16BECC102 **ENGINEERING MATHEMATICS I** **3 2 0 4**

OBJECTIVES:

- To develop analytical skills for solving different engineering problems.
- To understand the concepts of Matrices, sequences and series.
- To solve problems by applying Differential Calculus and Differential equations.

INTENDED OUTCOMES:

The student will be able to

- apply advanced matrix knowledge to Engineering problems.
- improve their ability in solving geometrical applications of differential calculus problems
- solve engineering problems involving hyperbolic functions, Beta and Gamma functions
- expose the concept of sequences and series

UNIT I MATRICES
(12)
Review of Matrix Algebra - Characteristic equation – Eigenvalues and Eigenvectors of a real matrix – Properties – Cayley-Hamilton theorem (excluding proof) – Orthogonal transformation of a symmetric matrix to diagonal form – Quadratic forms – Reduction to canonical form through orthogonal reduction.

UNIT II DIFFERENTIAL CALCULUS
(12)
Overview of Derivatives - Curvature in Cartesian co-ordinates – Centre and radius of curvature – Circle of curvature – Evolutes – Envelopes- Evolutes as Envelope of normals – Maxima and Minima of functions of two or more Variables – Method of Lagrangian Multipliers

UNIT III DIFFERENTIAL EQUATIONS
(11)
Linear Differential equations of second and higher order with constant coefficients - Euler's form of Differential equations – Method of variation parameters.

UNIT -IV FUNCTIONS OF SEVERAL VARIABLES **(12)**
Partial derivatives – Euler's theorem for homogeneous functions – Total derivatives – Differentiation of implicit functions – Jacobians –Maxima and Minima of functions of two or more Variables - Method of Lagrangian multipliers.

UNIT V SEQUENCES AND SERIES
(13)
Sequences: Definition and examples – **Series:** Types and Convergence – Series of positive terms – Tests of convergence: Comparision test, Integral test and D'Alembert's ratio test – Alternating series – Leibnitz's test – Series of positive and negative terms – Absolute and conditional convergence.

Total: 60

TEXT BOOKS:

S. No.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAR OF PUBLICATION
1	Hemamalini. P.T	Engineering Mathematics	McGraw Hill Education (India) Private Limited, New Delhi.	2014
2	Sundaram, V. Lakhminarayan,K.A. & Balasubramanian,R.	Engineering Mathematics for first year.	Vikas Publishing Home, New Delhi.	2006

REFERENCES:

S. No.	AUTHOR(S) NAME	TITLE OF THE BOOK	PUBLISHER	YEAR OF PUBLICATION
1	Grewel . B. S.	Higher Engineering Mathematics	Khanna Publications, New Delhi.	2014
2	Bhaskar Rao. P. B, Sri Ramachary SKVS, Bhujanga Rao. M	Engineering Mathematics I	BS Publications, India.	2010
3	Ramana. B.V	Higher Engineering Mathematics	Tata McGraw Hill Publishing Company, New Delhi.	2007
4	Shahnaz Bathul	Text book of Engineering Mathematics(Special Functions and Complex Variables)	PHI Publications, New Delhi.	2009
5	Michael D. Greenberg	Advanced Engineering Mathematics	Pearson Education, India	2009

WEBSITES :

1. www.efunda.com
2. www.mathcentre.ac.uk
3. www.intmath.com/matrices-determinants
4. www. Intmath.com/calculus/calculus-intro.php

KARPAGAM UNIVERSITY

Deemed to be University Established Under Section 3 of UGC Act 1956)

COIMBATORE-641 021

DEPARTMENT OF SCIENCE AND HUMANITIES

FACULTY OF ENGINEERING

B.E - (Regular) - I Semester

LESSON PLAN

SUBJECT: Engineering Mathematics –I

SUB CODE: 16BECC102 / 16BTAS102 / 16BTPE102 / 16BTCE102

S.NO	Topics covered	No. of hours
	UNIT-I MATRICES	
1	Introduction of Matrix Algebra	1
2	Characteristic Equation - Eigen values and Eigen vectors	1
3	Characteristic Equation - Eigen values and Eigen vectors	1
4	Tutorial 1: Characteristic Equation - Eigen values and Eigen vectors	1
5	Problems based on Properties	1
6	Problems based on Cayley – Hamilton theorem	1
7	Problems based on Cayley – Hamilton theorem	1
8	Tutorial 2: Problems based on Properties and Cayley – Hamilton theorem	1
9	Orthogonal transformation of a symmetric matrix to diagonal form	1
10	Tutorial 3: Orthogonal transformation of a symmetric matrix to diagonal form	1
11	Quadratic forms and Reduction to canonical form through orthogonal reduction	1
12	Canonical form through orthogonal reduction	1
13	Tutorial 4: Canonical form through orthogonal reduction	1
	Total	13
	UNIT II DIFFERENTIAL CALCULUS	
14	Introduction of Derivatives and Curvature in Cartesian co-ordinates	1
15	Curvature in Cartesian co-ordinates, Radius of curvature	1
16	Problems based on Centre of curvature	1
17	Tutorial 5: Radius and Centre of curvature	
18	Problems based on Circle of curvature	1
19	Tutorial 6: Circle of curvature	
20	Evolute – Problems	1
21	Evolute – Problems	1
22	Tutorial 7: Problems based on Evolute	1
23	Problems based on Envelope and Problems on Evolutes as Envelope of normal	1
24	Problems based on Envelope and Problems on Evolutes as Envelope of normal	
25	Tutorial 8: Problems based on Envelope	1
	Total	12
	UNIT – III DIFFERENTIAL EQUATIONS	
26	Introduction of Differential Equations and Equations of the first order and Higher Order	1
27	Linear Differential equations of second order with constant coefficients	1

28	Linear Differential equations of higher order with constant coefficients	1
29	Tutorial 9: Linear Differential equations of higher order with constant coefficients	1
30	Euler's form of Differential equations	1
31	Euler's form of Differential equations	1
32	Tutorial 10: Euler's form of Differential equations	1
33	Method of variation parameters	1
34	Method of variation parameters	1
35	Tutorial 11: Method of variation parameters	1
36	Tutorial 12: Method of variation parameters	1
	Total	11
	UNIT – IV FUNCTIONS OF SEVERAL VARIABLES	
37	Introduction of Partial derivatives	1
38	Problems - Euler's theorem for homogeneous functions	1
39	Problems - Total derivatives	1
40	Tutorial 13: Problems - Euler's theorem for homogeneous functions and total derivative	1
41	Differentiation of implicit functions	1
42	Definition- Jacobians and problems	1
43	Tutorial 14: Problems based on implicit functions and Jacobians	1
44	Maxima and Minima of functions of two or more Variables	1
45	Maxima and Minima of functions of two or more Variables	1
46	Tutorial 15: Maxima and Minima of functions of two or more Variables	1
47	Method of Lagrangian multipliers.	1
48	Tutorial 16: Method of Lagrangian multipliers.	1
	Total	12
	UNIT V SEQUENCES AND SERIES	
49	Introduction of Sequences and Series - Definition - examples	1
50	Series: Types and Convergence - Series of positive terms	1
51	Tutorial 17: Problems based on Convergence	1
52	Introduction of Tests of convergence - Comparison test and Integral test	1
53	D' Alembert's ratio test	1
54	Tutorial 18: Problems based on Comparison test and Integral test	1
55	Alternating series - Problems	1
56	Leibnitz's test - Problems	1
57	Tutorial 19: Problems based on Leibnitz's test	1
58	Series of positive and negative terms and Absolute and conditional convergence	1
59	Tutorial 20: Problems based on Absolute and conditional convergence	1
60	Discussion of previous years ESE Questions	1
	Total	12
	TOTAL	**60**

Faculty Incharge **HoD**

3 Cayley–Hamilton Theorem

Chapter Outline

- Introduction
- Cayley–Hamilton Theorem

3.1 ❑ INTRODUCTION

This theorem provides an alternative method for finding the inverse of a matrix, and any positive integral power of A can be expressed as a linear combination of those of lower degree.

3.2 ❑ CAYLEY–HAMILTON THEOREM

Every square matrix satisfies its own characteristic equation.

Application

The Cayley–Hamilton theorem can be used to find

- The power of a matrix, and
- The inverse of an $n \times n$ matrix A, by expressing these as polynomials in A of degree $< n$.

SOLVED EXAMPLES

Example 1 Verify that the matrix $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation and, hence, find A^4. **[KU May 2010, AU Jan. 2010]**

Solution The characteristic equation is $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

i.e., $$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

According to Cayley–Hamilton theorem, to prove $A^3 - 6A^2 + 8A - 3I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

Hence, $A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the given matrix A satisfies its own characteristic equation, i.e., $A^3 - 6A^2 + 8A - 3I = 0$

Multiplying on both sides by A, we get

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$A^4 = \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$ **Ans.**

Example 2 Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and, hence, find A^{-1} and A^4. **[KU Nov. 2010]**

Solution The characteristic equation is $|A - \lambda I| = 0$,

i.e., $$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$

To prove $A^3 - 3A^2 - 9A - 5I = 0$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$$

$$\therefore \quad A^3 - 3A^2 - 9A - 5I = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix} - \begin{bmatrix} 27 & 24 & 24 \\ 24 & 27 & 24 \\ 24 & 24 & 27 \end{bmatrix} - \begin{bmatrix} 9 & 18 & 18 \\ 18 & 9 & 18 \\ 18 & 18 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the Cayley–Hamilton theorem is verified.

$$A^3 - 3A^2 - 9A - 5I = 0 \qquad (1)$$

To find A^{-1}

$\div$ by $A \Rightarrow A^2 - 3A - 9I - 5A^{-1} = 0$

i.e., $-5A^{-1} = -A^2 + 3A + 9I$

$$-5A^{-1} = \begin{bmatrix} -9 & -8 & -8 \\ -8 & -9 & -8 \\ -8 & -8 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & 6 \\ 6 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 9 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix}$$

$$-5A^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = -\frac{1}{5}\begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

To find A^4, multiply (1) by A

$$A^4 - 3A^3 - 9A^2 - 5A = 0$$

i.e., $A^4 = 3A^3 + 9A^2 + 5A$

$$= \begin{bmatrix} 123 & 126 & 126 \\ 126 & 123 & 126 \\ 126 & 126 & 123 \end{bmatrix} + \begin{bmatrix} 81 & 72 & 72 \\ 72 & 81 & 72 \\ 72 & 72 & 81 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 209 & 208 & 208 \\ 208 & 209 & 208 \\ 208 & 208 & 209 \end{bmatrix}$$

Ans.

EXERCISE

Part A

1. State Cayley–Hamilton theorem.
2. Give two uses of the Cayley–Hamilton theorem.
3. If $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, write A^2 in terms of A and I, using Cayley–Hamilton theorem.
4. Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$.
5. Using Cayley–Hamilton theorem, find the inverse of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
6. Verify Cayley–Hamilton theorem for $\begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}$.
7. Verify Cayley–Hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.
8. Using Cayley–Hamilton theorem, find the inverse of $\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$
9. The Cayley–Hamilton theorem is used to find ____________
   (a) Eigen values (b) Eigen vectors
   (c) inverse and higher powers of A (d) quadratic form

Part B

1. Using Cayley–Hamilton theorem, find A^4 if $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$\left(\textbf{Ans.} \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix} \right)$$

2. Using Cayley–Hamilton theorem, find the inverse of the matrix

$A = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}$ $\left(\textbf{Ans.} \begin{bmatrix} 8 & 0 & -3 \\ -43 & 1 & 17 \\ 3 & 0 & -1 \end{bmatrix} \right)$

3. Find the characteristic equation of the matrix $A=\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}$. Show that the equation is satisfied by A and, hence, obtain the inverse of the given matrix. **[KU April 2011]**

$$\left(\textbf{Ans. } \lambda^3-4\lambda^2-20\lambda-35=0;\ A^{-1}=\frac{1}{35}\begin{bmatrix}-4 & 11 & -5\\ -1 & -6 & 25\\ 6 & 1 & -10\end{bmatrix}\right)$$

4. Find the characteristic equation of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}$. Show that the equation is satisfied by A. (**Ans.** $\lambda^3+\lambda^2-18\lambda-40=0$)

5. Using Cayley–Hamilton theorem, find the inverse of (i) $\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$ (ii) $\begin{bmatrix}7 & -1 & 3\\ 6 & 1 & 4\\ 2 & 4 & 8\end{bmatrix}$

$$\left(\textbf{Ans. } \text{(i)} \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}\ \text{(ii)}\ \frac{1}{50}\begin{bmatrix}-8 & 20 & -7\\ -40 & 50 & -10\\ 22 & -30 & 13\end{bmatrix}\right)$$

6. Find the characteristic equation of the matrix $A=\begin{bmatrix}3 & 1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$. Verify Cayley–Hamilton theorem for this matrix. Hence, find A^{-1}.

$$\left(\textbf{Ans. } A^{-1}=\frac{1}{20}\begin{bmatrix}7 & -2 & -3\\ 1 & 4 & 1\\ -2 & 2 & 8\end{bmatrix}\right)$$

7. Use Cayley–Hamilton theorem to find the inverse of the matrix

$A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ $\left(\textbf{Ans. } A^{-1}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\right)$

8. Using Cayley–Hamilton theorem, find A^{-1} given that $A=\begin{bmatrix}2 & -1 & 3\\ 1 & 0 & 2\\ 4 & -2 & 1\end{bmatrix}$

$$\left(\textbf{Ans. } A^{-1}=-\frac{1}{5}\begin{bmatrix}4 & -5 & -2\\ 7 & -10 & -1\\ -2 & 0 & 1\end{bmatrix}\right)$$

9. Using Cayley–Hamilton theorem, find the inverse of the matrix

$$A=\begin{bmatrix}5 & -1 & 5\\ 0 & 2 & 0\\ -5 & 3 & -15\end{bmatrix}.$$

$$\left(\textbf{Ans. } A^{-1}=\frac{1}{10}\begin{bmatrix}3 & 0 & 1\\ 0 & 5 & 0\\ -1 & 1 & -1\end{bmatrix}\right)$$

10. Find the characteristic equation of the matrix $A=\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}$ and show that the equation is also satisfied by A. (**Ans.** $\lambda^3-4\lambda^2-20\lambda-35=0$)

11. Verify Cayley–Hamilton theorem and hence find the inverse of the matrix

$$A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}.$$

$$\left(\textbf{Ans. }\begin{bmatrix}\frac{1}{10} & \frac{3}{10} & \frac{1}{5}\\ \frac{21}{10} & \frac{-7}{20} & \frac{-2}{5}\\ \frac{-9}{10} & \frac{3}{10} & \frac{1}{5}\end{bmatrix}\right)$$

4 Diagonalization of Square Matrices

Chapter Outline

- Introduction
- Diagonalization of Square Matrices
- Diagonalization by Orthogonal Transformation or Orthogonal Reduction

4.1 ❑ INTRODUCTION

Two square matrices A and B are said to be **similar** if there exists a nonsingular matrix C such that $B = C^{-1}AC$. The transformation A to $C^{-1}AC$ is called **similarity transformation**. The determinant, rank and Eigen values are preserved under similarity transformation. A matrix is said to be diagonalizable if it is similar to a diagonal matrix. The determinant of a diagonal matrix is simply the product of the diagonal elements; the rank is the number of nonzero diagonal elements and the Eigen values are the diagonal elements. Hence, it is very easy to deal with diagonal matrices.

4.2 ❑ DIAGONALIZATION OF SQUARE MATRICES

The process of finding a matrix M such that $M^{-1}AM = D$, where D is a diagonal matrix, is called diagonalization of the matrix A. As $M^{-1} AM = D$ is a similarity transformation, the matrices A and D are similar and, hence, A and D have the same Eigen values. The Eigen values of D are its diagonal elements. Thus, if we find a matrix M such that $M^{-1} AM = D$, D is a diagonal matrix whose diagonal elements are the Eigen values of A. A square matrix which is not diagonalizable is called **defective**.

Application

The direct application of diagonalization is that it gives us an easy way to compute large powers of a matrix A. The Eigen values of a system determine sometimes whether the system is stable or not. This has all to do with diagonalizing matrices. In quantum mechanical and quantum chemical computations, matrix diagonalization is one of the most frequently applied numerical processes.

➢ **Note**

(i) M is called the modal matrix of A whose elements are the Eigen vectors of A.

(ii) For this diagonalization process, A need not necessarily have distinct Eigen values. Even if two or more Eigen values of A are equal, the process holds good provided the Eigen vectors of A are linearly independent.

4.3 ❑ DIAGONALIZATION BY ORTHOGONAL TRANSFORMATION OR ORTHOGONAL REDUCTION

The process of finding a normalized modal matrix N such that $N^{-1}AN = D$ where D is a diagonal matrix is called orthogonal transformation or orthogonal reduction. The elements of N are the normalized Eigen vectors of A and it can be proved that N is an orthogonal matrix (i.e. $N^{-1} = N^T$). It is important to note that diagonalization by orthogonal transformation is possible only for a real symmetric matrix.

SOLVED EXAMPLES

Example 1 Reduce the matrix $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$ to diagonal form. **[AU Jan. 2010]**

Solution Let $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

Here, $D_1 = 17, D_2 = 42, D_3 = 0$.

∴ the characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$.

i.e., $$\lambda(\lambda^2 - 17\lambda + 42) = 0$$
$$\lambda(\lambda - 14)(\lambda - 3) = 0$$

$\Rightarrow$ $$\lambda = 0, 14, 3$$

∴ the Eigen values are 0, 14, 3.

To find the Eigen vectors, $[A - \lambda I]X = 0$.

i.e., $$\begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(10 - \lambda)x_1 - 2x_2 - 5x_3 = 0$$
$$-2x_1 + (2 - \lambda)x_2 + 3x_3 = 0$$
$$-5x_1 + 3x_2 + (5 - \lambda)x_3 = 0$$

$\lambda = 0$ gives $10x_1 - 2x_2 - 5x_3 = 0$; $-2x_1 + 2x_2 + 3x_3 = 0$; $-5x_1 + 3x_2 + 5x_3 = 0$.
Consider first two equations, which gives $x_1 = 1$, $x_2 = -5$, $x_3 = 4$.

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

$\lambda = 14$ gives

$$\begin{aligned} -4x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 - 12x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 - 9x_3 &= 0 \end{aligned}$$

Considering first two equations gives $x_1 = -3$, $x_2 = 1$, $x_3 = 2$.

$$\therefore \qquad X_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

$\lambda = 3$ gives

$$\begin{aligned} 7x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 - x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}$$

$\Rightarrow x_1 = 1$, $x_2 = 1$, $x_3 = 1$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \qquad M = \begin{bmatrix} 1 & -3 & 1 \\ -5 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$M^{-1} = \dfrac{1}{|M|}\text{Adj}M$ provided $|M| \neq 0$

$$|M| = -42$$

To find Adj$M$,
Co-factor of 1 = –1, Co-factor of –3 = 9, Co-factor of 1 = –14, Co-factor of 1 = –14, Co-factor of 1 = –3, Co-factor of –5 = 5
Co-factor of 4 = –4, Co-factor of 2 = –6, Co-factor of 1 = –14

$$\therefore \qquad \text{Adj}M = \begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}$$

$$\Rightarrow \qquad M^{-1} = -\frac{1}{42}\begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}$$

Consider

$$M^{-1}AM = -\frac{1}{42}\begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 & 1 \\ -5 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= -\frac{1}{42}\begin{bmatrix} -1 & 5 & -4 \\ 9 & -3 & -6 \\ -14 & -14 & -14 \end{bmatrix}\begin{bmatrix} 0 & -42 & 3 \\ 0 & 14 & 3 \\ 0 & 28 & 3 \end{bmatrix}$$

$$= -\frac{1}{42}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -588 & 0 \\ 0 & 0 & -126 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Proved.

Example 2 Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by orthogonal transformation.

[KU April 2011]

Solution The characteristic equation is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2-\lambda)(\lambda^2 - 2\lambda - 3) - (-\lambda - 1) - (-\lambda - 1) = 0$

$\Rightarrow$ $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$\Rightarrow$ $(\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$

$\therefore$ The Eigen values are –1, 1, 4.
The Eigen vectors are given by $(A - \lambda I)X = 0$.
when $\lambda = -1$

The Eigen vector is given by $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow$ $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

When $\lambda = 1$, the Eigen vector is given by $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow$ $X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

When $\lambda = 4$, the Eigen vector is given by $\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\Rightarrow \qquad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Hence, the modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore$ normalized modal matrix is,

$$N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

To prove $N^{-1} AN = D$, since N is an orthogonal matrix, it satisfies $N^{-1} = N^T$.
$\therefore$ it is enough to prove that $N^{-1} AN = D$.
Consider

$$N^{-1}AN = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

Proved.

EXERCISE

Part A

1. When are two matrices said to be similar?
2. Define diagonalizing a matrix.
3. What is the difference between diagonalization of a matrix by similarity and orthogonal transformations?
4. Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
5. Is it possible to diagonalize the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$?

   [**Ans:** The Eigen values $\lambda = 0, 0$ but there is only one Eigen vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So the matrix cannot be diagonalized.]
6. What type of matrices can be diagonalized using (i) similarity transformation, and (ii) orthogonal transformation?
7. In the orthogonal transformation $N^T AN = D$, D refers to a/an __________ matrix.
   (i) diagonal (ii) orthogonal
   (iii) symmetric (iv) skew-symmetric
8. In a modal matrix, the columns are the Eigen vectors of __________
   (i) A^{-1} (ii) A^2 (iii) A (iv) adj A
9. If $X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0,$ the Eigen vectors are said to be __________
   (i) dependent (ii) pairwise orthogonal
   (iii) skew-symmetric (iv) independent
10. If A is an orthogonal matrix, show that A^{-1} is also orthogonal.

Part B

1. Find the modal matrix of the following matrices.

   (i) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

   $\left[\textbf{Ans.} \text{ (i)} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ (ii)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \right]$

2. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

   (**Ans.** $A + 5I$)

3. Show that $A^T = A^{-1}$ for $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$.

4. Diagonalize the following matrices:

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\left[\textbf{Ans.}\ \text{(i)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \text{(ii)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(iii)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right]$

5. A square matrix A is defined by $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Find the modal matrix M and the resulting diagonal matrix D of A.

$\left(\textbf{Ans.}\ M = \begin{bmatrix} -1 & 1+\sqrt{5} & 1-\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}\right)$

6. Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Find a matrix M such that $M^{-1}AM$ is a diagonal matrix.

$\left(\textbf{Ans.}\ M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)$

7. Obtain the modal matrix and diagonalize the following matrices:

(i) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\left[\textbf{Ans.}\ \text{(i)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{(ii)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right]$

8. Diagonalize the matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$. $\left(\textbf{Ans.} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right)$

9. Diagonalize $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ by similarity transformation. $\left(\textbf{Ans.} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}\right)$

10. Diagonalize the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$. $\left(\textbf{Ans.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$

11. Diagonalize the following matrices by orthogonal transformation:

(i) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$\left[\textbf{Ans.}\ \text{(i)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{(ii)} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$

12. Diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation.

$\left(\textbf{Ans.} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$

13. Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ by orthogonal transformation.

$\left(\textbf{Ans.} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right)$

14. Diagonalize $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by orthogonal transformation.

$\left(\textbf{Ans.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)$ **[AU May 2011]**

2 Eigen Values, Eigen Vectors and the Characteristic Equation

Chapter Outline

- Introduction
- Characteristic Equation of a Matrix
- Important Properties of Eigen Values
- Linear Dependence and Independence of Vectors
- Properties of Eigen Vectors

2.1 ❑ INTRODUCTION

In this chapter, we shall discuss mainly square matrices A and throughout the ensuing discussion, any new facts and developments will be based on the determination of a vector X (to be called characteristic vector or Eigen vector) and a scalar λ (to be called characteristic value or Eigen value) such that $AX = \lambda X$. Based on these concepts of Eigen values and Eigen vectors, we shall indicate the conditions on A under which a nonsingular matrix P can be selected such that $P^{-1}AP$ is diagonal, i.e., A is similar to a diagonal matrix.

2.2 ❑ CHARACTERISTIC EQUATION OF A MATRIX

Characteristic Matrix

For a given matrix A, $A - \lambda I$ matrix is called the characteristic matrix, where λ is a scalar and I is the unit matrix.

Let $$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 3 & 1-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Characteristic Polynomial

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call the characteristic polynomial of the matrix A.

For example,

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 3 & 1-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - 3\lambda) - 2(-3\lambda + 5) + 1(\lambda + 5)$$
$$= -\lambda^3 + 5\lambda^2 + \lambda - 5$$

Characteristic Equation

The equation $|A - \lambda I| = 0$ is known as the characteristic equation of A and its roots are called the **characteristic roots** or **latent roots** or **Eigen values** or **characteristic values** or **latent values** or **proper values** of A.

Spectrum of A

The set of all Eigen values of the matrix A is called the spectrum of A.

Eigen-value Problem

The problem of finding the Eigen values of a matrix is known as an Eigen-value problem.

Characteristic Vector

Any nonzero vector X is said to be a characteristic vector of a matrix A if there exists a number λ such that $AX = \lambda X$, where λ is a characteristic root of a matrix A.

2.3 ❑ IMPORTANT PROPERTIES OF EIGEN VALUES

(i) Any square matrix A and its transpose A^T have the same Eigen values.

(ii) The sum of the Eigen values of a matrix is equal to the trace of the matrix.
[**Note**: The sum of the elements on the principal diagonal of a matrix is called the trace of the matrix.]

(iii) The product of the Eigen values of a matrix A is equal to the determinant of A.

(iv) If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the Eigen values of A then the Eigen values of

- (a) KA are $k\lambda_1, k\lambda_2 \ldots k\lambda_n$
- (b) A^m are $\lambda_1^m, \lambda_2^m \ldots \lambda_n^m$
- (c) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \cdots \frac{1}{\lambda_n}$.

(iv) The Eigen values of a real symmetric matrix (i.e. a symmetric matrix with real elements) are real.

2.4 ❑ LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

n-dimensional Vector or n-vector

An ordered set of n elements x_i of a field F written as

$$A = [x_1, x_2 \dots x_n] \tag{2.1}$$

is called an n-dimensional vector or n-vector over F and the elements $x_1, x_2 \dots x_n$ are called the first, second … nth components of A.

We find it more convenient to write the components of a vector in a column as

$$A^T = [x_1, x_2, x_3 \dots x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \tag{2.2}$$

Equation (2.1) is called a **row-vector** and Eq. (2.2) is called a **column-vector**.

Linear Dependence and Independence of Vectors

The vectors $A_1 = [x_{11}, x_{12}, x_{13} \dots x_{1m}]$, $A_2 = [x_{21}, x_{22}, x_{23} \dots x_{2m}] \dots A_n = [x_{n1}, x_{n2}, x_{n3} \dots x_{nm}]$ are called **linearly dependent** over F if there exists a set of n elements $\lambda_1, \lambda_2 \dots \lambda_n$ of F, λ_i's being not all zero, such that $\lambda_1 A_1 + \lambda_2 A_2 + \dots \lambda_n A_n = 0$.

Otherwise the n-vectors are called **linearly independent** over F.

2.5 ❑ PROPERTIES OF EIGEN VECTORS

(i) The Eigen vector X of a matrix A is not unique.

(ii) If $\lambda_1, \lambda_2 \dots \lambda_n$ be distinct Eigen values of an $n \times n$ matrix then the corresponding Eigen vectors $X_1, X_2 \dots X_n$ form a linearly independent set.

(iii) If two or more Eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to the equal roots.

(iv) Two Eigen vectors X_1 and X_2 are called orthogonal vectors if $X_1^T X_2 = 0$

(v) Eigen vectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

Applications

The Eigen-value and Eigen-vector method is useful in many fields because it can be used to solve homogeneous linear systems of differential equations with constant coefficients. Furthermore, in chemical engineering, many models are formed on the basis of systems of differential equations that are either linear or can be linearized and solved using the Eigen-value, Eigen-vector method. In general, most ordinary differential equations can be linearized and, therefore, solved by this method. Initial-value problems can also be solved by using the Eigen-value and Eigen-vector method.

Eigen-value analysis is also used in the designing of car stereo systems so that the sounds are directed appropriately for the listening pleasure of both the drivers and the passengers. Eigen-value analysis can indicate what needs to be changed to reduce the vibration of the car due to the music being played.

Oil companies frequently use Eigen-value analysis to explore land for oil. Oil, dirt and other substances give rise to linear systems which have different Eigen values, so Eigen-value analysis can give a good indication of where oil reserves are located.

Eigen values and Eigen vectors are used widely in science and engineering, particularly in physics. Rigid physical bodies have a preferred direction of rotation, about which they can rotate freely. For example, if someone were to throw a football, it would rotate around its axis while flying through the air. If someone were to hit the ball in the air, the ball would be likely to flop in a less simple way. Although this may seem like common sense, even rigid bodies with more complicated shapes will have preferred directions of rotation. These are called **axes of inertia**, and they are calculated by finding the Eigen vectors of a matrix called the **inertia tensor**. The Eigen values are also important and they are called **moments of inertia**.

SOLVED EXAMPLES

Example 1 Find the characteristic roots of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2-0 & 3-0 \\ 0-0 & 2-\lambda & 3-0 \\ 0-0 & 0-0 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)^2$$

$\therefore$ the characteristic equation of the matrix A is $(1 - \lambda)(2 - \lambda)^2 = 0$ and its roots are 1, 2, 2. **Ans.**

Example 2 Find the characteristic roots and corresponding characteristic vectors for the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution The characteristic equation is $|A - \lambda I| = 0$,

i.e.,
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(8 - \lambda)[(7 - \lambda)(3 - \lambda) - 16] + 6[-6(3 - \lambda) + 8] + 2[24 - 2(7 - \lambda)] = 0$

$\Rightarrow$ $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\Rightarrow$ $\lambda(-\lambda^2 + 18\lambda - 45) = 0$

$\Rightarrow$ $\lambda = 0, 3, 15$ are the characteristic roots of the matrix.

The characteristic vector X is obtained from $(A - \lambda I)X = 0$.

Case (i) $\lambda = 0$

If x, y, z are the components of a characteristic vector corresponding to the characteristic root 0, we have

$$(A - 0I)X = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$8x - 6y + 2z = 0$$
$$-6x + 7y - 4z = 0$$
$$2x - 4y + 3z = 0$$

$\therefore$ $$\frac{x}{21-16} = \frac{-y}{-18+8} = \frac{z}{24-8}$$

$\Rightarrow$ $$\frac{x}{5} = \frac{-y}{-10} = \frac{z}{10}$$

i.e., $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

$$(A - 3I)X = 0 \Rightarrow \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e.,
$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $$5x - 6y + 2z = 0$$
$$-6x + 4y - 4z = 0$$
$$2x - 4y = 0$$

$\therefore \quad \dfrac{x}{0-16} = \dfrac{-y}{0+8} = \dfrac{z}{24-8}$

$\Rightarrow \quad \dfrac{x}{-16} = \dfrac{-y}{8} = \dfrac{z}{16}$

$\Rightarrow \quad \dfrac{x}{-2} = \dfrac{y}{-1} = \dfrac{z}{2}$

$\therefore \quad X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Case (iii) $\lambda = 15$

$$(A-15I)\,X = 0 \Rightarrow \begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e., $\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\Rightarrow \quad -7x - 6y + 2z = 0$

$-6x - 8y - 4z = 0$

$2x - 4y - 12z = 0$

$\therefore \quad \dfrac{x}{96-16} = \dfrac{-y}{72+8} = \dfrac{z}{24+16}$

$\Rightarrow \quad \dfrac{x}{80} = \dfrac{-y}{80} = \dfrac{z}{40}$

$\therefore \quad \dfrac{x}{2} = \dfrac{y}{-2} = \dfrac{z}{1}$

$\therefore \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Hence, $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ **Ans.**

➢ **Note**

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the characteristic equation is given by $|A - \lambda I| = 0$ or $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$ where $D_1 = a_{11} + a_{22} + a_{33}$ (sum of the diagonals of A (or) trace of a matrix A)

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

= sum of the second-order minors of A whose principal diagonals lie along the principal diagonal of A.

$D_3 = |A|$ = determinant of A.

Example 3 Find the characteristic roots and corresponding characteristic vectors of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. **[KU Nov. 2010]**

Solution The characteristic equation is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

where $D_1 = 6 + 3 + 3 = 12$

$$D_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$= (18 - 4) + (18 - 4) + (9 - 1)$

$= 14 + 14 + 8$

$= 36$

$$D_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 6(9 - 1) + 2(-6 + 2) + 2(2 - 6)$

$= 48 - 8 - 8$

$= 32$

$\therefore$ the characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ and the roots are 2, 2, 8.

Case (i) $\lambda = 2$ (twice)

$$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e.,
$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$
$4x - 2y + 2z = 0$

$-2x + y - z = 0$

$2x - y + z = 0$

which are equivalent to a single equation . There is one equation in three unknowns.
$\therefore$ taking two of the unknowns, say $x = 1$ and $y = 0$, we get $z = -2$ and taking $x = 0$ and $y = 1$, we get $z = 1$.

$\therefore$
$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 8$

$$(A-8I)X = 0 \Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e.,
$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$
$$2x - y - 5z = 0$$

$\therefore$
$$\frac{x}{25-1} = \frac{-y}{10+2} = \frac{z}{2+10}$$
$$\frac{x}{24} = \frac{y}{-12} = \frac{z}{12}$$

$\therefore$
$$X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ **Ans.**

Example 4 The matrix A is defined as $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. Find the Eigen values of $3A^3 + 5A^2 - 6A + 2I$.

Solution The characteristic equation is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)(3-\lambda)(-2-\lambda) = 0$

i.e., $\lambda = 1, 3, -2$

Eigen values of $A^3 = 1, 27, -8$

Eigen values of $A^2 = 1, 9, 4$

Eigen values of $A = 1, 3, -2$

Eigen values of $I = 1, 1, 1$

$\therefore$ Eigen values of $3A^3 + 5A^2 - 6A + 2I$

First Eigen value $= 3(1)^3 + 5(1)^2 - 6(1) + 2 = 4$

Second Eigen value $= 3(27) + 5(9) - 6(3) + 2(1) = 110$

Third Eigen value $= 3(-8) + 5(4) - 6(-2) + 2(1) = 10$

$\therefore$ Required Eigen values are 4, 110, 10. **Ans.**

Example 5 Find the Eigen values and Eigen vectors of the matrix

$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. **[KU May 2010]**

Solution The characteristic equation is given by $|A - \lambda I| = 0$.

i.e., $$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow$ $(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 1, 2, 3$

To find Eigen vectors for the corresponding Eigen values, we will consider the matrix equation $(A - \lambda I)X = 0$.

Case (i) $\lambda = 1$

$$(A - \lambda I)X = 0 \quad \Rightarrow \quad \begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $-z = 0$

$\Rightarrow$ $x + y + z = 0$

$\Rightarrow$ $2x + 2y + 2z = 0$

Let $x = 1 \quad \Rightarrow \quad y = -1$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Case (ii) $\lambda = 2$

$$(A - \lambda I)X = 0 \quad \Rightarrow \quad \begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $-x - z = 0$

$x + z = 0$

$2x + 2y + z = 0$

$\therefore$ $$\frac{x}{-2} = \frac{y}{1} = \frac{z}{2}$$

$\therefore$ $$X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Case (iii) $\lambda = 3$

$$(A - \lambda I)X = 0 \quad \Rightarrow \quad \begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $-2x - z = 0$

$x - y + z = 0$

$2x + 2y = 0$

$\therefore$ $$\frac{x}{-2} = \frac{-y}{-2} = \frac{z}{4}$$

$$\therefore \qquad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Hence, the Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ **Ans.**

EXERCISE

Part A

1. If 1, 5 are the Eigen values of a matrix A, find the value of det A.
2. Find the constants a and b such that the matrix $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 and –2 as its Eigen values.
3. If the sum of two Eigen values and trace of a 3×3 matrix A are equal, find $|A|$.
4. What do you understand by the characteristic equation of the matrix A?
5. What is Eigen-value problem?
6. Find latent vectors of the matrix $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.
7. Define linearly dependent and linearly independent set of vectors.
8. Show that the set of vectors $X_1 = [1, 2, 3]$, $X_2 = [1, 0, 1]$ and $X_3 = [0, 1, 0]$ are linearly independent.
9. Prove that the set of vectors $X_1 = [1, 2, 3]$, $X_2 = [1, 0, 1]$ and $X_3 = [0, 1, 0]$ are linearly independent.
10. Define spectrum of a matrix.
11. Prove that any square matrix A and its transpose A^T have the same Eigen values.
12. Find the sum and product of the Eigen values of the matrix $A = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$.
13. Given $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, find the Eigen values of A^2.
14. Find the sum of the squares of the Eigen values of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.
15. Find the sum of the Eigen values of the inverse $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.
16. If A and B are 2 square matrices then what can you say about the characteristic roots of the matrices AB and BA?

17. If two of the Eigen values of a 3×3 matrix, whose determinant equals 4, are –1 and +2, what will be the third Eigen value of the matrix?

18. The matrix A is defined as $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Find the Eigen values of A^2.

19. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, find the Eigen values of $A^3 + 5A = 8I$.

20. The Eigen values of a matrix A are 1, – 2, 3. Find the Eigen values of $3I - 2A + A^2$.

Part B

1. Find the Eigen values of the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$. (**Ans.** 0, 1, – 2)

2. The matrix A is defined as $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$. Find the Eigen values of $3A^3 + 5A^2 + 6A + I$. (**Ans.** 15, –15, –53)

3. Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

   $\left(\textbf{Ans.} -1, 1, 2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right)$

4. Show that the vectors [1, 2, 0], [8, 13, 0] and [2, 3, 0] are linearly dependent.
5. Show the set of vectors [1, 1, 1], [1, 2, 3] and [2, 3, 8] are linearly independent.
6. Given that $A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$, verify that the sum and product of the Eigen values of A are equal to the trace of A and $|A|$ respectively.

7. Find the Eigen values and Eigen vectors of (adj$A$), where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

   $\left(\textbf{Ans.}\ 1, 4, 4, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$.

8. Verify that the Eigen vectors of the real symmetric matrix $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$ are orthogonal in pairs.

(**Hint:** Prove that $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$)

9. Find the Eigen values and Eigen vectors of the following matrices:

(i) $\begin{bmatrix}2 & -2 & 2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$ $\left(\textbf{Ans.} -2, 2, 2, \begin{bmatrix}-4\\ -1\\ 7\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}\right)$

(ii) $\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$ $\left(\textbf{Ans.} 1, 1, 5, \begin{bmatrix}1\\ 2\\ -5\end{bmatrix}, \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\right)$

(iii) $\begin{bmatrix}4 & 2 & -2\\ -5 & 3 & 2\\ -2 & 4 & 1\end{bmatrix}$ $\left(\textbf{Ans.} 1, 2, 5, \begin{bmatrix}2\\ 1\\ 4\end{bmatrix}, \begin{bmatrix}1\\ 1\\ 2\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right)$

(iv) $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 3 & 2 & 3\end{bmatrix}$ $\left(\textbf{Ans.} 0, 1, 5, \begin{bmatrix}0\\ -1\\ 1\end{bmatrix}, \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}4\\ 5\\ 11\end{bmatrix}\right)$

(v) $\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ **[KU April 2012]**

$\left(\textbf{Ans.} 5, -3, -3, \begin{bmatrix}1\\ 2\\ -1\end{bmatrix}, \begin{bmatrix}2\\ -1\\ 0\end{bmatrix}, \begin{bmatrix}3\\ 0\\ 1\end{bmatrix}\right)$

10. Find the Eigen values and Eigen vectors of (adj$A$), given that the matrix $A=\begin{bmatrix}2 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 2\end{bmatrix}$ **[KU May 2010]**

$\left(\textbf{Ans.} 1, 2, 3, \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}\right)$

Part I

Unit I Matrices

Characteristic equation; Eigen values and Eigen vectors of a real matrix; Properties; Cayley–Hamilton theorem (excluding proof); Orthogonal transformation of a symmetric matrix to diagonal form; Quadratic forms; Reduction to canonical form through orthogonal reduction.

Unit II Three-Dimensional Analytical Geometry

Direction ratios of the Line Joining two points; The plane; Plane through the intersection of two lines; The straight line; The plane and the straight line; Shortest distance between two skew lines; Equation of a sphere.

Unit III Geometrical Applications of Differential Calculus

Curvature in Cartesian coordinates; Centre and radius of curvature; Circle of curvature; Evolutes; Envelopes; Evolutes as envelope of normals.

Unit IV Functions of Several Variables

Partial derivatives; Euler's theorem for homogeneous functions; Total derivatives; Differentiation of implicit functions; Jacobians; Maxima and minima of functions of two or more variables; Method of Lagrangian multipliers.

Unit V Differential Equations

Equations of the first order and higher degree; Linear differential equations of second and higher order with constant coefficients; Euler's homogeneous linear differential equations; Mathematica software demonstration.

Unit I

Matrices

1 Matrices

Chapter Outline

- Introduction
- Definition of a Matrix
- Special type of Matrices
- Properties of Matrix Addition and Scalar Multiplication
- Properties of Matrix Transposition
- Determinants
- Simultaneous Linear Equations

1.1 ❑ INTRODUCTION

Matrices were invented about a century ago in connection with the study of simple changes and movements of geometric figures in coordinate geometry.

J J Sylvester was the first to use the Latin word "*matrix*" in 1850 and later on in 1858, **Arthur Cayley** developed the theory of matrices in a systematic way.

Matrices are powerful tools of modern mathematics and their study is becoming important day by day due to their wide applications in almost every branch of science and especially in physics (atomic) and engineering. These are used by sociologists in the study of dominance within a group, by demographers in the study of births and deaths, mobility and class structure, etc., by economists in the study of inter-industry economics, by statisticians in the study of 'design of experiments' and 'multivariate analysis', by engineers in the study of 'network analysis' used in electrical and communication engineering.

Matrix is an essential tool for engineers and scientists to solve a large number of problems in the branches of engineering such as in (i) electrical engineering, where the problems with electrical circuits are modelled with the help of matrix equations; (ii) structural engineering, where the problems are modelled in the form of matrix equations and then solved; (iii) a neural network, where a set of matrices represents a neural network and its activity can be explained with the help of matrix operations and also the knowledge gathered from a set of observations is stored in matrix form; (iv) image processing, where an image is considered as a big matrix and the templates for image processing operators like edge detection, thinning, filtering etc are basically matrices and the image-processing operations are directly or indirectly matrix operations; (v) graph theory, where a graph is represented by a matrix and the problem related to the graph can be solved using matrix algebra; (vi) control engineering, where the control problems are modelled using matrix or matrix differential equations; (vii) compiler design, where the grammar of a programming language may be expressed in terms of Boolean matrices and then the precedence of the operators used is the operator precedence grammar are computed; (viii) automata, where state transitions can be expressed using matrix theory.

Rectangular Array

Before we come to the formal definition of 'matrices' and to understand the same, let us consider the following example:

In an inter-university debate, a student can speak either of the five languages: Hindi, English, Bangla, Marathi and Tamil. A certain university, say, A sent 25 students of which 7 offered to speak in Hindi, 8 in English, 2 in Bangla, 5 in Marathi and the rest in Tamil; another university, say B, sent 20 students of which 10 spoke in Hindi, 7 in English and 3 in Marathi. Out of 25 students from the third university, say C, 5 spoke in Hindi, 10 in English, 6 in Bangla and 4 in Tamil.

The information given in the above example can be put in a compact way if we present it in a tabular form as follows:

University	*Number of speakers in*				
	Hindi	*English*	*Bangla*	*Marathi*	*Tamil*
A	7	8	2	5	3
B	10	7	0	3	0
C	5	10	6	0	4

The numbers in the above arrangement form is known as a **rectangular array**. In this array, the lines down the page are called **columns** whereas those across the page are called **rows**. Any particular number in this arrangement is known as an **entry** or an **element**. Thus, in the above arrangement, we find that there are 3 rows and 5 columns and we observe that there are 5 elements in each row and so the total number of elements = 3×5, i.e., 15.

If the data given in the above arrangement is written without lines enclosed by a pair of square brackets, i.e., in the form $\begin{bmatrix} 7 & 8 & 2 & 5 & 3 \\ 10 & 7 & 0 & 3 & 0 \\ 5 & 10 & 6 & 0 & 4 \end{bmatrix}$ then this is called a matrix.

1.2 ❑ DEFINITION OF A MATRIX

A system of any mn numbers arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix (which is read as m by n matrix).

For example, $\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mn} \end{bmatrix}$ (↓ Column) ← row is an $m \times n$ matrix where the symbols a_{ij} represent any numbers (a_{ij} lies in the ith row and jth column) and $\begin{bmatrix} 1 & 5 & 2 \\ 3 & -6 & 4 \end{bmatrix}$ is a 2×3 matrix.

➢ **Note**

(i) A matrix may be represented by the symbols $[a_{ij}]$, (a_{ij}), $||a_{ij}||$. Generally, the first system is adopted.

(ii) Each of the mn numbers constituting an $m \times n$ matrix is known as an **element of the matrix.**

The elements of a matrix may be scalar or vector quantities.

(iii) When $m = n$, the matrix is square, and is called a matrix of order n or an n – **square matrix.**

(iv) The plural of 'matrix' is 'matrices'.

1.3 ❑ SPECIAL TYPES OF MATRICES

Row Matrix

Any $1 \times n$ matrix which has only one row is called a row matrix or a row vector.

The matrix $A = [a_{11}, a_{12} \ldots a_{1n}]$ is a row matrix.

Column Matrix

Any $m \times 1$ matrix which has only one column is called a column matrix or a column vector.

The matrix $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix}$ is a column matrix.

Null Matrix or Zero Matrix

If the elements of a matrix are all zero, it is called a null or zero matrix. A zero matrix of order $m \times n$ is denoted by $0_{m,n}$ or simply by 0. A zero matrix may be rectangular or square.

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices which are square and rectangular respectively.

Diagonal Matrix

A square matrix with all the elements equal to zero except those in the leading diagonal is called a diagonal matrix.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix.

Scalar Matrix

A diagonal matrix all of whose diagonal elements are equal is called a scalar matrix.

For example, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 3.

Unit Matrix

A square matrix of order n which has unity for all its elements in the leading diagonal and whose all other elements are zero is called the unit matrix or the identity matrix of order n and is denoted by I_n. In other words, if each diagonal element of a scalar matrix is unity, the matrix is called a unit matrix.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 2 and 3 respectively.

Triangular Matrices (Echelon Form)

A square matrix in which all the elements below the leading diagonal are zero is called an **upper triangular matrix**. A square matrix in which all the elements above the leading diagonal are zero is called a **lower triangular matrix**.

For example, $\begin{bmatrix} a_{11} & 0 & . & . & 0 \\ a_{21} & a_{22} & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix}$ is lower triangular and $\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ 0 & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & a_{nn} \end{bmatrix}$

is upper triangular.

Transpose of a Matrix

The matrix got from any given matrix A by interchanging its rows and columns is called the transpose of A and is denoted by A' or A^T.

For example, if $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 5 & 6 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 \\ -1 & 5 \\ 3 & 6 \end{bmatrix}$ clearly $(A')' = A$.

Conjugate of a Matrix

If A is an $m \times n$ matrix then the $m \times n$ matrix obtained by replacing each element of A by its complex conjugate is called the conjugate matrix of A and is denoted by $\overline{A}$.

Thus, if $A = [a_{ij}]$ then $\overline{A} = \left[\overline{a_{ij}}\right]$ where $\overline{a_{ij}}$ is the complex conjugate of a_{ij}.

For example, if $A = \begin{bmatrix} 3+i & 5-i & 7 \\ 6 & 3+i & 2-i \\ 2+7i & 8 & 9 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 3-i & 5+i & 7 \\ 6 & 3-i & 2+i \\ 2-7i & 8 & 9 \end{bmatrix}$

➢ **Note**

(i) If the elements of A are over the field of real numbers then the conjugate of A coincides with A, i.e., $\overline{A} = A$.

(ii) The conjugate of the conjugate of a matrix coincides with itself, i.e., $\overline{(\overline{A})} = A$.

Symmetric Matrices

A square matrix $A = [a_{ij}]$ is said to be **symmetric** if $A = A^T$, i.e., $a_{ij} = a_{ji}$, and **skew-symmetric** if $A = -A^T$, i.e., $a_{ij} = -a_{ji}$, where i and j vary from 1 to n.

The matrices $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ are respectively symmetric and skew-symmetric.

➢ **Note**

In a symmetric matrix, all the elements placed symmetrically about the main diagonal are equal and in a skew-symmetric matrix, they differ by a multiple of –1.

Hermitian Matrices and Skew-Hermitian Matrices

A square matrix $A = [a_{ij}]$ is said to be **Hermitian** if $a_{ij} = \overline{a_{ji}}$, i.e., the (i, j)th element is the conjugate complex of the (j, i)th element.

A square matrix $A = [a_{ij}]$ is said to be **skew-Hermitian** if $a_{ij} = -\overline{a_{ji}}$, i.e., (i, j)th element is the negative conjugate of the (j, i)th element.

For example, $\begin{bmatrix} 1 & 1-4i \\ 1+4i & 2 \end{bmatrix}$ and $\begin{bmatrix} 3i & 2+i \\ -2+i & i \end{bmatrix}$ are respectively, Hermitian and skew-Hermitian matrices.

Trace of a Square Matrix

The sum of the main diagonal elements of a square matrix A is called the trace of A and is denoted by tr A.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} \text{ then}$$

trace $(A) = tr\ A = a_{11} + a_{22} + \ldots + a_{nn}$

➢ **Note**

(i) If A and B are of the same order then $tr(A + B) = tr\ A + tr\ B$
(ii) If A be of order $m \times n$ and B of order $n \times m$, then $tr\ AB = tr\ BA$.

1.4 ❑ PROPERTIES OF MATRIX ADDITION AND SCALAR MULTIPLICATION

Property (i) $A + B = B + A$
Property (ii) $(A + B) + C = A + (B + C)$
Property (iii) $\alpha(A + B) = \alpha A + \alpha B$
Property (iv) $(\alpha + \beta)A = \alpha A + \beta A$
Property (v) $(\alpha\beta)\ A = \alpha(\beta A)$

Thus, the matrix addition is **commutative** [Property (i)] and **associative [**Property (ii)**]**; and the scalar multiplication of a matrix is **distributive** over matrix addition [Property (iii)].

1.5 ❑ PROPERTIES OF MATRIX TRANSPOSITION

If A and B are two matrices, and 'α' is a scalar then
Property (i) $(A^T)^T = A$
Property (ii) $(A + B)^T = A^T + B^T$
Property (iii) $(\alpha A)^T = \alpha A^T$
Property (iv) $(AB)^T = B^T A^T$

1.6 ❑ DETERMINANTS

With each square matrix A, we can associate a determinant which is denoted by the symbol $|A|$ or det A or Δ. When A is a square matrix of order n, the corresponding determinant $|A|$ is said to be a determinant of order n. A matrix is just an arrangement and has no numerical value. A determinant has numerical value. In fact, every square matrix has its determinant and while finding inverse, rank, etc., of a matrix or solving the linear equations by matrix method, we come across it.

Further, $\begin{bmatrix} 2 & 5 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 5 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 5 \\ 6 & 2 \end{bmatrix}$ and $\begin{bmatrix} 9 & 6 \\ 5 & 2 \end{bmatrix}$ are different matrices but the corresponding determinants have the same value (–12). In matrices, numbers are enclosed by brackets or parenthesis or double bars. In determinants, numbers are enclosed by a pair of vertical lines (bars).

Determinants were first introduced for solving linear systems and have important engineering applications in systems of differential equations, electrical networks, Eigen-value problems, and so on. Many complicated expressions occurring in electrical and mechanical systems can be simplified by expressing them in the form of determinants.

The differences between matrices and determinants are as follows:

Matrices	*Determinants*
1. Number of rows and number of columns can be equal or unequal.	1. Number of rows and number of columns are equal.
2. Elements are enclosed by brackets or parentheses or double bars.	2. Elements are enclosed by a pair of vertical lines (bars).
3. A matrix has no numerical value.	3. A determinant has a numerical value.
4. Matrices are arrangements. By interchanging rows and columns in a matrix, a new matrix is obtained.	4. Even after interchanging rows and columns in a determinant, the value of the determinant is unaltered.

Properties of Determinants

The following properties can be used in evaluating determinants.

(i) A determinant is unaltered if the corresponding rows and columns are interchanged.

(ii) If each element of a row or column be multiplied by a constant, the value of the determinant is multiplied by the same constant.

(iii) If two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed.

(iv) If two rows (or columns) are identical, the value of the determinant is zero.

(v) A determinant is unaltered if the elements of any row (or column) be multiplied by a constant and added to the corresponding element of any other row (or column).

(vi) The determinant of a diagonal matrix is equal to the product of the elements in the diagonal.

(vii) The determinant of the product of two matrices is equal to the product of the determinants of the two matrices,

i.e., $|AB| = |A| \cdot |B|$

Minors of a Matrix

The determinant of every square submatrix of a given matrix A is called a minor of the matrix A.

For example, if $A = \begin{bmatrix} 5 & 2 & 10 \\ -1 & 3 & 7 \\ 6 & 4 & 6 \end{bmatrix}$

Some of the minors are $\begin{vmatrix} 5 & 2 & 10 \\ -1 & 3 & 7 \\ 6 & 4 & 6 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ -1 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 7 \\ 4 & 6 \end{vmatrix}$, 3, 6, etc.

Singular and Nonsingular Matrices

A square matrix A is said to be **singular** if its determinant is zero.

A square matrix A is said to be **nonsingular** if its determinant is not equal to zero. For example,

consider
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(6-16) - 2(18-8) + 3(12-2) \\ &= -10 - 20 + 30 \\ &= 0 \end{aligned}$$

$\therefore$ A is a singular matrix.

Consider
$$B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} |B| &= 2(6-1) - 1(4-1) + 3(2-3) \\ &= 10 - 3 - 3 \\ &= 4 \end{aligned}$$

Since $|B| = 4 \neq 0$, B is a nonsingular matrix.

Adjoint of a Square Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The adjoint of A is defined to be the transpose of the co-factor matrix of A and is denoted by adj$A$.

$$\text{adj}A = (A_{ij})^T, \text{ where } A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\therefore \quad \text{adj}A = (A_{ij})^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Reciprocal Matrix or Inverse of a Matrix

- **Definition**

If A be any matrix then a matrix B, if it exists such that $AB = BA = I$, B is called the inverse of A; I being a unit matrix.

For the products AB, BA to be both defined and equal, it is necessary that A and B are both square matrices of the same order. Thus, nonsquare matrices cannot possess inverses. Also, we can at once show that the inverse of a matrix, in case it exists, must be unique.

Nonsingular and Singular Matrices

A square matrix A is said to be nonsingular or singular according as $|A| \neq 0$ or $|A| = 0$.

Thus, only nonsingular matrices possess inverses.

➢ **Note**

(i) If A, B be two nonsingular matrices of the same order then the product AB is nonsingular and $(AB)^{-1} = B^{-1} A^{-1}$.
(ii) If A be a nonsingular matrix and k a positive integer then $A^{-k} = (A^k)^{-1}$.
(iii) The operations of transposing and inverting are commutative,
i.e., $(A^T)^{-1} = (A^{-1})^T$
(iv) The operations of conjugate transpose and inverse are commutative,
i.e., $(A^\theta)^{-1} = (A^{-1})^\theta$.

Orthogonal Matrix

A square matrix A is said to be orthogonal if $AA^T = A^T A = I$

But we know that $A \cdot A^{-1} = A^{-1} \cdot A = I$

Hence, we note that $A^T = A^{-1}$.

Hence, an orthogonal matrix can also be defined as follows:

A square matrix A is said to be orthogonal if $A^T = A^{-1}$

For example, if $$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

then $$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is orthogonal.

Rank of a Matrix

A number r is defined as the rank of an $m \times n$ matrix A provided,

(i) A has at least one minor of order r which does not vanish, and
(ii) there is no minor of order $(r + 1)$ which is not equal to zero.

➢ **Note**

(i) The rank of a matrix A is denoted by $\rho(A)$ (or) simply $R(A)$.
(ii) The rank of a zero matrix by definition is 0 (i.e.) $\rho(0) = 0$.
(iii) The rank of a matrix remains unaltered by the application of elementary row or column operations, i.e., all equivalent matrices have the same rank.

(iv) From the definition of rank of a matrix, we conclude that:
   (a) If a matrix A does not possess any minor of order $(r + 1)$ then $\rho(A) \leq r$.
   (b) If at least one minor of order r of the matrix A is not equal to zero then $\rho(A) \geq r$.

(v) If every minor of order p of a matrix A is zero then every minor of order higher than p is definitely zero.

Idempotent Matrix

A matrix such that $A^2 = A$ is called an idempotent matrix.

For example, if $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Periodic Matrix

A matrix A will be called a periodic matrix if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer, for which $A^{k+1} = A$, then k is said to be the period of A. If we choose $k = 1$, we get $A^2 = A$ and we call it the idempotent matrix.

Nilpotent Matrix

A matrix A will be called a nilpotent matrix if $A^k = 0$ (null matrix) where k is a positive integer; if however k is the least positive integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

For example, if $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$,

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Here, A is a nilpotent matrix whose index is 2.

Involuntary Matrix

A matrix A will be called an involuntary matrix if $A^2 = I$ (unit matrix). Since $I^2 = I$ always, the unit matrix is involuntary.

Equal Matrices

Two matrices are said to be equal if
(i) they are of the same order, and
(ii) the elements in the corresponding positions are equal.

Thus, if $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$

Here, $A = B$.

1.7 ❑ SIMULTANEOUS LINEAR EQUATIONS

The concepts and operations in matrix algebra are extremely useful in solving simultaneous linear equations.

Let the equations be

$$a_1x + a_2y + a_3z = d_1 \quad b_1x + b_2y + b_3z = d_2 \quad c_1x + c_2y + c_3z = d_3$$

$$\Rightarrow \quad \begin{bmatrix} a_1x & a_2y & a_3z \\ b_1x & b_2y & b_3z \\ c_1x & c_2y & c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\therefore \quad AX = B$$

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Hence, to solve linear equations, write down the coefficient matrix A and find its inverse A^{-1}. Then find $A^{-1}B$. This gives the value X which is the solution for the given linear equations.

Consistency of a System of Simultaneous Linear Equations

A system of simultaneous linear equations is $AX = B$ in matrix form. Consider the coefficient matrix A. Augment A by writing the constants vector as the last column. The resulting matrix is called an **augmented matrix** and is denoted by $(A : B)$ or $(A \vdots B)$ or simply $[A, B]$.

A system of simultaneous linear equations is **consistent** if the ranks of the coefficient matrix and the augmented matrix are equal,

i.e., $\rho(A) = \rho(A : B)$ (or) $R[A] = R[A, B]$.

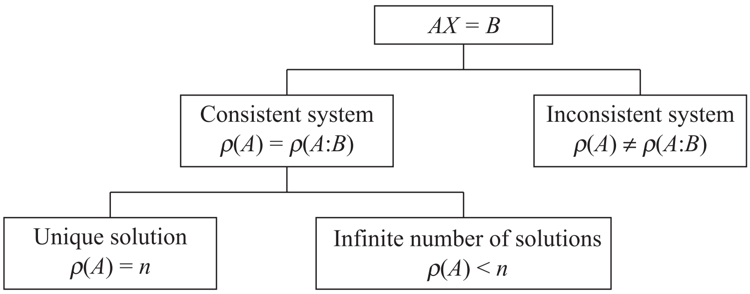
There are two possibilities:

(i) When $\rho(A) = \rho(A : B) = n$ (the number of unknowns), the system has a **unique solution.**

(ii) When $\rho(A) = \rho(A : B) < n$ (the number of unknowns), the system has **infinite solutions.** Let $\rho(A) = \rho(A : B) = r < n \cdot (n - r)$ of the unknowns are to be assigned values arbitrarily and the remaining r unknowns can then be obtained in terms of those $(n - r)$ values.

On the contrary, a system of simultaneous linear equations is **inconsistent** if the ranks of the coefficient matrix and the augmented matrix are not equal, i.e., $\rho(A) \neq \rho(A : B)$

These different possibilities are presented in a chart as follows:



5 Quadratic Forms

Chapter Outline

- Definition
- Quadratic Forms Expressed in Matrices
- Linear Transformation of Quadratic Form
- Canonical Form
- Index and Signature of the Quadratic Form
- Nature of Quadratic Forms
- Determination of the Nature of Quadratic Form (QF) without Reduction to Canonical Form

5.1 ❑ DEFINITION

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

For example,

(i) $ax^2 + 2hxy + by^2$

(ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$

(iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

5.2 ❑ QUADRATIC FORM EXPRESSED IN MATRICES

Quadratic form can be expressed as a product of matrices.

Quadratic form = X^TAX.

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (symmetric matrix)

X^T is the transpose of X.

$$X^T AX = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \quad a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \quad a_{13}x_1 + a_{23}x_2 + a_{33}x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{21}x_1x_2 + a_{31}x_1x_3 + a_{12}x_1x_2 + a_{22}x_2^2 + a_{32}x_2x_3 + a_{13}x_1x_3 + a_{23}x_2x_3 + a_{33}x_3^2$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{23} + a_{32})x_2x_3 + (a_{31} + a_{13})x_1x_3$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3$$

(As $a_{21} = a_{12}$, $a_{32} = a_{23}$, $a_{31} = a_{13}$ in a symmetric matrix, in general, $a_{ij} = a_{ji} = \frac{1}{2}$ coefficient of x_{ij} if $i \neq j$.)

5.3 ❑ LINEAR TRANSFORMATION OF QUADRATIC FORM

Let the given quadratic form in n variables be X^TAX where A is a symmetric matrix.

Consider the linear transformation $X = PY$.

Then $\qquad X^T = (PY)^T = Y^TP^T$.

$\therefore \qquad X^TAX = (Y^TP^T)A(PY) = Y^T(P^TAP)Y = Y^TBY$

where $\qquad B = P^TAP$.

Therefore, Y^TBY is also a quadratic form in n variables. Hence, it is a linear transformation of the quadratic form X^TAX under the linear transformation $X = PY$ and $B = P^TAP$.

5.4 ❑ CANONICAL FORM

If a real quadratic form be expressed as a sum or difference of the squares of new variables by means of any real nonsingular linear transformation then the latter quadratic expression is called a canonical form of the given quadratic form.

5.5 ❑ INDEX AND SIGNATURE OF THE QUADRATIC FORM

When the quadratic form X^TAX is reduced to the canonical form, it will contain only r terms, if the rank of A is r. The terms in the canonical form may be positive, zero or negative.

The number (p) of positive terms in the canonical form is called the **index** of the quadratic form.

Number of positive terms – Number of negative terms, i.e., $p - (r - p) = 2p - r$ is called **signature** of the quadratic form.

5.6 ❑ NATURE OF QUADRATIC FORMS

Definite, Semi-definite and Indefinite Real Quadratic Forms

Let X^TAX be a real quadratic form in n – variables $x_1, x_2, \ldots x_n$ with rank r and index p.

Then we say that the quadratic form is

(i) positive definite if $r = n$, $p = r$.
(ii) negative definite if $r = n$, $p = 0$.
(iii) positive semi-definite if $r < n$, $p = r$.
(iv) negative semi-definite if $r < n$, $p = 0$.

If the canonical form has both positive and negative terms, the quadratic form is said to be indefinite.

Examples:

(i) $x_1^2 + x_2^2$ is positive definite.
(ii) $-x_1^2 - x_2^2$ is negative definite.
(iii) $(x_1 - x_2)^2$ is positive semi-definite.
(iv) $-(x_1 - x_2)^2$ is negative semi-definite.
$x_1^2 - x_2^2$ is indefinite.

➢ **Note**

If X^TAX is positive definite then $|A| > 0$.

5.7 ❑ DETERMINATION OF THE NATURE OF QUADRATIC FORM (QF) WITHOUT REDUCTION TO CANONICAL FORM

Consider the quadratic form

$$X^T AX = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $D_1 = |a_{11}|$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The QF is

(i) positive definite if $D_i > 0$ for $i = 1, 2, 3$;
(ii) negative definite if $D_2 > 0$ and $D_1 < 0$, $D_3 < 0$;
(iii) positive semi-definite if $D_i > 0$ and at least one $D_i = 0$;
(iv) negative semi-definite if some of the determinants are zero in case (ii); and
(v) indefinite in all other cases.

Criteria for the Nature of Quadratic Form (or Value Class) in Terms of Nature of Eigen Values

Value Class	*Nature of Eigen Values*
Positive definite	Positive Eigen values
Positive semi-definite	Positive Eigen values and at least one is zero
Negative definite	Negative Eigen values
Negative semi-definite	Negative Eigen values and at least one is zero
Indefinite	Positive as well as negative Eigen values

SOLVED EXAMPLES

Example 1 Discuss the nature of the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$. **[KU April 2011]**

Solution The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$D_1 = |8| = 8 > 0,\ D_2 = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 20 > 0 \text{ and } D_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 0$$

∴ the QF is positive semi-definite. **Ans.**

Example 2 Write down the matrix of the quadratic form $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_2 + 5x_2x_3$

Solution

$$x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3 \qquad (1)$$

Coefficient of $x_1^2 = 1 = a_{11}$,

Coefficient of $x_2^2 = 2 = a_{22}$,

Coefficient of $x_3^2 = -7 = a_{33}$,

$\frac{1}{2}$ coefficient of $x_1x_2 = \frac{1}{2}(-4) = -2 = a_{12}$

$\frac{1}{2}$ coefficient of $x_1x_3 = \frac{1}{2}(8) = 4 = a_{13}$

$\frac{1}{2}$ coefficient of $x_2x_3 = \frac{1}{2}(5) = \frac{5}{2} = a_{23}$

∴ Eq. (1) can be expressed as X^TAX, where

$$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}=\begin{bmatrix}1&-2&4\\-2&2&\frac{5}{2}\\4&\frac{5}{2}&-7\end{bmatrix}$$

$\therefore$ given quadratic form = $[x_1 \quad x_2 \quad x_3]\begin{bmatrix}1&-2&4\\-2&2&\frac{5}{2}\\4&\frac{5}{2}&-7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ **Ans.**

Example 3 Write down the quadratic form corresponding to the matrix

$$A=\begin{bmatrix}1&2&5\\2&0&3\\5&3&4\end{bmatrix}.$$

Solution Quadratic form = X^TAX

$$=[x_1 \quad x_2 \quad x_3]\begin{bmatrix}1&2&5\\2&0&3\\5&3&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$=[x_1+2x_2+5x_3 \quad 2x_1+3x_3 \quad 5x_1+3x_2+4x_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$=x_1^2+2x_1x_2+5x_3x_1+2x_1x_2+3x_2x_3+5x_1x_3+3x_2x_3+4x_3^2$$

$$=x_1^2+4x_3^2+4x_1x_2+10x_1x_3+6x_2x_3.$$ **Ans.**

Example 4 Reduce the quadratic forms $6x_1^2+3x_2^2+14x_3^2+4x_1x_2+4x_2x_3+18x_3x_1$ and $2x_1^2+5x_2^2+4x_1x_2+2x_3x_1$ simultaneously to canonical forms by a real nonsingular transformation. **[KU May 2010]**

Solution The matrix of the first quadratic form is $A=\begin{bmatrix}6&2&9\\2&3&2\\9&2&14\end{bmatrix}$

The matrix of the second quadratic form is $B=\begin{bmatrix}2&2&1\\2&5&0\\1&0&0\end{bmatrix}$

The characteristic equation is $|A-\lambda B|=0$.

i.e., $$\begin{vmatrix}6-2\lambda&2-2\lambda&9-\lambda\\2-2\lambda&3-5\lambda&2\\9-\lambda&2&14\end{vmatrix}=0$$

$$\Rightarrow \qquad 5\lambda^3 - \lambda^2 - 5\lambda + 1 = 0$$

i.e.,
$$(\lambda - 1)(5\lambda - 1)(\lambda + 1) = 0$$

$$\Rightarrow \qquad \lambda = -1, \frac{1}{5}, 1$$

When $\lambda = -1$, $(A - \lambda B)X = 0$, given the equations,

$$8x_1 + 4x_2 + 10x_3 = 0;\ 4x_1 + 8x_2 + 2x_3 = 0;\ 10x_1 + 2x_2 + 14x_3 = 0$$

by solving, $X_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

When $\lambda = \frac{1}{5}$, $(A - \lambda B)X = 0$ gives

$$28x_1 + 8x_2 + 44x_3 = 0;\ 8x_1 + 10x_2 + 10x_3 = 0;\ 44x_1 + 10x_2 + 70x_3 = 0$$

by solving, $X_2 = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

When $\lambda = 1$, $(A - \lambda B)X = 0$ gives

$$4x_1 + 8x_3 = 0;\ -2x_2 + 2x_3 = 0;\ 8x_1 + 2x_2 + 14x_3 = 0$$

$$\Rightarrow \qquad X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

Since X_1, X_2, X_3 are not pairwise orthogonal, consider the modal matrix P.

Now, $P = \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

$$P^T AP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}\begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the quadratic form X^TAX is reduced to the canonical form $y_1^2 + y_2^2 + y_3^2$.

Now
$$P^T BP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the quadratic form X^TBX is reduced to the canonical form $y_1^2 + 5y_2^2 + y_3^2$. **Ans.**

Example 5 Reduce $6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_3x_1$ into canonical form. Find its nature, rank, index and signature.

[KU Nov. 2010, AU Jan. 2010, KU April 2012]

Solution The matrix of the quadratic form is $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$

The characteristic roots are given by $|A-\lambda I|=0$

i.e., $$\begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{vmatrix}=0$$

$\Rightarrow$ $\lambda^3-12\lambda^2+36\lambda-32=0$

$\therefore$ the Eigen values are $\lambda=8, 2, 2$

The Eigen vectors are obtained by $(A-\lambda I)X=0$

When $\lambda=8$, $(A-\lambda I)X=0$ gives

$$\begin{bmatrix}-2 & -2 & 2\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$\Rightarrow$ $-2x_1-2x_2+2x_3=0;\ -2x_1-5x_2-x_3=0;\ 2x_1-x_2-5x_3=0$

$\Rightarrow$ $X_1=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$

When $\lambda=2$, $(A-\lambda I)X=0$ reduces to a single equation $2x_1-x_2+x_3=0$

Putting $x_1=0$, we get $X_2=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$

Again, by putting $x_2=0$, we get $X_3=\begin{bmatrix}1\\ 0\\ -2\end{bmatrix}$

Now $X_1=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}, X_2=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$ and $X_3=\begin{bmatrix}1\\ 0\\ -2\end{bmatrix}$

Here, X_1, X_2, X_3 are not pairwise orthogonal.

(i.e., $X_1^TX_2=0, X_2^TX_3\neq 0, X_3^T X_1=0$)

X_3 is orthogonal to X_2, only when $X_3=\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}$, so that $X_1^TX_2=X_2^TX_3=X_3^TX_1=0$

$\therefore$ the normalized modal matrix is $P = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$

Consider

$$P^T AP = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the quadratic form X^TAX is transformed to the canonical form $8y_1^2 + 2y_2^2 + 2y_3^2$
Here, rank of the quadratic form = 3, index = 3, signature = 3.
$\therefore$ it is positive definite. **Ans.**

EXERCISE

Part A

1. If the canonical form of a quadratic form is $5y_1^2 + 6y_2^2$ then the rank is ______.
   (i) 5 (ii) 0 (iii) 2 (iv) 1
2. The nonsingular linear transformation used to transform the quadratic form to canonical form is ______
   (i) $X = N^TY$ (ii) $X = NY$ (iii) $Y = NX$ (iv) $Y = X$
3. Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$.
4. Define a quadratic form and give an example in two and three variables.
5. What do you mean by canonical form of a quadratic form?
6. Define index and signature of a quadratic form.
7. Discuss the nature of the quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$.
8. Discuss the nature of the quadratic form $2xy + 2yz + 2zx$.
9. Determine the nature of the following quadratic forms without reducing them to canonical forms:

(i) $x_1^2+3x_2^2+6x_3^2+2x_1x_2+2x_2x_3+4x_3x_1$

(ii) $2x_1^2+x_2^2-3x_3^2+12x_1x_2-8x_2x_3-4x_3x_1$

10. Find the index and signature of the quadratic form, $2x_1^2-5x_2^2+7x_3^2$.
11. State the conditions for a quadratic form to be positive definite and positive semi-definite.
12. Write down the matrices of the following quadratic forms:
    (i) $2x^2+3y^2+6xy$
    (ii) $2x^2+5y^2-6z^2-2xy-yz+8zx$
    (iii) $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$
    (iv) $x_1^2+2x_2^2+3x_3^2+4x_4^2+2x_1x_2+4x_1x_3-6x_1x_4-4x_2x_3+8x_2x_4-12x_3x_4$
13. Write down the quadratic forms corresponding to the following matrices.

(i) $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

14. Write down the matrix of the QF

$$3x_1^2+5x_2^2+5x_3^2-2x_1x_2+2x_2x_3+6x_3x_1$$

15. Define pairwise orthogonal.

Part B

1. Reduce the QF $8x_1^2+7x_2^2+3x_3^2-12x_1x_2-8x_2x_3+4x_3x_1$ to the canonical form through an orthogonal transformation and, hence, show that it is positive definite. Find also a nonzero set of values for x_1, x_2, x_3 that will make the QF zero.

$$\left(\textbf{Ans. } P=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}; Q=3y_2^2+15y_3^2; x_1=1, x_2=2, x_3=2\right)$$

2. Reduce the QF $10x_1^2+2x_2^2+5x_3^2+6x_2x_3-10x_3x_1-4x_1x_2$ to a canonical form by orthogonal reduction. Find also a set of nonzero values of x_1, x_2, x_3 which will make the QF zero.

$$\left(\textbf{Ans. } P=\begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}; Q=3y_2^2+14y_3^2; x_1=1, x_2=-5, x_3=4\right)$$

3. Find the value of λ so that the quadratic form
   $\lambda(x_1^2 + x_2^2 + x_3^2) + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ may be positive definite. (**Ans.** $\lambda > 2$)
4. Reduce the following quadratic forms to canonical forms or to sum of squares by orthogonal transformation. Write also rank, index and signature.
   (i) $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$
   (ii) $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$
   (iii) $3x^2 - 2x^2 - z^2 - 4xy + 8xz + 12yz$
   (iv) $x^2 + 3y^2 + 3z^2 - 2yz$
   [**Ans.** (i) $2y_1^2 + 3y_2^2 + 6y_3^2$; rank = 3, index = 3, signature = 3
   (ii) $4y_1^2 + y_2^2 + y_3^2$; rank = 3, index = 3, signature = 3
   (iii) $3y_1^2 + 6y_2^2 - 9y_3^2$; rank = 3, index = 2, signature = 1
   (iv) $y_1^2 + 2y_2^2 + 4y_3^2$; rank = 3, index = 3, signature = 3]
5. Reduce the QF $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ to the canonical form by an orthogonal transformation. (**Ans.** $y_1^2 + y_2^2 - 2y_3^2$)
6. Reduce the QF $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ into the canonical by an orthogonal transformation.
   (**Ans.** $y_1^2 + 2y_2^2 + 4y_3^2$)
7. Reduce the QF $y^2 + 2xy$ into the canonical form by an orthogonal reduction and state the nature of the QF. (**Ans.** $-y_1^2 + y_2^2 + y_3^2$; indefinite)
8. Discuss the nature of the following quadratic forms:
   (i) $2x^2 + 3z^2 + 2xy$
   (ii) $11x_1^2 + 14x_1y_1 + 14x_1z_1 + 8y_1z_1$
   (iii) $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$
   [**Ans.** (i) Positive definite (ii) Indefinite (iii) Positive semi-definite]
9. Reduce the following quadratic forms to canonical forms by orthogonal transformation. State the nature.
   (i) $16x_1x_2 - x_3^2$
   (ii) $7x_1^2 + 6x_2^2 + 5x_3^2 - 4x_1x_2 - 4x_2x_3$
   (iii) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_2x_3$
   [**Ans.** (i) $8y_1^2 - y_3^2 - 8y_3^2$; indefinite (ii) $9y_1^2 + 6y_2^2 + 3y_3^2$; positive definite
   (iii) $5y_1^2 + 2y_2^2 - y_3^2$; indefinite]
10. Find the nature of the following:
    (i) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$
    (ii) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$
    (iii) $5x^2 + 26y^2 + 10z^2 + 4yz + 14xz + 6xy$
    [**Ans.**(i) Indefinite (ii) Positive definite (iii) Positive semi-definite]

5. For the curve $r\theta = a$, find the value of ds/dr. (Marathwada '98)

6. With the usual meanings for r, s, θ and ϕ for the polar curve $r = f(\theta)$, show that $\dfrac{d\phi}{d\theta} + r\operatorname{cosec}^2\theta\,\dfrac{d^2r}{ds^2} = 0$. (V.T.U., 2000)

4.13. CURVATURE

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angles ψ and $\psi + \delta\psi$ with the x-axis, so that the angle between the tangents at P and $Q = \delta\psi$, (Fig. 4.9).

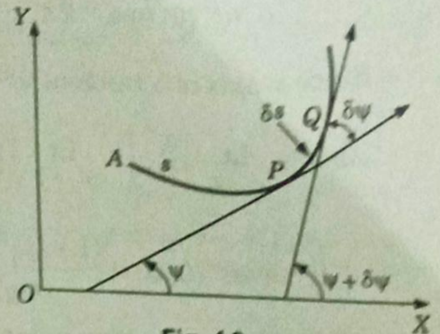


Fig. 4.9.

In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called the *total bending or total curvature* of the arc PQ.

$\therefore$ The *average curvature* of arc $PQ = \dfrac{\delta\psi}{\delta s}$

The limiting value of average curvature when Q approaches P (i.e. $\delta s \to 0$) is defined as the curvature of the curve at P.

Thus **curvature** K (at P) $= \dfrac{d\psi}{ds}$

Obs. Since $\delta\psi$ is measured in radians, the *unit of curvature is radians per unit length e.g.* radians per centimetre.

(2) **Radius of curvature.** *The reciprocal of the curvature of a curve at any point P is called the* **radius of curvature** *at P and is denoted by* ρ, so that $\rho = ds/d\psi$.

(3) **Centre of curvature.** *A point C on the normal at any point P of a curve distant* ρ *from it, is called the* **centre of curvature** *at P.*

(4) **Circle of curvature.** *A circle with centre C (centre of curvature at P) and radius* ρ *is called the* **circle of curvature** *at P.*

4.14. (1) Radius of curvature for cartesian curve $y = f(x)$, *is given by*

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

We know that $\tan\psi = dy/dx = y_1$ or $\psi = \tan^{-1}(y_1)$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1 + y_1^2}\cdot\frac{d(y_1)}{dx} = \frac{y_2}{1 + y_1^2}$$

$$\therefore \qquad \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = \sqrt{(1 + y_1^2)}\cdot\frac{1 + y_1^2}{y_2} = \frac{(1 + y_1^2)^{3/2}}{y_2} \qquad ...(1)$$

(2) **Radius of curvature for parametric equations**

$$x = f(t),\ y = \phi(t).$$

Denoting differentiations with respect to t by dashes,

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = y'/x'.$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx} = \frac{x'y'' - y'x''}{(x')^2}\cdot\frac{1}{x'}$$

Substituting the values of y_1 and y_2 in (1)

$$\rho = \left[1 + \left(\frac{y'}{x'}\right)^2\right]^{3/2} \Bigg/ \left[\frac{x'y'' - y'x''}{(x')^3}\right] = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

(3) Radius of curvature at the origin. Newton's formulae

(i) If x-axis is tangent to a curve at the origin, then

$$\rho \text{ at } (0, 0) = \underset{x \to 0}{\text{Lt}} \left(\frac{x^2}{2y}\right)$$

Since x-axis is a tangent at (0, 0), $(dy/dx)_0$ or $(y_1)_0 = 0$

Also $$\underset{x \to 0}{\text{Lt}} \left(\frac{x^2}{2y}\right) = \underset{x \to 0}{\text{Lt}} \left(\frac{2x}{2dy/dx}\right) = \underset{x \to 0}{\text{Lt}} \frac{1}{d^2y/dx^2} = \frac{1}{(y_2)_0} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$\therefore \quad \rho \text{ at } (0, 0) = \frac{[1 + (y_1^2)_0]^{3/2}}{(y_2)_0} = \frac{1}{(y_2)_0} = \underset{x \to 0}{\text{Lt}} \frac{x^2}{2y} \qquad \text{[From (1)]}$$

(ii) Similarly, *if y-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \underset{x \to 0}{\text{Lt}} \left(\frac{y^2}{2x}\right)$$

(iii) In case the curve passes through the origin but neither x-axis nor y-axis is tangent at the origin, we write the equation of the curve as

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots \qquad \text{[By Maclaurin's series]}$$

$$= px + qx^2/2 + \ldots\ldots \qquad [\because f(0) = 0$$

where $p = f'(0)$ and $q = f''(0)$

Substituting this in the equation $y = f(x)$, we find the values of p and q by equating coefficients of like powers of x. Then $\rho(0, 0) = (1 + p^2)^{3/2}/q$.

Obs. *Tangents at the origin to a curve are found by equating to zero the lowest degree terms in its equation.*

Example 4·42. *Find the radius of curvature at the point (3a/2, 3a/2) of the Folium* $x^3 + y^3 = 3axy$.

(Mangalore, 1999)

Differentiating with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$

or $$(y^2 - ax)\frac{dy}{dx} = ay - x^2 \qquad \ldots(i) \qquad \therefore \quad \frac{dy}{dx} \text{ at } (3a/2, 3a/2) = -1.$$

Differentiating (i),

$$\left(2y\frac{dy}{dx} - a\right)\frac{dy}{dx} + (y^2 - ax)\frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x \qquad \therefore \quad \frac{d^2y}{dx^2} \text{ at } (3a/2, 3a/2) = -32/3a$$

Hence $$\rho \text{ at } (3a/2, 3a/2) = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (-1)^2]^{3/2}}{-32/3a} = \frac{3a}{8\sqrt{2}} \quad \text{(in magnitude)}$$

Example 4·43. *Show that the radius of curvature at any point of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *is* $4a\cos\theta/2$.

(Madras, 1998 S)

We have $\frac{dx}{d\theta} = a\,(1 + \cos\theta),\ \frac{dy}{d\theta} = a \sin\theta$

$\therefore$ $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a \sin\theta}{a\,(1+\cos\theta)} = \frac{2 \sin\theta/2 \cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2} \cdot \frac{1}{a\,(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2} \cdot \frac{1}{2a\cos^2\theta/2} = \frac{1}{4a}\sec^4\frac{\theta}{2}.$$

$\therefore$ $$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{4a\,(1+\tan^2\theta/2)^{3/2}}{\sec^4\theta/2}$$

$$= 4a \cdot (\sec^2\theta/2)^{3/2} \cdot \cos^4\theta/2 = 4a\cos\theta/2.$$

Example 4·44. *Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.* *(Patna, 1997 S)*

The parametric equation of the curve is

$$x = a\cos^3 t,\ y = a\sin^3 t.$$

$$x'(= dx/dt) = -3a\cos^2 t \sin t,\ y' = 3a \sin^2 t \cos t.$$

$\therefore$ $$x'' = -3a\,(\cos^3 t - 2\cos t \sin^2 t) = 3a\cos t\,(2\sin^2 t - \cos^2 t)$$

$$y'' = 3a\,(2\sin t\cos^2 t - \sin^3 t) = 3a \sin t\,(2\cos^2 t - \sin^2 t)$$

$$x'^2 + y'^2 = 9a^2(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t) = 9a^2 \sin^2 t \cos^2 t$$

$$x'y'' - y'x'' = -9a^2\cos^2 t \sin^2 t\,(2\cos^2 t - \sin^2 t)$$
$$- 9a^2 \cos^2 t \sin^2 t\,(2\sin^2 t - \cos^2 t) = -9a^2 \sin^2 t \cos^2 t$$

$\therefore$ $$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27a^3 \sin^3 t \cos^3 t}{-9a^2 \sin^2 t \cos^2 t} = -3a \sin t \cos t.$$ *(Madras, 1992)*

Since $dy/dx = y'/x' = -\tan t,$

$\therefore$ Equation of the tangent at $(a\cos^3 t, a\sin^3 t)$ is $y - a\sin^3 t = -\tan t\,(x - a\cos^3 t)$

i.e. $$x\tan t + y - a\sin t = 0 \quad \text{...}(i)$$

p, length of $\perp$ from (0, 0) on $(i) = \frac{0 + 0 - a\sin t}{\sqrt{(\tan^2 t + 1)}} = -a\sin t\cos t$. Thus $\rho = 3p$.

Example 4·45. *Find ρ at the origin for the curves*

(i) $y^4 + x^3 + a\,(x^2 + y^2) - a^2 y = 0$ *(Marathwada, 1998)* (ii) $y - x = x^2 + 2xy + y^2$

(i) Equating to zero the lowest degree terms, we get $y = 0$.

$\therefore$ x-axis is the tangent at the origin. Dividing throughout by y, we have

$$y^3 + x \cdot \frac{x^2}{y} + a\left(\frac{x^2}{y} + y\right) - a^2 = 0$$

Let $x \to 0$, so that $\underset{x\to 0}{\text{Lt}}\ (x^2/2y) = \rho.$

$\therefore$ $$0 + 0 \cdot 2\rho + a\,(2\rho + 0) - a^2 = 0 \text{ or } \rho = a/2.$$

(ii) Equating to zero the lowest degree terms, we get $y = x$, as the tangent at the origin, which is neither of the coordinate axes.

$\therefore$ Putting $y = px + qx^2/2 + \ldots\ldots$ in the given equation, we get

$$px + qx^2/2 + \ldots\ldots - x = x^2 + 2x\,(px + qx^2/2 + \ldots\ldots) + (px + qx^2/2 + \ldots\ldots)^2$$

Equating coefficients of x and x^2,

$$p - 1 = 0,\ q/2 = 1 + 2p + p^2 \text{ i.e. } p = 1 \text{ and } q = 2 + 4 \cdot 1 + 2 \cdot 1^2 = 8.$$

$$\therefore \quad \rho\,(0, 0) = (1 + p^2)^{3/2}/q = (1 + 1)^{3/2}/8 = 1/2\sqrt{2}.$$

Problems 4·14

1. Find the radius of curvature at any point
   (i) $(at^2, 2at)$ of the parabola $y^2 = 4ax$. *(Madurai, 1990)*
   (ii) (o, c) of the catenary $y = c \cosh x/c$. *(Madras, 1993)*
2. Show that for the rectangular hyperbola $xy = c^2$, $\rho = \dfrac{(x^2 + y^2)^{3/2}}{2c^2}$ *(Madras, 2000 PT)*
3. Show that the radius of curvature at
   (i) $(a, 0)$ on the curve $y^2 = a^2 (a - x)/x$ is $a/2$. *(V.T.U., 2000 S)*
   (ii) $(a/4, a/4)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a/\sqrt{2}$. *(Osmania, 2000 S)*
4. Find the radius of curvature at any point on the
   (i) *ellipse* : $x = a \cos \theta, y = b \sin \theta$. (ii) *cycloid* : $x = a\,(\theta - \sin \theta), y = a\,(1 - \cos \theta)$. *(Mangalore, 1999)*
   (iii) *curve* $x = a\,(\cos t + t \sin t), y = a\,(\sin t - t \cos t)$. *(Kuvempu, 1998 ; Madras, 1996)*
5. Show that the radius of curvature at the point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a \sin \theta \cos \theta$. *(Madras, 2000 S ; Rewa, 1998)*
6. If ρ be the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$. *(J.N.T.U., 1998)*
7. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$. *(A.M.I.E., 1996 S)*
8. Prove that for the ellipse $x^2/a^2 + y^2/b^2 = 1$, $\rho = a^2 b^2/p^3$, p being the perpendicular from the centre on the tangent at (x, y). *(Nagpur, 1997)*
9. Show that the radius of curvature at an end of the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the semi-latus rectum. *(Osmania, 2000 S)*
10. Show that the radius of curvature ρ at P on an ellipse $x^2/a^2 + y^2/b^2 = 1$ is given by $\rho = CD^3/ab$ where CD is the semi-diameter conjugate to CP. *(Raipur, 1998)*
11. Find the radius of curvature at the origin for
    (i) $x^3 + y^3 - 2x^2 + 6y = 0$ (ii) $2x^4 + 3y^4 + 4x^2 y + xy - y^2 + 2x = 0$ *(Marathwada, 1992)*
    (iii) $y^2 = x^2\,(a + x)/(a - x)$.

(4) **Radius of curvature for polar curve** $r = f(\theta)$, *is given by*

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

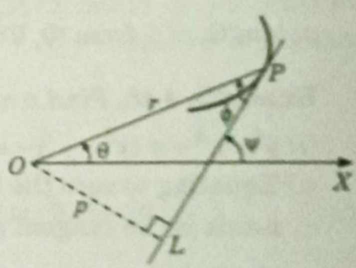


Fig. 4·10.

With the usual notations, we have from Fig. 4·10.

$$\psi = \theta + \phi$$

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \qquad ...(1)$$

Also we know that

$$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{r_1} \text{ or } \phi = \tan^{-1}\left(\frac{r}{r_1}\right) \qquad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating w.r.t. θ,

$$\frac{d\phi}{d\theta} = \frac{1}{1+(r/r_1)^2} \cdot \frac{r_1 \cdot r_1 - rr_2}{r_1^2} = \frac{r_1^2 - rr_2}{r^2 + r_1^2} \quad ...(2)$$

Also $$\frac{ds}{d\theta} = \sqrt{(r^2 + r_1^2)} \quad ...(3)$$

Substituting the value from (2) and (3) in (1),

$$\frac{1}{\rho} = \frac{1}{\sqrt{(r^2 + r_1^2)}} \cdot \left(1 + \frac{r_1^2 - rr_2}{r^2 + r_1^2}\right)$$

Hence $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

(5) **Radius of curvature for pedal curve** $p = f(r)$ *is given by*

$$\rho = r \frac{dr}{dp}$$

With the usual notation (Fig. 4·10), we have $\psi = \theta + \phi$

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \quad ...(1)$$

Also we know that $p = r \sin \phi$

$$\therefore \quad \frac{dp}{dr} = \sin \phi + r \cos \phi \frac{d\phi}{ds}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds} \cdot \frac{d\phi}{dr} \quad \text{[By (3) and (4) of § 4·12 (2)]}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = \frac{r}{\rho} \quad \text{[By (1)]}$$

Hence $$\rho = r\frac{dr}{dp}.$$

Example 4·46. *Show that the radius of curvature at any point of the cardioid* $r = a(1 - \cos\theta)$ *varies as* $\sqrt{r}$.

Differentiating w.r.t. θ, we get

$$r_1 = a \sin\theta, \ r_2 = a\cos\theta$$

$$\therefore \quad (r^2 + r_1^2)^{3/2} = [a^2(1-\cos\theta)^2 + a^2 \sin^2\theta]^{3/2} = a^3 [2(1-\cos\theta)]^{3/2}$$

$$r^2 - rr_2 + 2r_1^2 = a^2(1-\cos\theta)^2 - a^2(1-\cos\theta)\cos\theta + 2a^2\sin^2\theta = 3a^2(1-\cos\theta)$$

Thus $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2} = \frac{a^3\, 2\sqrt{2}\,(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}}{3} a (1-\cos\theta)^{1/2} = \frac{2\sqrt{2a}}{3}\left(\frac{r}{a}\right)^{1/2} \propto \sqrt{r}.$$

Otherwise : The pedal equation of this cardioid is $2ap^2 = r^3$ ...(*i*)

Differentiating w.r.t. p, we get

$$4ap = 3r^2\frac{dr}{dp} \text{ whence } \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4ar^{3/2}}{3r \cdot \sqrt{(2a)}} \propto \sqrt{r}. \quad [\because \ p = r^{3/2}/\sqrt{(2a)} \text{ from } (i)$$

Problems 4.15

1. Find the radius of curvature at the point (r, θ) on each of the curves :
   (i) $r = a(1 - \cos\theta)$ (ii) $r^n = a^n \cos n\theta$. (Bangalore, 1992 S)
2. For the cardioid $r = a(1 + \cos\theta)$, show that ρ^2/r is constant.
3. Find the radius of curvature for the parabola $2a/r = 1 + \cos\theta$.
4. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = 16a^2/9$. (Kurukshetra, 1998)
5. For any curve $r = f(\theta)$, prove that
$$\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right)$$
(Bhopal, 1991)

4.15. (1) CENTRE OF CURVATURE *at any point $P(x, y)$ on the curve $y = f(x)$ is given by*

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2}, \quad \bar{y} = y + \frac{1 + y_1^2}{y_2}.$$

Let $C(\bar{x}, \bar{y})$ be the centre of curvature and ρ the radius of curvature of the curve at $P(x, y)$ (Fig. 4.11). Draw PL and CM ⊥s to OX and $PN \perp CM$. Let the tangent at P make an $\angle\psi$ with the x-axis. Then $\angle NCP = 90° - \angle NPC = \angle NPT = \psi$

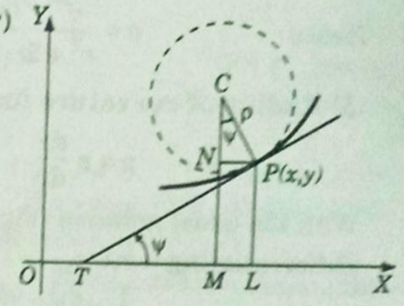


Fig. 4.11.

$$\therefore \quad \bar{x} = OM = OL - ML = OL - NP$$
$$= x - \rho\sin\psi = x - \frac{(1 + y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{(1 + y_1^2)}} \quad \left[\because \tan\psi = y_1, \ \therefore \sin\psi = \frac{y_1}{\sqrt{(1 + y_1^2)}}\right]$$
$$= x - \frac{y_1(1 + y_1^2)}{y_2}$$

and
$$\bar{y} = MC = MN + NC = LP + \rho\cos\psi \quad \left[\because \sec\psi = \sqrt{(1 + \tan^2\psi)} = \sqrt{(1 + y_1^2)}\right]$$
$$= y + \frac{(1 + y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{(1 + y_1^2)}} = y + \frac{1 + y_1^2}{y_2}$$

Cor. Equation of the circle of curvature at P is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

(2) Evolute. *The locus of the centre of curvature for a curve is called its* **evolute** *and the curve is called an* **involute** *of its evolute.*

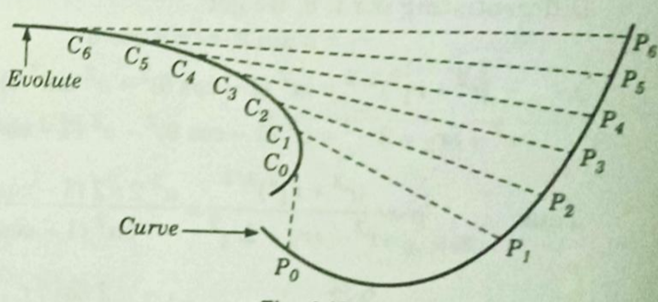


Fig. 4.12.

Example 4.47. *Find the coordinates of the centre of curvature at any point of the parabola $y^2 = 4ax$.*

Hence show that its evolute is

$$27ay^2 = 4(x - 2a)^3. \quad \text{(V.T.U., 2000)}$$

We have $2yy_1 = 4a$ *i.e.* $y_1 = 2a/y$

and
$$y_2 = -\frac{2a}{y^2} \cdot y_1 = -\frac{4a^2}{y^3}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2} = x - \frac{2a/y\,(1 + 4a^2/y^2)}{-4a^2/y^3}$$

$$= x + \frac{y^2 + 4a^2}{2a} = x + \frac{4ax + 4a^2}{2a} = 3x + 2a \qquad [\because y^2 = 4ax \qquad ...(i)$$

and
$$\bar{y} = y + \frac{1 + y_1^2}{y_2} = y + \frac{1 + 4a^2/y^2}{-4a^2/y^2}$$

$$= y - \frac{y(y^2 + 4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}} \qquad ...(ii)$$

To find the evolute, we have to eliminate x from (i) and (ii)

$$\therefore \quad (\bar{y})^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x} - 2a}{3}\right)^3 \text{ or } 27a(\bar{y})^2 = 4(\bar{x} - 2a)^3.$$

Thus the locus of $(\bar{x}, \bar{y})$ *i.e.* evolute, is $27ay^2 = 4(x - 2a)^3$.

Example 4·48. *Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another equal cycloid.* *(Pondicherry, 1998 S)*

We have $\quad y_1 = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \cot\frac{\theta}{2}$.

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right) \cdot \frac{d\theta}{dx}$$

$$= -\text{cosec}^2\frac{\theta}{2} \cdot \frac{1}{2} \cdot \frac{1}{a(1 - \cos\theta)} = -\frac{1}{4a\sin^4\theta/2}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2} = a(\theta - \sin\theta) + \cot\frac{\theta}{2}\left(-4a\sin^4\frac{\theta}{2}\right)\left(1 + \cot^2\frac{\theta}{2}\right)$$

$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2} \cdot 4a\sin^4\frac{\theta}{2} \cdot \text{cosec}^2\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 4a\sin\theta/2\cos\theta/2 = a(\theta - \sin\theta) + 2a\sin\theta = a(\theta + \sin\theta)$$

$$\bar{y} = y + \frac{1 + y_1^2}{y_2} = a(1 - \cos\theta) + \left(1 + \cot^2\frac{\theta}{2}\right)\left(-4a\sin^4\frac{\theta}{2}\right)$$

$$= a(1 - \cos\theta) - 4a\sin^4\theta/2 \cdot \text{cosec}^2\theta/2$$

$$= a(1 - \cos\theta) - 4a\sin^2\theta/2$$

$$= a(1 - \cos\theta) - 2a(1 - \cos\theta) = -a(1 - \cos\theta)$$

Hence the locus of $(\bar{x}, \bar{y})$ *i.e.* the evolute, is given by

$x = a(\theta + \sin\theta)$, $y = -a(1 - \cos\theta)$ which is another equal cycloid.

Problems 4·16

1. Find the coordinates of the centre of curvature at $(at^2, 2at)$ on the parabola $y^2 = 4ax$. *(V.T.U., 2000 S)*
2. Show that the equation of the evolute of the parabola $x^2 = 4ay$ is $4(y - 2a)^3 = 27ax^2$. *(J.N.T.U., 1998)*
3. Show that the evolute of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ (*i.e.* $x^2/a^2 + y^2/b^2 = 1$) is
   $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$. *(Madurai, 1998 S ; Triputi, 1998 S)*
4. Show that the evolute of the rectangular hyperbola $xy = c^2$ (*i.e.* $x = ct$, $y = c/t$) is the curve
   $(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}$. *(Madras, 2000 S)*
5. Show that the evolute of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is the curve
   $x = a(t - \sin t)$, $y - 2a = a(1 + \cos t)$.
6. Find the evolute of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ *i.e.* $x^{2/3} + y^{2/3} = a^{2/3}$. *(Mangalore, 1999)*

7. Show that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is $x^2 + y^2 = a^2$.
(Kuvempu, 1996)

8. Find the circle of curvature at the point $(a/4, a/4)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$. (Hamirpur, 1996 S)

9. Find the circle of curvature for the curve $x^3 + y^3 = 3xy$ at the point $(3/2, 3/2)$ on it.

4.16. (1) ENVELOPE

The equation $x\cos\alpha + y\sin\alpha = 1$ ...(1)

represents a straight line for a given value of α. If different values are given to α, we get different straight lines. All these straight lines thus obtained are said to constitute a family of straight lines.

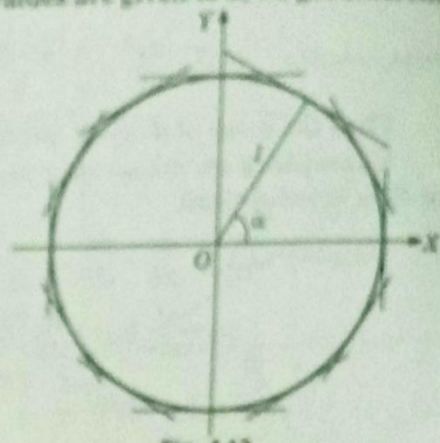


Fig. 4.13.

In general, *the curves corresponding to the equation $f(x, y, \alpha) = 0$ for different values of α, constitute a* **family of curves** *and α is called the* **parameter** *of the family.*

The envelope of a family of curves is the curve which touches each member of the family. For example, we know that all the straight lines of the family (1) touch the circle

$$x^2 + y^2 = 1 \qquad ...(2)$$

i.e. the envelope of the family of lines (1) is the circle (2)—Fig. 4.13, which may also be seen as the locus of the ultimate points of intersection of the consecutive members of the family of lines (1). This leads to the following :

Def. *If $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + \delta\alpha) = 0$ be two consecutive members of a family of curves, then the locus of their ultimate points of intersection is called the* **envelope** *of that family.*

(2) Rule to find the envelope of the family of curves $f(x, y, \alpha) = 0$:

Eliminate α from $f(x, y, \alpha) = 0$ and $\dfrac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$

Example 4.49. *Find the envelope of the family of lines $y = mx + \sqrt{(1 + m^2)}$, m being the parameter.* (Madras, 1990 S)

We have $\qquad (y - mx)^2 = 1 + m^2$ ...(i)

Differentiating (i) partially with respect to m,

$$2(y - mx)(-x) = 2m \quad \text{or} \quad m = xy/(x^2 - 1) \qquad ...(ii)$$

Now eliminate m from (i) and (ii).

Substituting the value of m in (i), we get

$$\left(y - \frac{x^2 y}{x^2 - 1}\right)^2 = 1 + \left(\frac{xy}{x^2 - 1}\right)^2 \quad \text{or} \quad y^2 = (x^2 - 1)^2 + x^2 y^2$$

or $\qquad x^2 + y^2 = 1$ **which is the required equation of the envelope.**

Obs. Sometimes the equation to the family of curves contains two parameters which are connected by a relation. In such cases, we eliminate one of the parameters by means of the given relation, then proceed to find the envelope.

Example 4.50. *Find the envelope of a system of concentric and coaxial ellipses of constant area.* (Tirupati, 1998 S)

Taking the common axes of the system of ellipses as the coordinate axes, the equation to an ellipse of the family is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ where } a \text{ and } b \text{ are the parameters.} \quad ...(i)$$

The area of the ellipse $= \pi ab$ which is given to be constant, say $= \pi c^2$.

$$\therefore \quad ab = c^2 \text{ or } b = c^2/a. \quad ...(ii)$$

Substituting in (i), $\frac{x^2}{a^2}+\frac{y^2}{(c^2/a)^2}=1$ or $x^2a^{-2}+(y^2/c^4)a^2=0$ ...(iii)

which is the given family of ellipses with a as the only parameter.

Differentiating partially (iii) with respect to a,

$$-2x^2a^{-3}+2(y^2/c^4)a=0 \text{ or } a^2=c^2x/y \quad ...(iv)$$

Eliminate a from (iii) and (iv).

Substituting the value of a^2 in (iii), we get

$$x^2(y/c^2x)+(y^2/c^4)(c^2x/y)=1 \text{ or } 2xy=c^2$$

which is the required equation of the envelope.

(3) Evolute *of a curve is the envelope of the normals to that curve* (Fig. 4·12).

Example 4·51. *Find the evolute of the parabola* $y^2=4ax$. (*Madras, 1996*)

Any normal to the parabola is $y=mx-2am-am^3$ ...(i)

Differentiating it with respect to m partially,

$$0=x-2a-3am^2 \text{ or } m=[(x-2a)/3a]^{1/2}$$

Substituting this value of m in (i),

$$y=\left(\frac{x-2a}{3a}\right)^{1/2}\left[x-2a-a\cdot\frac{x-2a}{3a}\right]$$

Squaring both sides, we have

$$27ay^2=4(x-2a)^3$$

which is the evolute of the parabola. (*cf.* Example 4·47).

Problems 4·17

Find the envelope of the following family of lines :

1. $y=mx+am^2$, m being the parameter. (*Madras, 1998 S*)
2. $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$, α being the parameter. (*Coimbatore, 1990*)
3. $y=mx-2am-am^3$. (*Madurai, 1998 S*)
4. $y=mx+\sqrt{(a^2m^2+b^2)}$, m being the parameter. (*Madras 1996*)
5. Find the envelope of the family of parabolas $y=x\tan\alpha-\frac{gx^2}{2u^2\cos\alpha}$, α being the parameter. (*Madras, 1998 S*)
6. Find the envelope of the straight line $x/a+y/b=1$, where the parameters a and b are connected by the relation : (i) $a+b=c$. (ii) $ab=c^2$ (*J.N.T.U., 1998*) (iii) $a^2+b^2=c^2$
7. Find the envelope of the family of ellipses $x^2/a^2+y^2/b^2=1$ for which $a+b=c$. (*Madras, 2000 S*)

Prove that the evolute of the

8. *ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

9. *hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$. *(Osmania, 1995)*

10. *parabola* $x^2 = 4by$ is $27bx^2 = 4(y - 2b)^3$

4.17. (1) INCREASING AND DECREASING FUNCTIONS

In the function $y = f(x)$, *if y increases as x increases* (as at A), *it is called an* **increasing function** *of x*.

On the contrary, *if y decreases as x increases* (as at C), *it is called a* **decreasing function** *of x*.

Let the tangent at any point on the graph of the function make an $\angle \psi$ with the x-axis (Fig. 4.14) so that

$$dy/dx = \tan \psi$$

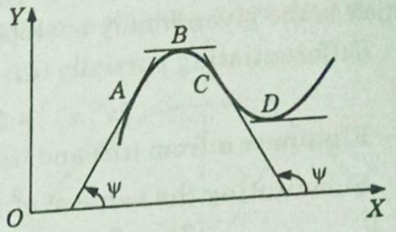


Fig. 4.14.

At any point such as A, where the function is increasing $\angle \psi$ is acute *i.e.* dy/dx is positive. At a point such as C, where the function is decreasing $\angle \psi$ is obtuse *i.e.* dy/dx is negative. Hence *the derivative of an increasing function is* + *ve*, and *the derivative of a decreasing function is* – *ve*.

Obs. *If the derivative is zero* (as at B or D), *then y is neither increasing nor decreasing*. In such cases, we say *that the function is* **stationary.**

(2) Concavity, Convexity and Point of inflexion

(*i*) *If a portion of the curve on both sides of a point, however small it may be, lies above the tangent (as at D), then the curve is said to be* **cancave upards** at D where d^2y/dx^2 is positive.

(*ii*) *If a portion of the curve on both sides of a point lies below the tangent* (as at B), *then the curve is said to be* **Convex upwards** *at B* where d^2y/dx^2 is negative.

(*iii*) *If the two portions of the curve lie on different sides of the tangent thereat* (*i.e.* the curve crosses the tangent (as at C), *then the point C* is said to be a **point of inflexion** of the curve.

At a point of inflexion $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$.

4.18. (1) MAXIMA AND MINIMA

Consider the graph of the continuous function $y = f(x)$ in the interval (x_1, x_2) (Fig. 4.15). Clearly the point P_1 is the highest in its own immediate neighbourhood. So also is P_3. At each of these points P_1, P_3 the function is said to have a *maximum* value.

On the other hand, the point P_2 is the lowest in its own immediate neighbourhood. So also is P_4. At each of these points P_2, P_4 the function is said to have a *minimum* value.

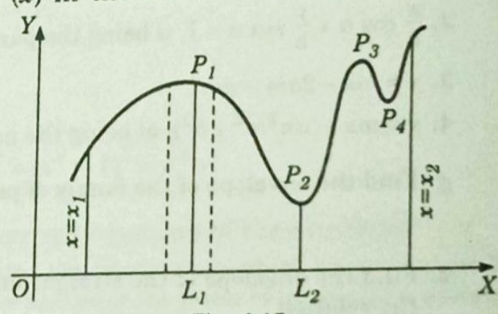


Fig. 4.15.

Thus, we have

Def. *A function f(x) is said to have a* **maximum** *value at x = a, if there exists a small number h, however small, such that f(a) > both f(a – h)* and *f(a + h)*

A function f(x) is said to have a **minimum** *value at x = a, if there exists a small number h, however small, such that f(a) < both f(a – h) and f(a + h).*

Obs. 1. *The maximum and minimum values of a function taken together are called its* **extreme** *values and the points at which the function attains the extreme values are called the* **turning points** *of the function.*

Obs. 2. *A maximum or minimum value of a function is not necessarily the greatest or least value of the function in any finite interval.* The maximum value is simply the greatest value in the immediate neighbourhood of the maxima point or the minimum value is the least value in the immediate neighbourhood of the minima point. In fact, *there may be several maximum and minimum values of a function in an interval and a minimum value may be even greater than a maximum value.*

Obs. 3. It is seen from the Fig. 4·15 that *maxima and minima values occur alternately.*

(2) **Conditions for maxima and minima.** At each point of extreme value, it is seen from the Fig. 4·15 that the tangent to the curve is parallel to the x-axis, *i.e.* its slope ($= dy/dx$) is zero. Thus if *the function is maximum or minimum at $x = a$, then $(dy/dx)_a = 0$.*

Around a maximum point say, P_1 ($x = a$), the curve is increasing in a small interval $(a - h, a)$ before L_1 and decreasing in $(a, a + h)$ after L_1 where h is positive and small.

i.e., in $(a - h, a)$, $dy/dx \geq 0$; at $x = a$, $dy/dx = 0$ and in $(a, a + h)$, $dy/dx \leq 0$.

Thus dy/dx (which is a function of x) changes sign from positive to negative in passing through P_1, *i.e.* it is a decreasing function in the interval $(a - h, a + h)$ and therefore, its derivative d^2y/dx^2 is negative at P_1 ($x = a$).

Similarly around a minimum point say P_2, dy/dx changes sign from negative to positive in passing through P_2, *i.e.* it is an increasing function in the small interval around L_2 and therefore its derivative d^2y/dx^2 is positive at P_2.

Hence (*i*) *$f(x)$ is maximum at $x = a$ if $f'(a) = 0$ and $f''(a)$ is – ve [i.e. $f'(a)$ changes sign from + ve to – ve]*

(*ii*) *$f(x)$ is minimum at $x = a$, if $f'(a) = 0$ and $f''(a)$ is + ve [i.e. $f'(a)$* changes sign from *– ve to +ve]*

Obs. A maximum or a minimum value is a stationary value but a stationary value may neither be a maximum nor a minimum value.

(3) **Procedure for finding maxima and minima.**

(*i*) *Put the given function $= f(x)$*

(*ii*) *Find $f'(x)$ and equate it to zero. Solve this equation and let its roots be a, b, c, ...*

(*iii*) *Find $f''(x)$ and substitute in it by turns $x = a, b, c$, ...*

If $f''(a)$ is – ve, $f(x)$ is maximum at $x = a$.

If $f''(a)$ is + ve, $f''(x)$ is minima at $x = a$.

(*iv*) Sometimes $f''(x)$ may be difficult to find out or $f''(x)$ may be zero at $x = a$. In such cases, see if *$f'(x)$ changes sign from + ve to – ve as x passes through a, then $f(x)$ is maximum at $x = a$.*

If $f'(x)$ changes sign from – ve to + ve as x passes through a, $f(x)$ is minimum at $x = a$.

If $f'(x)$ does not change sign while passing through $x = a$, $f(x)$ is neither maximum nor minimum at $x = a$.

Example 4·52. *Find the maximum and minimum values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval (0, 2).*

Let $\quad f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

Then $\quad f'(x) = 12x^3 - 6x^2 - 12x + 6 = 6(x^2 - 1)(2x - 1)$

$\therefore \quad f'(x) = 0$ when $x = \pm 1, \frac{1}{2}$.

So in the interval (0, 2), $f(x)$ can have maximum or minimum at $x = \frac{1}{2}$ or 1.

Now $\quad f''(x) = 36x^2 - 12x - 12 = 12(3x^2 - x - 1)$ so that $f''\left(\frac{1}{2}\right) = -9$ and $f''(1) = 12$.

$\therefore f(x)$ has a maximum at $x = \frac{1}{2}$ and a minimum at $x = 1$.

Thus the maximum value $= f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 1 = 2\frac{7}{16}$

and the minimum value $= f(1) = 3(1)^4 - 2(1)^3 - 6(1)^2 + 6(1) + 1 = 2$.

Example 4.53. *Show that $\sin x(1 + \cos x)$ is a maximum when $x = \pi/3$.*

Let $\quad f(x) = \sin x(1 + \cos x)$

Then $\quad f'(x) = \cos x(1 + \cos x) + \sin x(-\sin x)$

$= \cos x(1 + \cos x) - (1 - \cos^2 x) = (1 + \cos x)(2\cos x - 1)$

$\therefore \quad f'(x) = 0$ when $\cos x = \frac{1}{2}$ or -1 *i.e.* when $x = \pi/3$ or π.

Now $\quad f''(x) = -\sin x(2\cos x - 1) + (1 + \cos x)(-2\sin x) = -\sin x(4\cos x + 1)$

so that $\quad f''(\pi/3) = -3\sqrt{2}/2$ and $f''(\pi) = 0$.

Thus $f(x)$ has a maximum at $x = \pi/3$.

Since $f''(\pi)$ is 0, let us see whether $f'(x)$ changes sign or not.

When x is slightly $< \pi$, $f'(x)$ is $-$ve, when x is slightly $> \pi$, $f'(x)$ is again $-$ ve *i.e.* $f'(x)$ does not change sign as x passes through π. So $f(x)$ is neither maximum nor minimum at $x = \pi$.

Problems 4.18

1. Find the maximum and minimum values of $x^5 - 5x^4 + 5x^3 - 1$.
2. Find the extreme values of the function $(x - 1)^3(x + 1)^3$
3. The function $f(x)$ defined by $f(x) = a/x + bx$, $f(2) = 1$, has an extremum at $x = 2$. Determine a and b. Is this point (2, 1), a point of maximum or minimum on the graph of $f(x)$?
4. Show that the function $\sin 3x - 3\sin x$ is minimum when $x = \pi/2$ and maximum when $x = 3\pi/2$.
5. If a beam of weight w per unit length is built-in horizontally at one end A and rests on a support O at the other end, the deflection y at a distance x from O is given by
$$EIy = \frac{w}{48}(2x^4 - 3lx^3 + l^3x),$$
where l is the distance between the ends. Find x for y to be maximum.
6. The horse-power developed by an aircraft travelling horizontally with velocity v feet per second is given by
$$H = \frac{aw^2}{v} + bv,$$
where a, b and w are constants. Find for what value of v the horse-power is maximum.
7. The power output of a radio valve is proportional to $x/(x + r)^2$ where r, the valve resistance is constant and x is a variable impedance. Find x for the output to be maximum. *(Andhra, 1990)*
8. The velocity of waves of wave-length λ on deep water is proportional to $\sqrt{(\lambda/a + a/\lambda)}$, where a is a certain constant, prove that the velocity is minimum when $\lambda = a$.
9. In a submarine telegraph cable, the speed of signalling varies as $x^2 \log_e(1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1/\sqrt{e}$. *(Marathwada, 1990)*
10. The efficiency e of a screw-jack is given by $e = \tan\theta/\tan(\theta + \alpha)$, where α is a constant. Find θ if this efficiency is to be maximum. Also find the maximum efficiency. *(Ranchi, 1998)*

4.19. PRACTICAL PROBLEMS

In many problems, the function (whose maximum or minimum value is required) is not directly given. It has to be formed from the given data. If the function contains two variables, one of them has to be eliminated with the help of the other conditions of the problem. A number of problems deal with triangles, rectangles, circles, spheres, cones, cylinders etc. The student is therefore, advised to remember the formulae for areas, volumes, surfaces etc. of such figures.

Example 4·54. *A window has the form of a rectangle surmounted by a semi-circle. If the perimeter is 40 ft., find its dimensions so that the greatest amount of light may be admitted.* *(Madras, 2000 S)*

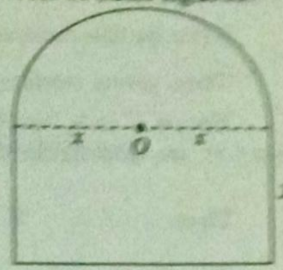


Fig. 4·16.

The greatest amount of light may be admitted means that the area of the window may be maximum.

Let x ft. be the radius of the semi-circle so that one side of the rectangle is $2x$ ft. (Fig. 4·16). Let the other side of the rectangle be y ft. Then the perimeter of the whole figure

$$= \pi x + 2x + 2y = 40 \text{ (given) and the area } A = \tfrac{1}{2}\pi x^2 + 2xy. \quad ...(i)$$

Here A is a function of two variables x and y. To express A in terms of one variable x (say), we substitute the value of y from (i) in it.

$$\therefore \quad A = \tfrac{1}{2}\pi x^2 + x\,[40 - (\pi + 2)\,x] = 40\,x - \left(\frac{\pi}{2} + 2\right)x^2$$

Then $$\frac{dA}{dx} = 40 - (\pi + 4)\,x$$

For A to be maximum or minimum, we must have $dA/dx = 0$ *i.e.* $40 - (\pi + 4)\,x = 0$

or $$x = 40/(\pi + 4)$$

$\therefore$ from (i), $y = \tfrac{1}{2}[40 - (\pi + 2)\,x] = \tfrac{1}{2}[40 - (\pi + 2)\,40/(\pi + 4)] = 40/(\pi + 4)$ *i.e.* $x = y$

Also $$\frac{d^2A}{dx^2} = -(\pi + 4), \text{ which is negative.}$$

Thus the area of the window is maximum when the radius of the semi-circle is equal to the height of the rectangle.

Example 4·55. *A rectangular sheet of metal of length 6 metres and width 2 metres is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. Find approximately, the height of the box, such that the volume of the box is maximum.*

Let the side of each of the squares cut off be x m. so that the height of the box is x m. and the sides of the base are $6 - 2x$, $2 - 2x$ m. (Fig. 4·17).

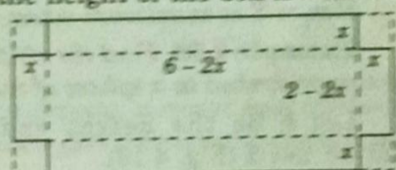


Fig. 4·17.

$\therefore$ Volume V of the box

$$= x\,(6 - 2x)\,(2 - 2x) = 4\,(x^3 - 4x^2 + 3x)$$

Then $$\frac{dV}{dx} = 4\,(3x^2 - 8x + 3)$$

For V to be maximum or minimum, we must have

$$dV/dx = 0 \quad i.e. \quad 3x^2 - 8x + 3 = 0$$

$$\therefore \quad x = \frac{8 \pm \sqrt{[64 - 4 \times 3 \times 3]}}{6} = 2{\cdot}2 \text{ or } {\cdot}45 \text{ m.}$$

The value $x = 2{\cdot}2$ m. is inadmissible, as no box is possible for this value.

Also $$\frac{d^2V}{dx^2} = 4\,(6x - 8), \text{ which is } -\text{ ve for } x = {\cdot}45 \text{ m.}$$

Hence the volume of the box is maximum when its height is 45 cm.

Example 4·56. *Show that the right circular cylinder of given surface (including the ends) and maximum volume is such that its height is equal to the diameter of the base.*

Let r be the radius of the base and h, the height of the cylinder.

Then given surface $S = 2\pi rh + 2\pi r^2$ ...(i) and the volume $V = \pi r^2 h$ ...(ii)

Hence V is a function of two variables r and h. To express V in terms of one variable only (say r), we substitute the value of h from (i) in (ii).

Then $$V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{1}{2} Sr - \pi r^3 \quad \therefore \quad \frac{dV}{dr} = \frac{1}{2} S - 3\pi r^2.$$

For V to be maximum or minimum, we must have $dV/dr = 0$,

i.e. $$\tfrac{1}{2} S - 3\pi r^2 = 0 \text{ or } r = \sqrt{(S/6\pi)}.$$

Also $\dfrac{d^2V}{dr^2} = -6\pi r$, which is negative for $r = \sqrt{(S/6\pi)}$.

Hence V is maximum for $r = \sqrt{(S/6\pi)}$.

i.e. for $6\pi r^2 = S = 2\pi rh + 2\pi r^2$ *i.e.* for $h = 2r$, which proves the required result. [by (i)

Example 4·57. *Show that the diameter of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is equal to the radius of the cone.*

Let r be the radius OA of the base and α the semi-vertical angle of the given cone (Fig. 4·18). Inscribe a cylinder in it with base-radius $OL = x$.

Then the height of the cylinder LP

$$= LA \cot\alpha = (r - x)\cot\alpha$$

$\therefore$ The curved surface S of the cylinder

$$= 2\pi x . LP = 2\pi x (r - x)\cot\alpha$$
$$= 2\pi \cot\alpha\, (rx - x^2)$$

$\therefore$ $$\frac{dS}{dx} = 2\pi \cot\alpha\, (r - 2x) = 0 \text{ for } x = r/2.$$

and $$\frac{d^2S}{dx^2} = -4\pi \cot\alpha$$

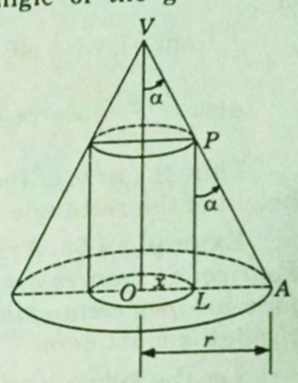


Fig. 4·18.

Hence S is maximum when $x = r/2$.

Example 4·58. *Find the altitude and the semi-vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a.*

Let h be the height and α the semi-vertical angle of the cone so that its radius $BD = h\tan\alpha$ (Fig. 4·19).

$\therefore$ The volume V of the cone is given by

$$V = \frac{1}{3}\pi (h\tan\alpha)^2 h = \tfrac{1}{3}\pi h^3 \tan^2\alpha.$$

Now we must express $\tan\alpha$ in terms of h.

In the rt. $\angle d\ \Delta AEO$,

$$EA = \sqrt{(OA^2 - a^2)} = \sqrt{[(h-a)^2 - a^2]} = \sqrt{(h^2 - 2ha)}$$

$\therefore$ $$\tan\alpha = \frac{EO}{EA} = \frac{a}{\sqrt{(h^2 - 2ha)}}$$

Thus $$V = \frac{1}{3}\pi h^3 \cdot \frac{a^2}{h^2 - 2ha} = \frac{1}{3}\pi a^3 \cdot \frac{h^2}{h - 2a}$$

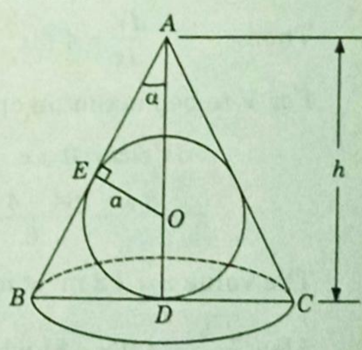


Fig. 4·19.

$$\therefore \quad \frac{dV}{dh} = \frac{1}{3}\pi a^2 \cdot \frac{(h-2a)\,2h - h^2 \cdot 1}{(h-2a)^2} = \frac{1}{3}\pi a^2 \cdot \frac{h\,(h-4a)}{(h-2a)^2}$$

Thus $\frac{dV}{dh} = 0$ for $h = 4a$, the other value $(h = 0)$ being not possible.

Also dV/dh is – ve when h is slightly $< 4a$, and it is + ve when h is slightly $> 4a$.

Hence V is minimum (*i.e.* least) when $h = 4a$

and
$$\alpha = \sin^{-1}\left(\frac{a}{OA}\right) = \sin^{-1}\left(\frac{a}{3a}\right) = \sin^{-1}\frac{1}{3}.$$

Example 4·59. *Find the volume of the largest possible right-circular cylinder that can be inscribed in a sphere of radius a.* (*Gorakhpur, 1991*)

Let O be the centre of the sphere of radius a. Construct a cylinder as shown in the Fig. 4·20. Let $OA = r$.

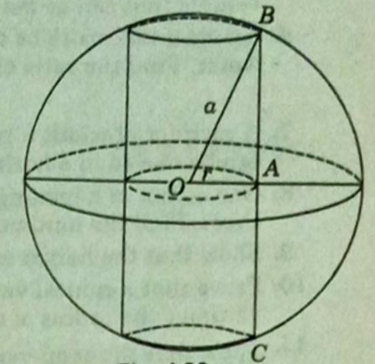


Fig. 4·20.

Then $AB = \sqrt{(OB^2 - OA^2)} = \sqrt{(a^2 - r^2)}$

$\therefore$ Height h of the cylinder $= 2 \cdot AB = 2\sqrt{(a^2 - r^2)}$.

Thus volume V of the cylinder

$$= \pi r^2 h = 2\pi r^2 \sqrt{(a^2 - r^2)}$$

$$\therefore \quad \frac{dV}{dr} = 2\pi\,\{2r\sqrt{(a^2 - r^2)} + r^2 \cdot \frac{1}{2}(a^2 - r^2)^{-1/2}(-2r)\}$$

$$= \frac{2\pi r\,(2a^2 - 3r^2)}{\sqrt{(a^2 - r^2)}}$$

Thus $dV/dr = 0$ when $r^2 = 2a^2/3$, the other value $(r = 0)$ being not admissible.

Now
$$\frac{d^2V}{dr^2} = 2\pi\,\frac{\sqrt{(a^2 - r^2)}\,(2a^2 - 9r^2) - r\,(2a^2 - 3r^2) \times \frac{1}{2}(a^2 - r^2)^{-1/2} \cdot (-2r)}{(a^2 - r^2)}$$

$$= 2\pi\,\frac{(a^2 - r^2)\,(2a^2 - 9r^2) + r^2\,(2a^2 - 3r^2)}{(a^2 - r^2)^{3/2}}$$ which is – ve for $r^2 = 2a^2/3$.

Hence V is maximum for $r^2 = 2a^2/3$ and maximum volume

$$= 2\pi r^2 \sqrt{(a^2 - r^2)} = 4\pi a^3/3\sqrt{3}.$$

Example 4·60. *Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is* $\frac{3}{2}c$ *miles per hour.*

Let v m.p.h. be the velocity of the boat so that its velocity relative to water (when going against the current) is $(v - c)$ m.p.h.

$\therefore$ time required to cover a distance of s miles $= \frac{s}{v - c}$ hours.

Since the petrol burnt per hour $= kv^3$, k being a constant.

$\therefore$ the total petrol burnt, y, is given by

$$y = k\,\frac{v^3 s}{v - c} = ks\,\frac{v^3}{v - c}. \quad \therefore \quad \frac{dy}{dv} = ks \cdot \frac{(v - c)\,3v^2 - v^3 \cdot 1}{(v - c)^2}$$

$$= ks \cdot \frac{v^2(2v - 3c)}{(v - c)^2}$$

CHAPTER 1.5

LINEAR PARTIAL DIFFERENTIAL EQUAITONS OF SECOND AND HIGHER ORDER WITH CONSTNAT COEFFICENTS

A linear homogeneous partial differential equation of nth order with constant coefficients is represented as:

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a^n \frac{\partial^n z}{\partial y^n} = f(x, y) \quad \dots(1)$$

NOTE

1. As the derivatives involved are of same order, the equation is called homogeneous.

2. If $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$

$$(D^n + a_1 D^{n-1} D' + \dots + a_n D'^n) z = f(x, y)$$

$$F(D, D') z = f(x, y) \quad \dots (2)$$

3. The complete solution of (2) consists of two parts, complementary function (C.F.) and the particular integral (P.I).

Complete solution is:

$$Z = C.F. + P.I. \quad \dots (3)$$

Procedure to find the complementary function (C.F.)

Consider a third order P.D.E.

$$a_0 \frac{\partial^3 z}{\partial x^3} + a_1 \frac{\partial^3 z}{\partial x^2 \partial y} + a_2 \frac{\partial^3 z}{\partial x \partial y^2} + a_3 \frac{\partial^3 z}{\partial y^3} = f(x, y) \quad \dots (1)$$

Put R.H.S = 0

$$a_0 \frac{\partial^3 z}{\partial x^3} + a_1 \frac{\partial^3 z}{\partial x^2 \partial y} + a_2 \frac{\partial^3 z}{\partial x \partial y^2} + a_3 \frac{\partial^3 z}{\partial y^3} = 0$$

Put $\frac{\partial}{\partial x} = D, \frac{\partial^2}{\partial x^2} = D^2 \dots$

$$\frac{\partial}{\partial y} = D', \frac{\partial^2}{\partial y^2} = D'^2 \dots$$

$$(a_o D^3 + a_1 D^2 D' + a_2 D D'^2 + a_3 D'^3) z = 0 \quad \dots (2)$$

$$f(D, D') z = 0 \quad \dots (3)$$

Put $D = m, D' = 1, f(m, 1) = 0$ or $f(D, D') = 0$ ... (4)

(4) is called the auxiliary equation or A.E. m_1, m_2, m_3 are the roots of (4).

Case (i)

If $m_1 \neq m_2 \neq m_3$ (real or complex and different) Solution is:

$$z = f_1 (y + m_1 x) + f_2 (y + m_2 x) + f_3 (y + m_3 x) \quad \dots (i)$$

Case (ii)

If $m_1 = m_2 = m_3$ (real and equal)

$$z = f_1 (y + m_1 x) + x f_2 (y + m_1 x) + x^2 f_3 (y + m_1 x) \quad \dots (ii)$$

UNIT 1

SOLVED EXAMPLES

Example 1

$$\frac{\partial^2 z}{\partial x^2}-\frac{5\partial^2 z}{\partial x\partial y}+6\frac{\partial^2 z}{\partial y^2}=0$$

SOLUTION: $(D^2-5DD'+6D'^2)z=0$

$$f(D,D')z=0$$

Put $D=m$, $D'=1$, the A.E. is

$$m^2-5m+6=0$$
$$(m-2)(m-3)=0$$
$$m=2,\ m=3$$

The complementary function is

C.F. is: $z=f_1(y+2x)+f_2(y+3x)$

Example 2

Solve $\frac{\partial^2 z}{\partial x^2}-6\frac{\partial^2 z}{\partial x\partial y}+9\frac{\partial^2 z}{\partial y^2}=0$

SOLUTION: $(D^2-6DD'+9D'^2)Z=0$

$$D=m,\ D'=1$$

A.E.: (Auxilary equation) $m^2-6m+9=0$

$$m=3,3$$

Complementary function is:

$$Z=f_1(y+3x)+x f_2(y+3x)$$

Example 3

Solve $\frac{\partial^3 z}{\partial x^3}-3\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial y^3}=0$

SOLUTION: $(D^3-3D^2D'+4D'^3)z=0$

$$f(D,D')z=0$$

UNIT 1

Put $D=m, D'=1$

A.E.: $m^3-3m^2+4=0$

Put: $m=-1,\ -1-3+4=0\quad \therefore m=-1$ is a root

$$\begin{array}{r|l} m=-1 & m^3-3m^2+4 \quad (m^2-4m+4 \\ (m+1) & m^3+m^2 \\ \hline & -4m^2+4 \\ & -4m^2-4m \\ \hline & 4m+4 \\ & 4m+4 \\ \hline & 0 \end{array}$$

Solve

$$m^2-4m+4=0$$
$$(m-2)(m-2)=0$$
$$m=2,2.$$

roots are $m=2,2,-1$

$\therefore$ Solution is $z=f_1(y+2x)+x f_2(y+2x)+f_3(y-x)$

Example 4 **(Anna Uni. Oct/Nov. 1996)**

Solve $(D^3-4D^2D'+4DD'^2]z=0.$

SOLUTION: Given $(D^3-4D^2D'+4DD'^2)z=0.$

A.E: $m^3-4m^2+4m=0.$

$$m=0,\ 2,\ 2$$

$\therefore$ Solutionis $z=f_1(y+0\cdot x)+f_2(y+2x)+x f_3(y+2x)$

Example 5 **(Anna Uni. April/May 2001)**

Solve $4\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0.$

SOLUTION: $(4D^2-D'^2)z=0.$

A.E: $4m^2-1=0;\ m=\pm\frac{1}{2}$

Solution is: $z=f_1(y+0.5x)+f_2(y-0.5x)$

Example 6 (Anna Uni. April/May 2003)

Solve $(D^3 - 3DD'^2 + 2D'^3)z = 0$

SOLUTION: $m^3 - 3m + 2 = 0.$

$$m = 1,\ 1,\ -2$$

Solution is $$z = f_1(y - 2x) + f_2(y + x) + xf_3(y + x)$$

Example 7 (Anna Uni. Nov/Dec. 2003)

Find the general solution of $4\dfrac{\partial^2 z}{\partial x^2} - 12\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{9\,\partial^2 z}{\partial y^2} = 0.$

SOLUTION: $4m^2 - 12m + 9 = 0$

$$m = \frac{3}{2},\ \frac{3}{2}$$

$$z = f_1(y + 1.5x) + xf_2(y + 1.5x).$$

Example 8 (Anna Uni. April/May 2005)

Solve $(D^3 + DD'^2 - D^2D' - D'^3)z = 0.$

SOLUTION: $m^3 - m^2 + m - 1 = 0.$

$$m = 1,\ m = \pm i$$

$$z = f_1(y + x) + f_2(y + ix) + f_3(y - ix)$$

Example 9 (Anna Uni. May 1996)

[Type: Non. homogeneous]

Solve $(D^2 - DD' + D' - 1)z = 0.$ ... (1)

SOLUTION: Take the solution as:

$$z = ce^{hx + ky} \quad \text{... (2)}$$

Replace D by h, D' by $k \Rightarrow$

$$h^3 - hk + k - 1 = 0.$$

$$h = 1,\ h = k - 1$$

complete solution is

$$z = \Sigma c_1 e^{x + ky} + \Sigma c_2 e^{(k-1)x + ky}.$$

$$z = e^x \phi_1(y) + e^{-x}\phi_2(y + x)$$

UNIT 1

Type

R.H.S $= f(x, y)$, if $R.H.S. \neq 0$, we need to find the particular Integral (P.I.) Here there are three cases.

Case (i)

$$\text{R.H.S} = e^{ax + by}$$

$$\Rightarrow \quad P.I = \frac{e^{ax+by}}{f(D, D')} = \frac{e^{ax+by}}{f(a, b)}$$

Provided $f(a, b) \neq 0$, if $f(a, b) = 0$, it is a case of failure.

Case (ii)

(a) R.H.S $= \sin(ax + by)$ (or) $\cos(ax + by)$

If $f(D^2, DD', D'^2)z = \sin(ax + by)$, then

$$\text{P.I} = \frac{\sin(ax + by)}{f(-a^2, -ab, -b^2)},\ \text{provided } f(-a^2, -ab, -b^2) \neq 0$$

If R.H.S. $= \cos(ax + by)$

$$\text{P.I} = \frac{\cos(ax + by)}{f(-a^2, -ab, -b^2)},\ \text{provided } f(-a^2, -ab - b^2) \neq 0$$

Case (iii)

R.H.S. $= x^p y^q$ (p, q being positive integers), then

$$\text{P.I} = \frac{1}{f(D, D')}x^p y^q = [f(D, D')]^{-1} x^p y^q,$$

Expand $f(D, D')$ in ascending powers of D, D' and then operate o
$x^p y^q$

NOTE *If the above cases fails then we need to prefer the General method for finding P.I.*

Example 10

Solve $\dfrac{\partial^2 z}{\partial x^2} + 7\dfrac{\partial^2 z}{\partial x \partial y} + 12\dfrac{\partial^2 z}{\partial y^2} = e^{2x+5y}$

SOLUTION: Given $\dfrac{\partial^2 z}{\partial x^2} + 7\dfrac{\partial^2 z}{\partial x \partial y} + 12\dfrac{\partial^2 z}{\partial y^2} - e^{2x+5y}$

R.H.S = 0: $(D^2 + 7DD' + 12D'^2)\, z = 0$

$$f(D, D')\, z = 0$$

A.E. (put $D = m, D' = 1$)

$$m^2 + 7m + 12 = 0$$

$$m = -3, -4$$

C.F. $= f_1(y - 3x) + f_2(y - 4x)$ ... (1)

R.H.S $= e^{2x+5y} = e^{ax+by}$ (Using R.H.S find P.I)

$a = 2, b = 5$

$f(D, D') = D^2 + 7DD' + 12D'^2$

$f(a, b) = f(2, 5) = 4 + 70 + 300 = 374 \neq 0$

$$\text{P.I} = \frac{e^{ax+by}}{f(a, b)} = \frac{e^{2x+5y}}{374} \quad ... (2)$$

Complete solution is: $z = CF + P.I$

$$z = f_1(y - 3x) + f_2(y - 4x) + \frac{e^{2x+5y}}{374}$$

Example 11

Solve $\dfrac{\partial^3 z}{\partial x^3} - 5\dfrac{\partial^3 z}{\partial x^2 \partial y} + 6\dfrac{\partial^3 z}{\partial y^3} = e^{4x+y}$

SOLUTION: $(D^3 - 5D^2 D' + 6D'^3)\, z = e^{4x+y}$ ... (1)

$$f(D, D') \cdot z = e^{4x+y}$$

R.H.S = 0, put $D = m, D' = 1$

$$m^3 - 5m^2 + 6 = 0$$

$m = -1: -1 - 5 + 6 = 0 \quad \therefore m = -1$ is a root

$m + 1$) $m^3 - 5m^2 + 6$ ($m^2 - 6m + 6$

$m^3 + m^2$
$-6m^2 + 6$
$-6m^2 - 6m$
$6m + 6$
$6m + 6$
0

$m^2 - 6m + 6 = 0$

$$m = \frac{6 \pm \sqrt{36 - 24}}{2(1)}$$

$$m = \frac{6 \pm 2\sqrt{3}}{2}$$

$$m = \frac{6 \pm 2\sqrt{3}}{2}$$

$$m = 3 \pm \sqrt{3}$$

roots are $m = -1, 3 + \sqrt{3}, 3 - \sqrt{3}$

C.F $= f_1(y - x) + f_2(y + (3 + \sqrt{3})x) + f_3(y + (3 - \sqrt{3})x)$ ... (2)

Using R.H.S. find the P.I

R.H.S $= e^{4x+y} = e^{ax+by} \Rightarrow a = 4, b = 1$

$f(D, D') = D^2 - 5D^2 D' + 6D'^3$

$f(a, b) = f(4, 1) = 64 - 80 + 6 = -10 \neq 0.$

$$\text{P.I} = \frac{e^{ax+by}}{f(a, b)} = \frac{e^{4x+y}}{-10} \quad ... (3)$$

Solution is:

$$z = \text{C.F} + \text{P.I}$$

UNIT 1

$$z = f_1(y-x) + f_2\left(y + (3+\sqrt{3})x\right) + f_3\left(y + (3-\sqrt{3})x\right) - \frac{e^{4x+y}}{10}$$

This is the Complete solution.

Example 12

Solve $(D-2D')(D-D')^3 z = e^{3x+2y}$

SOLUTION: Given $(D-2D')(D-D')^3 z = e^{3x+2y}$ ... (1)

Put **R.H.S = 0**

$$(D-2D')(D-D')^3 z = 0$$

$$f(D, D') \cdot z = 0$$

Put $D = m,\ D' = 1$

A.E: $(m-2)(m-1)^3 = 0$

$m = 2,\ m = 1, 1, 1.$

C.F. $= f_1(y+2x) + f_2(y+x) + xf_3(y+x) + x^2 f_4(y+x)$ ... (2)

R.H.S.$= e^{3x+2y} = e^{ax+by}$

$a = 3,\ b = 2$

$f(a, b) = (3-4)(3-2)^3 = -1$

$$\text{P.I} = \frac{e^{ax+by}}{f(a,b)} = \frac{e^{3x+2y}}{-1}$$ ... (3)

Complete solution is

$z = \text{C.F.} + \text{P.I}$

$$z = f_1(y+2x) + f_2(y+x) + xf_3(y+x) + x^2 f_4(y+x) - e^{3x+2y}$$

Example 13

Solve $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$

SOLUTION: Given $(D^2 - 2DD' + D'^2) z = \sin(2x+3y)$ ... (1)

R.H.S. = 0

A.E.: $m^2 - 2m + 1 = 0,\ m = 1, 1$

C.F. $= f_1(y+x) + x f_2(y+x)$ ... (2)

Using R.H.S find P.I

$\sin(ax+by) = \sin(2x+3y) \Rightarrow a = 2, b = 3$

Substitute: $D^2 = -a^2 = -4$

$DD' = -ab = -6$

$D'^2 = -b^2 = -9$

$$\text{P.I} = \frac{\sin(2x+3y)}{D^2 - 2DD' + D'^2} = \frac{\sin(2x+2y)}{-4+12-9}$$

$= -\sin(2x+3y)$ ... (3)

Complete solution is

$z = CF + P.I$

$z = f_1(y+x) + x f_2(y+x) - \sin(2x+3y)$

Example 14

Solve $(D^3 - 4D^2 D' + 4DD'^2) z = 12 \sin(2x+3y)$

SOLUTION: Given $(D^3 - 4D^2 D' + 4\ \ ^2) z = 12\sin(2x+3y)$ ... (1)

R.H.S. = 0: $(D^3 - 4D^2 D' + 4DD'^2) z = 0$

$f(D, D') \cdot z = 0$

A.E.: put $D = m,\ D' = 1 \Rightarrow m^3 - 4m^2 + 4m = 0$

$m(m^2 - 4m + 4) = 0$

$m = 0, 2, 2$

C.F. $= f_1(y + 0 \cdot x) + f_2(y+2x) + x f_3(y+2x)$ ... (2)

R.H.S $\Rightarrow \sin(ax+by) = \sin(2x+3y)$

$a = 2, b = 3$

$D^2 = -a^2 = -4 \mid DD' = -ab = -6 \mid D'^2 = -b^2 = -9$

$$\text{P.I} = 12 \cdot \frac{\sin(2x+3y)}{D^3 - 4D^2 D' + 4DD'^2} = 12 \frac{\sin(2x+3y)}{DD^2 - 4D^2 D' + 4DD'^2}$$

$$P.I = 12\frac{\sin(2x+3y)}{-4D+16D'-36D} = 12\cdot\frac{\sin(2x+3y)}{16D'-40D}$$

(multiply in numerator and Denominator by $40D + 16D'$)

$$P.I = \frac{-12\sin(2x+3y)}{40D-16D'} = \frac{-12(40D+16D')\sin(2x+3y)}{1600D^2-256D'^2}$$

$$= -12\frac{[40D(\sin(2x+3y))+16D'(\sin(2x+3y))]}{-6400+2304}$$

$$= \frac{-12}{-4096}(40\cos(2x+3y)\cdot 2+16\cos(2x+3y)\cdot 3)$$

$$P.I. = \frac{1536}{4096}\cos(2x+3y) \qquad \text{... (3)}$$

Complete solution is

$z = CF + P.I$

$$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + \frac{1536}{4096}\cos(2x+3y)$$

Example 15

Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(3x+2y)$

☞ **SOLUTION:** $(D^2 + DD' - 6D'^2)z = \cos(3x+2y)$

$f(D, D')z = \cos(3x+2y)$

R.H.S. = 0, Put $D = m, D' = 1$

$m^2 + m - 6 = 0$

$m = -3, 2$

C.F. $= f_1(y-3x) + f_2(y+2x)$ ... (1)

R.H.S. $\Rightarrow \cos(ax+by) = \cos(3x+2y)$

$a = 3, b = 2$

Put $D^2 = -a^2 = -9, DD' - ab = -6, D'^2 = -b^2 = -4$

$$P.I = \frac{\cos(3x+2y)}{D^2+DD'-6D'^2} = \frac{\cos(3x+2y)}{-9-6+24} = \frac{\cos(3x+2y)}{9} \qquad \text{... (2)}$$

Complete solution is $z = f_1(y-3x) + f_2(y+2x) + \dfrac{\cos(3x+2y)}{9}$

Example 16

Solve $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x \partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = \sin 4x \cos 3y$

☞ **SOLUTION:** Given $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x\, \partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = \sin 4x \cos 3y$

RHS = 0

$(D^2 - 5DD' + 6D'^2)z = 0$

A.E: $m^2 - 5m + 6 = 0$

$m = 2, 3$

C.F. $= f_1(y+2x) + f_2(y+3x)$ ... (1)

(using RHS find the P.I)

RHS$= \sin 4x \cos 3y = \frac{1}{2}[\sin(4x+3y) + \sin(4x-3y)]$

$f(D, D') = D^2 - 5DD'^1 + 6D'^2$

(i) $\sin(4x+3y)$

$a = 4,\ b = 3$

$D^2 = -16, DD' = -12, D'^2 = -9$

$$P.I_1 = \frac{\sin(4x+2y)}{-16+60-54}$$

$$PI_1 = \frac{\sin(4x+2y)}{-10} \qquad \text{...(2)}$$

(ii) $\sin(4x-3y)$

$a = 4, b = -3$

$D^2 = -16, DD' = 12, D'^2 = -9$

$$P.I_2 = \frac{\sin(4x-3y)}{-16-60-54}$$

$$PI_2 = \frac{\sin(4x-3y)}{-130} \qquad \text{...(3)}$$

Solution is $z = CF + PI_1 + PI_2$

$$z = f_1(y+2x) + f_2(y+3x) - \frac{1}{20}\sin(4x+2y) - \frac{1}{260}\sin(4x-3y)$$

NOTE

Solve $(D^2 - D'^2)z = \sin 2x \sin 3x$
(Anna Uni - April 1996)

Example 17

Solve $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{3x+4y} + \sin(4x-3y)$

SOLUTION: Given:

$$(D^2 - 3DD' + 2D'^2)\,z = e^{3x+4y} + \sin(4x-3y)$$

R.H.S $= 0 \Rightarrow (D^2 - 3DD' + 2D'^2)\,z = 0$

A.E: $m^2 - 3m + 2 = 0,\ m = 1, 2$

$\therefore$ C.F. $= f_1(y+x) + f_2(y+2x)$ ... (1)

$f(D, D') = D^2 - 3DD' + 2D'^2$

R.H.S.

$e^{3x+4y} = e^{ax+by}$	$\sin(4x-3x) = \sin(ax+by)$
$a = 3,\ b = 4$	$a = 4,\ b = -3$
$f(a, b) = 9 - 36 + 32$	$D^2 = -a^2 = -16$
	$DD' = -ab = 12$
$f(a, b) = 5$	$D'^2 = -b^2 = -9$
$PI_1 = \dfrac{e^{3x+4y}}{5}$ ...(2)	$PI_2 = \dfrac{\sin(4x-3y)}{-16-36-18} = \dfrac{\sin(4x-3y)}{-70}$ ...(3)

$\therefore$ Solution is

$z = CF + PI_1 + PI_2$

$$z = f_1(y+x) + f_2(y+2x) + \frac{e^{3x+4y}}{5} - \frac{\sin(4x-3y)}{70}$$

Example 18 (Anna Uni. April 2000, April/May 2004)

Solve $(D^3 - 7DD'^2 - 6D'^3)\,z = \sin(x+2y) + e^{2x+y}$

SOLUTION: R.H.S= 0

A.E: $m^3 - 7m - 6 = 0$

$m = -1, -2, 3$

CF $= f_1(y-x) + f_2(y-2x) + f_3(y+3x)$ ... (1)

R.H.S.

(i) $\sin(ax+by) = \sin(x+2y)$

$a = 1,\ b = 2$

$D^2 = -a^2 = -1,\ DD' = -ab = -2,\ D'^2 = -b^2 = -4$

$$\text{P.I}_1 = \frac{\sin(x+2y)}{D^3 - 7DD'^2 - 6D'^3} = \frac{\sin(x+2y)}{27D + 24D'}$$

$$= \frac{1}{3}\frac{\sin(x+2y)}{(9D+8D')} = \frac{1}{3}\frac{(9D-8D')\sin(x+2y)}{(81D^2 - 64D'^2)}$$

$$= \frac{(9D-8D')(\sin(x+2y))}{3\,(175)}$$

$$= \frac{1}{525}[\,9\cos(x+2y) - 16\cos(x+2y)\,]$$

$$\text{P.I}_1 = \frac{-7}{525}\cos(x+2y) = \frac{-1}{75}\cos(x+2y) \quad \text{... (2)}$$

(ii) $e^{ax+by} = e^{2x+y} \Rightarrow a = 2, b = 1$

$f(D, D') = D^3 - 7DD'^2 - 6D'^3$

$f(a, b) = f(2, 1) = -12$

$$\text{P.I}_2 = \frac{e^{2x+y}}{-12} \quad \text{... (3)}$$

Complete Solution is

$z = CF + PI_1 + PI_2$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) - \frac{e^{2x+y}}{12}$$

Example 19

Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = x + y$

SOLUTION: R.H.S. = 0

$$(D^2 + DD' - 6D'^2)\,z = 0$$

UNIT 1

A.E: $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$C.F = f_1(y-3x) + f_2(y+2x)$

$$P.I = \frac{x+y}{(D^2 + DD' - 6D'^2)} \quad \text{... (1)}$$

$$= \frac{(x+y)}{D^2\left[1 + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right]}$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}\left[x + y - \frac{1}{D}(0+1)\right]$$

$$PI = \frac{1}{D^2}[x + y - x] = \frac{1}{D^2}(y) = \frac{x^2 y}{2}$$

$$PI = \frac{x^2 y}{2} \quad \text{... (2)}$$

Complete solution is:

$$z = f_1(y-3x) + f_2(y+2x) + \frac{x^2 y}{2}$$

$D = \frac{\partial}{\partial x}$

$D' = \frac{\partial}{\partial y}$

$\frac{1}{D} = \int dx$

$\frac{1}{D'} = \int dy$

Example 20

Solve $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = y^2 + x$

☞ **SOLUTION:** R.H.S= 0

$(D^2 + 4DD' - 5D'^2)z = 0$

$m^2 + 4m - 5 = 0$

$m = -5, 1$

$C.F. = f_1(y-5x) + f_2(y+x) \quad \text{... (1)}$

$$P.I = \frac{(x+y^2)}{(D^2 + 4DD' - 5D'^2)} = \frac{1}{D^2\left(1 + 4\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)}(x+y^2)$$

$$= \frac{1}{D^2}\left(1 + \left(\frac{4D'}{D} - 5\frac{D'^2}{D^2}\right)\right)^{-1}(x+y^2)$$

$$= \frac{1}{D^2}\left(1 - \left(\frac{4D'}{D} - 5\frac{D'^2}{D^2}\right) + \frac{16D'^2}{D^2}\right)(y^2 + x)$$

$$PI = \frac{y^2 x^2}{2} + \frac{x^3}{6} - 4\frac{x^3 y}{3} + \frac{7x^4}{4}$$

Complete solution is $z = CF + PI$.

$$z = f_1(y-5x) + f_2(y+x) + \frac{x^2 y^2}{2} + \frac{x^3}{6} - 4\frac{x^3 y}{3} + \frac{7x^4}{4}$$

Example 21

Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$

☞ **SOLUTION:** $m^3 - 2m^2 = 0$

$m = 0, 0, 2$

$C.F = f_1(y) + x f_2(y) + f_3(y+2x) \quad \text{... (1)}$

$$P.I = \frac{1}{D^3 - 2D^2 D'}(2e^{2x}) + \frac{1}{D^3 - 2D^2 D'}(3x^2 y)$$

$$= \frac{2e^{2x}}{2^3 - 2.2^2 . 0} + \frac{3}{D^3}\left(1 - 2\frac{D'}{D}\right)(x^2 y)$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3}\left(x^2 y + 2\frac{1}{D}x^2\right)$$

$$\text{P.I.} = \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60} \quad \dots (2)$$

Solution is

$$z = \text{C.F} + \text{P.I}$$

$$z = f_1(y) + x f_2(y) + f_3(y + 2x) + \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Example 22

Solve $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$

☞ **SOLUTION:** R.H.S= 0

$(D^2 - 6DD' + 9D'^2)z = 0$

A.E: $m^2 - 6m + 9 = 0$

$$m = 3, 3$$

$$\text{C.F} = f_1(y + 3x) + x f_2(y + 3x) \quad \dots (2)$$

$$\text{P.I} = \frac{1}{(D^2 - 6DD' + 9D'^2)}(6x + 2y)$$

$$= \frac{1}{D^2\left(1 - \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)\right)}(6x + 2y)$$

$$= \frac{1}{D^2}\left(1 - \left(6\frac{D'}{D} - 9\frac{D'^2}{D^2}\right)\right)^{-1}(6x + 2y)$$

(As the function is $6x + 2y$, go upto D, D' is enough, higher order in D and D' may be neglected)

$$= \frac{1}{D^2}\left[1 + 6\frac{D'}{D}\right](6x + 2y) \qquad D = \frac{\partial}{\partial x}\ D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2}\left[(6x + 2y) + 6\frac{1}{D}D'(6x + 2y)\right] \qquad \frac{1}{D} = \int dx,\ \frac{1}{D_1} = \int dy$$

$$= \frac{1}{D^2}\left[6x + 2y + 6\frac{1}{D}(2)\right]$$

$$= \frac{1}{D^2}\left[6x + 2y + 12\frac{1}{D}(1)\right]$$

$$= \frac{1}{D^2}[6x + 2y + 12x].$$

$$= \frac{1}{D^2}[18x + 2y] = \frac{1}{D}\left(18\frac{x^2}{2} + 2xy\right)$$

$$= 9\frac{x^3}{3} + 2\frac{x^2}{2}y$$

$$\text{P.I} = 3x^3 + x^2 y = x^2(3x + y) \quad \dots (2)$$

Complete solution is

$$z = CF + P.I$$

$$z = f_1(y + 3x) + x f_2(y + 3x) + x^2(3x + y)$$

General Method for finding P.I.

NOTE

If the above cases fails to find P.I. as well as if the R.H.S function is in different form, we need to use the general method to find the P.I. $\frac{1}{D - mD'}f(x, y) = \int f(x, c - mx)\,dx$, *in which C is to be replaced by* $y + mx$ *after integration.*

Example 23

Solve $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$

☞ **SOLUTION:** Given $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$

A.E: $m^2 - m - 2 = 0$

$$m = -1, 2$$

$$\text{C.F} = f_1(y - x) + f_2(y + 2x) \quad \dots (1)$$

$$\text{P.I} = \frac{(y - 1)e^x}{(D^2 - DD' - 2D'^2)} = \frac{1}{(D - 2D')(D + D')}\left((y - 1)e^x\right)$$

$(D = mD') = D + D'$

$m = -1,\ f(x, y) = (y-1)\, e^x$

replace y by $c - mx \Rightarrow$ put $y = c + x$

W.K.: $f(x, y) = (y-1)\, e^x$

$f(x, c-x) = (c + x - 1)\, e^x$

$$\text{P.I} = \frac{1}{D - 2D'} \int \underset{u}{(c + x - 2)}\ \underset{v}{e^x}\, dx$$

$$\text{P.I} = \frac{1}{(D - 2D')} \left[e^x (c + x - 2) \right] \quad \dots \text{(I)}$$

Substitute $c = y + mx = y - x$ in (I)

$$\text{P.I} = \frac{1}{D - 2D'} \left((y-2)\, e^x \right)$$

(Again use the same formula)

$$\text{P.I} = \int (c - 2x - 2)\, e^x\, dx$$

$\text{P.I} = (c - 2x)\, e^x$ (II) (replace c by $y + 2x$ in (II))

$\text{P.I} = ye^x$ ... (2)

Complete Solution is $z = f_1(y - x) + f_2(y + 2x) + e^x y$

Example 24

Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\, \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y \cos x$

☞ **SOLUTION:** R.H.S = 0, $(D^2 + DD' - 6D'^2)\, z = 0$

A.E: $m^2 + m - 6 = 0$

$m = 2, -3$

$\text{CF} = f_1(y + 2x) + f_2(y - 3x)$ ... (1)

$$\text{P.I} = \frac{1}{(D^2 + DD' - 6D'^2)} (y \cos x)$$

$$= \frac{1}{(D + 3D')(D - 2D')} (y \cos x)$$

$$= \frac{1}{(D + 3D')} \frac{1}{(D - 2D')} (y \cos x)$$

$$= \frac{1}{(D + 3D')} \int (c - 2x) \cos x\, dx \quad \text{(replacing } y \text{ by } c - 2x)$$

$$= \frac{1}{(D + 3D')} \left((c - 2x) \sin x - (2)(-\cos x) \right)$$

$$= \frac{1}{(D + 3D')} \left((c - 2x) \sin x - 2 \cos x \right)$$

$$= \frac{1}{(D + 3D')} (y \sin x - 2 \cos x) \quad \text{(replacing } (c - 2x \text{ by } y)$$

$$= \int \left((c + 3x) \sin x - 2 \cos x \right) dx \quad \text{(replacing } y \text{ by } (c + 3x)$$

$= (c + 3x)(-\cos x) + \sin x$

$\text{P.I} = -y \cos x + \sin x$ (replace $c + 3x$ by y)

$\therefore$ complete solution is

$$z = f_1(y + 2x) + f_2(y - 3x) - y \cos x + \sin x$$

Example 25 **(Anna Uni. April/May 2003)**

Solve $(D^2 - 2DD' + D'^2)\, z = 8e^{x+2y}$

☞ **SOLUTION:** Given $(D^2 - 2DD' + D'^2)\, z = 8e^{x+2y}$ ... (1)

R.H.S = 0

$(D^2 - 2DD' + D'^2)\, z = 0.$

A.E: $m^2 - 2m + 1 = 0.$

$m = 1, 1$

$\text{C.F} = f_1(y + x) + x f_2(y + x)$ ... (2)

$$\text{P.I} = 8 \frac{e^{x+2y}}{D^2 - 2DD' + D'^2} = 8 \cdot \frac{e^{x+2y}}{1 - 4 + 4}$$

$\text{P.I} = 8e^{x+2y}$ ... (3)

$\therefore$ complete solution is

$$z = f_1(y + x) + x f_2(y + x) + 8\, e^{x+2y}$$

Example 26 (Anna Uni. Oct/Nov. 1996)

Solve $(D^2 + DD' - 6D'^2)\, z = \cos(2x + y) + e^{x-y}$

SOLUTION: Given

$$(D^2 + DD' - 6D'^2)\, z = \cos(2x + y) + e^{x-y}$$

$$\text{R.H.S} = 0 \Rightarrow (D^2 + DD' - 6D'^2)\, z = 0.$$

$$\text{A.E} : m^2 + m - 6 = 0$$

$$m = -3, 2$$

$$\text{C.F} = f_1(y - 3x) + f_2(y + 2x) \quad \dots (2)$$

$$\text{P.I}_1 = \frac{\cos 2x + y}{D^2 + DD' - 6D'^2} = \frac{x}{5}\sin(2x + y)$$

$$\text{P.I}_2 = \frac{e^{x-y}}{D^2 + DD' + 6D'^2} = \frac{-e^{x-y}}{6} \quad \dots (3)$$

complete solution is:

$$z = \text{C.F.} + \text{P.I}_1 + \text{P.I}_2$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) - \frac{1}{6}e^{x-y}$$

Example 27 (Anna Uni. March 1996)

Solve $(D^3 - 7DD'^2 - 6D'^3)\, z = \cos(x + 2y) + x$

SOLUTION: Given $(D^3 - 7DD'^2 - 6D'^3)\, z = \cos(x + 2y) + x \quad \dots (1)$

put R.H.S. = O

$$(D^3 - 7DD'^2 - 6D'^3)\, z = 0$$

$$\text{A.E} : m^3 - 7m - 6 = 0$$

$$m = -1, -2, 3.$$

$$\text{C.F} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) \quad \dots (2)$$

$$\text{P.I}_1 = \frac{\cos(x + 2y)}{(D^3 - 7DD'^2 - 6D'^3)} \Rightarrow \text{Replace } D^2 = -1,\ DD' = -2,\ D'^2 = -4$$

$$= \frac{\cos(x + 2y)}{(38D' - D)} = \frac{(38D' + D)(\cos x + 2y)}{1444D'^2 - D^2}$$

$$\text{P.I}_1 = \frac{\sin(x + 2y)}{75} \quad \dots (3)$$

$$\text{P.I}_2 = \frac{1}{(D^3 - 7DD'^2 - 6D'^3)}(x) = \frac{1}{D^3\left[1 - \left(\frac{7D'}{D^2} + \frac{6D'^3}{D^3}\right)\right]^{-1}}(x)$$

$$= \frac{1}{D^3}\left[1 - \left(\frac{7D'}{D^2} + \frac{6D'^3}{D^3}\right)\right]^{-1}(x)$$

$$\text{P.I}_2 = \frac{1}{D^3}[x] = \frac{x^4}{24}$$

$\therefore$ complete solution

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{\sin(x + 2y)}{75} + \frac{x^4}{24}$$

Example 28 (Anna Uni. April/May 2003)

Solve $(D^2 - DD' - 20D'^2)\, z = e^{5x+y} + \sin(4x - y)$

SOLUTION: Given $(D^2 - DD' - 20D'^2)\, z = e^{5x+y} + \sin(4x - y) \quad \dots (1)$

R.H.S = 0:

$$(D^2 - DD' - 20D'^2)\, z = 0$$

$$\text{A.E:} \quad m^2 - m - 20 = 0$$

$$m = 5, -4.$$

$$\text{C.F} = f_1(y + 5x) + f_2(y - 4x) \quad \dots (2)$$

$$\text{P.I}_1 = \frac{e^{5x+y}}{(D^2 - DD' - 20D'^2)} = \frac{xe^{5x+y}}{9} \quad (\text{Replace } D = 5,\ D' = 1)$$

$$\text{P.I}_2 = \frac{\sin(4x - y)}{(D^2 - DD' - 20D'^2)} = \frac{-x\cos(4x - y)}{9}$$

$$(\text{Replace } D^2 = -16,\ DD' = -4,\ D'^2 = -1) \quad \dots (4)$$

UNIT 1

∴ complete solution is:

$y = C.F + P.I_1 + P.I_2$

$$y = f_1(y+5x) + f_2(y-4x) + \frac{xe^{5x+y}}{9} - \frac{x\cos(4x-y)}{9}$$

Example 29 (Anna Uni. April 2003, 2005)

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + \cos(x+y)$

☞ **SOLUTION:** Given:

$$(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + \cos(x+y) \quad \ldots (1)$$

R.H.S: $(D^3 + D^2D' - DD'^2 - D'^3)z = 0.$

A.E: $m^3 + m^2 - m - 1 = 0.$

$m = 1, -1, -1$

$$C.F: = f_1(y+x) + f_2(y-x) + xf_3(y-x) \quad \ldots (2)$$

$$P.I_1 = \frac{e^{2x+y}}{D^3 + D^2D' - DD'^2 - D'^3} = \frac{e^{2x+y}}{9} \quad \ldots (3)$$

$$P.I_2 = \frac{\cos(x+y)}{(D-D')(D^2 + 2DD' + D'^2)}$$

$$= \frac{\cos(x+y)}{(D-D')(-1-2-1)}$$

$$= \frac{-1}{4}\frac{1}{D-D'}R.P\left[e^{i(x+y)}(1)\right]$$

$$= \frac{-1}{4}R.P(e^{ix+iy})\frac{1}{D-D'}(e^{ox+oy})$$

[Put $D^2 = -1$, $DD' = -1$, $D'^2 = -1$]

$$P.I_2 = \frac{-x}{4}\cos(x+y) \quad \ldots (4)$$

∴ solution is

$z = CF + P.I_1 + P.I_2$

$$z = f_1(y+x) + f_2(y-x) + xf_3(y-x) + xf_3(y-x) + \frac{e^{2x+y}}{9} - \frac{x}{4}\cos(x+y)$$

Example 30 (Anna Uni. Nov/Dec 2003)

Solve $(D^2 + 4DD' - 5D'^2)z = 3e^{2x-y} + \sin(x-2y)$

☞ **SOLUTION:** $(D^2 + 4DD' - 5D'^2)z = 3e^{2x-y}$

R.H.S= 0

$(D^2 + 4DD' - 5D'^2)z = 0$

A.E : $m^2 + 4m - 5 = 0$

$m = 1, -5$

$$C.F = f_1(y+x) + f_2(y-5x) \quad \ldots (2)$$

$$P.I_1 = \frac{1}{(D^2 + 4DD' - 5D'^2)}(3e^{2x-y}) = \frac{-e^{2x-y}}{3} \quad \ldots (3)$$

$$P.I_2 = \frac{\sin(x-2y)}{D^2 + 4DD' - 5D'^2}$$

$$P.I_2 = \frac{\sin(x-2y)}{-1+8+20} = \frac{\sin(x-2y)}{27} \quad \ldots (4)$$

[Put $D^2 = -a^2 = -1$, $DD' = -ab = 2$, $D'^2 = -4$]

∴ complete solution is

$z = C.F + P.I_1 + P.I_2$

$$z = f_1(y-5x) + f_2(y+x) - \frac{e^{2x-y}}{3} + \frac{\sin(x-2y)}{27}$$

Example 31 (Anna Uni. April/May 2004)

Solve $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 8\sin(x+3y)$

☞ **SOLUTION:** Given $(D^2 - 3DD' + 2D'^2)z = 8\sin(x+3y)$

R.H.S = 0.

$m^2 - 3m + 2 = 0,\ m = 2, 1$

$C.F = f_1(y+2x) + f_2(y+x)$

$$P.I = 8 \frac{1}{(D^2 - 3DD' + 2D'^2)} \sin(x+3y)$$

Replace : $D^2 = -1$, $DD' = -3$, $DD' = -3$, $D'^2 = -9$

$$P.I = \frac{8 \sin(x+3y)}{-10}$$

$\therefore$ solution is:

$$z = f_1(y+x) + f_2(y+2x) - \frac{4}{5}\sin(x+3y)$$

EXERCISE 1.5

Solve the following P.D.E

1. $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 Z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$
2. $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$
3. $(D^3 - 4D^2D' + 4DD'^2)Z = 6\sin(3x+2y)$
4. $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x+y$
5. $(Dx + Dy)^2 Z = e^{x-y}$
6. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + \cos(x+y)$
7. $(D^2 + D'^2)z = \frac{8}{x^2+y^2}$
8. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(3x+y)$
9. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$
10. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x+y$

11. $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = x^2$
12. $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 3x^2y$
13. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x-2y)$
14. $(D - 2D')(D - D')^2 z = e^{x+y}$
15. $(D^2 - 2DD' + D'^2)z = \sin(x-2y) + e^x(x+2y)$
16. $(2D^2 - 2DD' - D'^2)z = 2e^{3y} + e^{x+y} + y^2$
17. $2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 1$
18. $(D^2 - 3DD' + D'^2)z = \sin x \cos y$
19. $(D^3 - 7DD'^2 - 6D'^3)z = x^2y + \sin(x+2y)$
20. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$

UNIT 1

NOTE

The notation for partial derivatives are

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2}\ s=\frac{\partial^2 z}{\partial x\,\partial y}\quad t=\frac{\partial^2 z}{\partial y^2}$$

Definition: *A partial differential equation is an equation, which involves partial derivatives such as* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$ ... etc. *Which can be simply denoted as (P.D.E). Which can also be denoted as* $F(x, y, t, z, \ldots U_x, U_y, \ldots U_{xt} \ldots) = 0$

Example: $4\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}+z=x^2$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$$

(i) The order of the P.D.E is the order of the highest partial derivative occur in it.

(ii) The degree of the P.D.E is the degree of the highest order derivative occur in it.

Example: $\frac{\partial^3 z}{\partial x^3}+4\frac{\partial^2 z}{\partial x^2}+6\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=x^2+y^2$

order = 3; degree = 1

(iii) The solution of a PDE is a function of independent variables, which satisfies the P.D.E.

(iv) The general solution of the P.D.E contains arbitrary constants, or arbitrary functions or both.

(v) If the number of constants to be eliminated is equal to the number of independent variables, it will produce P.D.E of first order.

(vi) If the number of constants more than number of independent variables, if will produce P.D.E of higher order.

(vii) The order of the P.D.E is equal to number of arbitrary functions to be eliminated

INTRODUCTION

(i) Consider a Single variable function $y = f(x)$, for example: $y = 3x^2 + 2x + 11$, $y = \sin x$, $y = \cos x + x^2$, $y = \log x \ldots etc.$ Then $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ... are said to be differential coefficients or differentials. An equation which involves the above differentials are said to be ordinary differential equations (O.D.E).

Example: $\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + y = \sin x$

$$(x^2 D^2 + 4xD + 3)\, y = e^x \ldots etc.$$

(ii) Consider a multiple variable or several variable function ((function having two or more independent variable), $z = f(x, x, \ldots x)$, and $u = f(x, y, z \ldots)$) which are common occurrence in so many engineering application problems; particularly in Theory of Vibration, Heat transfer, Fluid mechanics, Thermodynamics....etc.

Example: $u = \sin(2x + 4y - 5z)$

$$z = e^{2x - 3y} \ldots \text{etc}$$

Then $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\, \partial y}$, $\frac{\partial^2 z}{\partial y^2}$

are said to be partial derivatives (or) partial differential coefficients.

1.1.1 FORMATION OF P.D.E BY ELIMINATING ARBITRARY CONSTANTS

SOLVED EXAMPLES

UNIT 1

Example 1

Form the P.D.E from $z = ax + by + \sqrt{a^2 + b^2}$

☞ **SOLUTION:** Given $z = ax + by + \sqrt{a^2 + b^2}$ ... (1)

differentiate w.r.t x: $\frac{\partial z}{\partial x} = p = a + 0 + 0$

$$p = a$$

differentiate w.r.t y: $\frac{\partial z}{\partial y}$ ... (3)

$$q = b$$

substitute a and b in (1)

$$z = px + qy + \sqrt{p^2 + q^2}$$

This is the P.D.E of first order

Example 2 **(Anna Uni. Nov/Dec 2004)**

Form the PD from $z = (x^2 + a^2)(y^2 + b^2)$

☞ **SOLUTION:** Given $z = (x^2 + a^2)(y^2 + b^2)$ ... (1)

differentiate w.r.tx : $p = (2x)(y^2 + b^2)$ ... (2)

differentiate w.r.ty : $q = (2y)(x^2 + a^2)$ ... (3)

substitute (2) and (3) in (1)

$$z = \frac{p}{2x} \cdot \frac{q}{2y} \Rightarrow 4xyz = pq$$

This is the P.D.E. of first order.

Example 3 **(Anna Uni. Nov/Dec. 2003)**

From the P.D.E from $(x - a)^2 + (y - b)^2 + z^2 = 1$

☞ **SOLUTION:** Given $(x - a)^2 + (y - b)^2 + z^2 = 1$ ... (1)

UNIT-4 (Differential Equations)

Def. A equation which involves differential co-efficient is called a differential equation.

Differential equations

Ordinary Differential Equations (ODEs)	Partial Differential Equations (PDEs)
A differential equation involving derivatives with respect to a single independent variable is called an ODE.	A differential equation involving partial derivatives with respect to more than one independent variable is called a PDE.
eg: 1) $\frac{dy}{dx} = 2\sin x$, 2) $\frac{d^2y}{dx^2} + m^2 y = 0$	eg: 1) $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2}$ 2) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$.

Def. ★ The order of a diff eqn is the order of the highest differential coefficient present in the eqn.

★ The degree of a diff eqn is the degree of the highest derivative after removing the radicals and fractions.

eg: 1) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

here the order is 2 and degree is 2.

2) $\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} + \frac{y}{a} = \sin x$.

here the order is 2 and degree is 1.

Solution of a differential equation.

★ Any relation between dependent and independent variables which, when substituted in the diff eqn, reduce it to an identity [(or) satisfies] is called a solution (primitive) of the differential eqn. [free from derivative]

★ The solution, in which the number of arbitrary constants occurring is the same as the order of the eqn is called the general solution or complete primitive.

★ Any Solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a Particular Integral.

★ General Solution = Complementary fn + Particular Integral
(ie) ($y = CF + PI$)

Linear differential Equations of Second and higher order with constant coefficients

★ The general form of linear eqn of 2nd order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{—— (1)}$$

where ~~P, Q, R~~ P, Q are constants and R is a fn of x or a constant.

Let D be the differential operator.

Then $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$.

$\therefore$ ① $\Rightarrow$ $D^2y + PDy + Qy = R$ (or) $(D^2+PD+Q)y = R$

* The general form of a linear differential eqn of the n^{th} order with constant coefficients is

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = X \quad \text{—②}$$

where $a_0(\neq 0), a_1, a_2, \dots, a_n$ are constants and X is a function of x.

And by using the differential operator ② becomes,

$$(a_0D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n)y = X$$

(or) $f(D)\,y = X$, where $f(D)$ is a polynomial in D.

* Homogenous eqn if $f(D)y = 0$. (Sol: $y = CF$).
* Non-homogenous eqn if $f(D)y = X$ where $X \neq 0$ (Sol: $y = CF + PI$)

To find Complementary function.

Let $f(D)\,y = 0$ —③.

First find the auxiliary eqn (A.E). (i.e) $f(m) = 0$ in ③ and later the roots are found from this eqn

Case (i): [The roots of A.E are real & distinct.]

C.F of ③ $= c_1e^{m_1x} + c_2e^{m_2x} + \dots + c_ne^{m_nx}$.

where $m_1, m_2, \dots m_n$ are the roots of the A.E & $c_1, c_2, \dots c_n$ are ~~the~~ arbitrary constants.

Case (ii): [The A.E has got real roots, some of which are equal.]

(i.e) If $m_1 = m_2 = m_3$, then

C.F of ③ $= y = (c_1x^2 + c_2x + c_3)e^{m_1x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}$.

Case (iii): [Two roots of the A.E are complex]

let $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$.

Then $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \dots + c_ne^{m_nx}$

Case (iv): [If $m_1 = m_3 = \alpha + i\beta$ & $m_2 = m_4 = \alpha - i\beta$]

$\therefore y = e^{\alpha x}[(c_1x + c_2)\cos\beta x + (c_3x + c_4)\sin\beta x] + c_5e^{m_5x} + \dots + c_ne^{m_nx}$.

To find Particular Integral (P.I)

Let $f(D)\,y = X$ ——— (4)

$P.I = \frac{1}{f(D)} X$ [(ie) P.I depends on X]

Rule 1: when $X = e^{ax}$, where a is a constant.

* $P.I = \frac{1}{f(D)} e^{ax}$, Replace D by a if $f(a) \neq 0$.
* if $f(a) = 0$ (and) $(D-a)^r$ is a factor of $f(D) = (D-a)^r \phi(D)$ where $\phi(a) \neq 0$.

Then $P.I = \frac{1}{\phi(a)} \cdot \frac{x^r}{r!} e^{ax}$.

Rule 2: when $X = \sin ax$ (or) $\cos ax$, where a is constant,

* Replace D^2 by $-a^2$ in P.I provided $\phi(-a^2) \neq 0$.
* if $\phi(-a^2) = 0$ then.

[$e^{iax} = \cos ax + i\sin ax$]

$\frac{1}{D^2+a^2} \sin ax = \frac{1}{D^2+a^2} \times$ Imaginary part of e^{iax}

$= \ldots\ldots = -\frac{x\cos ax}{2a}$.

Rule 3: when $X = x^m$, $m > 0$.

$P.I = \frac{1}{f(D)} x^m = \{f(D)\}^{-1} x^m$.

And use binomial expression for $\{f(D)\}^{-1}$.

note:

1) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
2) $(1+x)^{-1} = 1 - x + x^2 - \ldots$
3) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$
4) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
5) $(a+x)^n = a^n + n a^{n-1} x^1 + \frac{n(n-1)}{2!} a^{n-2} x^2 + \ldots + x^n$.

Rule 4: If $X = e^{ax} V$, where V is any fn of x.

$P.I = \frac{1}{f(D)} \cdot e^{ax} V = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$

(evaluated by rule (3) or (4))

Rule 5: If $X = x \cdot V(x)$

$P.I = x \frac{1}{f(D)} V(x) - \frac{f'(D)}{\{f(D)\}^2} V(x)$.

Rule 6: If X is any other fn of x.

$P.I = \frac{1}{f(D)} X = \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)} X$

Then use partial fractions method and solve them by using the above rules.

Problems.

1) If $y = x^2$ find $\frac{d^2y}{dx^2}$.

Sol. $y = x^2$,

$\frac{dy}{dx} = 2x$.

$\frac{d^2y}{dx^2} = 2$.

2) Find the Solution of $(D^2 - 8D + 16)y = 0$.

Sol. The auxiliary equation is $m^2 - 8m + 16 = 0$.

$\Rightarrow m^2 - 4m - 4m + 16 = 0$.

$\Rightarrow m(m-4) - 4(m-4) = 0$.

$\Rightarrow (m-4)^2 = 0$.

$\Rightarrow m = 4, 4$.

$\therefore y = (A + Bx)e^{4x}$ //.

3). Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$ $\Rightarrow (D^2y - 5Dy + 4y) = 0$ $\Rightarrow (D^2 - 5D + 4)y = 0$.

Sol.: The A.E is $m^2 - 5m + 4 = 0$.

$\Rightarrow (m-4)(m-1) = 0$

$\Rightarrow m = 4, m = 1$.

$\therefore y = C.F = Ae^{x} + Be^{4x}$.

4) Solve $(D^2 - 1)y = 0$.

Sol A.E is $m^2 - 1 = 0$. $\Rightarrow m^2 = 1$.

$\Rightarrow m = \pm 1$

$\therefore y = Ae^{-x} + Be^{x}$.

5) Solve $(D^2 + D + 1)y = 0$.

Sol. A.E is $m^2 + m + 1 = 0$.

$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{3}\,i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore y = e^{-\frac{x}{2}}\left[A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x\right]$

H.W 6) $(D^2 - 4D + 4)y = 0$.

7) $(D^2 + 6D + 8)y = 0$.

8) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$. $\rightarrow m = 5, 2$.

9) $\frac{d^4y}{dx^4} - y = 0$. $\Rightarrow (D^4 - 1)y = 0$.

Sol. A.E is $m^4 - 1^4 = 0$. ~~$\Rightarrow m^4 - 1$~~

$\Rightarrow (m^2)^2 - (1^2)^2 = 0$. $a^2 - b^2 = (a+b)(a-b)$

$\Rightarrow (m^2 + 1^2)(m^2 - 1^2) = 0$.

$\Rightarrow (m^2 + 1)(m+1)(m-1) = 0$. $m^2 = -1$, $m = \pm\sqrt{-1} = \pm i$

$\Rightarrow m = 1, -1, \pm i$

$\therefore y = Ae^{x} + Be^{-x} + e^{0x}(C \cos x + D \sin x)$

Solve $(D^2+1)y = e^{2x} \Rightarrow y = \frac{1}{(D^2+1)} \cdot e$

A.E is $m^2+1=0$.

$m^2 = -1 \Rightarrow m = \pm i$.

$y = C.F = A\cos x + B\sin x$.

$P.I = \frac{1}{(D^2+1)} \cdot e^{2x}$ $a=2$

$= \frac{1}{(4+1)} e^{2x} = \frac{e^{2x}}{5}$.

$\therefore$ The general sol, $y = C.F + P.I$

$= A\cos x + B\sin x + \frac{e^{2x}}{5}$.

11) Solve $(D^2+4D+13)y = 2e^{-x}$.

Sol A.E is $m^2+4m+13=0$.

$m = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$

$= \frac{-4 \pm 6i}{2} = -2 \pm 3i$. $[\alpha = -2 \;\&\; \beta = 3]$

$C.F = e^{-2x}[A\cos 3x + B\sin 3x]$.

$P.I = \frac{1}{(D^2+4D+13)} \cdot 2e^{-x}$

$= \frac{2}{(-1)^2+4(-1)+13} \cdot e^{-x}$

$= \frac{2}{10} e^{-x} = \frac{e^{-x}}{5}$.

$\therefore y = C.F + P.I = e^{-2x}[A\cos 3x + B\sin 3x] + \frac{e^{-x}}{5}$

12) Solve $(3D^2+D+14)y = 13e^{2x}$.

13) $(D^2-D-2)y = e^{2x} + e^x$.

Sol. A.E is $m^2-m-2=0$.

$\Rightarrow m^2-2m+m-2=0$

$\Rightarrow (m-2)(m+1)=0$.

$\Rightarrow m = 2, -1$.

$C.F = Ae^{-x} + Be^{2x}$.

$P.I = \frac{1}{(D^2-D-2)} \cdot (e^{2x}+e^x)$

$= \frac{1}{D^2-D-2} \cdot e^{2x} + \frac{1}{(D^2-D-2)} \cdot e^x$.

$= \frac{1}{(D-2)(D+1)} \cdot e^{2x} + \frac{1}{(1-1-2)} \cdot e^x$

$= \frac{1}{3} \cdot \frac{1}{(D-2)} \cdot e^{2x} + \frac{e^x}{(-2)}$

$= \frac{1}{3} x e^{2x} - \frac{e^x}{2}$.

$\therefore y = Ae^{-x} + Be^{2x} + \frac{1}{3}xe^{2x} - \frac{e^x}{2}$ //.

12) $(3D^2+D-14)y = 13e^{2x}$

Sol. $y = Ae^{2x} + Be^{-7x/3} + xe^{2x}$

14) $(D^2+4)y = \sin 3x$.

Sol A.E is $m^2+4=0$. $\Rightarrow m^2=-4 \Rightarrow m=\pm 2i$

$C.F = A\cos 2x + B\sin 2x$.

$\sin ax, \cos ax$

$P.I = \frac{1}{(D^2+4)} \cdot \sin 3x$

here $a=3$. replace D^2 by $-a^2$ (i.e. $D^2=-9$)

$= \frac{1}{(-9+4)} \cdot \sin 3x$

$= -\frac{\sin 3x}{5}$.

$\therefore y = A\cos 2x + B\sin 2x - \frac{\sin 3x}{5}$ //.

(x) 15) $(D^2+4D+4)y = 4\sin 2x$.

Sol. A.E is $m^2+4m+4=0 \Rightarrow (m+2)^2=0$

$\Rightarrow m=-2,-2$.

$\therefore C.F = (A+Bx)e^{-2x}$.

$P.I = \frac{1}{(D^2+4D+4)} \cdot 4\sin 2x$

$D^2=-4$.

$= 4 \cdot \frac{1}{(-4+4D+4)} \cdot \sin 2x$

$\frac{1}{D} = \int$.

$= \frac{4}{4}\int \sin 2x\, dx$

$= -\frac{\cos 2x}{2}$

$\therefore y = (A+Bx)e^{-2x} - \frac{\cos 2x}{2}$.

16) Solve $(D^2+16)y = e^{-3x} + \cos 4x$.

Sol. A.E is $m^2+16=0 \Rightarrow m^2=-16 \Rightarrow m=\pm 4i$.

$C.F = A\cos 4x + B\sin 4x$.

$P.I = \frac{1}{(D^2+16)}(e^{-3x}+\cos 4x)$

$= \frac{1}{(D^2+16)}e^{-3x} + \frac{1}{(D^2+16)}\cos 4x$.

here $D^2=-16$. $e^{iax} = \cos ax + i\sin bx$

$= \frac{e^{-3x}}{25} + \frac{1}{(D^2+16)}$ Realpart of e^{i4x}

$= \frac{e^{-3x}}{25} +$ Real part of $\frac{1}{(D+4i)(D-4i)} \cdot e^{i4x}$

$(D+4i)(D-4i) = D^2-4iD+4iD+16$, $i^2=-1$

$= \frac{e^{-3x}}{25} +$ Realpart of $\frac{1}{8i} \times \frac{i}{i} \cdot xe^{i4x}$

$= \frac{e^{-3x}}{25} +$ Realpart of $\left(-\frac{i}{8}\right)x[\cos 4x + i\sin 4x]$

$= \frac{e^{-3x}}{25} +$ Real part of $\left[-\frac{ix}{8}\cos 4x + \frac{x}{8}\sin 4x\right]$.

$= \frac{e^{-3x}}{25} + \frac{x}{8}\sin 4x$.

$\therefore y = A\cos 4x + B\sin 4x + \frac{e^{-3x}}{25} + \frac{x}{8}\sin 4x$ //.

[x form]

17) Solve $(D^2-1)y = x$.

Sol A.E is $m^2-1=0 \Rightarrow m=\pm 1$.

$C.F = Ae^{-x} + Be^{x}$.

$P.I = \frac{1}{(D^2-1)} x$.

$= \frac{1}{(-1)(1-D^2)} x$.

$= -(1-D^2)^{-1} x$

$= -[1+D^2+D^4+\cdots]x$

$= -[x + D^2x + D^4x + \cdots]$

$= -x$

$(1-x)^{-1} = 1+x+x^2+\cdots$

x

$Dx = 1.$

$D^2x = 0.$

$\vdots$

$\therefore y = Ae^{-x} + Be^{x} - x$ //.

18) Solve. $(D^3+8)y = x^4+2x+1$.

Sol. A.E is $m^3+8=0$.

	m^3	m^2	m	c
$m=-2$	1	0	0	8
	0	-2	4	-8
	1	-2	4	0

$\therefore (m+2)(m^2-2m+4)=0$.

$\Rightarrow m=-2$ & $m^2-2m+4=0$.

(ie) $m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2\pm\sqrt{-12}}{2}$

$= 1 \pm \sqrt{3}\, i$.

$\therefore m = -2, 1 \pm \sqrt{3} i$.

$C.F = Ae^{-2x} + e^{x}[B\cos\sqrt{3}x + C\sin\sqrt{3}x]$.

$P.I = \frac{1}{(D^3+8)}(x^4+2x+1)$.

$= \frac{1}{8(1+\frac{D^3}{8})}(x^4+2x+1)$

$= \frac{1}{8}(1+\frac{D^3}{8})^{-1}(x^4+2x+1)$.

$= \frac{1}{8}[1 - \frac{D^3}{8} + \frac{D^6}{8^2} - \cdots](x^4+2x+1)$

$= \frac{1}{8}[(x^4+2x+1) - \frac{D^3}{8}(x^4+2x+1) + \cdots]$

$= \frac{1}{8}[x^4+2x+1 - \frac{24x}{8}]$

$= \frac{1}{8}[x^4 - x + 1]$

$(1+x)^{-1} = 1-x+x^2-x^3\cdots$

x^4+2x+1

$D(x^4+2x+1) = 4x^3+2$

$D^2(x^4+2x+1) = 12x^2$

$D^3(x^4+2x+1) = 24x$.

$\therefore y = C.F + P.I$

$= Ae^{-2x} + e^{x}[B\cos\sqrt{3}x + C\sin\sqrt{3}x] + \frac{1}{8}[x^4-x+1]$.

19) Solve $[D^4 - 2D^3 + D^2]y = x^3$.

Sol. A.E is $m^4-2m^3+m^2=0$.

$\Rightarrow m^2(m^2-2m+1)=0$.

$\Rightarrow m^2(m-1)^2=0$.

$\Rightarrow m = 0, 0, 1, 1.$

C.F $= (A+Bx)e^{0x} + (C+Dx)e^{x}$

$= A + Bx + (C+Dx)e^{x}.$

P.I $= \dfrac{1}{(D^4 - 2D^3 + D^2)} \cdot x^3$

$= \dfrac{1}{D^2[1 + (D^2 - 2D)]} x^3$

$= \dfrac{1}{D^2} \cdot [1 + (D^2 - 2D)]^{-1} x^3$

x^3
$D(x^3) = 3x^2$
$D^2(x^3) = 3D(x^2) = 6x$
$D^3(x^3) = 6.$
$D^4(x^3) = 0.$

$= \dfrac{1}{D^2}[1 - (D^2 - 2D) + (D^2 - 2D)^2 - (D^2-2D)^3 \ldots] x^3$

$= \dfrac{1}{D^2}[1 - D^2 + 2D + (D^4 - 4D^3 + 4D^2) - (D^6 - 6D^5 + 6D^4 - 8D^3) + \ldots] x^3$

$= \dfrac{1}{D^2}[x^3 - D^2(x^3) + 2D(x^3) + D^4(x^3) - 4D^3(x^3) + 4D^2(x^3) - D^6(x^3) + 6D^5(x^3) - 6D^4(x^3) + 8D^3(x^3) + \ldots]$

$= \dfrac{1}{D^2}[x^3 - 6x + 6x^2 + 0 - 24 + 24x - 0 + 0 - 0 + 48]$

$= \dfrac{1}{D^2}[x^3 + 18x + 6x^2 + 24].$

$= \dfrac{1}{D}\int (x^3 + 18x + 6x^2 + 24]\,dx$

$= \dfrac{1}{D}\left[\dfrac{x^4}{4} + \dfrac{18x^2}{2} + \dfrac{6x^3}{3} + 24x\right].$

$= \int\left[\dfrac{x^4}{4} + 9x^2 + 2x^3 + 24x\right]dx.$

$= \dfrac{x^5}{20} + \dfrac{9x^3}{3} + \dfrac{2x^4}{4} + \dfrac{24x^2}{2}$

$= \dfrac{x^5}{20} + \dfrac{x^4}{2} + 3x^3 + 12x^2.$

$\therefore y = A + Bx + (C+Dx)e^{x} + \dfrac{x^5}{20} + \dfrac{x^4}{2} + 3x^3 + 12x^2.$

HW

20) $(D^2 - 2D + 1)y = (e^{x} + 1)^2.$

21) $(D^2 - 2D + 1)y = 2e^{-x} + x^2 + 3.$

[$e^{ax} \times$ form]. $PI = e^{ax} \dfrac{1}{f(D+a)} \cdot X$

22). Find the P.I of $(D^2 + 1)y = xe^{x}.$

Sol P.I $= \dfrac{1}{(D^2+1)} \cdot xe^{x}.$

$= e^{x} \cdot \dfrac{1}{((D+1)^2 + 1)} \cdot x$

x
$D(x) = 1$
$D^2(x) = 0$

$= e^{x} \cdot \dfrac{1}{[1 + D^2 + 2D + 1]} x = e^{x} \dfrac{1}{[2 + D^2 + 2D]} x$

$= \dfrac{e^{x}}{2} \cdot \dfrac{1}{\left[1 + \left(\dfrac{D^2 + 2D}{2}\right)\right]} x.$

$$= \frac{e^x}{2}\left[1+\left(\frac{D^2+2D}{2}\right)\right]^{-1} x$$

$$= \frac{e^x}{2}\left[1-\left(\frac{D^2+2D}{2}\right)+\left(\frac{D^2+2D}{2}\right)^2 - \cdots\right] x$$

$$= \frac{e^x}{2}\left[x - \frac{D^2(x)}{2} - D(x)\right]$$

$$= \frac{e^x}{2}\left[x - 1\right] //$$

23) Solve $(D+2)^2 y = e^{-2x}\sin x$.

Sol. A.E is $(m+2)^2 = 0 \Rightarrow m = -2, -2$.

$\therefore C.F = (A+Bx)e^{-2x}$.

$$P.I = \frac{1}{(D+2)^2} \cdot e^{-2x}\sin x$$

$$= e^{-2x}\frac{1}{((D-2)+2)^2}\sin x$$

$$= e^{-2x} \cdot \frac{1}{D^2}\sin x \qquad \left| \begin{array}{l} a=1 \\ D^2 = -a^2 \\ \quad = -1. \end{array}\right.$$

$$= -e^{-2x}\sin x.$$

$$y = (A+Bx)e^{-2x} - e^{-2x}\sin x //$$

Applications.

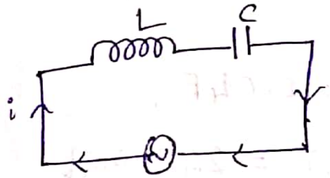
→ LC circuit. $i = \frac{dq}{dt}$ (or $i = q'$) (or) $q = \int i\, dt$

Elements in an RLC circuit.

Name	Symbol	notation	unit	Voltage Drop.
ohm's resistor	-WW-	R ohm's resistance.	ohms (Ω)	RI → Ri
Inductor	-ooo-	L Inductance	henrys (H)	$L\frac{dI}{dt}$
Capacitor	-||-	C capacitance	farads (F)	Q/C (or q/C)

Q (or) q → quantity of electricity (coulombs). (or capacitor charge).

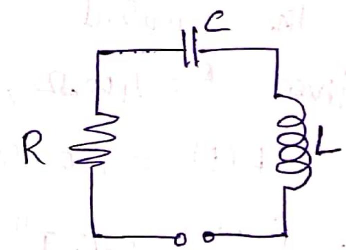
LC-circuit $V_R + V_L + V_C = V_S$.



$$L\frac{d^2I}{dt^2} + \frac{I}{c} = E'$$

$$L\frac{d^2q}{dt^2} + \frac{q}{c} = E_0\sin\omega t \;(= E)$$

RLC-circuit

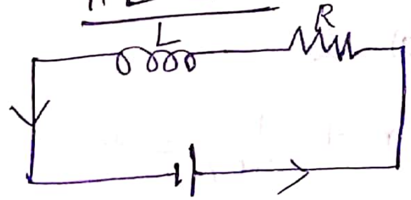


$E(t) = E_0\sin\omega t$.

$$LI'' + RI' + \frac{I}{c} = E'(t)$$

(or) $LQ'' + R.Q' + \frac{Q}{C} = E$ where $I = Q'$.

RL-circuit.



$$Ri + L\frac{di}{dt} = E \quad \text{(Kirchhoff's voltage law)}$$

$$\Rightarrow \frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

CR-circuit

By voltage law,

$$Ri + \frac{q}{c} = E$$

$$\Rightarrow R\frac{dq}{dt} + \frac{q}{C} = E.$$

at $Q(0) = 0$

(3) ⇒ $0 = -\frac{A}{1500.626} - \frac{B}{1832.707} - 0 - \frac{0.0092}{2}$.

⇒ $0.0007A + 0.0005B + 0.0046 = 0$.

⇒ $0.0007A + 0.0005B = -0.0046$ — (5).

(4) ⇒ $A + B = 0.2997$

(4) × 0.0007 ⇒ $0.0007A + 0.0007B = 0.0002$ — (6).

(5) − (6).

⇒ $0.0007A + 0.0005B = -0.0046$.

(−) $0.0007A$ (−) $+ 0.0007B$ (−) $= 0.0002$

$-0.0002B = -0.0048$

⇒ $B = \frac{0.0048}{0.0002} = 24$.

∴ (4) ⇒ $A = 0.2997 - 24 = -23.7$

∴ (3) ⇒ $I(t) = -23.7e^{-1500.626t} + 24e^{-1832.707t} - 0.2997\cos 2t + 0.0092\sin 2t$.

) Find the current in the RLC circuit, assuming zero initial current and capacitor charge with the following data $R = 450\Omega$, $L = 0.95H$, $C = 0.07F$, $E(t) = e^{-t}\sin^2 3t$ V.

Sol. Given: $R = 450\Omega$, $L = 0.95H$, $C = 0.07F$,

$E(t) = e^{-t}\sin^2 3t$ V. $= e^{-t}\left(\frac{1-\cos 6t}{2}\right)$ $= \frac{1}{2}(e^{-t} - e^{-t}\cos 6t)$

$1 - \cos 2t = 2\sin^2 t$

⇒ $\sin^2 t = \frac{1-\cos 2t}{2}$

⇒ $\sin^2 3t = \frac{1-\cos 6t}{2}$

$E'(t) = \frac{-e^{-t}}{2} - \frac{1}{2}\left[6e^{-t}\sin 6t + (-1)e^{-t}\cos 6t\right]$

$= \frac{-e^{-t}}{2} + 3e^{-t}\sin 6t + \frac{1}{2}e^{-t}\cos 6t$.

The differential eqn for the given problem is

$0.95I'' + 450I' + \frac{I}{0.07} = \frac{-e^{-t}}{2} + 3e^{-t}\sin 6t + \frac{1}{2}e^{-t}\cos 6t$

with $I(0) = 0$ & $Q(0) = 0$ as the initial conditions. — (1)

A.E of (1) is $\left(0.95m^2 + 450m + \frac{1}{0.07}\right) = 0$.

$m = \frac{-450 \pm \sqrt{202500 - 54.2857}}{1.9}$

$= -0.03175, -473.652$

C.F $= Ae^{-0.03175t} + Be^{-473.652t}$.

P.I $= \frac{1}{\left(0.95D^2 + 450D + \frac{1}{0.07}\right)}\left(\frac{-e^{-t}}{2}\right) + \frac{3}{\left(0.95D^2 + 450D + \frac{1}{0.07}\right)}e^{-t}\sin 6t + \frac{1}{2\left(0.95D^2 + 450D + \frac{1}{0.07}\right)}e^{-t}\cos 6t$

$$P.I = 0.00115e^{-t} - 0.000189e^{-t}\sin 6t - 0.00108e^{-t}\cos 6t + 0.00018050e^{-t}\sin 6t - 0.0000315e^{-t}\cos 6t$$

$$= 0.00115e^{-t} - 0.0000085e^{-t}\sin 6t - 0.001111e^{-t}\cos 6t$$

$$y = C.F + P.I$$

Find the current in the RLC circuit, assuming zero initial current and capacitor charge with the following data, $R = 400\Omega$, $L = 0.12H$, $C = 0.04F$, $E(t) = 120 \sin 2t$ V.

Sol. The differential equation for an RLC circuit is $LI'' + RI' + \frac{I}{C} = E'(t)$

Given $R = 400\Omega$, $L = 0.12H$, $C = 0.04F$,

$E(t) = 120 \sin 2t$ V,

$$E'(t) = 120 \cos 2t \times 2 = 240 \cos 2t$$

∴ The diff eqn becomes

$$0.12 I'' + 400 I' + \frac{I}{0.04} = 240 \cos 2t \quad \text{—(1)}$$

with $I(0) = 0$ & $Q(0) = 0$ as initial conditions.

$$(1) \Rightarrow \quad 0.12 I'' + 400 I' + 25I = 240 \cos 2t.$$

$$\Rightarrow (0.12D^2 + 400D + 25) I = 240 \cos 2t$$

A.E is $0.12m^2 + 400m + 25 = 0$.

$$\Rightarrow m = \frac{-400 \pm \sqrt{400^2 - 4(0.12)25}}{2(0.12)}$$

$$m = \frac{-400 \pm \sqrt{160000 - 12}}{0.24}$$

$$= \frac{-400 \pm 399.985}{0.24}$$

$$= -0.0625 \quad , \quad -3333.271$$

$$C.F = A e^{-0.0625t} + B e^{-3333.271t}$$

$$P.I = \frac{240}{(0.12D^2 + 400D + 25)} \cos 2t$$

here. $a = 2$. Replace $D^2 = -a^2 = -4$.

$$= \frac{240}{(0.12)(-4) + 400D + 25} \cos 2t$$

$$= \frac{240}{400D + 24.52} \cos 2t.$$

$$= \frac{240 \, (400D - 24.52)}{400(400) D^2 - (24.52)^2} \cos 2t$$

$$= \frac{96000 D (\cos 2t) - 5884.8 \cos 2t}{160000(-4) - 601.2304}$$

$$= \frac{-96000 \sin 2t \times 2 - 5884.8 \cos 2t}{-640601.2304}$$

$$P.I. = 0.299718 \sin 2t + 0.009186 \cos 2t$$

$$\therefore I(t) = C.F + P.I$$

$$= A e^{-0.0625t} + B e^{-3333.271t} + 0.299718 \sin 2t + 0.009186 \cos 2t \quad \text{—— (2)}$$

Gn that $I(0) = 0$.

$\therefore (2) \Rightarrow \quad 0 = A + B + 0.009186 \quad$ ($\because \sin 0 = 0$, $\cos 0 = 1$ & $e^0 = 1$)

$$\Rightarrow A + B = -0.009186 \quad \text{—— (3)}$$

W.K.T, $Q(t) = \int I(t)\, dt$.

$$\therefore Q(t) = \int \left[A e^{-0.0625t} + B e^{-3333.271t} + 0.299718 \sin 2t + 0.009186 \cos 2t \right] dt \quad \text{—— (4)}$$

~~at Q(0) = 0, (4) ⇒~~

~~0 = A~~

~~$Q(t) = \frac{A e^{-0.0625t}}{-0.0625} - \frac{B e^{3333.271t}}{3333.271}$~~

~~$+ 0.599436 \cos 2t$~~

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$$Q(t) = \frac{A e^{-0.0625t}}{-0.0625} - \frac{B e^{-3333.271t}}{3333.271}$$

$$+\frac{0.299718}{2} \cos 2t + \frac{0.009186}{2} \sin 2t$$

at $Q(0) = 0$, the above eqn becomes

$$0 = -\frac{A}{0.0625} - \frac{B}{3333.271} - \frac{0.299718}{2}$$

$$\Rightarrow 16A + 0.0003B = -0.149859 \quad \text{--- (4)}$$

(3) × 16 − (4)

$$\Rightarrow 16A + 16B = -0.146976$$

$$(-)\ 16A \ (-)+0.0003B = (+)-0.149859$$

$$15.9997B = 0.002883$$

$$\therefore B = 0.00018$$

$\therefore$ (3) $\Rightarrow$ $A = -0.009186 - 0.00018$

$$= -0.0094$$

$\therefore$ (2) $\Rightarrow$ $I(t) = -0.0094 e^{-0.0625t} + 0.00018 e^{-3333.2t}$

$$+0.299718 \sin 2t + 0.009186 \cos 2t$$

Partial Differentiation and Its Applications

1. Functions of two or more variables. 2. Partial derivatives. 3. Which variable is to be treated as constant. 4. Homogeneous functions—Euler's theorem. 5. Total derivative—Diff. of implicit functions. 6. Geometrical interpretation—Tangent plane and normal to a surface. 7. Change of variables. 8. Jacobians. 9. Taylor's theorem for functions of two variables. 10. Errors and approximations. 11. Total differential. 12. Maxima and minima of functions of two variables. 13. Lagrange's method of undetermined multipliers. 14. Differentiation under the integral sign—Leibnitz Rule.

5.1. (1) FUNCTIONS OF TWO OR MORE VARIABLES

We often come across quantities which depend on two or more variables. For example, the area of a rectangle of length x and breadth y is given by $A = xy$. For a given pair of values of x and y, A has a definite value. Similarly, the volume of a parallelopiped $(= xyh)$ depends on the three variables x(= length), y(= breadth) and h(= height).

Def. *A symbol z which has a definite value for every pair of values of x and y is called a function of two independent variables x and y* and we write $z = f(x, y)$ *or* $\phi(x, y)$.

We may interpret (x, y) as the coordinates of a point in the XY-plane and z as the height of the surface $z = f(x, y)$. We have come across several examples of such surfaces in chapter 4.

The set R of points (x, y) such that any two points P_1 and P_2 of R can be so joined that any arc P_1P_2 wholly lies in R, is called as *region* in the XY-plane. A region is said to be a *closed region* if it includes all the points of its boundary, otherwise it is called an *open region.*

A set of points lying within a circle having centre at (a, b) and radius $\delta > 0$, is said to be *neighbourhood* of (a, b) in the circular region $R : (x - a)^2 + (y - b)^2 < \delta^2$.

When z is a function of three or more variables $x, y, t, \ldots$, we represent the relation by writing $z = f(x, y, t, \ldots)$. For such functions, no geometrical representation is possible. However, the concepts of a region and neighbourhood can easily be extended to functions of three or more variables.

(2) **Limits.** *The function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$* and we write

$$\underset{\substack{x \to a \\ y \to b}}{Lt}\ f(x, y) = l$$

In terms of a circular neighbourhood, we have the following *definition of the limit :*

The function $f(x, y)$ defined in a region R, is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number ε, there exists another positive number δ such that $|f(x, y) - l| < \varepsilon$ for $0 < (x - a)^2 + (y - b)^2 < \delta^2$ for every point (x, y) in R.

(3) **Continuity.** *A function $f(x, y)$ is said to be continuous at the point (a, b) if*

$$\underset{\substack{x \to a \\ y \to b}}{Lt}\ f(x, y) \text{ exists and } = f(a, b)$$

If a function is continuous at all points of a region, then it is said to be *continuous in that region*. A function which is not continuous at a point is said to be *discontinuous* at that point.

Obs. *Usually, the limit is the same irrespective of the path along which the point (x, y) approaches (a, b) and*

$$\underset{x \to a}{\text{Lt}} \left\{ \underset{y \to b}{\text{Lt}} f(x, y) \right\} = \underset{y \to b}{\text{Lt}} \left\{ \underset{x \to a}{\text{Lt}} f(x, y) \right\}$$

But it is not always so, as the following examples show :

$$\text{Lt} \left(\frac{x - y}{x + y} \right) \text{ as } (x, y) \to (0, 0) \text{ along the line } y = mx$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x - mx}{x + mx} = \frac{1 - m}{1 + m}$$ which is different for lines with different slopes.

Also $\underset{x \to 0}{\text{Lt}} \left[\underset{y \to 0}{\text{Lt}} \left(\frac{x - y}{x + y} \right) \right] = \underset{x \to 0}{\text{Lt}} \left(\frac{x}{x} \right) = 1.$ whereas $\underset{y \to 0}{\text{Lt}} \left[\underset{x \to 0}{\text{Lt}} \left(\frac{x - y}{x + y} \right) \right] = \underset{y \to 0}{\text{Lt}} \left(\frac{-y}{y} \right) = -1.$

∴ As (x, y) is made to approach (0, 0) along different paths, $f(x, y)$ approaches different limits. Hence the two repeated limits are not equal and $f(x, y)$ is discontinuous at the origin.

Also the function is not defined at (0, 0) since $f(x, y) = 0/0$ for $x = 0, y = 0$.

(4) *As in the case of functions of one variable, the following results hold:*

I. If $\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y) = l$ and $\underset{\substack{x \to a \\ y \to b}}{Lt} g(x, y) = m$,

then (*i*) $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x, y) \pm g(x, y)] = l \pm m$ (*ii*) $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x, y) \cdot g(x, y)] = l \cdot m$

(*iii*) $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x, y)/g(x, y)] = l/m$ $(m \neq 0)$

II. *If $f(x, y)$, $g(x, y)$ are continuous at (a, b) then so also are the functions*

$f(x, y) \pm g(x, y)$, $f(x, y) . g(x, y)$ and $f(x, y)/g(x, y)$

provided $g(x, y) \neq 0$ in the last case.

Problems 5.1

Evaluate the following limits :

1. $\underset{\substack{x \to 1 \\ y \to 2}}{Lt} \dfrac{2x^2 y}{x^2 + y^2 + 1}$ 2. $\underset{\substack{x \to 0 \\ y \to 0}}{Lt} \dfrac{xy}{x^2 + y^2}$ 3. $\underset{\substack{x \to \infty \\ y \to 2}}{Lt} \dfrac{xy + 1}{x^2 + 2y^2}$ 4. $\underset{\substack{x \to 1 \\ y \to 1}}{Lt} \dfrac{x(y - 1)}{y(x - 1)}$

5. If $f(x, y) = \dfrac{x - y}{2x + y}$, show that $\underset{x \to 0}{Lt} \left[\underset{y \to 0}{Lt} f(x, y) \right] \neq \underset{y \to 0}{Lt} \left[\underset{x \to 0}{Lt} f(x, y) \right]$

Also show that the function is discontinuous at the origin.

6. Show that the function $f(x, y) = x^2 + 2y$, $(x, y) \neq (1, 2)$
$= 0$ $(x, y) = (1, 2)$

is discontinuous at (1, 2).

7. Investigate the continuity of the function

$f(x, y) = xy/(x^2 + y^2)$, $(x, y) \neq (0, 0)$
$= 0$ $(x, y) = (0, 0)$

at the origin.

Note. In whatever follows, all the functions considered are continuous and their partial derivatives (as defined below) exist.

5.2. PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y.

If we keep y as constant and vary x alone, then z is a function of x only. The derivative of z with respect to x, treating y as constant, is called the *partial derivative of z with respect to x* and is denoted by one of the symbols

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f. \qquad \text{Thus} \quad \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of z with respect to y, keeping x as constant, is called the *partial derivative of z with respect to y* and is denoted by one of the symbols

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f. \qquad \text{Thus} \quad \frac{\partial z}{\partial y} = \underset{\delta y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Similarly, if z is a function of three or more variables $x_1, x_2, x_3, \ldots$ *the partial derivative of z with respect to x_1, is obtained by differentiating z with respect to x_1, keeping all other variables constant* and is written as $\partial z / \partial x_1$.

In general f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y.

Thus
$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}^{*}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}, \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Sometimes we use the following notation

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

Example 5.1. *Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.*

We have $\quad z = x^3 + y^3 - 3axy.$

$$\therefore \quad \frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \text{ and } \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \text{ and } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a.$$

We observe that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

Example 5.2. *If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$,*

show that $\quad \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}.$ (*Mysore, 1997 ; Marathwada, 1994*)

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* It is important to note that in the subscript notation the subscripts are written in the same order in which we differentiate whereas in the '∂' notation the order is opposite.

and $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ (Madras, 2000 ; A.M.I.E., 1994)

We have $$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} - \left\{2y \cdot \tan^{-1}\frac{x}{y} + y^2 \cdot \frac{1}{1+(x/y)^2}\left(-\frac{x}{y^2}\right)\right\}$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = x - 2y\tan^{-1}\frac{x}{y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left\{x - 2y\tan^{-1}\frac{x}{y}\right\} = 1 - 2y \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Similarly, $\frac{\partial u}{\partial x} = 2x\tan^{-1} y/x - y$

and $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{2x\tan^{-1}\frac{y}{x} - y\right\} = \frac{x^2-y^2}{x^2+y^2}.$$ Hence the result.

Example 5.3. *If $z = f(x+ct) + \phi(x-ct)$, prove that*

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}.$$

We have $$\frac{\partial z}{\partial x} = f'(x+ct) \cdot \frac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \qquad ...(i)$$

Again $$\frac{\partial z}{\partial t} = f'(x+ct)\frac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial t}(x-ct) = cf'(x+ct) - c\phi'(x-ct)$$

and $$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)] \qquad ...(ii)$$

From (*i*) and (*ii*), it follows that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$.

Obs. This is an important partial differential equation, known as *wave equation* (§ 18.4).

Example 5.4. If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$? (Calicut, 1994)

We have $$\frac{\partial\theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \left(\frac{-2r}{4t}\right) = -\frac{r}{2}t^{n-1}e^{-r^2/4t}$$

$$\therefore \quad r^2\frac{\partial\theta}{\partial r} = -\frac{r^3}{2}t^{n-1}e^{-r^2/4t}$$

and $$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{3r^2}{2}t^{n-1}e^{-r^2/4t} - \frac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}\left(-\frac{2r}{4t}\right)$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$$

Also $$\frac{\partial\theta}{\partial t} = nt^{n-1} \cdot e^{-r^2/4t} + t^n \cdot e^{-r^2/4t} \cdot \frac{r^2}{4t^2} = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$$

Since $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$,

$$\therefore \quad \left(-\frac{3}{2}t^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t} = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$$

or $$\left(n+\frac{3}{2}\right)t^{n-1}e^{-r^2/4t}=0.$$

Hence $$n=-3/2.$$

Example 5·5. *If* $v=(x^2+y^2+z^2)^{-1/2}$, *prove that*

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0. \text{ (Laplace equation)*}$$ *(J.N.T.U., 2000 ; Mangalore, 1997)*

We have $\dfrac{\partial v}{\partial x}=-\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x=-x(x^2+y^2+z^2)^{-3/2}$

and $$\frac{\partial^2 v}{\partial x^2}=-[1\cdot(x^2+y^2+z^2)^{-3/2}+x(-3/2)(x^2+y^2+z^2)^{-5/2}\cdot 2x]$$

$$=-(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2]=(x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly $\dfrac{\partial^2 v}{\partial y^2}=(x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2)$ and $\dfrac{\partial^2 v}{\partial z^2}=(x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$

Hence $$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=(x^2+y^2+z^2)^{-5/2}\cdot(0)=0.$$

Obs. *A function v satisfying the Laplace equation is said to be a* **harmonic function.**

Example 5·6. *If* $u=x^y$, *show that* $\dfrac{\partial^3 u}{\partial x^2\partial y}=\dfrac{\partial^3 u}{\partial x\partial y\partial x}$.

We have $\dfrac{\partial u}{\partial y}=x^y\log_e x$ and $\dfrac{\partial^2 u}{\partial x\partial y}=yx^{y-1}\cdot\log x+x^y\cdot\dfrac{1}{x}=x^{y-1}(y\log x+1)$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2\partial y}=\frac{\partial}{\partial x}[x^{y-1}(y\log x+1)] \quad ...(i)$$

Again $$\frac{\partial u}{\partial x}=yx^{y-1} \text{ and } \frac{\partial^2 u}{\partial y\partial x}=1\cdot x^{y-1}+y\left(\frac{1}{x}x^y\log x\right)=x^{y-1}(1+y\log x)$$

$$\therefore\quad \frac{\partial^3 u}{\partial x\partial y\partial x}=\frac{\partial}{\partial x}[x^{y-1}(y\log x+1)] \quad ...(ii)$$

From (*i*) and (*ii*) follows the required result.

Problems 5·2

1. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$, if
   (*i*) $z=x^2y-x\sin xy$; (*ii*) $z=\log(x^2+y^2)$; (*iii*) $z=\tan^{-1}[(x^2+y^2)/(x+y)]$; (*iv*) $x+y+z=\log z$.
2. If $z(x+y)=x^2+y^2$, show that $\left(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)^2=4\left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)$ *(Mysore, 1994)*
3. If $z=e^{ax+by}f(ax-by)$, prove that $b\dfrac{\partial z}{\partial x}+a\dfrac{\partial z}{\partial y}=2abz$. *(V.T.U., 2000)*
4. Given $u=e^{r\cos\theta}\cos(r\sin\theta)$, $v=e^{r\cos\theta}\sin(r\sin\theta)$; prove that $\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta}$ and $\dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}$.
5. Prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ if (*i*) $u=\tan^{-1}a\log(x^2+y^2)+b\tan^{-1}\left(\dfrac{y}{x}\right)$; (*ii*) $u=\tan^{-1}\left[\dfrac{2xy}{x^2-y^2}\right]$.
6. If $z=\tan(y+ax)-(y-ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2}=a^2\dfrac{\partial^2 z}{\partial y^2}$. *(Karnataka, 1990)*

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* See footnote p. 33.

7. Verify that $f_{xy} = f_{yx}$, when f is equal to (*i*) $\sin^{-1}(y/x)$; (*ii*) $\log x \tan^{-1}(x^2 + y^2)$.

8. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial f}{\partial y}\right] = 0$.

(*Hamirpur*, 1995)

9. If $v = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t}$, prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$. (*Nagpur*, 1997)

10. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u = Ae^{-gx} \sin(nt - gx)$, where A, g, n are positive constants then $g = \sqrt{(n/2\mu)}$.

11. Find the value of n so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0.$$ (*Marathwada*, 1990

12. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$.

13. If $u = \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

14. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $V_{xx} + V_{yy} + V_{zz} = m(m + 1) r^{m-2}$.

15. If $\frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$.

(*A.M.I.E.*, 199

16. If $v = \log(x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 1$. (*Mysore*, 1997

17. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$ (*A.M.I.E.*, 1996

18. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -9(x + y + z)^{-2}$.

(*Delhi*, 1997 ; *Hamirpur*, 1994

19. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$. (*Gauhati*, 1999; *A.M.I.E.*, 19

5·3. WHICH VARIABLE IS TO BE TREATED AS CONSTANT

(1) Consider the equation $x = r\cos\theta$, $y = r\sin\theta$ ...

To find $\partial r/\partial x$, we need a relation between r and x. Such a relation will contain one m variable θ or y, for we can eliminate only one variable out of four from the relations (1). Thus two possible relations are

$$r = x \sec\theta$$ ..

and

$$r^2 = x^2 + y^2$$ ..

Now we can find $\partial r/\partial x$ either from (2) by treating θ as constant or from (3) be regarding y constant. And there is no reason to suppose that the two values of $\partial r/\partial x$ so found, are equal avoid confusion as to which variable is regarded constant, we introduce the following :

Notation : $(\partial r/\partial x)_\theta$ *means the partial derivative of r with respect to x keeping θ constan* a relation expressing r as a function of x and θ.

Thus from (2), $(\partial r/\partial x)_\theta = \sec\theta$.

When no indication is given regarding the variable to be kept constant, then accordin convention $(\partial/\partial x)$ always means $(\partial/\partial x)_y$ and $\partial/\partial y$ means $(\partial/\partial y)_x$. Similarly $\partial/\partial r$ means $(\partial/$ and $\partial/\partial\theta$ means $(\partial/\partial\theta)_r$.

(2) In thermodynamics, we come across ten variables such as p (pressure), v (volume), T (temperature), W (work), ϕ (entropy) etc. Any one of these can be expressed as a function of other two variables *e.g.* $T = f(p, v)$, $T = g(p, \phi)$

As we shall see, these respectively give rise to the following results :

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial v} dv \quad ...(i)$$

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial \phi} d\phi \quad ...(ii)$$

Now, $\partial T/\partial p$ appearing in (i), has been obtained from T as function of p and v, treating v as constant, we write it as $(\partial T/\partial p)_v$.

Similarly, $\partial T/\partial p$ occuring in (ii), is written as $(\partial T/\partial p)_\phi$.

Example 5·7. *If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ (*Mangalore, 1997 ; Andhra, 1990 ; Rewa, 1990*)

We have $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \left(\frac{\partial r}{\partial x}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial x^2}$

Similarly, $\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left(\frac{\partial r}{\partial y}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial y^2}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

Now to find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$ etc., we write $r = (x^2 + y^2)^{1/2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r}$$

and $$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \partial r/\partial x}{r^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$.

Substituting the values of $\partial r/\partial x$ etc. in (i), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r} f'(r).$$

Problems 5·3

1. If $x = r \cos \theta, y = r \sin \theta$, show that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ and $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$

2. If $x^2 = au + bv, y^2 = au - bv$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$

3. If $u = lx + my, v = mx - ly$, show that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}, \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$

   (*Marathwada, 1990*)

4. If $x = r \cos \theta, y = r \sin \theta$, prove that (i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ (*Mysore, 1994 S*)

(ii) $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \ (x \neq 0, y \neq 0).$

5. If $u = f(r)$ where $r = \sqrt{(x^2 + y^2 + z^2)}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.

5.4. (1) HOMOGENEOUS FUNCTIONS

An expression of the form $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots + a_n y^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as

$$x^n \, [a_0 + a_1 (y/x) + a_2 (y/x)^2 + \ldots\ldots + a_n \, (y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n \, \phi \, (y/x)$, is called a* **homogeneous function** of degree n in x and y.

For instance, $x^3 \cos (y/x)$ is a homogeneous function of degree 3, in x and y.

In general, a function $f(x, y, z, t, \ldots\ldots)$ is said to be a homogeneous function of degree n in x, $y, z, t, \ldots\ldots$, if it can be expressed in the form $x^n \phi \, (y/x, z/x, t/x, \ldots\ldots)$.

(2) **Euler's theorem on homogeneous functions.*** *If u be a homogeneous function of degree n in x and y, then*

$$\mathbf{x} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{y} \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = \mathbf{nu}.$$

Since u is a homogeneous function of degree n in x and y, therefore,

$$u = x^n f(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = n x^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot y \left(-\frac{1}{x^2}\right) = n x^{n-1} f\left(\frac{y}{x}\right) - y x^{n-2} f'\left(\frac{y}{x}\right)$$

and $\quad \frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$. Hence $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n x^n f\left(\frac{y}{x}\right) = nu$.

In general, if u be a homogeneous function of degree n in $x, y, z, t, \ldots\ldots$, then,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + t \frac{\partial u}{\partial t} \ldots\ldots = nu.$$

Example 5.8. *Show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \log u$ where $\log u = (x^3 + y^3)/(3x + 4y)$.*

(*Mysore, 1998*)

Since $z = \log u = \frac{x^3 + y^3}{3x + 4y} = x^2 \cdot \frac{1 + (y/x)^3}{3 + 4\,(y/x)}$,

$\therefore$ z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z \qquad \ldots(i)$$

But $\quad \frac{\partial z}{\partial x} = \frac{1}{u} \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{1}{u} \frac{\partial u}{\partial y}$

Hence (i) becomes

$$x \cdot \frac{1}{u} \frac{\partial u}{\partial x} + y \cdot \frac{1}{u} \frac{\partial u}{\partial y} = 2 \log u \quad \text{or} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \log u$$

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* After an enormously creative Swiss mathematician *Leonhard Euler* (1707–1783). He studied unde[r] *John Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totall[y] blind in 1771, he contributed to almost all branches of mathematics.

Example 5·9. *If* $u = \sin^{-1}\dfrac{x+2y+3z}{x^8+y^8+z^8}$ *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$.

(*Marathwada, 1998*)

Here u is not a homogeneous function. We, therefore, write

$$w = \sin u = \frac{x+2y+3z}{x^8+y^8+z^8} = x^{-7}\,\frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}$$

Thus w is a homogeneous function of degree -7 in x, y, z. Hence by Euler's theorem

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = (-7)\,w \qquad ...(i)$$

But $\dfrac{\partial w}{\partial x} = \cos u\,\dfrac{\partial u}{\partial x},\ \dfrac{\partial w}{\partial y} = \cos u\,\dfrac{\partial u}{\partial y},\ \dfrac{\partial w}{\partial z} = \cos u\,\dfrac{\partial u}{\partial z}$

$\therefore$ (*i*) becomes $\quad x\cos u\,\dfrac{\partial u}{\partial x} + y\cos u\,\dfrac{\partial u}{\partial y} + z\cos u\,\dfrac{\partial u}{\partial z} = -7\sin u$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u.$$

Example 5·10. *If z is a homogeneous function of degree n in x and y, show that*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial z^2} = n(n-1)\,z.$$ (*Mangalore, 1997 ; C.S.E., 1995*)

By Euler's theorem, $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad ...(i)$$

Differentiating (*i*) partially w.r.t. x, we get $x\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial z}{\partial x} + y\dfrac{\partial^2 z}{\partial x\partial y} = n\dfrac{\partial z}{\partial x}$

i.e. $$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y} = (n-1)\frac{\partial z}{\partial x} \qquad ...(ii)$$

Again differentiating (*i*) partially w.r.t. y, we get $x\dfrac{\partial^2 z}{\partial y\partial x} + \dfrac{\partial z}{\partial y} + y\dfrac{\partial^2 z}{\partial y^2} = n\dfrac{\partial z}{\partial y}$

i.e. $$x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y} \qquad ...(iii)$$

Multiplying (*ii*) by x and (*iii*) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = n(n-1)\,z. \qquad [\text{by } (i)$$

Problems 5·4

1. Show that $f(x, y, z) = (4x^3 + 2y^2z)/(x + 2y + 3z)$ is homogeneous. What is its degree ?
2. Verify Euler's theorem, when (*i*) $f(x, y) = ax^2 + 2hxy + by^2$ (*Osmania, 1995*)
   (*ii*) $f(x, y) = x^2(x^2 - y^2)^3/(x^2 + y^2)^3$ (*Bhopal, 1991*)
   (*iii*) $f(x, y) = 3x^2yz + 5xy^2z + 4z^4$ (*J.N.T.U., 1999*)
3. If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$. (*Mysore, 1994 S*)
4. If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$ (*J.N.T.U., 1999 ; Mangalore, 1997*)
5. If $\sin u = \dfrac{x^2y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$ (*Karnataka, 1993*)
6. If $u = \cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$. (*Gorakhpur, 1991*)

7. If $z = \log(x^2 + xy + y^2)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$. (Gauhati, 1999)

8. (a) If u is homogeneous function of nth degree in x, y, z, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$.

(b) If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. (V.T.U., 2000 S)

(c) Show that $xu_x + yu_y + zu_z = 2\tan u$, where $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$ (Mysore, 1997 S)

9. If $z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$. (Kuvempu, 1998 S, A.M.I.E., 1997 W)

10. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$. (Mysore, 1995)

and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$. (Andhra, 2000, Assam, 1998; Ranchi, 1998)

11. Given $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$, prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$.

(Marathwada 1994)

12. If $u = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. (Bangalore, 1990)

13. If $u = \tan^{-1}(y^2/x)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin^2 u \sin 2u$. (V.T.U., 2001)

14. If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$

(Marathwada, 1990; Gujarat, 1990)

5.5. (1) TOTAL DERIVATIVE

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x, y)$. Thus we can find the ordinary derivative du/dt which is called the *total derivative* of u to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find du/dt without actually substituting the values of x and y in $f(x, y)$, we establish the following formula :

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad \ldots(i)$$

Proof. We have $u = f(x, y)$

Giving increment δt to t, let the corresponding increments of x, y and u be δx, δy and δu respectively.

Then $u + \delta u = f(x + \delta x, y + \delta y)$

Subtracting, $\delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)]$$

$$\therefore \frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\cdot\frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y}\cdot\frac{\delta y}{\delta t}$$

Taking limits as $\delta t \to 0$, δx and δy also $\to 0$, we have

$$\frac{du}{dt} = \underset{\delta y \to 0}{\text{Lt}}\left[\underset{\delta x \to 0}{\text{Lt}}\left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\right\}\right]\frac{dx}{dt} + \underset{\delta y \to 0}{\text{Lt}}\left\{\frac{f(x, y + \delta y) - f(x, y)}{\delta y}\right\}\frac{dy}{dt}$$

$$= \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{\partial f(x, y + \delta y)}{\partial x} \right\} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$

[Supposing $\partial f(x, y)/\partial x$ to be a continuous function of y]

$$= \frac{\partial f(x, y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$ which is the desired formula.

Cor. Taking $t = x$, (*i*) becomes, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$ ...(*ii*)

Obs. If $u = f(x, y, z)$, where x, y, z are all functions of a variable t, then we can similarly prove that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \quad ...(iii)$$

(2) **Differentiation of implicit functions.** If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x, then (*ii*) becomes

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

This gives the *important formula* $\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$ $\left[\frac{\partial f}{\partial y} \neq 0\right]$

for the first differential coefficient of an implicit function.

Example 5·11. *Given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$, find du/dt as a function of t. Verify your result by direct substitution.* (*Calicut, 1999*)

We have $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = \left(\cos\frac{x}{y}\right)\frac{1}{y} \cdot e^t + \left(\cos\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)2t$

$$= \cos(e^t/t^2) \cdot e^t/t^2 - 2\cos(e^t/t^2) \cdot e^t/t^3 = \{(t-2)/t^3\}e^t \cos(e^t/t^2)$$

Also $u = \sin(x/y) = \sin(e^t/t^2)$

$\therefore$ $$\frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2e^t - e^t \cdot 2t}{t^4} = \frac{t-2}{t^3} e^t \cos\left(\frac{e^t}{t^2}\right)$$ as before.

Example 5·12. *If x increases at the rate of 2 cm/sec at the instant when $x = 3$ cm. and $y = 1$ cm., at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing ?*

Let $u = 2xy - 3x^2y$, so that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt} \quad ...(i)$$

When $x = 3$ and $y = 1$, $dx/dt = 2$, and u is neither increasing nor decreasing, *i.e.* $du/dt = 0$.

$\therefore$ (*i*) becomes $0 = (2 - 6 \times 3)2 + (2 \times 3 - 3 \times 9)\frac{dy}{dt}$

or $\frac{dy}{dt} = -\frac{32}{21}$ cm/sec. Thus y is decreasing at the rate of 32/21 cm/sec.

Example 5·13. *If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find du/dx.*

From $f(x, y) = x^3 + y^3 + 3xy - 1$, we have

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x} \quad ...(i)$$

Also $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = (1 \cdot \log xy + x \cdot 1/x) + (x/y) \cdot dy/dx.$

Hence $du/dx = 1 + \log xy - x(x^2 + y)/y(y^2 + x)$ [by (*i*)]

Example 5·14. *Formula for the second differential coefficient of an implicit function.*

If $f(x, y) = 0$, show that

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$ (J.N.T.U., 1998)

We have $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{p}{q} \qquad \ldots(i)$$

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{p}{q}\right) = -\frac{q\,(dp/dx) - p\,(dq/dx)}{q^2} \qquad \ldots(ii)$$

Using the notations : $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial q}{\partial x}$, $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y}$,

we have $$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \cdot \frac{dy}{dx} = r + s\,(-p/q) = \frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} \cdot \frac{dy}{dx} = s + t\,(-p/q) = \frac{qs - pt}{q}$$

Substituting the values of dp/dx and dq/dx in (*ii*), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr - ps}{q}\right) - p\left(\frac{qs - pt}{q}\right)\right] = -\frac{q^2r - 2pqs + p^2t}{q^3}$$

Problems 5·5

1. If $u = \sin^{-1}(x - y)$, $x = 3t$ and $y = 4t^3$, show that $du/dt = 3(1 - t^2)^{-1/2}$.

   (Gauhati, 1999; Kuvempu, 1998 S)

2. If $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$, find du/dx.

3. Find the total differential coefficient of x^2y with respect to x when x and y are connected by the relation $x^2 + xy + y^2 = 1$.

   (Andhra, 1999)

4. At a given instant the sides of a rectangle are 4 ft. and 3 ft. respectively and they are increasing at the rate of 1·5 ft./sec. and 0·5 ft./sec. respectively, find the rate at which the area is increasing at that instant.

5. If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm. per second and it passes through the value $x = 3$ cm., show that if y is passing through the value $y = 1$ cm., y must be decreasing at the rate of $2\frac{2}{15}$ cm. per second, in order that z shall remain constant.

   (Rewa, 1990)

6. If $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.

   (Mysore, 1997 S)

7. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$.

8. If the curves $f(x, y) = 0$ and $\phi(x, y) = 0$ touch, show that at the point of contact, $\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}$.

9. If $f(x, y) = 0$, show that $\left(\frac{\partial f}{\partial y}\right)^3 \frac{d^2y}{dx^2} = 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x \partial y}\right) - \left(\frac{\partial f}{\partial y}\right)^2 \frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2 \left(\frac{\partial^2 f}{\partial y^2}\right)$ (A.M.I.E., 1997 W)

5·6. (1) GEOMETRICAL INTERPRETATION

If $P(x, y, z)$ be the coordinates of a point referred to rectangular axes OX, OY, OZ then the equation $z = f(x, y)$ represents a *surface*.

Let a plane $y = b$ parallel to the XZ-plane pass through P cutting the surface along the curve APB given by

$$z = f(x, b).$$

As y remains equal to b and x varies then P moves along the curve APB and $\partial z/\partial x$ is the ordinary derivative of $f(x, b)$ w.r.t. x.

Hence *$\partial z/\partial x$ at P is the tangent of the angle which the tangent at P to the section of the surface $z = f(x, y)$ by a plane through P parallel to the plane XOZ, makes with a line parallel to the x-axis.*

Similarly, *$\partial z/\partial y$ at P is the tangent of the angle which the tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$, makes with a line parallel to the y-axis.*

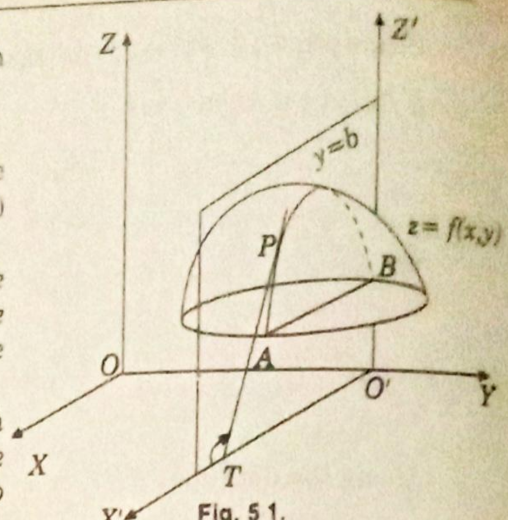

Fig. 5.1.

(2) **Tangent plane and Normal to a surface.** Let $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ be two neighbouring points on the surface $F(x, y, z) = 0$. ...(*i*)

Let the arc PQ be δs and the chord PQ be δc, so that (as for plane curves)

$$\operatorname*{Lt}_{Q \to P} (\delta s/\delta c) = 1.$$

The direction cosines of PQ are $\dfrac{\delta x}{\delta c}, \dfrac{\delta y}{\delta c}, \dfrac{\delta z}{\delta c}$ *i.e.* $\dfrac{\delta x}{\delta s} \cdot \dfrac{\delta s}{\delta c}, \dfrac{\delta y}{\delta s} \cdot \dfrac{\delta s}{\delta c}, \dfrac{\delta z}{\delta s} \cdot \dfrac{\delta s}{\delta c}$

When $\delta s \to 0$, $Q \to P$ and PQ tends to tangent line PT. Then noting that the co-ordinates of any point on arc PQ are functions of s only, the direction cosines of PT are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \qquad \text{...(ii)}$$

Differentiating (*i*) with respect to s, we obtain $\dfrac{\partial F}{\partial x} \cdot \dfrac{dx}{ds} + \dfrac{\partial F}{\partial y} \cdot \dfrac{dy}{ds} + \dfrac{\partial F}{\partial z} \cdot \dfrac{dz}{ds} = 0.$

This shows that the tangent line whose direction cosines are given by (*ii*), is perpendicular to the line having direction ratios

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \qquad \text{...(iii)}$$

Since we can take different curves joining Q to P, we get a number of tangent lines at P and the line having direction ratios (*iii*) will be perpendicular to *all* these tangent lines at P. Thus all the tangent lines at P lie in a plane through P perpendicular to line (*iii*).

Hence *the equation of the tangent plane to (i) at the point P is*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{X} - \mathbf{x}) + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{Y} - \mathbf{y}) + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{Z} - \mathbf{z}) = 0$$

where (X, Y, Z) are the current co-ordinates of any point on this tangent plane.

Also *the equation of the normal to the surface at P* (*i.e.* the line through P, perpendicular to the tangent plane at P) is

$$\frac{\mathbf{X} - \mathbf{x}}{\partial \mathbf{F}/\partial \mathbf{x}} = \frac{\mathbf{Y} - \mathbf{y}}{\partial \mathbf{F}/\partial \mathbf{y}} = \frac{\mathbf{Z} - \mathbf{z}}{\partial \mathbf{F}/\partial \mathbf{z}}.$$

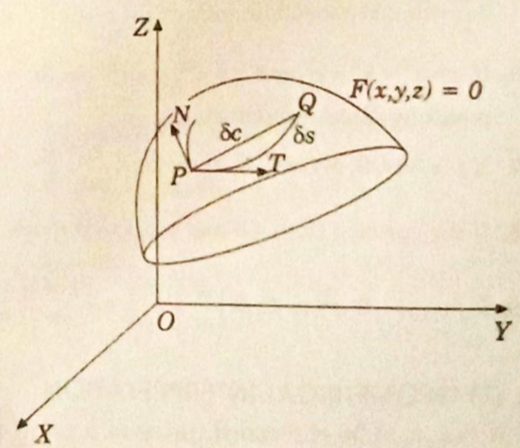

Fig. 5.2.

Example 5.15. *Find the equations of the tangent plane and the normal to the surface* $z^2 = 4(1 + x^2 + y^2)$ *at* (2, 2, 6). (*Bangalore*, 1993)

We have $F(x, y, z) = 4x^2 + 4y^2 - z^2 + 4$.

$\partial F/\partial x = 8x$, $\partial F/\partial y = 8y$, $\partial F/\partial z = -2z$, and at the point (2, 2, 6)

$\partial F/\partial x = 16$, $\partial F/\partial y = 16$, $\partial F/\partial z = -12$

Hence the equation of the tangent plane at (2, 2, 6) is $16(X - 2) + 16(Y - 2) - 12(Z - 6) = 0$

i.e. $$4X + 4Y - 3Z + 2 = 0 \quad ...(i)$$

Also the equation of the normal at (2, 2, 6) [being perpendicular to (*i*)] is

$$\frac{X-2}{4} = \frac{Y-2}{4} = \frac{Z-6}{-3}.$$

Problems 5.6

Find the equations of the tangent plane and the normal to each of the following surfaces at the given points :

1. $\frac{x^2}{2} - \frac{y^2}{3} = Z$ at (2, 3, –1). (*Andhra*, 1994)
2. $2x^2 + y^2 = 3 - 2z$ at (2, 1, – 3) (*Assam*, 1998)
3. $xyz = a^2$ at (x_1, y_1, z_1).
4. $2xz^2 - 3xy - 4x = 7$ at (1, – 1, 2) (*Mangalore*, 1997)
5. Show the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact. (*Andhra*, 1990)
6. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$, if $\frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$.

5.7. CHANGE OF VARIABLES

If $$u = f(x, y) \quad ...(1)$$

where $$x = \phi(s, t) \text{ and } y = \psi(s, t), \quad ...(2)$$

it is often necessary to change expressions involving u, x, y, $\partial u/\partial x$, $\partial u/\partial y$ etc. to expressions involving u, s, t, $\partial u/\partial s$, $\partial u/\partial t$ etc.

The necessary formulae for the change of variables are easily obtained. If t is regarded as a constant, then x, y, u will be functions of s alone. Therefore, by (*i*) of page 203, we have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \quad ...(3)$$

where the ordinary derivatives have been replaced by the partial derivatives because x, y are functions of two variables s and t.

Similarly regarding s as constant, we obtain $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$ ...(4)

On solving (3) and (4) as simultaneous equations in $\partial u/\partial x$ and $\partial u/\partial y$, we get their values in terms of $\partial u/\partial s$, $\partial u/\partial t$, u, s, t.

If instead of the equations (2), s and t are given in terms of x and y, say: $s = \xi(x, y)$ and $t = \eta(x, y)$, ...(5)

then it is easier to use the formulae $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$ ...(6)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} \quad ...(7)$$

The higher derivatives of u can be found by repeated application of the formulae (3) and (4) or of (6) and (7).

Example 5.16. [illegible]

[illegible] [illegible]

We have [illegible]

[illegible] ...(i)

[illegible] ...(ii)

(i) − (ii) gives [illegible]

Example 5.17. [illegible]

[illegible] [illegible]

[illegible]

[illegible]

[illegible] ...(i)

Similarly, [illegible] ...(ii)

and [illegible] ...(iii)

Adding (i), (ii) and (iii), we get the required result.

Example 5.18. *If* $x + y = 2e^{\theta} \cos \phi$ *and* $x - y = 2ie^{\theta} \sin \phi$, *show that*

[illegible] [illegible]

We have [illegible] [illegible]

and [illegible]

Since [illegible] is a composite function of θ and ϕ

[illegible]

[illegible]

or [illegible] ...(i)

Also [illegible]

[illegible]

[illegible] ...(ii)

Using the operator (*i*), we have

$$\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)=e^{2\theta}\left(\frac{\partial}{\partial x}\cdot e^{i\phi}-\frac{\partial}{\partial y}\cdot e^{-i\phi}\right)\left(\frac{\partial u}{\partial x}e^{i\phi}+\frac{\partial u}{\partial y}e^{-i\phi}\right)$$

$$=e^{2\theta}\left\{e^{2i\phi}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+e^{-2i\phi}\frac{\partial^2 u}{\partial y^2}\right\} \quad ...(iii)$$

$$\frac{\partial^2 u}{\partial \phi^2}=\frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right)=(ie^{\theta})^2\left(\frac{\partial}{\partial x}\cdot e^{i\phi}-\frac{\partial}{\partial y}\cdot e^{-i\phi}\right)\left(\frac{\partial u}{\partial x}e^{i\phi}-\frac{\partial u}{\partial y}e^{-i\phi}\right)$$

$$=-e^{2\theta}\left(e^{2i\phi}\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+e^{-2i\phi}\frac{\partial^2 u}{\partial y^2}\right) \quad ...(iv)$$

Adding (*iii*) and (*iv*), we get

$$\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial \phi^2}=e^{2\theta}\left(4\frac{\partial^2 u}{\partial x\partial y}\right)=4xy\frac{\partial^2 u}{\partial x\partial y} \qquad [\because\ xy=e^{2\theta}$$

Example 5·19. *Transform the equation* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *into polar co-ordinates.* (*Delhi, 1997*)

We have $x=r\cos\theta$, $y=r\sin\theta$ and $r=\sqrt{(x^2+y^2)}$, $\theta=\tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{(x^2+y^2)}}=\cos\theta \text{ and } \frac{\partial \theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}$$

Thus $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$

i.e. $\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$ Similarly $\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$.

$$\therefore \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \quad ...(i)$$

and

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\sin^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial r} \quad ...(ii)$$

Adding (*i*) and (*ii*), we get $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}$.

Hence the transformed equation is $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$.

Problems 5·7

1. If $z=\log(u^2+v)$, $u=e^{x^2+y^2}$, $v=x^2+y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. (*J.N.T.U., 1990*)
2. If $u=f(r,s)$, $r=x+at$, $s=y+bt$ and x, y, t are independent variables, show that $\frac{\partial u}{\partial t}=a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}$. (*Madras, 1996 S*)

3. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

(V.T.U., 2000 ; A.M.I.E., 1997 W ; Andhra, 1994 S)

4. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$ (S. Gujarat, 1990)

5. If $u = f(r, s, t)$ and $r = x/y$, $s = y/z$, $t = z/x$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(Mysore, 1994 S ; J.N.T.U., 1990)

6. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

(Marathwada, 1990)

7. Given that $z^3 + xy - y^2z = 6$, obtain the expressions for $\partial y/\partial x$, $\partial z/\partial x$ in terms of x, y, z and find their values at the point $(0, 1, 2)$.

$\left[\textbf{Hint.} \text{ If } f(x, y, z) = 0, \text{ then } \frac{\partial y}{\partial x} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} \text{ and } \frac{\partial z}{\partial x} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial z}\right]$

8. Given that $u(x, y, z) = f(x^2 + y^2 + z^2)$ where $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$, find $\frac{\partial u}{\partial \theta}$ and $\frac{\partial u}{\partial \phi}$.

9. If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$, show that

$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1.$$

10. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

(Delhi, 1997)

5·8. (1) JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$ is called the *Jacobian** of u, v with respect to x, y

and is written as $\frac{\partial(u, v)}{\partial(x, y)}$ or $J\left(\frac{u, v}{x, y}\right)$.

Similarly the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Likewise, we can define Jacobians of four or more variables. An important application of Jacobians is in connection with the change of variables in multiple integrals (§ 7.7).

(2) **Properties of Jacobians.** We give below two of the important properties of Jacobians. For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

I. *If* $J = \partial(u, v)/\partial(x, y)$ *and* $J' = \partial(x, y)/\partial(u, v)$ *then* $JJ' = 1$.

Let $u = f(x, y)$ and $v = g(x, y)$.

Suppose, on solving for x and y, we get $x = \phi(u, v)$ and $y = \psi(u, v)$.

---

* Called after the German mathematician *Carl Gustav Jacob Jacobi* (1804–1851), who made significant contributions to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

Thus

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} \qquad ...(i)$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}$$

and

$$\therefore\; JJ' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

[Interchanging rows and columns of the IInd determinant]

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$ [By virtue of (i)]

II. *If u, v are functions of r, s and r, s are functions of x, y, then* $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{\partial(u, v)}{\partial(r, s)}\cdot\dfrac{\partial(r, s)}{\partial(x, y)}$

$$\frac{\partial(u, v)}{\partial(r, s)}\cdot\frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

[Interchanging rows and columns of the IInd det.]

$$= \begin{vmatrix} \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial y} \\ \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}$$

Example 5·20. (*i*) *In polar co-ordinates,* $x = r\cos\theta,\; y = r\sin\theta$, *show that*

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$ (Andhra, 2000 ; Gulbarga, 1998 S)

(*ii*) *In cylindrical co-ordinates* (Fig. 5·26), $x = \rho\cos\phi,\; y = \rho\sin\phi,\; z = z$, *show that*

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

(*iii*) *In spherical polar co-ordinates* (Fig. 5·26), $x = r\sin\theta\cos\phi,\; y = r\sin\theta\sin\phi,\; z = r\cos\theta$, *show that*

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$$ (Gauhati, 1999)

(*i*) We have $\dfrac{\partial x}{\partial r} = \cos\theta,\; \dfrac{\partial x}{\partial \theta} = -r\sin\theta$ and $\dfrac{\partial y}{\partial r} = \sin\theta,\; \dfrac{\partial y}{\partial \theta} = r\cos\theta$

$$\therefore\; \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(*ii*) We have $\dfrac{\partial x}{\partial \rho} = \cos\phi,\; \dfrac{\partial x}{\partial \phi} = -\rho\sin\phi,\; \dfrac{\partial x}{\partial z} = 0,$

$\dfrac{\partial y}{\partial \rho} = \sin\phi,\; \dfrac{\partial y}{\partial \phi} = \rho\cos\phi,\; \dfrac{\partial y}{\partial z} = 0$ and $\dfrac{\partial z}{\partial \rho} = 0,\; \dfrac{\partial z}{\partial \phi} = 0,\; \dfrac{\partial z}{\partial z} = 1$

$$\therefore\; \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

(*iii*) We have $\dfrac{\partial x}{\partial r} = \sin\theta\cos\phi,\; \dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\; \dfrac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,$

$$\frac{\partial y}{\partial r} = \sin\theta \sin\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,$$

$$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial z}{\partial \phi} = 0.$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Problems 5.8

1. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\partial(r, \theta)/\partial(x, y)$, $\partial(x, y)/\partial(r, \theta)$ and prove that $|\partial(r, \theta)/\partial(x, y)| \cdot |\partial(x, y)/\partial(r, \theta)| = 1$. *(A.M.I.E., 1997 W)*
2. If $x = u(1 - v)$, $y = uv$, prove that $JJ' = 1$. *(V.T.U., 2000 S ; Andhra, 1998)*
3. If $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, show that $\partial(x, y) / \partial(\xi, \eta) = \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta)$. *(Ranchi, 1998)*
4. If $u = \dfrac{x + y}{1 - xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. *(Marathwada, 1990)*
5. If $u = yz/x$, $v = zx/y$, $w = xy/z$, show that $\partial(u, v, w)/\partial(x, y, z) = 4$. *(J.N.T.U., 1998; Mysore, 1997)*
6. If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$. *(Gorakhpur, 1991)*
7. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\partial(F, G, H)/\partial(u, v, w)$. *(Kuvempu, 1996 ; Marathwada, 1994)*
8. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2v$. *(V.T.U., 2000 ; A.M.I.E., 1997)*

9. TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Considering $f(x + h, y + k)$ as a function of a single variable x, we have by Taylor's theorem*

$$f(x + h, y + k) = f(x, y + k) + h\frac{\partial f(x, y + k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y + k)}{\partial x^2} + \dots \quad \dots(i)$$

Now expanding $f(x, y + k)$ as a function of y only,

$$f(x, y + k) = f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \dots\dots$$

$\therefore$ (i) takes the form $f(x + h, y + k) = f(x, y) + k\dfrac{\partial f(x, y)}{\partial y} + \dfrac{k^2}{2!}\dfrac{\partial^2 f(x, y)}{\partial y^2} + \dots\dots$

$$+ h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \dots\dots\right\} + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial(x, y)}{\partial y} + \dots\dots\right\}$$

Hence $$f(x + h, y + k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \dots \quad \dots(1)$$

In symbols we write it as $f(x + h, y + k) = f(x, y) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \dots\dots$

Taking $x = a$ and $y = b$, (1) becomes

$$f(a + h, b + k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \dots$$

---

* Named after an English mathematician *Brook Taylor* (1685–1731).

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$, $k = y - b$, we get

$$f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)]$$
$$+ \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \ldots\ldots \quad \ldots(2)$$

This is Taylor's expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$. It is used to expand $f(x, y)$ in the neighbourhood of (a, b).

Cor. Putting $a = 0$, $b = 0$, in (2), we get

$$f(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \ldots\ldots$$

This is Maclaurins expansion of $f(x, y)$. ...(3)

Example 5·21. *Expand $e^x \log(1 + y)$ in powers of x and y upto terms of third degree.*

(*Andhra, 1999*)

Here $f(x, y) = e^x \log(1 + y)$ $\therefore$ $f(0, 0) = 0$

$f_x(x, y) = e^x \log(1 + y)$ $\quad f_x(0, 0) = 0$

$f_y(x, y) = e^x \dfrac{1}{1 + y}$ $\quad f_y(0, 0) = 1$

$f_{xx}(x, y) = e^x \log(1 + y)$ $\quad f_{xx}(0, 0) = 0$

$f_{xy}(x, y) = e^x \dfrac{1}{1 + y}$ $\quad f_{xy}(0, 0) = 1$

$f_{yy}(x, y) = -e^x (1 + y)^{-2}$ $\quad f_{yy}(0, 0) = -1$

$f_{xxx}(x, y) = e^x \log(1 + y)$ $\quad f_{xxx}(0, 0) = 0$

$f_{xxy}(x, y) = e^x \dfrac{1}{1 + y}$ $\quad f_{xxy}(0, 0) = 1$

$f_{xyy}(x, y) = -e^x (1 + y)^{-2}$ $\quad f_{xyy}(0, 0) = -1$

$f_{yyy}(x, y) = 2e^x (1 + y)^{-3}$ $\quad f_{yyy}(0, 0) = 2$

Now Maclaurin's expansion of $f(x, y)$ gives

$$f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2!}\{x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\}$$
$$+ \frac{1}{3!}\{x^3 f_{xxx}(0, 0) + 3x^2 y\, f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\} + \ldots\ldots$$

$$\therefore \quad e^x \log(1 + y) = 0 + x(0) + y(1) + \frac{1}{2!}\{x^2(0) + 2xy(1) + y^2(-1)\}$$
$$+ \frac{1}{3!}\{x^3(0) + 3x^2 y(1) + 3xy^2(-1) + y^3(2)\} + \ldots\ldots.$$
$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2 y - xy^2) + \frac{1}{3}y^3 + \ldots\ldots$$

Example 5·22. *Expand $f(x, y) = \tan^{-1}(y/x)$ in powers of $(x - 1)$ and $(y - 1)$ upto third-degree terms. Hence compute $f(1{\cdot}1, 0{\cdot}9)$ approximately.*

Here $a = 1$, $b = 1$ and $f(1, 1) = \tan^{-1}(1) = \pi/4$.

$f_x = \dfrac{-y}{x^2 + y^2}$, $\quad f_x(1, 1) = -\dfrac{1}{2}$; $\quad f_y = \dfrac{x}{x^2 + y^2}$, $\quad f_y(1, 1) = \dfrac{1}{2}$

5.11. TOTAL DIFFERENTIAL

If u is a function of two variables x and y, the *total differential* of u is defined as

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \qquad (1)$$

The differentials dx and dy are respectively the increments δx and δy in x and y. If x and y are not independent variables but functions of another variable t even then the formula (1) holds and we write

$dx = \frac{dx}{dt}\,dt$ and $dy = \frac{dy}{dt}\,dt$. Similar definition can be given for a function of three or more variables.

Problems 5.9

1. Expand the following functions in powers of x and y as far as terms of third degree :

   (i) $\sin x \sin y$ (ii) $e^x \sin y$ (Gauhati, 1999 ; Andhra, 1998 ; Mysore, 1994 S)

2. Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$ using Taylor's theorem.

   (Assam, 1998, A.M.I.E., 1997 ; Kuvempu, [illegible])

3. Expand $f(x, y) = \sin xy$ in powers of $(x - 1)$ and $(y - \pi/2)$ upto the second degree terms.
4. If $f(x, y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately. (*Hamirpur, 1994 S*)
5. If the kinetic energy $k = wv^2/2g$, find approximately the change in the kinetic energy as w changes from 49 to 49.5 and v change from 1600 to 1590. (*Mysore, 1995*)
6. Find the possible percentage error in computing the resistance r from the formula $1/r = 1/r_1 + 1/r_2$, if r_1, r_2 are both in error by 2%.
7. The voltage V across a resistor is measured with an error h, and the resistance R is measured with an error k. Show that the error in calculating the power $W(V, R) = V^2/R$ generated in the resistor, is $VR^{-2}(2Rh - Vk)$.
8. Find the percentage error in the area of an ellipse if one per cent error is made in measuring the major and minor axes. (*V.T.U., 2000 S*)
9. The time of oscillation of a simple pendulum is given by the equation $T = 2\pi\sqrt{(l/g)}$. In an experiment carried out to find the value of g, errors of 1.5% and 0.5% are possible in the values of l and T respectively. Show that the error in the calculated value of g is 0.5%.
10. If $pv^2 = k$ and the relative errors in p and v are respectively 0.05 and 0.025, show that the error in k is 10%. (*Mysore, 1999*)
11. If the H.P. required to propel a steamer varies as the cube of the velocity and square of the length. Prove that a 3% increase in velocity and 4% increase in length will require an increase of about 17% in H.P.
12. The range R of a projectile which starts with a velocity v at an elevation α is given by $R = (v^2 \sin 2\alpha)/g$. Find the percentage error in R due to an error of 1% in v and an error of $\frac{1}{2}$% in α.
13. The deflection at the centre of a rod of length l and diameter d supported at its ends, loaded at the centre with a weight w varies at wl^3d^{-4}. What is the increase in the deflection corresponding to p% increase in w, q% decrease in l and r% increase in d?
14. The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $(s^2D^{2/3}/t^2)$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance diminished by 2%.
15. The indicated horse power I of an engine is calculated from the formula $I = PLAN/33{,}000$, where $A = \pi d^2/4$. Assuming that error of r per cent may have been made in measuring P, L, N and d, find the greatest possible error in I.
16. At a distance of 50 metres from the foot of the tower the elevation of its top is 30°. If the possible errors in measuring the distance and elevation are 2 cm and 0.05 degrees, find the approximate error in calculating the height.
17. If the sides of a plane triangle ABC vary in such a way that its circum-radius remains constant, prove that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$ (*Hamirpur, 1995 ; Mysore, 1994 S*)

[**Hint.** The circumradius R is given by $R = a/2 \sin A = b/2 \sin B = c/2 \sin C$.]

18. If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{(1-x^2)}} + \dfrac{dy}{\sqrt{(1-y^2)}} + \dfrac{dz}{\sqrt{(1-z^2)}} = 0.$

(*A.M.I.E., 1996 W ; Hamirpur, 1996 S*)

[**Hint.** $2(x - yz)\,dx + 2(y - zx)\,dy + 2(z - xy)\,dz = 0$. Also $(x - yz)^2 = (1 - y^2)(1 - z^2)$, ...]

5.12. (1) MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Def. *A function $f(x, y)$ is said to have a* **maximum** *or* **minimum** *at $x = a$, $y = b$, according as*

$$f(a + h, b + k) < \text{or} > f(a, b),$$

for all positive or negative small values of h and k.

In other words, if $\Delta = f(a+h, b+k) - f(a, b)$ *is of the same sign for all small values of* h, k, *and if this sign is negative, then* $f(a, b)$ *is a maximum. If this sign is positive,* $f(a, b)$ *is a minimum.*

Considering $z = f(x, y)$ as a surface, maximum value of z occurs at the top of an elevation (e.g. a dome) from which the surface descends in every direction and a minimum value occurs at the bottom of a depression (e.g. a bowl) from which the surface ascends in every direction. Sometimes the maximum or minimum value may form a *ridge* such that the surface descends or ascends in all directions except that of the ridge. Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface falls for displacement in certain directions and rises for displacements in other directions. Such a point is called a **saddle point.**

Note. *A maximum or minimum value of a function is called its* **extreme value.**

(2) Conditions for f(x, y) to be maximum or minimum.

Using Taylor's theorem page 212, we have $\Delta = f(a+h, b+k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{a,b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots\ldots \quad ...(i)$$

For small values of h and k, the second and higher order terms are still smaller and hence may be neglected. Thus

sign of Δ = sign of $[hf_x(a, b) + kf_y(a, b)]$.

Taking $h = 0$ we see that the right hand side changes sign when k changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at (a, b) unless $f_y(a, b) = 0$.

Similarly taking $k = 0$, we find that $f(x, y)$ cannot have a maximum or minimum at (a, b) unless $f_x(a, b) = 0$.

Hence *the necessary conditions for* $f(x, y)$ *to have a maximum or a minimum at* (a, b) *are that*

$$\mathbf{f_x(a, b) = 0, f_y(a, b) = 0.}$$

If these conditions are satisfied, then for small value of h and k, (*i*) gives

sign of Δ = sign of $\left[\frac{1}{2!}(h^2r + 2hks + k^2t)\right]$ where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$ and $t = f_{yy}(a, b)$.

Now $\quad h^2r + 2hks + k^2t = \frac{1}{r}\left[h^2r^2 + 2hkrs + k^2rt\right] = \frac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$

Thus sign of Δ = sign of $\frac{1}{2r}\left[(hr + ks)^2 + k^2(rt - s^2)\right] \quad ...(ii)$

In (*ii*), $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δ will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x, y)$ *has a maximum or a minimum at* (a, b) *according as* $r <$ or > 0.

If $rt - s^2 < 0$, then Δ will change with h and k and hence there is no maximum or minimum at (a, b) *i.e.* it is a *saddle point.*

If $rt - s^2 = 0$, further investigation is required to find whether there is a maximum or minimum at (a, b) or not.

Note. Stationary value. $f(a, b)$ *is said to be a stationary value of* $f(x, y)$, *if* $f_x(a, b) = 0$ *and* $f_y(a, b) = 0$ *i.e.* the function is stationary at (a, b).

Thus every extreme value is a stationary value but the converse may not be true.

(3) Working rule to find the maximum and minimum values of f(x, y).

1. Find $\partial f/\partial x$ *and* $\partial f/\partial y$ *and equate each to zero. Solve these as simultaneous equations in* x *and* y. Let $(a, b), (c, d), \ldots\ldots$ be the pairs of values.

2. *Calculate the value of* $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$, $t = \partial^2 f/\partial y^2$ *for each pair of values.*

3. (*i*) *If* $rt - s^2 > 0$ *and* $r < 0$ *at* (a, b), $f(a, b)$ *is a max. value.*

(*ii*) *If* $rt - s^2 > 0$ and $r > 0$ *at* (a, b), $f(a, b)$ *is a min. value.*

(*iii*) *If* $rt - s^2 < 0$ *at* (a, b), $f(a, b)$ *is not an extreme value, i.e.* (a, b) *is a saddle point.*

(*iv*) *If* $rt - s^2 = 0$ *at* (a, b), *the case is doubtful and needs further investigation.*

Similarly examine the other pairs of values one by one.

Example 5·27. *Examine the following function for extreme values :*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2. \qquad \text{(Delhi, 1997)}$$

We have $\qquad f_x = 4x^3 - 4x + 4y \; ; \; f_y = 4y^3 + 4x - 4y$,

and $\qquad r = f_{xx} = 12x^2 - 4, \; s = f_{xy} = 4, \; t = f_{yy} = 12y^2 - 4 \qquad ...(i)$

Now $f_x = 0$, $f_y = 0$ give $x^3 - x + y = 0, \quad ...(i) \qquad y^3 + x - y = 0 \qquad ...(ii)$

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Putting $y = -x$ in (*i*), we obtain $x^3 - 2x = 0$, *i.e.* $x = \sqrt{2}, -\sqrt{2}, 0$.

∴ Corresponding values of y are $-\sqrt{2}, \sqrt{2}, 0$.

At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +$ve and r is also + ve. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

At $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and r are + ve.

Hence $f(-\sqrt{2}, \sqrt{2})$ is also a minimum value.

At $(0, 0)$, $rt - s^2 = 0$ and, therefore, further investigation is needed.

Now $f(0, 0) = 0$ and for points along the x-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin.

Again for points along the line $y = x$, $f(x, y) = 2x^4$ which is positive.

Thus in the neighbourhood of $(0, 0)$ there are points where $f(x, y) < f(0, 0)$ and there are points where $f(x, y) > f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value.

Example 5·28. *Discuss the maxima and minima of* $f(x, y) = x^3 y^2 (1 - x - y)$.

(*Gauhati, 1999; Ranchi, 1998*)

We have $f_x = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 \; ; \; f_y = 2x^3 y - 2x^4 y - 3x^3 y^2$

and $\qquad r = f_{xx} = 6xy^2 - 12x^2 y^2 - 6xy^3 \; ; \; s = f_{xy} = 6x^2 y - 8x^3 y - 9x^2 y^2 ; \; t = f_{yy} = 2x^3 - 2x^4 - 6x^3 y.$

When $f_x = 0$, $f_y = 0$, we have $x^2 y^2 (3 - 4x - 3y) = 0$, $x^3 y (2 - 2x - 3y) = 0$

Solving these, the stationary points are $(1/2, 1/3)$, $(0, 0)$.

Now $rt - s^2 = x^4 y^2 [12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

At $(1/2, 1/3)$, $rt - s^2 = \dfrac{1}{16} \cdot \dfrac{1}{9} \left[12\left(1 - 1 - \dfrac{1}{3}\right)\left(1 - \dfrac{1}{2} - 1\right) - (6 - 4 - 3)^2 \right] = \dfrac{1}{14} > 0$

Also $r = 6\left(\dfrac{1}{2} \cdot \dfrac{1}{9} - \dfrac{2}{4} \cdot \dfrac{1}{9} - \dfrac{1}{2} \cdot \dfrac{1}{27}\right) = -\dfrac{1}{9} < 0$

Hence $f(x, y)$ has a maximum at $(1/2, 1/3)$ and maximum value $= \dfrac{1}{8} \cdot \dfrac{1}{9}\left(1 - \dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{1}{432}$.

At $(0, 0)$, $rt - s^2 = 0$ and therefore further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5 (1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$ *i.e.* in the neighbourhood of $(0, 0)$, there are points where $f(x, y) > f(0, 0)$ and there are points where $f(x, y) < f(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

Example 5·29. *In a plane triangle, find the maximum value of* $\cos A \cos B \cos C$. (*Raipur, 1998*)

We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$\therefore \cos A \cos B \cos C = \cos A \cos B \cos [\pi - (A + B)]$

$= -\cos A \cos B \cos (A + B) = f(A, B)$, say.

We get $\frac{\partial f}{\partial A} = \cos B [\sin A \cos (A + B) + \cos A \sin (A + B)]$

$= \cos B \sin (2A + B)$

and $\frac{\partial f}{\partial B} = \cos A \sin (A + 2B)$

$\frac{\partial f}{\partial A} = 0, \frac{\partial f}{\partial B} = 0$ only when $A = B = \pi/3$.

Also $r = \frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos (2A + B), t = \frac{\partial^2 f}{\partial B^2} = 2 \cos A \cos (A + 2B)$

$s = \frac{\partial^2 f}{\partial A \partial B} = -\sin B \sin (2A + B) + \cos B \cos (2A + B) = \cos (2A + 2B)$

When $A = B = \pi/3, r = -1, s = -\frac{1}{2}, t = -1$ so that $rt - s^2 = 3/4$.

These show that $f(A, B)$ is maximum for $A = B = \pi/3$.

Then $C = \pi - (A + B) = \pi/3$.

Hence $\cos A \cos B \cos C$ is maximum when each of the angles is $\pi/3$ *i.e.* triangle is equilateral and its maximum value = 1/8.

5·13. LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method* proves very convenient. Now we explain this method.

Let $u = f(x, y, z)$ ...(1)

be a function of three variables x, y, z which are connected by the relation.

$$\phi(x, y, z) = 0 \quad ...(2)$$

For u to have stationary values, it is necessary that

$$\partial u/\partial x = 0, \partial u/\partial y = 0, \partial u/\partial z = 0.$$

$$\therefore \quad \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0 \quad ...(3)$$

Also differentiating (2), we get $\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi = 0$ ...(4)

Multiply (4) by a parameter λ and add to (3). Then

$$\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

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* See footnote page 102.

This equation will be satisfied if $\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0, \frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0, \frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0.$

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working rule : 1. *Write* $F = f(x, y, z) + \lambda\phi(x, y, z)$

2. *Obtain the equations* $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0.$

3. *Solve the above equations together with* $\phi(x, y, z) = 0$.

The values of x, y, z so obtained will give the stationary value of $f(x, y, z)$.

Obs. Although the Lagrange's method is often very useful in application yet the drawback is that we cannot determine the nature of the stationary point. This can sometimes, be determined from physical considerations of the problem.

Example 5·30. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (*J.N.T.U.*, *1998*)

Let x, y and z ft. be the edges of the box and S be its surface.

Then $$S = xy + 2yz + 2zx \quad ...(i)$$

and $$xyz = 32 \quad ...(ii)$$

Eliminating z from (*i*) with the help of (*ii*), we get $S = xy + 2(y+x)\frac{32}{xy} = xy + 64\left(\frac{1}{x}+\frac{1}{y}\right)$

$\therefore \partial S/\partial x = y - 64/x^2 = 0$ and $\partial S/\partial y = x - 64/y^2 = 0$.

Solving these, we get $x = y = 4$.

Now $r = \partial^2 S/\partial x^2 = 128/x^3$, $s = \partial^2 S/\partial x\partial y = 1$, $t = \partial^2 S/\partial y^2 = 128/y^3$.

At $x = y = 4$, $rt - s^2 = 2 \times 2 - 1 = +$ ve and r is also + ve.

Hence S is minimum for $x = y = 4$. Then from (*ii*), $z = 2$.

Otherwise (*by Lagrange's method*) :

Write $$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

Then $$\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \quad ...(iii)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \quad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \quad ...(v)$$

Multiplying (*iii*) by x and (*iv*) by y and subtracting, we get $2zx - 2zy = 0$ or $x = y$.

[The value $z = 0$ is neglected, as it will not satisfy (*ii*)]

Again multiplying (*iv*) be y and (*v*) be z and subtracting, we get $y = 2z$.

Hence the dimensions of the box are $x = y = 2z = 4$ ...(*vi*)

Now let us see what happens as z increases from a small value to a large one. When z is small, the box is flat with a large base showing that S is large. As z increases, the base of the box decreases rapidly and S also decreases. After a certain stage, S again starts increasing as z increases. Thus S must be a minimum at some intermediate stage which is given by (*vi*). Hence S is minimum when $x = y = 4$ ft and $z = 2$ ft.

Example 5·31. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere* $x^2 + y^2 + z^2 = 1$. (*Kanpur*, *1996*)

Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x, y, z), \text{ say} \qquad ...(i)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \qquad ...(ii)$$

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

Then $\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x, \ \frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y, \ \frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$

$\therefore$ $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$ give

$$x - 3 + \lambda x = 0, \ y - 4 + \lambda y = 0, \ z - 12 + \lambda z = 0 \qquad ...(iii)$$

which give $\lambda = -\frac{x-3}{x} = -\frac{y-4}{y} = -\frac{z-12}{z}$

$$= \pm\frac{\sqrt{[(x-3)^2 + (y-4)^2 + (z-12)^2]}}{\sqrt{(x^2+y^2+z^2)}} = \pm\frac{\sqrt{f}}{1}$$

Substituting for λ in (iii), we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}}, \ y = \frac{4}{1 \pm \sqrt{f}}, \ z = \frac{12}{1 \pm \sqrt{f}}$$

$$\therefore \quad x^2 + y^2 + z^2 = \frac{9 + 16 + 144}{(1 \pm \sqrt{f})^2} = \frac{169}{(1 \pm \sqrt{f})^2}$$

Using (ii), $1 = \frac{169}{(1 \pm \sqrt{f})^2}$ or $1 \pm \sqrt{f} = \pm 13, \sqrt{f} = 12, 14.$

[We have left out the negative values of $\sqrt{f}$, because $\sqrt{f} = AP$ is +ve by (i)]

Hence maximum $AP = 14$ and minimum $AP = 12$.

Example 5.32. *A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.*

Let V be the volume enclosed by the tent and S be its surface area.

Then V = Cuboid $(ABCD, A'B'C'D')$ + Pyramid $(K, A'B'C'D')$

$$= x^2 y + \frac{1}{3}x^2 h = x^2(y + h/3)$$

$$S = 4(ABGF) + 4\Delta KGH = 4xy + 4\cdot\frac{1}{2}(x \cdot KM)$$

$$= 4xy + x\sqrt{(x^2 + 4h^2)} \quad [\because KM = \sqrt{(KL^2 + LM^2)} = \sqrt{[h^2 + (x/2)^2]}]$$

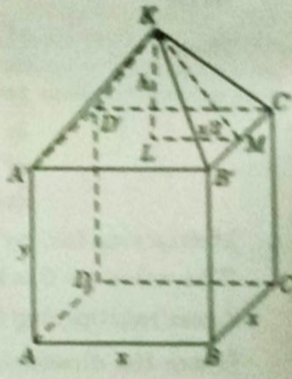

Fig. 5.4

For constant V, we have

$$\delta V = 2x(y + h/3)\,\delta x + x^2(\delta y) + \frac{x^2}{3}\,\delta h = 0$$

For minimum S, we have

$$\delta S = [4y + \sqrt{(x^2 + 4h^2)} + x\cdot\frac{1}{2}(x^2 + 4h^2)^{-1/2}\cdot 2x]\,\delta x + 4x\delta y + x\cdot\frac{1}{2}(x^2 + 4h^2)^{-1/2}\cdot 8h\delta h = 0$$

By Lagrange's method,

$$[4y + \sqrt{(x^2 + 4h^2)} + x^2(x^2 + 4h^2)^{-1/2}] + \lambda\cdot 2x(y + h/3) = 0 \qquad ...(i)$$

$$4x + \lambda\cdot x^2 = 0 \qquad ...(ii)$$

$$4hx(x^2 + 4h^2)^{-1/2} + \lambda\cdot x^2/3 = 0 \qquad ...(iii)$$

(ii) gives $\lambda = -4/x$. Then (iii) becomes

$$4hx(x^2 + 4h^2)^{-1/2} - 4x/3 = 0 \quad \text{or} \quad x = \sqrt{5}h$$

CHAPTER

39

INFINITE SERIES

39.1 SEQUENCE

A *sequence* is a succession of numbers or terms formed according to some definite rule. The nth term in a sequence is denoted by u_n.

For example, if $u_n = 2n + 1$.

By giving different values of n in u_n, we get different terms of the sequence.

Thus, $u_1 = 3,\ u_2 = 5,\ u_3 = 7, \ldots$

A sequence having unlimited number of terms is known as an *infinite sequence*.

39.2 LIMIT

If a sequence tends to a limit l, then we write $\lim\limits_{n \to \infty} (u_n) = l$

39.3 CONVERGENT SEQUENCE

If the limit of a sequence is finite, the sequence is convergent. If the limit of a sequence does not tend to a finite number, the sequence is said to be *divergent*.

e.g., $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2} + \ldots$. is a convergent sequence.

$3, 5, 7, \ldots, (2n + 1), \ldots$ is a divergent sequence.

39.4 BOUNDED SEQUENCE

$u_1, u_2, u_3 \ldots, u_n \ldots$ is a bounded sequence if $u_n < k$ for every n.

39.5 MONOTONIC SEQUENCE

The sequence is either increasing or decreasing, such sequences are called *monotonic*.

e.g., $1, 4, 7, 10, \ldots$ is a monotonic sequence.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is also a monotonic sequenece.

$1, -1, 1, -1, 1, \ldots$ is not a monotonic sequence.

A sequence which is monotonic and bounded is a convergent sequence.

EXERCISE 39.1

Determine the general term of each of the following sequence. Prove that the following sequences are convergent.

1. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ **Ans.** $\frac{1}{2^n}$

2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **Ans.** $\frac{n}{n+1}$

3. $1, -1, 1, -1, \ldots$ **Ans.** $(-1)^{n-1}$

4. $\frac{1^2}{1!}, \frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \frac{5^2}{5!}, \ldots$ **Ans.** $\frac{n^2}{n!}$

Which of the following sequences are convergent ?

5. $u_n = \frac{n+1}{n}$ **Ans.** Convergent

6. $u_n = 3n$ **Ans.** Divergent

7. $u_n = n^2$ **Ans.** Divergent

8. $u_n = \frac{1}{n}$ **Ans.** Convergent

39.6 REMEMBER THE FOLLOWING LIMITS

(*i*) $\lim_{n \to \infty} x^n = 0$ if $x < 1$ and $\lim_{n \to \infty} x^n = \infty$ if $x > 1$

(*ii*) $\lim_{n \to \infty} \frac{x^n}{n!} = 0$ for all values of x

(*iii*) $\lim_{n \to \infty} \frac{\log n}{n} = 0$

(*iv*) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

(*v*) $\lim_{n \to \infty} (n)^{1/n} = 1$

(*vi*) $\lim_{n \to \infty} [n!]^{1/n} = \infty$

(*vii*) $\lim_{n \to \infty} \left[\frac{(n!)}{n}\right]^{1/n} = \frac{1}{e}$

(*viii*) $\lim_{n \to \infty} n\, x^n = 0$ if $x < 1$

(*ix*) $\lim_{n \to \infty} n^h = \infty$

(*x*) $\lim_{n \to \infty} \frac{1}{n^k} = 0$

(*xi*) $\lim_{x \to \infty} \left[\frac{a^x - 1}{x}\right] = \log a$ or $\lim_{n \to \infty} \frac{a^{1/n} - 1}{1/n} = \log a$

(*xii*) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(*xiii*) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

39.7 SERIES

A *series* is the sum of a sequence.

Let $u_1, u_2, u_3, \ldots\ldots\ldots, u_n, \ldots\ldots\ldots$ be a given sequence. Then, the expression $u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n + \ldots\ldots\ldots$ is called the series associated with the given sequence.

For example, $1 + 3 + 5 + 7 + \ldots$ is a series.

If the number of terms of a series is limited, the series is called *finite*. When the number of terms of a series are unlimited, it is called an *infinite series*.

$$u_1 + u_2 + u_3 + u_4 + \ldots + u_n + \ldots \infty$$

is called an infinite series and it is denoted by $\sum_{n=1}^{\infty} u_n$ or $\Sigma\, u_n$. The sum of the first n terms of a series is denoted by S_n.

39.8 CONVERGENT, DIVERGENT AND OSCILLATORY SERIES

Consider the infinite series $\Sigma\, u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Three cases arise:

(*i*) If S_n tends to a finite number as $n \to \infty$, the series $\Sigma\, u_n$ is said to be *convergent*.

(*ii*) If S_n tends to infinity as $n \to \infty$, the series $\Sigma\, u_n$ is said to be *divergent*.

(*iii*) If S_n does not tend to a unique limit, finite or infinite, the series $\Sigma\, u_n$ is called *oscillatory*.

39.9 PROPERTIES OF INFINITE SERIES

1. The nature of an infinite series does not change:
   (i) by multiplication of all terms by a constant k.
   (ii) by addition or deletion of a finite number of terms.
2. If two series Σu_n and Σv_n are convergent, then $\Sigma (u_n + v_n)$ is also convergent.

Example 1. *Examine the nature of the series* $1 + 2 + 3 + 4 + \ldots + n + \ldots \infty$.

Solution. Let $S_n = 1 + 2 + 3 + 4 + \ldots + n = \dfrac{n(n+1)}{2}$ [Series in A.P.]

Since $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \dfrac{n(n+1)}{2} \Rightarrow \infty$

Hence, this series is divergent. **Ans.**

Example 2. *Test the convergence of the series* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$

Solution. Let $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$ [Series in G.P.]

$$= \frac{1}{1 - \frac{1}{2}} = 2 \qquad \left(S_n = \frac{a}{1-r}\right)$$

$$\lim_{n \to \infty} S_n = 2$$

Hence, the series is convergent. **Ans.**

Example 3. *Prove that the following series:*

$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots$ *is convergent and find its sum.* (M.U. 2008)

Solution. Here,
$$u_n = \frac{n+1}{(n+2)!} = \frac{n+2-1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{1}{(n+2)!}$$
$$= \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$
$$S_n = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots$$
$$+ \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right) = \frac{1}{2!} - \frac{1}{(n+2)!}$$
$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{1}{2!} - \frac{1}{(n+2)!}\right] = \frac{1}{2}$$

$\therefore$ Σu_n converges and its limit is $\frac{1}{2}$. **Ans.**

Example 4. *Discuss the nature of the series* $2 - 2 + 2 - 2 + 2 - \ldots \infty$.

Solution. Let $S_n = 2 - 2 + 2 - 2 + 2 - \ldots \infty$
$= 0$ if n is even
$= 2$ if n is odd.

Hence, S_n does not tend to a unique limit, and, therefore, the given series is oscillatory. **Ans.**

Properties of geometric series.

The series $1 + r + r^2 + r^3 + \ldots \infty$ is
i) Convergent if $|r| < 1$ ii) divergent if $r \geq 1$
iii) Oscillatory if $r \leq -1$.

necessary conditions for convergent series.

For every convergent series Σu_n. $\lim_{n\to\infty} u_n = 0$

Cauchy's fundamental test for divergence

If $\lim_{n\to\infty} u_n \neq 0$, the series is divergent.

eg: Test the convergence of the series $1+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots+\frac{n}{n+1}+\cdots\infty$.

ans. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0$.

hence by the Cauchy's fundamental test for divergence, the series is divergent.

EXERCISE 39.4

Examine for convergence:

1. $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{4}{\sqrt{17}}+\ldots+\frac{2^n}{\sqrt{4^n+1}}+\ldots\infty$ **Ans.** Divergent
2. $\sum_{n=1}^{\infty}\frac{n}{n+1}$ **Ans.** Divergent
3. $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}$ **Ans.** Divergent
4. $\sum \cos\frac{1}{n}$ **Ans.** Divergent
5. $1+\frac{1}{2}+2+\frac{1}{3}+3+\frac{1}{4}+4+\ldots$ **Ans.** Divergent
6. $\Sigma\,(6-n^2)$ **Ans.** Divergent
7. $\Sigma\,(-2^n)$ **Ans.** Divergent
8. $\Sigma\,3^{n+1}$ **Ans.** Divergent

39.14 *p*-SERIES

The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\infty$ *is* (*i*) *convergent if* $p > 1$ (*ii*) *Divergent if* $p \le 1$.

(*MDU, Dec. 2010*)

Solution. Case 1: $(p > 1)$

The given series can be grouped as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\frac{1}{11^p}+\frac{1}{12^p}+\frac{1}{13^p}+\frac{1}{14^p}+\frac{1}{15^p}\right)+\ldots$$

Now

$$\frac{1}{1^p} = 1 \qquad \ldots(1)$$

$$\frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p} = \frac{2}{2^p} \qquad \ldots(2)$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} < \frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} = \frac{4}{4^p} \qquad \ldots(3)$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p} < \frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p} = \frac{8}{8^p} \qquad \ldots(4)$$

On adding (1), (2), (3) and (4), we get:

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots$$

$$< \frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots$$

$$< 1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{2}\right)^{2p-2}+\left(\frac{1}{2}\right)^{3p-3}+\ldots$$

$$< \frac{1}{1-\left(\frac{1}{2}\right)^{p-1}} \qquad \left[\text{G.P., } r = \left(\frac{1}{2}\right)^{p-1},\ S = \frac{1}{1-r}\right]$$

$<$ Finite number if $p > 1$

Hence, the given series is convergent when $p > 1$.

Case 2: $p = 1$

When $p = 1$, the given series becomes

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \dots$$

$$1 + \frac{1}{2} = 1 + \frac{1}{2} \qquad \dots(1)$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \qquad \dots(2)$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2} \qquad \dots(3)$$

$$\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16} = \frac{8}{16} = \frac{1}{2} \qquad \dots(4)$$

On adding (1), (2), (3) and (4), we get

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \dots$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$$> 1 + \frac{n}{2} \quad (n \to \infty)$$

$$> \infty$$

Hence, the given series is divergent when $p = 1$.

Case 3: $p < 1$

$$\frac{1}{2^p} > \frac{1}{2}, \qquad \frac{1}{3^p} > \frac{1}{3}, \qquad \frac{1}{4^p} > \frac{1}{4} \text{ and so on}$$

$$\text{Therefore, } \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$> \text{divergent series } (p = 1)$ [From Case 2]

$$\left[\text{As the series on R.H.S. } \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) \text{ is divergent}\right]$$

Hence, the given series is divergent when $p < 1$.

39.15 COMPARISON TEST

If two positive terms Σu_n and Σv_n be such that

$\lim_{n \to \infty} \frac{u_n}{v_n} = k$ **(finite number), then both series converge or diverge together.**

Proof. By definition of limit there exists a positive number ε, however small, such that

$$\left|\frac{u_n}{v_n} - k\right| < \varepsilon \text{ for } n > m \qquad i.e., \; -\varepsilon < \frac{u_n}{v_n} - k < +\varepsilon$$

$$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \text{ for } n > m$$

Ignoring the first m terms of both series, we have

$$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \text{ for all } n. \quad ...(1)$$

Case 1. Σv_n is convergent, then

$$\lim_{n \to \infty} (v_1 + v_2 + ... v_n) = h \text{ (say)} \qquad \text{where } h \text{ is a finite number.}$$

From (1), $u_n < (k + \varepsilon)\, v_n$ for all n.

$$\lim_{n \to \infty} (u_1 + u_2 + ... + u_n) < (k + \varepsilon) \lim_{n \to \infty} (v_1 + v_2 + ... + v_n) = (k + \varepsilon)h$$

Hence, Σu_n is also convergent.

Case 2. Σv_n is divergent, then

$$\lim_{n \to \infty} (v_1 + v_2 + ... + v_n) \to \infty \quad ...(2)$$

Now from (1) $\qquad k - \varepsilon < \dfrac{u_n}{v_n}$

$$u_n > (k - \varepsilon)v_n \text{ for all } n$$

$$\lim_{n \to \infty} (u_1 + u_2 + ... + u_n) > (k - \varepsilon) \lim_{n \to \infty} (v_1 + v_2 + ... + v_n)$$

From (2), $\lim_{n \to \infty} (u_1 + u_2 + ... + u_n) \to \infty$

Hence, Σu_n is also divergent.

Note. For testing the convergence of a series, this Comparison Test is very useful. We choose Σv_n (p-series) in such a way that

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \text{finite number.}$$

Then the nature of both the series is the same. The nature of Σv_n (p-series) is already known, so the nature of Σu_n is also known.

Example 8. *Test the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+10}$ *for convergence or divergence.*

Solution. Here, $\qquad u_n = \dfrac{1}{n+10}$

Let $\qquad v_n = \dfrac{1}{n}$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n}{n+10} = \lim_{n \to \infty} \frac{1}{1 + \frac{10}{n}} = 1 = \text{finite number.}$$

According to Comparison Test both series converge or diverge together, but Σv_n is divergent as $p = 1$.

$\therefore \quad \Sigma u_n$ is also divergent. **Ans.**

Example 9. *Test the convergence of the following series:*

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots \qquad \textit{(M.D.U., 2000)}$$

Solution. Here, we have

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{\sqrt{1+0}-0}{(1-0)^3-0} = 1$$

Which is finite and non-zero.

$\therefore \sum u_n$ and $\sum v_n$, converge or diverge together since $\sum v_n = \sum \frac{1}{n^{\frac{5}{2}}}$ is of the form

$\sum \frac{1}{n^p}$. where $P = \frac{5}{2} > 1$.

$\therefore \sum v_n$ is convergent $\Rightarrow \sum u_n$ is convergent. **Ans.**

Example 13. *Test the convergence and divergence of the following series.*

$$\sum_{n=1}^{\infty}\frac{2n^2+3n}{5+n^5}$$ *(Gujarat, I Semester, Jan. 2009)*

Solution. Here, $u_n = \frac{2n^2+3n}{5+n^5} = \frac{n^2\left(2+\frac{3}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\,\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1}$

Let $v_n = \frac{1}{n^3}$

By Comparison Test

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^3\left(2+\frac{3}{n}\right)}{n^3\left(\frac{5}{n^5}+1\right)} = \lim_{n\to\infty}\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1} = 2 = \text{Finite number.}$$

According to comparison test both series converge or diverge together but Σv_n is convergent as $p = 2$.

Hence, the given series is convergent. **Ans.**

Example 14. *Test the following series for convergence* $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

Solution. Given series is $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

Here $u_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$

Let $v_n = \frac{1}{n^{p-1}} \quad \therefore \quad \frac{u_n}{v_n} = \frac{1+\frac{1}{n}}{n^{p-1}} \times \frac{n^{p-1}}{1} = 1+\frac{1}{n}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = 1$$

Therefore, both the series are either convergent or divergent.

But Σv_n is convergent if $p - 1 > 1$, *i.e.*, if $p > 2$ (P series)

and is divergent if $p - 1 \le 1$, *i.e.*, if $p \le 2$

$\therefore$ The given series is convergent if $p > 2$ and divergent if $p \le 2$. **Ans.**

$\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\ldots$

$u_n = \frac{1}{(2n+1)^p} = \frac{1}{n^p\left(2+\frac{1}{n}\right)^p}$

$v_n = \frac{1}{n^p}$

$\frac{u_n}{v_n} = \frac{n^p}{n^p(2+\frac{1}{n})^p} = \frac{1}{2^p}$

$\lim_{n\to\infty}\frac{u_n}{v_n}$

Σv_n is convergent if $p>1$. & divergent if $p\le 1$.

$\therefore$ The given series is convergent if $p>1$ & divergent if $p\le 1$

EXERCISE 39.5

Examine the convergence or divergence of the following series:

1. $2 + \frac{3}{2}\cdot\frac{1}{4} + \frac{4}{3}\cdot\frac{1}{4^2} + \frac{5}{4}\cdot\frac{1}{4^3} + \ldots \infty$ **Ans.** Convergent

2. $1 + \frac{1.2}{1.3} + \frac{1.2.3}{1.3.5} + \frac{1.2.3.4}{1.3.5.7} + \ldots \infty$ **Ans.** Convergent

3. $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \ldots \infty$ **Ans.** Divergent

4. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots\ldots \infty$ **Ans.** Convergent *(M.D. University, Dec. 2004)*

5. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots \infty$ **Ans.** Convergent

6. $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \ldots$ **Ans.** Convergent *(M.D. University, 2001)*

7. $\frac{1}{3} + \frac{2!}{3^2} + \frac{3!}{3^3} + \ldots \infty$ **Ans.** Convergent

8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$ **Ans.** Divergent

9. $\sum_{n=1}^{\infty} \frac{2n^3 + 5}{4n^5 + 1}$ **Ans.** Convergent

10. $\sum_{n=1}^{\infty} \frac{a^n}{x^n + n^a}$ **Ans.** If $x > a$, convergent; if $x \le a$, Divergent

11. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ **Ans.** Convergent

12. $\sum_{n=1}^{\infty} \sqrt{(n^2+1)} - n$ **Ans.** Divergent

13. $\sum_{n=1}^{\infty} \left[\sqrt{(n^4+1)} - \sqrt{(n^4-1)}\right]$ **Ans.** Convergent

14. $\sum_{n=1}^{\infty} \frac{2^n + 1}{3^n + n}$ **Ans.** Convergent

15. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **Ans.** Convergent

16. $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ **Ans.** Convergent

39.16 D'ALEMBERT'S RATIO TEST

Statement. ***If Σu_n is a positive term series such that $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = k$ then***

(i) the series is convergent if $k < 1$ (ii) the series is divergent if $k > 1$

Solution. k = 1, ratio test fails.

Case I. When $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = k < 1$

By definition of a limit, we can find a number r (< 1) such that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n \ge m \qquad \left[\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r \ldots\right]$$

Omitting the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \ldots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots\right) = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots \infty\right)$$

$$< u_1 (1 + r + r^2 + r^3 + \ldots \infty) \qquad (r < 1)$$

$$= \frac{u_1}{1-r}, \text{ which is a finite quantity.}$$

Hence, Σu_n is convergent.

Case 2. When $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = k > 1$

By definition of limit, we can find a number m such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$

$$\frac{u_2}{u_1} > 1, \quad \frac{u_3}{u_2} > 1, \quad \frac{u_4}{u_3} > 1$$

Ignoring the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \ldots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots\right) = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots \infty\right)$$

$$\geq u_1 (1 + 1 + 1 + 1 \ldots \text{ to } n \text{ terms}) = nu_1$$

$$\left[\because \lim_{n\to\infty} (u_1 + u_2 + \ldots + u_n) = nu_1\right]$$

$$\lim_{n\to\infty} S_n \geq \lim_{n\to\infty} nu_1 = \infty$$

Hence, Σu_n is divergent.

Note. When $\frac{u_{n+1}}{u_n} = 1 \quad (k = 1)$

The ratio test fails.

For Example. Consider the series whose n^{th} term $= \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \quad \ldots(1)$$

Consider the second series whose n^{th} term is $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1 \quad \ldots(2)$$

Thus, from (1) and (2) in both cases $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

But we know that the first series is divergent as $p = 1$.
The second series is convergent as $p = 2$.

Hence, when $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, the series may be convergent or divergent.

Thus, ratio test fails when $k = 1$.

Example 15. *Test for convergence of the series whose n^{th} term is* $\frac{n^2}{2^n}$.

Solution. Here, we have $u_n = \frac{n^2}{2^n}, \quad u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

By D'Alembert's Test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

Hence, the series is convergent by D'Alembert's Ratio Test. **Ans.**

Example 16. *Test for convergence the series whose n^{th} term is* $\frac{2^n}{n^3}$.

Solution. Here, we have $u_n = \frac{2^n}{n^3}$, $\quad u_{n+1} = \frac{2^{n+1}}{(n+1)^3}$

By D'Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} = \frac{2}{\left(1+\frac{1}{n}\right)^3} \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^3} = 2 > 1$$

Hence, the series is divergent. **Ans.**

Example 17. *Discuss the convergence of the series:*

$$\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n \quad (x > 0)$$

(M.D.University, Dec., 2001)

Solution. Here, we have

$$u_n = \sqrt{\frac{n}{n^2+1}}\, x^n$$

$$\therefore \quad u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}}\, .x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1}} \cdot \sqrt{\frac{n^2+2n+2}{n^2+1}} \cdot \frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)}} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)}} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$, *i.e.* $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, the Ratio Test fails.

When $x = 1$, $u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{1}{n^2}}}$

$$v_n = \frac{1}{\sqrt{n}},$$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{1}{n^2}}} \cdot \frac{\sqrt{n}}{1} = \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

Which is finite and non-zero.

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$.

$\sum v_n$ diverges $\Rightarrow \sum u_n$ diverges.

Hence, the given series $\sum u_n$ converges is $x < 1$ and diverges if $x \geq 1$. **Ans.**

EXERCISE 39.6

Test the convergence for series:

1. $\sum_{n=1}^{n} \frac{n^2}{3^n}$ **Ans.** Convergent

2. $\sum_{n=1}^{n} \frac{n!}{n^n}$ **Ans.** Convergent

3. $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \dots \infty$ **Ans.** Convergent

4. $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \dots \infty$ **Ans.** Convergent

5. $\sum_{n=1}^{n} \frac{n!.2^n}{n^n}$ **Ans.** Convergent

6. $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n.3^n}$ **Ans.** Convergent if $x > 3$, Divergent if $x < 3$

7. Prove that, if $u_{n+1} = \frac{k}{1+u_n}$, where $k > 0$, $u_1 > 0$, then the series Σu_n converges to the positive root of the equation $x^2 + x = k$.

39.17 RAABE'S TEST (HIGHER RATIO TEST)

> *If Σu_n is a positive term series such that* $\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, *then*
> *(i) the series is convergent if $k > 1$* *(ii) the series is divergent if $k < 1$.*

Proof. Case I. $k > 1$

Let p be such that $k > p > 1$ and compare the given series Σu_n with $\sum \frac{1}{n^p}$ which is convergent as $p > 1$.

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \quad \text{or} \quad \left(\frac{u_n}{u_{n+1}}\right) > \left(1 + \frac{1}{n}\right)^p > 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \dots$$

(Binomial Theorem)

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!}\frac{1}{n} + \dots.$$

If $$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

and $k > p$ which is true as $k > p > 1$; Σu_n is convergent when $k > 1$.

Case II. $k < 1$ Same steps as in Case I.

Notes:

1. Raabe's Test fails if $k = 1$
2. Raabe's Test is applied only when D'Alembert's Ratio Test fails.

CHAPTER 6

FOURIER SERIES

Definition : A function $f(x)$ is said to be periodic if and only if $f(x+p)=f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x is called *the period of the function*.

For example, for any integer n, $\sin(x+2n\pi)=\sin x$ for all x. Therefore, $\sin x$ is periodic. For $n=1$, $\sin(x+2\pi)=\sin x$. There is no positive number 'a' which is less than 2π such that $\sin(a+x)=\sin x$ for all x. Therefore, 2π is the period of $\sin x$. Similarly, 2π is the period for $\cos x$. But $\tan(\pi+x)=\tan x$ and π is the least positive value such that $\tan(\pi+x)=\tan x$.

So, $\tan x$ is periodic of period π.

Sin nx, cos nx are periodic functions of period $\frac{2\pi}{n}$.

Standard Results in integrals. If m, n are integers,

1. If $n \neq 0$, $\int_c^{c+2\pi} \sin nx\,dx = 0 \therefore \int_0^{2\pi} \sin nx\,dx = 0$

   If $n=0$ $\int_c^{c+2\pi} \sin nx\,dx = \int_c^{c+2\pi} 0\,dx = 0$

2. If $n \neq 0$, $\int_c^{c+2\pi} \cos nx\,dx = 0 \therefore \int_0^{2\pi} \cos nx\,dx = 0$

3. $\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0 \therefore \int_0^{2\pi} \sin mx \cos nx\,dx = 0$

4. If $m \neq n$. $\int_c^{c+2\pi} \sin mx \sin nx\,dx = 0 \therefore \int_0^{2\pi} \sin mx \sin nx\,dx = 0$ if $m \neq n$

5. If $n \neq 0$, $\int_c^{c+2\pi} \sin^2 nx\,dx = \pi \therefore \int_0^{2\pi} \sin^2 nx\,dx = \pi$

6. If $n \neq 0$, $\int_c^{c+2\pi} \cos^2 nx\,dx = \pi \therefore \int_0^{2\pi} \cos^2 nx\,dx = \pi$

7. If $n \neq m$, $\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0$

8. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx] + k$

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx] + k$$

9. **Bernoulli's generalised formula of integration by parts.**

Integration by parts : $\int u\,dv = uv - \int v\,du$.

We extend this result.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\ldots$$

where suffix denotes the integration and primes denote the differentiation.

Example. *Evaluate (i)* $\int (x^2+7x+5)\cos 3x\,dx$ *(ii)* $\int x^2 e^{2x}\,dx$

Here take u = polynomial $= x^2+7x+5$

and $v = \cos 3x$

(i)
$$\int uv\,dx = \int (x^2+7x+5)\cos 3x\,dx$$
$$= (x^2+7x+5)\left(\frac{\sin 3x}{3}\right) - (2x+7)\left(-\frac{\cos 3x}{9}\right) + (2)\left(-\frac{\sin 3x}{27}\right) + c.$$

(ii)
$$\int x^2 e^{2x}\,dx = (x^2)\left(\frac{e^{2x}}{2}\right) - (2x)\left(\frac{e^{2x}}{4}\right) + (2)\left(\frac{e^{2x}}{8}\right) + c.$$

Some results. If n is any integer, $\sin n\pi = 0$, $\cos n\pi = (-1)^n$.

Fourier series of $f(x)$:

If $f(x)$ is defined in $(0, 2\pi)$ and if $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then the R.H.S. series of sines and cosines is called the Fourier series of $f(x)$ in the interval $(0, 2\pi)$.

Theorem. If $f(x)$ is defined in $(0, 2\pi)$ and if $f(x)$ can be represented by the trigonometric series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \ldots(i)$$

then
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

then
$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Taking $c = 0$, we get previous results.
Taking $c = -\pi$, we get that in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Even and odd functions : In $(-\infty, \infty)$,

(i) If $f(-x) = f(x)$ for all x then $f(x)$ is even.

(ii) If $f(-x) = -f(x)$ for all x, then $f(x)$ is odd.

Also,

(i) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even.

(ii) $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd.

(iii) $\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx$

Example 1. *Find the Fourier series of periodicity* 2π *for* $f(x) = x^2$ *in* $(0, 2\pi)$.

Deduce $\sum_1^{\infty}\frac{1}{n^2} = \pi^2/6$.

Sol. Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad ...(1)$$

where
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$\therefore$
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi} = \frac{1}{3\pi}\left[8\pi^3 - 0\right] = \frac{8}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$
$$= \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{2(2\pi)}{n^2}\cos 2n\pi\right] \quad \text{since other terms vanish}$$
$$= \frac{4}{n^2} \quad \text{since } \cos 2n\pi = 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$
$$= \frac{1}{\pi}\left[(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3}(\cos 2n\pi - 1)\right]$$
$$= -\frac{4\pi}{n}$$

Substituting a_0, a_n, b_n values in (1)

$$\therefore \quad f(x) = x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$x = 0$ is an end point of the range.

Value of Fourier series at x = 0 is $\frac{f(0)+f(2\pi)}{2}$

$$\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{0^2+4\pi^2}{2}=2\pi^2$$

$$\sum\frac{4}{n^2} = 2\pi^2-\frac{4\pi^2}{3}=\frac{2\pi^2}{3}$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e., $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \pi^2/6.$

Example 2. *Find Fourier series of $f(x) = x$ in $(0, 2\pi)$ of periodicity*

Sol. Let $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}x\,dx=\frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi}=2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(1-1)\right]=0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\right]=-\frac{2}{n}$$

Substituting a_0, a_n, b_n in (1), we get

$$f(x) = x=\pi-2\sum_1^{\infty}\frac{1}{n}\sin nx.$$

Example 3. *Find Fourier series of* $f(x) = \frac{(\pi-x)^2}{4}$ *in* $(0, 2\pi)$ *of periodicity* 2π.

Sol. Let $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{(-3)}\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi}\left[-\pi^3-\pi^3\right]$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right)+2(\pi-x)\left(-\frac{\cos nx}{n^2}\right)+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{2}{n^2}\{+\pi+\pi\}\right]=\frac{1}{n^2}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)+2(\pi-x)\left(-\frac{\sin nx}{n^2}\right)+(2)\left(-\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^2}{n}+\frac{\pi^2}{n}\right]=0$$

$$\therefore \quad \frac{(\pi-x)^2}{4} = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

Example 4. *Expand* $f(x) = \frac{1}{2}(\pi-x)$ in $(0, 2\pi)$ *as a Fourier series of periodicity* 2π.

Deduce $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots=\frac{\pi}{4}$

Sol. Let $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ ...(1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{2\pi}$$

$$= \frac{1}{\pi}[0] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2}(1-1)\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] = \frac{1}{n}$$

$$\therefore \quad f(x) = \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

Put $x = \pi/2$. $f(x)$ is continuous at $x = \pi/2$

$$\therefore \quad \frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin\frac{4\pi}{2} + \ldots = \frac{\pi}{4}$$

i.e., $1 - \frac{1}{3} + \frac{1}{5}$ .... to $\infty = \frac{\pi}{4}$.

Example 5. *Obtain the Fourier series of periodicity* 2π *for* $f(x) = e^{-x}$ *in the interval* $0 < x < 2\pi$. *Hence deduce the value of* $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$.

Sol. Let $\quad f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad$ ...(1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left(-e^{-x}\right)_0^{2\pi}$$

$$= \frac{1}{\pi}(-e^{-2\pi} + 1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-1) - (-1)\right]$$

$$= \frac{1 - e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{-2\pi} + n\right]$$

$$= \frac{n}{\pi(1+n^2)}(1 - e^{-2\pi})$$

Substituting a_0, a_n, b_n in (1), we get

$$e^{-x} = \frac{(1 - e^{-2\pi})}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)\right] \quad \ldots(2)$$

In $(0, 2\pi)$, e^{-x} is continuous. Therefore, at $x = \pi$, the value of the Fourier series equals the value of the function. Hence replacing x by π in (2),

$$e^{-\pi} = \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\right]$$

$$\frac{\pi e^{-\pi}}{1 - e^{-2\pi}} = \frac{1}{2} + \left(-\frac{1}{2}\right) + \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore \quad \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{e^{\pi} - e^{-\pi}} = \frac{\pi}{2}\cdot\frac{2}{e^{\pi} - e^{-\pi}} = \frac{\pi}{2}\,\text{cosech}\,\pi$$

Find the half range cosine series for $f(x)=(\pi-x)^2$ in $(0,\pi)$. Hence deduce $\frac{1}{1^2}+\frac{1}{2^2}+ \cdots$

sol Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi-x)^2 dx = \frac{2}{\pi}\left(\frac{(\pi-x)^3}{-3}\right)_0^{\pi}$$

$$= -\frac{2}{3\pi}\left[(\pi-\pi)^3 - (\pi-0)^3\right] = \frac{2}{3\pi}\pi^3 = \frac{2}{3}\pi^2$$

$$= -\frac{2}{3\pi}\left[0-\pi^3\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left((\pi-x)^2\frac{\sin nx}{n}\right)_0^{\pi} + \int_0^{\pi}\frac{\sin nx}{n}2(\pi-x)dx\right]$$

Put $(\pi-x)^2 = u$

$2(\pi-x)(-dx) = du$

$\cos nx\,dx = dv$

$\frac{\sin nx}{n} = v$

$$= \frac{2}{\pi}\left[0-0+\frac{2}{n}\int_0^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$= \frac{4}{n\pi}\int_0^{\pi}(\pi-x)\sin nx\,dx$$

$$= \frac{4}{n\pi}\left[\left((\pi - x)\left(-\frac{\cos nx}{n}\right)\right)_0^{\pi} - \frac{1}{n}\int_0^{\pi} \cos nx \, dx\right]$$

$u = \pi - x$

$du = -dx$

$dv = \sin nx \, dx$

$v = -\frac{\cos nx}{n}$

$$= \frac{4}{n\pi}\left[\left(0 + \frac{\pi \cos 0}{n}\right) - \frac{1}{n}\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{4\pi}{n^2\pi} - \frac{4}{n^3\pi}(\sin n\pi - \sin 0)$$

$$= \frac{4\pi}{n^2\pi} = \frac{4}{n^2}$$

$$f(x) = \frac{1}{2} \cdot \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

$$(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

Put $x = 0$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$4\sum \frac{1}{n^2} = \pi^2 - \pi^2/3 = \frac{2}{3}\pi^2$$

$$4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{2}{3}\pi^2$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{2}{3 \times 4}\pi^2 = \frac{\pi^2}{6}$$

3. Find the half range sine series for $f(x) = k(l-x)$ in $(0, l)$.

Sol. The half range sine series of

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^l k(l-x) \sin\frac{n\pi x}{l}\,dx$$

Put $l - x = u$, $-dx = du$

$dv = \sin\frac{n\pi x}{l}\,dx$

$v = -\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}$

$$= k\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l \mp \frac{l}{n\pi}\int_0^l \cos\frac{n\pi x}{l}(-dx)\right]$$

$$= k\left[0 + \frac{l^2}{n\pi}\cos 0\right]$$

$$= \frac{kl^2}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{k l^2}{n\pi} \sin \frac{n\pi x}{l}$$

Questions	opt1	opt2	opt3	opt4	opt5
The sum of the main diagonal elements of a matrix is called------	trace of a matrix	quadratic form	eigen value	canonical form	
Every square matrix satisfies its own ----------	characteristic polynomial	characteristic equation	orthogonal transformati on	canonical form	
The orthogonal transformation used to diagonalise the symmetric matrix A is----	$N^T AN$	$X^T AX$	NAN^{-1}	NA	
If $\lambda_1, \lambda_2, \lambda_3,\dots\dots\dots \lambda_n$ are the eigen values of A ,then $k\lambda_1$,$k\lambda_2$, $k\lambda_3,\dots\dots\dots$,$k\lambda_n$ are the eigen values of --------------	kA	kA^2	kA^{-1}	A^{-1}	
Diagonalisation of a matrix by orthogonal reduction is true only for a ----- n	diagonal	triangular	real symmetric	scalar	
In a modal matrix, the columns are the -----------	eigen vectors of A	eigen vectors of adj A	eigen vectors of inverse ofA	eigen values of A	
If atleast one of the eigen values of A is zero, then det A = -----	0	1	10	5	
If the canonical form of a quadratic form is $5y_1^2 - 6y_2^2$, then the index is ---------	4	0	2	1	
det (A- λI) represents------	characteristic polynomial	characteristic equation	quadratic form	canonical form	
If $\lambda_1, \lambda_2, \lambda_3,\dots\dots\dots \lambda_n$ are the eigen values of A ,then $1/\lambda_1$,$1/\lambda_2$, $1/\lambda_3,\dots\dots\dots$,$1/\lambda_n$ are the eigen values of --------------	A^{-1}	A	A^n	A^p	
If $\lambda_1, \lambda_2, \lambda_3,\dots\dots\dots \lambda_n$ are the eigen values of A ,then λ_1^p , λ_2^p , ………. λ_n^p are the eigen values of	A^{-1}	A^2	A^{-p}	A^p	
Cayley -Hamilton theorem is used to find ------------	inverse and higher powers of A	eigen values	eigen vectors	quadratic form	
The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are ------------	linearly dependent	orthogonal	singular	not unique	

If all the eigen values of a matrix are distinct, then the corresponding eigen vectors are----------	linearly dependent	unique	not unique	linearly independent
The eigen values of a ---------------------- matrix are its diagonal elements	diagonal	symmetric	skew-matrix	triangular
In the orthogonal transformation $N^T AN = D$, D refers to a ---------- matrix.	diagonal	orthogonal	symmetric	skew-symmetric
In a modal matrix, the columns are the eigen vectors of----------	A^{-1}	A^2	A	adj A
If the eigen values of $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ are 0,3 & 15, then its nature is-----------	positive definite	positive semidefinite	indefinite	negative definite
The elements of the matrix of the quadratic form $x_1^2 + 3x_2^2 + 4x_1x_2$ are ------------	$a_{11} = 1, a_{12} = 2$, $a_{21} = 2$, $a_{22} = 3$	$a_{11} = -1$, $a_{12} = -2$, $a_{21} = 2$, $a_{22} = 3$	$a_{11} = 1$, $a_{12} = 4$, $a_{21} = 4$, $a_{22} = 3$	$a_{11} = 1$, $a_{12} = 4$, $a_{21} = 3$, $a_{22} = 1$
If the sum of two eigen values and trace of a 3x3 matrix A are equal, then det A = ---------	$\lambda_1 \lambda_2 \lambda_3$	0	1	2
If 1,5 are the eigen values of a matrix A, then det A = -------	5	0	25	6
If the canonical form of a quadratic form is $5y_1^2 + 6y_2^2$, then the rank is --------	4	0	2	1
The non –singular linear transformation used to transform the quadratic form to canonical form is ----------	$X = N^TY$	X= NY	Y= NX	NXA
The eigen vector is also known as-------	latent value	latent vector	column value	orthogonal value
If 1,3,7 are the eigen values of A, then the eigen values of 2A are ------------	1,3,7	1,9,21	2,6,14	1,9,49
If the eigen values of 2A are 2, 6, 8 then eigen values of A are __________	1,3,4	2,6,8	1,9,16	12,4,3
The number of positive terms in the canonical form is called the _________	rank	index	Signature	indefinite
If all the eigenvalues of A are positive then it is called as________	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite
If all the eigenvalues of A are negative then it is called as________	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite
A homogeneous polynomial of the second degree in any number of variables is called the _______	characteristic polynomial	characteristic equation	quadratic form	canonical form

The Set of all eigen values of the matrix A is called the ____________ of A	rank	index	Signature	spectrum
A Square matrix A and its transpose have ________ eigen values.	different	Same	Inverse	Transpose
The sum of the ___________ of a matrix A is equal to the sum of the principal diagonal elements of A.	characteristic polynomial	characteristic equation	eigen values	eigen vectors
The product of the eigenvalues of a matrix A is equal to__________	Sum of main diagonal	Determinant of A	Sum of minors of Main diagonal	Sum of the cofactors of A
The eigenvectors of a real symmetric are ________	equal	unequal	real	symmetric
When the quadratic form is reduced to the canonical form, it will contain only r terms, if the _____ of A is r.	rank	index	Signature	spectrum
The excess of the number of positive terms over the number of negative terms in the canonical form is called the ____________ of the quadratic form.	rank	index	Signature	spectrum
If all the eigen values of A are less than zero and atleast one eigen value is zero then the quadratic form is said to be ____________	Positive definite	Negative defin	Positive semi	Negative semidefinite
If all the eigen values of A are greater than zero and atleast one eigen value is zero then the quadratic form is said to be ____________	Positive definite	Negative definite	Positive semidefinite	Negative semidefinite
If the quadratic form has both positive and negative terms then it is said to be ____________	Positive definite	Negative definite	Positive semidefinite	indefinite

opt6	Answer
	trace of a matrix
	characteristic equation
	$N^T AN$
	kA
	real symmetric
	eigen vectors of A
	0
	1
	characteristic equation
	A^{-1}
	A^p
	inverse and higher powers of A
	orthogonal

linearly
independent
triangular
diagonal

A

positive
semidefinite

$a_{11} = 1, a_{12} = 2$, $a_{21} = 2$, $a_{22} = 3$

0

5
2

X= NY

latent vector

2,6,14

1,3,4
index
Positive
definite
Negative
definite
quadratic
form

spectrum

Same
eigen values

Determinant
of A

real
rank

Signature

Negative
semidefinite
Positive
semidefinite
indefinite

Questions	opt1	opt2	opt3	opt4	opt5
A polynomial fucntion in R	is never continuous in R	may or may not be continuous in R	is always continuous in R	is continuous in R except at x = 0	
The function f(x)=|x| is	continuous for all x	discontinuous at x=0 only	continuous at x = 0 only	discontinuous for all x	
Which of the following is continuous at x = 0 ?	f(x) = 1/x	f(x) = |x| / x	f(x) = |x|	x = x / |x|	
If f is finitely derivable at c, then f is also ______ at c	discontinuous	continuous	derivative	limit	
A function f is said to be ___ in an interval [a, b] if it is continuous at each and every point of the interval	discontinuous	continuous	derivative	limit	
A function f is said to be continuous in an interval [a, b] if it is____ at each and every point of the interval	discontinuous	continuous	derivative	limit	
The exponential function is ___ at all points of R	discontinuous	continuous	derivative	limit	
Which of the following is continuous function?	e^x	sin x	cos x	e^x, sinx, cosx	
Every differentiable function is ____	constant	discontinuous	algebraic	continuous	
Every polynomical function of degree n is ___	constant	discontinuous	algebraic	continuous	
The derivative of (log x) is	1/x	x	x^2	0	
The derivative of (e^x) is	1/x	x	x^2	e^x	
The derivative of constant is	1/x	0	x^2	x	
The derivative of (sin x) is	cos x	0	x^2	x	
The derivative of (cos x) is	(cos x)	(- sin x)	tan x	(-x)	
The derivative of (tan x) is	(cos x)	(- sin x)	tan x	(sec^2 x)	
The derivative of (cosec x) is	(-cos x)	(- cosec x. cot	tan x	(sec^2x)	
The derivative of (sec x) is	(sec x tan x)	(- cosec x. cot	tan x	(sec^2x)	
The derivative of (cot x) is	(-cos x)	(- cosec^2 x)	tan x	(sec^2x)	
The derivative of (x^3) is	3x^2	3x^3	3x	3	
The derivative of (5x) is	5x	5	1	0	
The derivative of (10) is	0	2	3	10	
The derivative of (5x^2) is	10	0	10x	5x	

The derivative of (e^3x) is	6 e^3x	3 e^x	3 e^3x	e^3x
The derivative of (sin 4x) is	(4cos 4x)	(- 4sin x)	tan4 x	(cos 4x)
The derivative of (cos 2x) is	(- 2sin x)	(- 2sin 2x)	tan x	(- sin 2x)
The derivative of (cos 5x) is	(- 5sin x)	(- 5sin 2x)	tan x	(- sin 5x)
Find the first derivative of 6x^3	18x^2	18x	18	6x^2
Find the second derivative of 6x^3	36	18x^2	36x	18x
Find the third derivative of 6x^3	36	18x^2	36x	18x
Find the first derivative of (x^3+2)	x^2+2	x^2	3x^2	3x
Find the second derivative of (x^3+2)	x^2+2	6x	3x^2	3x
Find the third derivative of (x^3+2)	x^2+2	6x	3x^2	6
Find the first derivative of (log x+2)	1/x	x	x^2	0
Find the first derivative of (e^x+2x)	e^x	e^x+2	e^x	0
Find the second derivative of (e^x+2x)	e^x	e^x+2	e^x	0
Find the first derivative of (kx)	kx	x	k	1
Find the second derivative of (kx)	kx	x	k	0
Find the derivative of y = (x^2) with respect to x	x	2x	x^2	0
Find the derivative of y = (sin 5x) with respect to x	5 cos 5x	(-5 cos 5x)	cos 5x	5 cos x

opt6	Answer
	is always continuous in R
	continuous at x = 0 only
	f(x) = |x| continuous
	continuous
	continuous
	continuous
	e^x, sinx, cosx
	continuous
	continuous
	1/x
	e^x
	0
	cos x
	(- sin x)
	(sec^2 x)
	(- cosec x. cot x)
	(sec x tan x)
	(- cosec^2 x)
	3x^2
	5
	0
	10x

3 e^3x
(4cos 4x)
(- 2sin 2x)
(- 5sin 2x)
18x^2
36x
36
3x^2
6x
6
1/x
e^x+2
e^x
k
0
2x
5 cos 5x

Questions	opt1	opt2
An equation involving one dependent variable and its derivatives with respect to independent variable is called ___________	Ordinary Differential Equation	Partial Differential Equation
The ODE of the first order can be written as	F(x,y,s,t)	F(x,y,z,p,q)
C.F+P.I is called_____solution	Singular	Complete
The roots of the A.E of D.E, (D^2-2D+1)y=0 are	(0 1)	(3 2)
The quadratic equation of roots are real and distinct. What is the Complementary function?	$C.F = Ae^{m_1x} + Be^{m_2x}$	
The order of the (D^2+D)y=0 is	2	1
The roots of the A.E of D.E, (D^4-1)y=0 are	(1 ,1, 1, 1)	(1, 1, -1, 1)
The roots of the A.E of D.E, (D^3-D^2+D-1)y=0 are	(1,-i, i)	(i, i, -i)
The roots of the A.E of D.E, (D^3-7D-6)y=0 are	(1, 2, 3)	(1, -2, 3)

opt3	opt4	opt5	opt6	Answer	
Difference Equation	Integral Equation			Ordinary Differential Equation	
F(x,y,z)	F(x,y,y')=0			F(x,y,y')=0	
General	particular			General	
(1 2)	(1 1)			(1 1)	
				$C.F = Ae^{m_1 x} + Be^{m_2 x}$	
0	-1			2	
(1 ,-1, 1, -1)	(1, -1, i, -i)			(1, -1, i, -i)	
(1, i, -i)	(1, 1, 1)			(1, -i, i)	
(3, 2, -1)	(-1, -2, 3)			(-1, -2, 3)	

The degree of the (D^2+2D+2) y=0 is	1	3
The particular integral of (D^2-2D+1)y=e^x is__________	((x^2)/2) e^x	(x/2) e^x
The roots of the A.E of D.E, (D^2-4D+4)y=0 are	(2, 1)	(2, 2)
If y=ax+b then differentiating with respect to x=________	a	a+b
A Differential Equation is said to be _____________ if the dependent variable and its differential co-efficient occur only in the first degree.	Linear equation	Non-Linear equation
The P.I of the Differential equation (D^2 -3D+2) y=12 is________	1 / 2	1 / 7
If f(D)=D^2 -2, (1/f(D))e^2x=______	(1 / 2) e^x	(1 / 4) e^2x
If f(D)=D^2 +5, (1/f(D)) sin 2x =______	sin x	cos x
To transform (xD^2+D+7)y=1/x into a linear differential equation with constant coefficient. Put x=______	e^(-t)	e^(2t)

0	2			1		
((x^2)/4) e^x	((x^3)/3) e^x			((x^2)/2) e^x		
(2, -2)	(-2, 2)			(2, 2)		
b	ab			a		
Homogeneous equation	Non-Homogeneous equation			Linear equation		
6	10			6		
(1 / 2) e^(-2x)	(1 / 2) e^2x			(1 / 2) e^2x		
sin 2x	-sin 2x			sin 2x		
e^(t)	e^(-2t)			e^(t)		

The particular integral of (D^2 +19D+60)y= e^x is_______	(-e^(-x))/80	(e^(-x))/80
The particular integral of (D^2+25) y= cosx is __________	(cosx)/24	(cosx)/25
The particular integral of (D^2+25) y= sin4x is ___________	(-sin4x)/9	(sin4x)/9
The particular integral of (D^2+1) y= sinx is ____________	xcosx/2	(-xcosx)/2
The particular integral of (D^2 -9D+20)y=e^(2x) is____________	e^(2x) /6	e^(2x) /(-6)
The particular integral of (D^2 +D-72)y= e^(7x) is ___________	e^(7x)/16	e^(-7x)/16
The particular integral of (D^2-1) y= sin2x is __________	(-sin2x)/5	sin2x/5
The particular integral of (D^2+2) y= cosx is __________	(-cosx)	(-sinx)
In a PDE, there will be one dependent variable and ____ independent variables	only one	two or more

(e^x)/80	(-e^x)/80			(e^x)/80	
(-cosx)/24	(-cosx)/25			cosx/24	
(sin4x)/41	(-sin4x)/41			(sin4x)/9	
(-xsinx)/2	xsinx/2			(-xcosx)/2	
e^(2x) /12	e^(2x) /(-12)			e ^ (2x) /6	
e^(7x)/(-16)	e^(-7x)/(-16)			e^(7x)/(-16)	
sin2x/3	(-sin2x)/3			(-sin2x)/5	
cosx	sinx			cosx	
no	infinite number of			two or more	

The ______ of a PDE is that of the highest order derivative occurring in it	degree	power
The degree of the a PDE is ______of the higest order derivative	power	ratio
Afirst order PDE is obtained if______	Number of arbitrary constants is equal Number of independent variables	Number of arbitrary constants is lessthan Number of independent variables
In the form of PDE, f(x,y,z,a,b)=0. What is the order?	1	2
What is form of the z=ax+by+ab by eliminating the arbitrary constants?	z=qx+py+pq	z=px+qy+pq
A solution obtained from the complete integral by giving paticular values to the arbitrary constant is called a _____ solution.	complete	general
The solution f(x,y,z,a,b)= 0 of the first order PDE, Which contains two arbitrary constants is called a ____ solution.	complete	general
General solution of PDE F(x,y,z,p,q)=0 is any arbitray function F of specific functions u,v is____satisfying given PDE	F(u,v)=0	F(x,y,z)=0
The Lagrange's linear PDE is of the form _______	Pp+Qq=r	Pp+Qq= R

order	ratio			order	
degree	order			power	
Number of arbitrary constants is greater than Number of independent variables	Number of arbitrary constants is not equal to Number of independent variables			Number of arbitrary constants= Number of independent variables	
3	4			1	
z=px+qy+p	z=py+qy+q			z=px+qy+pq	
particular	singular			particular	
particular	singular			complete	
F(x,y)=0	F(p,q)=0			F(u,v)=0	
Pp+Qp= R	Pq+Qq= R			Pp+Qq= R	

_________ is of the form of the Lagrange's auxiliary equation	dx/P=dy/Q=dz/R	dx/Q=dy/P=dz/R
The complete solution of the PDE, pq=1 is____	z=ax+(1/a)y+b	z=ax+y+b
The order and degree of the solution of the PDE is y=f(y+x)+g(y+x)+e^2x_______	1 and 2	2 and 1
The complete solution of clairaut's equation is _____	z=bx+ay+f(a,b)	z=ax+by+f(a,b)
The clairaut's equation can be written in the form	z=px+qy+f(p,q)	z= py+qx
From the PDE by eliminating the arbitrary function from z=f(x^2 -y^2) is	xp+yq=0	p=-(x/y)
Which of the following is the type f(z,p,q)=0 ?	p(1+q)=qx	p(1+q)=qz
The equation (D^2 z+2xy(Dz)^2+D'=5 is of order ____and degree____	2 and 2	2 and 1
The complementry function of (D^2 -4 DD'+4D'^2)z=x+y is	f(y+2x)+xg(y+2x)	f(y+x)+xg(y+2x)

dx/R=dy/Q=dz/P	dx/P=dy/R=dz/Q			dx/P=dy/Q=dz/R	
z= ax+(1-2x)/y+c	z=ax+b			z=ax+(1/a)y+b	
0 and 1	1 and 1			2 and 1	
z=ax+by	z=f(a,b)			z=ax+by+f(a,b)	
z=px+f(a,b)	z=py+qy+f(p,q)			z=px+qy+f(p,q)	
q=yp/x	yp+xq=0			yp+xq=0	
p(1+q)=qy	p=2x f(y+2x)			p(1+q)=qz	
1 and 1	0 and 1			2 and 1	
f(y+x)+xg(y+x)	f(y+4x)+xg (y+4x)			f(y+2x)+xg(y+2x)	

The solution of xp+yq=z is ______	f(x^2,y^2)=0	f(xy,yz)
The solution of p+q=z is _____	f(xy,ylogz)=0	f(x+y, y+logz)=0
The roots of the PDE(D^2-2DD'+D' ^2)z=0 are	0,1	i,-i
The particular integral of e^(ax+by)/ (D-(aD' /b))^2 is ------	e^(ax+by)	(x2/2) e^(ax+by)
The particular integral of e^(ax+by)/ (D-(aD' /b)) is ---------	ax-by+c	e^(ax+by)
The subsidiary equations of the Lagrange's equation $(z-y)p+(x-z)q=y-x$ is____	$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}$	

f(x,y)=0	f(x/y ,y/z)=0			f(x/y ,y/z)=0	
f(x-y, y-logz)=0	f(x-y,y+logz)=0			f(x-y, y-logz)=0	
1,2	1,1			1,1	
ax-by+c	ax+by			(x2/2)e^(ax+by)	
ax+by	xe^(ax+by)			xe^(ax+by)	
$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{z-x}$	$\frac{dx}{x-y}=\frac{dy}{z-y}=\frac{dz}{y-x}$			$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}$	

UNIT - IV - FUNCTIONS OF SEVERAL VARIABLES					
QUESTIONS	**OPTION 1**	**OPTION 2**	**OPTION 3**	**OPTION 4**	**ANSWER**
The partial differentiation is a function of__________ or more variables .	two	zero	one	three	two
If z=f(x,y) where x and y are ______ function of another variable t	continuous	differential	two	one	continuous
If f(x,y)=0 then xand y are said to be an __________ function	implicit	extremum	explicit	differential	implicit
The concept of jacobian is used when we change the variables in_________	multiple integrals	single integrals	diffenential	function	multiple integrals
The extreme values of f(x,y,z) in such a situation are called_________values	extreme	constrained extreme	boundry values	initial	constrained extreme
The jacobian were introduced by__________	C.G.Jacobi	johon	Gauss	Euler	C.G.Jacobi
f(a,b) is said to be extreme value of f(x,y) if it is either a________________	maximum or minimum	zero	minimum	maximum	maximum or minimum
The Lagrange multiplier is denoted by __________________	l	m	n	p	l
Every extremum value is a stationary value but a stationary value need not be an ____________ value.	infimum	minimum	maximum	extremum	extremum
F is differentiable and where not all of its first differential derivatives vanish simultaneously then the functions u1,u1…..un are said to be functionally_____	independent	dependent	explicit	implicit	dependent
f(a,b) is a maximum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood ________	f(a,b)>f(a+h,b+k)	f(a,b)<f(a+h, b+k)	f(a,b)<0	f(a,b)>0	f(a,b)>f(a+h,b+k)
f(a,b) is a minimum value of f(x,y) if there exists some neighbourhood of the point (a,b) such that for every point (a+h,b+k) of the neighbourhood ________	f(a,b)>f(a+h,b+k)	f(a,b)<f(a+h, b+k)	f(a,b)<0	f(a,b)>0	f(a,b)<f(a+h,b+k)
The necessary condition for maxima is ___________	∂f/∂x (a,b)=0	∂f/∂x (a,b)= 1	∂f/∂y (a,b)=5	∂f/∂y (a,b)=1	∂f/∂x (a,b)=0
The necessary condition for minimum is _____________	∂f/∂x (a,b)=0	∂f/∂y (a,b)=0	∂f/∂x (a,b)=1	∂f/∂y (a,b)=1	∂f/∂y (a,b)=0

f(a,b) is said to be a stationary value of f(x,y) if (x,y) is___________	∂f/∂x (a,b)=0 and ∂f/∂y (a,b)=0	∂f/∂x (a,b)=1	∂f/∂y (a,b)=0	∂f/∂y (a,b)=1	∂f/∂x (a,b)=0 and ∂f/∂y (a,b)=0
If f(a,b) is said to be _________of f(x,y) if it is either maximum or minimum.	extremum value	boundary value	end	power	extremum value
If u be a _____ of degree n in x and y.	linear	homogeneou s	non-homogeneous	polynonmial	homogeneous
The _______ differentiation is a function of two or more variables.	ODE	PDE	partial	total	partial
The ______ were introduced by C.G.Jacobi.	jacobian	millian	taylor	Gauss	jacobian
The concept of ________ is used when we change the variables in multiple integrals	taylor	gauss	maculaurin	jacobian	jacobian
If the function u,v,w of three independent variables x,y,z are not independent then the Jacobian of u,v,w with respect to x,y,z is always equal to	1	0	Infinity	Jacobian of x,y,z with respect ro u,v,w	0
The function f(x)=10+x^6	is a decrasing function of x	has a minimum at x=0	has neither a maximum nor a minimum at x=0	saddle point	has neither a maximum nor a minimum at x=0
The function f(x,y)=2x^2+2xy-y^3 has	only one stationary point at (0,0)	two stationary points at (0,0)and (1/6,1/3)	two stationary point at (0,0) and (1,-1)	not stationary points	two stationary points at (0,0)and (1/6,1/3)
If(a/3,a/3) is an extreme point on xy(a-x-y), the maxima is	a^3/27	a/27	a^3/9	a/9	a^3/27
Any function of the type f(x,y)=c is called an _______function	Implicit	Explicit	Constant	composite	Implicit
If u=f(x,y) ,where x=pi(t),y=si(t) then u is a function of t and is called the ____ function	Implicit	Explicit	Constant	composite	composite
The point at which function f(x,y) is either maximum or minimum is known as ______ point	Stationary	Saddle point	extremum	implicit	Stationary

If rt-s^2>0 and r<0 at (a,b) the f(x,y) is maximum at (a,b) and the_______ value of the function(a,b)	Maximum	Minimum	maximum or minimum	zero	Maximum
If rt-s^2>0 and r>0 at (a,b) the f(x,y) is minimum at (a,b) and the_______ value of the function(a,b)	Maximum	Minimum	maximum or minimum	zero	Minimum
If rt-s^2>0 at (a,b) the f(x,y) is neither maximum nor minimum at (a,b) such point is known as _______ point	Stationary	Saddle point	extremum	implicit	Saddle point
If f(x,y) is a function of two variables x,y then _________	lim f(x,y)=1	lim f(x,y=0	lim f(x,y)>0	lim f(x,y)<0	lim f(x,y)=1

Questions	opt1	opt2	opt3
The Taylor,s series of f(x,y) at the point (0,0) is ___________ series.	Maclaurins	Taylor	power
The expansion of f(x,y) by Taylor series is __________	zero	unique	minimum
The period of *cos nx*, where n is the positive integer is ________.	$2\pi/n$	$n/2\pi$	2π
f(x,y) =e^x siny at (1,π/2)then ___________	f=0	f=1	f=2
f(x,y) = e^xy at(1,1) then __________	f=1	f=e	f=0
Which of the following functions has the period 2π?	cos x	sin nx	tan nx
1/π ∫ f(x) sinnx dx between the limits c to c+2π gives the Fourier coefficient_____	a_0	a_n	b_n
If f(x) = -x for -π< x< 0 then its Fourier coefficient a0 is_______-	(π^2)/2	π/2	π/3
If a function satisfies the condition f(-x) = f (x) then which is true?	a_0 = 0	a_n = 0	a_0 = a_n = 0

opt4	opt5	opt6	Answer
binomial			Maclaurins
maximum			unique
nπ			2π/n
f=e			f=e
f=2			f=2
tan x			cos x
b_1			b_n
π			π
b_n = 0			b_n = 0

If a function satisfies the condition f(-x) = -f (x) then which is true?	a0 = 0	an = 0	a_0 = a_n = 0
Which of the following is an odd function?	sin x	cos x	x^2
Which of the following is an even function?	x^3	cos x	sin x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)
The function f(x) is said to be an even function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)
∫f(x) dx = 2∫f(x) dx between the limits -a to a if f(x) is ------	even	continuous	odd
∫f(x) dx = 0 between the limits -a to a if f(x) is ------	even	continuous	odd
If a periodic function f(x) is odd, it's Fourier expansion contains no ------ terms.	coefficient an	sine	coefficient a0
If a periodic function f(x) is even, it's Fourier expansion contains no ------ terms.	cosine	sine	coefficient a_0

b_n = 0			a_0 = a_n = 0
x^4			sin x
sin^2x			cos x
f(-x) = f(-x)			f(-x) = - f(x)
f(-x) = f(-x)			f(-x) = f(x)
discontinues			even
discontinues			odd
cosine			cosine
coefficient a_n			sine

In dirichlet condition, the function f(x) has only a ----- number of maxima and minima.	uncountable	continuous	infinite
In Fourier series, the function f(x) has only a finite number of maxima and minima. This condition is known as -------	Dirichlet	Kuhn Tucker	Laplace
In dirichlet condition, the function f(x) has only a ----- number of discontinuities .	uncountable	continuous	infinite
A sequence {2^n} is	Convergent	divergent	Oscillatory
A sequence (-1)^n+2 is	Convergent	divergent	Oscillatory
A sequence {2n+1/3n-2} is	Convergent	divergent	Oscillatory
A sequence {2n^2+n/3n^2-3} is	Convergent	divergent	Oscillatory
A sequence 5+(-1)^n is	Convergent	divergent	Oscillatory
The series ∑ cos(1/n) is	Convergent	divergent	Oscillatory

finite			finite
Cauchy			Dirichlet
finite			finite
unique			divergent
unique			Oscillatory
unique			Convergent
unique			Convergent
unique			Oscillatory
unique			Convergent

The series ∑ x^n/(n^3+1) at x=1 is	Convergent	divergent	Oscillatory
The series 1-(1/2^2)+(1/3^2)-(1-4^2)+… is	Convergent	divergent	Oscillatory
The series 2-(3/2)+(4/3)-(5/4)+… is	Convergent but not absolutely	divergent	absolutely Convergent
The series 1+(1/√2)+(1/√3)+… is	Convergent but not absolutely	Oscillatory	divergent
In a series positive terms ∑ u_n if limit n tends to ∞ u_n/u_n+1 is not equal to zero then the series ∑ u_n is	Convergent	divergent	not Convergent
The series 1-(1/2)+1-(3/4)+1-(7/8)+… is	Convergent	conditionally Convergent	absolutely Convergent
The series (1/(a+1) -(1/(a+2) +(1/a+3)-(1 /a+4)+...convergent if	a>0	a<0	a<-1
The series 1-2x+3x^2-4x^3+... where 0<x<1 is	Convergent	divergent	Oscillatory
The series 1/(1+2^(-1)) +1/(1+2^(-2))+1/(1+2^(-3))… is	Convergent	divergent	Oscillatory

Not unique			Convergent
Not unique			Convergent
Oscillates finitely			Oscillates finitely
absolutely Convergent			divergent
Oscillatory			not Convergent
Oscillatory			Oscillatory
$a \leq 0$			$a > 0$
unique			Convergent
unique			divergent

The series whose nth term is ∑ sin (1/n) is	Convergent	divergent	Oscillatory
The series 2+(3/4)+(4/9)+(5/16)+…+(n+1)/n^2 +… is	Convergent	divergent	Oscillatory
If p and q are positive real number, then the series 2^p/1^q+3^p/2^q+4^p/3^q+… converges	p<q-1	p<q+1	p≥q-1
An ordered set of real number a_1,a_2,…a_n is called a_____________	Series	sequence	Montonic sequence
If a sequence has a ________,it is called a convergent sequence	Finite limit	Infinite limit	limit
A sequence is said to be bounded above if there exists a number k, such that _____ for every n.	a_n>k	a_n≥k	a_n≤k
Both increasing and decreasing sequence are called ______ sequence.	Convergent	Montonic	Bounded
If limit n tends to ∞ a_n is equal to _______then the sequence is said to be Convergent	finite and unique	Infinite	unique
If u1,u2,….un,…be an infinite sequence or real numbers,then u1+u2+….+un+…is called__________	infinite series	finite series	finite terms

Not unique			Convergent
Not unique			divergent
$p \geq q+1$			$p<q-1$
Montonic sequence			sequence
Bounded			Finite limit
$a_n<k$			$a_n \leq k$
divergent			Montonic
not unique			finite and unique
infinite terms			infinite series

The series 1+2+3+ +n+…+...∞ is	Convergent	divergent	Oscillatory
Every absolutely convergent series is a ______ series	Convergent	divergent	Oscillatory
Any convergent series of ________ terms is also absolutely convergent	negative	positive	zero
If limit n tends to infinite u_n/u_n+1 = m is a series of positive terms ∑ u_n is convergent if _____	m>0	m<1	m>1
If limit n tends to ∞ u_n/u_n+1 = m is a series of positive terms ∑ u_n is divergent if _____	m>0	m<1	m>1
If limit n tends to ∞ u_n/u_n+1 = m is a series of positive terms .when the ratio test fails	m>0	m<1	m>1

not unique			divergent
not unique			Convergent
unique			positive
m=1			m>1
m=1			m<1
m=1			m=1