KARPAGAM ACADEMY OF HIGHER EDUCATION
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	SEMESTER VI	**L T P C**
15PHU603A	**MATHEMATICAL PHYSICS**	**5 - - 5**

Scope: Mathematics is the back-born of science and engineering. So, it is necessary for a physics student to be familiar with different methods in mathematics.
Objectives: The objective of this paper is to give a basic idea about different methods of mathematics, used in Physics.

UNIT - I

Operations with Del operator - Gradient of scalar field, physical interpretation -Divergence of a vector function - curl of a vector - curl of the curl - The Laplacian operator - Line, surface and volume integrals - Important vector identities - Gauss divergence theorem - Problems in Gauss divergence theorem - Stoke's theorem and its proof with simple problems - Classification of vector fields - Orthogonal, curvilinear coordinates, differential operators interms of orthogonal curvilinear coordinates - gradient, curl and Laplacian in spherical polar coordinates and cylindrical coordinates.
Tensors – Contrvariant and covariant tensor

UNIT - II

Matrices-Special types of matrices -Transpose of a matrix - Conjugate of a matrix - Conjugate transpose of a matrix-symmetric and antisymmetric matrices - Hermitian and skew - Hermitian matrices - Determinant of a matrix - Adjoint of a matrix - Inverse of a matrix -Unitary matrices - Rank of a matrix and simple problems - Characteristic matrix and characteristic equation - Characteristic vector - Methods of finding the Eigen values and Eigen vectors of a matrix.

UNIT - III

Differential Equations: Introduction – Solution in simple cases of ordinary differential equations of second order – Simple problems from Physics – Partial Differential equations – Special types of differential equations arising in Physics.
Group Theory: Introduction in sets, mappings and binary operations – groups – elementary properties of groups – The centre of a group – Cosets or consents of a subgroup – cyclic group.

UNIT - IV

Functions of a complex variable – single and multivalued functions – Cauchy – Riemann differential equation – analytical – line integrals of complex function – Cauchy's integral theorem and integral formula – derivatives of an analytic function – Taylor's variable – Residue and Cauchy's residue theorem – application to the equation of definite integrals – conformal transformation – Invariance of the Laplacian.

UNIT – V

Arithmetic mean - Median - Quartiles - Deciles - Percentiles - Mode - Empirical relation between mean, median and mode - Geometric mean, harmonic mean - Relation between arithmetic mean, geometric mean and harmonic mean - Range - Range meanor average deviation - Standard deviation - Variance and mean square deviation.

Text Book

1. SathyaPrakash, 2002, Mathematical Physics, 4th Edition, S. Chand & Company, New Delhi.
2. Gupta. B.D., 2002, Mathematical Physics, 2nd Edition, Vikas Publishing house Pvt Ltd, New Delhi.

REFERENCES

1. Gerald C.F., 1998, Applied Numerical Analysis, 5th Edition, Addison Wesley, California.
2. Rajput. B.S., 2003, Mathematical Physics, 16th Edition, PragatiPrakasan, Meerut.
3. Pipes L.A. and L.R. Harwill, 'Applied Mathematics for Engineers and Physicists', McGrawhill, 1970.

KARPAGAM ACADEMY OF HIGHER EDUCATION

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LECTURE PLAN
DEPARTMENT OF PHYSICS

STAFF NAME: N.GEETHA
SUBJECT NAME: MATHEMATICAL PHYSICS SUB.CODE:15PHU603A
SEMESTER: VI CLASS: III B.Sc (PHY)

UNIT-I

S. No	Lecture Duration (Hr)	Topics to be covered	Support Material
1	1hr	Operations with Del operator	T1:18
2	1hr	Gradient of scalar field, physical interpretation	T1:18
3	1hr	Divergence of a vector function	T1:24
4	1hr	curl of a vector - curl of the curl	T1:28-29
5	1hr	The Laplacian operator	T1:29
6	1hr	Line, surface and volume integrals - Important vector identities	T1:22;32
7	1hr	Gauss divergence theorem - Problems in Gauss divergence theorem	T1:36
8	1hr	Stoke's theorem and its proof with simple problems	T1:47
9	1hr	Classification of vector fields	T1:59
10	1hr	Orthogonal, curvilinear coordinates, differential operators interms of orthogonal curvilinear coordinates	T1:64
11	1hr	Continuation	
12	1hr	Gradient, curl and Laplacian in spherical polar coordinates and cylindrical coordinates.	T!:66
13	1hr	Tensors – Contravariant and covariant tensor	T1:187
14	1hr	Revision	
Total no. of hours planned for unit-I			14

UNIT-II

S. No	Lecture Duration (Hr)	Topics to be covered	Support Material
1	1hr	Introduction	
2	1hr	Matrices-Special types of matrices	T1:109
3	1hr	Transpose of a matrix	
4	1hr	Conjugate of a matrix - Conjugate transpose of a matrix	T1:112,114
5	1hr	symmetric and antisymmetric matrices	T1:115
6	1hr	Hermitian and skew - Hermitian matrices	T1:117
7	1hr	Determinant of a matrix	T1:119,124
8	1hr	Adjoint of a matrix	
9	1hr	Inverse of a matrix -Unitary matrices	T1:124,130
10	1hr	Rank of a matrix and simple problems	T1:134
11	1hr	Characteristic matrix and characteristic equation -	T1:138,139
12	1hr	Characteristic vector	
13	1hr	Methods of finding the Eigen values and Eigen vectors of a matrix.	T1:159
14	1hr	Revision	
		Total no. of hours planned for unit-II	14

UNIT-III

S.No	Lecture Duration (Hr)	Topics to be covered	Support Material
1	1hr	Differential Equations: Introduction	T1:394
2	1hr	Solution in simple cases of ordinary differential equations of second order	T1:395
3	1hr	– Simple problems from Physics	
4	1hr	Partial Differential equations	
5	1hr	Special types of differential equations arising in Physics.	
6	1hr	Continuation	
7	1hr	Group Theory	T1:814
8	1hr	Introduction in sets, mappings and binary operations	
9	1hr	Continuation	
10	1hr	groups – elementary properties of groups	T1:815
11	1hr	The centre of a group	
12	1hr	Cosets or consents of a subgroup	T1:821
13	1hr	Continuation	
14	1hr	cyclic group	T1:816
15	1hr	Revision	
Total no. of hours planned for unit-III			15

UNIT-IV

UNIT-IV

S.No	Lecture Duration (Hr)	Topics to be covered	Support Material
1	1hr	Introduction	
2	1hr	Functions of complex variable	T1:293
3	1hr	Single and multi valued functions	T1:294
4	1hr	Cauchy Riemann differential equation,Analytical line integrals of complex function	T1:296
5	1hr	Cauchy's integral theorem and its formula	T1:309,318
6	1hr	Derivatives of an analytic function	
7	1hr	Taylor's variable	T1:319
8	1hr	Residue and Cauchy residue theorem	
9	1hr	Application to the equation of definite integrals	T1:324
10	1hr	Conformal transformation	T1:332
11	1hr	continuation	T1:341
12	1hr	Invariance of the laplacian	
13	1/2hr	continuation	T1:342
	1/2hr	Revision	
Total no. of hours planned for unit-IV			13

UNIT-V

S. No	Lecture Duration (Hr)	Topics to be covered	Support Material
1	1hr	Introduction	
2	1hr	Arithmetic mean	T1:766
3	1hr	Median	T1:768
4	1hr	Quartiles	
5	1hr	Deciles	
6	1hr	Percentiles	
7	1hr	Mode	T1:768
8	1hr	Empirical relation between mean, median and mode	
9	1hr	Geometric mean, harmonic mean	T1:767
10	1hr	Relation between arithmetic mean, geometric mean and harmonic mean	
11	1hr	Continuation	
12	1hr	Range	T1:770
13	1hr	Range mean or average deviation	T1:770
14	1hr	Standard Deviation	T1:771
15	1hr	Variance and mean square deviation	
16	1hr	Revision	
17	1hr	Old question paper discussion	
18	1hr	Old question paper discussion	
19	1hr	Old question paper discussion	
Total no. of hours planned for unit-v			19

Textbooks : T1-Sathya Prakash,2002, Mathematical physics, fourth edition, S. chand & company, New Delhi.
R1-Gupta.B.D,2002,Mathematical Physics,second Edition, Vikas publishing house pvt Ltd,New Delhi.

UNIT-I

SYLLABUS

Operations with Del operator - Gradient of scalar field, physical interpretation - Divergence of a vector function - curl of a vector - curl of the curl - The Laplacian operator - Line, surface and volume integrals - Important vector identities - Gauss divergence theorem - Problems in Gauss divergence theorem - Stoke's theorem and its proof with simple problems - Classification of vector fields - Orthogonal, curvilinear coordinates, differential operators interms of orthogonal curvilinear coordinates - gradient, curl and Laplacian in spherical polar coordinates and cylindrical coordinates.
Tensors – Contravariant and covariant tensor

GRADIENT OF SCALAR FIELD,PHYSICAL INTERPRETATION:

The gradient of a scalar field is a vector field and whose magnitude is the rate of change and which points in the direction of the greatest rate of increase of the scalar field. If the vector is resolved, its components represent the rate of change of the scalar field with respect to each directional component. Hence for a two-dimensional scalar field $\varnothing$(x,y).

$$\text{grad}\,\phi(x,y) = \nabla\,\phi(x,y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$$

And for a three-dimensional scalar field $\varnothing$(x, y, z)

$$\text{grad}\,\phi(x,y,z) = \nabla\,\phi(x,y,z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

The gradient of a scalar field is the derivative of f in each direction. Note that the gradient of a scalar field is a vector field. An alternative notation is to use the *del* or *nabla* operator, ∇ f = grad f.

For a three dimensional scalar, its gradient is given by:

$$grad(V) = \underline{a}_n \frac{dV}{dn} = \nabla V$$

Gradient is a vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar.

$dV = (\nabla V) \cdot dl$, where $dl = a_i \cdot dl$

In Cartesian

$$\nabla \equiv \underline{a}_x \frac{\partial}{\partial x} + \underline{a}_y \frac{\partial}{\partial y} + \underline{a}_z \frac{\partial}{\partial z}$$

In Cylindrical

$$\nabla \equiv \underline{a}_r \frac{\partial}{\partial r} + \underline{a}_\phi \frac{\partial}{r \cdot \partial \phi} + \underline{a}_z \frac{\partial}{\partial z}$$

In Spherical

$$\nabla \equiv \underline{a}_R \frac{\partial}{\partial R} + \underline{a}_\theta \frac{\partial}{R \partial \theta} + \underline{a}_\phi \frac{\partial}{R \cdot \sin\theta \cdot \partial \phi}$$

Properties of gradient

· We can change the vector field into a scalar field only if the given vector is differential. The given vector must be differential to apply the gradient phenomenon.

· The gradient of any scalar field shows its rate and direction of change in space.

Example 1: For the scalar field $\varnothing(x,y) = 3x + 5y$,calculate gradient of $\varnothing$.

Solution 1: Given scalar field $\varnothing(x,y) = 3x + 5y$

$$\nabla\ \phi(x,y)=\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)=(3,5)$$

Example 2: For the scalar field $\varnothing(x,y) = x^4yz$,calculate gradient of $\varnothing$.

Solution: Given scalar field $\varnothing(x,y) = x^4yz$

$$\nabla\ \phi(x,y,z)=\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right)$$
$$=\left(4x^3yz,x^4z,x^4y\right)$$

Example 3: For the scalar field $\varnothing(x,y) = x^2 sin5y$,calculate gradient of $\varnothing$.

Solution: Given scalar field $\varnothing(x,y) = x^2 sin5y$

$$\nabla\ \phi(x,y)=\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)$$
$$=\left(2x\sin(5y),5x^2\cos(5y)\right)$$

DIVERGENCE OF A VECTOR FUNCTION:

A **vector** is a quantity that has a *magnitude* in a certain *direction*. Vectors are used to model forces, velocities, pressures, and many other physical phenomena. A **vector field** is a function that assigns a vector to every point in space. Vector fields are used to model force fields (gravity, electric and magnetic fields), fluid flow, etc.

The divergence of a vector field F = <P,Q,R> is defined as the partial derivative of P with respect to x plus the partial derivative of Q with respect to y plus the partial derivative of R with respect to z.

$$divF=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

The divergence of a vector field is also given by:

$$div(\underline{A}) \triangleq \lim_{\Delta v \to 0} \frac{\oint_s \underline{A} \cdot d\underline{s}}{\Delta v}$$

We define the divergence of a vector field at a point, as the net outward flux of per volume as the volume about the point tends to zero.

$\nabla \cdot A = divA$

In Cartesian

$\nabla \cdot A \equiv \partial A_x/\partial_x + \partial A_y/\partial_y + \partial A_z/\partial_z$

In Cylindrical

$\nabla \cdot A \equiv \partial(r \cdot Ay)/(r \cdot \partial_{r)} + \partial A_ø/(r \cdot \partial_{ø)} + \partial A_z/\partial_z$

In Spherical

$\nabla \cdot A \equiv \partial(R^2 \cdot A_{R)}/(R^2 \cdot \partial_{R)} + \partial(A_ø \cdot sin\theta)/(R \cdot sin\theta \cdot \partial\theta_{)} + \partial A_ø/(R \cdot sin\theta \cdot \partial_ø)$

Example 1: Compute the divergence of $F(x, y) = 3x^2 i + 2yj$.

Solution: The divergence of $F(x, y)$ is given by $\nabla \bullet F(x, y)$ which is a dot product.

$$\begin{aligned}\nabla \bullet F(x,y) &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2y) \\ &= 6x + 2\end{aligned}$$

Example 2: Calculate the divergence of the vector field $G(x,y,z) = e^x i + ln(xy)j + e^{xyz}k$.

Solution: The divergence of $G(x,y,z)$ is given by $\nabla \bullet G(x,y,z)$ which is a dot product. Its components are given by:

$G_1 = e^x$

$G_2 = ln(xy)$

$G_3 = e^{xyz}$

and its divergence is:

$$\nabla \bullet G(x,y,z) = \frac{\partial}{\partial x}(G_1) + \frac{\partial}{\partial y}(G_2) + \frac{\partial}{\partial z}(G_3)$$
$$= \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(\ln(xy)) + \frac{\partial}{\partial z}(e^{xyz})$$
$$= e^x + \frac{\partial}{\partial y}((\ln x) + (\ln y)) + e^{xyz}\frac{\partial}{\partial z}(xyz)$$
$$= e^x + \frac{1}{y} + e^{xyz}xy$$

Example 3: Calculate the divergence of the vector field $G(x, y, z) = 4y/x^2 \cdot i + (\sin y)j + 3k$

Solution: The divergence of $G(x, y, z)$ is given by $\nabla \bullet G(x,y,z)$ which is a dot product. Its components are given by:

$G_1 = 4y/x^2$

$G_2 = (\sin y)$

$G_3 = 3$

and its divergence is

$$\nabla \bullet G(x,y,z) = \frac{\partial}{\partial x}(G_1) + \frac{\partial}{\partial y}(G_2) + \frac{\partial}{\partial z}(G_3)$$
$$= \frac{\partial}{\partial x}(\frac{4y}{x^2}) + \frac{\partial}{\partial y}(\sin y) + \frac{\partial}{\partial z}(3)$$
$$= 4y \times \frac{\partial}{\partial x}(x^{-2}) + (\cos y) + 0$$
$$= -8yx^{-3} + \cos y$$

CURL OF A VECTOR:

The curl of a vector field *A*, denoted by curl *A* or ∇ x *A*, is a vector whose magnitude is the maximum net circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum!.

In Cartesian

$$\nabla \times \underline{A} = \begin{vmatrix} \underline{a}_x & \underline{a}_y & \underline{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \underline{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \underline{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \underline{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

In Cylindrical

$$\nabla \times \underline{A} = \frac{1}{r}\begin{vmatrix} \underline{a}_r & \underline{a}_\phi r & \underline{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & A_\phi & A_z \end{vmatrix}$$

$$= \underline{a}_r\left(\frac{\partial A_z}{r\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \underline{a}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \underline{a}_z\frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)$$

In Spherical

$$\nabla \times \underline{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \underline{a}_R & \underline{a}_\phi R & \underline{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\phi & (R\sin\theta)A_\phi \end{vmatrix} =$$

$$\underline{a}_R \frac{1}{R\sin\theta}\left(\frac{\partial(A_\phi \sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) + \underline{a}_\theta \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial A_\phi}{\partial R}(RA_\phi)\right)$$

$$+\underline{a}_\phi \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)$$

Given a vector field F(x, y, z) = Pi + Qj + Rk in space. The curl of **F** is the new vector field

$$curlF = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k.$$

This can be remembered by writing the curl as a "determinant"

$$curlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Theorem: Let **F** be a three dimensional differentiable vector field with continuous partial derivatives. Then Curl **F** = 0, if and only if **F** is conservative.

Example 1: Determine if the vector field **F** = yz^2**i** + (xz^2 + 2) **j** + (2xyz - 1) **k** is conservative.

Solution:

$$curlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2+2 & 2xyz\ -1 \end{vmatrix}$$

$$= (2xz-2xz)i-(2yz-2yz)j+(z^2-z^2)k$$

$$=0$$

Therefore the given vector field F is conservative.

Example 2: Find the curl of F(x, y, z) = $3x^2$**i** + 2z**j** – x**k**.

Solution:

$$curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 2z & -x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-x)-\frac{\partial}{\partial z}(2z)\right)i-\left(\frac{\partial}{\partial x}(-x)-\frac{\partial}{\partial z}(3x^2)\right)j+\left(\frac{\partial}{\partial x}(2z)-\frac{\partial}{\partial y}(3x^2)\right)k$$

$$= (0-2)i-(-1-0)j+(0-0)k$$

$$= -2i+j$$

Example 3: What is the curl of the vector field **F** = ($x + y + z$, $x - y -$ z, $x^2 + y^2 + z^2$)?

Solution:

$$curl F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & x-y-z & x^2+y^2+z^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x-y-z)\right)i - \left(\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x+y+z)\right)j +$$

$$\left(\frac{\partial}{\partial x}(x-y-z) - \frac{\partial}{\partial y}(x+y+z)\right)k$$

$$= (2y+1)i - (2x-1)j + (1-1)k$$

$$= (2y+1)i + (1-2x)j + 0k$$

$$= (2y+1, 1-2x, 0)$$

Example 4: Find the curl of $F = (x^2 - y)i + 4zj + x^2k$.

Solution:

$$curl F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y) & 4z & x^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(4z)\right)i - \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(x^2-y)\right)j +$$

$$\left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2-y)\right)k$$

$$= (0-4)i - (2x-0)j + (0+1)k$$

$$= (-4)i - (2x)j + 1k$$

$$= (-4, -2x, 1)$$

GAUSS DIVERGENCE THEOREM:

Let **B** be a solid region in $\boldsymbol{R^3}$ and let **S** be the surface of **B**, oriented with outwards pointing normal vector. Gauss Divergence theorem states that for a $\boldsymbol{C^1}$ vector field **F,** the following equation holds:

$$\iint_S F \cdot ds = \iiint_B (\nabla \cdot F) dV$$

Note that for the theorem to hold, the orientation of the surface must be pointing outwards from the region **B**, otherwise we'll get the minus sign in the above equation. Note that since **S** is the boundary of**B**, then it is always a closed surface ie: it has no boundary. In other words, the integral of a continuously differentiable vector field across a boundary (flux) is equal to the integral of the divergence of that vector field within the region enclosed by the boundary.

Applications of Gauss Theorem:

- The Aerodynamic Continuity Equation

1. The surface integral of mass flux around a control volume without sources or sinks is equal to the rate of mass storage.
2. If the flow at a particular point is incompressible, then the net velocity flux around the control volume must be zero.
3. As net velocity flux at a point requires taking the limit of an integral, one instead merely calculates the divergence.
4. If the divergence at that point is zero, then it is incompressible. If it is positive, the fluid is expanding, and vice versa.

- Gauss's Theorem can be applied to any vector field which obeys an inverse-square law (except at the origin) such as gravity, electrostatic attraction, and even examples in quantum physics such as probability density.

Example 1: Use the divergence theorem to calculate $\iint_S F \cdot ds$, where **S** is the surface of the box **B**with vertices (±1, ±2, ±3) with outwards pointing normal vector and $F(x, y, z) = (x^2z^3, 2xyz^3, xz^4)$.

Solution: Note that the surface integral will be difficult to compute, since there are six different components to parameterize (corresponding to the six sides of the box) and so one would have to compute six different integrals. Instead, using Gauss Theorem, it is easier to compute the integral (∇ ·F)of **B.**

First, we compute $(\nabla \cdot F) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$. Now we integrate this function over the region B bounded by S:

$$\iint_S F \cdot ds = \iiint_B (\nabla \cdot F) dV$$
$$= \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} 8xz^3\, dx dy dz$$
$$= 0$$

which is easy to verify.

Example 2: Evaluate $\iint_S (3x\,\mathbf{i} + 2y\,\mathbf{j}) \cdot dA$, where **S** is the sphere given by $x^2 + y^2 + z^2 = 9$.

Solution: We could parametrize the surface and evaluate the surface integral, but it is much faster to use the divergence theorem. Since

$$\text{div}(3x\,\mathbf{i} + 2y\,\mathbf{j}) = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(0) = 3 + 2 = 5$$

The divergence theorem gives:

$$\iint_S (3x\,\mathbf{i} + 2y\,\mathbf{j}) \cdot dA = \iiint_R 5dV = 5 \times (\text{Volume of the sphere}) = 180\pi$$

Example 3: Let **R** be the region in $\boldsymbol{R^3}$ by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$ and let **S** be the boundary of the region **R.** Evaluate $\iint_S (y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}) \cdot dA$

Solution:

Since $\operatorname{div}(y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}) = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2) = 2z$

The divergence theorem gives:

$$\iint_S z^2\,\mathbf{k} \cdot dA = \iiint_R 2z dV$$

It is easiest to set up the triple integral in cylindrical coordinates:

$$\iiint_R 2zdV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 2z \cdot r \cdot dzdrd\theta$$

$$= 2\pi \int_0^1 \left[z^2 r\right]_{z=r^2}^1 dr$$

$$= 2\pi \int_0^1 \left[r - r^5\right] dr$$

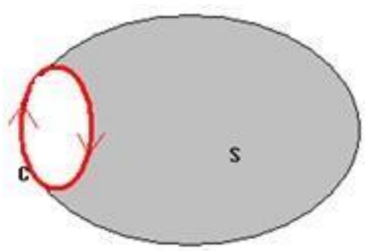
$$= 2\pi \int_0^1 \left(\frac{1}{2} - \frac{1}{6}\right) dr = \frac{2\pi}{3}$$

STOKES THEOREM:

The Stokes's Theorem is given by:

$$\int_S (\nabla \times \underline{A}) \cdot d\underline{s} = \oint_C \underline{A} \cdot d\underline{l}$$

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.



THE LAPLACIAN OPERATOR:

The Laplacian for a scalar function ϕ is a scalar differential operator defined by

$$\nabla^2 \phi = \frac{1}{h_1\, h_2\, h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2\, h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1\, h_3}{h_2} \frac{\partial}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1\, h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right] \phi, \qquad (1)$$

where the h_i are the scale factors of the coordinate system (Weinberg 1972, p. 109; Arfken 1985, p. 92).

Note that the operator ∇^2 is commonly written as Δ by mathematicians (Krantz 1999, p. 16).

The Laplacian is extremely important in mechanics, electromagnetics, wave theory, and quantum mechanics, and appears in Laplace's equation

$$\nabla^2 \phi = 0, \qquad (2)$$

the Helmholtz differential equation

$$\nabla^2 \psi + k^2\, \psi = 0, \qquad (3)$$

the wave equation

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}, \qquad (4)$$

and the Schrödinger equation

$$i \hbar \frac{\partial \Psi (x, y, z, t)}{\partial t} = \left[-\frac{\hbar^2}{2 m} \nabla^2 + V (x) \right] \Psi (x, y, z, t). \qquad (5)$$

The analogous operator obtained by generalizing from three dimensions to four-dimensional spacetime is denoted $\Box^2$ and is known as the d'Alembertian. A version of the Laplacian that operates on vector functions is known as thevector Laplacian, and a tensor Laplacian can be similarly defined. The square of the Laplacian $(\nabla^2)^2 = \nabla^4$ is known as the biharmonic operator.

A vector Laplacian can also be defined, as can its generalization to a tensor Laplacian.

The following table gives the form of the Laplacian in several common coordinate systems.

coordinate system	$\nabla^2 f$
Cartesian coordinates	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
cylindrical coordinates	$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$
parabolic coordinates	$\frac{1}{u\,v\,(u^2+v^2)} \left[\frac{\partial}{\partial u}\left(u\,v \frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(u\,v \frac{\partial f}{\partial v}\right)\right] + \frac{1}{u^2\,v^2} \frac{\partial^2 f}{\partial \theta^2}$
parabolic cylindrical coordinates	$\frac{1}{u^2+v^2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{\partial^2 f}{\partial z^2}$
spherical coordinates	$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi}\left(\sin \phi \frac{\partial f}{\partial \phi}\right)$

The finite difference form is

$$\nabla^2 \psi(x, y, z) = \frac{1}{h^2}[\psi(x+h, y, z) + \psi(x-h, y, z) + \psi(x, y+h, z) + \psi(x, y-h, z) + \psi(x, y, z+h) + \psi(x, y, z-h) - 6\psi(x, y, z)]. \quad (6)$$

For a pure radial function $g(r)$,

$$\nabla^2 g(r) \equiv \nabla \cdot [\nabla g(r)] \quad (7)$$

$$= \nabla \cdot \left[\frac{\partial g(r)}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial g(r)}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial g(r)}{\partial \phi}\hat{\boldsymbol{\phi}}\right] \quad (8)$$

$$= \nabla \cdot \left(\hat{\mathbf{r}}\frac{d g}{d r}\right). \quad (9)$$

Using the vector derivative identity

$$\nabla \cdot (f\,\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + (\nabla f) \cdot \mathbf{A}, \quad (10)$$

so

$$\nabla^2 g(r) \equiv \nabla \cdot [\nabla g(r)] \quad (11)$$

$$= \frac{d g}{d r}\nabla \cdot \hat{\mathbf{r}} + \nabla\left(\frac{d g}{d r}\right) \cdot \hat{\mathbf{r}} \quad (12)$$

$$= \frac{2}{r}\frac{d g}{d r} + \frac{d^2 g}{d r^2}. \quad (13)$$

Therefore, for a radial power law,

$$\nabla^2 r^n = \frac{2}{r} n\, r^{n-1} + n(n-1) r^{n-2} \quad (14)$$

$$= [2n + n(n-1)]\, r^{n-2} \quad (15)$$

$$= n(n+1)\, r^{n-2}. \quad (16)$$

An identity satisfied by the Laplacian is

$$\nabla^2 \|\mathbf{x}\,\mathsf{A}\| = \frac{\|\mathsf{A}\|_{HS}^2 - \left\|(\mathbf{x}\,\mathsf{A})\,\mathsf{A}^{\mathrm{T}}\right\|^2}{\|\mathbf{x}\,\mathsf{A}\|^3}, \quad (17)$$

where $\|A\|_{HS}$ is the Hilbert-Schmidt norm, $\mathbf{x}$ is a row vector, and A^T is the transpose of A.

To compute the Laplacian of the inverse distance function $1/r$, where $r \equiv |\mathbf{r} - \mathbf{r}'|$, and integrate the Laplacian over a volume,

$$\int_V \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3\,\mathbf{r}. \tag{18}$$

This is equal to

$$\int_V \nabla^2 \frac{1}{r}\, d^3\,\mathbf{r} = \int_V \nabla \cdot \left(\nabla \frac{1}{r} \right) d^3\,\mathbf{r} \tag{19}$$

$$= \int_S \left(\nabla \frac{1}{r} \right) \cdot d\,\mathbf{a} \tag{20}$$

$$= \int_S \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \hat{\mathbf{r}} \cdot d\,\mathbf{a} \tag{21}$$

$$= \int_S -\frac{1}{r^2}\, \hat{\mathbf{r}} \cdot d\,\mathbf{a} \tag{22}$$

$$= -4\pi \frac{R^2}{r^2}, \tag{23}$$

where the integration is over a small sphere of radius R. Now, for $r > 0$ and $R \to 0$, the integral becomes 0. Similarly, for $r = R$ and $R \to 0$, the integral becomes -4π. Therefore,

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta^3 (\mathbf{r} - \mathbf{r}'), \tag{24}$$

where $\delta(\mathbf{x})$ is the delta function.

LINE INTEGRAL:

The line integral of a vector field $\mathbf{F}(\mathbf{x})$ on a curve σ is defined by

$$\int_\sigma \mathbf{F}\cdot d\,\mathbf{s} = \int_a^b \mathbf{F}(\sigma(t))\cdot\sigma'(t)\,d\,t, \qquad (1)$$

where $\mathbf{a}\cdot\mathbf{b}$ denotes a dot product. In Cartesian coordinates, the line integral can be written

$$\int_\sigma \mathbf{F}\cdot d\,\mathbf{s} = \int_C F_1\,d\,x + F_2\,d\,y + F_3\,d\,z, \qquad (2)$$

where

$$\mathbf{F} \equiv \begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ F_3(\mathbf{x}) \end{bmatrix}. \qquad (3)$$

For z complex and $\gamma : z = z(t)$ a path in the complex plane parameterized by $t \in [a, b]$,

$$\int_\gamma f\,d\,z = \int_a^b f(z(t))\,z'(t)\,d\,t. \qquad (4)$$

Poincaré's theorem states that if $\nabla\times\mathbf{F} = \mathbf{0}$ in a simply connected neighborhood $U(\mathbf{x})$ of a point $\mathbf{x}$, then in this neighborhood, $\mathbf{F}$ is the gradient of a scalar field $\phi(\mathbf{x})$,

$$\mathbf{F}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) \qquad (5)$$

for $\mathbf{x} \in U(\mathbf{x})$, where ∇ is the gradient operator. Consequently, the gradient theorem gives

$$\int_\sigma \mathbf{F}\cdot d\,\mathbf{s} = \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) \qquad (6)$$

for any path σ located completely within $U(\mathbf{x})$, starting at $\mathbf{x}_1$ and ending at $\mathbf{x}_2$.

This means that if $\nabla \times \mathbf{F} = \mathbf{0}$ (i.e., $\mathbf{F}(\mathbf{x})$ is an irrotational field in some region), then the line integral is path-independent in this region. If desired, a Cartesian path can therefore be chosen between starting and ending point to give

$$\int_{(a,b,c)}^{(x,y,z)} F_1\,dx + F_2\,dy + F_3\,dz = \int_{(a,b,c)}^{(x,b,c)} F_1\,dx + \int_{(x,b,c)}^{(x,y,c)} F_2\,dy + \int_{(x,y,c)}^{(x,y,z)} F_3\,dz. \tag{7}$$

If $\nabla \cdot \mathbf{F} = 0$ (i.e., $\mathbf{F}(\mathbf{x})$ is a divergenceless field, a.k.a. solenoidal field), then there exists a vector field $\mathbf{A}$ such that

$$\mathbf{F} = \nabla \times \mathbf{A}, \tag{8}$$

where $\mathbf{A}$ is uniquely determined up to a gradient field (and which can be chosen so that $\nabla \cdot \mathbf{A} = 0$).

SURFACE INTEGRAL:

For a scalar function f over a surface parameterized by u and v, the surface integral is given by

$$\Phi = \int_S f\,da \tag{1}$$

$$= \int_S f(u, v)\,|\mathbf{T}_u \times \mathbf{T}_v|\,du\,dv, \tag{2}$$

where $\mathbf{T}_u$ and $\mathbf{T}_v$ are tangent vectors and $\mathbf{a} \times \mathbf{b}$ is the cross product.

For a vector function over a surface, the surface integral is given by

$$\Phi = \int_S \mathbf{F} \cdot d\mathbf{a} \tag{3}$$

$$= \int_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, da \qquad (4)$$

$$= \int_S f_x\, dy\, dz + f_y\, dz\, dx + f_z\, dx\, dy, \qquad (5)$$

where $\mathbf{a} \cdot \mathbf{b}$ is a dot product and $\hat{\mathbf{n}}$ is a unit normal vector. If $z = f(x, y)$, then $d\mathbf{a}$ is given explicitly by

$$d\mathbf{a} = \pm\left(-\frac{\partial z}{\partial x}\hat{\mathbf{x}} - \frac{\partial z}{\partial y}\hat{\mathbf{y}} + \hat{\mathbf{z}}\right) dx\, dy. \qquad (6)$$

If the surface is surface parameterized using u and v, then

$$\Phi = \int_S \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv. \qquad (7)$$

VOLUME INTEGRAL:

A triple integral over three coordinates giving the volume within some region G,

$$V = \iiint_G dx\, dy\, dz.$$

ORTHOGONAL CO-ORDINATES:

In elementary geometry, orthogonal is the same as perpendicular. Two lines or curves are orthogonal if they are perpendicular at their point of intersection. Two vectors v and w of the real plane $\mathbb{R}^2$ or the real space $\mathbb{R}^3$ are orthogonal iff their dot product $\mathbf{v} \cdot \mathbf{w} = 0$. This condition has been exploited to define orthogonality in the more abstract context of the n-dimensional real space $\mathbb{R}^n$.

More generally, two elements v and w of an inner product space E are called orthogonal if the inner product of v and w is 0. Two subspaces V and W of E are called orthogonal if every element of V is orthogonal to every element of W. The same definitions can be applied to any symmetric or differential *k*-form and to any Hermitian form.

CURVILINEAR CO-ORDINATES:

A coordinate system composed of intersecting surfaces. If the intersections are all at right angles, then the curvilinear coordinates are said to form an orthogonal coordinate system. If not, they form a skew coordinate system.

A general metric $g_{\mu\nu}$ has a line element

$$d s^2 = g_{\mu\nu}\, d u^\mu\, d u^\nu, \quad (1)$$

where Einstein summation is being used. Orthogonal coordinates are defined as those with a diagonal metric so that

$$g_{\mu\nu} \equiv \delta^\mu_\nu\, h^2_\mu, \quad (2)$$

where δ^μ_ν is the Kronecker delta and h_μ is a so-called scale factor. Orthogonal curvilinear coordinates therefore have a simple line element

$$d s^2 = \delta^\mu_\nu\, h^2_\mu\, d u^\mu\, d u^\nu \quad (3)$$

$$= h^2_\mu\, (d u^\mu)^2, \quad (4)$$

which is just the Pythagorean theorem, so the differential vector is

$$d\,\mathbf{r} = h_\mu\, d u_\mu\, \hat{\mathbf{u}}_\mu, \quad (5)$$

or

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} d u_1 + \frac{\partial \mathbf{r}}{\partial u_2} d u_2 + \frac{\partial \mathbf{r}}{\partial u_3} d u_3, \quad (6)$$

where the scale factors are

$$h_i \equiv \left| \frac{\partial \mathbf{r}}{\partial u_i} \right| \quad (7)$$

and

$$\hat{\mathbf{u}}_i \equiv \frac{\frac{\partial \mathbf{r}}{\partial u_i}}{\left| \frac{\partial \mathbf{r}}{\partial u_i} \right|} \quad (8)$$

$$= \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_i}. \quad (9)$$

Equation (◇) may therefore be re-expressed as

$$d\mathbf{r} = h_1\, d u_1\, \hat{\mathbf{u}}_1 + h_2\, d u_2\, \hat{\mathbf{u}}_2 + h_3\, d u_3\, \hat{\mathbf{u}}_3. \quad (10)$$

TENSORS:

An nth-rank tensor in m-dimensional space is a mathematical object that has n indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of space. However, the dimension of the space is largely irrelevant in most tensor equations (with the notable exception of the contracted Kronecker delta). Tensors are generalizations of scalars (that have no indices), vectors (that have exactly one index), and matrices (that have exactly two indices) to an arbitrary number of indices.

Tensors provide a natural and concise mathematical framework for formulating and solving problems in areas of physics such as elasticity, fluid mechanics, and general relativity.

The notation for a tensor is similar to that of a matrix (i.e., $\mathsf{A} = (a_{ij})$), except that a tensor $a_{ijk\ldots}$, $a^{ijk\ldots}$, $a_i{}^{jk}\ldots$, etc., may have an arbitrary number of indices. In addition, a tensor with rank $r+s$ may be of mixed type (r, s), consisting of r so-called "contravariant" (upper) indices and s "covariant" (lower) indices. Note that the positions of the slots in which contravariant and covariant indices are placed are significant so, for example, $a_{\mu\nu}{}^{\lambda}$ is distinct from $a_{\mu}{}^{\nu\lambda}$.

While the distinction between covariant and contravariant indices must be made for general tensors, the two are equivalent for tensors in three-dimensional Euclidean space, and such tensors are known as Cartesian tensors.

Objects that transform like zeroth-rank tensors are called scalars, those that transform like first-rank tensors are called vectors, and those that transform like second-rank tensors are called matrices. In tensor notation, a vector $\mathbf{v}$ would be written v_i, where $i = 1, \ldots, m$, and matrix is a tensor of type $(1, 1)$, which would be written $a_i{}^j$ in tensor notation.

Tensors may be operated on by other tensors (such as metric tensors, the permutation tensor, or the Kronecker delta) or by tensor operators (such as the covariant derivative). The manipulation of tensor indices to produce identities or to simplify expressions is known as index gymnastics, which includes index lowering and index raisingas special cases. These can be achieved through multiplication by a so-called metric tensor g_{ij}, g^{ij}, $g_i{}^j$, etc., e.g.,

$$g^{ij} A_j = A^i \qquad (1)$$

$$g_{ij} A^j = A_i \qquad (2)$$

(Arfken 1985, p. 159).

Tensor notation can provide a very concise way of writing vector and more general identities. For example, in tensor notation, the dot product $\mathbf{u} \cdot \mathbf{v}$ is simply written

$$\mathbf{u}\cdot\mathbf{v} = u_i\, v^i, \tag{3}$$

where repeated indices are summed over (Einstein summation). Similarly, the cross product can be concisely written as

$$(\mathbf{u}\times\mathbf{v})_i = \epsilon_{i\,j\,k}\, u^j\, v^k, \tag{4}$$

where $\epsilon_{i\,j\,k}$ is the permutation tensor.

Contravariant second-rank tensors are objects which transform as

$$A'^{\,i\,j} = \frac{\partial x_i'}{\partial x_k}\,\frac{\partial x_j'}{\partial x_l}\,A^{k\,l}. \tag{5}$$

Covariant second-rank tensors are objects which transform as

$$C'_{i\,j} = \frac{\partial x_k}{\partial x_i'}\,\frac{\partial x_l}{\partial x_j'}\,C_{k\,l}. \tag{6}$$

Mixed second-rank tensors are objects which transform as

$$B'_{\ j}{}^{i} = \frac{\partial x_i'}{\partial x_k}\,\frac{\partial x_l}{\partial x_j'}\,B^{k}{}_{l}. \tag{7}$$

If two tensors A and B have the same rank and the
same covariant and contravariant indices, then they can be added in the obvious way,

$$A^{i\,j} + B^{i\,j} = C^{i\,j} \tag{8}$$
$$A_{i\,j} + B_{i\,j} = C_{i\,j} \tag{9}$$
$$A^{i}{}_{j} + B^{i}{}_{j} = C^{i}{}_{j}. \tag{10}$$

The generalization of the dot product applied to tensors is called tensor contraction, and consists of setting two unlike indices equal to each other and then summing using the Einstein summation convention. Various types of derivatives can be taken of tensors, the most common being the comma derivative and covariant derivative.

If the components of any tensor of any tensor rank vanish in one particular coordinate system, they vanish in all coordinate systems. A transformation of the variables of a tensor changes the tensor into another whose components are linear homogeneous functions of the components of the original tensor.

A tensor space of type (r, s) can be described as a vector space tensor product between r copies of vector fieldsand s copies of the dual vector fields, i.e., one-forms. For example,

$$T^{(3,1)} = T\,M \otimes T\,M \otimes T\,M \otimes T^*\,M \tag{11}$$

is the vector bundle of $(3, 1)$-tensors on a manifold M, where $T\,M$ is the tangent bundle of M and $T^*\,M$ is its dual. Tensors of type (r, s) form a vector space. This description generalized to any tensor type, and an invertible linear map $J : V \to W$ induces a map $\tilde{J} : V \otimes V^* \to W \otimes W^*$, where V^* is the dual vector space and J the Jacobian, defined by

$$\tilde{J}\,(v_1 \otimes v_2^*) = \left(J\,v_1 \otimes \left(J^{\mathrm{T}}\right)^{-1}\,v_2^*\right), \tag{12}$$

where J^{T} is the pullback map of a form is defined using the transpose of the Jacobian. This definition can be extended similarly to other tensor products of V and V^*. When there is a change of coordinates, then tensors transform similarly, with J the Jacobian of the linear transformation.

COVARIANT TENSORS:

A covariant tensor, denoted with a lowered index (e.g., a_μ) is a tensor having specific transformation properties. In general, these transformation properties differ from those of a contravariant tensor.

To examine the transformation properties of a covariant tensor, first consider the gradient

$$\nabla\phi \equiv \frac{\partial\phi}{\partial x_1}\hat{\mathbf{x}}_1 + \frac{\partial\phi}{\partial x_2}\hat{\mathbf{x}}_2 + \frac{\partial\phi}{\partial x_3}\hat{\mathbf{x}}_3, \qquad (1)$$

for which

$$\frac{\partial\phi'}{\partial x_i'} = \frac{\partial\phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'}, \qquad (2)$$

where $\phi(x_1, x_2, x_3) = \phi'(x_1', x_2', x_3')$. Now let

$$A_i \equiv \frac{\partial\phi}{\partial x_i}, \qquad (3)$$

then any set of quantities A_j which transform according to

$$A_i' = \frac{\partial x_j}{\partial x_i'} A_j \qquad (4)$$

or, defining

$$a_i^j \equiv \frac{\partial x_j}{\partial x_i'}, \qquad (5)$$

according to

$$A_i' = a_i^j A_j \qquad (6)$$

is a covariant tensor.

Contravariant tensors are a type of tensor with differing transformation properties, denoted a^ν. To turn acontravariant tensor a^ν into a covariant tensor a_μ (index lowering), use the metric tensor $g_{\mu\nu}$ to write

$$g_{\mu\nu}\, a^{\nu} = a_{\mu}\,. \qquad (7)$$

Covariant and contravariant indices can be used simultaneously in a mixed tensor.

In Euclidean spaces, and more generally in flat Riemannian manifolds, a coordinate system can be found where the metric tensor is constant, equal to Kronecker delta

$$g_{\mu\nu} = \delta_{\mu\nu}\,. \qquad (8)$$

Therefore, raising and lowering indices is trivial, hence covariant and contravariant tensors have the same coordinates, and can be identified. Such tensors are known as Cartesian tensors.

A similar result holds for flat pseudo-Riemannian manifolds, such as Minkowski space, for which covariant and contravariant tensors can be identified. However, raising and lowering indices changes the sign of the temporal components of tensors, because of the negative eigenvalue in the Minkowski metric.

CONTRAVARIANT TENSORS:

A contravariant tensor is a tensor having specific transformation properties (cf., a covariant tensor). To examine the transformation properties of a contravariant tensor, first consider a tensor of rank 1 (a vector)

$$d\,\mathbf{r} = d\,x_1\,\hat{\mathbf{x}}_1 + d\,x_2\,\hat{\mathbf{x}}_2 + d\,x_3\,\hat{\mathbf{x}}_3\,, \qquad (1)$$

for which

$$d\,x_i' = \frac{\partial x_i'}{\partial x_j}\,d\,x_j\,. \qquad (2)$$

Now let $A_i \equiv d\,x_i$, then any set of quantities A_j which transform according to

$$A'_i = \frac{\partial x'_i}{\partial x_j} A_j, \quad (3)$$

or, defining

$$a_{ij} \equiv \frac{\partial x'_i}{\partial x_j}, \quad (4)$$

according to

$$A'_i = a_{ij} A_j \quad (5)$$

is a contravariant tensor. Contravariant tensors are indicated with raised indices, i.e., a^μ.

Covariant tensors are a type of tensor with differing transformation properties, denoted a_ν. However, in three-dimensional Euclidean space,

$$\frac{\partial x_j}{\partial x'_i} = \frac{\partial x'_i}{\partial x_j} \equiv a_{ij} \quad (6)$$

for $i, j = 1$, 2, 3, meaning that contravariant and covariant tensors are equivalent. Such tensors are known asCartesian tensor. The two types of tensors do differ in higher dimensions, however.

Contravariant four-vectors satisfy

$$a^\mu = \Lambda^\mu_\nu a^\nu, \quad (7)$$

where Λ is a Lorentz tensor.

To turn a covariant tensor a_ν into a contravariant tensor a^μ (index raising), use the metric tensor $g^{\mu\nu}$ to write

$$g^{\mu\nu} a_\nu = a^\mu. \quad (8)$$

Covariant and contravariant indices can be used simultaneously in a mixed tensor.

POSSIBLE QUESTIONS

8 MARK:

- **State the Gauss's divergence theorem.**
- **State Stoke's theorem with its analytical proof.**
- **Show that the vector V = (sin y + z) i + (x cos y – z) j + (x – y) k is irrotational.**
- **Determine the constant 'a' so that the vector V = (x + 3y) i + (y – 2z) j + (x + az) k**
- **is solenoidal.**
- **Find curl curl f for f = x 2y i – 2xz j + 2yz k**
- **If a is a constant vector find (i) div (r x a) and ii) curl (r x a)**
- **Explain the Gauss divergence theorem.**
- **Find the constants a, b, c so that the vector F = (x +2y + az) i + (bx – 3y – z) j +(4x + cy + 2z) is irrotational.**
- **Write a short note on classification of vector field.**
- **Prove that i) curl r = 0 and ii) div r = 3**
- **Write differential operators in terms of orthogonal curvilinear co-ordinates.**
- **Taking F = x 2 y i + xz j + 2yz k verify that div curl F = 0.**

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 21

DEPARTMENT OF PHYSICS

CLASS: III B. Sc., PHYSICS BATCH: 2015-2018

MATHEMATICAL PHYSICS (15PHU603A)

UNIT I

QUESTIONS	CHOICE 1	CHOICE 2	CHOICE 3	CHOICE 4	ANSWER
Ñ x (Ñ f) =	1	0	2	-1	0
Ñ^{2} (r^{m}) is equal to	mr^{m-1}	m^{2} r^{m-2}	m(m+1) r^{m-2}	(m+1) m r^{m-1}	m(m+1) r^{m-2}
The divergence theorem enables to convert a surface integral on a closed surface into a ------	line integral	volume integral	surface integral	None	volume integral
If A is solenoidal , then	div A=0	curl A =0	|A| = 0	div (curlA) =0	div A=0
If r is position vector , then Ñx r =	3	2	1	0	0
If f=4xi+yj-2k then Ñ.f=?	1	0	3	2	3
The function f is said to satisfify the laplace equation if	Ñ^{2} f	Ñf	Ñ^{4}f	Ñ^{3}f	Ñ^{2} f
If r is position vector , then Ñ.r =	0	1	2	3	3
If A is irrotational , then	| A | = 1	ÑxA =0	|A| =0	Ñ.A=0	ÑxA =0
The divergence of the position vector r is	1	2	r	3	3
If r = xi +yj +zk , then Ñ.(ar) is equal to	a	r	0	3a	0
Which of the following is a scalar function ?	Ñ.A	Ñf	Ñ(Ñ.A)	ÑxA	Ñ.A
Given that f = x^{2} +y^{2} + z^{2} , then Ñ^{2} f is	1	3	6	0	6
If i, j and k are the unit vectors along the coordinate axes , then (i .i) is	0	1	p	j	1
If i , j and k are the unit vectors along the coordinate axes , then (jxj) is	1	k	0	p	0
If i and j are the unit vectors along x and y projections, then (i.j) is	0	1	k	3	0
If r = xi +yj +zk, then Ñ.r =?	1	2	0	3	3
If i , j and k are the unit vectors along the x, y, z axes , then jxk is equal to	0	1	I	3	I
If r = 2xi -yj +2zk, then Ñ.r =?	0	4	2	3	3
If A = 3i-5j+2k and B = 4i+3j then A.B is equal to	-3	19	-14	11	-3
If A is irrotational , then	Ñ.A =0	ÑxA=0	Ñ.A¹0	ÑxA ¹0	ÑxA=0
A vector A is said to be solenoidal if	ÑxA ¹0	ÑxA=0	Ñ.A¹0	Ñ.A =0	Ñ.A =0
If F is solenoidal ,then	Ñ.F =0	ÑxF=0	Ñ^{2}F=0	Ñ . ÑXF =0	Ñ.F =0
In Stoke's theorem, ò$_{c}$ A.dr =	òò (Ñ.A) ds	òò (ÑxA) ds	0	òò (ÑA) ds	òò (ÑxA) ds
Ñ (kf) =------	Ñ x k(f)	k (Ñ x f)	k (Ñ f)	Ñ . k(f)	k (Ñ f)
Ñ (f + Y) = -----	Ñ f +Ñ Y	Ñ f -Ñ Y	none	Ñ f * Ñ Y	Ñ f +Ñ Y
Ñ (fY) =	(Ñf)Y - f(ÑY)	Ñ(fY) + f(ÑY)	Ñ(fY) - f(ÑY)	(Ñf)Y + f(ÑY)	(Ñf)Y + f(ÑY)
Ñ (f/Y) =	[(Ñf)Y - f(ÑY)]/ Y^{2}	[(Ñf)Y + f(ÑY)]/ Y^{2}	[(Ñf)Y + f(ÑY)]/ Y^{3}	[(Ñf)Y * f(ÑY)]/ Y^{2}	[(Ñf)Y - f(ÑY)]/ Y^{2}

Ñ(A + B) = -------	Ñ . A - Ñ . B	Ñ . A + Ñ . B	Ñ . A x Ñ . B	Ñ . B / Ñ . A	Ñ . A + Ñ . B
Ñ. (kA) =------	Ñ x k(A)	k (Ñ x A)	k (Ñ . A)	Ñ . k(A)	k (Ñ . A)
Ñ . (fA) =------	(Ñf) .A + f ((Ñ . A) .A	k (Ñ x f)	k (Ñ f)	(Ñf) .A - f ((Ñ . A) .A	(Ñf) .A + f ((Ñ . A) .A
Ñ x (A + B) = ------	Ñ x A - Ñ x B	Ñ x A + Ñ x B	Ñ x A . Ñ x B	Ñ x A / Ñ x B	Ñ x A + Ñ x B
Ñx(kA) = ------	Ñ . k(A)	k x(Ñ x A)	k x(Ñ . A)	k .(Ñ x A)	k .(Ñ x A)
Ñ x (fA) =------	(Ñf) x A + f ((Ñ x A)	(Ñf) . A + f ((Ñ . A)	(Ñxf) x A + f x(Ñ x A)	(Ñf) x A - f ((Ñ x A)	(Ñf) x A + f ((Ñ x A)
The operator Ñ defined by	¶/¶x -¶/¶y -¶/¶z	¶/¶x +¶/¶y -¶/¶z	¶/¶x -¶/¶y +¶/¶z	¶/¶x +¶/¶y +¶/¶z	¶/¶x +¶/¶y +¶/¶z
The operator Ñ^{2} defined by ------	¶^{2}/¶x^{2} +¶^{2}/¶y^{2} +¶^{2}/¶z^{2}	¶^{2}/¶x^{2} +¶^{2}/¶y^{2} -¶^{2}/¶z^{2}	¶^{2}/¶x^{2} -¶^{2}/¶y^{2} +¶^{2}/¶z^{2}	¶^{2}/¶x^{2} -¶^{2}/¶y^{2} -¶^{2}/¶z^{2}	¶^{2}/¶x^{2} +¶^{2}/¶y^{2} +¶^{2}/¶z^{2}
A single valued function f(x,y,z) is said to be a hormonic function if its second partial derivatives exist and are continuous and if the function satisfies the ------ equation	Integral	Laplace	continuous	Differential	Laplace
If r = xi + yj + zk then, Ñ (1/ r) = -----	–r/r^{2}	r/r^{3}	1/r^{3}	–r/r^{3}	–r/r^{3}
The divergence of a curl of a vector is ---	one	three	zero	two	zero
If A =A_1i +A_2j + A_3k, where A1, A2, A3 have continuous second partials, then Ñ . (Ñ x A) =---	2	1	-1	-2	2
If Ñ . V = 0, then the vector V is said to be	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Solenoidal vector
If Ñ x V = 0, then the vector V is said to be	Irrotational vector	Position vector	Zero vector	Solenoidal vector	Irrotational vector
The vector A = x^2z^2i + xyz^2j – xz^3k is	Irrotational vector	Solenoidal vector	Zero vector	Position vector	Solenoidal vector
If f is a hormonic function, then Ñf is	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Zero vector
If A and B are irrotational, then AxB is	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Zero vector
If A is irrotational, then ÑxA is	1	-1	2	0	0
If A is solenoidal, then Ñ.A is	1	-1	0	-2	0
div (curl A) = ----	0	1	-1	2	0
Curl (grad f) = ---	1	non zero	2	0	1
Curl (A+B) = -----	Curl A – Curl B	curl A + Curl B	curl A * Curl B	curl A / Curl B	curl A + Curl B
d/dt(A.B) = -----	A.dA/dt +dB/dt. B	A.dB/dt +dA/dt .B	A.dA/dt - dB/dt	A.dA/dt + dB/dt	A.dA/dt - dB/dt
If F is constant vector, then curl F = ----	1	2	non zero	0	0
Ñ . (Ñf) =------	Ñ^{2}f	0	Ñf	f	Ñ^{2}f
Grad r^n = -----	nr^{n-1} r	nr^{n-2} r	$(n-1)r^{n-2}$ r	r^{n-1} r	nr^{n-2} r
.------------ is a vector quantity.	Mass	.pressure	volume	force	force
If f is a scalar function and f(t) is a vector function then (ff)	ff ' - f'f.	.ff ' + f'f.	ff ' x f'f	ff ' / f'f.	ff ' - f'f.
Gradient of a constant is -----------.	Constant	1	0	gradiant	0
The derivative of the sum of two derivable vector functions f(t) and g(t) of the scalar variable t, is equal to the---------- of their derivatives.	sum	difference	multiple	division	sum

Any scalar function f which satisfies the partial differential equation	harmonic function	Homogeneous function	Nonharmonic function	Nonhomogeneous function	harmonic function
.If f=(x+y)i+xj+zk and s is the surface of the cube bounded by the planes x=0,x=1,y=0,y=1,z=0 and z=1 then the surface integral is ----------	1	2	4	6	6
.∭ Δf dv=--------	. ∬ f.n ds	∭ f.n ds	∬ fxn ds	∭ fxn ds	. ∬ f.n ds
.∭ ΔxB dv=--------	∬ nxB ds	. ∭ nxB ds	∬ n.B ds	. ∭ n.B ds	∬ nxB ds
In a Gauss divergence theorem,f is a vector point function --- and ----- in a region v of space	finite and differentiable	infinite and differentiable	finite andNon differentiable	in finite and Nondifferentiable	finite and differentiable

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UNIT-II

SYLLABUS

Matrices-Special types of matrices -Transpose of a matrix - Conjugate of a matrix - Conjugate transpose of a matrix-symmetric and antisymmetric matrices - Hermitian and skew - Hermitian matrices - Determinant of a matrix - Adjoint of a matrix - Inverse of a matrix -Unitary matrices - Rank of a matrix and simple problems - Characteristic matrix and characteristic equation - Characteristic vector - Methods of finding the Eigen values and Eigen vectors of a matrix.

MATRICES:

A rectangular array of numbers is called a matrix. We shall mostly be concerned with matrices having real numbers as entries. The horizontal arrays of a matrix are called its ROWS and the vertical arrays are called its COLUMNS. A matrix having *m* rows and *n* columns is said to have the order $m \times n$.

A matrix ***A*** of order $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column.

In a more concise manner, we also denote the matrix ***A*** by $[a_{ij}]$ by suppressing its order.

Note: Some books also use

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

to represent a matrix.

A matrix having only one column is called a column vector and a matrix with only one row is called a row vector. Whenever a vector is used, it should be understood from the context whether it is a row vector or a column vector.

Here are a couple of examples of different types of matrices:

Symmetric:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -5 \\ 3 & -5 & 6 \end{bmatrix}$$

Diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Upper Triangular

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Lower Triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 0 \\ 12 & 5 & 3 \end{bmatrix}$$

Zero

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 1:

Let $A = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 3 & 7 \\ 4 & 6 & 0 \end{bmatrix}$, list out the a_{ij}'s values in $\boldsymbol{A}$.

Solution:

$a_{11} = 1$, $a_{12} = 3$, $a_{13} = 6$

$a_{21} = 2$, $a_{22} = 3$, $a_{23} = 7$

$a_{31} = 4$, $a_{32} = 4$, $a_{33} = 0$

Example 2:

State the a_{ij}'s values in $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \\ 4 & 6 & 8 \end{bmatrix}$.

Solution:

$a_{11} = 9, a_{12} = 8, a_{13} = 7$

$a_{21} = 6, a_{22} = 5, a_{23} = 4$

$a_{31} = 3, a_{32} = 2, a_{33} = 1$

$a_{41} = 4, a_{42} = 6, a_{43} = 8$

Example 3: Provide two examples of column and row matrices each.

Solution:

We know that, a matrix having only one column is called a column vector or column matrix and a matrix with only one row is called a row vector or row column.

$$1. A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 & 9 & 0 \end{bmatrix}$$

Example 4: Is the following matrix classified under the category of matrices?

$$A = \begin{bmatrix} 9 & 7 & 8 \\ 8 & 7 & 9 \\ 6 & 9 & \end{bmatrix}$$

Solution:

A is not a matrix because column three or we can say row three is incomplete.

Example 5: What is the order of the following matrix?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution:

We know that a matrix having *m* rows and *n* columns is said to have the order $m \times n$, therefore the order of $\boldsymbol{A}$ is 4×3.

Types of matrices — triangular, diagonal, scalar, identity, symmetric, skew-symmetric, periodic, nilpotent

Upper triangular matrix. A square matrix in which all the elements below the diagonal are zero i.e. a matrix of type:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Lower triangular matrix. A square matrix in which all the elements above the diagonal are zero i.e. a matrix of type

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Diagonal matrix. A square matrix in which all of the elements are zero except for the diagonal elements i.e. a matrix of type

$$D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

It is often written as D = diag(a_{11}, a_{22}, a_{33}, ... , a_{nn})

Scalar matrix. A diagonal matrix in which all of the diagonal elements are equal to some constant "k" i.e. a matrix of type

$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & k \end{bmatrix}$$

Identity matrix. A diagonal matrix in which all of the diagonal elements are equal to "1" i.e. a matrix of type

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

An identity matrix of order nxn is denoted by $\mathbf{I_n}$ **.**

Transpose of a matrix.

The matrix resulting from interchanging the rows and columns in the given matrix. The transpose of

$$A = \begin{bmatrix} 3 & 2 & 8 \\ 1 & 5 & 4 \end{bmatrix}$$

is

$$A^T = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 8 & 4 \end{bmatrix}$$

The first row of A becomes the first column of A^T, the second row of A becomes the second column of A^T, etc.. It corresponds to a "flip"of the matrix about the diagonal running down from the upper left corner.

Symmetric matrix.

A square matrix in which corresponding elements with respect to the diagonal are equal; a matrix in which $a_{ij} = a_{ji}$ where a_{ij} is the element in the i-th row and j-th column; a matrix which is equal to its transpose; a square matrix in which a flip about the diagonal leaves it unchanged. Example:

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 5 \\ -3 & 5 & 6 \end{bmatrix}$$

Skew-symmetric matrix.

A square matrix in which corresponding elements with respect to the diagonal are negatives of each other; a matrix in which $a_{ij} = -a_{ji}$ where a_{ij} is the element in the i-th row and j-th column; a matrix which is equal to the negative of its transpose. The diagonal elements are always zeros. Example:

$$\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$$

Direct Sum. Let $A_1, A_2, \ldots, A_s$ be square matrices of respective orders $m_1, m_2, \ldots, m_s$. The generalization

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_s \end{bmatrix} = Diag(A_1, A_2, \ldots, A_s)$$

of the diagonal matrix is called the direct sum of the A_i.

Inverse of a matrix.

If A and B are square matrices such that AB = BA = I where I is the identity matrix , then B is called the inverse of A and we write $B = A^{-1}$. The matrix B also has A as an inverse and we can write $A = B^{-1}$.

Commutative and anti-commutative matrices. If A and B are square matrices such that AB = BA, then A and B are called commutative or are said to commute. If AB = -BA, the matrices are said to **anti-commute.**

Periodic matrix. A matrix A for which $\mathbf{A^{k+1} = A}$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then A is said to be of**period k**. If k = 1, so that $A^2 = A$, then A is called **idempotent**.

Nilpotent matrix. A matrix A for which $\mathbf{A^p = 0}$, where p is some positive integer. If p is the least positive integer for which $A^p = 0$, then A is said to be**nilpotent of index p**.

DETERMINANT OF A MATRIX:

Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations. As shown by Cramer's rule, a nonhomogeneous system of linear equations has a unique solution iff the determinant of the system's matrix is nonzero (i.e., the matrix is nonsingular). For example, eliminating x, y, and $z$from the equations

$$a_1 x + a_2 y + a_3 z = 0 \quad (1)$$

$$b_1 x + b_2 y + b_3 z = 0 \quad (2)$$

$$c_1 x + c_2 y + c_3 z = 0 \quad (3)$$

gives the expression

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1 = 0, \quad (4)$$

which is called the determinant for this system of equation. Determinants are defined only for square matrices.

If the determinant of a matrix is 0, the matrix is said to be singular, and if the determinant is 1, the matrix is said to be unimodular.

The determinant of a matrix A,

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \end{vmatrix} \tag{5}$$

is commonly denoted $\det(\mathsf{A})$, $|\mathsf{A}|$, or in component notation as $\sum(\pm a_1\, b_2\, c_3 \cdots)$, $D(a_1\, b_2\, c_3 \cdots)$, or $|a_1\, b_2\, c_3 \ldots|$(Muir 1960, p. 17). Note that the notation $\det(\mathsf{A})$ may be more convenient when indicating the absolute value of a determinant, i.e., $|\det(\mathsf{A})|$ instead of $||\mathsf{A}||$. The determinant is implemented in the Wolfram Language as Det[*m*].

A 2×2 determinant is defined to be

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv a\,d - b\,c. \tag{6}$$

A $k\times k$ determinant can be expanded "by minors" to obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k2} & a_{k3} & \cdots & a_{kk} \end{vmatrix}$$
$$- a_{12} \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k3} & \cdots & a_{kk} \end{vmatrix} + \ldots \pm a_{1k} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{k(k-1)} \end{vmatrix}. \tag{7}$$

A general determinant for a matrix A has a value

$$|\mathsf{A}| = \sum_{i=1}^{k} a_{ij}\, C_{ij}, \tag{8}$$

with no implied summation over j and where C_{ij} (also denoted a^{ij}) is the cofactor of a_{ij} defined by

$$C_{ij} \equiv (-1)^{i+j} M_{ij}. \quad (9)$$

and M_{ij} is the minor of matrix A formed by eliminating row i and column j from A. This process is calleddeterminant expansion by minors (or "Laplacian expansion by minors," sometimes further shortened to simply "Laplacian expansion").

A determinant can also be computed by writing down all permutations of $\{1, \ldots, n\}$, taking each permutation as the subscripts of the letters a, b, ..., and summing with signs determined by $\epsilon_p = (-1)^{i(p)}$, where $i(p)$ is the number ofpermutation inversions in permutation p (Muir 1960, p. 16), and $\epsilon_{n_1 n_2 \cdots}$ is the permutation symbol. For example, with $n = 3$, the permutations and the number of inversions they contain are 123 (0), 132 (1), 213 (1), 231 (2), 312 (2), and 321 (3), so the determinant is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1. \quad (10)$$

If a is a constant and A an $n \times n$ square matrix, then

$$|a \mathsf{A}| = a^n |\mathsf{A}|. \quad (11)$$

Given an $n \times n$ determinant, the additive inverse is

$$|-\mathsf{A}| = (-1)^n |\mathsf{A}|. \quad (12)$$

Determinants are also distributive, so

$$|\mathsf{A}\,\mathsf{B}| = |\mathsf{A}|\,|\mathsf{B}|. \quad (13)$$

This means that the determinant of a matrix inverse can be found as follows:

$$|I| = |A\,A^{-1}| = |A|\,|A^{-1}| = 1, \qquad (14)$$

where I is the identity matrix, so

$$|A| = \frac{1}{|A^{-1}|}. \qquad (15)$$

Determinants are multilinear in rows and columns, since

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \qquad (16)$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}. \qquad (17)$$

The determinant of the similarity transformation of a matrix is equal to the determinant of the original matrix

$$|B\,A\,B^{-1}| = |B|\,|A|\,|B^{-1}| \qquad (18)$$

$$= |B|\,|A|\,\frac{1}{|B|} \qquad (19)$$

$$= |A|. \qquad (20)$$

The determinant of a similarity transformation minus a multiple of the unit matrix is given by

$$|B^{-1}\,A\,B - \lambda\,I| = |B^{-1}\,A\,B - B^{-1}\,\lambda\,I\,B| \qquad (21)$$

$$= |B^{-1}\,(A - \lambda\,I)\,B| \qquad (22)$$

$$= |B^{-1}|\,|A - \lambda\,I|\,|B| \qquad (23)$$

$$= |A - \lambda\,I|. \qquad (24)$$

The determinant of a transpose equals the determinant of the original matrix,

$$|\mathsf{A}| = \left|\mathsf{A}^{\mathrm{T}}\right|, \qquad (25)$$

and the determinant of a complex conjugate is equal to the complex conjugate of the determinant

$$|\bar{\mathsf{A}}| = \overline{|\mathsf{A}|}. \qquad (26)$$

Let ϵ be a small number. Then

$$|\mathsf{I} + \epsilon \mathsf{A}| = 1 + \epsilon \operatorname{Tr}(\mathsf{A}) + O\left(\epsilon^2\right), \qquad (27)$$

where $\operatorname{Tr}(\mathsf{A})$ is the matrix trace of A. The determinant takes on a particularly simple form for a triangular matrix

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{k\,1} \\ 0 & a_{22} & \cdots & a_{k\,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k\,k} \end{vmatrix} = \prod_{n=1}^{k} a_{n\,n}. \qquad (28)$$

Important properties of the determinant include the following, which include invariance under elementary row and column operations.

1. Switching two rows or columns changes the sign.

2. Scalars can be factored out from rows and columns.

3. Multiples of rows and columns can be added together without changing the determinant's value.

4. Scalar multiplication of a row by a constant c multiplies the determinant by c.

5. A determinant with a row or column of zeros has value 0.

6. Any determinant with two rows or columns equal has value 0.

Property 1 can be established by induction. For a 2×2 matrix, the determinant is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2 \quad (29)$$

$$= -(b_1 a_2 - a_1 b_2) \quad (30)$$

$$= -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} \quad (31)$$

For a 3×3 matrix, the determinant is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= -\left(a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right) = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}$$

$$= -\left(-a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}\right) = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \quad (32)$$

$$= -\left(a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}\right) = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}.$$

Property 2 follows likewise. For 2×2 and 3×3 matrices,

$$\begin{vmatrix} k\,a_1 & b_1 \\ k\,a_2 & b_2 \end{vmatrix} = k\,(a_1 b_2) - k\,(b_1 a_2) \quad (33)$$

$$= k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad (34)$$

and

$$\begin{vmatrix} k\,a_1 & b_1 & c_1 \\ k\,a_2 & b_2 & c_2 \\ k\,a_3 & b_3 & c_3 \end{vmatrix} = k\,a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} k\,a_2 & c_2 \\ k\,a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} k\,a_2 & b_2 \\ k\,a_3 & b_3 \end{vmatrix} \quad (35)$$

$$= k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (36)$$

Property 3 follows from the identity

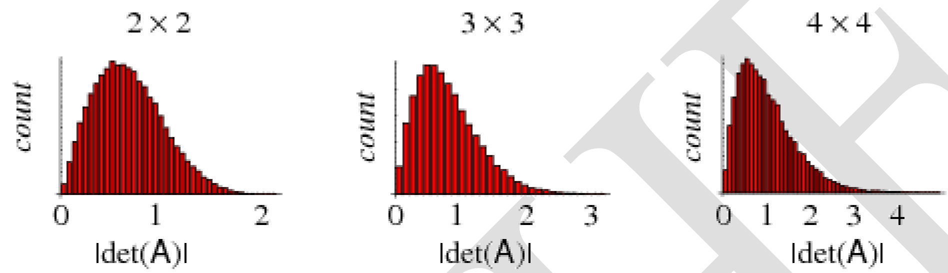
$$\begin{vmatrix} a_1 + k\,b_1 & b_1 & c_1 \\ a_2 + k\,b_2 & b_2 & c_2 \\ a_3 + k\,b_3 & b_3 & c_3 \end{vmatrix} = (a_1 + k\,b_1) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 + k\,b_2 & c_2 \\ a_3 + k\,b_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 + k\,b_2 & b_2 \\ a_3 + k\,b_3 & b_3 \end{vmatrix}. \quad (37)$$

If a_{ij} is an $n \times n$ matrix with a_{ij} real numbers, then $\det[a_{ij}]$ has the interpretation as the oriented n-dimensionalcontent of the parallelepiped spanned by the column vectors $[a_{i,1}]$, ..., $[a_{i,n}]$ in $\mathbb{R}^n$. Here, "oriented" means that, up to a change of $+$ or $-$ sign, the number is the n-dimensional content, but the sign depends on the "orientation" of the column vectors involved. If they agree with the standard orientation, there is a $+$ sign; if not, there is a $-$ sign. Theparallelepiped spanned by the n-dimensional vectors $\mathbf{v}_1$ through $\mathbf{v}_i$ is the collection of points

$$t_1\,\mathbf{v}_1 + \ldots + t_i\,\mathbf{v}_i, \quad (38)$$

where t_j is a real number in the closed interval $[0, 1]$.

Several accounts state that Lewis Carroll (Charles Dodgson) sent Queen Victoria a copy of one of his mathematical works, in one account, *An Elementary Treatise on Determinants*. Heath (1974) states, "A well-known story tells how Queen Victoria, charmed by *Alice in Wonderland*, expressed a desire to receive the author's next work, and was presented, in due course, with a loyally inscribed copy of *An Elementary Treatise on Determinants*," while Gattegno (1974) asserts "Queen Victoria, having enjoyed *Alice* so much, made known her wish to receive the author's other books, and was sent one of Dodgson's mathematical works." However, in *Symbolic Logic* (1896), Carroll stated, "I take this opportunity of giving what publicity I can to my contradiction of a silly story, which has been going the round of the papers, about my having presented certain books to Her Majesty the Queen. It is so constantly repeated, and is such absolute fiction, that I think it worth while to state, once for all, that it is utterly false in every particular: nothing even resembling it has occurred" (Mikkelson and Mikkelson).



Hadamard (1893) showed that the absolute value of the determinant of a complex $n \times n$ matrix with entries in theunit disk satisfies

$$|\det A| \leq n^{n/2} \qquad (39)$$

(Brenner 1972). The plots above show the distribution of determinants for random $n \times n$ complex matrices with entries satisfying $|a_{ij}| < 1$ for $n = 2$, 3, and 4.

RANK OF A MATRIX:

The DETERMINANT of a matrix det *A* or $|A|$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$|A| = a_{11} - a_{12}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$

$(a_{22}a_{33}-a_{23}a_{32})$ is called the minor of a_{11} and is usually denoted $|A_{ij}|$ - in this case $|A_{11}|$

$$A = \begin{bmatrix} 3 & 5 & 4 \\ 6 & 9 & 7 \\ 2 & 8 & 1 \end{bmatrix}$$

$$|A| = +3\begin{vmatrix} 9 & 7 \\ 8 & 1 \end{vmatrix} - 5\begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 6 & 9 \\ 2 & 8 \end{vmatrix}$$

= 3(9 – 56) - 5(6 – 14) + 4(48 – 18)

= -141 + 40 + 120

= 19

The COFACTOR of the elements of a_{ij} denoted by c_{ij}
is

$c_{ij} = (-1)^{i+j} \left| A_{ij} \right|$

$$\left|A\right| = \sum_{j=1}^{n} a_{ij} c_{ij} = \sum_{i}^{n} a_{ij} c_{ij}$$

PROPERTIES OF DETERMINANTS

1. $\left| A^T \right| = \left| A \right|$

2. $\begin{vmatrix} a_{11} & ka_{12} \\ a_{21} & ka_{22} \end{vmatrix} = ka = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

3. If A is (n x n) then $\left| kA \right| = k^n \left| A \right|$

4. If a square matrix has two equal rows or columns its determinant is zero

5. If any row (or column) is the multiple of any other row (or column) then its determinant is zero

6. The value of a determinant is unchanged if a multiple of one row (or column) is added to another row (or column)

7. If A is a diagonal matrix of order n then its determinant is $a_{11}a_{22} \ldots. a_{nn}$

8. If A is a triangular matrix of order n then its determinant is $a_{11}a_{22} \ldots. a_{nn}$

9. If B is the matrix obtained from a square matrix A by interchanging any two rows (or columns) then det B = -det A

10. If A and B are square matrices of the same order then $|AB| = |A|\,|B|$

11. If $A_1, A_2, \ldots, A_s$ are square matrices then $|\text{diag}(A_1, A_2, \ldots, A_s)| = |A_1|\,|A_2| \ldots |A_s|$

12. In general $|A + B|$ does not equal $|A| + |B|$

The RANK of a matrix is equal to the highest order non-zero determinant that can be formed from its sub-matrices

$$A = \begin{bmatrix} 4 & 5 & 2 & 14 \\ 3 & 9 & 6 & 21 \\ 8 & 10 & 7 & 28 \\ 1 & 2 & 9 & 5 \end{bmatrix}$$

det A = 0

$$\begin{vmatrix} 4 & 5 & 2 \\ 3 & 9 & 6 \\ 8 & 10 & 7 \end{vmatrix} = 63$$

Rank of A = 3

The rank of a matrix can also be measured by the maximum number of linearily independent columns of A

This also equals the maximum number of linearily independent rows

$$1\begin{bmatrix}4\\3\\8\\1\end{bmatrix}+2\begin{bmatrix}5\\9\\10\\2\end{bmatrix}+0\begin{bmatrix}2\\6\\7\\9\end{bmatrix}+(-1)\begin{bmatrix}14\\21\\28\\5\end{bmatrix}=0$$

$c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4 = 0$

A FULL COLUMN RANK matrix has the same number of linearily independent columns (rows) equal to the number of columns

A FULL ROW RANK matrix has the same number of linearily independent rows (columns) equal to the number of rows

If A does not have full row and column rank it is SINGULAR
If A does have full row and column rank it is NON-SINGULAR

rank (I_n) = n
rank (kA) = rank (A)
rank (A^T) = rank (A)
If A is (m x n) then rank (A) is ≤ min {m, n}
rank AB ≤ min{rank (A), rank (B)}

INVERSES

If A and B are matrices of order n such that $AB = BA = I_n$ then B is called the inverse of A

A has an inverse iff it is of full column and row rank – non-singular

$A^{-1} = C^T / |A|$

C^T is the transpose of the matrix of co-factors

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$|A| = 4 - 6 = -2$

$c_{11} = 4$ $c_{12} = -3$ $c_{21} = -2$ $c_{22} = 1$

$$A^{-1} = \frac{-1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T$$

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

PROPERTIES OF INVERSES

1. $I^{1 =} I$
2. $(A^{-1})^{-1} = A$
3. $AB = I$ $BA = I$

4. *A* non-singular A^{-1} non-singular
5. *A* and *B* non-singular $(AB)^{-1} = B^{-1}A^{-1}$

LEFT INVERSE of a (m x n) matrix *A* is the (n x m) matrix *B* such that $BA = I_n$

RIGHT INVERSE of a (m x n) matrix *A* is the (n x m) matrix *C* such that $AC = I_m$

(T x k) design matrix *X* which has rank k < T

has an infinite number of left inverses including

$(X^T X)^{-1} X^T$

IDEMPOTENT MATRICES

$AA = A$

$$A = \begin{bmatrix} 0.4 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

Consider $M = [I - X(X^T X)^{-1} X^T]$

KRONECKER PRODUCT

$$A \otimes B$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 1\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} & 3\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \\ 2\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} & 0\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 & 6 & 6 & 0 \\ 1 & 0 & 3 & 3 & 0 & 9 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 6 & 0 & 0 & 0 \end{bmatrix}$$

EIGEN VALUE AND EIGEN VECTORS:

Eigenvalues are a special set of scalars associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic roots, characteristic values (Hoffman and Kunze 1971), proper values, or latent roots (Marcus and Minc 1988, p. 144).

The determination of the eigenvalues and eigenvectors of a system is extremely important in physics and engineering, where it is equivalent to matrix diagonalization and arises in such common applications as stability analysis, the physics of rotating bodies, and small oscillations of vibrating systems, to name only a few. Each eigenvalue is paired with a corresponding so-called eigenvector (or, in general, a corresponding right eigenvectorand a corresponding left eigenvector; there is no analogous distinction between left and right for eigenvalues).

The decomposition of a square matrix A into eigenvalues and eigenvectors is known in this work as eigen decomposition, and the fact that this decomposition is always possible as long as the matrix consisting of the eigenvectors of A is square is known as the eigen decomposition theorem.

The Lanczos algorithm is an algorithm for computing the eigenvalues and eigenvectors for large symmetric sparse matrices.

Let A be a linear transformation represented by a matrix A. If there is a vector $\mathbf{X} \in \mathbb{R}^n \neq \mathbf{0}$ such that

$$\mathsf{A}\mathbf{X} = \lambda \mathbf{X} \tag{1}$$

for some scalar λ, then λ is called the eigenvalue of A with corresponding (right) eigenvector $\mathbf{X}$.

Letting A be a $k \times k$ square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \tag{2}$$

with eigenvalue λ, then the corresponding eigenvectors satisfy

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \tag{3}$$

which is equivalent to the homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4}$$

Equation (4) can be written compactly as

$$(\mathsf{A} - \lambda \mathsf{I})\mathbf{X} = \mathbf{0}, \tag{5}$$

where I is the identity matrix. As shown in Cramer's rule, a linear system of equations has nontrivial solutions iff thedeterminant vanishes, so the solutions of equation (5) are given by

$$\det(A - \lambda I) = 0. \qquad (6)$$

This equation is known as the characteristic equation of A, and the left-hand side is known as the characteristic polynomial.

For example, for a 2×2 matrix, the eigenvalues are

$$\lambda_{\pm} = \frac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{4\, a_{12}\, a_{21} + (a_{11} - a_{22})^2}\,\right], \qquad (7)$$

which arises as the solutions of the characteristic equation

$$x^2 - x\,(a_{11} + a_{22}) + (a_{11}\, a_{22} - a_{12}\, a_{21}) = 0. \qquad (8)$$

If all k eigenvalues are different, then plugging these back in gives $k-1$ independent equations for the $k$components of each corresponding eigenvector, and the system is said to be nondegenerate. If the eigenvalues are n-fold degenerate, then the system is said to be degenerate and the eigenvectors are not linearly independent. In such cases, the additional constraint that the eigenvectors be orthogonal,

$$\mathbf{X}_i \cdot \mathbf{X}_j = |\mathbf{X}_i|\,|\mathbf{X}_j|\,\delta_{ij}, \qquad (9)$$

where δ_{ij} is the Kronecker delta, can be applied to yield n additional constraints, thus allowing solution for theeigenvectors.

Eigenvalues may be computed in the Wolfram Language using Eigenvalues[*matrix*]. Eigenvectors and eigenvalues can be returned together using the command Eigensystem[*matrix*].

Assume we know the eigenvalue for

$$A\,\mathbf{X} = \lambda\,\mathbf{X}. \qquad (10)$$

Adding a constant times the identity matrix to A,

$$(\mathsf{A} + c\,\mathsf{I})\,\mathbf{X} = (\lambda + c)\,\mathbf{X} \equiv \lambda'\,\mathbf{X}, \tag{11}$$

so the new eigenvalues equal the old plus c. Multiplying A by a constant c

$$(c\,\mathsf{A})\,\mathbf{X} = c\,(\lambda\,\mathbf{X}) \equiv \lambda'\,\mathbf{X}, \tag{12}$$

so the new eigenvalues are the old multiplied by c.

Now consider a similarity transformation of A. Let $|\mathsf{A}|$ be the determinant of A, then

$$\left|\mathsf{Z}^{-1}\,\mathsf{A}\,\mathsf{Z} - \lambda\,\mathsf{I}\right| = \left|\mathsf{Z}^{-1}\,(\mathsf{A} - \lambda\,\mathsf{I})\,\mathsf{Z}\right| \tag{13}$$

$$= |\mathsf{Z}|\,|\mathsf{A} - \lambda\,\mathsf{I}|\,\left|\mathsf{Z}^{-1}\right| \tag{14}$$

$$= |\mathsf{A} - \lambda\,\mathsf{I}|, \tag{15}$$

so the eigenvalues are the same as for A.

POSSIBLE QUESTIONS

8 MARK

- Show that the matrix A = $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is orthogonal.
- Show that the matrix B = $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ is unitary.
- Find the eigen values of matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$
- Show that the matrix A = $\begin{pmatrix} 0 & 1+i \\ -1-i & 0 \end{pmatrix}$ is Skew – symmetric but not Skew Hermitian.
- Explain the different types of matrices.(any 5)
- Show that the matrix A = $\begin{pmatrix} 0 & 1+i \\ -1-i & 0 \end{pmatrix}$ is Skew – symmetric but not Skew Hermitian.
- Find the Eigen values of the matrix A = $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$
- Find the Rank of following matrices i) $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
- Find the eigen values of matrix A = $\begin{pmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{pmatrix}$
- Find the Rank of following matrices i) $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
- If matrix A = $\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} -5 & 1 & 3 \\ 4 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ Find A.B
- Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

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UNIT II

QUESTIONS	CHOICE 1	CHOICE 2	CHOICE 3	CHOICE 4	ANSWER
A diagonal matrix in which all the diagonal elements are equal is called --------.	scalar matrix	diagonal matrix	unit matrix	null matrix	scalar matrix
[3 8 9 -2] is a row matrix of order-------------------	1x 4	4 x 1	1x 1	4 x 4	1x 4
in a square matrix A, a_{ij} = 0 for i <j, then it is called a --------------- matrix.	lower triangular	upper triangular	diagonal	triangular	lower triangular
The matrix multiplication of two matrices A and B is possible only if ----------------.	order of A is m x n and order of B is m x n	order of A is m x n and order of B is n x p	order of A is m xm and order of B is n x n	order of A is m x n and order of B is p x n	order of A is m x n and order of B is n x p
If the order of the matrix A is 4 x 5 and the order of the matrix B is 2 x 4 then the resultant matrix BA has the order-----------.	2 x 5	2 x 4	4 x 5	4 x 4	2 x 5
A Square matrix such that A’ = – A is called --------------	symmetric	skew symmetric	hermit ion	scalar	skew symmetric
The sum of the diagonal elements in a matrix A is the ------------.	Trace of A	unit of A	Transpose of A	inverse of A	Trace of A
If a square matrix A of order n is of the form $A^n = 0$, then it is an ----------- matrix.	Identity	Idempotent	Nilpotent	Orthogonal	Nilpotent
If a square matrix A of order n is of the form $A^n = A$, then it is an ----------- matrix.	Identity	Idempotent	Nilpotent	Orthogonal	Idempotent
$(AB)^{-1}$ = ---------------.	A^{-1}	B^{-1}	$A^{-1}B^{-1}$	$B^{-1}A^{-1}$	$B^{-1}A^{-1}$
$(A^T)^{-1}$ = ---------------.	(A^T)	$(A^{-1})^T$	(A^{-1})	$(A^T)^T$	$(A^{-1})^T$
The ------------ of a square matrix A is the transpose of the matrix formed by replacing the elements of A by their corresponding cofactors.	Transpose	Inverse	Cofactor	Ad joint	Ad joint
The formula for solving the system of simultaneous linear equations by matrix inversion method is ---------------	X = A B	$X = A B^{-1}$	$X = A^{-1} B$	$X = B A^{-1}$	$X = A^{-1} B$
The transpose of a matrix A is getting by ----------------.	Interchanging rows into columns only.	Interchanging columns into rows only	Interchanging rows into columns and columns into rows	Taking the same matrix without interchanging the rows and columns.	Interchanging rows into columns only.

The subtraction of any two matrices A and B are possible only if ----.	A and B have same elements	A and B have same order	A and B have different order	A and B have different elements	A and B have same order
Two matrices A and B said to be equal if ---------------.	A and B have same elements	A and B have same order	A and B have different order	A and B have same elements and same order	A and B have same elements and same order
Every matrix is a ------------------------ of it self.	sub matrix	unit matrix	equal matrix	none	sub matrix
The co – factor of an element A_{ij} is defined as ----------------.	$(-1)^{i+j}$ * Determinant obtained by deleting i th row and j th column of A	(-1) * Determinant obtained by deleting i th row and j th column of A	Determinant obtained by deleting i th row and j th column of A	$(-1)^{i+j}$ * Determinant obtained by deleting j th row and i th column of A	$(-1)^{i+j}$ * Determinant obtained by deleting i th row and j th column of A
$A^{-1}A$ = ---------------.	0	1	Identity matrix	Zero matrix	Identity matrix
I*A = ---------------.	A	0	Identity matrix	Zero matrix	A
An identity matrix is also found as a ------------- matrix	Scalar	Diagonal	triangular	scalar and diagonal	scalar and diagonal
In a 3x3 square matrix the minor and the cofactor of the element a_{23} have ---------.	Same sign and same value	same sign and different values	Opposite sign and same value	opposite sign and different values	Opposite sign and same value
If every element of a matrix is multiplied by a constant k ,then the determinant value of the matrix is multiplied by-----------------	k	k-1	k^2	k+1	k
The determinant value of the unit matrix of order 2 is ------------------	1	0	-1	2	1
If any one of the row or column of a matrix is zero then the determinant value of the matrix is ------------------	0	positive	negative	one	0
If A is singular ,its inverse is ----------------	null matrix	does not exists	1/Adj A	1/|A|	does not exists
A rectangular matrix will not possesses--------------	inverse	cofactor	determinant	transpose	inverse
For any square matrix (adj A).A = A. (adj A) is --------------	|A| .I	|A |	1 / |A|	A^T	|A| .I
For any two square matrices A and B (adj AB) = ----------------	(adj A). (adj B)	(adj B). (adj A)	(adj BA)	(adj B +adj A)	(adj B). (adj A)

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UNIT-III

SYLLABUS

Differential Equations: Introduction – Solution in simple cases of ordinary differential equations of second order – Simple problems from Physics – Partial Differential equations – Special types of differential equations arising in Physics.

Group Theory: Introduction in sets, mappings and binary operations – groups – elementary properties of groups – The centre of a group – Cosets or consents of a subgroup – cyclic group.

DIFFERENTIAL EQUATIONS:

A differential equation is an equation that involves the derivatives of a function as well as the function itself. If partial derivatives are involved, the equation is called a partial differential equation; if only ordinary derivatives are present, the equation is called an ordinary differential equation. Differential equations play an extremely important and useful role in applied math, engineering, and physics, and much mathematical and numerical machinery has been developed for the solution of differential equations.

ORDINARY DIFFERENTIAL EQUATION:

An ordinary differential equation (frequently called an "ODE," "diff eq," or "diffy Q") is an equality involving a function and its derivatives. An ODE of order n is an equation of the form

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0, \qquad (1)$$

where y is a function of x, $y' = dy/dx$ is the first derivative with respect to x,

and $y^{(n)} = d^n y / dx^n$ is the $n$thderivative with respect to x.

Nonhomogeneous ordinary differential equations can be solved if the general solution to the homogenous version is known, in which case the undetermined coefficients method or variation of parameters can be used to find the particular solution.

Many ordinary differential equations can be solved exactly in the Wolfram Language using DSolve[*eqn*, *y*, *x*], and numerically using NDSolve[*eqn*, *y*, {*x*, *xmin*, *xmax*}].

An ODE of order n is said to be linear if it is of the form

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \ldots + a_1(x)\,y' + a_0(x)\,y = Q(x). \quad (2)$$

A linear ODE where $Q(x) = 0$ is said to be homogeneous. Confusingly, an ODE of the form

$$y' = f\left(\frac{y}{x}\right) \quad (3)$$

is also sometimes called "homogeneous."

In general, an nth-order ODE has n linearly independent solutions. Furthermore, any linear combination of linearly independent functions solutions is also a solution.

Simple theories exist for first-order (integrating factor) and second-order (Sturm-Liouville theory) ordinary differential equations, and arbitrary ODEs with linear constant coefficients can be solved when they are of certain factorable forms. Integral transforms such as the Laplace transform can also be used to solve classes of linear ODEs. Morse and Feshbach (1953, pp. 667-674) give canonical forms and solutions for second-order ordinary differential equations.

While there are many general techniques for analytically solving classes of ODEs, the only practical solution technique for complicated equations is to use numerical methods (Milne 1970, Jeffreys and Jeffreys 1988). The most popular of these is the Runge-Kutta method, but many others have been developed, including the collocation method and Galerkin method. A vast amount of research and huge numbers of publications have been devoted to the numerical solution of differential equations, both ordinary and partial (PDEs) as a result of their importance in fields as diverse as physics, engineering, economics, and electronics.

The solutions to an ODE satisfy existence and uniqueness properties. These can be formally established byPicard's existence theorem for certain classes of ODEs. Let a system of first-order ODE be given by

$$\frac{d x_i}{d t} = f_i (x_1, \ldots, x_n, t), \qquad (4)$$

for $i = 1$, ..., n and let the functions $f_i (x_1, \ldots, x_n, t)$, where $i = 1$, ..., n, all be defined in a domain D of the $(n+1)$-dimensional space of the variables x_1, ..., x_n, t. Let these functions be continuous in D and have continuous firstpartial derivatives $\partial f_i / \partial x_j$ for $i = 1$, ..., n and $j = 1$, ..., n in D. Let $(x_1^0, \ldots, x_n^0)$ be in D. Then there exists a solution of (4) given by

$$x_1 = x_1 (t), \ldots, x_n = x_n (t) \qquad (5)$$

for $t_0 - \delta < t < t_0 + \delta$ (where $\delta > 0$) satisfying the initial conditions

$$x_1 (t_0) = x_1^0, \ldots, x_n (t_0) = x_n^0. \qquad (6)$$

Furthermore, the solution is unique, so that if

$$x_1 = x_1^* (t), \ldots, x_n = x_n^* (t) \qquad (7)$$

is a second solution of (◇) for $t_0 - \delta < t < t_0 + \delta$ satisfying (◇),
then $x_i (t) \equiv x_i^* (t)$ for $t_0 - \delta < t < t_0 + \delta$. Because every nth-order ODE can be expressed as a system of n first-order ODEs, this theorem also applies to the single nth-order ODE.

An exact first-order ordinary differential equation is one of the form

$$p (x, y) \, d x + q (x, y) \, d y = 0, \qquad (8)$$

where

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \qquad (9)$$

An equation of the form (◇) with

$$\frac{\partial p}{\partial y} \neq \frac{\partial q}{\partial x} \qquad (10)$$

is said to be nonexact. If

$$\frac{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}{q} = f(x) \qquad (11)$$

in (◇), it has an x-dependent integrating factor. If

$$\frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{x\,p - y\,q} = f(x\,y) \qquad (12)$$

in (◇), it has an $x\,y$-dependent integrating factor. If

$$\frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{p} = f(y) \qquad (13)$$

in (◇), it has a y-dependent integrating factor.

Other special first-order types include cross multiple equations

$$y\,f(x\,y)\,d\,x + x\,g(x\,y)\,d\,y = 0, \qquad (14)$$

homogeneous equations

$$y' = f\left(\frac{y}{x}\right), \qquad (15)$$

linear equations

$$y' + p(x)\, y = q(x), \qquad (16)$$

and separable equations

$$y' = X(x)\, Y(y). \qquad (17)$$

Special classes of second-order ordinary differential equations include

$$y'' = f(y, y') \qquad (18)$$

(x missing) and

$$y'' = f(x, y') \qquad (19)$$

(y missing). A second-order linear homogeneous ODE

$$y'' + P(x)\, y' + Q(x)\, y = 0 \qquad (20)$$

for which

$$\frac{Q'(x) + 2\, P(x)\, Q(x)}{2\,[Q(x)]^{3/2}} = [\text{constant}] \qquad (21)$$

can be transformed to one with constant coefficients.

The undamped equation of simple harmonic motion is

$$y'' + \omega_0^2 y = 0, \tag{22}$$

which becomes

$$y'' + \beta y' + \omega_0^2 y = 0 \tag{23}$$

when damped, and

$$y'' + \beta y' + \omega_0^2 y = A \cos(\omega t + \delta) \tag{24}$$

when both forced and damped.

Systems with constant coefficients are of the form

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\,\mathbf{x}(t) + \mathbf{p}(t). \tag{25}$$

The following are examples of important ordinary differential equations which commonly arise in problems of mathematical physics.

Abel's differential equation

$$y' = f_0(x) + f_1(x)\,y + f_2(x)\,y^2 + f_3(x)\,y^3 + \ldots \tag{26}$$

$$[g_0(x) + g_1(x)\,y]\,y' = f_0(x) + f_1(x)\,y + f_2(x)\,y^2 + f_3(x)\,y^3. \tag{27}$$

Airy differential equation

$$y'' - x\,y = 0. \tag{28}$$

Anger differential equation

$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = \frac{x - \nu}{\pi x^2} \sin(\nu\pi). \tag{29}$$

Baer differential equations

$$(x - a_1)(x - a_2) y'' + \tfrac{1}{2}[2x - (a_1 + a_2)] y' - \left(p^2 x + q^2\right) y = 0, \tag{30}$$

$$(x - a_1)(x - a_2) y'' + \tfrac{1}{2}[2x - (a_1 + a_2)] y' - \left(k^2 x^2 - p^2 x + q^2\right) y = 0. \tag{31}$$

Bernoulli differential equation

$$y' + p(x) y = q(x) y^n. \tag{32}$$

Bessel differential equation

$$x^2 y'' + x y' + \left(\lambda^2 x^2 - n^2\right) y = 0. \tag{33}$$

Binomial differential equation

$$(y')^m = f(x, y). \tag{34}$$

Bôcher equation

$$y'' + \frac{1}{2}\left[\frac{m_1}{x - a_1} + \ldots + \frac{m_{n-1}}{x - a_{n-1}}\right] y'$$
$$+ \frac{1}{4}\left[\frac{A_0 + A_1 x + \ldots + A_l x^l}{(x - a_1)^{m_1} (x - a_2)^{m_2} \cdots (x - a_{n-1})^{m_{n-1}}}\right] y = 0. \tag{35}$$

Briot-Bouquet equation

$$x^m y' = f(x, y). \quad (36)$$

Chebyshev differential equation

$$\left(1 - x^2\right) y'' - x y' + \alpha^2 y = 0. \quad (37)$$

Clairaut's differential equation

$$y = x y' + f(y'). \quad (38)$$

Confluent hypergeometric differential equation

$$x y'' + (c - x) y' - a y = 0. \quad (39)$$

d'Alembert's equation

$$y = x f(y') + g(y'). \quad (40)$$

Duffing differential equation

$$y'' + \omega_0^2 y + \beta y^3 = 0. \quad (41)$$

Eckart differential equation

$$y'' + \left[\frac{\alpha\eta}{1+\eta} + \frac{\beta\eta}{(1+\eta)^2} + \gamma\right] y = 0, \quad (42)$$

where $\eta = e^{\delta x}$.

Emden-Fowler differential equation

$$(x^p\, y')' \pm x^\sigma\, y^n = 0. \tag{43}$$

Euler differential equation

$$x^2\, y'' + a\, x\, y' + b\, y = S(x). \tag{44}$$

Halm's differential equation

$$\left(1 + x^2\right)^2 + y'' + \lambda\, y = 0. \tag{45}$$

Hermite differential equation

$$y'' - 2\, x\, y' + \lambda\, y = 0. \tag{46}$$

Heun's differential equation

$$w'' + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) w' + \frac{\alpha\beta\, x - q}{x\,(x-1)\,(x-a)}\, w = 0, \tag{47}$$

where $w' = d\,w/d\,x$.

Hill's differential equation

$$y'' + \left[\theta_0 + 2 \sum_{n=1}^{\infty} \theta_n \cos(2\, n\, z)\right] y = 0. \tag{48}$$

Hypergeometric differential equation

$$x\,(x-1)\, y'' + [(1 + \alpha + \beta)\, x - \gamma]\, y' + \alpha\beta\, y = 0. \tag{49}$$

Jacobi differential equation

$$\left(1-x^2\right) y'' + [\beta - \alpha - (\alpha + \beta + 2) x] y' + n (n + \alpha + \beta + 1) y = 0. \quad (50)$$

Laguerre differential equation

$$x y'' + (1-x) y' + \lambda y = 0. \quad (51)$$

Lamé's differential equation

$$\left(x^2 - b^2\right)\left(x^2 - c^2\right) z'' + x\left(x^2 - b^2 + x^2 - c^2\right) z' - \left[m (m+1) x^2 - \left(b^2 + c^2\right) p\right] z = 0, \quad (52)$$

where $z' = dz/dx$.

Lane-Emden differential equation

$$\frac{1}{\xi^2} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0. \quad (53)$$

Legendre differential equation

$$\left(1-x^2\right) y'' - 2 x y' + \alpha (\alpha + 1) y = 0. \quad (54)$$

Linear constant coefficients

$$a_0 y^{(n)} + \ldots + a_{n-1} y' + a_n y = p(x). \quad (55)$$

Lommel differential equation

$$x^2 y'' + x y' - \left(x^2 + \nu^2\right) y = k x^{\mu+1}. \quad (56)$$

Löwner's differential equation

$$y' = -y\,\frac{1+\kappa(x)\,y}{1-\kappa(x)\,y}. \tag{57}$$

Malmstén's differential equation

$$y'' + \frac{r}{z}\,y' = \left(A\,z^m + \frac{s}{z^2}\right)y. \tag{58}$$

Mathieu differential equation

$$V'' + [a - 2\,q\cos(2\,v)]\,V = 0, \tag{59}$$

where $V' = d\,V/d\,v$.

Modified Bessel differential equation

$$x^2\,y'' + x\,y' - \left(x^2 + n^2\right)y = 0. \tag{60}$$

Modified spherical Bessel differential equation

$$r^2\,R'' + 2\,r\,R' - \left[k^2\,r^2 + n\,(n+1)\right]R = 0, \tag{61}$$

where $R' = d\,R/d\,r$

Rayleigh differential equation

$$y'' - \mu\left(1 - \tfrac{1}{3}\,y'^2\right)y' + y = 0. \tag{62}$$

Riccati differential equation

$$w' = q_0\,(x) + q_1\,(x)\,w + q_2\,(x)\,w^2. \tag{63}$$

Riemann P-Differential Equation

$$u'' + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right]u' + \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right] \frac{u}{(z-a)(z-b)(z-c)} = 0, \quad (64)$$

where $u' = du/dz$.

Sharpe's differential equation

$$z y'' + y' + (z + A) y = 0. \quad (65)$$

Spherical Bessel differential equation

$$r^2 R'' + 2 r R' + \left[k^2 r^2 - n(n+1)\right] R = 0, \quad (66)$$

where $R' = dR/dr$.

Struve differential equation

$$z^2 y'' + z y' + \left(z^2 - \nu^2\right) y = \frac{4\left(\frac{1}{2} z\right)^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)}. \quad (67)$$

Sturm-Liouville equation

$$\frac{d}{dx}\left[p(x) y'\right] + \left[\lambda w(x) - q(x)\right] y = 0. \quad (68)$$

Gegenbauer differential equation

$$\left(1-x^2\right)y''-(2\alpha+1)\,x\,y'+n\,(n+2\,\alpha)\,y=0. \quad (69)$$

van der Pol equation

$$y''-\mu\left(1-y^2\right)y'+y=0. \quad (70)$$

Weber differential equation

$$y''+\left(n+\tfrac{1}{2}-\tfrac{1}{4}\,z^2\right)y=0, \quad (71)$$

where $y'=d\,y/d\,z$.

Whittaker differential equation

$$u''+u'+\left(\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right)u=0, \quad (72)$$

where $u'=d\,u/d\,z$.

FIRST ORDER DIFFERENTIAL EQUATIONS:

Given a first-order ordinary differential equation

$$\frac{d\,y}{d\,x}=F(x,y), \quad (1)$$

if $F(x,y)$ can be expressed using separation of variables as

$$F(x,y)=X(x)\,Y(y), \quad (2)$$

then the equation can be expressed as

$$\frac{dy}{Y(y)} = X(x)\,dx \tag{3}$$

and the equation can be solved by integrating both sides to obtain

$$\int \frac{dy}{Y(y)} = \int X(x)\,dx. \tag{4}$$

Any first-order ODE of the form

$$\frac{dy}{dx} + p(x)\,y = q(x) \tag{5}$$

can be solved by finding an integrating factor $\mu = \mu(x)$ such that

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + y \frac{d\mu}{dx} \tag{6}$$

$$= \mu q(x). \tag{7}$$

Dividing through by μy yields

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{\mu}\frac{d\mu}{dx} = \frac{q(x)}{y}. \tag{8}$$

However, this condition enables us to explicitly determine the appropriate μ for arbitrary p and q. To accomplish this, take

$$p(x) = \frac{1}{\mu}\frac{d\mu}{dx} \tag{9}$$

in the above equation, from which we recover the original equation (◇), as required, in the form

$$\frac{1}{y}\frac{dy}{dx}+p(x)=\frac{q(x)}{y}. \tag{10}$$

But we can integrate both sides of (9) to obtain

$$\int p(x)\,dx=\int\frac{d\mu}{\mu}=\ln\mu+c \tag{11}$$

$$\mu=e^{\int p(x)\,dx}. \tag{12}$$

Now integrating both sides of (◇) gives

$$\mu y=\int\mu q(x)\,dx+c \tag{13}$$

(with μ now a known function), which can be solved for y to obtain

$$y=\frac{\int\mu q(x)\,dx+c}{\mu}=\frac{\int e^{\int^x p(x')\,dx'}q(x)\,dx+c}{e^{\int^x p(x')\,dx'}}, \tag{14}$$

where c is an arbitrary constant of integration.

Given an nth-order linear ODE with constant coefficients

$$\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1\frac{dy}{dx}+a_0 y=Q(x), \tag{15}$$

first solve the characteristic equation obtained by writing

$$y\equiv e^{rx} \tag{16}$$

and setting $Q(x)=0$ to obtain the n complex roots.

$$r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \ldots + a_1 r e^{rx} + a_0 e^{rx} = 0 \quad (17)$$

$$r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0. \quad (18)$$

Factoring gives the roots r_i,

$$(r - r_1)(r - r_2) \cdots (r - r_n) = 0. \quad (19)$$

For a nonrepeated real root r, the corresponding solution is

$$y = e^{rx}. \quad (20)$$

If a real root r is repeated k times, the solutions are degenerate and the linearly independent solutions are

$$y = e^{rx}, y = x e^{rx}, \ldots, y = x^{k-1} e^{rx}. \quad (21)$$

Complex roots always come in complex conjugate pairs, $r_\pm = a \pm i b$. For nonrepeated complex roots, the solutions are

$$y = e^{ax} \cos(b x), y = e^{ax} \sin(b x). \quad (22)$$

If the complex roots are repeated k times, the linearly independent solutions are

$$y = e^{ax} \cos(b x), y = e^{ax} \sin(b x), \ldots, y = x^{k-1} e^{ax} \cos(b x), y = x^{k-1} e^{ax} \sin(b x). \quad (23)$$

Linearly combining solutions of the appropriate types with arbitrary multiplicative constants then gives the complete solution. If initial conditions are specified, the constants can be explicitly determined. For example, consider the sixth-order linear ODE

$$\left(\tilde{D}-1\right)\left(\tilde{D}-2\right)^3\left(\tilde{D}^2+\tilde{D}+1\right)y=0, \tag{24}$$

which has the characteristic equation

$$(r-1)(r-2)^3\left(r^2+r+1\right)=0. \tag{25}$$

The roots are 1, 2 (three times), and $\left(-1\pm\sqrt{3}\ i\right)/2$, so the solution is

$$y=A\,e^x+B\,e^{2x}+C\,x\,e^{2x}+D\,x^2\,e^{2x}+E\,e^{-x/2}\cos\left(\tfrac{1}{2}\sqrt{3}\ x\right)+F\,e^{-x/2}\sin\left(\tfrac{1}{2}\sqrt{3}\ x\right). \tag{26}$$

If the original equation is nonhomogeneous ($Q(x)\neq 0$), now find the particular solution y^* by the method ofvariation of parameters. The general solution is then

$$y(x)=\sum_{i=1}^{n}c_i\,y_i(x)+y^*(x), \tag{27}$$

where the solutions to the linear equations are $y_1(x)$, $y_2(x)$, ..., $y_n(x)$, and $y^*(x)$ is the particular solution.

SECOND ORDER DIFFERENTIAL EQUATIONS:

An ordinary differential equation of the form

$$y''+P(x)\,y'+Q(x)\,y=0. \tag{1}$$

Such an equation has singularities for finite $x=x_0$ under the following conditions: (a) If either $P(x)$ or $Q(x)$diverges as $x\to x_0$, but $(x-x_0)\,P(x)$ and $(x-x_0)^2\,Q(x)$ remain finite as $x\to x_0$, then x_0 is called a regular or nonessential singular point. (b) If $P(x)$ diverges faster than $(x-x_0)^{-1}$ so that $(x-x_0)\,P(x)\to\infty$ as $x\to x_0$, or $Q(x)$ diverges faster than $(x-x_0)^{-2}$ so that $(x-x_0)^2\,Q(x)\to\infty$ as $x\to x_0$, then x_0 is called an irregular or essential singularity.

Singularities of equation (1) at infinity are investigated by making the substitution $x \equiv z^{-1}$, so $dx = -z^{-2}\,dz$, giving

$$\frac{dy}{dx} = -z^2 \frac{dy}{dz} \tag{2}$$

$$\frac{d^2 y}{dx^2} = -z^2 \frac{d}{dz}\left(-z^2 \frac{dy}{dz}\right) \tag{3}$$

$$= -z^2\left(-2z\frac{dy}{dz} - z^2 \frac{d^2 y}{dz^2}\right) \tag{4}$$

$$= 2z^3 \frac{dy}{dz} + z^4 \frac{d^2 y}{dz^2}. \tag{5}$$

Then (3) becomes

$$z^4 \frac{d^2 y}{dz^2} + \left[2z^3 - z^2 P\left(z^{-1}\right)\right]\frac{dy}{dz} + Q\left(z^{-1}\right) y = 0. \tag{6}$$

Case (a): If

$$\alpha(z) \equiv \frac{2z - P\left(z^{-1}\right)}{z^2} \tag{7}$$

$$\beta(z) \equiv \frac{Q\left(z^{-1}\right)}{z^4} \tag{8}$$

remain finite at $x = \pm\infty$ ($z = 0$), then the point is ordinary. Case (b): If either $\alpha(z)$ diverges no more rapidly than $1/z$ or $\beta(z)$ diverges no more rapidly than $1/z^2$, then the point is a regular singular point. Case (c): Otherwise, the point is an irregular singular point.

Morse and Feshbach (1953, pp. 667-674) give the canonical forms and solutions for second-order ordinary differential equations classified by types of singular points.

For special classes of linear second-order ordinary differential equations,
variable coefficients can be transformed into constant coefficients. Given a second-order linear ODE with variable coefficients

$$\frac{d^2 y}{d x^2} + p(x) \frac{d y}{d x} + q(x) y = 0. \qquad (9)$$

Define a function $z \equiv y(x)$,

$$\frac{d y}{d x} = \frac{d z}{d x} \frac{d y}{d z} \qquad (10)$$

$$\frac{d^2 y}{d x^2} = \left(\frac{d z}{d x}\right)^2 \frac{d^2 y}{d z^2} + \frac{d^2 z}{d x^2} \frac{d y}{d z} \qquad (11)$$

$$\left(\frac{d z}{d x}\right)^2 \frac{d^2 y}{d z^2} + \left[\frac{d^2 z}{d x^2} + p(x) \frac{d z}{d x}\right] \frac{d y}{d z} + q(x) y = 0 \qquad (12)$$

$$\frac{d^2 y}{d z^2} + \left[\frac{\frac{d^2 z}{d x^2} + p(x) \frac{d z}{d x}}{\left(\frac{d z}{d x}\right)^2}\right] \frac{d y}{d z} + \left[\frac{q(x)}{\left(\frac{d z}{d x}\right)^2}\right] y \qquad (13)$$

$$\equiv \frac{d^2 y}{d z^2} + A \frac{d y}{d z} + B y = 0. \qquad (14)$$

This will have constant coefficients if A and B are not functions of x. But we are free to set B to an arbitrary positiveconstant for $q(x) \geq 0$ by defining z as

$$z \equiv B^{-1/2} \int [q(x)]^{1/2} \, d x. \qquad (15)$$

Then

$$\frac{d z}{d x} = B^{-1/2} [q(x)]^{1/2} \qquad (16)$$

$$\frac{d^2 z}{d x^2} = \frac{1}{2} B^{-1/2} [q(x)]^{-1/2} q'(x), \tag{17}$$

and

$$A = \frac{\frac{1}{2} B^{-1/2} [q(x)]^{-1/2} q'(x) + B^{-1/2} p(x) [q(x)]^{1/2}}{B^{-1} q(x)} \tag{18}$$

$$= \frac{q'(x) + 2 p(x) q(x)}{2 [q(x)]^{3/2}} B^{1/2}. \tag{19}$$

Equation (◇) therefore becomes

$$\frac{d^2 y}{d z^2} + \frac{q'(x) + 2 p(x) q(x)}{2 [q(x)]^{3/2}} B^{1/2} \frac{d y}{d z} + B y = 0, \tag{20}$$

which has constant coefficients provided that

$$A \equiv \frac{q'(x) + 2 p(x) q(x)}{2 [q(x)]^{3/2}} B^{1/2} = [\text{constant}]. \tag{21}$$

Eliminating constants, this gives

$$A' \equiv \frac{q'(x) + 2 p(x) q(x)}{[q(x)]^{3/2}} = [\text{constant}]. \tag{22}$$

So for an ordinary differential equation in which A' is a constant, the solution is given by solving the second-order linear ODE with constant coefficients

$$\frac{d^2 y}{d z^2} + A \frac{d y}{d z} + B y = 0 \tag{23}$$

for z, where z is defined as above.

A linear second-order homogeneous differential equation of the general form

$$y'' + P(x)\, y' + Q(x)\, y = 0 \tag{24}$$

can be transformed into standard form

$$z'' + q(x)\, z = 0 \tag{25}$$

with the first-order term eliminated using the substitution

$$\ln y \equiv \ln z - \tfrac{1}{2}\int P(x)\, dx. \tag{26}$$

Then

$$\frac{y'}{y} = \frac{z'}{z} - \tfrac{1}{2} P(x) \tag{27}$$

$$\frac{y\, y'' - y'^2}{y^2} = \frac{z\, z'' - z'^2}{z^2} - \tfrac{1}{2} P'(x) \tag{28}$$

$$\frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2} P'(x) \tag{29}$$

$$\frac{y''}{y} = \left[\frac{z'}{z} - \tfrac{1}{2} P(x)\right]^2 + \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2} P'(x) \tag{30}$$

$$= \frac{z'^2}{z^2} - \frac{z'}{z} P(x) + \tfrac{1}{4} P^2(x) + \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2} P'(x), \tag{31}$$

so

$$\frac{y''}{y} + P(x)\frac{y'}{y} + Q(x) = -\frac{z'}{z} P(x) + \tfrac{1}{4} P^2(x) + \frac{z''}{z} - \tfrac{1}{2} P'(x) + P(x)\left[\frac{z'}{z} - \tfrac{1}{2} P(x)\right] + Q(x) \tag{32}$$

$$= \frac{z''}{z} - \tfrac{1}{2} P'(x) - \tfrac{1}{4} P^2(x) + Q(x) = 0. \tag{33}$$

Therefore,

$$z'' + \left[Q(x) - \tfrac{1}{2} P'(x) - \tfrac{1}{4} P^2(x)\right] z \equiv z''(x) + q(x) z = 0, \tag{34}$$

where

$$q(x) \equiv Q(x) - \tfrac{1}{2} P'(x) - \tfrac{1}{4} P^2(x). \tag{35}$$

If $Q(x) = 0$, then the differential equation becomes

$$y'' + P(x) y' = 0, \tag{36}$$

which can be solved by multiplying by

$$\exp\left[\int^x P(x')\, dx'\right] \tag{37}$$

to obtain

$$0 = \frac{d}{dx}\left\{\exp\left[\int^x P(x')\, dx'\right] \frac{dy}{dx}\right\} \tag{38}$$

$$c_1 = \exp\left[\int^x P(x')\, dx'\right] \frac{dy}{dx} \tag{39}$$

$$y = c_1 \int^x \frac{dx}{\exp\left[\int^x P(x')\, dx'\right]} + c_2. \tag{40}$$

For a nonhomogeneous second-order ordinary differential equation in which the x term does not appear in the function $f(x, y, y')$,

$$\frac{d^2 y}{d x^2} = f(y, y'), \tag{41}$$

let $v \equiv y'$, then

$$\frac{d v}{d x} = f(v, y) = \frac{d v}{d y}\frac{d y}{d x} = v\frac{d v}{d y}. \tag{42}$$

So the first-order ODE

$$v\frac{d v}{d y} = f(y, v), \tag{43}$$

if linear, can be solved for v as a linear first-order ODE. Once the solution is known,

$$\frac{d y}{d x} = v(y) \tag{44}$$

$$\int \frac{d y}{v(y)} = \int d x. \tag{45}$$

On the other hand, if y is missing from $f(x, y, y')$,

$$\frac{d^2 y}{d x^2} = f(x, y'), \tag{46}$$

let $v \equiv y'$, then $v' = y''$, and the equation reduces to

$$v' = f(x, v), \tag{47}$$

which, if linear, can be solved for v as a linear first-order ODE. Once the solution is known,

$$y = \int v(x)\, d x. \tag{48}$$

Nonhomogeneous ordinary differential equations can be solved if the general solution to the homogenous version is known, in which case variation of parameters can be used to find the particular solution. In particular, the particular solution $y^*(x)$ to a nonhomogeneous second-order ordinary differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{49}$$

can be found using variation of parameters to be given by the equation

$$y^*(x) = -y_1(x)\int \frac{y_2(x)\,g(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)\,g(x)}{W(x)}\,dx, \tag{50}$$

where $y_1(x)$ and $y_2(x)$ are the homogeneous solutions to the unforced equation

$$y'' + p(x)y' + q(x)y = 0 \tag{51}$$

and $W(x)$ is the Wronskian of these two functions.

PARTIAL DIFFERENTIAL EQUATIONS:

A partial differential equation (PDE) is an equation involving functions and their partial derivatives; for example, thewave equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}. \tag{1}$$

Some partial differential equations can be solved exactly in the Wolfram Language using DSolve[*eqn*, *y*, {*x1*, *x2*}], and numerically using NDSolve[*eqns*, *y*, {*x*, *xmin*, *xmax*}, {*t*, *tmin*, *tmax*}].

In general, partial differential equations are much more difficult to solve analytically than are ordinary differential equations. They may sometimes be solved using a Bäcklund transformation, characteristics, Green's function,integral transform, Lax pair, separation of variables, or--when all else fails (which it frequently does)--numerical methods such as finite differences.

Fortunately, partial differential equations of second-order are often amenable to analytical solution. Such PDEs are of the form

$$A\, u_{xx} + 2\, B\, u_{xy} + C\, u_{yy} + D\, u_x + E\, u_y + F = 0. \qquad (2)$$

Linear second-order PDEs are then classified according to the properties of the matrix

$$\mathsf{Z} \equiv \begin{bmatrix} A & B \\ B & C \end{bmatrix} \qquad (3)$$

as elliptic, hyperbolic, or parabolic.

If Z is a positive definite matrix, i.e., $\det(\mathsf{Z}) > 0$, the PDE is said to be elliptic. Laplace's equation and Poisson's equation are examples. Boundary conditions are used to give the constraint $u(x, y) = g(x, y)$ on $\partial\Omega$, where

$$u_{xx} + u_{yy} = f\left(u_x, u_y, u, x, y\right) \qquad (4)$$

holds in Ω.

If det$(\mathsf{Z}) < 0$, the PDE is said to be hyperbolic. The wave equation is an example of a hyperbolic partial differential equation. Initial-boundary conditions are used to give

$$u(x, y, t) = g(x, y, t) \text{ for } x \in \partial\Omega,\ t > 0 \qquad (5)$$

$$u(x, y, 0) = v_0(x, y) \text{ in } \Omega \quad (6)$$

$$u_t(x, y, 0) = v_1(x, y) \text{ in } \Omega, \quad (7)$$

where

$$u_{xy} = f(u_x, u_t, x, y) \quad (8)$$

holds in Ω.

If det $(Z) = 0$, the PDE is said to be parabolic. The heat conduction equation equation and other diffusion equations are examples. Initial-boundary conditions are used to give

$$u(x, t) = g(x, t) \text{ for } x \in \partial\Omega, t > 0 \quad (9)$$

$$u(x, 0) = v(x) \text{ for } x \in \Omega, \quad (10)$$

where

$$u_{xx} = f\left(u_x, u_y, u, x, y\right) \quad (11)$$

holds in Ω.

The following are examples of important partial differential equations that commonly arise in problems of mathematical physics.

Benjamin-Bona-Mahony equation

$$u_t + u_x + u\,u_x - u_{xxt} = 0. \quad (12)$$

Biharmonic equation

$$\nabla^4 \phi = 0. \qquad (13)$$

Boussinesq equation

$$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}. \qquad (14)$$

Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (15)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \qquad (16)$$

Chaplygin's equation

$$u_{xx} + \frac{y^2}{1 - \frac{y^2}{c^2}} u_{yy} + y\, u_y = 0. \qquad (17)$$

Euler-Darboux equation

$$u_{xy} + \frac{\alpha\, u_x - \beta\, u_y}{x - y} = 0. \qquad (18)$$

Heat conduction equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T. \qquad (19)$$

Helmholtz differential equation

$$\nabla^2 \psi + k^2 \psi = 0. \qquad (20)$$

Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial x^2}-\mu^2\psi. \qquad (21)$$

Korteweg-de Vries-Burgers equation

$$u_t+2\,u\,u_x-\nu\,u_{xx}+\mu\,u_{xxx}=0. \qquad (22)$$

Korteweg-de Vries equation

$$u_t+u_{xxx}-6\,u\,u_x=0. \qquad (23)$$

Krichever-Novikov equation

$$\frac{u_t}{u_x}=\frac{1}{4}\frac{u_{xxx}}{u_x}-\frac{3}{8}\frac{u_{xx}^2}{u_x^2}+\frac{3}{2}\frac{p(u)}{u_x^2}, \qquad (24)$$

where

$$p(u)=\tfrac{1}{4}\left(4u^3-g_2\,u-g_3\right). \qquad (25)$$

Laplace's equation

$$\nabla^2\psi=0. \qquad (26)$$

Lin-Tsien equation

$$2\,u_{tx}+u_x\,u_{xx}-u_{yy}=0. \qquad (27)$$

Sine-Gordon equation

$$v_{tt} - v_{xx} + \sin v = 0. \quad (28)$$

Spherical harmonic differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right]u = 0. \quad (29)$$

Tricomi equation

$$u_{yy} = y\, u_{xx}. \quad (30)$$

Wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}. \quad (31)$$

GROUP:

Suppose that we take an equilateral triangle and look at its symmetry group. There are two obvious sets of symmetries. First one can rotate the triangle through 120° . Suppose that we choose clockwise as the positive direction and denote rotation through 120° as R. It is natural to represent rotation through 240° as R^2, where we think of R^2 as the effect of applying R twice.

If we apply R three times, represented by R^3, we would be back where we started. In other words we ought to include the trivial symmetry I, as a symmetry of the triangle (in just the same way that we think of zero as being a number). Note that the symmetry rotation through 120° anticlockwise, could be represented as R^{-1} . Of course this is the same as rotation through 240° clockwise, so that $R^{-1} = R^2$.

The other obvious sets of symmetries are flips. For example one can draw a vertical line through the top corner and flip about this line. Call this operation $F = F_1$. Note that $F^2 = I$, representing the fact that flipping twice does nothing.

There are two other axes to flip about, corresponding to the fact that there are three corners. Putting all this together we have

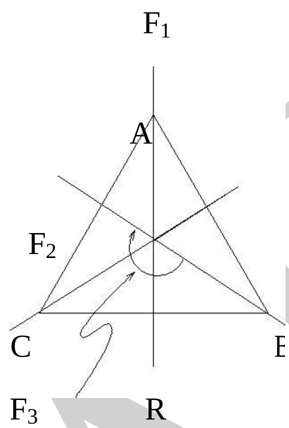

Figure 1. Symmetries of an equilateral triangle

The set of symmetries we have created so far is then equal to

$$\{ I, R, R^2, F_1, F_2, F_3 \}.$$

Is this all? The answer is yes, and it is easy to see this, once one notices the following fact; any symmetry is determined by its action on the vertices of the triangle. In fact a triangle is determined by its vertices, so this is clear. Label the vertices A, B and C, where A starts at the top, B is the bottom right, and C is the bottom left.

Now in total there are at most six different permutations of the letters A, B and C. We have already given six different symmetries, so we must in fact have exhausted the list of symmetries.

Note that given any two symmetries, we can always consider what happens when we apply first one symmetry and then another. However note that the notation RF is ambiguous. Should we apply R first and then F or F first and then R? We will adopt the convention that RF means first apply F and then apply R.

Now RF is a symmetry of the triangle and we have listed all of them. Which one is it? Well the action of RF on the vertices will take

$$A \longrightarrow A \longrightarrow B$$

$$B \longrightarrow C \longrightarrow A$$

$$C \longrightarrow B \longrightarrow C.$$

In total then A is sent to B, B is sent to A and C is sent to C. As this symmetry fixes one of the vertices, it must be a flip. In fact it is equal to F_3.

Let us now compute the symmetry FR. Well the action on the vertices is as follows

$$A \longrightarrow B \longrightarrow C$$

$$B \longrightarrow C \longrightarrow B$$

$$C \longrightarrow A \longrightarrow A.$$

So in total the action on the vertices is given as A goes to C, B goes to B and C goes to A. Again this symmetry fixes the vertex B and so it is equal to F_2.

Thus $F_3 = RF = FR = F_2$.

Let us step back a minute and consider what (algebraic) structure these examples give us. We are given a set (the set of symmetries) and an operation on this set, that is a rule that tells us how to multiply (in a formal sense) any two elements. We have an identity (the symmetry that does nothing). As this symmetry does nothing, composing with this symmetry does nothing (just as multiplying by the number one does nothing).

Finally, given any symmetry there is an inverse symmetry which undoes the action of the symmetry (R represents rotation through 120° clockwise, and R^{-1} represents rotation through 120° anticlockwise, thus undoing the action of R).

Definition 1.1. A group G is a set together with two operations (or more simply, functions), one called multiplication $m: G \times G \longrightarrow G$ and the other called the inverse $i : G \longrightarrow G$. These operations obey the following rules

(1) Associativity: For every g, h and k ∈ G,

$$m(m(g, h), k) = m(g, m(h, k)).$$

(2) Identity: There is an element e in the group such that for every g ∈ G

$$m(g, e) = g$$

and

$$m(e, g) = g.$$

(3) Inverse: For every g ∈ G,

$$m(g, i(g)) = e = m(i(g), g).$$

It is customary to use different (but equivalent) notation to denote the operations of multiplication and inverse. One possibility is to use the ordinary notation for multiplication

$$m(x, y) = xy.$$

The inverse is then denoted

$$i(g) = g^{-1}.$$

The three rules above will then read as follows

(1)

$$(gh)k = g(hk).$$

(2)

$$ge = g = eg$$

(3)

$$gg^{-1} = eg^{-1} g.$$

Another alternative is to introduce a slight different notation for the multiplication rule, something like *. In this case the three rules come out as

(1)

$$(g * h) * k = g * (h * k).$$

(2)

$$g * e = g = e * g$$

(3)

$$g * g^{-1} = e = g^{-1} * g.$$

The key thing to realise is that the multiplication rule need not have any relation to the more usual mutliplication rule of ordinary numbers.

Let us see some examples of groups. Can we make the empty set into a group? How would we define the multiplication? Well the answer is that there is nothing to define, we just get the empty map. Is this empty map associative? The answer is yes, since there is nothing to check. Does there exist an identity? No, since the empty set does not have any elements at all.

Thus there is no group whose underlying set is empty.

Now suppose that we take a set with one element, call it a. The definition of the multiplication rule is obvious. We only need to know how to multiply a with a,

$$m(a, a) = aa = a^2 = a * a = a.$$

Is this multiplication rule associative? Well suppose that g, h and k are three elements of G. Then g = h = k = a. We compute the LHS,

$$m(m(a, a), a) = m(a, a) = a.$$

Similarly the RHS is

$$m(a, m(a, a)) = m(a, a) = a.$$

These two are equal and so this multiplication rule is associative. Is their an identity? Well there is only one element of the group, a. We have to check that if we multiply e = a by any other element g of the group then we get back g. The only possible choice for g is a.

$$m(g, e) = m(a, a) = a = g,$$

and

$$m(e, g) = m(a, a) = a = g.$$

So a acts as an identity. Finally does every element have an inverse?

Pick an element g of the group G. In fact g = a. The only possiblity for an inverse of g is a.

$$m(g, g^{-1}) = m(a, a) = a = e.$$

Similarly

$$g^{-1} g = aa = a = e.$$

So there is a unique rule of multiplication for a set with one element, and with this law of multiplication we get a group.

Consider the set {a, b} and define a multiplication rule by

$$aa = a \quad ab = b$$
$$ba = b \quad bb = a$$

Here a plays the role of the identity. a and b are their own inverses. It is not hard to check that associativity holds and that we therefore get a group.

To see some more examples of groups, it is first useful to prove a general result about associativity.

Lemma 1.2. Let $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h: C \longrightarrow D$ be three functions.

Then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof. Both the LHS and RHS are functions from $A \longrightarrow D$. To prove that two such functions are equal, it suffices to prove that they give the same value, when applied to any element $a \in A$.

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a)))$$

Similarly

$$((h \circ g) \circ f))(a) = (h \circ g)(f(a)) = h(g(f(a))).$$ D

The set $\{I, R, R^2, F_1, F_2, F_3\}$ is a group, where the multiplication rule is composition of symmetries. Any symmetry, can be interpreted as a function $R^2 \longrightarrow R^2$, and composition of symmetries is just composition of functions. Thus this rule of multiplication is associative by (1.2).

I plays the role an identity. Since we can undo any symmetry, every element of the group has an inverse.

Definition 1.3. The dihedral group D_n of order n is the group of symmetries of a regular n-gon.

With this notation, D_3 is the group above, the set of symmetries of an equilateral triangle. The same proof as above shows that D_n is a group.

Definition 1.4. We say that a group G is abelian, if for every g and h in G,

$$gh = hg.$$

The groups with one or two elements above are abelian. However D_3 as we have already seen is not abelian. Thus not every group is abelian.

Consider the set of whole numbers W = {1, 2, . . . } under addition. Is this a group?
Lemma 1.5. Addition and multiplication of complex number is asso- ciative.

Proof. Well-known.

So addition of whole numbers is certainly associative. Is there an identity? No. So W is not a group under addition, since there is no identity.

How about if we enlarge this set by adding 0, to get the of natural numbers N? In this case there is an identity, but there are no inverses. For example 1 has no inverse, since if you add a non-negative number to 1 you get something at least one.

On the other hand (Z, +) is a group under addition, where Z is the set of integers. Similiarly Q, R, C are all groups under addition.

How about under multiplication? First how about Z. Multiplication is associative, and there is an identity, one. However not every element has an inverse. For example, 2 does not have an inverse.
What about Q under multiplication? Associativity is okay. Again one plays the role of the identity and it looks like every element has an

inverse. Well not quite, since 0 has no inverse.

Once one removes zero to get Q^* , then we do get a group under multiplication. Similarly R^* and C^* are groups under multiplication.
All of these groups are abelian.

We can create some more interesting groups using these examples. Let $M_{m,n}(C)$ denote m × n matrices, with entries in C. The multipli- cation rule is addition of matrices (that is add corresponding entries). This operation is certainly associative, as this can be checked entry by entry. The zero matrix (that is the matrix with zeroes everywhere) plays the role of the identity.

Given a matric A, the inverse matrix is −A, that is the matrix ob- tained by changing the sign of every entry. Thus $M_{m,n}(C)$ is a group under addition, which is easily seen to be abelian. We can the replace complex numbers by the reals, rationals or integers.

$GL_n(C)$ denotes the set of n × n matrices, with non-zero determi- nant. Multiplication is simply matrix multiplication. We check that this is a group. First note that a matrix corresponds to a (linear) func- tion $R^n \longrightarrow R^n$, and under this identification, matrix multiplication corresponds to composition of functions.

Thus matrix multiplication is associative. The matrix with one's on the main diagonal and zeroes everywhere else is the identity matrix.

SUBGROUP:

Consider the chain of inclusions of groups

$$Z \subset Q \subset R \subset C.$$

where the law of multiplication is ordinary addition.

Then each subset is a group, and the group laws are obviously compatible. That is to say that if you want to add two integers together, it does not matter whether you consider them as integers, rational num- bers, real numbers or complex numbers, the sum always comes out the same.

Definition 2.1. Let G be a group and let H be a subset of G. We say that H is a subgroup of G, if the restriction to H of the rule of multiplication and inverse makes H into a group.

Notice that this definition hides a subtlety. More often than not, the restriction to $H \times H$ of m, the rule of multiplication of G, won't even define a rule of multiplication on H itself, because there is no a priori reason for the product of two elements of H to be an element of H.

For example suppose that G is the set of integers under addition, and H is the set of odd numbers. Then if you take two elements of H and add them, then you never get an element of H, since you will always get an even number.

Similarly, the inverse of H need not be an element of H. For example take H to be the set of natural numbers. Then H is closed under addition (the sum of two positive numbers is positive) but the inverse of every element of H does not lie in H.

Definition 2.2. Let G be a group and let S be subset of G.

We say that S is closed under multiplication, if whenever a and b are in

S, then the product of a and b is in S.

We say that S is closed under taking inverses, if whenever a is in S, then the inverse of a is in S.

For example, the set of even integers is closed under addition and taking inverses. The set of odd integers is not closed under addition (in a big way as it were) and it is closed under inverses. The natural numbers are closed under addition, but not under inverses.

Proposition 2.3. Let H be a non-empty subset of G.

Then H is a subgroup of G iff H is closed under multiplication and taking inverses. Furthermore, the identity element of H is the identity element of G and the inverse of an element of H is equal to the inverse element in G.

If G is abelian then so is H.

Proof. If H is a subgroup of G, then H is closed under multiplication and taking inverses by definition.

So suppose that H is closed under multiplication and taking inverses. Then there is a well defined product on H. We check the axioms of a group for this product.

Associativity holds for free. Indeed to say that the multiplication on H is associative, is to say that for all g, h and k ∈ H, we have

$$(gh)k = g(hk).$$

But g, h and k are elements of G and the associative rule holds in G. Hence equality holds above and multiplication is associative in H.

We have to show that H contains an identity element. As H is non-empty we may pick a ∈ H. As H is closed under taking inverses, $a^{-1} \in H$. But then

$$e = aa^{-1} \in H$$

as H is closed under multiplication. So e ∈ H. Clearly e acts as an identity element in H as it is an identity element in G. Suppose that h ∈ H. Then $h^{-1} \in H$, as H is closed under taking inverses. But h^{-1} is clearly the inverse of h in H as it is the inverse in G.

Finally if G is abelian then H is abelian. The proof follows just like the proof of associativity. D

Example 2.4. (1) The set of even integers is a subgroup of the set of integers under addition. By (2.3) it suffices to show that the even integers are closed under addition and taking inverses, which is clear.

(3) The set of natural numbers is not a subgroup of the group of integers under addition. The natural numbers are not closed under taking inverses.

(4) The set of rotations of a regular n-gon is a subgroup of the group D_n of symmetries of a regular n-gon. By (2.3) it suffices to check that the set of rotations is closed under multiplication

and inverse. Both of these are obvious. For example, suppose that R_1 and R_2 are two rotations, one through θ radians and the

other through φ. Then the product is a rotation through θ + φ. On the other hand the inverse of R_1 is rotation through $2\pi - \theta$.

(5) The group D_n of symmetries of a regular n-gon is a subgroup of the set of invertible two by two matrices, with entries in R. Indeed any symmetry can be interpreted as a matrix. Since we have already seen that the set of symmetries is a group, it is in fact a subgroup.

(5) The following subsets are subgroups.

$$M_{m,n}(Z) \subset M_{m,n}(Q) \subset M_{m,n}(R) \subset M_{m,n}(C).$$

(6) The following subsets are subgroups.

$$GL_n(Q) \subset GL_n(R) \subset GL_n(C).$$

(7) It is interesting to enumerate the subgroups of D_3. At one extreme we have D_3 and at the other extreme we have {I}. Clearly the set of rotations is a subgroup, $\{I, R, R^2\}$. On the other hand $\{I, F_i\}$ forms a subgroup as well, since $F_i^2 = I$. Are these the only subgroups?

Suppose that H is a subgroup that contains R. Then H must contain R^2 and I, since H must contain all powers of R. Sim- ilarly if H contains R^2 , it must contain $R^4 = (R^2)^2$. But

$R^4 = R^3R = R.$

Suppose that in addition H contains a flip. By symmetry, we may suppose that this flip is $F = F_1$. But $RF_1 = F_3$ and $FR = F_2$. So then H would be equal to G.

The final possibility is that H contains two flips, say F_1 and F_2. Now $F_1R = F_2$, so

$$R = F_1^{-1} F_2 = F_1F_2.$$

So if H contains F_1 and F_2 then it is forced to contain R. In this case H = G as before.

Here are some examples, which are less non-trivial.

Definition-Lemma 2.5. Let G be a group and let $g \in G$ be an element of G.

The centraliser of $g \in G$ is defined to be

$$C_g = \{ h \in G \mid hg = gh \}.$$

Then C_g is a subgroup of G.

Proof. By (2.3) it suffices to prove that C_g is closed under multiplication and taking inverses.

Suppose that h and k are two elements of C_g. We show that the product hk is an element of C_g. We have to prove that (hk)g = g(hk).

(hk)g	= h(kg)	by associativity
	= h(gk)	as $k \in C_g$
	= (hg)k	by associativity
	= (gh)k	as $h \in C_g$
	= g(hk)	by associativity.

Thus $hk \in C_g$ and C_g is closed under multiplication.

Now suppose that $h \in G$. We show that the inverse of h is in G. We have to show that $h^{-1}g = gh^{-1}$. Suppose we start with the equality

$$hg = gh.$$

Multiply both sides by h^{-1} on the left. We get

$$h^{-1}(hg) = h^{-1}(gh),$$

so that simplifying we get

$$g = (h^{-1} g)h.$$

Now multiply both sides of this equality by h^{-1} on the right,

$$gh^{-1} = (h^{-1} g)(hh^{-1}).$$

Simplifying we get

$$ghg^{-1} = g^{-1}h,$$

which is what we want. Thus $h^{-1} \in C_g$. Thus C_g is closed under taking inverses and C_g is a subgroup by (2.3). D

Lemma 2.6. Let G be a finite group and let H be a non-empty finite set, closed under multiplication.

Then H is a subgroup of G.

Proof. It suffices to prove that H is closed under taking inverses. Let $a \in H$. If a = e then $a^{-1} = e$ and this is obviously in H. So we may assume that a = e. Consider the powers of a,

$$a, a^2, a^3, \ldots .$$

As H is closed under products, it is obviously closed under powers (by an easy induction argument). As H is finite and this is an infinite sequence, we must get some repetitions, and so for some m and n distinct positive integers

$$a^m = a^n.$$

Possibly switching m and n, we may assume m < n. Multiplying both sides by the inverse a^{-m} of a^m, we get

$$a^{n-m} = e.$$

As a 6==e, n−m 6= 1. Set k = n−m−1. Then k > 0 and $b = a^k \in H$. But

$$ba = a^k a = a^{n-m-1} a = a^{n-m} = e.$$

Similarly

$$ab = e.$$

Thus b is the inverse of a. Thus H is closed under taking inverses and so H is a subgroup of G by (2.3). D

CO-SETS:

Consider the group of integers Z under addition. Let H be the subgroup of even integers. Notice that if you take the elements of H and add one, then you get all the odd elements of Z. In fact if you take the elements of H and add any odd integer, then you get all the odd elements.

On the other hand, every element of Z is either odd or even, and certainly not both (by convention zero is even and not odd), that is, we can partition the elements of Z into two sets, the evens and the odds, and one part of this partition is equal to the original subset H.

Somewhat surprisingly this rather trivial example generalises to the case of an arbitrary group G and subgroup H, and in the case of finite groups imposes rather strong conditions on the size of a subgroup.

To go further, we need to recall some basic facts abouts partitions and equivalence relations.

Definition 3.1. Let X be a set. An equivalence relation $\sim$ is a relation on X, which is

(6) (reflexive) For every $x \in X$, $x \sim x$.

(7) (symmetric) For every x and $y \in X$, if $x \sim y$ then $y \sim x$.

(8) (transitive) For every x and y and $z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

Example 3.2. Let S be any set and consider the relation

$$a \sim b \quad \text{if and only if} \quad a = b.$$

A moments thought will convince the reader this is an equivalence relation.

Let S be the set of people in this room and let

$$a \sim b \quad \text{if and only if a and b have the same colour top.}$$

Then $\sim$ is an equivalence relation.

Let S = R and

$$a \sim b \quad \text{if and only if} \quad a \geq b.$$

Then $\sim$ is reflexive and transitive but not symmetric. It is not an equivalence relation.

Lemma 3.3. Let G be a group and let H be a subgroup. Let $\sim$ be the relation on G defined by the rule

$$a \sim b \quad \text{if and only if} \quad b^{-1} a \in H.$$

Then $\sim$ is an equivalence relation.

Proof. There are three things to check. First we check reflexivity. Suppose that $a \in G$. Then $a^{-1}a = e \in H$, since H is a subgroup. But then $a \sim a$

by definition of ∼ and ∼ is reflexive.

Now we check symmetry. Suppose that a and b are elements of G

and that a ∼ b. Then $b^{-1}a \in H$. As H is closed under taking inverses, $(b^{-1}a)^{-1} \in H$. But

$$(b^{-1}a)^{-1} = a^{-1}(b^{-1})^{-1}$$

$$= a^{-1}b.$$

Thus $a^{-1}b \in H$. But then by definition b ∼ a. Thus ∼ is symmetric.

Finally we check transitivity. Suppose that a ∼ b and b ∼ c.

Then $b^{-1}a \in H$ and $c^{-1}b \in H$. As H is closed under multiplication $(c^{-1}b)(b^{-1}a) \in H$. On the other hand

$$(c^{-1}b)(b^{-1}a) = c^{-1}(bb^{-1})a$$

$$= c^{-1}(ea) = c^{-1}a.$$

Thus $c^{-1}a \in H$. But then a ∼ c and ∼ is transitive.

As ∼ is reflexive, symmetric and transitive, it is an equivalence relation. D

On the other hand if we are given an equivalence relation, the natural thing to do is to look at its equivalence classes.

Definition 3.4. Let ∼ be an equivalence relation on a set X. Let $a \in X$ be an element of X. The equivalence class of a is

$$[a] = \{ b \in X \mid b \sim a \}.$$

Example 3.5. In the examples (3.2), the equivalence classes in the first example are the singleton sets, in the second example the equivalence classes are the colours.

Definition 3.6. Let X be a set. A partition P of X is a collection of subsets A_i, $i \in I$, such that

(1) The A_i cover X, that is,

$$A_i = X.$$

$$i \in I$$

(2) The A_i are pairwise disjoint, that is, if i = j then

$$A_i \cap A_j = \emptyset .$$

Lemma 3.7. Given an equivalence relation ∼ on X there is a unique partition of X. The elements of the partition are the equivalence classes of ∼ and vice-versa. That is, given a partition P of X we may construct an equivalence relation ∼ on X such that the partition associated to ∼ is precisely P .

Concisely, the data of an equivalence relation is the same as the data of a partition.

Proof. Suppose that ∼ is an equivalence relation. Note that $x \in [x]$ as $x \sim x$. Thus certainly the set of equivalence classes covers X. The only thing to check is that if two equivalence classes intersect at all, then in fact they are equal.

We first prove a weaker result. We prove that if x ∼ y then [x] = [y]. Since y ∼ x, by symmetry, it suffices to prove that [x] ⊂ [y]. Suppose that a ∈ [x]. Then a ∼ x. As x ∼ y it follows that a ∼ y, by transitivity. But then a ∈ [y]. Thus [x] ⊂ [y] and by symmetry [x] = [y].

So suppose that x ∈ X and y ∈ X and that z ∈ [x] ∩ [y]. As z ∈ [x], z ∼ x. As z ∈ [y], z ∼ y. But then by what we just proved [x] = [z] = [y].

Thus if two equivalence classes overlap, then they coincide and we have a partition.

Now suppose that we have a partition

$$P = \{ A_i \mid i \in I \}.$$

Define a relation ∼ on X by the rule x ∼ y iff x ∈ A_i and y ∈ A_i (same i, of course). That is, x and y are related iff they belong to the same part. It is straightforward to check that this is an equivalence relation, and that this process reverses the one above. Both of these things are left as an exercise to the reader. D

Example 3.8. Let X be the set of integers. Define an equivalence relation on Z by the rule x ∼ y iff x − y is even.

Then the equivalence classes of this relation are the even and odd numbers.

More generally, let n be an integer, and let nZ be the subset consisting of all multiples of n,

$$nZ = \{ an \mid a \in Z \}.$$

Since the sum of two multiples of n is a multiple of n,

$$an + bn = (a + b)n,$$

and the inverse of a multiple of n is a multiple of n,

$$-(an) = (-a)n,$$

nZ is closed under multiplication and inverses. Thus nZ is a subgroup of Z.

The equivalence relation corresponding to nZ becomes $a \sim b$ iff $a-b \in nZ$, that is, $a - b$ is a multiple of n. There are n equivalence classes,

$$[0], [1], [2], [3], \ldots [n - 1].$$

Definition-Lemma 3.9. Let G be a group, let H be a subgroup and let $\sim$ be the equivalence relation defined in (3.3). Let $g \in G$. Then

$$[g] = gH = \{ gh \mid h \in H \}.$$

gH is called a left coset of H.

Proof. Suppose that $k \in [g]$. Then $k \sim g$ and so $g^{-1}k \in H$. If we set $h = g^{-1}k$, then $h \in H$. But then $k = gh \in gH$. Thus $[g] \subset gH$.

Now suppose that $k \in gH$. Then $k = gh$ for some $h \in H$. But then $h = g^{-1}k \in H$. By definition of $\sim$, $k \sim g$. But then $k \in [g]$. D

In the example above, we see that the left cosets are

$$(8) = \{ an \mid a \in Z \}$$

$$(9) = \{ an + 1 \mid a \in Z \}$$

$$(10) = \{ an + 2 \mid a \in Z \}$$

$$\ddots$$

$$[n - 1] = \{ an - 1 \mid a \in Z \}.$$

It is interesting to see what happens in the case $G = D_3$. Suppose we take $H = \{I, R, R^2\}$. Then

$$[I] = H = \{I, R, R^2\}.$$

Pick $F_1 \notin H$. Then

$$[F_1] = F_1H = \{F_1, F_2, F_3\}.$$

Thus H partitions G into two sets, the rotations, and the flips,

$$\{\{I, R, R^2\}, \{F_1, F_2, F_3\}\}.$$

Note that both sets have the same size.

Now suppose that we take $H = \{I, F_1\}$ (up to the obvious symme- tries, this is the only other interesting example).

In this case

$$[I] = IH = H = \{I, F_1\}.$$

Now R is not in this equivalence class, so

$$[R] = RH = \{R, RF_1\} = \{R, F_2\}.$$

Finally look at the equivalence class containing R^2 .

$$[R^2] = R^2H = \{R^2, R^2F_1\} = \{R^2, F_3\}.$$

The corresponding partition is

$$\{\{I, F_1\}, \{R, F_2\}, \{R^2, F_3\}\}.$$

Note that, once again, each part of the partition has the same size.

Definition 3.10. Let G be a group and let H be a subgroup.

The index of H in G, denoted [G : H], is equal to the number of left cosets of H in G.

Note that even though G might be infinite, the index might still be finite. For example, suppose that G is the group of integers and let H be the subgroup of even integers. Then there are two cosets (evens and odds) and so the index is two.

We are now ready to state our first Theorem.

Theorem 3.11. (Lagrange's Theorem) Let G be a group. Then

$$|H|[G : H] = |G|.$$

In particular if G is finite then the order of H divides the order of

G.

Proof. Since G is a disjoint union of its left cosets, it suffices to prove that the cardinality of each coset is equal to the cardinality of H.

Suppose that gH is a left coset of H in G. Define a map

$$A : H \longrightarrow gH,$$

by sending $h \in H$ to $A(h) = gh$. Define a map

$$B : gH \longrightarrow H,$$

by sending $k \in gH$ to $B(k) = g^{-1}k$. These maps are both clearly well-defined.

We show that B is the inverse of A. We first compute

$$B \circ A: H \longrightarrow H.$$

Suppose that $h \in H$, then

$$(B \circ A)(h) = B(A(h)) = B(gh)$$

$$= g^{-1}(gh)$$
$$= h.$$

Thus $B \circ A: H \longrightarrow H$ is certainly the identity map. Now consider

$$A \circ B : gH \longrightarrow gH.$$

Suppose that $k \in gH$, then

$$(A \circ B)(k) = A(B(k)) = A(g^{-1}k) = g(g^{-1}k) = k.$$

Thus B is indeed the inverse of A. In particular A must be a bijection and so H and gH must have the same cardinality. D

CYCLIC GROUP:

Lemma 4.1. Let G be a group and let H_i, $i \in I$ be a collection of subgroups of G.

Then the intersection

$$H = \bigcap_{i \in I} H_i,$$

is a subgroup of G

Proof. First note that H is non-empty, as the identity belongs to every H_i. We have to check that H is closed under products and inverses.

Suppose that g and h are in H. Then g and h are in H_i, for all i. But then $hg \in H_i$ for all i, as H_i is closed under products. Thus $gh \in H$.

Similarly as H_i is closed under taking inverses, $g^{-1} \in H_i$ for all $i \in I$.
But then $g^{-1} \in H$.

Thus H is indeed a subgroup. D

Definition-Lemma 4.2. Let G be a group and let S be a subset of G. The subgroup H = (S) generated by S is equal to the smallest subgroup of G that contains S.

Proof. The only thing to check is that the word smallest makes sense. Suppose that H_i, $i \in I$ is the collection of subgroups that contain S.

By (4.1), the intersection H of the H_i is a subgroup of G.
On the other hand H obviously contains S and it is contained in each H_i.
Thus H is the smallest subgroup that contains S. D

Lemma 4.3. Let S be a non-empty subset of G.

Then the subgroup H generated by S is equal to the smallest subset of G, containing S, that is closed under taking products and inverses.

Proof. Let K be the smallest subset of G, closed under taking products and inverses.

As H is closed under taking products and inverses, it is clear that H must contain K. On the other hand, as K is a subgroup of G, K

must contain H.
But then H = K. D

Definition 4.4. Let G be a group. We say that a subset S of G gen- erates G, if the smallest subgroup of G that contains S is G itself.

Definition 4.5. Let G be a group. We say that G is cyclic if it is generated by one element.

Let G = (a) be a cyclic group. By (4.3)

$$G = \{ a^i \mid i \in Z \}.$$

Definition 4.6. Let G be a group and let $g \in G$ be an element of G. The order of g is equal to the cardinality of the subgroup generated by g.

Lemma 4.7. Let G be a finite group and let g ∈ G.

Then the order of g divides the order of G.

Proof. Immediate from Lagrange's Theorem. D

Lemma 4.8. Let G be a group of prime order.

Then G is cyclic.

Proof. If the order of G is one, there is nothing to prove. Otherwise pick an element g of G not equal to the identity. As g is not equal to the identity, its order is not one. As the order of g divides the order of G and this is prime, it follows that the order of g is equal to the order of G.

But then G = (g) and G is cyclic. D

It is interesting to go back to the problem of classifying groups of finite order and see how these results change our picture of what is going on.

Now we know that every group of order 1, 2, 3 and 5 must be cyclic. Suppose that G has order 4. There are two cases. If G has an element a of order 4, then G is cyclic.

We get the following group table.

*	e	a	a^2	a^3

e	e	a	a	a
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
$_a3$	a^3	e	a	$_a2$

Replacing a^2 by b, a^3 by c we get

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Now suppose that G does not contain any elements of order 4. Since the order of every element divides 4, the order of every element must be 1, 2 or 4. On the other hand, the only element of order 1 is the identity element. Thus if G does not have an element of order 4, then every element, other than the identity, must have order 2.

In other words, every element is its own inverse.

*	e	a	b	c
e	e	a	b	c
a	a	e	?	
b	b		e	
c	c			e

Now ? must in fact be c, simply by a process of elimination. In fact we must put c somewhere in the row that contains a and we cannot put it in the last column, as this already contains c. Continuing in this way, it turns out there is only one way to fill in the whole table

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

So now we have a complete classification of all finite groups up to order five (it easy to see that there is a cyclic group of any order; just take the rotations of a regular n-gon). If the order is not four, then the only possibility is a cyclic group of that order. Otherwise the order is four and there are two possibilities.

Either G is cyclic. In this case there are two elements of order 4 (a and a^3) and one element of order two (a^2). Otherwise G has three elements of order two. Note however that G is abelian.
So the first non-abelian group has order six (equal to D_3).

One reason that cyclic groups are so important, is that any group G contains lots of cyclic groups, the subgroups generated by the ele- ments of G. On the other hand, cyclic groups are reasonably easy to understand. First an easy lemma about the order of an element.

Lemma 4.9. Let G be a group and let $g \in G$ be an element of G. Then the order of g is the smallest positive number k, such that
$a^k = e$.

Proof. Replacing G by the subgroup (g) generated by g, we might as well assume that G is cyclic, generated by g.

Suppose that $g^l = e$. I claim that in this case

$$G = \{ e, g, g^2, g^3, g^4, \ldots, g^{l-1} \}.$$

Indeed it suffices to show that the set is closed under multiplication and taking inverses.

Suppose that g^i and g^j are in the set. Then $g^i g^j = g^{i+j}$. If $i + j < l$ there is nothing to prove. If $i + j \geq l$, then use the fact that $g^l = e$ to rewrite g^{i+j} as g^{i+j-l}. In this case $i + j - l > 0$ and less than l. So the set is closed under products.

Given g^i, what is its inverse? Well $g^{l-i} g^i = g^l = e$. So g^{l-i} is the inverse of g^i. Alternatively we could simply use the fact that H is finite, to conclude that it must be closed under taking inverses.

Thus $|G| \leq l$ and in particular $|G| \leq k$. In particular if G is infinite, there is no integer k such that $g^k = e$ and the order of g is infinite and the smallest k such that $g^k = e$ is infinity. Thus we may assume that the order of g is finite.

Suppose that $|G| < k$. Then there must be some repetitions in the
set

$$\{ e, g, g^2, g^3, g^4, \ldots, g^{k-1} \}.$$

Thus $g^a = g^b$ for some $a = b$ between 0 and $k - 1$. Suppose that $a < b$. Then $g^{b-a} = e$. But this contradicts the fact that k is the smallest integer such that $g^k = e$. D

Lemma 4.10. Let G be a finite group of order n and let g be an element of G.

Then $g^n = e$.

Proof. We know that $g^k = e$ where k is the order of g. But k divides n. So n = km. But then

$$g^n = g^{km} = (g^k)^m = e^m = e. \qquad D$$

Lemma 4.11. Let G be a cyclic group, generated by a. Then

(9) G is abelian.

(10) If G is infinite, the elements of G are precisely

$$. . a^{-3}, a^{-2}, a^{-1}, e, a, a^2, a^3, . . .$$

(11) If G is finite, of order n, then the elements of G are precisely

$$e, a, a^2, . . . , a^{n-2}, a^{n-1},$$

and $a^n = e$.

Proof. We first prove (1). Suppose that g and h are two elements of G. As G is generated by a, there are integers m and n such that $g = a^m$
and $h = a^n$. Then

$$gh = a^m a^n$$
$$= a^{m+n}$$
$$= a^{n+m}$$
$$= hg.$$

Thus G is abelian. Hence (1).

(2) and (3) follow from (4.9). D

Note that we can easily write down a cyclic group of order n. The group of rotations of an n-gon forms a cyclic group of order n. Indeed any rotation may be expressed as a power of a rotation R through $2\pi/n$. On the other hand, $R^n = 1$.

However there is another way to write down a cyclic group of order n. Suppose that one takes the integers Z. Look at the subgroup nZ. Then we get equivalence classes modulo n, the left cosets.

$$[0], [1], [2], [3], \dots , [n - 1].$$

I claim that this is a group, with a natural method of addition. In fact I define

$$[a] + [b] = [a + b].$$

in the obvious way. However we need to check that this is well-defined. The problem is that the notation

$$[a]$$

is somewhat ambiguous, in the sense that there are infinitely many numbers a' such that

$$[a'] = [a].$$

In other words, if the difference $a' - a$ is a multiple of n then a and a' represent the same equivalence class. For example, suppose that n = 3. Then [1] = [4] and [5] = [−1]. So there are two ways to calculate

$$[1] + [5].$$

One way is to add 1 and 5 and take the equivalence class. [1] + [5] = [6]. On the other hand we could compute [1] + [5] = [4] + [−1] = [3]. Of course [6] = [3] = [0] so we are okay.

So now suppose that a' is equal to a modulo n and b' is equal to b modulo n. This means

$$a' = a + pn$$

and

$$b' = b + qn,$$

where p and q are integers. Then

$$a' + b' = (a + pn) + (b + qn) = (a + b) + (p + q)n.$$

So we are okay

$$[a + b] = [a' + b'],$$

and addition is well-defined. The set of left cosets with this law of addition is denote Z/nZ, the integers modulo n. Is this a group? Well associativity comes for free. As ordinary addition is associative, so is addition in the integers modulo n.

[0] obviously plays the role of the identity. That is

$$[a] + [0] = [a + 0] = [a].$$

Finally inverses obviously exist. Given [a], consider [−a]. Then

$$[a] + [-a] = [a - a] = [0].$$

Note that this group is abelian. In fact it is clear that it is generated by [1], as 1 generates the integers Z.

How about the integers modulo n under multiplication? There is an obvious choice of multiplication.

$$[a] \cdot [b] = [a \cdot b].$$

Once again we need to check that this is well-defined. Exercise left for the reader.

Do we get a group? Again associativity is easy, and [1] plays the role of the identity. Unfortunately, inverses don't exist. For example [0] does not have an inverse. The obvious thing to do is throw away zero. But even then there is a problem. For example, take the integers modulo 4. Then

$$[2] \cdot [2] = [4] = [0].$$

So if you throw away [0] then you have to throw away [2]. In fact given n, you should throw away all those integers that are not coprime to n, at the very least. In fact this is enough.

Definition-Lemma 4.12. Let n be a positive integer.

The group of units, U_n, for the integers modulo n is the subset of Z/nZ of integers coprime to n, under multiplication.

Proof. We check that U_n is a group.

First we need to check that U_n is closed under multiplication. Sup- pose that $[a] \in U_n$ and $[b] \in U_n$. Then a and b are coprime to n. This means that if a prime p divides n, then it does not divide a or b. But then p does not divide ab. As this is true for all primes that divide n, it follows that ab is coprime to n. But then $[ab] \in U_n$. Hence multiplication is well-defined.

This rule of multiplication is clearly associative. Indeed suppose that [a], [b] and $[c] \in U_n$. Then

$$([a] \cdot [b]) \cdot [c] = [ab] \cdot c$$
$$= [(ab)c]$$
$$= [a(bc)]$$
$$= [a] \cdot [bc]$$
$$= [a] \cdot ([b] \cdot [c]).$$

So multiplication is associative.

Now 1 is coprime to n. But then $[1] \in U_n$ and this clearly plays the role of the identity.

Now suppose that $[a] \in U_n$. We need to find an inverse of [a]. We want an integer b such that

$$[ab] = 1.$$

This means that

$$ab + mn = 1,$$

for some integer m. But a and n are coprime. So by Euclid's algorithm, such integers exist. D

Definition 4.13. The Euler φ function is the function ϕ(n) which assigns the order of U_n to n.

Lemma 4.14. Let a be any integer, which is coprime to the positive integer n.

Then $a^{\varphi(n)} = 1 \bmod n$.

Proof. Let g = [a] ∈ U_n. By (4.10) $g^{\varphi(n)} = e$. But then

$$[a^{\varphi}(n)] = [1].$$

Thus

$$a^{\varphi(n)} = 1 \mod n. \qquad D$$

Given this, it would be really nice to have a quick way to compute ϕ(n).

Lemma 4.15. The Euler ϕ function is multiplicative. That is, if m and n are coprime positive integers,

$$\phi(mn) = \phi(m)\phi(n).$$

Proof. We will prove this later in the course

Given (4.15), and the fact that any number can be factored, it suffices to compute $\phi(p^k)$, where p is prime and k is a positive integer.

Consider first $\phi(p)$. Well every number between 1 and $p - 1$ is automatically coprime to p. So $\phi(p) = p - 1$.

Theorem 4.16. (Fermat's Little Theorem) Let a be any integer. Then $a^p = a \bmod p$. In particular $a^{p-1} = 1 \bmod p$ if a is coprime to p.

Proof. Follows from (4.14). D

How about $\phi(p^k)$? Let us do an easy example.

Suppose we take $p = 3$, $k = 2$. Then of the eight numbers between 1 and 8, two are multiples of 3, 3 and $6 = 2 \cdot 3$. More generally, if a number between 1 and $p^k - 1$ is not coprime to p, then it is a multiple of p. But there are $p^{k-1} - 1$ such multiples,

$$p = 1 \cdot p, 2p, 3p, \ldots (p^{k-1} - 1)p.$$

Thus $(p^k - 1) - (p^{k-1} - 1) - p^k - p^{k-1}$ numbers between 1 and p^k are coprime to p. We have proved

Lemma 4.17. Let p be a prime number. Then

$$\phi(p^k) = p^k - p^{k-1} .$$

Example 4.18. What is the order of U_{5000}?

$$5000 = 5 \cdot 1000 = 5 \cdot (10)^3 = 5^4 \cdot 2^3 .$$

Now

$$\phi(2^3) = 2^3 - 2^2 = 4,$$

and

$$\phi(5^4) = 5^4 - 5^3 = 5^3(4) = 125 \cdot 4.$$

As the Euler-phi function is multiplicative, we get

$$\phi(5000) = 4 \cdot 4 \cdot 125 = 2^4 \cdot 5^3 = 2000.$$

It is also interesting to see what sort of groups one gets. For example, what is U_6?

$\phi(6) = \phi(2)\phi(3) = 1 \cdot 2 = 2$. Thus we get a cyclic group of order 2. In fact 1 and 5 are the only numbers coprime to 6.

$$5^2 = 25 = 1 \mod 6.$$

How about U_8? Well

$$\phi(8) = 4.$$

So either U_8 is either cyclic of order 4, or every element has order 2. 1, 3, 5 and 7 are the numbers coprime to 2. Now

$$3^2 = 9 = 1 \mod 8,$$

$$5^2 = 25 = 1 \mod 8,$$

and

$$7^2 = 49 = 1 \mod 8.$$

So

$$[3]^2 = [5]^2 = [7]^2 = [1]$$

and every element of U_8, other than the identity, has order two. But then U_8 cannot be cyclic.

POSSIBLE QUESTIONS:

8 MARK:

- Derive the concept of subgroup and abelian..
- write a short note on cosets and cyclic group.
- Explain the concept of group and its properties with examples.
- Derive the laplace equation in terms of Cartesian co-ordinates.
- Explain the concept of group theory.
- Explain the concept of group theory and its properties with example.
- Derive the general solution for the first order differential equation
- Derive the general solution for second order differential equation.

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 21

DEPARTMENT OF PHYSICS

CLASS: III B. Sc., PHYSICS BATCH: 2015-2018

MATHEMATICAL PHYSICS (15PHU603A)

UNIT III

QUESTIONS	CHOICE 1	CHOICE 2	CHOICE 3	CHOICE 4	ANSWER
For partial differential equation, if $b^2 - 4ac = 0$ then equation is called	None of these	hyperbolic	parabolic	elliptic	parabolic
Boundary condition which include direct boundary value is	Dirichlet boundary condition	Neumann boundary condition	forced boundary condition	discrete boundary condition	Dirichlet boundary condition
Region of flow trailing a body where effect of that body is felt on velocity field is called	flow region	wake	trailing region	velocity region	wake
Measure of circulation of fluid is called	stability	vorticity	viscosity	None of these	vorticity
Flow in which each particle of fluid follows an irregular path is called	laminar flow	turbulent flow	mixed flow	None of these	turbulent flow
A partial differential equation has	one independent variable	two or more independent variables	more than one dependent variable	equal number of dependent and independent variables	two or more independent variables
At Mach number Ma < 1, pressure disturbances travel	faster than flow velocity	slower than flow velocity	equal to flow velocity	None of these	faster than flow velocity
Dividing line between subsonic and super sonic regime is called	subsonic line	supersonic line	sonic line	reference line	sonic line
Truncation error becomes zero as mesh spacing tends to	maximum	minimum	zero	None of these	zero
Difference between exact solution to mathematical model and discretized equations used to approximate it is called	modeling error	discretization error	convergence error	None of these	discretization error
Euler number indicates relationship between	pressure drop	temperature drop	velocity drop	viscosity drop	pressure drop
{ - 3 n : n ε Z } IS an abelian group under	subtraction	division	multiplication	addition	addition
If G = { 1, -1, ι, - ι } is group under multiplication, then inverse of - ι is	1	−1	ι	None of Above	ι
A monoid is always a	A group	a commutative group	a non abelian group	groupoid	groupoid
If a, b are elements of a group G, then $(ba)^{-1}$ =	a-1 b-1	b-1 a-1	a-1 b	b-1 a	a-1 b-1
A monoid is always a	A group	a commutative group	a non abelian group	semi - group	semi - group
The solution of a differential equation which is not obtainted from the general solution is known as	particular solution	singular solution	complete solution	Auxiliary solution	singular solution
The differential equation dy/dx = y^2 is	linear	non-linear	Quasilinear	None of these	non-linear
Ratio between longest side and shortest side of mesh is called	mesh orthogonality	mesh skewness	mesh aspect ratio	mesh smoothness	mesh aspect ratio

When Reynolds number Re > 4000, flow is	turbulent	transient	laminar	None of these	turbulent
Path of fluid particles can not be tracked in	turbulent flow	laminar flow	mixed flow	None of these	turbulent flow
When mach number is in between 1.2 - 5, flow is in	subsonic regime	super sonic regime	sonic regime	transonic regime	super sonic regime
Artificial node is added for	Dirichlet boundary condition	Neumann boundary condition	forced boundary condition	discrete boundary condition	Neumann boundary condition
A space of interest where mass can cross boundary is	Control volume	Control surface	control system	control boundary	Control volume
When fluid properties does not change with time, flow is called	steady	unsteady	viscous	non viscous	steady
Boundary of control volume is called	Control volume	Control surface	control system	control boundary	Control surface
A process in which flow in boundary layer can no longer stay attached to surface & separates from surface is called	Force separation	boundary separation	flow separation	surface separation	flow separation
Finite difference method is	exact solution	approximate solution method	unique solution	None of these	approximate solution method
difference between mathematical model and real world it is trying to represent is called	modeling error	discretization error	convergence error	None of these	modeling error

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UNIT-IV

SYLLABUS:

Functions of a complex variable – single and multivalued functions – Cauchy – Riemann differential equation – analytical – line integrals of complex function – Cauchy's integral theorem and integral formula – derivatives of an analytic function – Taylor's variable – Residue and Cauchy's residue theorem – application to the equation of definite integrals – conformal transformation – Invariance of the Laplacian.

I. Introduction

The complex number system is merely a logical extension of the real number system. The set of complex numbers includes the real numbers and still more. All complex numbers are of the form

$$x + iy$$

where $i = \sqrt{-1}$. In other words $i^2 = -1$. If y = 0, then the complex number x + iy becomes the real number x. This is why we say that the complex numbers are still "more" than the reals. The real numbers form a proper subset of the reals. We do not mean that the complex numbers are more numerous. We simply mean that they subsume the reals.

Because there are two real numbers (x and y) associated with each complex number, we are able to depict complex numbers using a plane, as opposed to the reals which are depicted on a line. Unlike the real number system, complex numbers are not ordered. This means that it is not meaningful to say $z_1 < z_2$ in the complex number system, even though such a thing is posssible in the reals.

It is possible to define addition and multiplication of complex numbers in the following intuitive ways:

Addition: $(x_1+iy_1) + (x_2 + iy_2) = (x_1+x_2) + i(y_1+y_2)$

Multiplication: $(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2+x_2y_1)$

The complex number 0 + i0 is the complex counterpart of zero in the reals. It is the complex additive identity. We will at times simply denote it as 0. The multiplicative identity is equal to 1 + i0, which we will at times denote as 1.

A complex number can be written as z, so long as we understand that z = x + iy. It is possible to discuss subtracting and dividing complex numbers. For example

$$z_1 - z_2 = (x_1+iy_1) + (-x_2 + i(-y_2)) = (x_1 - x_2) + i(y_1 - y_2)$$

$$\frac{z_1}{z_2} = (x_1 + iy_1)(\frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2}) = 1$$

In addition to the basic operations of addition, subtraction, multiplication, and division, we can also perform more complicated operations – such as taking the square root.

$$\sqrt{z_1} = a + ib \text{ where } (a+ib)(a+ib) = z_1 = x_1 + iy_1 .$$

Example: Find $\sqrt{3+i4}$

Note that there are two solutions: $\sqrt{3+i4}$ =2+i and $\sqrt{3+i4}$ = -2-i

To check this we note that (2+i)(2+i) = 3 + i4 as is the case with –2-i.

It is not hard to show that there will be *exactly* two complex square roots for any (nonzero) complex number.

The complex conjugate of a complex number z = x + iy is denoted $\bar{z}$ and is defined as (x – iy). The modulus of a complex number is defined as $|z| = \sqrt{z\bar{z}}$.

Representation in the 2-Dimensional Plane

Each complex number can be written as z = x+iy. This means that we can associate an ordered pair (x,y) with each and every complex number z = x+iy. Luckily, this gives us a graphical representation of the complex number system. We can visualize the complex numbers.

The 2-dimensional plane that represents the complex numbers is sometimes called the **Argand plane** but was first employed by Gauss. The horizontal axis represents the real numbers which is a 1-dimensional subspace of the plane. The vertical axis represents "pure" complex numbers; or numbers which have no real part, x. A good question is whether or not the complex numbers (field) is isomorphic to R^2 under addition and multiplication.

The unit circle plays an important role in complex numbers since any (non-zero) complex number can be written as a scalar multiple of its corresponding point on the unit circle. For example, the point z = x + iy is a $(x^2 + y^2)$ - multiple of

$$\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

which lies on the unit circle. Seen in this way, the unit circle can generate the entire set of complex numbers through the appropriate multiplication of scalars to points on the unit circle. This is analogous to a point on the real number line (e.g., 1 and –1) being able to generate any other number on the real number line by the appropriate multiplication of a scalar. The unit circle is not unique in this regard, though. Other types of geometric objects containing the origin can do this as well. The value of the circle is that each point on the circle is equidistant from the origin and this distance is equal to 1. Note that the *real* numbers 1 and –1 have distance from zero equal to unity and can generate any real number through multiplication of a scalar. It is interesting, in this regard, that the inverse of a complex number involves the normalization of both the real and (negative) imaginary parts of the number. That is, the denominator is the formula for a circle.

The unit circle separates the plane into two regions. The set of points that are strictly inside the circle (called the open unit disk) and the set of points on and outside the unit circle. The interior of the unit circle (i.e. the unit disk) is particularly important to the stability of certain difference equations. It is similarly involved in determining whether a time series is covariance stationary.

III. Functions of a Complex Variable

We can define complex valued functions of a complex variable. That is, the domain of the function is the complex variable field and the range is also the complex field. We can write this as

$$w = f(z)$$

where both w and z are complex numbers. Such functions have complex numbers as parameters, as well. For example, we can write the following function

$$w = f(z) = \frac{z_o}{z} = \frac{1-i2}{x+iy} = \frac{(x-2y)+i2(x+y)}{x^2+y^2} = u+iv$$

Clearly, w is complex, as is z. The parameter z_o is also complex, but it is a fixed complex number. Note how that u and v have become real multivariate functions of x and y. That is,

$$u = u(x,y) = \frac{x-2y}{x^2+y^2} \quad \text{and} \quad v = v(x,y) = \frac{2(x+y)}{x^2+y^2}$$

As x and y run over all the values in R^2, both u and v are determined, and hence z is determined accordingly. This complex z then determines the value of w.

One of the most useful of all the complex functions is the exponential function. This function has a straigtforward relation to the trignometric functions. We can understand this relation by using a MacLaurin series for the e^x function. To begin with

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Which, if we substitute iθ for x, we get

$$e^{\theta i} = 1 + \frac{(\theta i)}{1!} + \frac{(\theta i)^2}{2!} + \cdots$$

$$= 1 + \frac{\theta i}{1!} - \frac{\theta^2}{2!} - \frac{\theta^3 i}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5 i}{5!} - \frac{\theta^6}{6!} \cdots$$

$$= \{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots\} + \{\frac{\theta i}{1!} - \frac{\theta^3 i}{3!} + \frac{\theta^5 i}{5!} - \frac{\theta^7 i}{7!} \cdots\}$$

$$= \cos(\theta) + i\sin(\theta)$$

Note that the point (cos(θ), sin(θ)) is on the unit circle and as θ runs from 0 to 2π, the point moves completely around the circle in a counterclockwise fashion. We can therefore write any complex number as a scalar multiple of $e^{i\theta}$. Usually this is written as

$$z = x + iy = r\cos(\theta) + i\, r\sin(\theta) = re^{i\theta}$$

with $\theta = \arctan(y/x)$ and $r^2 = x^2 + y^2$.

The complex exponential and its relation to the trigonometric functions is of the greatest imporance in of mathematics. It is incredibly useful and leads to some rather extraordinary and unexpected results.

For example, it allows us to easily compute the following *real* number

$$z = i^i = \frac{1}{\sqrt{e^{\pi}}} \approx 0.207$$

where $i = \sqrt{-1}$. It also allows us to write out the logarithm of a negative number, which was a great controversy during the time of Euler and Leibnitz. That is, we can write

$$z = \ln(-1) = i\pi$$

from which all other negative logarithms can be derived.[1] The logarithm of a complex number can also be derived using this relation. Hence, we have

$$\ln(z) = \ln(x+iy) = \ln(re^{i\theta}) = \ln(r) + i\theta$$

where θ is the angle formed by vectors (x, 0) and (0, y) and where $r^2 = x^2 + y^2$.

[1] The logarithmic function defined on complex and negative numbers is a multi-valued function and in fact our result above only holds provided we specify the "branch" on which we are evaluating the log. One can see this since log(1) = log(-1) + log(-1) = 2πi according to the branch we have decided to use. The value log(1) = 0 corresponds to yet another branch.

Polynomial equations, even simple ones, have solutions which are surprising to those who look only for real solutions. For example, even the very simple equation

$$z^4 + 1 = 0$$

has FOUR distinct roots (consider $z^2 = i$ and $z^2 = -i$). In general, the Fundamental Theorem of Algebra tells us that there will be exactly n complex roots (possibly repeated) which solve an n^{th} order polynomial equation. Once again, it is important to remember that the coefficients on these polynomial equations can also be complex numbers, as well.

The familiar trigonometric functions of sin(x) and cos(x) can be defined for complex numbers. This is done in the perfectly logical manner as follows:

$$\sin(z) = \sin(x+iy) = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos(z) = \cos(x+iy) = \frac{e^{ix} + e^{-ix}}{2},$$

where we remember that $\frac{1}{i} = -i$.

IV. Limits and Derivatives of Complex Functions

Limits in the complex system are complicated by the fact that z = x + iy depends on (x, y) and therefore one can approach $z_o = x_o + iy_o$ along infinitely many paths. For example,

$$z_n = (x_o + \frac{1}{n}) + i(y_o + \frac{1}{n}) \text{ and } z_n^{'} = (x_o - \frac{1}{n}) + i(y_o - \frac{1}{n})$$

both limit to $z_o = x_o + iy_o$, but do so along different paths. Obviously, other more complicated paths are possible. This makes it a little more difficult to define a derivative, which makes use of limits in its defintion.

The derivative of w = f(z), if it exists, is defined by the unique limit

$$\frac{dw}{dz} = \lim_{\delta \to (0+i0)} \frac{f(z+\delta) - f(z)}{\delta}$$

where $\delta \to (0 + i0) = 0$, the origin, along *ANY* path.

Example: w = f(z) = z^2 is differentiable. To see this, assume (g(n), h(n)) are functions parametrized such that they limit to the origin as $n \to \infty$. The ordered pair we have assumed maps out any path to the origin and is perfectly general. It is not difficult to show that

$$f'(z) = \lim_{n\to\infty} \frac{\{(x + g(n)) + i(y + h(n)\}^2 - \{x + iy\}^2}{g(n) + ih(n)}$$

$$= \lim_{n\to\infty} [2\{x + iy\} + \{g(n) + ih(n)\}]$$

$$=$$

and thus, regardless of the path we take to the origin, the limit remains the same and thus the derivative of f(z) is equal to f '(z) = 2z. ■

V. The Cauchy-Riemann Equations and Complex Differentiation

Suppose that we consider f(z) = z^2 and substitute into this z = x+iy. We can therefore write this function again in the following way:

$$f(z) = z^2 = F(x,y) = (x+iy)^2 = (x^2 - y^2) + i2xy = u(x,y) + iv(x,y)$$

where $u(x,y) = (x^2 - y^2)$ and where $v(x,y) = 2xy$. Now since z = x+iy, we know that

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

from which it follows that $\frac{\partial x}{\partial z} = \frac{1}{2}$ and $\frac{\partial y}{\partial z} = \frac{1}{2i}$. Now consider the complex derivative f '(z).

$$f'(z) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = (2x - 2yi)(\frac{1}{2}) + (-2y + 2xi)(\frac{1}{2i})$$

This of course reduces to $f'(z) = 2z$ and the result agrees with the derivative computed in the previous section using limits.

Now suppose that F(x,y) = u(x,y) + iv(x,y) is *any* differentiable complex function. What must be true about the functions u and v ? This is the subject of the Cauchy - Riemann equations.

First, suppose that z changes by x changing alone. Then, assume that z changes by y changing alone. This would give us two expressions for the derivative of

f(z) = F(x,y).

The first (*holding y constant*) can be written as

$$f'(z)|_{y\,constant} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + i\frac{\partial v}{\partial x}\frac{\partial x}{\partial z} = \frac{1}{2}\{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\}$$

while the second (*holding x constant*) can be written as

$$f'(z)|_{x\,constant} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} + i\frac{\partial v}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\{-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\}.$$

Now, the derivative of f(z) cannot depend on which way that z is changing (either by x changing alone or alternatively by y changing alone) and so the two expressions for $f'(z)$ must be equal if the derivative exists. This implies that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and that } -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

These two equalities are known as the Cauchy-Riemann Equations.

VI. Complex Integration

The first thing to note about complex integration is that it is done in the plane and therefore uses contour or line integration as its basis. A strong understanding of line integration is therefore useful when discussing complex variables. This is not so unexpected since the 2 dimensional plane in complex variables now operates analogously to the real line in elementary calculus. Instead of integrating over some interval, or collection of intevals, we must integrate along some curve or line in 2-space.

The second thing to note about complex integration is that integration no longer implies the measurement of some area. That is, one does not necessarily get a “clean” real number associated with an integral in complex variables. What this inplies is that integrals cannot be ordered by size as they can be in real integration. One cannot say that *this area is larger than that area*. This is because there is no ordering of the complex numbers, unlike the reals. Indeed, the integral of a complex function of a complex variable typically yields a complex number.

Here is a simple example to show complex integration:

Example: Let f(z) = z, where z = x+iy. It is obvious that the image of the function f is not a real number; it is complex. Now suppose that we integrate this in the x-y plane along the line y = x from (0,0) to (1,1). We are integrating the function f(z) along the ray from the origin to the point (1,1). This directed line segment is sometimes denoted C for curve, even though it is a straight line segment. We assume that we move from the point (0,0) to the point (1,1), since direction is important. Now let’s actually do the integration.

$$\int_C f(z)\,dz = \int_C z\,dz = \int_C (x+iy)(dx+idy)$$

$$= \int_0^1 (x+ix)(1+i)dx$$

$$= \int_0^1 x(1+i)^2\,dx$$

$$= i\int_0^1 2x\,dx$$

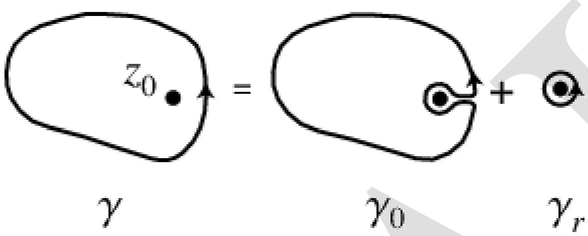
$$= ix^2\,\Big|_0^1$$

$$= i \qquad \blacksquare$$

This example shows that the complex definite integral of the complex function

f(z) = z is itself a complex number; in fact, it is equal to z = i.

CAUCHY'S INTEGRAL FORMULA:



Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{z - z_0}, \qquad (1)$$

where the integral is a contour integral along the contour γ enclosing the point z_0.

It can be derived by considering the contour integral

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0}, \qquad (2)$$

defining a path γ_r as an infinitesimal counterclockwise circle around the point z_0, and defining the path γ_0 as an arbitrary loop with a cut line (on which the forward and reverse contributions cancel each other out) so as to go around z_0. The total path is then

$$\gamma = \gamma_0 + \gamma_r, \quad (3)$$

so

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0} = \oint_{\gamma_0} \frac{f(z)\,dz}{z - z_0} + \oint_{\gamma_r} \frac{f(z)\,dz}{z - z_0}. \quad (4)$$

From the Cauchy integral theorem, the contour integral along any path not enclosing a pole is 0. Therefore, the first term in the above equation is 0 since γ_0 does not enclose the pole, and we are left with

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0} = \oint_{\gamma_r} \frac{f(z)\,dz}{z - z_0}. \quad (5)$$

Now, let $z \equiv z_0 + r e^{i\theta}$, so $dz = i r e^{i\theta} d\theta$. Then

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0} = \oint_{\gamma_r} \frac{f(z_0 + r e^{i\theta})}{r e^{i\theta}} i r e^{i\theta} d\theta \quad (6)$$

$$= \oint_{\gamma_r} f(z_0 + r e^{i\theta})\, i\, d\theta. \quad (7)$$

But we are free to allow the radius r to shrink to 0, so

$$\oint_\gamma \frac{f(z)\,dz}{z - z_0} = \lim_{r \to 0} \oint_{\gamma_r} f(z_0 + r e^{i\theta})\, i\, d\theta \quad (8)$$

$$\oint_{\gamma_r} f(z_0)\, i\, d\theta \quad (9)$$

$$= i f(z_0) \oint_{\gamma_r} d\theta \quad (10)$$

$$= 2\pi i f(z_0), \quad (11)$$

giving (1).

If multiple loops are made around the point z_0, then equation (11) becomes

$$n(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\, dz}{z - z_0}, \tag{12}$$

where $n(\gamma, z_0)$ is the contour winding number.

A similar formula holds for the derivatives of $f(z)$,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{13}$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \left[\oint_\gamma \frac{f(z)\, dz}{z - z_0 - h} - \oint_\gamma \frac{f(z)\, dz}{z - z_0} \right] \tag{14}$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \oint_\gamma \frac{f(z)\, [(z - z_0) - (z - z_0 - h)]\, dz}{(z - z_0 - h)(z - z_0)} \tag{15}$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \oint_\gamma \frac{h\, f(z)\, dz}{(z - z_0 - h)(z - z_0)} \tag{16}$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\, dz}{(z - z_0)^2}. \tag{17}$$

Iterating again,

$$f''(z_0) = \frac{2}{2\pi i} \oint_\gamma \frac{f(z)\, dz}{(z - z_0)^3}. \tag{18}$$

Continuing the process and adding the contour winding number n,

$$n(\gamma, z_0) f^{(r)}(z_0) = \frac{r!}{2\pi i} \oint_\gamma \frac{f(z)\, dz}{(z - z_0)^{r+1}}. \tag{19}$$

TAYLOR'S VARIABLE:

The Taylor's series specifies the value of a function at one point in terms of the value of the function and its derivatives at a reference point say . It is an expansion in powers of the change in variable here say (z - a).

Fig.53: Contouri *C* defined in the anticlockwise direction, with parametric equation |z - a| < R

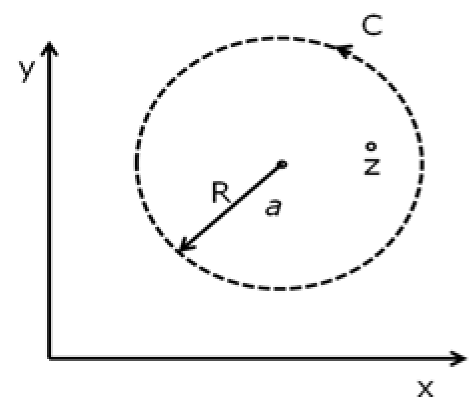
Let f(z) be an analytic function inside and on a simple closed curve *C*, with the center as a and radius R. Then at each point inside the contour *C* we define

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots +$$

The power series here converges to f(z) when |z -a| < R.

Here R will be the radius of convergence which is defined as the distance from the reference point to the nearest singularity of the function f(z). On |z -a| = R the series may or may not converge while for|z -a| >R the series diverges. If the nearest singularity of f(z) is at ∞ , the radius of convergence is at , i.e. the series converges for all values of z.

If a = 0 then the resulting series is often called the **Maclaurin series**.



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Proof of the Taylor's series

Let us consider an analytic function $f(z)$ in a neighborhood of a point z = a . Also let *C* be a circle which lies in this neighborhood and has the center a. Then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)ds}{(s-z)} \quad ----(1)$$

where,

z: any arbitrary fixed point inside the contour *C*

s: be the complex variable of integration.

The radius of convergence of the Taylor series is at least equal to the shortest distance from a to the boundary *C*. It may be larger but then the series may not represent $f(z)$ at all points of *C*which lie in the interior of the circle of convergence.

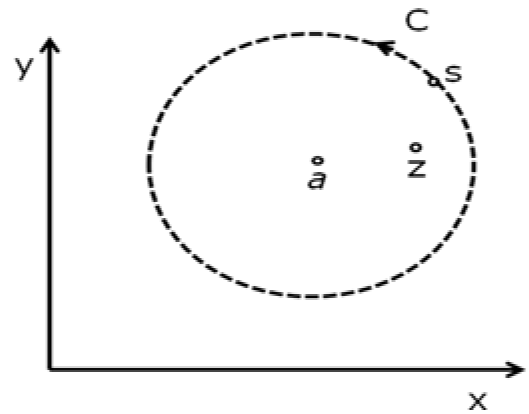

Fig.54: Contour C, with center at a and having the variable of integration as s and any point in the interior as z

We shall first develop $\frac{1}{(s-z)}$ in powers z-a of

$$\frac{1}{(s-z)} = \frac{1}{s-a-(z-a)} = \frac{1}{(s-a)\left(1-\frac{z-a}{s-a}\right)} \quad ----(2)$$

Since s is on C while z is inside C, therefore

$$\left|\frac{z-a}{s-a}\right| < 1 \quad ----(3)$$

Now, from the geometric progression for |q|<1

$$1+q+q^2+q^n = \frac{1-q^{n+1}}{1-q} \qquad q \neq 1$$

We obtain the relation,

$$\frac{1}{1-q} = 1+q+q^2+q^n+\frac{q^{n+1}}{1-q}$$

Here,

$$q = \frac{z-a}{s-a}$$

Then

$$\frac{1}{1-\left(\frac{z-a}{s-a}\right)} = 1 + \frac{z-a}{s-a} + \left(\frac{z-a}{s-a}\right)^2 + \cdots + \left(\frac{z-a}{s-a}\right)^n + \frac{\left(\frac{z-a}{s-a}\right)^{n+1}}{1-\left(\frac{z-a}{s-a}\right)}$$

Using 2

$$= 1 + \frac{z-a}{s-a} + \left(\frac{z-a}{s-a}\right)^2 + \cdots + \left(\frac{z-a}{s-a}\right)^n + \frac{\left(\frac{z-a}{s-a}\right)^{n+1}}{\left(\frac{s-z}{s-a}\right)} \quad ----(A)$$

Inserting (A) in (2) which is then put back in (1) to get

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)} + \frac{(z-a)}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)^2} + \cdots + \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)^{n+1}} + R_n(z) \quad --(4)$$

Since z and a are constants we take the powers of z-a out from under the integral sign and the last term is

$$R_n(z) = \frac{(z-a)^{n+1}}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)^{n+1}(s-z)} \quad ---(5)$$

Using Cauchy's integral formula

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^{n+1}} \qquad n = 1,2,3,\ldots$$

We comprehend the series as

$$f(z) = f(a) + \frac{(z-a)}{1\,!} f'(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + R_n(z) \quad ---(6)$$

This representation is called the Taylor's series

Rn (z)is called the Remainder. Since the analytic functions has derivatives to all orders we may take n as large as possible i.e. As $n \to \infty$ we write

$$f(z) = \sum_{m=0}^{\infty} f^m(a) \frac{(z-a)^m}{m!} \quad -----(7)$$

This is the Taylor's series for the function f(z) with center at a. The series in (7) above will clearly converge and represent f(z) if and only if

$$\lim_{n\to\infty} R_n(z) = 0 \qquad ---(8)$$

This we can prove as s is on the contour C while z is inside C thus we will have |s-z| > 0. Since f(z) is analytic inside C and on C it follows that the absolute value of $\frac{f(s)}{(s-z)}$ is bounded thus we can say

$$\left|\frac{f(s)}{(s-z)}\right| < M, \quad for\ all\ values\ of\ s\ on\ C$$

Let r be the radius of C then |s-a|=r, for all s on C and C has the length as $2\pi r$

Therefore,

$$|R_n(z)| = \left|\frac{(z-a)^{n+1}}{2\pi i} \oint_C \frac{f(s)ds}{(s-a)^{n+1}(s-z)}\right|$$

$$\leq \frac{|(z-a)|^{n+1}}{2\pi} \left|\frac{M\ 2\pi r}{r^{n+1}}\right| = \frac{M\ r\ |(z-a)|^{n+1}}{r^{n+1}}$$

As n→ ∞ then the right hand side is zero.

Example: Express the Taylor series for *f*(z) = e^z^

Soln.: The $f(z)$ is an analytic function. We can express the Taylor series about z=0. In general an expansion about z=a is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots +$$

The power series here converges to $f(z)$ when $|z - a| < R$. While about z=0 can be written as

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^n(0)}{n!}z^n + \cdots +$$

$$f(z) = e^z = e^0 = 1$$

$$f'(z) = e^z = e^0 = 1$$

$$f''(z) = e^z = e^0 = 1$$

Thus,

$$f(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots + \qquad ; \quad |z| < \infty$$

The first singularity of this function lies at infinity thus the radius of convergence is infinity.

Example: Express the Taylor series for (i) *f*(z) = sin z (ii) *f*(z) = cos z (iii) *f*(z) = sinh z (iv) *f*(z) = cosh z

Soln.: The given functions in all the cases are analytic functions. The first singularity of these function lies at infinity thus the radius of convergence is infinity. We can express the Taylor series about z=0 as

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 \ldots + \frac{f^n(0)}{n!} z^n + \cdots + \quad ; |z| < \infty$$

(i) $f(z) = \sin z$

$f(z) = \sin\ z = \sin 0 = 0$

$f'(z) = \cos z = \cos 0 = 1$

$f''(z) = -\sin z = -\sin 0 = 0$

$f'''(z) = -\cos z = -\cos 0 = -1$

Thus, we find that $f^{2n}(0) = 0$ while $f^{2n+1}(0) = (-1)^n$ So,

$$f(z) = 0 + \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \ldots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots + \quad ; |z| < \infty$$

$$\sin z = (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad ; |z| < \infty \quad ---(A)$$

Differentiating each side of (A) with respect to z and interchanging the symbols for differentiation and summation we have we can write the expansion as

(ii)

$$\cos z = (-1)^n \frac{z^{2n}}{(2n)!} \quad ; |z| < \infty \quad ---(B)$$

Such a case is possible as term by term differentiation of a power series is allowed. Here the power series is a Maclaurin or Taylor series.

(iii) We can use the identity sin iz = i sinh z, So using (A) by replacing every z by iz

$$\sin i z = (-1)^n \frac{(iz)^{2n+1}}{(2n+1)!}$$

$$\text{i} \sinh \text{z} = (\text{i i})^n \frac{\text{i}^{2n+1}\ \text{z}^{2n+1}}{(2n+1)!} = \frac{\text{i} \quad \text{z}^{2n+1}}{(2n+1)!}$$

The first singularity of this function lies at infinity thus the radius of convergence is infinity.

$$sinh\, z = \frac{z^{2n+1}}{(2n+1)!} \;;\quad |z| < \infty \,,\quad ----(C)$$

(iv) A term by term differentiation of equation (C) yields

$$\cosh z = \frac{z^{2n}}{(2n)!} \;;\quad |z| < \infty \,,\quad ----(C)$$

Example: Find the geometric series for the function

(i) $f(z) = \frac{1}{1-z}$ **(ii)** $f(z) = \frac{1}{1+z}$

Soln. : The function $f(z)$ is singular at z=1, this point lies on the circle of convergence

Derivative of the function $f(z) = \frac{1}{1-z}$ are of type

$$f^n(z) = \frac{n!}{(1-z)^{n+1}} \quad ;n = 0,1,2,\ldots$$

So the series and in particular the Maclaurin series has

$f^n(0)$ =n!

Thus,

$f^n(0) = n!$

Thus,

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 \dots + \frac{f^n(0)}{n!}z^n + \dots + \quad ; |z| < 1$$

$$\frac{1}{1-z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^n}{n!} + \dots + \quad ; |z| < 1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad ; |z| < 1$$

Replace every z by (-z) we get

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad ; |z| < 1$$

RESIDUE AND CAUCHY'S RESIDUE THEOREM:

Residue

We recall that if a function $f(z)$ is analytic in a simply connected region, then according to Cauchy's integral theorem the value of the integral

$$\oint_C f(z)dz = 0$$

is always zero, where C is a closed contour lying wholly in R.

If, on the other hand, the function $f(z)$ fails to be analytic at a finite number of points in the interior of the contour C in R, then there is a specific number called the **residue**, which each of these points (points of singularity) contribute to the value of the integral.

We note that a point 'a' is an isolated singularity if the function fails to be analytic at that point and in addition there is some neighborhood throughout which the function is analytic except at the point itself. ***The contribution of the singularity towards the integral is the residue.*** Thus for a non-analytic function the integral

$$\oint_C f(z)dz \neq 0$$

In this case we may represent the function by a Laurent series which converges in the domain 0<|z-a|<R, where R is the distance from 'a' to the nearest singular point of the function f(z). Thus in general we can write

$$f(z) = \sum_{n=-\infty}^{\infty} A_n\,(z-a)^n$$

$$= \cdots + A_n(z-a)^n + \cdots A_2(z-a)^2 + A_1(z-a) + A_0 + \frac{A_{-1}}{z-a} + \frac{A_{-2}}{(z-a)^2} + \frac{A_{-3}}{(z-a)^3} + \cdots + \frac{A_{-n}}{(z-a)^n} + \cdots \qquad ---(i)$$

where we can write

$$A_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^{n+1}} \qquad ; n = 0, \pm 1, \pm 2, \pm 3$$

For the special case n=-1 we have

$$\mathbf{2\pi i\, A_{-1}} = \oint_C \boldsymbol{f(z)dz}$$

The coefficient A_{-1} which is the coefficient of $\frac{1}{z-a}$ in the above expansion of the Laurent series is called the RESIDUE of the function at the isolated singular point 'a'. Thus

$A_{-1} = \underset{z=a}{Res} f(z) = Residue$

Example: Find the integral of the function $f(z) = \frac{\sin z}{z^4}$ **around the unit circle C in the counterclockwise sense.**

Soln.

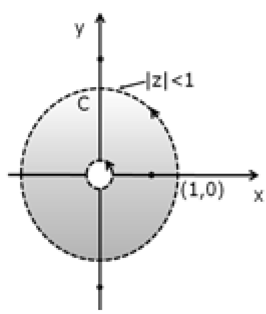

Fig.61: Isolated singular point at z=0 inside a unit circle.

We obtain the Laurent series for an isolated singularity at z=0 of the function as

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

Comparing it with (i) we find that the residue is

$A_{-1} = -\frac{1}{3!}$

Hence,

$$\oint_C f(z)dz = 2\pi i\, A_{-1} = 2\pi i \left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

Example: Show that the integral of the function $f(z) = e^{1/z^2}$ around the circle C, |z|=2 described in the counterclockwise sense is zero.

Soln.

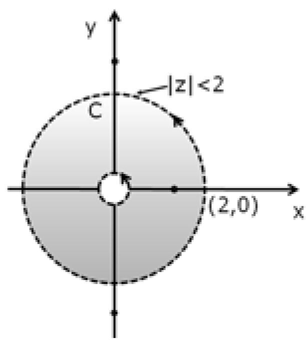

Fig.62: Isolated singular point at z=0 inside a circle |z|<2.

The isolated singularity at point z=0 lies interior to the contour C. By Maclaurin series we know that

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Thus we can write a series for $f(z) = e^{1/z^2}$

$$f(z) = e^{1/z^2} = \sum_{n=0}^{\infty} \frac{1}{z^{2n}\ \ n!} = 1 + \frac{1}{z^2\ \ 1!} + \frac{1}{z^4\ \ 2!} + \frac{1}{z^6\ \ 3!} + \cdots$$

and comparing it with (i) we find that the residue is $A_{-1} = 0$

Hence,

$$\oint_C f(z)dz = 2\pi i\, A_{-1} = 0$$

Example: Find the residue at simple poles for the function $f(z) = \frac{2-5z}{z^2-z}$

Soln. Using the definition

$$\operatorname*{Res}_{z=a} f(z) = A_{-1} = \lim_{z\to a}(z-a)f(z)$$

To evaluate the residue we see that the function $f(z) = \frac{2-5z}{z^2-z}$ has simple poles at z=0 and z=1. Thus, we have at these simple poles

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}(z-0)\frac{2-5z}{z^2-z} = -2$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)\frac{2-5z}{z^2-z} = -3$$

Example: Find the residue for the function $f(z) = \frac{z}{(z-2)(z+1)^2}$

Soln. In this function $f(z) = \frac{z}{(z-2)(z+1)^2}$ we see that we have simple pole at z=2 and a pole of order 2 at z=-1. So, for the simple pole we calculate the residue as

$$\underset{z=1}{Res} f(z) = \lim_{z\to 2}(z-2)\frac{z}{(z-2)(z+1)^2} = \frac{2}{9}$$

For the pole of order k=2 we find the residue at z=-1 by using the general formula

$$A_{-1} = \lim_{z\to a}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\{(z-a)^k f(z)\} = \lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\{(z+1)^2\frac{z}{(z-2)(z+1)^2}\}$$

$$= \lim_{z\to -1}\frac{d}{dz}\{\frac{z}{z-2}\} = \lim_{z\to -1}\frac{z-2-z}{z-2^2} = -\frac{2}{9}$$

Example: Obtain the Laurent series expansion of the following functions in the neighborhood of the singular points and calculate the residues: (i) (ii)

$f(z) = \frac{\cos z}{z}$ (ii) $f(z) = \frac{e^z}{(z-1)^2}$

Solution

(i) Note that z=0 is the isolated singular point of the function $f(z) = \frac{\cos z}{z}$

We recall the series expansion of cos z as

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \qquad -\infty < z < \infty$$

Thus,

$$f(z) = \frac{\cos z}{z} = \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \left(\frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots\right)$$

We note that in this series the coefficient of the 1/z term is 1. Thus, $A_{-1} = 1$, therefore is the residue of the given function.

(ii) The function has a double pole at z=1. We expand the exponential series in powers of (z-1)

$$f(z) = \frac{e^z}{(z-1)^2} = \frac{e\ e^{z-1}}{(z-1)^2} = \frac{e}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

$$= \frac{e}{(z-1)^2} + \frac{e}{(z-1)} + \frac{e}{2!} + \frac{e(z-1)}{3!} + \cdots$$

This is the required Laurent series expansion. We observe the coefficient of 1/(z-1) is the residue

What have we achieved so far? We agree that now we can evaluate the residue of a function $f(z)$ with one singular point in a contour using Laurent series expansion. However, how do we proceed when encloses more than one isolated singular points? In such a situation, we have to extend the concept of residue developed so far to more than one singularity. The theorem of residues deals with such a general case and we discuss it in the following section.

Residue Theorem:

We extend the concept of residue developed so far to the case when the integrand has several singularities. Let us consider a positively oriented simple closed contour C, within and on which a function is analytic except for a finite number of singular points $z_1, z_2, z_3, \ldots z_n$ interior to C. If $A_1, A_2, A_3 \ldots, A_n$ denote the residues of 'f(z)' at those respective points then

$$\oint_C f(z)dz = 2\pi i\ (A_1 + A_2 + A_3 + \cdots A_n)$$

Proof:

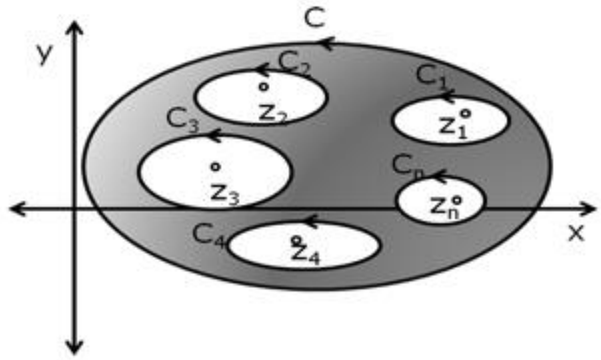

Fig.63: A finite number of singular points z_1, z_2, z_3 … z_n interior to C.

Let the singular points z_j(j=1,2,…n) be the centers of positively oriented circles C_j which are interior to C and are so small that no two of the circles have points in common. That is to say that all the singularities are isolated by small contours.

The circles C_j along with the simple closed contour C form the boundary of a closed region. As we can see in the figure the function is analytic throughout the shaded region, which is a multiply connected domain. Hence, by Cauchy Goursat theorem

$$\oint_C f(z)dz - \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz - \cdots - \oint_{C_n} f(z)dz = 0$$

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$

$$= 2\pi i\, Res[(f(z)_{C_1}) + (f(z)_{C_2}) + \cdots + (f(z)_{C_n}) = 2\pi i\,[A_1 + A_2 + A_3 + \cdots A_n] = 2\pi i \sum_{j=1}^{n} A_j$$

We note that **this theorem is valid only for isolated singularities.** The immense utility of this theorem stems from the fact that it facilitates calculation of a contour integral indirectly through the residues of f (z) at the singularities inside C.

Example: Use the residue theorem to evaluate the integral

$$\oint_C \frac{5z-2}{z(z-1)}dz$$

Where, C is the circle |z|=2 described in counterclockwise sense. Verify your result using (i) Laurent series (ii) partial fractions.

Solution.: The integrand has two singularities at z=0 and z=1 and both of which are interior to the contour C as we see in the figure. We need to find the residues for both the singularities.

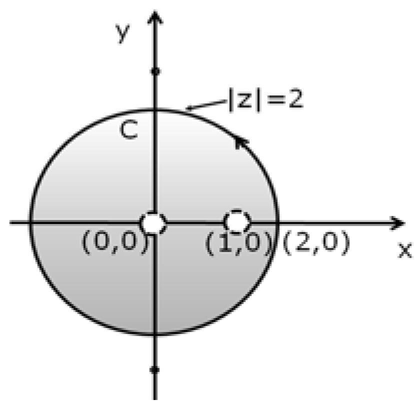

Fig.64: Contour of radius 2, having singularities at z=0 and z=1

(i) Laurent Series approach.

We know that

$$\frac{1}{(1-z)} = 1 + z + z^2 + z^3 + \cdots + \qquad ; |z| < 1$$

So we expand the function in the different domains and observe in 0<|z|<1

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\left(\frac{-1}{(1-z)}\right) = \frac{5z-2}{z}(-1-z-z^2-z^3-\cdots)$$

$$= \frac{2}{z} - 3 - 3z - \cdots$$

The coefficient of 1/z is the residue $A_{-1} = \underset{z=0}{Res} f(z) = 2$.

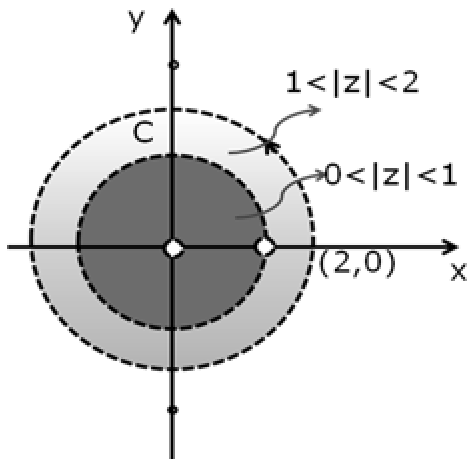

Fig.65: Different domains. Darker circle is for |z|<1 case. The lighter is the annular domain 1<|1|<2

Now we observe in the domain 1<|z|<2

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1}\left(\frac{1}{1+(z-1)}\right)$$

$$\left[5 + \frac{3}{z-1}\right](1-(z-1)+(z-1)^2-\cdots \quad) \quad ;0 < |z-1| < 1$$

The coefficient of 1/(z-1) is the residue $A_{-1} = \underset{z=1}{Res} f(z) = 3$.

Thus by residue theorem

$$\oint_C \frac{5z-2}{z(z-1)}dz = 2\pi i\,(\underset{z=0}{Res} f(z) + \underset{z=1}{Res} f(z)) = 2\pi i(2+3) = 10\pi i$$

(ii) Partial fractions approach

The given function can be expressed as

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{(z-1)}$$

Thus we can write the integral clearly as

$$\oint \frac{5z-2}{z(z-1)}\,dz = \oint \frac{2}{z}dz + \oint \frac{3}{(z-1)}dz = 2\pi i(2+3) = 10\pi i$$

Both the results lead to the same result.

Example: Use the residue theorem to evaluate the integral

$$\oint_C \frac{dz}{z^3(z+3)}$$

where, C is the circle (i)|z|=2 or (ii) |z+2|=3 described in counterclockwise sense

Soln.:The given function is $f(z) = \frac{1}{z^3(z+3)}$. It has singularities at z=0, which is a pole of order 3 and a simple pole at z=-3

(i) We pick the case |z|=2. In this we see that the simple pole z=-3 does not lie in the concerned region. Thus there exists only one singularity at z=0. We evaluate that using the standard formula of the residue for order m

$$A_{-1} = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$
$$= \lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\{(z)^3 \frac{1}{z^3(z+3)}\} = \lim_{z\to 0} \frac{1}{2!} \frac{d}{dz}\{\frac{-1}{(z+3)^2}\} = \lim_{z\to 0} \frac{1}{2!} \frac{2}{(z+3)^3} = \frac{1}{27}$$

Thus using the residue theorem

$$\oint_C \frac{1}{z^3(z+3)}dz = 2\pi i \left(\underset{z=0}{Res} f(z)\right) = \frac{2\pi i}{27}$$

(ii) We pick the case |z+2|=3. In this we see that the simple pole z=-3 as well as pole of order 3 at z=0 lie inside the contour. We have already calculated the residue at z=0 in the above part. We need to evaluate not the residue at z=-3 only. It is a simple pole thus we have

$$A_{-1} = \lim_{z\to a} \{(z-a)\ \ f(z)\}$$

$$= \lim_{z\to -3} \{(z+3)\ \ \frac{1}{z^3(z+3)}\} = \lim_{z\to -3} \{\frac{1}{z^3}\} = \frac{-1}{27}$$

Thus using the residue theorem

$$\oint_C \frac{1}{z^3(z+3)}dz = 2\pi i \left(\underset{z=0}{Res} f(z) + \underset{z=-3}{Res} f(z)\right) = \frac{2\pi i}{27} + \frac{-2\pi i}{27} = 0$$

Applications of Residues

The residue theorem is one of the most powerful results of complex variable theory since it finds so many and so varied applications in mathematical analysis and physical sciences. Actually, the usefulness of the *residue theorem* stems from the fact that, even when the integrand is not innocent-looking, it *facilitates the evaluation of the integral by way of a rather straight forward calculation of residues at the singular points of the given function*. Further, the detailed shape of the contour is not relevant, except in so far as it encloses certain singular points. From pragmatic considerations, the residue theorem is of special importance in the evaluation of real integrals. It is not possible to cover all the applications in an elementary course such as this. Therefore, we shall concentrate only on the evaluation of some types of definite integrals using the method of residues. Once you get familiar with the basic principles of this method, you should be able to apply these to more advanced applications.

POSSIBLE QUESTIONS

8 MARK

- State and Prove the Cauchy's integral theorem.
- Find the value of the integral ,(i)along the straight line from z=0 to z=1+I; (ii)along real axis from z=0 to z=1 and then along a line parallel to the imaginary axis from z=1 to z=1+i.
- State and prove Cauchy's integral formula.
- Find the different values of (1+i) 1/3
- Mention the properties of Moduli and Arguments.
- Write a short note on complex conjugates.
- Test the analyticity of the function w=sin z and hence derive that (sin z)=cos z
- Derive the Cauchy's Riemann differential equations.
- Find the value of the intergral (i)along y=x 2 ,having (0,0)(3,9)end points (ii)along y=3x between the same points.Do the values depend upon path.
- State and prove the Cauchy's integral theorem

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 21

DEPARTMENT OF PHYSICS

CLASS: III B. Sc., PHYSICS BATCH: 2015-2018

MATHEMATICAL PHYSICS (15PHU603A)

UNIT IV

QUESTIONS	CHOICE 1	CHOICE 2	CHOICE 3	CHOICE 4	ANSWER
The exponential form of a complex number is	**$z = re^{iq}$**	$z = e^{iq}$	z = cos q / r	z = r / cos q	**$z = re^{iq}$**
The construction of analytic function using………………method	integral	differential	Milne-Thomson	analytics	Milne-Thomson
Ifw=f(z)is continuous at every point of aregion of R,it is said to be ………………..in the entire region R.	continuous	dis continuous	piecewise continuous	uniform continuous	continuous
If Ur=(1/r) VΘ and Vr=-(1/r) UΘ are the Cauchy-Riemann equation in……………………form	cartesian	polar	integral	differential	polar
All polynomials in z are……………………..	continuous	differentiable	analytic	dis continuous	analytic
If f(z) derivative only at the origin, it is not……………………nowhere	analytic	argument	modulus	real	analytic
An analytic function with constant realpart is …………………	finite	infinite	constant	continuous	constant
An analytic function with constant modulus is …………………	finite	infinite	constant	zero	constant
Any function which has continuous second order partial derivatives and which satifies Laplace's equation is called……………………..function	Laplace	continuous	harmonic	integral	harmonic
Any function which satisfies the Laplace equation is known as	harmonic function	conjugate function	single function	analytic function	harmonic function
A single valued function f(z) which is differentiable at z = zo it is said to be	irregular function	analytic function	periodic function	all the above	analytic function
If a given number is wholly real, it is found in/on	a real axis	imaginary	x-y plane	space	imaginary
A set which entirely consists of interior points is known as	a null set	a bounded set	a closed set	an open set	an open set
The symbol i with the property $i^2 = -1$ was introduced by	Euler	Gauss	Cauchy	Reimann	Euler
In the Argand diagram, the fourth roots of unity forms a -------	rectangle	square	cube	none	square
The Conjugate of 1/i is	–i	i	1	-1	–i
The value of $i^2 + i^3 + i^4$ is	irregular function	regular function	infinite	none	none
The sum of n^{th} roots of unity are ------	0	**1**	2	3	0

In the Argand diagram, the fourth roots of unity forms a -------	square	rectangle	circle	rombus	square
A complex number can be represented as an ordered pair of …………… numbersx and y.	real	imaginary	whole	complex	real
A ………... number can be represented as an ordered pair of real numbers x and y.	natural	whole	complex	rational	complex
A complex number z=x+iy is zero if x=0 and y=…………	0	1	2	3	0
A complex number z=x+iy is ……………..if x=0 and y=0.	finite	infinite	zero	one	zero
If z=x+iy is a complexnumber, then Ž=x-iy is called ………………..of the complex number z	real	imaginary	conjugate	whole	conjugate
Two complex numbers z1=x1+iy1 & z2=x2+iy2 are equal if x1=x2 and y1……….y2	equal	not equal	greater than	less than	equal
The sum of two complex numbers are also…………………..numbers	real	imaginary	whole	complex	complex
The differeenceof two complex numbers are also…………………numbers	real	imaginary	complex	rational	complex
The product of two complex numbers are also ……………numbers	imaginary	complex	real	irrational	complex
The quotient of two complex numbers are also …………….numbers	real	imaginary	whole	complex	complex
A complex number z is represented by a point P in a plane,such a plane is called ………………….plane	real	cartesian	argand	polar	argand
In a complex number r=√(x2+y2) is called the ……………..of z	range	real	modulus	minus	modulus
The angleof a complex number is called the ………………of z	angle	argument	modulus	rational	argument
A single valued function w=f(z)of a complex variable z is said to be……………… at a point z0	one-one	onto	analytic	rational	analytic
A ………. valued function w=f(z)of a complex variable z is said to be analytic at a point z0	zero	single	double	triple	single
A single valued function w=f(z)of a complex variable z is said to be analytic at a point z0 if it has a …………….derivative at z0	zero	unique	second order	higher order	unique
The function f(z) is………………..in a region R if it has a derivative at every point of R.	analytic	argument	modulus	real	analytic
The function f(z) is analytic in a region R if it has a …………. at every point of R.	integral	derivative	zero	finite	derivative

The function f(z) is analytic in a region R if it has a derivative at every point of …………..	Z	R	Q	W	R
A point at which the function w=f(z) fails to be analytic is calld a ……………….of f(z)	singular	non singular	rational	irrational	singular
A point at which the function w=f(z) ……………... to be analytic is calld a singular of f(z)	correct	fails	zero	one	fails
A point at which the function w=f(z) fails to be ………………….is calld a singular point of f(z)	modulus	analytic	argument	continuous	analytic
If w=f(z) is an analytic function z, then the four……………..should exists	ordinary derivative	partial derivative	Total derivative	higher order derivative	partial derivative
If w=f(z) is an …………... function z, then the four partial order derivatives should exists	analytic	argument	modulus	real	analytic
The Cauchy-Riemann equations are otherwise called as………………..equation	C-R	C-E	R-E	C-I	C-R
The C-R equations satisfy …………….. And ……………………..	Ux=Vy & Vx=-Uy	Ux=Vy & Vx=-Uy	Ux=-Vy & Vx=-Uy	Ux=Vy & Vx=Uy	Ux=Vy & Vx=-Uy
If f(z)=u+iv is an analytic function of z=x+iy ,then u and v satisfy ………………..equation	integral	differential	Laplace	C-R equation	Laplace
If f(z)=u+iv is an ………...function of z=x+iy ,then u and v satisfy Laplce equation	real	analytic	imaginary	rational	analytic
Any function of x and y which possesses continuous first and second order partial derivatives and satifies………..	integral	differential	Laplace	C-R equation	Laplace
Any function of x and y which possesses …………... first and second order partial derivatives and satifies Laplace equation	uniform continuous	discontinuous	continuous	piecewise continuous	continuous
Any function of x and y which possesses continuous first and second order …………. derivatives and satifies Laplace equation	partial	ordinary	total	higher	partial
Any function of x and y which possesses continuous first and second order partial derivatives and satifies laplace equation is called a ………..equation	Laplace	integral	differential	harmonic	harmonic
The real and imaginary parts of an analytic function satisfy …………….equation	Laplace	integral	differential	harmonic	laplace

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UNIT-V

SYLLABUS

Arithmetic mean - Median - Quartiles - Deciles - Percentiles - Mode - Empirical relation between mean, median and mode - Geometric mean, harmonic mean - Relation between arithmetic mean, geometric mean and harmonic mean - Range - Range mean or average deviation - Standard deviation - Variance and mean square deviation.

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ARITHMETIC MEAN:

The arithmetic mean of a set of values is the quantity commonly called "the" mean or the average. Given a set of samples $\{x_i\}$, the arithmetic mean is

$$\bar{x} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i. \quad (1)$$

It can be computed in the Wolfram Language using Mean[*list*].

The arithmetic mean is the special case M_1 of the power mean and is one of the Pythagorean means.

When viewed as an estimator for the mean of the underlying distribution (known as the population mean), the arithmetic mean of a sample is called the sample mean.

For a continuous distribution function, the arithmetic mean of the population, denoted μ, $\bar{x}$, $\langle x \rangle$, or $A(x)$ and called the population mean of the distribution, is given by

$$\mu \equiv \int_{-\infty}^{\infty} P(x) f(x)\, dx, \quad (2)$$

where $\langle x \rangle$ is the expectation value. Similarly, for a discrete distribution,

$$\mu \equiv \sum_{n=1}^{N} P(x_n) f(x_n). \quad (3)$$

The arithmetic mean satisfies

$$\langle f(x) + g(x) \rangle = \langle f(x) \rangle + \langle g(x) \rangle \quad (4)$$

$$\langle c\, f(x) \rangle = c \langle f(x) \rangle, \quad (5)$$

and

$$\langle f(x)\, g(y) \rangle = \langle f(x) \rangle \langle g(y) \rangle \quad (6)$$

if x and y are independent statistics. The "sample mean," which is the mean estimated from a statistical sample, is an unbiased estimator for the population mean.

Hoehn and Niven (1985) show that

$$A(a_1 + c, a_2 + c, \ldots, a_n + c) = c + A(a_1, a_2, \ldots, a_n) \quad (7)$$

for any constant c. For positive arguments, the arithmetic mean satisfies

$$A \geq G \geq H, \quad (8)$$

where G is the geometric mean and H is the harmonic mean (Hardy *et al.* 1952, Mitrinović 1970, Beckenbach and Bellman 1983, Bullen *et al.* 1988, Mitrinović *et al.* 1993, Alzer 1996). This can be shown as follows. For $a, b > 0$,

$$\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right)^2 \geq 0 \quad (9)$$

$$\frac{1}{a} - \frac{2}{\sqrt{a\,b}} + \frac{1}{b} \geq 0 \qquad (10)$$

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{a\,b}} \qquad (11)$$

$$\sqrt{a\,b} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}} \qquad (12)$$

$$G \geq H, \qquad (13)$$

with equality iff $b = a$. To show the second part of the inequality,

$$\left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{a\,b} + b \geq 0 \qquad (14)$$

$$\frac{a+b}{2} \geq \sqrt{a\,b} \qquad (15)$$

$$A \geq G, \qquad (16)$$

with equality iff $a = b$. Combining (◇) and (◇) then gives (◇).

Given n independent random normally distributed variates X_i, each with population mean $\mu_i = \mu$ and variance $\sigma_i^2 = \sigma^2$,

$$\bar{x} \equiv \frac{1}{N} \sum_{i=1}^{N} x_i \qquad (17)$$

$$\langle \bar{x} \rangle = \frac{1}{N} \left\langle \sum_{i=1}^{N} x_i \right\rangle \qquad (18)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \langle x_i \rangle \qquad (19)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\mu \quad (20)$$

$$= \frac{1}{N}(N\mu) \quad (21)$$

$$= \mu, \quad (22)$$

so the sample mean is an unbiased estimator of the population mean. However, the distribution of $\bar{x}$ depends on the sample size. For large samples, $\bar{x}$ is approximately normal. For small samples, Student's *t*-distribution should be used.

The variance of the sample mean is independent of the distribution, and is given by

$$\text{var}(\bar{x}) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{N}x_i\right) \quad (23)$$

$$= \frac{1}{N^2}\text{var}\left(\sum_{i=1}^{N}x_i\right) \quad (24)$$

$$= \frac{1}{N^2}\sum_{i=1}^{n}\text{var}(x_i) \quad (25)$$

$$= \left(\frac{1}{N^2}\right)\sum_{i=1}^{N}\sigma^2 \quad (26)$$

$$= \frac{\sigma^2}{N}. \quad (27)$$

For small samples, the sample mean is a more efficient estimator of the population mean than the statistical median, and approximately $\pi/2$ less (Kenney and Keeping 1962, p. 211). Here, an estimator of a parameter of a probability distribution is said to be more efficient than another one if it has a smaller variance. In this case, the variance of the sample mean is generally less than the variance of the sample median. The relative efficiency of two estimators is the ratio of this variance.

A general expression that often holds approximately is

$$\text{mean} - \text{mode} \approx 3\,(\text{mean} - \text{median}) \qquad (28)$$

MEAN, MEDIAN, MODE AND RANGE:

Mean, median, and mode are three kinds of "averages". There are many "averages" in statistics, but these are, I think, the three most common, and are certainly the three you are most likely to encounter in your pre-statistics courses, if the topic comes up at all.

The "mean" is the "average" you're used to, where you add up all the numbers and then divide by the number of numbers. The "median" is the "middle" value in the list of numbers. To find the median, your numbers have to be listed in numerical order from smallest to largest, so you may have to rewrite your list before you can find the median. The "mode" is the value that occurs most often. If no number in the list is repeated, then there is no mode for the list.

The "range" of a list a numbers is just the difference between the largest and smallest values.

- ***Find the mean, median, mode, and range for the following list of values:***

  13, 18, 13, 14, 13, 16, 14, 21, 13

The mean is the usual average, so I'll add and then divide:

(13 + 18 + 13 + 14 + 13 + 16 + 14 + 21 + 13) ÷ 9 = 15

Note that the mean, in this case, isn't a value from the original list. This is a common result. You should not assume that your mean will be one of your original numbers.

The median is the middle value, so first I'll have to rewrite the list in numerical order:

13, 13, 13, 13, 14, 14, 16, 18, 21

There are nine numbers in the list, so the middle one will be the (9 + 1) ÷ 2 = 10 ÷ 2 = 5th number:

13, 13, 13, 13, 14, 14, 16, 18, 21

So the median is 14.

The mode is the number that is repeated more often than any other, so 13 is the mode.

The largest value in the list is 21, and the smallest is 13, so the range is 21 – 13 = 8.

mean: 15
median: 14
mode: 13
range: 8

Note: The formula for the place to find the median is "([the number of data points] + 1) ÷ 2", but you don't have to use this formula. You can just count in from both ends of the list until you meet in the middle, if you prefer, especially if your list is short. Either way will work.

- ***Find the mean, median, mode, and range for the following list of values:***

1, 2, 4, 7

The mean is the usual average:

(1 + 2 + 4 + 7) ÷ 4 = 14 ÷ 4 = 3.5

The median is the middle number. In this example, the numbers are already listed in numerical order, so I don't have to rewrite the list. But there is no "middle" number, because there are an even number of numbers. Because of this, the median of the list will be the mean (that is, the usual average) of the middle two values within the list. The middle two numbers are 2 and 4, so:

(2 + 4) ÷ 2 = 6 ÷ 2 = 3

So the median of this list is 3, a value that isn't in the list at all.

The mode is the number that is repeated most often, but all the numbers in this list appear only once, so there is no mode.

The largest value in the list is 7, the smallest is 1, and their difference is 6, so the range is 6.

mean: 3.5
median: 3
mode: none
range: 6

The values in the list above were all whole numbers, but the mean of the list was a decimal value. Getting a decimal value for the mean (or for the median, if you have an even number of data points) is perfectly okay; don't round your answers to try to match the format of the other numbers.

- ***Find the mean, median, mode, and range for the following list of values:***

8, 9, 10, 10, 10, 11, 11, 11, 12, 13

The mean is the usual average, so I'll add up and then divide:

(8 + 9 + 10 + 10 + 10 + 11 + 11 + 11 + 12 + 13) ÷ 10 = 105 ÷ 10 = 10.5

The median is the middle value. In a list of ten values, that will be the (10 + 1) ÷ 2 = 5.5-th value; the formula is reminding me, with that "point-five", that I'll need to average the fifth and sixth numbers to find the median. The fifth and sixth numbers are the last 10 and the first 11, so:

(10 + 11) ÷ 2 = 21 ÷ 2 = 10.5

The mode is the number repeated most often. This list has two values that are repeated three times; namely, 10 and 11, each repeated three times.

The largest value is 13 and the smallest is 8, so the range is 13 – 8 = 5.

mean: 10.5
median: 10.5
modes: 10 and 11
range: 5

As you can see, it is possible for two of the averages (the mean and the median, in this case) to have the same value. But this is *not* usual, and you should *not* expect it.

- ***A student has gotten the following grades on his tests: 87, 95, 76, and 88. He wants an 85 or better overall. What is the minimum grade he must get on the last test in order to achieve that average?***

The minimum grade is what I need to find. To find the average of all his grades (the known ones, plus the unknown one), I have to add up all the grades, and then divide by the number of grades. Since I don't have a score for the last test yet, I'll use a variable to stand for this unknown value: "x". Then computation to find the desired average is:

$(87 + 95 + 76 + 88 + x) \div 5 = 85$

Multiplying through by 5 and simplifying, I get:

$87 + 95 + 76 + 88 + x = 425$

$346 + x = 425$

$x = 79$

He needs to get at least a 79 on the last test.

QUARTILES:

Quartiles are values that divide a sample of data into four equal parts. With them you can quickly evaluate a data set's spread and central tendency, which are important first steps in understanding your data.

Quartile	**Description**
1st quartile (Q1)	25% of the data are less than or equal to this value.
2nd quartile (Q2)	The median. 50% of the data are less than or equal to this value.
3rd quartile (Q3)	75% of the data are less than or equal to this value.
Interquartile range	The distance between the 1st and 3rd quartiles (Q3-Q1); thus, it spans the middle 50% of the data.

For example, for the following data: 7, 9, 16, 36, 39, 45, 45, 46, 48, 51

- Q1 = 14.25
- Q2 (median) = 42
- Q3 = 46.50

- Interquartile range = 14.25 to 46.50, or 32.25

NOTE

Quartiles are calculated values, not observations in the data. It is often necessary to interpolate between two observations to calculate a quartile accurately.

Because they are not affected by extreme observations, the median and interquartile range are a better measure of central tendency and spread for highly skewed data than are the mean and standard deviation.

DECILES:

Deciles are the partition values which divide the set of observations into ten equal parts. There are nine deciles: D1,D2,D3,…,D9D1,D2,D3,…,D9. The first decile is D1D1, which is a point which has 10% of the observations below it.

D1=Value of (n+110)thitem

D2=Value of 2(n+110)thitem

D3=Value of 3(n+110)thitem

⋮

D9=Value of 9(n+110)thitem

Quartile for a Frequency Distribution (Discrete Data)

D1=Value of (n+110)thitem(n=∑f)

D2=Value of 2(n+110)thitem

D3=Value of 3(n+110)thitem

⋮

D9=Value of 9(n+110)thitem

Quartile for Grouped Frequency Distribution

D1 = l+hf(n10−c) (n=∑f)

D2=l+hf(2n10−c)

D3=l+hf(3n10−c)

⋮

D9=l+hf(9n10−c)D1 = l+hf(n10−c) (n=∑f)D2=l+hf(2n10−c)D3=l+hf(3n10−c)⋮D9=l+hf(9n10−c
)

PERCENTILES:

Percentiles are the points which divide the set of observations into one hundred equal parts. These points are denoted by P1,P2,P3,…,P99P1,P2,P3,…,P99, and are called the first, second, third... ninety ninth percentile. The percentiles are calculated for a very large number of observations like workers in factories and the populations in provinces or countries. Percentiles are usually calculated for grouped data. The first percentile denoted by P1P1 is calculated as P1=Value of (n100)thitemP1=Value of (n100)thitem. We find the group in which the (n100)th(n100)th item lies and then P1P1 is interpolated from the formula.

P1 = l+hf(n100−c) (n=∑f)

P2=l+hf(2n100−c)

P3=l+hf(3n100−c)

⋮

P99=l+hf(99n100−c)

GEOMETRIC MEAN:

The geometric mean of a sequence $\{a_i\}_{i=1}^{n}$ is defined by

$$G(a_1, \ldots, a_n) \equiv \left(\prod_{i=1}^{n} a_i\right)^{1/n}. \quad (1)$$

Thus,

$$G(a_1, a_2) = \sqrt{a_1 a_2} \quad (2)$$

$$G(a_1, a_2, a_3) = (a_1 a_2 a_3)^{1/3}, \quad (3)$$

and so on.

The geometric mean of a list of numbers may be computed using GeometricMean[*list*] in the Wolfram Languagepackage DescriptiveStatistics` .

For $n = 2$, the geometric mean is related to the arithmetic mean A and harmonic mean H by

$$G = \sqrt{A H} \quad (4)$$

(Havil 2003, p. 120).

The geometric mean is the special case M_0 of the power mean and is one of the Pythagorean means.

Hoehn and Niven (1985) show that

$$G(a_1 + c, a_2 + c, \ldots, a_n + c) > c + G(a_1, a_2, \ldots, a_n) \quad (5)$$

for any positive constant c.

HARMONIC MEAN:

The harmonic mean $H(x_1, ..., x_n)$ of n numbers x_i (where $i=1$, ..., n) is the number H defined by

$$\frac{1}{H} \equiv \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}. \tag{1}$$

The harmonic mean of a list of numbers may be computed in the Wolfram Language using HarmonicMean[*list*].

The special cases of $n=2$ and $n=3$ are therefore given by

$$H(x_1, x_2) = \frac{2 x_1 x_2}{x_1 + x_2} \tag{2}$$

$$H(x_1, x_2, x_3) = \frac{3 x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3}, \tag{3}$$

and so on.

The harmonic means of the integers from 1 to n for $n=1$, 2, ... are 1, 4/3, 18/11, 48/25, 300/137, 120/49, 980/363, ... (OEIS A102928 and A001008).

For $n=2$, the harmonic mean is related to the arithmetic mean A and geometric mean G by

$$H = \frac{G^2}{A} \tag{4}$$

The harmonic mean is the special case M_{-1} of the power mean and is one of the Pythagorean means. In older literature, it is sometimes called the subcontrary mean.

The volume-to-surface area ratio for a cylindrical container with height h and radius r and the mean curvature of a general surface are related to the harmonic mean.

Hoehn and Niven (1985) show that

$$H(a_1 + c, a_2 + c, \ldots, a_n + c) > c + H(a_1, a_2, \ldots, a_n) \quad (5)$$

for any positive constant c .

RELATION BETWEEN ARITHMETIC MEAN, GEOMETRIC MEAN AND HARMONIC MEAN:

For two numbers x and y, let x, a, y be a sequence of three numbers. If x, a, y is an arithmetic progression then 'a' is called *arithmetic mean*. If x, a, y is a geometric progression then 'a' is called *geometric mean*. If x, a, y form a harmonic progression then 'a' is called *harmonic mean*.

Let AM = arithmetic mean, GM = geometric mean, and HM = harmonic mean. The relationship between the three is given by the formula

AM×HM=GM2AM×HM=GM2

Below is the derivation of this relationship.

Derivation of AM × HM = GM^2

Arithmetic mean:

x,AM,yx,AM,y → arithmetic progression

Taking the common difference of arithmetic progression,

AM−x=y−AMAM−x=y−AM

x+y=2AMx+y=2AM → Equation (1)

Geometric Progression

x,GM,yx,GM,y → geometric progression

The common ratio of this geometric progression is

GMx=yGMGMx=yGM

xy=GM2xy=GM2 → Equation (2)

Harmonic Progression

x,HM,yx,HM,y → harmonic progression

1x,1HM,1y1x,1HM,1y → the reciprocal of each term will form an arithmetic progression

The common difference is

1HM−1x=1y−1HM1HM−1x=1y−1HM

2HM=1y+1x2HM=1y+1x

2HM=x+yxy2HM=x+yxy → Equation (3)

Substitute x + y = 2AM from Equation (1) and xy = GM^2 from Equation (2) to Equation (3)

2HM=2AMGM22HM=2AMGM2

GM2=AM×HMGM2=AM×HM

Range mean or average deviation , Standard deviation , Variance and mean square deviation:

Mean is a measure of central tendency. It measures what the majority of the data are doing toward the middle of a set. The mean is often referred to as the **average** of a data set. As an example, an algebra class has 10 students. Their grades on the last test were 85, 90, 87, 93, 100, 53, 78, 85, 99 and 82. What is the average grade for the students? To find mean, simply add all the numbers in a data set and divide by the number of items in the set:

85 + 90 + 87 + 93 + 100 + 53 + 78 + 85 + 99 + 82 = 852 852 / 10 = 85.2

The average, or mean, test grade in the class is 85.2.

Mode Occurs Most

Mode is another measure of central tendency. The mode is just the number that occurs most frequently. It's easy to remember because **mode** and **most** sound alike. Using the algebra class example, what grade occurred most frequently among the students? To answer, put the values in order:

53, 78, 82, 85, 85, 87, 90, 93, 99, 100

The only grade that occurred more than once is 85. Since 85 occurred **most**, the mode is 85.

Median Is the Middle, Range Is the Spread

Median is another measure of central tendency. The median is simply the **middle**number of a set. Put the numbers in order and look for one in the middle. If there is no middle number, add the two in the center and divide by 2. In the algebra class example, what is the median grade? To answer, put the values in order:

53, 78, 82, 85, 85, 87, 90, 93, 99, 100

Since there are an even number of test grades, there is no middle number. The two test grades in the middle are 85 and 87. Add them and divide by 2:

85 + 87 = 172 172 / 2 = 86

The median, or middle grade, is 86.

Range is a quick calculation. Range is simply the largest value minus the smallest. It shows you how spread out the numbers are. For these grades, subtract 53 from 100 to get the range of 47.

Find Variance Before Standard Deviation

Standard deviation is the square root of the variance, so you must find the variance first. **Variance** is the average of the squared difference of each number from the mean. That may sound confusing, but it's pretty simple to do. Take each number in the set and subtract if from the mean. Then square it. Add those values together, and divide by the number of items in your set. Working with the algebra class grades again, subtract each one from the mean:

85.2 - 53 = 32.2 85.2 - 78 = 7.2 85.2 - 82 = 3.2 85.2 - 85 = 0.2 85.2 - 85 = 0.2 85.2 - 87 = -1.8 85.2 - 90 = -4.8 85.2 - 93 = -7.8 85.2 - 99 = -13.8 85.2 - 100 = 14.8

Square each of those values, then add them together:

1,036.84 + 51.84 + 10.24 + 0.04 + 0.04 + 3.24 + 23.04 + 60.84 + 190.44 + 219.04 = 1,595.6

Finally, divide that sum by the number of items in the set, in this case 10:

1,595.6 / 10 = 159.56

The variance for this data set is 159.56.

Standard Deviation Measures Spread

Standard deviation is the measure of how spread out the numbers are from the center of a data set. A small standard deviation means a lot of the numbers are grouped around the middle of the set. A large standard deviation means that the number are spread out with some very high and low numbers. With the algebra grades, use this equation:

square root (159.56) = 12.63

The standard deviation for this data set is 12.63.

POSSIBLE QUESTIONS

8 MARK:

- . Following table gives the weight of 31 persons in a sample survey. Calculate geometric mean.

Weight	130	135	140	145	146	148	149	150	157
No. of persons	3	4	6	6	3	5	2	1	1

- The monthly incomes of 10 families in rupees in a certain village are given below. Calculate harmonic mean.

Family	1	2	3	4	5	6	7	8	9	10
Income	85	70	10	75	500	8	42	250	40	36

- Calculate geometric mean of the following

S.No	1	2	3	4	5
Values	50	72	54	82	93

- 10 students of B.Com class of a college have obtained the following marks in statics out of 100 marks. Calculate the standard deviation.

S.No	1	2	3	4	5	6	7	8	9	10
Marks	5	10	20	25	40	42	45	48	70	80

- Compute the mode from the following series

Size of item	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
frequency	20	24	32	28	20	16	34	10	8

- The arithmetic mean of following data is 28. Find the missing frequency and median.

Profits per shop	0-10	10-20	20-30	30-40	40-50	50-60
No of shops	12	18	27	7	17	6

- Calculate mean from the following data

Value	1	2	3	4	5	6	7	8	9	10
Frequency	21	30	28	40	26	34	40	9	15	57

- Find median for the following data

Roll No	1	2	3	4	5	6	7
Marks	45	32	18	57	65	28	46

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 21

DEPARTMENT OF PHYSICS

CLASS: III B. Sc., PHYSICS BATCH: 2015-2018

MATHEMATICAL PHYSICS (15PHU603A)

UNIT V

QUESTIONS	CHOICE 1	CHOICE 2	CHOICE 3	CHOICE 4	ANSWER
A measure of ……………...helps to get a single representative value for a set of unequal values.	central tendency	dispersion	final tendency	initial tendency	central tendency
A measure of central tendency helps to get a single representative value for a set of ………….. values.	equal	unequal	greater	smaller	unequal
……………….. is the total of the values of the items divided by their number	Mean	Median	Mode	Range	Mean
Arithmetic Mean is the ……………...of the values of the items divided by their number	sum	difference	equal	total	total
Arithmetic Mean is the total of the values of the items ……………. by their number	sum	difference	product	divided	divided
The ………….. of the deviations of the values from their arithemetic mean is zero.	sum	difference	product	divided	sum
The sum of the deviations of the values from their ……………….is zero.	arithmetic mean	arithmetic median	arithmetic mode	arithmetic range	arithmetic mean
The sum of the deviations of the values from their arithemetic mean is ………….....	zero	one	two	three	zero
……………... is the value of the middle most item when all the items are in order of magnitude.	Mean	Median	Mode	Range	Median
Median is the value of the ……………... most item when all the items are in order of magnitude.	initial	final	middle	higher	middle
……….. is the value which has the greatest frequency density.	Mean	Median	Mode	Range	Mode
Mode is the value which has the …………….frequency density.	smallest	greatest	initial	final	greatest
………………. mean is the appropriate root of the product of the values of the items.	arithmetic	geometric	harmonic	standard	geometric
Geometric mean is the appropriate……………….of the product of the values of the items.	sum	difference	root	quotient	root
Geometric mean is the appropriate root of the …………….of the values of the items.	sum	difference	product	divided	product
……………... is the reciprocal of the mean of reciprocals of the values of the items	arithmetic	geometric	harmonic	standard	harmonic
Harmonic mean is the ……………..of the mean of reciprocals of the values of the items	sum	difference	root	reciprocal	reciprocal
Harmonic mean is the reciprocal of the ………… of reciprocals of the values of the items	Mean	Median	Mode	Range	Mean

In symmetical distributions the relation is ………………	mean=median=mode	mean≠median=mode	mean=median≠mode	mean≠median≠mode	mean=median=mode
The relation between the means is …………	A.M< G.M< H.M	A.M = G.M =H.M	A.M > G.M > H.M	A.M≠ G.M≠ H.M	A.M > G.M > H.M
…………...are positional values.	quartile	mean	median	standard	quartile
…………divide the total frequency into ten equal parts and hence their name.	quartile	deciles	percentiles	mean	deciles
Deciles divide the ……………frequency into ten equal parts and hence their name.	sum	difference	equal	total	total
Deciles divide the total frequency into ……….. equal parts and hence their name.	zero	five	ten	twenty	ten
………… divide the total frequency into hundred equal parts and hence their name.	quartile	deciles	percentiles	mean	percentiles
Percentiles divide the total frequency into ……….. parts and hence their name.	ten	twenty	fifty	hundred	hundred
………….measures give pure numbers which are free form the units of measurements of data.	Relative	absolute	possibility	finite	Relative
Relative measures give …………...which are free form the units of measurements of data.	real numbers	pure numbers	complex numbers	imaginary numbers	pure numbers
Relative measures give pure numbers which are free form the …………. of measurements of data.	scale	value	units	range	units
…………………..and ………………measures are two kinds of measures of dispersion.	absolute and possibility	finite and infinite	non relative and relative	absolute and relative	absolute and relative
……………... is the difference between the greatest and smallest of the values.	Median	Mean	Range	Mode	Range
Range is the …………... between the greatest and smallest of the values.	sum	difference	product	quotient	difference
Range is the difference between the ………………….. of the values.	smallest and greatest	greatest and smallest	finite and infinite	greatest and infinite	greatest and smallest
…………………. is used in statistical quality control.	Median	Mean	Range	Mode	Range
Range is used in statistical ……………….. control.	units	constant	quality	value	quality
……………deviation is half of the difference between first and third quartiles.	quartile	mean	median	standard	quartile
Quartile deviation is ……………..of the difference between first and third quartiles.	one fourth	half	one third	three fourth	half
Quartile deviation is half of the difference between ………………..quartiles.	first and third	first and two	two and third	third and fourth	first and third
There are …………….. kinds of mean deviations	one	two	three	four	three
…………………….. the root mean square deviation of the values from their arithmetic mean	mean	median	mode	standard deviation	standard deviation
Standard deviation the……………….deviation of the values from their arithmetic mean	root mean square	root median square	root mode square	root range square	root mean square

Standard deviation the root mean square deviation of the values from their arithmetic …………	mean	median	mode	standard deviation	mean
…………... deviationof the values from the arithmetic mean is known as variance.	Mean square	root mean square	range square	standard deviation	Mean square
Mean square deviationof the values from the arithmetic mean is known as variance.	arithmetic range	arithmetic mode	arithmetic median	arithmetic mean	arithmetic mean
Mean square deviationof the values from the arithmetic mean is known as ………………	mean	median	variance	standard deviation	variance
………………….. is the positive square root of variance.	mean	median	variance	standard deviation	standard deviation
Standarad deviation is the positive …………...of variance.	square root	cubic root	fourth root	fifth root	square root
Standarad deviation is the positive square root of ………..	mean	median	variance	standard deviation	variance
The formula for range is ………..	L-S	L+S	L*S	L/S	L-S

PREPARED BY N.GEETHA ,ASSISTANT PROFESSOR, DEPARTMENT OF PHYSICS, KAHE.

Reg. No : -----------------

[15PHU603A]

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Under Section 3 of UGC Act 1956)

COIMBATORE -641 021

(For the candidates admitted from 2015 onwards)

B.Sc. DEGREE EXAMINATIONS,

Sixth Semester

I INTERNAL EXAMINATION

PHYSICS

Mathematical Physics

Duration: 2 hrs **Max Marks: 50**

PART – A

(20 x 1 = 20 marks)

1. The scalar is the quantity having__________.
   a)magnitude b) direction c) both magnitude and direction d) magnitude but no direction.
2. The vector is the quantity having ___________.
   a) magnitude b) direction c) both magnitude and direction d) magnitude but no direction.
3. Kronecker delta symbol is
   a) covariant tensor b) a contravariant tensor c) an invariant d) a mixed tensor
4. The example for scalar quantity are ___________.
   a)mass b)force c) acceleration d) displacement
5. The example for vector quantity are ___________.
   a)velocity b) time c) volume d) work
6. The vector whose magnitude are different but they have the same direction and sense is called _________.
   a) like vector b) unlike vector c) Equal vector d) unit vector
7. The vector whose magnitude is equal but have opposite sense is called __________.
   a) Negative vector b) unlike vector c) Equal vector d) unit vector
8. The vector whose magnitude is zero is called __________.
   a) like vector b) Zero vector c) Equal vector d) unit vector
9. The vector whose magnitude is unity is called ___________.
   a) like vector b) Zero vector c) Equal vector d) unit vector
10. The vector associated with a linear directional affect are called __________.
    a) like vector b) Zero vector c) Equal vector d)polar vector
11. The vector associated with rotation about an axis are called ___________.
    a) like vector b) Zero vector c) Equal vector d)axial vector
12. The example for polar vector is ___________.
    a) torque b) force c) angular velocity d)angular momentum
13. The example for axial vector is ___________.
    a) torque b) force c) linear velocity d)linear momentum
14. The scalar product is also called as ____________.
    a) dot product b) cross product c)del product d)nabla product
15. The vector product is also called as ____________.
    a) dot product b) cross product c) del product d)nabla product
16. A matrix may be defined as a ____________ array of numbers.

a) square b) rectangle c) square or rectangle
d)both square and rectangle

17. A matrix having the same number of rows and columns are called as __________.
    a) diagonal matrix b) square matrix
    c) scalar matrix d)unit matrix
18. If all the elements in the square matrix is zero except in the leading diagonal is called _____.
    a) diagonal matrix b) square matrix
    c) scalar matrix d)unit matrix
19. A scalar matrix in which each diagonal element is unity is called ___________
    a) diagonal matrix b) square matrix
    c) scalar matrix d)unit matrix
20. A diagonal matrix in which all diagonal element are equal is called ___________
    a) diagonal matrix b) square matrix
    c) scalar matrix d)unit matrix

PART – B

Answer all the following questions

(3 x 10 = 30 marks)

21. a. State and derive Gauss divergence theorem.

(OR)

b. Show that (i)div (φ A) = φ div A + A .grad φ
(ii)curl (φ A)= φ curl A + (grad φ) x A

22. a. Show that r^n r is an irrotational vector for any value of n, but is solenoidal only if n= - 3 (r is position vector of a point).

(OR)

b. Show that (i)div (AXB) =B.curl A – A.curl B.
(ii) curl (AXB)=(B. ∇)A-(A. ∇)B +A div B – B div A.
(iii) curl curl A= grad div A- ∇^2 . A.

23. a Find the Eigen values of the matrix A = $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(OR)

b.Explain the gradient of the scalar field with its physical significance.