B.E Electronics and Communication Engineering **2018-2019**

Semester-III

18BEEC305 **SIGNALS AND SYSTEMS** **3H-3C**

Instruction Hours/week: L:3 T:0 P:0 **Marks:** Internal**:40** External**:60** Total**:100**

End Semester Exam:3 Hours

Course Objective

- Objective of the course is to understand signal types, properties and analysis, demonstrate and understand the fundamental properties of linear time-invariant systems.

Course Outcomes

At the end of this course students will demonstrate the ability to

- Analyze different types of signals
- Represent continuous and discrete systems in time and frequency domain using different transforms
- Apply Fourier series and Transforms on signals
- Investigate whether the system is stable
- Sample and reconstruct a signal
- Apply Laplace and Z Transforms on signals

UNIT I INTRODUCTION TO SIGNALS AND SYSTEMS

Energy and power signals, continuous and discrete time signals, continuous and discrete amplitude signals. System properties: linearity: additivity and homogeneity, shift-invariance, causality, stability, realizability.

UNIT II LTI SYSTEMS AND ANALYSIS

Linear shift-invariant (LSI) systems, impulse response and step response, convolution, input-output behavior with aperiodic convergent inputs. Characterization of causality and stability of linear shift invariant systems. System representation through differential equations and difference equations.

UNIT III FOURIER SERIES AND FOURIER TRANSFORM

Periodic and semi-periodic inputs to an LSI system, the notion of a frequency response and its relation tothe impulse response, Fourier series representation, the Fourier Transform, convolution/multiplication and their effect in the frequency domain, magnitude and phase response, Fourier domain duality. The Discrete-Time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT). Parseval's Theorem. The idea of signal space and orthogonal bases,

UNIT IV LAPLACE TRANSFORM ANALYSIS

The Laplace Transform, notion of eigen functions of LSI systems, a basis of eigen functions, region of convergence, poles and zeros of system, Laplace domain analysis, solution to differential equations and system behavior.

UNIT V Z TRANFORM AND SAMPLING

The z-Transform for discrete time signals and systems- eigen functions, region of convergence, z-domain analysis. State-space analysis and multi-input, multi-output representation. The state-transition matrix and its role. The Sampling Theorem and its implications- Spectra of sampled signals. Reconstruction: ideal interpolator, zero-order hold, first-order hold, and so on. Aliasing and its effects. Relation between continuous and discrete time systems.

Suggested Readings

1. A.V. Oppenheim, A.S. Willsky and I.T. Young, "Signals and Systems", Prentice Hall, 1983.
2. R.F. Ziemer, W.H. Tranter and D.R. Fannin, "Signals and Systems - Continuous and Discrete", 4th edition, Prentice Hall, 1998.
3. Papoulis, "Circuits and Systems: A Modern Approach", HRW, 1980.
4. B.P. Lathi, "Signal Processing and Linear Systems", Oxford University Press, c1998.
5. Douglas K. Lindner, "Introduction to Signals and Systems", McGraw Hill International Edition: c1999.
6. Simon Haykin, Barry van Veen, "Signals and Systems", John Wiley and Sons (Asia) Private Limited, c1998.
7. Robert A. Gabel, Richard A. Roberts, "Signals and Linear Systems", John Wiley and Sons, 1995.
8. M. J. Roberts, "Signals and Systems - Analysis using Transform methods and MATLAB", TMH, 2003.
9. Ashok Ambardar,"Analog and Digital Signal Processing", 2nd Edition, Brooks/ Cole Publishing Company (An international Thomson Publishing Company), 1999.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Pollachi Main Road, Eachanari Post,

Coimbatore – 641 021

FACULTY OF ENGINEERING

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

LECTURE PLAN

NAME OF THE STAFF : Mr.G.ARAVINDH

DESIGNATION : ASSISTANT PROFESSOR

CLASS : B.E-II YEAR ECE

SUBJECT : SIGNALS AND SYSTEMS

SUBJECT CODE : 18BEEC305

S.No	TOPICS TO BE COVERED	TIME DURATION	SUPPORTING MATERIALS	TEACHING AIDS
UNIT-I INTRODUCTION TO SIGNALS AND SYSTEMS				
1	Energy and power signals	01	T1 Pg.No :139	BB
2	Continuous and discrete time signals	01	T1 Pg.No :140-152	BB
3	Continuous and discrete amplitude signals	01	T1 Pg.No :172-174	BB
4	System properties: linearity: additivity and homogeneity, shift-invariance, causality, stability, realizability.	02	T1 Pg.No :174-177	BB
UNIT-II LTI SYSTEMS AND ANALYSIS				
8	Linear shift-invariant (LSI) systems	01	T1 Pg.No :378-391,392-406	BB
9	Impulse response and step response	02	T1 Pg.No :419	BB
10	Convolution, input-output behavior with aperiodic convergent inputs.	01	T1 Pg.No :442-458	BB

11	Characterization of causality and stability of linear shift invariant systems	01	T1 Pg.No :436-441	BB
12	System representation through differential equations	02	T1 Pg.No :467	BB
13	System representation through difference equations.	02	T1 Pg.No :475	BB
UNIT-III FOURIER SERIES AND FOURIER TRANSFORM				
17	Periodic and semi-periodic inputs to an LSI system and the	01	T1 Pg.No :236-259	BB
18	The notion of a frequency response and its relation to the impulse response	01	T1 Pg.No :274	BB
19	Fourier series representation	01	T1 Pg.No :274-275	BB
20	The Fourier Transform, convolution/multiplication and their effect in the frequency domain	02	T1 Pg.No :280-296	BB
21	Magnitude and phase response, Fourier domain duality.	01	T1 Pg.No :306	BB
22	The Discrete-Time Fourier Transform (DTFT)	02	T1 Pg.No :311	BB
23	Discrete Fourier Transform (DFT).	02	T1 Pg.No :315	BB
24	Parseval's Theorem and the idea of signal space and orthogonal bases,	01	T1 Pg.No :290,292	BB
UNIT-IV LAPLACE TRANSFORM ANALYSIS				
26	The Laplace Transform, notion of eigen functions of LSI systems	02	T1 Pg.No :727-738	BB
27	A basis of eigen functions, region of convergence	02	T1 Pg.No :1261-1265	BB
28	Poles and zeros of system Laplace domain analysis	02	T1 Pg.No :1230-1246	BB
29	Solution to differential equations and system behavior.	03	T1 Pg.No :893	BB
UNIT-V Z TRANFORM AND SAMPLING				
32	The z-Transform for discrete time signals and systems- eigen functions, region of convergence, z-domain analysis. - Aliasing and its effects. Relation between continuous	02	T1 Pg.No :899-902	BB

	and discrete time systems.			
33	State-space analysis and multi-input, multi-output representation.	02	T1 Pg.No :68-72,77-78	BB
34	The state-transition matrix and its role.	01	T1 Pg.No :81	BB
35	The Sampling Theorem and its implications, Spectra of sampled signals.	02	T1 Pg.No :1088-1125	BB
36	Reconstruction: ideal interpolator, zero-order hold, first-order hold, and so on.	02	T1 Pg.No :819-824	BB
37	Aliasing and its effects. Relation between continuous and discrete time systems.	02	T1 Pg.No :833-844	BB

Total No of Lecture Hours Planned: 45 Hrs

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TEXT BOOKS:

S.NO.	**Author(s) Name**	**Title of the book**	**Publisher**	**Year of the publication**
1	A.V. Oppenheim, A.S. Willsky and I.T. Young	"Signals and Systems"	Prentice Hall	1983

SUGGEST READINGS:

S.NO.	**Author(s) Name**	**Title of the book**	**Publisher**	**Year of the publication**

1	R.F. Ziemer, W.H. Tranter and D.R. Fannin	"Signals and Systems - Continuous and Discrete"	4th edition, Prentice Hall,	1998
2	M. J. Roberts	"Signals and Systems - Analysis using Transform methods and MATLAB"	TMH	2003
3	Douglas K. Lindner	"Introduction to Signals and Systems"	McGraw Hill International	1999
4	J. Nagrath, S. N. Sharan, R. Ranjan, S. Kumar	"Signals and Systems"	TMH	2001
5	Robert A. Gabel, Richard A. Roberts	"Signals and Linear Systems"	John Wiley and Sons	1995

STAFF IN-CHARGE **HOD/ECE**

Basics of Signals and Systems

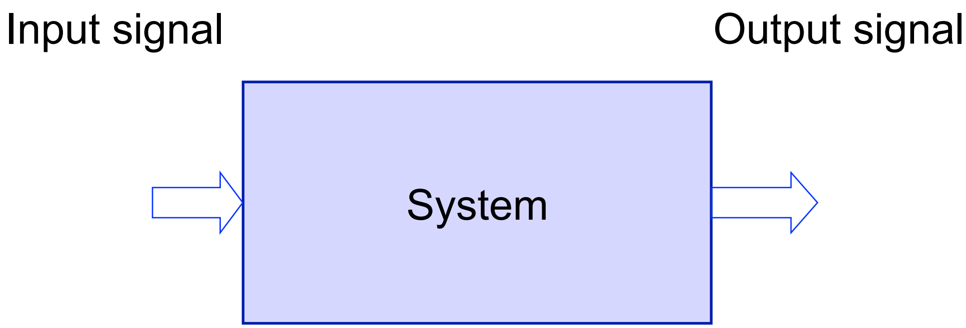
Gloria Menegaz

AA 2011-2012

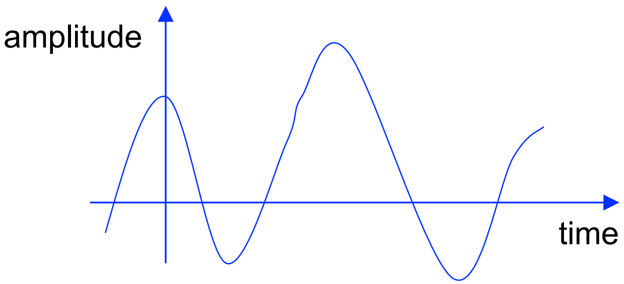
Didactic material

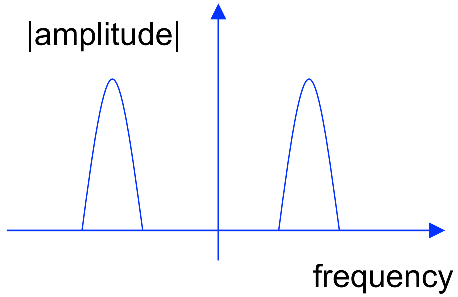
- Textbook
  - Signal Processing and Linear Systems, B.P. Lathi, CRC Press
- Other books
  - Signals and Systems, Richard Baraniuk's lecture notes, available on line
  - Digital Signal Processing (4th Edition) (Hardcover), John G. Proakis, Dimitris K Manolakis
  - Teoria dei segnali analogici, M. Luise, G.M. Vitetta, A.A. D'Amico, McGraw-Hill
  - Signal processing and linear systems, Schaun's outline of digital signal processing
- All textbooks are available at the library
- Handwritten notes will be available on demand

Signals&Systems

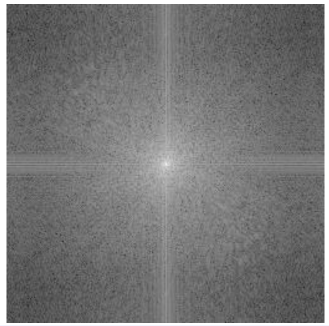


Linear time invariant systems (LTIS)





LTIS perform any kind of processing on the input data to generate output data



Contents

Signals

- Signal classification and representation
  - Types of signals
  - Sampling theory
  - Quantization
- Signal analysis
  - Fourier Transform
    - Continuous time, Fourier series, Discrete Time Fourier Transforms, Windowed FT
  - Spectral Analysis

Systems

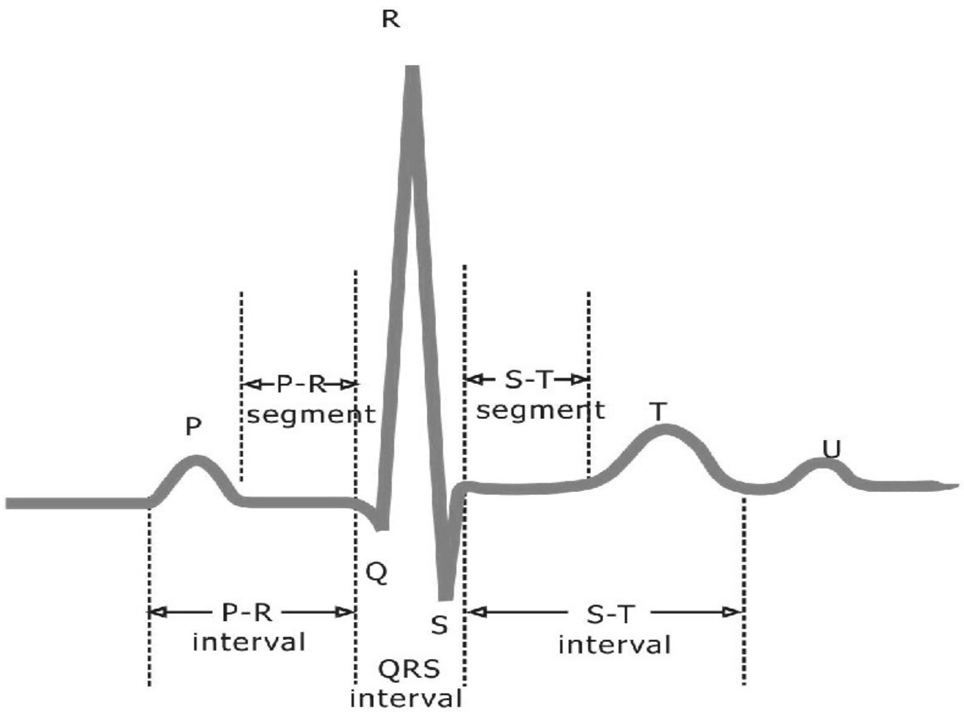
- Linear Time-Invariant Systems
  - Time and frequency domain analysis
  - Impulse response
  - Stability criteria
- Digital filters
  - Finite Impulse Response (FIR)
- Mathematical tools
  - Laplace Transform
    - Basics
  - Z-Transform
    - Basics

Applications in the domain of Bioinformatics

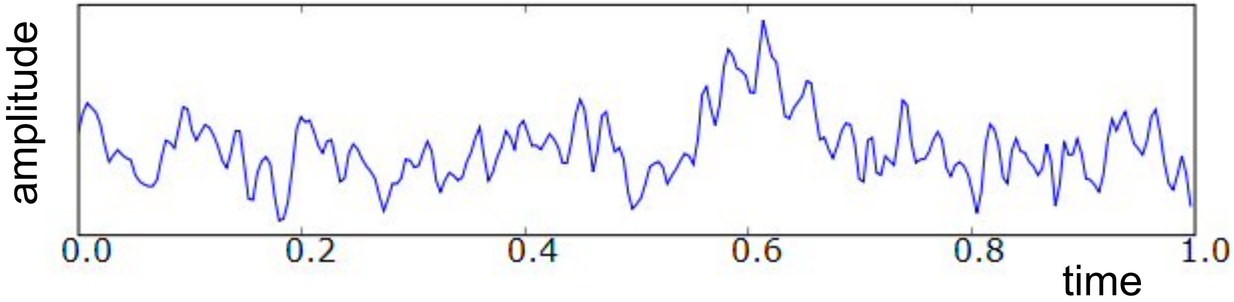
What is a signal?

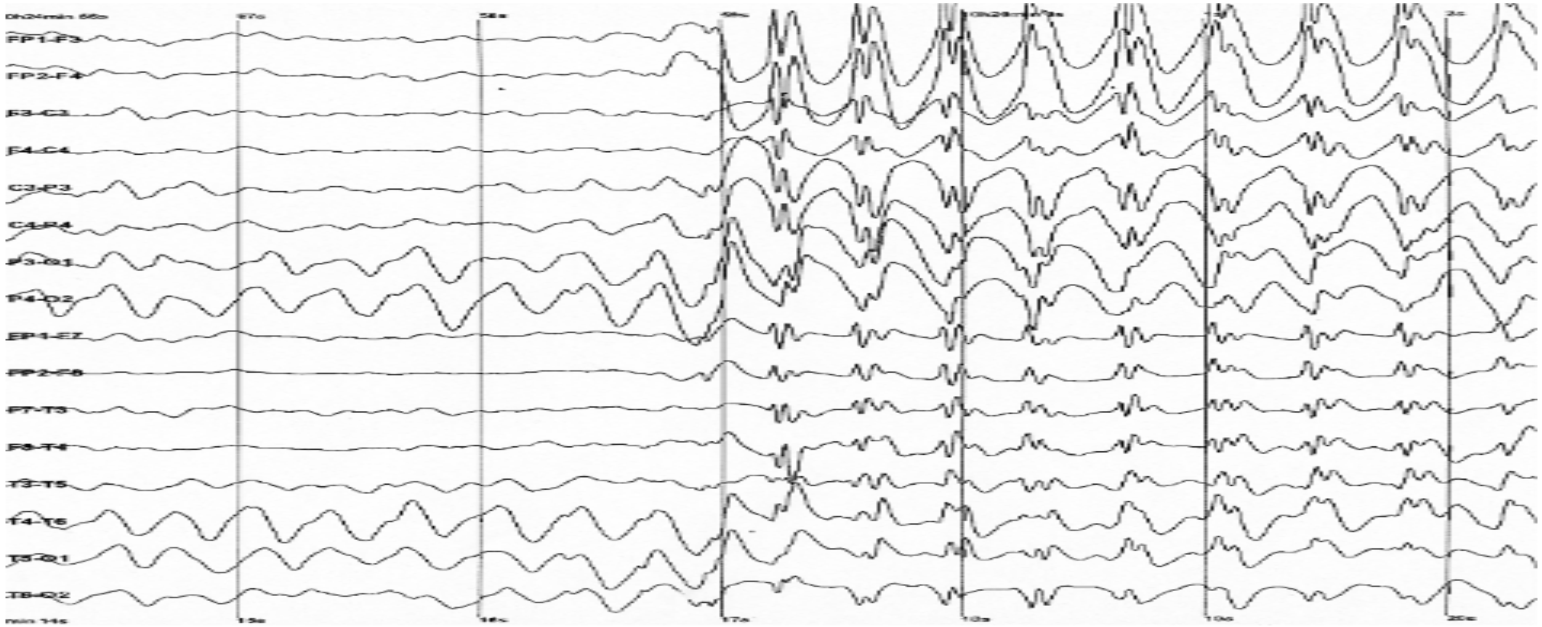
- A signal is a set of information of data
  - Any kind of physical variable subject to variations represents a signal
  - Both the independent variable and the physical variable can be either scalars or vectors
    - Independent variable: time (t), space (x, $\mathbf{x}=[x_1,x_2]$, $\mathbf{x}=[x_1,x_2,x_3]$)
    - Signal:
    - Electrochardiography signal (EEG) 1D, voice 1D, music 1D
    - Images (2D), video sequences (2D+time), volumetric data (3D)

Example: 1D biological signals: ECG



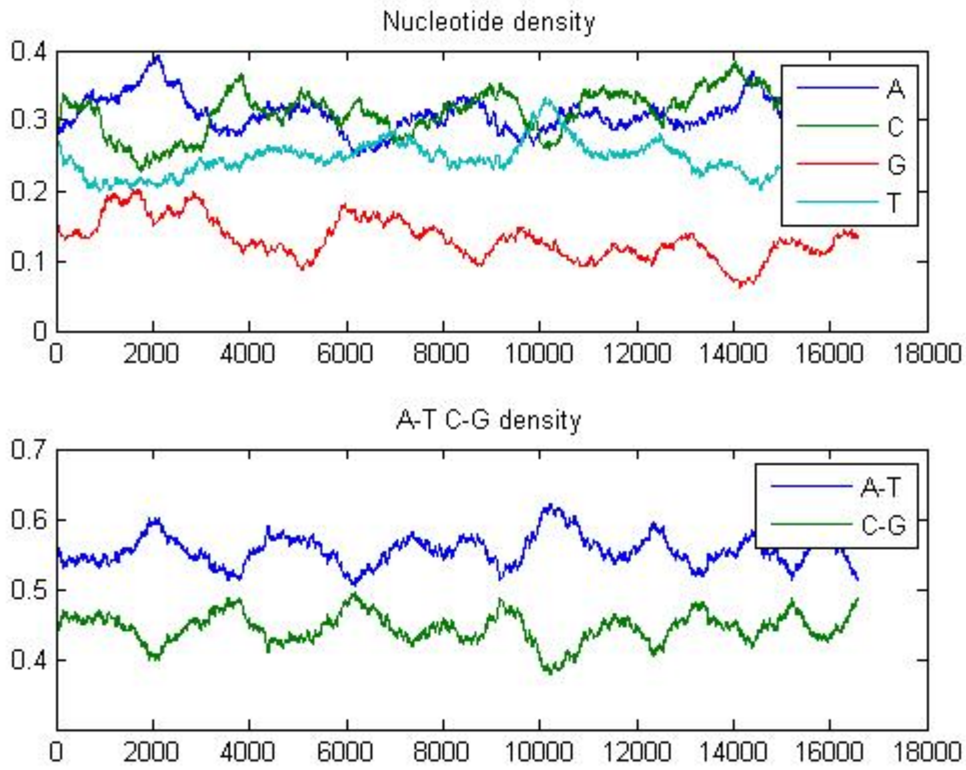
Example: 1D biological signals: EEG





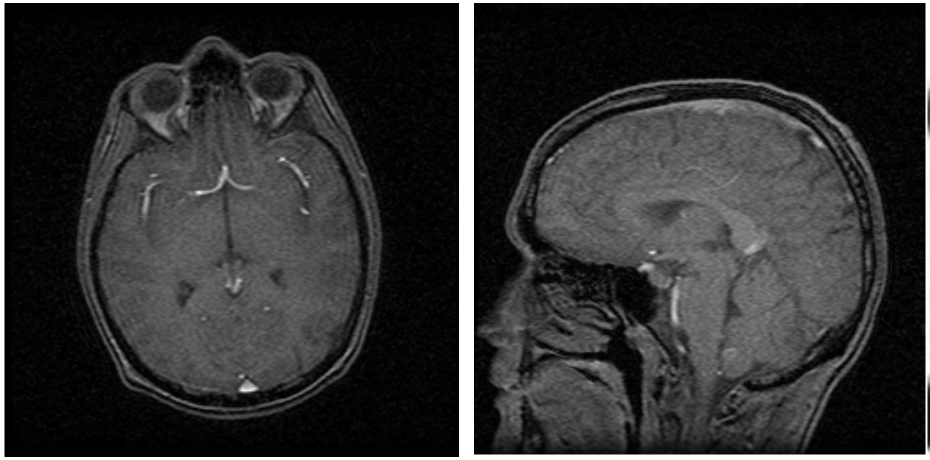
1D biological signals: DNA sequencing

GATCACAGGTCTATCACCCTATTAACCACTCACGGGAGCTCTCCATG……

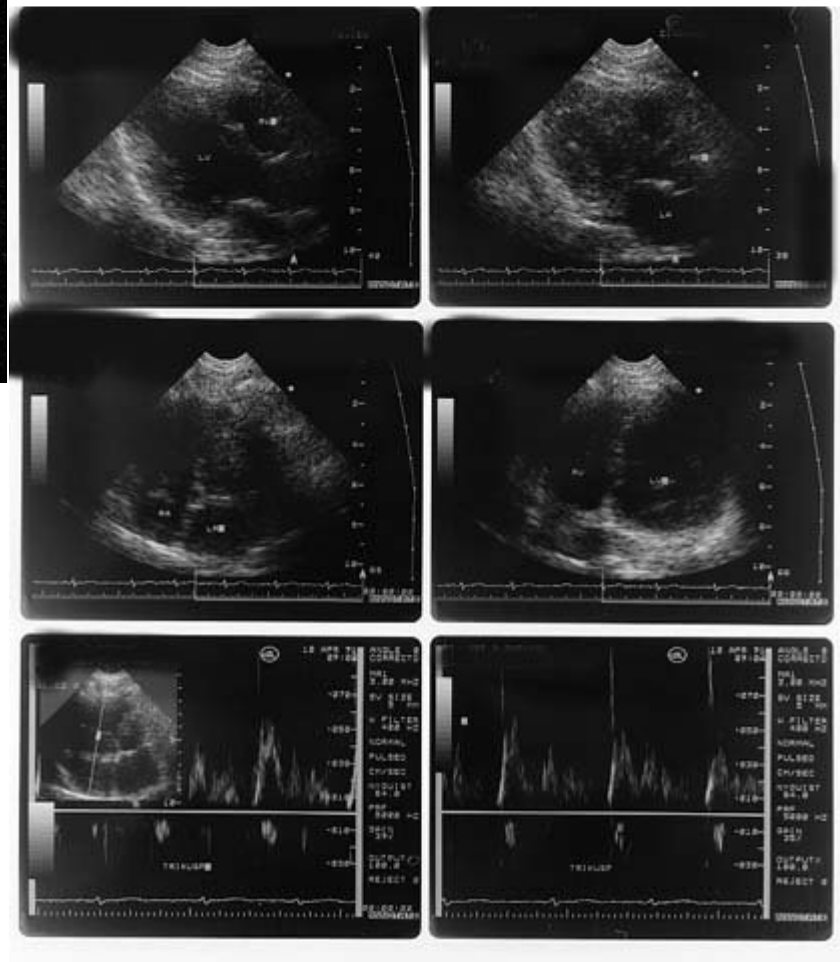


Example: 2D biological signals: MI

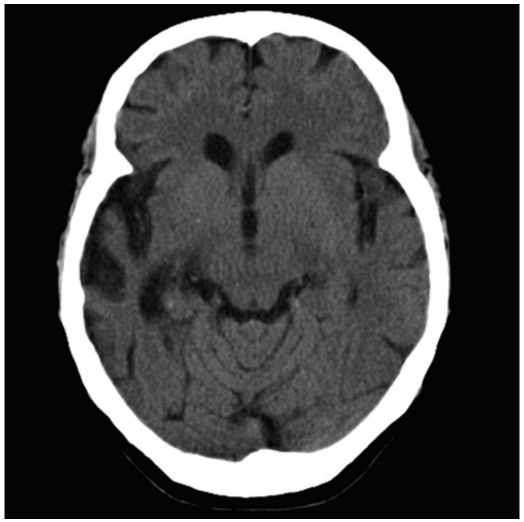
MRI



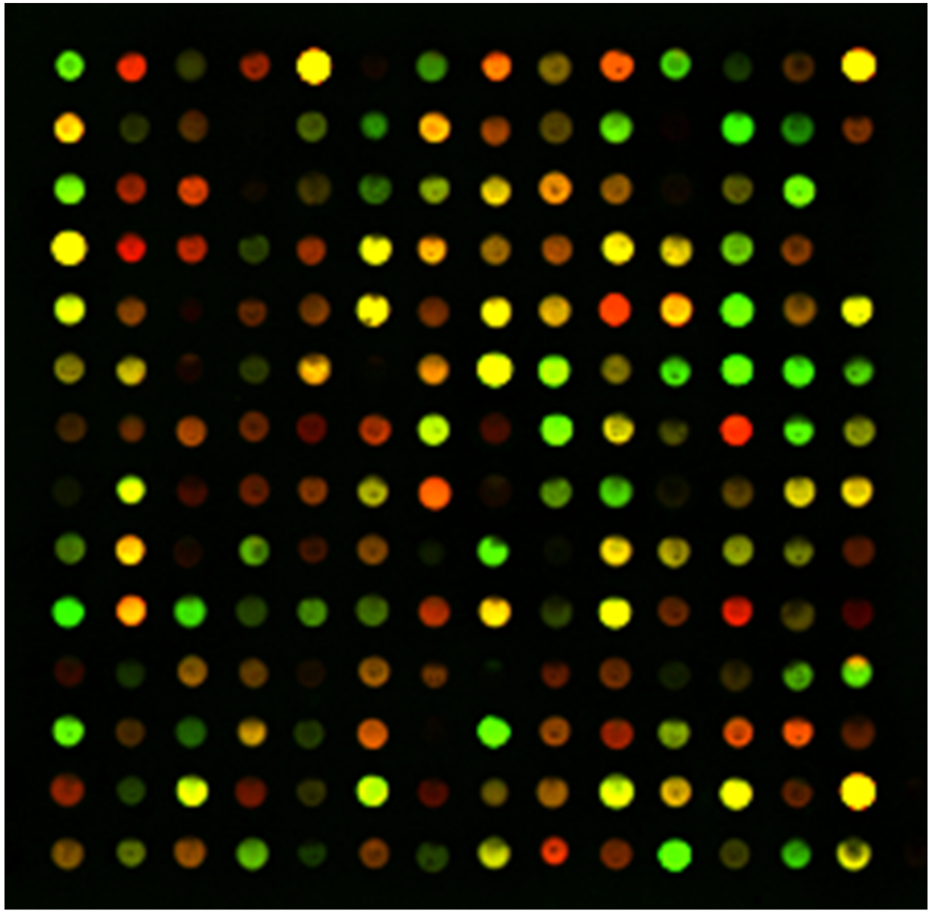
US



CT



Example: 2D biological signals: microarrays



Signals as functions

- Continuous functions of real independent variables
  - 1D: $f=f(x)$
  - 2D: $f=f(x,y)$ x,y
  - Real world signals (audio, ECG, images)
- Real valued functions of discrete variables
  - 1D: $f=f[k]$
  - 2D: $f=f[i,j]$
  - *Sampled* signals
- Discrete functions of discrete variables
  - 1D: $f^d=f^d[k]$
  - 2D: $f^d=f^d[i,j]$
  - *Sampled and quantized* signals

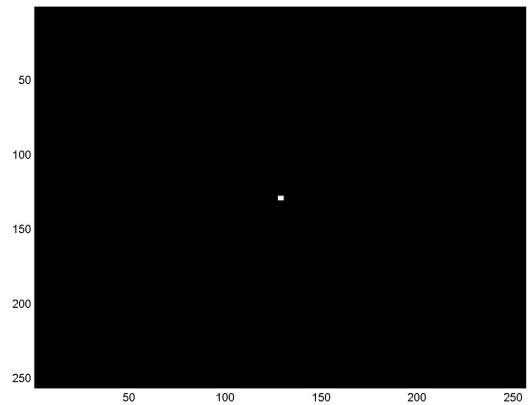
Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of (x,y) values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations
  - Set of values taken by the function : gray levels

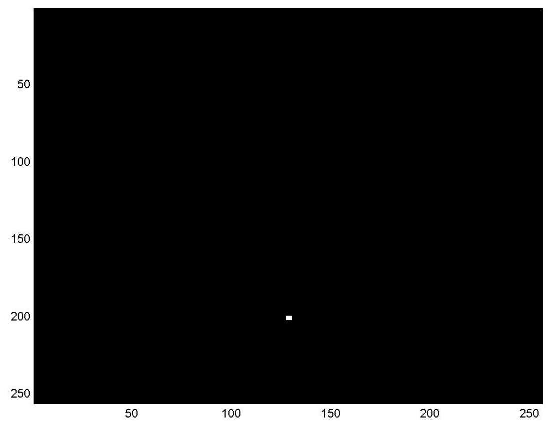
- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - I,J: number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position $[i,j]$

Example 1: δ function

$$\delta[i,j] = \begin{cases} 1 & i = j = 0 \\ 0 & i,j \neq 0; i \neq j \end{cases}$$



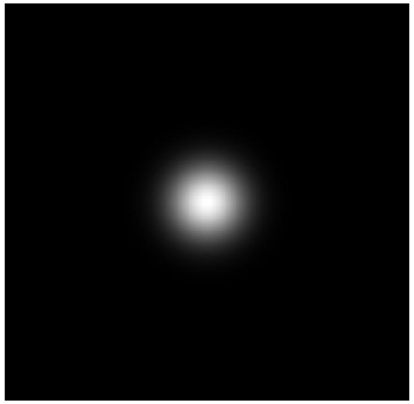
$$\delta[i,j-J] = \begin{cases} 1 & i = 0; j = J \\ 0 & otherwise \end{cases}$$



Example 2: Gaussian

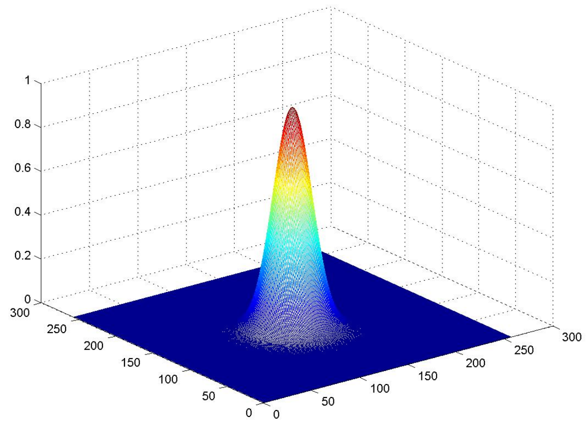
Continuous function

$$f(x,y)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{x^2+y^2}{2\sigma^2}}$$

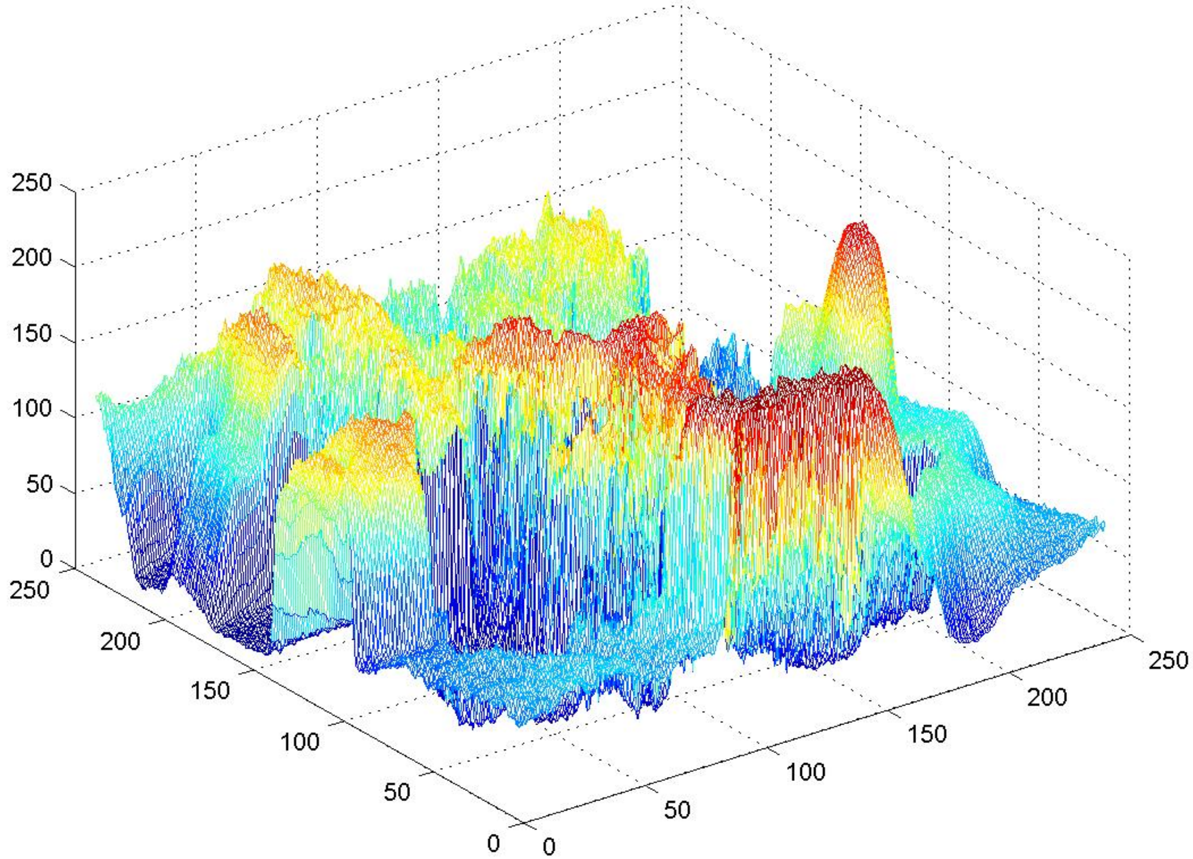


Discrete version

$$f[i,j]=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{i^2+j^2}{2\sigma^2}}$$



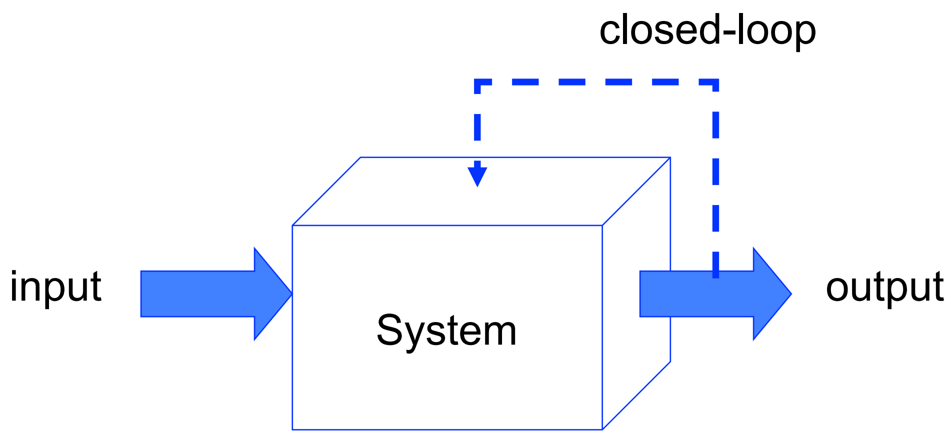
Example 3: Natural image



Example 3: Natural image

What is a system?

- Systems process signals to
  - Extract information (DNA sequence analysis)
  - Enable transmission over channels with limited capacity (JPEG, JPEG2000, MPEG coding)
  - Improve security over networks (encryption, watermarking)
  - Support the formulation of diagnosis and treatment planning (medical imaging)
  - …….



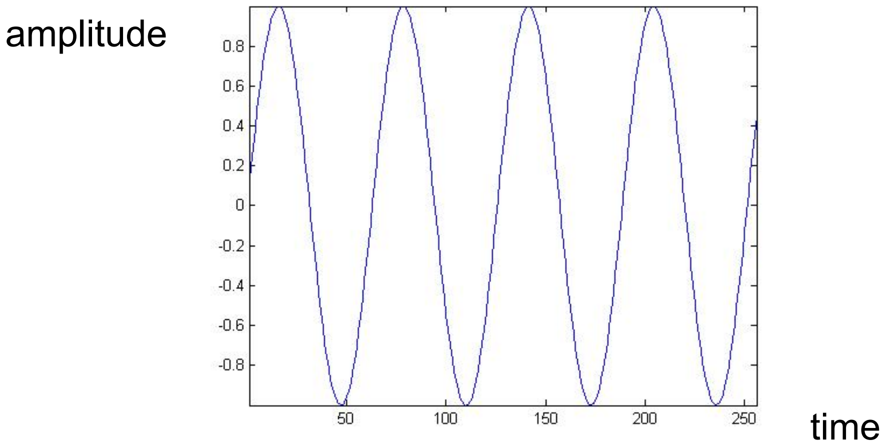
The function linking the output of the system with the input signal is called transfer function and it is typically indicated with the symbol $h(\bullet)$

Classification of signals

- Continuous time – Discrete time
- Analog – Digital (numerical)
- Periodic – Aperiodic
- Energy – Power
- Deterministic – Random (probabilistic)
- Note
  - Such classes are not disjoint, so there are digital signals that are periodic of power type and others that are aperiodic of power type etc.
  - Any combination of single features from the different classes is possible

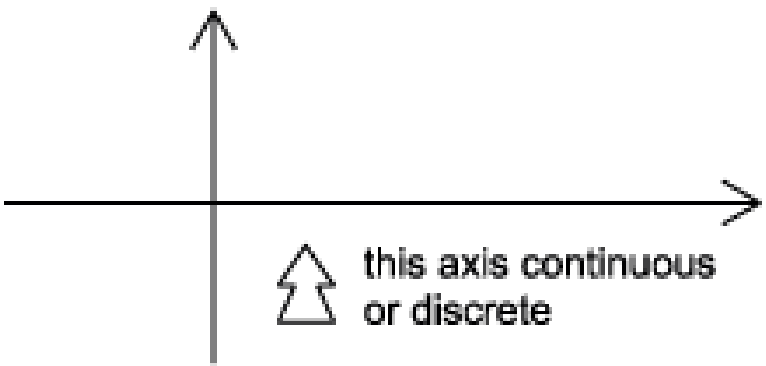
Continuous time – discrete time

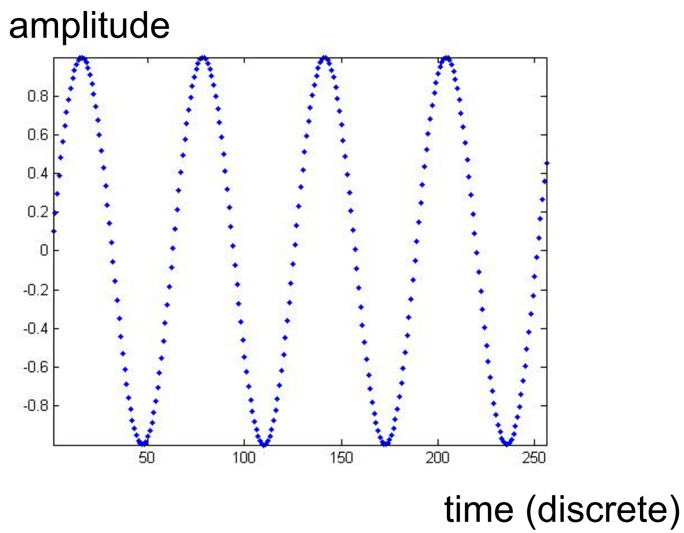
- Continuous time signal: a signal that is specified for every real value of the independent variable
  - The independent variable is continuous, that is it takes any value on the real axis
  - The domain of the function representing the signal has the cardinality of real numbers
    - Signal ↔ f=f(t)
    - Independent variable ↔ time (t), position (x)
    - For continuous-time signals: $t \in \mathbb{R}$



Continuous time – discrete time

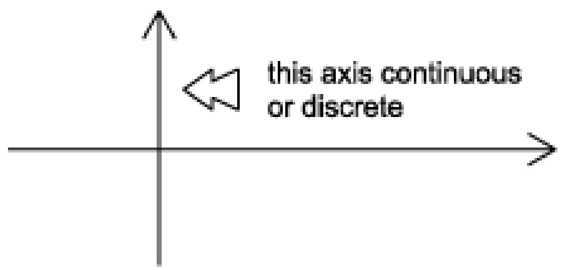
- Discrete time signal: a signal that is specified only for *discrete values* of the independent variable
  - It is usually generated by *sampling* so it will only have values at *equally spaced* intervals along the time axis
  - The domain of the function representing the signal has the cardinality of integer numbers
    - Signal ↔ f=f[n], also called "sequence"
    - Independent variable ↔ n
    - For discrete-time functions: $t \in \mathbf{Z}$





Analog - Digital

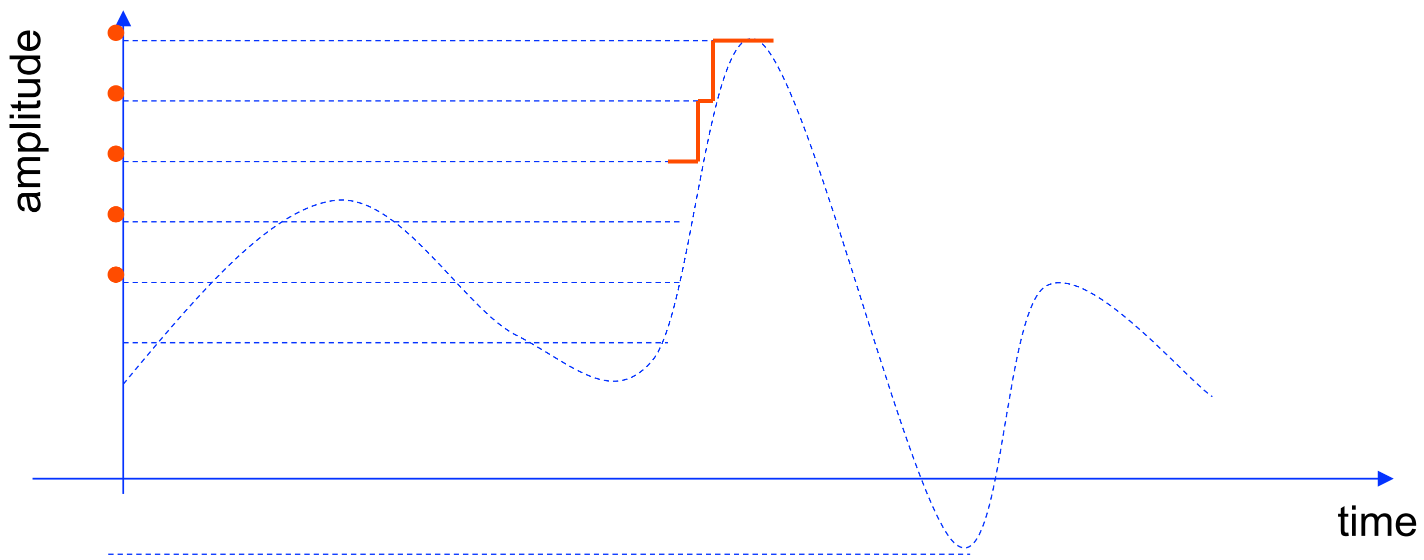
- **Analog signal**: signal whose amplitude can take on any value in a continuous range
  - The amplitude of the function f(t) (or f(x)) has the cardinality of real numbers
    - The difference between analog and digital is similar to the difference between continuous-time and discrete-time. In this case, however, the difference is with respect to the value of the function (y-axis)
  - Analog corresponds to a continuous y-axis, while digital corresponds to a discrete y-axis



- *Here we call digital what we have called quantized in the EI class*
- An analog signal can be both continuous time and discrete time

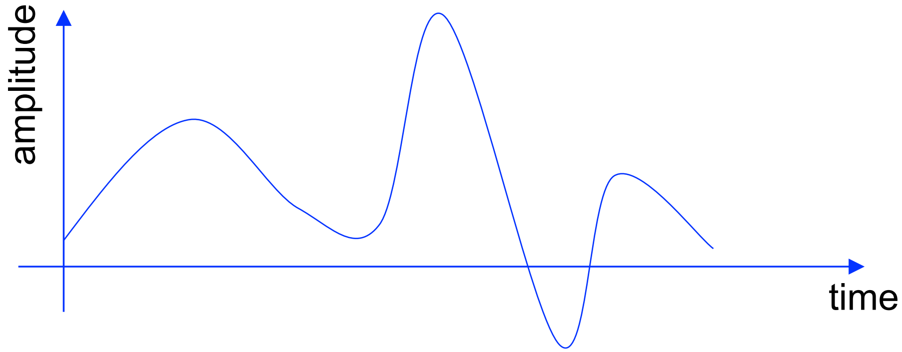
Analog - Digital

- **Digital signal**: a signal is one whose amplitude can take on only a finite number of values (thus it is quantized)
  - The amplitude of the function f() can take only a finite number of values
  - A digital signal whose amplitude can take only M different values is said to be M-ary
    - Binary signals are a special case for M=2

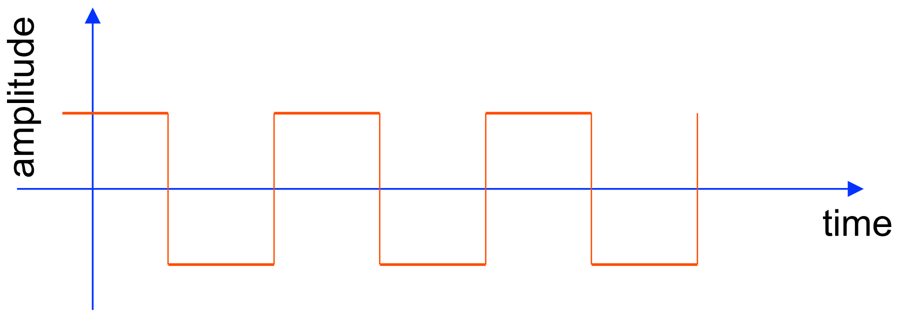


Example

– Continuous time analog

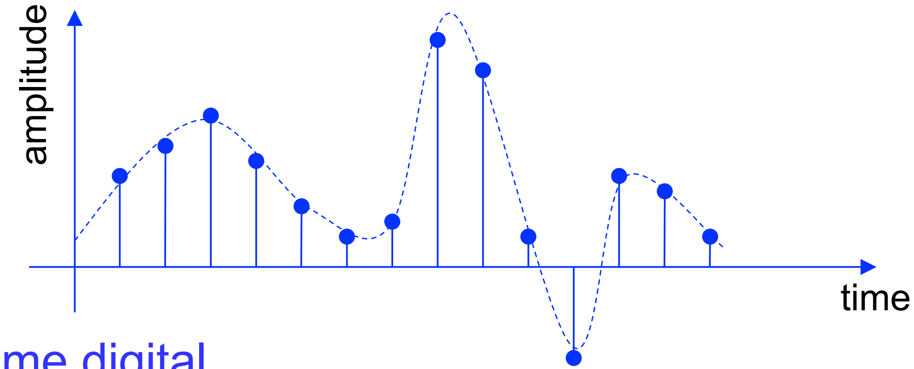


– Continuous time digital (or quantized)
  - binary sequence, where the values of the function can only be one or zero.

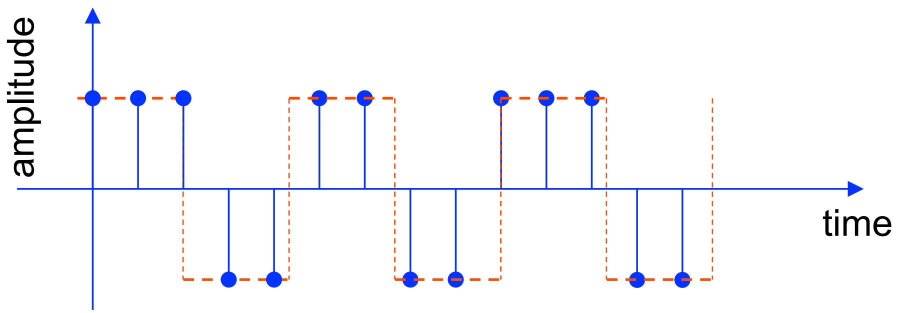


Example

- Discrete time analog



- Discrete time digital
  - binary sequence, where the values of the function can only be one or zero.



Summary

Signal amplitude/ Time or space	Real	Integer
Real	Analog Continuous-time	Digital Continuous-time
Integer	Analog Discrete-time	Digital Discrete time

Note

- In the image processing class we have defined as digital those signals that are both quantized and discrete time. It is a more restricted definition.
- The definition used here is as in the Lathi book.

Periodic - Aperiodic

- A signal f(t) is *periodic* if there exists a positive constant T_0 such that

$$f(t+T_0) = f(t) \qquad \forall t$$

  – The *smallest* value of T_0 which satisfies such relation is said the *period* of the function f(t)
  – A periodic signal remains unchanged when *time-shifted* of integer multiples of the period
  – Therefore, by definition, it starts at minus infinity and lasts forever

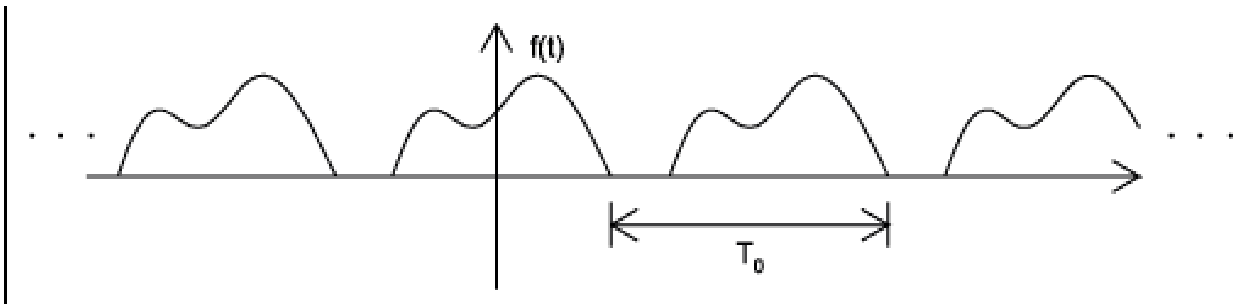
$$-\infty \leq t \leq +\infty \qquad t \in {}^\circ$$

$$-\infty \leq n \leq +\infty \qquad n \in \mathbf{Z}$$

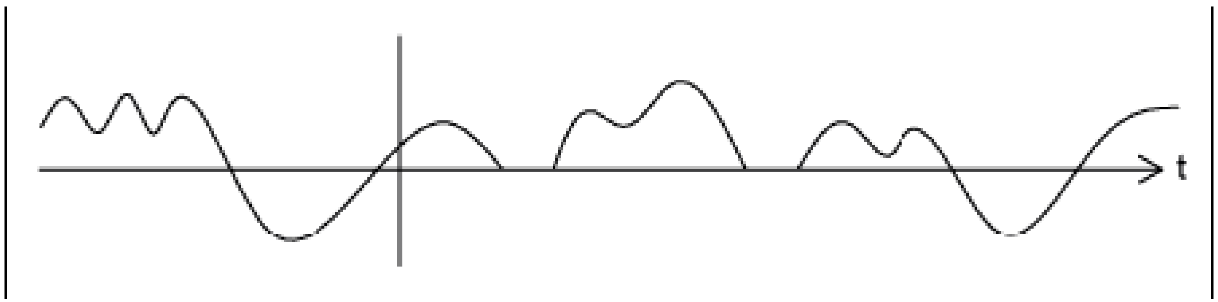
  – Periodic signals can be generated by *periodical extension*

Examples

- Periodic signal with period T_0

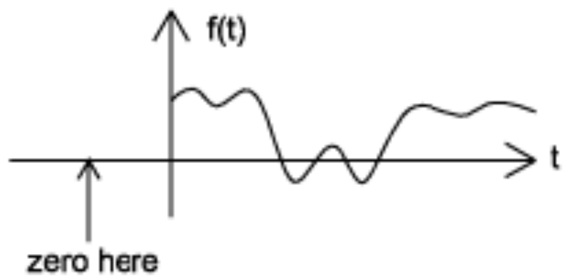


- Aperiodic signal

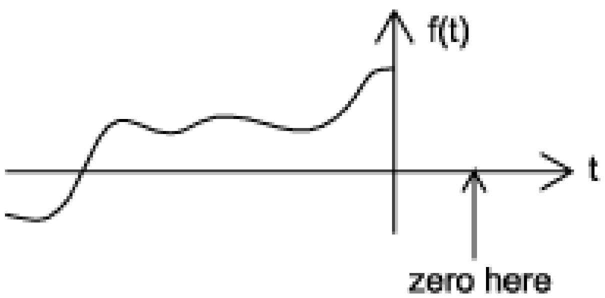


Causal and non-Causal signals

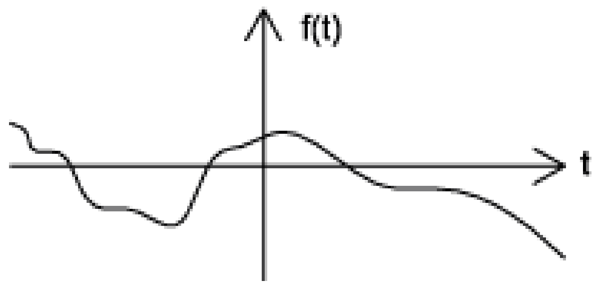
- *Causal* signals are signals that are *zero for all negative time (or spatial positions)*, while



- *Anticausal* are signals that are zero for all *positive time (or spatial positions)*.



- *Noncausal* signals are signals that have nonzero values in both positive and negative time



Causal and non-causal signals

- Causal signals

$$f(t) = 0 \qquad t < 0$$

- Anticausals signals

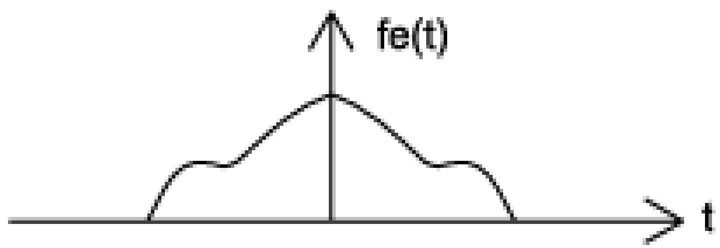
$$f(t) = 0 \qquad t \geq 0$$

- Non-causal signals

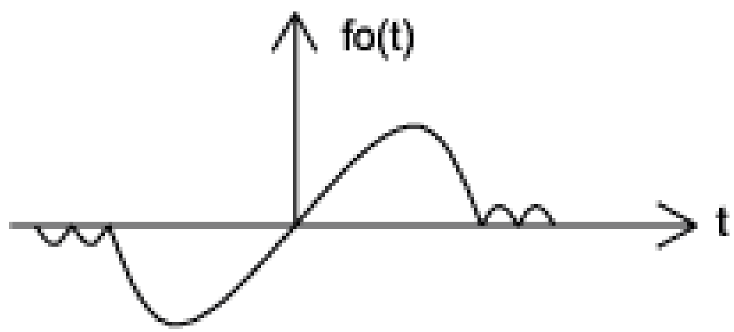
$$\exists t_1 < 0 : \qquad f(t_1) = 0$$

Even and Odd signals

- An even signal is any signal f such that f (t) = f (-t). Even signals can be easily spotted as they are symmetric around the vertical axis.



- An odd signal, on the other hand, is a signal f such that f (t)= - (f (-t))



Decomposition in even and odd components

- Any signal can be written as a combination of an even and an odd signals
  - Even and odd components

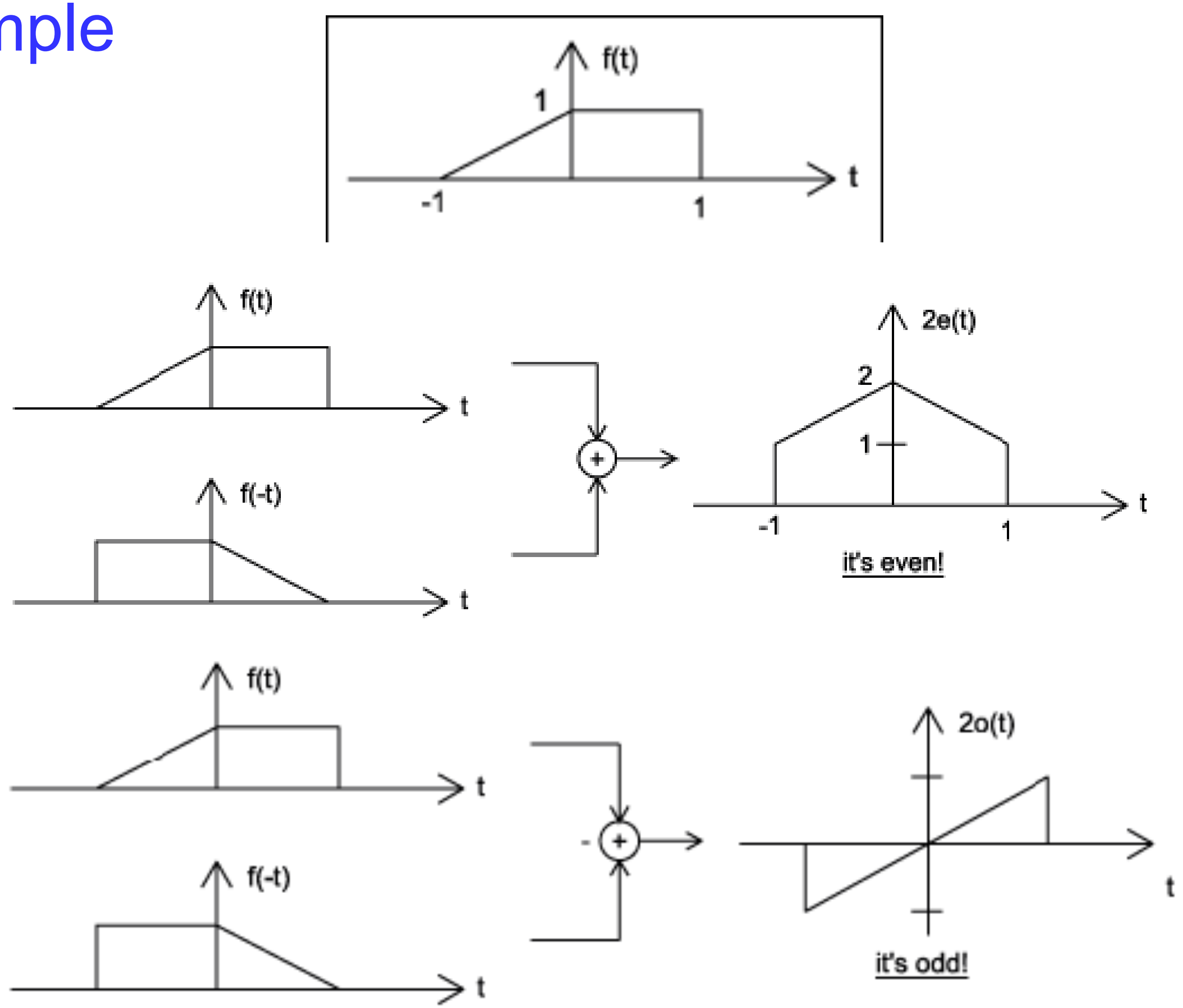
$$f(t)=\frac{1}{2}\left(f(t)+f(-t)\right)+\frac{1}{2}\left(f(t)-f(-t)\right)$$

$$f_e(t)=\frac{1}{2}\left(f(t)+f(-t)\right) \quad \text{even component}$$

$$f_o(t)=\frac{1}{2}\left(f(t)-f(-t)\right) \quad \text{odd component}$$

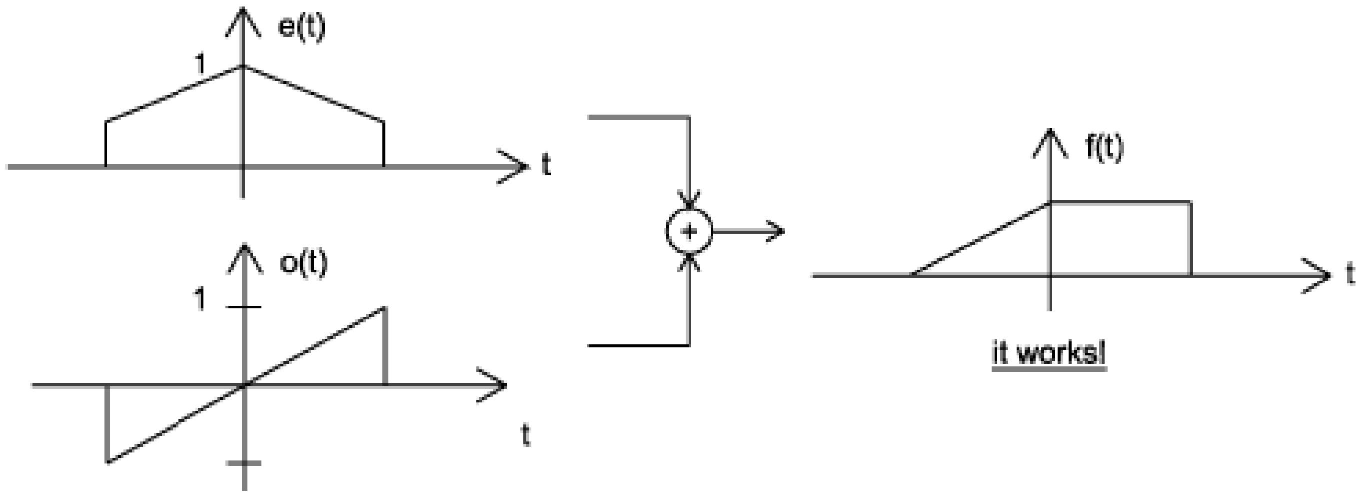
$$f(t)=f_e(t)+f_o(t)$$

Example



Example

- Proof



Some properties of even and odd functions

- even function x odd function = odd function
- odd function x odd function = even function
- even function x even function = even function
- Area

$$\int_{-a}^{a} f_e(t)\,dt = 2\int_{0}^{a} f_e(t)\,dt$$

$$\int_{-a}^{a} f_e(t)\,dt = 0$$

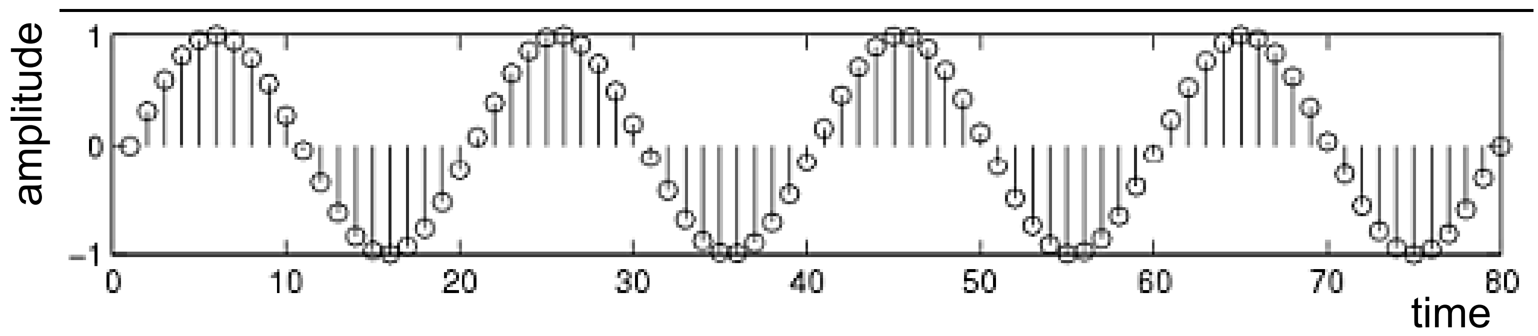
Deterministic - Probabilistic

- Deterministic signal: a signal whose *physical description* in known completely
- A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table.
- Because of this the future values of the signal can be calculated from past values with complete confidence.
  - There is *no uncertainty* about its amplitude values
  - Examples: signals defined through a mathematical function or graph

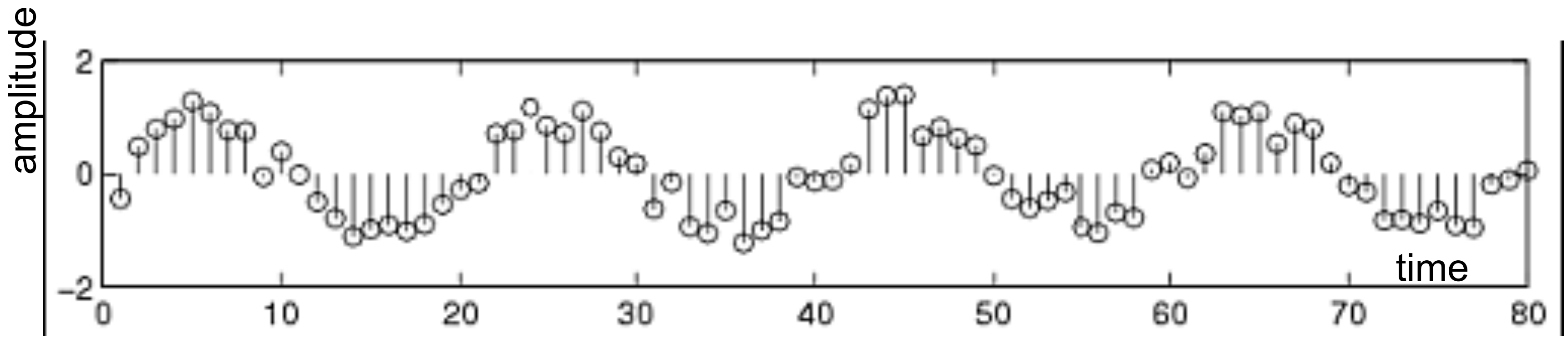
- Probabilistic (or random) signals: the amplitude values *cannot be predicted precisely* but are known only in terms of probabilistic descriptors
- The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals
  - They are realization of a stochastic process for which a model could be available
  - Examples: EEG, evocated potentials, noise in CCD capture devices for digital cameras

Example

- Deterministic signal



- Random signal



Finite and Infinite length signals

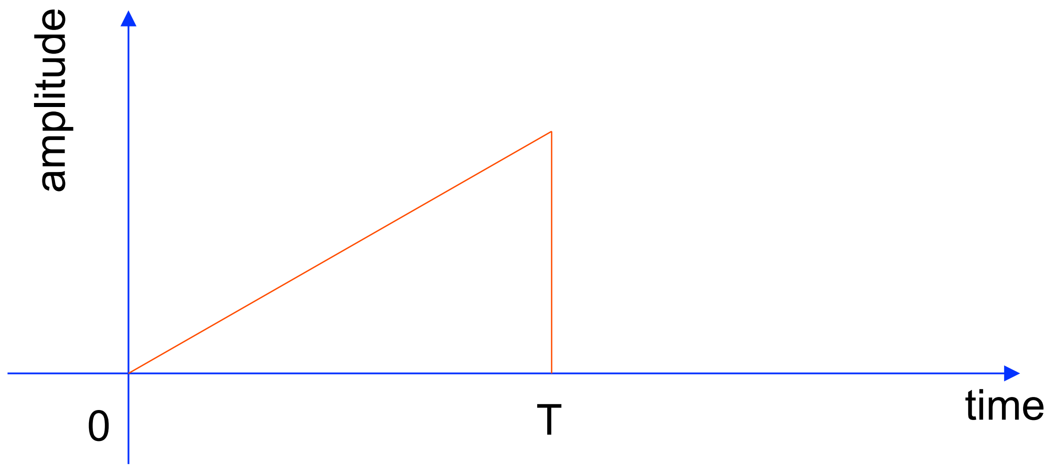
- A finite length signal is non-zero over a finite set of values of the independent variable

$$f = f(t), \forall t : t_1 \leq t \leq t_2$$
$$t_1 > -\infty, t_2 < +\infty$$

- An infinite length signal is non zero over an infinite set of values of the independent variable
  – For instance, a sinusoid $f(t)=\sin(\omega t)$ is an infinite length signal

Size of a signal: Norms

- "Size" indicates largeness or strength.
- We will use the mathematical concept of the norm to quantify this notion for both continuous-time and discrete-time signals.
- The energy is represented by the area under the curve (of the squared signal)



Energy

- Signal energy

$$E_f = \int_{-\infty}^{+\infty} f^2(t)dt$$

$$E_f = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt$$

- Generalized energy : L_p norm
  - For p=2 we get the energy (L_2 norm)

$$\|f(t)\| = \left(\int \left(|f(t)|\right)^p dt\right)^{1/p}$$

$$1 \leq p < +\infty$$

Power

- Power
  - The power is the time average (mean) of the squared signal amplitude, that is the *mean-squared* value of f(t)

$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^2(t)dt$$

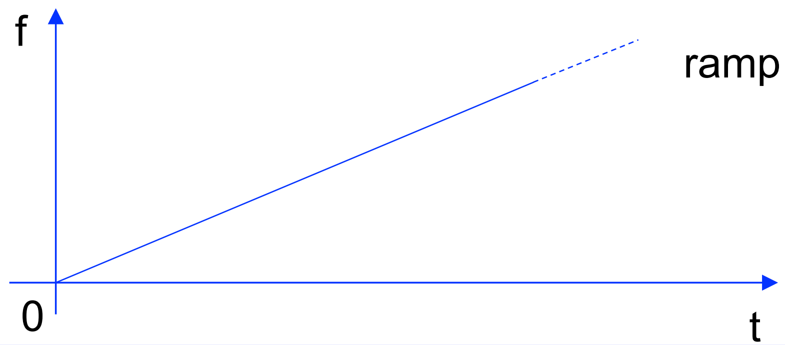
$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 \, dt$$

Power - Energy

- The square root of the power is the root mean square (*rms*) value
  - This is a very important quantity as it is the most widespread measure of similarity/dissimilarity among signals
  - It is the basis for the definition of the Signal to Noise Ratio (SNR)

$$SNR = 20\log_{10}\left(\sqrt{\frac{P_{signal}}{P_{noise}}}\right)$$

  - It is such that a constant signal whose amplitude is =rms holds the same power content of the signal itself
- There exists signals for which neither the energy nor the power are finite

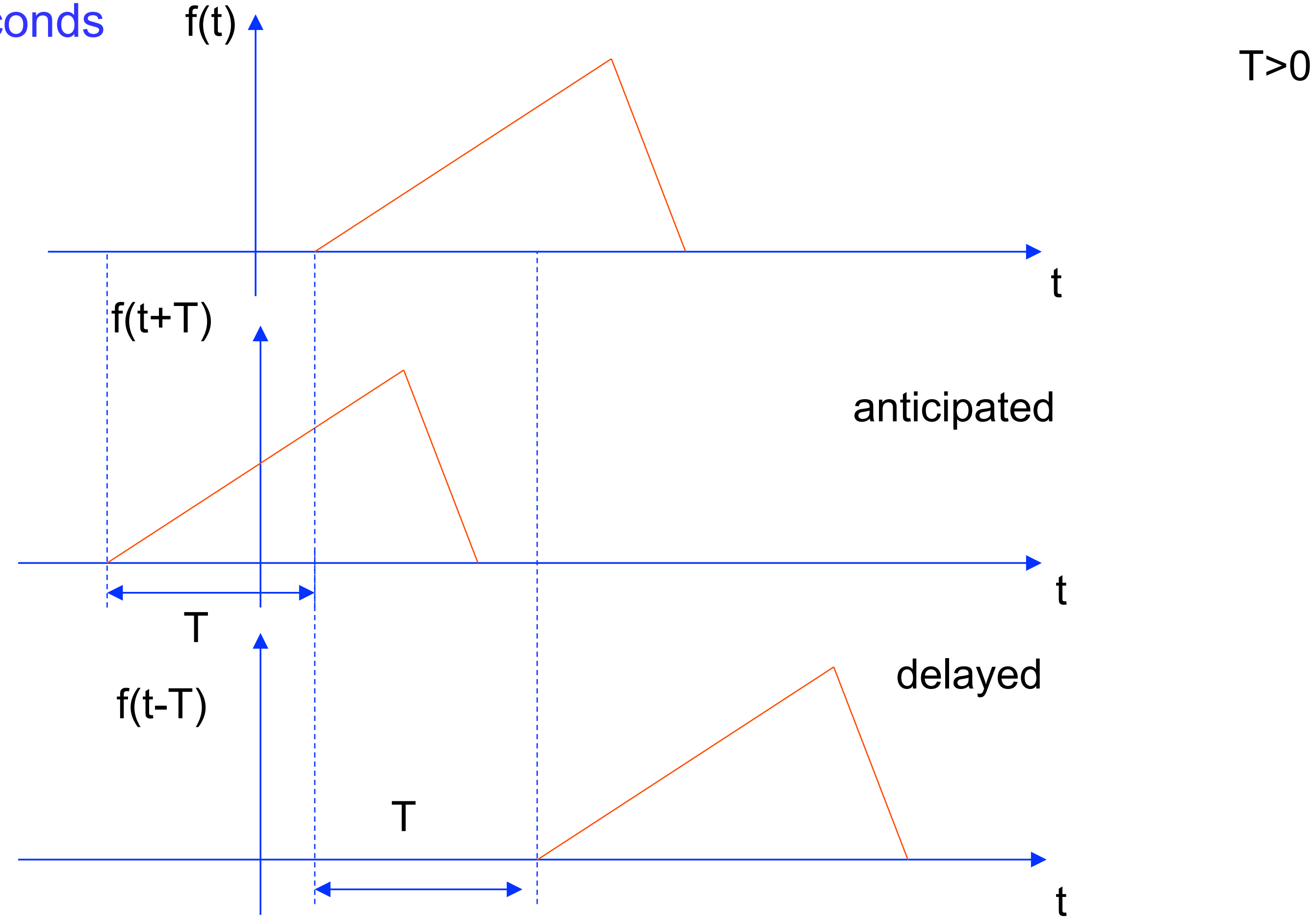


Energy and Power signals

- A signal with finite energy is an energy signal
  - Necessary condition for a signal to be of energy type is that the amplitude goes to zero as the independent variable tends to infinity
- A signal with finite and different from zero power is a power signal
  - The mean of an entity averaged over an infinite interval exists if either the entity is periodic or it has some statistical regularity
  - A power signal has infinite energy and an energy signal has zero power
  - There exist signals that are neither power nor energy, such as the ramp
- All practical signals have finite energy and thus are energy signals
  - It is impossible to generate a real power signal because this would have infinite duration and infinite energy, which is not doable.

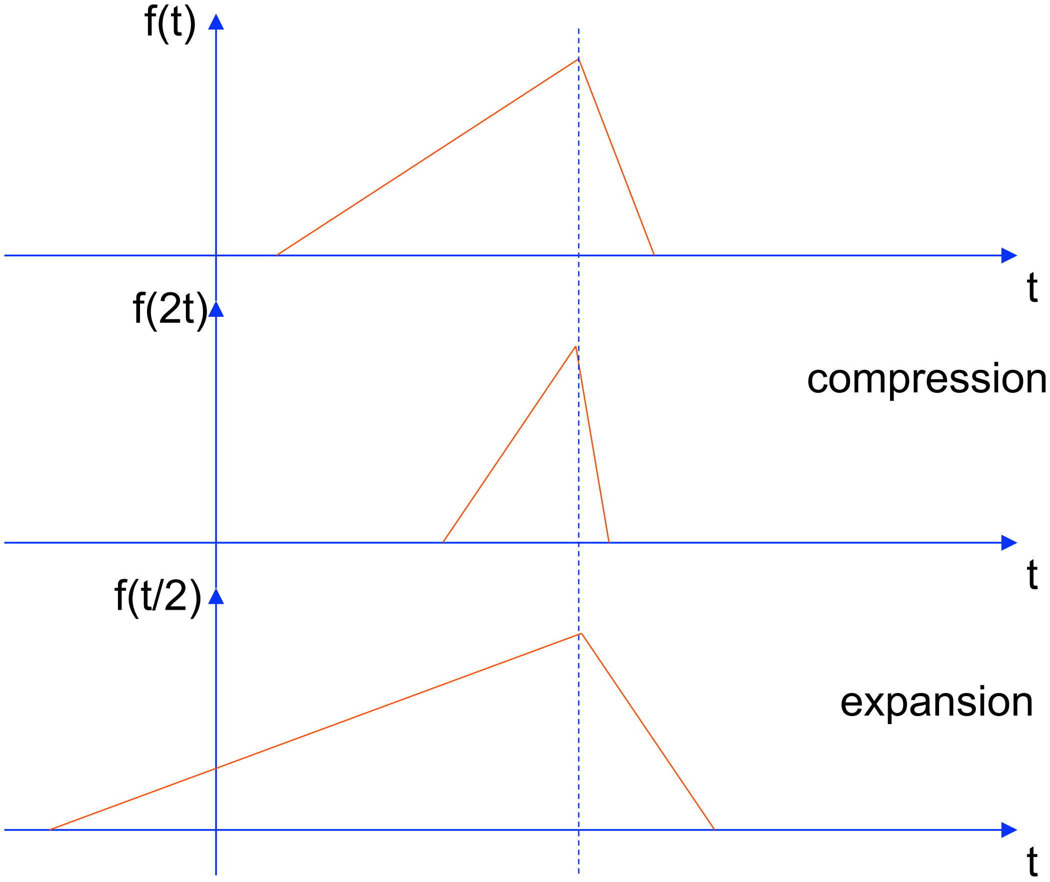
Useful signal operations: shifting, scaling, inversion

- **Shifting**: consider a signal f(t) and the same signal delayed/anticipated by T seconds



Useful signal operations: shifting, scaling, inversion

- (Time) Scaling: compression or expansion of a signal in time



compression $\varphi(t) = f(2t)$

expansion $\varphi(t) = f(t/2)$

Useful signal operations: shifting, scaling, inversion

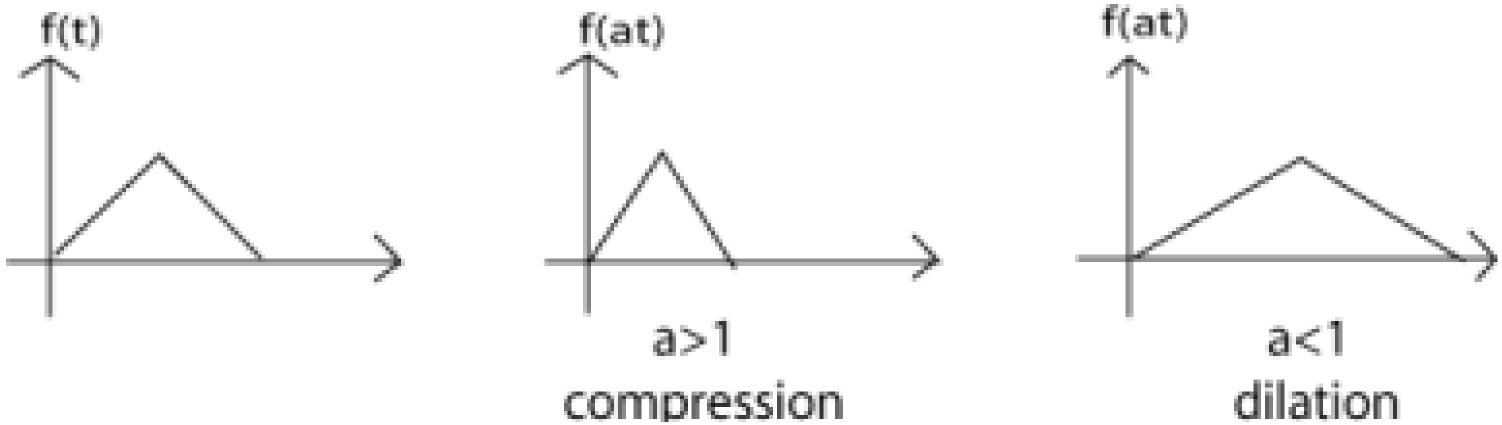
- Scaling: generalization

$$a > 1$$

$$\varphi(t) = f(at) \rightarrow \text{compressed version}$$

$$\varphi(t) = f\left(\frac{t}{a}\right) \rightarrow \text{dilated (or expanded) version}$$

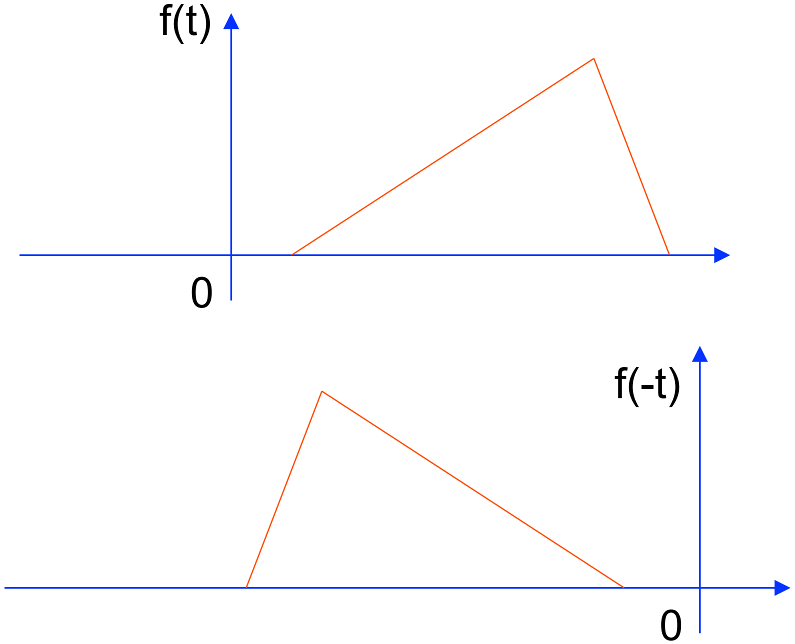
Viceversa for $a < 1$

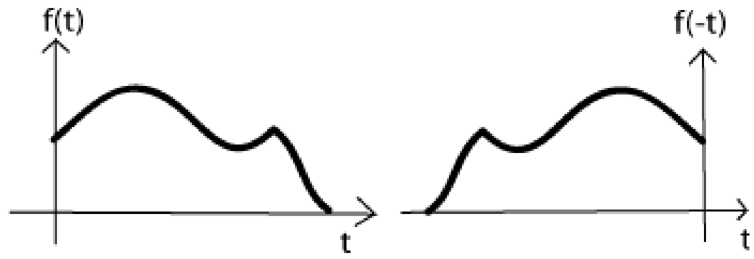


Useful signal operations: shifting, scaling, inversion

- (Time) inversion: mirror image of f(t) about the vertical axis

$$\varphi(t) = f(-t)$$





Useful signal operations: shifting, scaling, inversion

- Combined operations: $f(t) \rightarrow f(at\text{-}b)$
- Two possible sequences of operations

1. *Time shift f(t) by to obtain f(t-b). Now time scale the shifted signal f(t-b) by a to obtain f(at-b).*
2. *Time scale f(t) by a to obtain f(at). Now time shift f(at) by b/a to obtain f(at-b).*
   - *Note that you have to replace t by (t-b/a) to obtain f(at-b) from f(at) when replacing t by the translated argument (namely t-b/a))*

Useful functions

- Unit step function
  - Useful for representing causal signals

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

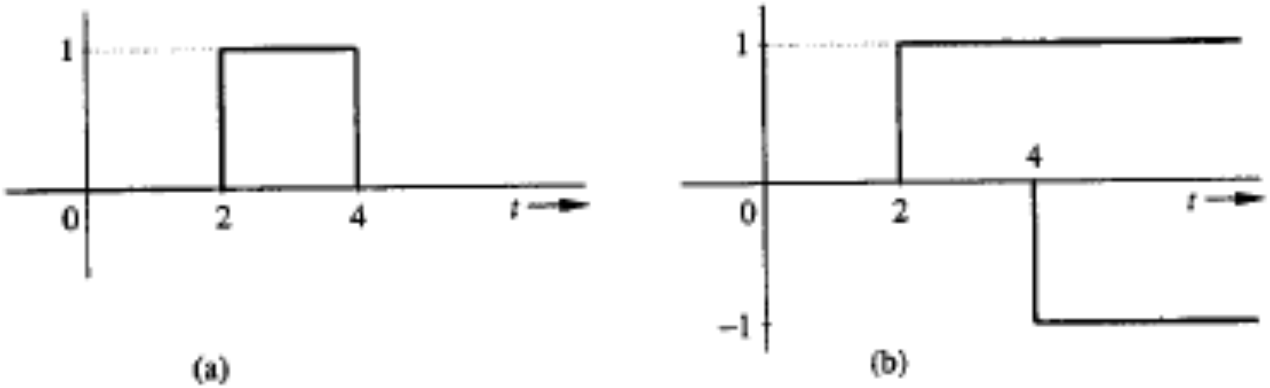
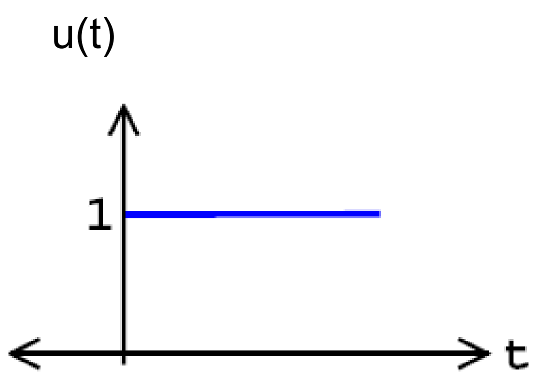


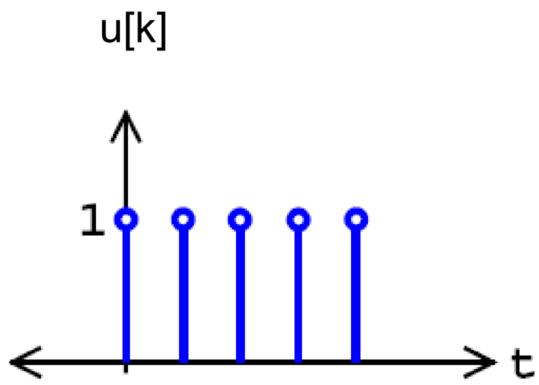
Fig. 1.15 Representation of a rectangular pulse by step functions.

$$f(t) = u(t-2) - u(t-4)$$

Useful functions

- Continuous and discrete time unit step functions

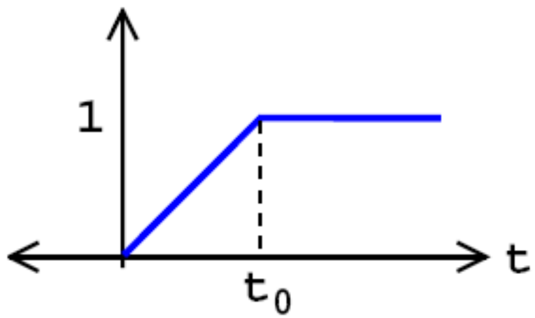




Useful functions

- Ramp function (continuous time)

$$r(t) = \begin{cases} 0 \text{ if } t < 0 \\ \frac{t}{t_0} \text{ if } 0 \leq t \leq t_0 \\ 1 \text{ if } t > t_0 \end{cases}$$

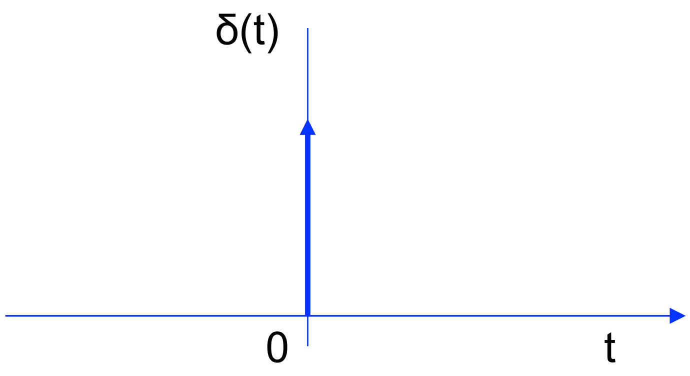


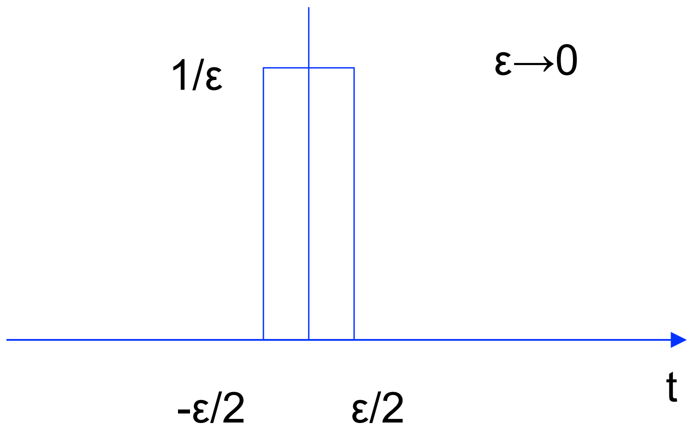
Useful functions

- Unit impulse function

$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$





Properties of the unit impulse function

- Multiplication of a function by impulse

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$
$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

- Sampling property of the unit function

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t)\,dt = \int_{-\infty}^{+\infty} \phi(0)\delta(t)\,dt = \phi(0)\int_{-\infty}^{+\infty} \delta(t)\,dt = \phi(0)$$
$$\int_{-\infty}^{+\infty} \phi(t)\delta(t-T)\,dt = \phi(T)$$

  – The area under the curve obtained by the product of the unit impulse function shifted by T and $\varphi(t)$ is the value of the function $\varphi(t)$ for t=T

Properties of the unit impulse function

- The unit step function is the integral of the unit impulse function

$$\frac{du}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t)$$

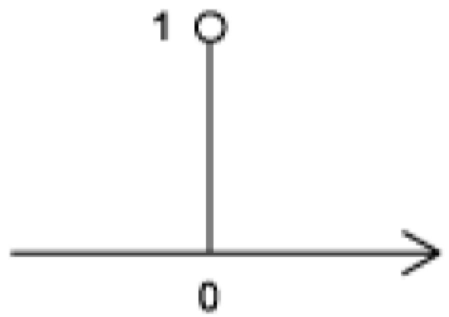
  – Thus

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Properties of the unit impulse function

- Discrete time impulse function

$$\delta[n] = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases}$$



Useful functions

- Continuous time complex exponential

$$f(t) = Ae^{j\omega t}$$

- Euler's relations

$$Ae^{j\omega t} = Acos(\omega t) + j(Asin(\omega t))$$

$$cos(\omega t) = \frac{e^{j\omega t} + e^{-(j\omega t)}}{2}$$

$$sin(\omega t) = \frac{e^{j\omega t} - e^{-(j\omega t)}}{2j}$$

$$e^{j\omega t} = cos(\omega t) + jsin(\omega t)$$

- Discrete time complex exponential
  - k=nT

$$\begin{aligned} f[n] &= Be^{snT} \\ &= Be^{j\omega nT} \end{aligned}$$

Useful functions

- Exponential function e^{st}
  - Generalization of the function $e^{j\omega t}$

$$s = \sigma + j\omega$$

Therefore

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos\omega t + j\sin\omega t) \tag{1.30a}$$

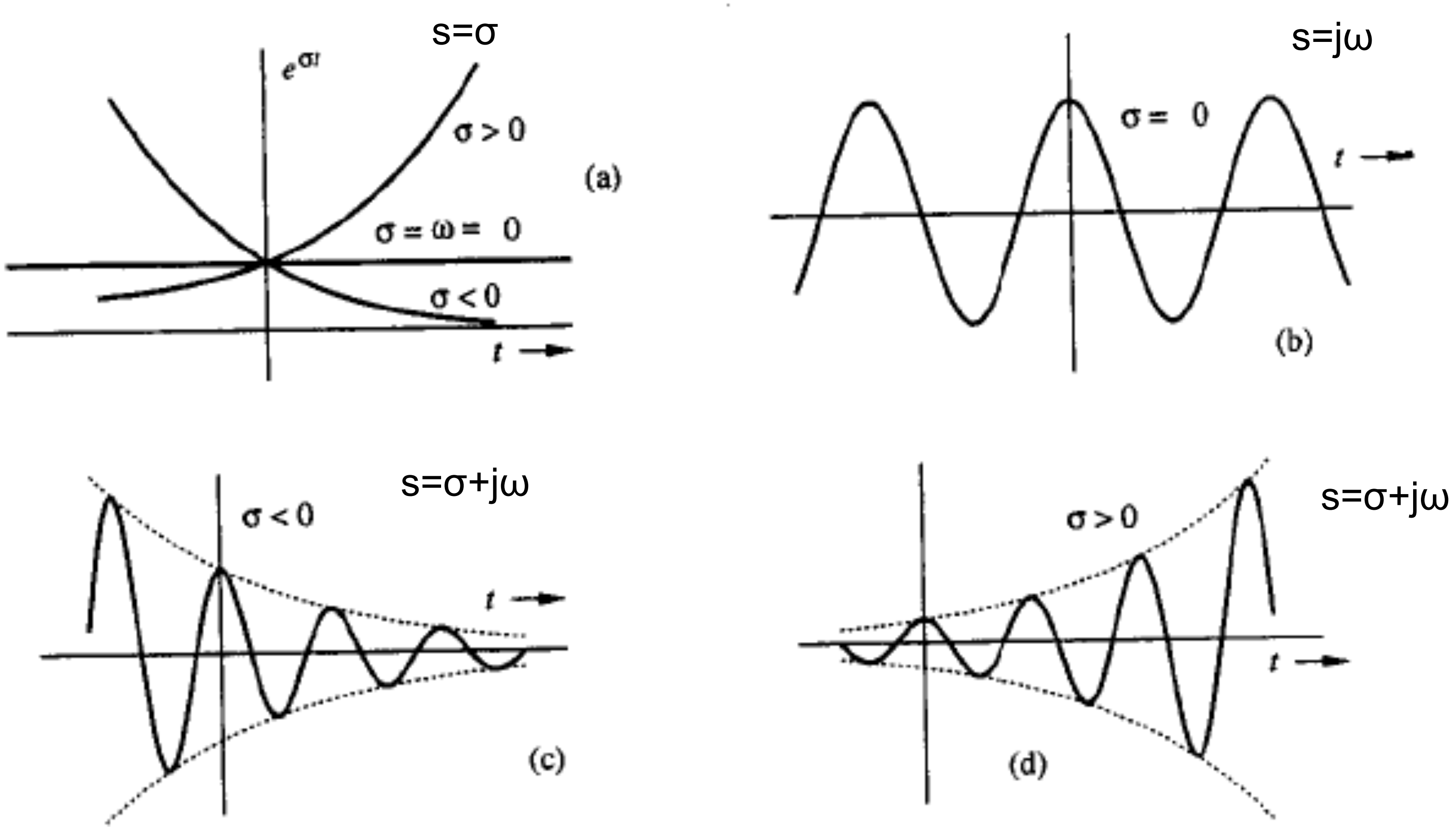
If $s^* = \sigma - j\omega$ (the conjugate of s), then

$$e^{s^*t} = e^{\sigma - j\omega} = e^{\sigma t}e^{-j\omega t} = e^{\sigma t}(\cos\omega t - j\sin\omega t) \tag{1.30b}$$

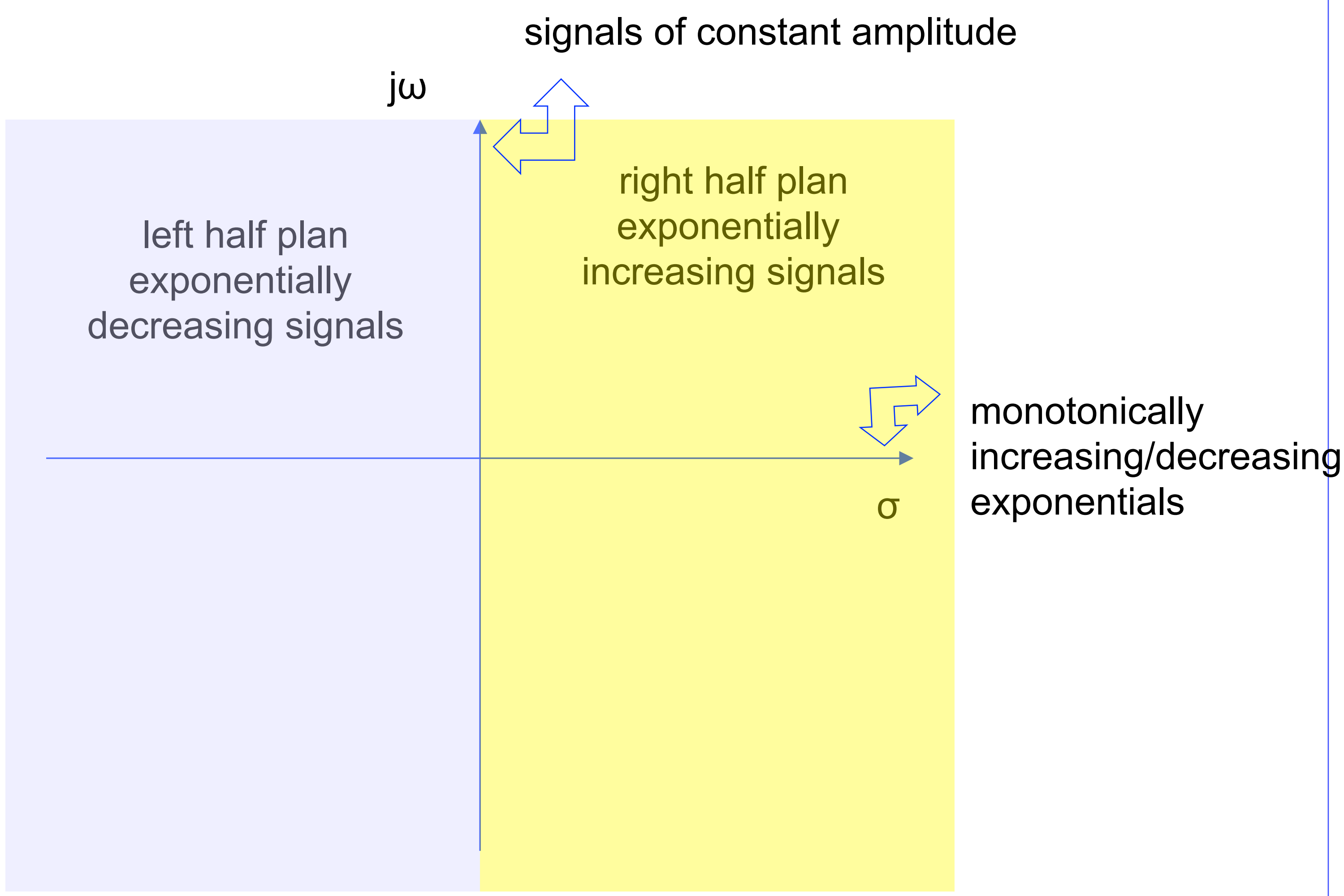
and

$$e^{\sigma t}\cos\omega t = \frac{1}{2}(e^{st} + e^{s^*t}) \tag{1.30c}$$

The exponential function



Complex frequency plan

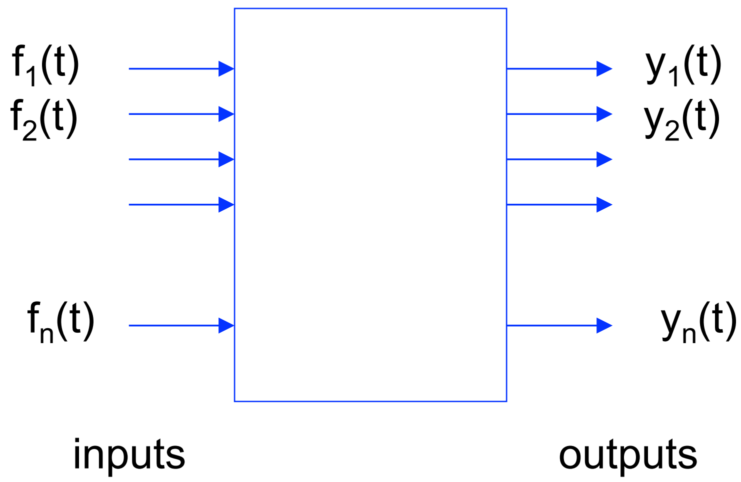


Basics of Linear Systems

2D Linear Systems

Systems

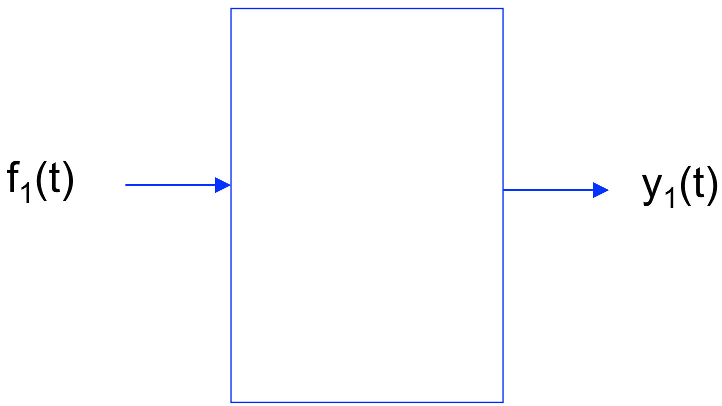
- A system is characterized by
  - inputs
  - outputs
  - rules of operation (mathematical model of the system)

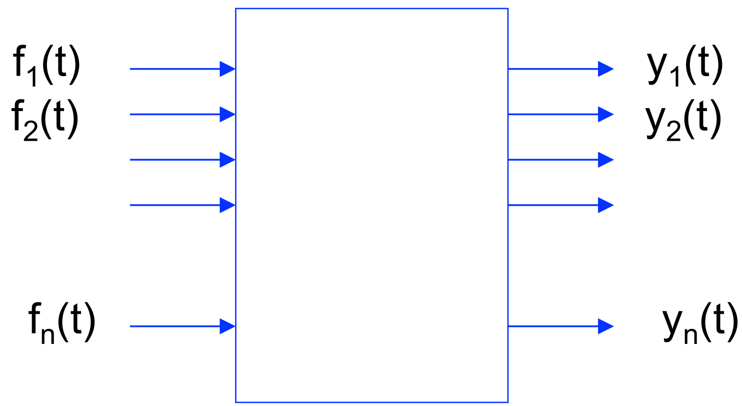


Systems

- Study of systems: mathematical modeling, analysis, design
  - Analysis: how to determine the system output given the input and the system mathematical model
  - design or synthesis: how to design a system that will produce the desired set of outputs for given inputs

- SISO: single input single output - MIMO: multiple input multiple output



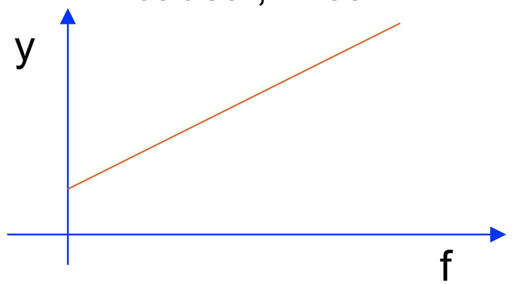


inputs outputs inputs outputs

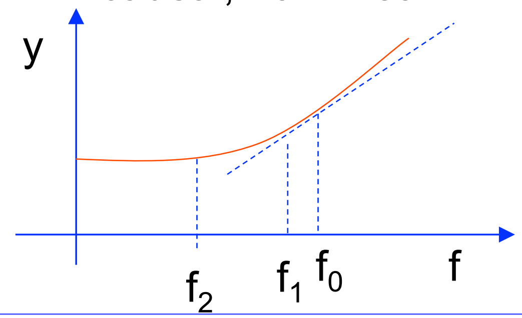
Response of a linear system

- Total response = Zero-input response + Zero-state response
  - The output of a system for t≥0 is the result of two independent causes: the initial conditions of the system (or system state) at t=0 and the input f(t) for t≥0.
  - Because of linearity, the total response is the sum of the responses due to those two causes
  - The zero-input response is only due to the initial conditions and the zero-state response is only due to the input signal
  - This is called decomposition property
- Real systems are *locally* linear
  - Respond linearly to small signals and non-linearly to large signals

causal, linear



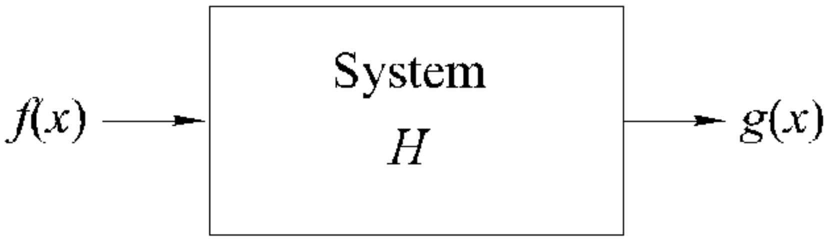
causal, non linear



locally linear around f_0

Review: Linear Systems

- We define a system as a unit that converts an input function into an output function

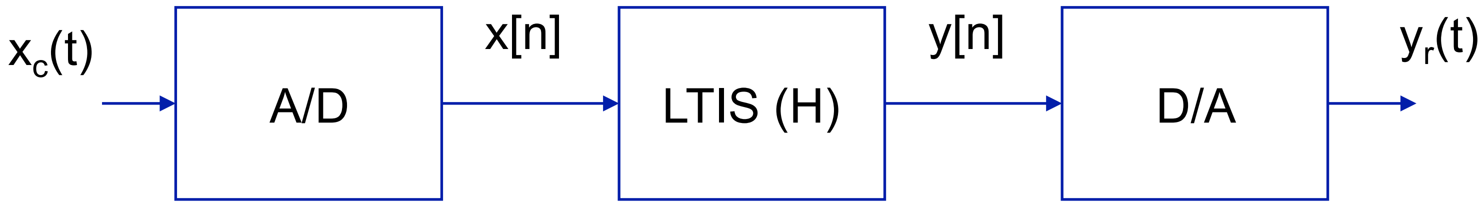


$$\boxed{g(x) = H[f(x)]}$$

Independent variable

System operator or Transfer function

Linear Time Invariant Discrete Time Systems



$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y_r(j\Omega) = H(j\Omega)X_c(j\Omega)$$

$$H(j\Omega) = \begin{cases} H(j\Omega) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$

IF
- The input signal is bandlimited
- The Nyquist condition for sampling is met
- The digital system is linear and time invariant

THEN
The overall continuous time system is equivalent to a LTIS whose frequency response is H.

Overview of Linear Systems

- Let $g_i(x) = H[f_i(x)]$

  where $f_i(x)$ is an arbitrary input in the class of all inputs $\{f(x)\}$, and $g_i(x)$ is the corresponding output.

- If

$$
\begin{aligned}
H[a_i f_i(x) + a_j f_j(x)] &= a_i H[f_i(x)] + a_j H[f_{ji}(x)] \\
&= a_i g_i(x) + a_j g_j(x)
\end{aligned}
$$

  Then the system H is called a *linear system*.

- A linear system has the properties of *additivity* and *homogeneity*.

Linear Systems

- The system H is called *shift invariant* if

$$\boxed{g_i(x) = H[f_i(x)] \text{ implies that } g_i(x+x_0) = H[f_i(x+x_0)]}$$

for all $f_i(x) \in \{f(x)\}$ and for all x_0.

- This means that offsetting the independent variable of the input by x_0 causes the same offset in the independent variable of the output. Hence, the input-output relationship remains the same.

Linear Systems

- The operator *H* is said to be *causal*, and hence the system described by *H* is a *causal system*, if there is no output before there is an input. In other words,

$$\boxed{f(x) = 0 \text{ for } x < x_0 \text{ implies that } g(x) = H[f(x)] = 0 \text{ for } x < x_0.}$$

- A linear system *H* is said to be *stable* if its response to any bounded input is bounded. That is, if

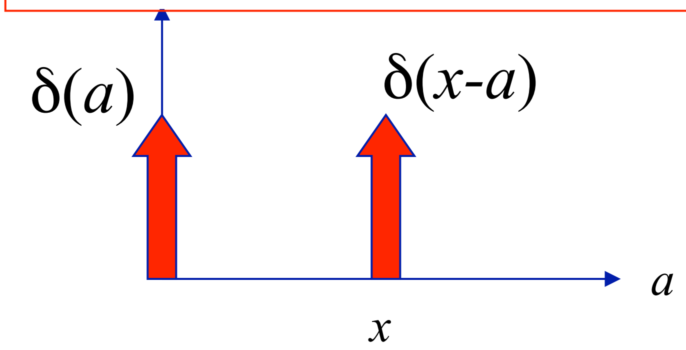
$$\boxed{|f(x)| < K \text{ implies that } |g(x)| < cK}$$

where *K* and *c* are constants.

Linear Systems

- A *unit impulse function*, denoted $\delta(a)$, is *defined* by the expression

$$\boxed{\int_{-\infty}^{\infty} f(\alpha)\delta(x-\alpha)d\alpha = f(x).}$$



- The response of a system to a unit impulse function is called the *impulse response* of the system.

$$h(x) = H[\delta(x)]$$

Linear Systems

- If *H* is a linear shift-invariant system, then we can find its response to any input signal f(x) as follows:

$$\boxed{g(x) = \int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha.}$$

- This expression is called the *convolution integral*. It states that the response of a linear, fixed-parameter system is completely characterized by the convolution of the input with the system impulse response.

Linear Systems

- Convolution of two functions of a continuous variable is defined as

$$\boxed{f(x)*h(x)=\int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha}$$

- In the discrete case

$$\boxed{f[n]*h[n]=\sum_{m=-\infty}^{\infty} f[m]h[n-m]}$$

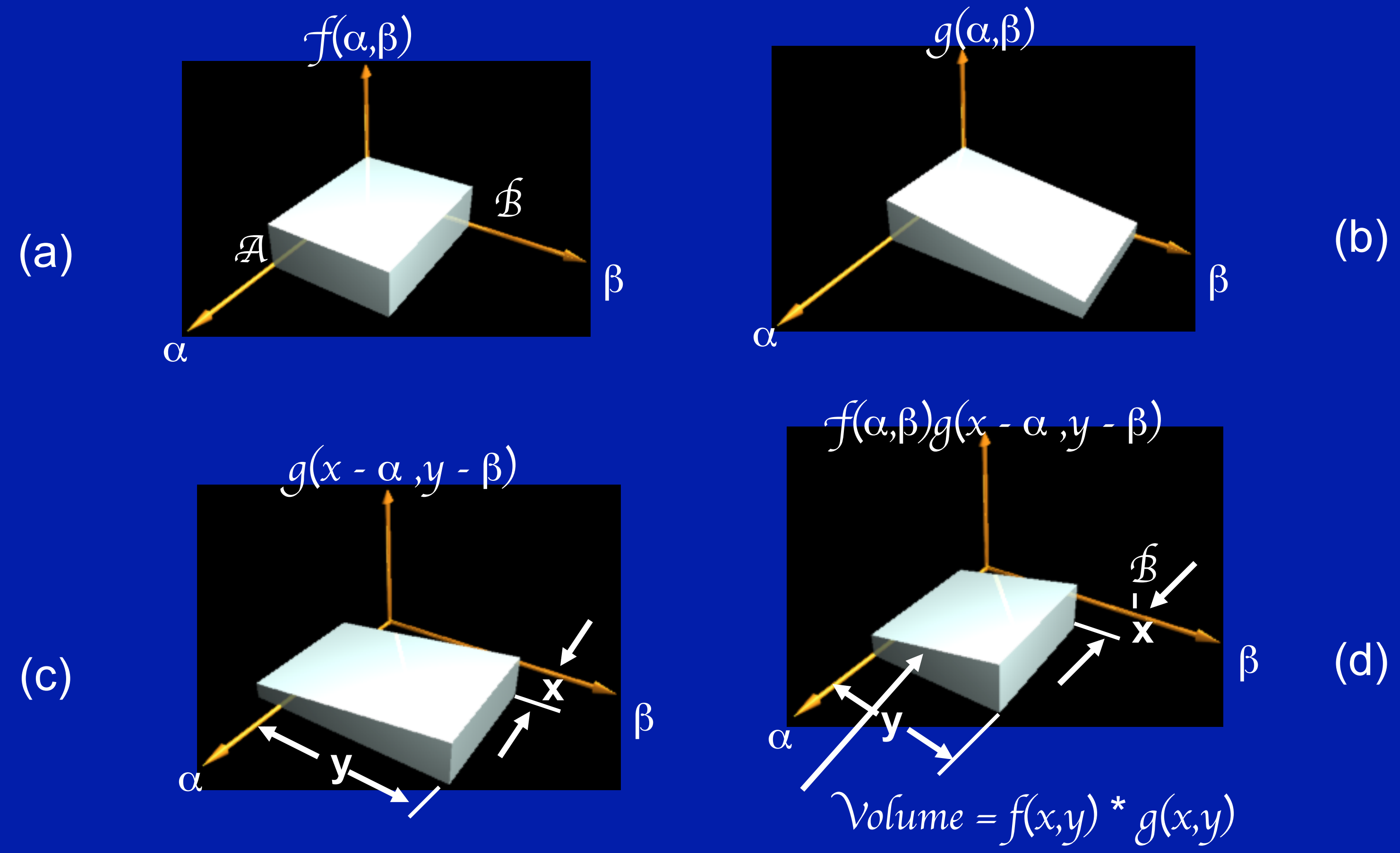
Linear Systems

- In the 2D discrete case

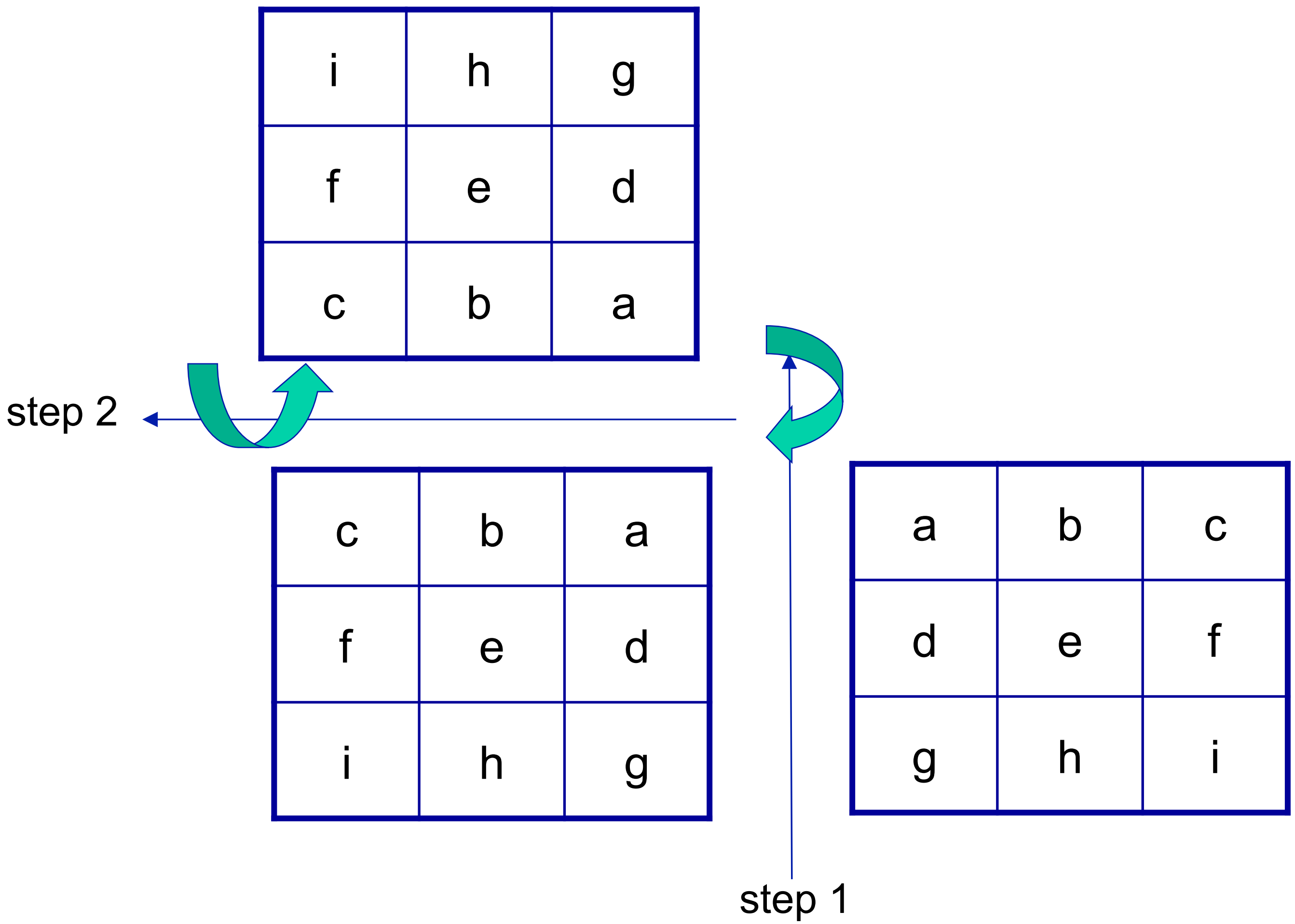
$$f[n_1, n_2] * h[n_1, n_2] = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f[m_1, m_2] h[n_1 - m_1, n_2 - m_2]$$

$h[n_1, n_2]$ is a linear filter.

Illustration of the folding, displacement, and multiplication steps needed to perform two-dimensional convolution



Matrix perspective



Convolution Example

$$f[n_1,n_2]**h[n_1,n_2]=\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty} f[m_1,m_2]h[n_1-m_1,n_2-m_2]$$

h

1	-1	-1
1	2	-1
1	1	1

Rotate

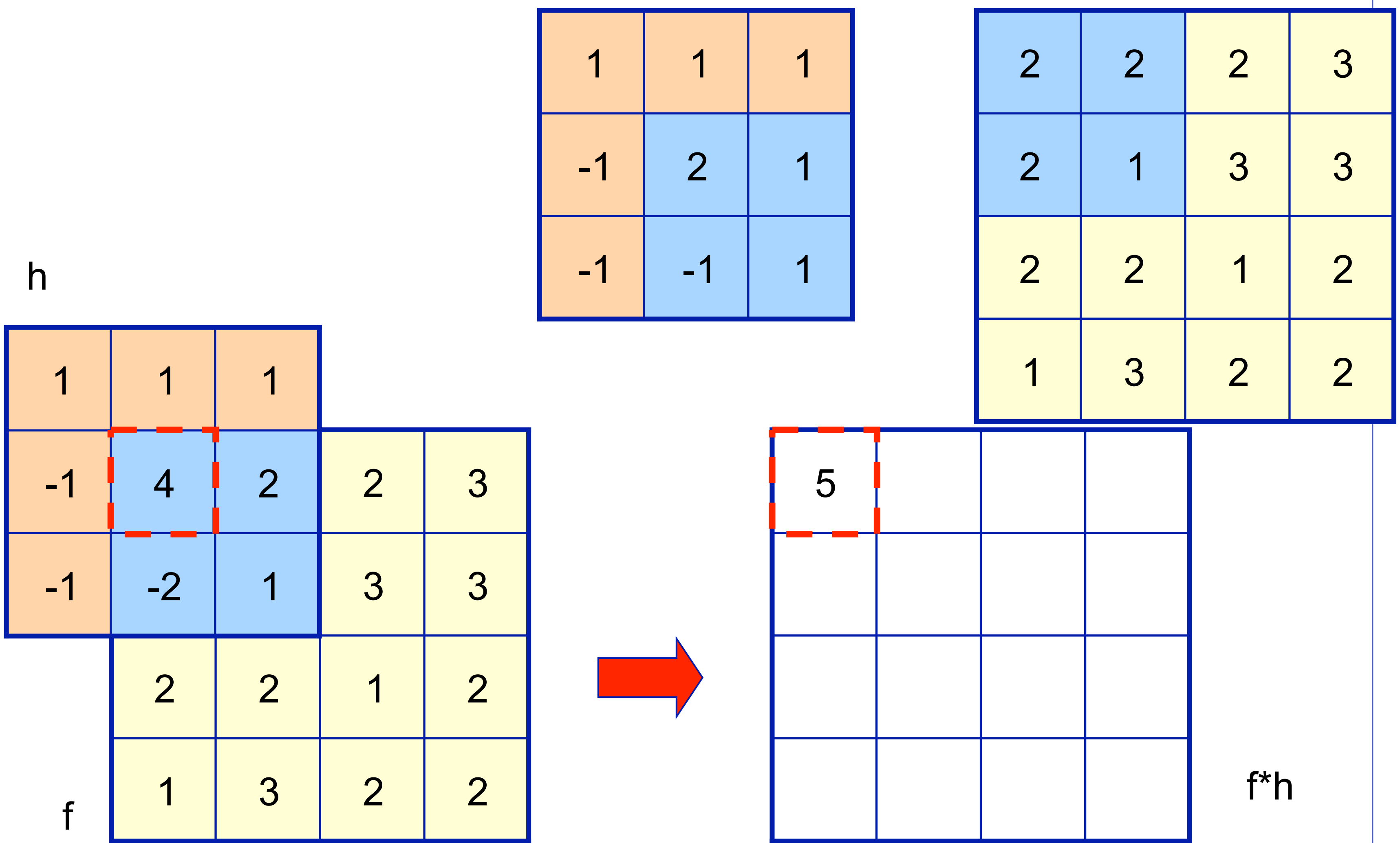
1	1	1
-1	2	1
-1	-1	1

f

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

From C. Rasmussen, U. of Delaware

Convolution Example

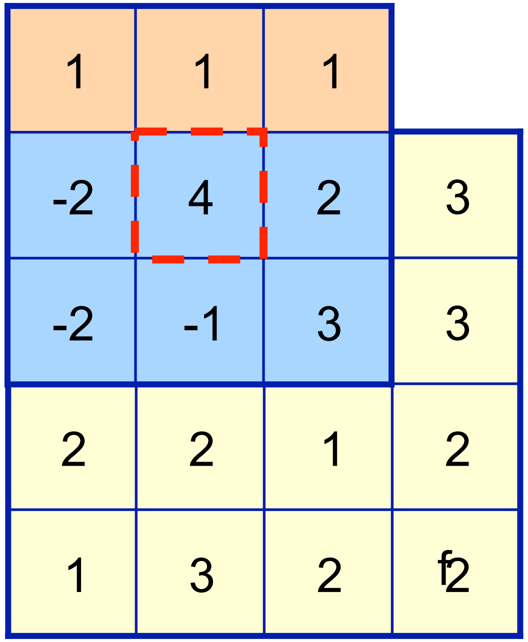


Convolution Example

h

1	1	1
-1	2	1
-1	-1	1

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2



5	4		

f*h

Convolution Example

1	1	1
-1	2	1
-1	-1	1

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

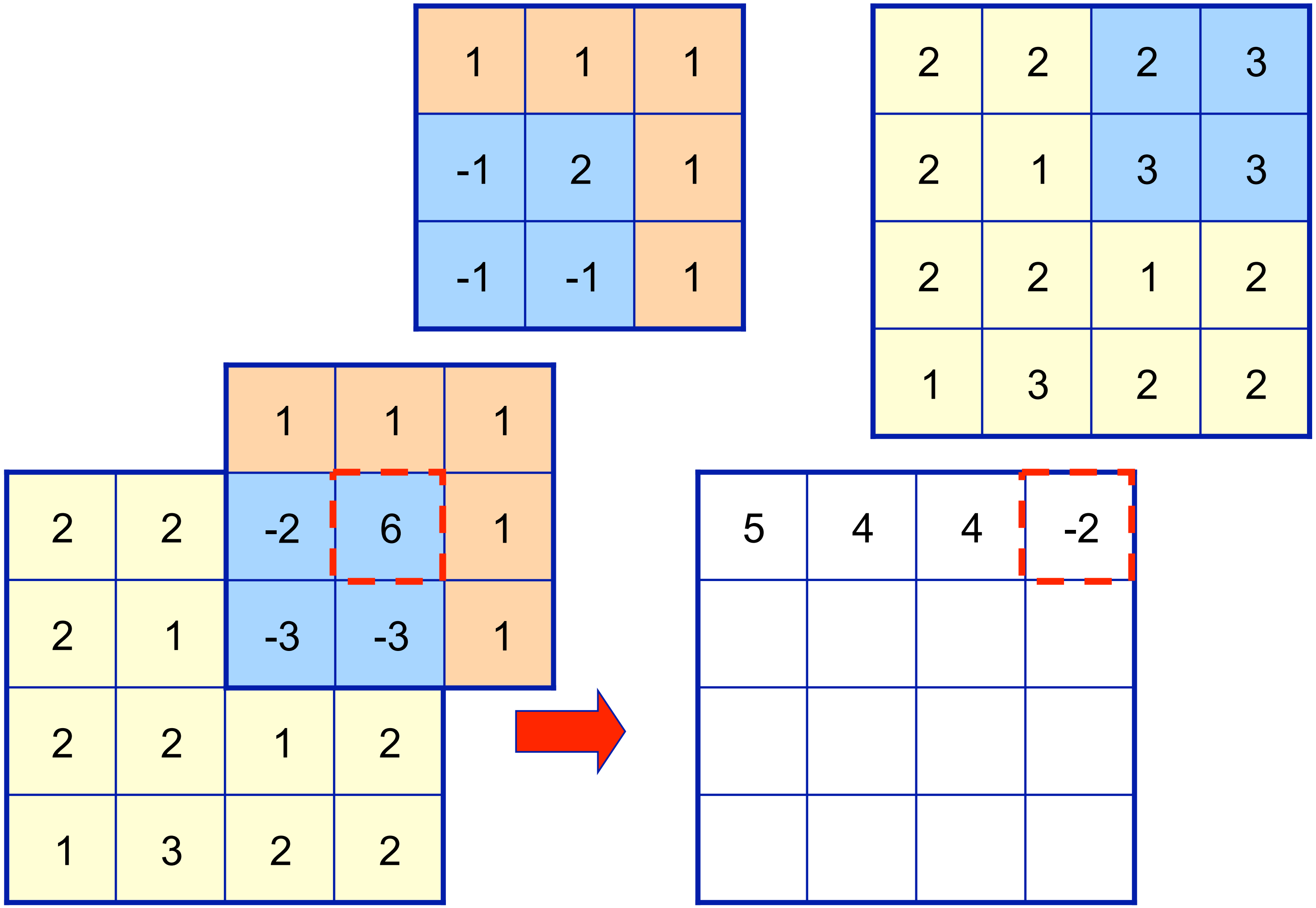
h	1	1	1
2	-2	4	3
2	-1	-3	3
2	2	1	2
1	3	2	2

f

5	4	4	

f*h

Convolution Example



f

f*h

Convolution Example

1	1	1
-1	2	1
-1	-1	1

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

h

1	2	2	2	3
-1	4	1	3	3
-1	-2	2	1	2
	1	3	2	2

f

5	4	4	-2
9			

f*h

Convolution Example

1	1	1
-1	2	1
-1	-1	1

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

h

2	2	2	3
-2	2	3	3
-2	-2	1	2
1	3	2	2

f

5	4	4	-2
9	6		

f*h

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Linear, Shift-invariant Systems and Fourier Transforms

- Linear systems underly much of what happens in nature and are used in instrumentation to make measurements of various kinds.
- We will define linear systems formally and derive some properties.
- We will show that exponentials are natural basis functions for describing linear systems.
- Fourier transforms (CT/CA), Fourier Series (CT/CA + periodic in time), and Discrete Fourier Transforms (DT/CA + periodic in time and in frequency) will be defined.
- We will look at an application that demonstrates:
  1. Definition of a power spectrum from the DFT.
  2. Statistics of the power spectrum and how we generally can derive statistics for any *estimator* or *test statistic*.
  3. The notion of an *ensemble* or parent population from which a given set of measurements is drawn (a *realization* of the process).
  4. Investigate a "detection" problem (finding a weak signal in noise) and assess the false-alarm probability.

Types of Signals

By “signal” we simply mean a quantity that is a function of some independent variable. For simplicity, we will often consider a single independent variable (time) e.g. x(t). Later we will consider 2 or more dimensions of general variables. .

A signal is characterized by an *amplitude* as a function of *time* and 4 kinds of signals can be defined depending on whether the time and amplitude are discrete or continuous.

	TIME	
AMPLITUDE	discrete	continuous
discrete	Digital Signals (m bits per sample)	CT, DA (m bits)
continuous	DT, CA (∞ bits per sample)	Analog Signals (∞ bits per sample)

Quantum mechanics says there are only DT, DA signals but much of what we will do is in the classical regime.

Examples

CT/CA	Light intensity from a star (ignore photons and counting statistics)
CT/DA	Earth's human population
DT/CA	Intensity of the moon at times of the full moon $\lvert t_{j+1} - t_j \rvert \sim 28$ days
DT/DA	Earth's population at times of the full moon

Approach taken in the course

Theoretical treatments (analytical results) will generally be applied to DT/CA signals, for simplicity.

For the most part, we will consider *analog signals* and *DT/CA signals*, the latter as an approximation to digital signals. For most analyses, the discreteness in time is a strong influence on what we can infer from the data. Discreteness in amplitude is not so important, except insofar as it represents a source of error (*quantization noise*). However, we will consider the case of extreme quantization into *one* bit of information and derive estimators of the autocovariance.

Generically, we refer to a DT signal as a *time series* and the set of all possible analyses as *"time series analysis"*. However, most of what we do is applicable to any sequence of data, regardless of what the independent variable is.

Often, but not always, we can consider a DT signal to be a sampled version of a CT signal (counter examples: occurrence times of discrete events such as clock ticks, heartbeats, photon impacts, etc.).

Nonuniform sampling often occurs and has a major impact on the structure of an algorithm.

We will consider the effects of quantization in digital signals.

Linear Systems

Consider a linear differential equation in y

$$f(y, y', y'', \ldots) = x(t), \qquad y' \equiv \frac{dy}{dt}, \text{etc.}$$

whose solutions include a complete set of orthogonal functions. We can represent the relationship of $x(t)$ (the driving function) and $y(t)$ (the output) in transformational form:

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{system}\\ h(t)\end{matrix}} \longrightarrow y(t)$$

where $h(t)$ describes the action of the system on the input x to produce the output y. We *define* $h(t)$ to be the response of the system to a δ-function input. **Thus, $h(t)$ is the "impulse response" or Green's function of the system.**

We wish to impose *linearity* and *shift invariance* on the systems we wish to consider:

Linearity:

If $x_1 \longrightarrow y_1$ and $x_2 \longrightarrow y_2$ then $ax_2 + bx_2 \longrightarrow ay_1 + by_2,$ for any a, b

E.g. $y = x^2$ is *not* a linear operation.

Time or shift invariance (stationarity)

If $x(t) \longrightarrow y(t),$ then $x(t + t_0) \longrightarrow y(t + t_0)$ for any t_0

The output "shape" depends on the "shape" of the input, not on the time of occurrence.

Singularity Functions

We need some useful singularity "functions":

1. $\delta(t)$ defined as a functional

$$z(t) \equiv \int dt' \; \delta(t'-t) \; z(t') \qquad \text{and} \qquad \int_a^b dt' \; \delta(t'-t) = \begin{cases} 1 & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

2. Loosely speaking, $\delta(0) \longrightarrow \infty, \delta(t \neq 0) \longrightarrow 0$; So $\delta(t)$ has finite (unit) area.

3. $U(t)$ unit step function (or Heaviside function)

$$U(t) \; = \; \int_0^\infty dt' \; \delta(t'-t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad \text{and} \qquad \frac{dU(t)}{dt} = \delta(t) \tag{2}$$

$$\Rightarrow U(t-t_0) \; = \; \int_{t_0}^\infty dt' \; \delta(t'-t) = \begin{cases} 1 & t \geq t_0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Convolution theorem

By definition

$$\delta(t) \longrightarrow h(t)$$

Using *linearity* we have

$$a\,\delta(t) \longrightarrow a\,h(t)$$

Let $a = x(t')$ then

$$x(t')\,\delta(t) \longrightarrow x(t')\,h(t)$$

By *shift invariance* we have

$$\delta(t - t') \longrightarrow h(t - t')$$

Combining L + SI,

$$x(t')\,\delta(t - t') \longrightarrow x(t')\,h(t - t')$$

But, again by linearity, we can sum many terms of this kind. So, integrating over all t':

$$\int_{-\infty}^{\infty} dt'\,x(t')\,\delta(t - t') \longrightarrow \int_{-\infty}^{\infty} dt'\,x(t')\,h(t - t')$$

But by definition of $\delta(t), \mathrm{LHS} = x(t)$, so

$$x(t) \longrightarrow \int_{-\infty}^{\infty} dt'\,x(t')\,h(t - t') = y(t)$$

By a change of variable on the RHS to $\tilde{t} = t - t'$ we also have

$$x(t) \longrightarrow \int_{-\infty}^{\infty} dt'\,x(t - t')\,h(t') = y(t)$$

Any linear, shift invariant system can be described as the convolution of its impulse response with an arbitrary input.

Using the notation $*$ to represent the integration, we therefore have

$$y(t) = x * h = h * x$$

Properties:

1. Convolution commutes:
$$\int dt'\, h(t')x(t-t') = \int dt'\, h(t-t')x(t')$$
2. Graphically, convolution is “invert, slide, and sum”
3. The general integral form of $*$ implies that, usually, information about the input is lost since $h(t)$ can “smear out” or otherwise preferentially weight portions of the input.
4. Theoretically, if the system response $h(t)$ is known, the output can be ‘deconvolved’ to obtain the input. But this is unsuccessful in many practical cases because: a) the system $h(t)$ is not known to arbitrary precision or, b) the output is not known to arbitrary precision.

Why are linear systems useful?

1. Filtering (real time, offline, analog, digital, causal, acausal)
2. Much *signal processing and data analysis* consists of the application of a linear operator (smoothing, running means, Fourier transforms, generalized channelization, ...)
3. Natural processes can often be described as linear systems:
   - Response of the Earth to an earthquake (propagation of seismic waves)
   - Response of an active galactic nucleus swallowing a star (models for quasar light curves)
   - Calculating the radiation pattern from an ensemble of particles
   - Propagation of electromagnetic pulses through plasmas
   - Radiation from gravitational wave sources (in weak-field regime)

We want to be able to attack the following kinds of problems:

1. *Algorithm development:* Given $h(t)$, how do we get $y(t)$ given $x(t)$ ("how" meaning to obtain *efficiently*, hardware vs. software, etc.) $t\ vs.\ f$ domain?
2. *Estimation:* To achieve a certain kind of output, such as parameter estimates subject to "constraints"(e.g. minimum square error), how do we design $h(t)$? (least squares estimation, prediction, interpolation)
3. *Inverse Theory:* Given the output (e.g. a measured signal) and assumptions about the input, how well can we determine $h(t)$ (parameter estimation)? How well can we determine the original input $x(t)$? Usually the output is corrupted by *noise*, so we have

$$y(t) = h(t) * x(t) + \epsilon(t).$$

The extent to which we can determine h and x depends on the *signal-to-noise ratio*:
$\langle (h * x)^2 \rangle^{1/2} / \langle \epsilon^2 \rangle^{1/2}$ where $\langle\ \rangle$ denotes averaging brackets.

We also need to consider *deterministic*, *chaotic* and *stochastic* systems:

- Deterministic $\Rightarrow$ predictable, precise (noiseless) functions
- Chaotic $\Rightarrow$ deterministic but apparently stochastic processes
- Stochastic $\Rightarrow$ not predictable (random)
- Can have systems with stochastic input and/or stochastic system response $h(t) \longrightarrow$ stochastic output.

Not all processes arise from linear systems but linear concepts can still be applied, along with others.

Natural Basis Functions for Linear Systems

In analyzing LTI systems we will find certain basis functions, **exponentials**, to be specially useful. Why is this so?

Again consider an LTI system $y = h * x$. Are there input functions that are unaltered by the system, apart from a multipicative constant? Yes, these correspond to the **eigenfunctions** of the associated differential equation.

We want those functions $\phi(t)$ for which

$$y(t) = \phi * h = H\phi \qquad \text{where } H \text{ is just a number}$$

That is, we want

$$y(t) = \int dt'\, h(t')\, \phi(t - t') = H\, \phi(t)$$

This can be true if $\phi(t - t')$ is *factorable*:

$$\phi(t - t') = \phi(t)\psi(t')$$

where $\psi(t')$ is a constant in t but can depend on t'.

We constrain $\psi(t')$ with:

$$i)\ \phi(t-t')_{t'=0} \ \equiv \ \phi(t) = \phi(t)\psi(0) \Rightarrow \psi(0) = 1$$

$$ii)\ \phi(t-t')_{t=t'} \ \equiv \ \phi(0) = \phi(t)\psi(t) \Rightarrow \psi(t) = \frac{\phi(0)}{\phi(t)}$$

$$iii)\ \phi(t-t')_{t=0} \ \equiv \ \phi(-t') = \phi(0)\psi(t') \Rightarrow \psi(t') = \frac{\phi(-t')}{\phi(0)}$$

Now ii) and iii) automatically satisfy i). With no loss of generality we can set

$$\phi(0) = 1 \Rightarrow \psi(t) = \frac{1}{\phi(t)} = \phi(-t)$$

We want functions whose time reverses are their reciprocals. These are exponentials (or $2^{st},\ a^{st}$, etc):

$$\phi(t) = e^{st}$$

Check that e^{st} behaves as required:

$$\begin{aligned} y = \phi * h &= \int dt' \, \phi(t-t') h(t') \\ &= \int dt' \, e^{s(t-t')} h(t') \\ &= e^{st} \int dt' \, e^{-st'} h(t') \\ &= e^{st} \, H(s) \end{aligned}$$

So $\phi \longrightarrow \phi \, H(s)$

$\phi =$ eigenvector $\qquad H =$ eigenvalue

Note $H(s)$ depends on s and h.

Two kinds of systems

Causal

$h(t) = 0$ for $t < 0$

output now depends only on past values of input

$$H(s) = \int_0^\infty dt'\, e^{-st'} h(t') \quad \textbf{Laplace transform}$$

Acausal

$h(t)$ not necessarily 0 for $t < 0$

$$H(s) = \int_{-\infty}^\infty dt'\, e^{-st'} h(t')|_{s=i\omega} \quad \textbf{Fourier transform}$$

Exponentials are useful for describing the action of a linear system because they "slide through" the system. If we can describe the actual input function in terms of exponential functions, then determining the resultant output becomes trivial. This is, of course, the essence of Fourier transform treatments of linear systems and their underlying differential equations.

Convolution Theorem in the Transform Domain

Consider input $\longrightarrow$ output

$$a\, e^{i\omega t} \longrightarrow a\, H(i\omega)\, e^{i\omega t} \qquad \textbf{linearity}$$

We can choose an arbitrary a, so let's use

$$\tilde{X}(\omega)\, e^{i\omega t} \longrightarrow \tilde{X}(\omega)\, H(i\omega)\, e^{i\omega t} \tag{4}$$

By linearity we can superpose these inputs. So <u>integrate</u> over ω with a judicious choice of normalization $(1/2\pi)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{X}(\omega)\, e^{i\omega t} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{X}(\omega)\, H(i\omega)\, e^{i\omega t}$$

Let's call $LHS\, x(t)$ and the $RHS\, y(t)$:

$$x(t) \equiv \frac{1}{2\pi}\int d\omega\, \tilde{X}(\omega)\, e^{i\omega t} \qquad y(t) = \frac{1}{2\pi}\int d\omega\, \tilde{X}(\omega)\, H(i\omega)\, e^{i\omega t}$$

What is the relationship of $\tilde{X}(\omega)$ to $x(t)$?

Multiply $x(t)$ by $e^{-i\omega' t}$ and integrate to get

$$\int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega' t} = \boxed{\frac{1}{2\pi}\int dw\, \tilde{X}(\omega)\, \int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega')t}} \tag{5}$$

Now the integral over t on the RHS gives

$$\int_{-\infty}^{\infty} dt\ e^{i(\omega-\omega')t} \longrightarrow \begin{cases} 0 & \omega \neq \omega' \\ \infty & \omega = \omega' \end{cases} \tag{6}$$

i.e. just like a delta function. So (invoking the correct weighting factor, or normalization)

$$\int_{-\infty}^{\infty} dt\ e^{i(\omega-\omega')t} = 2\pi\ \delta(\omega-\omega') \tag{7}$$

Therefore the boxed RHS becomes

$$\int dw\ \tilde{X}(\omega)\, \delta(\omega-\omega') = \tilde{X}(\omega'). \tag{8}$$

Therefore we have

$$\tilde{X}(\omega') = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega' t}. \tag{9}$$

and the inverse relation

$$x(t) = \frac{1}{2\pi} \int dw\ \tilde{X}(\omega)\ e^{-i\omega t}.$$

We say that $x(t)$ and $\tilde{X}(\omega)$ are a Fourier transform pair.

Going back to equation **??** it is clear that the FT of $y(t)$ is the integrand on the RHS so

$$\tilde{Y}(\omega) = \tilde{X}(\omega)\ H(i\omega).$$

Usually we rewrite this as $\tilde{H}(\omega) \equiv H(i\omega)$ so

$$\tilde{Y}(\omega) = \tilde{X}(\omega)\ \tilde{H}(\omega)$$

Therefore, we have shown that

$$\begin{aligned} y(t) &= x(t) * h(t) \quad \text{convolution} \\ \tilde{Y}(\omega) &= \tilde{X}(\omega)\, \tilde{H}(\omega) \quad \text{multiplication} \end{aligned}$$

This product relation is extremely useful for

1. Deriving impulse responses of composite systems.
2. In discrete form (i.e. digitially) for implementing convolutions: ω - domain multiplications can be much faster than t - domain convolutions

Fourier Transform Relations

Here we summarize the Fourier transform relations for a variety of signals. Let $f(t)$ be a continous, aperiodic function and $\tilde{F}(f)$ be its Fourier transform. We denote their relations

$$f(t) = \int_{-\infty}^{\infty} df\, \tilde{F}(f) e^{+2\pi i f t}$$

$$\tilde{F}(f) = \int_{-\infty}^{\infty} dt\, f(t) e^{-2\pi i f t},$$

as $f(t) \Longleftrightarrow \tilde{F}(f)$.

We need to consider the following functions: 1. The Dirac delta ‘function’

$$\delta(t)$$

2. A periodic train of delta functions (‘bed of nails’) with period Δ:

$$s(t, \Delta) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$$

3. The periodic extension $f_p(t)$ of a function $f(t)$ defined using the bed of nails function:

$$f_p(t) = f(t) * s(t, \Delta) \qquad * \text{ denotes convolution}$$

4. An aperiodic function $f(t)$ sampled at intervals Δt:

$$f_s(t) = f(t) \times s(t, \Delta t)$$

5. The sampled and periodically extended signal:

$$f_{ps}(t) = f_p(t) \times s(t, \Delta t)$$

1D Fourier Transform Theorems

Function		Fourier transform	
1	$\Longleftrightarrow$	$\delta(f)$	
$\delta(t)$	$\Longleftrightarrow$	1	
$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$	$\Longleftrightarrow$	$\tilde{S}(f) = \Delta^{-1} \sum_{-\infty}^{\infty} \delta(f - k/\Delta)$	Bed of nails function
$y(t) = x(t) * h(t)$	$\Longleftrightarrow$	$\tilde{X}(f)\tilde{H}(f)$	Convolution
$C_x(\tau) \equiv \int dt\, x^*(t)x(t+\tau)$	$\Longleftrightarrow$	$\|\tilde{X}(f)\|^2$	Correlation
$x(t - t_0)$	$\Longleftrightarrow$	$e^{-i\omega t_0}\tilde{X}(f)$	Shift theorem
$e^{+i2\pi f_0 t}x(t)$	$\Longleftrightarrow$	$\tilde{X}(f - f_0)$	Shift theorem
$x(at)$	$\Longleftrightarrow$	$a^{-1}\tilde{X}(f/a)$	Scaling theorem
$\tilde{X}(t)$	$\Longleftrightarrow$	$x(-f)$	duality theorem
$x^*(t)$	$\Longleftrightarrow$	$\tilde{X}^*(-f)$	Conjugation
$x^*(t) = x(t)$	$\Longleftrightarrow$	$\tilde{X}^*(-f) = \tilde{X}(f)$	Hermiticity
$\int_{-\infty}^{\infty} dt\, \|x(t)\|^2$	=	$\int_{-\infty}^{\infty} df\, \|\tilde{X}(f)\|^2$	Parseval's theorem
$\frac{dx}{dt}$	$\Longleftrightarrow$	$2\pi i f \tilde{X}(f)$	Derivative theorem
$\int dt'\, X(t')$	$\Longleftrightarrow$	$(2\pi i f)^{-1}\tilde{X}(f)$	Integration theorem
$x(t) = \sum_m x_m \frac{\sin 2\pi\Delta f(t - m\delta t)}{2\pi\Delta f(t - m\delta t)}$	$\Longleftrightarrow$	$\sum_m x_m e^{-2\pi i m f \delta t}\Pi\left(\frac{f}{2\Delta f}\right)$	Sampling theorem Bandlimited Δf = half BW. $\Pi(x)$ = rectangle function
$x_p(t) = x(t) * s(t)$	$\Longleftrightarrow$	$\tilde{X}(f)\tilde{S}(f)$	Periodic in time
$x_p(t) = \sum_k a_k e^{2\pi i k t/\Delta}$ where $a_k \equiv \Delta^{-1}\tilde{X}(k/\Delta)$	$\Longleftrightarrow$	$\Delta^{-1}\sum_k \tilde{X}(k/\Delta)\delta(f - k/\Delta)$	Fourier series

Points

1. You can bootstrap from a few basic FT pairs by using the FT theorems
2. Narrow functions in one domain are wide in another (Uncertainty Principle, related to the scaling theorem).
3. Functions with sharp edges in one domain are oscillatory in the other (Gibbs phenomenon)
4. **Derivative theorem**:

$$\begin{aligned} f(t) &\Longleftrightarrow \tilde{F}(f) \\ \frac{df}{dt} &\Longleftrightarrow 2\pi i f \tilde{F}(f). \end{aligned} \tag{10}$$

5. **Integration theorem**:

$$\begin{aligned} f(t) &\Longleftrightarrow \tilde{F}(f) \\ \int_t dt' f(t' &\Longleftrightarrow (2\pi i f)^{-1} \tilde{F}(f). \end{aligned} \tag{11}$$

6. Consider a noisy signal, like white noise (which has a constant average FTbut a realization of white noise is noisy in both domains). Differentiation of the noise **increases** the high-frequency components and thus increases the noise relative to any signal.
7. Integration of the noise **reduces** the high frequency components. "Smoothing" (low-pass filtering) of data is closely related to integration and in fact reduces high-frequency components.

Gaussian Functions

Why useful and extraordinary?

1. We have the fundamental FT pair:
$$e^{-\pi t^2} \Longleftrightarrow e^{-\pi f^2}$$
This can be obtained using the FT definition and by completing the square. Once you know this FT pair, many situations can be analyzed without doing a single integral.

2. The Gaussian is one of the few functions whose shape is the same in both domains.

3. The width in the time domain (FWHM = full width at half maximum) is
$$\Delta t = \frac{2\sqrt{\ln 2}}{\sqrt{\pi}} = 0.94$$

4. The width in the frequency domain $\Delta\nu$ is the same.

5. Then
$$\Delta t \Delta\nu = \frac{4\ln 2}{\pi} = 0.88 \sim 1.$$

6. Now consider a scaled version of the Gaussian function: Let $t \to t/T$. The scaling theorem then says that
$$e^{-\pi(t/T)^2} \Longleftrightarrow Te^{-\pi(fT)^2}.$$
The time-bandwidth product is the same as before since the scale factor T cancels. After all, $\Delta t\Delta\nu$ is dimensionless!

7. The Gaussian function has the smallest time-bandwidth product (minimum uncertainty wave packet in QM)

8. Central Limit Theorem: A quantity that is the sum of a large number of statistically independent quantities has a probability density function (PDF) that is a Gaussian function. We will state this theorem more precisely when we consider probability definitions.

9. Information: The Gaussian function, as a PDF, has maximum entropy compared to any other PDF. This plays a role in development of so-called maximum entropy estimators.

Chirped Signals

Consider the chirped signal $e^{i\omega t}$ with $\omega = \omega_0 + \alpha t$, (a linear sweep in frequency).
We write the signal as:

$$v(t) = e^{i\omega t} = e^{i(\omega_0 t + \alpha t^2)}.$$

The name derives from the sound that a swept audio signal would make.

1. Usage or occurrence:
   (a) wave propagation through dispersive media
   (b) objects that spiral in to an orbital companion, producing chirped gravitational waves
   (c) swept frequency spectrometers, radar systems
   (d) dedispersion applications (pulsar science)
2. We can use the convolution theorem to write
$$\begin{aligned}\tilde{V}(f) &= \mathrm{FT}\left\{e^{i(\omega_0 t + \alpha t^2)}\right\} \\ &= \mathrm{FT}\left\{e^{i\omega_0 t}\right\} * \mathrm{FT}\left\{e^{i(\alpha t^2)}\right\} \\ &= \delta(f - f_0) * \mathrm{FT}\left\{e^{i(\alpha t^2)}\right\}.\end{aligned}$$
3. The FT pair for a Gaussian function would suggest that the following is true:
$$e^{-i\pi t^2} \Longleftrightarrow e^{-i\pi f^2}.$$
4. Demonstrate that this is true!
5. Within constants and scale factor, the FT of the chirped signal is therefore
$$\tilde{V}(f) \propto e^{i(\pi(f-f_0)^2)}$$

Three Classes of Fourier Transform

Fourier Transform (FT): applies to continuous, aperiodic functions:

$$
\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} df\, e^{2\pi i f t} \tilde{F}(f) \\
\tilde{F}(f) &= \int_{-\infty}^{\infty} dt\, e^{-2\pi i f t} f(t)
\end{aligned}
$$

Basis functions $e^{2\pi i f t}$ are orthornomal on$[-\infty, \infty]$

$$
\int_{\infty}^{\infty} dt\, e^{2\pi i f t} e^{-2\pi i f t} = \delta(t)
$$

Fourier Series: applies to continuous, periodic functions with period P:

$$
\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} e^{2\pi i (n/P) t} \tilde{F}_n \\
\tilde{F}_n &= \frac{1}{P} \int_0^P dt\, e^{-2\pi i (n/P) t} f(t)
\end{aligned}
$$

$f(t)$ periodic with period P, $\quad$ orthonormal on $[0, P]$

$$
\int_0^P dt\, e^{2\pi i (n/P) t} e^{-2\pi i (n'/P) t} = \delta_{n,n'}
$$

Discrete Fourier Transform (DFT): applies to discrete time and discrete frequency functions:

$$
\begin{aligned}
f_k &= \sum_{n=0}^{\infty} e^{2\pi i n k/N} \tilde{F}_n \\
\tilde{F}_n &= \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i n k/N} f_k
\end{aligned}
$$

$$f_k, \tilde{F}_n \text{ periodic with period } N, \quad \text{orthonormal on } [0, N]$$

$$\sum_{n=0}^{N-1} e^{2\pi i n k/N} e^{-2\pi i n k'} = \delta_{k,k'}$$

The Fourier transform is the most general because the other two can be derived from it. The DFT is not "just" a sampled version of the FT. Nontrivial consequences take place upon digitization, as we shall see.

2
Fourier Series and Fourier Transform

2.1 INTRODUCTION

Fourier series is used to get frequency spectrum of a time-domain signal, when signal is a periodic function of time. We have seen that the sum of two sinusoids is periodic provided their frequencies are integer multiple of a fundamental frequency, w_0.

2.2 TRIGONOMETRIC FOURIER SERIES

Consider a signal $x(t)$, a sum of sine and cosine function whose frequencies are integral multiple of w_0

$$x(t) = a_0 + a_1 \cos(w_0 t) + a_2 \cos(2w_0 t) + \cdots$$
$$b_1 \sin(w_0 t) + b_2 \sin(2w_0 t) + \cdots$$
$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nw_0 t) + b_n \sin(nw_0 t)) \tag{1}$$

$a_0,\ a_1, \ldots,\ b_1,\ b_2, \ldots$ are constants and w_0 is the fundamental frequency.

Evaluation of Fourier Coefficients

To evaluate a_0 we shall integrate both sides of eqn. (1) over one period $(t_0, t_0 + T)$ of $x(t)$ at an arbitrary time t_0

$$\int_{t_0}^{t_0+T} x(t)dt = \int_{t_0}^{t_0+T} a_0 dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos(nw_0 t)dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin(nw_0 t)dt$$

Since $\int_{t_0}^{t_0+T} \cos(nw_0 dt) = 0$

$$\int_{t_0}^{t_0+T} \sin(nw_0 dt) = 0$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)dt \tag{2}$$

To evaluate a_n and b_n, we use the following result:

$$\int_{t_0}^{t_0+T} \cos(nw_0 t) \cos(mw_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \neq 0 \end{cases}$$

Multiply eqn. (1) by sin (mw_0t) and integrate over one period

$$\int_{t_0}^{t_0+T} x(t)\sin(mw_0t)dt = a_0\int_{t_0}^{t_0+T}\sin(mw_0t)dt + \sum_{n=1}^{\infty} a_n\int_{t_0}^{t_0+T}\cos(nw_0t)\sin(mw_0t)dt + \sum_{n=1}^{\infty} b_n\int_{t_0}^{t_0+T}\sin(mw_0t)\sin(nw_0t)dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(nw_0t)dt \tag{4}$$

Example 1:

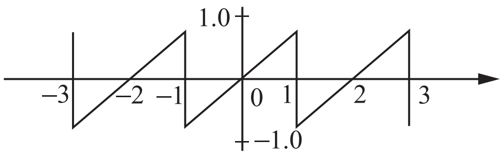


Fig. 2.1.

$$T \to -1 \text{ to } 1 \qquad T = 2 \qquad w_0 = \pi \qquad x(t) = t, -1 < t < 1$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} t\,dt = \frac{1}{4}(1-1) = 0$$

$$a_n = 0$$

$$b_n = \int_{-1}^{1} t\sin \pi nt dt = \left[\frac{-t\cos \pi nt}{n\pi} - \frac{\cos \pi nt}{n\pi}\right]_{-1}^{1}$$

$$= \frac{-1}{n\pi}[t\cos \pi nt + \cos \pi nt]_{-1}^{1} = -\frac{1}{n\pi}[2\cos\pi + \cos\pi - \cos\pi]$$

$$b_n = \frac{-2}{n\pi}\cos n\pi = \frac{2}{\pi}\left[\frac{-(-1)^n}{n}\right]$$

$$\begin{array}{cccccc} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ \frac{2}{\pi} & \frac{-2}{2\pi} & \frac{2}{3\pi} & \frac{-2}{4\pi} & \frac{2}{5\pi} & \frac{-2\cdots}{6\pi} \end{array}$$

$$x(t) = \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{-(-1)^n}{n}\right]\sin n\pi t$$

$$= \frac{2}{\pi}\left[\sin\pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t - \frac{1}{4}\sin 4\pi t + \cdots\right]$$

Example 2:

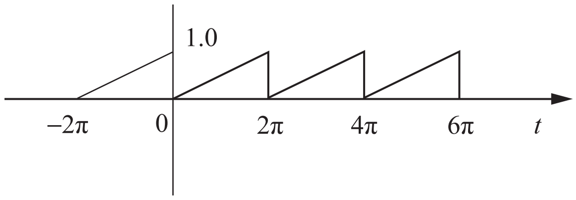


Fig. 2.2.

$$x(t) = \frac{t}{2\pi} \qquad T = 2\pi \qquad w_0 = \frac{2\pi}{T} = 1$$

$$a_0 = \frac{1}{T}\int_0^{2\pi} x(t)dt = \frac{1}{4\pi^2}\left[\frac{1}{2}t^2\right]_0^{2\pi} = \frac{1}{2}$$

$$a_n = \frac{2}{4\pi^2}\int_0^{2\pi} t\cos ntdt = \frac{1}{2\pi^2}\left[\frac{t\sin t}{n} + \frac{\sin nt}{n}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{2\pi\sin 2n\pi}{n} + \frac{\sin 2n\pi}{n}\right] = 0$$

$$b_n = \frac{2}{4\pi^2}\int_0^{2\pi} t\sin ntdt = \frac{-1}{2\pi^2}\left[\frac{t\cos nt}{n} + \frac{\cos nt}{n}\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi^2}\left[\frac{2\pi\cos 2n\pi}{n} + \frac{\cos 2n\pi}{n} - \frac{1}{n}\right]$$

$$b_n = \frac{-1}{n\pi}$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-1}{n\pi}\right)\sin nt = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\,(nt + \pi/2)$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \cdots\right]$$

Example 3:

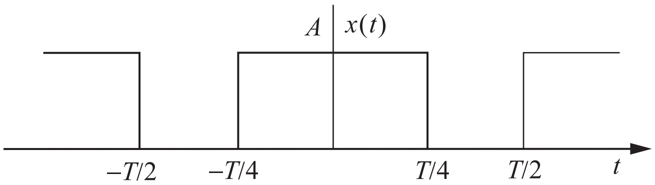


Fig. 2.3. Rectangular waveform

Figure shows a periodic rectangular waveform which is symmetrical to the vertical axis. Obtain its F.S. representation.

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\, nw_0 t + b_n \sin\, nw_0 t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\,(nw_0 t) \quad bn = 0$$

$$x(t) = 0 \quad \text{for } \frac{-T}{2} < t < \frac{-T}{4}$$

$$+A \quad \text{for } \frac{-T}{4} < t < \frac{T}{4}$$

$$0 \quad \text{for } \frac{T}{4} < t < \frac{T}{2}$$

$$a_0 = \frac{1}{T} \int_{-T/4}^{T/4} A dt = \frac{A}{2}$$

$$a_n = \frac{2}{T} \int_{-T/4}^{T/4} A \cos\,(nw_0 t) dt = \frac{2A}{Tnw_0} \left[\sin\, nw_0 \frac{T}{4} + \sin\, nw_0 \frac{T}{4} \right]$$

$$a_n = \frac{4A}{2\pi n} \sin\left(\frac{n\pi}{2}\right) = \frac{2A}{\pi n} \sin\left(\frac{n\pi}{2}\right) \quad w_0 = \frac{2\pi}{T}$$

$$a_1 = \frac{4A}{2\pi} = \frac{2A}{\pi}$$

$$a_2 = 0$$

$$a_3 = \frac{2A}{3\pi} \sin\frac{3\pi}{2} = \frac{2A}{3\pi}(-1) = \frac{-2A}{3\pi}$$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \left(\cos\, w_0 t - \frac{1}{3} \cos\, 3w_0 t + \frac{1}{5} \cos\, 5w_0 t + \cdots \right)$$

Example 4: Find the trigonometric Fourier series for the periodic signal $x(t)$.

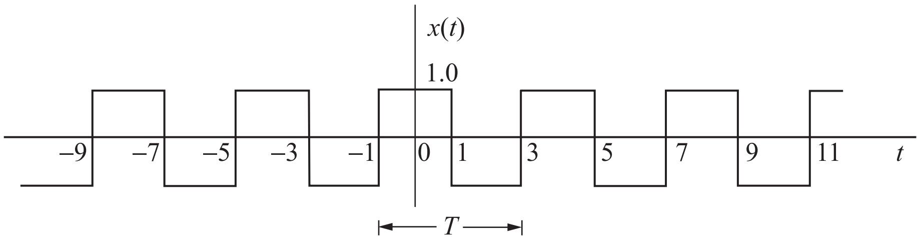


Fig. 2.4.

SOLUTION:

$$b_n = 0 \quad x(t) = \begin{cases} 1 & -1 < t < 1 \\ -1 & 1 < t < 3 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-1}^{3} x(t)dt = \frac{1}{T}\left[\int_{-1}^{1} dt + \int_{t}^{3}(-1)dt\right] \qquad T = 4$$

$$= \frac{1}{T}[2-2] = 0 \qquad \therefore \quad w_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$a_n = \frac{2}{T}\left[\int_{-1}^{1} \cos\,(nw_0t)dt + \int_{1}^{3} \cos\,(nw_0t)dt\right]$$

$$= \frac{2}{2\pi n}\left\{\left[2\sin\frac{\pi n}{2}\right] - \left[\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right]\right\}$$

$$= \frac{1}{n\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right] \qquad \sin\frac{3n\pi}{2} = \sin\left(\pi + \frac{n\pi}{2}\right) = -\sin\frac{n\pi}{2}$$

$$a_n = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$a_n = \begin{cases} 0 & n = \text{even} \\ \frac{4}{n\pi} & n = 1, 5, 9, 13 \\ \frac{-4}{n\pi} & n = 3, 7, 11, 15 \end{cases}$$

$$x(t) = \frac{4}{\pi}\cos\left(\frac{\pi}{2}t\right) - \frac{4}{3\pi}\cos\left(\frac{3\pi}{2}t\right) + \frac{4}{5\pi}\cos\left(\frac{5\pi}{2}t\right) - \frac{4}{7\pi}\cos\left(\frac{7\pi}{2}t\right) + \cdots$$

$$x(t) = \frac{4}{\pi}\left[\cos\left(\frac{\pi}{2}t\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2}t\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}t\right)\cdots\right]$$

Example 5: Find the F.S.C. for the continuous-time periodic signal

$$x(t) = 1.5 \qquad 0 \le t < 1$$
$$= -1.5 \qquad 1 \le t < 2$$

with fundamental freq. $w_0 = \pi$

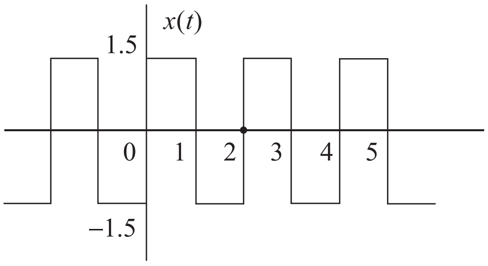


Fig. 2.5.

SOLUTION:

$$T = \frac{2\pi}{w_0} = 2,\ w_0 = \pi$$

$$a_0 = a_n = 0$$

$$b_n = \int_0^1 1.5 \sin n\pi t\, dt - \int_1^2 1.5 \sin n\pi t\, dt$$

$$= \frac{1.5}{n\pi}\left\{[-\cos n\pi + 1] + [\cos 2n\pi - \cos n\pi]\right\}$$

$$b_n = \frac{3}{n\pi}[1 - \cos n\pi]$$

$$x(t) = \frac{3}{\pi}\left[2 \sin \pi t + \frac{2}{3}\sin 3\pi t + \frac{2}{5}\sin 5\pi t + \cdots\right]$$

$$\frac{6}{\pi}\left[\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \cdots\right]$$

$$C_0 = \frac{1}{2}\left[\int_0^1 1.5 dt - 1.5\int_1^2 dt\right] = 0$$

OR

By using complex exponential Fourier series

$$C_n = \frac{1}{2}\left[\int_0^1 1.5 e^{-jn\pi t} dt - 1.5 \int_1^2 e^{-jn\pi t} dt\right]$$

$$C_n = \frac{3}{-4jn\pi}\left[e^{-jn\pi t}\Big|_0^1 - e^{-jn\pi t}\Big|_1^2\right]$$

$$= \frac{-3}{4jn\pi}\left[e^{-jn\pi} - 1 - e^{-j2n\pi} + e^{-jn\pi}\right]$$

$$= \frac{3}{2jn\pi}\left[1 - e^{-jn\pi}\right] = \frac{3}{2jn\pi}[1 - \cos n\pi]$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{-jn\pi t}$$

$$\sum_{n=-\infty}^{\infty} \frac{3}{2jn\pi}\left[1 - e^{-jn\pi}\right] e^{jn\pi t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{3}{2jn\pi}\left[e^{jn\pi t} - e^{jn\pi t} \cos \pi n\right]$$

for $n = 1$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin t \sin t dt = \frac{A}{2\pi}\int_0^{\pi}(1-\cos 2t)dt$$

$$= \frac{A}{2\pi}[\pi] = \frac{A}{2}$$

When n is even

$$= \frac{A}{2\pi}\left[\frac{2}{n+1} - \frac{2}{1-n}\right] = \frac{2A}{\pi(1-n^2)}$$

Example 7:

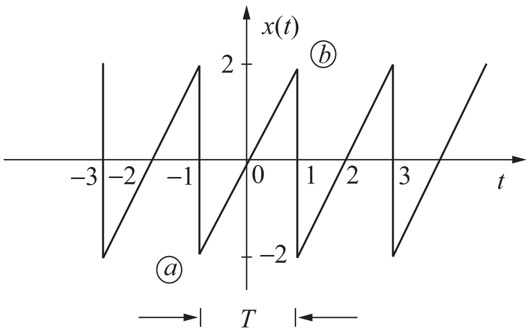


Fig. 2.7.

SOLUTION:

$$T = 2 \quad w_0 = \frac{2\pi}{T} = \pi$$

$$x(t) = \begin{cases} 2t & -1 < t < 1 \\ 0 & \end{cases}$$

Point (a) $(-1,-2)$
Point (b) $(1,2)$

$$y-(-2) = \frac{2-(-2)}{1-(-1)}(x-(-1))$$

$$y+2 = \frac{4}{2}(x+1)$$

$$y+2 = 2x+2$$

$$y = 2x$$

$$x(t) = 2t$$

Since function is an odd function

$$a_n = 0,\ a_0 = \frac{1}{T}\int_{-1}^{1} 2t dt = \frac{1}{2}\times 0 = 0$$

$$b_n = \frac{2}{T}\int_{-1}^{1} t \sin(n\pi t)dt = \frac{2}{T}\left[\frac{-t\cos n\pi t}{n\pi}\Bigg|_{-1}^{1} + \frac{1}{n^2\pi^2}\cos n\pi t\Bigg|_{-1}^{1}\right]$$

2.3 CONVERGENCE OF FOURIER SERIES – DIRICHLET CONDITIONS

Existence of Fourier Series: The conditions under which a periodic signal can be represented by an F.S. are known as Dirichlet conditions. F.P. → Fundamental Period

(1) The function $x(t)$ has only a finite number of maxima and minima, if any within the F.P.
(2) The function $x(t)$ has only a finite number of discontinuities, if any within the F.P.
(3) The function $x(t)$ is absolutely integrable over one period, that is

$$\int_0^T |x(t)|dt < \infty$$

2.4 PROPERTIES OF CONTINUOUS FOURIER SERIES

(1) Linearity: If $x_1(t)$ and $x_2(t)$ are two periodic signals with period T with F.S.C. C_n and D_n then F.C. of linear combination of $x_1(t)$ and $x_2(t)$ are given by

$$\text{FS}\,[Ax_1(t)+Bx_2(t)] = AC_n + BD_n$$

Proof: If $z(t) = Ax_1(t)+Bx_2(t)$

$$a_n = \frac{1}{T}\int_{t_0}^{t_0+T}[Ax_1(t)+Bx_2(t)]\,e^{-jnw_0t} = \frac{A}{T}\int_T x_1(t)e^{-jnw_0t}dt + \frac{B}{T}\int_T x_2(t)e^{-jnw_0t}dt$$

$$a_n = AC_n + BD_n$$

(2) Time shifting: If the F.S.C. of $x(t)$ are C_n then the F.C. of the shifted signal $x(t-t_0)$ are

$$\text{FS}\,[x(t-t_0)] = e^{-jnw_0\ t_0}C_n$$

$$\text{Let } t - t_0 = \tau$$

$$dt = d\tau$$

$$B_n = \frac{1}{T}\int_T x(t-t_0)e^{-jnw_0t}dt$$

$$= \frac{1}{T}\int_T x(\tau)e^{-jnw_0(t_0+\tau)}d\tau = \frac{1}{T}\int_T x(\tau)e^{-jnw_0\tau}d\tau \cdot e^{-jnw_0t_0}$$

$$B_n = e^{-jnw_0\,t_0}\cdot C_n$$

(3) Time reversal: $\text{FS}[x(-t)] = C_{-n}$

$$B_n = \frac{1}{T}\int_T x(-t)e^{-jnw_0t}dt = \frac{1}{T}\int_T x(-t)e^{-j(-n)w_0T}dt$$

$$-t = \tau$$

$$dt = -d\tau$$

$$= \frac{1}{T}\int_{-T} x(\tau)e^{-j(-n)w_0\tau}d\tau = C_{-n}$$

Example 8: Compute the exponential series of the following signal.

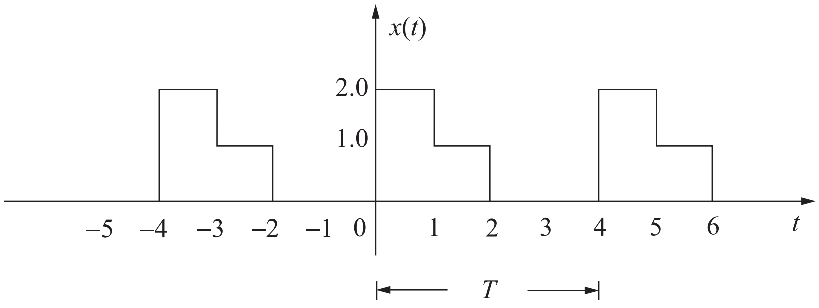


Fig. 2.8.

SOLUTION:

$$T = 4 \quad w_0 = \frac{\pi}{2}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{4}\left[\int_0^1 2dt + \int_1^2 dt\right] = \frac{3}{4}$$

$$C_n = \frac{1}{4}\left[\int_0^1 2e^{-jn\frac{\pi t}{2}}dt + \int_1^2 e^{-jn\frac{\pi t}{2}}dt\right]$$

$$= \frac{1}{4}\left\{\frac{-4}{jn\pi}\left[e^{-jn\frac{\pi}{2}} - 1\right] - \frac{2}{jn\pi}\left[e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right]\right\}$$

$$= \frac{-1}{2jn\pi}\left[2e^{\frac{-jn\pi}{2}} - 2 + e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right] = \frac{-1}{2jn\pi}\left[e^{-jn\frac{\pi}{2}} + e^{-jn\frac{\pi}{2}} - 2\right]$$

$$= -\frac{1}{jn\pi}\left[1 - \frac{1}{2}(-1)^n - \frac{1}{2}e^{-jn\frac{\pi}{2}}\right] \quad x(t) = \frac{3}{4} + \sum_{n=-\infty}^{\infty}\frac{1}{jn\pi}\left[e^{jn\frac{\pi}{2}} - \frac{1}{2}(-1)^n e^{jn\frac{\pi}{2}} - \frac{1}{2}\right]$$

Example 9:

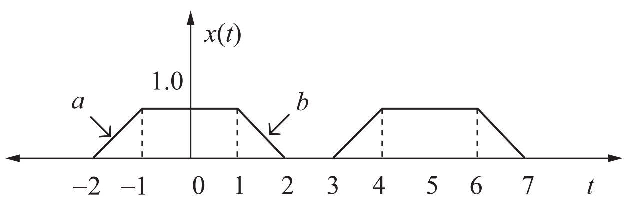


Fig. 2.9.

SOLUTION:

$$T = 5 \quad w_0 = \frac{2\pi}{5}$$

$$x(t) = \begin{cases} t+2 & -2 < t < -1 \\ 1.0 & -1 < t < 1 \\ 2-t & 1 < t < 2 \end{cases}$$

(a)

$$(-2,0)(-1,1)$$

$$(y-1) = \frac{-1}{-1}(x+1)$$

$$y = t+2$$

(b) $(1,1)(2,0)$

$$y - 0 = \frac{1}{-1}(x-2)$$

$$y = -x+2 = -t+2$$

$$C_0 = \frac{1}{5}\left[\int_{-2}^{-1}(t+2)dt + \int_{-1}^{1} dt + \int_{1}^{2}(2-t)dt\right]$$

$$C_0 = \frac{3}{5}$$

$$C_n = \frac{1}{5}\left[\underbrace{\int_{-2}^{-1}(t+2)e^{-j\frac{2n\pi}{5}}dt}_{A} + \underbrace{\int_{-1}^{1} e^{-j\frac{2n\pi}{5}}dt}_{B} + \underbrace{\int_{1}^{2}(2-t)e^{-j\frac{2n\pi}{5}}dt}_{C}\right]$$

$$A = \int_{-2}^{-1} e^{-j\frac{2n\pi}{5}t}dt + \int_{-2}^{-1} 2e^{-j\frac{2n\pi}{5}t}dt$$

$$A = -\frac{1}{j\phi}\left\{te^{-j\phi}\int_{-2}^{-1}\right\} + \frac{1}{\phi^2}e^{-j\phi}\int_{-2}^{-1} + \frac{2}{-j\phi}e^{j}\frac{2n\pi}{5}\int_{-2}^{-1}$$

$$= \frac{5}{j\,2n\pi}\left(-e^{j\frac{2n\pi}{5}} + 2e^{j\frac{4n\pi}{5}}\right) + \frac{25}{4n^2\pi^2}\left(e^{j\frac{2n\pi}{5}} - e^{j\frac{4n\pi}{5}}\right) - \frac{10}{2n\pi j}$$

$$A = \frac{5}{j2n\pi}\left(-e^{j\frac{2n\pi}{5}} + 4e^{j\frac{4n\pi}{5}}\right) + \frac{25}{4n^2\pi^2}\left(e^{j\frac{2n\pi}{5}} - e^{j\frac{4n\pi}{5}}\right)$$

$$B = \frac{e^{j\frac{2n\pi}{5}} - e^{-j\frac{2n\pi}{5}}}{j\frac{2n\pi}{5}} = \frac{5}{j\,2n\pi}\left(e^{j\frac{2n\pi}{5}} - e^{-j\frac{2n\pi}{5}}\right)$$

$$C = \frac{-10}{j\,2n\pi}\left(e^{-j\frac{4n\pi}{5}} - e^{-j\frac{2n\pi}{5}}\right) + \frac{10}{j\,2n\pi}e^{-j\frac{4n\pi}{5}} - \frac{5}{j\,2n\pi}e^{-j\frac{2n\pi}{5}} - \frac{25}{4n^2\pi^2}e^{-j\frac{4n\pi}{5}} + \frac{25}{4n^2\pi^2}e^{j\frac{n2\pi}{5}}$$

$$C_n = \frac{1}{5}\left[\frac{25}{n^2 4\pi^2}\left(e^{\frac{j2n\pi}{5}} - e^{\frac{j4n\pi}{5}}\right) - \frac{25}{4n^2\pi^2}\left(e^{-\frac{j4n\pi}{5}} - e^{-\frac{j2n\pi}{5}}\right)\right]$$

$$C_n = \frac{5}{2n^2\pi^2}\left[\cos\left(\frac{2\pi n}{5}\right) - \cos\left(\frac{4\pi n}{5}\right)\right]$$

Example 10: For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

Determine the fundamental frequency w_0 and the Fourier series coefficients C_n such that

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

SOLUTION:

Given

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

The time period of the signal $\cos\left(\frac{2\pi}{3}t\right)$ is

$$T_1 = \frac{2\pi}{w_1} = \frac{2\pi}{2\frac{\pi}{3}} = 3\,\text{sec}$$

The time period of the signal $\sin\left(5\frac{\pi}{2}t\right)$ is

$$T_2 = 2\frac{\pi}{w_2} = \frac{2\pi}{5\frac{\pi}{3}} = \frac{6}{5}\,\text{sec}$$

$\frac{T_1}{T_2} = \frac{3}{\frac{6}{5}} = \frac{5}{2}$ ratio of two integers, rational number, hence periodic.

$$2T_1 = 5T_2$$

The fundamental period of the signal $x(t)$ is

$$T = 2T_1 = 5T_2 = 6\,\text{sec}$$

and the fundamental frequency is

$$w_0 = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$$= 2 + \cos(2w_0 t) + 4\sin(5w_0 t)$$

$$= 2 + \frac{\left(e^{j2w_0 t} + e^{-j2w_0 t}\right)}{2} + \frac{4\left(e^{j5w_0 t} - e^{-j5w_0 t}\right)}{2j}$$

$$= 2 + 0.5\left(e^{j2w_0 t} + e^{-j2w_0 t}\right) - 2j\left(e^{j5w_0 t} - e^{-j5w_0 t}\right)$$

$$x(t) = 2je^{+j(-5)w_0 t} + 0.5e^{+j(-2)w_0 t} + 2 + 0.5e^{+j2w_0 t} - 2je^{+j5w_0 t}$$

$$a_n = \frac{2}{T}\int_{-\pi}^{\pi} x(t)\cos nt dt = \frac{4}{T}\int_{-\pi}^{0}\left(\frac{2t}{\pi}+1\right)\cos nt\ dt$$

$$= \frac{4}{2\pi}\left\{\frac{2t}{n\pi}\sin t + \frac{\sin nt}{n} - \int_{-\pi}^{0}\frac{2}{\pi}\sin nt\ dt\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2t}{\pi}\sin nt + \sin nt + \frac{2}{n^2\pi}\cos nt\int_{-\pi}^{0}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2}{n^2\pi} + \frac{2}{n^2\pi}\cos nt\right\} = \frac{4}{n^2\pi^2}\left\{1-\cos n\pi\right\} = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$a_n = \begin{cases} 0 & n \text{ even } 2,\ 4,\ 6,\ 8,\cdots \\ \frac{8}{n^2\pi^2} & n \text{ odd } 1,\ 3,\ 5,\ 7,\cdots \end{cases}$$

2.5 FOURIER TRANSFORM

2.5.1 Definition

Let $x(t)$ be a signal which is a function of time t. The Fourier transform of $x(t)$ is given as

$$\mathrm{X}\ (jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \tag{1}$$

Fourier transform or

$$\mathrm{X}\ (if) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{2}$$

Since $w = 2\pi f$

Similarly, $x(t)$ can be recovered from its Fourier transform $X(jw)$ by using Inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw \tag{3}$$

$$x(t) = \int_{-\infty}^{\infty} X(if)e^{j2\pi ft}dt \tag{4}$$

Fourier transform $X(jw)$ is the complex function of frequency w. Therefore, it can be expressed in the complex exponential form as follows:

$$X(jw) = |X(jw)|e^{j\,\underline{|X(jw)}}$$

Here $|X(jw)|$ is the amplitude spectrum of $x(t)$ and $\underline{|X(jw)}$ is phase spectrum.

For a real-valued signal

(1) Amplitude spectrum is symmetric about vertical axis c (even function.)
(2) Phase spectrum is anti-symmetrical about vertical axis c (odd function.)

2.5.2 Existence of Fourier transform (Dirichlet's condition)

The following conditions should be satisfied by the signal to obtain its F.T.

(1) The function $x(t)$ should be single valued in any finite time interval T.

(2) The function $x(t)$ should have at the most finite number of discontinuities in any finite time interval T.

(3) The function $x(t)$ should have finite number of maxima and minima in any finite time interval T.

(4) The function $x(t)$ should be absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

- These conditions are sufficient, but not necessary for the signal to be Fourier transformable.
- A physically realizable signal is always Fourier transformable. Thus, physical realizability is the sufficient condition for the existence of F.T.
- All energy signals are Fourier transformable.

$$j\frac{d}{dw}X(jw) = FT(tx(t))$$

$$FT(tx(t)) = j\frac{d}{dw}X(jw)$$

Example 12: Obtain the F.T. of the signal $e^{-at}u(t)$ and plot its magnitude and phase spectrum.

SOLUTION:

$x(t) = e^{-at}u(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(a+j2\pi f)t}dt$$

$$X(f) = \frac{1}{a + j2\pi f}$$

To obtain the magnitude and phase spectrum:

$$|X(f)| = \frac{a - j2\pi f}{a^2 + (2\pi f)^2} = \left(\frac{a}{a^2 + 4\pi^2 f^2}\right)A - j\left(\frac{2\pi f}{a^2 + 4\pi^2 f^2}\right)B$$

$$|X(f)| = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}} = \frac{1}{\sqrt{a^2 + w^2}}$$

$$|X(f)| = \tan^{-1}\left[\frac{-2\pi f}{a}\right] = -\tan^{-1}\left(\frac{w}{a}\right)$$

$$\text{for } a = 1,\ |X(f)| = \frac{1}{\sqrt{1 + w^2}},\ \underline{|X(f)} = -\tan^{-1} w$$

w	0	1	2	3	4	5	10	15	25	8
$\lvert X(w)\rvert$	1	.707	0.447	0.316	0.242	0.196	0.09	0.066	0.03	0
$\underline{\lvert X(w)}$	0	45°	−63.4	−71.5	−75.9	−78.6	−84.2	−86.2	−87.7	−90°

(ii) $x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t > 0 \end{cases}$

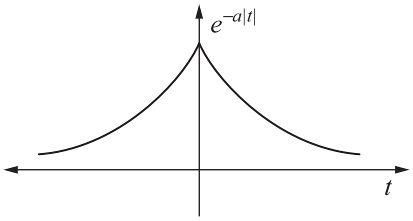


Fig. 2.13. Graphical representation of $e^{-a|t|}$

$$x(w) = \frac{1}{a+jw} + \frac{1}{a-jw} = \frac{2a}{a^2+w^2}$$

$$\text{for } a = 1\ X(w) = \frac{2}{1+w^2}$$

$$|X(w)| = \frac{2}{1+w^2}\ \ \frac{|X(w)}{} = 0$$

w (in radians)	$-\infty$	-10	-5	-3	-2	-1	0	1	2	3	4	5	10	∞
$\|X(w)\|$	0	0.019	0.0769	0.2	0.4	1	2	1	0.4	0.2	.1176	0.0769	0.019	0

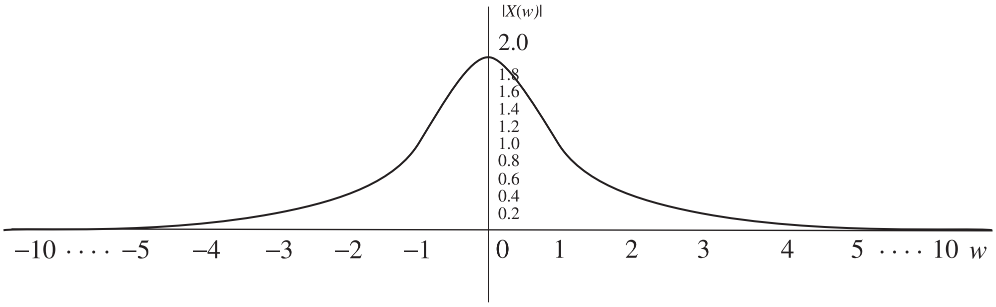


Fig. 2.14. Magnitude plot

(iii) $x(t) = e^{-a|t|}\ \text{sgn}(t)$

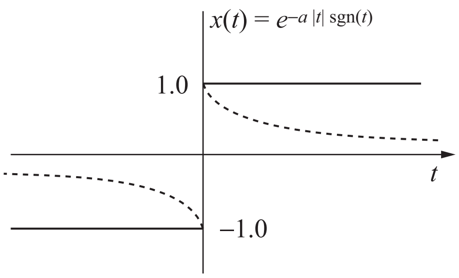


Fig. 2.15. Graphical representation of $e^{-a|t|}\text{sgn}(t)$

(ii)
$$x(t) = 1$$
$$X(w) = \int_{-\infty}^{\infty} e^{-jwt} dt = \infty$$

This means Dirichlet condition is not satisfied. But its F.T. can be calculated with the help of duality property.

$$\delta(t) \xleftrightarrow{FT} 1$$

Duality property states that: $x(t) \xleftrightarrow{FT} X(w)$ then

$$X(t) \xleftrightarrow{FT} 2\pi x(-w)$$

Here $X(t) = 1$, then $x(-w)$ will be

$$x(t) = \delta(t); \quad X(w) = 1$$

then $X(t) = 1$; $\quad 1 \xleftrightarrow{FT} 2\pi\delta(-w)$

We know that $\delta(w)$ will be an even function of w, since it is impulse function.

Hence, $\delta(-w) = \delta(w)$. Then above equation becomes

$$1 \xleftrightarrow{FT} 2\pi\delta(-w)$$

Thus, if $x(t) = 1$, then $X(w) = 2\pi\delta(w)$

(iii) $x(t) = \text{sgn}(t) \qquad \text{sgn}(t) = \left\{ \begin{array}{ll} 1 & t > 0 \\ -1 & t < 0 \end{array} \right\}$

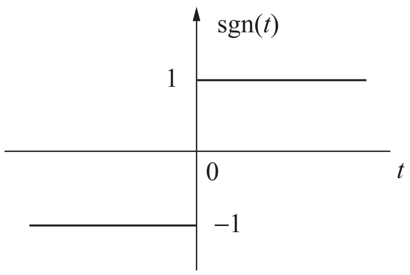


Fig. 2.17. Graphical representation of sgn(t)

$$x(t) = 2u(t) - 1$$

Differentiating both the sides

$$\frac{d}{dt}x(t) = 2\frac{d}{dt}u(t) = 2\delta(t)$$

Taking the F.T. of both sides

$$F\left[\frac{d}{dt}x(t)\right] = 2F[\delta(t)]$$
$$jwX(w) = 2$$
$$X(w) = \frac{2}{jw}$$
$$X(w) = \int_{0}^{\infty} e^{-jwt} dt - \int_{-\infty}^{0} e^{-jwt} dt$$

(iv)
$$x(t) = u(t)$$
$$\text{sgn}(t) = 2u(t) - 1$$
$$2u(t) = 1 + \text{sgn}(t)$$

Taking F.T. of both sides

$$2F[2u(t)] = F(1) + F[\text{sgn}(t)] = 2\pi\delta(w) + \frac{2}{jw}$$
$$2u(t) \overset{FT}{\longleftrightarrow} 2\pi\delta(w) + \frac{2}{jw}$$
$$u(t) \overset{FT}{\longleftrightarrow} \pi\delta(w) + \frac{1}{jw}$$

Properties of unit impulse:

(1) $\int\limits_{-\infty}^{\infty} x(t)\delta(t) = x(0)$

(2) $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

(3) $\int\limits_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$

(4) $\delta(at) = \frac{1}{|a|}\delta(t)$

(5) $\int\limits_{-\infty}^{\infty} x(\tau)\delta(t-x)dt = x(t)$

(6) $\delta(t) = \frac{d}{dt}u(t)$

Example 15: Obtain the F.T. of a rectangular pulse shown in Fig. 2.18.

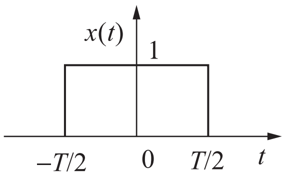


Fig. 2.18. Rectangular pulse

SOLUTION:

$$X(w) = \int\limits_{\frac{-T}{2}}^{\frac{T}{2}} e^{-jwt}dt = \frac{-1}{jw}\left[e^{-jw\frac{T}{2}} - e^{jw\frac{T}{2}}\right] = \frac{2}{w}\sin\left(\frac{wT}{2}\right)$$
$$X(w) = T\frac{\sin\left(\pi\frac{wT}{2\pi}\right)}{\pi\frac{wT}{2\pi}} = \sin c\left(\frac{wT}{2\pi}\right) = T\frac{\sin\left(\pi\frac{wT}{2\pi}\right)}{\pi\frac{wT}{2\pi}}$$

Sampling function or interpolating function or filtering function denoted by $S_a(x)$ or $\sin c(x)$ as shown in figure.

$$\sin c(x) = \frac{\sin \pi x}{\pi x}$$

(1) $\sin c(x) = 0$ when $x = \pm n\pi$
(2) $\sin c(x) = 1$ when $x = 0$ (using L'Hospital's rule)
(3) $\sin c(x)$ is the product of an oscillating signal $\sin x$ of period 2π and a decreasing signal $\frac{1}{x}$. Therefore, $\sin c(x)$ is making sinusoidal of oscillations of period 2π with amplified decreasing continuously as $\frac{1}{x}$.

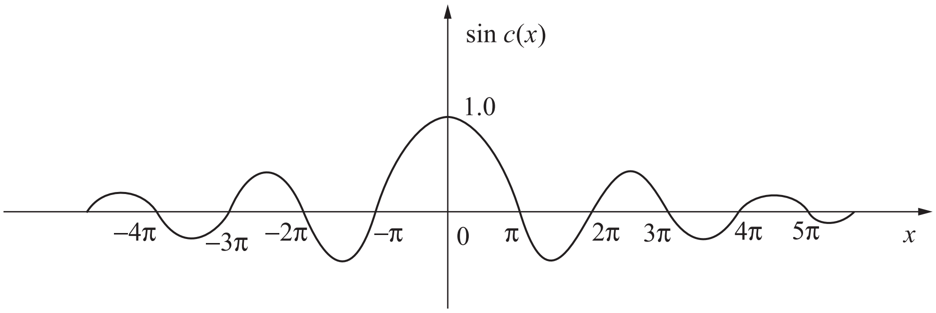


Fig. 2.19. Sine function

$$\begin{aligned}
\sin cx &= \frac{\sin \pi x}{\pi x}; & \sin c(0) &= \frac{0}{0} = 1 \text{ L'Hospital rule}\\
\sin c(1) &= \frac{\sin \pi}{\pi} = 0; & \sin c(-1) &= 0\\
\sin c(2) &= 0; & \sin c(-2) &= 0\\
\sin c(1/4) &= 0.9 & \sin c(-1/4) &= 0.9\\
\sin c(2/4) &= .6366 & \sin c(-0.5) &= .6366\\
\sin c(3/4) &= 0.3 & \sin c(-7.5) &= .3\\
\sin c(1.5) &= -.2122 & \sin c(-1.5) &= -.2122\\
\sin c(2.5) &= .1273 & \sin c(2.5) &= .1273
\end{aligned}$$

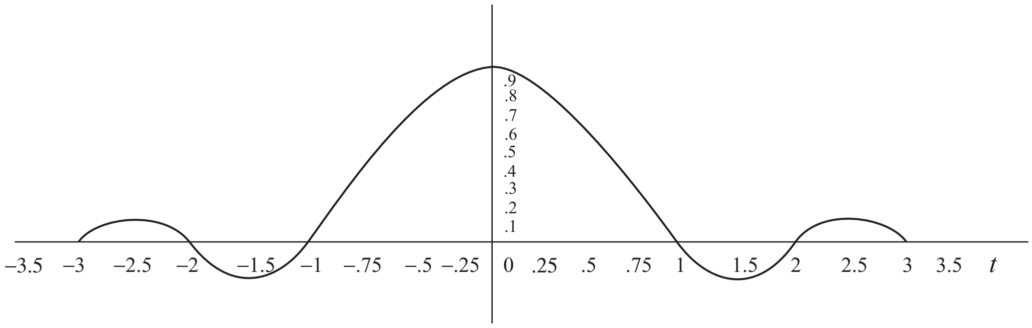


Fig. 2.20. Sine function

Example 17: Obtain F.T. and spectrums of following signals:

(i) $x(t) = \cos w_0 t$ (ii) $x(t) = \sin w_0 t$

SOLUTION:

(i)
$$x(t) = \cos w_0 t = \frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t}$$

$$1 \overset{FT}{\longleftrightarrow} 2\pi\delta(w);\ \frac{1}{2} \overset{FT}{\longleftrightarrow} \pi\delta(w)$$

Frequency shifting property states that $e^{j\beta t}x(t) \overset{FT}{\longleftrightarrow} X(w-\beta)$

$$\frac{1}{2}e^{jw_0 t} \overset{FT}{\longleftrightarrow} \pi\delta(w-w_0)$$

$$\frac{1}{2}e^{-jw_0 t} \overset{FT}{\longleftrightarrow} \pi\delta(w+w_0)$$

$$F[x(t)] = FT\left\{\frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t}\right\}$$

$$X(w) = \pi[\delta(w-w_0) + \delta(w+w_0)]$$

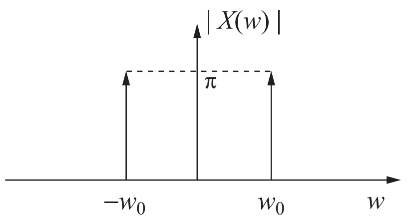


Fig. 2.22. Magnitude plot of $\cos w_0 t$

(ii)
$$x(t) = \sin w_0 t$$

$$X(w) = \frac{\pi}{j}[\delta(w-w_0) - \delta(w+w_0)]$$

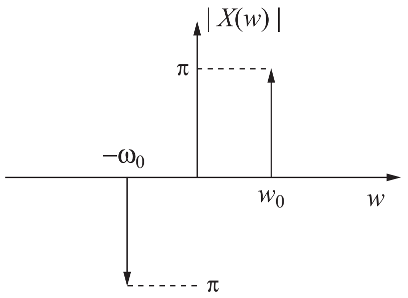


Fig. 2.23. Magnitude plot of $\sin w_0 t$

Example 18: Obtain the F.T. of

$$x(t) = te^{-at}u(t)$$

from property of Fourier transform $\mathrm{FT}[tx(t)] = j\frac{d}{dw}X(w)$

$$\mathrm{FT}\left[e^{-at}\right] = \frac{1}{a+jw}$$

$$\mathrm{FT}(te^{-at}) = j\frac{d}{dw}\left(\frac{1}{a+jw}\right) = j\frac{(a+jw)\frac{d}{dw}(1) - 1\frac{d}{dw}(a+jw)}{(a+jw)^2} = \frac{1}{(a+jw)^2}$$

Inverse Fourier Transform: (IFT)

Example 19: Find the IFT of

(i) $X(w) = \frac{2jw+1}{(jw+2)^2}$ by partial fraction expansions

(ii) $X(w) = \frac{1}{(a+jw)^2}$ by convolution property

(iii) $X(w) = e^{-|w|}$

(iv) $X(w) = e^{-2w}u(w)$

SOLUTION:

(i)

$$X(w) = \frac{A}{jw+2} + \frac{B}{(jw+2)^2};\ 2jw+1 = A(jw+2)+B \quad A=2 \quad 2A+B=1 \quad B=-3$$

$$X(w) = \frac{2}{jw+2} - \frac{3}{(jw+2)^2}$$

$$x(t) = 2e^{-2t}u(t) - 3te^{-2t}u(t)$$

(ii)

$$X(w) = \frac{1}{(a+jw)^2} = \frac{1}{(a+jw)(a+jw)} = X_1(w)X_2(w)$$

$$X_1(w) = \frac{1}{a+jw},\ X_2(w) = \frac{1}{a+jw}$$

$$x_1(t) = e^{-at}u(t),\ x_2(t) = e^{-at}u(t)$$

Using convolution property

$$x(t) = x_1(t)^* x_2(t)$$

$$x(t) \xleftrightarrow{\mathrm{FT}} X(w)$$

$$x_1(t)^* x_2(t) \xleftrightarrow{\mathrm{FT}} X_1(w)X_2(w)$$

$$x(t) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-a(t-\tau)}u(t-\tau)d\tau \qquad \begin{cases} u(\tau) = 1\ \tau \le 0 \\ u(t-\tau) = 1 \quad t \le \tau \end{cases}$$

$$= \int_0^t e^{-at}d\tau = te^{-at}u(t)$$

Example 20: Find the F.T. of the function

$$x(t-t_0) = e^{-(t-t_0)}u(t-t_0)$$

SOLUTION:

$$\text{If } F[x(t)] = X(w)$$

$$\text{then } \mathrm{FT}[x(t-t_0)] = e^{-jwt_0}X(w)$$

$$F\left[e^{-t}u(t)\right] = \frac{1}{1+jw}$$

$$F\left[e^{-(t-t_0)}u(t-t_0)\right] = \frac{e^{-jwt_0}}{1+jw}$$

Example 21: Find the F.T. of the function

$$x(t) = [u(t+1)-u(t-1)]\cos 2\pi t$$

SOLUTION:

$$\mathrm{FT}(\cos 2\pi t) = \mathrm{FT}\left(\frac{e^{j2\pi t}+e^{-j2\pi t}}{2}\right)$$

$$\mathrm{FT}[1] = 2\pi\delta(w)$$

$$\mathrm{FT}[e^{jw_0t}] = 2\pi\delta(w-w_0)$$

$$F[\cos 2\pi t] = \pi\delta(1w-2\pi)+\pi\delta(w+2\pi) \tag{1}$$

$$F[u(t+1)-u(t-1)] = \int_{-1}^{1} e^{-jwt}dt = -\frac{1}{jw}\left(e^{-jw}-e^{jw}\right) = \frac{2\sin w}{w} \tag{2}$$

$$F[x(t)] = F[\{u(t+1)-u(t-1)\}\cos 2\pi t]$$

$x(t)$ is multiplication of (1) and (2), so by using multiplication property

$$x(t)y(t) \xleftrightarrow{\mathrm{FT}} \frac{1}{2\pi}X_1(w)^*Y_1(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)Y(w-\tau)d\tau$$

$$X(w) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\frac{2\sin\tau}{\tau}\pi\delta(w-2\pi-\tau)+\delta(w+2\pi-\tau)\right]d\tau$$

$$X(w) = \int_{-\infty}^{\infty}\frac{\sin\tau}{\tau}\delta(w-2\pi-\tau)d\tau + \int_{-\infty}^{\infty}\frac{\sin\tau}{\tau}\delta(w+2\pi-\tau)d\tau$$

$$\text{Since} \quad \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

$$X(w) = \sin(w-2\pi)/(w-2\pi)+\sin(w+2\pi)/(w+2\pi)$$

Example 22: Determine the Fourier transform of a triangular function as shown in figure.

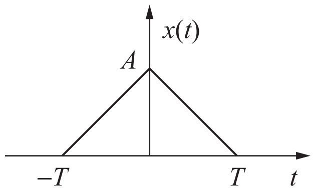
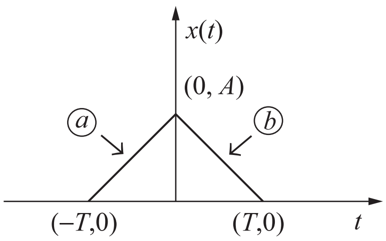


Fig. 2.24. Triangular pulse

SOLUTION:



Equation of line (a) is

$$x(t) = A\left(\frac{t}{T}+1\right)$$

Equation of line (b) is

$$x(t) = A\left(1-\frac{t}{T}\right)$$

Mathematically, we can write $x(t)$ as

$$x(t) = A\left(\frac{t}{T}+1\right)[u(t+T)-u(t)] + A\left(1-\frac{t}{T}\right)[u(t)-u(t-T)]$$

$$x(t) = \frac{A}{T}(t+T)[u(t+T)-u(t)] + \frac{A}{T}(T-t)[u(t)-u(t-T)]$$

$$x(t) = \frac{A}{T}\Big\{(t+T)u(t+T)-(t+T)u(t)\Big\} + \frac{A}{T}\Big\{[(T-t)u(t)-(T-t)u(t-T)]\Big\}$$

$$x(t) = \frac{A}{T}\Big\{r(t+T)-tu(t)-Tu(t)\Big\} + \frac{A}{T}\Big\{Tu(t)-tu(t)+r(t-T)\Big\}$$

$$= \frac{A}{T}\Big\{r(t+T)-r(t)-Tu(t)\Big\} + \frac{A}{T}\Big\{Tu(t)-r(t)+r(t-T)\Big\}$$

$$= \frac{A}{T}\left[\Big\{r(t+T)-2r(t)+r(t-T)\Big\}\right]$$

$$X(jw) = \frac{A}{T}\left[\frac{e^{jwT}}{(jw)^2} - \frac{2}{(jw)^2} + \frac{e^{-jwT}}{(jw)^2}\right]$$

$$\Pi(t) = \text{rect}(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{rect}(t-5) = \begin{cases} 1 & -\frac{1}{2} \le t-5 < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{rect}(t-5) = \begin{cases} 1 & \frac{9}{2} \le t \le \frac{11}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = \int_{-\infty}^{\infty} rect(t-5)e^{-jwt}dt$$

$$= \int_{9/2}^{11/2} e^{-jwt}dt = \left.\frac{e^{-jwt}}{-jw}\right|_{9/2}^{11/2}$$

$$= \frac{e^{-\frac{j11w}{2}} - e^{-\frac{9jw}{2}}}{-jw} = \frac{e^{-9j\frac{w}{2}} - e^{-11j\frac{w}{2}}}{jw}$$

$$= \frac{e^{-5jw}e^{jw/2} - e^{-5jw}e^{-jw/2}}{jw} = \frac{2e^{-5jw}\left(e^{jw/2} - e^{-jw/2}\right)}{w2j}$$

$$= \frac{2e^{-5jw}}{w}\sin\frac{w}{2} = e^{-5jw}\left(\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right)$$

$$X(jw) = e^{-5jw}S_a\left(\frac{w}{2}\right)$$

2.6 PROPERTIES OF CONTINUOUS-TIME FOURIER TRANSFORM

(1) Linearity

If FT $(x_1(t)) = X_1(jw)$
and FT $(x_2(t)) = X_2(jw)$
Then linearity property states that

$$\text{FT}(Ax_1(t) + Bx_2(t)) = AX_1(jw) + BX_2(jw)$$

where A and B are constants.

Proof:

$$\text{Let } r(t) = Ax_1(t) + Bx_2(t)$$

$$\text{FT}(r(t)) = R(jw) = \int_{-\infty}^{\infty} r(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} (Ax_1(t) + Bx_2(t))\, e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j(-w)\tau} d\tau$$

$$F(x(t)) = X(-jw)$$

(4) Time shifting

If FT $(x(t)) = X(jw)$
then FT $(x(t-t_0)) = e^{-jwt_0} X(jw)$

Proof:

$$\text{Let } r(t) = x(t-t_0)$$

$$R(jw) = \int_{-\infty}^{\infty} r(t) e^{-jwt} dt = \int_{-\infty}^{\infty} x(t-t_0) e^{-jwt} dt$$

$$R(jw) = \text{FT}(x(t-t_0)) = \int_{-\infty}^{\infty} x(t-t_0) e^{-jwt} dt$$

Let $t - t_0 = \tau$ $\quad dt = d\tau$

$$\text{FT } (x(t-t_0)) = \int_{-\infty}^{\infty} x(\tau) e^{-jw(t_0+\tau)} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-jwt} e^{-jwt_0} d\tau$$

$$= e^{-jwt_0} \int_{-\infty}^{\infty} x(\tau) e^{-jw\tau} d\tau$$

$$\text{FT } (x(t-t_0)) = e^{-jwt_0} X(jw). \text{ Similarly, FT } (x(t+t_0)) = e^{jwt_0} X(jw)$$

$$\text{So FT } (x(t \pm t_0)) = e^{\pm jwt_0} X(jw)$$

(5) Frequency shifting

$$\text{If FT } (x(t)) = X(jw)$$

$$\text{FT } (e^{jw_0 t} x(t)) = X(j(w-w_0))$$

$$\text{Let } r(t) = e^{jw_0 t} x(t)$$

$$\text{FT } (r(t)) = \text{FT}\left(e^{jw_0 t} x(t)\right) = R(jw) = \int_{-\infty}^{\infty} e^{jw_0 t} x(t) e^{-jwt} dt$$

$$\text{FT}\left(e^{jw_0 t} x(t)\right) = \int_{-\infty}^{\infty} x(t) e^{-j(w-w_0)t} dt$$

$$\text{Let } w - w_0 = w'$$

$$= \int_{-\infty}^{\infty} x(t)e^{-jw't}dt$$

$$\text{FT}\left(e^{jw_0t}x(t)\right) = X(jw') = X(j(w - w_0))$$

$$\text{Similarly, FT}(e^{-jw_0t}x(t)) = X(j(w + w_0))$$

$$\text{We can write as FT}\left(e^{\pm jw_0t}x(t)\right) = X(j(w \mp w_0))$$

(6) Duality or symmetry property

If FT $(x(t)) = X(jw)$

then FT $(x(t)) = 2\pi x(-jw)$

Proof:

We know that $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$

Replacing t by $-t$, we get

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$$

$$2\pi\, x(-t) = \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$$

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$$

Interchanging t by jw

$$2\pi\, x(-jw) = \int_{-\infty}^{\infty} X(t)e^{-jwt}dt$$

$$2\pi\, x(-jw) = \text{FT}(X(t))$$

(7) Convolution in time domain

If FT $(x_1(t)) = X_1(jw)$ and FT $(x_2(t)) = X_2(jw)$

then FT $(x_1(t)^* x_2(t)) = X_1(jw)X_2(jw)$

i.e., convolution in time domain becomes multiplication in frequency domain.

Proof:

$$r(t) = x_1(t)^* x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$\text{FT}(r(t)) = R(jw) = \int_{-\infty}^{\infty} r(t) e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right) e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \, e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) d\tau \int_{-\infty}^{\infty} x_2(t-\tau) \, e^{-jwt} dt$$

Let $t - \tau = \propto$ so $dt = d \propto$

$$\text{FT}\left[x_1(t)^* x_2(t)\right] = \int_{-\infty}^{\infty} x_1(t) d\tau \int_{-\infty}^{\infty} x_2(\propto) \, e^{-jw(\propto+\tau)} d \propto$$

$$= \int_{-\infty}^{\infty} x_1(\tau) d\tau \int_{-\infty}^{\infty} x_2(\propto) \, e^{-jw\propto} \, e^{-jw\tau} d \propto$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \, e^{-jw\tau} d\tau \int_{-\infty}^{\infty} x_2(\propto) \, e^{-jw\propto} d \propto$$

$$\text{FT}\left[x_1(t)^* x_2(t)\right] = X_1(jw) \, X_2(jw)$$

(8a) Integration in time domain

If FT $(x(t)) = X(jw)$
then FT $\left(\int_{-\infty}^{t} x(\tau) d\tau\right) = \frac{1}{jw} \times (jw)$

Proof: Let $r(t) = \int_{-\infty}^{t} x(\tau) d\tau$
Differentiating w.r.t. t

$$\frac{dr(t)}{dt} = x(t) \Rightarrow \text{FT}(x(t)) = \text{FT}\left(\frac{d}{dt} r(t)\right)$$

From differentiation in time domain

$$X(jw) = jwX(jw)$$

$$R(jw) = \frac{1}{jw} X(jw)$$

$$\text{FT}(r(t)) = \text{FT}\left(\int_{-\infty}^{t} x(\tau) d\tau\right) = \frac{1}{jw} X(jw)$$

(8b) Differentiation in time domain

If FT $(x(t)) = X(jw)$

then $\left(\frac{d}{dt}x(t)\right) = jw \times (jw)$

Proof: We know that $x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(jw)e^{jwt}dw$. Differentiating both sides w.r.t. t

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(jw)\left(\frac{d}{dt}e^{jwt}\right)dw$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} jwX(jw)e^{jwt}dw$$

$$= j\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} (wX(jw))e^{jwt}dw$$

$$\frac{d}{dt}x(t) = j\ \mathrm{FT}^{-1}(wX(jw))$$

yields FT $\left(\frac{d}{dt}x(t)\right) = jwX(jw)$. On generalizing we get FT $\left(\frac{d^n}{dt^n}x(t)\right) = (jw)^n X(jw)$

(9) Differentiation in frequency domain

If FT $(x(t)) = X(jw)$

then FT $(tx(t)) = j\frac{d}{dw}X(jw)$

Proof: We know that $X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

On differentiating both sides w.r.t. w

$$\frac{d}{dw}X(jw) = \int\limits_{-\infty}^{\infty} x(t)\left(\frac{d}{dw}e^{-jwt}\right)dt = -\int\limits_{-\infty}^{\infty} j\ t\ x(t)e^{-jwt}dt$$

Multiplying both sides by j

$$j\frac{d}{dw}X(jw) = \int\limits_{-\infty}^{\infty} (tx(t))e^{-jwt}dt \quad \text{since } j^2 = -1 \text{ or } -j^2 = 1$$

$$j\frac{d}{dw}X(jw) = \mathrm{FT}\left[t\ x(t)\right]$$

$$\mathrm{FT}\left[t\ x(t)\right] = j\ \frac{d}{dw}\ X(jw)$$

(10) Convolution in frequency domain (multiplication in time domain (multiplication theorem))

$$\text{If } \mathrm{FT}(x_1(t)) = X_1(jw) \text{ and } \mathrm{FT}\left[x_2(t)\right] = X_2(jw)$$

$$\mathrm{FT}(x_1(t)x_2(t)) = \frac{1}{2\pi}(X_1(jw)^* X_2(jw))$$

Proof:

$$E = \int_{-\infty}^{\infty} \left|x(t)^2\right| dt = \int_{-\infty}^{\infty} \left|x(t)x^*(t)dt \tag{1}$$

$$\text{We know that } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{+jwt}dw$$

$$\text{So } x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw \tag{2}$$

on putting (1)

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(jw)e^{-jwt}dw\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(jw)\int_{-\infty}^{\infty} x(t)e^{-jwt}dt\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)X^*(jw)dw$$

$$= \int_{-\infty}^{\infty} \left|x(t)^2\right|dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X(jw)\right|^2 dw$$

Relation between Laplace Transform and Fourier Transform

Fourier transform $X(jw)$ of a signal $x(t)$ is given as

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \tag{1}$$

F.T. can be calculated only if $x(t)$ is absolutely integrable

$$= \int_{-\infty}^{\infty} \left|x(t)\right|dt < \infty \tag{2}$$

Laplace transform $X(s)$ of a signal $x(t)$ is given as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{3}$$

We know that $s = \sigma + jw$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+jw)t}dt$$

$$X(s) = \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right]e^{-jwt}dt \tag{4}$$

Comparing (1) and (4), we find that L.T. of $x(t)$ is basically the F.T. of $[x(t)e^{-\sigma t}]$.

If $s = jw$, i.e. $\sigma = 0$, then eqn. (4) becomes $X(s) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = X(jw)$

Thus, $X(s) = X(jw)$ when $\sigma = 0$ or $s = jw$

This means L.T. is same as F.T. when $s = jw$. The above equation shows that F.T. is special case of L.T. Thus, L.T. provides broader characterization compared to F.T., $s = jw$ indicates imaginary axis in complex s-plane.

2.7 APPLICATIONS OF FOURIER TRANSFORM OF NETWORK ANALYSIS

Example 24: Determine the voltage $V_{out}(t)$ to a current source excitation $i(t) = e^{-t}u(t)$ for the circuit shown in figure.

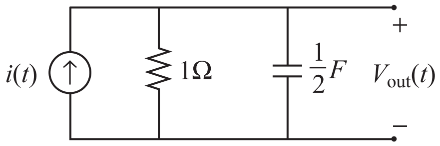
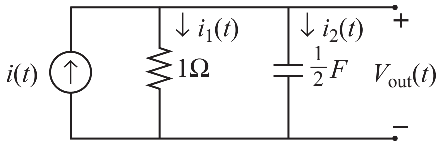


Fig. 2.26.

SOLUTION:



$$i(t) = i_1(t) + i_2(t)$$

$$i(t) = \frac{V_{out}(t)}{1} + \frac{1}{2}\frac{dV_{out}(t)}{dt}$$

$$\left\{\begin{array}{l} \text{since } i = \frac{V}{R} \\ \text{and } i = c\frac{dv}{dt} \text{ or } v = \frac{1}{c}\int i dt \end{array}\right.$$

$$e^{-t}u(t) = V_{out}(t) + \frac{1}{2}\frac{dV_{out}(t)}{dt} \qquad (1)$$

On taking the z-transform on both sides

$$\frac{1}{1+jw} = V_{out}(jw)\left\{1 + \frac{jw}{2}\right\} = \frac{(2+jw)}{2}V_{out}(jw)$$

$$V_{out}(jw) = \frac{2}{(1+jw)(2+jw)} = \frac{A}{1+jw} + \frac{B}{2+jw}$$

$$V_{out}(jw) = \frac{2}{1+jw} - \frac{2}{2+jw}$$

$$\left\{\begin{array}{l} A(2+jw) + B(1+jw) = 2 \\ 2A + B = 2 \\ A + B = 0 \; s_0 \; A = -B \\ 2A - A = 2; \quad A = 2, B = -2 \end{array}\right.$$

$$V_0(jw) = \frac{2}{6(jw)^2 + 7(jw) + 1} = \frac{2}{(6jw+1)(jw+1)}$$

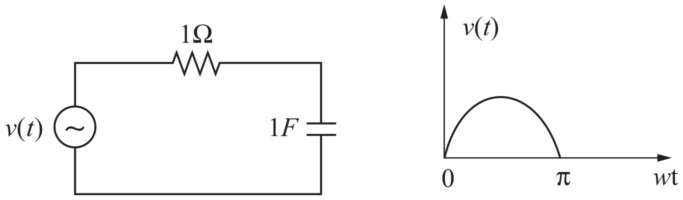
$$V_0(jw) = \frac{1/3}{(jw+1/6)(jw+1)} = \frac{A}{\frac{1}{6}+jw} + \frac{B}{1+jw}$$

$$V_0(jw) = \frac{2}{5\left(\frac{1}{6}+jw\right)} - \frac{2}{5(1+jw)} \tag{5}$$

Taking inverse Fourier transform, we get

$$V_0(t) = \frac{2}{5}\left(e^{-t/6} - e^{-t}\right)u(t) \tag{6}$$

Example 26: Determine the response of current in the network shown in Fig. 2.28(a) when a voltage having the waveform shown in Fig. 2.28(b) is applied to it by using the Fourier transform.



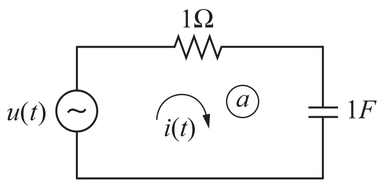
(a) **(b)**

Fig. 2.28.

SOLUTION:

Waveform $V(t)$ is defined as

$$V(t) = \sin t(u(t) - u(t-\pi)) \tag{1}$$



Let $i(t)$ be the current in the loop. Applying KVL in loop

$$V(t) = 1 \cdot i(t) + \frac{1}{1}\int_0^t i(t)dt = i(t) + \int_0^t i(t)dt \tag{2}$$

On taking Fourier transform of

$$V(jw) = \frac{1}{(jw)^2+1} + \frac{e^{-j\pi w}}{(jw)^2+1}$$

Since $$\text{FT}\left[\sin t u(t)\right] = \frac{1}{(jw)^2+1}$$

$$\text{FT}\left[\sin t u(t-\pi)\right] = \frac{e^{-j\pi w}}{(jw)^2+1}$$

Solve using F.T. formula

$$V(jw) = \frac{1+e^{-j\pi w}}{(jw)^2+1} \tag{3}$$

$$V(jw) = I(jw) + \frac{1}{jw} I(jw)$$

$$V(jw) = \left(1+\frac{1}{jw}\right) I(jw) = \frac{jw+1}{jw} I(jw)$$

$$I(jw) = \frac{jw}{jw+1} V(jw) \tag{4}$$

$$I(jw) = \frac{jw}{jw+1} \cdot \frac{(1+e^{-j\pi w})}{((jw)^2+1)} \quad \text{From (3)}$$

$$I(jw) = \frac{jw}{jw+1} \cdot \left\{ \frac{1}{(jw)^2+1} + \frac{e^{-j\pi w}}{(jw)^2+1} \right\}$$

$$= \underbrace{\frac{jw}{(jw+1)} \cdot \frac{1}{((jw)^2+1)}}_{I_1(jw)} + \underbrace{\frac{jw}{(jw+1)} \cdot \frac{1}{((jw)^2+1)} \cdot e^{-j\pi w}}_{I_2(jw)}$$

$$I_1(jw) = \frac{A}{jw+1} + \frac{Bjw+c}{((jw)^2+1)}$$

$$= \frac{-1/2}{(jw+1)} + \frac{\frac{1}{2}(jw+1)}{((jw)^2+1)}$$

$$i_1(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}\cos tut + \frac{1}{2}\sin t\delta t + \frac{1}{2}\sin tu(t)$$

Since IFT $\left\{\frac{1}{(jw)^2+1}\right\} = \sin tu(t)$

so IFT $\left(\frac{jw}{(jw)^2+1}\right) = \frac{d}{dt}\sin tu(t)$

Using differential in time domain property

$$\text{IFT}\left[\frac{jw}{(jw)^2+1}\right] = \cos tu(t) + \sin t\delta(t)$$

$$I_2(jw) = \frac{jw}{(jw+1)} \cdot \frac{1}{((jw)^2+1)} \cdot e^{-j\pi w}$$

$$I_2(jw) = I_3(jw) \cdot e^{-j\pi w}$$

Since $\quad I_3 = I_1(jw)$

so $\quad i_3(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}\cos tu(t) + \frac{1}{2}\sin t\delta(t) + \frac{1}{2}\sin tu(t)$

From time shifting property FT $(x(t \pm t_0)) = e^{\pm jwt_0} \times (jw)$

$$i_2(t) = i_3(t-\pi)$$

$$= -\frac{1}{2}e^{-(t-\pi)}u(t-\pi) + \frac{1}{2}\cos(t-\pi)u(t-\pi) + \frac{1}{2}\sin(t-\pi)\delta(t-\pi) + \frac{1}{2}\sin(t-\pi)u(t-\pi)$$

so $$i(t) = \frac{1}{2} - \left[-e^{-t} + \cos t + \sin t\right]u(t) + \frac{1}{2}\sin t\,\delta(t) + \frac{1}{2}\left[-e^{-(t-\pi)} + \cos(t-\pi) + \sin(t-\pi)\right]u(t-\pi) + \frac{1}{2}\sin(t-\pi)\delta(t-\pi)$$

Example 27: For the *RC* circuit shown in figure.

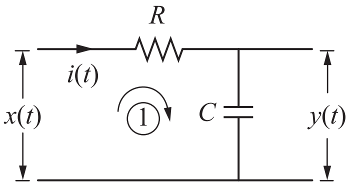


Fig. 2.29.

(a) Determine frequency response of the circuit.

(b) Find impulse response.

(c) Plot the magnitude and phase response for $RC = 1$.

SOLUTION:

Applying KVL in loop (1)

$$x(t) - Ri(t) - \frac{1}{C}\int_{-\infty}^{t} i(t)dt = 0$$

$$x(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt \tag{1}$$

$$\begin{cases} Since \\ V_R = iR \\ V_c = \frac{1}{C}\int i(t)dt \end{cases}$$

and $$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt \tag{2}$$

$$A = B - C$$

$$|H(jw)| = \frac{1}{\sqrt{1+w^2}} \tag{8}$$

$$H(jw) = 1 - (1 + jw)$$

$$= \tan^{-1}\frac{0}{1} - \tan^{-1} w = -\tan^{-1} w \tag{9}$$

For different values of w, we find $|H(jw)|$ and $H(jw)$

S. No	w	$\lvert H(jw)\rvert$	$H(jw)$
1–	$-\infty$	0	90°
2–	−50	0.0199	88.9°
3–	−20	0.0499	87.1°
4–	−10	0.099	84.3°
5–	−5	0.196	78.7°
6–	−2	0.447	63.4°
7–	−1	0.707	45°
8–	0	1	0
9–	1	0.707	−45°
10–	2	0.447	−63.4°
11–	5	0.196	−78.7°
12–	10	0.099	−84.3°
13–	20	0.0499	−87.1°
14–	50	0.0199	−88.9°
15–	∞	0	−90°

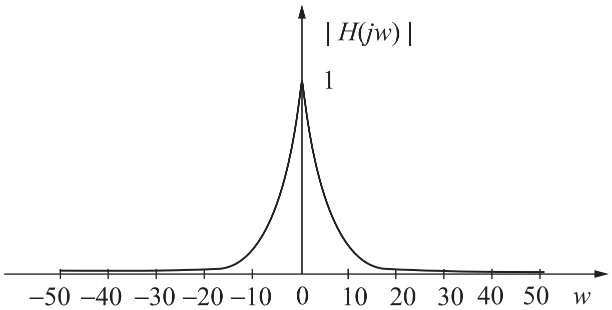


Fig. 2.30. Magnitude plot frequency response of the circuit

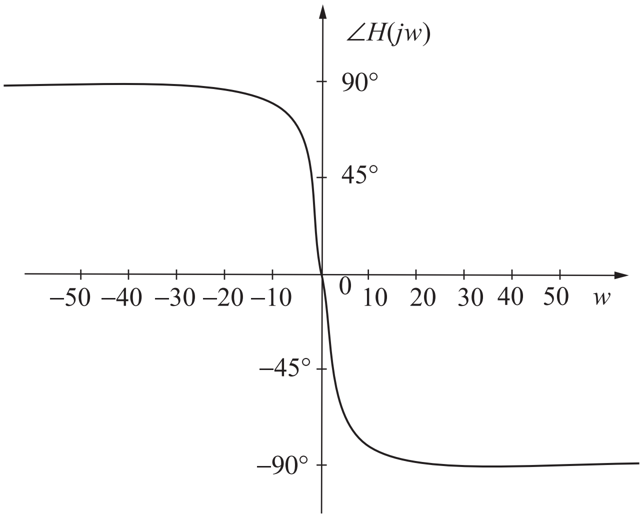


Fig. 2.31. Phase plot

Example 28: For the circuit shown in figure, determine the output voltage $V_0(t)$ to a voltage source excitation $V_{i(t)} = e^{-t}u(t)$ using Fourier transform

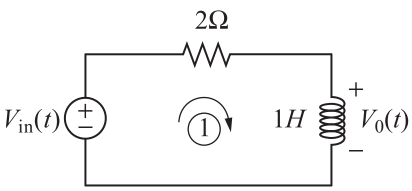


Fig. 2.32.

SOLUTION:

$$\text{Since } V_{in(t)} = e^{-t}u(t) \tag{1}$$

$$V_{in(jw)} = \frac{1}{1+jw} \tag{2}$$

Applying KVL in loop (1)

$$V_{in(t)} = 2i(t) + 1 \cdot \frac{di(t)}{dt}$$

$$V_{in(t)} = 2i(t) + \frac{di(t)}{dt} \tag{3}$$

$$V_0(t) = 1 \cdot \frac{di(t)}{dt}$$

$$V_0(t) = \frac{di(t)}{dt} \tag{4}$$

Q3: (i) State and prove the following properties of Fourier series:
(a) Time shifting property (b) Frequency shifting property
(ii) What are Dirichlet's conditions?

Q4: Find the fundamental period T, the fundamental frequency w_0 and the Fourier series coefficients a_n of the following periodic signal;

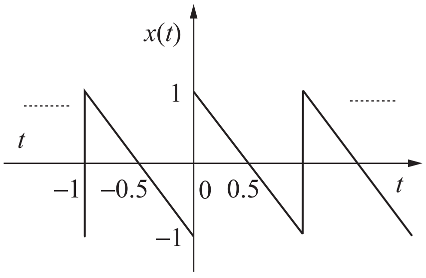


Fig. 2.3 P.

Q5: Obtain the Fourier series component of the periodic square wave signals.

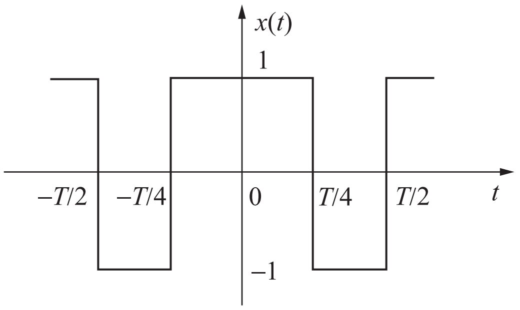


Fig. 2.4 P.

Q6: Determine the Fourier transform of the Gate function

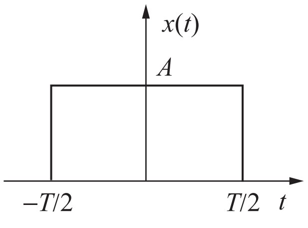


Fig. 2.5 P.

Q7: Determine the Fourier series representation of the signal

$$x(t) = \begin{cases} t - t^2 \text{ for } -\pi \le t \le \pi \\ 0 \quad \text{elsewhere} \end{cases}$$

Q8: For the continuous-time periodic signal

$$x(t) = 2 + \cos[2\pi t/3] + 4\sin[5\pi t/3]$$

determine the fundamental frequency w_0 and the Fourier series coefficients C_n such that

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

Q9: Find the Fourier transform of the following signals:

(a) $x(t) = \delta(t)$ (b) $x(t) = 1$ (c) $x(t) = \text{sgn}\,(t)$ (d) $x(t) = u(t)$

(e) $x(t) = \exp(-at)u(t)$ (f) $x(t) = \cos\,[w_0 t]\sin\,[w_0 t]$

Q10: Show that the Fourier transform of rect $(t-5)$ is $Sa(w/2)\exp(j5w)$. Sketch the resulting amplitude and phase spectrum.

Q11: Find the inverse Fourier transform of spectrum shown in figure.

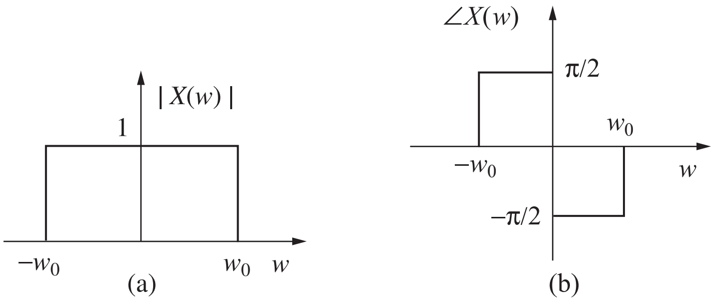


Fig. 2.6 P.

Q12: Find the Fourier transform of the following waveform.

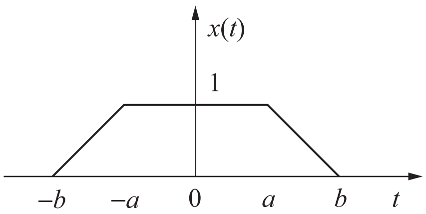


Fig. 2.7 P.

Q13: State and prove duality property of CTFT.

Q14: Determine the Fourier transform of the signal
$x(t) = \{tu(t)^*[u(t) - u(t-1)]\}$, where $u(t)$ is unit step function and * denotes the convolution operation.

Q15: Show that the frequency response of a CTLTIS is $Y(w) = H(w)X(w)$

where $X(w) =$ Fourier transform of the signal $x(t)$

$H(w) =$ Fourier transform of LTIS response $h(t)$

Q16: Find the Fourier transform of the signal $x(t)$ shown in figure below.

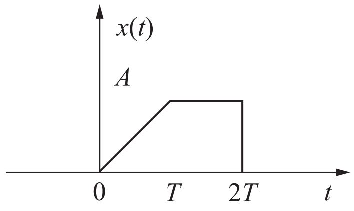


Fig. 2.8 P.

Q17: Determine the frequency response $H(jw)$ and impulse response $h(t)$ for a stable CTLTIS characterized by the linear constant coefficient differential equation given as

$$d^2y(t)/dt^2 + 4dy(t)/dt + 3y(t) = dx(t)/dt + 2x(t)$$

Q18: Find the Fourier transform of the signal $x(t)$ shown in figure below.

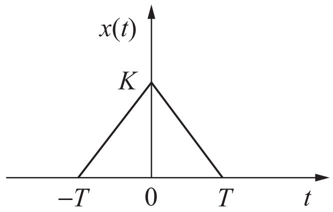


Fig. 2.9 P.

Q19: If $g(t)$ is a complex signal given by $g(t) = g_r(t) + jg_i(t)$ where $g_r(t)$ and $g_i(t)$ are the real and imaginary parts of $g(t)$ respectively. If $G(f)$ is the Fourier transform of $g(t)$, express the Fourier transform of $g_r(t)$ and $g_i(t)$ in terms of $G(f)$.

Q20: Find the coefficients of the complex exponential Fourier series for a half wave rectified sine wave defined by

$$x(t) = \begin{cases} A\sin(w_0 t), \ 0 \leq t \leq T_0/2 \\ 0, \ T_0/2 \leq t \leq T_0 \end{cases}$$

with $x(t) = x(t + T_0)$

Q21: (a) Show that the Fourier transform of the convolution of two signals in the time domain can be given by the product of the Fourier transform of the individual signals in the frequency domain.

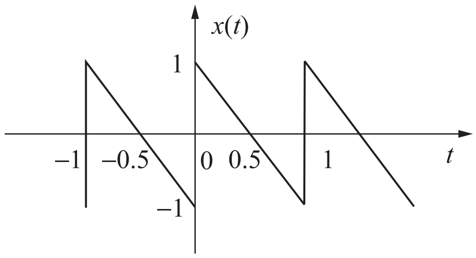
(b) Determine the Fourier transform of the signal

$$x(t) = \frac{1}{2}\left[\delta(t+1) + \delta(t-1) + \delta\left(t + \frac{1}{2}\right)\delta + \left(t - \frac{1}{2}\right)\right]$$

$$a_n = \frac{1-(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{n\pi}$$

Q3:



$$T = 1$$

$$w_0 = 2\pi \text{ rad/ sec}$$

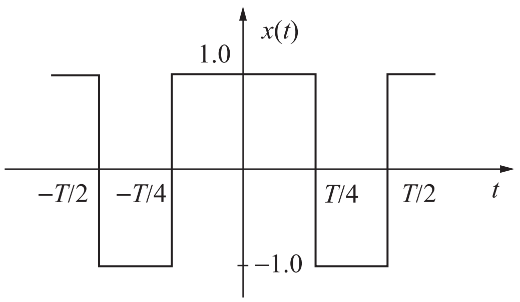
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$x(t) = -2t + 1$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n w_0 t\, dt$$

$$a_n = 0$$

Q4:



$$\frac{T}{2} - \left(-\frac{T}{4}\right) = \frac{3T}{4};\ w_0 = \frac{2\pi}{\frac{3T}{4}} = \frac{8\pi}{3T}$$

$$x(t) = \begin{cases} 1\left(-\frac{T}{4} \le t \le \frac{T}{4}\right) \\ -1\left(\frac{T}{4} \le t \le \frac{T}{2}\right) \end{cases}$$

$$a_0 = \frac{1}{\frac{3T}{4}}\left\{\int_{-\frac{T}{4}}^{\frac{T}{4}} dt + \int_{\frac{T}{4}}^{\frac{T}{2}} (-1)dt\right\} = \frac{4}{3T}\frac{T}{4} = \frac{1}{3}$$

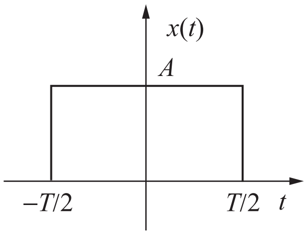
$$a_n = \frac{8}{3T}\left\{\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos\frac{8n\pi}{3T}dt - \int_{\frac{T}{4}}^{\frac{T}{2}} \cos\frac{8n\pi}{3T}t\,dt\right\}$$

$$a_n = \frac{1}{n\pi}\left[3\sin\frac{2n\pi}{3} - \sin\frac{4n\pi}{3}\right]$$

$$b_n = 0,\ \text{since even function}$$

$$x(t) = \frac{1}{3} + \frac{1}{\pi}\left[3\sin\frac{2\pi}{3} - \sin\frac{4\pi}{3} + \frac{3}{2}\sin\frac{4\pi}{3} - \frac{1}{2}\sin\frac{8\pi}{3} + \cdots\right]$$

Q5:



$$x(t) = \begin{cases} \mathrm{A} & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$X(jw) = A\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-jwt}dt = \frac{2A}{w}\sin\frac{wT}{2} = \frac{AT}{\frac{wT}{2}}\sin\frac{wT}{2}$$

$$X(if) = AT\,\mathrm{sinc}\,fT$$

Q6:

$$T_0 = 2\pi;$$

$$w_0 = 1;$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(t - t^2\right)dt = \frac{-\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(t - t^2\right)\cos nt\,dt = \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(t - t^2\right)\sin nt\,dt = \frac{-2(-1)^n}{n}$$

Taking inverse Fourier transform

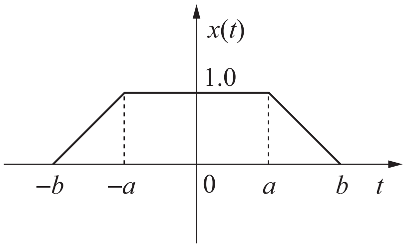
$$x_1(t) = \frac{1}{2\pi}\int_0^{w_0} -j\, e^{jwt} dw = \frac{1 - e^{jw_0 t}}{2\pi t}$$

$$x_2(t) = \frac{1}{2\pi}\int_{-w_0}^{0} j\, e^{jwt} dw = \frac{1 - e^{-jw_0 t}}{2\pi t}$$

$$x(t) = x_1(t) + x_2(t) = \frac{1}{2\pi t}(1 - e^{jw_0 t} + 1 - e^{-jw_0 t})$$

$$= \frac{1}{2\pi t}(2 - 2\cos w_0 t) = \frac{2\sin^2 \frac{w_0 t}{2}}{\pi t}$$

Q11:



$$x(t) = \begin{cases} \frac{t+b}{b-a} & \text{for} - b < t < -a \\ 1 & \text{for} - a < t < a \\ \frac{t-b}{a-b} & \text{for} \quad a < t < b \end{cases}$$

$$X(jw) = \frac{2}{w^2(b-a)}(\cos wa - \cos wb)$$

Q12:

$$x(t) = tu(t)^*[u(t) - u(t-1)]$$

$$x_1(t) = tu(t) \qquad x_2(t) = u(t) - u(t-1)$$

Differentiating in frequency domain property

$$\mathrm{FT}(tx(t)) = j\frac{d}{dw}X(jw)$$

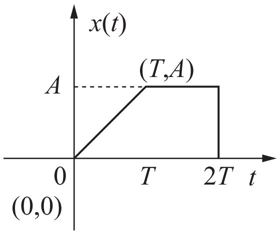
$$X_1(jw) = \frac{1}{(jw)^2}$$

$$X_2(jw) = \int_0^1 1.e^{-jwt} dt = \frac{1}{jw}(1 - e^{-jw})$$

$$X(jw) = X_1(jw)X_2(jw) = \frac{1}{(jw)^3}(1 - e^{-jw})$$

Q13: Prove convolution in time domain property.

Q14:



$$x(t) = \begin{cases} \frac{A}{T}t & 0 < t < T \\ A & T < t < 2T \end{cases}$$

$$X(jw) = \frac{A}{T}\int_0^T te^{-jwt}dt + A\int_T^{2T} e^{-jwt}dt$$

$$X(jw) = \frac{A}{T}\left[\frac{te^{-jwt}}{-jw}\int_0^T - \int_0^T \frac{e^{-jwt}}{jw}dt\right] + A\left[\frac{e^{-jwt}}{-jw}\bigg|\begin{matrix}2T\\T\end{matrix}\right]$$

$$= \frac{A}{T}\left\{\frac{Te^{jwt}}{-jw} + \frac{1}{w^2}\left(e^{-jwT} - 1\right)\right\} + A\left\{\frac{e^{-j2wT} - e^{-jwT}}{-jw}\right\}$$

$$= A\left\{\frac{e^{-jwt}}{-jw} + \frac{1}{w^2 T}\left(e^{-jwT} - 1\right)\right\} - \frac{A}{jw}e^{-jwT}\left(e^{-jwT} - 1\right)$$

$$= \frac{Ae^{-jw}T}{jw} + \frac{A}{w^2 T}e^{-jwT} - \frac{A}{w^2 T} - \frac{A}{jw}e^{-j2wT} + \frac{A}{jw}e^{-jwT}$$

$$= \frac{A}{wT}\left(\frac{1}{w}e^{-jwT} - \frac{1}{w} + jTe^{-2jwT}\right)$$

Q15:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \tag{1}$$

Taking Fourier transform on both sides

$$(jw)^2 Y(jw) + 4(jw)Y(jw) + 3Y(jw) = (jw)X(jw) + 2X(jw)$$

$$\left((jw)^2 + 4(jw) + 3\right)Y(jw) = ((jw) + 2)X(jw) \tag{2}$$

Frequency response $H(jw) = \dfrac{Y(jw)}{X(jw)} = \dfrac{2 + jw}{(jw)^2 + 4jw + 3}$ (3)

$$H(jw) = \frac{2 + jw}{(3 + jw)(1 + jw)} = \frac{A}{3 + jw} + \frac{B}{1 + jw}$$

$$x(t) = A\sin w_0 t \text{ for } 0 \le t \le \frac{T_0}{2}$$

$$= 0 \qquad \text{for } \frac{T_0}{2} \le t \le T_0$$

$$C_0 = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} A\sin w_0 t\,dt = \frac{A}{T_0}\left(\frac{-\cos w_0 t}{w_0}\,\Bigg|_0^{\frac{T_0}{2}}\right)$$

$$= -\frac{A}{T_0\cdot\frac{2\pi}{T_0}}\left[\cos w_0\cdot\frac{T_0}{2} - 1\right] = -\frac{A}{2\pi}[\cos\pi - 1] = \frac{A}{2}$$

$$C_n = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} A\sin w_0 t e^{-jnw_0 t}\,dt$$

$$= \frac{A}{2jT_0}\int_0^{\frac{T_0}{2}} (e^{jw_0 t} - e^{-jnw_0 t})e^{-jnw_0 t}\,dt$$

$$= \frac{A}{2jT_0}\int_0^{\frac{T_0}{2}} \left(e^{jw_0 t(1-n)} - e^{-jw_0 t(n+1)}\right)dt$$

$$= \frac{A}{2jT_0}\left(\frac{e^{jw_0 t(1-n)}}{jw_0(1-n)} - \frac{e^{-jw_0 t(n+1)}}{-jw_0(n+1)}\,\Bigg|_0^{\frac{T_0}{2}}\right)$$

$$= \frac{A}{2jT_0 w_0}\left(\frac{e^{jw_0(1-n)\frac{T_0}{2}}}{1-n} + \frac{e^{-jw_0(n+1)\frac{T_0}{2}}}{(n+1)} - \frac{1}{1-n} - \frac{1}{n+1}\right)$$

$$= -\frac{A}{4\pi}\left[\frac{e^{j\pi(1-n)}}{1-n} + \frac{e^{-j\pi(n+1)}}{n+1} - \frac{1}{1-n} - \frac{1}{n+1}\right]$$

$$= -\frac{A}{4\pi}\left(\frac{e^{j\pi}e^{-jn\pi}}{1-n} + \frac{e^{-jn\pi}\cdot e^{-j\pi}}{n+1} - \frac{1}{1-n} - \frac{1}{n+1}\right)$$

$$= -\frac{A}{4\pi}\left(\frac{-e^{-jn\pi}}{1-n} - \frac{e^{-jn\pi}}{n+1} - \frac{1}{1-n} - \frac{1}{n+1}\right) \qquad \text{Since } e^{j\pi} = -1$$

$$= \frac{A}{4\pi}\left(\frac{2e^{-jn\pi}}{1-n^2} + \frac{2}{1-n^2}\right)$$

$$= \frac{A}{2\pi(1-n^2)}(e^{-jn\pi} + 1)$$

Q19:

$$x(t) = \frac{1}{2}\left(\delta(t+1) + \delta(t-1) + \delta\left(t+\frac{1}{2}\right) + \delta\left(t-\frac{1}{2}\right)\right)$$

Taking Fourier transform on both sides

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \qquad (1)$$

$$X(jw) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\delta(t+1) + \delta(t-1) + \delta\left(t+\frac{1}{2}\right) + \delta\left(t-\frac{1}{2}\right)\right)e^{-jwt}dt$$

$$X(jw) = \frac{1}{2}\left(\int_{-\infty}^{\infty} \delta(t+1)e^{-jwt}dt + \int_{-\infty}^{\infty} \delta(t-1)e^{-jwt}dt + \int_{-\infty}^{\infty} \delta\left(t+\frac{1}{2}\right)e^{-jwt}dt \right.$$
$$\left. + \int_{-\infty}^{\infty} \delta\left(t-\frac{1}{2}\right)e^{-jwt}dt\right)$$

Since $\mathrm{FT}(\delta(t)) = 1$

So $\mathrm{FT}(\delta(t \pm t_0)) = e^{\pm jwt_0}dt$ $\qquad$ {using time shifting property}

$$X(jw) = \frac{1}{2}\left(e^{jw} + e^{-jw} + e^{j\frac{w}{2}} + e^{-j\frac{w}{2}}\right)$$

$$X(jw) = \frac{e^{jw} + e^{-jw}}{2} + \frac{e^{j\frac{w}{2}} + e^{-j\frac{w}{2}}}{2}$$

$$X(jw) = \cos\, w + \cos\frac{w}{2}$$

OBJECTIVE TYPE QUESTIONS

Q1: If the Fourier transform of a function $x(t)$ is $X(jw)$, then $X(jw)$ is defined as

(a) $\int_{-\infty}^{\infty} x(t)e^{jwt}dt$ (b) $\int_{-\infty}^{\infty} \frac{dx(t)}{dt}e^{-jwt}dt$
(c) $\int_{-\infty}^{\infty} x(t)dt$ (d) $\int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

Q2: If $X(jw)$ be the Fourier transform of $x(t)$, then

(a) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$ (b) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$
(c) $x(t) = \frac{1}{2\pi}\int_{0}^{\infty} X(jw)e^{jwt}dw$ (d) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{-jwt}dw$

Q3: Fourier transform of $x(t) = 1$ is

(a) $2\pi\, \delta(w)$ (b) $\pi\, \delta(w)$ (c) $3\pi\, \delta(w)$ (d) $4\pi\, \delta(w)$

Q4: Fourier transform of $x(t - t_0)$ is

(a) $e^{-jwt}{}_0X(jw)$ (b) $e^{jwt}{}_0X(jw)$ (c) $\frac{1}{t_0}X(jw)$ (d) $t_0e^{-jwt}{}_0X(jw)$

Q19: The trigonometric Fourier series of a periodic time function have

(a) sine terms (b) cosine term

(c) both (a) and (b) (d) DC term

Q20: Fourier series is defined as

$$x(t) = a_\circ + \sum_{n=1}^{\infty} (a_n \cos nw_0 t + b_n \sin w_0 t)$$

(a) True (b) False

Answers:

(1) d	(2) a	(3) a	(4) a	(5) b
(6) c	(7) a	(8) a	(9) a	(10) d
(11) c	(12) b	(13) b	(14) a	(15) a
(16) a	(17) e	(18) c	(19) c	(20) a

UNSOLVED PROBLEMS

Q1: Show that the Fourier transform of $x(t) = \delta(t+2) + \delta(t) + \delta(t-2)$ is $(1 + 2\cos 2w)$.

Q2: Show that the inverse Fourier transform of $X(jw) = 2\pi\delta(w) + \pi\delta(w-4\pi) + \pi\delta(w+4\pi)$ is $x(t) = 1 + \cos\ 4\pi t$.

Q3: Calculate the Fourier transform of $te^{-|t|}$, using the F.T. pair, FT $\left[e^{-|t|}\right] = \frac{2}{1+w^2}$. Also find the Fourier transform of $\frac{4t}{(1+t^2)^2}$ using duality property.

Q4: $X(jw) = \delta(w) + \delta(w-\pi) + \delta(w-5)$; find IFT $x(t)$ and show that $x(t)$ is non-periodic.

Q5: Find the Fourier transform of the triangular pulse as shown in figure.

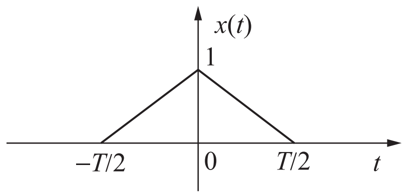


Fig. 2.10 P.

Ans. $X(jw) = \frac{T}{2} \sin c^2(\frac{wt}{4})$

Q6: Find the Fourier transform of $x(t) = \text{rect}(t/2)$. Ans. $X(jw) = 2\sin cw$

Q7: Find the Fourier transform of the signal $x(t) = \cos w_0 t$ by using the frequency shifting property.

Ans: $X(jw) = \pi[\delta(w-w_0) + \delta(w+w_0)]$

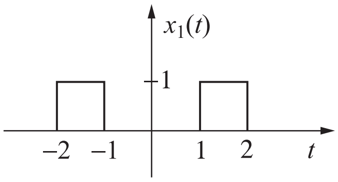
Q8: Show that FT $[\sin w_0 t u(t)] = \frac{w_0}{w_0^2 - w^2} + \frac{\pi j}{2}[\delta(w+w_0) - \delta(w-w_0)]$.

Q9: Find inverse Fourier transform of $X(jw) = \frac{jw}{(1+jw)^2}$

Ans. $x(t) = \frac{d}{dt}[te^{-t}u(t)]$

Q10: Sketch and then find the Fourier transform of following signals

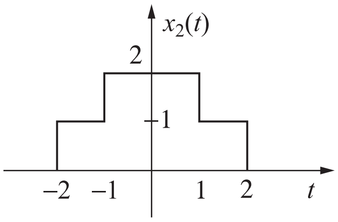
(a) $x_1(t) = \pi\left(t + \frac{3}{2}\right) + \pi\left(t - \frac{3}{2}\right)$ Ans. (a)



$X_1(jw) = 2\sin c\frac{w}{2}\cos 3\frac{w}{2}$

Fig. 2.11 P.

(b) $x_2(t) = \pi\left(\frac{t}{4}\right) + \pi\left(\frac{t}{2}\right)$ Ans. (b)



$X_2(jw) = 4\sin c2w + 2\sin cw$

Fig. 2.12 P.

Q11: Find the frequency response $x(jw)$ of the *RC* circuit shown in figure. Plot the magnitude and phase response for $RC = 1$

$$x(jw) = \frac{y(jw)}{x(jw)} = \frac{1}{1 + jwRC}$$

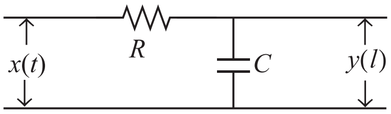


Fig. 2.13 P.

$$\text{Ans. } |x(jw)| = \frac{1}{\sqrt{1+w^2}}$$
$$x(jw) = -\tan^{-1} w$$

Q12: Find the Fourier series of the waveform shown in figure.

$$x(t) = \frac{2A}{jn\pi}\text{for } n = 1,3,5,7$$

$$x(t\pi)t = \frac{1}{\pi} + \frac{1}{2}\cos 5\pi t - \frac{2}{3\pi}\cos 10$$
$$- \frac{2}{8\pi}\cos 15\pi t$$

Q15: The output of a system is given by

$$x(t) = \begin{cases} A\sin w_0 t & \text{for} \quad 0 \le t \le \pi \\ 0 & \text{for} \quad \pi \le t \le 2\pi \end{cases}$$

Determine trigonometric form of Fourier series of $x(t)$

Ans. $\left[x(t) = \frac{A}{\pi} + \frac{A}{2}\cos(nt - \frac{\pi}{2}) + \sum_{n=2}^{\infty} \frac{2A}{\pi(1-n^2)}\cos nt\right]$

Signals and Systems
Lecture 7: Laplace Transform

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Introduction

- We had defined e^{st} as a basic function for CT LTI systems,s.t. $e^{st} \to H(s)e^{st}$
- In Fourier transform $s = j\omega$
- In Laplace transform $s = \sigma + j\omega$
- By Laplace transform we can
  - Analyze wider range of systems comparing to Fourier Transform
  - Analyze both stable and unstable systems
- The bilateral Laplace Transform is defined:

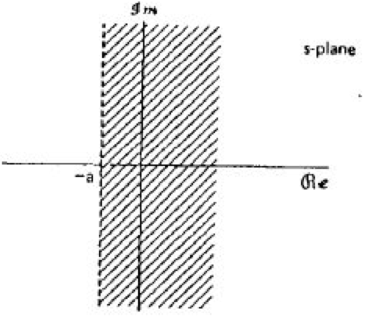
$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
\Rightarrow X(\sigma + j\omega) &= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt \\
&= \mathcal{F}\{x(t)e^{-\sigma t}\}
\end{aligned}
$$

Region of Convergence (ROC)

- Note that: $X(s)$ exists only for a specific region of s which is called Region of Convergence (ROC)
- ROC: is the $s=\sigma+j\omega$ by which $x(t)e^{-\sigma}$ converges:
  $ROC : \{s=\sigma+j\omega \ s.t. \ \int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt<\infty\}$
  - Roc does not depend on ω
  - Roc is absolute integrability condition of $x(t)e^{-\sigma t}$
- If $\sigma=0$, i,e, $s=j\omega \rightsquigarrow X(s)=\mathcal{F}\{x(t)\}$
- ROC is shown in s-plane
- The coordinate axes are $\mathcal{R}e\{s\}$ along the horizontal axis and $\mathcal{I}m\{s\}$ along the vertical axis.

Example

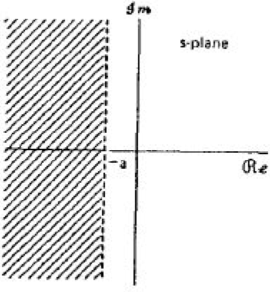
- Consider $x(t) = e^{-at}u(t)$
- $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \frac{-1}{s+a}e^{-(s+a)t}|_0^{\infty} = \frac{-1}{s+a}(e^{-(s+a)\infty} - 1)$
- If $\mathcal{R}e(s+a) > 0 \rightsquigarrow \mathcal{R}e(s) = \sigma > -\mathcal{R}e(a), X(s)$ is bounded
- $\therefore X(s) = \frac{1}{s+a},\ ROC : \mathcal{R}e(s) > -\mathcal{R}e(a)$



(a)

Example

- Consider $x(t) = -e^{-at}u(-t)$
- $X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = \frac{1}{s+a}e^{-(s+a)t}|_{-\infty}^{0} = \frac{1}{s+a}(1 - e^{(s+a)\infty})$
- If $\mathcal{R}e(s+a) < 0 \rightsquigarrow \mathcal{R}e(s) = \sigma < -\mathcal{R}e(a), X(s)$ is bounded
- $\therefore X(s) = \frac{1}{s+a},\ ROC : \mathcal{R}e(s) < -\mathcal{R}e(a)$



- In the recent two examples two different signals had similar Laplace transform but with different Roc
- To obtain unique $x(t)$ both $X(s)$ and ROC is required
- If $x(t)$ is defined as a linear combination of exponential functions, $\rightsquigarrow$ its Laplace transform $(X(s))$ is rational
- In LTI expressed in terms of linear constant-coefficient differential equations, Laplace Transform of its impulse response (its transfer function) is rational
- $X(s) = \frac{N(s)}{D(s)}$
  - Roots of $N(s)$ zeros of X(s); They make X(s) equal to zero.
  - Roots of $D(s)$ poles of X(s); They make X(s) to be unbounded.

- To study the stability of LTI systems zeros and poles are illustrated in s-plane (pole-zero plot)
- number of poles and zeros are equal for $-\infty$ to ∞
  - Consider degree of $D(s)$ (# of poles): m; degree of $N(s)$ (# of zeros): n
  - If $m < n \rightsquigarrow$ There are $n - m = k$ poles in ∞
  - If $m > n \rightsquigarrow$ There are $m - n = k$ zeros in ∞

ROC Properties

- ROC only depends on σ
  - In s-plane Roc is strips parallel to $j\omega$ axis
- If $X(s)$ is rational, Roc does not contain any pole
  - Since $D(s) = 0$, makes $X(s)$ unbounded
- If $x(t)$ is finite duration and is absolutely integrable, then ROC is entire s-plane
- If $x(t)$ is right sided and $\mathcal{R}e\{s\} = \sigma_0 \in$ ROC then $\forall s \;\; \mathcal{R}e\{s\} > \sigma_0 \in$ ROC
- If $x(t)$ is left sided and $\mathcal{R}e\{s\} = \sigma_0 \in$ ROC then $\forall s \;\; \mathcal{R}e\{s\} \leq \sigma_0 \in$ ROC
- If $x(t)$ is two sided and $\mathcal{R}e\{s\} = \sigma_0 \in$ ROC then ROC is a strip in s-plane including $\mathcal{R}e\{s\} = \sigma_0$

ROC Properties

- If $X(s)$ is rational
  - the ROC is bounded between poles or extends to infinity,
  - no poles of $X(s)$ are contained in ROC
  - If $x(t)$ is right sided, then ROC is in the right of the rightmost pole
  - If $x(t)$ is left sided, then ROC is in the left of the leftmost pole
- If ROC includes $j\omega$ axis then $x(t)$ has FT

Inverse of Laplace Transform (LT)

- By considering σ fixed, inverse of LT can be obtained from inverse of FT:
- $x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\underbrace{\sigma + j\omega}_{s})e^{j\omega t}d\omega$
- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}d\omega$
- assuming σ is fixed $\rightsquigarrow ds = jd\omega$
- $\therefore x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}ds$
- If $X(s)$ is rational , we can use expanding the rational algebraic into a linear combination of lower order terms and then one may use
    - $X(s) = \frac{1}{s+a} \rightsquigarrow x(t) = -e^{-at}u(-t)$ if $\mathcal{R}e\{s\} < -a$
    - $X(s) = \frac{1}{s+a} \rightsquigarrow x(t) = e^{-at}u(t)$ if $\mathcal{R}e\{s\} > -a$
- Do not forget to consider ROC in obtaining inverse of LT!

LT Properties

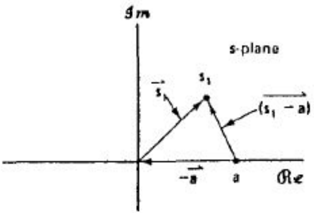
- Linearity: $ax_1(t) + bx_2(t) \Leftrightarrow aX_1(s) + bX_2(s)$
  - ROC contains: $R_1 \bigcap R_2$
  - If $R_1 \bigcap R_2 = \emptyset$ it means that LT does not exit
  - By zeros and poles cancelation ROC can be larger than $R_1 \bigcap R_2$
- Time Shifting:$x(t-T) \Leftrightarrow e^{-sT}X(s)$ with ROC$=R$
- Shifting in S-Domain: $e^{s_0 t}x(t) \Leftrightarrow X(s-s_0)$ with ROC$= R + \mathcal{R}e\{s_0\}$
- Time Scaling: $x(at) \Leftrightarrow \frac{1}{|a|}X(\frac{s}{a})$ with ROC $= \frac{R}{a}$
- Differentiation in Time-Domain: $\frac{dx(t)}{dt} \Leftrightarrow sX(s)$ with ROC containing R
- Differentiation in the s-Domain: $-tx(t) \Leftrightarrow \frac{dX(s)}{ds}$ with ROC $= R$
- Convolution: $x_1(t) * x_2(t) \Leftrightarrow X_1(s)X_2(s)$ with ROC containing $R_1 \cap R_2$

Analyzing LTI Systems with LT

- LT of impulse response is $H(s)$ which is named transfer function or system function.
- Transfer fcn can represent many properties of the system:
  - Causality: $h(t) = 0$ for $t < 0 \rightsquigarrow$ It is right sided
    - ROC of a causal system is a right-half plane
    - Note that the converse is not always correct
    - Example: $H(s) = \frac{e^s}{s+1},\ \mathcal{R}e\{s\} > -1 \rightsquigarrow h(t) = e^{-(t+1)}u(t+1)$ it is none zero for $-1 < t < 0$
    - For a system with rational transfer fcn, causality is equivalent to ROC being the right-half plane to the right of the rightmost pole
  - Stability: $h(t)$ should be absolute integrable $\rightsquigarrow$ its FT converges
    - An LTI system is stable iff its ROC includes $j\omega$ axis ($0 \in$ ROC)
  - A causal system with rational $H(s)$ is stable iff all the poles of $H(s)$ have negative real-parts (are in left-half plane)

Geometric Evaluation of FT by Zero/Poles Plot

- Consider $X_1(s) = s - a$

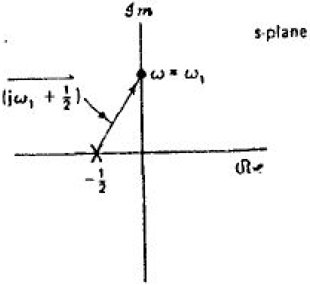


  - $|X_1|$: length of X_1
  - $\measuredangle X_1$: angel of X_1
- Now consider $X_2(s) = \frac{1}{s-a} = \frac{1}{X_1(s)}$
  - $log X_2 = -log X_1$
  - $\measuredangle X_2 = -\measuredangle X_1$

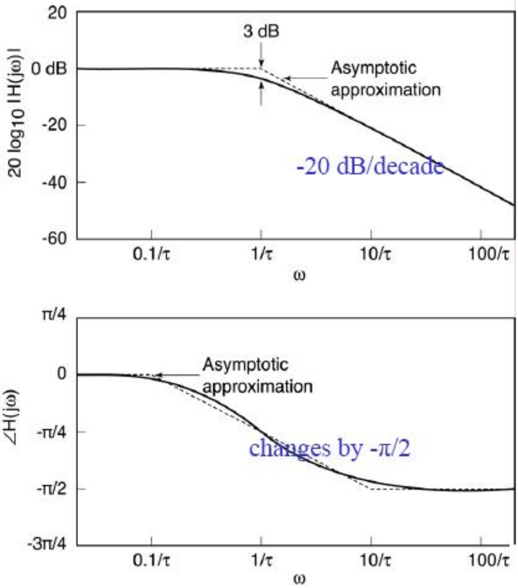
- For higher order fcns:

  $X(s) = M\frac{\prod_{i=1}^{R}(s-\beta_i)}{\prod_{j=1}^{P}(s-\alpha_j)}$
  - $|X(s)| = |M|\frac{\prod_{i=1}^{R}|s-\beta_i|}{\prod_{j=1}^{P}|s-\alpha_j|}$
  - $\measuredangle X(s) = \measuredangle M + \sum_{i=1}^{R}\measuredangle(s-\beta_i) - \sum_{j=1}^{R}\measuredangle(s-\alpha_j)$
- Example:

  $H(s) = \frac{1/2}{s+1/2}, \quad \mathcal{R}e\{s\} > \frac{-1}{2}$
  - $h(t) = \frac{1}{2}e^{-t/2}u(t)$
  - $s(t) = [1 - e^{-t/2}]u(t)$
  - $H(j\omega) = \frac{1/2}{j\omega+1/2}$
  - $|H(j\omega)|^2 = \frac{(1/2)^2}{w^2+(1/2)^2}$
  - $\measuredangle H(j\omega) = -\tan^{-1}2\omega$
  - $0 < \omega < \infty \rightsquigarrow -\pi/2 < \measuredangle H(j\omega) < 0$
  - $\omega\uparrow \rightsquigarrow |H|\downarrow, \measuredangle H(j\omega)\downarrow$



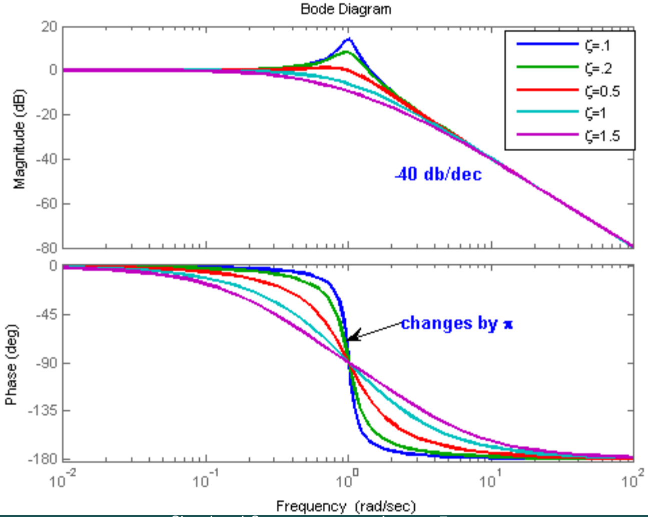
- Now let us substitute 2 with τ in the previous example
- $H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$
- $|H(j\omega)|^2 = \frac{(1/\tau)^2}{w^2 + (1/\tau)^2} = \begin{cases} 1 & \omega = 0 \\ \frac{1}{\sqrt{2}} & \omega = \frac{1}{\tau} \\ \frac{1}{\tau\omega} & \omega \gg \frac{1}{\tau} \end{cases}$
- $\measuredangle H(j\omega) = -\tan^{-1}\tau\omega = \begin{cases} 0 & \omega = 0 \\ \frac{-\pi}{4} & \omega = \frac{1}{\tau} \\ \frac{-\pi}{2} & \omega \gg \frac{1}{\tau} \end{cases}$
- Relation between real part of poles and response of the systems
  - τ is time constant of first order systems which control response speed of the systems
  - Poles are located at $-\frac{1}{\tau}$
  - The farther the poles from $j\omega$ axis $\rightsquigarrow$ cut-off freq. $\uparrow$, $\tau\downarrow$, the faster decaying the impulse response, the faster rise time of step response


20
0 dB
-20
-40
-60
20 log10 |H(jω)|
3 dB
Asymptotic approximation
-20 dB/decade
0.1/τ
1/τ
10/τ
100/τ
ω
π/4
0
-π/4
-π/2
-3π/4
∠H(jω)
Asymptotic approximation
changes by -π/2
0.1/τ
1/τ
10/τ
100/τ
ω

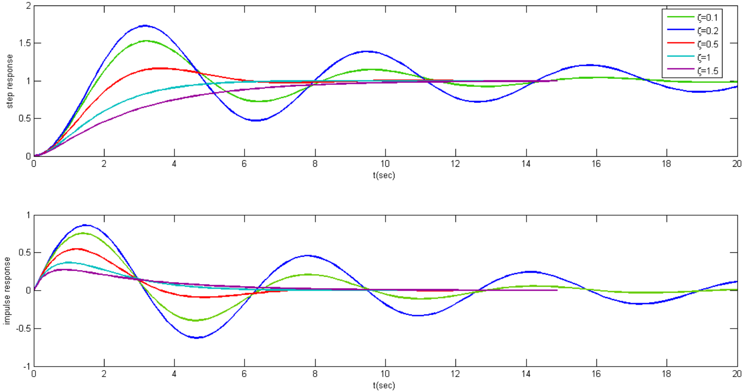
Response for Second Order system

- $h(t) = M(e^{c_1 t} - e^{c_2 t})u(t)$
- $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$
- $C_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- $0 < \zeta < 1$: under damp (two complex poles)
- $\zeta = 1$ critically damp ($s = -\omega_n$)
- $\zeta > 1$: Over damp (two negative real poles)
- For fixed ω_n, $\zeta \uparrow\uparrow \rightsquigarrow$, settling time for step response $\uparrow$

Bode Plot of $H(j\omega)$

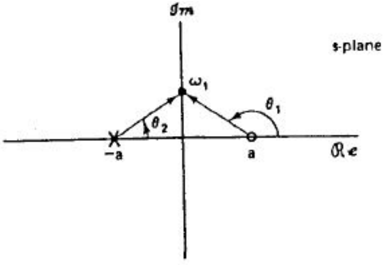


Impulse and Step Response of the second order system

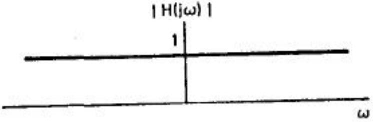


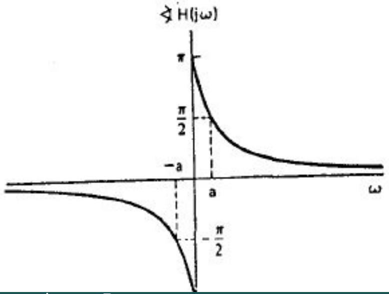
All Pass Filters

- Passes the signal in all freqs. with a little decreasing/increasing the magnitude
- Why do we use all-pass filters?
- $H(s) = \frac{s-a}{s+a} \quad \mathcal{R}e\{s\} > -a, \ a > 0$
- $|H(\jmath\omega)| = 1$
- $\measuredangle H(j\omega) = \theta_1 - \theta_2 = \pi - 2\theta_2 = \pi - 2tan^{-1}(\frac{\omega}{a}) = \begin{cases} \pi & \omega = 0 \\ \frac{\pi}{2} & \omega = a \\ 0 & \omega \gg a \end{cases}$


$\mathcal{I}m$
s-plane
ω_1
θ_1
θ_2
$-a$
a
$\mathcal{R}e$

(a)


$|H(j\omega)|$
1
ω


$\sphericalangle H(j\omega)$
π
$\frac{\pi}{2}$
$-a$
a
ω
$-\frac{\pi}{2}$

Stability Analysis by Routh-Hurwitz

- Remind: A system with rational transfer fcn is causal and stable if all of its poles are in LHP.
- $H(s) = \frac{N(s)}{D(s)}$, $D(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$
- How can we verify the stability of this system?
  - Method 1: Find the roots of $D(s)$
    - If n is large, it is difficult to find: –(
  - Method 2: Routh-Hurwitz method
    - Provide the following table

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\ \vdots & \vdots & & & \\ s^0 & h_{n-1} & & & \end{array}$$

    - First row includes odd coefficients of $D(s)$
    - Second row includes even coefficients of $D(s)$

Stability Analysis by Routh-Hurwitz

- b_i, c_i are defined as follows:

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \quad b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}, \quad c_{n-3} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}$$

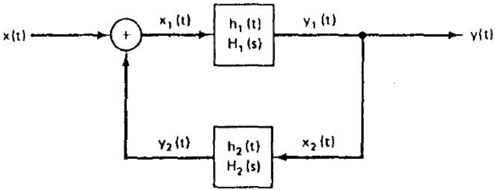
- Follow the same rule for other rows parameters
- # of RHP root os $D(s)$ equals to # of signs changing in the first column of the table
- Necessary condition for using Routh-Horwitz method is that all coefficients of $D(s)$ should exist and have similar sign(otherwise there are more than one pole on imaginary axis, it is not stable)
- Necessary and Sufficient conditions for stability is that no signs changing appears in the first column of the Routh-Horwitz table

- Initial Value Theorem: If $x(t) = 0$ for $t < 0$ and $x(t)$ does not contain any impulse or higher order singularities at the origin then $x(0^+) = \lim_{s\to\infty} sX(s)$
  - $X(s)$ may include a simple pole at the origin which represents a step signal.
  - More than one pole at the origin and in $j\omega$ axis make the signal oscillating
- Final Value Theorem: If $x(t) = 0$ for $t < 0$ and $x(t)$ is bounded when $t \to \infty$ then $x(\infty) = \lim_{s\to 0} sX(s)$
- Consider $H(s) = \frac{N(s)}{D(s)}$, n is degree of $N(s)$, d is degree of $D(s)$:
  - $H(0^+) = \begin{cases} 0 & d > n+1 \\ \text{constant value} \neq 0 & d = n+1 \\ \infty & d < n+1 \end{cases}$

LTI Systems Description

- $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$
- $\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$
- $H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$
- ROC depends on
  - placement of poles
  - boundary conditions (right sided, left sided, two sided,...)

- Feedback Interconnection of two LTI systems:
  - $Y(s) = Y_1(s) = X_2(s)$
  - $X_1(s) = X(s) + Y_2(s) = X(s) + H_2(s)Y(s)$
  - $Y(s) = H_1(s)X_1(s) = H_1(s)[X(s) + H_2(s)Y(s)]$
  - $\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 - H_2(s)H_1(s)}$
  - ROC: is determined based on roots of $1 - H_2(s)H_1(s)$



Block Diagram Representation for Causal LTI Systems

- We can represent a transfer fcn by different methods:
- Example: $H(s) = \frac{2s^2+4s-6}{s^2+3s+2}$
  1. $H(s) = (2s^2+4s-6)\frac{1}{s^2+3s+2}$
  2. Assuming it is causal so it is at initial rest
     - $W(s) = \frac{1}{s^2+3s+2}X(s) \Leftrightarrow \frac{d^2w}{dt^2} + 3\frac{dw}{dt} + 2w = x(t)$
     - $Y(s) = (2s^2+4s-6)W(s) \Leftrightarrow y(t) = 2\frac{dw^2}{dt^2} + 4\frac{dw}{dt} - 6w$
  3. $H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1}$
  4. $H(s) = \frac{2(s-1)}{s+2}\frac{s+3}{s+1}$

Unilateral LT

- It is used to describe causal systems with nonzero initial conditions: $\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt = \mathcal{UL}\{x(t)\}$
- If $x(t) = 0$ for $t < 0$ then $\mathcal{X}(s) = X(s)$
- Unilateral LT of $x(t)$ = Bilateral LT of $x(t)u(t^-)$
- If $h(t)$ is impulse response of a causal LTI system then $H(s) = \mathcal{H}(s)$
- ROC is not necessary to be recognized for unilateral LT since it is always a right-half plane
- For rational $\mathcal{X}(s)$, ROC is in right of the rightmost pole

Similar Properties of Unilateral and Bilateral LT

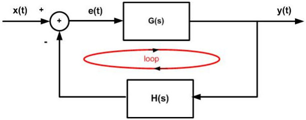
- Convolution: Note that for unilateral LT, If both $x_1(t)$ and $x_2(t)$ are zero for $t < 0$, then $\mathcal{X}(s) = \mathcal{X}_1(s)\mathcal{X}_2(s)$
- Time Scaling
- Shifting in s domain
- Initial and Finite Theorems: they are indeed defined for causal signals
- Integrating: $\int_{0^-}^{t} x(\tau)d\tau = x(t) * u(t) \overset{UL}{\Leftrightarrow} \mathcal{X}(s)\mathcal{U}(s) = \frac{1}{s}\mathcal{X}(s)$
- The main difference between $\mathcal{UL}$ and LT is in time differentiation:
  - $\mathcal{UL}\{\frac{dx(t)}{dt}\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$
  - Use the rule $\int fdg = fg - \int gdf$
  - $\therefore \mathcal{UL}\{\frac{dx(t)}{dt}\} = s\int_{0^-}^{\infty} x(t)e^{-st}dt + x(t)e^{-st}|_{0^-}^{\infty} = s\mathcal{X}(s) - x(0^-)$
  - $\mathcal{UL}\{\frac{dx(t)}{dt}\} = s\mathcal{X}(s) - x(0^-)$
  - $\mathcal{UL}\{\frac{d^2x(t)}{dt^2}\} = \mathcal{UL}\{\frac{d}{dt}\{\frac{dx(t)}{dt}\}\} = s(s\mathcal{X}(s) - x(0^-)) - \acute{x}(0^-) = s^2\mathcal{X}(s) - sx(0^-) - \acute{x}(0^-)$
  - Follow the same rule for higher derivatives

Example

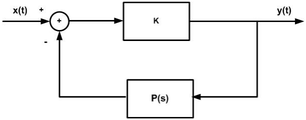
- Consider $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t)$, where
  $y(0^-) = \beta = 3,\ \acute{y}(0^-) = \gamma = -5,\ x(t) = \alpha = 2u(t)$
- Take $\mathcal{UL}$:
  - $s^2\mathcal{Y}(s) - \beta\mathcal{Y}(s) - \gamma + 3(s\mathcal{Y}(s) - \beta) + 2\mathcal{Y}(s) = \mathcal{X}(s)$
  - $\mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)+\gamma}{s^2+3s+2}}_{ZIR} + \underbrace{\frac{\mathcal{X}(s)}{s^2+3s+2}}_{ZSR}$
  - Zero State Response (ZSR): is a response in absence of initial values
    - $\mathcal{H}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)}$
    - Transfer fcn is ZSR
    - ZSR: $\mathcal{Y}_1(s) = \frac{\alpha}{s(s+1)(s+2)} = \frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}$
    - $y_1(t) = (1 - 2e^{-t} + e^{-2t})u(t)$
  - Zero Input Response (ZIR): is a response in absence of input $(x(t) = 0)$
  - ZIR: $\mathcal{Y}_2(s) = \frac{3(s+3)-5}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{2}{s+2}$
  - $y_2(t) = (e^{-t} + 2e^{-2t})u(t)$
- $y(t) = y_1(t) + y_2(t)$

Feed Back Applications

- Closed loop Transfer fcn: $Q(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{Open\ \ loop\ \ Gain}{1-Loop\ \ Gain}$

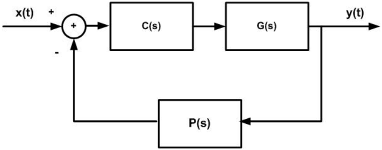


1. Inverting



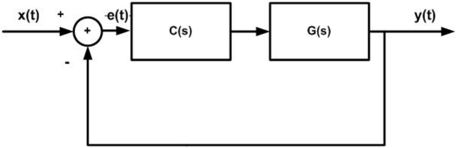
  - $Q(s) = \frac{K}{1+Kp(s)}$
  - If choose K s.t. $Kp(s) \gg 1$ then $Q(s) \simeq \frac{1}{p(s)}$
  - Example: For a capacitor, consider i as output and v as input, it is a differentiator
  - By using the above interconnection, we can make an integrator

2. Stabilizing Unstable Systems



- $G(s)$ is unstable
- We should define $P(s)$ and $C(s)$ to make closed-loop system stable (poles of closed-loop system be in LHP)
- $Q(s) = \frac{C(s)G(s)}{1+C(s)P(s)G(s)}$
- Example 1: $G(s) = \frac{1}{s-2}, \ C(s) = K, \ P(s) = 1$
- $Q(s) = \frac{K}{s-2+K}$
- Choosing $K > 2$ make it stable
- Example 2: $G(s) = \frac{1}{s^2-4}$
- By $C(s) = K$ cannot be stabilized
- Choose $C(s) = K_1 + K_2 s$, $K_2 > 0$, and $K_1 > 4$ can stabilize the closed-loop system

3. Tracking



- Objective: Defining $C(s)$ s.t. $e(t) = x(t) - y(t) \to 0$ as $t \to \infty$
- $E(s) = \frac{1}{1+C(s)G(s)} X(s)$
- Consider $x(t)$ as unite step
- $\lim e(t)_{t\to\infty} = \lim sE(s)_{s\to 0} = \lim_{s\to 0} \frac{s}{1+C(s)G(s)} \frac{1}{s}$
- If we choose $C(s)$ s.t. $C(s)G(s) \gg 1$ then $e(t) \to 0$ as $t \to \infty$

4. Decreasing effect of disturbance

5. Decreasing Sensitivity to uncertainties

State Space Representation

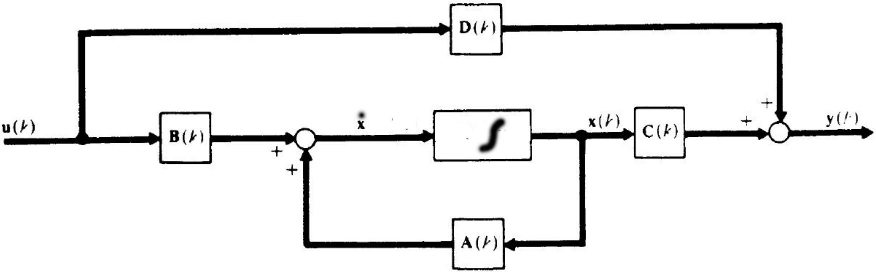
- Previously we learnt that for a LTI system with $y(t)$: output signal, $u(t)$: input signal, and $h(t)$ impulse response
  $y(t) = h(t) * u(t) \rightsquigarrow Y(s) = H(s)U(s)$
- These representation of the system only express I/O relation
- It does not give us internal specification of the system.
- State space representation not only provide us information on I/O but also gives us good view on internal specification of the system
- States of a system at time t_0 includes min required information to express the system situation at time t_0
- They are first degree equations

- State space representation of LTI system

$$\begin{aligned} \dot{X}(t) &= AX(t) + BU(t) \text{ state equations} \\ Y(t) &= CX(t) + DU(t) \text{ output equations} \end{aligned}$$

  - $X \in R^n$: state vector
  - $U \in R^m$: input vector
  - $Y \in R^p$ output vector
  - $A^{n \times n}$: System Matrix
  - $B^{n \times m}$: input matrix
  - $C^{p \times n}$: output matrix
  - $D^{p \times m}$: coupling matrix
- Number of state usually equals to degree of the system
  - It usually equals to number of active elements in the system (# of capacitors and inductors in RLC circuits)
  - However in some cases like having cut-set of inductors and loop of capacitors degree of the system would be less than # of active elements
  - One could choose number of the states greater than n in such case some modes are not observable or controllable
- Set of states is not unique for a system

Block Diagram of State Space Representation



Solving State Equations by UL

- Assuming x is causal $\rightsquigarrow$ we are using UL
- $\dot{x} = Ax + Bu \overset{UL}{\Leftrightarrow} sX(s) - x(0) = AX(s) + BU(s)$
- $X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$
- Let us define $\phi(t) = L^{-1}\{(sI - A)^{-1}\}$: Transition Matrix
- $x(t) = \underbrace{\phi(t)x(0)}_{ZIR} + \underbrace{\int_0^t \phi(\tau)Bu(t-\tau)d\tau}_{ZSR}$
- For LTI systems $\phi(t) = e^{At}$

Methods to Find Transition Matrix

1. $\phi(t) = L^{-1}(sI - A)^{-1}$
   - Example: $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$
   - $\phi(t) = L^{-1}(sI - A)^{-1} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$
   - For large A, finding inverse matrix is time consuming and complicated

Methods to Find Transition Matrix

2. Approximate by Infinite Power Series

- The transition matrix is system specification and input does not affect on it:

$$\begin{aligned} \dot{x} &= Ax(t) \quad (1) \\ x(t) &= \Phi(t)x(0) \quad (2) \end{aligned}$$

- Let us represent transition matrix by an infinite power series:

$$x(t) = (k_0 + k_1 t + k_2 t^2 + \ldots)x_0 \quad (3)$$

- $\dot{x}(t) = (k_1 + 2k_2 t + \ldots)x_0$
- $\therefore (k_1 + 2k_2 t + 3k_3 t^2 + \ldots)x_0 = A(k_0 + k_1 t + \ldots)x_0$
- $k_1 = Ak_0,\ k_2 = A\frac{k_1}{2},\ k_3 = A\frac{k_2}{3}$
- Substitute $t = 0$ in (3): $k_0 = I$
- $k_0 = I,\ k_1 = A,\ k_2 = \frac{A^2}{2!},\ k_3 = \frac{A^3}{3!}$
- $\phi(t) = e^{At} = I + At + A^2\frac{t^2}{2!} + \ldots$

Methods to Find Transition Matrix

3. By Cayley Hamilton Theorem
   - Reminder: Eigne Value of Matrix A is a scalar value λ s.t.
     - $Av = \lambda v$
     - where v is a vector named Eigne vector
   - To find eigne values:
     - $|\lambda I - A| = 0 \rightsquigarrow \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$
     - The above equation is named characteristic equation of matrix A
   - Considering Cayley Hamilton Theorem result in [?]: $e^{At} = a_0(t)I + a_1(t)A + \ldots + a_{n-1}(t)A^{n-1}$
   - Eigne vector of matrix A is eigne vector ofe^{At}
     $$\left.\begin{array}{r} Av_i = \lambda_i v_i \\ A^2 v_i = \lambda_i^2 v_i \\ \vdots A^n v_i = \lambda_i^n v_i \end{array}\right\} \Rightarrow e^{\lambda_i t} v_i = (a_0(t)I + a_1(t)\lambda_i + a_2(t)\lambda_i^2 + \ldots + a_{n-1}(t)\lambda^{n-1})v_i$$
   - By assuming n distinct eigne values and solving n equations all coefficients $a_i(t)$ are obtained

Example

- $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$
- $\lambda_1 = -2,\ \lambda_2 = -3$
- $e^{-2t} = a_0(t) - 2a_1(t)$
- $e^{-3t} = a_0(t) - 3a_1(t)$
- $a_1(t) = e^{-2t} - e^{-3t}$
- $a_0(t) = 3e^{-2t} - 2e^{-3t}$
- $\phi(t) = e^{At} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$

Defining Transfer Function from State Space Eq.

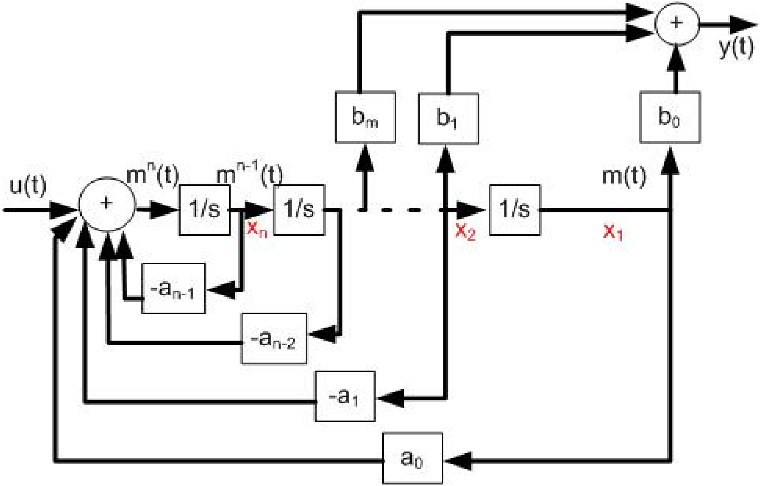
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- Transfer fcn: $H(s) = \frac{Y(s)}{X(s)}$
- Transfer fcn is ZIR:
  $sX(s) = AX(s) + BU(s) \rightsquigarrow X(s) = (sI - A)^{-1}BU(s)$
- $Y(s) = [C(sI - A)^{-1}B + D]U(s)$
- $H(s) = C(sI - A)^{-1}B + D = C\frac{adj(sI-A)}{det(sI-A)}B + D$
- Poles of a system are eigne values of matrix A
- BUT all eigne values of A are not poles of the system (due to zero-pole cancelation)
  - If an unstable poles is canceled by a zero the system is not internally stable anymore

State Space Realizations

- Several state space realization can be obtained from a transfer fcn. two of them are introduced here.
  1. Controllable Canonical Form
     - Consider $H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0} = \frac{b(s)}{a(s)}$, $n > m$
     - If $n = m$ then we can define $H(s) = b_n + \frac{\bar{b}_m s^m + \bar{b}_{m-1} s^{m-1} + \ldots + \bar{b}_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0}$
     - Let us define a axillary fcn $M(s)$
     - $\frac{Y(s)}{U(s)} = \frac{Y(s)}{M(s)} \cdot \frac{M(s)}{U(s)} = b(s) . \frac{1}{a(s)}$
     - $M(s)a(s) = U(s) \rightsquigarrow M(s)(s^n + a_{n-1}s^{n-1} + \ldots + a_0) = U(s)$
     - $m^n(t) = -a_{n-1} m^{n-1}(t) - \ldots - a_0 m(t) + u(t)$
     - $Y(s) = b(s)M(s) \rightsquigarrow y(t) = b_m m(t)^m + \ldots + b_1 \dot{m}(t) + b_0 m(t)$

Controllable Canonical Form



- (assume $m = n-1$) By defining output of integrators as states:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + u \\
y &= b_0 x_1 + b_1 x_2 + \ldots + b_{n-1} x_n
\end{aligned}$$

- $\therefore A = \begin{bmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$

  $C = [b_0 \; b_1 \; \ldots \; \ldots \; b_{n-1}], D = 0$

- **Example:** $H(s) = \frac{s^3+6s^2+5s+2}{s^3+7s^2+3s+1}$

2. Diagonal Form and Jordan Form

- Consider characteristic equation has n separate roots:$H(s) = \frac{\beta_1}{s-P_1} + \frac{\beta_2}{s-P_2} + \frac{\beta_3}{s-P_3} + \ldots + \frac{\beta_n}{s-P_n}$
- $\therefore A = \begin{bmatrix} P_1 & \ldots & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & P_n \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$
  $C = [\beta_1 \ \beta_2 \ \ldots \ \beta_n], D = 0$

2. Diagonal Form and Jordan Form

- If there are frequent poles, for example if there are three similar poles $:H(s) = \frac{\beta_1}{s-P_1} + \frac{\beta_2}{(s-P_2)^3} + \frac{\beta_3}{(s-P_2)^2} + \frac{\beta_4}{s-P_2}$, matrices A, B, and C are modified as follows:

$$A = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & P_2 & 1 & 0 \\ 0 & 0 & P_2 & 1 \\ 0 & 0 & 0 & P_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4], D = 0$$

- **Example:** $H(s) = \frac{s^2+3s+1}{(s+1)^2(s+3)}$

Chapter 3

The z-transform and Analysis of LTI Systems

Contents

Primary points
- Convolution of discrete-time signals simply becomes multiplication of their z-transforms.
- Systematic method for finding the **impulse response** of LTI systems described by **difference equations**: **partial fraction expansion**.
- Characterize LTI discrete-time systems in the z-domain

Secondary points
- Characterize discrete-time signals
- Characterize LTI discrete-time systems and their response to various input signals

3.1

The z-transform

We focus on the bilateral z-transform.

3.1.1 The bilateral z-transform

The **direct z-transform** or **two-sided z-transform** or **bilateral z-transform** or just the **z-transform** of a discrete-time signal $x[n]$ is defined as follows.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \quad \text{or} \quad X(\cdot) = Z\left\{x[\cdot]\right\} \quad \text{or shorthand:} \quad x[n] \overset{Z}{\leftrightarrow} X(z)\,.}$$

- Note capital letter for transform.
- In the math literature, this is called a **power series**.
- It is a mapping from the space of discrete-time signals to the space of functions defined over (some subset of) the complex plane.
- We will also call the complex plane the z-plane.

We will discuss the **inverse z-transform** later.

Convergence

Any time we consider a summation or integral with infinite limits, we must think about **convergence**.

We say an infinite series of the form $\sum_{n=-\infty}^{\infty} c_n$ **converges** [1, p. 141] if there is a $c \in \mathbb{C}$ such that $\lim_{N\to\infty} \left| c - \sum_{n=-N}^{N} c_n \right| = 0$.

- Some infinite series do converge to a finite value, *e.g.*, $1 + 1/2 + 1/4 + 1/8 + \cdots = \frac{1}{1-1/2} = 2$,
  since $\left| 2 - \sum_{n=0}^{N} (1/2)^n \right| = \left| 2 - \frac{1-(1/2)^{N+1}}{1-1/2} \right| = (1/2)^N \to 0$ as $N \to \infty$.
- One can also extend the notion of convergence to include "convergence to ∞" [2, p. 37].
  Example. The infamous **harmonic series** is an infinite series that converges to infinity: $1 + 1/2 + 1/3 + 1/4 + \cdots = \infty$.
- Some infinite series simply do not converge, *e.g.*, $1 - 1 + 1 - 1 + \cdots = \ ?$

The z-transform of a signal is an infinite series for each possible value of z in the complex plane. Typically only some of those infinite series will converge. We need terminology to distinguish the "good" subset of values of z that correspond to convergent infinite series from the "bad" values that do not.

Definition of ROC

On *p. 152*, the textbook, like many DSP books, defines the **region of convergence** or **ROC** to be:
"the set of all values of z for which $X(z)$ attains a finite value."

Writing each z in the polar form $z = r\, e^{\jmath\phi}$, on *p. 154*, the book says that: "finding the ROC for $X(z)$ is equivalent to determining the range of values of r for which the sequence $x[n]\, r^{-n}$ is **absolutely summable**."

Unfortunately, that claim of equivalence is *incorrect* if we use the book's definition of ROC on *p. 152*. There are examples of signals, such as $x[n] = \frac{1}{n}\, u[n-1]$, for which certain values of z lead to a convergent infinite series, but yet $x[n]\, r^{-n}$ is *not* absolutely summable.

So we have two possible *distinct* definitions for the ROC: "the z values where $X(z)$ is finite," or, "the z values where $x[n]\, z^{-n}$ is absolutely summable." Most DSP textbooks are not rigorous about this distinction, and in fact either definition is fine from a practical perspective. The definitions are compatible in the case of z-transforms that are rational, which are the most important type for practical DSP use. To keep the ROC properties (and Fourier relations) simple, we adopt the following definition.

$$\boxed{\begin{array}{c}\text{The ROC is the set of values } z \in \mathbb{C} \text{ for which the sequence } x[n]\, z^{-n} \text{ is absolutely summable, } \textit{i.e.}, \\ \left\{ z \in \mathbb{C} : \sum_{n=-\infty}^{\infty} \left| x[n]\, z^{-n} \right| < \infty \right\}.\end{array}}$$

All absolutely summable sequences have convergent infinite series [1, p. 144]. But there are some sequences, such as $(-1)^n/n$, that are not absolutely summable yet have convergent infinite series. These will not be included in our definition of ROC, but this will not limit the practical utility.

Skill: ***Finding a z-transform completely, including both $X(z)$ and the ROC.***

Example. $x[n] = \delta[n]$. $X(z) = 1$ and ROC = $\mathbb{C}$ = entire z-plane.

Example. $x[n] = \delta[n-k]$. $X(z) = z^{-k}$ and

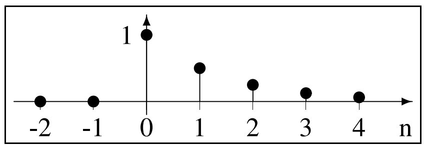
$$\text{ROC} = \begin{cases} \mathbb{C}, & k = 0 \\ \mathbb{C} - \{0\}, & k > 0 \\ \mathbb{C} - \{\infty\}, & k < 0. \end{cases} \qquad \boxed{\delta[n-k] \overset{Z}{\leftrightarrow} z^{-k}}$$

Example. $x[n] = \{4, \underline{3}, 0, \pi\}$. $X(z) = 4z + 3 + \pi z^{-2}$, ROC = $\mathbb{C} - \{0\} - \{\infty\}$

For a **finite-duration signal**, the ROC is the entire z-plane, possibly excepting $z = 0$ and $z = \infty$.

Why? Because for $k > 0$: z^k is infinite for $z = \infty$ and z^{-k} is infinite for $z = 0$; elsewhere, polynomials in z and z^{-1} are finite.

Example. $x[n] = p^n\, u[n]$. **Skill:** ***Combining terms to express as geometric series.***



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = \sum_{n=0}^{\infty} (pz^{-1})^n = 1 + \left(\frac{p}{z}\right) + \left(\frac{p}{z}\right)^2 + \left(\frac{p}{z}\right)^3 + \cdots = \frac{1}{1 - pz^{-1}}.$$

The series converges iff $\left|pz^{-1}\right| < 1$, *i.e.*, if $\{|z| > |p|\}$.

$$\boxed{p^n\, u[n] \overset{Z}{\leftrightarrow} \frac{1}{1 - pz^{-1}}, \text{ for } |z| > |p|}$$ ***Picture 3.2 shading outside circle radius*** $|p|$

Smaller $|p|$ means faster decay means larger ROC.

Example. Important special case: $p = 1$ leaves just the unit step function. $\boxed{u[n] \overset{Z}{\leftrightarrow} U(z) = \frac{1}{1 - z^{-1}},\ |z| > 1}$

Example. $x[n] = -p^n\, u[-n-1]$ for $p \neq 0$. ***Picture*** . An **anti-causal** signal.

$$X(z) = \sum_{n=-\infty}^{-1} -p^n z^{-n} = -\sum_{k=1}^{\infty} (p^{-1}z)^k = -(p^{-1}z) \sum_{k=0}^{\infty} (p^{-1}z)^k = -p^{-1}z \frac{1}{1 - p^{-1}z} = \frac{1}{1 - pz^{-1}}.$$

The series converges iff $\left|p^{-1}z\right| < 1$, *i.e.*, if $|z| < |p|$. ***Picture 3.3 shading inside circle radius*** $|p|$

Note that the last two examples *have the same formula* for $X(z)$. The ROC is essential for resolving this ambiguity!

Laplace analogy

$$\begin{aligned} \mathrm{e}^{\lambda t}\, u(t) \quad &\overset{\mathcal{L}}{\leftrightarrow} \quad \frac{1}{s - \lambda}, \qquad \text{real}(s) > \text{real}(\lambda) \\ -\,\mathrm{e}^{\lambda t}\, u(-t) \quad &\overset{\mathcal{L}}{\leftrightarrow} \quad \frac{1}{s - \lambda}, \qquad \text{real}(s) < \text{real}(\lambda) \end{aligned}$$

General shape of ROC

In the preceding two examples, we have seen ROC's that are the interior and exterior of circles. What is the general shape?

$$\boxed{\text{The ROC is always an annulus, } \textit{i.e.}, \{r_2 < |z| < r_1\}.}$$

Note that r_2 can be zero (possibly with $\leq$) and r_1 can be ∞ (possibly with $\leq$).

Explanation. Let $z = r\,e^{\jmath\theta}$ be polar form.

$$\begin{aligned} |X(z)| &= \left|\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]|\, r^{-n} \text{ by triangle inequality} \\ &= \sum_{n=-\infty}^{-1} |x[n]|\; r^{-n} + \sum_{n=0}^{\infty} |x[n]|\; r^{-n} \\ &= \sum_{n=1}^{\infty} |x[-n]|\, r^{n} + \sum_{n=0}^{\infty} \frac{|x[n]|}{r^n}. \end{aligned}$$

The ROC is the subset of $\mathbb{C}$ where *both* of the above sums are finite.

If the right sum (the "causal part") is finite for some z_2 with magnitude $r_2 = |z_2|$,
then that sum will also be finite for any z with magnitude $r \geq r_2$, since for such an r each term in the sum is smaller.
So the ROC for the right sum is the subset of $\mathbb{C}$ for which $|z| > r_2$, which is the exterior of some circle.

Likewise if the left sum (the "anti-causal part") is finite for some z_1 with magnitude $r_1 = |z_1|$,
then that sum will also be finite for any z with magnitude $r \leq r_1$, since for such an r each term in the sum is smaller.
So the ROC for the left sum is the subset of $\mathbb{C}$ for which $|z| < r_1$, for some r_1, which is the interior of some circle.

$$\boxed{\text{The ROC of a causal signal is the exterior of a circle of some radius } r_2.}$$

$$\boxed{\text{The ROC of an anti-causal signal is the interior of a circle of some radius } r_1.}$$

For a general signal $x[n]$, the ROC will be the *intersection* of the ROC of its causal and noncausal parts, which is an annulus.
If $r_2 < r_1$, then that intersection is a (nonempty) annulus. Otherwise the z-transform is undefined (does not exist).

Simple example of a signal which has empty ROC?
$x[n] = 1 = u[n] + u[-n-1]$.
Recall $u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1-z^{-1}}$ for $\{|z| > 1\}$.
ROC for the causal part is $\{|z| > 1\}$,
ROC for the anti-causal part is $\{|z| < 1\}$.

TABLE 3.1 - discuss here

Table shows signals decreasing away from zero, since for non-decreasing signals the z-transform is usual undefined (empty ROC).
Energy signals must eventually diminish to zero.

Subtleties in defining the ROC (optional reading!)

We elaborate here on why the two possible definitions of the ROC are *not* equivalent, contrary to to the book's claim on *p. 154*.

Consider the **harmonic series** signal $x[n] = \frac{1}{n} u[n-1]$. (A signal with no practical importance.)
The z-transform of this signal is

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}.$$

Consider first the exterior of the unit circle. If $r = |z| > 1$ then

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} z^{-n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{r} \right)^n < \sum_{n=1}^{\infty} \left(\frac{1}{r} \right)^n < \infty.$$

So $\{|z| > 1\}$ will be included in the ROC, by either definition.

Now consider the interior of the unit circle. If $r = |z| < 1$ then

$$\sum_{n=1}^{N} \left| \frac{1}{n} z^{-n} \right| = \sum_{n=1}^{N} \frac{1}{n} \left(\frac{1}{r} \right)^n > \sum_{n=1}^{N} \frac{1}{n} \to \infty.$$

So $\{|z| < 1\}$ will not be in the ROC, by the "absolutely summable" definition.

Now consider the point $z = 1$. At this point $X(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. So there is a pole at $z = 1$ and $z = 1$ is not in the ROC by either definition.

Now consider the point $z = -1$. At this point $X(-1) = \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n = -\log 2$, which is a well-defined finite value!
See `http://mathworld.wolfram.com/HarmonicSeries.html` for more information.
It is easy to verify that sum using the Taylor expansion of log around 1, evaluated at 2.

- So the point $z = -1$ would be included in the ROC defined by the "attains a finite value" definition.
- However, at $z = -1$ the series $\frac{1}{n}(-1)^{-n}$ is not absolutely summable, since $\sum_{n=1}^{\infty} \left| \frac{1}{n} (-1)^{-n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. So the point $z = -1$ is not included in the "absolutely summable" definition of the ROC.

Furthermore, there are other points around the unit circle where the z-transform series is convergent but not absolutely summable.

Consider $z = e^{\jmath 2\pi M/N}$, with N even and M odd.

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{\jmath 2\pi (M/N) n} = \sum_{k=0}^{\infty} \sum_{l=1}^{N} \frac{1}{Nk + l} e^{\jmath 2\pi (M/N)(Nk+l)} = \sum_{k=0}^{\infty} \sum_{n=1}^{N} \frac{1}{Nk + n} e^{\jmath 2\pi (M/N) n}.$$

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{Nk + n} e^{\jmath 2\pi (M/N) n} &= \sum_{n=1}^{N/2} \left[\frac{1}{Nk + n} e^{\jmath 2\pi (M/N) n} + \frac{1}{Nk + n + N/2} e^{\jmath 2\pi (M/N)(n+N/2)} \right] \\
&= \sum_{n=1}^{N/2} \left[\frac{1}{Nk + n} - \frac{1}{Nk + n + N/2} \right] e^{\jmath 2\pi (M/N) n} = \sum_{n=1}^{N/2} \frac{N/2}{(Nk + n)(Nk + n + N/2)} e^{\jmath 2\pi (M/N) n}.
\end{aligned}$$

This is like $1/k^2$, so it will be convergent.

3.1.2

The inverse z-transform

One method for determining the inverse is contour integration using the **Cauchy integral theorem**. ***See 3.4.***

Key point: we want to avoid this! By learning z-transform properties, can expand small table of z-transforms into a large set.

3.2

Properties of the z-transform

For each property must consider both "what happens to formula $X(z)$" *and* what happens to ROC.

Linearity

If $x_1[n] \stackrel{Z}{\leftrightarrow} X_1(z)$ and $x_2[n] \stackrel{Z}{\leftrightarrow} X_2(z)$ then

$$x[n] = a_1\, x_1[n] + a_2\, x_2[n] \stackrel{Z}{\leftrightarrow} a_1\, X_1(z) + a_2\, X_2(z)$$

Follows directly from definition.

Very useful for finding z-transforms and inverse z-transforms!

The ROC of the sum contains at least as much of the z-plane as the intersection of the two ROC's.

Example: $x[n] = \cos(\omega_0 n + \phi)\, u[n]$ (causal sinusoid).
By Euler's identity, $x[n] = \frac{1}{2}\left(e^{\jmath(\omega_0 n+\phi)} + e^{-\jmath(\omega_0 n+\phi)}\right) u[n] = \frac{1}{2}\, e^{\jmath\phi} \left(e^{\jmath\omega_0}\right)^n u[n] + \frac{1}{2}\, e^{-\jmath\phi} \left(e^{-\jmath\omega_0}\right)^n u[n]$.
Applying previous example with "$p = e^{\pm\jmath\omega_0}$" and linearity:

$$X(z) = \frac{\frac{1}{2}\, e^{\jmath\phi}}{1 - e^{\jmath\omega_0}\, z^{-1}} + \frac{\frac{1}{2}\, e^{-\jmath\phi}}{1 - e^{-\jmath\omega_0}\, z^{-1}} = \frac{\frac{1}{2}\, e^{\jmath\phi}\left(1 - e^{-\jmath\omega_0}\, z^{-1}\right) + \frac{1}{2}\, e^{-\jmath\phi}\left(1 - e^{\jmath\omega_0}\, z^{-1}\right)}{(1 - e^{\jmath\omega_0}\, z^{-1})(1 - e^{-\jmath\omega_0}\, z^{-1})} = \frac{\cos\phi - z^{-1}\cos(\omega_0 - \phi)}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}.$$

What is the ROC? $\{|z| > |p| = 1\}$, as one expects since $|\cos(\omega n)| \le 1$.

Time shifting

If $x[n] \stackrel{Z}{\leftrightarrow} X(z)$, then $x[n-k] \stackrel{Z}{\leftrightarrow} z^{-k}\, X(z)$.
Simple proof by change of index variable.
ROC is unchanged, except for adding or deleting $z = 0$ or $z = \infty$.

Now clear why unit delay was labeled z^{-1}.

Scaling the z-domain, aka modulation

If $x[n] \stackrel{Z}{\leftrightarrow} X(z)$ with ROC = $\{r_1 < |z| < r_2\}$, then $a^n\, x[n] \stackrel{Z}{\leftrightarrow} X\left(a^{-1}z\right)$ with ROC = $\{|a|r_1 < |z| < |a|r_2\}$.

Example. Decaying sinusoid: $x[n] = \frac{1}{2^n}\cos(\omega_0 n)\, u[n]$.

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}\cos\omega_0}{1 - z^{-1}\cos\omega_0 + \frac{1}{4}z^{-2}}$$

with ROC = $\left\{|z| > \frac{1}{2}\right\}$.

Time reversal

If $x[n] \stackrel{Z}{\leftrightarrow} X(z)$ with ROC = $\{r_1 < |z| < r_2\}$, then $x[-n] \stackrel{Z}{\leftrightarrow} X\left(z^{-1}\right)$ with ROC = $\{1/r_2 < |z| < 1/r_1\}$.

Simple proof by change of summation index, since positive powers of z become negative and vice versa.

Differentiation in z-domain

If $x[n] \stackrel{Z}{\leftrightarrow} X(z)$ then $n\, x[n] \stackrel{Z}{\leftrightarrow} -z\frac{d}{dz}\, X(z)$. The ROC is unchanged.

Proof:

$$-z\frac{d}{dz}\, X(z) = -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = -z\sum_{n=-\infty}^{\infty} x[n]\,(-n)\, z^{-n-1} = \sum_{n=-\infty}^{\infty} (n\, x[n])\, z^{-n} = Z\left\{n\, x[n]\right\}.$$

Caution for derivative when $n = 0$.

Example: $x[n] = n\, u[n]$ (unit ramp signal). We know $U(z) = 1/(1 - z^{-1})$ for $\{|z| > 1\}$. So

$$X(z) = -z\frac{d}{dz}U(z) = -z\frac{-z^{-2}}{(1 - z^{-1})^2} = \frac{z^{-1}}{(1 - z^{-1})^2}, \qquad \{|z| > 1\}.$$

Convolution

$$\boxed{\text{If } x_1[n] \overset{Z}{\leftrightarrow} X_1(z) \text{ and } x_2[n] \overset{Z}{\leftrightarrow} X_2(z) \text{ then } x[n] = x_1[n] * x_2[n] \overset{Z}{\leftrightarrow} X(z) = X_1(z)\, X_2(z)}$$

Proof:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \right] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-n} \right] \\ &= \sum_{k=-\infty}^{\infty} x_1[k]\, z^{-k}\, X_2(z) \\ &= X_1(z)\, X_2(z) \end{aligned}$$

The ROC of the convolution contains *at least as much* of the z-plane as the intersection of the ROC of $X_1(z)$ and the ROC of $X_2(z)$.

Recipe for convolution without tears:
- Compute both z-transforms
- Multiply
- Find inverse z-transform. (Hopefully already in table...)

Example. $x[n] = u[n] * u[n-1]$

$$X(z) = \frac{1}{1-z^{-1}} \cdot z^{-1} \frac{1}{1-z^{-1}} = \frac{z^{-1}}{(1-z^{-1})^2}$$

using the time-shift property. So $x[n] = n\, u[n]$ from previous example.
Contrast with continuous-time: $u(t) * u(t) = t u(t)$.

ROC for both $u[n]$ and $u[n-1]$ is $\{|z| > 1\}$. Same ROC for their convolution.

Convolution and LTI systems

If $x[n] \to \boxed{\text{LTI } h[n]} \to y[n]$, then since $y[n] = x[n] * h[n]$, $Y(z) = H(z)\, X(z)$.

Example: where ROC after convolution is larger than intersection.
$h[n] = \delta[n] - \delta[n-1]$ (discrete-time differentiator).
$x[n] = u[n-2]$ (delayed step function).
$H(z) = 1 - z^{-1}$ for $z \neq 0$.
$X(z) = \frac{z^{-2}}{1-z^{-1}}$ for $\{|z| > 1\}$. (Why?)

$y[n] = x[n] * h[n]$, so

$$Y(z) = H(z)\, X(z) = (1 - z^{-1}) \frac{z^{-2}}{1-z^{-1}} = z^{-2}$$

which has ROC $= \mathbb{C} - \{0\}$, which is "bigger" than intersection of ROC_X and ROC_H.
What is $y[n]$? $y[n] = \delta[n-2]$.

Correlation of two sequences

If $x[n] \overset{Z}{\leftrightarrow} X(z)$ and $y[n] \overset{Z}{\leftrightarrow} Y(z)$ ***are both real*** then

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-l] \overset{Z}{\leftrightarrow} R_{xy}(z) = X(z)\, Y(z^{-1})$$

since $r_{xy}[l] = x[l] * y[-l]$ and by convolution and time-reversal properties.

The ROC is at least as large as the intersection of the ROC of $X(z)$ with the ROC of $Y(z^{-1})$.

Multiplication of two sequences ***(mention only)***

If $x_1[n] \overset{Z}{\leftrightarrow} X_1(z)$ and $x_2[n] \overset{Z}{\leftrightarrow} X_2(z)$ then

$$x[n] = x_1[n]\, x_2[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{2\pi \jmath} \oint X_1(v)\, X_2^*\left(\frac{z}{v}\right) v^{-1}\, \mathrm{d}v$$

Read about ROC

Parseval's relation ***(mention only)***

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \frac{1}{2\pi \jmath} \oint X_1(z)\, X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, \mathrm{d}z$$

provided that $r_{1l} r_{2l} < 1 < r_{1u} r_{2u}$

Initial value theorem

If $x[n]$ is **causal**, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Proof: simple from definition: $X(z) = x[0] + x[1]\, z^{-1} + x[2]\, z^{-2} + \cdots$

Final value theorem

If $x[n]$ is causal then

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1)\, X(z)\,.$$

The limit exists provided the ROC of $(z-1)\, X(z)$ includes the unit circle.

Comparison to Laplace properties

Compared to corresponding properties for Laplace transform, there are some missing.
Which ones?

Conjugation

$x^*[n] \overset{Z}{\leftrightarrow} X^*(z^*)$

So if $x[n]$ is real, then $X(z) = X^*(z^*)$.
(For later: If in addition, $X(z)$ is rational, then the polynomial coefficients are real.)

Laplace properties for which z-transform analogs are less obvious because time index n is an integer in DT.

Property	Continuous-Time Laplace transform	Discrete-Time z-transform
Time scaling	$f(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right),\ a \neq 0.$	?
Differentiation/difference in the time domain	$\frac{\mathrm{d}}{\mathrm{d}t} x_{\mathrm{a}}(t) \overset{\mathcal{L}}{\leftrightarrow} s\, X_{\mathrm{a}}(s)$	$x[n] - x[n-1] \overset{Z}{\leftrightarrow} (1 - z^{-1})\, X(z)$
Integration/summation in the time domain	$\int_{-\infty}^{t} x_{\mathrm{a}}(\tau)\, \mathrm{d}\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s} X_{\mathrm{a}}(s)$	$\sum_{k=-\infty}^{n} x[k] \overset{Z}{\leftrightarrow} \frac{1}{1-z^{-1}}\, X(z)$

In discrete time, the analog of **time scaling** is **up-sampling** and **down-sampling**.

Time expansion (up-sampling)

Define the M-times **upsampled** version of $x[n]$ as follows:

$$\begin{aligned} y[n] &= \begin{cases} x[n/M], & \text{if } n \text{ is a multiple of } M \\ 0, & \text{otherwise} \end{cases} \\ \text{for } M = 2: &= \{\ldots, 0, x[-2], 0, x[-1], 0, x[0], 0, x[1], 0, x[2], 0, \ldots\}. \end{aligned}$$

Then $Y(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-nM} = X\left(z^M\right)$, with $\mathrm{ROC}_Y = \left\{z \in \mathbb{C} : z^M \in \mathrm{ROC}_X\right\}$.

$$\boxed{x[n] \uparrow M \overset{Z}{\leftrightarrow} X\left(z^M\right)}$$

Example. Find z-transform of $y[n] = \left\{\underline{1}, 0, 0, 1/8, 0, 0, 1/8^2, \ldots\right\}$. The brute-force way to solve this problem is as follows:

$$Y(z) = 1 + (1/8)z^{-3} + (1/8)^2 z^{-6} + \cdots = \sum_{k=0}^{\infty} (1/8)^k z^{-3k} = \sum_{k=0}^{\infty} \left(\frac{1}{8z^3}\right)^k = \frac{1}{1 - (1/8)z^{-3}},$$

if $|(1/8)z^{-3}| < 1$ *i.e.*, $|z| > 1/2$ = ROC.

The alternative approach is to use upsampling properties. $y[n]$ is formed by upsampling by a factor of $m = 3$ the signal $x[n] = (1/8)^n\, u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1-(1/8)z^{-1}}$ for ROC = $\{|z| > 1/8\}$. Thus $Y(z) = X\left(z^3\right) = \frac{1}{1-(1/8)z^{-3}}$ for ROC = $\left\{|z^3| > 1/8\right\}$.

Down-sampling

One way to "down sample" is to zero out all samples except those that are multiples of m: Define

$$\begin{aligned} y[n] &= \begin{cases} x[n], & n \text{ not a multiple of } m \\ 0, & \text{otherwise} \end{cases} \\ \text{for } m = 2: &= \{\ldots, 0, x[-4], 0, x[-2], 0, x[0], 0, x[2], 0, x[4], 0, \ldots\}. \end{aligned}$$

General case left as exercise.

Example: $m = 2$.

Trick: write $y[n] = \frac{1}{2}(1 + (-1)^n)\, x[n] = \frac{1}{2}\, x[n] + \frac{1}{2}(-1)^n\, x[n]$.
Using linearity and z-domain scaling property: $Y(z) = \frac{1}{2}[X(z) + X(-z)]$.
ROC of $Y(z)$ is at least as large as ROC of $X(z)$.

Formula that is useful for such derivations:

$$\cdots + g[-2] + g[0] + g[2] + g[4] + \cdots = \sum_{n=-\infty}^{\infty} g[2n] = \sum_{m=-\infty}^{\infty} \frac{1}{2}\,(1 + (-1)^m)\, g[m]\,.$$

3.3

Rational z-transforms

All of the above examples had z-transforms that were rational functions, *i.e.*, a ratio of two polynomials in z or z^{-1}.

$$X(z) = \frac{B(z)}{A(z)} = g \frac{\prod_k (z - z_k)}{\prod_k (z - p_k)}.$$

This is a very important class (*i.e.*, for LTI systems described by difference equations).

3.3.1

Poles and zeros

- The **zeros** of a z-transform $X(z)$ are the values of z where $X(z) = 0$.
- The **poles** of a z-transform $X(z)$ are the values of z where $X(z) = \infty$. (*cf.* mesh plot of $X(z)$)

If $X(z)$ is a **rational** function, *i.e.*, a ratio of two polynomials in z, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Without loss of generality, we assume $a_0 \neq 0$ and $b_0 \neq 0$, so we can rewrite

$$X(z) = \frac{b_0}{a_0} \frac{z^{-M}}{z^{-N}} \frac{z^M + \frac{b_1}{b_0} z^{M-1} + \cdots + \frac{b_M}{b_0}}{z^N + \frac{a_1}{a_0} z^{N-1} + \cdots + \frac{a_N}{a_0}} \triangleq \frac{b_0}{a_0} z^{N-M} \frac{N'(z)}{D'(z)}$$

$N'(z)$ has M finite roots at $z_1, \ldots, z_M$, and $D'(z)$ has N finite roots at $p_1, \ldots, p_N$. So we can rewrite $X(z)$:

$$X(z) = \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)}$$

or

$$\boxed{X(z) = G z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)},}$$

where $G \triangleq \frac{b_0}{a_0}$. Thus

- $X(z)$ has M finite zeros at $z_1, \ldots, z_M$
- $X(z)$ has N finite poles at $p_1, \ldots, p_N$
- If $N > M$, $X(z)$ has $N - M$ zeros at $z = 0$
- If $N < M$, $X(z)$ has $M - N$ poles at $z = 0$
- There can also be poles or zeros at $z = \infty$, depending if $X(\infty) = \infty$ or $X(\infty) = 0$
- Counting all of the above, there will be the same number of poles and zeros.

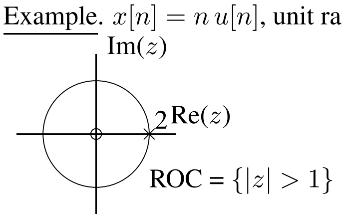
Because of the boxed form above, $X(z)$ is *completely determined by is pole-zero locations* up to the scale factor G. The scale factor only affects the *amplitude* (or units) of the signal or system, whereas the poles and zeros affect the *behavior*.

A **pole-zero plot** is a graphic description of rational $X(z)$, up to the scale factor. Use $\circ$ for zeros and $\times$ for poles. Multiple poles or zeros indicated with adjacent number.

$$\boxed{\text{By definition, the ROC will not contain any poles.}}$$

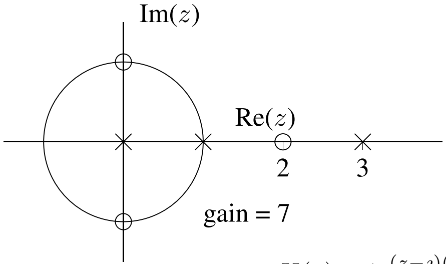
Skill: *Go from $x[n]$ to $X(z)$ to pole-zero plot.*

Example. $x[n] = n\, u[n]$, unit ramp signal. Previously showed that $X(z) = \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$, $\{|z| > 1\}$.



Skill: *Go from pole-zero plot to $X(z)$ to $x[n]$.*

Example. What are possible ROC's in following case? Answer: $\{|z| < 1\}$, $\{1 < |z| < 3\}$, or $\{3 < |z|\}$.



$X(z) = 7\frac{(z-\jmath)(z+\jmath)(z-2)}{(z-0)(z-1)(z-3)} = 7\frac{(1-\jmath z^{-1})(1+\jmath z^{-1})(1-2z^{-1})}{(1-z^{-1})(1-3z^{-1})}$. But what is $x[n]$? (PFE soon...)

3.3.2

Pole location and time-domain behavior for causal signals

The roots of a polynomial with real coefficients (the usual case) are either real or complex conjugate pairs. Thus we focus on these cases.

Single real pole

$$x[n] = p^n\, u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1-pz^{-1}} = \frac{z}{z-p}.$$

Fig. 3.11

- signal decays if pole is inside unit circle
- signal blows up if pole is outside unit circle
- signal alternates sign if pole is in left half plane, since $(-|p|)^n = (-1)^n |p|^n$

Double real pole

$$x[n] = np^n\, u[n] \overset{Z}{\leftrightarrow} X(z) = -z\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{1-pz^{-1}} = \frac{pz^{-1}}{(1-pz^{-1})^2} = \frac{pz}{(z-p)^2}$$

Fig. 3.12

Generalization to multiple real poles?

Pair of complex-conjugate poles

From *Table 3.3*:

$$a^n \sin(\omega_0 n)\, u[n] \overset{Z}{\leftrightarrow} \frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}} = \frac{az\sin\omega_0}{z^2 - 2az\cos\omega_0 + a^2} = \frac{az\sin\omega_0}{(z-a\,\mathrm{e}^{\jmath\omega_0})(z-a\,\mathrm{e}^{-\jmath\omega_0})},$$

where a is assumed real. The roots of the denominator polynomial are

$$z = \frac{2a\cos\omega_0 \pm \sqrt{(2a\cos\omega_0)^2 - 4a^2}}{2} = a\cos\omega_0 \pm a\sqrt{\cos^2\omega_0 - 1} = a[\cos\omega_0 \pm \sqrt{-\sin^2\omega_0}] = a[\cos\omega_0 \pm j\sin\omega_0] = a\,\mathrm{e}^{\pm\jmath\omega_0}.$$

Thus the poles of the transform of the above signal are at $p = a\,\mathrm{e}^{\jmath\omega_0}$ and $p^* = \mathrm{e}^{-\jmath\omega_0}$.

Thus the following signal has a pair of complex-conjugate poles:

$$x[n] = a^n \sin(\omega_0 n)\, u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{az\sin\omega_0}{(z-p)(z-p^*)}.$$

(Also see (3.6.43).)
Fig. 3.13

What determines the rate of oscillation? ω_0

Qualitative relationship with Laplace: $z \equiv e^{sT}$, in terms of pole-zero locations.

3.3.3

The system function of a LTI system

As noted previously: $x[n] \to \boxed{\text{LTI } h[n]} \to y[n] = x[n] * h[n] \overset{Z}{\leftrightarrow} \boxed{Y(z) = H(z)\,X(z)\,.}$

- Forward direction: transform $h[n]$ and $x[n]$, multiply, then inverse transform.
- Reverse engineering: put in known signal $x[n]$ with transform $X(z)$; observe output $y[n]$; compute transform $Y(z)$. Divide the two to get the **system function** or **transfer function** $\boxed{H(z) = Y(z)\,/\,X(z)\,.}$
  If you can choose any input $x[n]$, what would it be? Probably $x[n] = \delta[n]$ since $X(z) = 1$, so output is directly the impulse response.
- The third rearrangement $\boxed{X(z) = Y(z)\,/\,H(z)}$ is also useful sometimes.

Now apply these ideas to the analysis of LTI systems that are described by general linear constant-coefficient difference equations (LCCDE) (or just **diffeq** systems):

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]\,.$$

Goal: find impulse response $h[n]$. Not simple with time-domain techniques. Systematic approach uses z-transforms.

Applying linearity and shift properties taking z-transform of both sides of the above:

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k}\, Y(z) + \sum_{k=0}^{M} b_k z^{-k}\, X(z)$$

so

$$\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] Y(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right] X(z)$$

so, defining $a_0 \triangleq 1$,

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}},}$$

What is the name for this type of system function? It is a **rational** system function. (Ratio of polynomials in z.)

Now we can see why "$-$" sign in difference equation.
We can also see why studying rational z-transforms is very important.

The system function for a LCCDE system is rational.

Skill: ***Convert between LCCDE and system function.***

What about irrational system functions? (optional reading)

Although all diffeq systems have rational z-transforms, diffeq systems are just a (particularly important) type of system within the broader family of LTI systems. There do exist (in principle at least) LTI systems that do not have rational system functions.

Example. Consider the LTI system having the impulse response $h[n] = \frac{1}{n}\, u[n]$.
The system function for this (IIR) system is $H(z) = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n} = \log z^{-1} = -\log z$, which certainly is not rational.

However, this system does not have any known practical use, and would be entirely impractical to implement!

All zero system

If $N = 0$ or equivalently $a_1 = \cdots = a_N = 0$, then the system function simplifies to

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k} = \frac{\prod_{k=1}^{M}(z - z_k)}{z^M}.$$

The M poles at $z = 0$ are called **trivial poles**.

Why are they called trivial poles? One reason is that they correspond only to a time shift. The other is that if a system has a pole outside the unit circle, then certain bounded inputs will produce an unbounded output (unstable). But a pole at zero does not cause this unstable behavior, so its effect is in some sense trivial.

Then there are M roots of the "numerator" polynomial that are nontrivial zeros. Thus this is called a **all-zero system**.
The impulse response is FIR:

$$h[n] = \sum_{k=0}^{N} b_k \, \delta[n-k] \, .$$

All pole system

If $M = 0$ or equivalently $b_1 = \cdots = b_M = 0$, then the system function reduces to

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}} = b_0 \frac{z^N}{\prod_{k=1}^{N}(z - p_k)},$$

where $a_0 \stackrel{\triangle}{=} 1$. This system function has N **trivial zeros** at $z = 0$ that are relatively unimportant, and the denominator polynomial has N roots that are the poles of $H(z)$. Thus this is called a **all-pole system**.
The impulse response is IIR.

Otherwise the impulse response is called a **pole-zero system**, and the impulse response is IIR.

Skill: ***Find impulse response*** $h[n]$ ***for rational system function*** $H(z)$**.**

Example. Find impulse response $h[n]$ for a system described by the following input-output relationship: $y[n] = -\, y[n-2] + x[n]$.

Recall that earlier we found the impulse response of $y[n] = y[n-1] + x[n]$ by a "trick."
Now we can approach such problems systematically.

Do not bother using above formulas, just use the *principle* of going to the transform domain.

Write z-transforms: $Y(z) = -z^{-2}\, Y(z) + X(z)$, so $(1 + z^{-2})\, Y(z) = X(z)$ and $H(z) = \dfrac{1}{1 + z^{-2}}$.

From *Table 3.3*: $\cos(n\pi/2)\, u[n] \stackrel{Z}{\leftrightarrow} \dfrac{1}{1 + z^{-2}}$, so $h[n] = \cos(n\pi/2)\, u[n] = \{\underline{1}, 0, -1, 0, \ldots\}$.

Note that there is more than one choice (causal and anti-causal) for the inverse z-transform since ROC never discussed.
Why did I choose the causal sequence? Because all LTI systems described by difference equations are **causal**.

In the above case, we could work from Table 3.3 to find $h[n]$ from $H(z)$. But what if the example were $y[n] = y[n-3] + x[n]$?
Looks simple, should be do-able. By same approach, $H(z) = \dfrac{1}{1 - z^{-3}}$, which is not in our table.
So what do we do? We need inverse z-transform method(s)!

Summary

The above concepts are very important!

3.4

Inversion of the z-transform

Skill: *Choosing and performing simplest approach to inverting a z-transform.*

Methods for inverse z-transform
- Table lookup (already illustrated), using *properties*
- Contour integration
- Series expansion into powers of z and z^{-1}
- Partial-fraction expansion and table lookup

Practical problems requiring inverse z-transform?
- Given a system function $H(z)$, *e.g.*, described by a pole-zero plot, find $h[n]$.
  This is particularly important since we will *design* filters "in the z-domain."
- When performing convolution via z-transforms: $Y(z) = H(z)\, X(z)$, leading to $y[n]$.

3.4.1

The inverse z-transform by contour integration

$$\boxed{x[n] = \frac{1}{2\pi \jmath} \oint X(z)\, z^{n-1}\, \mathrm{d}z}$$

The integral is a **contour integral** over a **closed path** C that must
- enclose the origin,
- lie in the ROC of $X(z)$.

Typically C is just a circle centered at the origin and within the ROC.

Cauchy residue theorem. ***skip*** : ***see text***

$$\frac{1}{2\pi j} \oint z^{n-1-k}\, \mathrm{d}z = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} = \delta[n-k]\,.$$

The rest of this section might be called "how to avoid using this integral."

3.4.2

The inverse z-transform by power series expansion, aka "**coefficient matching**"

If we can expand the z-transform into a power series (considering its ROC), then "by the uniqueness of the z-transform:"

$$\boxed{\text{if } \; X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \text{ then } \; x[n] = c_n,}$$

i.e., the signal sample values in the time-domain are the corresponding coefficients of the power series expansion.

Example. Find impulse response $h[n]$ for system described by $y[n] = 2\,y[n-3] + x[n]$.

By the usual Y/X method, we find $H(z) = \frac{1}{1-2z^{-3}}$.
From the diffeq, we this is a causal system. Do we want an expansion in terms of powers of z or z^{-1}? We want z^{-1}.

Using geometric series: $H(z) = \frac{1}{1-2z^{-3}} = \sum_{k=0}^{\infty}(2z^{-3})^k = \sum_{k=0}^{\infty} 2^k z^{-3k} = 1 + 2z^{-3} + 2^2 z^{-6} + \ldots$.
Thus $h[n] = \{\underline{1}, 0, 0, 2, 0, 0, 4, \ldots\} = \sum_{k=0}^{\infty} 2^k\,\delta[n-3k]$.

This case was easy since the power series was just the familiar geometric series.
In general one must use tedious **long division** if the power series is not easy to find.

Very useful for checking the first few coefficients!

Example. Find the impulse response $h_2[n]$ for the system described by $y[n] = 2\,y[n-3] + x[n] + 5\,x[n-1]$.
We have

$$H_2(z) = \frac{1+5z^{-1}}{1-2z^{-3}} = \frac{1}{1-2z^{-3}} + \frac{5z^{-1}}{1-2z^{-3}} = H(z) + 5z^{-1}\,H(z) \Longrightarrow h_2[n] = h[n] + 5\,h[n-1] = \{\underline{1}, 5, 0, 2, 10, 0, 4, 20, \ldots\}\,.$$

Example. What if we knew we had an anti-causal system? (*e.g.*, $y[n] = 2\,y[n+3] + x[n+1]$).

Rewrite $H(z) = z/(1-2z^3) = z\sum_{k=0}^{\infty}(2z^3)^k = \sum_{k=0}^{\infty} 2^k z^{3k+1} \Longrightarrow$
$h[n] = \sum_{k=0}^{\infty} 2^k\,\delta[n+(3k+1)] = \{\ldots, 4, 0, 0, 2, 0, 0, 1, \underline{0}, 0, 0, \ldots\}$.

But we still need a systematic method for general cases.

To PFE or not to PFE?

Before delving into the PFE, it is worth noting that there are often multiple mathematically equivalent answers to discrete-time inverse z-transform problems.

Example. Find the impulse response $h[n]$ of the causal system having system function $H(z) = \dfrac{1+5z^{-1}}{1-2z^{-1}}$.

Approach 1: expand $H(z)$ into two terms and use linearity and shift properties:

$$H(z) = \frac{1}{1-2z^{-1}} + 5z^{-1}\frac{1}{1-2z^{-1}} \Longrightarrow h[n] = 2^n\,u[n] + 5\cdot 2^{n-1}\,u[n-1]\,.$$

Approach 2: perform "**long division**:"

$$H(z) = -\frac{5}{2} + \left[\frac{1+5z^{-1}}{1-2z^{-1}} + \frac{5}{2}\right] = \frac{5}{2} + \frac{\frac{7}{2}}{1+2z^{-1}} \Longrightarrow h[n] = -\frac{5}{2}\,\delta[n] \underbrace{+\frac{7}{2}2^n\,u[n]}_{\text{due to pole}}.$$

Which answer is correct for $h[n]$? Both!
(Equality is not immediately obvious, but one can show that they are equal using $\delta[n] = u[n] - u[n-1]$.)

However, the second form is preferable because this system has one pole, at $z = 2$, so it is preferable to use the form that has exactly one term for each pole. The asymptotic (large n) behavior is more apparent in the second form.

3.4.3

The inverse z-transform by partial-fraction expansion

General strategy: suppose we have a "complicated" z-transform $X(z)$ for which we would like to find the corresponding discrete-time signal $x[n]$. If we can express $X(z)$ as a linear combination of "simple" functions $\{X_k(z)\}$ whose inverse z-transform is known, then we can use linearity to find $x[n]$. In other words:

$$X(z) = \alpha_1 X_1(z) + \cdots + \alpha_K X_K(z) \Longrightarrow x[n] = \alpha_1 x_1[n] + \cdots + \alpha_K x_K[n]\,.$$

In principle one can apply this strategy to any $X(z)$. But whether "simple" $X_k(z)$'s can be found will depend on the particular form of $X(z)$.

Fortunately, for the class of **rational** z-transforms, a decomposition into simple terms is *always* possible, using the **partial-fraction expansion** (**PFE**) method.

What are the "simple forms" we will try to find? They are the "single real pole," "double real pole," and "complex conjugate pair" discussed previously, summarized below.

Type	$X(z)$	$x[n]$
polynomial in z	$\sum_k c_k z^{-k}$	$\sum_k c_k\, \delta[n-k]$
single real pole	$\dfrac{1}{1-pz^{-1}}$	$p^n\, u[n]$
double real pole	$\dfrac{pz^{-1}}{(1-pz^{-1})^2}$	$np^n\, u[n]$
double real pole	$\dfrac{1}{(1-pz^{-1})^2}$	$(n+1)p^n\, u[n]$
triple real pole	$\dfrac{1}{(1-pz^{-1})^3}$	$\dfrac{(n+2)(n+1)}{2} p^n\, u[n]$
complex conjugate pair	$\dfrac{az\sin\omega_0}{(z-a\,\mathrm{e}^{\jmath\omega_0})(z-a\,\mathrm{e}^{-\jmath\omega_0})}$	$a^n \sin(\omega_0 n)\, u[n]$
complex conjugate pair $p = \lvert p\rvert\, \mathrm{e}^{\jmath\omega_0}$	$\dfrac{r}{1-pz^{-1}} + \dfrac{r^*}{1-p^*z^{-1}}$	$2\lvert r\rvert\, \lvert p\rvert^n \cos(\omega_0 n + \angle r)\, u[n]$

Step 1: Decompose $X(z)$ into proper form + polynomial

As usual, we assume $a_0 = 1$, without loss of generality, so we can write the rational z-transform as follows:

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}.$$

Such a rational function is called **proper** iff $a_N \neq 0$ and $M < N$. We want to work with proper rational functions.

We can always rewrite an improper rational function ($M \geq N$) as the sum of a polynomial and a proper rational function.

$$\boxed{\text{If } M \geq N, \text{ then } \frac{P_M(z^{-1})}{P_N(z^{-1})} = P_{M-N}(z^{-1}) + \frac{P_{N-1}(z^{-1})}{P_N(z^{-1})}.}$$

Example.

$$\begin{aligned}
X(z) &= \frac{1+z^{-2}}{1+2z^{-1}} = \frac{1}{2}z^{-1} + \left[\frac{1+z^{-2}}{1+2z^{-1}} - \frac{1}{2}z^{-1}\right] = \frac{1}{2}z^{-1} + \frac{1+z^{-2} - \frac{1}{2}z^{-1}\left(1+2z^{-1}\right)}{1+2z^{-1}} = \frac{1}{2}z^{-1} + \frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}} \\
&= \frac{1}{2}z^{-1} - \frac{1}{4} + \left[\frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}} + \frac{1}{4}\right] = -\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{1-\frac{1}{2}z^{-1} + \frac{1}{4}\left(1+2z^{-1}\right)}{1+2z^{-1}} = -\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{\frac{5}{4}}{1+2z^{-1}}.
\end{aligned}$$

In general this is always possible using long division.
The polynomial part is trivial to invert. Therefore, from now on we focus on **proper** rational functions.

Step 2: Find roots of denominator (poles)

The MATLAB `roots` command is useful here, or the quadratic formula when $N = 2$.

We call the roots $p_1, \ldots, p_N$, since these roots are the **poles** of $X(z)$.

Step 3a: PFE for distinct roots

One can use z or z^{-1} for PFE. The book chooses z. We choose z^{-1} to match MATLAB's `residuez`.

If the poles $p_1, \ldots, p_N$ are all different (distinct) then the expansion we seek has the form

$$X(z) = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}}, \tag{3-1}$$

where the r_k's are real or complex numbers called **residues**.

For distinct roots:

$$\boxed{r_k = (1 - p_k z^{-1})\, X(z) \Big|_{z=p_k}}$$

Proof:

$$(1 - p_k z^{-1})\, X(z) = (1 - p_k z^{-1}) \frac{r_1}{1 - p_1 z^{-1}} + \cdots + r_k + \cdots + (1 - p_k z^{-1}) \frac{r_N}{1 - p_N z^{-1}},$$

and evaluate the LHS and RHS at $z = p_k$.

Step 4a: inverse z-transform

Assuming $x[n]$ is causal (*i.e.*, ROC = $\{|z| > \max_k |p_k|\}$):

$$\boxed{x[n] = r_1 p_1^n\, u[n] + \cdots + r_N p_N^n\, u[n]\,.}$$

The discrete-time signal corresponding to a rational function in proper form *with distinct roots* is a weighted sum of geometric progression signals.

Complex conjugate pairs

In the usual case where the polynomial coefficients are real, any complex poles occur in conjugate pairs. Furthermore, the corresponding residues in the PFE also occur in complex-conjugate pairs.

PFE residues occur in complex-conjugate pairs for complex-conjugate roots.

skip Proof (for the distinct-root case with real coefficients):
Let p and p^* denote a complex-conjugate pair of roots. Suppose $X(z) = \frac{Y(z)}{(1-pz^{-1})(1-p^*z^{-1})}$ where $Y(z)$ is a ratio of polynomials in z with real coefficients. Then

$$\begin{aligned} r_1 &= (1 - pz^{-1})\, X(z) \Big|_{z=p} = \frac{Y(z)}{1 - p^* z^{-1}} \Big|_{z=p} = \frac{Y(p)}{1 - p^*/p} \\ r_2 &= (1 - p^* z^{-1})\, X(z) \Big|_{z=p^*} = \frac{Y(z)}{(1 - pz^{-1})} \Big|_{z=p^*} = \frac{Y(p^*)}{1 - p/p^*} = \left[\frac{Y^*(p^*)}{1 - p^*/p}\right]^* = \left[\frac{Y(p)}{1 - p^*/p}\right]^* = r_1^* \end{aligned}$$

since $Y^*(p) = Y(p^*)$ because $Y(z)$ has real coefficients.

Example.

$$X(z) = \frac{r}{1 - pz^{-1}} + \frac{r^*}{1 - p^* z^{-1}}$$

thus

$$x[n] = [rp^n + r^*(p^*)^n]\, u[n]\,.$$

Since this is of the form $a + a^*$, it must be real, so it is useful to express it using real quantities.

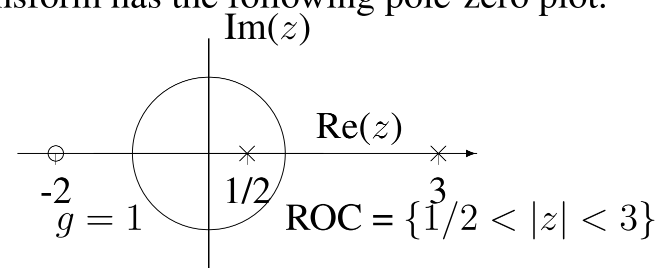
$$x[n] = 2\,\mathrm{real}(rp^n)\, u[n] = 2\,\mathrm{real}\big(|r|\, e^{\jmath\phi}\, |p|^n\, e^{\jmath\omega_0 n}\big)\, u[n] = 2\,|r|\,|p|^n \cos(\omega_0 n + \phi)\, u[n]$$

where $p = |p|\, e^{\jmath\omega_0}$ and $r = |r|\, e^{\jmath\phi}$. Note the different roles of $\angle p = \omega_0$ (frequency) and $\angle r = \phi$ (phase).

3.4.3

Example. Inverse z-transform by PFE

Find the signal $x[n]$ whose z-transform has the following pole-zero plot.



Find the formula for $X(z)$ and manipulate it (resorting to long division if necessary) to put in "proper form:"

$$
\begin{aligned}
X(z) &= \frac{z+2}{(z-3)(z-1/2)} \\
&= \frac{z^{-1}+2z^{-2}}{(1-3z^{-1})(1-\frac{1}{2}z^{-1})} && \text{(negative powers of } z \text{ in denominator)} \\
&= \frac{4}{3} + \left[\frac{z^{-1}+2z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} - \frac{2}{3/2}\right] && \text{(avoiding long division)} \\
&= \frac{4}{3} + \frac{z^{-1}+2z^{-2}-\frac{4}{3}\left[1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}\right]}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} \\
&= \frac{4}{3} + \frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{(1-3z^{-1})(1-\frac{1}{2}z^{-1})} && \text{(proper form!)} \\
&= \frac{4}{3} + \frac{r_1}{1-3z^{-1}} + \frac{r_2}{1-\frac{1}{2}z^{-1}} && \text{(PFE)}
\end{aligned}
$$

residue values:

$$
r_1 = \left.\frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=3} = \frac{2}{3}, \qquad r_2 = \left.\frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{1-3z^{-1}}\right|_{z=1/2} = -2
$$

$$
X(z) = \frac{4}{3} + \frac{\frac{2}{3}}{1-3z^{-1}} + \frac{-2}{1-\frac{1}{2}z^{-1}} \qquad \text{(could multiply out to check).}
$$

Considering the ROC, we conclude

$$
x[n] = \frac{4}{3}\,\delta[n] \underbrace{-\frac{2}{3}3^n\,u[-n-1]}_{\text{anti-causal}} -2\left(\frac{1}{2}\right)^n u[n]\,.
$$

MATLAB approach: `[r p k] = residuez([0 1 2], [1 -7/2 3/2])`
returns (in decimals): `r = [2/3 -2], p = [3 1/2], k = 4/3.`

General PFE formula for single poles, for proper form[1] with $M < N$:

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots b_M z^{-M}}{\prod_{k=1}^N (1 - p_k z^{-1})} = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}}$$

where the **residue** is given by:

$$\boxed{r_j = (1 - p_k z^{-1})\, X(z) \Big|_{z=p_k} = \left. \frac{B(z)}{\prod_{l \neq k} (1 - p_l z^{-1})} \right|_{z=p_k}.}$$

If p_k is a **repeated (2nd-order) pole**:

$$\boxed{\begin{aligned} X(z) &= \cdots + \frac{r_{k,1}}{1 - p_k z^{-1}} + \frac{r_{k,2}}{(1 - p_k z^{-1})^2} + \cdots \\ r_{k,1} &= \frac{1}{-p_k} \frac{\mathrm{d}}{\mathrm{d}z^{-1}} (1 - p_k z^{-1})^2 \, X(z) \Big|_{z=p_k} \\ r_{k,2} &= (1 - p_k z^{-1})^2 \, X(z) \Big|_{z=p_k}. \end{aligned}}$$

In general, if p_k is an L**th-order repeated pole**, then

$$X(z) = \cdots + \sum_{l=1}^{L} \frac{r_{k,l}}{(1 - p_k z^{-1})^l} + \cdots$$

where

$$r_{k,l} = \frac{1}{(L-l)!\,(-p_k)^{(L-l)}} \frac{d^{L-l}}{d(z^{-1})^{L-l}} (1 - p_k z^{-1})^L \, X(z) \Big|_{z=p_k}, \quad l = 1, \ldots, L.$$

Rarely would one do this by hand for $L > 2$. Use `residuez` instead!

Fact. For real signals, any complex poles appear in complex conjugate pairs, and the corresponding residues come in complex conjugate pairs:

$$X(z) = \cdots + \frac{r}{1 - p z^{-1}} + \frac{r^*}{1 - p^* z^{-1}} + \cdots$$

Letting $p = |p|\, \mathrm{e}^{\jmath\omega_0}$ and $r = |r|\, \mathrm{e}^{\jmath\phi}$ (note the difference in meaning of the angles!):

$$\begin{aligned} x[n] &= r p^n \, u[n] + r^* (p^*)^n \, u[n] \\ &= |r|\, \mathrm{e}^{\jmath\phi} \left(|p|\, \mathrm{e}^{\jmath\omega_0}\right)^n u[n] + |r|\, \mathrm{e}^{-\jmath\phi} \left(|p|\, \mathrm{e}^{-\jmath\omega_0}\right)^n u[n] \\ &= |r| \left(|p|^n \, \mathrm{e}^{\jmath\phi}\, \mathrm{e}^{\jmath\omega_0 n} + \mathrm{e}^{-\jmath\phi}\, \mathrm{e}^{-\jmath\omega_0 n}\right) u[n] \\ &= 2\, |r|\, |p|^n \cos(\omega_0 n + \phi)\, u[n]\,. \end{aligned}$$

[1]If not in proper form, then first do long division.

Example. Finding the impulse response of a diffeq system.

Find the impulse response of the system described by the following diffeq:

$$y[n] = \frac{4}{3}\,y[n-1] - \frac{7}{12}\,y[n-2] + \frac{1}{12}\,y[n-3] + x[n] - x[n-3]\,.$$

Step 0: Find the system function. ________________________ (linearity, shift property)

$$Y(z) = \frac{4}{3}z^{-1}\,Y(z) - \frac{7}{12}z^{-2}\,Y(z) + \frac{1}{12}z^{-3}\,Y(z) + X(z) - z^{-3}\,X(z)$$
$$\left[1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}\right] Y(z) = \left[1 - z^{-3}\right] X(z)$$

so (by convolution property):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}.$$

Step 1: Decompose system function into proper form + polynomial. ________________

In this case we can see by comparing the coefficients of the z^{-3} terms that the coefficient for the 0th-order term will be 1/(1/12) = 12.

$$H(z) = 12 + \left[\frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12\right] = 12 + P(z)$$

where

$$\begin{aligned} P(z) &= \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12 \\ &= \frac{1 - z^{-3} - 12\left[1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}\right]}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} \\ &= \frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}. \end{aligned}$$

Note that $P(z)$ is a proper rational function.
Since $H(z) = 12 + P(z)$, we see that $h[n] = 12\,\delta[n] + p[n]$.
We now focus on finding $p[n]$ from $P(z)$ by PFE.

Step 2: Find poles (roots of denominator).

The MATLAB command `roots([1 -4/3 7/12 -1/12])` returns `0.5 0.5 0.33`, so we check and verify that the denominator can be factored:

$$1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3} = \left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right),$$

so in factored form:

$$P(z) = \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right)}.$$

Step 3: Find PFE

Since there is one repeated root, the PFE form is

$$P(z) = \frac{r_{1,1}}{1 - \frac{1}{2}z^{-1}} + \frac{r_{1,2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} + \frac{r_2}{1 - \frac{1}{3}z^{-1}}. \tag{3-2}$$

For a single pole at $z = p_k$, we find the residue using this formula:

$$r_k = (1 - p_k z^{-1})\, P(z)\Big|_{z=p_k}.$$

Thus for the single pole at $z = 1/3$:

$$r_2 = (1 - \frac{1}{3}z^{-1})\, P(z)\Big|_{z=1/3} = \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2}\Bigg|_{z=1/3} = -104.$$

For a double pole at $z = p_k$, the residues are given by

$$r_{k,1} = \frac{1}{-p_k}\frac{\mathrm{d}}{\mathrm{d}z^{-1}}(1 - p_k z^{-1})^2\, P(z)\Big|_{z=p_k}, \quad \text{and} \quad r_{k,2} = (1 - p_k z^{-1})^2\, P(z)\Big|_{z=p_k}.$$

Thus for the double pole at $z = 1/2$:

$$\begin{aligned} r_{1,1} &= \frac{1}{-1/2}\,\frac{d}{dz^{-1}}(1 - \frac{1}{2}z^{-1})^2\, P(z)\Big|_{z=1/2} = -2\,\frac{d}{dz^{-1}}\frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{1}{3}z^{-1}}\Bigg|_{z=1/2} \\ &= -2\,\frac{(1 - \frac{1}{3}z^{-1})(16 - 14z^{-1}) - (-11 + 16z^{-1} - 7z^{-2})(-\frac{1}{3})}{(1 - \frac{1}{3}z^{-1})^2}\Bigg|_{z=1/2} = 114, \end{aligned}$$

and

$$r_{1,2} = (1 - \frac{1}{2}z^{-1})^2\, P(z)\Big|_{z=1/2} = \frac{-11 + 16z^{-1} - 7z^{-2}}{(1 - \frac{1}{3}z^{-1})}\Bigg|_{z=1/2} = -21.$$

Substituting in these residues into equation (3-2):

$$P(z) = \frac{114}{1 - \frac{1}{2}z^{-1}} + \frac{-21}{(1 - \frac{1}{2}z^{-1})^2} + \frac{-104}{1 - \frac{1}{3}z^{-1}}.$$

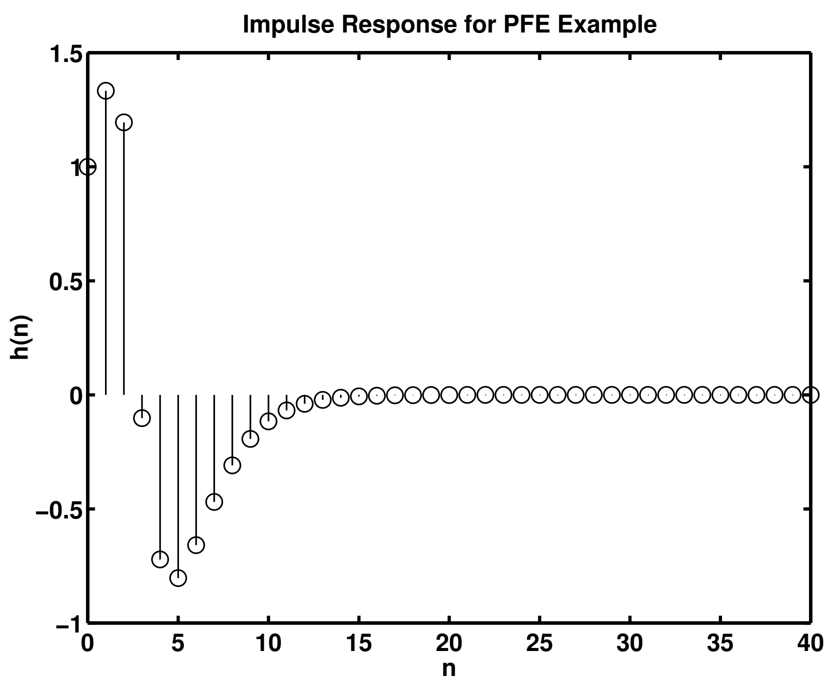
Step 4: Inverse z-transform (using table lookup)

$$p[n] = 114\left(\frac{1}{2}\right)^n u[n] - 21(n+1)\left(\frac{1}{2}\right)^n u[n] - 104\left(\frac{1}{3}\right)^n u[n]\,.$$

Substituting into proper form decomposition above yields our final answer:

$$\boxed{h[n] = 12\,\delta[n] + \left[(114 - 21(n+1))\left(\frac{1}{2}\right)^n - 104\left(\frac{1}{3}\right)^n\right] u[n]\,.}$$

The Resulting Impulse Response



Sanity check: $h[0] = 1$, as it should because for this system $y[0] = x[0]$ for a causal input.

Using MATLAB for PFE

Most of the above work is built into the following MATLAB command:

```
[r p k] = residuez([1 0 0 -1], [1 -4/3 7/12 -1/12])
```

which returns

- `r = [114 -21 -104]` (residues)
- `p = [0.5 0.5 0.3333]` (poles)
- `k = [12]` (direct terms)

Furthermore, using MATLAB's `impz` command, one can compute values of $h[n]$ directly from $\{b_k\}$ and $\{a_k\}$ (but it does not provide a *formula* for $h[n]$).

skim

3.4.4
Decomposition of rational z-transforms

If $a_0 = 1$ then

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}.$$

Product form.

Combine complex-conjugate pairs

$b_{1k} = -2\,\mathrm{real}(z_k)$

$b_{2k} = |z_k|^2$

$a_{1k} = -2\,\mathrm{real}(p_k)$

$a_{2k} = |p_k|^2$

$$X(z) = b_0 \frac{\prod_{k=?}^{?} (1 - z_k z^{-1})}{\prod_{k=?}^{?} (1 - p_k z^{-1})} \frac{\prod_{k=?}^{?} (1 + b_{1k} z^{-1} + b_{2k} z^{-k})}{\prod_{k=?}^{?} (1 + a_{1k} z^{-1} + a_{2k} z^{-k})}.$$

useful for implementing, see Ch 7, 8

just skim for now!

3.5
The One-Sided z-transform

skim

Useful for analyzing response of non-relaxed systems.

Definition:

$$X^+(z) \triangleq \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

3.5.1

3.5.2 **Solution of difference equations with nonzero initial conditions**

3.6

Analysis of LTI Systems in the z-domain

Main goals:
- Characterize response to inputs.
- Characterize system properties (stability, causality, etc.) in z-domain.

3.6.1

Response of systems with rational system functions

$X(z) \to \boxed{H(z)} \to Y(z)$. Goal: characterize $y[n]$

Assume
- $H(z)$ is a pole-zero system, *i.e.*, $H(z) = B(z) \,/\, A(z)$.
- Input signal has a rational z-transform of the form $X(z) = N(z)/Q(z)$.

Then
$$Y(z) = H(z)\, X(z) = \frac{B(z)\, N(z)}{A(z)\, Q(z)}.$$

So the output signal also has a rational z-transform.

How do we find $y[n]$? Since $Y(z)$ is rational, we use PFE to find $y[n]$.

Assume
- Poles of system $p_1, \ldots, p_N$ are unique
- Poles of input signal $q_1, \ldots, q_L$ are unique
- Poles of system and input signal are all different
- Zeros of system and input signal differ from all poles (so no pole-zero cancellation)
- Proper form
- Causal input sequence and causal LTI system

Then
$$X(z) = \sum_{k=1}^{L} \frac{\alpha_k}{1 - q_k z^{-1}} \xrightarrow{\mathcal{T}} Y(z) = \sum_{k=1}^{N} \frac{r_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{s_k}{1 - q_k z^{-1}}$$

so (assuming a causal system) the response is:

$$y[n] = \underbrace{\sum_{k=1}^{N} r_k p_k^n \, u[n]}_{\text{natural}} + \underbrace{\sum_{k=1}^{L} s_k q_k^n \, u[n]}_{\text{forced}}.$$

The output signal for a causal pole-zero system with input signal having rational z-transform is a weighted combination of geometric progression signals.

If there are repeated poles, then of course the PFE has terms of the form $np^n \, u[n]$ etc.

The output signal has two parts
- The p_k terms are the **natural response** $y_{\text{nr}}[n]$ of the system. (The input signal affects only the residues r_k).
  Each term of the form $p_k^n \, u[n]$ is called a **mode** of the system.
- The q_k terms are the **forced response** $y_{\text{fr}}[n]$ of the system. (The system affects "only" the residues s_k.)

Transient response from pole-zero plot

What about systems that are not necessarily in proper form?
There may be additional $k_l \, \delta[n-l]$ terms in the impulse response.

From the pole-zero plot corresponding to $H(z)$, we can identify how many $k_l \, \delta[n-l]$ terms will occur in the impulse response.
For causal systems:
- If there are one or more zeros at $z = 0$, then there will be no $\delta[n]$ terms in $h[n]$.
- If there are no poles or zeros at $z = 0$, then there will be one term of the form $k_0 \, \delta[n]$ in the the impulse response.

- If there are $N_1 \geq 1$ poles at $z = 0$, then $h[n]$ will include $N_1 + 1$ terms of the form $k_l \, \delta[n - l]$.

For IIR filters, the δ terms are less important than the terms in the impulse response (and in the transient response) that correspond to nonzero poles.

3.6.2

Response of pole-zero systems with nonzero initial conditions

skim

3.6.3

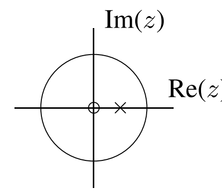
Transient and steady-state response

Define $y_{\mathrm{nr}}[n]$ to be the natural response of the system, *i.e.*, $y_{\mathrm{nr}}[n] = \sum_{k=1}^{N} r_k p_k^n \, u[n]$.

- If all the poles have magnitude less than unity, then this response decays to zero as $n \to \infty$.
- In such cases we also call the natural response the **transient response**.
- Smaller magnitude poles lead to faster signal decay. So *the closer the pole is to the unit circle, the longer the transient response*.

The forced response has the form $y_{\mathrm{fr}}[n] = \sum_{k=1}^{L} s_k q_k^n \, u[n]$.

- If all of the input signal poles are within the unit circle, then the forced response will decay towards zero as $n \to \infty$.
- If the input signal has a pole *on the unit circle* then there is a persistent sinusoidal component of the input signal. The forced response to such a sinusoid is also a persistent sinusoid.
- In this case, the forced response is also called the **steady-state** response.



Example. System (initially relaxed) described by diffeq: $y[n] = \frac{1}{2}\, y[n-1] + x[n]$.

What are the poles of the system? At $p = 0.5$. $H(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}$.

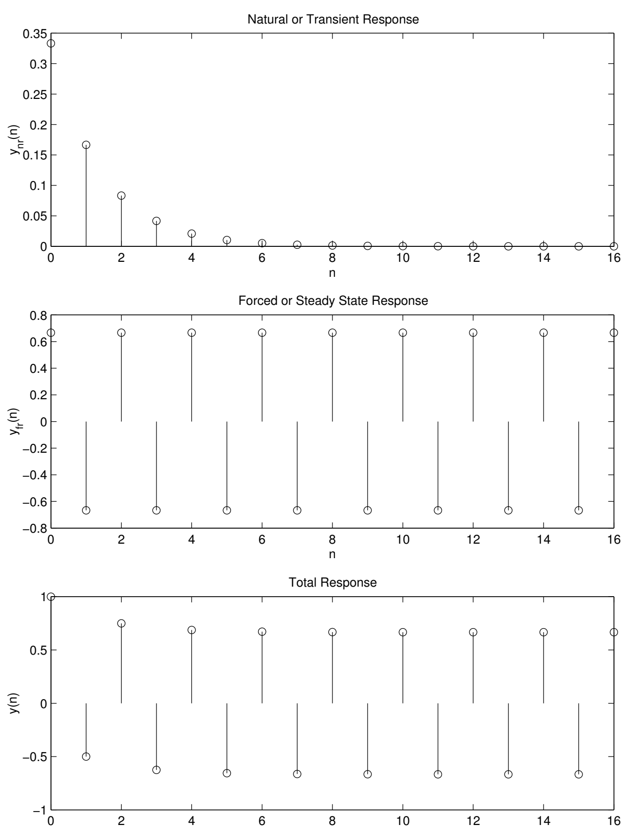
Signal: $x[n] = (-1)^n \, u[n]$. Pole at $q = -1$. $X(z) = \dfrac{1}{1 + z^{-1}}$.

$$Y(z) = H(z)\, X(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 + z^{-1})} = \frac{1}{1 + \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2}} = \frac{1/3}{1 - \frac{1}{2} z^{-1}} + \frac{2/3}{1 + z^{-1}}$$

where I found the PFE using `[r p k] = residuez(1, [1 1/2 -1/2])`. So

$$y[n] = \underbrace{\frac{1}{3}\left(\frac{1}{2}\right)^n u[n]}_{\text{natural / transient}} + \underbrace{\frac{2}{3}(-1)^n \, u[n]}_{\text{forced / steady state}} .$$

Where did 2/3 come from? $H(-1) = 2/3$.



Geometric progression signals are "almost" eigenfunctions of LTI systems

Fact: the forced response of an LTI system with rational system function $H(z)$ that is driven by a geometric progression input signal $x[n] = q^n u[n]$ is that same geometric progression scaled by $H(q)$, *i.e.*,

$$x[n] = q^n u[n] \xrightarrow{\mathcal{T}} y[n] = y_{\mathrm{nr}}[n] + H(q)\, q^n u[n],$$

if no poles in system at $z = q$.

$$Y(z) = H(z)\, X(z) = H(z) \frac{1}{1 - qz^{-1}} = \frac{B(z)}{A(z)(1 - qz^{-1})} = \frac{P(z)}{A(z)} + \frac{r}{1 - qz^{-1}}$$

by PFE if no roots of $A(z)$ at $z = q$.

Residue:

$$r = (1 - qz^{-1})\, Y(z)\big|_{z=q} = (1 - qz^{-1})\, H(z) \left.\frac{1}{1 - qz^{-1}}\right|_{z=q} = H(q)$$

so

$$y[n] = y_{\mathrm{nr}}[n] + H(q)\, q^n u[n]\,.$$

In particular, if $q = \mathrm{e}^{\jmath\omega_0}$, then the input signal is a **causal sinusoid**, and the forced response is a steady-state response. And if the LTI system is stable, then it has no poles on the unit circle, so the condition that $A(z)$ have no roots at $z = q$ is satisfied. So the steady-state response is

$$y_{\mathrm{fr}}[n] = H(\mathrm{e}^{\jmath\omega_0})\, \mathrm{e}^{\jmath\omega_0 n} u[n] = |H(\mathrm{e}^{\jmath\omega_0})|\, \mathrm{e}^{\jmath(\omega_0 n + \angle H(\mathrm{e}^{\jmath\omega_0}))} u[n]$$

which is a causal sinusoidal signal.

Thus the interpretation of $H(\mathrm{e}^{\jmath\omega_0})$ as a **frequency response** is entirely appropriate, even in the case of non-eternal sinusoidal signals!

Note that if the system is stable, then the poles are inside the unit circle so the natural response will be a transient response in this case, so eventually the output just looks essentially like the steady-state sinusoidal response.

3.6.4

Causality and stability

We previously described six system properties: **linearity**, **invertibility**, **stability**, **causality**, **memory**, **time-invariance**.

- We first described these properties in general.
- We then characterized these properties in terms of the impulse response $h[n]$ of an LTI system, because any LTI system is described completely by its impulse response $h[n]$.
  - **causality**: $h[n] = 0 \; \forall \, n < 0$.
  - **stability**: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
- Now we characterize these properties in the z-domain.
  If it exists, the system function $H(z)$ (including its ROC) also describes completely an LTI system, since we can find $h[n]$ from $H(z)$, *i.e.*, we can determine the output $y[n]$ for any input signal $x[n]$ if we know $H(z)$ and its ROC.

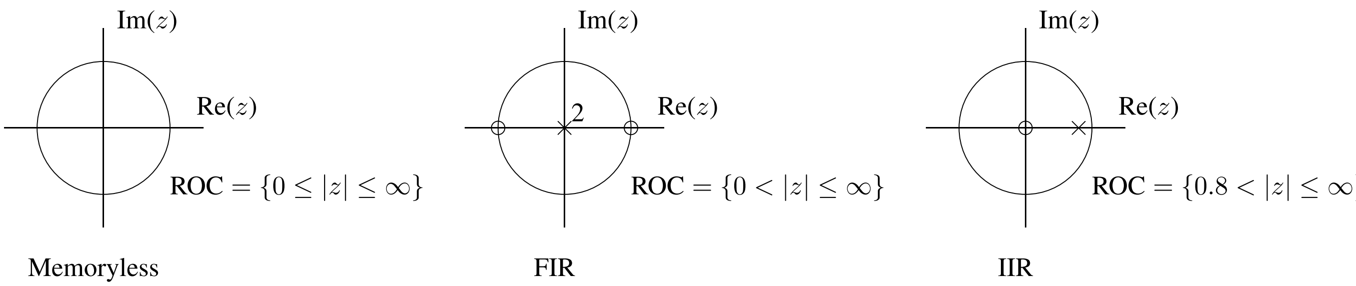
Skill: *Examine conditions for causality, stability, invertibility, memory in the z-domain.*

Memory

What is the system function and ROC of a memoryless system?

An LTI system is memoryless iff $h[n] = b_0 \, \delta[n]$. So $H(z) = b_0$. So $H(z)$ has no poles or zeros, and ROC $= \mathbb{C}$.

In terms of dynamic systems, recall that previously we noted that FIR systems are "all zero" systems (poles at origin only).



Causality

Previous time-domain condition for causality: LTI system is causal iff its impulse response $h[n]$ is 0 for $n < 0$.

How can we express this in the z-domain?
We showed earlier that the ROC of the z-transform of a right-sided signal is the exterior of a circle.
But is ROC = "exterior of a circle" enough? No!

Example. $h[n] = u[n+1] \overset{Z}{\leftrightarrow} \dfrac{z}{1-z^{-1}} = \dfrac{z^2}{z-1}$ for $\{1 < |z| < \infty\}$.
The ROC is a circle's exterior, and $h[n]$ is right-sided, but the system is *not* causal.

For a causal system, the system function (assuming it exists) has a series expansion that involves only *non-positive* powers of z:

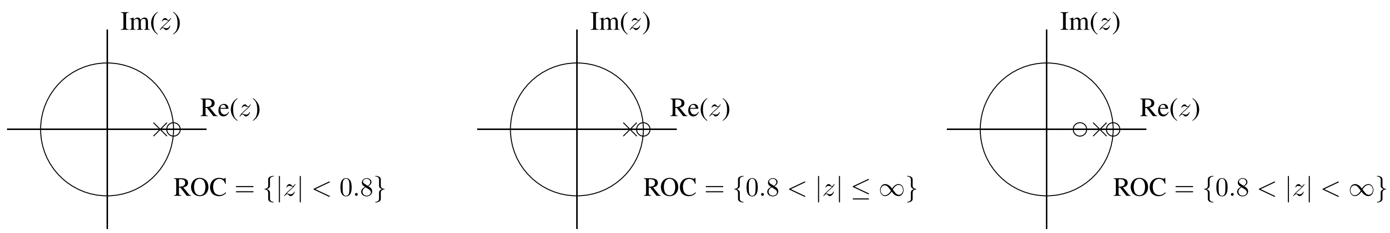
$$H(z) = \sum_{n=0}^{\infty} h[n]\, z^{-n} = h[0] + h[1]\, z^{-1} + h[2]\, z^{-2} + \cdots .$$

So the ROC of such an $H(z)$ will include $|z| = \infty$. (In fact, $\lim_{z\to\infty} H(z) = h[0]$, which must be finite.)

An LTI system with impulse response $h[n]$ is **causal** iff the ROC of the system function is the exterior of a circle of radius $r < \infty$ *including* $z = \infty$, *i.e.*, ROC = $\{r < \|z\| \le \infty\}$, or, in the trivial case of a memoryless system, ROC = $\{0 \le \|z\| \le \infty\}$.

Example. (***skip***) Is the LTI system with system function $H(z) = z^2 - z^{-1}$ causal? The ROC is $\mathbb{C} - \{\infty\} - \{0\}$, which is the exterior of a circle of radius 0, excluding ∞. Thus noncausal, which we knew since $h[n] = \delta[n+2]$.

Example. Which of the following pole-zero plots correspond to causal systems?



Only the middle one. For the right one $H(z) = g\frac{(z-1)(z-1/2)}{z-0.8}$ which is infinite at $z = \infty$. It is noncausal.

A given pole-zero plot for a rational system function corresponds to a causal LTI system iff there are at least as many (finite) poles as (finite) zeros and the ROC is the exterior of the circle intersecting the outermost pole.

Stability

Recall time-domain condition for stability: an LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
How to express in the z-domain?
Recall definition of the ROC of a system function:

$$z \in \text{ROC iff } \{h[n]\, z^{-n}\} \text{ is absolutely summable, } \textit{i.e.}, \; S(z) = \sum_{n=-\infty}^{\infty} |h[n]|\, |z|^{-n} < \infty.$$

- Suppose system is stable. What can we say about ROC?
  If the system is stable, then on the unit circle, where $|z| = 1$, we see $S(z) < \infty$.
  Thus BIBO stable system $\Longrightarrow$ ROC includes unit circle.
- Conversely, if the ROC includes the unit circle, then it includes the point $z = 1$, so $S(1) < \infty$, which implies $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ so the system is BIBO stable.

$$\boxed{\text{An LTI system is BIBO stable iff the ROC of its system function includes the unit circle.}}$$

Example. Suppose an LTI system has a pole on the unit circle at $z = e^{\jmath\omega_0}$. If we apply the bounded input $e^{\jmath\omega_0 n}\, u[n]$, then the steady state response (see 3.6.6 below) will include a term like $n\, e^{\jmath\omega_0 n}\, u[n]$, which is unbounded.

So poles on the unit circle preclude stability.

Example. $y[n] = -\,y[n-1] + x[n] \Longrightarrow H(z) = \frac{1}{1+z^{-1}} = \frac{z}{z+1}$ which has a pole at $z = -1$ so this system is unstable.

In general causality and stability are unrelated properties.

However, for a causal system we can narrow the condition for stability.

For a causal system, the ROC is the exterior of a circle. For it to be stable as well, the ROC must include the unit circle, so the radius r for the ROC must be less than 1. There cannot be any poles in the ROC, so all the poles must be inside (or on the boundary of) the circle of radius $r < 1$, which are thus inside the unit circle.

$$\boxed{\text{A causal LTI system is BIBO stable iff all of the poles of its system function are } \textit{inside} \text{ the unit circle.}}$$

Example. Accumulator: $y[n] = y[n-1] + x[n]$ has $H(z) = \frac{1}{1-z^{-1}}$. Stable? No: causal but pole at $z = 1$ so unstable.

Recall earlier pictures showing that causal signals with poles outside unit circle are blowing up.

Intuition: signals with poles on the unit circle are the most "persistent" of the bounded signals, since they are oscillatory with no decay. So for the system to have bounded output for such bounded input signals, its ROC must include the unit circle.

skip *3.6.6* Multiple-order poles and stability

Can poles of system function lie on the unit circle and still have the system be stable? No.

Example. Consider $h[n] = u[n]$, so $H(z) = 1/(1 - z^{-1})$, which has a pole at $z = 1$. Now consider the input $x[n] = u[n]$, which is certainly a bounded input. The output $y[n] = (n+1)\, u[n]$, as we derived long ago. So the output is not bounded.

This can happen anywhere on the unit circle.
Therefore for a causal system to be stable, all the poles of its system function must lie strictly inside the unit circle.

3.6.5 Pole-zero cancellations

When a system has a pole and a zero at *exactly* the same location, they cancel each other out.

Example. Is the system $y[n] = 3\, y[n-1] + x[n]$ stable? No, since it hsa a pole at $z = 3$.

Example. Find system function and pole-zero plot and assess stability for diffeq system $y[n] = 3\, y[n-1] + \frac{1}{3}\, x[n] - x[n-1]$. Since $[1 - 3z^{-1}]\, Y(z) = [\frac{1}{3} - z^{-1}]\, X(z)$, the system function is $H(z) = Y(z) \,/\, X(z) = \frac{\frac{1}{3} - z^{-1}}{1 - 3z^{-1}} = \frac{1}{3}$ and $h[n] = \frac{1}{3}\, \delta[n]$.

The pole and zero at $z = 3$ cancel, so yes, theoretically this is a stable LTI system.

picture of direct form I implementation $H_1(z) = \frac{1}{3} - z^{-1}$, $H_2(z) = \frac{1}{1 - 3z^{-1}}$.

In practice there may be imperfect pole-zero cancellation. For example, in binary representation,

$$1/3 = .010101\ldots = \sum_{k=0}^{\infty} 2^{-(2k+1)} = 1/4 + 1/16 + 1/64 + \cdots$$

which cannot be represented exactly with a finite number of bits. With 8 bits (.01010101), we get 0.333251953125 not $1/3$.

Invertibility

In time domain, an LTI system with impulse response $h[n]$ is invertible iff there exists an LTI system having some impulse response $h_I[n]$ that satisfies: $h[n] * h_I[n] = \delta[n]$.

In z-domain: $H(z)\, H_I(z) = 1$, so $H_I(z) = \frac{1}{H(z)}$.

Example. $H(z) = \frac{7}{5} \frac{z-2}{z-1/2} \Longrightarrow H_I(z) = \frac{5}{7} \frac{z-1/2}{z-2}$.

So the poles becomes zeros and the zeros become poles.

Thus, in principle, any LTI system with rational system function is invertible.

However, in practice usually we want a *stable* inverse.

A causal, stable LTI system has a causal stable inverse iff all of its poles and zeros are within the unit circle.

3.6.7
The Schur-Cohn stability test ***skip***

We now have two valid procedures for checking stability of *causal* LTI systems:
- Check if $\sum_{n=0}^{\infty} |h[n]| < \infty$.
- Check if poles of system lie inside unit circle.

To perform either one of these checks, generally one needs a concrete expression for $h[n]$ or for $H(z)$.

For a rational system function $H(z) = B(z) \,/\, A(z)$, the poles are the roots of the denominator polynomial: $A(z) = 1 + a_1 z^{-1} + \cdots a_N z^{-N}$. Given concrete numerical values for the a_k coefficients, the usual approach to testing stability would be to just use the MATLAB `roots` function and check the magnitudes of the roots.

But in the design process, often we have ranges of possible values for the coefficients, and we cannot check all of them using a numerical root-finding routine. And for degrees greater than 2, there is no simple method for analytically finding the roots.

The Schur-Cohn test provides a method for verifying stability of discrete-time LTI systems having rational system functions *without explicitly finding the roots* of the denominator polynomial. This is important practically since generally we want stable systems.

This test is the analog of the **Routh-Hurwitz criterion** used for testing stability of continuous-time systems.

The Schur-Cohn Stability Test

The Schur-Cohn test provides a method for verifying stability of LTI systems with rational system functions *without explicitly finding the roots* of the denominator polynomial. This is very important practically since generally we want stable systems.

Procedure.

- Initialization: $A_N(z) = A(z) = \sum_{k=0}^{N} a_k z^{-k}$, $a_N(k) = a_k$
- Define: $A_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k}$, where $a_m(0) = 1$
- Define: $B_m(z) = z^{-m} A_m(z^{-1}) = \sum_{k=0}^{m} a_m(m-k) z^{-k}$.
  This is called the **reverse polynomial**, since order of coefficients are reversed.
- Define: $K_m = a_m(m),\ m = 1, \ldots, N$
- Recursion: $\boxed{A_{m-1}(z) = \dfrac{A_m(z) - K_m\, B_m(z)}{1 - K_m^2}}$ for $m = N, N-1, \ldots, 1$
- Test: The roots of $A(z)$ are all inside the unit circle iff $\boxed{|K_m| < 1}$ for $m = 1, 2, \ldots, N$.

The following second-order analysis serves as an "example."

3.6.8 Stability of second-order systems

For first-order systems $y[n] = a\, y[n-1] + x[n]$, stability is trivial: check if $|a| < 1$.

Next interesting case is second-order systems:

$$y[n] = -a_1\, y[n-1] - a_2\, y[n-2] + b_0\, x[n] \Longrightarrow H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$

Question. What values of a_1 and a_2 lead to a stable system?

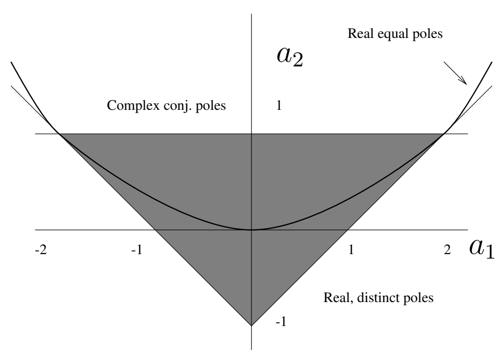
In this 2nd order case we could determine the roots using the quadratic formula.
That is not feasible for $N > 2$, so we use the Schur-Cohn method as an example.

$A_2(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$ so $K_2 = a_2(2) = a_2$

$$A_1(z) = \frac{A_2(z) - K_2\, B_2(z)}{1 - K_2^2} = \frac{1 + a_1 z^{-1} + a_2 z^{-2} - a_2(a_2 + a_1 z^{-1} + z^{-2})}{1 - a_2^2} = \frac{1 - a_2^2 + a_1(1 - a_2) z^{-1}}{1 - a_2^2} = 1 + \frac{a_1}{1 + a_2} z^{-1},$$

so $K_1 = \dfrac{a_1}{1 + a_2}$. Thus $H(z)$ is stable iff $\boxed{|a_2| < 1}$ and $\boxed{\left|\dfrac{a_1}{1 + a_2}\right| < 1}$ or $|1 + a_2| > |a_1|$.

When $|a_2| < 1$, $|1 + a_2| = 1 + a_2$, so we need $1 + a_2 > |a_1|$, *i.e.*, $-(1 + a_2) < a_1 < 1 + a_2$.



Restricting our designs to coefficients in that triangle will ensure stability, without explicitly finding the roots.

In this 2nd order case the roots are given by the quadratic formula: $p = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2 - 4a_2}{4}}$
- Real and equal poles when $a_1^2 = 4a_2$, *i.e.*, on the parabola $a_2 = a_1^2/4$ that touches corners of triangle.
- Real and distinct poles when $a_1^2 > 4a_2$, which is below parabola.
- Complex poles otherwise, above parabola.

The book derives the corresponding impulse response for each case.

3.7

Summary
- z-transform and its properties
- convolution property for z-domain convolution
- system function of LTI systems
- finding impulse response of diffeq system having rational system function
- characterizing properties of output signals (forced, natural, transient, steady-state response)
- characterizing system properties (causality and stability) in z-domain

We now have many representations of systems:
- time domain:
  - block diagram
  - impulse response
  - difference equation
- transform domain:
  - system function
  - pole-zero plot
  - frequency response (soon)

Skill: ***Convert between these six system representations.*** (See diagram.)
- Use z for going between $H(z)$ and pole-zero plot.
- Use z^{-1} for PFE and for finding diffeq coefficients.

Where is 2D and image processing examples? Although 2D z-transform's have been studied, *e.g.*, [3], they are not particularly useful in image processing, especially compared to the Fourier transform. In contrast, the 1D z-transform is the foundation for 1D filter design.

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[2] H. L. Royden. *Real analysis*. Macmillan, New York, 3 edition, 1988.

[3] J. S. Lim. *Two-dimensional signal and image processing*. Prentice-Hall, New York, 1990.

[4] A. K. Jain. *Fundamentals of digital image processing*. Prentice-Hall, New Jersey, 1989.

Discrete-time systems described by difference equations (FIR and IIR)

Difference equation:

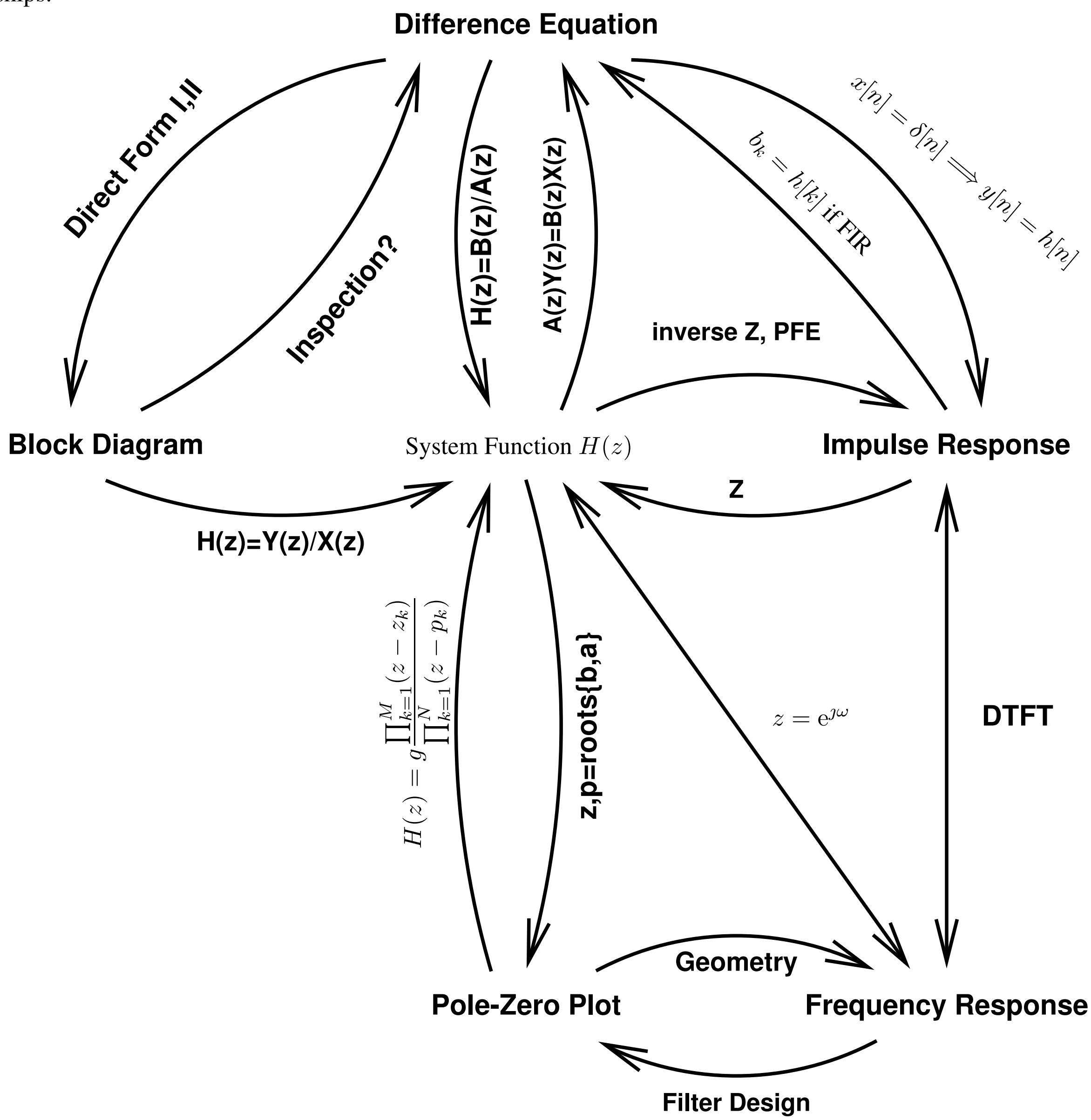
$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

System function (in expanded polynomial and in factored polynomial forms):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=1}^{N} a_k z^{-k}} = b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

Frequency magnitude response: $|\mathcal{H}(\omega)| = b_0 \frac{\prod_k |e^{\jmath\omega} - z_k|}{\prod_k |e^{\jmath\omega} - p_k|}$

Relationships:



Each representation corresponds to a type of input/output relationship, *e.g.*, convolution.

z-Transforms

Chapter 7

In the study of discrete-time signal and systems, we have thus far considered the time-domain and the frequency domain. The z-domain gives us a third representation. All three domains are related to each other.

A special feature of the z-transform is that for the signals and system of interest to us, all of the analysis will be in terms of ratios of polynomials. Working with these polynomials is relatively straight forward.

Definition of the z-Transform

- Given a finite length signal $x[n]$, the z-transform is defined as

$$\boxed{X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} x[k](z^{-1})^{k}} \tag{7.1}$$

  where the sequence support interval is $[0, N]$, and z is any complex number

- This transformation produces a new representation of $x[n]$ denoted $X(z)$

- Returning to the original sequence (*inverse z-transform*) $x[n]$ requires finding the coefficient associated with the nth power of z^{-1}

- Formally transforming from the time/sequence/*n*-domain to the z-domain is represented as

$$n\text{-Domain} \overset{z}{\leftrightarrow} z\text{-Domain}$$

$$x[n] = \sum_{k=0}^{N} x[k]\delta[n-k] \overset{z}{\leftrightarrow} X(z) = \sum_{k=0}^{N} x[k]z^{-k}$$

- A sequence and its z-transform are said to form a *z-transform pair* and are denoted

$$x[n] \overset{z}{\leftrightarrow} X(z) \tag{7.2}$$

  – In the sequence or n-domain the independent variable is n

  – In the z-domain the independent variable is z

Example: $x[n] = \delta[n - n_0]$

- Using the definition

$$X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} \delta[k-n_0]z^{-k} = z^{-n_0}$$

- Thus,

$$\delta[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}$$

Example: $x[n] = 2\delta[n] + 3\delta[n-1] + 5\delta[n-2] + 2\delta[n-3]$

- By inspection we find that

$$X(z) = 2 + 3z^{-1} + 5z^{-2} + 2z^{-3}$$

Example: $X(z) = 4 - 5z^{-2} + z^{-3} - 2z^{-4}$

- By inspection we find that

$$x[n] = 4\delta[n] - 5\delta[n-2] + \delta[n-3] - 2\delta[n-4]$$

- What can we do with the z-transform that is useful?

The z-Transform and Linear Systems

- The z-transform is particularly useful in the analysis and design of LTI systems

The z-Transform of an FIR Filter

- We know that for any LTI system with input $x[n]$ and impulse response $h[n]$, the output is

$$y[n] = x[n] * h[n] \tag{7.3}$$

- We are interested in the z-transform of $h[n]$, where for an FIR filter

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \tag{7.4}$$

- To motivate this, consider the input

$$x[n] = z^n, -\infty < n < \infty \tag{7.5}$$

- The output $y[n]$ is

$$\begin{aligned} y[n] &= \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k z^{n-k} \\ &= \sum_{k=0}^{M} b_k z^n z^{-k} = \left(\sum_{k=0}^{M} b_k z^{-k} \right) z^n \end{aligned} \tag{7.6}$$

- The term in parenthesis is the z-transform of $h[n]$, also known as the *system function* of the FIR filter
- Like $H(e^{j\omega})$ was defined in Chapter 6, we define the system function as

$$\boxed{H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}} \tag{7.7}$$

- The z-transform pair we have just established is

$$h[n] \overset{z}{\leftrightarrow} H(z)$$

$$\sum_{k=0}^{M} b_k \delta[n-k] \overset{z}{\leftrightarrow} \sum_{k=0}^{M} b_k z^{-k}$$

- Another result, similar to the frequency response result, is

$$y[n] = h[n] * z^n = H(z) z^n \tag{7.8}$$

- Note if $z = e^{j\hat{\omega}}$, we in fact have the frequency response result of Chapter 6

- The system function is an Mth degree polynomial in complex variable z

- As with any polynomial, it will have M roots or *zeros*, that is there are M values z_0 such that $H(z_0) = 0$
  - These M zeros completely define the polynomial to within a gain constant (scale factor), i.e.,

$$\begin{aligned} H(z) &= b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \\ &= (1 - z_1 z^{-1})(1 - z_2 z^{-1}) \cdots (1 - z_M z^{-1}) \\ &= \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{z^M} \end{aligned}$$

where z_k, $k = 1, \ldots, M$ denote the zeros

Example: Find the Zeros of

$$h[n] = \delta[n] + \frac{1}{6}\delta[n-1] - \frac{1}{6}\delta[n-2]$$

- The z-transform is

$$\begin{aligned} H(z) &= 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2} \\ &= \left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right) \\ &= \left(z + \frac{1}{2}\right)\left(z - \frac{1}{3}\right) / z^2 \end{aligned}$$

- The zeros of $H(z)$ are -1/2 and +1/3
- The difference equation

$$y[n] = 6x[n] + x[n-1] - x[n-2]$$

has the same zeros, but a different scale factor;

<u>proof</u>:

Properties of the *z*-Transform

- The z-transform has a few very useful properties, and its definition extends to infinite signals/impulse responses

The Superposition (Linearity) Property

$$\boxed{ax_1[n] + bx_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z)} \tag{7.9}$$

<u>proof</u>

$$\begin{aligned} X(z) &= \sum_{n=0} (ax_1[n] + bx_2[n])z^{-1} \\ &= a\sum_{n=0}^{N} x_1[n]z^{-1} + b\sum_{n=0}^{N} x_2[n]z^{-1} \\ &= aX_1(z) + bX_2(z) \end{aligned}$$

The Time-Delay Property

$$x[n-1] \overset{z}{\leftrightarrow} z^{-1}X(z) \tag{7.10}$$

and

$$x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}X(z) \tag{7.11}$$

proof: Consider

$$X(z) = \alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_N z^{-N}$$

then

$$\begin{aligned} x[n] &= \sum_{k=0}^{N} \alpha_k \delta[n-k] \\ &= \alpha_0 \delta[n] + \alpha_1 \delta[n-1] + \cdots + \alpha_N \delta[n-N] \end{aligned}$$

Let

$$\begin{aligned} Y(z) &= z^{-1}X(z) \\ &= \alpha_0 z^{-1} + \alpha_1 z^{-2} + \cdots + \alpha_N z^{-N-1} \end{aligned}$$

so

$$\begin{aligned} y[n] &= \alpha_0 \delta[n-1] + \alpha_1 \delta[n-2] + \cdots + \alpha_N \delta[n-N-1] \\ &= x[n-1] \end{aligned}$$

Similarly

$$\begin{aligned} Y(z) &= z^{-n_0}X(z) \\ \Rightarrow y[n] &= x[n-n_0] \end{aligned}$$

A General z-Transform Formula

- We have seen that for a sequence $x[n]$ having support interval $0 \le n \le N$ the z-transform is

$$X(z) = \sum_{n=0}^{N} x[n] z^{-n} \tag{7.12}$$

- This definition extends for doubly infinite sequences having support interval $-\infty \le n \le \infty$ to

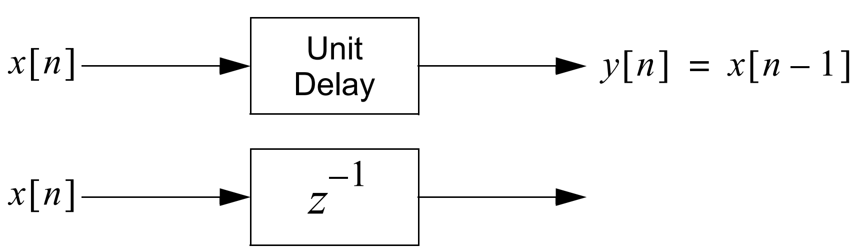
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{7.13}$$

  – There will be discussion of this case in Chapter 8 when we deal with infinite impulse response (IIR) filters

The z-Transform as an Operator

The z-transform can be considered as an operator.

Unit-Delay Operator



- In the case of the unit delay, we observe that

$$y[n] = \underset{\substack{\uparrow \\ \text{unit delay operator}}}{z^{-1}}\{x[n]\} = x[n-1] \tag{7.14}$$

which is motivated by the fact that $Y(z) = z^{-1}X(z)$

- Similarly, the filter

$$y[n] = x[n] - x[n-1]$$

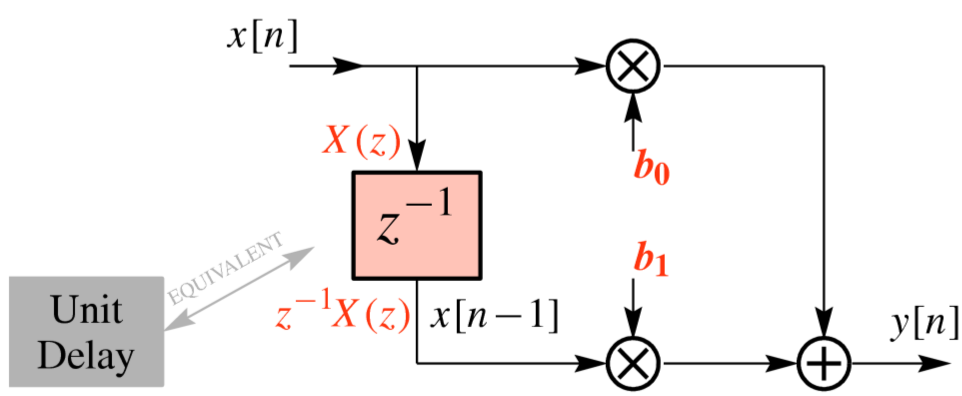
can be viewed as the operator

$$y[n] = (1 - z^{-1})\{x[n]\} = x[n] - x[n-1]$$

since

$$Y(z) = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z)$$

Example: Two-Tap FIR



- Using the operator convention, we can write by inspection that

$$Y(z) = b_0X(z) + b_1z^{-1}X(z)$$

$$y[n] = b_0x[n] + b_1x[n-1]$$

Convolution and the z-Transform

- The impulse response of the unity delay system is

$$h[n] = \delta[n-1]$$

and the system output written in terms of a convolution is

$$y[n] = x[n] * \delta[n-1] = x[n-1]$$

- The system function (z-transform of $h[n]$) is

$$H(z) = z^{-1}$$

and by the previous unit delay analysis,

$$Y(z) = z^{-1}X(z)$$

- We observe that

$$\boxed{Y(z) = H(z)X(z)} \tag{7.15}$$

<u>proof</u>:

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k] \tag{7.16}$$

We now take the z-transform of both sides of (7.16) using superposition and the general delay property

$$\begin{aligned} Y(z) &= \sum_{k=0}^{M} h[k](z^{-k}X(z)) \\ &= \left(\sum_{k=0}^{M} h[k]z^{-k}\right)X(z) = H(z)X(z) \end{aligned} \tag{7.17}$$

- Note: For the case of $x[n]$ a finite duration sequence, $X(z)$ is a polynomial, and $H(z)X(z)$ is a product of polynomials in z^{-1}

<u>Example</u>: Convolving Finite Duration Sequences

- Suppose that

$$\begin{aligned} x[n] &= 2\delta[n] - 3\delta[n-2] + 4\delta[n-3] \\ h[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

- We wish to find $y[n]$ by first finding $Y(z)$
- We begin by z-transforming each of the sequences

$$\begin{aligned} X(z) &= 2 - 3z^{-2} + 4z^{-3} \\ H(z) &= 1 + 2z^{-1} + z^{-2} \end{aligned}$$

- We find $Y(z)$ by direct multiplication

$$\begin{aligned} Y(z) &= (2 - 3z^{-2} + 4z^{-3})(1 + 2z^{-1} + z^{-2}) \\ &= 2 + 4z^{-1} - z^{-2} - 2z^{-3} + 5z^{-4} + 4z^{-5} \end{aligned}$$

- We find $y[n]$ using the delay property on each of the terms of $Y(z)$

$$\begin{aligned} y[n] = {} & 2\delta[n] + 4\delta[n-1] - \delta[n-2] \\ & - 2\delta[n-3] + 5\delta[n-4] + 4\delta[n-5] \end{aligned}$$

Convolve directly?

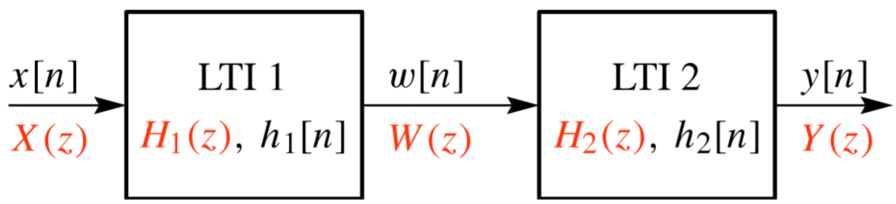
- This section has established the very important result that polynomial multiplication can be used to replace sequence convolution, when we work in the z-domain, i.e.,

z-Transform Convolution Theorem

$$y[n] = h[n]*x[n] \overset{z}{\leftrightarrow} H(z)X(z) = Y(z)$$

Cascading Systems

- We have seen cascading of systems in the time-domain and the frequency domain, we now consider the z-domain



- We know from the convolution theorem that

$$W(z) = H_1(z)X(z)$$

- It also follows that

$$Y(z) = H_2(z)W(z)$$

so by substitution

$$\begin{aligned} Y(z) &= [H_2(z)H_1(z)]X(z) \\ &= [H_1(z)H_2(z)]X(z) \end{aligned} \tag{7.18}$$

- In summary, when we cascade two LTI systems, we arrive at the cascade impulse response as a cascade of impulse responses in the time-domain and a product of the z-transforms in the z-domain

$$\boxed{h[n] = h_1[n]*h_2[n] \overset{z}{\leftrightarrow} H_1(z)H_2(z) = H(z)}$$

Factoring z-Polynomials

- Multiplying z-transforms creates a cascade system, so factoring must create subsystems

---

Example: $H(z) = 1 + 3z^{-1} - 2z^{-2} + z^{-3}$

- Since $H(z)$ is a third-order polynomial, we should be able to factor it into a first degree and second degree polynomial
- We can use the MATLAB function `roots()` to assist us

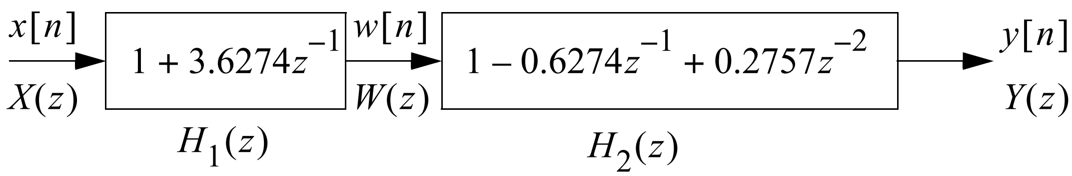
```
>> p = roots([1 3 -2 1])

p = -3.6274
     0.3137 + 0.4211i
     0.3137 - 0.4211i

>> conv([1 -p(2)],[1 -p(3)])

ans =  1.0000    -0.6274      0.2757 - 0.0000i
```

- With one real root, the logical factoring is to create two polynomials as follows

$$H_1(z) = 1 + 3.6274z^{-1}$$

$$\begin{aligned} H_2(z) &= (1 - (0.3137 + j0.4211)z^{-1}) \\ &\quad (1 - (0.3137 - j0.4211)z^{-1}) \\ &= 1 - 0.6274z^{-1} + 0.2757z^{-2} \end{aligned}$$

- The cascade system is thus:



- As a check we can multiply the polynomials

```
>> conv([1 -p(1)],conv([1 -p(2)],[1 -p(3)]))

ans = 1.0000, 3.0000, -2.0000-0.0000i, 1.0000-0.0000i
```

- The difference equations for each subsystem are

$$\begin{aligned} w[n] &= x[n] + 3.6274x[n-1] \\ y[n] &= w[n] - 0.6274w[n-1] + 0.2757w[n-2] \end{aligned}$$

Deconvolution/Inverse Filtering

- In a two subsystems cascade can the second system undo the action of the first subsystem?
- For the output to equal the input we need $H(z) = 1$
- We thus desire

$$H_1(z)H_2(z) = 1 \text{ or } H_2(z) = \frac{1}{H_1(z)}$$

Example: $H_1(z) = 1 - az^{-1}, |a| < 1$

- The inverse filter is

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - az^{-1}}$$

- This is no longer an FIR filter, it is an infinite impulse response (IIR) filter, which is the topic of Chapter 8
- We can approximate $H_2(z)$ as an FIR filter via long division

$$\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad\quad a^2z^{-2} \\
 & \qquad\quad \underline{a^2z^{-2} - a^3z^{-3}} \\
 & \qquad\qquad\qquad a^3z^{-3}
\end{array}$$

- An $M + 1$ term approximation is

$$H_2(z) = \sum_{k=0}^{M} a^k z^{-k}$$

  – Recall the deconvolution filter of Lab 8?

Relationship Between the z-Domain and the Frequency Domain

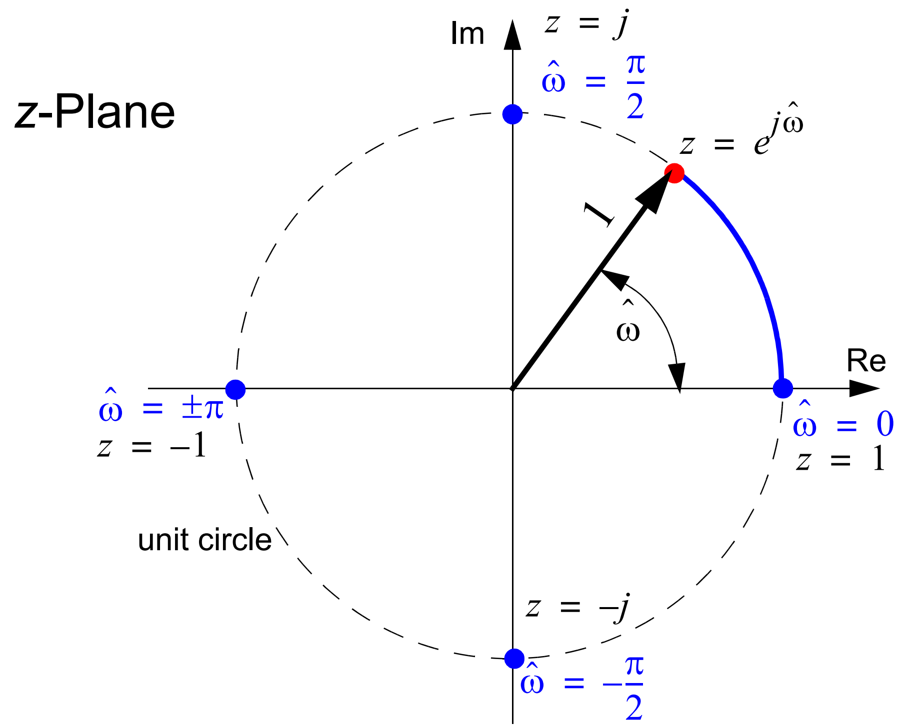
$\hat{\omega}$ - Domain		z - Domain
$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$	versus	$H(z) = \sum_{k=0}^{M} b_k z^{-k}$

- Comparing the above we see that the connection is setting $z = e^{j\hat{\omega}}$ in $H(z)$, i.e.,

$$\boxed{H(e^{j\hat{\omega}}) = H(z)\Big|_{z = e^{j\hat{\omega}}}} \tag{7.19}$$

The z-Plane and the Unit Circle

- If we consider the z-plane, we see that $H(e^{j\hat{\omega}})$ corresponds to evaluating $H(z)$ on the unit circle



- From this interpretation we also can see why $H(e^{j\hat{\omega}})$ is periodic with period 2π
  - As $\hat{\omega}$ increases it continues to sweep around the unit circle over and over again

The Zeros and Poles of *H*(*z*)

- Consider

$$H(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} \tag{7.20}$$

  where we have assumed that $b_0 = 1$

- Factoring $H(z)$ results in

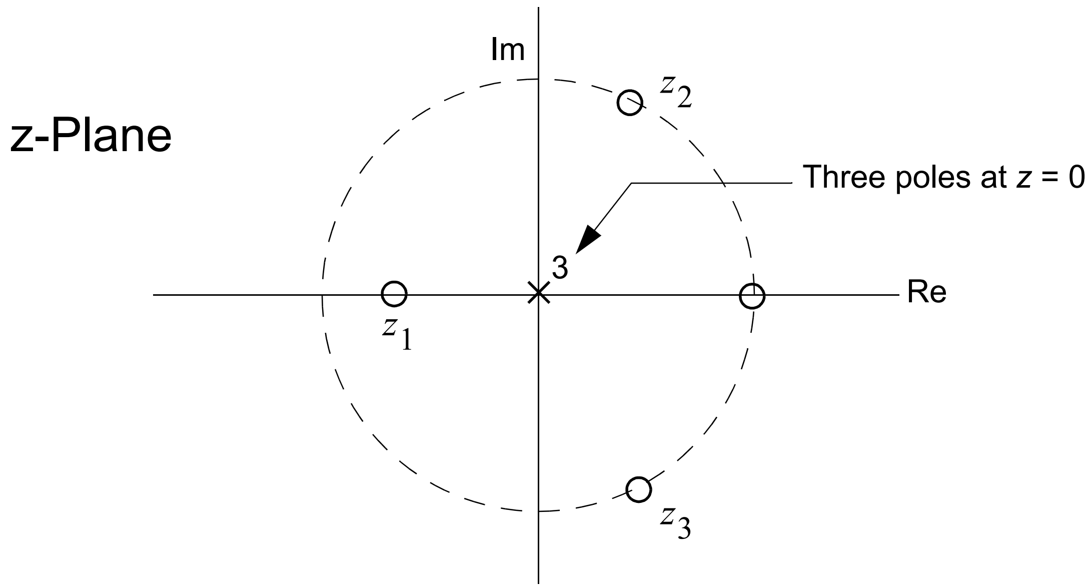
$$H(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1}) \tag{7.21}$$

- Multiplying by z^3/z^3 allows to write $H(z)$ in terms of positive powers of z

$$\begin{aligned} H(z) &= \frac{z^3 + b_1 z^2 + b_2 z^1 + b_3 z^0}{z^3} \\ &= \frac{(z - z_1)(z - z_2)(z - z_3)}{z^3} \end{aligned} \tag{7.22}$$

- The *zeros* are the locations where $H(z) = 0$, i.e., z_1, z_2, z_3
- The *poles* are where $H(z) \to \infty$, i.e., $z \to 0$
- <u>Note</u> that the poles and zeros only determine $H(z)$ to within a constant; recall the example on page 7-5

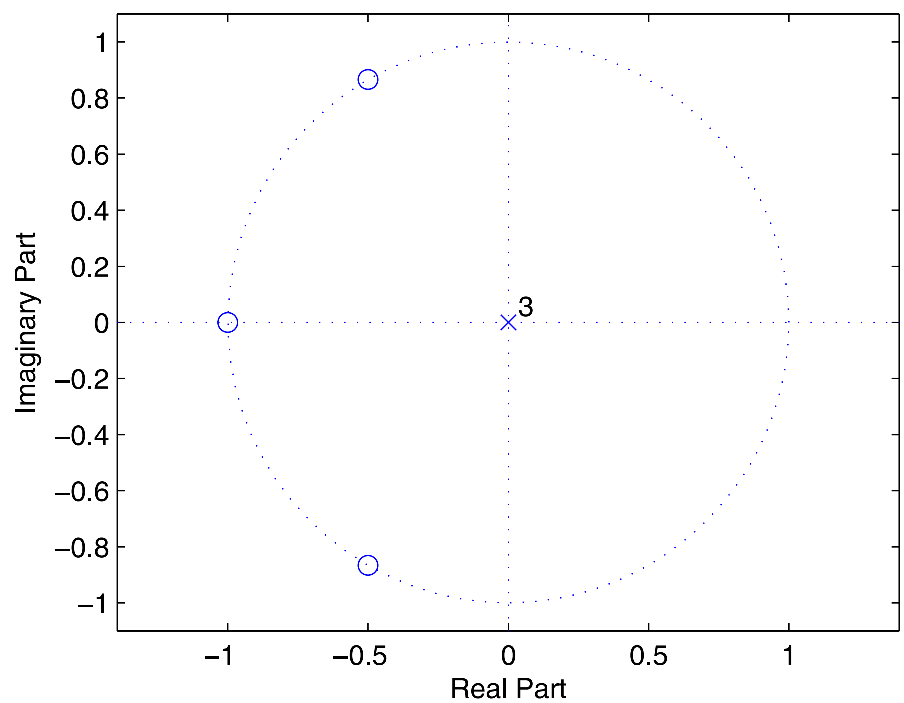
- A *pole-zero plot* displays the pole and zero locations in the z-plane



Example: $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

- MATLAB has a function that supports the creation of a pole-zero plot given the system function coefficients

```
>> zplane([1 2 2 1],1)
```



The Significance of the Zeros of $H(z)$

- The difference equation is the actual time domain means for calculating the filter output for a given filter input
- The difference equation coefficients are the polynomial coefficients in $H(z)$
- For $x[n] = z_0^n$ we know that

$$y[n] = H(z_0)z_0^n, \tag{7.23}$$

so in particular if z_0 is one of the zeros of $H(z)$, $H(z_0) = 0$ and the output $y[n] = 0$

- <u>If a zero lies on the unit circle</u> then the output will be zero for a sinusoidal input of the form

$$x[n] = z_0^n = (e^{j\hat{\omega}_0})^n = e^{j\hat{\omega}_0 n} \tag{7.24}$$

where $\hat{\omega}_0$ is the angle of the zero relative to the real axis, which is also the frequency of the corresponding complex sinusoid; why?

$$y[n] = \left(H(z)\Big|_{z = e^{j\hat{\omega}_0}}\right)e^{j\hat{\omega}_0 n} = 0 \tag{7.25}$$

Nulling Filters

- The special case of zeros on the unit circle allows a filter to *null*/block/*annihilate* complex sinusoids that enter the filter at frequencies corresponding to the angles the zeros make with respect to the real axis in the z-plane

- The nulling property extends to real sinusoids since they are composed of two complex sinusoids at $\pm\hat{\omega}_0$, and zeros not on the real axis will always occur in conjugate pairs if the filter coefficients are real
- This nulling/annihilating property is useful in rejecting unwanted jamming and interference signals in communications and radar applications

---

Example: $H(z) = 1 - 2\cos(\hat{\omega}_0)z^{-1} + z^{-2}$, $x[n] = \cos(\hat{\omega}_0 n)$

- Factoring $H(z)$ we find that

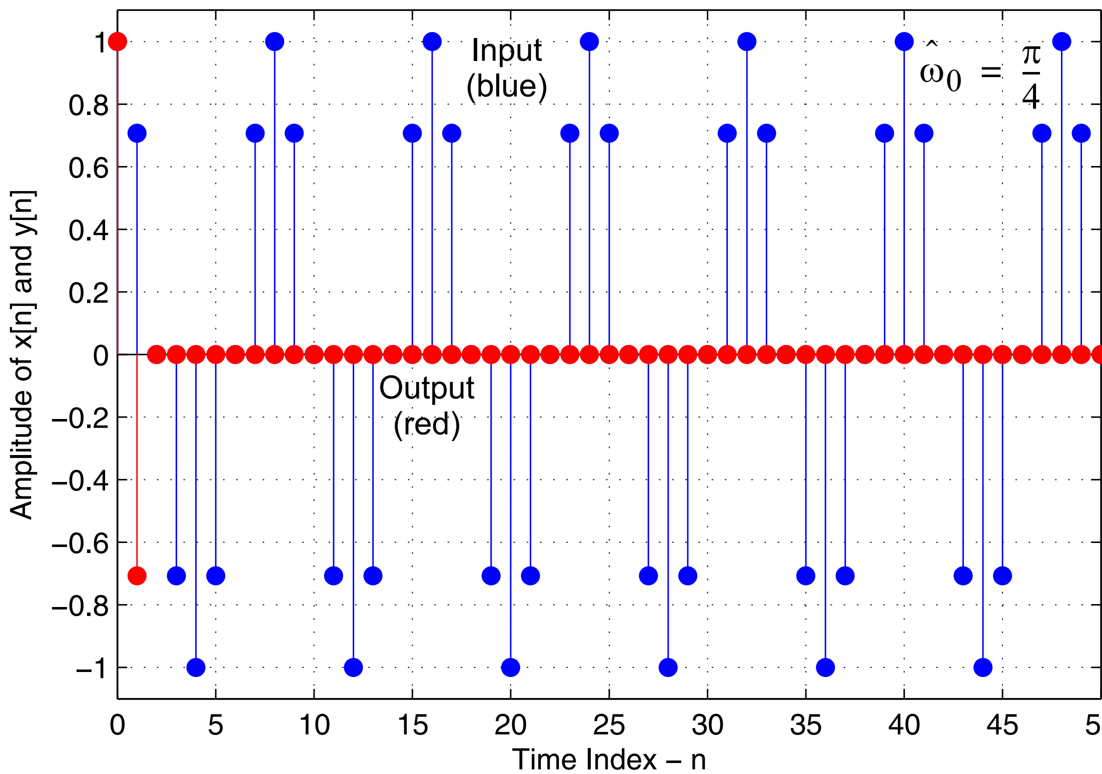
$$H(z) = \Big(1 - \underbrace{e^{j\hat{\omega}_0}}_{z_1} z^{-1}\Big)\Big(1 - \underbrace{e^{-j\hat{\omega}_0}}_{z_2} z^{-1}\Big)$$

- Expanding $x[n]$ we see that

$$x[n] = \frac{1}{2}e^{-j\hat{\omega}_0 n} + \frac{1}{2}e^{j\hat{\omega}_0 n}$$

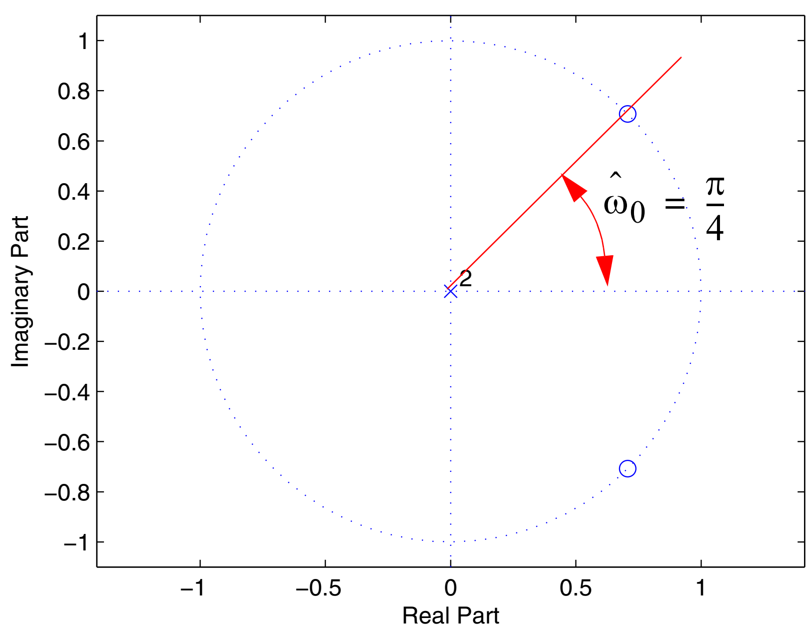
- The nulling action of $H(z)$ at $\pm\hat{\omega}_0$ will remove the signal from the filter output
- We can set up a simple simulation in MATLAB to verify this

```
>> n = 0:100;
>> w0 = pi/4;
>> x = cos(w0*n);
>> y = filter([1 -2*cos(w0) 1],1,x);
>> stem(n,x,'filled')
>> hold
Current plot held
>> stem(n,y,'filled','r')
>> axis([0 50 -1.1 1.1]); grid
```



- Since the input is applied at $n = 0$, we see a small transient while the filter settles to the final output, which in this case is zero

```
>> zplane([1 -2*cos(w0) 1],1)% check the pole-zero plot
```



Graphical Relation Between z and $\hat{\omega}$

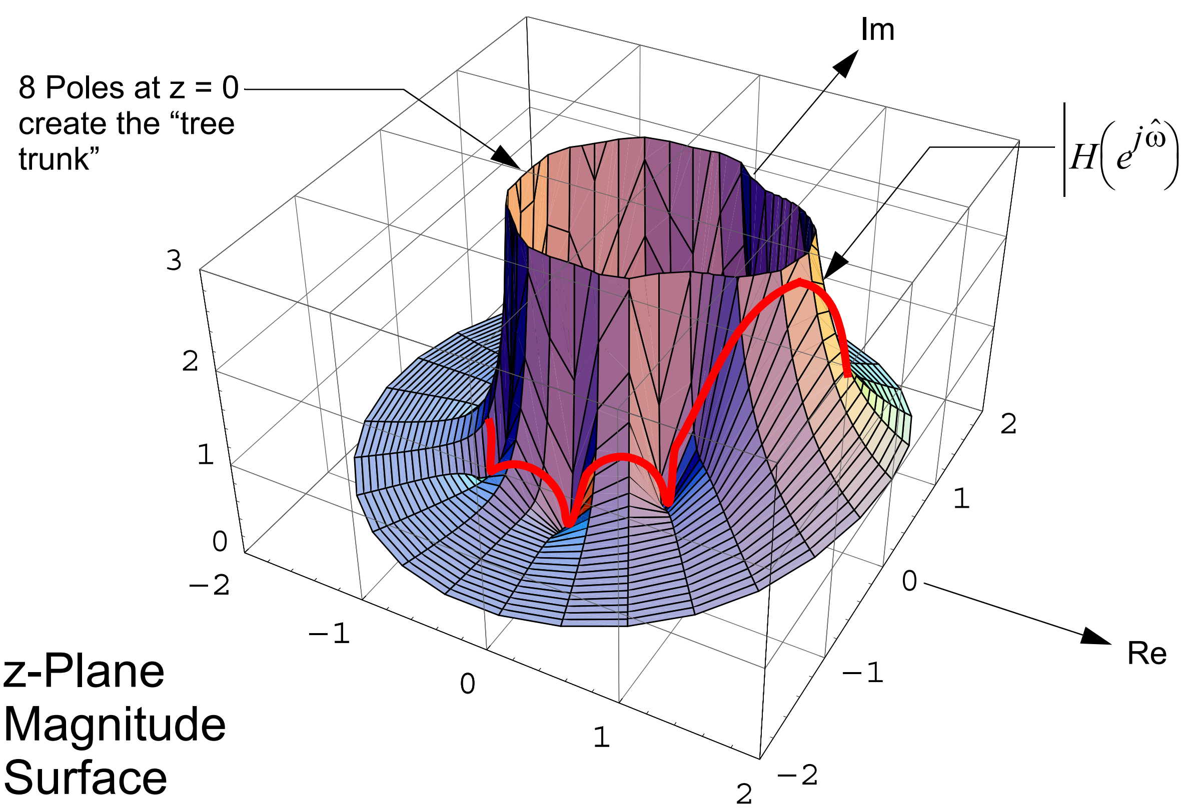
- When we make the substitution $z = e^{j\hat{\omega}}$ in $H(z)$ we know that we are evaluating the z-transform on the unit circle and thus obtain the frequency response
- If we plot say $|H(z)|$ over the entire z-plane we can visualize how cutting out the response on just the unit circle, gives us the frequency response magnitude

---

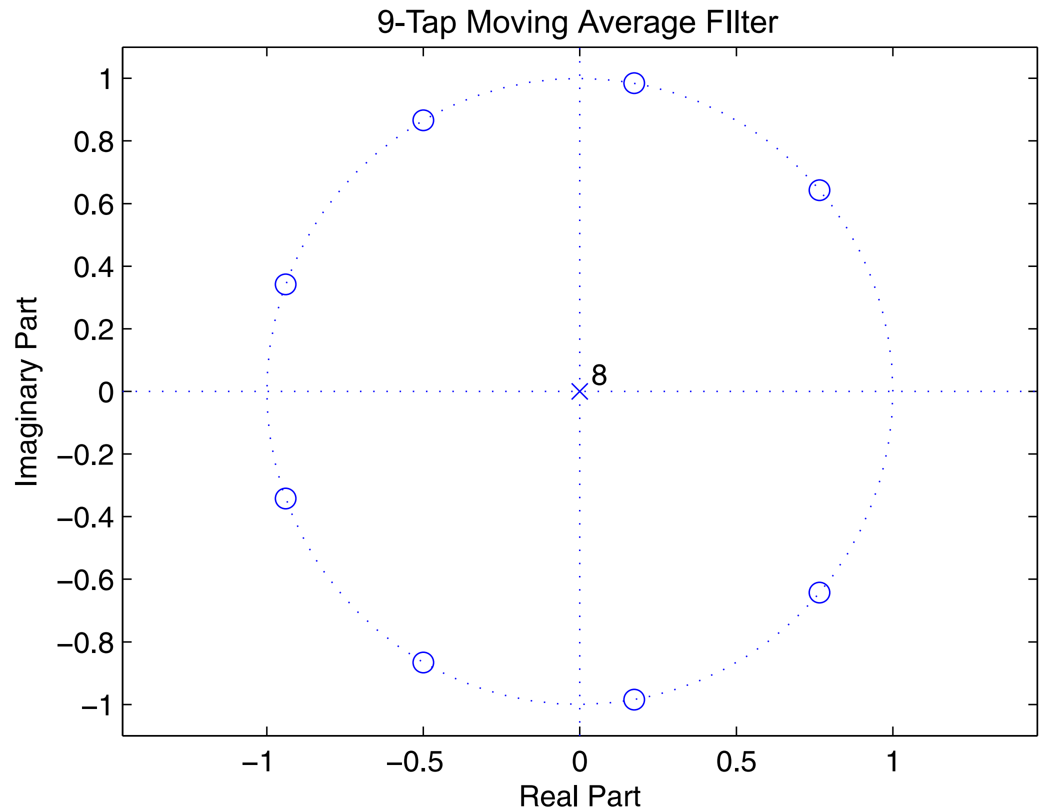
Example: $L = 9$ Moving Average Filter (9 taps/8th-order)

- Here we have

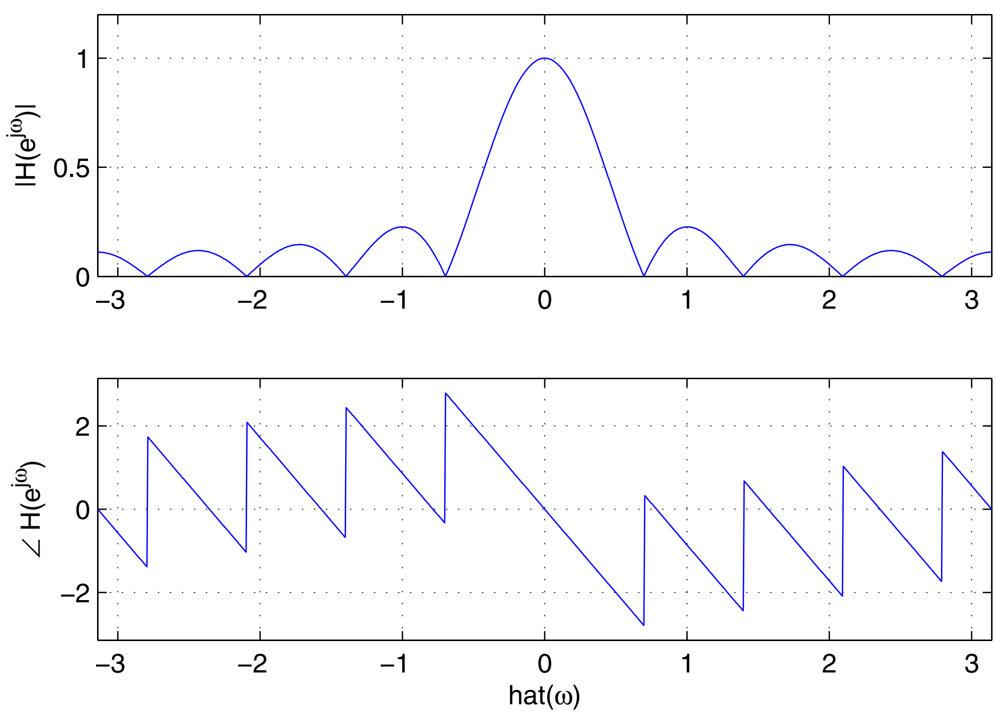
$$H(z) = \frac{1}{9}\sum_{k=0}^{9-1} z^{-k} = \frac{1}{9}\prod_{k=1}^{8}(1 - e^{-j2\pi k/9}z^{-1})$$



```
>> zplane([ones(1,9)]/9,1)
```




```
>> w = -pi:(pi/500):pi;
>> H = freqz([ones(1,9)/9],1,w);
```



Useful Filters

The L-Point Moving Average Filter

- The L-point moving average (running sum) filter has

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \tag{7.26}$$

and system function (z-transform of the impulse response)

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} \tag{7.27}$$

- The sum in (7.27) can be simplified using the geometric series sum formula

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} = \frac{1}{L}\cdot\frac{1-z^{-L}}{1-z^{-1}} = \frac{1}{L}\cdot\frac{z^{L}-1}{z^{L-1}(z-1)} \tag{7.28}$$

- Notice that the zeros of $H(z)$ are determined by the roots of the equation

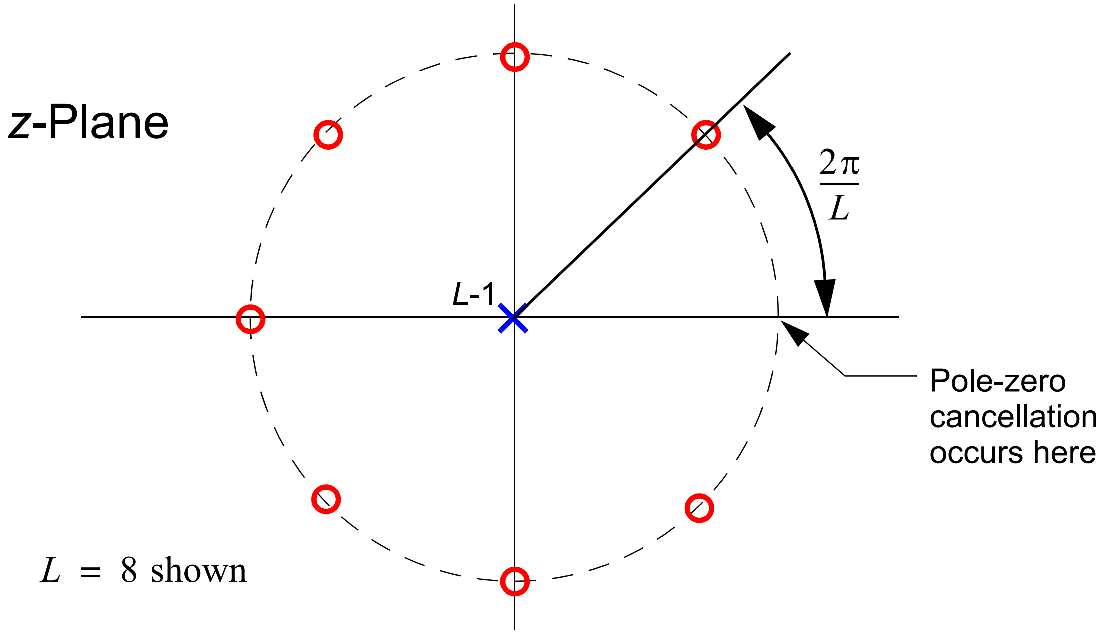
$$z^{L} - 1 = 0 \Rightarrow z^{L} = 1 \tag{7.29}$$

- The roots of this equation can be found by noting that $e^{j2\pi k} = 1$ for k any integer, thus the roots of (7.29) (zeros of (7.28)) are

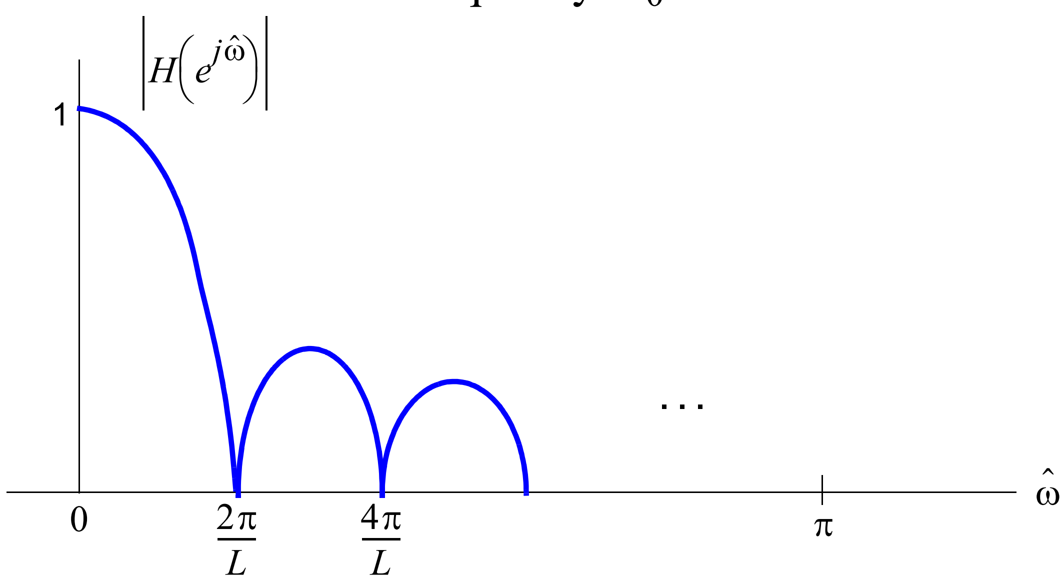
$$z_k = e^{j2\pi k/L},\ k = 0, 1, 2, \ldots, L-1 \tag{7.30}$$

- These roots are referred to as the L roots of unity

- One of the zeros sits at $z = 1$, but there is also a pole at $z = 1$, so there is a pole-zero cancellation, meaning that the pole-zero plot of $H(z)$ corresponds to the L-roots of unity, less the root at $z = 0$



- We have seen the frequency response of this filter before
- The first null occurs at frequency $\hat{\omega}_0 = 2\pi/L$



A Complex Bandpass Filter

see text

A Bandpass Filter with Real Coefficients

see text

Practical Filter Design

- Here we will use `fdatool` from the MATLAB signal processing toolbox to design an FIR filter

Properties of Linear-Phase Filters

- A class of FIR filters having symmetrical coefficients, i.e., $b_k = b_{M-k}$ for $k = 0, 1, \ldots, M$ has the property of linear phase

The Linear Phase Condition

- For a filter with symmetrical coefficients we can show that $H(e^{j\hat{\omega}})$ is of the form

$$H(e^{j\hat{\omega}}) = R(e^{j\hat{\omega}})e^{-j\omega M/2} \tag{7.31}$$

  where $R(e^{j\hat{\omega}})$ is a real function

- The fact that $R(e^{j\hat{\omega}})$ is real means that the phase of $H(e^{j\hat{\omega}})$ is a linear function of frequency plus the possibility of $\pm\pi$ phase jumps whenever $R(e^{j\hat{\omega}})$ passes through zero

Example: $H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_1 z^{-3} + b_0 z^{-4}$

- By factoring out z^{-2} we can write

$$H(z) = [b_0(z^2 + z^{-2}) + b_1(z^1 + z^{-1}) + b_2]z^{-2}$$

- We now move to the frequency response by letting $z \to e^{j\hat{\omega}}$

$$H(e^{j\hat{\omega}}) = [2b_0\cos(2\hat{\omega}) + 2b_1\cos(\hat{\omega}) + b_2]e^{-j\hat{\omega}4/2}$$

- Note that here we have $M = 4$, so we see that the linear phase term is indeed of the form $e^{-j\hat{\omega}M/2}$ and the real function $R(e^{j\hat{\omega}})$ is of the form

$$R(e^{j\hat{\omega}}) = b_2 + 2[b_0\cos(2\hat{\omega}) + b_1\cos(\hat{\omega})]$$

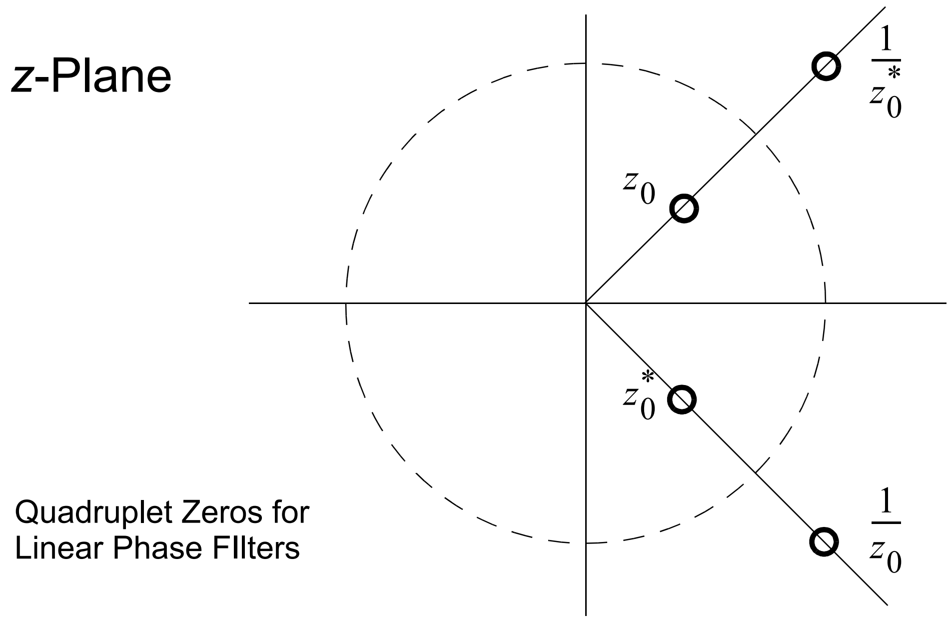
Locations of the Zeros of FIR Linear-Phase Systems

- Further study of $H(z)$ for the case of symmetric coefficients reveals that

$$H(1/z) = z^M H(z) \tag{7.32}$$

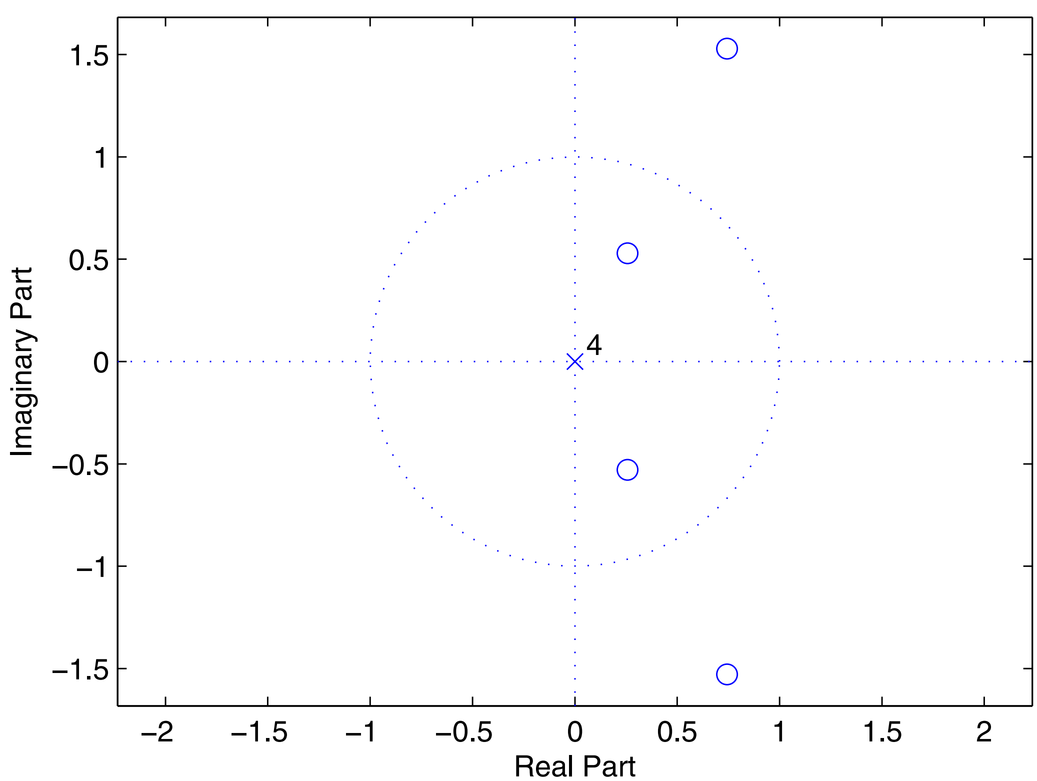
- A consequence of this condition is that for $H(z)$ having a zero at z_0 it will also have a zero at $1/z_0$
- Assuming the filter has real coefficients, complex zeros occur in conjugate pairs, so the even symmetry condition further implies that the zeros occur as quadruplets

$$\left\{z_0, z_0^*, \frac{1}{z_0}, \frac{1}{z_0^*}\right\}$$



Example: $H(z) = 1 - 2z^{-1} + 4z^{-2} - 2z^{-3} + z^{-4}$

```
>> zplane([1 -2 4 -2 1],1)
```



questions	opt1	opt2	opt3	opt4	opt5
A signal is a function which represents	. time variation	velocity variation	voltage variation	current variation	
A signal with real value of time is a	Discrete signal	Digital signal	Real signal	.Continuous signal	
ECG signal is a	. random & multichannel signal	random & multidimensional signal	deterministic & multidimensional signal	deterministic & multichannel signal	
If x(n) = 0,n<0, Pe is the even part of x(n) & Po is the odd part. Which of the following is true?	Po ≥ Pe	. Pe ≥ Po	Pe = Po	Pe ≠ Po	
A memory in a DT system is analog of	. energy storage in a continuous time system	memory in a CT system	sampled memory of a CT system	sampled in a CT system	
Which of the following is not true? A. u(-n) = 1; B. u(-n) = 0	both A & B	.only A	only B	neither A nor B	
Choose the correct answer 1. ramp A. u(t) = 1 2. step B. δ(t) = 1 3. pulse C. r(t) = t 4. Impulse D. p(t) = 1	.1-C; 2-A; 3-D;4 – B	1-B; 2-C; 3-D;4 – A	1-B; 2-D; 3-A;4 – C	1-D; 2-A; 3-B;4 – C	
Impulse function is	δ(t) = 1	∑ δ(t) dt = 1	. ∫ δ(t) dt = 1	∫ δ(t) dt= 0	
A sampled signal is	Analog signal	Digital signal	. Discrete signal	Continuous signal	
DT signal has	constant amplitude and constant time period	.varying amplitude and constant time period	constant amplitude and varying time period	varying amplitude and varying time period	
Choose the best answer A. s = α 1. Decreasing sinusoidal exponential B. s = - α 2. Increasing sinusoidal exponential C. s = - α + jβ 3. Decreasing exponential D. s = α + jβ 4. Increasing exponential	A - 4; B - 3; C – 2; D – 1	A - 3; B - 1; C – 4; D – 2	. A - 4; B - 3; C – 1; D – 2	A - 3; B - 4; C – 2; D – 1	

A discrete time signal can be represented	only in tabular form	only in sequence form	only in graphical form	. all forms
A general form of complex exponential signal is	$a^{n}e^{-j\omega on}$	$a\text{-}^{n}e^{j\omega on}$	. $a^{n}e^{j\omega on}$	$a\text{-}^{n}e^{-j\omega on}$
A time reversed signal is	y(t) = x(t)	. y(t) = x(-t)	y(-t) = x(-t)	y(t) = x(t -T)
u(-t+1) is	time reversed signal	time shifted signal	.time reversed & shifted signal	time reversed & scaled signal
A time scaled signal x(at)	diminishes 0<a<1	expands a>0	. expands 0<a<1	diminishes a<0
A pulse train is A. Discrete signal B. Periodic signal C.Digital signal D. Aperiodic signal	A & C only	. B & C only	A & D only	C & D only
Noise signal is a	certain signal with finite time	uncertain signal with finite time	certain signal with infinite time	. uncertain signal with infinite time
The auto-correlation function of the white noise is	constant	step function	Impulse function.	pulse function
EEG signal is a	periodic signal	random signal.	even signal	deterministic signal
In a periodic signal, the fundamental period T is	smallest positive value.	highest positive value	smallest negative value	highest negative value
The odd part of the signal is	½ [x(t) + x(-t)]	½ [x(t) - x(-t)].	½[x(-t) + x(t)]	½ [x(-t) - x(-t)]
The fundamental period N is	(2πm / ω).	(2π / ωm)	(ω/ 2πm)	(ωm / 2π)
Time shifting operation	increases the signal	reverses the signal	scales the signal	moves the signal
Folding operation gives	. reflected image	refractive image	magnified image	minimized image

Voice signal is a one dimensional signal	Generated by single source	Detected by single sensor	. Depends on a single variable	varies continuously
A periodic signal must have the fundamental period to be	logarithmic	integer	. rational	exponential
Tv picture signal is a	one dimensional signal	two dimensional signal	. three dimensional signal	multi dimensional signal
cos t is a	symmetric signal.	Anti-smmetric signal	Aperiodic signal	periodic signal
The fundamental period of sin50πt is	1/ 50 sec	. 1/ 25 sec	1/75 sec	1 sec
2u(t) + 2 sin2t is	a. periodic signal	b. Aperiodic signal	c. inverted signal	d. discrete signal
$e^{j2\pi n/3} + e^{j3\pi n/4}$ has a period of	3	8	24	10
cos(2πn) is	periodic signal.	Aperiodic signal	continuous signal	random signal
The even components of a signal are the same for both positive and negative values of n (Assertion). The discretion is	depend only on positive instants	depend only on negative instants	depend on both positive and negative instants.	does not depend on both positive and negative instants
The instantaneous power p(t) is	a. $1 / (v^2(t)R)$	b. $R / v^2(t)$	c. $v^2(t)R$	$v^2(t) / R$.
Lt $t\to\infty \int\|x(t)\|^2\, dt$ is a	Power signal	Energy signal.	both power and energy signal	neither a power nor an energy signal
Lt $t\to\infty\ (1 / 2N + 1) \sum \|x(n)\|^2$ represents a	Power signal.	Energy signal	both power and energy signal	neither a power nor an energy signal

A signal is an energy signal if	total energy is infinite	total energy is finite.	total power is finite	total power is infinite
Which of the following is true? A. Non-periodic & Deterministic signals are energy signals B. Periodic & random signals are power signals C. Periodic & Deterministic signals are energy signals D. Non- periodic & random signals are power signals	A & C	B & D	C & D	A & B.
A power signal has energy	0	10	∞.	100
An energy signal has power of	10	0	∞	100
A signal with infinite power and infinite energy is	Only power signal	only energy signal	both power & energy signal	
A signal with finite power and infinite energy is	Only power signal.	only energy signal	both power & energy signal	Neither a power nor energy signal
A signal with 0 power and finite energy is	Only power signal	only energy signal.	both power & energy signal	Neither a power nor energy signal
δ(t-1) exists only at	0	1	n	t
y(t) = x(t-2) shifts a signal by 2 units	right .	left	above	below
u(n)-u(n-1) is	u(n)	u(n-1)	δ(n).	δ(-n)
Relation between δ(t) & u(t) is	δ(t) = d/dt u(t).	u(t)= d/dt δ(t)	δ(t) = ∫ u(t)dt	u(t)= ∫ δ(t)dt
y(t) = ax(t) is	time scaled signal	amplitude scaled signal.	time shifted signal	amplitude shifted signal
The power of 10cos5tcos10t is	5W	10W	25W.	50W
e^{-3t} u(t) is a	power signal	energy signal.	pulse signal	step signal
A system that depends on the past and present input is	static system	. causal system	Non-causal system	Dynamic system
Stable systems will satisfy	homogeneity	superposition	BOBI	BIBO

$y(n) = x(n^3)$ is a	.linear system	non-linear system	non-causal system	unstable system
$y(t) = tx(t)+3$ is a ________ system	Linear & Time Invariant	.Non – Linear & Time Invariant	Linear & Time variant	Non – Linear & Time Variant
$y(n) = x(n)\cos\omega n$ is a __________ system	Dynamic & Causal	Static & Non – causal	. Static & Causal	Dynamic & Non – causal
Homogeneity and superposition principles are satisfied by	Static systems	Causal systems	. Linear systems	Stable systems
$y(n) = x^2n$ is	linear	. non-linear	unstable	dynamic
The signal which exists only at t=0 is	Time	Level	series	. impulse
The general form of CT exponential signal is	$C\ e^{t}$	. $C\ e^{at}$	$C\ e^{a2t2}$	$C^2\ e^{at2}$

opt6	answer
	. time variation
	.Continuous signal
	. random & multichannel signal
	. Pe ≥ Po
	. energy storage in a continuous time system
	.only A
	1-C; 2-A; 3-D;4 – B
	. ∫ δ(t) dt = 1
	. Discrete signal
	.varying amplitude and constant time period
	. A - 4; B - 3; C – 1; D – 2

. all
forms

. $a^n e^{j\omega on}$

. y(t) =
x(-t)
.time
reversed
&
shifted
signal
.
expands
0<a<1
. B & C
only

.
uncertain
signal
with
infinite
time
Impulse
function.
random
signal.

smallest
positive
value.
½ [x(t) -
x(-t)].
(2πm /
ω).
moves
the signal

.
reflected
image

.
Depends
on a
single
variable
. rational

three
dimensio
nal signal
symmetri
c signal

. 1/ 25
sec
1/ 25 sec

24

periodic
signal.

depend
on both
positive
and
negative
instants.

$v^2(t) / R$.

Energy
signal.

Power
signal.

total energy is finite.

A & B.

∞.

0

Neither a power nor energy signal.

Only power signal.

only energy signal.

1

right .

δ(n).

δ(t) = d/dt u(t).

amplitude scaled signal.

25W.

energy signal.

. causal system

BIBO

.linear
system

.Non –
Linear &
Time
Invariant
. Static
& Causal

. Linear
systems
. non-
linear
. impulse
. C e^{at}

questions	opt1	opt2	opt3
Fourier Series is used to represent	Continuous time signals	Discrete time signals	Continuous time periodic signals
For Fourier series to converge _____________ condition has to be satisfied.	Fourier	Euler	Lagrange
Which among the following is true?A.Over a period x(t) must be absolutely integrable.B.x(t) can infinite number of maxima and minimaC.x(t) can have many values at a certain instantD.x(t) must have finite number of discontinuities.	A & D	B & C	A & C
Choose the best answerA. Time Shifting1. jkω0X(K)B. Frequency Shifting2. X(K)C. Differentiation3. e -jk ω0 t0 X(K)D. Time Scaling4. X(k – K0)	A – 4; B – 3; C – 2; D - 1	A – 4; B – 3; C – 1; D – 2	A – 3; B – 4; C – 1; D - 2
If x(t) on FS gives X(k), then x*(t) is	X*(k)	X (-k)	X* (-k)
The product of 2 signals on FS will be	composition	convolution	multiplication
Parsevals equation of a Fourier series relates ________signal in frequency domain.	even signal	odd signal	energy signal
An odd signal in frequency domain will satisfy	–X(-k)	X(-k)	–X(k)
CT Fourier transform is used to represent	periodic signal	aperiodic signals	all CT signals
FT representation of a signal x(t) is	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega 0t}d\omega$	$\frac{1}{2\pi}\oint_{-\infty}^{\infty} X(j\omega)e^{-j\omega 0t}d\omega$	$\frac{1}{\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega 0}$
The synthesis equation of FT plays a role for	periodic signal	aperiodic signal	power signal
The FT of e –atu(t) is	$X(j\omega) = \frac{1}{a - j\omega}$	$X(j\omega) = \frac{2A}{A^2 + \omega^2}$	$X(j\omega) = \frac{1}{a + j}$
The F[δ(t)] is	1	0	∞
F[e –at] is	$X(j\omega) = \frac{1}{a - j\omega}$	$X(j\omega) = \frac{2A}{A^2 + \omega^2}$	$X(j\omega) = \frac{1}{a + j}$

FT of rectangular pulse in time domain is	$X(j\omega) = \frac{\sin \omega T_1}{\omega}$	$X(j\omega) = \frac{2\sin \omega T_1}{T_1}$	$X(j\omega) = \frac{\sin \omega T_1}{T_1}$
FT of rectangular pulse in frequency domain in	$X(\omega) = \frac{\omega}{\pi} \sin c\left(\frac{\omega t}{\pi}\right)$	$X(\omega) = \frac{\omega}{\pi} \sin c\left(\frac{\pi}{\omega t}\right)$	$X(\omega) = \sin c\left(\right.$
A increase in time of a signalA. broadens $X(j\omega)$ and narrows x(t)B. broadens x(t) and narrows $X(j\omega)$ C. broadens both x(t) and $X(j\omega)$	A only	B only	C only
The linearity property of FS is	aX(jω) + bY(jω)	aX(jω) • bY(jω)	aX(k) • bY(k)
LT of a1f1(t) + a2f2(t)	a1F1(js) + a2F2(js)	a1F1(jω) + a2F2(jω)	a1F1(s) + a2F2(s)
The trigonometric FS of a periodic function can have only	cosine term	sine term	cosine and sine terms
If the FT of a deterministic signal g(t) is G(f), then the FT of g(t-2) is	G(f)e –j4πf	G(2 – f)	2G(2f)
The inverse FS of frequency shifted X(k-5) is	e jkωt x(t)	e j5ωt x(t)	e j5kω x(t)
A waveform with discontinuities is always characterized by	converging Fourier spectra	strong harmonics	half wave harmonics
Conversion of analog signal to discrete signal is known as	quantization	companding	discretization
The error that occurs on sampling a signal below Nyquist rate is	guarding error	quantization error	aliasing error
The value of ak in exponential Fourier series is	$a_k = \frac{2}{T}\int_T x(t) e^{-jk\omega_0 t} dt$	$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$	$a_k = \frac{2}{T}\int_T x(t) e^{j}$
The Laplace transform of sinhat is	$\frac{a}{s^2 - a^2}$	$\frac{s}{s^2 - a^2}$	$\frac{a}{s^2 + a^2}$
The LT of tsinat is	$\frac{as}{(s^2 + a^2)^2}$	$\frac{2s}{(s^2 + a^2)^2}$	$\frac{2as}{(s^2 + a^2)^2}$
The Inverse LT of is	5e –t sin2t	5e –t sin5t	2e –t sin2t
The unilateral LT of a function x (t) is	$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$	$X(s) = \int_{-\infty}^{\infty} x(t) e^{-s\omega_0 t} dt$	$X(s) = \int_{0-}^{\infty} x(t) e^{-}$
The inverse Laplace transform of X(s) is	$x(t) = \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} dt$	$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} dt$	j

Magnitude and phase of Frequency co-efficient is obtained by	$\sqrt{x^2+y^2}, \tan^{-1}(\frac{y}{x})$	$\sqrt{x^2+y^2}, \tan^{-1}(\frac{x}{y})$	$\sqrt{(x+y)^2}, \tan^{-}$
In a complex exponential e - jkω0 t which of the following is true when the values of ‘k’ are	0, 1 & 2 respectively.	constant, I harmonic, II harmonic	fundamental, I harmonic, constant thereafter
The LT is mainly used for	dynamic signals	linear signals	time invariant signals
e - iθ is	cos θ + j sin θ	cos θ - sinθ	cos θ j sin θ
The Nyquist rate of a signal is	200 Hz	800 Hz	100 Hz
The Nyquist sampling rate for a signal g(t) = 10 cos(50πt) cos2(150πt), when t is in seconds is	150 samples / s	200 samples / s	250 samples / s
In sampling a signal, guard time is provided when sampling time is	< 1 / 2 fmax	. = 1 / 2 fmax	> 1/ 2 fmax
Assertion: A signal must be sampled at least twice the highest frequency (Nyquist rate) Dissertion:	Recovery of signal is worst	Guard band is less	Recovery of signal is easier
In sampling, overlapping of frequency components will occur if sampling time is same as	< 1 / 2 fmax	= 1 / 2 fmax	> 1/ 2 fmax
The Nyquist rate of the signal is	fmax	2 fmax	1 / 2 f
A signal of maximum frequency 10 KHz is sampled at Nyquist rate. The time interval between 2 successive samples is	50 µs	100 µs	1000 µs
In communication, the sampling technique leads to	higher efficiency	higher speed of communication	costly equipment
In order to get back the original signal from the sampled signal it is necessary to use	low pass filter	high pass filter	band pass filter
Sinusoidal functions and exponential functions are examples of	singular functions	Gaussian functions	orthogonal functions

Amount of information in a CT signal is	0	2 bits	2 bauds
FT of f(t) = 1 is	$\pi\delta(\omega)$	$3\pi\delta(\omega)$	$2\pi\delta(\omega)$
Match A. e - jωt 01. Multiplied exponential function B. 2. a constant C. kδ(ω)3. rectangular pulse D. 4. shifted impulse function	A – 3; B – 4; C – 2; D – 1	A – 4; B – 3; C – 1; D – 2	A – 4; B – 3; C – 2; D – 1
The FT of unit step function is	$\frac{1}{j\omega}$	$\pi\delta(\omega)$	$\frac{1}{j\omega} + 2\pi\delta(\omega)$
The advantages of LT are A. It gives total solution systematicallyB. It gives solution in frequency domain onlyC. The initial conditions cannot be incorporated	A & C	B & C	A & B
The LT of e – at sinωt will be	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{1}{(s+a)^2+\omega^2}$	$\frac{s}{(s+a)^2+\omega^2}$
Match A. t1. B. u(t) 2. C. e at3. D. sinωt4.	A – 2; B – 3; C – 4; D - 1	A – 3; B – 2 C – 4; D - 1	A – 3; B – 4; C – 2; D - 1
The initial value theorem is	$\mathop{Lt}_{t\to\infty} f(t) = \mathop{Lt}_{s\to 0} sF(s)$	$\mathop{Lt}_{t\to 0} f(t) = \mathop{Lt}_{s\to 0} F(s)$	$\mathop{Lt}_{t\to 0} f(t) = \mathop{Lt}_{s\to\infty} sF(s)$
The final value theorem is	$\mathop{Lt}_{t\to\infty} f(t) = \mathop{Lt}_{s\to 0} sF(s)$	$\mathop{Lt}_{t\to 0} f(t) = \mathop{Lt}_{s\to 0} F(s)$	$\mathop{Lt}_{t\to 0} f(t) = \mathop{Lt}_{s\to\infty} sF(s)$

opt4	opt5	opt6	answer
Discrete time periodic signals.			Continuous time periodic signals
Dirichlet			Dirichlet
B & D			A & D
A – 4; B – 1; C – 3; D – 2			A – 4; B – 3; C – 1; D – 2
X (k)			X* (-k)
conjugation			convolution
power signal			power signal
–X*(k)			–X(-k)
all DT signals			all CT signals
$^{t}\, d\omega$		$\frac{1}{\pi}\int_{-\infty}^{\infty} X(j\omega) e^{-j\omega 0t} d\omega$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega 0t} d\omega$
all CT signals			aperiodic signal
$\overline{\omega}$		$X(j\omega)=\frac{A}{A^2+\omega^2}$	$X(j\omega)=\frac{1}{a+j\omega}$
δ(t)			1
$\overline{\omega}$		$X(j\omega)=\frac{A}{A^2+\omega^2}$	$X(j\omega)=\frac{2A}{A^2+\omega^2}$

$X(j\omega) = \frac{2\sin\omega T_1}{\omega}$ $\left.\frac{\omega t}{\pi}\right)$	$X(j\omega) = \frac{2\sin\omega T_1}{\omega}$ $X(\omega) = \text{sinc}\left(\frac{\pi}{\omega t}\right)$ $X(\omega) = \frac{\omega}{\pi}\sin c\left(\frac{\omega t}{\pi}\right)$
A, B, C	B only
aX(k) + bY(k)	aX(k) + bY(k)
a1F1(ω) + a2F2(ω)	a1F1(s) + a2F2(s)
sinc term	cosine and sine terms
f(-2)	G(f)e –j4πf
e jk5t x(t)	e j5ωt x(t)
quarter wave harmonics	strong harmonics
sampling	sampling
granular error	aliasing error
$^{jk\omega 0t}dt$ $a_k = \frac{1}{T}\int_T x(t)e^{jk\omega 0t}dt$	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega 0t}dt$
$\frac{s}{s^2+a^2}$	$\frac{a}{s^2-a^2}$ $\frac{2a}{(s^2+a^2)^2}$
2e –t sin5t	5e –t sin2t
^{-st}dt $X(s) = \int_{0-}^{\infty} x(t)e^{-s\omega 0t}dt$	$X(s) = \int_{0-}^{\infty} x(t)e^{-st}dt$
$\int_{-\infty}^{\infty} X(s)e^{st}dt$	$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{-st}dt$

$^{-1}(\frac{y}{x})$ $\sqrt{(x+y)^2}, \tan^{-1}(\frac{x}{y})$	$\sqrt{x^2+y^2}, \tan^{-1}(\frac{y}{x})$
I harmonic, II harmonic, constant thereafter	constant, fundamental, I harmonic
causal signals	causal signals
cos θ + sinθ	cos θ j sin θ
400 Hz	400 Hz
350 samples / s	350 samples / s
1/ fmax	. = 1 / 2 fmax
Recovery of signal is best	Recovery of signal is best
1/ fmax	> 1/ 2 fmax
max	2 fmax
5 µs	50 µs
loss of data	higher speed of communication
band- reject filter	low pass filter
periodic functions	periodic functions

∞

∞

nπδ(ω)

A – 4; B – 2; C – 3; D – 1

2πδ(ω)

A – 4; B – 3; C – 1; D – 2

$\frac{1}{j\omega}$ πδ(ω)

A, B & C

$\frac{1}{j\omega}$

A & B

$\frac{\omega}{(s+a)^2+\omega^2}$

A – 2; B – 4; C – 1; D – 3

$\frac{\omega}{(s+a)^2+\omega^2}$

A – 2; B – 3; C – 4; D - 1

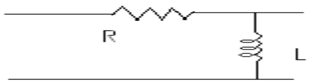
$F(s)$

$\underset{t\to 0}{Lt} f(t) = \underset{s\to\infty}{Lt} sF(s)$

$\underset{t\to\infty}{Lt} f(t) = \underset{s\to\infty}{Lt} sF(s)$

$\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt} sF(s)$

questions	opt1	opt2	opt3
The impulse response of the two systems in cascade are h1 (t)=e-2tu(t) and h2 (t)=2e-tu(t).The impulse h(t) of the overall system is	2(e-t – e-2t)u(t)	(2e-t – e-2t)u(t)	(e-t – 2e-2t)u(t)
The overall system described above is	Causal	Unstable	Stable
Assertion: A transfer function realization using differentiators is not preferable Reason	Amplifies low frequency signal	Amplifies high frequency noise signal	Amplifies low frequency noise signal
The ULT is applied for	stable signals	noncausal signals	causal signals
The L.T of unit ramp function is	2/s2	1/(s+s)	s2
The impulse response of a LTI system h(t)=(10sin(5πt))/πt	noncausal	stable	causal
The frequency response of an LTI system characterized by the differential equation d/dt y(t)+ay(t)=x(t) is	H(ω)	H(f)	(a) and (b)
The eigen function and eigen value respectively are	est and e-st	est and H(s)	est and H(f)
The differential equation is useful in obtaining	frequency response	impulse response	frequency and impulse response
Mark the wrong statement	x1(t)*x2(t)=x2(t)*x1(t)	x1(t)*[x2(t)*x3(t)]= x1(t)*x2(t)+ x1(t)*x3(t)	x1(t)*[x2(t)*x3(t)]=[x1(t)*x2(t)]*x3(t)
Mark the correct statement	x1(t)*x2(t-T)=z(t-T)	x1(t)*x2(t-T)=z(T)	x1(t)* x2(t-T)=z(t)

The convolution of x1(t)=u(t) and x2(t)=u(t) is	t u(t)	u(t)	u(t)/t
The step response of the circuit is 	1 / (S + L / R)	1 / (L + S / R)	1 / (1 + L / R)
The convolution of x1(t)=sint u(t) and x2(t)=u(t)	(1-sint)	(1-sint)u(t)	(1-cost)u(t)
.The convolution of x1(t)=tu(t) and x2(t)=u(t) is	(t2/2) u(t)	(t3/2) u(t)	tu(t)
The impulse response of two systems connected in parallel is	h1(t)+h2(t)	x(t)*h(t)	h1(t)*h2(t)
The impulse response of two systems connected in series is	h1(t)*h2(t)	x(t)*h(t)	(a) and (b)
The system is static or memory less for	$h(\tau)=0\ \tau \neq 0$	$h(\tau)=0\ \ \tau = 0$	$h(\tau)= c\delta(t)\ t = 0$
The system is casual for	$h(\tau)=0\ \tau=0$	$h(\tau)=0\ \tau<0$	$h(\tau)=0\ \tau>0$
The convolution system is stable if the impulse response is	absolutely integrable	absolutely differentiable	both integrable and differentiable
The impulse response h(t)=e2tu(t-1) is	stable	unstable	absolutely integrable
The impulse response for h(t)=(1/RC)e-t/RC u(t) is	Infinity	One	Zero
The impulse response for h(t)=e-2tu(-t-1) exists for	Negative values of t & is causal	Positive values of t & is causal	Negative values of t & is noncausal

The impulse response h(t)=(t-1)u(t-1) exists for	Negative values of t & is noncausal	Positive values of t & is causal	Negative values of t & is causal
For the natural response of differential equation	Output produced due to initial conditions	Output produced due to input	Output produced due to initial conditions and input=0
For the forced response of differential equation	output produced due to input and initial conditions = ∞	output produced due to initial conditions and input =0	output produced due to initial conditions
The natural response of the system 10dy(t)/dt + 2y(t) =x(t) with y(0)=2 is	2e-0.2t	e-0.2t	2e-t
An LTI system is causal if the impulse response is	zero for positive t	positive for positive t	negative for negative t
The system h(t)=te-tu(t) is	unstable.	stable	causal and unstable.
The system h(t)=e-4tu(t+10)	noncausal and stable.	noncausal and unstable.	causal and stable.
Match the following.InputParticular Solution.(i)1A) k1cosωt + k2sinωt.(ii) e-at B) k.(iii) cos(ωt+φ)C) ke-at	i - C, ii – A, iii – B.	i - A, ii – C, iii – B.	i – B, ii – C, iii – A.
The direct form – I implementation of 2nd order system needs	Three integrators and three summers.	Four integrators and four summers.	Four integrators and three summers.

The frequency response of LTI-CT system are also called as	transfer function.	system transfer function.	system function.
If the response of LTI continuous time system to unit step signal is ½ - ½ e-2t, then impulse response of the system is___________	½ - ½ e-2t	e-2t	1- e-2t
Which property is not true for convolution integral________?	h1(t) * h2(t) = h2(t) * h1(t)	[h1(t) + h2(t)] * h3(t) = h1(t) * h3(t) + h2(t) * h3(t)	[h1(t) + h2(t)] * h3(t) = h1(t)h3(t) + h2(t)h3(t)
Which signal is anticasual?	x(t) = 0, t < 0	x(t) = 0, t > 0	x(n) = 0, n < 0
Mark the correct statement	x(t) * δ(t - t0) = x(t0)	x(t) * δ(t) = 1	x(t) * δ(t - t0) = x(t - t0)
Mark the wrong statement	x(t) * δ(t) = x(t)	x(t) * δ(t - λ) = x(t - λ)	x(t- λ) * δ(t – λ2) = x(t – λ1– λ2)
The response y(t) of linear system is____	Zero input response	Zero state response	Zero input response + Zero state response
Bilateral and unilateral Laplace transform differs in terms of________	Lower limit of integration	Upper limit of integration	They are same
For casual continuous – time LTI system, ROC is in the_________	Left of all system poles	Right of all system poles	Right of all zeros

If the system is casual and stable, the system poles must lie	On the jω axis	On the left half of s-plane	On the right half of s- plane
Inverse Laplace transform of 1 / (s-a); ROC <a is ________	e at u(-t)	e -at u(-t)	e -at u(t)
Zero input response is due to	input to the system.	depends on system transfer function.	due to system state.
Zero state response is due to	input to the system.	depends on system transfer function.	due to system state.
In memoryless system	Zero state response is zero.	Zero input response is zero.	both responses are zero.
The transfer function of a single loop system is	T(s) = G(s) / (1 - G(s)H(s))	T(s) = H(s) / (1 - G(s)H(s))	T(s) = H(s) / (1 + G(s)H(s))
The impulse response of the system having transfer function H(s) = 1/(s2(s+1)) is	(t2*e-2t)u(t)	(t2e-t)u(t)	(te-t)u(t)
Let y(t) = x(t) *h(t). Then	x(t-t1) * h(t-t2) = y(t-t1-t2)	x(t) * h(t-t2) = y(t-t1-t2)	x(t-t1)*h(t) = y(t-t1-t2)
If x1(t) and x2(t) are both periodic signals with a common period To, the convolution of x1(t) and x2(t)	does not converge.	Converge.	Periodic convolution of x1(t) and x2(t) converge.
For a stable continuous-time LTI system with impulse response h(t) that is real and even	cosωt is an eigen function.	sinωt is an eigen function.	cosωt and sinωt are eigen function with different eigen values.
Consider a CT LTI system whose step response is s(t)=e-tu(t). The output of this system to the input x(t) = u(t-1) – u(t-3) is	e(t-1)u(t-1) – e(t-3)u(t-3)	e(t-1)u(t-1) + e(t-3)u(t-3)	e-(t-1)u(t-1) – e-(t-3)u(t-3)

Consider the system dy(t)/dt +ay(t) = x(t). The system is	not linear if y(0) = y0 ≠ 0.	is linear if y(0) = 0	satisfies a and b.
A system can be realized using. A. Indirect form. B. Cascade form. C. Direct form	A & B only	B & C only	A & C only
If h(t) is the impulse response of casual, linear, time invariant, continuous system. Then output y(t) of the system for an input of x(t), is________	$\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$	$\int_{0}^{+\infty} x(\tau)h(t-\tau)d\tau$	$\int_{0}^{t} x(\tau)h(t-\tau$

opt4	opt5	opt6	answer
(2e-t – 2e-2t)u(t)			2(e-t – e-2t)u(t)
Noncausal and unstable			Stable
Amplifies high frequency signal			Amplifies high frequency noise signal
unstable signals			causal signals
1/s2			1/s2
none of the above.			causal
None of the above.			(a) and (b)
e-st and H(s)			est and H(s)
step response			frequency and impulse response
x1(t)*x2(t)= x1(t)+ x2(t)			x1(t)*x2(t)= x1(t)+ x2(t)
x1(t)*x2(t-T)=z(t+T)			x1(t)*x2(t-T)=z(t-T)

t/u(t)	t u(t)
1 / (S + R / L)	1 / (S + R / L)
(1-cost)	(1-cost)u(t)
u(t)	(t2/2) u(t)
(a) and (b)	(a) and (b)
(a) or (b)	x(t)*h(t)
h(τ)= cδ(t) t ≠ 0	h(τ)=0 τ ≠ 0
h(τ)=0 τ≠0	h(τ)=0 τ<0
none of the above	absolutely integrable
(a) and (c)	unstable
e-t	One
Positive values of t & is noncausal	Negative values of t & is noncausal

Positive values of t & is noncausal	Positive values of t & is causal
Output produced due to input and initial condition=0	Output produced due to initial conditions and input=0
output produced due to input	output produced due to initial conditions and input =0
2et	2e-0.2t
zero for negative t	zero for negative t
both a & c.	stable
causal and unstable.	noncausal and stable.
i – B, ii – A, iii – C.	i – B, ii – C, iii – A.
Three integrators and four summers..	Four integrators and four summers.

impulse function	system transfer function.
Constant	e-2t
$[h1(t) * h2(t)] * h3(t) = h1(t) * h2(t) * h3(t)$	$[h1(t) + h2(t)] * h3(t) = h1(t)h3(t) + h2(t)h3(t)$
$x(t) = 1, t < 0$	$x(t) = 0, t > 0$
$x(t) * \delta(t - t0) = x(t + t0)$	$x(t) * \delta(t - t0) = x(t - t0)$
$\delta(t - \lambda1) * \delta(t - \lambda2) = \delta(\lambda1 - \lambda2)$	$\delta(t - \lambda1) * \delta(t - \lambda2) = \delta(\lambda1 - \lambda2)$
Zero input response - Zero state response	Zero input response + Zero state response
Bilateral transform does not exist	Lower limit of integration
Left of all zeros	Right of all zeros

a and b	On the left half of s-plane
e at u(t)	e at u(-t)
(b) and (c)	(b) and (c)
(a) and (b)	(a) and (b)
both responses are finite.	Zero state response is zero.
T(s) = G(s) / (1 + G(s)H(s))	T(s) = G(s) / (1 + G(s)H(s))
(t*e-t)u(t)	(t*e-t)u(t)
x(t-t1) * h(t-t2) = y(t-t1).y(t-t2)	x(t-t1) * h(t-t2) = y(t-t1-t2)
(a) and (c).	(a) and (c).
None of the above.	None of the above.
e-(t-1)u(t-1) + e-(t-3)u(t-3)	e-(t-1)u(t-1) – e-(t-3)u(t-3)

not linear if y(0) = y0.	satisfies a and b.
All A, B & C	B & C only

$\tau)d\tau$ $\int_{-t}^{t} x(\tau)h(t-\tau)d\tau$ $\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

questions	opt1	opt2	opt3	opt4	opt5
Fourier Series co-effecients for a continuous time periodic wave is _____________ of envelope function.	Samples	Equations	Functions	Limits	
Fourier transform representation is convergence of Fourier series representation of a signal when the period approaches	Zero	Infinity	One	Two	
Spectrum is the ____________________ transform of CT / DTsignal	Laplace	Z	Fourier	Y	
$X(e^{j\omega})$ is the ______________________ of the signal	Continuous Time Fourier transform	Laplace transform	Z transform	Discrete Time Fourier transform	
DTFT of u(n) may be	$\frac{1}{1+ae^{-j\omega}}$	$\frac{e^{j\omega^1}}{1+e^{-j\omega}}$	$\frac{\cdot e^{j\omega^1}}{1-e^{-j\omega}}$	$\frac{1}{1-e^{-j\omega}}$	
DTFT of δ(n-k) is	$.e^{-j\omega k}$	$e^{-j\omega nk}$	$e^{+j\omega nk}$	$e^{+j\omega k}$	
DTFT of {1, -1, 2, 2} is	$1 - e^{j\omega} + 2e^{j2\omega} + 2e^{j3\omega}$	$1 - e^{-j\omega} + 2e^{-j2\omega} - 2e^{-j3\omega}$	$1 + e^{j\omega} + 2e^{j2\omega} + 2e^{j3\omega}$	$.1 - e^{-j\omega} + 2e^{-j2\omega} + 2e^{-j3\omega}$	
$X(e^{j\omega}) = e^{-j\omega}$, then x(n) is	$\frac{\sin\pi(n-1)}{(n-1)}$	$\frac{\cos\pi(n-1)}{\pi(n-1)}$	$\frac{\sin\pi(n+1)}{\pi(n-1)}$	$\frac{\sin(n-2)}{\pi(n-1)}$	
Match A. Differentiation in frequency domain 1. Time convolution 2. C. Parsevals Theorem 3. Frequency Convolution 4.	.A – 3; B – 4; C – 2; D – 1	A – 3; B – 3; C – 4; D – 1	A – 3; B – 4; C – 1; D – 2	A – 4; B – 3; C – 2; D – 1	
The periodicity property of Discrete Fourier transform satisfies the relation	$X(\omega) = X(\omega + 2\pi n)$	$X(\omega) = X(\omega + 2n)$	$X(\omega) = X(\omega + 2\pi)$	. $X(\omega) = X(\omega + 2\pi k)$	
The correlation property in DTFT gives	$X_1(e^{j\omega}) X_2(e^{-j\omega})$	$X_1(e^{j\omega}) X_2(e^{j\omega})$	$1/2\ X_1(e^{j\omega}) X_2(e^{-j\omega})$	$1/2\ X_1(e^{j\omega}) X_2(e^{j\omega})$	
Discrete Fourier Transform is defined only for sequences with	infinite length	finite length	both	none	

$X_1(e^{j\omega})$ is a _______________ function of frequency	discrete	digital	continuous	sequential
DFT is a powerful tool for	Time domain analysis	Frequency domain analysis	K domain analysis	S domain analysis
Z – transform is used to take a DT time-domain signal in	frequency domain	s-domain	. complex variable frequency domain	complex variable time domain
ROC is a condition for which	Continuous Time Fourier Series converges	Continuous Time Fourier Transform converges	Discrete Time Fourier Series converges	. Z-transform converges
For $X(Z) = 1 + 2Z^{-1} + 3Z^{-2}$ the ROC is	at $Z = 0$	at $Z = \infty$	. except at $Z = 0$	except at $Z = \infty$
27. If x(n) on Z – transform is X(Z), then anx(n) on Z-transform is	$X(a/Z)$, ROC $\|a\|r_1 < \|Z\| < \|a\|r_2$	. $X(Z/a)$, ROC $\|a\|r_1 < \|Z\| < \|a\|r_2$	$X(a/Z)$, ROC $\|a\|r_2 < \|Z\| < \|a\|r_1$	$X(Z/a)$, ROC $\|a\|r_2 < \|Z\| < \|a\|r_1$
Z-transform of a unit sample is	0	. 1	∞	
Is Parsevals Theorem in	Z-Transform	Fourier Transform	Laplace Transform	Fourier Series
The basic principle of Z- transform is to design	Low Pass Filter	High Pass Filter	Analog Filter Design	.Digital Filter Design
Choose the correct answer The significance of ROC is A. ROC is used to determine the causality B. ROC is used to determine the stability C. ROC is used to deternmine the linearity D. ROC is used to determine the variance.	A & C	.A & B	C & D	A & D
Z – transform of the signal {1,3,2,0,0,0. . . . } is	$1 + 2/Z + 3/Z^2$	$1 + 3/Z$	. $1 + 3/Z + 2/Z^2$	$1 + 2/Z^2$

The ROC of {1,3,2,0,0,0. . . . } is	whole of Z – plane	whole of the Z – plane except Z = ∞	. whole of the Z-plane except Z = 0	at Z = 0
Z transform of $\delta(n)$ is	. 1	z	1 / z	z^2
If x(n) and y(n) are two finite sequences, then x(n)*y(n) is	X(z) / Y(z)	Y(z) / X(z)	X2(z)Y(z)	. X(z)Y(z)
If x(n) X(Z), then the valid one is	x(-n) X(z)	x(-n) zX(z)	x(-n) X(z) / z	. x(-n) X(1/ z)
If u(n) z / (z – 1), then Z[u(-n)] is	z / (1-z)	1 / 1-z	. 1/z-1	z^2 / z-1
If x(n) = 0 for n<0 and x(n) = 3^n for n>0, then the Z – transform of sequence x(n) is	1/z-3	.Z/z-3	$1/(z-3)^{-1}$	$1/(z-1)^2$
The sequence x(n) = {2,3,4,3} is	circularly odd	. circularly even	neither a nor b	partly a and partly b
MatchA. a nx(n) 1. X(z -1) B. x(-n) 2. X(Z*)]* C. nx(n) 3. X(Z/a) D. x*(n) 4. –Z d{X(Z)} / dz	. A – 3; B- 1; C- 4; D- 2	A – 3; B- 4; C- 1; D- 2	A – 1; B- 3; C- 4; D- 2	A – 3; B- 4; C- 2; D- 1
If x(n) X(Z) then Im[x(n)]	1/2j[X(Z) + X*(Z*)]	1/2j[X(Z) – X (Z*)]	1/2j[X*(Z) – X*(Z*)]	. 1/2j[X(Z) – X*(Z*)]
Find the correct meaning of $x((n+k))_N$ from the following	. Sequence x(n) shifted clockwise by k samples	sequence x(n) shifted anti – clockwise by k samples	Sequence x(n) shifted anti – clockwise by N samples	sequence x(n) shifted clockwise by N samples
Which of the following is true	$W_N^{k+N/2} = W_N^k$	. $W_N^{k+N/2} = -W_N^k$	$W_N^{k+N/2} = W_N^{N/2}$	$W_N^{k+N/2} = -W_N^{N/2}$
The relationship between DFT and Z-transform is X(k) = X(Z) when	$Z = e^{-j2\pi kn/N}$	$Z = e^{j2\pi k/N}$	. $Z = e^{-j2\pi k/N}$	$Z = e^{j2\pi k/N}$

Assertion: DFT and IDFT are linear transformations on s(k) and S(K)Reason	.DFT is obtained by sampling operation in both time and frequency domains	DFT is obtained by interpolation operation in both time and frequency domains	DFT is obtained by sampling operation in time and interpolation in frequency domain	DFT is obtained by sampling operation in frequency and interpolation in time domain
The left sided exponential sequence is	$a^{n}u(n)$ for n >= 0	$-a^{n}u(n)$ for n >= 0	$-a^{n}u(-n)$ for n <= 0	. $-a^{n}u(-n-1)$ for n <= 0
if $x(n) = 3^{m}$ for n <0 and x(n) = 0 for n > 0 then Z – transform of x(n) is	3 / Z – 3	. 3 / 3 – Z	1 / Z – 3	1 / 3 – Z
If $X(Z) = 3aZ^{-1} / (1 – aZ^{-1})^{3}$ and $\lvert a\rvert < \lvert Z\rvert$, then the initial value of x(n) is	2	1	. 0	∞
The Z – transform of a^{n} is	$1 / (1 + aZ^{-1})$	. $1 / (1 – aZ^{-1})$	$Z / (1 – aZ^{-1})$	$Z / (1 + aZ^{-1})$
If x(n) X(Z), then	x(n-2) x(-2) + $x(-1)Z^{-1}$ + $x(0)z^{-2}$	x(n-2) x(-2) + x(-1)Z + z X(z)	x(n-2) x(-2) + $x(-2)Z^{-1}$ + z(-2)X(z)	. x(n-2) x(-2) + $x(-1)Z^{-1}$ + $z^{-2}X(z)$
In $X(Z) = 5 / (1 – z^{-1}) + (-4) / (1 – 0.8 Z^{-1})$ if ROC is $\lvert Z\rvert > 1$, then x(n)	$[4 – 4(0.8)^{n}]$ u(n)	$[4 – 5(0.8)^{n}]$ u(n)	. $[5 – 4(0.8)^{n}]$ u(n)	$[5 – 5(0.8)^{n}]$ u(n)
In $X(Z) = 5 / (1 – z^{-1}) + (-4) / (1 – 0.8 Z^{-1})$ if ROC is $0.8 < \lvert Z\rvert < 1$, then x(n)	$-5(1)^{n}$ u(-n) – $4(0.8)^{n}$ u(n)	. $-5(1)^{n}$ u(-n-1) – $4(0.8)^{n}$ u(n)	$-5(1)^{n}$ u(-n-1) – $4(0.8)^{n}$ u(-n-1)	$-5(1)^{n}$ u(n) – $4(0.8)^{n}$ u(n)

opt6 answer

Samples

Infinity

Fourier

Ans: d

. $\frac{1}{1-e^{-j\omega}}$

.$e^{-j\omega k}$

.$1 - e^{-j\omega} + 2 e^{-j2\omega} + 2 e^{-j3\omega}$

. $\frac{\sin \pi(n-1)}{\pi(n-1)}$

.A – 3; B – 4; C – 2; D – 1

$X(\omega) = X(\omega + 2\pi k)$

1 / 2

$X_1(e^{j\omega}) X_2(e^{-j\omega})$

finite length

continuo
us

Frequenc
y domain
analysis

complex
variable
frequency
domain

Z-
transform

converge
s

except at
Z = 0

. X(Z/a),
ROC
$|a|r_1 < |Z|$
$< |a|r_2$

. 1

Z-
Transfor
m

.Digital
Filter
Design

.A & B

. $1 + 3 / Z + 2 / Z^2$

. whole of the Z-plane except Z = 0

. 1

.

X(z)Y(z)

. x(-n) X(1/ z)

. 1/z-1

$\longleftrightarrow$

.Z/z-3

.

circularly even

. A – 3; B- 1; C- 4; D- 2

.

1/2j[X(Z) – X*(Z*)]

Sequence x(n) shifted clockwise by k samples

. $W_N^{k+N/2} = -W_N^k$

. $Z = e^{-j2\pi k/N}$

.DFT is obtained by sampling operation in both time and frequency domains

. $-a^{n}u(-n-1)$ for $n <= 0$

. $3 / 3 – Z$

. 0

. $1 / (1 – aZ^{-1})$

. $x(n-2)$ $x(-2) + x(-1)Z^{-1} + z^{-2}X(z)$

. $[5 – 4(0.8)^{n}] u(n)$

. $-5(1)^{n} u(-n-1) – 4(0.8)^{n} u(n)$

questions	opt1	opt2
Examples of shift invariant system are A. Thermal systemB. Noise EffectsC. Printing documents by the printer	A & B	A & C
y(n) = cos x(n) is a	linear & stable system	linear & unstable system
y(n) = sgn x(n) is a _____, _______ system	static, causal	dynamic, causal
Sampling and truncation systems are examples of	linear & shift invariant systems	nonlinear & shift variant systems
An LTI – DT system will be stable if the unit sample response is	absolutely integrable	absolutely summable
The impulse response of x1(n) = {1,-3, 2} and x2(n) = {1,2,1} is	{ 1, 1, 3, 1,2}	{ 1, 3, 1, 1, 2}
The relationship between Z & S Plane is	$\sigma > 0, \Leftrightarrow |z| <1$	$\sigma < 0, \Leftrightarrow |z| <1$
Z- transform of unit exponential sequence is	$1 / (1- e^{-\alpha T} Z^{-1})$	$1 / (1- e^{-T} Z^{-1})$
Select the appropriate comment on Z- transform	Good for analysis.	Difference equations help in easy computer program.
Assertion: Unstable systems cannot be cascaded Reason	Perfect cancellation is very difficult	Unstable pole can be excited by other inputs

Instability can be determined from the	poles of the open loop transfer function	Zeroes of the open loop transfer function
The ROC of $(1/3)^n [u(-n) – u(n-8)]$ is	$\|Z\| < 1/3$	$\|Z\| > 1/3$
If $x(n)$ is $-5(1)^n u(-n-1) – 4(0.8)^n u(n)$ will have ROC as	$\|Z\| > 1$	$\|Z\| < 0.8$
The poles are the values for which X(Z) is	0	1 / ∞
If N is the no. of poles, M is the no. of Zeroes and if N > M has	N+ M Zeroes at the origin	N – M Zeroes at the origin
If N is the no. of poles, M is the no. of Zeroes and if N < M has	N+ M poles at the origin	N – M poles at the origin
If X(Z) = ∞, then there is a	Zero at ∞	Zero at 0
If X(Z) = 0, then there is a	Zero at ∞	Zero at 0
The system $h(n) = -2(3)^n u(-n-1) – (0.5)^n u(n)$ is stable if	$\|Z\| > 3$	$\|Z\| < 0.5$
The ROC of {0,0,1,2,4} is the entire Z – plane	except at Z = ∞	except at Z = 0
Convolution between 2 signals can be done using	Graphical Method	Tabular Method
Match the following Roots of the equationNatural Response A. Real & Distinct 1. $c\,\alpha^n$ B complex 2 Kr n C. α^n 3. $C_0 + C_1 n + \ldots C_p n^p$ D. n^p 4. $r^n[K_1 \cos(n\Omega) + K_2 \sin(n\Omega)$	A – 3; B – 1; C – 4; D - 2	A – 2; B – 3; C – 4; D - 1
Match the following Roots of the equationNatural Response Real , repeated 1. C_1 coswt + C_2 sin wt Cos(ωt + β) 2. $e^{at}(C_0 + C_1 t)$ N 3. $r^n\{K_0 + K_1 n + \ldots + K_p n^p]$ te ^{at} 4. $C_0 + C_1 n$	A – 3; B – 1; C – 4: D – 2	A – 1; B – 2; C –3; D –4
The forced response of y(n) – 0.4y(n-1) = u(n) is	1.33	1. 44
The natural response of y9n) + 0.1y(n-1) – 0.3y(n-2) = 2 u(n) is	$K_1 (0.6)^n + K_2 (-0.6)^n$	$K_1 (0.3)^n + K_2 (-0.6)^n$

For the system $(1 - z^{-1} - 2 z^{-2})y(n) = x(n)$ is	y(n) – y(n-1) – 2y(n-2) = x(n) – x(n-1)	y(n) – y(n-1) – 2y(n-2) = x(n)
The system y(n) – y(n-1) = x(n)is	causal	noncausal
Assertion: Two systems x(n) and x^2(n)are connected in cascade. The response change when they are reversed.	Reason	Squaring system is not an LTI system
Choose the correct answer	Possible to confirm the input – output relations	Possible to optimize the system
State is the knowledge of the variables at _________ to determine the behaviour of the system.	$t = t_1$	t = T
The state equation is	Q' = AQ + BX	Q = AQ + BX
The output equation is	Y = CQ' + BX	Y' = CQ + BX
For the state equations X'(t) = Px(t) + Qu(t) Y(t) = Rx(t) + Su(t) Match the following List I List II A. P1. n x p B. Q2. q x n C. R3. n x n D. S4. V x P	A – 4; B – 1; C – 2; D - 3	A – 1; B – 3; C – 4; D - 2
For the state space representation from the transfer function the system is represented in	Direct – I form	Direct – II form
Which of the statements is true? Each block diagram representation of a system can be translated directly into computer algorithm but it needs	sampling	quantizing
The Z^{-1} block is a representation of	differentiator	Integrator
Laplace Transform and Z transform replace time domain operation into	algebraic equation	Differentiation equation

For a rectangular signal x(n) = { 1 0< n <5}, g(n) = x(n) – x(n-1) the Z-transform is	$G(Z) = 1 – Z^{-3}$, $\|Z\| >0$	$G(Z) = 1 – Z^{-2}$, $\|Z\| >0$
In a RLC network, ______________ are chosen as state variables	energy	Inductor voltage
The Inductor current of an electric network are considered as	series voltage	shunt voltage
A system has the following Zeroes (0,-1) and poles at 0.5 + j0.5, 0.5 –j0.5. The system is	stable	unstable
The system $Z (Z+1) / Z^{2} – Z + 0.5$ has poles at	0.5 + j 0.5	0.5 + j 0.5, 0.5 – j0.5
The input to the integrators are	present state	past state
The state variables are the	input of delay element	output of delay element
The state transition matrix determines	the transition of the final state at t = 0	the transition of the final state at t = ∞
The state variables are	smallest set of variables that determine the stability of a system	smallest set of variables that determine the state of a system
The state equation is given by	x(k) = Ax(k) + Bu(k)	x(k+1) = Ax(k)
If x(k) be the input of a delay element, the output will be	x(k+1)	x(k)
The transfer function of a SISO system is	$h(Z) = C(ZI – A)^{-1}B$	$H(Z) = B(ZI – A)^{-1} + D$

Total response of the system is	ZIR x ZSR	ZIR – ZSR

opt3	opt4	opt5	opt6	answer
B & C	A, B & C			A & C
nonlinear & stable system	nonlinear & unstable system			nonlinear & stable system
static, non-causal	dynamic, noncausal			static, causal
static & stable systems	non causal & stable systems			static & stable systems
either A or B	Both A & B			absolutely summable
{ 1,-1,-3,1,2}	{ 1,-1,-3,-1,2}			{ 1,-1,-3,1,2}
$\sigma > 0$, $\Leftrightarrow$ $\lvert z \rvert = 1$	$\sigma < 0$, $\Leftrightarrow$ $\lvert z \rvert > 1$			$\sigma < 0$, $\Leftrightarrow$ $\lvert z \rvert < 1$
$1 / (1- e^{T} Z^{-1})$	$1 / (1+e^{-\alpha T} Z^{-1})$			$1 / (1- e^{-\alpha T} Z^{-1})$
Difference equations can be solved easily.	Perfect Pole-Zero Plots can be obtained			Difference equations help in easy computer program.
Stable pole can be excited by other ports	Cancellation is perfect			Perfect cancellation is very difficult

poles of the closed loop transfer function	Zeroes of the closed loop transfer function	poles of the closed loop transfer function
$\lvert Z\rvert > (1/3)^n$	$\lvert Z\rvert < (1/3)^n$	$\lvert Z\rvert > 1/3$
$0.8 < \lvert Z\rvert < 1$	$\lvert Z\rvert < 1$	$0.8 < \lvert Z\rvert < 1$
1	∞	∞
M – N Zeroes at the origin	M Zeroes at the origin	N – M Zeroes at the origin
M – N poles at the origin	N poles at the origin	M – N poles at the origin
pole at ∞	pole at 0	pole at ∞
pole at ∞	pole at 0	Zero at ∞
$\lvert Z\rvert >$ unit circle	$0.5 < \lvert Z\rvert < 3$	$0.5 < \lvert Z\rvert < 3$
except at Z = 1	except at Z > 1	except at Z = 0
Matrix Method	All of the above	All of the above
A – 2; B – 4; C – 1; D - 3	A – 2; B – 4; C – 3; D – 1	A – 2; B – 4; C – 1; D - 3
A – 3; B – 4; C – 4: D – 1	A – 2; B – 3; C – 4: D – 2	A – 3; B – 1; C – 4: D – 2
1.66	1.55	1.66
$K_1 (0.5)^n + K_2 (-0.5)^n$	$K_1 (0.5)^n + K_2 (-0.6)^n$	$K_1 (0.5)^n + K_2 (-0.6)^n$

y(n) – 2y(n-2) = x(n)-x(n-1) unstable	y(n) – 2y(n-1) – y(n-2) = x(n) stable	y(n) – y(n-1) – 2y(n-2) = x(n) stable
Squaring system is an LTI system	Squaring system is not a linear system	Reason
Possible not to include the initial condition	Possible to analyze only linear systems	Possible to optimize the system
$t = t_\infty$	$t = t_0$	$t = t_0$
Q’ = AQ’ + BX	Q’ = AQ’ + BX’	Q’ = AQ + BX
Y’ = CQ’ + BX	Y = CQ + BX	Y = CQ + BX
A – 3; B – 1; C – 4; D - 2	A – 3; B – 4; C – 2; D – 1	A – 3; B – 1; C – 4; D - 2
Cascade form coding	Parallel form modulation	Direct – II form quantizing
Multiplier Difference Equation	Unit Delay Convolution Equation	Unit Delay algebraic equation

$G(Z) = 1 - Z^{-6}$, $\|Z\| > 0$	$G(Z) = 1 - Z^{-1}$, $\|Z\| > 0$	$G(Z) = 1 - Z^{-6}$, $\|Z\| > 0$
capacitor voltage	Capacitor current	capacitor voltage
next state variable	state variables	state variables
time variant	causal	stable
0.6 + j 0.5, 0.6 – j0.5	0.5 + j 1, 0.5 – j1	0.5 + j 0.5, 0.5 – j0.5
future state	no states	future state
input to summer	output to summer	output of delay element
the transition of the initial state at t = ∞	the transition of the initial state at t = 0	the transition of the initial state at t = 0
smallest set of variables that determine the causality of a system	smallest set of variables that determine the linearity of a system	Ans: b
$x(k+1) = Ax(k) + Bu(k)$	$x(k) = Au(k)$	$x(k+1) = Ax(k) + Bu(k)$
x(k-1)	x(k/2)	x(k-1)
$H(Z) = C(ZI - A)^{-1}B + C$	$H(Z) = D(ZI - A)^{-1}B + D$	$H(Z) = D(ZI - A)^{-1}B + D$

ZIR / ZSR	ZIR + ZSR	ZIR x ZSR