}}$$

$$\text{cosec } hx = \frac{1}{\sin hx} = \frac{2}{e^{x} - e^{-x}}$$

(v) **Relations between Hyperbolic and Circular Functions**

$$\sin ix = i \sin hx$$
$$\cos ix = \cos hx$$
$$\tan ix = i \tan hx$$

(vi) $\cos h^2x - \sin h^2x = 1$, $\sec h^2x + \tan h^2x = 1$

$\cot h^2x - \text{cosec } h^2x = 1$

(vii) $\sin h(x \pm y) = \sin hx \cos hy \pm \cos hx \sin hy$

$\cos h(x \pm y) = \cos hx \cos hy \pm \sinh x \sinh y$

$$\tan h(x \pm y) = \frac{\tan hx \pm \tan hy}{1 + \tan hx \tan hy}$$

(viii) $\sin h2x = 2 \sin hx \cosh x$

$\cos h2x = \cos h^2x + \sin h^2x = 2 \cos h^2x - 1 = 1 + 2 \sin h^2x$

$$\tan h2x = \frac{2 \tan hx}{1 + \tan h^2x}$$

(ix) $\sin h3x = 3\sin hx + 4\sin h^3x$

$\cos h3x = 4\cos h^3x - 3\cos hx$

$$\tan h3x = \frac{3\tan hx + \tan h^3x}{1 + 3\tan h^2x}$$

(x)

$$\sin hx + \sin hy = 2\sin h\frac{x+y}{2}\cos h\frac{x-y}{2}$$

$$\sin hx - \sin hy = 2\cos h\frac{x+y}{2}\sin h\frac{x-y}{2}$$

$$\cos hx + \cos hy = 2\cos h\frac{x+y}{2}\cos h\frac{x-y}{2}$$

$$\cos hx - \cos hy = 2\sin h\frac{x+y}{2}\sin h\frac{x-y}{2}$$

(xi) $\cos h^2x - \sin h^2x = 1$

(xii) Complex trigonometric functions satisfy the same identities as real trigonometric functions.

$$\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z$$

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2\sin z\cos z \quad \text{and} \quad \cos 2z = \cos^2 z - \sin^2 z$$

$$\sin \bar{z} = \overline{\sin z}$$

$\sin(z + 2n\pi) = \sin z$, n is any integer

$\cos(z + 2n\pi) = \cos z$, n is any integer

(xiii) **Inverse Trigonometric and Hyperbolic Functions**

Complex inverse trigonometric functions are defined by the following:

$$\cos^{-1} z = -i\log(z + \sqrt{z^2+1})$$

$$\sin^{-1} z = -i\log(iz + \sqrt{1-z^2})$$

$$\tan^{-1} z = -\frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right) = \frac{i}{2}\log\frac{i+z}{i-z},\ z \neq \pm i$$

$$\operatorname{cosec}^{-1} z = \sin^{-1}\left(\frac{1}{z}\right) = -i\log\left(\frac{1+\sqrt{z^2-1}}{z}\right),\ z \neq 0$$

$$\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right) = -i\log\left(\frac{1+\sqrt{1-z^2}}{z}\right),\ z \neq 0$$

$$\cot^{-1} z = \tan^{-1}\left(\frac{1}{z}\right) = \frac{-i}{2}\log\left(\frac{z+i}{z-i}\right),\ z \neq \pm i$$

Complex inverse hyperbolic functions are defined by the following:

$$\cosh^{-1} z = \log(z + \sqrt{z^2 - 1}),\ \sinh^{-1} z = \log(z + \sqrt{z^2 + 1})$$

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\ z \neq \pm 1$$

$$\text{cosech}^{-1} z = \sinh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right),\ z \neq 0$$

$$\text{sech}^{-1} z = \cosh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right),\ z \neq 0$$

$$\coth^{-1} z = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right),\ z \neq \pm 1$$

21.4 ❑ LIMIT OF A FUNCTION

A function $f(z)$ is said to have the **limit** 'b' as z approaches a point 'a', written $\lim_{z \to a} f(z) = b$, if f is defined in a neighborhood of 'a' (except perhaps at 'a' itself) and if the values of f are close to 'b' for all z close to 'a', i.e., the number b is called the **limit** of the function $f(z)$ as $z \to a$, if the absolute value of the difference $f(z) - b$ remains less than any preassigned positive number $\in$ every time the absolute value of the difference $z - a$ for $z \neq a$, is less than some positive number δ (dependent on $\in$).

More briefly, the number b is the limit of the function $f(z)$ as $z \to a$, if the absolute value $|f(z) - b|$ is arbitrarily small when $|z - a|$ is sufficiently small.

21.5 ❑ DERIVATIVE

A function $f(z)$ is said to be **differentiable** at a point $z = z_0$ if the limit $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists. This limit is then called the derivative of $f(z)$ at the point $z = z_0$ and is denoted by $f'(z_0)$.

If we write $z = z_0 + \Delta z$ then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

21.6 ❑ ANALYTIC FUNCTIONS

A function defined at a point z_0 is said to be **analytic** at z_0, if it has a derivative at z_0 and at every point in some neighborhood of z_0. It is said to be analytic in a region R, if it is analytic at every point of R. Analytic functions are otherwise named **holomorphic** or **regular** functions.

A point at which a function $f(z)$ is not analytic is called a **singular point** or **singularity** of $f(z)$.

21.7 ❑ CAUCHY–RIEMANN EQUATIONS

The necessary condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain R is that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ must exist and satisfy the Cauchy–Riemann equations, namely,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The sufficient condition for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic at the point $z = x + iy$ of a domain R is that the four partial derivatives u_x, u_y, v_x and v_y exist, are continuous and satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ at each point of R.

➢ **Note**

(i) The two partial differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called the **Cauchy–Riemann equations** and they may be written as $u_x = v_y$ and $u_y = -v_x$

(ii) The Cauchy–Riemann equations are referred as C-R equations

(iii) C-R equations in polar form are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

21.8 ❑ HARMONIC FUNCTION

A real function of two variables x and y that possesses continuous second-order partial derivatives and satisfies the Laplace equation is called a **harmonic function**.

If u and v are harmonic functions such that $u + iv$ is analytic then each is called the **conjugate harmonic function** of the other.

➢ **Note**

(i) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the **Laplacian operator** and is denoted by ∇^2.

(ii) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$ is known as **Laplace equation** in two dimensions.

21.9 ❑ PROPERTIES OF ANALYTIC FUNCTIONS

Property 1

The real and imaginary parts of an analytic function $f(z) = u + iv$ satisfy the Laplace equation in two dimensions.

● **Proof**

Since $f(z) = u + iv$ is an analytic function, it satisfies C-R equations,

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and (21.1)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{21.2}$$

Differentiating both sides of (21.1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{21.3}$$

Differentiating both sides of (21.2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \partial x} \tag{21.4}$$

By adding (21.3) and (21.4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\text{since } \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}, \text{ when they are continuous}\right)$$

$\Rightarrow u$ satisfies Laplace equation.
Now differentiating both sides of (21.1) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \tag{21.5}$$

Differentiating both sides of (21.2) partially with respect to x we get

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2} \tag{21.6}$$

Subtracting (21.5) and (21.6),

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

i.e.,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore v$ satisfies Laplace equation.
Hence, if $f(z)$ is analytic then both real and imaginary parts satisfy Laplace's equation.

➢ **Note**

If $f(z) = u + iv$ is analytic then u and v are harmonic. Conversely, when u and v are any two harmonic functions then $f(z) = u + iv$ need not be analytic.

Property 2

If $f(z) = u + iv$ is an analytic function then the curves of the family $u(x, y) = C_1$ cut orthogonally the curves of the family $v(x, y) = C_2$ where C_1 and C_2 are constants.

- **Proof**

Given $u(x, y) = C_1$
Taking differentials on both sides, we get

$$du = 0$$

i.e.,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\therefore\ \frac{dy}{dx} = -\frac{\left(\dfrac{\partial u}{\partial x}\right)}{\left(\dfrac{\partial u}{\partial y}\right)} = m_1$$ (say), where m_1 is the slope of the curve $u(x, y) = C_1$ at (x, y)

From the second curve $v(x, y) = C_2$, we get $\dfrac{dy}{dx} = -\dfrac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)} = m_2,$ where m_2 is the slope of the curve $v(x, y) = C_2$ at (x, y).

Now,
$$m_1 m_2 = \frac{\left(\dfrac{\partial u}{\partial x}\right)}{\left(\dfrac{\partial u}{\partial y}\right)} \cdot \frac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)}$$

$$= \frac{\left(\dfrac{\partial v}{\partial y}\right)}{-\left(\dfrac{\partial v}{\partial x}\right)} \cdot \frac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)}$$ (as $f(z)$ is analytic, it satisfies C-R equation)

$\Rightarrow m_1 m_2 = -1$

Hence, the curves cut each other orthogonally.

Here, the two families are said to be **orthogonal trajectories** of each other.

21.10 ❑ CONSTRUCTION OF ANALYTIC FUNCTIONS (MILNE–THOMSON METHOD)

To find f(z) when u is given

We know that $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$.

i.e., $$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(By C-R equations)} \tag{21.7}$$

Let $$\frac{\partial u(x, y)}{\partial x} = \phi_1(x, y) \text{ and then calculate } \phi_1(z, 0) \tag{21.8}$$

and $$\frac{\partial u(x, y)}{\partial y} = \phi_2(x, y) \text{ and then calculate } \phi_2(z, 0) \tag{21.9}$$

Substituting (21.8) and (21.9) in (21.7), we get

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

Integrating, we get $\int f^1(z)dz = \int \phi_1(z, 0)dz - i \int \phi_2(z, 0)dz$

i.e., $$f(z) = \int \phi_1(z, 0)dz - i\int \phi_2(z, 0)dz.$$

To find f(z) when v is given

We know that $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \tag{21.10}$$

Let $$\frac{\partial v(x, y)}{\partial y} = \phi_1(z, 0) \tag{21.11}$$

and $$\frac{\partial v(x, y)}{\partial x} = \phi_2(z, 0) \tag{21.12}$$

Substituting (21.11) and (21.12) in (21.10), we get

$$f'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

Integrating, we get $\int f'(z)dz = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$

i.e., $$f(z) = \int \phi_1(z, 0)dz + i\int \phi_2(z, 0)dz$$

21.11 ❑ APPLICATIONS

Irrotational Flows

A flow in which the fluid particles do not rotate about their own axes while flowing is said to be irrotational.

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u = \frac{-\partial \phi}{\partial x}, v = \frac{-\partial \phi}{\partial y} \tag{21.13}$$

In two-dimensional flow, the stream function ψ always exists such that

$$u = \frac{-\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x} \tag{21.14}$$

From (21.13) and (21.14), we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x} \tag{21.15}$$

which are the well-known **Cauchy–Riemann equations**. Hence, $\phi + i\psi$ is an analytic function of $z = x + iy$. Moreover, ϕ and ψ are known as conjugate functions.

On multiplying and rewriting, (21.15) gives

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0 \tag{21.16}$$

showing that the families of curves given by ϕ = constant and ψ = constant intersect orthogonally. Thus, the curves of equi-velocity potential and the stream lines intersect orthogonally.

Differentiating the equation given in (21.15) with respect to x and y respectively, we get $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y}$ and $\frac{\partial^2 \phi}{\partial y^2} = \frac{-\partial^2 \psi}{\partial x \partial y}$. (21.17)

Since $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$, (21.17) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{21.18}$$

Again differentiating Eq. (21.15) with respect to y and x respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial^2\psi}{\partial y^2} \text{ and } \frac{\partial^2\phi}{\partial x\partial y} = \frac{-\partial^2\psi}{\partial x^2}$$

Subtracting these, $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$ (21.19)

Equations (21.18) and (21.19) show that ϕ and ψ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.

Complex Potential

Let $w = \phi + i\psi$ be taken as a function of $x + iy$

Thus, suppose that $w = f(z)$

i.e., $\phi + i\psi = f(x = iy)$ (21.20)

Differentiating (21.20) with respect to x and y respectively, we get

$$\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = f'(x + iy)$$ (21.21)

and $$\frac{\partial\phi}{\partial y} + i\frac{\partial\psi}{\partial y} = i\,f'(x + iy)$$

or $$\frac{\partial\phi}{\partial y} + i\frac{\partial\psi}{\partial y} = i\left(\frac{\partial\phi}{\partial x} + \frac{i\partial\psi}{\partial x}\right) \text{ by}$$ (21.22)

Equating real and imaginary parts, we get

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x}$$

which are C-R equations. Then w is an analytic function of z and w is known as the complex potential.

Conversely, if w is an analytic function of z then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion. The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called **equipotential lines** and **stream lines** respectively.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **equipotential lines** and **lines of force**.

In heat-flow problems, the curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are respectively called **isothermals** and **heat-flow lines.**

SOLVED EXAMPLES

Example 1 Prove that the function $f(z) = |z|^2$ is differentiable only at the origin.

Solution Given $f(z) = |z|^2$

i.e., $$u + iv = |x + iy|^2 = [\sqrt{x^2 + y^2}]^2 \qquad (\text{as } z = x + iy \text{ and } f(z) = u + iv)$$
$$= x^2 + y^2$$

$$\Rightarrow \quad u = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$

$$v = 0$$

$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

If $f(z)$ is differentiable then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad 2x = 0 \quad \Rightarrow \quad x = 0$$

Also,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad 2y = 0 \quad \Rightarrow \quad y = 0$$

$\therefore$ C-R equations are satisfied only when $x = 0$, $y = 0$

Hence, $f(z) = |z|^2$ is differentiable only at the origin (0, 0). **Proved.**

Example 2 Prove that the function $f(z) = z\bar{z}$ is not analytic except at $z = 0$.

Solution Given $f(z) = z\bar{z}$

i.e.,

$$u + iv = (x + iy)(x - iy)$$

$$u + iv = x^2 + y^2$$

Equating real and imaginary parts.

$$u = x^2 + y^2$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y$$

$$v = 0$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C-R equations are not satisfied

$\therefore f(z) = z\bar{z}$ is not analytic except at $z = 0$. **Proved.**

Example 3 Show that (i) an analytic function with a constant real part is a constant, and (ii) an analytic function with a constant modulus is also a constant.

[KU Nov. 2010, April 2012; AU Nov. 2010, Nov. 2011]

Solution Let $f(z) = u + iv$ be an analytic function.

(i) Let $u = C_1$ (a constant)

Then $\frac{\partial u}{\partial x} = u_x = 0$ and $\frac{\partial u}{\partial y} = u_y = 0$.

Since $f(z)$ is an analytic function, by C-R equations $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow \quad v_y = 0$ and $v_x = 0$.

As $v_x = 0$ and $v_y = 0$, v must be independent of x and y and must be a constant C_2.

$\therefore \; f(z) = u + iv = C_1 + iC_2$ which is a constant.

(ii) Let $f(z) = u + iv$ be an analytic function.

Given $|f(z)| = \sqrt{u^2 + v^2} = k$ (a constant)

Differentiating partially with respect to x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and $$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Since $f(z)$ is an analytic function, it satisfies C-R equations.

$\therefore$ the above two equations may be written as,

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$

and $$v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

By solving, we get $\frac{\partial u}{\partial x} = u_x = 0$ and $\frac{\partial u}{\partial y} = u_y = 0$.

By C-R equations, it implies that $\frac{\partial v}{\partial x} = v_x = 0$ and $\frac{\partial v}{\partial y} = v_y = 0$.

Thus, $f(z) = u + iv$ is a constant. **Proved.**

Example 4 If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

[AU May 2006, KU Nov. 2011, KU April 2013]

Solution Let $f(z) = u(x, y) + iv(x, y)$

Then $|f(z)|^2 = u^2 + v^2$ and $|f'(z)|^2 = u_x^2 + v_x^2$

To prove $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2) = 4(u_x^2 + v_x^2)$

Now, $\frac{\partial}{\partial x}(u^2) = 2uu_x$ and $\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x)$

$$= 2[uu_{xx} + u_x u_x] = 2uu_{xx} + u_x^2]$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2[uu_{yy} + u_y^2]$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2u[u_{xx} + u_{yy}] + 2[u_x^2 + u_y^2]$$

$$= 2[u_x^2 + u_y^2] \qquad \text{(since } u_{xx} + u_{yy} = 0) \qquad (1)$$

Again, $$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x^2]$$

and $$\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(v^2) = 2v(v_{xx}+v_{yy})+2(v_x^2+v_y^2)$$

$$= 2(v_x^2+v_y^2) \qquad (\text{since } v_{zz}+v_{yy}=0) \qquad (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 2[u_x^2+u_y^2+v_x^2+v_y^2]$$

$$= 2[u_x^2+v_x^2+v_x^2+u_x^2] \qquad (\text{by using C-R equations}) \quad = 4[u_x^2+v_x^2].$$

Hence, $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$ **Proved.**

Example 5 Show that if $f(z)$ is a regular function of z then $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|$ = 0. **[AU May 2012]**

Solution $\operatorname{Log}|f(z)| = \frac{1}{2}\log|f(z)|^2 = \frac{1}{2}\log(u^2+v^2)$

$$\therefore \quad \frac{\partial}{\partial x}\log|f(z)| = \frac{1}{2}\left[\frac{2uu_x+2v\cdot v_x}{u^2+v^2}\right] = \frac{uu_x+vv_x}{u^2+v^2}$$

$$\frac{\partial^2}{\partial x^2}\log|f(z)| = \frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_{xx}+v_x^2)-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{1}{u^2+v^2}[uu_x+vv_{xx}+u_x^2+v_x^2] - \frac{2}{(u^2+v^2)^2}(uu_x+vv_x)^2 \qquad (1)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\log|f(z)| = \frac{1}{u^2+v^2}[uu_{yy}+vv_{yy}+u_y^2+v_y^2] - \frac{2}{(u^2+v^2)^2}(uu_y+vv_y)^2 \qquad (2)$$

Adding (1) and (2), we get $-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|$

$$= \frac{1}{u^2+v^2}[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+u_x^2+v_x^2+u_y^2+v_y^2] - \frac{2}{(u^2+v^2)^2}$$

$$[(uu_x+vv_x)^2+(uu_y+vv_y)^2]$$

$$= \frac{1}{(u^2+v^2)}[2(u_x^2+v_x^2)] - \frac{2}{(u^2+v^2)^2}[(uu_x+vv_x)^2+(-uv_x+vu_x)^2]$$

$$= \frac{2(u_x^2+v_x^2)}{u^2+v^2} - \frac{2}{(u^2+v^2)^2}[u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)]$$

$$= \frac{2(u_x^2+v_x^2)}{u^2+v^2} - \frac{2(u^2+v^2)(u_x^2+v_x^2)}{(u^2+v^2)^2}$$

$$= 0$$ **Proved.**

Example 6 Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and determine its conjugate. Also find $f(z)$. **[KU May 2010, KU April 2013]**

Solution Given $u = \frac{1}{2}\log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}; \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}; \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

Hence, u satisfies Laplace's equation.
$\therefore$ u is harmonic.

To find conjugate of *u*

We know that $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \frac{-\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)} = \frac{x\,dy - y\,dx}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}d\left(\frac{y}{x}\right)$$

$$\int dv = \int \frac{d(y/x)}{1+(y/x)^2}$$

i.e., $$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore$ the required analytic function is $f(z) = u + iv$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

i.e., $f(z) = \log z$ **Ans.**

Example 7 If $u(x, y) = e^x(x\cos y - y\sin y)$, find $f(z)$ so that $f(z)$ is analytic.

Solution Given $u = e^x(x\cos y - y\sin y)$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \cos y(xe^x + e^x) - y\sin y e^x$$

$$\therefore \quad \phi_1(z, 0) = ze^z + e^z \qquad (1)$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = -xe^x \sin y - e^x(\sin y + y \cos y)$$

$\therefore \qquad \phi_2(z, 0) = 0 \qquad (2)$

By Milne–Thomson method,

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= ze^z + e^z + 0$$
$$= e^z(z + 1)$$

$\therefore \qquad f(z) = \int e^z(z + 1)dz$

$$= ze^z - e^z + e^z + C$$

i.e., $\qquad f(z) = ze^z + C$ **Ans.**

Example 8 Find the analytic function $f(z) = u + iv$ given that $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

[AU May 2006]

Solution Given $u + iv = f(z) \qquad (1)$

$\therefore \qquad iu - v = i\,f(z) \qquad (2)$

Adding (1) and (2), we get

$$(u - v) + i(u + v) = (1 + i)f(z)$$

Let $u - v = U$,

$$u + v = V \quad \text{and} \quad F(z) = (1 + i)\,f(z)$$

$$\frac{\partial V}{\partial x} = \frac{(\cos h2y - \cos 2x)2\cos 2x - \sin 2x \cdot 2\sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(x, y) = \frac{\partial V}{\partial x}$$

$$= \frac{2\cos 2x \cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cos h2y - \cos 2x)^2}$$

$$\phi_1(x, y) = \frac{\partial V}{\partial y} = \frac{-\sin 2x(2\sin h2y)}{(\cos h2y - \cos 2x)^2}$$

$$= \frac{-2\sinh 2y \sin 2x}{(\cos h2y - \cos 2x)^2}$$

By Milne–Thomson method, we have

$$F'(z) = \phi_1(z, 0) + i\phi_2(z, 0)$$

$$\phi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$\phi_1(z, 0) = 0$$

and $\qquad F'(z) = i\dfrac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$

$$= i\frac{-2}{1 - \cos 2z} = i\frac{-1}{\frac{1 - \cos 2z}{2}}$$

$$= i\frac{-1}{\sin^2 z} = -i\,\text{cosec}^2 z$$

$\therefore \qquad f(z) = -\dfrac{i}{1+i}\displaystyle\int \operatorname{cosec}^2 z\, dz$

i.e., $\qquad f(z) = \dfrac{i+1}{2}\cot z + C$ **Ans.**

Example 9 Find the analytic function $f(z) = u + iv$ if $u + v = \dfrac{x}{x^2+y^2}$ and $f(1) = 1$.

[AU Nov. 2010]

Solution Given $u + iv = f(z)$ (1)

$iu - v = if(z)$ (2)

Adding (1) and (2), we get

$(u - v) + i(u + v) = (1 + i)f(z)$

i.e., $\qquad U + iV = F(z)$ (3)

where $\qquad U = u - v,\ V = u + v = \dfrac{x}{x^2+y^2},\ F(z) = (1 + i)\,f(z)$ (4)

$$V = \frac{x}{x^2+y^2}$$

$$\phi_1(x, y) = \frac{\partial V}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$\therefore \qquad \phi_1(z, 0) = 0$ (5)

$$\phi_2(x, y) = \frac{\partial V}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore \qquad \phi_2(z, 0) = \dfrac{-z^2}{z^4} = -\dfrac{1}{z^2}$ (6)

By Milne's method, we have

$$F'(z) = \phi_1(z_1 0) + i\phi_2(z, 0)$$

$$= 0 - i\frac{1}{z^2}$$

$$F(z) = -i\int \frac{1}{z^2}\, dz$$

$\therefore \qquad = -i\left(-\dfrac{1}{z}\right) + C$

$F(z) = \dfrac{i}{z} + C$ (7)

But $F(z) = (1 + i)\, f(z)$ [from (4) and (8)]

From (7) and (8), we get

$$(1+i)\, f(z) = \frac{i}{z} + C$$

$$f(z) = \frac{i}{z(1+i)} + \frac{C}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)z} + C_1, \text{ where } C_1 = \frac{C}{1+i}$$

$$f(z) = \frac{1+i}{2z} + C_1$$

Given $f(1) = 1$

i.e., $$f(1) = \frac{1+i}{2} + C_1 = 1$$

$\Rightarrow$ $$C_1 = 1 - \frac{(1+i)}{2}$$
$$= \frac{1-i}{2}$$

$\therefore$ $$f(z) = \frac{1+i}{2z} + \frac{1-i}{2}$$ **Ans.**

Example 10 Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. **[AU Nov. 2010]**

Solution

Let $z = x + iy$ (1)

$\therefore$ $\bar{z} = x - iy$ (2)

From (1) and (2), we get

$$x = \frac{z+\bar{z}}{2}, \; y = \frac{z-\bar{z}}{2i} = \frac{-i}{2}(z-\bar{z})$$

Now, $$\frac{\partial x}{\partial z} = \frac{1}{2}, \; \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$$
$$\frac{\partial y}{\partial z} = \frac{-i}{2}, \; \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

Now, $$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} \quad (3)$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} \quad (4)$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$\therefore$ $$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$\Rightarrow$ $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$ **Proved.**

Example 11 If $f(z) = u + iv$ is analytic, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$. **[AU Nov. 2010]**

Solution We know that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\partial\bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\cdot\frac{1}{2}\log|f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\log[f'(z)\,f'(\bar{z})]$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}[\log f'(z)+\log f'(\bar{z})]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right]=0$$

Proved.

Example 12 If $u = x^2 - y^2$ and $v = -\dfrac{y}{x^2+y^2}$, prove that both u and v satisfy Laplace's equation but that $u + iv$ is not a regular function of z. **[KU Nov. 2011]**

Solution Given $u = x^2 - y^2$

Then $$\frac{\partial u}{\partial x} = u_x = 2x;\ \frac{\partial^2 u}{\partial x^2} = u_{xx} = 2;\ \frac{\partial u}{\partial y} = u_y = -2y;\ \frac{\partial^2 u}{\partial y^2} = u_{yy} = -2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

i.e., u satisfies Laplace's equation.

$$v = -\frac{y}{x^2+y^2}$$

Then $$\frac{\partial v}{\partial x} = v_x = \frac{2xy}{(x^2+y^2)^2};\ v_{xx} = 2y\left[\frac{(x^2+y^2)\cdot - x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = v_y = -\left[\frac{(x^2+y^2)\cdot 1 - 2y^2}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = v_{yy} = \frac{(x^2+y^2)^2\,2y - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

i.e., v satisfies Laplace's equation.
Now, $u_x \neq v_y$ and $u_y \neq -v_x$
i.e., C-R equations are not satisfied by u and v.
Hence, $u + iv$ is not an analytic (regular) function of z. **Ans.**

Example 13 Show that the function $u(x, y) = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find a function $v(x, y)$ such that $u + iv$ is an analytic function.

[AU June 2010]

Solution Let $f(z) = u + iv$ be an analytic function with $u(x, y) = 3x^2y + x^2 - y^3 - y^2$

Then
$$\frac{\partial u}{\partial x} = u_x = 6xy + 2x; \frac{\partial^2 u}{\partial x^2} = u_{xx} = 6y + 2;$$
$$\frac{\partial u}{\partial y} = u_y = 3x^2 - 3y^2 - 2y; \frac{\partial^2 u}{\partial y^2} = u_{yy} = -6y - 2$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence, $u(x, y)$ is a harmonic function.

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy = \frac{-\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -u_y dx + u_x dy$$

$\therefore dv = (-3x^2 + 2y + 3y^2)dx + (6xy + 2x)dy$ where the RHS is a perfect differential equation.

$$dv = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$$
$$= -\int (3x^2 - 3y^2 - 2y)\, dx + \int (6xy + 2x)\, dy$$

$\therefore$ $v = (3xy^2 + 2xy - x^3) + C$

$\therefore$ $f(z) = u + iv = 3x^2y + x^2 - y^3 - y^2 + i(3xy^2 + 2xy - x^3 + C)$

$= -i[x^3 + 3x^2(iy) + 3xi^2y^2 + i^3y^3] + [x^2 + 2xiy + i^2y^2] + iC$

$= -i[x + iy]^3 + [x + iy]^2 + iC$

$\therefore$ $f(z) = iz^3 + z^2 + iC$ **Ans.**

EXERCISE

Part A

1. Define analytic function of a complex variable.
2. State any two properties of an analytic function.
3. Define a harmonic function with an example.
4. Verify whether the function $\phi(x, y) = e^x \sin y$ is harmonic or not.
5. Find the constant '*a*' so that $u(x, y) = ax^2 - y^2 + xy$ is harmonic.
6. Is $f(z) = z^3$ analytic? Justify.
7. What do you mean by a conjugate harmonic function? Find the conjugate harmonic of x.
8. Show that an analytic function with a constant real part is constant.
9. Write down the necessary condition for $w = f(z) = f(re^{i\theta})$ to be analytic.
10. Show that the function $u = \tan^{-1}\left(\frac{y}{x}\right)$ is harmonic.
11. Show that xy^2 cannot be the real part of an analytic function.
12. $f(z) = u + iv$ is such that u and v are harmonic. Is $f(z)$ analytic always? Justify.

13. State C-R equations in Cartesian coordinates.
14. Prove that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is a harmonic function.
15. Show that the function $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ satisfies Cauchy–Riemann equations.
16. Show that the real part u of an analytic function satisfies the equation $\nabla^2 u = 0$.
17. Check whether the function $\frac{1}{z}$ is analytic or not.
18. Test the analyticity of the function $2xy + i(x^2 - y^2)$.
19. State the basic difference between the limit of a function of a real variable and that of a complex variable.
20. Find the analytic function $f(z) = u + iv$, given that (i) $u = y^2 - x^2$, (ii) $v = \sin hx \sin y$, and (iii) $u = \dfrac{x}{x^2 + y^2}$.

Part B

1. Prove that the following functions are not differentiable (and, hence, not analytic) at the origin.

   (i) $f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

   (ii) $f(z) = \begin{cases} \dfrac{xy^2(x + iy)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

2. Prove that for the following function, C-R equations are satisfied at the origin but $f(z)$ is not analytic there.

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

3. Show that $f(z) = \sin \bar{z}$ is not an analytic function of z.
4. Find whether the Cauchy–Riemann equations are satisfied for the following functions where $w = f(z)$.

   (i) $w = 2xy + i(x^2 - y^2)$ **(Ans.** No**)**

   (ii) $w = \dfrac{x - iy}{x^2 + y^2}$ **(Ans.** No**)**

   (iii) $w = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$ **(Ans.** Yes**)**

   (iv) $w = \cos x \sin hy$ **(Ans.** Yes**)**

   (v) $w = z^3 - 2z^2$ **(Ans.** Yes**)**

5. Show that an analytic function with a constant imaginary part is constant.
6. Show that $u + iv = \dfrac{x - iy}{x - iy + a}$, where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

7. Find an analytic function $w = u + iv$ whose real part is given by
   (i) $u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$ **[Ans.** $e^{-x}(x - iy)^2 (\cos y - i \sin y)$**]**
   (ii) $u = \dfrac{x}{x^2 + y^2}$ $\left(\textbf{Ans. } \dfrac{1}{z} + C\right)$
   (iii) $u = e^x(x \cos y - y \sin y)$ **(Ans.** $ze^z + C$**)**
   (iv) $u = x^4 - 6x^2y^2 + y^4$ **(Ans.** $z^4 + C$**)**
   (v) $u = -\sin x \sin hy$ **(Ans.** $-i \cos z + iC$**)**
8. Find an analytic function $w = u + iv$ whose imaginary part is given by
   (i) $v = e^x(x \cos y + y \sin y)$ **(Ans.** $ize^{-z} + C$**)**
   (ii) $v = -2 \sin x(e^y - e^{-y})$ **(Ans.** $\log z + C$**)**
   (iii) $v = \dfrac{\sin x \sin hy}{\cos 2x + \cos h2y}$ $\left(\textbf{Ans. } \dfrac{1 + \sec z}{2}\right)$
   (iv) $v = x^2 - y^2 + 2xy - 3x - 2y$ **[Ans.** $z^2 - 2z + i(z^2 - 3z)$**]**
   (v) $v = x^3 - 3x^2y + 2x + 1 + y^3 - 3xy^2$ **[Ans.** $(i - 1)z^3 + 2z + C$**]**
9. Show that the following functions are harmonic and find their harmonic conjugates.
   (i) $u = \cos x \cos hy$ **(Ans.** $-\sin x \sin hy + C$**)**
   (ii) $u = e^x(\cos y - \sin y)$ **(Ans.** Not harmonic**)**
   (iii) $u = e^{-x}(y \cos y - x \sin y)$ (**Ans.** $e^x(x \cos y + y \sin y) + C$)
   (iv) $u = e^x \cos y$ (**Ans.** $e^x \sin y + C$)
   (v) $u = 2xy + 3xy^2 - 2y^3$ **(Ans.** Not harmonic**)**
10. Find $f(z) = u + iv$, if $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, given that $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$.
    $\left[\textbf{Ans. } f(z) = \cot\left(\dfrac{z}{2}\right) + \left(\dfrac{1 - i}{2}\right)\right]$
11. Find $f(z) = u + iv$ if $2u - 3v = 3y^2 - 2xy - 3x^2 + 3y - x$ and $f(0) = 0$.
    (**Ans.** $f(z) = iz^2 - z$**)**
12. If $f(z) = u + iv$ is a regular function of z, then show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 |f(z)|^{p-2}|f'(z)|^2$.
13. If $u = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$, find $f(z)$ such that $f(z)$ is analytic.
    (Ans. $f(z) = \cot z + C$**)**
14. Show that $\phi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ can represent the velocity potential in an incompressible fluid flow. Also find the corresponding stream function and complex potential. $\left[\textbf{Ans. } \psi = 2xy - \dfrac{y}{x^2 + y^2} + C;\ f(z) = z^2 + \dfrac{1}{z} + iC\right]$

22 Conformal Mapping

Chapter Outline

- Introduction
- Conformal Transformation
- Conformal Mapping by Elementary Transformations
- Some Standard Transformations
- Bilinear Transformation

22.1 ❑ INTRODUCTION

Many physical problems involving ideal fluid flow, steady-state heat flow, electrostatics, magnetism, current flow etc., can be solved using conformal mapping techniques. These problems generally involve Laplacian in three-dimensional coordinates and also divergence and are of three-dimensional vector functions.

Geometrical Representation

To draw the curve of a complex variable (x, iy), we take two axes, i.e., the first one is the real axis and the other is the imaginary axis. A number of points (x, y) are plotted on the z-plane, by taking different values of z (different values of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called an **Argand diagram**.

Let $w = f(z) = f(x + iy) = u + iv$.

To draw a curve of w, we take the u-axis and v-axis. By plotting different points (u, v) on the w-plane and joining them, we get a curve C on the w-plane.

Transformation

For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this **transformation or mapping of z-plane into w-plane.** If a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.

If the point $P(x, y)$ moves along the curve C in the z-plane, the point $P'(u, v)$ will move along a corresponding curve C_1 in the w-plane. We then say that a curve C in the z-plane is mapped into the corresponding curve C_1 in the w-plane by the relation $w = f(z)$.

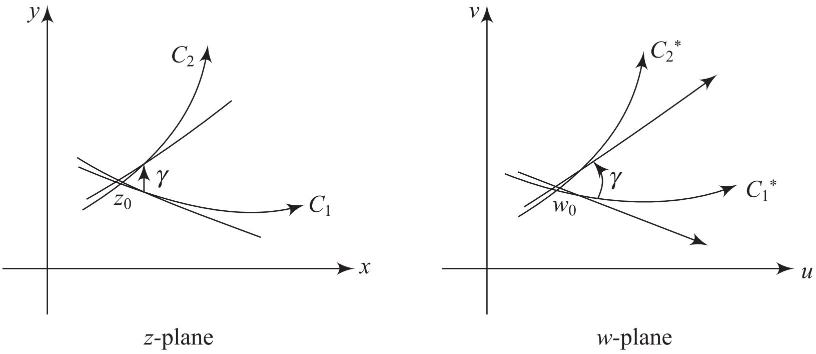


Fig. 22.1

22.2 ❑ CONFORMAL TRANSFORMATION (OR CONFORMAL MAPPING)

A mapping $w = f(z)$ is said to be **conformal** if the angle between any two smooth curves C_1, C_2 in the z-plane intersecting at the point z_0 is equal in magnitude and sense to the angle between their images C_1^*, C_2^* in the w-plane at the point $w_0 = f(z_0)$.

Thus, **conformal mapping** preserves angles both in magnitude and sense (which is also known as conformal mapping of the first kind). If only the magnitude of the angle is preserved, then the mapping is known as **isogonal mapping** (or conformal mapping of the second kind).

Conformal mapping is used to map complicated regions conformally onto simpler, standard regions such as circular disks, half-planes and strips for which the boundary-value problems are easier.

Given two mutually orthogonal one-parameter family of curves, say $\phi(x, y) = C_1$ and $\phi(x, y) = C_2$. Their image curves in the w-plane $\phi(u, v) = C_3$ and $\phi(u, v) = C_4$ under a conformal mapping are also mutually orthogonal. Thus, conformal mapping preserves the property of mutual orthogonality of a system of curves in the plane.

➢ **Note**

(i) **Critical point** of a function $w = f(z)$ is a point z_0, where $f'(z_0) \neq 0$.
(ii) A mapping $w = f(z)$ is conformal at each point z_0 where $f(z)$ is analytic and $f'(z_0) \neq 0$
(iii) An analytic function $f(z)$ is conformal everywhere except at its critical points where $f'(z) \neq 0$.
(iv) Solutions of Laplace's equation are invariant under conformal transformation.
(v) Conjugate functions remain conjugate functions after conformal transformation. This is the main reason for the great importance of conformal transformations in applications.

22.3 ❑ CONFORMAL MAPPING BY ELEMENTARY TRANSFORMATIONS

General linear transformation, or simply transformation, is defined by the function

$$w = f(z) = az + b \tag{22.1}$$

where $a \neq 0$ and b are arbitrary complex constants. The function maps conformally the extended complex z-plane onto the extended w-plane, since this function is analytic and $f'(z) = a \neq 0$ for any z. If $a = 0$ (22.1) reduces to a constant function.

22.4 ❑ SOME STANDARD TRANSFORMATIONS

Translation

The transformation $w = z + c$, where c is a complex constant, represents a translation. Consider the transformation $w = z + c$, where $c = a + ib$.

i.e., $\quad u + iv = (x + iy) + (a + ib)$

$\Rightarrow \quad u = x + a \quad$ and $\quad v = y + b$

i.e., $\quad x = u - a \quad$ and $\quad y = v - b$

On substituting the values of x and y in the equation of the curve to be transformed, we get the equation of the image in the w-plane.

The point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + a, y + b)$ in the w-plane. Similarly, other points of the z-plane are mapped onto the w-plane. Thus, if the w-plane is superposed on the z-plane, the figure of the w-plane is shifted through a vector c.

In other words, the transformation is a mere **translation** of the axes.

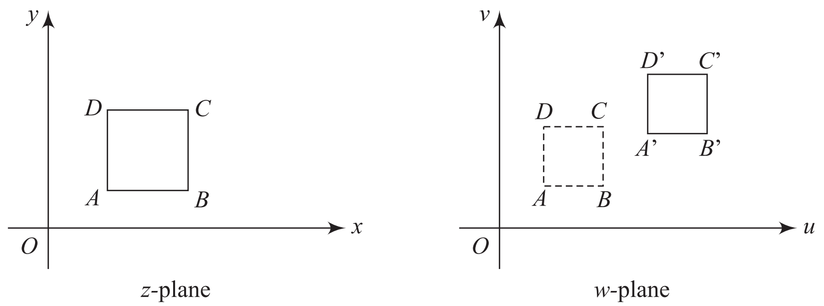


Fig. 22.2

Magnification and Rotation

Consider the transformation $w = cz$ (22.2)

where c, z, w are all complex numbers.

Let $z = re^{i\theta}$, $w = \text{R}e^{i\phi}$, $c = ae^{i\alpha}$

Substituting these values in (22.2), we have

$$\text{R}e^{i\phi} = (ae^{i\alpha})(re^{i\theta}) = ar\, e^{i(\theta + \alpha)}$$

i.e., $\quad R = ar \quad$ and $\quad \phi = \theta + \alpha$

Thus, we see that the transformation $w = cz$ corresponds to a rotation together with magnification.

Algebraically, $\quad w = cz \quad$ or $\quad u + iv = (a + ib)(x + iy)$

$$u + iv = ax - by + i(ay + bx)$$

$\Rightarrow \quad u = ax - by \quad$ and $\quad v = ay + bx$.

On solving these equations, we can get the values of x and y.

i.e., $$x = \frac{au + bv}{a^2 + b^2};\ y = \frac{-bu + av}{a^2 + b^2}$$

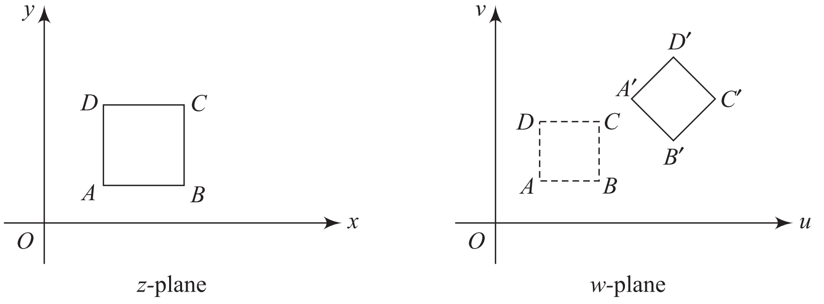


Fig. 22.3

On putting the values of x and y in the equation of the curve to be transformed, we get the equation of the image.

Inversion and Reflection **[KU April 2012]**

Consider the transformation $w = \frac{1}{z}$ (22.3)

$$z = re^{i\theta} \quad \text{and} \quad w = \text{R}e^{i\phi}$$

Substituting these values in (22.3), we get

$$\text{R}e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\Rightarrow \qquad R = \frac{1}{r} \text{ and } \phi = -\theta$$

Thus, the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'\left(\frac{1}{r}, -\theta\right)$ in the w-plane. Hence, the transformation is an inversion of z followed by reflection into the real axis. The points inside the unit circle $|z| = 1$ map onto points outside it, and points outside the unit circle into points inside it.

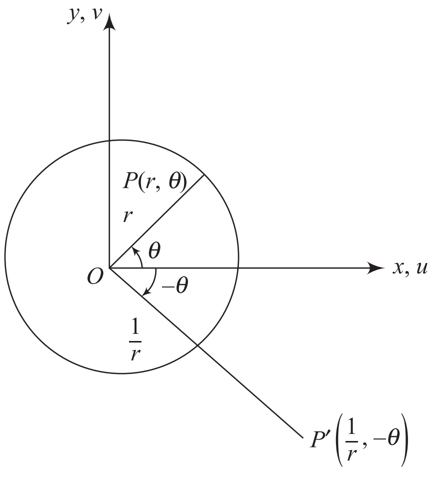


Fig. 22.4

Now consider the transformation $w = \frac{1}{z}$ or $z = \frac{1}{w}$.

i.e., $$x + iy = \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{(u + iv)(u - iv)} = \frac{u - iv}{u^2 + v^2}$$

$$\therefore \qquad x = \frac{u}{u^2+v^2}, \; y = \frac{-v}{u^2+v^2}$$

Let the circle $a(x^2 + y^2) + bx + cy + d = 0$ (22.4)
be in the z-plane.
If $a \neq 0$, (22.4) represents a circle and if $a = 0$, it represents a straight line.
On substituting the values of x and y in (22.4), we get

$$\frac{a}{u^2+v^2} + \frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} + d = 0$$

$$\Rightarrow \qquad d(u^2+v^2) + bu - cv + a = 0 \qquad (22.5)$$

If $d \neq 0$ Eq. (22.5) represents a circle and if $d = 0$ it represents a straight line.
The various cases are discussed as follows:

- **When $a \neq 0$, $d \neq 0$**

The transformation $w = \frac{1}{z}$ transforms circles not passing through the origin into circles not passing through the origin.

- **When $a \neq 0$, $d = 0$**

The transformation $w = \frac{1}{z}$ transforms circles passing through the origin in the z-plane and maps into the straight lines not passing through the origin in the w-plane.

- **When $a = 0$, $d \neq 0$**

The transformation $w = \frac{1}{z}$ transforms straight lines in the z-plane not passing through the origin into circles through the origin in the w-plane.

- **When $a = 0$, $d = 0$**

The transformation $w = \frac{1}{z}$ transforms straight lines through the origin in the z-plane into straight lines through the origin in the w-plane.

22.5 ❑ BILINEAR TRANSFORMATION (OR MÖBIUS TRANSFORMATION)

The transformation $w = f(z) = \dfrac{az+b}{cz+d}$ (22.8)

where a, b, c, d are complex or real constants subject to $ad - bc \neq 0$ is known as bilinear transformation.
Differentiating (22.8), we get

$$\frac{dw}{dz} = \frac{(cz+d)a - (az+b)c}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2}$$

If $ad - bc \neq 0$ then $\frac{dw}{dz} \neq 0$ for any z and, therefore, bilinear transformation is conformal for all z, i.e., it maps the z-plane conformally onto the w-plane

If $ad - bc = 0$ then $\frac{dw}{dz} = 0$ for any z. Then every point of the z-plane is critical and the function is not conformal.

From (22.8), we get $w(cz + d) = az + b$,

i.e., $$cwz + dw - az - b = 0 \quad (22.9)$$

Equation (22.9) is linear in z and linear in w or bilinear in z and w. Bilinear transformation is also known as **linear fractional transformation** or **Mobius transformation.**

For a choice of the constants a, b, c, d, we get special cases of bilinear transformation as

(i) $w = z + b$ (Translation)
(ii) $w = az$ (Rotation)
(iii) $w = az + b$ (Linear transformation)
(iv) $w = \frac{1}{z}$ (Inversion in the unit circle)

Thus, bilinear transformation can be considered as a combination of these transformations.

Fixed Points (or Invariant Points)

Fixed (or invariant) points of a function $w = f(z)$ are points which are mapped onto themselves, i.e., $w = f(z) = z$.

- **Example**

$w = z$ has every point as a fixed point.
$w = \bar{z}$ infinitely many.
$w = \frac{1}{z}$ has two.
$w = z + b$ has no fixed point.

The fixed points of the bilinear transformation $w = \frac{az + b}{cz + d}$ are given by $\frac{az + b}{cz + d} = z$.

As this is quadratic in z, we will get two fixed points for the bilinear transformation.

Cross-ratio

The **cross-ratio,** or **anharmonic ratio,** of four numbers z_1, z_2, z_3, z_4 is the linear function given by $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$.

➢ **Note**

(i) The cross-ratio of four points is invariant under a bilinear transformation, i.e., if w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation, then $\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_1 - w_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$.

(ii) The bilinear transformation that maps three given points z_2, z_3, z_4 onto three given points w_2, w_3, w_4 is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w-w_3)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z-z_3)}$$

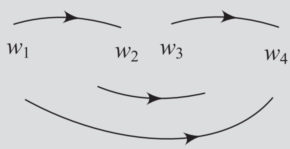


Fig. 22.5

SOLVED EXAMPLES

Example 1 Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$.

Solution Let $z = x + iy$; $w = u + iv$

Given $w = z + 3 + 2i$

i.e., $u + iv = (x + iy) + (3 + 2i)$

$\Rightarrow$ $u = x + 3$; $v = y + 2$

Given the circle $|z| = 2$

i.e., $x^2 + y^2 = 4$

i.e., $(u-3)^2 + (v-2)^2 = 4$

Hence, the circle $x^2 + y^2 = 4$ maps into $(u-3)^2 + (v-2)^2 = 4$ in the w-plane which is also a circle with centre at (3, 2) and radius of 2 units. **Ans.**

Example 2 Find the image of the triangular region in the z-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ under the transformation $w = 2z$. **[KU May 2010]**

Solution Given $w = 2z$. i.e., $u + iv = 2(x + iy)$

$\therefore$ $u = 2x$ and $v = 2y$

When $x = 0$, $u = 0$, the line $x = 0$ is transformed into the line $u = 0$ in the w-plane.
When $y = 0$, $v = 0$, the line $y = 0$ is transformed into the line $v = 0$ in the w-plane.
When $x + y = 1$, we get

$$\frac{u}{2}+\frac{v}{2}=1$$

$\Rightarrow$ $u + v = 2$

$\therefore$ the line $x + y = 1$ is transformed into the line $u + v = 2$ in the w-plane.

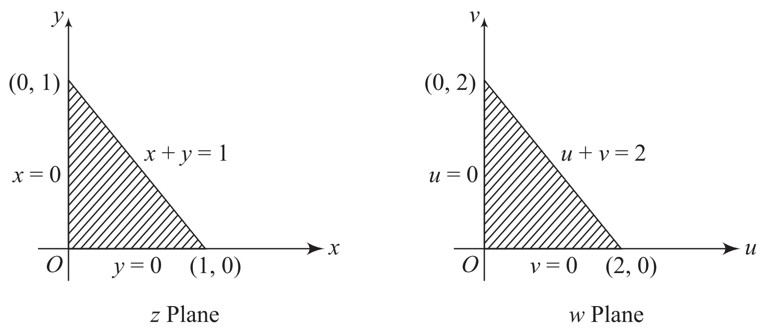


Fig. 22.6

Example 3 Find the image of the circle $|z-1|=1$ in the complex plane under the mapping $w=\frac{1}{z}$.

Solution The given transformation is $w=\frac{1}{z}$

i.e., $z=\frac{1}{w}$

The equation of the circle is $|z-1|=1$

i.e., $|x+iy-1|=1$

$$(x-1)^2+y^2=1 \quad \Rightarrow \quad x^2-2x+y^2=0 \tag{1}$$

Now, $w=u+iv$

$$\therefore \quad z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

$$\therefore \quad x=\frac{u}{u^2+v^2} \tag{2}$$

and

$$y=\frac{-v}{u^2+v^2} \tag{3}$$

Substituting (2) and (3) in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2-2\left(\frac{u}{u^2+v^2}\right)+\left(\frac{-v}{u^2+v^2}\right)^2=0$$

i.e., $u^2-2u(u^2+v^2)+v^2=0$

$(u^2+v^2)(1-2u)=0$

$\Rightarrow$ $1-2u=0$ (since $u^2+v^2\neq 0$)

i.e., $2u-1=0$ which is a straight line in the w-plane. Hence, the circle $|z-1|=1$ is mapped into a straight line under the transformation $w=\frac{1}{z}$. **Ans.**

Example 4 Find the image of the infinite strips (i) $\frac{1}{4}<y<\frac{1}{2}$; and (ii) $0<y<\frac{1}{2}$ under the transformation $w=\frac{1}{z}$. **[KU April 2013]**

Solution Let $w=u+iv$, $z=x+iy$.

Given $w=\frac{1}{z}$

i.e., $$u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$$

i.e., $$u=\frac{x}{x^2+y^2} \tag{1}$$

$$v=\frac{-y}{x^2+y^2} \tag{2}$$

Now, $\frac{u}{v} = \frac{-x}{y}$.

i.e., $$x = \frac{-uy}{v} \tag{3}$$

Substituting (3) in (2), we get

$$v = \frac{-y}{\frac{u^2y^2}{v^2} + y^2} = \frac{-v^2}{(u^2+v^2)\cdot y}$$

or $$y = \frac{-v}{u^2+v^2} \tag{4}$$

(i) Consider a strip $\frac{1}{4} < y < \frac{1}{2}$.

When $y = \frac{1}{4}$,

From (4), $\frac{1}{4} = \frac{-v}{u^2+v^2}$

i.e., $u^2 + v^2 + 4v = 0$ or $u^2 + (v+2)^2 = 4$.

which is a circle whose centre is at (0, –2) in the w-plane and radius is 2 units.

When $y = \frac{1}{2}$,

From (4), $\frac{-v}{u^2+v^2} = \frac{1}{2}$

i.e., $u^2 + (v+1)^2 = 1$.

which is a circle whose centre is at (0, –1) in the w-plane and the radius is 1 unit.

Hence, the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region common to the circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w-plane.

(ii) Consider a strip $0 < y < \frac{1}{2}$.

When $y = 0$,

from (4), we get $v = 0$.

When $y = \frac{1}{2}$,

from (4), we get $\frac{1}{2} = \frac{-v}{u^2+v^2}$.

i.e., $u^2 + v^2 + 2v = 0$

i.e., $u^2 + (v+1)^2 - 1 = 0$

which is a circle whose centre is at (0, –1) in the w-plane and radius is 1 unit.

$\therefore$ the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half-plane. **Ans.**

Example 5 Find the invariant points of the transformation $w = -\dfrac{2z+4i}{iz+1}$.

Solution The invariant points of the transformation are given by $z = -\dfrac{2z+4i}{iz+1}$

$\Rightarrow \quad iz^2 + 3z + 4i = 0$

i.e., $\quad z^2 - 3iz + 4 = 0$

i.e., $\quad (z-4i)(z+i) = 0$

i.e., $z = 4i, -i$ are the invariant points. **Ans.**

Example 6 Find the image of $|z+2i| = 2$ under the transformation $w = \dfrac{1}{z}$.

[AU May 2010]

Solution The given transformation is $w = \dfrac{1}{z}$

i.e., $\quad z = \dfrac{1}{w}$

Given $\quad |z+2i| = 2$

$|x + iy + 2i| = 2$

i.e., $\quad |x + i(y+2)| = 2$

$\Rightarrow \quad x^2 + (y+2)^2 = 4$

i.e., $\quad x^2 + y^2 + 4y = 0 \qquad (1)$

Now, $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

i.e., $\quad x + iy = \dfrac{u-iv}{u^2+v^2}$

$\Rightarrow \quad x = \dfrac{u}{u^2+v^2}, \qquad (2)$

and $\quad y = \dfrac{-v}{u^2+v^2} \qquad (3)$

Substituting (2) and (3) in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 + 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$u^2 + v^2 - 4v(u^2+v^2) = 0$$

$$(u^2+v^2)(1-4v) = 0$$

$\Rightarrow \quad 1 - 4v = 0 \quad (\text{as } u^2 + v^2 \neq 0)$

which is a straight line in the w-plane. **Ans.**

Example 7 Find the bilinear transformation that maps the points $z_1 = -i$, $z_2 = 0$, $z_3 = i$ into the points $w_1 = -1$, $w_2 = i$, $w_3 = 1$ respectively. **[AU Oct. 2009, KU Nov. 2010]**

Solution Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \quad (1)$$

Given $z_1 = -i, z_2 = 0, z_3 = 0; w_1 = -1, w_2 = i, w_3 = 1$ (2)

Substituting (2) in (1), we get

$$\frac{(w+1)(i-1)}{(-1-i)(1-w)} = \frac{(z+i)(0-i)}{(-i-0)(i-z)}$$

i.e.,
$$\frac{(w+1)}{(w-1)}\frac{(i-1)(i-1)}{(i+1)(i-1)} = \frac{-(z+i)}{(z-i)}$$

i.e.,
$$\frac{w+1}{w-1}\cdot\frac{-2i}{-2} = \frac{-(z+i)}{(z-i)}$$

$$\frac{w+1}{w-1} = \frac{i(z+i)}{z-i}$$

By componendo and dividendo,

$$\frac{(w+1)+(w-1)}{(w+1)-(w-1)} = \frac{i(z+i)+(z-i)}{i(z+i)-(z-i)}$$

$$\frac{2w}{2} = \frac{z(1+i)-(1+i)}{z(i-1)-(1-i)}$$

$$w = \frac{(1+i)(z-1)}{(i-1)(z+1)}$$

$$= \frac{(1+i)(-i-1)}{(i-1)(-i-1)}\cdot\frac{(z-1)}{(z+1)}$$

$\Rightarrow$
$$w = -\left(\frac{z-1}{z+1}\right)$$ **Ans.**

Example 8 Find the bilinear transformation which maps the points $z_1 = -1, z_2 = 0, z_3 = 1$ into the points $w_1 = 0, w_2 = i, w_3 = 3i$ respectively.

[AU Nov. 2010, KU April 2012]

Solution Let the bilinear translation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \quad (1)$$

Given $z_1 = -1, z_2 = 0, z_3 = 1; w_1 = 0, w_2 = i, w_3 = 3i$ (2)

Substituting (2) in (1), we get

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{w(-2i)}{-i(3i-w)} = \frac{(z+1)}{1-z}$$

$$\frac{-2iw}{(w-3i)i} = -\left(\frac{z+1}{z-1}\right)$$

i.e., $$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2w(z-1) = (z+1)(w-3i)$$
$$= zw - 3iz + w - 3i$$

$\Rightarrow$ $$w[2(z-1) - (z+1)] = -3i(z+1)$$

or $$w = -3i\frac{(z+1)}{z-3}$$ **Ans.**

Example 9 Show that under the mapping $w = \frac{i-z}{i+z}$, the image of the circle $x^2 = y^2 < 1$ is the entire half of the w-plane to the right of the imaginary axis.

[AU Nov. 2011]

Solution Given $w = \frac{i-z}{i+z}$

i.e., $(i + z)w = i - z$

$iw + zw = i - z$

i.e., $z(w + 1) = i(1 - w)$

$\Rightarrow$ $$z = \frac{i(1-w)}{1+w}$$

Also given $x^2 + y^2 < 1$

i.e., $|z| < 1$, i.e., $\left|\frac{i(1-w)}{1+w}\right| < 1$

i.e., $|i|\ |1 - w| < |1 + w|$, i.e., $|1 - u - iv| < |1 + u + iv|$ [as $|i| = 1$]

i.e., $(1 - u)^2 + v^2 < (1 + u)^2 + v^2$

i.e., $1 + u^2 - 2u + v^2 < 1 + u^2 + 2u + v^2$

$\Rightarrow$ $4u > 0$

or $u > 0$

Hence, the circle $x^2 + y^2 < 1$, i.e., $|z| < 1$ is mapped into the entire half of the w-plane to the right of the imaginary axis.

When $|z| = 1$ i.e., $x^2 + y^2 = 1$ which is the unit circle, we get $u = 0$ which is the imaginary axis of the w-plane. **Proved.**

EXERCISE

Part A

1. Define conformal mapping.
2. When is a transformation said to be isogonal? Prove that the mapping $w = \bar{z}$ is isogonal.
3. Define critical point of a transformation.

4. Find the images of the circle $|z| = a$ under the transformations (i) $w = z + 2 + 3i$, and (ii) $w = 2z$.
5. Under the transformation $w = iz + i$, show that the half-plane $x > 0$ maps into the half-plane $w > 1$.
6. Find the invariant point of the bilinear transformation $w = \frac{1+z}{1-z}$.
7. Find the fixed points of $w = \frac{3z-4}{z-1}$.
8. Define Mobius transformation.
9. Find the invariant point of the transformation $w = \frac{1}{z-2i}$.
10. Find the image of $x^2 + y^2 = 4$ under the transformation $w = 3z$.
11. Find the image of the circle $|z - \alpha| = r$ by the mapping $w = z + c$ where c is a constant.
12. Find the fixed points of the transformation $w = \frac{1}{z+2i}$.
13. Find the invariant points of the transformation $w = \frac{1+z}{1-z}$.
14. Find the image of the circle $|z| = 3$ under the transformation $w = 2z$.
15. Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$.
16. Find the image of the real axis of the z-plane by the transformation $w = \frac{1}{z+i}$.
17. Define cross-ratio of four points in a complex plane.
18. Prove that a bilinear transformation has at most two fixed points.

Part B

1. For the mapping $w = \frac{1}{z}$, find the image of the family of circles $x^2 + y^2 = ax$, where a is real. $\left(\textbf{Ans. } u = \frac{1}{a}, \text{ is a straight line}\right)$
2. Determine the region of the w-plane into which the region bounded by $x = 1$, $y = 1$, $x + y = 1$ is mapped by the transformation $w = z^2$.
   (**Ans.** $4u + v^2 = 4$, $4u - v^2 = -4$, $u^2 = 2$, $v^2 = 1$)
3. Determine the images of the regions under $w = \frac{1}{z}$. (i) $x > 1, y > 0$ (ii) $0 < y < \frac{1}{2c}$.
   $\left[\textbf{Ans. } \text{(i) } \left|w - \frac{1}{2}\right| < \frac{1}{2} \text{ (ii) } u^2 + (v+c)^2 > c^2\right]$
4. Find an analytic function $w = f(z)$ which maps the half-plane $x \geq 0$ onto the region $u \geq 2$ such that $z = 0$ corresponds to $w = 2 + i$.
   (*Hint*: $w_1 = z$, $w_2 = w_1 + 2$, $w = w_2 + \text{i}$)
   (**Ans.** $w = z + 2 + i$)
5. Determine and plot the images of the regions under the transformation $w = z^2$.
   (i) $|z| = 2$ (ii) $|\arg z| \leq \frac{\pi}{2}$ (iii) $\frac{1}{2} < |z| < 2, \text{Re } z \geq 0$
   $\left[\textbf{Ans. } \text{(i) } 1w > 4 \text{ (ii) } |\arg w| \leq \pi \text{ (iii) } \frac{1}{4} < |w| < 4, -\pi \leq \phi \leq \pi\right]$

6. Find the invariant (fixed) points of the transformation:

   (i) $w = \dfrac{z-1}{z+1}$ (ii) $w = z^2$ (iii) $w = \dfrac{2z-5}{z+4}$ (iv) $w = (z-i)^2$

   $$\left[\textbf{Ans.}\ \text{(i)}\ z = \pm i\ \ \text{(ii)}\ z = 0, 1\ \ \text{(iii)}\ z = -1+2i\ \ \text{(iv)}\ z = \frac{(1+2i) \pm \sqrt{1+4i}}{2}\right]$$

7. Find the bilinear transformation that maps z_1, z_2, z_3 onto w_1, w_2, w_3 respectively.
   (i) $z = -1, 0, 1$ onto $w = 0, i, 3i$
   (ii) $z = 0, -i, -1$ onto $w = i, 1, 0$
   (iii) $z = 1, i, -1$ onto $w = 2, i, -2$
   (iv) $z = \infty, i, 0$ onto $w = 0, i, \infty$
   (v) $z = 1, 0, -1$ onto $w = i, 1, \infty$

   $$\left[\begin{array}{l}\textbf{Ans.}\ \text{(i)}\ w = \dfrac{-3i(z+1)}{z-3},\ \text{(ii)}\ w = -i\left(\dfrac{z+1}{z-1}\right)\ \text{(iii)}\ w = \dfrac{-6z+2i}{iz-3} \\ \quad \text{(iv)}\ w = -\dfrac{1}{z}\ \text{(v)}\ w = \dfrac{(-1+2i)z+1}{z+1}\end{array}\right]$$

8. Verify that the equation $w = \dfrac{1+iz}{1+z}$ maps the exterior of the circle $|z| = 1$ into the upper half-plane $v > 0$.
9. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. **(Ans.** i, $2i$**)**
10. Show that the transformation $w = \dfrac{i(1-z)}{1+z}$ maps the circle $|z| = 1$ into the real axis of the w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane.
11. Show that the transformation $w = \dfrac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$.
12. Show that transformation $w = \dfrac{i-z}{i+z}$ maps the circle $|z| = 1$ onto the imaginary axis of the w-plane. Find also the images of the interior and exterior of this circle.

S.No	**Questions**	Opt 1	Opt 2	Opt 3	Opt 4	Answer
1	An example of single valued function of z is ________.	w = z^2	w = z^(1/2)	w=SQRT(z)	w=z^-1	w = z^2
2	An example of multiple valued function of z is _______.	w = z^2	w = z^(1/2)	w=SQRT(z)	w=z^-1	w = z^(1/2)
3	The distance between two points z and z_0 is	$\|z-z_0\|$	$\|z+z_0\|$	z	z_0	$\|z-z_0\|$
4	A circle of radius 1 with centre at origin can be represented by _______.	$\|z\|>1$	$\|z\| < 1$	$\|z\|=1$	$\|z\| = 0$	$\|z\| = 1$
7	If f(z) is differentiable at z_0 then f(z) is ________ at $z_{0.}$	discontinuous	continuous	regular	irregular	continuous
8	A function is said to be _________ at a point if its derivative exists not only at point but also in some neighborhood of that point.	entire function	integral function	analytic	continuous	analytic
9	A function which is analytic everywhere in the finite plane is called ___________.	analytic function	holomorphic functio	regular function	entire function	entire function
11	The necessary condition for f(z) to be analytic is_________	$u_x = v_y$ and $v_x = -u_y$	$u_x = -v_y$ and $v_x =$	$u_x = v_y$ and $v_x = u\_$	$u_x = -v_y$ and $v_x = -u_y$	$u_x = v_y$ and $v_x = -u_y$
12	A real function of two variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called __________.	analytic function	regular function	holomorphic function	harmonic function	harmonic function
13	If u and v are harmonic functions such that u+iv is analytic then each is called the _________ of the other.	conjugate harmonic	analytic	entire function	not analytic	conjugate harmonic

14	A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is called _______ at that point.	Conformal	isogonal	entire function	unconformal	Conformal
15	A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be __________ at that point.	Conformal	isogonal	entire function	unconformal	isogonal
16	A mapping w = f(z) is said to be conformal at $z = z_0$ if __________.	$f'(z_0) = 0$	$f'(z_0) = f(z)$	$f'(z_0) \neq 0$	$f'(z_0) \neq f(z)$	$f'(z_0) \neq 0$
17	The point at which the mapping w = f(z) is not conformal, that is, f'(z) = 0 is called ________ of the mapping.	common	fixed	invariant	critical	critical
18	A _________ point of a mapping w = f(z) are points that are mapped onto themselves, are kept fixed under the mapping.	common	fixed	critical	variant	fixed
19	The transformation w = a+z where a is a complex constant, represents a __________.	translation	magnification	rotation	reflection	translation
20	The transformation _________ where a is a complex constant represents a translation.	w = az	w = az+b	w = a+z	w = 1/z	w = a+z
21	The transformation _________ where a is a real constant represents magnification.	w = a+z	w = 1/z	w = az+b	w = az	w = az
22	The transformation w = az where a is a real constant represents __________.	translation	magnification	reflection	inversion	magnification
23	In general linear transformation, w = az+b where a and b are complex constants represents _______.	magnification	rotation	translation	magnification, rotation and translation	magnification, rotation and translation
24	The transformation w=(az+b)/(cz+d), where a, b, c, d are complex numbers is called a ______.	Linear transformation	bilinear transformation	fractional transformation	translation	bilinear transformation
25	A bilinear transformation is also called a _______.	linear transformation	inversion	fractional transformation	linear fractional transformation	linear fractional transformation
26	The value of i =	SQRT(-1)	SQRT(1)	-1	1	SQRT(-1)
27	_________ represents the interior of the circle excluding its circumference.	$\|z - z_0\| >$ delta	$\|z - z_0\| <$ delta	$\|z - z_0\| \geq$ delta	$\|z - z_0\| \leq$ delta	$\|z - z_0\| <$ delta
28	_________ represents the interior of the circle including its circumference.	$\|z - z_0\| >$ delta	$\|z - z_0\| <$ delta	$\|z - z_0\| \geq$ delta	$\|z - z_0\| \leq$ delta	$\|z - z_0\| \leq$ delta
29	_________ represents the exterior of the circle.	$\|z - z_0\| >$ delta	$\|z - z_0\| <$ delta	$\|z - z_0\| \geq$ delta	$\|z - z_0\| \leq$ delta	$\|z - z_0\| >$ delta
30	Cauchy-Riemann equations are necessary conditions for a function w = f(z) to be an _____.	entire function	integral function	analytic function	continuous function	analytic function
31	Cauchy-Riemann equations are	$u_x = v_y$ and $v_x = -u_y$	$u_x = -v_y$ and $v_x =$	$u_x = v_y$ and $v_x = u\_$	$u_x = -v_y$ and $v_x = -u_y$	$u_x = v_y$ and $v_x = -u_y$
32	The real and imaginary parts of an analytic function f(z) = u+iv satisfies the ______ equation in two dimensions.	Cauchy-Riemann	Homogeneous	Laplace	Euler	Laplace
33	An analytic function with a constant real part is __________.	a variable	a constant	an analytic function	an entire function	a constant
34	An analytic function with a constant modulus is __________.	a variable	a constant	an analytic function	an entire function	a constant
35	A fixed point is also called as _________.	invariant points	critical points	common point	origin	invariant points
36	The fixed point of w=(5z+4)/(z+5) is	2,1	1,-1	-2, 2	0, 1	-2, 2
37	The critical point of z=(2z+1)/(z+2) is	1, 1	1, -1	1,2	0,1	1, -1
38	Solutions of Laplace's equation are ________ under conformal transformation	common	fixed	invariant	critical	invariant
39	If f(z) is analytic, and f'(z)=0 everywhere, then f(z) is _____	a variable	a constant	an analytic function	an entire function	a constant
40	An analytic function with a constant imaginary part is __________.	a variable	a constant	an analytic function	an entire function	a constant
41	If u+iv is analytic, then v-iu is _______	entire function	integral function	analytic	continuous	analytic
44	w=z has every point as a _______ point	fixed	critical	invariant	common	fixed
45	w=1/z has _______ fixed points	1	2	3	4	2
46	w=z+b has ________ fixed points	0	1	2	3	0

Unit X

Complex Integration

23 Complex Integration

Chapter Outline

23.1 ❑ INTRODUCTION

Integration of functions of a complex variable plays a very important role in many areas of science and engineering. The advantage of complex integration is that certain complicated real integrals can be evaluated and properties of analytical functions can be established. Using integration, we shall prove a very important result in the theory of analytic functions:

If a function $f(z)$ is analytic in a domain D then it possesses derivatives of all orders in D, that is $f'(z)$, $f''(z)$ … are all analytic functions in D.

Such a result does not exist in the real-variable theory. Also, the complex-integration approach can be used to evaluate many improper integrals of a real variable, which cannot be evaluated using real integral calculus. The concept of definite integral for functions of a real variable does not directly extend to the case of complex variables.

In the case of a real variable, the path of integration in the definite integral $\int_a^b f(x)dx$ is along a straight line. In complex integration, the path could be along any curve from $z = a$ to $z = b$.

23.2 ❑ LINE INTEGRAL IN COMPLEX PLANE

- **Continuous Arc**

The set of points (x, y) defined by $x = \phi(t)$, $y = \psi(t)$, with parameter t in the interval (a, b), defines a continuous arc provided ϕ and ψ are continuous functions.

- **Smooth Arc**

If ϕ and ψ are differentiable, the arc is said to be smooth.

- **Simple Curve**

It is a curve having no self-intersections, i.e., no two distinct values of t correspond to the same point (x, y).

- **Closed Curve**

It is one in which end points coincide, i.e., $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$.

- **Simple Closed Curve**

It is a curve having no self-intersections and with coincident end points.

- **Contour**

It is a continuous chain of a finite number of smooth arcs.

- **Closed Contour**

It is a piecewise smooth closed curve without points of self-intersection.

23.3 ❑ LINE INTEGRAL

Definite integral or complex line integral or simply line integral of a complex function $f(z)$ from z_1 to z_2 along a curve C is defined as

$$\begin{aligned}\int_C f(z)dz &= \int_C (u+iv)(dx+i\,dy)\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\end{aligned}$$

Here, C is known as path of integration. If it is a closed curve, the line integral is denoted by $\oint_C$.

When the direction is in positive sense, it is indicated as $\int_{C+}$ or simply, $\int_C$ while negative direction is denoted by $\int_C$. Counter integral is an integral along a closed contour.

23.4 ❑ BASIC PROPERTIES OF LINE INTEGRALS

(i) Linearity: $\int_C (k_1 f(z) + k_2 g(z))\,dz = k_1 \int_C f(z)dz + k_2 \int_C g(z)dz$

(ii) Sense reversal: $\int_a^b f(z)dz = -\int_b^a f(z)dz$

(iii) Partitioning of path: $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$
where the curve C consists of the curves C_1 and C_2.

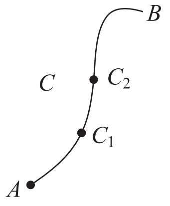


Fig. 23.1

➢ **Note**

Although real definite integrals are interpreted as area, no such interpretation is possible for complex definite integrals.

23.5 ❑ SIMPLY CONNECTED REGION AND MULTIPLY CONNECTED REGION

A simply connected region R is a domain such that every simple closed path in R contains only points of R.

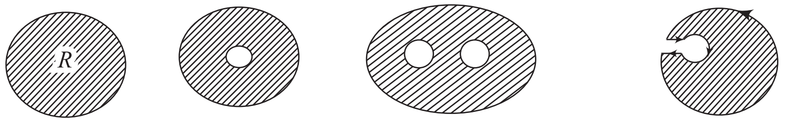
● **Example**

Interior of a circle, rectangle, triangle, ellipse, etc.

A multiply connected region is one that is not simply connected.

● **Example**

Annulus region, region with holes.



Simply connected region | Doubly connected region | Triply connected region | Simply connected region (or) Multiply connected region converted into simply connected region by cross-cuts.

Fig. 23.2

23.6 ❑ EVALUATION OF A COMPLEX INTEGRAL

To evaluate the integral $\int_C f(z)dz$, we have to express it in terms of real variables.

Let $f(z) = u + iv$ where $z = x + iy$, $dz = dx + idy$

$$\therefore \quad \int_C f(z)dz = \int_C (u + iv)dz$$
$$= \int_C (u + iv)(dx + idy)$$
$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

23.7 ❑ CAUCHY'S INTEGRAL THEOREM OR CAUCHY'S FUNDAMENTAL THEOREM

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\int_C f(z)dz = 0$.

- **Proof**

Let the region enclosed by a curve C be R and let

$$f(z) = u + iv,\ z = x + iy,\ dz = dx + i\,dy$$

$$\int_C f(z)dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy \quad \text{(by Green's theorem)}$$

Replacing $-\dfrac{\partial v}{\partial x}$ by $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ by $\dfrac{\partial u}{\partial x}$, we get

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)dx\,dy$$

$$= 0 + i0 = 0$$

or $\int_C f(z)dz = 0$

➢ **Note**

(i) Cauchy's integral theorem is also known as Cauchy's theorem.
(ii) Cauchy's theorem without the assumption that f' is continuous is known as the **Cauchy–Goursat theorem.**
(iii) Simple connectedness is essential.

23.8 ❑ EXTENSION OF CAUCHY'S INTEGRAL THEOREM TO MULTIPLY CONNECTED REGIONS

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

- **Proof**

By Cauchy's integral theorem, we know that $\int_C f(z)dz = 0$ where the path of integration is along AB and the curve C_2 in clockwise direction, and BA and along C_1 in anticlockwise direction,

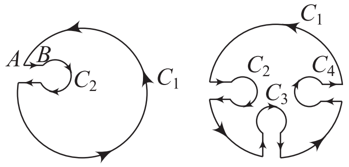


Fig. 23.3

i.e., $\int_{AB} f(z)dz + \int_{C_2} f(z)dz + \int_{BA} f(z)dz + \int_{C_1} f(z)dz = 0$

or $\int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0$ (since $\int_{AB} f(z)dz = -\int_{BA} f(z)dz$)

Reversing the direction of the integral around C_2, we get

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

➢ **Note**

By introducing as many cross-cuts as the number of inner boundaries, we can give the proof in a similar manner for the extension of Cauchy's integral theorem.

23.9 ❑ CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve C and if a is any point within C then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$.

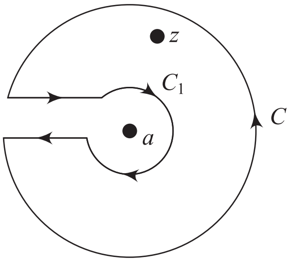


Fig. 23.4

• **Proof**

Consider the function $\frac{f(z)}{z-a}$, which is analytic at all points within C except $z = a$.

With a point a as centre and radius r, draw a small circle C_1 lying entirely within C. Now, $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1;

Hence, by Cauchy's integral theorem for a multiply connected region, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)-f(a)+f(a)}{z-a}dz$$

$$= \int_{C_1} \frac{f(z)-f(a)}{z-a}dz + f(a)\int_{C_1} \frac{dz}{z-a} \qquad (23.1)$$

For any point on C_1

Now, $$\int_{C_1} \frac{f(z)-f(a)}{z-a}dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}} ire^{i\theta}d\theta$$

[as $z-a = re^{i\theta}$ and $dz = ire^{i\theta}\,d\theta$]

$$= \int_0^{2\pi} [f(a+re^{i\theta})-f(a)]id\theta = 0$$ (where r tends to zero]

$$\int_{C_1} \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = \int_0^{2\pi} id\theta = i[0]_0^{2\pi i} = 2\pi i$$

Putting the values of the integrals of RHS in (23.1), we have

$$\int_C \frac{f(z)}{z-a}dz = 0 + f(a)(2\pi i)$$

or $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

23.10 ❑ CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

If a function $f(z)$ is analytic in a region R then its derivative at any point $z = a$ of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

where C is any closed curve in R surrounding the point $z = a$.

- **Proof**

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \tag{23.2}$$

Differentiating (23.2) with respect to a, we get

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right)\cdot dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

Similarly, $$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz$$

$$f^n(a) = \frac{n!}{2\pi !}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

SOLVED EXAMPLES

Example 1 Use Cauchy's integral formula to evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)}dz$, where C is the circle $|z| = 4$.

[AU June 2009, April 2011; KU Nov. 2011]

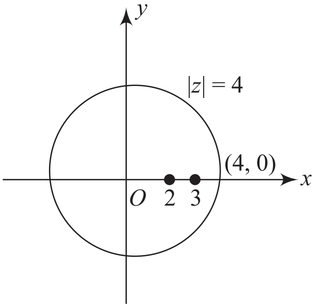


Fig. 23.5

Solution

$$\frac{1}{(z-2)(z-3)} = \frac{1}{(z-3)} - \frac{1}{(z-2)}$$

∴ given integral

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3}dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}dz$$

$$= \int_C \frac{f(z)}{(z-3)}dz - \int_C \frac{f(z)}{(z-2)}dz \tag{1}$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic on and inside C.

The points $z = 2$ and $z = 3$ lie inside C.

$\therefore$ by Cauchy's integral formula, from (1), we get,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$$
$$= 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=3} - 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=2}$$
$$= 2\pi i(\sin 9\pi + \cos 9\pi) - 2\pi i(\sin 4\pi + \cos 4\pi)$$
$$= -2\pi i - 2\pi i = -4\pi i$$ **Ans.**

Example 2 Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$, using Cauchy's integral formula. **[AU May 2012]**

Solution $|z-2| = \frac{1}{2}$ is the circle with centre at $z = 2$ and radius equal to $\frac{1}{2}$.

The point $z = 2$ lies inside the circle $|z-2| = \frac{1}{2}$

The given integral can be rewritten as

$$\int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2} dz = \int_C \frac{f(z)}{(z-2)^2} dz \text{ (say)}$$

$f(z) = \frac{z}{z-1}$ is analytic on and inside C and the point $z = 2$ lies inside C.

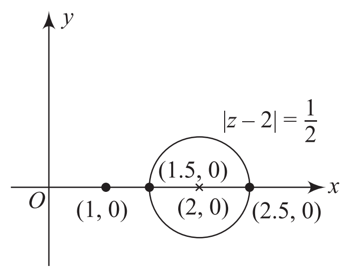


Fig. 23.6

$\therefore$ by Cauchy's integral formula,

$$\int_C \frac{z}{(z-1)(z-2)^2} dz = \frac{2\pi i}{1!} f'(2)$$
$$= 2\pi i \left\{\frac{d}{dz}\left(\frac{z}{z-1}\right)\right\}_{z=2}$$
$$= 2\pi i \left\{\frac{-1}{(z-1)^2}\right\}_{z=2} = -2\pi i$$ **Ans.**

Example 3 Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $|z+1+i| = 2$ using Cauchy's integral formula. **[AU Nov. 2011]**

Solution $|z+1+i| = 2$ is the circle whose centre is $-1-i$ and radius is 2 units.

Consider $\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$

$\therefore$ the integral is not analytic at $z = -1-2i$ and $-1+2i$.

The point $z = -1-2i$ lies inside C.

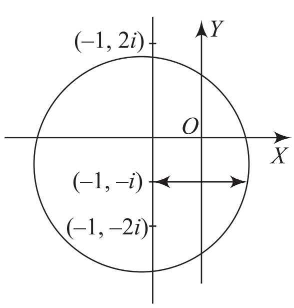


Fig. 23.7

We rewrite the given integral as

$$\int_C \frac{\left(\dfrac{z+4}{z+1-2i}\right)}{z+1+2i}dz = \int_C \frac{f(z)}{z-(-1-2i)}dz \text{ (say)}$$

$f(z)$ is analytic on and inside C and the point $(-1, -2i)$ lies inside C.
$\therefore$ by Cauchy's integral formula,

$$\int_C \frac{z+4}{z^2+2z+5}dz = 2\pi i\, f(-i-2i)$$

$$= 2\pi i\left\{\frac{-1-2i+4}{-1-2i+1-2i}\right\}$$

$$= \frac{-\pi}{2}(3-2i)$$

Ans.

EXERCISE

Part A

1. The value of the integral $\int_C \frac{dz}{z^2-2z}$ where C is the circle $|z-2| = 1$, traversed in the counter-clockwise sense is
   (i) $-\pi i$ (ii) $2\pi i$ (iii) πi (iv) 0
2. The value of the integral $\int_C \frac{z^2-z+1}{z-1}dz$, where C is the circle $|z| = \frac{1}{2}$ is
   (i) 0 (ii) πi (iii) $-\pi i$ (iv) $-2\pi i$
3. What is the value of $\int_C e^z\, dz$ if $c: |z| = 1$?
4. State Cauchy's integral formula.
5. Evaluate $\int_C \frac{dz}{z-2}$ where C is the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).
6. Evaluate $\int_C \frac{3z^2+7z+1}{(z-3)}dz$ where $C: |z| = 2$.
7. Evaluate $\int_C \frac{dz}{z^2-5z+6}$ where C is the circle $|z-1| = \frac{1}{2}$.
8. State Cauchy's formula for the first derivative of an analytic function.
9. State Cauchy's fundamental theorem.
10. Evaluate $\int_C \frac{z\,dz}{z-2}$ where $C: |z| = 1$.
11. Evaluate $\int_C \frac{2}{(z+3)}dz$ where $C: |z| = 2$.
12. Evaluate $\int_C \frac{1}{2z-3}dz$ where $C: |z| = 1$.

13. Evaluate $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}dz$ where C is $|z| = 4$ using Cauchy's integral formula.
14. Evaluate $\int_C \frac{dz}{(z-3)^2}$ where $C: |z| = 1$.
15. State the Cauchy–Goursat theorem.

Part B

1. Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i| = 2$. $\left(\textbf{Ans.}\ \frac{-2\pi i}{9}\right)$
2. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$ using Cauchy's integral formula. where C is the circle $|z| = \frac{3}{2}$. **(Ans.** $2\pi i$**)**
3. Find the value of $\int_C \frac{2z^2+z}{z^2-1}dz$. **(Ans.** $3\pi i$**)**
4. Evaluate the following:

   (i) $\int_C \frac{dz}{(z^2+4)^2}$, where C is $|z-i| = 2$

   (ii) $\int_C \frac{z^3+z+1}{z^2-7z+6}dz$ where C is the ellipse $4x^2 + 9y^2 = 1$

   (iii) $\int_C \frac{z^3+1}{z^2-3iz}dz$ where C is $|z| = 1$. $\left[\textbf{Ans.}\ \text{(i)}\ \frac{\pi}{16}, \text{(ii)}\ 0, \text{(iii)}\ -\frac{2\pi}{3}\right]$
5. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}dz$ where C is $|z| = 3$. (**Ans.** $-4\pi i$)
6. If $f(a) = \int_C \frac{4z^2+z+5}{z-a}dz$ where C is $|z| = 2$, find the values of $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$. (**Ans.** $20\pi i$; $2\pi(i-1)$; $-14\pi i$; $16\pi i$)
7. Evaluate $\int_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1). **(Ans.** $-1 + i$**)**
8. Evaluate $\int_C \frac{z^2+1}{z^2-1}dz$ where (i) $C: |z-1| = 1$, (ii) $C: |z+1| = 1$, and (iii) $C: |z-i| = 1$. **[Ans.** (i) $2\pi i$ (ii) $-2\pi i$ (iii) 0**]**
9. Evaluate $\int_C \frac{\sin 2z}{(z+3)(z+1)^2}dz$ where C is the rectangle with vertices at $3 + i$, $-2 + i$, $-2 - i$, $3 - i$. $\left[\textbf{Ans.}\ \pi i \frac{(4\cos 2 + \sin 2)}{2}\right]$
10. Evaluate $\int_C \frac{z^4-3z^2+6}{(z+i)^3}dz$ where $C: |z| = 2$. **(Ans.** $-18\pi i$**)**

24 Taylor and Laurent Series Expansions

Chapter Outline

- Introduction
- Taylor's Series
- Laurent's Series

24.1 ❑ INTRODUCTION

Power Series

A power series in powers of $(z - z_0)$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0) + \cdots \tag{24.1}$$

Here, a_0, a_1, a_2 ... are complex (or real) constants known as coefficients of the series. z is a complex variable and z_0 is called the centre of the series. Equation (24.1) is also known as the power series about the point z_0.

Power series in powers of z is

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

obtained as a particular case with $z_0 = 0$ in (24.1). The **region of convergence** of a series is the set of all points z for which the series converges.

Three distinct possibilities exist regarding the region of convergence of a power series (24.1).

(i) The series converges only at the point $z = z_0$.

(ii) The series converges everywhere inside a circular disk $|z - z_0| < R$ and diverges everywhere outside the disk $|z - z_0| > R$. Here, R is known as the **radius of convergence** and the circle $|z - z_0| = R$ as the **circle of convergence.**

➢ **Note**

(i) The series may converge or diverge at the points on the circle of convergence.

(ii) **Geometric Series:** $\sum_{m=0}^{\infty} z^m = 1 + z + z^2 + \cdots$ converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. (i.e., $R = 1$)

(iii) **Power series:** $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z. (i.e., $R = \infty$)

Power series play an important role in complex analysis, since they represent analytic functions and conversely every analytic function has a power series representation called Taylor series similar to Taylor series in real calculus.

Analytic functions can also be represented by another type of series called **Laurent series,** which consist of positive and negative integral powers of the independent variable. They are useful for evaluating complex and real integrals.

24.2 ❑ TAYLOR'S SERIES (TAYLOR'S THEOREM)

If a function $f(z)$ is analytic at all points inside a circle C with its centre at the point a and radius R then at each point z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

• Proof

Take any point z inside C. Draw a circle C_1 with centre a, enclosing the point z. Let w be a point on the circle C_1.

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)}$$

$$= \frac{1}{(w-a)} \frac{1}{\left(1 - \frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

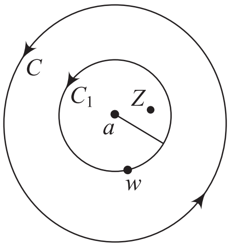


Fig. 24.1

Applying the binomial theorem,

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right]$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^{n+1}} + \cdots \quad (24.2)$$

As $|z-a| < |w-a|$ or $\frac{|z-a|}{|w-a|} < 1$,

so the series converges uniformly. Hence, the series is integrable.

Multiplying (24.2) by $f(w)$,

$$\frac{f(w)}{w-z}=\frac{f(w)}{w-a}+(z-a)\frac{f(w)}{(w-a)^2}+(z-a)^2\frac{f(w)}{(w-a)^3}+\cdots+(z-a)^n\frac{f(w)}{(w-a)^{n+1}}+\cdots$$

On integrating with respect to w, we get

$$\int_{C_1}\frac{f(w)}{w-z}dw=\int_{C_1}\frac{f(w)}{w-a}dw+(z-a)\int_{C_1}\frac{f(w)}{(w-a)^2}dw+\cdots$$
$$+(z-a)^n\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw+\cdots \tag{24.3}$$

We know that

$$\int_{C_1}\frac{f(w)}{(w-z)}dz=2\pi i\, f(z),\ \int_{C_1}\frac{f(w)}{w-a}dw=2\pi i\, f(a)$$

$$\int_{C_1}\frac{f(w)}{(w-a)^2}dw=2\pi i\, f'(a)\text{ , and so on.}$$

Substituting these values in (24.3), we get

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^n(a)}{n!}(z-a)^n+\cdots$$

➢ **Note**

(i) Putting $a=0$ in the Taylor's series, we get $f(z)=f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\cdots$

This series is called the **McLaurin's series** of $f(z)$.

(ii) **Standard McLaurin's Series**

(a) $e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$ for $|z|<\infty$

(b) $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$ for $|z|<\infty$

(c) $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}\cdots$ for $|z|<\infty$

(d) $\sin hz=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$ for $|z|<\infty$

(e) $\cos hz=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots$ for $|z|<\infty$

(f) $(1-z)^{-1}=1+z+z^2+z^3+\cdots$ for $|z|<1$

(g) $(1+z)^{-1}=1-z+z^2-z^3+\cdots$ for $|z|<1$

(h) $(1-z)^{-2}=1+2z+3z^2+\cdots$ for $|z|<1$

(iii) Expansion of a function $f(z)$ about a singular point $z=h$ means, expansion of $f(z)$ in powers of $(z-h)$.

24.3 ❑ LAURENT'S SERIES (LAURENT'S THEOREM)

If $f(z)$ is analytic on C_1 and C_2 and the annular region bounded by the two concentric circles C_1 and C_2 of radii r_1 and $r_2(r_2 < r_1)$ and with centre at a then for all in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots$$

where
$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, n = 0, 1, 2, 3...$$

$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, n = 1, 2, 3...$$

- **Proof**

By introducing a cross-cut AB, the multiply connected region R is converted to a simply connected region. Now, $f(z)$ is analytic in this region.

Now by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}$$

$$\int_{AB} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_{BA} \frac{f(w)}{w-z} dw$$

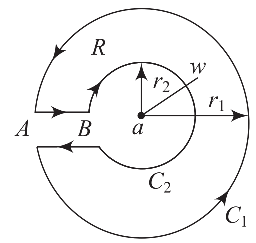


Fig. 24.2

Integral along c_2 is clockwise, so it is negative.

$$\therefore \quad f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw \tag{24.4}$$

For the first integral, $\frac{f(w)}{w-z}$ can be expended exactly as in Taylor's series since w lies on C_1,

$$|z-a| \le |w-a| \text{ or } \frac{|z-a|}{|w-a|} \le 1$$

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-a} dw + \frac{(z-a)}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^2} dw$$

$$+ \frac{(z-a)^2}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^3} dw + \cdots$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \tag{24.5}$$

$$\left[\text{as } a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw\right]$$

In the second integral, w lies on C_2

$$\therefore \quad |w-a| < |z-a| \text{ or } \frac{|w-a|}{|z-a|} < 1$$

So here,
$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)} = \frac{-1}{(z-a)} \cdot \frac{1}{1-\dfrac{w-a}{z-a}}$$

$$= -\frac{1}{z-a}\left[1 - \frac{w-a}{z-a}\right]^{-1}$$

$$= -\frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \cdots + \left(\frac{w-a}{z-a}\right)^{n+1} + \cdots\right]$$

Multiplying by $\frac{-f(w)}{2\pi i}$, we get

$$-\frac{1}{2\pi i}\frac{f(w)}{w-z} = \frac{1}{2\pi i}\frac{f(w)}{z-a} + \frac{1}{2\pi i}\frac{(w-a)}{(z-a)^2}f(w) + \frac{1}{2\pi i}\frac{(w-a)^2}{(z-a)^3}f(w) + \cdots$$

$$= \left(\frac{1}{z-a}\right)\frac{1}{2\pi i}f(w) + \frac{1}{(z-a)^2}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-1}} + \frac{1}{(z-a)^3}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-2}} + \cdots$$

Integrating, we have

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw = \frac{1}{(z-a)}\frac{1}{2\pi i}\int_{C_2} f(w)\,dw + \frac{1}{(z-a)^2}\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-1}}dw$$

$$+ \frac{1}{(z-a)^3}\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-2}}dw + \cdots$$

$$= \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \cdots \qquad (24.6)$$

$$\left[\text{as } b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw\right]$$

Substituting the values of both integrals from (24.5) and (24.6) in (24.4), we get

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + b_1(z-a)^{-1} + b_2(z-a)^{-2} + \cdots$$

or
$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

➢ **Note**

(i) If $f(z)$ is analytic at all points inside C_1 (i.e., no singular points inside C_2) then by Cauchy's theorem, $b_n = 0$ for all $n - 1 \geq 0$. Hence, the Laurent series reduces to Taylor series. Thus, Laurent's series expansion about an analytic point a is Taylor series expansion about a.

(ii) The region of convergence of Laurent's series is the annulus region $R_1 < |z - a| < R_2$.

(iii) If $f(z)$ has more than one singular point then several (more than one) Laurent series expansions can be obtained about the same singular point by appropriately considering analytic regions about (centred) at a.

(iv) The part $\sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of $(z - a)$ is called the **analytic part** of the Laurent's series, while $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z - a)$ is called the **principal part** of the Laurent's series.

SOLVED EXAMPLES

Example 1 Obtain Taylor's series expansion to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z|<2$. **[KU Nov. 2010]**

Solution Let $f(z)=\frac{z^2-1}{(z+2)(z+3)}$

$$=1+\frac{-5z-7}{(z+2)(z+3)} \quad (1)$$

Consider $$\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put $z=-3 \Rightarrow B=-8$

Put $z=-2 \Rightarrow A=3$

$\therefore$ $$\frac{-5z-7}{(z+2)(z+3)}=\frac{3}{z+2}-\frac{8}{z+3}$$

$\therefore$ $$(1) \Rightarrow f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$

Given $|z|<2$, i.e., $\frac{|z|}{2}<1$, so clearly $\frac{|z|}{3}<1$

i.e., $$\left|\frac{z}{2}\right|<1 \text{ and } \left|\frac{z}{3}\right|<1$$

$\therefore$ $$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

By using binomial theorem,

$$f(z)=1+\frac{3}{2}\left[1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right]$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n}-\frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n}$$

$$=1+\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right]z^n$$ **Ans.**

Example 2 Expand $\frac{1}{(z-1)(z-2)}$ in Laurent's series valid for $|z|<1$ and $1<|z|<2$. **[AU Nov. 2010]**

Solution Let $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

(i) Given $|z| < 1$ obviously $\dfrac{|z|}{2} < 1$, i.e., $\left|\dfrac{z}{2}\right| < 1$

$$\therefore \quad \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= -\frac{1}{2\left(1-\dfrac{z}{2}\right)} + \frac{1}{1-z}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right] + [1+z+z^2+\cdots]$$

i.e.,
$$f(z) = \frac{1}{2} + \frac{3z}{4} + \frac{7}{8}z^2 + \cdots$$

(ii) Given $1 < |z| < 2$

$$1 < |z| \;\Rightarrow\; \frac{1}{|z|} < 1, \text{ i.e., } \left|\frac{1}{z}\right| < 1$$

$$|z| < 2 \;\Rightarrow\; \frac{|z|}{2} < 1, \text{ i.e., } \left|\frac{z}{2}\right| < 1$$

$$\therefore \quad f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$ **Ans.**

Example 3 If $0 < |z-1| < 2$, express $f(z) = \dfrac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of $z-1$. **[AU April 2011]**

Solution Let $z - 1 = u$

$\therefore$ $0 < |z-1| < 2$ becomes $0 < |u| < 2$

Now,
$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$z = A(z-3) + B(z-1)$$

Put $z = 1, \quad \Rightarrow \quad A = -\frac{1}{2}$

Put $z = 3, \quad \Rightarrow \quad B = \frac{3}{2}$

$$\therefore \quad \frac{z}{(z-1)(z-3)} = \frac{-\frac{1}{2}}{z-1} + \frac{\frac{3}{2}}{z-3}$$

(or) $\frac{u+1}{u(u-2)} = -\frac{1}{2u} + \frac{3}{2(u-2)}$ (as $z - 1 = u \Rightarrow z = u + 1$)

So instead of expanding $\frac{z}{(z-1)(z-3)}$ in powers of $(z - 1)$, it is enough to expand $\frac{u+1}{u(u-2)}$ in powers of u.

$$\frac{u+1}{u(u-2)} = -\frac{1}{2u} + \frac{3}{2(u-2)}$$

Since $|u| < 2$, we have $\frac{|u|}{2} < 1$. i.e., $\left|\frac{u}{2}\right| < 1$.

$$\therefore \quad \frac{u+1}{u(u-2)} = \frac{-1}{2u} - \frac{3}{4\left(1 - \frac{u}{2}\right)}$$

$$= \frac{-1}{2u} - \frac{3}{4}\left(1 - \frac{u}{2}\right)^{-1}$$

$$= \frac{-1}{2u} - \frac{3}{4}\left[1 + \frac{u}{2} + \left(\frac{u}{2}\right)^2 + \cdots\right]$$

$$= \frac{-1}{2u} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n$$

$$\therefore \quad \frac{z}{(z-1)(z-3)} = \frac{-1}{2(z-1)} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$ **Ans.**

Example 4 Obtain the Laurent's expansion for $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ which are valid in (i) $1 < |z| < 4$, and (ii) $|z| > 4$. **[AU Nov. 2011]**

Solution Let $f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$

$$\Rightarrow \quad f(z) = 1 + \frac{-5z-8}{(z+1)(z+4)} \quad (1)$$

(since the degrees of z in both numerator and in denominator are equal, divide it)

Consider $\frac{-5z-8}{(z+1)(z+4)} = \frac{A}{(z+1)} + \frac{B}{(z+4)}$

$$-5z-8 = A(z+4)+B(z+1)$$

Put $z=-1 \Rightarrow A=-1$

Put $z=-4 \Rightarrow B=-4$

$$\therefore \quad \frac{-5z-8}{(z+1)(z+4)} = \frac{-1}{(z+1)} - \frac{-4}{(z+4)} \tag{2}$$

Substituting (2) in (1), we get

$$f(z) = 1 - \frac{1}{(z+1)} - \frac{4}{(z+4)}$$

(i) Given $1 < |z| < 4$

$$1 < |z| \Rightarrow \frac{1}{|z|} < 1, \text{ i.e., } \left|\frac{1}{z}\right| < 1$$

$$|z| < 4 \Rightarrow \frac{|z|}{4} < 1, \text{ i.e., } \left|\frac{z}{4}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - 4\frac{1}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \left[-\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \left[-\frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{z^n} - \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{z}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z^n} - \left(\frac{z}{4}\right)^n\right]$$

(ii) Given $|z| > 4$

$$\frac{4}{|z|} < 1, \text{ i.e., } \left|\frac{4}{z}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{1}{1+z} - \frac{4}{z+4}$$

$$= 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{z\left(1+\frac{4}{z}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right] - \frac{4}{z}\left[1 - \frac{4}{z} + \left(\frac{4}{z}\right)^2 - \cdots\right]$$

$$= 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n} - \frac{4}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{4}{z}\right)^n$$

$$= 1 - \sum_{n=0}^{\infty}(-1)^n \left[\frac{1}{z^{n+1}} + \left(\frac{4}{z}\right)^{n+1}\right]$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}}(1 + 4^{n+1})$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n (1 + 4^n)\cdot\frac{1}{z^n} \qquad \textbf{Ans.}$$

Example 5 Find the Laurent's series of $f(z) = \dfrac{1}{z(1-z)}$ valid in the region (i) $|z+1| < 1$, (ii) $1 < |z+1| < 2$, and (iii) $|z+1| > 2$. **[KU May 2010, Nov. 2011]**

Solution Let $z + 1 = u$ or $z = u - 1$

$$\therefore \qquad f(z) = \frac{1}{z(1-z)} = \frac{1}{(u-1)(2-u)} = \frac{1}{u-1} + \frac{1}{2-u} \qquad (1)$$

(i) Given $|z+1| < 1 \Rightarrow |u| < 1$

$$\therefore \qquad f(z) = \frac{-1}{1-u} + \frac{1}{2\left(1-\frac{u}{2}\right)}$$

$$= -(1-u)^{-1} + \frac{1}{2}\left(1 - \frac{u}{2}\right)^{-1}$$

$$= -[1 + u + u^2 + \cdots] + \frac{1}{2}\left[1 + \left(\frac{u}{2}\right) + \left(\frac{u}{2}\right)^2 + \cdots\right]$$

$$= -\sum_{n=0}^{\infty}u^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^{n+1}}\right)u^n$$

i.e.,
$$f(z) = \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^{n+1}}\right)(z+1)^n$$

(ii) Given $1 < |z+1| < 2$. i.e., $1 < |u| < 2$

$$1 < |u| \;\Rightarrow\; \frac{1}{|u|} < 1, \text{ i.e., } \left|\frac{1}{u}\right| < 1$$

$$|u| < 2 \;\Rightarrow\; \frac{|u|}{2} < 1 \text{ i.e., } \left|\frac{u}{2}\right| < 1$$

Consider (1), $f(z) = \dfrac{1}{u-1} + \dfrac{1}{2-u}$

$$= \frac{1}{u\left(1-\dfrac{1}{u}\right)} + \frac{1}{2\left(1-\dfrac{u}{2}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right] + \frac{1}{2}\left[1+\left(\frac{u}{2}\right)+\left(\frac{u}{2}\right)^2+\cdots\right]$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{u^{n+1}} + \sum_{n=0}^{\infty}\frac{u^n}{2^{n+1}}$$

i.e.,
$$f(z) = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z+1)^n$$

(iii) $|z+1| > 2$, i.e., $|u| > 2 \Rightarrow \left|\dfrac{2}{u}\right| < 1$

$\therefore$
$$f(z) = \frac{1}{u\left(1-\dfrac{1}{u}\right)} - \frac{1}{u\left(1-\dfrac{2}{u}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1-\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right] - \frac{1}{u}\left[1+\frac{2}{u}+\left(\frac{2}{u}\right)^2+\cdots\right]$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} - \frac{1}{u}\sum_{n=0}^{\infty}\frac{2^n}{u^n}$$

$$= \sum_{n=0}^{\infty}(1-2^n)\frac{1}{u^{n+1}}$$

or
$$f(z) = \sum_{n=0}^{\infty}(1-2^n)\frac{1}{(z+1)^{n+1}}$$ **Ans.**

EXERCISE

Part A

1. Define radius and circle of convergence of power series.
2. State Taylor's theorem and Laurent's theorem.
3. State McLaurin's series.
4. Give some standard McLaurin's series.
5. What do you mean by analytic part and principal part of Laurent's series of a function of z?
6. Expand $\dfrac{1}{z(z-1)}$ as Laurent's series about $z = 0$ in the annulus $0 < |z| < 1$.
7. Find the Laurent's series expansion of $f(z)=\dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$.
8. Expand $f(z) = e^z$ in a Taylor's series about $z = 0$.
9. Expand $\cos z$ at $z=\dfrac{\pi}{4}$ in a Taylor's series.
10. In the power series $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$, z_0 is called the ______ of the series.

Part B

1. Find the Taylor's series expansion of $f(z)=\dfrac{z}{z(z+1)(z+2)}$ about $z = i$.

   State also the region of convergence of the series.

   $$\left[\textbf{Ans. } \sum_{n=0}^{\infty}(-1)^n\left\{\frac{2}{(2+i)^{n+1}}-\frac{1}{(1+i)^{n+1}}\right\}(z-i)^n\right]$$

2. Find the Laurent's series expansion of $f(z)=\dfrac{z^2-1}{z^2+5z+6}$ valid in the region (i) $|z| < 2$, (ii) $2 < |z| < 3$, and (iii) $|z| > 3$ **[KU April 2013]**

   $$\left[\begin{array}{l}\textbf{Ans. } \text{(i) } 1+\sum_{n=0}^{\infty}(-1)^n\left\{\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right\}z^n \text{ (ii) } 1+3\sum(-1)^n\frac{2^n}{z^{n+1}}-8\sum(-1)^n\frac{z^n}{3^{n+1}}\\ \text{(iii) } 1+\sum(-1)^n\{3.2^n-8.3^n\}1/z^{n+1}\end{array}\right]$$

3. Find the Laurent's series expansion of $f(z)=\dfrac{z}{(z-1)(z-2)}$, valid in the region (i) $|z + 2| < 3$, (ii) $3 < |z +2| < 4$, and (iii) $|z + 2| > 4$.

   $$\left[\begin{array}{l}\textbf{Ans. } \text{(i) } \sum_{n=0}^{\infty}\left[-\frac{1}{2.4^n}+\frac{1}{3^{n+1}}\right](z+2)^n \text{ (ii) } -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z+2)^n}{4^n}-\sum_{n=0}^{\infty}\frac{3^n}{(z+2)^{n+1}}\\ \text{(iii) } \sum_{n=0}^{\infty}(2.4^n-3^n)\cdot\frac{1}{(z+2)z^{n+1}}\end{array}\right]$$

4. Expand $\dfrac{z^2-6z-1}{(z-1)(z+2)(z-3)}$ in $3<|z+2|<5$.

$$\left[\textbf{Ans. } \frac{2}{z+2}+\frac{3}{(z+2)^2}+\frac{3^2}{(z+2)^3}+\cdots+\frac{1}{5}\left[1+\frac{z+2}{5}+\frac{(z+2)^2}{5^2}+\frac{(z+2)^3}{5^3}+\cdots\right]\right]$$

5. Find Laurent's series of $f(z)=\dfrac{e^z}{z(1-z)}$ about $z=1$. Find the region of convergence.

$$\left[\begin{array}{l}\textbf{Ans. } f(z)=\dfrac{1}{e}\left[-\dfrac{1}{z-1}-\dfrac{3}{2}(z-1)+\dfrac{1}{3}(z-1)^2+\cdots\right] \\ \text{Region of convergence is } |z-1|<1\end{array}\right]$$

6. Obtain the Laurent's series expansion for $f(z)=\dfrac{1}{z(z-1)}$ for (i) $0<|z|<1$, and (ii) $0<|z-1|<1$. $\left[\textbf{Ans. (i) } -\dfrac{1}{z}(1+z+z^2+\cdots)\text{ (ii) }\dfrac{1}{z-1}(1-(z-1)+(z-1)^2\ldots)\right]$

7. Find Laurent's series about the indicated singularity. (i) $\dfrac{e^{2z}}{(z-1)^3}, z=1$ (ii) $\dfrac{z}{(z+1)(z+2)}, z=-2$ (iii) $\dfrac{1}{z^2(z-3)^2}, z=3$

$$\left[\begin{array}{l}\textbf{Ans. (i) } \dfrac{e^2}{(z-1)^3}+\dfrac{2e^2}{(z-1)^2}+\dfrac{2e^2}{(z-1)}+\dfrac{4e^2}{3}+\dfrac{2e^2}{3}(z-1)+\cdots \\ \text{(ii) } \dfrac{2}{2+z}+1+(z+2)+(z+2)^2+\cdots \\ \text{(iii) } \dfrac{1}{9(z-3)^2}-\dfrac{2}{27(z-3)}+\dfrac{1}{27}-\dfrac{4(z-3)}{243}+\cdots\end{array}\right]$$

25 Theory of Residues

Chapter Outline

- Introduction
- Classification of Singularities
- Residues
- Cauchy's Residue Theorem
- Evaluation of Real Definite Integrals by Contour Integration

25.1 ❑ INTRODUCTION

The residue theorem is a very powerful and elegant theorem in complex integration. Using the residue theorem, many complicated real integrals can be evaluated. It is also used to sum a real convergent series and to find the inverse of a Laplace transform.

25.2 ❑ CLASSIFICATION OF SINGULARITIES

A point at which a function $f(z)$ is not analytic is known as a **singular point** or **singularity** of the function.

- **Example**

The function $f(z) = \dfrac{1}{z-5}$ has a singular point at $z - 5 = 0$ or $z = 5$.

If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$ then $z = a$ is said to be an **isolated singularity** of the function $f(z)$. Otherwise, it is called **non-isolated**.

- **Example**

(i) The function $\dfrac{1}{(z-2)(z-7)}$ has two isolated singular points, namely, $z = 2$ and $z = 7$ [since $(z-2)(z-7) = 0$ or $z = 2, 7$].

(ii) The function $\dfrac{1}{\sin\frac{\pi}{z}}$ is not analytic at the points where $\sin\frac{\pi}{z} = 0$, i.e., at the points $\frac{\pi}{z} = n\pi$.

i.e., at the points $z = \frac{1}{n}(n = 1, 2, 3\ldots)$.

Thus, $z = 1, \frac{1}{2}, \frac{1}{3} \ldots, z = 0$ are the points of singularity. But $z = 0$ is the non-isolated singularity of the function $\dfrac{1}{\sin\frac{\pi}{z}}$ because in the neighbourhood $z = 0$, there are infinite number of other singularities $z = \frac{1}{n}$, where n is very large.

Let a function $f(z)$ have an isolated singular point $z = a$. $f(z)$ can be expanded in a Laurent's series expansion around $z = a$ as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}} + \cdots$$

In some cases, it may happen that the coefficients $b_{m+1} = b_{m+2} = \ldots = 0$,

Then the series reduces to

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}$$

Then $z = a$ is said to be **a pole of order *m*** of the function $f(z)$.

When $m = 1$, the pole is said to be a **simple pole**.

In this case, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \dfrac{b_1}{(z-a)}$.

If the number of terms of negative powers in the above expansion are infinite then $z = a$ is called an **essential singular point** of $f(z)$.

If a single-valued function $f(z)$ is not defined at $z = a$, but $\lim_{z\to a} f(z)$ exists then $z = a$ is called a **removable singularity**.

- **Example**

$z = 0$ is a removable singularity of $f(z) = \dfrac{\sin z}{z}$, since $f(0)$ is not defined, but $\lim_{z\to 0}\left(\dfrac{\sin z}{z}\right) = 1$.

25.3 ❑ RESIDUES

Residue of an analytic function $f(z)$ at an isolated singular point $z = a$ is the coefficient say b_1 of $(z - a)^{-1}$ in the Laurent's series expansion of $f(z)$ about a. Residue of $f(z)$ at a is denoted by $\operatorname{Res}_{z=a} f(z)$. From Laurent's series, we know that the coefficient b_1 is given by $b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$.

Thus, the residue of $f(z)$ at $z = a$, $= \operatorname{Res}_{z=a} f(z) = b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$.

where C is any closed contour enclosing a (and such that f is analytic on and within C).

Calculation of Residue at Simple Pole

(i) If $f(z)$ has a simple pole at $z = a$, then $\operatorname{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)\,f(z)$.

(ii) Suppose $f(z) = \frac{P(z)}{Q(z)}$ has a simple pole at a such that $P(a) \neq 0$.

Then $\operatorname{Res}_{z=a} f(z) = \operatorname{Res}_{z=a}\frac{P(z)}{Q'(z)} = \frac{P(a)}{Q'(a)}$

Calculation of Residue at a Multiple Pole

If $f(z)$ has a pole of order n at $z = a$, then

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{(n-1)!}\lim_{z\to a}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}$$

25.4 ❑ CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic within and on a simple closed curve C except at a finite number of poles within C then $\oint_C f(z)dz = 2\pi i$ (sum of residues at the poles within C).

Proof Let $C_1, C_2, C_3 \ldots C_n$ be the non-intersecting circles with centre at $a_1, a_2 \ldots a_n$ respectively and radii so small that they lie entirely within the closed curve C. Then $f(z)$ is analytic in the multiply connected legion lying between the curves C and $C_1, C_2 \ldots C_n$. Applying Cauchy's theorem,

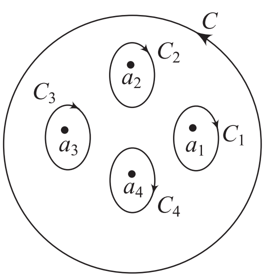


Fig. 25.1

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$

$$= 2\pi i \operatorname{Res}_{z=a_1} f(z) + 2\pi i \operatorname{Res}_{z=a_2} f(z) \cdots + 2\pi i \operatorname{Res}_{z=a_n} f(z)$$

$$= 2\pi i\left[\operatorname{Res}_{z=a_1} f(z) + \operatorname{Res}_{z=a_2} f(z) \cdots + \operatorname{Res}_{z=a_n} f(z)\right]$$

$\therefore \oint_c f(z)dz = 2\pi i$ (sum of residues at the poles within C)

25.5 ❑ EVALUATION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated using Cauchy's theorem of residues. For finding the integrals, we take a closed curve C, find the poles of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

Then using Cauchy's theorem of residues, we have $\int_C f(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at the poles within C)

We call the curve a contour and the process of integration along a contour as contour integration.

Type 1

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where f is a rational function of $\cos\theta$ and $\sin\theta$

In this type of integrals, put $z = e^{i\theta}$

Differentiating with respect to θ, we get,

$$dz = ie^{i\theta}\, d\theta, \text{ i.e., } d\theta = \frac{dz}{iz}$$

We know that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

i.e., $$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and $$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\therefore \quad \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \int_C f\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz}$$

where C is the unit circle $|z| = 1$

$$= \frac{1}{i}\int_C f\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\frac{dz}{z}$$

$$= \int_C \phi(z)\, dz \text{ (say)}$$

Clearly, $\phi(z)$ is a rational function of z.

Hence, by the residue theorem, $\int_C \phi(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at its poles inside C).

Type 2

Consider the integral $\int_C \phi(z)dz$, where C is the positively oriented semicircle Γ, $|z| = R$, Im $z \geq 0$ together with the line segment $L : [-R, R]$. Such integrals can be evaluated by integrating $f(z)$ round a contour C consisting of a semicircle Γ of radius R large enough to include all the poles of $f(z)$

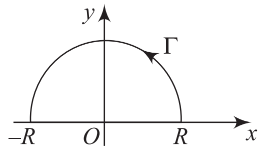


Fig. 25.2

and the part of the real axis from $x = -R$ to $x = R$. Here, the only singularities of $f(z)$ in the upper half-plane are poles.

When $\phi(z)$ has singularities on the real axis then $\int_C \phi(z)dz = \int_{-R}^{R} \phi(x)dx + \int_\Gamma \phi(z)dz$.

By the residue theorem, we have $\int_C \phi(z)dz = 2\pi i$ (sum of the residues of the function $\phi(z)$ at its poles in the upper half-plane).

i.e., $\int_{-R}^{R} \phi(x)dx + \int_\Gamma \phi(z)dz = 2\pi i$ (sum of the residues of the function $\phi(z)$ at its poles within C).

Putting $R \to \infty$ we get, $\int_{-\infty}^{\infty} \phi(x)dx,$ provided $\int_\Gamma \phi(z)dz \to 0$.

Type 3

Integrals of the form $\int_{-\infty}^{\infty} (\sin ax)\, f(x)dx$ or $\int_{-\infty}^{\infty} (\cos ax)\, f(x)dx$. $a > 0$ where $f(z)$ is such that $f(z) \to 0$ as $z \to \infty$ and it does not have a pole on the real axis.

SOLVED EXAMPLES

Example 1 Find the residue of $f(z) = \dfrac{1}{(z^2+1)^2}$ about each singularity.

Solution Given $f(z) = \dfrac{1}{(z^2+1)^2} = \dfrac{1}{[(z-i)(z+i)]^2}$

$$= \frac{1}{(z-i)^2(z+i)^2}$$

Here, $z = i, -i$ are poles of order 2.

Now,
$$[\text{Res}\, f(z)]_{z=i} = \underset{z\to i}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}[(z-i)^2 f(z)]$$

$$= \underset{z\to i}{\text{Lt}} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{1}{(z-i)^2(z+i)^2}\right]$$

$$= \underset{z\to i}{\text{Lt}} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$$

$$= \underset{z\to i}{\text{Lt}} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$= \frac{-i}{4}$$

$$[\text{Res } f(z)]_{z=-i} = \underset{z\to -i}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}[(z+i)^2 f(z)]$$
$$= \underset{z\to -i}{\text{Lt}} \frac{d}{dz}\left[(z+i)^2 \cdot \frac{1}{(z-i)^2(z+i)^2}\right]$$
$$= \underset{z\to -i}{\text{Lt}} \frac{d}{dz}\left[\frac{1}{(z-i)^2}\right]$$
$$= \underset{z\to -i}{\text{Lt}} \frac{-2}{(z-i)^3} = \frac{-2}{8i} = \frac{i}{4}$$ **Ans.**

Example 2 Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C is the circle $|z-i|=2$.

[AU June 2009, May 2012]

Solution Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

Here, $z = -1$ is a pole of order 2.
And $z = 2$ is a simple pole.
Clearly, $z = 2$ lies outside the circle $|z-i| = 2$

$\therefore$ $[\text{Res } f(z)]_{z=2} = 0$

Now,
$$[\text{Res } f(z)]_{z=-1} = \underset{z\to -1}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}[(z+1)^2 f(z)]$$
$$= \underset{z\to -1}{\text{Lt}} \frac{d}{dz}\left[(z+1)^2 \cdot \frac{(z-1)}{(z+1)^2(z-2)}\right]$$
$$= \underset{z\to -1}{\text{Lt}} \frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$
$$= \underset{z\to -1}{\text{Lt}} \left[\frac{(z-2)-(z-1)}{(z-2)^2}\right]$$
$$= \underset{z\to -1}{\text{Lt}} \left[\frac{-2+1}{(z-2)^2}\right] = \underset{z\to -1}{\text{Lt}} \left[-\frac{1}{(z-2)^2}\right]$$
$$= \frac{-1}{(-1-2)^2} = -\frac{1}{9}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i \ \text{[sum of the residues]}$$
$$= 2\pi i\left(-\frac{1}{9}\right) = \frac{-2\pi i}{9}$$ **Ans.**

Example 3 Evaluate $\int_C \frac{dz}{c(z^2+9)^3}$, where C is $|z-i| = 3$ by using Cauchy's residue theorem.

[KU Nov. 2011]

Solution Let $f(z) = \frac{1}{(z^2+9)^3}$

The singularities of $f(z)$ are obtained by $z^2 + 9 = 0$
$\Rightarrow z = \pm 3i$, of which $z = 3i$ lies inside the circle $|z - i| = 3$
$z = 3i$ is a triple pole of $f(z)$.

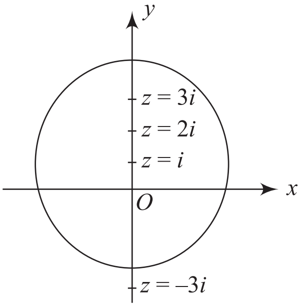


Fig. 25.3

$$\therefore \qquad [\text{Res } f(z)]_{z=3i} = \frac{1}{2!}\left[\frac{d^2}{dz^2}\frac{1}{(z+3i)^3}\right]_{z=3i}$$

$$= \frac{1}{2!}\left[\frac{12}{(z+3i)^5}\right]_{z=3i}$$

$$= \frac{6}{6^5 i^5} = \frac{1}{1296i}$$

By Cauchy's residue theorem,

$$\int_C \frac{dz}{(z^2+9)^3} = 2\pi i \times \frac{1}{1296i} = \frac{\pi}{648}$$ **Ans.**

Example 4 Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, a > b > 0$.

[KU May 2010; AU Nov. 2010, Nov. 2011, April 2013]

Solution Let $z = e^{i\theta}$

$$\Rightarrow \qquad d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\therefore \qquad \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{a + \frac{1}{2}b\left(z + \frac{1}{z}\right)} \text{ where } C \text{ is } |z| = 1$$

$$= \frac{1}{i}\int_C \frac{dz}{z\left[a + \frac{1}{2}b\left(z + \frac{1}{z}\right)\right]}$$

$$= \frac{1}{i}\int_C \frac{dz}{z\left[\frac{2az + bz^2 + b}{2z}\right]}$$

$$\text{i.e.,} \qquad \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{i}\int_C \frac{dz}{bz^2 + 2az + b}$$

$$= \frac{2}{i}\int_C f(z)dz \qquad (1)$$

The poles of $f(z)$ are given by the roots of $bz^2 + 2az + b = 0$

$$\therefore \qquad z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

i.e.,
$$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a-\sqrt{a^2-b^2}}{b}$$

Let
$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}; \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Since $a > b > 0$, $|\beta| > 1$

But the modulus of the product of the roots $|\alpha\beta| = 1$ (since if $az^2 + b + c = 0$, product of the roots $|\alpha\beta| = \frac{c}{a}$).

Since $|\beta| > 1$ and $|\alpha\beta| = 1$, we get $|\alpha| < 1$ so that $z = \alpha$ is the only simple pole inside C.

Since $z = \alpha$ and $z = \beta$ are the roots of $bz^2 + 2az + b = 0$, we can write $bz^2 + 2az + b = b(z-\alpha)(z-\beta)$

Hence,
$$f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$$

Now,
$$[\text{Res } f(z)]_{z=\alpha} = \underset{z\to\alpha}{\text{Lt}} (z-\alpha)\cdot f(z)$$
$$= \underset{z\to\alpha}{\text{Lt}} (z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)}$$
$$= \underset{z\to\alpha}{\text{Lt}} \frac{1}{b(z-\beta)} = \frac{1}{b(\alpha-\beta)}$$
$$= \frac{1}{b\left[\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right]}$$
$$= \frac{1}{b\frac{2\sqrt{a^2-b^2}}{b}}$$
$$= \frac{1}{2\sqrt{a^2-b^2}}$$

From (1), since $|\beta| > 1$,
β lies outside the circle $|z| = 1$

$\therefore$
$$[\text{Res } f(z)]_{z=\beta} = 0$$

Hence, (1) $\Rightarrow$
$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{i}\int_C f(z)dz$$
$$= \frac{2}{i}\,[2\pi i \times (\text{sum of the residues})]$$
$$= \frac{2}{i}\cdot 2\pi i\left[\frac{1}{2\sqrt{a^2-b^2}}\right]$$

$\therefore$
$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$ **Ans.**

Example 5 Evaluate $\int_0^{\pi} \frac{a d\theta}{a^2 + \sin^2\theta}, a > 0$. **[KU Nov. 2010]**

Solution Let

$$I = \int_0^{\pi} \frac{a d\theta}{a^2 + \sin^2\theta}$$

$$= \int_0^{\pi} \frac{a d\theta}{a^2 + \left(\frac{1 - \cos 2\theta}{2}\right)}$$

$$= \int_0^{\pi} \frac{2a d\theta}{2a^2 + 1 - \cos 2\theta}$$

Put $2\theta = \phi \Rightarrow 2d\theta = d\phi$
When $\theta = 0, \phi = 0$ and when $\theta = \pi, \phi = 2\pi$

$$\therefore \quad I = \int_0^{2\pi} \frac{2a\left(\frac{d\phi}{2}\right)}{2a^2 + 1 - \cos\phi}$$

$$= \int_0^{2\pi} \frac{a d\phi}{2a^2 + 1 - \cos\phi} \qquad (1)$$

Put $z = e^{i\phi}$, then $d\phi = \frac{dz}{iz}$

$$\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Then

$$(1) \Rightarrow I = \int_C \frac{a \cdot \frac{dz}{iz}}{\left[2a^2 + 1 - \frac{1}{2}\left(z + \frac{1}{z}\right)\right]}$$

where C is the unit circle $|z| = 1$

$$= \frac{a}{i}\int_C \frac{dz}{\left[2a^2 + 1 - \frac{1}{2}\left(\frac{z^2+1}{z}\right)\right]}$$

$$= \frac{a}{i}\int_C \frac{dz}{\left[\frac{4a^2z + 2z - z^2 - 1}{2z}\right]}$$

$$= \frac{2a}{i}\int_C \frac{dz}{(4a^2 + 2) - z^2 - 1}$$

$$= -\frac{2a}{i}\int_C \frac{dz}{z^2 - (4a^2 + 2)z + 1}$$

$$= 2ai\int_C \frac{dz}{z^2 - (4a^2 + 2)z + 1}$$

$\therefore I = \int_C f(z)dz$, where $f(z) = \frac{2ai}{z^2 - (4a^2 + 2)z + 1}$

The poles of $f(z)$ are the solutions of

$$z^2 - (4a^2 + 2)z + 1 = 0$$

$$z^2 - (4a^2 + 2)z + 1 = 0$$

$$\therefore \quad z = \frac{(4a^2+2) \pm \sqrt{(4a^2+2)^2 - 4}}{2}$$

$$= \frac{2(2a^2+1) \pm 4a\sqrt{a^2+1}}{2}$$

$$= (2a^2+1) \pm 2a\sqrt{a^2+1}$$

$$\Rightarrow \quad z = (2a^2+1) + 2a\sqrt{a^2+1} \text{ or } (2a^2+1) - 2a\sqrt{a^2+1}$$

Let $\alpha = (2a^2+1) + 2a\sqrt{a^2+1}$ and $\beta = (2a^2+1) - 2a\sqrt{a^2+1}$

Since α, β are the roots of $z^2 - (4a^2 + 2)z + 1 = 0$, the product of the roots $\alpha\beta = 1$

Since $a > 0$, $\alpha > 1$ also, $\beta < 1$.

$\therefore$ out of the two poles α and β, $z = \beta$ lies within the unit circle $|z| = 1$ (since $|\beta| < 1$)

Now,
$$[\operatorname{Res} f(z)]_{z=\beta} = \underset{z\to\beta}{\text{Lt}} (z-\beta) \cdot f(z)$$

$$= \underset{z\to\beta}{\text{Lt}} (z-\beta) \cdot \frac{2ai}{(z-\alpha)(z-\beta)}$$

$$= \frac{2ai}{\beta - \alpha}$$

$$= \frac{2ai}{(2a^2+1-2a\sqrt{a^2+1}) - (2a^2+1-2a\sqrt{a^2+1})}$$

$$= \frac{2ai}{-4a\sqrt{a^2+1}} = \frac{-i}{2\sqrt{a^2+1}}$$

$$\therefore \quad I = \int_C f(z)dz$$

$$= 2\pi i \text{ [sum of the residues of } f(z) \text{ at its poles]}$$

$$= 2\pi i \left[\frac{-i}{2\sqrt{a^2+1}}\right]$$

$$\therefore \quad \int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{a^2+1}}$$ **Ans.**

Example 6 Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx, a > 0, b > 0$.

[KU May 2010, Nov. 2011]

Solution Let $\int_C \phi(z)dz = \int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz$

where C consists of the semicircle Γ and the bounding diameter $[-R, R]$.

Now,
$$\int_C \phi(z)dz = \int_{-R}^{R} \phi(x)dx + \int_{\Gamma} \phi(z)dz \quad (1)$$

Now, $$\phi(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$
$$= \frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

Here, the poles are $z = ia, -ia, ib, -ib$

Here, $z = ia$ and $z = ib$ lie in the upper half-plane while $z = -ia$ and $z = -ib$ lie in the lower half-plane.

We have to find the residues of $\phi(z)$ at each of its poles which lies in the upper half-plane.

$$\therefore \quad [\text{Res}\, f(z)]_{z=ia} = \underset{z\to ia}{\text{Lt}}\, (z-ia)\cdot\phi(z)$$
$$= \underset{z\to ia}{\text{Lt}}\, (z-ia)\frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}$$
$$= \underset{z\to ia}{\text{Lt}}\, \frac{z^2}{(z-ia)(z^2+b^2)}$$
$$= \underset{z\to ia}{\text{Lt}}\, \frac{(ia)^2}{(ia+ia)((ia)^2+b^2)}$$
$$= \frac{-a^2}{2ia(-a^2+b^2)}$$
$$= \frac{a}{2i(a^2-b^2)}$$

$$[\text{Res}\, f(z)]_{z=ib} = \underset{z\to ib}{\text{Lt}}\, (z-ib)\phi(z)$$
$$= \underset{z\to ib}{\text{Lt}}\, (z-ib)\frac{z^2}{(z^2+a^2)(z+ib)(z-ib)}$$
$$= \underset{z\to ib}{\text{Lt}}\, \frac{z^2}{(z^2+a^2)(z+ib)}$$
$$= \frac{(ib)^2}{[(ib)^2+a^2][ib+ib]}$$
$$= \frac{-b^2}{(a^2-b^2)2ib} = \frac{-b}{2i(a^2-b^2)}$$

In (1), making $R \to \infty$, we get
$$\int_C \phi(z)dz = \int_{-\infty}^{\infty}\phi(x)dx + \int_\Gamma \phi(z)dz$$

When $R \to \infty$, $|z| \to \infty$ and $\phi(z) \to 0$

$$\therefore \quad \int_C \phi(z)dz = \int_{-\infty}^{\infty}\phi(x)dx \quad \text{[from (1)]}$$

$$\therefore \quad \int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty}\frac{z^2dx}{(z^2+a^2)(z^2+b^2)}$$
$$= 2\pi i$$

[sum of the residues of $\phi(z)$ at each pole in the upper half-plane]

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right]$$

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)}\right] = 2\pi i\left[\frac{a-b}{2i(a-b)(a+b)}\right]$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$ **Ans.**

Example 7 Evaluate $\int_0^\infty \frac{dx}{x^4+1}$. **[KU Nov. 2010]**

Solution Consider $\int_0^\infty \frac{dx}{x^4+1}$

$$\int_0^\infty \frac{dx}{x^4+1} = \int_0^\infty \frac{dx}{z^4+1}$$

i.e.,
$$2\int_0^\infty \frac{dx}{x^4+1} = \int_{-\infty}^\infty \frac{dx}{z^4+1}$$

The poles are the roots of $z^4 + 1 = 0$

i.e., $z^4 = -1$

$\Rightarrow$ $z = (-1)^{\frac{1}{4}}$

$$= \left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right] \text{ where } n = 0, 1, 2, 3$$

When $n = 0$, $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

When $n = 1$, $z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = e^{\frac{i3\pi}{4}}$

When $n = 2$, $z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = e^{\frac{i5\pi}{4}}$

When $n = 3$, $z = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = e^{\frac{i7\pi}{4}}$

Hence, the poles are $z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$.

Out of these poles, $z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}$ lies in the upper half-plane.

$$\therefore \qquad [\text{Res}\,\phi(z)]_{z=e^{\frac{i\pi}{4}}} = \underset{z\to e^{\frac{i\pi}{4}}}{\text{Lt}} \frac{z-e^{\frac{i\pi}{4}}}{z^4+1}$$

$$= \underset{z\to e^{\frac{i\pi}{4}}}{\text{Lt}} \frac{1}{4z^3} = \frac{1}{4\left(e^{\frac{i\pi}{4}}\right)^3} \text{ (applying L'Hospital's rule)}$$

$$= \frac{1}{4e\dfrac{i3\pi}{4}}$$

$$[\text{Res}\,\phi(z)]_{z=e^{\frac{i3\pi}{4}}} = \underset{z \to e^{\frac{i3\pi}{4}}}{\text{Lt}} \frac{z - e^{\frac{i3\pi}{4}}}{z^4+1}$$

$$= \underset{z \to e^{\frac{i3\pi}{4}}}{\text{Lt}} \frac{1}{4z^3} = \frac{1}{4\left(e^{\frac{i3\pi}{4}}\right)^3}$$

$$= \frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$\therefore \quad 2\int_0^{\infty} \frac{dx}{x^4+1} = \int_{-\infty}^{\infty} \frac{dz}{z^4+1}$$

$= 2\pi i$ [sum of the residues at each pole in the upper half-plane]

$$= 2\pi i\left[\frac{1}{4e^{\frac{i3\pi}{4}}} + \frac{1}{4e^{\frac{i9\pi}{4}}}\right]$$

$$= \frac{\pi i}{2}\left[e^{-\frac{i3\pi}{4}} + e^{-\frac{i9\pi}{4}}\right]$$

$$= \frac{\pi i}{2}\left[\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + \left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right)\right]$$

$$= \frac{\pi i}{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = \frac{\pi i}{2}\left[\frac{-2i}{\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

$$\therefore \quad \int_0^{\infty} \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{z^4+1} = \frac{1}{2}\frac{\pi}{\sqrt{2}}$$

Ans.

EXERCISE

Part A

1. Define essential singularity with an example.
2. Define removable singularity with an example.
3. Define simple pole and multiple pole of a function $f(z)$. Give one example for each.
4. Define the residue of a function at an isolated singularity.
5. State the formula for finding the residue of a function at a multiple pole.
6. Find the residues at the isolated singularities of each of the following:

   (i) $\dfrac{z}{(z+1)(z-2)}$ (ii) $\dfrac{ze^z}{(z-1)^2}$ (iii) $\dfrac{z\sin z}{(z-\pi)^3}$

7. Evaluate the following integrals using Cauchy's residue theorem:

   (i) $\displaystyle\int_C \frac{z+1}{z(z-1)}dz$ where $C : |z| = 2$

(ii) $\int_C \frac{e^{-z}}{z^2} dz$ where C : $|z| = 1$

8. Explain how to convert $\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$ into a contour integral, where f is a rational function.
9. Obtain the poles of $\frac{z+4}{z^2+2z+5}$.
10. By using residue theorem, find the value of $\int_C \frac{z-2}{z-1} dz$ where C is $|z| = 2$.
11. Find the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ at $z = -2$.
12. Find the singularities of $f(z) = \frac{z+4}{z^2+2z+2}$.
13. Find the residue of $f(z) = \frac{z}{z^2+1}$ about $z = i$.
14. Find the residue of $f(z) = \frac{1}{(z^2+a^2)^2}$ at $z = ai$
15. Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.
16. Find the poles of $f(z) = \frac{1}{\sin\frac{1}{z-a}}$.
17. Find the singularities of the function $f(z) = \frac{\cot \pi z}{(z-a)^3}$.
18. Give the forms of the definite integrals which can be evaluated using the infinite semicircular contour above the real axis.
19. Define Cauchy's residue theorem.
20. Find the residue of $\frac{1}{(z^3-1)^2}$ at $z = 1$.

Part B

1. Evaluate the following using Cauchy's residue theorem:

(i) $\int_C \frac{1-2z}{z(z-1)(z-2)} dz, C:|z| = \frac{3}{2}$

(ii) $\int_C \frac{2z-1}{z(z+2)(2z+1)} dz, C:|z| = 1$

(iii) $\int_C \frac{e^{-z}}{z^2} dz, C:|z| = 1$

(iv) $\int_C \frac{12z-7}{(z-1)^2(2z+3)} dz, C:|z+i| = \sqrt{3}$

$\left[\textbf{Ans.}\ \text{(i)}\ 3\pi i\ \text{(ii)}\ \frac{5\pi i}{3}\ \text{(iii)}\ -2\pi i\ \text{(iv)}\ 4\pi i\right]$

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

3. Evaluate $\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta}$. $\left(\textbf{Ans. } \frac{2\pi}{15}\right)$

4. Evaluate $\int_0^{\infty} \frac{dx}{x^4+a^4}$. $\left(\textbf{Ans. } \frac{\pi}{a^3\cdot\sqrt{2}}\right)$

5. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

6. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$. $\left(\textbf{Ans. } \frac{\pi}{4a^3}, a>0\right)$

7. Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx$. $\left(\textbf{Ans. } \frac{1}{2}\pi e^{-a}\right)$

8. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx$. $\left(\textbf{Ans. } \frac{\pi}{a}e^{-a}\right)$

9. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$.

10. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx$. $\left(\textbf{Ans. } \frac{\pi}{6}\right)$

11. Evaluate the integral $\int_0^{\infty} \frac{x^2}{x^4+1}dx$ using contour integration.

12. Evaluate $\int_0^{\infty} \frac{\cos x}{(1+x^2)^2}dx$. $\left(\textbf{Ans. } \frac{\pi}{2e}\right)$

Questions	opt1	opt2	opt3	opt4	opt5
A curve is called a_____________ if it does not intersect itself	Simple closed curve	multiple curve	simply connected region	multiple connected region	
A curve is called ________ if it is not a simple closed curve	connected region	multiple curve	simply connected region	multiple connected region	
If f(z) is analytic in a simply connected domain D and C is any simple closed path then ∫(from c)f(z)dz =	1	2πi	0	πi	
If f(z) is analytic inside on a simple closed curve C and a be any point inside C then ∫(from c)f(z)dz /(z-a)=	2πi f(a)	2πi	0	πi	
The value of ∫(from c) [(3z^2+7z+1)/(z+1)] dz where C is |z| = 1/2 is	2πi	-6πi	πi	πi/2	
The value of ∫(from c) (cos πz/z-1) dz if C is |z| =2	2πi	-2πi	πi	πi/3	
The value of ∫(from c) (1/z-1) dz if C is |z| =2	2πi	3πi	πi	πi/4	
The value of ∫(from c) (1/z-3) dz if C is |z| =1	3πi	πi	πi/4	0	
The value of ∫(from c) (1/(z-3)^3) dz if C is |z| =2	3πi	πi	πi/5	0	
The Taylor's series of f(z) about the point z=0 is called ____________series	Maclaurin's	Laurent's	Geometric	Arithmetic	
The value of ∫(from c) (1/z+4) dz if C is |z| =3	3πi	πi	πi/4	0	
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the _____ part	regular	principal	real	imaginary	
In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the _____ part	regular	principal	real	imaginary	
The poles of the function f(z) = z/((z-1)(z-2)) are at z = _______	1, 2	2,3	1,-1	3,4	
The poles of cotz are_______	2nπ	nπ	3nπ	4nπ	
The poles of the function f(z) = cos z/((z+3)(z-4)) are at z = ______	- 3, 4	2,3	1,-1	3,4	
The isolated singular point of f(z) = z/((z-4)(z-5))	1,2	2,3	0,2	4,5	
The isolated singular point of f(z) = z/((z(z-3))	1,3	2,4	0,3	4,5	
A simple pole is a pole of order _______	1	2	3	4	
The order of the pole z= 2 for f(z) = z/((z+1)(z-2)^2)	1	2	3	4	
Residue of (cosz / z) at z = 0 is	0	1	2	4	
The residue at z = 0 of ((1 + e^z) / (zcosz+sinz)) is	0	1	2	4	

The residue of f(z) = cot z at z= 0 is_____	0	1	2	4
The singularity of f(z) = z / ((z-3)^3) is______	0	1	2	3
A point z=a is said to be a _______point of f(z), if f(z) is not analytic at z=a	Singular	isolated singular	removable	essential singular
A point z=a is said to be a _______point of f(z), if f(z) is analytic except at z=a	Singular	isolated singular	removable	essential singular
In Laurent's series of f(z) about z=a, the terms containing the negative powers is called the ____point	Singular	isolated singular	removable singular	essential singular
In Laurent's series of f(z) about z=a, the terms containing the positive powers is called the ____point	Singular	isolated singular	removable singular	essential singular
In contour integration, cos θ=______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz
In contour integration, sin θ=______	(z^2+1)/2z	(z^2+1)/2iz	(z^2-1)/2z	(z^2-1)/2iz

opt6	Answer
	Simple closed curve
	multiple curve
	0
	$2\pi i f(a)$
	$-6\pi i$
	$-2\pi i$
	$2\pi i$
	0
	0
	Maclaurin's
	0
	regular
	principal
	1, 2
	$n\pi$
	- 3, 4
	4,5
	0,3
	1
	2
	1
	1

1
3

Singular

isolated
singular

essential
singular

removable
singular

(z^2+1)/2
z
(z^2-
1)/2iz