14BECS502 **DISCRETE MATHEMATICS** **3 1 0 4 100**

INTENDED OUTCOMES:

- To acquire the knowledge needed to test the logic of a program.
- To gain knowledge in the applications of expert system, in data base.
- To provide adequate knowledge in class of functions, groups and graph theory.

UNIT- I SET THEORY

Basic concepts – Notations – Subset – Algebra of sets – The power set – Ordered pairs and Cartesian product – Relations on sets –Types of relations and their properties – Relational matrix and the graph of a relation – Partitions – Equivalence relations – Partial ordering – Poset – Hasse diagram – Lattices and their properties – Sublattices – Boolean algebra – Homomorphism.

UNIT -II FUNCTIONS

Definitions of functions – Classification of functions –Type of functions - Examples – Composition of functions – Inverse functions – Binary and n-ary operations – Characteristic function of a set – Hashing functions – Recursive functions – Permutation functions.

UNIT- IV PROPOSITIONAL CALCULUS

Propositions – Logical connectives – Compound propositions – Conditional and biconditional propositions – Truth tables – Tautologies and contradictions – Contrapositive – Logical equivalences and implications – DeMorgan's Laws - Normal forms – Principal conjunctive and disjunctive normal forms – Rules of inference – Arguments - Validity of arguments.

UNIT -V PREDICATE CALCULUS

Predicates – Statement function – Variables – Free and bound variables – Quantifiers – Universe of discourse – Logical equivalences and implications for quantified statements – Theory of inference – The rules of universal specification and generalization – Validity of arguments.

UNIT – V GRAPH THEORY

Graphs and graph models - Graph terminology and special types of graphs - Representing graphs and graph-isomorphism - connectivity

TEXT BOOK:

S. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
1	Trembly, J. P. and Manohar, R	Discrete Mathematical Structures with Applications to Computer Science	Tata McGraw–Hill Pub. Co. Ltd, New Delhi.	2008

REFERENCES:

S. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
1	Bernard Kolman, Robert, C., Busby and Sharan Cutler Ross	Discrete Mathematical Structures	Pearson Education Pvt. Ltd, New Delhi	2003
2	Kenneth H Rosen	Discrete Mathematics and its Applications	Tata McGraw - Hill Pub. Co. Ltd, New Delhi.	2003
3	Manikavasagampillai T.K. and Others	Algebra Vol.1	Vishwanathan Publisher, Tamil Nadu	2000
4	Sundaresan V. , Ganapathy Subramanian, K.S. and Ganesan, K.	Discrete Mathematics	A.R. Publications, Tamil Nadu.	2002

WEBSITES:

1. www.mhhe.com/rosen
2. www.siam.org/books/series/dt.php
3. www.abeliangroup.com
4. www.dmtcs.org/dmtcs-ojs/index.php/dmtcs

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education

Faculty of Engineering

Department of Science and Humanities

LECTURE PLAN

Name : M.Kokilamani **Class : III CSE**

Subject: Discrete Mathematics **Subject Code: 14BECS502**

S.No.	NAME OF THE TOPICS	HOURS
	UNIT I – SET THEORY	
1.	Basic concepts – Notations - subset	1
2.	Algebra of sets – The power set	1
3.	Ordered pairs and Cartesian product – Relations on sets	1
4.	Types of relations and their properties	1
5.	Relational matrix and the graph of a relation - Partitions	1
6.	Tutorial -1 (Subset, Algebra of sets, power set, Ordered pairs and Cartesian product, Relations on sets)	1
7.	Equivalence relations	1
8.	Partial ordering	1
9.	Poset - Hasse diagram	1
10.	Lattices and their properties	1
11.	Sublattices - Boolean algebra	1
12.	Tutorial -2 (Equivalence relations, Partial ordering, Hasse diagram, Lattices, Sublattices)	1
13.	Homomorphism	1
	Unit I total hours	**13**
	UNIT II - FUNCTIONS	
14.	Introduction – Definition of functions	1
15.	Classification of functions	1
16.	Type of functions	1
17.	Examples – Composition of functions	1
18.	Inverse functions	1
19.	Tutorial – 3 (Type of functions, Composition of functions, Inverse functions)	1
20.	Binary and n-ary operations	1
21.	Characteristic function of a set	1
22.	Hashing functions	1
23.	Recursive functions	1
24.	Permutation functions	1
25.	Tutorial – 4 (Hashing functions, Recursive functions, Permutation functions)	1
	Unit II total hours	**12**
	UNIT III - PROPOSITIONAL CALCULUS	
26.	Introduction to Propositions-Logical connectives-Compound propositions – Concepts and Examples	1
27.	Concepts of Conditional and biconditional propositions – Problems	1
28.	Problems based on Truth tables, Tautologies and contradictions	1
29.	Contrapositive, Logical equivalences and Implications – Concepts and Examples	1
30.	Tutorial – 5 (Problems based on Logical equivalences and Implications, Truth tables, Tautologies and contradictions)	1
31.	DeMorgan’s Laws, Normal forms and Concepts of Principal	1

	conjunctive normal forms & Principal disjunctive normal forms	
32.	Principal conjunctive normal form (PCNF)– Problems	1
33.	Principal disjunctive normal form (PDNF) - Problems	1
34.	Problems based on PCNF & PDNF	1
35.	Concepts of Rules of inference, Arguments and Validity of arguments	1
36.	Introduction of Validity of arguments and Problems based on Direct & Indirect methods	1
37.	Tutorial -6 (Problems based on PCNF & PDNF Problems, Direct and Indirect methods)	1
	Unit III total hours	**12**
	UNIT IV – PREDICATE CALCULUS	
38.	Introduction to Predicates –statement function- variables – Concepts and Examples	1
39.	Concepts of Free and bound variables – Problems	1
40.	Problems based on Quantifiers	1
41.	Problems based on Universe of discourse	1
42.	Tutorial – 7(Problems based on Free and bound variables, Quantifiers, Universe of discourse)	1
43.	Logical equivalences and Implications for quantified statements– Concepts and Examples	1
44.	Theory of inference - Concepts	1
45.	Theory of inference - Problems	1
46.	The rules of universal specification and generalization – concepts	1
47.	The rules of universal specification and generalization - problems	1
48.	Tutorial – 8 (Logical equivalences and Implications for quantified statements, Theory of inference, The rules of universal specification and generalization)	1
49.	Validity of arguments	1
	Unit IV total hours	**12**
	UNIT V - GRAPH THEORY	
50.	Introduction to Graph theory – Examples and Concepts of Connectivity	1
51.	Idea of trees, Spanning trees, Cut vertices and edges – Examples	1
52.	Abstract of Covering – Examples	1
53.	Concept of Matching – Theorem and proof	1
54.	Matching – Examples	1
55.	Tutorial -9 (Problems based on Matching and Colouring Connectivity and Spanning trees)	1
56.	Independent sets – Problems	1
57.	Introduction to Colouring and Examples	1
58.	Concepts of Planar graph, Planarity and Isomorphism	1
59.	Tutorial -10 (Planarity Problems based on Colouring in Graph theory)	1
60.	Previous 5 years ESE question paper – Discussion and Revision	1
	Unit V total hours	**11**
	Grand Total	**50+10**

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2	Kenneth Hoffman, Ray Kunze.	Linear Algebra	Prentice Hall India 2nd Edition	2003

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WEBSITES:

1. www.siam.org/books/series/dt.php
2. www.mathword.com
3. www.dmtcs.org/dmtcs-ojs/index.php/dmtcs

Staff In-charge **HOD / S&H**

UNIT-I

SET THEORY

①

Introduction:

The theory of sets is the most fundamental concept in modern Mathematics. Sets & mathematical logic are now basic to, much of the design of Computers and electrical circuits.

SETS: Define a member with an ex

2I (A set is a collection of well defined objects. The objects in a set can be anything: numbers, letters etc. Any object belonging to a set is called a member or an element of that set)

Ex.

- (The set of all Indians.)
- The set of rivers in India
- The set of all vowels of the alphabet
- $N = \{1, 2, 3 \cdots\}$
- $W = \{0, 1, 2 \cdots\}$
- $Z = \{0, \pm 1, \pm 2 \cdots\}$
- Q = rational nos.
- R = Real nos.
- C = Complex nos.

Cardinality: The no. of distinct elements in a set is called the cardinality of the set and is denoted by $\# A$ or $|A|$.

Eg. • $A = \{a, b, c\}$, $n(A) = |A| = 3$

• $A = \{x \mid x$ is a even no. which is less

$= \{2, 4, 6, 8\}$.

TYPES:

1. Finite & Infinite sets:-

2I (A set is said to be finite

it consists of a specific no. of diffe

elements. Otherwise a set is Infinite.

Eg:. • Let A be the set of the days of

week. Then A is finite.

• Let $B = \{1, 3, 5, \ldots\}$. Then B is in

2. Null set: A set which does not conta

2I any element is called an EMPTY or NU

set and is denoted by ϕ or $\{\}$.

Eg: $A = \{x : x$ is an integer in $0 < x < 1$

3. Universal set: A set is called a UNIVE

2I set if it includes every set under discu

and is denoted by U.

Eg: In plane geometry, the unive

set consists of all points in the pl

• The collection of all elemen

needed for a particular problem in ha

is called universal set

4. Subsets: Let A & B be any two sets. We

2I that A is contained in B and write A

if every element of A is also an elt of

If $A \subseteq B$, then A is called a SUBSET of B & B is called a SUPERSET of A and can be read as B contains A. The relation '$\subseteq$' is usually called set inclusion.

Eg: $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6\}$

Then $A \subseteq B$ & $B \nsubseteq A$.

Proper subset:

If $A \subseteq B$ & $A \neq B$, then we say that A is a PROPER SUBSET of B & we write $A \subset B$ or $A \subsetneq B$.

Equivalent ~~& Equal sets~~:

Let A & B be two sets.

The set A & B are called as

(i) equivalent sets, if $n(A) = n(B)$.

(ii) equal sets, if the sets have the same elements. (OR) Two sets A & B are EQUAL iff $A \subseteq B$ & $B \subseteq A$.

Eg: $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{a, c, b\}$

A & B are equivalent whereas A & C are equal sets.

Singleton set: A set having exactly one element is called SINGLETON SET

Power set:

The power set of a set A is a collection or family of all subsets of A & is denoted as $P(A)$.

Eg: $A = \{1, 2, 3\}$

$2^3 = 8$
$2^4 = 16$
$2^5 = 32$

$P(A) = \{ \{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2$

Note: If A is a set consisting of 'n' elements then its power set $P(A)$ consists of 2^n elements.

Operations on sets:-

1. **Union:** The union of two sets A & B, denoted by $A \cup B$, is defined as the set of all elements which belongs to either A or B or both. The symbol '$\cup$' stands for union.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Eg: $A = \{2, 4, 6, 8, 10\}$ & $B = \{4, 8, 12\}$

$A \cup B = \{2, 4, 6, 8, 10, 12\}$.

2. **Intersection:** The intersection of any two sets A & B denoted by $A \cap B$, is defined as the set of elements which belongs to both A & B

ie, $A \cap B = \{x : x \in A \text{ \& } x \in B\}$.

Eg: $A = \{x : x$ is an english alphabet$\}$

$B = \{x : x$ is a vowel in English alphabet$\}$

$A \cap B = \{a, e, i, o, u\}$

3. **Disjoint set:** Two sets are said to be DISJOINT if $A \cap B = \{\}$

Eg: $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$

Then $A \cap B = \{\}$

Difference Set:

The difference of two sets A & B, denoted by A − B, is defined as the set of all elements which are in A but not in B.

Mathematically, $A - B = \{x : x \in A \ \& \ x \notin B\}$

Complement of a set:

Let A be a subset of U. The Complement of A relative to U denoted by A^c is defined ~~by~~ to the set of all elements in U which are not in A.

i.e, $A^c = \{x : x \in U \ \& \ x \notin A\}$

6. Symmetric difference:

The Symmetric difference (Boolean sum) to two sets A & B is denoted by $A \Delta B$ (or $A \oplus B$) and is defined by, $A \Delta B = (A - B) \cup (B - A)$

i.e, $A \Delta B = \{x : x \in A \ \& \ x \notin B\}$ or $\{x \in B \ \& \ x \notin A\}$

Ordered pairs:

2 [An ordered pair consists of two objects in a given, fixed order. Note that an ordered pair is not a set consisting of two objects. The ordering of the two objects is important. The two objects need not be distinct. We shall denote an ordered pair by $\langle a, b \rangle$.

- The equality of two ordered pair $\langle a, b \rangle$ $\langle x, y \rangle$ is defined by $\langle a, b \rangle = \langle x, y \rangle \Leftrightarrow (x = a \ \& \ y = b)$

Cartesian product:

(Let A & B be two sets. Then Cartesian product of A & B denoted by $A \times B$ is set of all ordered pairs $\langle a, b\rangle$, where $a \in A$, $b \in B$.

In symbol,

$$A \times B = \{ \langle a, b\rangle \mid a \in A,\ b \in B \})$$

Eg:

If $A = \{1, 2\}$ $B = \{1, 2, 3\}$ Find $A \times B$, $B \times A$, $(A \times B) \cap (B \times A)$, $A \times (B \cap A)$, $(A \times B) \times A$

– $A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

– $B \times A = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

$(A \times B) \cap (B \times A) = \{(1,1), (1,2), (2,1), (2,2)\}$

$B \cap A = \{1, 2\}$

– $A \times (B \cap A) = \{(1,1), (1,2), (2,1), (2,2)\}$

– $(A \times B) \times A = \{((1,1), 1), ((1,2), 1), ((1,3), 1), ((2,1), 1), ((2,2), 1), ((2,3), 1), ((1,1), 2), ((1,2), 2), ((1,3), 2), ((2,1), 2), ((2,2), 2), ((2,3), 2) ...$

clearly, $A \times B \neq B \times A$

Relations:

(Let A & B be any two non-empty sets. A subset $A \times B$ is a called RELATION or BINARY RELATION.)

ie, Any set of ordered pairs define a binary relation. If a & b is in the relation then we write it as $\langle a, b\rangle \in R$ (or aRb.

Eg:-
1) Let $A = \{1, 2, 3, 4\}$ $B = \{1, 7, 8\}$. The relation R is defined such that "less than" then

~~AXB~~

$R = \{(1,7), (1,8), (2,7), (2,8), (3,7), (3,8), (4,7), (4,8)\}$

2) Let A & B are set of students in a class. we can take a relation R as "is taller than" or "is shorter than".

3) Let $A = \{2, 4, 6\}$. Define the relation R from A to A as aRb iff $a < b$, then

$R = \{(2,4), (2,6), (4,6)\}$.

Complementary relation:-

Let A & B are two finite sets & R be a relation from A to B, then the Complementary relation of R is defined as,

$$R^c = \{(a,b) \in A \times B \mid (a,b) \notin R\}$$

Inverse Relation:-

Let R be a relation from set A to B then the inverse relation of R is defined by

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

Eg: Let $A = \{1, 2, 3\}$ & $B = \{1, 4\}$ and consider the relation R s.t '<' then find R, R^c, R^{-1}.

$A \times B = \{(1,1), (1,4), (2,1), (2,4), (3,1), (3,4)\}$

$R = \{(1,4), (2,4), (3,4)\}$

$R^c = \{(1,1), (2,1), (3,1)\}$

$R^{-1} = \{(4,1), (4,2), (4,3)\}$

H.W Let $A = \{a, b, c\}$ & $B = \{1, 2, 4\}$ the relation

$R = \{(a,1), (a,4), (b,1), (b,4), (c,1), (c,2), (c,4)\}$

find R^c, R^{-1}.

Union & intersection of two relations:

Let R & S are 2 relations from a set A to B then $R \cup S$ & $R \cap S$ are defined as follows:

$$R \cup S = \{(a,b) \mid (a,b) \in R \text{ or } (a,b) \in S\}$$

$$R \cap S = \{(a,b) \mid (a,b) \in R \text{ and } (a,b) \in S\}$$

Eg:

Let $A = \{1, 2, 3\}$, $B = \{2, 6\}$ & the relation

$R = \{(1,2), (1,6), (2,6), (3,6)\}$

$S = \{(1,2), (1,6), (2,2), (2,6), (3,2), (3,6)\}$

Then

$R \cup S = \{(1,2), (1,6), (2,2), (2,6), (3,2), (3,6)\}$

$R \cap S = \{(1,2), (1,6), (2,6), (3,6)\}$.

Types of Relations:

1. Reflexive: A binary relation R in a set X is REFLEXIVE if for every $x \in X$, xRx, that is $(x,x) \in R$ or R is reflexive in X $\Longleftrightarrow$ $(x)(x \in X \rightarrow xRx)$.

2. **Symmetric:** A relation R in a set X is SYMMETRIC, if for every x & y in X, whenever xRy, then yRx.

Ex: In N, define aRb if $a+b=7$. This is symmetric for $a+b=7 \Rightarrow b+a=7$

i.e., R is symmetric in $X \Leftrightarrow$

$$(x)(y)(x \in X \wedge y \in X \wedge xRy \rightarrow yRx).$$

3. **Transitive:** A relation R in a set X is TRANSITIVE if, for every x, y and z in X, whenever xRy & yRz then xRz.

Ex: Let $A = \{1,2,3\}$. Then the relation $R = \{(1,2), (2,1), (1,1), (2,2)\}$ is transitive

i.e., R is transitive in $X \Leftrightarrow$

$$(x)(y)(z)(x \in X \wedge y \in X \wedge z \in X \wedge xRy \wedge yRz \rightarrow xRz)$$

4. **Irreflexive:** A relation R in a set X is irreflexive if, for every $x \in X$, $(x,x) \notin R$

Note: Any relation which is not reflexive is not necessarily irreflexive and vice-versa.

5. **Antisymmetric:** A relation R in a set X is ANTISYMMETRIC if, for every x & y in X, whenever xRy & yRx, then $x=y$.

Symbolically,

Ex: Let $A = \{1,2,3\}$. Then the relation $R = \{(1,1), (2,2), (3,2), (2,3)\}$ is not antisymmetric $\because (3,2)\ (2,3)$ both belong to R & $(3,2) \neq (2,3)$

R is antisymmetric in $X \Leftrightarrow$

$$(x)(y)(x \in X \wedge y \in X \wedge xRy \wedge yRx \longrightarrow x=y)$$

Equivalence Relation:

A relation R on a set A is called an equivalence relation if R is reflexive, Symmetric & transitive.

Partially Ordering Relation:

A binary relation R on a set A is said to be partially ordering relation (or partial order) if R is reflexive, anti-symmetric & transitive.

A set together with a partial order φ defined on it is called a partially ordered set (or) POSET and is denoted by (A, φ).

Defn:

Eg. Defn: A relation R on a set A is said to be

1. Reflexive if $(a,a) \in R \quad \forall \ a \in A$
2. Irreflexive if $(a,a) \notin R \quad \forall \ a \in A$
3. Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$
4. Antisymmetric if $a \neq b$ & $(a,b) \in R \Rightarrow (b,a) \notin$
5. Transitive if $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$
6. Asymmetric if $(a,b) \in R \Rightarrow (b,a) \notin R$.

Eg:

1. Set $A = \{1, 2, 3, 4\}$ &

$R = \{(1,1), (1,2), (2,1), (1,3), (3,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3$

Then R is symmetric but not reflexive ~~as $(4,4) \notin R$ and~~ also not transitive as

(3,1), (1,2) $\in R$ but $(3,2) \notin R$.

6

The relation S is symmetric, transitive but not reflexive as $(4,4) \notin S$.

3 Give an example of a relation defined on a suitable set which is (a) ref, sym, tran.
(b) ref, sym. but not transitive.
(c) sym. transitive but not reflexive.

Sol.

Let $A = \{1,2,3\}$

(a) The relation $R_1 = \{(1,1), (2,2), (3,3)\}$ is ref, sym. & Transitive

(b) The relation $R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$ is clearly ref. & sym. It is not transitive for $(2,1) \in R_2$, $(1,3) \in R_2$ but $(2,3) \notin R_2$

(c) The relation $R_3 = \{(1,1)\}$ is sym & trans. but not ref. $\because$ $(2,2)$ & $(3,3) \notin R_3$.

3. If $A = \{1,2,3,4\}$ then

(i) The relation $\{(1,2), (2,4)\}$ is not ref., not sym. & not transitive.

(ii) The relation $\{(1,1), (1,3), (3,1), (3,4), (4,3)\}$ is Sym. but neither ref. nor transitive.

(iii) The relation $\{(1,1), (2,2), (3,3), (4,4), (1,3), (3,2)\}$ is ref. but neither sym. nor trans.

(iv) The relation $\{(1,1),(1,3)\}$ is transitive but neither ref. nor sym.

(v) The relation $\{(1,1),(2,2),(3,3),(4,4),(1,3),(3,1),(3,4)$ is ref., sym. & but not trans.

(vi) The relation $\{(1,1),(2,2),(3,3),(4,4),(1,3)\}$ is ref., trans. but not sym.

(vii) The relation $\{(1,1),(2,2),(2,3),(3,2),(3,3)$ is sym., trans but not ref.

(viii) The relation $\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1)$ is ref., sym. & trans.

$\therefore$ This relation is Equivalence relation

4. Check whether the relation R is ref., irre sym., asym, antisym, trans.

(a) $R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),($

(b) $R=\{(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)\}$

(c) $R=\phi$

(d) $R=\{(1,2),(1,3),(3,1),(1,1),(3,3),(3,2),(1,4),(4,2),$

(e) $R=\{(1,1),(1,2),(1,3),(1,5),(2,3),(4,4),(4,2),(4,3),$

(f) $R=\{(1,2),(1,4),(2,1),(2,3),(2,4),(3,2),(4,1),(4,2)\}$

	Ref.	Irr.	Sym.	Asym.	Anti-Sym	Trans
(a)	✓	X	✓	X	X	✓
(b)	X	X	X	X	X	X
(c)	X	✓	✓	✓	✓	✓
(d)	X	X	X	X	X	✓
(e)	X	X	✓	X	✓	✓
(f)	X	X	✓	X	X	X

5. Give an exp. of a relation R on $A = \{1,2,3,4\}$ which is neither sym. nor antisym.

$$R = \{(1,2), (2,1), (2,4)\}$$

6. Give an exp. of a relation R on $A = \{1,2,3,4\}$ which is sym. trans. but not ref.

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

7. let $X = \{1,2 .. 7\}$ & $R = \{(x,y) \mid x-y$ is divisible by $3\}$
S.T R is an Equivalence relation.

Sol

$$R = \{(1,4), \cancel{(1,1)}, (1,7), (2,5), (3,6), (4,7), (4,1), (7,1), (5,2), (6,3), (7,4), (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)\}$$

$\because$ $(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7) \in R$

we have $x R x$, $\forall x = 1, .. 7$

$\therefore$ R is reflexive

$(1,4) \to (4,1)$, $(1,7) \to (7,1)$, $(2,5) \to (5,2)$, $(3,6) \to (6,3)$
$(4,7) \to (7,4)$ exist...

$\therefore x R y \Rightarrow y R x \quad \forall x, y \in X.$

$\therefore$ R is symmetric

And
$(1,4).(4,1) \to (1,1)$ ... exist
$(4,1).(1,7) \to (4,7)$
xRy & $yRz \Rightarrow xRz \quad \forall x, y, z \in X = \{1,2..\}$

$\therefore$ R is transitive

Hence R is an equivalence relation.

8. Prove that the relation "Congruence modulo over the set of the integers is an E. R.

Sol. Let Z^+ be the set of positive integers and 'm' be given +ve integer. we define the "Congruence modulo m" relation on Z^+

for $x, y \in Z^+$, $x \equiv y \pmod m \iff x-y$ is divisible by

i.e, $x - y = km$ for $k \in$

Let $x, y, z \in Z^+$ then,

(i) as $x - x = 0 \cdot m = 0$, so $x \equiv x \pmod m$ $\forall x \in$

$\therefore$ It is reflexive.

(ii) Claim: $x \equiv y \pmod m \Rightarrow y \equiv x \pmod m$

Let $x \equiv y \pmod m$

$\Rightarrow x - y = km$ for some int. k

$\Rightarrow -(x-y) = -km$

$\Rightarrow y - x = -km$

$\Rightarrow y \equiv x \pmod m$

$\therefore$ The relation is sym.

(iii) Claim:

$x \equiv y \pmod m$ & $y \equiv z \pmod m \Rightarrow x \equiv z \pmod m$.

Let $x \equiv y \pmod m$ & $y \equiv z \pmod m$.

$\Rightarrow x - y = km$ & $y - z = lm$; for int. l & k

Now,

$(x+y) + (y-z) = km + lm$

$\Rightarrow (x-z) = (k+l)m$

$\Rightarrow x \equiv z \pmod m$ as $k+l$ is also an int.

$\therefore$ The relation is transitive.

$\therefore$ Congruence modulo is ref, sym. & trans.

The relation "Congruence modulo m" is an E.R.

9. P.T intersection of 2 equivalence is again an E.R and what abt the union.

Sol.

Let ρ & σ are E.R on X

Claim: $\rho \cap \sigma$ is an E.R.

(i) Let $x \in X$

Then $x \rho x$ & $x \sigma x$ ($\because \rho, \sigma$ are ref.)

$\therefore x (\rho \cap \sigma) x$.

Hence $\rho \cap \sigma$ is ref.

(ii) Let $x (\rho \cap \sigma) y$

Then $x \rho y$ & $x \sigma y$

$\Rightarrow y \rho x$ & $y \sigma x$ ($\because \rho, \sigma$ are sym.)

Clearly the domain of a relation f[rom]
A to B is a subset of A and its [range]
is a subset of B

Eg 1

Let $A = \{1, 2, 3, 4\}$, $B = \{r, s, t\}$ &
$R = \{(1, r), (2, s), (3, r)\}$.

The domain of R is the set $\{1, 2, 3\}$ &
range of R is the set $\{r, s\}$.

H.W. Eg 2 Let R be the relation from $A = \{1, 3, ...$
to $B = \{2, 4, 6, 8\}$ which is defined as a[R b]
$\Leftrightarrow a > b$. List the elements of R &
find its domain & range.

Eg. 3 Let R be the relation from $A = \{2, 3 ...$
to $B = \{3, 6, 7, 10\}$ which is defined by t[he]
expression "x divides y" find (i) relation (ii) Range [&] do[main]

(i) $R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$

(ii) $D(R) = \{2, 3, 5\}$ & $R(R) = \{3, 6, 10\}$.

Composition of Relations:-

H.W 4. Let $P = \{(1, 2), (2, 4), (3, 3)\}$ & $Q = \{(1, 3), (2, 4), (4, 2) ...$
Find $P \cup Q$, $P \cap Q$, $D(P)$, $D(Q)$, $D(P \cup Q)$, $R(P)$, $R(Q)$,
$R(P \cap Q)$ & show that $D(P \cup Q) = D(P) \cup D(Q)$
$R(P \cap Q) \subseteq R(P) \cap R(Q)$.

5. Let L denote the relation "less than or equal to"
& D denote the relation "divides". where x D y means
"x divides y". Both L & D are defined on the set $\{1, 2, 3, 6\}$. Fi[nd]
L & D as sets ...

Composition of Binary relations:-

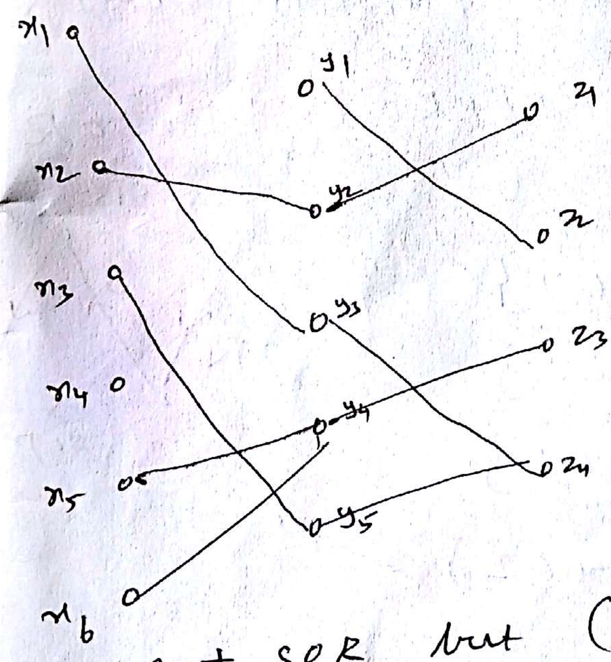
Let R be a relation from X to Y and S be a relation from Y to Z. Then a relation written as $R \circ S$ is called a COMPOSITE relation of R and S, where

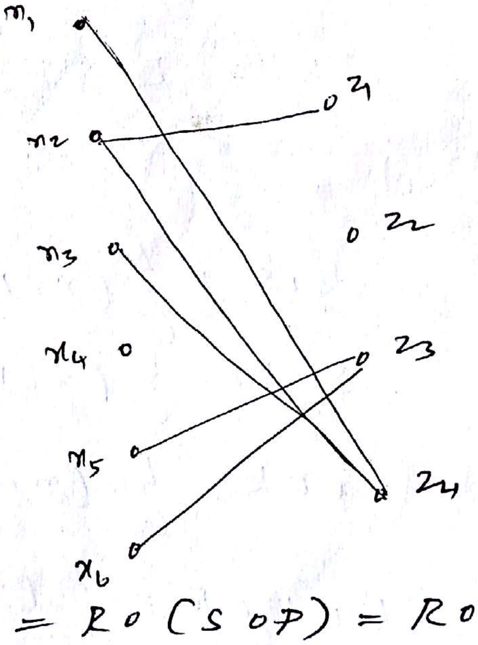
$$R \circ S = \{(x, z) \mid x \in X \wedge z \in Z \wedge (\exists y)(y \in Y \wedge (x, y) \in R \wedge (y, z) \in S)\}$$

The operation of obtaining $R \circ S$ from R and S is called COMPOSITION of relations.

For example in fig.

$$X \xrightarrow{R} Y \xrightarrow{S} Z \Rightarrow X \xrightarrow{R \circ S} Z$$





Note: $R \circ S \neq S \circ R$ but $(R \circ S) \circ P = R \circ (S \circ P) = R \circ S \circ P$

1. Let $R = \{(1,2), (3,4), (2,2)\}$ & $S = \{(4,2), (2,5), (3,1), (1,3)\}$. Find $R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$, $R \circ R$, $S \circ S$ & $R \circ R \circ R$. $S \circ S \circ S$

Sol

$RoS = \{(1,5), (3,2), (2,5)\}$

$SoR = \{(4,2), (3,2), (5,4)\} \neq RoS$

$Ro(SoR) = \{(3,2)\}$

$(RoS)oR = \{(3,2)\}$

42 25 31 13
42 25 31 13

$RoR = \{(1,2), (2,2)\}$

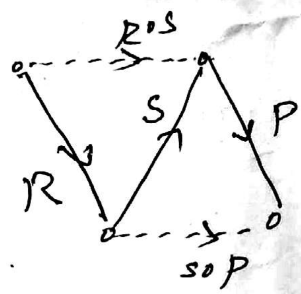
$SoS = \{(4,5), (3,3), (1,1)\}$

$RoRoR = \{(1,2), (2,2)\}$

Let R and S be two relations on a set of the integers I: $R = \{(x, 2x) \mid x \in I\}$,
$S = \{(x, 7x) \mid x \in I\}$.

Find RoS, RoR, RoRoR & RoSoR.

Sol



$RoS = \{(x, 14x) \mid x \in I\}$

$SoR = \{(x, 14x) \mid x \in I\}$

$RoR = \{(x, 4x) \mid x \in I\}$

$RoRoR = \{(x, 8x) \mid x \in I\}$

$RoSoR = \{(x, 28x) \mid x \in I\}$

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Relation Matrix and the graph of a Relation:-

A relation R from a finite set X to a finite set Y can also be represented by a matrix called the RELATION MATRIX (r_{ij}) of R.

Let $X = \{x_1, x_2, \dots x_m\}$, $Y = \{y_1, y_2, \dots y_n\}$ and R be a relation from X to Y.
The relation matrix is defined as

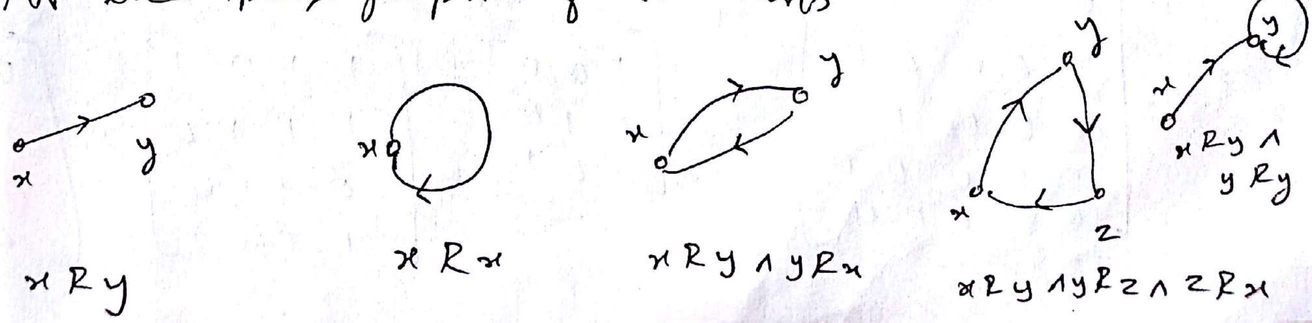
$$r_{ij} = \begin{cases} 1 & \text{if } x_i R y_j \\ 0 & \text{if } x_i \not R y_j \end{cases}$$

where r_{ij} is the element in the i^{th} row and j^{th} column. If X has m elts & Y has n elts then the relation matrix is an $m \times n$ matrix.

Note:

1. Let R be a relation in a set $X = \{x_1, \dots x_m\}$. The elements of X are represented by points or circles called NODES.
2. If $x_i R x_i$, we get an arc which starts from node x_i and returns to node x_i. Such an arc is called a LOOP.

For example, graphs of relations



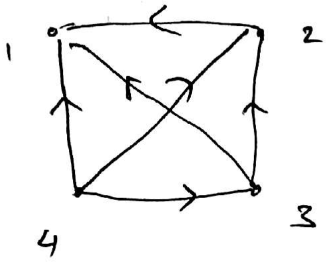
xRy $\quad$ xRx $\quad$ $xRy \wedge yRx$ $\quad$ $xRy \wedge yRz \wedge zRx$ $\quad$ $xRy \wedge yRy$

Eg 1. Let $X = \{1, 2, 3, 4\}$ & $R = \{(x, y) \mid x > y\}$. Draw the graph of R and also give its matrix.

Sol

The graph and the corresponding relation matrix for the relation

$R = \{(4,1), (4,2), (4,3), (3,1), (3,2), (2,1)\}$ is given in

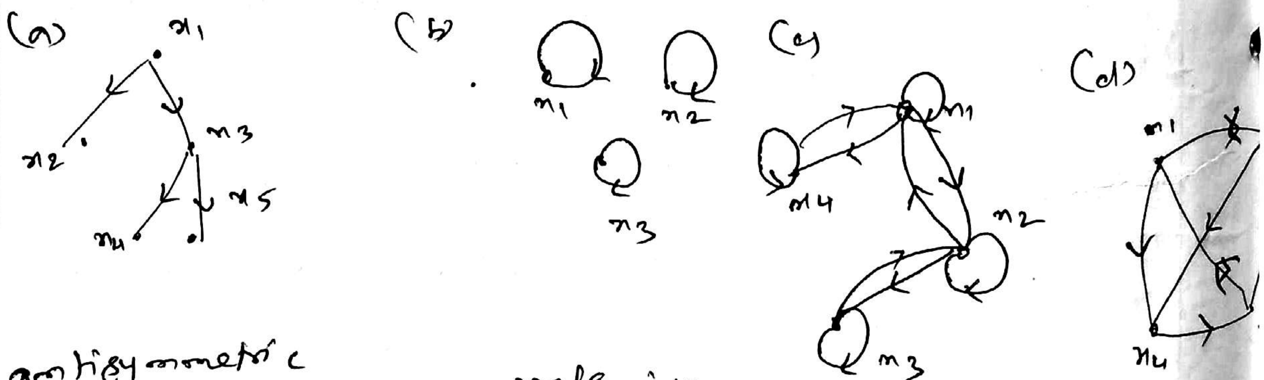


$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Eg 2. Determine the properties of the relations given by the graphs shown by the fig. and also write the corresponding relation matrices.

Sol.

The relation given by the graph in



antisymmetric   reflexive   ref. & Sym   transitive

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

H.W 1. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define R as '$<$' on A. Find the relation matrix.

2. Draw the relation graph for the following relations

(a) $R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$ on the set $X = \{1, 2, 3, 4\}$

(b) $R_1 = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$ on the set $Y = \{1, 2, 3\}$.

Q.3 Let R be the relation represented by $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find the relation matrix representing (a) R^{-1} (b) R^c (c) R^2

Sol

(a) To get the inverse relation M_R^{-1} of a relation matrix (M_R) just write the transpose of M_R.

$$M_R^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) To find the complement relation matrix, replace 0 by 1 & 1 by 0 in the given relation matrix

$$M_R^c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) To find the relation matrix of R^2

Now $R^2 = R \circ R$; $M_R^2 = M_R \cdot M_R$.

$$M_R^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Graphical Representation:

The relation array can be viewed graphically as elements of sets represented by nodes and an ordered pair is represented by an edge between the vertices that correspond to the pair elements, with an arrow pointing to the second element of the pair.

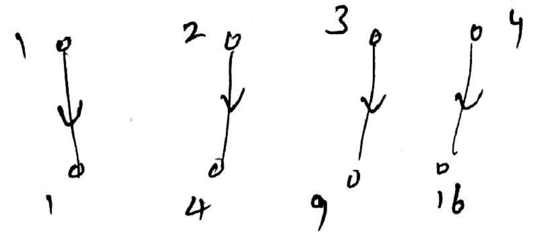
Eg 1: Let $A = \{1, 2, 3, 4\}$ & $B = \{1, 4, 9, 16\}$ and the relation $R = \{(1,1), (2,4), (3,4), (4,16)\}$. Draw the relation graph.

Sol

Relation matrix

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 4 & 9 & 16 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Relation graph



1. Let $A = \{a, b, c, d\}$, $B = \{a, e, f, d\}$ and let $R = \{(a,e), (a,f), (b,e), (c,f), (b,d), (d,d), (d,a)\}$ Draw the relation graph 13

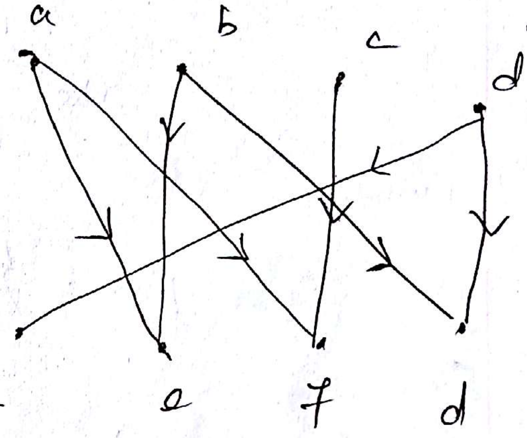
Sol.

Relation matrix

$$\begin{array}{c|cccc} & a & e & f & d \\ \hline a & & & & \\ b & & & & \\ c & & & & \\ d & & & & \end{array}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & e & f & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Relation graph



2. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 9, 10\}$, $C = \{5, 6, 7, 8\}$

$R = \{(1,1), (1,3), (2,9), (2,10), (3,3), (4,10)\}$ &

$S = \{(1,5), (3,7), (9,7), (10,8)\}$. Find $R \circ S$ and its relation graph.

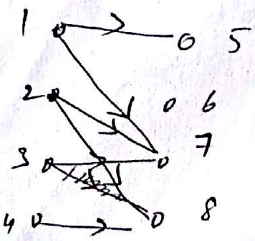
Sol

$R \circ S = \{(1,5), (1,7), (2,7), (2,8), (3,7), (4,8)\}$

The corresponding matrix is

$$M_{R \circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The corresponding relation graph ~~matrix~~ $G_{R \circ S}$ is



Equivalence classes: (or) R - equivalence class gen... by

If R is an equivalence relation on a set A, the set of all elements of A that are related to an element a of A is called the EQUIVALENCE CLASS of 'a' and denoted by $[a]_R$. (or) a/R

(OR)

The equivalence class of 'a' under the relation R is defined as:

$$[a] = \{ x \mid (a, x) \in R \}$$

Ex.

Consider the relation R defined on Z by $x R y \Leftrightarrow x - y$ is a multiple of 3.

Then

$$[0] = \{ y \in Z \mid y - 0 = 3k \text{; where } k \in Z \}$$
$$= \{ 0, \pm 3, \pm 6 \ldots \pm 3k \ldots \}$$

$$[1] = \{ 3k+1 \mid k \in Z \}$$
$$= \{ \ldots -5, -2, 1, 4, 7 \ldots \}$$

$$[2] = \{ 3k+2 \mid k \in Z \}$$
$$= \{ \ldots -4, -1, 2, 5, 8 \ldots \}$$

$$[3] = \{ 3k+3 \mid k \in Z \}$$
$$= \{ \ldots -3, 0, 3, 6 \ldots \}$$

(OR)

1/12 Let Z be the set of integers and let R be the relation called "Congruence modulo 3" defined by $R = \{(x,y) \mid x \in Z \wedge y \in Z \wedge (x-y) \text{ is divisible by } 3\}$ Determine the equivalence classes generated by the elements of Z.

Sol The equivalence classes are

$$[0]_R = \{\ldots -6, -3, 0, 3, 6 \ldots\}$$
$$[1]_R = \{\ldots -5, -2, 1, 4, 7 \ldots\}$$
$$[2]_R = \{\ldots -4, -1, 2, 5, 8 \ldots\}$$

$$Z/R = \{[0]_R, [1]_R, [2]_R\}.$$

2. Prove that any two equivalence classes are identical or disjoint.

Sol.

T.P. $(a,b) \in R \Rightarrow [a]_R = [b]_R$

Suppose $(a,b) \in R$

let $x \in [a] \Leftrightarrow (x,a) \in R$

$\Leftrightarrow (x,b) \in R$

($\because (x,a) \in R$ & $(a,b) \in R$ $\Rightarrow R$ is transitive)

$\Leftrightarrow x \in [b]$

$\Leftrightarrow [a] = [b]$

$\therefore [a] = [b]$.

Now suppose $[a], [b]$ are two equivalence classes

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Unit 1 Set theory

Questions	OPT 1	OPT 2	OPT3	OPT 4	ANSWERS
____ is a collection of well-defined objects.	element	member	set	finite set	set
{a,b,c} then cardinality of the set is ________	nullset	one	two	three	three
The two sets A and B are called as _________ if n(a) = n(B)	equal set	equivalent	null set	Subset	equivalent set
The two sets A and B are called as _________ if the sets have the same elements.	equal set	equalent se	null set	Subset	equal se
If every element of the set A is an element of the another set B then A is ________ of B	subset	superset	empty set	universal set	subset
If every element of the set A is an element of the another set B then B is ________ of A	subset	superset	empty set	universal set	superset
If the cardinality of the set is zero then the set is ______________	subset	superset	empty set	universal set	empty se
Empty set is a __________ of every set.	subset	superset	empty set	universal set	subset
Universal set is the _____________ of all the sets.	subset	superset	empty set	universal set	superset
If A = {1,2,3,4} and B = {2,4} then A intersection B =	{2,4}	{1,2,3,4}	{1,2}	{}	{2,4}
If A = {1,2,3,4} and B = {2,4} then A union B =	{2,4}	{1,2,3,4}	{1,2}	{}	{1,2,3,4}
Two sets are said to be disjoint if A intersection B =	A		B	A union B	{}
If n subsets of a set are given, then the number of __________ is 2 power n	min terms	minimax terms	sets	infinite sets	min terms
If n subsets of a set are given, then the number of __________ is 2 power n	max terms	minimax terms	sets	infinite sets	max terms
Every singleton subset constitutes a ___________	set	partition	min term	max term	partition
A×B ______ B×A	equal set	not equal	does not exist	exist	not equa
A __________ R from a set A to a set B is a subset R of the cartesian product A×B	Relation	Binary relation	duality principle	partition of a set	Binary relation
Let R be a relation on a set A then if aRa for all a in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	reflexive
Let R be a relation on a set A then if aRb then bRa for all a,b in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	symmetr
Let R be a relation on a set A then if aRb and bRc then aRc for all a,b,c in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	transitive
A relation R on a set A is called an equivalence relation if R is	reflexive , symmetric and transitive	reflexive , antisymmetric and transitive	irreflexive , symmetric and transitive	irreflexive, antisymmetric and transitive	reflexive , symmetric and transitive
A relation R on a set A is called an partial order relation if R is	reflexive , symmetric and transitive	reflexive , antisymmetric and transitive	irreflexive , symmetric and transitive	irreflexive, antisymmetric and transitive	reflexive , antisymmetric and transitive

A - B ______ B - A	equal set	not equal	does not exist	exist	not equa
A method which pairs elements of the set A with unique elements of the set B is called_________	Set	domain	codomai n	function	function
One to one function is also called as_________	injective function	surjective function	bijective function	inverse function	injectiv e function
Onto function is also called as __________	injective function	surjective function	bijective function	inverse function	surjecti ve function
A function whichn is one to one and onto is called as __________	injective function	surjective function	bijective function	inverse function	bijectiv e function
If every element of the domain is mapped to unique element of the codomain then the function is called as__________	injective function	surjective function	bijective function	constant function	constan t function
If atleast one element of the codomain is not mapped by any element of the domain then the function is called as _________	into function	surjective function	bijective function	constant function	into function
A function that assigns each element of a set into itself is called as __________	surjective function	identity function	constant function	into function	identity function
A one to one mapping of a set onto itself is sometimes called _________ of the set.	Constant function	Inverse function	permutati on function	Compos ition function	permut ation function
A bijective function is called invertible because we can define __________ of this function.	Constant function	Inverse function	permutati on function	Compos ition function	Inverse function
The commutative law does not hold for __________	Constant function	Inverse function	permutati on function	Compos ition function	Compo sition function
Operations of the set union are ___________ on the set of subsets of a universal set	Uninary operation	Binary operation	compositi on function	permuta tion function	Binary operati on
Operations of the set intersection are ___________ on the set of subsets of a universal set	Uninary operation	Binary operation	compositi on function	permuta tion function	Binary operati on
The absolute value of an integer n is a ________ on the set Z of integer.	Uninary operation	Binary operation	compositi on function	permuta tion function	Uninary operati on
The complement of a set is a __________ on the power set of any set.	Uninary operation	Binary operation	compositi on function	permuta tion function	Uninary operati on
If the identity for a binary operation on a set exists, then it is ________	unique	dual	zero	finite	unique

Over the set of real numbers the element __________ is the identity for addition	1	0	1 and 0	infinite	0

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LECTURE NOTES ON RELATIONS AND FUNCTIONS

PETE L. CLARK

Contents

1. Relations

1.1. **The idea of a relation.** Let X and Y be two sets. We would like to formalize the idea of a **relation** between X and Y. Intuitively speaking, this is a well-defined "property" R such that given any $x \in X$ and $y \in Y$, either x bears the property R to y, or it doesn't (and not both!). Some important examples:

Example 1.1. *Let X be a set of objects and let Y be a set of sets. Then "membership" is a relation R from X to Y: i.e., we have xRy if $x \in y$.*

Example 1.2. *Let S be a set, and let $X = Y = 2^S$, the power set of S (recall that this is the set of all subsets of S. Then containment, $A \subseteq B$ is a relation between X and Y. (Proper containment, $A \subsetneq B$, is also a relation.)*

Example 1.3. *Let $X = Y$. Then equality is a relation from X to Y: we say xRy iff $x = y$. Also inequality is a relation between X and Y: we say xRy iff $x \neq y$.*

Example 1.4. *Let $X = Y = \mathbb{R}$. Then $\leq, <, \geq, >$ are relations between $\mathbb{R}$ and $\mathbb{R}$.*

Example 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then we can define a relation from $\mathbb{R}$ to $\mathbb{R}$, by xRy if and only if $y = f(x)$.*

Date: April 15, 2016.

Example 1.6. *Let $X = Y = \mathbb{Z}$. Then divisibility is a relation between $\mathbb{Z}$ and $\mathbb{Z}$: we say xRy if $x \mid y$.*

Example 1.7. *Let $X = Y = \mathbb{Z}$. Then "having the same parity" is a relation between $\mathbb{Z}$ and $\mathbb{Z}$.*

In many of the above examples we have $X = Y$. This will often (but certainly not always!) be the case, and when it is we may speak of relations **on X**.

1.2. The formal definition of a relation.

We still have not given a formal definition of a relation between sets X and Y. In fact the above way of thinking about relations is easily formalized, as was suggested in class by Adam Osborne: namely, we can think of a relation R as a function from $X \times Y$ to the two-element set $\{\mathrm{TRUE}, \mathrm{FALSE}\}$. In other words, for $(x, y) \in X \times Y$, we say that xRy if and only if $f((x, y)) = \mathrm{TRUE}$.

This is a great way of thinking about relations. It has however one foundational drawback: it makes the definition of a relation depend on that of a function, whereas the standard practice for about one hundred years is the reverse: we want to define a function as a special kind of relation (c.f. Example 5 above). The familiar correspondence between logic and set theory leads us to the official definition:

Definition: A relation R between two sets X and Y is simply a **subset** of the Cartesian product $X \times Y$, i.e., a collection of ordered pairs (x, y).

(Thus we have replaced the basic logical dichotomy "TRUE/FALSE" with the basic set-theoretic dichotomy "is a member of/ is not a member of".) Note that this new definition has some geometric appeal: we are essentially identifying a relation R with its *graph* in the sense of precalculus mathematics.

We take advantage of the definition to adjust the terminology: rather than speaking (slightly awkwardly) of relations "from X to Y" we will now speak of relations **on X × Y**. When $X = Y$ we may (but need not!) speak of relations **on X**.

Example 1.8. *Any curve in $\mathbb{R}^2$ defines a relation on $\mathbb{R} \times \mathbb{R}$. E.g. the unit circle*

$$x^2 + y^2 = 1$$

is a relation in the plane: it is just a set of ordered pairs.

1.3. Basic terminology and further examples.

Let X,Y be sets. We consider the set of all relations on $X \times Y$ and denote it by $\mathcal{R}(X, Y)$. According to our formal definition we have

$$\mathcal{R}(X, Y) = 2^{X \times Y},$$

i.e., the set of all subsets of the Cartesian product $X \times Y$.

Example 1.9. *a) Suppose $X = \varnothing$. Then $X \times Y = \varnothing$ and $\mathcal{R}(X \times Y) = 2^{\varnothing} = \{\varnothing\}$. That is: if X is empty, then the set of ordered pairs (x, y) for $x \in X$ and $y \in Y$ is empty, so there is only one relation: the empty relation.*
b) Suppose $Y = \varnothing$. Again $X \times Y = \varnothing$ and the discussion is the same as above.

Example 1.10. *a) Suppose $X = \{\bullet\}$ consists of a single element. Then $X \times Y = \{(\bullet, y) \mid y \in Y\}$; in other words, $X \times Y$ is essentially just Y itself, since the first coordinate is always the same. Thus a relation R on $X \times Y$ corresponds to a subset of Y: formally, the set of all $y \in Y$ such that $\bullet Ry$.*
b) Suppose $Y = \{\bullet\}$ consists of a single element. The discussion is analogus to that of part a), and relations on $X \times Y$ correspond to subsets of X.

Example 1.11. *Suppose X and Y are finite sets, with $\#X = m$ and $\#Y = n$. Then $\mathcal{R}(X, Y) = 2^{X \times Y}$ is finite, of cardinality*

$$\#2^{X \times Y} = 2^{\#X \times Y} = 2^{\#X \cdot \#Y} = 2^{mn}.$$

The function 2^{mn} grows rapidly with both m and n, and the upshot is that if X and Y are even moderately large finite sets, the set of all relations on $X \times Y$ is very large. For instance if $X = \{a, b\}$ and $Y = \{1, 2\}$ then there are $2^{2 \cdot 2} = 16$ relations on $X \times Y$. It is probably a good exercise for you to write them all down. However, if $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ then there are $2^{3 \cdot 3} = 512$ relations on $X \times Y$, and – with apologies to the Jackson 5? – it is less easy to write them all down.

Exercise 1.1. *Let X and Y be nonempty sets, at least one of which is infinite. Show: $\mathcal{R}(X, Y)$ is infnite.*

Given two relations R_1 and R_2 between X and Y, it makes sense to say that $R_1 \subseteq R_2$: this means that R_1 is "stricter" than R_2 or that R_2 is "more permissive" than R_1. This is a very natural idea: for instance, if X is the set of people in the world, R_1 is the brotherhood relation – i.e., $(x, y) \in R_1$ iff x and y are brothers – and R_2 is the sibling relation – i.e., $(x, y) \in R_2$ iff x and y are siblings – then $R_1 \subsetneq R_2$: if x and y are brothers then they are also siblings, but not conversely.

Among all elements of $\mathcal{R}(X, Y)$, there is one relation $R_\varnothing$ which is the strictest of all, namely $R_\varnothing = \emptyset$:[1] that is, for *no* $(x, y) \in X \times Y$ do we have $(x, y) \in R_\varnothing$. Indeed $R_\varnothing \subset R$ for any $R \in \mathcal{R}(X, Y)$. At the other extreme, there is a relation which is the most permissive, namely $R_{X \times Y} = X \times Y$ itself: that is, for *all* $(x, y) \in X \times Y$ we have $(x, y) \in R_{X \times Y}$. And indeed $R \subset R_{X \times Y}$ for any $R \in \mathcal{R}(X, Y)$.

Example 1.12. *Let $X = Y$. The equality relation $R = \{(x, x) \mid x \in X\}$ can be thought of geometrically as the diagonal of $X \times Y$.*

The **domain**[2] of a relation $R \subseteq X \times Y$ is the set of $x \in X$ such that there exists $y \in Y$ with $(x, y) \in R$. In other words, it is the set of all elements in x which relate to at least one element of Y.

Example 1.13. *The circle relation $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ has domain $[-1, 1]$.*

Given a relation $R \subset X \times Y$, we can define the **inverse relation** $R^{-1} \subset Y \times X$ by interchanging the order of the coordinates. Formally, we put

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}.$$

Geometrically, this corresponds to reflecting across the line $y = x$.

[1] The notation here is just to emphasize that we are viewing $\varnothing$ as a relation on $X \times Y$.

[2] I don't like this terminology. But it is used in the course text, and it would be confusing to change it.

Example 1.14. *Consider the relation $R \subset \mathbb{R} \times \mathbb{R}$ attached to the function $f(x) = x^2$:*

$$R = \{(x, x^2) \mid x \in \mathbb{R}\}.$$

The graph of this relation is an upward-opening parabola: it can also be described by the equation $y = x^2$. The inverse relation R^{-1} is $\{(x^2, x) \mid x \in \mathbb{R}\}$, which corresponds to the equation $x = y^2$ and geometrically is a parabola opening rightward. Note that the domain of the original relation R is $\mathbb{R}$, whereas the domain of R^{-1} is $[0, \infty)$. Moreover, R^{-1} is not a function, since some values of x relate to more than one y-value: e.g. $(1, 1)$ and $(1, -1)$ are both in R^{-1}.

Example 1.15. *Consider the relation attached to the function $f(x) = x^3$: namely*

$$R = \{(x, x^3) \mid x \in \mathbb{R}\}.$$

This relation is described by the equation $y = x^3$; certainly it is a function, and its domain is $\mathbb{R}$. Consider the inverse relation

$$R^{-1} = \{(x^3, x) \mid x \in \mathbb{R}\},$$

which is described by the equation $x = y^3$. Since every real number has a unique real cube root, this is equivalent to $y = x^{\frac{1}{3}}$. Thus this time R^{-1} is again a function, and its domain is $\mathbb{R}$.

Later we will study functions in detail and one of our main goals will be to understand the difference between Examples 1.14 and 1.15.

1.4. **Properties of relations.**

Let X be a set. We now consider various properties that a relation R on X – i.e., $R \subset X \times X$ may or may not possess.

Reflexivity: For all $x \in X$, $(x, x) \in R$.

In other words, each element of X bears relation R to itself. Another way to say this is that the relation R contains the equality relation on X.

Exercise 1.2. *Which of the relations in Examples 1.1 through 1.15 are reflexive?*

Anti-reflexivity: For all $x \in X$, $(x, x) \not inR$.

Certainly no relation on X is both reflexive and anti-reflexive (except in the silly case $X = \varnothing$ when both properties hold vacuously) . However, notice that a relation need not be either reflexive or anti-reflexive: if there are $x, y \in X$ such that $(x, x) \in R$ and $(y, y) \notin R$, then neither property holds.

Symmetry: For all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Again, this has a geometric interpretation in terms of symmetry across the diagonal $y = x$. For instance, the relation associated to the function $y = \frac{1}{x}$ is symmetric since interchanging x and y changes nothing, whereas the relation associated to the function $y = x^2$ is not. (Looking ahead a bit, a function $y = f(x)$ is symmetric iff it coincides with its own inverse function.)

Exercise 1.3. *Which of the relations in Examples 1.1 through 1.15 are symmetric?*

Example 1.16. *Let V be a set. A* ***(simple, loopless, undirected) graph*** *– in the sense of graph theory, not graphs of functions! – is given by a relation E on V which is irreflexive and symmetric. Thus: for $x, y \in V$, we say that x and y are* ***adjacent*** *if $(x, y) \in E$. Moreover x is never adjacent to itself, and the adjacency of x and y is a property of the unordered pair $\{x, y\}$: if x is adjacent to y then y is adjacent to x.*

Anti-Symmetry: for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Exercise 1.4. *Which of the relations in Examples 1.1 through 1.16 are anti-symmetric?*

Transitivity: for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

"Being a parent of" is not transitive, but "being an ancestor of" is transitive.

Exercise 1.5. *Which of the relations in Examples 1.1 through 1.15 are transitive?*

Worked Exercise 1.6.
Let R be a relation on X. Show the following are equivalent:
(i) R is both symmetric and anti-symmetric.
(ii) R is a subrelation of the equality relation.
Solution*: Suppose that we have a relation R on X which is both symmetric and anti-symmetric. Then, for all $x, y \in R$, if $(x, y) \in R$, then by symmetry we have also $(y, x) \in R$, and then by anti-symmetry we have $x = y$. Thus we've shown that if (i) holds, the only possible elements $(x, y) \in R$ are those of the form (x, x), which means that R is a subrelation of the equality relation. Conversely, if R is a subrelation of equality and $(x, y) \in R$, then $y = x$, so $(y, x) \in R$. Similarly, if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$. So R is both symmetric and anti-symmetric.*

Now we makes two further defintions of relations with possess certain combinations of these basic properties. The first is the most important definition in this section.

An **equivalence relation** on a set X is a relation on X which is reflexive, symmetric and transitive.

A **partial ordering** on a set X is a relation on X which is reflexive, anti-symmetric and transitive.

Exercise 1.7. *Which of the relations in Examples 1.1 through 1.16 are equivalence relations? Which are partial orderings?*

We often denote equivalence relations by a tilde – $x \sim y$ – and read $x \sim y$ as "x is equivalent to y". For instance, the relation "having the same parity" on $\mathbb{Z}$ is an equivalence relation, and $x \sim y$ means that x and y are both even or both odd. Thus it serves to group the elements of $\mathbb{Z}$ into subsets which share some common property. In this case, all the even numbers are being grouped together and all the odd numbers are being grouped together. We will see shortly that this is a characteristic property of equivalence relations: every equivalence relation on a set X determines a **partition** on X and conversely, given any partition on X we can define an equivalence relation.

The concept of a partial ordering should be regarded as a "generalized less than or equal to" relation. Perhaps the best example is the containment relation $\subseteq$ on the power set $\mathcal{P}(S)$ of a set S. This is a very natural way of regarding one set as "bigger" or "smaller" than another set. Thus the insight here is that containment satisfies many of the formal properties of the more familiar $\leq$ on numbers. However there is one property of $\leq$ on numbers that does not generalize to $\subseteq$ (and hence not to an arbitrary partial ordering): namely, given any two real numbers x, y we must have either $x \leq y$ or $y \leq x$. However for sets this does not need to be the case (unless S has at most one element). For instance, in the power set of the positive integers, we have $A = \{1\}$ and $B = \{2\}$, so neither is it true that $A \subseteq B$ or that $B \subseteq A$. This is a much stronger property of a relation:

Totality: For all $x, y \in X$, either $(x, y) \in R$ or $(y, x) \in R$.

A **total ordering** (or **linear ordering**) on a set X is a partial ordering satisfying dichotomy.

Example 1.17. *The relation $\leq$ on $\mathbb{R}$ is a total ordering.*

There is an entire branch of mathematics – **order theory** – devoted to the study of partial orderings.[3] In my opinion order theory gets short shrift in the standard mathematics curriculum (especially at the advanced undergraduate and graduate levels): most students learn only a few isolated results which they apply frequently but with little context or insight. Unfortunately we are not in a position to combat this trend: partial and total orderings will get short shrift here as well!

1.5. Partitions and Equivalence Relations.

Let X be a set, and let $\sim$ be an equivalence relation on X.

For $x \in X$, we define the **equivalence class of x** as

$$[x] = \{y \in X \mid y \sim x\}.$$

For example, if $\sim$ is the relation "having the same parity" on $\mathbb{Z}$, then

$$[2] = \{\ldots, -4, -2, 0, 2, 4, \ldots\},$$

i.e., the set of all even integers. Similarly

$$[1] = \{\ldots - 3, -1, 1, 3, \ldots\}$$

is the set of all odd integers. But an equivalence class in general has many "representatives". For instance, the equivalence class $[4]$ is the set of all integers having the same parity as 4, so is again the set of all even integers: $[4] = [2]$. More generally, for any even integer n, we have $[n] = [0]$ and for any odd integer n we have $[n] = [1]$. Thus in this case we have partitioned the integers into two subsets: the even integers and the odd integers.

We claim that given any equivalence relation $\sim$ on a set X, the set $\{[x] \mid x \in X\}$ forms a partition of X. Before we proceed to demonstrate this, observe that we are now strongly using our convention that there is no "multiplicity" associated to membership in a set: e.g. the sets $\{4, 2 + 2, 1^1 + 3^0 + 2^1\}$ and $\{4\}$ are equal. The

[3] For instance, there is a journal called **Order**, in which a paper of mine appears.

above representation $\{[x] \mid x \in X\}$ is highly redundant: for instance in the above example we are writing down the set of even integers and the set of odd integers infinitely many times, but it only "counts once" in order to build the set of subsets which gives the partition.

With this disposed of, the verification that $\mathcal{P} = \{[x] \mid x \in X\}$ gives a partition of X comes down to recalling the definition of a partition and then following our noses. There are three properties to verify:

(i) That every element of $\mathcal{P}$ is nonempty. Indeed, the element $[x]$ is nonempty because it contains x! This is by reflexivity: $x \sim x$, so $x \in \{y \in X \mid y \sim x\}$.
(ii) That the union of all the elements of $\mathcal{P}$ is all of X. But again, the union is indexed by the elements x of X, and we just saw that $x \in [x]$, so every x in X is indeed in at least one element of $\mathcal{P}$.
(iii) Finally, we must show that if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$: i.e., any two elements of $\mathcal{P}$ which have a common element must be the same element. So suppose that there exists $z \in [x] \cap [y]$. Writing this out, we have $z \sim x$ and $z \sim y$. By symmetry, we have $y \sim z$; from this and $z \sim x$, we deduce by transitivity that $y \sim x$, i.e., $y \in [x]$. We claim that it follows from this that $[y] \subset [x]$. To see this, take any $w \in [y]$, so that $w \sim y$. Since $w \sim x$, we conclude $w \sim x$, so $w \in [x]$. Rerunning the above argument with the roles of x and y interchanged we get also that $[y] \subset [x]$, so $[x] = [y]$. This completes the verification.

Note that the key fact underlying the proof was that any two equivalence classes $[x]$ and $[y]$ are either disjoint or coincident. Note also that we did indeed use all three properties of an equivalence relation.

Now we wish to go in the other direction. Suppose X is a set and $\mathcal{P} = \{U_i\}_{i \in I}$ is a partition of X(here I is just an index set). We can define an equivalence relation $\sim$ on X as follows: we say that $x \sim y$ if there exists $i \in I$ such that $x, y \in U_i$. In other words, we are decreeing x and y to be equivalent exactly when they lie in the same "piece" of the partition. Let us verify that this is an equivalence relation. First, let $x \in X$. Then, since $\mathcal{P}$ is a partition, there exists some $i \in I$ such that $x \in U_i$, and then x and x are both in U_i, so $x \sim x$. Next, suppose that $x \sim y$: this means that there exists $i \in I$ such that x and y are both in U_i; but then sure enough y and x are both in U_i ("and" is commutative!), so $y \sim x$. Similarly, if we have x, y, z such that $x \sim y$ and $y \sim z$, then there exists i such that x and y are both in U_i and a possibly different index j such that y and z are both in U_j. But since $y \in U_i \cap U_j$, we must have $U_i = U_j$ so that x and z are both in $U_i = U_j$ and $x \sim z$.

Moreover, the processes of passing from an equivalence relation to a partition and from a partition to an equivalence relation are mutually inverse: if we start with an equivalence relation R, form the associated partition $\mathcal{P}(R)$, and then form the associated equivalence relation $\sim (\mathcal{P}(R))$, then we get the equivalence relation R that we started with, and similarly in the other direction.

1.6. **Examples of equivalence relations.**

Example 1.18. *(Congruence modulo n) Let $n \in \mathbb{Z}^+$. There is a natural partition of $\mathbb{Z}$ into n parts which generalizes the partition into even and odd. Namely, we put*

$$Y_1 = \{\ldots, -2n, -n, 0, n, 2n, \ldots\} = \{kn \mid k \in \mathbb{Z}\}$$

the set of all multiples of n,

$$Y_2 = \{\ldots, -2n+1, -n+1, 1, n+1, 2n+1\ldots\} = \{kn+1 \mid k \in \mathbb{Z}\},$$

and similarly, for any $0 \leq d \leq n-1$, we put

$$Y_d = \{\ldots, -2n+d, -n+d, d, n+d, 2n+d\ldots\} = \{kn+d \mid kin\mathbb{Z}\}.$$

That is, Y_d is the set of all integers which, upon division by n, leave a remainder of d. Earlier we showed that the remainder upon division by n is a well-defined integer in the range $0 \leq d < n$. Here by "well-defined", I mean that for $0 \leq d_1 \neq d_2 < n$, the sets Y_{d_1} and Y_{d_2} are disjoint. Recall why this is true: if not, there exist k_1, k_2 such that $k_1 n + d_1 = k_2 n + d_2$, so $d_1 - d_2 = (k_2 - k_1)n$, so $d_1 - d_2$ is a multiple of n. But $-n < d_1 - d_2 < n$, so the only multiple of n it could possibly be is 0, i.e., $d_1 = d_2$. It is clear that each Y_d is nonempty and that their union is all of $\mathbb{Z}$, so $\{Y_d\}_{d=0}^{n-1}$ gives a partition of $\mathbb{Z}$.

The corresponding equivalence relation is called **congruence modulo** n, and written as follows:

$$x \equiv y \pmod{n}.$$

What this means is that x and y leave the same remainder upon division by n.

Proposition 1.19. *For integers x, y, the following are equivalent:*
(i) $x \equiv y \pmod{n}$.
(ii) $n \mid x - y$.

Proof. Suppose that $x \equiv y \pmod{n}$. Then they leave the same remainder, say d, upon division by n: there exist $k_1, k_2 \in \mathbb{Z}$ such that $x = k_1 n + d$, $y = k_2 n + d$, so $x - y = (k_1 - k_2)n$ and indeed $n \mid x - y$. Conversely, suppose that $x = k_1 n + d_1$, $y = k_2 n + d_2$, with d_1 and d_2 distinct integers both in the interval $[0, n-1]$. Then, if n divides $x - y = (k_1 - k_2)n + (d_1 - d_2)$, then it also divides $d_1 - d_2$, which as above is impossible since $-n < d_1 - d_2 < n$. □

Example 1.20. *(Fibers of a function) Let $f : X \to Y$ be a function. We define a relation R on X by $(x_1, x_2) \in R$ iff $f(x_1) = f(x_2)$. This is an equivalence relation. The equivalence class of $[x]$ is called the* ***fiber over f(x)****.*

1.7. Extra: composition of relations.

Suppose we have a relation $R \subset X \times Y$ and a relation $S \subset Y \times Z$. We can define a **composite relation** $S \circ R \subset X \times Z$ in a way which will generalize compositions of functions. Compared to composition of functions, composition of relations is much less well-known, although as with many abstract concepts, once it is pointed out to you, you begin to see it "in nature'. This section is certainly optional reading.

The definition is simply this:

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y \text{such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

In other words, we say that x in the first set X relates to z in the third set Z if there exists at least one intermediate element y in the second set such that x relates to y and y relates to z.

In particular, we can always compose relations on a single set X. As a special case, given a relation R, we can compose it with itself: say

$$R^{(2)} = R \circ R = \{(x,z) \in X \times X \mid \exists y \in X \text{ such that } xRy \text{ and } yRz\}.$$

Proposition 1.21. *For a relation R on X, the following are equivalent:*
(i) R is transitive.
(ii) $R^{(2)} \subseteq R$.

Exercise 1.8. *Show that the composition of relations is associative.*

Exercise 1.9. *Show: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.*

Exercise 1.10. *Let $X = \{1, \ldots, N\}$. To a relation R on X we associate its **adjacency matrix** $M = M(R)$: if $(i,j) \in R$, we put $M(i,j) = 1$; otherwise we put $M(i,j) = 0$. Show that the adjacency matrix of the composite relation R^2 is the product matrix $M(R) \cdot M(R)$ in the sense of linear algebra.*

2. Functions

Let X and Y be sets. A **function** $f : X \to Y$ is a special kind of relation between X and Y. Namely, it is a relation $R \subset X \times Y$ satisfying the following condition: for all $x \in X$ there exists exactly one $y \in Y$ such that $(x,y) \in R$. Because element of y attached to a given element x of X is unique, we may denote it by $f(x)$.

Geometrically, a function is a relation which passes the **vertical line test**: every vertical line $x = c$ intersects the graph of the function in exactly one point. In particular, the domain of any function is all of X.

Example 2.1. *The equality relation $\{(x,x) \mid x \in X\}$ on X is a function: $f(x) = x$ for all x. We call this the **identity function** and denote it by 1_X.*

Example 2.2. *a) Let Y be a set. Then $\varnothing \times Y = \varnothing$, so there is a unique relation on $\varnothing \times Y$. This relation is – vacuously – a function.*
b) Let X be a set. Then $X \times \varnothing = \varnothing$, so there is a unique relation on $X \times \varnothing$, with domain $\varnothing$. If $X = \varnothing$, then we get the empty function $f : \varnothing \to \varnothing$. If $X \neq \varnothing$ then the domain is not all of X so we do not get a function.

If $f : X \to Y$ is a function, the second set Y is called the **codomain** of f. Note the asymmetry in the definition of a function: although every element x of the domain X is required to be associated to a unique element y of Y, the same is not required of elements y of the codomain: there may be multiple elements x in X such that $f(x) = y$, or there may be none at all.

The **image** of $f : X \to Y$ is $\{y \in Y$ such that $y = f(x)$ for some $x \in X.\}$[4]

In calculus one discusses functions with domain some subset of $\mathbb{R}$ and codomain $\mathbb{R}$. Moreover in calculus a function is usually (but not always...) given by some relatively simple algebraic/analytic expression, and the convention is that the domain is the largest subset of $\mathbb{R}$ on which the given expression makes sense.

[4]Some people call this the **range**, but also some people call the set Y (what we called the codomain) the range, so the term is ambiguous and perhaps best avoided.

Example 2.3.
a) The function $y = 3x$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Its range is all of $\mathbb{R}$.
b)The function $y = x^2$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Its range is $[0, \infty)$.
c) The function $y = x^3$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Its range is all of $\mathbb{R}$.
d) The function $y = \sqrt{x}$ is a function from $[0, \infty)$ to $\mathbb{R}$. Its range is $[0, \infty)$.
e) The arctangent $y = \arctan x$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Its range is $(\frac{-\pi}{2}, \frac{\pi}{2})$.

2.1. The set of all functions from X to Y.

Let X and Y be sets. We denote the set of all functions $f : X \to Y$ by Y^X. Why such a strange notation? The following simple and useful result gives the motivation. Recall that for $n \in \mathbb{Z}^+$, we put $[n] = \{1, 2, \ldots, n\}$, and we also put $[0] = \varnothing$. Thus $\#[n] = n$ for all $n \in \mathbb{N}$.

Proposition 2.4. *Let $m, n \in \mathbb{N}$. Then we have*

$$\#[m]^{[n]} = m^n.$$

In words: the set of all functions from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$ has cardinality m^n.

Proof. To define a function $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$, we must specify a sequence of elements $f(1), \ldots, f(n)$ in $\{1, \ldots, m\}$. There are m possible choices for $f(1)$, also m possible choices for $f(2)$, and so forth, up to m possible choices for $f(n)$, and these choices are independent. Thus we have $m \cdots m$ n times $= m^n$ choices overall. $\square$

2.2. Injective functions.

From the perspective of our course, the most important material on functions are the concepts *injectivity, surjectivity and bijectivity* and the relation of these properties with the existence of inverse functions.

A function $f : X \to Y$ is **injective** if every element y of the codomain is associated to at most one element $x \in X$. That is, f is injective if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Let us meditate a bit on the property of injectivity. One way to think about it is via a horizontal line test: a function is injective if and only if each horizontal line $y = c$ intersects the graph of f in **at most** one point. Another way to think about an injective function is as a function which entails no loss of information. That is, for an injective function, if your friend tells you $x \in X$ and you tell me $f(x) \in Y$, then I can, in principle, figure out what x is because it is uniquely determined.

Consider for instance the two functions $f(x) = x^2$ and $f(x) = x^3$. The first function $f(x) = x^2$ is not injective: if y is any positive real number then there are two x-values such that $f(x) = y$, $x = \sqrt{y}$ and $x = -\sqrt{y}$. Or, in other words, if $f(x) = x^2$ and I tell you that $f(x) = 1$, then you are in doubt as to what x is: it could be either $+1$ or -1. On the other hand, $f(x) = x^3$ is injective, so if I tell you that $f(x) = x^3 = 1$, then we can conclude that $x = 1$.

How can we verify in practice that a function is injective? One way is to construct an inverse function, which we will discuss further later. But in the special case when $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, the methods of calculus give useful criteria for injectivity.

Before stating the result, let us first recall the definitions of increasing and decreasing functions. A function $f : \mathbb{R} \to \mathbb{R}$ is **(strictly) increasing** if for all $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2 \implies f(x_1) < f(x_2)$. Similarly, f is **(strictly) decreasing** if for all $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2 \implies f(x_1) > f(x_2)$. Notice that a function which is increasing or decreasing is injective. The "problem" is that a function need not be either increasing or decreasing, although "well-behaved" functions of the sort one encounters in calculus have the property that their domain can be broken up into intervals on which the function is either increasing or decreasing. For instance, the function $f(x) = x^2$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

Theorem 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.*
a) If f is injective, then f is either increasing or decreasing.
b) If f is differentiable and either $f'(x) > 0$ for all $x \in \mathbb{R}$ or $f'(x) < 0$ for all $x \in \mathbb{R}$, then f is injective.

It is something of a sad reflection on our calculus curriculum that useful and basic facts like this are not established in a standard calculus course. However, the full details are somewhat intricate. We sketch a proof below.

Proof. We prove part a) by contraposition: that is, we assume that f is continuous and *neither* increasing nor decreasing, and we wish to show that it is not injective. Since f is not decreasing, there exist $x_1 < x_2$ such that $f(x_1) \leq f(x_2)$. Since f is not increasing, there exist $x_3 < x_4$ such that $f(x_3) \geq f(x_4)$. If $f(x_3) = f(x_4)$. We claim that it follows that there exist $a < b < c$ such that either
Case 1:$f(b) \geq f(a)$ and $f(b) \geq f(c)$, or
Case 2: $f(b) \leq f(a)$ and $f(b) \leq f(c)$.
This follows from a somewhat tedious consideration of cases as to in which order the four points x_1, x_2, x_3, x_4 occur, which we omit here. Now we apply the Intermediate Value Theorem to f on the intervals $[a, b]$ and $[b, c]$. In Case 1, every number smaller than $f(b)$ but sufficiently close to it is assumed both on the interval $[a, b]$ and again on the interval $[b, c]$, so f is not injective. In Case 2, every number larger than $f(b)$ but sufficiently close to it is assumed both on the interval $[a, b]$ and again on $[b, c]$, so again f is not injective.
As for part b), we again go by contraposition and assume that f is not injective: that is, we suppose that there exist $a < b$ such that $f(a) = f(b)$. Applying the Mean Value Theorem to f on $[a, b]$, we get that there exists c, $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0,$$

contradicting the assumption that $f'(x)$ is always positive or always negative. □

Remark: The proof shows that we could have replaced part b) with the apparently weaker hypothesis that for all $x \in \mathbb{R}$, $f'(x) \neq 0$. However, it can be shown that this is equivalent to f' always being positive or always being negative, a consequence of the **Intermediate Value Theorem For Derivatives**.

Example 2.6. *a) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \arctan x$. We claim f is injective. Indeed, it is differentiable and its derivative is $f'(x) = \frac{1}{1+x^2} > 0$ for all $x \in \mathbb{R}$. Therefore f is strictly increasing, hence injective.*
b) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = -x^3 - x$. We claim f is injective. Indeed, it is

differentiable and its derivative is $f'(x) = -3x^2 - 1 = -(3x^2 + 1) < 0$ *for all* $x \in \mathbb{R}$. *Therefore* f *is strictly decreasing, hence injective.*

Example 2.7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be given by* $f(x) = x^3$. *One meets this function in precalculus and calculus mathematics, and one certainly expects it to be injective. Unfortunately the criterion of Theorem 2.5 falls a bit short here: the derivative is* $f'(x) = 3x^2$, *which is always non-negative but is* 0 *at* $x = 0$.

We will show "by hand" that f *is indeed injective. Namely, let* $x_1, x_2 \in \mathbb{R}$ *and suppose* $x_1^3 = x_2^3$. *Then*

$$0 = x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2).$$

Seeking a contradiction, we suppose that $x_1 \neq x_2$. *Then* $x_1 - x_2 \neq 0$, *so we can divide through by it, getting*

$$0 = x_1^2 + x_1x_2 + x_2^2 = (x_1 + \frac{x_2}{2})^2 + \frac{3}{4}x_2^2.$$

Because each of the two terms in the sum is always non-negative, the only way the sum can be zero is if

$$(x_1 + \frac{x_2}{2})^2 = \frac{3}{4}x_2^2 = 0.$$

The second equality implies $x_2 = 0$, *and plugging this into the first inequality gives* $x_1^2 = 0$ *and thus* $x_1 = 0$. *So* $x_1 = 0 = x_2$: *contradiction.*

We gave a proof of the injectivity of $f : x \mapsto x^3$ to nail down the fact that Theorem 2.5 gives a sufficient but not necessary criterion for a differentiable function to be injective. But we would really like to able to *improve* Theorem 2.5 so as to handle this example via the methods of caclulus. For instance, let n be a positive integer. Then we equally well believe that the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^{2n+1}$ should be injective. It is possible to show this using the above factorization method....but it is real work to do so. The following criterion comes to the rescue to do this and many other examples easily.

Theorem 2.8. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a differentiable function.*
a) Suppose that $f'(x) \geq 0$ *for all* x *and that there is no* $a < b$ *such that* $f'(x) = 0$ *for all* $x \in (a, b)$. *Then* f *is strictly increasing (hence injective).*
b) Suppose that $f'(x) \leq 0$ *for all* x *and that there is no* $a < b$ *such that* $f'(x) = 0$ *for all* $x \in (a, b)$. *Then* f *is strictly decreasing (hence injective).*

Proof. We prove part a); the proof of part b) is identical. Again we go by contrapositive: suppose that f is not strictly increasing, so that there exists $a < b$ such that $f(a) \leq f(b)$. If $f(a) < f(b)$, then applying the Mean Value Theorem, we get a c in between a and b such that $f'(c) < 0$, contradiction. So we may assume that $f(a) = f(b)$. Then, by exactly the same MVT argument, $f'(x) \geq 0$ for all x implies that f is at least weakly increasing, i.e., $x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$. But a weakly increasing function f with $f(a) = f(b)$ must be constant on the entire interval $[a, b]$, hence $f'(x) = 0$ for all x in (a, b), contradicting the hypothesis. □

Worked Exercise 2.1. *We will show that for any* $n \in \mathbb{Z}^+$, *the function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $x \mapsto x^{2n+1}$ *is injective. Indeed we have* $f'(x) = (2n+1)x^{2n}$, *which is non-negative for all* $x \in \mathbb{R}$ *and is* 0 *only at* $x = 0$. *So Theorem 2.8a) applies to show that* f *is strictly increasing, hence injective.*

2.3. **Surjective functions.** A function $f : X \to Y$ if its image $f(X)$ is equal to the codomain Y. More plainly, for all $y \in Y$, there is $x \in X$ such that $f(x) = y$.

In many ways surjectivity is the "dual property" to injectivity. For instance, it can also be verified by a horizontal line test: a function f is surjective if and only if each horizontal line $y = c$ intersects the graph of f in **at least one point**.

Worked Exercise 2.2. *Let m and b be real numbers. Is $f(x) = mx + b$ surjective?*

Solution: It is surjective if and only if $m \neq 0$. First, if $m = 0$, then $f(x) = b$ is a constant function: it maps all of $\mathbb{R}$ to the single point b and therefore is at the opposite extreme from being surjective. Conversely, if $m \neq 0$, write $y = mx + b$ and solve for x: $x = \frac{y-b}{m}$. Note that this argument also shows that if $m \neq 0$, f is injective: given an arbitary y, we have solved for a unique value of x.

By the intermediate value theorem, if a continuous function $f : \mathbb{R} \to \mathbb{R}$ takes on two values $m \leq M$, then it also takes on every value in between. In particular, if a continuous function takes on arbitrarily large values and arbitrarily small values, then it is surjective.

Theorem 2.9. *Let $a_0, \ldots, a_n \in \mathbb{R}$ and suppose $a_n \neq 0$. Let $P : \mathbb{R} \to \mathbb{R}$ by*

$$P(x) = a_n x^n + \ldots + a_1 x + a_0.$$

Thus P is a polynomial of degree n. Then: P is surjective if and only if n is odd.

Proof. Suppose that n is odd. Then, if the leading term a_n is positive, then

$$\lim_{x\to\infty} P(x) = +\infty, \quad \lim_{x\to-\infty} P(x) = -\infty,$$

whereas if the leading term a_n is negative, then

$$\lim_{x\to\infty} P(x) = -\infty, \quad \lim_{x\to-\infty} P(x) = +\infty,$$

so either way P takes on arbitarily large and small values. By the Intermediate Value Theorem, its range must be all of $\mathbb{R}$.
Now suppose n is even. Then if a_n is positive, we have

$$\lim_{x\to\infty} P(x) = \lim_{x\to-\infty} P(x) = +\infty.$$

It follows that there exists a non-negative real number M such that if $|x| \geq M$, $P(x) \geq 0$. On the other hand, since the restriction of P to $[-M, M]$ is a continuous function on a closed interval, it is bounded below: there exists a real number m such that $P(x) \geq m$ for all $x \in [-M, M]$. Therefore $P(x) \geq m$ for all x, so it is not surjective. Similarly, if a_n is negative, we can show that P is bounded above so is not surjective. □

2.4. **Bijective functions.**

A function $f : X \to Y$ is **bijective** if it is both injective and surjective.

Exercise 2.3. *Show: or any set X, the identity function $1_X : X \to X$ by $1_X(x) = x$ is bijective.*

Exercise 2.4. *Determine which of the functions introduced so far in this section are bijective.*

A function is bijective iff for every $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$.

The following result is easy but of the highest level of importance.

Theorem 2.10. *For a function $f : X \to Y$, the following are equivalent:*
(i) f is bijective.
(ii) The inverse relation $f^{-1} : Y \to X = \{(f(x), x) \mid x \in X\}$ is itself a function.

Proof. Indeed, we need f to be surjective so that the domain of f^{-1} is all of Y and we need it to be injective so that each y in Y is associated to no more than one x value. □

2.5. **Composition of functions.**

Probably the most important and general property of functions is that they can, under the right circumstances, be *composed*.[5] For instance, in calculus, complicated functions are built up out of simple functions by plugging one function into another, e.g. $\sqrt{x^2+1}$, or $e^{\sin x}$, and the most important differentiation rule – the Chain Rule – tells how to find the derivative of a composition of two functions in terms of the derivatives of the original functions.

Let $f : X \to Y$ and $g : Y \to Z$: that is, the codomain of f is equal to the domain of g. Then we can define a new function $g \circ f : X \to Z$ by:

$$x \mapsto g(f(x)).$$

Remark: Note that $g \circ f$ means first perform f and then perform g. Thus function composition proceeds from right to left, counterintuitively at first. There was a time when this bothered mathematicians enough to suggest writing functions *on the right*, i.e., $(x)f$ rather than $f(x)$. But that time is past.

Remark: The condition for composition can be somewhat relaxed: it is not necessary for the domain of g to equal the codomain of f. What is precisely necessary and sufficient is that for every $x \in X$, $f(x)$ lies in the domain of g, i.e.,

$$\text{Range}(f) \subseteq \text{Codomain}(g).$$

Example: The composition of functions is generally *not* commutative. In fact, if $g \circ f$ is defined, $f \circ g$ need not be defined at all. For instance, suppose $f : \mathbb{R} \to \mathbb{R}$ is the function which takes every rational number to 1 and every irrational number to 0 and $g : \{0, 1\} \to \{a, b\}$ is the function $0 \mapsto b,\ 1 \mapsto a$. Then $g \circ f : \mathbb{R} \to \{a, b\}$ is defined: it takes every rational number to a and every irrational number to b. But $f \circ g$ makes no sense at all:

$$f(g(0)) = f(b) = ???.$$

Remark: Those who have taken linear algebra will notice the analogy with the multiplication of matrices: if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is defined, an $m \times p$ matrix. But if $m \neq p$, the product BA is not defined. (In fact this is more than an analogy, since an $m \times n$ matrix A can be viewed as a linear transformation $L_A : \mathbb{R}^n \to \mathbb{R}^m$. Matrix multiplication is indeed

[5]This is a special case of the composition of relations described in §$X.X$, but since that was optional material, we proceed without assuming any knowledge of that material.

a special case of composition of functions.)

Even when $g \circ f$ and $f \circ g$ are both defined – e.g. when $f, g : \mathbb{R} \to \mathbb{R}$, they need not be equal. This is again familiar from precalculus mathematics. If $f(x) = x^2$ and $g(x) = x + 1$, then

$$g(f(x)) = x^2 + 1, \text{ whereas } f(g(x)) = (x + 1)^2 = x^2 + 2x + 1.$$

On the other hand, function composition is always **associative**: if $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ are functions, then we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Indeed the proof is trivial, since both sides map $x \in X$ to $h(g(f(x)))$.[6]

Exercise: Let $f : X \to Y$.
a) Show that $f \circ 1_X = f$.
b) Show that $1_Y \circ f = f$.

2.6. Basic facts about injectivity, surjectivity and composition.

Here we establish a small number of very important facts about how injectivity, surjectivity and bijectivity behave with respect to function composition. First:

Theorem 2.11. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions.*
a) If f and g are injective, then so is $g \circ f$.
b) If f and g are surjective, then so is $g \circ f$.
c) If f and g are bijective, then so is $g \circ f$.

Proof. a) We must show that for all $x_1,\ x_2 \in X$, if $g(f(x_1)) = g(f(x_2))$, then $x_1 = x_2$. But put $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $g(y_1) = g(y_2)$. Since g is assumed to be injective, this implies $f(x_1) = y_1 = y_2 = f(x_2)$. Since f is also assumed to be injective, this implies $x_1 = x_2$.
b) We must show that for all $z \in Z$, there exists at least one x in X such that $g(f(x)) = z$. Since $g : Y \to Z$ is surjective, there exists $y \in Y$ such that $g(y) = z$. Since $f : X \to Y$ is surjective, there exists $x \in X$ such that $f(x) = y$. Then $g(f(x)) = g(y) = z$.
c) Finally, if f and g are bijective, then f and g are both injective, so by part a) $g \circ f$ is injective. Similarly, f and g are both surjective, so by part b) $g \circ f$ is surjective. Thus $g \circ f$ is injective and surjective, i.e., bijective, qed. □

Now we wish to explore the other direction: suppose we know that $g \circ f$ is injective, surjective or bijective? What can we conclude about the “factor” functions f and g?

The following example shows that we need to be careful.

Example: Let $X = Z = \{0\}$, let $Y = \mathbb{R}$. Define $f : X \to Y$ be $f(0) = \pi$ (or your favorite real number; it would not change the outcome), and let f be the constant function which takes every real number y to 0: note that this is the unique function from $\mathbb{R}$ to $\{0\}$. We compute $g \circ f$: $g(f(0)) = g(\pi) = 0$. Thus $g \circ f$ is the identity function on X: in particular it is bijective. However, both f and g are far

[6]As above, this provides a conceptual reason behind the associativity of matrix multiplication.

from being bijective: the range of f is only a single point $\{\pi\}$, so f is not surjective, whereas g maps every real number to 0, so is not injective.

On the other hand, something is true: namely the "inside function" f is injective, and the outside function g is surjective. This is in fact a general phenomenon.

Theorem 2.12. *(Green and Brown Fact) Let $f : X \to Y$ and $g : Y \to Z$ be functions.*
a) If $g \circ f$ is injective, then f is injective.
b) If $g \circ f$ is surjecitve, then g is surjective.
c) If $g \circ f$ is bijective, then f is injective and g is surjective.

Proof. a) We proceed by contraposition: suppose that f is not injective: then there exist $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$. But then $g(f(x_1)) = g(f(x_2))$, so that the distinct points x_1 and x_2 become equal under $g \circ f$: that is, $g \circ f$ is not injective.
b) Again by contraposition: suppose that g is not surjective: then there exists $z \in Z$ such that for no y in Y do we have $z = g(y)$. But then we certainly cannot have an $x \in X$ such that $z = g(f(x))$, because if so taking $y = f(x)$ shows that z is in the range of g, contradiction.
c) If $g \circ f$ is bijective, it is injective and surjective, so we apply parts a) and b). $\square$

Remark: The name of Theorem 2.12 comes from the Spring 2009 version of Math 3200, when I presented this result using green and brown chalk, decided it was important enough to have a name, and was completely lacking in inspiration.

2.7. **Inverse Functions.**

Finally we come to the last piece of the puzzle: let $f : X \to Y$ be a function. We know that the inverse relation f^{-1} is a function if and only if f is injective and surjective. But there is another (very important) necessary and sufficient condition for invertibility in terms of function composition. Before stating it, recall that for a set X, the identity function 1_X is the function from X to X such that $1_X(x) = x$ for all $x \in X$. (Similarly $1_Y(y) = y$ for all $y \in Y$.)

We say that a function $g : Y \to X$ is the **inverse function** to $f : X \to Y$ if both of the following hold:

(IF1) $g \circ f = 1_X$: i.e., for all $x \in X$, $g(f(x)) = x$.
(IF2) $f \circ g = 1_Y$: i.e., for all $y \in Y$, $f(g(y)) = y$.

In other words, g is the inverse function to f if applying one function and then the other – in either order! – brings us back where we started.

The point here is that g is supposed to be related to f^{-1}, the inverse relation. Here is the precise result:

Theorem 2.13. *Let $f : X \to Y$.*
a) The following are equivalent:
(i) f is bijective.
(ii) The inverse relation $f^{-1} : Y \to X$ is a function.

(iii) f has an inverse function g.
b) When the equivalent conditions of part a) hold, then the inverse function g is uniquely determined: it is the function f^{-1}.

Proof. a) We already know the equivalence of (i) and (ii): this is Theorem 2.10 above.
(ii) $\implies$ (iii): Assume (ii), i.e., that the inverse relation f^{-1} is a function. We claim that it is then the inverse function to f in the sense that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$. We just do it: for $x \in X$, $f^{-1}(f(x))$ is the unique element of X which gets mapped under f to $f(x)$: since x is such an element and the uniqueness is assumed, we must have $f^{-1}(f(x)) = x$. Similarly, for $y \in Y$, $f^{-1}(y)$ is the unique element x of X such that $f(x) = y$, so $f(f^{-1}(y)) = f(x) = y$.
(iii) $\implies$ (i): We have $g \circ f = 1_X$, and the identity function is bijective. By the Green and Brown Fact, this implies that f is injective. Similarly, we have $f \circ g = 1_Y$ is bijective, so by the Green and Brown Fact, this implies that f is surjective. Therefore f is bijective.[7]
b) Suppose that we have any function $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. By the proof of part a), we know that f is bijective and thus the inverse relation f^{-1} is a function such that $f^{-1} \circ f = 1_X$, $f \circ f^{-1} = 1_Y$. Thus

$$g = g \circ 1_Y = g \circ (f \circ f^{-1}) = (g \circ f) \circ f^{-1} = 1_X \circ f^{-1} = f^{-1}.$$

□

In summary, for a function f, being bijective, having the inverse relation (obtained by "reversing all the arrows") be a function, and having another function g which undoes f by composition in either order, are all equivalent.

[7]A very similar argument shows that g is bijective as well.

Unit 2 Functions

Questions	OPT 1	OPT 2	OPT3	OPT 4	ANSWERS
A method which pairs elements of the set A with unique elements of the set B is called_________	Set	domain	codomain	function	function
One to one function is also called as_________	injective function	surjective function	bijective function	inverse function	injective function
Onto function is also called as __________	injective function	surjective function	bijective function	inverse function	surjective function
A function whichn is one to one and onto is called as __________	injective function	surjective function	bijective function	inverse function	bijective function
If every element of the domain is mapped to unique element of the codomain then the function is called as__________	injective function	surjective function	bijective function	constant function	constant function
If atleast one element of the codomain is not mapped by any element of the domain then the function is called as _________	into function	surjective function	bijective function	constant function	into function
A function that assigns each element of a set into itself is called as __________	surjective function	identity function	constant function	into function	identity function
A one to one mapping of a set onto itself is sometimes called _________ of the set.	Constant function	Inverse function	permutation function	Composition function	permutation function
A bijective function is called invertible because we can define __________ of this function.	Constant function	Inverse function	permutation function	Composition function	Inverse function
The commutative law does not hold for __________	Constant function	Inverse function	permutation function	Composition function	Composition function
Operations of the set union are ___________ on the set of subsets of a universal set	Uninary operation	Binary operation	composition function	permutation function	Binary operation
Operations of the set intersection are ___________ on the set of subsets of a universal set	Uninary operation	Binary operation	composition function	permutation function	Binary operation
The absolute value of an integer n is a ________ on the set Z of integer.	Uninary operation	Binary operation	composition function	permutation function	Uninary operation
The complement of a set is a __________ on the power set of any set.	Uninary operation	Binary operation	composition function	permutation function	Uninary operation
If the identity for a binary operation on a set exists, then it is ________	unique	dual	zero	finite	unique

Over the set of real numbers the element _________ is the identity for addition	1	0	1 and 0	infinite	0
____ is a collection of well-defined objects.	element	member	set	finite set	set
{a,b,c} then cardinality of the set is ________	nullset	one	two	three	three
The two sets A and B are called as _________ if n(a) = n(B)	equal set	equivalent	null set	Subset	equivalent set
The two sets A and B are called as _________ if the sets have the same elements.	equal set	equalent se	null set	Subset	equal se
If every element of the set A is an element of the another set B then A is ________ of B	subset	superset	empty set	universal set	subset
If every element of the set A is an element of the another set B then B is ________ of A	subset	superset	empty set	universal set	superset
If the cardinality of the set is zero then the set is ______________	subset	superset	empty set	universal set	empty se
Empty set is a __________ of every set.	subset	superset	empty set	universal set	subset
Universal set is the _____________ of all the sets.	subset	superset	empty set	universal set	superset
If A = {1,2,3,4} and B = {2,4} then A intersection B =	{2,4}	{1,2,3,4}	{1,2}	{}	{2,4}
If A = {1,2,3,4} and B = {2,4} then A union B =	{2,4}	{1,2,3,4}	{1,2}	{}	{1,2,3,4}
Two sets are said to be disjoint if A intersection B =	A		B	A union B	{}
If n subsets of a set are given, then the number of __________ is 2 power n	min terms	minimax terms	sets	infinite sets	min terms
If n subsets of a set are given, then the number of __________ is 2 power n	max terms	minimax terms	sets	infinite sets	max terms
Every singleton subset constitutes a ___________	set	partition	min term	max term	partition
$A \times B$ ______ $B \times A$	equal set	not equal	does not exist	exist	not equa
A __________ R from a set A to a set B is a subset R of the cartesian product $A \times B$	Relation	Binary relation	duality principle	partition of a set	Binary relation
Let R be a relation on a set A then if aRa for all a in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	reflexive
Let R be a relation on a set A then if aRb then bRa for all a,b in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	symmetr
Let R be a relation on a set A then if aRb and bRc then aRc for all a,b,c in A then R is called___________	reflexive	symmetric	transitive	antisymmetric	transitive
A relation R on a set A is called an equivalence relation if R is	reflexive , symmetric and transitive	reflexive , antisymmetric and transitive	irreflexive , symmetric and transitive	irreflexive, antisymmetric and transitive	reflexive , symmetric and transitive
A relation R on a set A is called an partial order relation if R is	reflexive , symmetric and transitive	reflexive , antisymmetric and transitive	irreflexive , symmetric and transitive	irreflexive, antisymmetric and transitive	reflexive , antisymmetric and transitive

A - B _______ B - A	equal set	not equal	does not exist	exist	not equa

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UNIT-IV

Logics and Proofs

Propositional Logic

<u>Def</u>. A proposition (or statement) is a declarative sentence which is either true or false but not both.

<u>eg</u>. $1+6=7$ (True - T)

$2-7=5$ (False - F)

<u>Def</u>. A declarative sentence which cannot be further split up into simple sentences are called primary ~~sentences~~ statements (or atomic statements or primitive statements)

<u>Def</u>. New Statements can be formed from atomic statements through the use of sentimental connectives. The resulting statements are called molecular or compound statements.

<u>Def</u>. Negation ($\neg$ or $\sim$) [not]:

The negation of a statement is generally formed by introducing the word 'not' at a proper place in the given statement.

If p is any proposition, then the negation is denoted by $\neg p$.

Truth Table (for negation)

p	$\neg p$
T	F
F	T

eg: $p : 6 > 5$ (T).

$\neg p : 6 < 5$ (F).

Def: Conjunction ($\wedge$) [and]:

The conjunction of two statements P & Q is the statement denoted by $P \wedge Q$ read as P and Q.

If both P & Q have the truth values T, then $P \wedge Q$ has the truth value T. Otherwise $P \wedge Q$ has the truth value F.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

eg: $P : 3 < 5$ (T)

$Q : 2+3 = 6$ (F)

$P \wedge Q : 3<5$ and $2+3=6$ (F).

Def: Disjunction ($\vee$) [or]:

The disjunction of two statements P and Q is the statement denoted by $P \vee Q$. $P \vee Q$ has the truth value F only when both P & Q have the truth value F.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

eg: $P : 3+5 = 8$ (T).

$Q : 5 < 3$ (F).

$P \vee Q : 3+5 = 8$ or $5 < 3$ (T).

Def: Conditional Statement : [If .... then] [$\longrightarrow$] If P then Q

Let P & Q be any two statements. Then the statement $P \to Q$ is called a conditional statement.

(X)

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

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eg: P: $3+6=10$ (F)

Q: $2+6=9$ (F).

$P \rightarrow Q$: If $3+6=10$ then $2+6=9$. (T).

Def

Biconditional : [$\leftrightarrow$ or $\rightleftarrows$] [iff].

Let P & Q be any two statements.

Then $P \leftrightarrow Q$ is called biconditional statement.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

eg: P: $4<5$ (T)

Q: $6>7$ (F)

$P \leftrightarrow Q$: $4<5$ iff $6>7$ (F).

1) Construct the truth table ~~of~~ for

a) $\neg(P \wedge Q)$ b) $(\neg P) \vee (\neg Q)$

Sol a)

P	Q	$P \wedge Q$ (and)	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

b)

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

2) Construct the truth table for

a) $(\neg P) \wedge (\neg Q)$, b) $P \rightarrow Q$ → given in the def of conditional statement

c) $P \rightarrow \neg Q$.

Sol a)

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

c)

P	Q	$\neg Q$	$P \rightarrow \neg Q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

Construct the truth table for $S: (P \to Q) \leftrightarrow (\neg P \vee Q)$.

Sol.

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$	$(P \to Q) \leftrightarrow (\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Def. A Statement formula which is true always irrespective of the truth values of the individual variables is called a tautology.

eg: $P \vee \neg P$ is a Tautology.

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Def. A Statement formula which is always false is called a contradiction (or) absurdity.

eg: $P \wedge \neg P$ is a contradiction

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Def : A Statement formula which is neither tautology nor contradiction is called contigency.

eg: $P \leftrightarrow Q$ is a contigency.

1) Show that the proposition $P \vee \neg (P \wedge Q)$ is a tautology.

Sol

P	Q	$P \wedge Q$	$\neg (P \wedge Q)$	$P \vee \neg (P \wedge Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

$\therefore$ $P \vee \neg (P \wedge Q)$ is a Tautology since its has truth value ~~is~~ T.

2) Check the following proposition is tautology.

Sol $((P \to Q) \to R) \vee \neg P$

P	Q	R	$P \to Q$	$(P \to Q) \to R$	$\neg P$	$((P \to Q) \to R) \vee \neg P$
T	T	T	T	T	F	T
T	T	F	T	F	F	F
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

Since all the entries in the resulting column is not 'T', hence the given proposition is not a tautology.

Equivalence

Def. Two statement formulas A & B are equivalent iff $A \leftrightarrow B$ or $A \rightleftarrows B$ is a tautology. It is denoted by the symbol $A \Leftrightarrow B$.

Equivalent formulas

1) Idempotent laws.

$P \vee P \Leftrightarrow P$ & $P \wedge P \Leftrightarrow P$.

2) Associative laws

$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ & $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3) Commutative laws.

$P \vee Q \Leftrightarrow Q \vee P$ & $P \wedge Q \Leftrightarrow Q \wedge P$.

4) Distributive laws.

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ & $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5) $P \vee F \Leftrightarrow P$ & $P \wedge T \Leftrightarrow P$.

6) $P \vee T \Leftrightarrow T$ & $P \wedge F \Leftrightarrow F$

7) $P \vee \neg P \Leftrightarrow T$ & $P \wedge \neg P \Leftrightarrow F$

8) Absorptions laws

$P \vee (P \wedge Q) \Leftrightarrow P$ & $P \wedge (P \vee Q) \Leftrightarrow P$.

9) De Morgan's laws

$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ & $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$.

1) Prove $(P \to Q) \Leftrightarrow (\neg P \vee Q)$.

Sol.

P	Q	$\neg P$	$\neg P \vee Q$	$P \to Q$	$(P \to Q) \Leftrightarrow (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

2) Construct the truth table for the following proposition: $\neg(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$.

Sol.

P	Q	R	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$	$Q \wedge R$	$P \vee (Q \wedge R)$	$\neg(P \vee (Q \wedge R))$	$\neg(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$
T	T	T	T	T	T	T	T	F	F
T	T	F	T	T	T	F	T	F	F
T	F	T	T	T	T	F	T	F	F
T	F	F	T	T	T	F	T	F	F
F	T	T	T	T	T	T	T	F	F
F	T	F	T	F	F	F	F	T	F
F	F	T	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F	T	F

$\therefore \neg(P \vee (Q \wedge R)) \not\Leftrightarrow ((P \vee Q) \wedge (P \vee R))$.

But $(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$.

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$	$(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

Tautological implication

Def. A statement formula A is said to be tautologically imply a statement ~~form~~ formula B iff $A \to B$ is a tautology.

1) Show that $(P \wedge q) \Rightarrow (p \to q)$.

Sol. It is enough to prove that $(P \wedge q) \to (p \to q)$ is a tautology.

P	q	$P \wedge q$	$P \to q$	$(P \wedge q) \to (p \to q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Hence it is a tautology.

Def. Converse, contrapositive and inverse propositions:

Let $p \to q$ be the conditional proposition.

Then $q \to P$ is called its converse.

$\neg q \to \neg P$ is called its contrapositive.

$\neg P \to \neg q$ is called its inverse.

1) Find the contrapositive of the inverse of $p \to q$.

Sol. Given: $p \to q$ as the conditional proposition

Inverse of $P \to q$ is $\neg P \to \neg q$.

$\therefore$ The contrapositive of $\neg P \to \neg q$ is

$\neg(\neg q) \to \neg(\neg P)$ which is $q \to P$.

Normal forms

def: If we write the given statement formula in a particular form (interms of $\wedge$ (product), $\vee$ (sum) and $\neg$) then it is called normal form. Here given statement formula and its normal form are equivalent.

def. (Disjunctive normal forms :- (DNF).

DNF = (Elementry Product) $\vee$ (Elementry Product) $\vee$ ---- $\vee$ (Elementry Product)

Def: Conjunctive normal forms (CNF):-

$$CNF = \begin{pmatrix}\text{elementory}\\ \text{Sum}\end{pmatrix} \wedge \begin{pmatrix}\text{elementry}\\ \text{Sum}\end{pmatrix} \wedge \ldots \wedge \begin{pmatrix}\text{elementry}\\ \text{Sum}\end{pmatrix}$$

1) Obtain the desjunctive normal form of $P \wedge (P \to Q)$.

Sol.

	Reasons
$P \wedge (P \to Q)$	
$\Rightarrow P \wedge (\neg P \vee Q)$	$(\because P \to Q) \Rightarrow \neg P \vee Q$
$\Rightarrow (P \wedge \neg P) \vee (P \wedge Q)$	(Distributive rule)

$\therefore$ DNF of $P \wedge (P \to Q)$ is $(P \wedge \neg P) \vee (P \wedge Q)$.

(ie) the statement is written interms of sum of elementary product.

2) Obtain Conjunctive normal form of $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$.

Sol

$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$	
$\Rightarrow Q \vee ((P \vee \neg P) \wedge \neg Q)$	($\because$ distributive rule)
$\Rightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$	($\because$ distributive rule)
$\Rightarrow (P \vee \neg P \vee Q) \wedge (Q \vee \neg Q)$	(associative & Commutative rules)
This is required CNF.	

Principal normal forms.

* let P & Q be two variables, then the minterm are $P \wedge Q$, $P \wedge \neg Q$; $\neg P \wedge Q$ & $\neg P \wedge \neg Q$.

* The possible maxterms with 2 variables are $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$.

Def. Principal disjunctive normal forms (PDNF):

PDNF = (Minterms) $\vee$ (minterms) $\vee \ldots \vee$ (minterms)

Def Principal conjunctive normal forms (PCNF)

PCNF = (maxterms) $\wedge$ (maxterms) $\wedge \ldots \wedge$ (maxterms).

working rule to obtain PDNF

1) apply equivalence rules interms of $\vee, \wedge$ & $\neg$
2) Apply each fact $\wedge T$
3) Instead of T apply $P \vee \neg P$
4) apply distributive rules
5) " commutative "

1) obtain principal disjunctive normal forms of $\neg P \vee Q$.

Sol.

	Reasons
$\neg P \vee Q$	
$\Rightarrow (\neg P \wedge T) \vee (Q \wedge T)$ $Q \wedge T$	$(P \wedge T) \Leftrightarrow P$.
$\Rightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$	$(\because T \Leftrightarrow P \vee \neg P)$
$\Rightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$	by distributive rules.

$\Rightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$ (Commutative & Idempotent rules)

which is the required PDNF.

2) Obtain principal disjunctive normal forms of

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

Sol.

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Rightarrow ((P \wedge Q) \wedge T) \vee ((\neg P \wedge R) \wedge T) \vee ((Q \wedge R) \wedge T)$ $(\because P \wedge T \Leftrightarrow P)$

$\Rightarrow ((P \wedge Q) \wedge (R \vee \neg R)) \vee ((\neg P \wedge R) \wedge (Q \vee \neg Q)) \vee ((Q \wedge R) \wedge (P \vee \neg P))$ $(\because P \vee \neg P \Leftrightarrow T)$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$ (Distributive rule).

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$ (Commutative rule & Idempotent rule)

which is the required PDNF.

Working rule to obtain PCNF:

1) Write the gn. statement formula interms of $\vee, \wedge$ & $\neg$ alone
2) Apply $\vee F$ for each term $(\because P \vee F \Leftrightarrow P)$
3) Instead of F, apply $P \wedge \neg P$
4) Apply distributive rules
5) " commutative rules.

1) Obtain the principal conjunctive normal form of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$.

Sol.

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$\Leftrightarrow (\neg P \rightarrow R) \wedge ((Q \rightarrow P) \wedge (P \rightarrow Q))$

$\Rightarrow (P \vee R) \wedge ((\neg Q \vee P) \wedge (\neg P \vee Q))$ $(\because P \rightarrow Q \Leftrightarrow \neg P \vee Q)$

$\Rightarrow ((P \vee R) \vee F) \wedge ((\neg Q \vee P) \vee F) \wedge ((\neg P \vee Q) \vee F)$ $(\because P \vee F = P)$

$\Rightarrow ((P \vee R) \vee (Q \wedge \neg Q)) \wedge ((\neg Q \vee P) \vee (R \wedge \neg R)) \wedge ((\neg P \vee Q) \vee (R \wedge \neg R))$ $(\because P \wedge \neg P = F)$

$\Rightarrow ((P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$ (by distributive law)

$\Rightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$ (Commutative law)

This is the required PCNF.

Since the PCNF of a contradiction contains all the maxterms while its PDNF has none of the minterms, the PCNF of S is given by

$$S \Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$$

« Example : 5 »

Find the PCNF and PDNF of $\neg (P \vee Q) \leftrightarrow (P \wedge Q)$

Solution :

We first note that

$$P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

The PCNF is obtained as follows :

$\neg (P \vee Q) \leftrightarrow (P \wedge Q)$

$\Leftrightarrow (\neg (P \vee Q) \wedge (P \wedge Q)) \vee (\neg\neg(P \vee Q) \wedge \neg(P \wedge Q))$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q)$

$\Leftrightarrow (P \wedge \neg P \wedge Q \wedge \neg Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q)$

$\Leftrightarrow (F \wedge F) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow F \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$

The PDNF is obtained as follows :

$\neg (P \vee Q) \leftrightarrow (P \wedge Q)$

$\Leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\Leftrightarrow ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

$\Leftrightarrow F \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee F$

$\Leftrightarrow (Q \wedge \neg P) \vee (P \wedge \neg Q)$

$\Leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q)$

Alternately, the PDNF can be obtained as follows : Let S denote $\neg (P \vee Q) \leftrightarrow (P \wedge Q)$. The PCNF of $\neg S$ is

$$\neg S \Leftrightarrow (P \vee \neg Q) \wedge (\neg P \vee Q)$$

and hence

$S \Leftrightarrow \neg\neg S$

$\Leftrightarrow \neg ((P \vee \neg Q) \wedge (\neg P \vee Q))$

$\Leftrightarrow \neg (P \vee \neg Q) \vee \neg (\neg P \vee Q)$

$\Leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q).$

« Example : 6 »

Find the PCNF and PDNF of

$$S : (\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$$

Solution :

We know that

$$P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Therefore

$$P \leftrightarrow \neg Q \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

The PDNF is obtained as follows:

$(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

$\Leftrightarrow \neg (\neg P \vee \neg Q) \vee (P \leftrightarrow \neg Q)$

$\Leftrightarrow \neg (\neg P \vee \neg Q) \vee ((P \wedge \neg Q) \vee (\neg P \wedge Q))$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

($\because$ The remaining minterms in 2 variables)

« Example : 4 »

Obtain PDNF and PCNF $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

Solution :

P	Q	R	$\neg P$	$\neg P \rightarrow R$	$Q \leftrightarrow P$	$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$	Minterm	maxterm
F	F	F	T	F	T	F		$P \vee Q \vee R$
F	F	T	T	T	T	T	$\neg P \wedge \neg Q \wedge R$	—
F	T	F	T	F	F	F	—	$P \vee \neg Q \vee R$
F	T	T	T	T	F	F	—	$P \vee \neg Q \vee \neg R$
T	F	F	F	T	F	F	—	$\neg P \vee Q \vee R$
T	F	T	F	T	F	F	—	$\neg P \vee Q \vee \neg R$
T	T	F	F	T	T	T	$P \wedge Q \cancel{\vee} \wedge \neg R$	$\neg P \vee \neg Q \vee \neg R$
T	T	T	F	T	T	T	$P \wedge Q \wedge R$	—

The PDNF is $(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

The PCNF is $(P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

Predicate Calculus

eg. $\underbrace{\text{Ram}}_{\text{Subject (small letters)}}$ $\underbrace{\text{is a boy}}_{\text{Predicate (capital letters)}}$.

Mathematically, the above statement can be written as $B(r)$.
$B \to$ boy, $r \to$ ram

Definitions.

1) 1-place predicate.

If there is only one name of object associated with a predicate then it is known as 1-place predicate.

eg: Pavithra is rich.

This can be symbolized as $R(p)$.

2) 2-Place predicate.

If there are 2 names of objects associated with a predicate then it is known as 2-place predicate.

eg: Renuka is shorter than Manjula.

This can be symbolized as $S(r,m)$ where S represents predicate "is shorter than" and r,m represents subject "Renuka & Manjula".

3) 3-place predicate.

If there are 3 names of objects associated with a predicate then it is known as 3-place predicate.

eg: Mr. X sits between Mr. Y and Mr. Z.

This can be symbolized as $S(x,y,z)$.

4) 4-place predicate.

If there are 4 names of objects associated with a predicate then it is known as 4-place predicate.

eg: Green and Miller played bridge against Johnson and smith.

This can be symbolized as $P(g,m,j,s)$.

5) n-place predicate

In general, an n-place predicate is a predicate requiring n names of objects where $n>0$. It is denoted by $P(a_1,a_2,\ldots a_n)$ where $a_1,a_2,\ldots a_n$ are the names of objects associated with predicate & P is a predicate.

6) Statement Functions.

A simple statement functions of one variable is defined to be an expression consisting of a predicate symbol and an individual variable.

eg: $M(x)$: x is mortal.

note: $M(Sam)$: Sam is mortal represents a statement.

7) Compound statement function is obtained by combining one or more simple statement functions using logical connective.

eg: Let $M(x)$: x is a man
& $H(x)$: x is mortal
be the two simple statement functions.

Then we can form compound statement fns as

★ $M(x) \land H(x)$
★ $M(x) \lor H(x)$
★ $M(x) \rightarrow H(x)$
★ $\neg H(x)$
★ $M(x) \leftrightarrow \neg H(x)$.

8) Universal Quantifier.

The quantifier "for all x" is called the universal quantifier. It is denoted by the symbol "$(\forall x)$ or (x)". The universal quantifier is equivalent to each of the following phrases.

i) For all x
ii) For every x
iii) For each x
iv) Everything x is such that
v) Each thing x is such that

eg: "Every apple is red".

This statement can be re-written as "For all x, if x is an apple then x is red".

Define $A(x)$: x is an apple.
& $R(x)$: x is red.

Symbolically, we can write the above statement as

$(\forall x)\ (A(x) \rightarrow R(x))$.

9) Existential Quantifier.

The quantifier for "some x" is called the existential quantifier. It is denoted by the symbol "$(\exists x)$". The existential quantifier is also equivalent to each of the following phrases.

i) For some x
ii) Some x such that
iii) There exists an x such that
iv) There is an x such that
v) There is atleast one x such that.

eg: "Some men are clever".

This statement can be rewritten as "There is an x such that x is a man and x is clever" — (1)

Define $M(x)$: x is man.
& $C(x)$: x is clever.

(1) can be represented in symbolic form as

$(\exists x)\ (M(x) \land C(x))$.

Note:

	Symbols
and	$\land$
or	$\lor$
not	$\neg$
If ..... then	$\rightarrow$
If and only if (iff)	$\leftrightarrow$

Problems

1) Let $M(x)$: x is a mammal

XI $W(x)$: x is warm blooded,

Translate into formula : Every mammal is warm blooded.

Sol. The given statement "Every mammal is warm blooded" can be restated as

"For all x, if x is a mammal then x is warm blooded."

$\therefore$ Its symbolic form is

$$(\forall x)\ ((M(x)) \rightarrow W(x)).$$

2) Let $G(x,y)$: x is taller than y.

Translate the following into formula:

For any x and any y if x is taller than y then it is not true that y is taller than x.

Sol. Given $G(x,y)$: x is taller than y.

$\therefore$ $G(y,x)$: y is taller than x

$\therefore$ The Symbolic form of the given statement is

$$(\forall x)\ (\forall y)\ (G(x,y) \rightarrow \neg G(y,x)).$$

10) Free and bound variables.

i) The variable is said to be bound if it is concerned with either universal $(\forall x)$ or existential $(\exists x)$ quantifier.

ii) The Scope of the quantifier is the formulae immediately following the quantifier.

iii) The variable which is not concerned with any quantifier is called free variable.

Problem

1) Let $P(x)$: $x = x^2$ be the statement.

If the universe of discourse is the set of integers, what are the truth values of a) $P(-1)$

b) $(\forall x)\ P(x)$.

Sol. Given, $P(x)$: $x = x^2$.

Universe of discourse $= \{ \ldots\ldots -2, -1, 0, 1, 2, \ldots\ldots \}$

a) $P(-1)$: $-1 = (-1)^2$

$\Rightarrow -1 \neq 1$

$\therefore$ Truth value of $P(-1)$ is false.

b) when $x = 2$, $P(2)$: $2 \neq 4$.

$\therefore$ Truth value of $(\forall x)\ P(x)$ is false.

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The Theory of Inference (X) 14m

Implication Rules of Inference

1) Rule P (A premise may be introduced at any point in the derivation)
2) Rule T (A formula S may be introduced at any point in a derivation if S is tautologically implied by any one or more of the preceding formula)

Implication Rules

1) a) $P, P \to Q \Rightarrow Q$ (modus ~~Phones~~ Ponens)
   b) $\neg Q, P \to Q \Rightarrow \neg P$ (modus tollens)
   c) $\neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism)
2) $P \to Q, Q \to R \Rightarrow P \to R$ (hypothetical syllogism or Chain rule)
3) $P, Q \Rightarrow P \wedge Q$
4) $P \wedge Q \Rightarrow P, Q$
5) $P, Q \Rightarrow P \vee Q$
6) $P \wedge \neg Q \Leftrightarrow \neg(P \to Q)$

Problem

1) Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P.

Sol. Here given premises are $P \to Q$, $Q \to R$ & P.
Conclusion is R.

{1}	1) $P \to Q$	Rule P
{2}	2) P	Rule P
{1,2}	3) Q	Rule T $(P, P \to Q \Rightarrow Q)$
{4}	4) $Q \to R$	Rule P
{1,2,4}	5) R	Rule T $(P, P \to Q \Rightarrow Q)$

(or)

{1}	1) $P \to Q$	Rule P
{2}	2) $Q \to R$	Rule P
{1,2}	3) $P \to R$	Rule T $(P \to Q, Q \to R \Rightarrow P \to R)$
{4}	4) P	Rule P
{1,2,4}	5) R	Rule T $(P, P \to Q \Rightarrow Q)$

The Theory of Inference for predicate calculus

Rules

1) Rule P
2) Rule T
3) Universal Generalization : (UG)

$A(y) \Rightarrow (\forall x) A(x)$

4) Existential Generalization : (EG)

$A(y) \Rightarrow (\exists x)\, A(x)$

5) Universal Specification : (US)

$(\forall x)\, A(x) \Rightarrow A(y)$

6) Existential ~~Specialize~~ Specification : (ES)

$(\exists x)\, A(x) \Rightarrow A(y)$

Problems

1) Verify the validity of the following argument:
Lions are dangerous animals. There are lions. There are dangerous animals.

Sol. Let $L(x)$: x is a lion.

$D(x)$: x is a dangerous animal.

The given argument can be restated as
"For all x, if x is a lion then x is a dangerous animal. There are lions implies that there are dangerous animals".

From the given problem, we have
Premises as $(\forall x)(L(x) \rightarrow D(x))$, $(\exists x)(L(x))$ and the conclusion is $(\exists x) D(x)$.

∴ Symbolic form of the given argument is
$(\forall x)(L(x) \rightarrow D(x))$, $(\exists x)(L(x)) \Rightarrow (\exists x)(D(x))$

$\{1\}$	1) $(\exists x)(L(x))$	Rule P
$\{1\}$	2) $L(y)$	Rule ES (1)
$\{3\}$	3) $(\forall x)(L(x) \rightarrow D(x))$	Rule P
$\{3\}$	4) $L(y) \rightarrow D(y)$	Rule US (3)
$\{1,3\}$	5) $D(y)$	Rule T (∵ Modus Ponens $(P, P \rightarrow Q \Rightarrow Q)$) (2), (4)
$\{1,3\}$	6) $(\exists x)(D(x))$	Rule EG

Hence the argument is valid.

2) Show that $(\forall x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$.

Sol.

$\{1\}$	1) $(\forall x)(H(x) \rightarrow M(x))$	Rule P
$\{1\}$	2) $H(s) \rightarrow M(s)$	Rule US
$\{3\}$	3) $H(s)$	Rule P
$\{1,3\}$	4) $M(s)$	Rule (T) $(P, P \rightarrow Q \Rightarrow Q)$

3) Show that $(\forall x)(P(x) \to Q(x)) \wedge (\forall x)(Q(x) \to R(x)) \Rightarrow (\forall x)(P(x) \to R(x))$.

Sol.

{1}	1) $(\forall x)(P(x) \to Q(x))$	Rule P
{1}	2) $P(y) \to Q(y)$	Rule US
{3}	3) $(\forall x)(Q(x) \to R(x))$	Rule P
{3}	4) $Q(y) \to R(y)$	Rule US
{1,3}	5) $P(y) \to R(y)$	Rule T $(P \to Q, Q \to R \Rightarrow P \to R)$ (2),(4)
{1,3}	6) $(\forall x)(P(x) \to R(x))$	Rule UG.

4) Prove that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

Sol

{1}	1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
{1}	2) $P(y) \wedge Q(y)$	Rule US
{1}	3) $P(y)$	Rule T $(P \wedge Q \Rightarrow P)$ (2)
{1}	4) $Q(y)$	Rule T $(P \wedge Q \Rightarrow Q)$ (2)
{1}	5) $(\exists x)P(x)$	Rule EG (3)
{1}	6) $(\exists x)Q(x)$	Rule EG (4)
{1}	7) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule T $(P, Q \Rightarrow P \wedge Q)$ (5), (6)

5) Give an argument which will establish the validity of the following inference:

All integers are rational numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2.

Sol. Let $P(x)$: x is an integer.

$R(x)$: x is a rational number.

$S(x)$: x is a power of 2.

Then the given inverence pattern is

$(\forall x)(P(x) \to R(x))$,

$(\exists x)(P(x) \wedge S(x))$

$\Rightarrow (\exists x)(R(x) \wedge S(x))$.

{1}	1) $(\forall x)(P(x) \to R(x))$	Rule P
{1}	2) $P(y) \to R(y)$	Rule US
{3}	3) $(\exists x)(P(x) \wedge S(x))$	Rule P
{3}	4) $P(y) \wedge S(y)$	Rule US.
{3}	5) $P(y)$	Rule T $(P \wedge Q \Rightarrow P)$
{3}	6) $S(y)$	Rule T $(P \wedge Q \Rightarrow Q)$
{1, 3}	7) $R(y)$	Rule T $(P, P \to Q \Rightarrow P)$
{1, 3}	8) $R(y) \wedge S(y)$	Rule T $(P, Q \Rightarrow P \wedge Q)$
{1, 3}	9) $(\exists x)(R(x) \wedge S(x))$	Rule ES.

6) Show that $(\exists x)\, M(x)$ follows logically from the premises $(\forall x)\,(H(x) \rightarrow M(x))$ and $(\exists x)\, H(x)$.

Sol

$\{1\}$	1) $(\forall x)\,(H(x) \rightarrow M(x))$	Rule P
$\{1\}$	2) $H(y) \rightarrow M(y)$	Rule US
$\{3\}$	3) $(\exists x)\, H(x)$	Rule P
$\{3\}$	4) $H(y)$	Rule ES
$\{1,3\}$	5) $M(y)$	Rule T $(P, P\rightarrow Q \Rightarrow Q)$ (2), (4)
$\{1,3\}$	6) $(\exists x)\, M(x)$	Rule EG

x———x

~~To~~ ~~for~~ In Inference theory, the process of derivation can be either direct method or indirect method of proof.

Indirect method of proof

Method of contradiction — method of contrapositive.

7) Show that $(\forall x)\,(P(x) \vee Q(x)) \Rightarrow (\forall x)\, P(x) \vee (\exists x)\, Q(x)$ using the indirect method.

Sol Method -1: Method of contradiction

Assume $\neg[(\forall x)\, P(x) \vee (\exists x)\, Q(x)]$ as an additional premises.

$\{1\}$	1) $\neg[(\forall x) P(x) \vee (\exists x) Q(x)]$	~~Ad~~ Assumed premises.
$\{1\}$	2) $(\exists x)\neg P(x) \wedge (\forall x)\neg Q(x)$	Rule T (Demorgan's law) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
$\{1\}$	3) $(\exists x)\, \neg P(x)$	Rule T $(P \wedge Q \Rightarrow P)$ (2)
$\{1\}$	4) $(\forall x)\, \neg Q(x)$	Rule T $(P \wedge Q \Rightarrow Q)$ (2)
$\{1\}$	5) $\neg P(y)$	Rule ES (3)
$\{1\}$	6) $\neg Q(y)$	Rule US (4)
$\{1\}$	7) $\neg P(y) \wedge \neg Q(y)$	Rule T $(P, Q \Rightarrow P \wedge Q)$ (5), (6)
$\{1\}$	8) $\neg(P(y) \vee Q(y))$	Rule T (Demorgan's law) (7)
$\{9\}$	9) $(\forall x)\,(P(x) \vee Q(x))$	Rule P
$\{9\}$	10) $P(y) \vee Q(y)$	Rule US
$\{1,9\}$	11) $[P(y) \vee Q(y)] \wedge \neg[P(y) \vee Q(y)]$	Rule T $(P, Q \Rightarrow P \wedge Q)$ (8), (10)

This is a false value. Therefore by method of contradiction we have

$(\forall x)\,(P(x) \vee Q(x)) \Rightarrow (\forall x)\, P(x) \vee (\exists x)\, Q(x)$.

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Method-2 : Method of contrapositive ($P \to Q$ means $\neg Q \to \neg P$ is the contrapositive)

{1}	1) $\neg[(\forall x) P(x) \vee (\exists (x)) Q(x)]$	Assumed premises.
{1}	2) $(\exists x) \neg P(x) \wedge (\forall x) \neg Q(x)$	Rule T (Demorgan's law)
{1}	3) $(\exists x) \neg P(x)$	Rule T $(P \wedge Q \Rightarrow P)$ (2)
{1}	4) $(\forall x) \neg Q(x)$	Rule T $(P \wedge Q \Rightarrow Q)$ (2)
{1}	5) $\neg P(y)$	Rule ES (3)
{1}	6) $\neg Q(y)$	Rule US (4)
{1}	7) $\neg P(y) \wedge \neg Q(y)$	Rule T $(P, Q \Rightarrow P \wedge Q)$ (5), (6)
{1}	8) $\neg(P(y) \vee Q(y))$	Rule T (Demorgan's law) (7)
{1}	9) $(\exists x) \neg(P(x) \vee Q(x))$	Rule EG (8)
	~~10) $\neg(\exists(x)$~~	
{1}	10) $\neg[(\forall x)(P(x) \vee Q(x))]$	Rule T (apply $\neg$) (9)

hence we have

$\neg[(\forall x) P(x) \vee (\exists (x)) Q(x)] \Rightarrow \neg[(\forall x) P(x) \vee Q(x)]$

$\therefore$ by the method of contrapositive we have,

$(\forall x)(P(x) \vee Q(x)) \Rightarrow (\forall x) P(x) \vee (\exists x) Q(x)$ //

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Unit -3 Propostional calculus

Questions	**OPT 1**	**OPT 2**	**OPT3**	**OPT 4**	**ANSWERS**
The equivalent statement for P and not P	F	T	F and T	none	F
The implications of P	P	not P	P or Q	P and Q	P or Q
The implications of P and Q is	P	Q	P or Q	not P	P
P or P "equivalent to" P is called as	idempotent	associative	closure	identity	idempotent
If P then Q is "equivalent to"	not P or Q	not P and Q	P and Q	P or Q	not P or Q
A statement which has true as the truth value for all the assignments is called	contradiction	tautology	either tautology or contradiction	implication	tautology
A statement which has false as the truth value for all the assignments is called	contradiction	tautology	either tautology or contradiction	implication	contradiction
If P has T and Q has F as their truth value, then P or Q has ----- as truth value	T	F	0	implication	T
A biconditional statement P if and only if Q is " equivalent to "	(Not P or Q) and (not Q or P)	(Not P or Q) or (not Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(Not P or Q) and (not Q or P)
A biconditional statement notP if and only if Q is " equivalent to "	(Not P or Q) and (not Q or P)	(Not P or Q) or (not Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(P or Q) and (not Q or P)
In the statement If P then Q the antecedent is	P	Q	notP	not Q	P
In the statement If P then Q the consequent is	P	Q	notP	not Q	Q
Out of the following which is the well formed formula	P and Q	(P or Q	if P then Q)	if (if P then Q) then Q)	P and Q
Elementary products are	P and not P	P	P andQ	not P	all of these
Elementary sum are	P	Not Q	P or Q	not P or P	all of these
pcnf contains	product of maxterms	sum of max terms	sum of minterms	product of min terms	product of maxterms
pdnf contains	product of maxterms	sum of max terms	sum of minterms	product of min terms	sum of minterms
P "exclusive or" Q is the negation of	if P then Q	if Q then P	P if and only if Q	Q if and only if P	P if and only if Q
The other name of tautology is	identically true	identically false	universally false	false	identically true

The other name of contradiction is	identically true	identically false	universally true	true	identically false
The converse of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	" If Q then P"
The contra positive of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	"if not Q then not P"
The inverse of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	" if not P then not Q"
A statement A is said to tautologically imply a statement B if and only if " if A then B "is a	tautology	contradiction	false	none	tautology
P and (P or Q) is	P	Q	P or Q	P and Q	P
For two variables the number of possible assignment of truth values is _______	2	2^n	n	2n	2^n
The substitution instance of a tautology is a ___________	tautology	contradiction	identically false	all of these	tautology
Equivalence is a ----- relation	reflexive	symmetric	transitive	asymmetric	symmetric
A statement "A" is said to imply another statement "B" if ---- is a tautology	if A then B	if B then A	if (not A) then B	if (not B) then A	if A then B
The other name for pcnf is	product of sums canonical form	sum of products canonical form	product of products canonical form	sum of sums canonical form	product of sums canonical form
The other name for pdnf is	product of sums canonical form	sum of products canonical form	product of products canonical form	sum of sums canonical form	sum of products canonical form
In the statement "The cricket ball is white", the predicate is	white	ball	cricket ball	both white and ball	white
In the statement "Every mammal is warm blooded", the predicate is	warm blooded	mammal	warm	all of these	warm blooded
In the statement "Every mammal is warm blooded", the object is	warm blooded	mammal	warm	all of these	mammal
Use quantifiers to say that √3 is not a rational number	negation (there exists x a rational number)(x^2=3)	(there exists x a rational number)(x^2=3)	negation (there exists x a rational number)(x^2≠=3)	all of these	negation (there exists x a rational number)(x^2=3)

Existential Specification is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	(there exists x)(A(x)) implies A(y)
Existential Generalisation is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	A(x) implies (there exists y)(A(y))
Universal Specification is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	(For all x) (A(x)) implies A(y)
Universal Generalisation is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	A(x) implies (For all y)(A(y))
Symbolize the statement" Every mammal is warm blooded"	(For all x) (M(x))→ W(x))	(there exists x) (M(x))→ W(x))	(For all x) (W(x))→ M(x))	(there exists x) (W(x))→ M(x))	(For all x) (M(x))→ W(x))

Math 3040 **Spring 2011**

The Predicate Calculus[†]

Contents

1. Introduction

The predicate calculus is an extension of the propositional calculus that includes the notion of quantification. Instead of dealing only with statements, which have a definite truth-value, we deal with the more general notion of *predicates*, which are assertions in which *variables* appear. These variables are presented as "ranging" over some given sets, and quantification is a process that applies to these variables. Statements are here viewed as special predicates in which there are either no variables at all or in which *all* variables have been quantified. All of the operations of the propositional calculus extend to predicates virtually without change. What needs to be understood, however, is how these operations interact with quantification

We shall begin by giving a few simple examples of variables and quantification and then turn to describing what we mean by these in general.

2. Some examples

Variables occur in all parts of mathematics, starting with basic algebra, set theory, number theory, and calculus.

For example, the concept of a *polynomial* includes the notion of a variable. When we write $x^3 + 17x + 10$, we think of x as a variable ranging over some set of numbers, say the real numbers. The polynomial itself is a recipe for performing certain algebraic operations on the variable x.

In set theory, we often consider two sets S and T and then, in some manner, define a function $f : S \to T$. We denote the function values by $f(s)$, where s denotes a variable ranging over S. Sometimes, we may wish to write $t = f(s)$ and call s the independent variable and t the dependent variable.

In calculus, we frequently do what the preceding paragraph just described. The notion of a variable is particularly well-suited to calculus, since it studies the relative rates of different variables, integration with respect to a variable, etc.

In these areas, we often make statements in which we refer to a certain set of values that the variable ranges over. For example, we may say that the polynomial expression $x^2 + 1$ assumes positive values *for all* values of the real variable x. Or we may say that $\cos^2(x) \leq 1$ *for all* real x. Alternatively, we may wish to say that a certain equation has a solution, as when we say "The polynomial equation $x^3 + 17x + 10 = 0$ has a real root." This may be expressed as "*There exists* an x such that $x^3 + 17x + 10 = 0$," provided it is understood that x ranges over the real numbers.

These two ways of qualifying the range of values that the variable is permitted to attain—namely, *for all* and *there exists* — are known as *quantification*: the first is so-called *universal* quantification, the second *existential* quantification.

We now make some of these ideas more precise.

3. General elements of sets.

We shall assume the ideas from set theory described in the earlier chapter *Set Theory*.

No matter how a set S is defined or presented, there is a notion of a *general element* of S. This is conceived as an object about which we know nothing other than it belongs to S. It has all of the properties needed to qualify as a member of S but nothing further. Put another way, a general element of S has exactly those properties shared by all elements of S.

For example, if S is the set of all positive reals, then all we know about a general element of S is that it is real and > 0. So, 3, $\sqrt{2}$, and π are positive reals, but they are not general elements of S because we know more about them than that they are real and > 0.

Clearly the notion of a general element of a set S is an abstract concept, since any concrete instance of an element in S has something particular about it which is what enables us to single it out. The whole point of the concept is to enable us to reason conveniently about an element of S *qua element of* S, without inadvertently introducing additional restrictive identifiers.

Sometimes mathematicians use the term *arbitrary element*, as a synonym for *general element*. For example, we may say "Let x be an arbitrary element of the set S." It is a common error for students to misunderstand this term and to then select a specific element of S rather than to attend to a general one.

Exercise 1. (a) How would you describe a general element of the set of lower-case alphabetic characters $\{a, b, c, d\}$?

(b) Is b a general element of the set $\{b\}$?

4. Variables and constants

The notion of a *variable* in logic or in mathematics is a linguistic construct. A *variable ranging over a set* S is a symbol that represents a general element of S. Sometimes we want to consider several variables ranging over S (or over various sets), and for this we use separate symbols for each. You are free to use whatever symbols you like for variables, provided your usage is consistent, both within itself and with respect to other sources you may be referencing. It is common to use lower case letters near the end of the alphabet for variables—such as $r, s, t, u, v, w, x, y, z$—but this is not mandatory. When many variables are used, their symbols often contain subscripts, as in x_1, x_2, x_3, y_1, y_2, etc.

The notion of a *constant* in S is closely analogous to that of a variable. It is also a linguistic construct. But in this case it is a symbol used to represent a *particular* element of S, which is assumed to be held fixed for the duration of the discussion about S, or for part of the discussion. Often letters from the early part of the alphabet are used for constants, such as a, b, k, B, C, etc., but again this is not mandatory.

Sometimes notation for some constant spreads from one practitioner to another and attains wide currency, even universal acceptance. This occurs when the mathematics in question is especially important and/or the originators of the usage are very influential. In this way, symbols such as "0", "1", "2", "π", "e", etc., have come to represent constants whose meaning is (nearly) universally understood.

5. Expressions

Variables and constants generally appear in larger linguistic constructs, the precise nature of which depends on the mathematical system that is being considered. Usually, such a system involves *elements* in one or more sets, various *relations* among the elements of the sets, *operations* on the elements of the sets, and possibly also various kinds of *functions* relevant to the system. Some particular elements, functions, operations, or relations may be singled out and denoted by special symbols, whereas others remain general. As we have discussed in *Set Theory*, it is important to assume that all of the sets considered in the mathematical system are subsets of some universal set $\mathcal{U}$.

Given a system with such elementary ingredients, we select symbols for representing elements, functions and operations (leaving relational symbols until later) and combine these symbols *according to certain formation rules* to obtain more and more complex strings of symbols that represent more complex objects of the system. Such well-formed strings are called *expressions*.

For an example of this, if one is studying integer arithmetic, one would consider the operations $+, -$, and $\times$, as well as, perhaps, the related operations of raising to the n^{th} power, for non-negative integers n. One would also want to consider the usual constants $0, \pm 1, \pm 2, \ldots$ etc., as mentioned above. Thus, one would expect expressions like $x+y$, $7x+1$, $3x^2 - 4x + 2$, $x^n - y^n$, $(x_1 + x_2 + x_3)(x_1 - x_2 + x_3)$, etc. Usually, parentheses are included among the allowed symbols in order to make the formation rule clear.

Exercise 2. Which of the following concatenations of symbols are valid expressions in integer arithmetic? (The concatenation does not include blanks or semicolons.) (a)1 ; (b) $((1) + (1))$; (c) $1 + \times 2$; (d)$x\sqrt{x}$;

If one is dealing with real numbers, then one would want to include all the expressions of integer arithmetic (now extended to refer to real numbers) as well as allowing division and using function expressions. Thus, expressions such as $1/x$, $\sqrt{x^2 + y^2}$, ye^x, $\ln(1 + x)$, $\sin(xyz)$, would be adjoined to the previous list.

It is also useful to allow expressions that involve *no* variables, that is, involving only constants and operations and functions applied to these. Thus, for example, the constants $0, 1, \sqrt{2}$ are expressions, as are $1 + 1$, or e^2, or $2\pi + 3$.

Each system has its own ingredients and rules of formation, but the general scheme of building complex expressions from simple pieces is common to all of these. All of this is defined precisely and in complete generality in a course in mathematical logic. We do not do this here, relying instead on the knowledge we have already gained from years of experience in working with mathematical expressions and with an informal understanding of the concept.

6. Predicates

A *predicate* P in a mathematical system is a declarative assertion involving a finite number of expressions and relations in the system. If no logical operations appear in the predicate, i.e., it contains only expressions and relations, we may say that P is a *simple predicate* or an *atomic predicate*. Just as in the case of statements in propositional calculus, more complex predicates are constructed by applying logical operations to simpler ones, starting with atomic predicates.

The variables in the expressions are drawn from a finite list, say $x_1, x_2, \ldots, x_n$, so we may write $P = P(x_1, x_2, \ldots, x_n)$ to emphasize the role of the variables. Not every expression contained in P need make use of every variable. We assume that $x_1, x_2, \ldots, x_n$ range over various sets: x_i ranges over the set S_i. It could be that all of the S_i are equal to one given set S, or it could be that some or all of them are distinct from the others. This depends on the mathematical system we are considering and on the predicate P. We sometimes say that P is a predicate *over* $S_1, S_2, \ldots, S_n$.

For example, in the theory of real numbers, we have simple predicates such as $e^x > 0$, $x^2 + y^2 = 1$, $\sin(x + y) = \sin x \cos y + \cos x \sin y$, and so on. In number theory, we have predicates such as $m|n$, $\ell + m \equiv n(\mod 17)$, $m > n$, and so on. In the first example, x and y are real variables, $e^x, \sin x, \cos y$, etc. are well-known functions, and the relations are $>$ and $=$. In the second example, ℓ, m, n are variables and $|$, $\equiv (\mod 17)$, and $>$ are relations.

It is assumed that a predicate P is a meaningful assertion when each variable x_i is interpreted as a general element of set S_i, $i = 1, 2, \ldots$, etc. For example, $x^2 + 2x - 3 = 0$ defines a predicate $P(x)$ over the real numbers. Note that it asserts something: namely, that the square of a real number plus twice that number minus 3 equals 0. However, *it is not a statement*, because its truth or falsity depends on the particular value of x.

The variables appearing in a predicate are of two kinds: those that have been quantified and those that have not. We describe quantification later. For now, we note only that quantified variables are often called *bound variables*, whereas unquantified ones are called *free variables.*

7. Logical operations and predicates

The logical operations of propositional calculus—negation, conjunction, disjunction, implication, identity, together with their iterations—can be applied to predicates just as they are to statements. Thus, for example, let $P = P(x_1, x_2, \ldots, x_k)$ and $Q = Q(y_1, y_2, \ldots, y_\ell)$ be predicates. We allow the possibility that some of the y_j's equal some of the x_i's. Then we can form new predicates

$$\neg P,\ P \wedge Q,\ P \vee Q,\ P \Rightarrow Q,\ P$$

involving the variables

$$\{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_\ell\}.$$

As stated in §6, these predicates are to be understood as declarative assertions about general elements of the sets over which the variables range. Note that these logical operations *do not* reduce the number of variables.

8. Specialization

Because predicates generally have variables, they allow logical operations that are not available to us in the case of statements. (More precisely, they are trivial in the case of statements.) The simplest such operation is that of *specialization of a variable.*

To describe this precisely, we begin with the idea of *specializing a variable in an expression.* Suppose that E is an expression in which the variable x appears, x ranging over the set S. We may then write $E(x)$ to emphasize the appearance of x in E (although other unexhibited variables may also appear in E). We shall say that x is *specialized to a value a in E* if *every* occurrence of x in E is replaced by the same symbol a, which represents a particular element S. (Note that it is important that this symbol *not* also be used in a different role in the expression!) A slightly different but similar locution is sometimes used for the same thing: namely, we may say that *E is specialized at $x = a$.*

If x is specialized to a in E, we may then denote the specialized expression by $E(a)$. If E is, indeed, an expression involving x and other variables $y, z, \ldots$, then $E(a)$ is simply an expression involving $y, z, \ldots$, but no longer involving x. If E involves only x at the outset, then $E(a)$ is an expression involving no variables, i.e., it represents a specific element of S or of some other set considered in the system.

For example, suppose S is the set $\mathbb{Z}$ of integers and $E = E(x)$ is the expression $y - x^2 - 2x$. Then $E(1) = y - 3$. For another example, suppose S is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, T is the set of positive reals, and $E(x) = e^{-x}$, where x ranges over $\mathbb{N}$. Then, $E(3) = e^{-3} = (1/e)^3 \in T$. And so on.

Often this process is described as *evaluating $E(x)$ at $x = a$.*

Of course, *we may specialize more than one variable at a time*, as long as we take care to substitute each occurrence of the variables by the specific elements to which they are being specialized. The elements may or may not be distinct from each other, as the case may dictate.

For example, when $E = y - x^2 - 2x$, as above, we may write E as $E(x, y)$ and then specialize to $E(-2, 0) = 0$ or to $E(1, 1) = -2$, etc.

If E involves exactly n variables, say $x_1, x_2, \ldots, x_n$, then it may be convenient to exhibit all of these in our notation, and we write $E = E(x_1, x_2, \ldots, x_n)$.

We now consider a predicate P in which a free variable x appears. That is, x appears as an unquantified variable in one or more expressions that appear in P. We may write $P(x)$ to focus attention on x. Let a be a particular element of the set S over which x ranges. We say that *x is specialized to a in P* if x is specialized to a in *every* expression in P that contains x. (Sometimes, instead, we may say that P is specialized at $x = a$.) We may write the result as $P(a)$.

This is now a predicate in which x no longer appears: it is a predicate in one fewer free variable, and we often say that the variable has been *eliminated.* If all the free variables in a predicate get eliminated, then, as we shall see, the result is a statement, with a definite truth-value.

This process may clearly be repeated any number of times, as long as there are free variables left. For example, consider the simple expression $E(x, y) : y - x^2 - 2x$ mentioned above, and form the predicate $P(x, y) : y - x^2 - 2x = 3$. Then, $P(5, y)$ is the predicate $y - 5^2 - 2 \cdot 5 = 3$, and $P(5, 17)$ is the (false) statement $17 - 5^2 - 2 \cdot 5 = 3$.

Much the same thing will be seen in the case of quantification.

9. Logical equivalence of predicates

Just as with statements in Propositional Calculus, we are interested in when two predicates P and Q can be reasonably said to be logically equivalent. This motivates the following considerations.

Two preconditions clearly make sense. First of all, the predicates should both be part of the same mathematical context or theory(e.g., calculus, number theory, linear algebra, etc.). Secondly, both should involve the same free variables ranging over the same sets. (If the symbols used for the free variables in one predicate are different from those in the other, we assume that the variables in one of them can be re-labeled so that they are the same as corresponding variables in the other.)

With those preconditions satisfied, we can now define what it means for P and Q to be logically equivalent. For notational simplicity, we assume that the only free variables involved are x and y, so that P and Q may be written as $P(x, y)$ and $Q(x, y)$, respectively.

Then, $P(x, y)$ and $Q(x, y)$ are logically equivalent—written $P \overset{\text{LE}}{\leftrightarrow} Q$—if and only if, for all possible specializations $x = a$ and $y = b$, the truth value of the statement $P(a, b)$ equals the truth value of $Q(a, b)$. Recall, in the chapter on the propositional calculus, we have seen that $P(a, b)$ and $Q(a, b)$ have the same truth value if and only if $P(a, b) \iff Q(a, b)$ is true, which we write more briefly as $P(a, b) \iff Q(a, b)$. So we may abbreviate the definition of logical equivalence of P and Q as follows: $P \overset{\text{LE}}{\leftrightarrow} Q$ if and only if $P(a, b) \iff Q(a, b)$, for all specializations $x = a$ and $y = b$.

Here are two simple examples. In both cases the variables are assumed to range over the real numbers. (a) Let $P(x, y)$ be the predicate $sin(x) = sin(y)$, and let $Q(x)$ be the predicate $x = y$. Clearly, the statements $P(a, a)$ and $Q(a, a)$ are both true, for every real number a. However, $P(0, \pi)$ is true, whereas $Q(0, \pi)$ is false. Therefore, it is not the case that $P \overset{\text{LE}}{\leftrightarrow} Q$. (b) Let $P(x)$ be the predicate $x^2 - 1 = 0$, and let $Q(x)$ be the predicate $(x = 1) \vee (x = -1)$. It is easy to see that both $P(x)$ and $Q(x)$ are true when x equals 1 or -1, and both are false otherwise. So $P \overset{\text{LE}}{\leftrightarrow} Q$.

Thus the notion of logical equivalence of predicates is based on that used for statements. And, therefore, not surprisingly, it has most of the same properties of the earlier notion. For one example, a general substitution law holds, as the following exercise illustrates.

Exercise 3. Suppose that K and L are logical operations of the propositional calculus, each operating on the two atomic statements P and Q. Suppose that the logical expressions $K(P, Q)$ and $L(P, Q)$ are logically equivalent, as defined in §7 of the Symbolic Logic I notes. Let A and B be any predicates, so that $K(A, B)$ and $L(A, B)$ are also predicates. Prove that $K(A, B) \overset{\text{LE}}{\leftrightarrow} L(A, B))$. (Hint: You may assume, for simplicity, that both A and B involve exactly the variables x and y. You must show, for any specializations $x = a$ and $y = b$, that $K(A(a, b), B(a, b))$ has the same truth value as $L(A(a, b), B(a, b))$.)

The following exercise shows that the relation of truth-equivalence of predicates depend on the choice of sets over which the variables range.

Exercise 4. Consider the following predicates:

$$P(x,y) : y = 5 - x^2 \quad \text{and} \quad y > 0.$$
$$Q(x,y) : y = 7 - 3x \quad \text{and} \quad 0 < x \le 2,$$

where the variables x and y are assumed to vary over the positive integers.

(a) Verify that $P \overset{\text{LE}}{\leftrightarrow} Q$.
(b) Suppose that, instead of assuming that the variables range over the positive integers, assume that the variables x, y range over the positive real numbers. Verify that $P \overset{\text{LE}}{\leftrightarrow} Q$ is false.

(Hint: For (a), you have to show that the set of all (a, b) that satisfy $P(x, y)$ is the same as the set of all (a, b) that satisfy $Q(x, y)$ (assuming that a and b are positive integers). For (b), you have to show that these two sets are different (assuming that a and b are positive real numbers).)

10. Quantification

Quantification is a logical operation applied to predicates P. The general scheme of things goes like this. A variable x is selected, and we quantify P with respect to x. If x does not appear in P, then the quantification operation is trivial: we get P again. If x does appear in P, then we get a new predicate in which x has been eliminated. We can now, if we choose, quantify this new predicate again with respect to some other variable. And so on.

In contrast to the operation of specialization, in which several variables may be specialized at once and the order of specialization is irrelevant, quantification is applied to one variable at a time and the order of quantification, in general, does make a difference—as we shall see in the next section.

In what follows, we must distinguish two kinds of variables in P: namely, those variables in P that we have already quantified earlier and those we have not. The former are called *bound variables*, the latter *free variables*. Quantification may be applied only to free variables, hence, no variable gets quantified more than once.

We now become more precise and describe the two kinds of quantification: *universal* and *existential.*

To fix notation, we suppose that P is a predicate containing a free variable x ranging over a set A. It is possible that P has other variables as well, some bound, some free, but to signify our present interest in x, we shall write P as $P(x)$. The so-called *scope* of x consists of all occurrences of x in P.

To quantify $P(x)$ with respect to x, we consider all possible specializations $x = a$ in P, each one giving a predicate $P(a)$ in the remaining variables. Quantification with respect to x is an assertion about these $P(a)$. There are two cases.

10.1. **Universal quantification.** For universal quantification, we form $(\forall x)P(x)$ (to be read "For all x, $P(x)$"). This asserts *all* the specialized $P(a)$.

If P contains free variables other than x, say $y, z, \ldots$, then $(\forall x)P(x)$ is *a predicate in these remaining free variables*, and again we say that the variable x has been *eliminated.* If x is the only free variable in P, then $(\forall x)P(x)$ has no free variables, so it is a statement. Indeed, it is precisely the statement that asserts *all of the statements P(a) simultaneously.*

Here are two examples:

- Let P be the predicate $x^2 + 1 \neq 0$, and suppose that x ranges over the set of real numbers. Then, $(\forall x)P(x)$ asserts: for every real number a, $a^2 + 1 \neq 0$. This is certainly a true statement.
- Let P be the predicate $3x + 4y = 5$, so that P has two variables x and y. We suppose that they both range over the set of rational numbers. Then, $(\forall x)P(x)$ is the assertion: for each rational number a, $\quad 3a + 4y = 5$. This is a predicate involving one variable y. If y is subsequently specialized to some rational value, say b, then the predicate becomes a *statement* asserting that, for each rational number a, $\quad 3a + 4b = 5$. Clearly, such an assertion is false no matter what specific rational number b represents.

Therefore, to repeat: With universal quantification we are asserting ***all*** *of the specialized assertions $P(a)$ at once. When x is the only free variable in P, we see that $(\forall x)P(x)$ is a true statement precisely when all of the $P(a)$ are true simultaneously.*

10.2. **Existential quantification.** For existential quantification, we form $(\exists x)P(x)$ (to be read, "There exists an x such that $P(x)$"). This is the predicate that asserts at least one of the predicates $P(a)$. (Note that it does not specify which ones.)

Again, if P contains other free variables $y, z, \ldots$, then $(\exists x)P(x)$ is a predicate in these remaining free variables. If not, then $(\exists x)P(x)$ is the statement that "At least one of the statements $P(a)$ is true."

Here are two examples of existential quantification:

- Let P be the predicate $x^2 + x + 1 = 0$, where x ranges over the real numbers. Then $(\exists x)P(x)$ asserts that the polynomial has a real root, which the student may easily check to be a false statement.
- Let P be the predicate $x^2 + y^2 + 2xy - y + 3 = 0$, where both x and y range over the reals. Then $(\exists x)P(x)$ is a predicate in the variable y. We may also write this as $(\exists x)P(x, y)$ to signal the role that y plays. In any case, let us write this new predicate as $Q(y)$. For any specialization $y = b$, $Q(y)$ clearly becomes $Q(b)$—i.e., $(\exists x)P(x, b)$—which is the statement that the polynomial $x^2 + b^2 + 2xb - b + 3 = 0$ has a real root. Using the quadratic formula, it is not hard to show that this statement is true when $b \geq 3$, and it is false when $b < 3$.

Therefore, to repeat: With existential quantification we are asserting ***at least one*** *of the specialized assertions $P(a)$ (but not specifying which one). When x is the only free variable in P, we see that $(\exists x)P(x)$ is a true statement precisely when at least one specialization $P(a)$ is a true statement.*

Exercise 5. Verify the last assertion in the second example above.

11. Changing the order of quantification

When we apply logical operations successively to predicates (or read a predicate that involves a succession of logical operations), we must be careful about the order in which the operations are applied. This is, for example, completely analogous to composing linear operators defined on a vector space, or to multiplying matrices. The order in which this is done will affect the answer.

In this section, we look at some examples of this in case the operations we exchange are both quantification operations. The following two sections show what happens when we try to exchange quantification with some of the elementary logical operations.

For our first example, let us deal with real variables again, and let R be the predicate $x+y<1$. Now consider the statement $(\forall x)(\exists y)R(x,y)$. To determine whether it is true or false, we begin at the left, just as with function composition. We must decide whether, for every real number r, the statement $(\exists y)R(r,y)$ is true. Now, in turn, this statement is true if we can find at least one real number s such that $R(r,s)$ is true: that is $r+s<1$. Clearly, this last is possible: just choose s to be any real number $<1-r$. So, we have shown that $(\forall x)(\exists y)R(x,y)$ is true.

But notice what happens when the quantifiers are interchanged. We obtain the statement $(\exists y)(\forall x)R(x,y)$. Just by reading the statement —"There is a y such that for all x, $R(x,y)$"—one can see that this is not the same as the previous statement. But, to clinch the matter, let us check its truth value. Again start on the left. To determine whether or not the statement is true, we must find at least one real number u such that $(\forall x)R(x,u)$ is true. And to demonstrate this last, we must show that for any real number v, $v+u<1$. But no matter what our choice of u, the real number $v=2-u$ violates the condition $v+u<1$. Therefore, the statement $(\exists y)(\forall x)R(x,y)$ is false.

There are, however, certain cases of successive quantification in which the order does not matter. Here is a rule that covers these cases: *Whenever two successive quantification are of the same type—i.e., both universal or both existential—then the order of quantification does not matter.*

For example, consider the predicate $P(x,y): x^2+2y-6<0$, defined for x and y ranging over the real numbers. Then $(\exists x)(\exists y)P(x,y)$ is logically equivalent to $(\exists y)(\exists x)P(x,y)$. For each of these is true exactly when $P(a,b)$ is true for some real numbers a and b (in fact, precisely when the real numbers a and b satisfy $a^2<6-2b$) and false otherwise. For another example, let $Q(x,y)$ be the predicate $e^{x+y}=e^x\cdot e^y$, where x and y range over the reals. Then $(\forall x)(\forall y)Q(x,y)$ and $(\forall y)(\forall x)Q(x,y)$ both assert (truthfully in this case) that the statement $Q(a,b)$is true for all real numbers a and b.

12. The interaction between quantification and negation

Let P be a predicate containing the free variable x ranging over the set S (and perhaps other variables). Then, the basic facts relating $\forall x$ to negation can be stated as two propositions:

1. **Proposition.** $\neg(\forall x)P(x) \iff (\exists x)(\neg P(x))$.

2. **Proposition.** $\neg(\exists x)P(x) \iff (\forall x)(\neg P(x))$.

Exercise 6. Prove each of the two propositions under the assumption that x is the only free variable in P. (Hint: Note that in this special case, the two predicates exhibited in the first proposition are actually statements. So, it suffices to show that these two statements have the same truth value. This can be derived directly from the definition of quantification. Similarly for the second proposition.)

Notice that in both propositions, on the left-hand side, we are negating first and then quantifying, whereas on the right-hand side we are quantifying first and then negating. Therefore, *to make a valid change of order in this case, we have to change the quantifier*: in the first proposition, we change the universal quantifier to an existential one; in the second proposition, we do the reverse.

Here are some special cases of the above propositions. In the first four, De Morgan's laws are also used. In the last two, one also makes use of the logical equivalence $\neg(A\Rightarrow B) \iff A\wedge\neg B$.

A word of advice (which applies to many rules and formulas in mathematics): It is best not to try to memorize the following list. Rather, remember the principles by which the list is constructed. In this case there are two principles: (i) Exchanging the order of quantification and negation changes a universal quantifier to an existential and an existential quantifier to a universal. (ii) Exchanging negation with an elementary logical operation follows the rules in the propositional calculus. Specifically, De Morgan's Laws show that a disjunction is changed to a conjunction and vice versa.

- $\neg(\forall x)(A(x) \wedge B(x)) \iff (\exists x)\neg(A(x) \wedge B(x)) \iff (\exists x)(\neg A(x) \vee \neg B(x))$.
- $\neg(\forall x)(A(x) \vee B(x)) \iff (\exists x)\neg(A(x) \vee B(x)) \iff (\exists x)(\neg A(x) \wedge \neg B(x))$.
- $\neg(\exists x)(A(x) \wedge B(x)) \iff (\forall x)\neg(A(x) \wedge B(x)) \iff (\forall x)(\neg A(x) \vee \neg B(x))$.
- $\neg(\exists x)(A(x) \vee B(x)) \iff (\forall x)\neg(A(x) \vee B(x)) \iff (\forall x)(\neg A(x) \wedge \neg B(x))$.
- $\neg(\forall x)(A(x) \Rightarrow B(x)) \iff (\exists x)(A(x) \wedge \neg B(x))$.
- $\neg(\exists x)(A(x) \Rightarrow B(x)) \iff (\forall x)(A(x) \wedge \neg B(x))$.

To be more concrete, let us take the third of these, letting A(x) be the predicate $x^2 - 3x + 2 = 0$ and B(x) the predicate $x^2 - 6x + 9 = 0$. Then $\neg \exists(x)(A(x) \wedge B(x))$ asserts that it is not the case that the two polynomial equations have a root in common, whereas $(\forall x)(\neg A(x) \vee \neg B(x))$ asserts that, for every real number x, either $x^2 - 3x + 2 \neq 0$ or $x^2 - 6x + 9 \neq 0$. Clearly these two statements assert the same thing, so they have the same truth-value.

13. The relationship between quantification and conjunction and disjunction

We begin with two cautionary examples.

First, suppose that $A = A(x)$ is the predicate $x^2 + 3x + 2 = 0$ and $B = B(x)$ is the predicate $4x - 7 = 0$, where the variable x ranges over, say, the complex numbers $\mathbb{C}$. Now consider the following statements:

$$(\exists x)(A(x) \wedge B(x)) \tag{1}$$

$$(\exists x)A(x) \wedge (\exists x)B(x). \tag{2}$$

Statement (1) asserts the existence of a real number which is a *simultaneous* root of the two equations, whereas statement (2) asserts only that each equation separately has a root. Clearly, these are two very different assertions, so it is not surprising that they have different truth values, statement (1) being false and statement (2) true.

Next, we'll use the predicates $C(x) : x < 3$ and $D(x) : x > 0$, where x is a variable ranging over the reals, and we consider

$$(\forall x)(C(x) \vee D(x)) \tag{3}$$

$$(\forall x)C(x) \vee (\forall x)D(x) \tag{4}$$

In statement (3), we are first quantifying and then performing a disjunction; in (4), we are reversing the order. Statement (3) asserts that every real number is either less than 3 or greater than 0, which is certainly true. Statement (4) asserts that *either* every real number is < 3 *or* that every real number is > 0. Neither of these is true, so certainly statement (4) is false.

If we look closely at what "went wrong" in these examples, we can see that the problem involves the *scope* of the quantification operations. In statement (1), there is one instance of quantification over the variable x, the scope of which involves both predicates A and B. In statement (2), there are two quantification operations, each with more limited scope (the one involving only A, the other only B). A similar observation applies to statements (3) and (4).

These examples show that we have to be careful about the scope of the quantifiers before trying to exchange them with conjunction or disjunction.

The observation about scope, however, shows us how we may do this in cases like those of statements (2) and (4). Let $\mathcal{Q}x$ denote either $\forall x$ or $\exists x$, and let $E(x)$ and $F(x)$ be two predicates involving the free variable x (and perhaps some other variables). We'll make use of the fact that we may change the name of the variable x without affecting truth-values. We can choose any symbol instead of x provided it does not already appear in the predicate. So, suppose y does not appear in F, and then replace x by y. Then, $(\mathcal{Q}x)F(x)$ asserts the same thing as $(\mathcal{Q}y)F(y)$. Now make sure that y is chosen not only to be different from any variable appearing in F originally, but also different from any variable appearing in E. Then, the two predicates

$$(\mathcal{Q}x)E(x) \wedge (\mathcal{Q}x)F(x), \text{ and} \tag{5}$$

$$(\mathcal{Q}x)E(x) \vee (\mathcal{Q}x)F(x) \tag{6}$$

can be replaced by

$$(\mathcal{Q}x)E(x) \wedge (\mathcal{Q}y)F(y), \text{ and} \tag{7}$$

$$(\mathcal{Q}x)E(x) \vee (\mathcal{Q}y)F(y), \tag{8}$$

respectively, without changing their meaning.

Having done this, we can now see that, for each predicate, since the two quantifications involve different variables, we may move the second quantifier to the left, without affecting the meaning or truth value):

$$(\mathcal{Q}x)(\mathcal{Q}y)(E(x) \wedge F(y)), \text{and} \tag{9}$$

$$(\mathcal{Q}x)(\mathcal{Q}y)(E(x) \vee F(y)). \tag{10}$$

Therefore, a quantifier may be "moved to the left", provided the quantified variable is given a new name different from those of other variables already appearing.

Note that if we apply this to statements (2) and (4), we do get all the quantifiers on the left, but there are still two quantifications required rather than just one as in statements (1) and (3). Statements (1) and (3) (which already have their single quantifier on the left) are not affected by this procedure.

The above procedure still works even when the second quantifier is not the same as the first. For example, we can replace $(\exists x)E(x) \wedge (\forall y)F(y)$ by $(\exists x)(\forall y)(E(x) \wedge F(y))$, etc.

Finally, the logical equivalence (f) in Exercise 11 of the propositional calculus notes

$$(P \Rightarrow Q) \qquad \Longleftrightarrow \qquad (\neg P \vee Q),$$

together with the fact that substituting a predicate by a logically equivalent one does not affect truth values, shows that the relationship between quantification and implication may be understood in terms of relationship between quantification, negation, and disjunction. We don't go any further into this, but we present the following exercise as an illustration.

Exercise 7. Let $P(x,y)$ be the predicate $x < y^{-1}$ and $Q(x,z)$ the predicate $-x = z^2$. Here we assume that x and z range over the real numbers and y ranges over the *positive* real numbers. Show that

$$\Big((\forall y)P(x,y) \Rightarrow (\exists z)Q(x,z)\Big) \iff (\exists y)(\exists z)\Big(P(x,y) \Rightarrow Q(x,z)\Big).$$

(Hint: There are two ways to approach this problem. The first way is to use the discussion above to change the implication in the left-hand predicate to an expression involving disjunction, then to move the quantifier $(\exists z)$ to the left in line what we discuss above, and finally, to change the resulting expression back to one involving implication. The other way is to use the definition of equivalence of predicates to show directly that both sides are equivalent — note that both sides are predicates in the free variable x.)

Exercise 8. Formulate the following statements using predicates and quantifiers:

(a) If a, b, and c denote the lengths of the sides of a planar triangle, then $a + b > c$.
(b) If a, b, and c denote the lengths of the sides of a planar right triangle, with c the largest, then $a^2 + b^2 = c^2$.
(c) If a, b, c and n are any positive integers and $n > 2$, then $a^n + b^n \neq c^n$.

Now reformulate each of the above, if necessary, so that only negation, disjunction, and quantification are used.

Statement (a) is a special case of the well-known Triangle Inequality that students often see in a linear algebra course. Statement (b) is, of course, the Pythagorean Theorem, which is one of the cornerstones of geometry. And statement (c) is known as Fermat's Last Theorem. It was proved in 1994 by Andrew Wiles, more than 350 years after it was conjectured by Pierre de Fermat.

Exercise 9. (a) Let f be a real-valued function defined on an open interval I of real numbers, and let a be an element of I. The following statement expresses the fact that f *is continuous at* a. Formulate that statement in terms of quantifiers and predicates: For every $\epsilon > 0$, there is a $\delta > 0$ such that, for every real x satisfying $0 < |x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. Now formulate this so that only quantification, negation and disjunction are used and so that all quantification occurs on the left.

(b) Let f be a real-valued function defined on an open interval I of real numbers, and let a be an element of I, as in the foregoing. Express the assertion that f is *not* continuous at a by applying negation to the last statement you obtained in the preceding exercise and exchanging negation with the other operations according to the rules we've derived. In the end, you should have a statement for which negation has been completely "distributed" (i.e., cannot be exchanged any further with other operations without reversing steps).

(c) Let f be a continuous real-valued function defined on a closed interval J. Use quantifiers (on the left) and predicates to express the fact that f achieves a maximum value on J.

(d) Suppose that g is a differentiable (hence continuous) real-valued function defined on an open interval I. Use quantifiers and predicates to express the fact that if g achieves a maximum value at some element $x \in I$, then the derivative $f'(x) = 0$.

14. Quantification and methods of proof

All of the methods of proof discussed in *The Propositional Calculus* carry over to the more general setting of the predicate calculus. Only a few additional points need to be noted now, in connection with quantification.

14.1. **Universal quantification.**

Suppose we wish to prove a statement of the form $(\forall x)P(x)$, where the free variable x ranges over a set S. As discussed earlier, this mean that we must prove the statements $P(s)$ for every choice of $s \in S$. When S is a small set, this may sometimes be possible by direct enumeration and checking. But often S is a large set, such as the set of natural numbers or the set of real numbers. In those cases a different approach is needed.

In the case of the natural numbers, a standard method is the *method of mathematical induction*. Students are already familiar with simple versions of this from, say, calculus courses, but in any case, we go into this method more fully in the chapter *The Natural Numbers*.

When induction does not seem appropriate or adequate, the usual method is to choose an *arbitrary* (i.e., general) element s in S and then prove $P(s)$. The choice of an arbitrary element s means that the information about s contained in the proof can only consist of properties that it has by virtue of its membership in S. Therefore, the proof applies simultaneously to every particular element in S.

Example: Let x be a variable ranging over the real numbers. Let us prove that

$$(\forall x)\big((x > 1) \Rightarrow (x^3 > x^2)\big).$$

We choose an arbitrary real number s. To prove that the implication $(s > 1) \Rightarrow (s^3 > s^2)$ is true, we need only show that it is not falsified. If $s \leq 1$, the implication is vacuously true, so we may then restrict attention to those cases in which $s > 1$. We must check that in those cases, $s^3 > s^2$. Since $s > 1$, we know that s is positive. Using facts about real multiplication, we multiply both sides of the inequality $s > 1$ by s and obtain $s^2 > s$. Now multiply this last inequality by s to obtain $s^3 > s^2$. Therefore the implication is true.

The reader may have noticed that a slight shortcut is possible. Namely, since in the cases in which s specializes to a number ≤ 1, the implication is vacuously true, we don't really need to bother checking anything. That is, we may as well assume the hypothesis $s > 1$ right away and proceed from there. This is a general feature of proofs of implications in which the hypothesis restricts the range of the variable.

The actual technique that you use to prove the universal statement will depend, of course, on the statement. Some suggestions about how to proceed appear in the chapter *The Propositional Calculus*.

Exercise 10. (a) Prove: For all positive, real numbers a and b,

$$\frac{1}{2}(a + b) \geq \sqrt{ab}.$$

(b) Prove: For all $x \geq 1$, $\quad x \geq 1 + \ln(x)$.

14.2. **Existential quantification.**

To prove a statement of the form $(\exists x)P(x)$, where again x ranges over S, we must show that, for some element $s \in S$, $P(s)$ is true. This may be accomplished by one of two methods.

In the first method, we produce such an element s explicitly, which we describe by saying that we *find* s or we *construct* s. *This itself is a two-step process*: (1) The element s must be found or constructed, and then (2) the truth of $P(s)$ must be demonstrated.

In the second method, the existence of s is proved, either directly as a consequence of some known result or indirectly. In the indirect proof, we assume that there is no such s and derive a contradiction. In symbolic terms, we prove that $\big((\forall x)\neg P(x)\big) \Rightarrow C$, where C is a suitably formulated contradiction.

Here are two examples that illustrate these methods.

Example: Let $P(x)$ be the predicate $x^2 - 2x - 3 = 0$, where x is a variable ranging over the real numbers. The statement $(\exists x)P(x)$ affirms that the equation has at least one real solution. It is this solution that we must find or construct. The student has already learned how to do this in a beginning algebra course. In general, one can try to factor the left hand side into linear factors. Or failing that, one can "complete-the-square" on the left-hand side. Or one can simply recall the quadratic formula and write down the roots. Even guessing is legitimate, though unless one has good reasons for a guess this method is not usually optimal. In the end, any of these methods should produce two possible values for a solution $x = s$: namely, $x = 3$ and $x = -1$. This is the first step.

It remains to verify that the predicate is true for one or another of these values. This can be done by evaluating the equation at 3 or at -1. Alternatively, often the method used to find the answer (e.g., factoring or completing the square) has steps that are reversible, so the verification can be done by just running the steps in reverse. This step is usually left out, since it is routine, but it is important to understand that, from a logical point of view, it needs to be there, even if just in background. So, after solving an equation as Step 1 in the proof of an existence statement, students in this course should either evaluate the equation at the found solution or say something like: *Since the foregoing steps are reversible, the value $x = s$ does satisfy the equation.*

Although the foregoing procedure is the simplest for this particular problem, there is also a method that proves $(\exists x)P(x)$ directly, a method that can be applied to more complicated equations for which there are no nice algebraic methods or formulas. This method uses the Intermediate Value Theorem from calculus. We write $f(x) = x^2 - 2x - 3$ and observe that this defines a continuous function of a real variable. We then compute $f(0) = -3$ and $f(5) = 12$: note that $f(0)$ is negative and $f(5)$ is positive. The Intermediate Value Theorem then implies that there is a real number s between 0 and 5 for which $f(s) = 0$.

Example: This example also uses facts from calculus, including the Intermediate Value Theorem. The student is assumed to be familiar with these and should assume them while digesting the argument. Let I denote the set of all real numbers x satisfying $0 \le x \le 1$. I is often called *the unit interval.* Suppose that $f : I \to I$ is any given continuous function. We shall prove that f has a *fixed point.* That is, we prove the statement $(\exists x)(f(x) = x)$, where we assume that the variable x ranges over I. In this case, we give an indirect proof. We assume that the statement is false and derive a contradiction.

By the earlier discussion, the negation of $(\exists x)(f(x) = x)$ is $(\forall x)(f(x) \neq x)$. Consider the function g given by the equation $g(x) = f(x) - x$, for all $x \in I$. Our assumption implies that $g(x) \neq 0$, for all x, hence the same is true for the absolute value $|g(x)|$. Finally, define the function $h : I \to \mathbb{R}$ by the equation $h(x) = g(x)/|g(x)|$, for all $x \in I$. Since the denominator is never zero, basic facts about continuous functions proved in a calculus course imply that h is a continuous function. Notice that, by definition, the value $h(x)$ is either equal to 1 (when $g(x) > 0$) or -1 (when $g(x) < 0$). h never assumes a value other than these two. So far we have not reached a contradiction, because it is possible that h is simply the function that is constantly equal to 1, or the function that is constantly equal to -1. However, this is not the case, as can be seen by evaluating $g(0)$ and $g(1)$. In fact, $g(0) = f(0) - 0 > 0$, because $f(0) \in I$ and $f(0) \neq 0$, by assumption. Further, $g(1) = f(1) - 1 < 0$, because $f(1) \in I$ and $f(1) \neq 1$, by assumption. It follows from what was said above that $h(0) = 1$ and $h(1) = -1$. The Intermediate Value Theorem then implies that there must be a real number $x \in I$ such that $h(x) = 0$. But we have seen that h assumes only the values ± 1. So, we have arrived at the promised contradiction.

This example is a special case of what is known as *Brouwer's Fixed Point Theorem*, which belongs to a branch of mathematics known as *topology*. We have left out a number of smaller steps for the sake of brevity; this can make it tougher going for the reader. Most published mathematics, whether in class notes, textbooks, or published papers make such omissions. The higher the level of mathematics, the more omissions there will be. Therefore, it is *very important* at this stage for the student to get in the habit of reading proofs accompanied by writing material (e.g., pencil or pen and paper). Each time there are steps in the proof that merit checking or filling in, the student should do so.

Even though the proof above is more complex and requires more ingenuity than the usual proof we shall encounter, it is a very good example of an indirect proof that proves an existence statement.

Exercise 11. Let $p(x)$ be a polynomial of degree n, with top-degree coefficient 1. Define $q(x)$ for $x \neq 0$ by the equation $q(x) = p(x)/x^n$.

(a) Prove that $\lim_{x\to\infty} q(x) = 1$ and $\lim_{x\to-\infty} q(x) = 1$.

(b) Prove that $p(x)$ and x^n have the same sign when $|x|$ is very large.

(c) Prove that $n \quad odd \Rightarrow (\exists x)(p(x) = 0)$.

14.3. **Concluding practical remarks.** The discussion above focused on only the simplest kinds of quantified statements that you may be called upon to prove. More complicated statements might have multiple quantifications or propositional connectives (i.e., conjunction, negation, etc.) intertwined with various predicates. These can always be untangled and reduced to a step-by-step consideration of the simpler forms we discuss above and in previous sections. Going into all these possibilities in detail would take us too far afield with not much practical benefit.

One practical problem that a student might confront is that statements that need to be proved are not always in nice explicitly quantified form even though there is implicit quantification occurring. For example, when you are asked to solve a certain equation, this is tantamount to proving an existential statement. Or, suppose you are given the following: "Let p be an odd prime. Prove that $2^p - 2$ is divisible by p." You are being asked to prove a universal statement (i.e., for *all* odd primes p, ... etc.) This is largely a matter of getting used to the style of mathematical writing. It is worth paying close attention to this, since existential statements require different methods of proof from those required by universal statements.

A remark related to the foregoing caveat occurs when you are asked to prove or disprove a certain general statement. We have already given a suggestion as to how to proceed in *The Propositional Calculus, §9 (c), "What to do when you are stuck," item (iv).* Here, we elaborate slightly.

As an example, you may be given what looks like it might be a trig. identity and you are asked to prove or disprove it. What do you do?

First recognize that the statement, say a trig identity or trig. identity look-alike, is a universal statement: e.g., For all real x and y, $\cos(x+y) = \cos(x)\cos(y) - \tan(x)\tan(y)$. Next, recognize that *to prove this universal result*, you need to work with *general* real numbers x and y, not with specifically chosen ones. Third, recognize that when you *disprove* this universal result, you are *proving the negation of a universally quantified statement.* Therefore, you can exchange the order of quantification and negation, but you need to change the universal quantifier to an existential one (cf. §12). This means that, *to disprove the universal assertion* you need to find or construct specific values of the variables for which the identity *fails*

Inspect the given statement to see if you can find any clues as to whether it is true or false. Maybe it's similar to something you've seen before. Maybe you have a hunch. That's good. Go with your hunches...at least for a certain amount of time. But, what do you do if you have no hunch? Or your earlier hunch failed? Well, it is usually much easier to calculate some specific values than it is to prove a general assertion. So, if you don't know what to do, try to "plug in" some values and see what happens. There are two possibilities: (1) After a small number of trials, the equation fails. Good. You've proved the negation. You're done. (2) After a small number of trials, the equation still holds. Okay. Not so bad. Don't panic. But don't stop here. *You haven't proved anything.* At least not what you need to prove. But, you have gathered some evidence that the equation *might* be universally true (particularly if you chose your values relative randomly). Therefore, you can proceed to attempt a proof of the identity with a certain amount of confidence that this is the direction to go. (Of course, none of this advice tells you anything about trigonometry. That will be your problem.:-))

Just for fun, consider the possible trig. identity above, taking the position that we don't know whether it's true or false. Let's choose some values for x and y. We need to pick values for which we can compute the expression, but the values shouldn't be too trivial (e.g., like 0), since then we might get an equality for trivial reasons...a false positive, so to speak. So, let's take $x = \pi/4$ and $y = \pi/4$. Then, $\cos(\pi/4) = 1/\sqrt{2}$, $\tan(\pi/4) = 1$, $\cos(\pi/4+\pi/4) = \cos(\pi/2) = 0$. Plugging in these values, we get $0 = (1/\sqrt{2})^2 - 1\cdot 1 = -1/2$, which is clearly false. This proves that the equation is not an identity. (By the way, if we had picked $x = 0 = y$, we would have obtained $1 = 1 \cdot 1 + 0 \cdot 0$, which is true but misleading.)

This concludes our brief foray into the foundations of the predicate calculus. For more advanced work in this area, consider taking Math 481 or Math 483. For further practical tips in dealing with quantifiers in proofs, see Solow's book, Chapters 4 through 7.

Unit -4 PREDICATE CALCULUS

Questions	OPT 1	OPT 2	OPT3	OPT 4	ANSWERS
In the statement "The cricket ball is white", the predicate is	white	ball	cricket ball	both white and ball	white
In the statement "Every mammal is warm blooded", the predicate is	warm blooded	mammal	warm	all of these	warm blooded
In the statement "Every mammal is warm blooded", the object is	warm blooded	mammal	warm	all of these	mammal
Use quantifiers to say that √3 is not a rational number	negation (there exists x a rational number)(x^2=3)	(there exists x a rational number)(x^2=3)	negation (there exists x a rational number)(x^2≠=3)	all of these	negation (there exists x a rational number)(x^2=3)
Existential Specification is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	(there exists x)(A(x)) implies A(y)
Existential Generalisation is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	A(x) implies (there exists y)(A(y))
Universal Specification is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	(For all x) (A(x)) implies A(y)

Universal Generalisation is a rule of the form	(For all x) (A(x)) implies A(y)	A(x) implies (For all y)(A(y))	(there exists x)(A(x)) implies A(y)	A(x) implies (there exists y)(A(y))	A(x) implies (For all y)(A(y))
Symbolize the statement" Every mammal is warm blooded"	(For all x) (M(x))→ W(x))	(there exists x) (M(x))→ W(x))	(For all x) (W(x))→ M(x))	(there exists x) (W(x))→ M(x))	(For all x) (M(x)) → W(x))
The equivalent statement for P and not P	F	T	F and T	none	F
The implications of P	P	not P	P or Q	P and Q	P or Q
The implications of P and Q is	P	Q	P or Q	not P	P
P or P "equivalent to" P is called as	idempotent	associative	closure	identity	idempotent
If P then Q is "equivalent to"	not P or Q	not P and Q	P and Q	P or Q	not P or Q
A statement which has true as the truth value for all the assignments is called	contradiction	tautology	either tautology or contradiction	implication	tautology
A statement which has false as the truth value for all the assignments is called	contradiction	tautology	either tautology or contradiction	implication	contradiction
If P has T and Q has F as their truth value, then P or Q has ----- as truth value	T	F	0	implication	T
A biconditional statement P if and only if Q is " equivalent to "	(Not P or Q) and (not Q or P)	(Not P or Q) or (not Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(Not P or Q) and (not Q or P)
A biconditional statement notP if and only if Q is " equivalent to "	(Not P or Q) and (not Q or P)	(Not P or Q) or (not Q or P)	(P or Q) and (not Q or P)	(Not P or Q) and (Q or P)	(P or Q) and (not Q or P)
In the statement If P then Q the antecedent is	P	Q	notP	not Q	P
In the statement If P then Q the consequent is	P	Q	notP	not Q	Q
Out of the following which is the well formed formula	P and Q	(P or Q	if P then Q)	if (if P then Q) then Q)	P and Q

Elementary products are	P and not P	P	P andQ	not P	all of these
Elementary sum are	P	Not Q	P or Q	not P or P	all of these
pcnf contains	product of maxterms	sum of max terms	sum of minterms	product of min terms	product of maxterms
pdnf contains	product of maxterms	sum of max terms	sum of minterms	product of min terms	sum of minterms
P "exclusive or" Q is the negation of	if P then Q	if Q then P	P if and only if Q	Q if and only if P	P if and only if Q
The other name of tautology is	identically true	identically false	universally false	false	identically true
The other name of contradiction is	identically true	identically false	universally true	true	identically false
The converse of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	" If Q then P"
The contra positive of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	"if not Q then not P"
The inverse of "if P then Q" is	" If Q then P"	" if not P then not Q"	"if not Q then not P"	all of these	" if not P then not Q"
A statement A is said to tautologically imply a statement B if and only if " if A then B "is a	tautology	contradiction	false	none	tautology
P and (P or Q) is	P	Q	P or Q	P and Q	P
For two variables the number of possible assignment of truth values is _______	2	2^n	n	2n	2^n
The substitution instance of a tautology is a ___________	tautology	contradiction	identically false	all of these	tautology
Equivalence is a ----- relation	reflexive	symmetric	transitive	asymmetric	symmetric

A statement "A" is said to imply another statement "B" if ---- is a tautology	if A then B	if B then A	if (not A) then B	if (not B) then A	if A then B
The other name for pcnf is	product of sums canonical form	sum of products canonical form	product of products canonical form	sum of sums canonical form	product of sums canonical form
The other name for pdnf is	product of sums canonical form	sum of products canonical form	product of products canonical form	sum of sums canonical form	sum of products canonical form

1-15

UNIT-V

GRAPH THEORY

Introduction:-

In the last three decades graph theory has established itself as a worthwhile mathematical discipline and there are many applications of graph theory to a wide variety of subjects which include Physics, Chemistry, Operations Research, Engineering, Computer science, Economics etc.

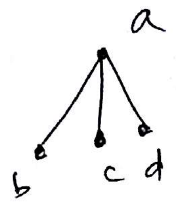
The development of many branches in Mathematics has been necessitated while considering certain real life problems arising in pratical life or problems arising in other sciences.

GRAPH: A graph G consists of a pair $(V(G), X(G))$, where $V(G)$ is a non-empty finite set whose elements are called POINTS (or) VERTICES and $X(G)$ (or) $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$.

The elements of $X(G)$ are called LINES (or) EDGES of a graph G.

In otherwords, a GRAPH with p-points an q-lines is called a (p,q) - graph.

for example, let $V = \{a, b, c, d\}$ &



$X = \{(a,b), (a,c), (a,d)\}$.

$\therefore$ $G = (V, X)$ is a $(4,3)$ - graph

Note ①: If $x = \{u, v\} \in X(G)$, the line 'x' is said to join u & v. we write $x = uv$. we say that the points u & v are ADJACENT.

② we also say that the point u & the line 'x' are INCIDENT with each oth

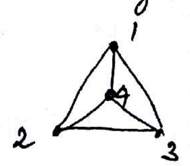
③ for our convenient, $V(G) = V$ & $X(G) = X$.

COMPLETE GRAPH: A graph in which any two distinct points are adjacent is call a complete graph.

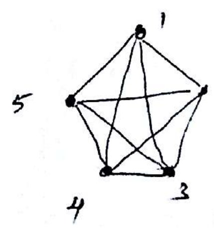
The complete graph with p-points i denoted by K_p.

Note ②, ① K_3 is called a triangle

② The K_4 is



③ The K_5 is



<u>NULL GRAPH:</u> A graph whose edge set is empty is called a <u>NULL GRAPH</u> (or) totally disconnected graph.

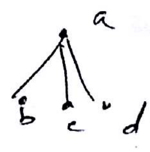
<u>Defn:</u> A graph G is called a <u>BIGRAPH</u> (or) BIPARTITE GRAPH if V can be partitioned into two disjoint subsets V_1 & V_2 such that every line of G joins a point of V_1 to a point of V_2.

<u>Note</u> ①: (V_1, V_2) is called a BIPARTITION of G.

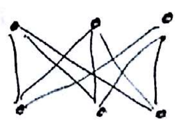
②: G Contains every line joining the points of V_1 to the points of V_2 then G is called a COMPLETE BIGRAPH.

③: If V_1 Contains 'm' -points and V_2 contains 'n' -points then the Complete bigraph G is denoted by $K_{m,n}$.

for example (i) $K_{1,3}$ is

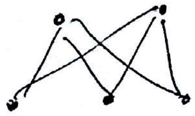


(ii) $K_{3,3}$ is

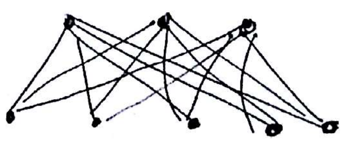


④ $K_{1,m}$ is called a <u>STAR</u> for $m \geq 1$.

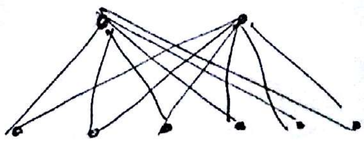
⑤



$K_{2,3}$



$K_{3,5}$



$K_{2,6}$

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DEGREE: The degree of a point v_i in a graph G is the number of lines inci. with v_i. The degree of v_i is denot by $d_G(v_i)$ (or) deg. v_i (or) simply d_i

Theorem 1:

Prove that the sum of the degrees all the vertices in a graph G is equ to twice the number of edges.

Proof

To prove : $\sum_{i=1}^{n} d(v_i) = 2E$

Let $G = (V, E)$ be any graph,

where V is the collection of vertices

E is the collection of edges.

Since, each edges contributes twic a degree, the sum of the degree of a vertices in G is twice as the number of edges in G.

i.e., $\sum_{i=1}^{n} d(v_i) = 2|E| = 2E$.

Theorem (2): Show that the number of vertices of odd degree in a graph is always even.

Proof

Let $v_1, v_2, \ldots v_p$ be the vertices in G.

Let $u_1, u_2 \ldots u_k$ be the vertices of odd degree & let $w_1, w_2 \ldots w_m$ be the vertices of even degree.

$\therefore \quad k + m = p$

$$\Rightarrow \sum_{i=1}^{n} d(v_i) = 2E.$$

$$\Rightarrow \sum_{i=1}^{k} d(u_i) + \sum_{j=1}^{m} d(w_j) = 2E \text{ is even}$$

$$\Rightarrow \sum_{i=1}^{k} d(u_i) = 2E - \sum_{j=1}^{m} d(w_j) \text{ is even}$$

$$\therefore \sum_{i=1}^{n} d(u_i) \text{ is even.}$$

Hence k must be even.

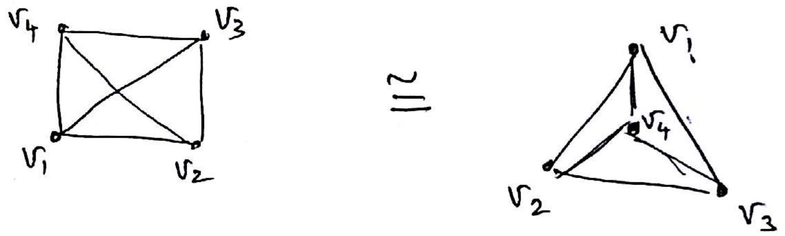
$\therefore$ we conclude that, the number of vertices of odd degree in a graph is always even.

ISOMORPHISM:

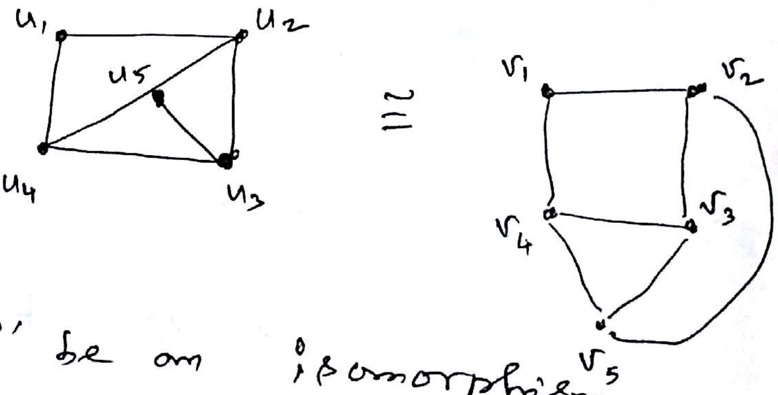
Two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V$ are said to be ISOMORPHIC, if there ex a bijection $f: V_1 \to V_2$ such that u, v are adjacent in G_1 if and only if $f(u), f($ are adjacent in G_2.

If G_1 is isomorphic to G_2, w write $G_1 \cong G_2$. The map f is called ISOMORPHISM from G_1 to G_2

For example, ①



②



Note ①: Let 'f' be an isomorphism of a graph $G_1 = (V_1, E_1)$ to the graph $G_2 = (V_1, E_2$ Let $v \in V_1$. Then $\deg v = \deg f(v)$.

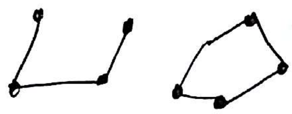
Note ②: Two isomorphic graphs have th same number of points and the same number of lines.

Defn:

Let $G=(V,E)$ be a graph. The COMPLEMENT $\bar{G}$ of G is defined to be the graph which has V as its set of points and two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

G is said to be a SELF COMPLEMENTARY graph, if G is isomorphic to $\bar{G}$.

For example,



Problem 1

Prove that any self complementary graphs has $4n$ or $4n+1$ points.

Sol.

Let $G=(V(G), E(G))$ be a self-complementary graph with p-points.

Since G is self complementary, G is isomorphic to $\bar{G}$.

$$\therefore |E(G)| = |E(\bar{G})|$$

Also, $|E(G)| + |E(\bar{G})| = \binom{p}{2} = \dfrac{p(p-1)}{2}$ (*) RESULT

$$\Rightarrow 2|E(G)| = \frac{p(p-1)}{2}$$

$$\Rightarrow |E(G)| = \frac{p(p-1)}{4}$$ is an integer

further one of p (or) $p-1$ is odd.

Hence p (or) $p-1$ is a multiple of 4.

$\therefore$ p is of the form $4n$ (or) $4n+$

Note:

1. A graph G is complete iff $\overline{G}$ is totally disconnected.

2. $deg_{\overline{G}}(v) = p-1 - deg_G(v)$

3. Upto isomorphism there are exact 4 graphs on 3 vertices.

INDEPENDENT SETS & COVERING:

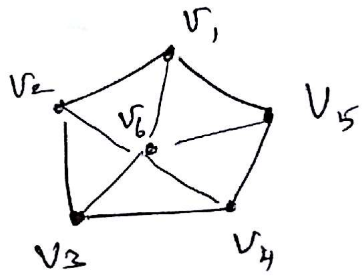
Defn: A COVERING of a graph $G=(V,E)$ is a subset K of V such that every line is incident with a vertex in K. A covering K is called a MINIMUM COVERING if G has no covering K' with $|K'| < |K|$.

A number of vertices in a minimum covering of G is called the COVERING NUMBER of G and is denoted by β.

A subset S of V is called an **INDEPENDENT SET** of G if no two vertices of S are adjacent in G. An independent set S is said to be **MAXIMUM** if G has no independent set S' with $|S'| > |S|$.

The number of vertices in a maximum independent set is called the **INDEPENDENCE NUMBER** of G and is denoted by α.

For example,



Here $\{v_6\}$ is an inde. set.

$\{v_1, v_3\}$ is a max. inde. set

$\{v_1, v_2, v_3, v_4, v_5\}$ is a covering &

$\{v_2, v_6, v_4, v_5\}$ is a min. covering.

Note: $\alpha + \beta = p$.

MATRICES

Defn ① Let $G=(V,E)$ be a (p,q)-graph.

Let $V=\{v_1, v_2 \ldots v_p\}$. The $p\times p$ matrix

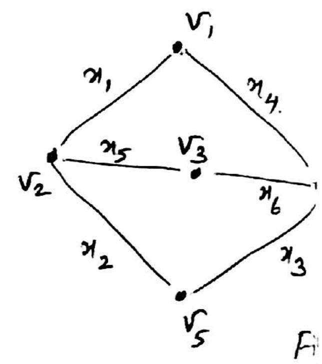
$A=(a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$ – is a

the ADJACENCY MATRIX of the graph G.

For example,

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$



Fi

Defn. ②:

Let $G=(V,E)$ be a (p,q)-graph.

Let $V=\{v_1, v_2 \ldots v_p\}$ & $E=\{x_1, x_2 \ldots x_q\}$.

The $p\times q$ matrix $B=(b_{ij})$, where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } x_j \\ 0 & \text{otherwise} \end{cases}$$ –

called the INCIDENCE MATRIX of the graph

For ex, fig. ①,

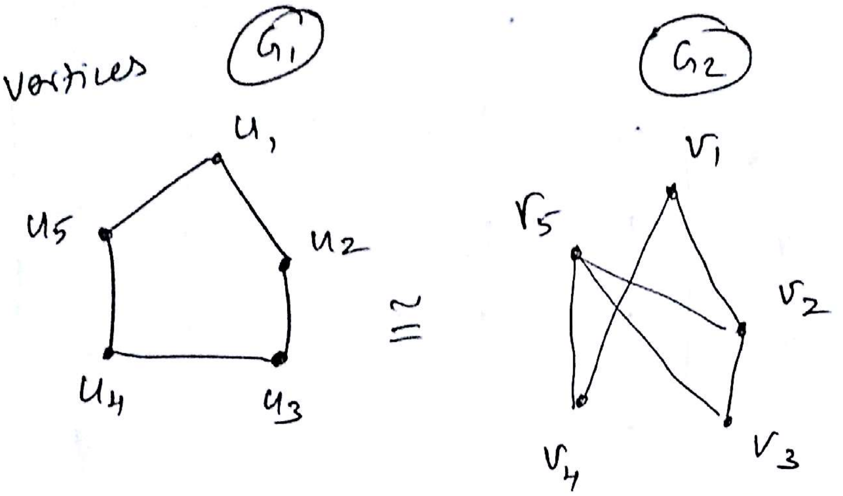
$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(i) Sm (1)

Let G_1 & G_2 be any two graphs with 5 vertices. Draw the graphs but the graph is isomorphic also find the adjacency matrices of the two graphs.

Sol

Let G_1 & G_2 be any two graphs with 5 vertices



We also note, they have got circuits of length 5 which pass through all vertices, namely $u_1-u_2-u_3-u_4-u_5-u_1$ & $v_5-v_3-v_2-v_1-v_4-v_5$.

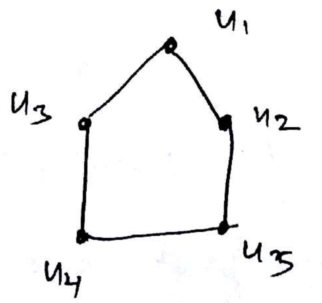
In both the circuits, the degrees of the ordered vertices are 3, 2, 3, 2, 2. The two graphs are isomorphic, as their adjacency matrices are the same.

$$AG_1 \cong \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

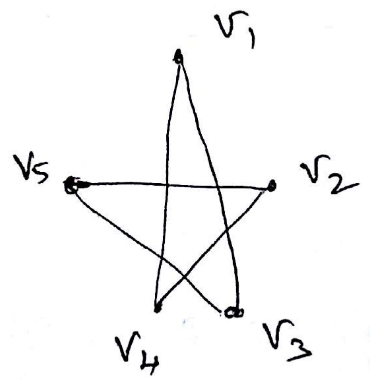
$$AG_2 \cong \begin{array}{c} \\ v_5 \\ v_3 \\ v_2 \\ v_1 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} v_5 & v_3 & v_2 & v_1 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ex ⑳. Give an ex. for the isomorphism graph.

Sol



G



$\overline{G}$

Note 1. The adjacency matrix A is symmetric

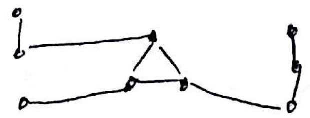
CONNTECTEDNESS

Defn 1. Two points u & v of a graph G are said to be CONNECTED if there exists a u-v path* in G. (* A PATH if all its points are distinct).

Furtherwards, A graph G is said to be CONNECTED if every pair of its points are connected.

Defn. 2
Let G_i denote the induced subgraph of G with vertex set V_i. Clearly the subgraphs $G_1, G_2 \dots G_n$ are connected and are called the COMPONENTS of G.

For ex,



Note 1. A graph which is not connected is said to be disconnected.

2. The union of two graphs is disconnected.

3. A graph G is connected iff it has exactly one component.

4.  gives a disconnected graph with 5 components.

(XX) Example ①. Show that the maximum number of edges in a simple disconnected graph G with n-vertices and k-components is $\frac{(n-k)(n-k+1)}{2}$.

Sol.

Let the number of vertices in the i^{th} component of G be n_i. $(n_i \geq 1)$.

Then, $n_1 + n_2 + \cdots + n_k = n$.

$$\Rightarrow \sum_{i=1}^{k} n_i = n \text{ ——— } (1)$$

Hence, $\sum_{i=1}^{k} (n_i - 1) = n - k$

$$\therefore \left\{\sum_{i=1}^{k} (n_i - 1)\right\}^2 = (n-k)^2$$

$$\Rightarrow \left\{\sum_{i=1}^{k} (n_i - 1)\right\}^2 = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i \neq j} (n_i - 1)(n_j - 1) = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2.$$

[$\because$ the second member in the l. side of (2) is ≥ 0, as each

$$\Rightarrow \sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k \text{ ——— } (3)$$

Now the maximum number of edges in the i^{th} component of $G = \frac{1}{2} n_i (n_i - 1)$ *.

∴ Maximum no. of edges of G

$$= \frac{1}{2} \sum_{i=1}^{K} n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^{K} n_i^2 - \frac{1}{2} n \quad (\because \text{by } ①)$$

$$\leq \frac{1}{2} (n^2 - 2nK + K^2 + 2n - K) - \frac{1}{2} n \quad [\because \text{by } ③)$$

$$\leq \frac{1}{2} (n^2 - 2nK + K^2 + n - K)$$

$$\leq \frac{1}{2} \left((n-K)^2 + (n-K) \right)$$

$$\leq \frac{1}{2} \left[(n-K)(n-K) + (n-K) \right]$$

$$\leq \frac{1}{2} (n-K)(n-K+1).$$

Hence proved.

Ex. ②: Prove that any 2 simple connected graphs with n-vertices all of degree 2 are isomorphic.

Sol

WKT, $\sum_{i=1}^{n} d(v_i) = 2|E|$.

then $|V|$ = no. of vertices 'n',

$|E|$ = no. of edges.

Further, the degree of every vertex is.

$\therefore \sum_{i=1}^{n} 2 = 2|E|$

$\Rightarrow 2[n-1+1] = 2|E|$

$\Rightarrow n = |E|$

$\therefore$ The no. of edges = no. of vertices.

$\therefore$ The graphs are cycle graphs.

Hence they are isomorphic.

Defn. Cut point (or) cut vertice

A cutpoint of a graph G is a point whose removal increases the no. components.

Note ①: A BRIDGE of a graph G is a whose removal increases the no. of comp

②. Clearly if v is a cut-point of a connected graph, G-v is disconne

Defn:

A graph G is said to be <u>n- connected</u> if $\kappa(G) \geq n$ and <u>n-line Connected</u> if $\lambda(G) \geq n$; where $\kappa(G)$ is Connectivity of a graph & $\lambda(G)$ is line Connectivity.

Note ① The Connectivity and line connectivity of a disconnected graph is <u>0</u>

② The connectivity of a Connected graph with a cutpoint is <u>1</u>.

③ The line Connectivity of a Connected graph with a bridge is <u>1</u>

④ A nontrivial graph is 1-Connected iff it is Connected.

⑤ If a graph is <u>n-connected</u> graph then it is <u>n-line Connected</u>

Example ① If G is a k-connected graph then prove that $q \geq \frac{pk}{2}$.

sol. Since G is k-connected, $k \leq \delta$. —— ①

$\because$ for any graph G, $k \leq \lambda \leq \delta$.

where k - vertex connectivity,

λ - edge (or) line Connectivity

δ - the minimum degree.

$$\therefore \; q = \frac{1}{2} \Sigma d(v)$$

$$\geq \frac{1}{2} \cdot p \cdot \delta \quad (\because d(v) \geq \delta \;\; \forall v)$$

$$\geq \frac{pk}{2} \quad (\because \text{by } ①)$$

② Prove that there is no 3-connected graph with 7 edges.

Sol

Suppose G is a 3-connected graph with 7 edges.

G has 7 edges $\Rightarrow$ $p \geq 5$

Now, by above problem $q \geq \frac{pk}{2}$

$$q \geq \frac{p \cdot 3}{2}$$

$$\Rightarrow q \geq \frac{15}{2} \quad (\because p = 5)$$

$\therefore q \geq 8$ which is contradiction.

Hence there is no 3-connected graph with 7 edges.

TREES

Defn: *. A WALK is called a TRAIL if all its lines are distinct.

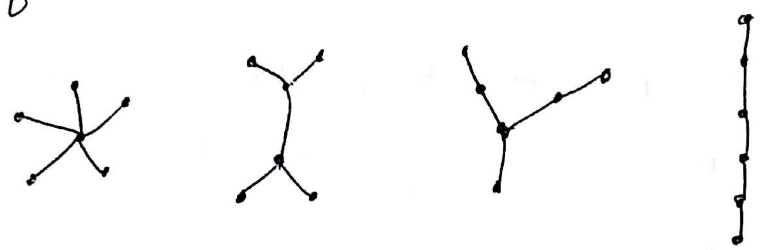
*. A closed walk $v_0, v_1, v_2 \ldots v_n = v_0$ in which $n \geq 3$ and $v_0, v_1 \ldots v_{n-1}$ are distinct is called a CYCLE of length 'n'.

*. A graph that contains no cycles is called an ACYCLIC graph.

*. A connected acyclic graph is called a TREE.

*. Any graph without cycles is also called a FOREST so that the components of a forest are trees.

For ex:,



It is gives all trees with 6 vertices.

Note ①. Every connected graph has a spanning tree

② Every tree with exactly 2 vertices of degree 1 is a path.

③ The origin & terminus of a longest path in a tree have degree is 1

④ Every tree is a bipartite graph.

⑤ Every block of a tree is K_2.

(6) In a tree every edge is a bridge

(7) Any connected (p, q) graph with $p+1 = q$

a tree

(8) The components of a forest are tree

(9) Any acyclic (p, q) graph with $p+1 = q$ is

(10) In a tree every point of degree > 1

cutpoint.

MATCHINGS

Defn:

*. Any set M of independent lines of graph G is called a MATCHING of G.

*. If $uv \in M$, we say that u & v matched under M. We also say that points u & v are M-saturated.

*. A matching 'M' is called a PERF MATCHING, if every point of G is M-satu

Ex. (1) for what values of n does the Complete graph K_n have perfect match

Sol.

Clearly any graph with 'p' odd has perfect matching.

Also the complete graph K_n has perfect matching if n is even.

for example, if $V(K_n) = \{1, 2, 3 \ldots n\}$ then $\{12, 34, \ldots (n-1)n\}$ is a perfect matching of K_n.

Thus, K_n has a perfect matching iff n is even.

Ex. (2): Find the number of perfect matchings in the complete graph K_{2n}.

Sol

Let $V(K_{2n}) = \{v_1, v_2 \ldots v_{2n}\}$.

The vertex v_1 can be saturated in $2n-1$ ways by choosing any line e_1 incident at v_1. without loss of generality, $e_1 = v_1 v_2$, $2n-3$ ways and $e_2 = v_3 v_4$, $2n-5$ ways,

proceeding like this, $e_1, e_2 \ldots e_k$ can be saturated in ~~k~~ $2n-(2k+1)$ ways.

Hence the number of perfect matchings in K_2

$$= 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-1)(2n)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

$$= \frac{(2n)!}{2^n n!}.$$

Note

① A tree has atmost one perfect match

② The number of perfect matchings in the complete bipartite graph $K_{n,n}$ is $\underline{n!}$

③ If G has a perfect matching, then G an even number of vertices.

④ K_6 has a perfect matching.

PLANARITY

Defn:

*. A graph is said to be embedded in surface S when it is drawn on S so no two edges intersect.

* A graph is called PLANAR, if it be drawn on a plane without intersecting edges.

*. A graph is called NON-PLANAR if it is not planar.

*. A graph that is drawn on the without intersecting edges is called a PLANE GRAPH.

Note ① K_1, K_2, K_3, K_4 are planar

② K_5 is non-planar

(x) State and prove the Euler's theorem for planar graph.

Statement: If G is a connected plane graph having V, E and F as the sets of vertices, edges and faces respectively, then $|V| - |E| + |F| = 2$.

Proof

The proof is by induction on the number of edges of G.

Let $|E| = 0$. Since G is connected, it is K_1 so that $|V| = 1$, $|F| = 1$ (the infinite face) and hence $|V| - |E| + |F| = 1 - 0 + 1 = 2$.

Now let G be a graph as in the theorem and suppose that the theorem is true for all connected plane graphs with atmost $|E| - 1$ edges.

If G is a tree, then $|E| = |V| - 1$ & $|F| = 1$ and hence $|V| - |E| + |F| = 2$.

If G is not a tree, let 'x' be an edge contained in some cycle of G.

Then $G' = G - x$ is a connected plane gra[ph]
such that $|V(G')| = |V|$, $|E(G')| = |E$
& $|F(G')| = |F| - 1$

Hence by the induction hypoth[esis]

$|V(G')| - |E(G')| + |F(G')| = 2$ so that

$|V| - (|E| - 1) + (|F| - 1) = 2$

$\Rightarrow |V| - |E| + |F| = 2$.

This completes the induction and th[e]

Ex ② Show that K_5 is non-planar.

sol WKT, $e \leq 3n - 6$ Here $n = 5$,

$\therefore 10 \leq 3(5) - 6$

$\Rightarrow 10 \leq 15 - 6$

$\Rightarrow 10 \leq 9$, which is false.

$\therefore K_5$ is non-planar.

Note ①. Every planar graph is isomorphic plane graph.

② $K_{3,3}$ is non-planar.

③ Any plane graph can be embedded on surface of a sphere.

④ Every subgraph of a planar graph is p[lanar]

⑤ The Petersen graph is non-planar.

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Q.1 Prove that in any connected plane (p,q) graph $(p \geq 3)$ with r faces $q \geq \frac{3r}{2}$ & $q \leq 3p-6$.

Sol

Case (i) Let G be a tree

Then $r=1$, $q=p-1$ & $p \geq 3 \Rightarrow p-1 \geq 2$

$\Rightarrow q \geq 2$

$\Rightarrow q \geq 3/2$

$\Rightarrow q \geq 3r/2$

Hence $q \geq \frac{3r}{2}$ & $q \leq 3p-6$, since $p-1 \leq 3p-6$ (as $p \geq 3$)

Case (ii) Let G have a cycle.

Let f_i, $(i=1,2 \ldots r)$ be the faces of G.

Since each edge lies on the boundary of almost two faces.

$2q \geq \sum_{i=1}^{r}$ (number of edges in the boundary of face f_i).

$\Rightarrow 2q \geq 3r$. ($\because$ each face is bounded by atleast 3 edges).

$\Rightarrow q \geq \frac{3r}{2}$ ——— (1)

By Euler's theorem $|V|-|E|+|F|=2$

ie, $p-q+r=2$

$\Rightarrow r=2+q-p$

Substituting for r in (1), we get

$$q \geq \frac{3}{2}(2+q-p),$$ ~~which~~

$$\Rightarrow q \leq 3p-6.$$

COLOURABILITY

Defn:

*. An assignment of colours to the vertices of a graph so that no two adjacent vertices get the same colour is called COLOURING of the graph.

*. For each colour, the set of all points which get that colour is independent and is called a colour class.

*. The chromatic number $\chi(G)$ of a graph G is the minimum no. of colours needed to colour G.

*. A graph G is called, n-colourable, if $\chi(G) \leq n$.

Note: ① Every planar graph is 5-colourable (5. colour theorem)

② Every planar graph is 4-colourable (4-colour theorem).

③ Let $f(G, \lambda)$ denote the number of different colourings of G from λ colours.

for ex, $f(K_1, \lambda) = \lambda$ & $f(K_2, \lambda) = \lambda^2$; this is called chromatic polynomial.

(X) Prob.)

Prove that $\lambda^4 - 3\lambda^3 + 3\lambda^2$ cannot be the chromatic polynomial of any gr

Sol.

Suppose there exists a graph G such

$f(G,\lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2$.

$\therefore$ The number of points in G is 4.

Also the number of lines in G is 3.

($\because$ if G is a (p,q)-graph, the coefficient λ^{p-1} in $f(G,\lambda)$ is $-q$).

Case (i) Suppose G is connected.

Since $q = 3 = p-1$; G is a tree

Hence, $f(G,\lambda) = \lambda(\lambda-1)^3$

[$\because$ If G is a tree, with n-point $n \geq 2$, then $f(G,\lambda) = \lambda(\lambda-1)^{n-1}$]

$\Rightarrow f(G,\lambda) = \lambda(\lambda^3 - 3\lambda^2 + 3\lambda - 1)$

$= \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$.

which is a contradiction.

Case (ii) Suppose G is NOT connecte

Then $G = K_3 \cup K_1$

$\therefore f(G,\lambda) = f(K_3,\lambda) \cdot f(K_1,\lambda)$

$= \left[\lambda\,(\lambda-1)^{3-1}\right] \cdot \left[\lambda \cdot (\lambda-1)^{1-1}\right]$

$= \lambda\,(\lambda-1)^2 \cdot \lambda$

$= \lambda\,(\lambda-1) \cdot (\lambda-1) \cdot \lambda$

$= \lambda^2\,(\lambda^2 - 2\lambda + 1)$

$= \lambda^4 - 2\lambda^3 + \lambda^2$, which is again a

Contradiction.

Hence the result is proved.

Note ① The chromatic no. of K_p is p

② The chromatic no. of any totally disconnected graph is 1

③ The chromatic no. of any nontrivial tree is 2

④ Any connected bipartite graph is uniquely 2- colourable

⑤ If G is a graph with p points its chromatic polynomial has degree p.

—— xxxxxx ——

Unit 5 Graphs

Questions	OPT 1	OPT 2	OPT3	OPT 4	ANSWERS
A graph witth p-points and q-lines is called a ___ graph.	(p,q)	(u,v)	(1,1)	(0,0)	(p,q)
A graph in which any two distinct points are adjacent is called a ____ graph	bipartite	complete	petersen	loop	complete
A graph in which any two distinct points are____ is called a complete graph	(u,v)	(p,q)	incident	adjacent	adjacent
A graph in which any ____ distinct points are adjacent is called a complete graph	0	1	2	3	2
The sum of the degrees of all the vertices in a graph G is equal to ___ the	twice	triple	half	5 times	twice
The number of vertices of odd degree in a graph is always ___	odd	even	either odd or even	neither odd nor even	even
Any self complementary graphs has 4n or ____ points	4n-1	4n+1	4n-2	4n+2	4n+1
A graph G is said to be a ___ graph, if G is isomorphic to G	self compleme ntary	complete	petersen	bipartite	self compleme ntary
A number of vertices in a minimum covering of G is called the ___ of G	covering	minimim covering	maximum covering	covering number	covering number
The number of vertices in a maximum independent set is called the __ of G.	independe nt set	independen ce number	maximum covering	minimum covering	independe nce number
The adjacency matirx 'A' is ___	transitive	reflexive	symmetric	incident	symmetric
A ___ if all its points are distinct	walk	path	trail	closed	path
Two points 'u' and 'v' of a graph G are said to be ____ if there exists a u-v	connected	disconnecte d	components	isomorphic	connected
The union of two graphs is___	connected	disconnecte d	components	isomorphic	disconnec ted
A graph G is connected iff it has exactly ___component	0	1	2	3	1
A graph G is _____ iff it has exactly one component	connected	disconnecte d	components	isomorphic	connected

Any 2 simple connected graphs with n-vertices all of degree 2 are ___	connected	disconnected	components	isomorphic	isomorphic
A ___ of a graph G is a point whose removal increases the number of components	cut point	cut edge	bridge	block	cut point
A ___ of a graph G is a line whose removal increases the number of components	cut point	cut edge	bridge	block	bridge
If 'v' is a cutpoint of a connected graph then G-v is _____	connected	disconnected	components	isomorphic	disconnected
The connectivity and line connectivity of a disconnected graph is __	3	2	1	0	0
The connectivity of a connected graph with a cutpoint is ___	3	2	1	0	1
The line connectivity of a conneced graph with a bridge is ___	3	2	1	0	1
A nontrivial graph is 1-connected iff it is _____	connected	disconnected	components	isomorphic	connected
A nontrivial graph is ___ iff it is connected.	2n-connected	n-connected	2-connected	1-connected	1-connected
If a graph is ___ graph then it is n-line connected.	2n-connected	n-connected	2-connected	1-connected	n-connected
There is no 3-connected graph has ___ edges	3	5	7	9	7
A ___ is called a trial if all its lines are distinct.	walk	path	trail	closed	walk
A graph that contains no cycles is called ___ graph	self complementary	complete	cycle	acyclic	acyclic
A connected acyclic graph is called a __	self complementary	complete	cycle	tree	tree
Any graph without cycles is also called a __	forest	complete	cycle	tree	forest
Every conneced graph has a _____	forest	spanning tree	cycle	tree	spanning tree
Every tree with exactly 2 vertices of degree 1 is a ___	walk	path	trail	closed	path

The origin and terminus of a longest path in a tree have degree is ___	0	1	2	3	1
Every tree is a ___ graph	self compleme ntary	complete	petersen	bipartite	bipartite
In a tree every edge is a ___	cut point	cut edge	bridge	block	bridge
Any connected (p,q) graph wih p+1=q is a ___	forest	spanning tree	cycle	tree	tree
The componets of a forest are ___	forest	spanning tree	cycle	trees	trees
A tree has atmost ___ perfect matching	4	3	2	1	1
A graph is called ___ if it can be drawn on a plane without intersecting edges.	matching	covering	colouring	planar	planar